Linear constraints on Face numbers of Polytopes

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Introduction

Polytopes show up naturally in many fields of mathematics, as pure geometric objects as well as in applied areas such as linear programming. Various aspects of polytopes have been studied in the past, including their metric properties, symmetry or—as in this thesis—combinatorial structure.

By the combinatorial structure of a polytope we mean the set of inclusion relations that are satisfied by the faces, in other words its face poset. An important invariant is the f-vector, whose entries count the numbers of faces of each dimension. Similarly, the flag vector of a polytope counts the numbers of incidences between faces of given dimensions, that is, chains of given types in the face poset.

A central question in this area is the characterisation of the f-, and more generally the flag vectors of polytopes. For the 3-dimensional case this problem was solved by Steinitz in 1906, but for polytopes of arbitrary dimension it is still open.

Many restrictions, mostly linear constraints, have been established for the entries of these vectors. The agenda for this thesis is to try to understand the interplay between these relations, present conclusions that can be drawn from them, obtain new examples with interesting, non-trivial properties, and search for further necessary conditions that might hold for f- and flag vectors of polytopes.

Overview and main results

The thesis starts with a short summary of basic definitions and notation in Chapter 1. After an account of the most important examples and constructions of polytopes, we mention the main tools and results from the literature, both geometric and combinatorial, that are used throughout. We introduce and give some examples of combinatorial techniques such as the $\mathtt{cd}$-index and convolution of linear forms.

Chapter 2 is devoted to a rather elementary approach to understanding the combinatorial structure of polytopes, namely, adding vertices successively. The most basic instance of this is stacking, which we generalise to some extent to the concept of pseudostacking. In both cases, the combinatorial structure of the resulting polytopes is discussed, as well as the effect on the f- and flag vector.

It turns out that major restrictions on the input are necessary for pseudostacking if one hopes to get to reasonably useful statements about the constructed polytopes. We therefore mainly restrict the discussion to some special cases that are used later. After presenting the general results, we conclude the chapter with the explicit statements in the 4-dimensional case.

In Chapter 3 we first discuss f- and flag vectors generally, without referring to any particular dimension. We review two bases of polytopes: the one introduced by Bayer and Billera [7], and the one given by Kalai [36]. The latter will be used again later, whereas for the first, a close examination of the original proof yields the following result:

Proposition 3.1.3. *The flag vectors of the basis polytopes of Bayer and Billera span the affine integer lattice* $\mathbb{Z}^{F_d-1}$.

We then turn our attention to linear inequalities for the flag vectors. We give a recipe to visualise and aid in analysing properties of f- and flag vectors, generalising a method used by Ziegler [59]. The central question is whether a given inequality defines a facet of the convex hull of the set of all flag vectors of polytopes, or if stronger restrictions can be found.

As an example of the presented method we apply it to the 3-dimensional case and illustrate the connection to Steinitz' characterisation of f-vectors of 3-polytopes [54]. Along the same lines we also give a characterisation of the f-vectors of centrally-symmetric 3-polytopes:

Theorem 3.3.6. *An integer vector* (f_0, f_1, f_2) *is the* f*-vector of a centrally-symmetric* 3*-polytope if and only if all entries are even, and the linear relations* $f_0 - f_1 + f_2 = 2$, $f_2 \leq 2f_0 - 4$, $f_0 \leq 2f_2 - 4$, *and* $f_0 + f_2 \geq 14$ *hold.*

The flag vectors of 4-polytopes are the focus of Chapter 4. They are contained in the cone defined by all currently known linear inequalities, given by Bayer [6]. Using the visualisation recipe from the previous chapter we obtain a new view of the cone and discuss examples of polytopes with extremal properties. Furthermore, we construct new families of such extremal polytopes which show that the cone is in some sense a rather close description of the convex hull of all flag vectors of 4-polytopes:

Theorem 4.1.6. *All of the six inequalities for flag vectors given by Bayer are facet-defining or asymptotically facet-defining for the convex hull of flag vectors of* 4*-polytopes.*

The first of the mentioned families leads to the following result, obtained in joint work with Andreas Paffenholz [43]:

Theorem 4.2.2. *For every integer* $k \geq 1$ *there is an elementary* 2*-simple,* 2*-simplicial* 4*-polytope on* $4k + 1$ *vertices.*

Another construction provides different interesting examples:

Theorem 4.3.7. *There exists a family of* 2*-simplicial polytopes that are not center-boolean and have asymptotically fewer vertices than facets.*

The given examples lead to new inequalities that might be valid for all flag vectors of 4-polytopes. Stated in terms of fatness and complexity, the most interesting of them reads as follows:

Conjecture 4.1.7. *Fatness* F *and complexity* C *of every* 4*-polytope satisfy* $4F - C \leq 20$*.*

In particular, this would imply that fatness of 4-polytopes is bounded by 9.

In Chapter 5 we return to the arbitrary-dimensional case, this time taking certain selected inequalities into account. The idea for the special structure of these inequalities comes from Braden [16], whose sketch of the proof of the following result we work out:

Proposition 5.2.8. *For dimensions* $d \leq 6$ *all non-trivial Braden sequences are facet-defining or asymptotically facet-defining for the convex hull of flag vectors of* d*-polytopes.*

Additionally, we prove that in high dimensions there are a large number of inequalities arising from Braden sequences that can at most be asymptotically facet-defining. We can show for some special cases that this property is in fact satisfied:

Propositions 5.3.4/5.3.5. *For* $k \geq 2$ *and dimensions* $3k - 1$*, respectively* $3k$*, the second-to-last, respectively third-to-last Braden sequence is asymptotically facet-defining for the convex hull of polytope flag vectors.*

In Chapter 6 we discuss f-vectors. The visualisation in the 4-dimensional case according to our general method coincides with Ziegler's original approach in [59]. We proceed with the analogous discussion for the 5-dimensional case which yields a 3-dimensional view and many interesting observations concerning 5-polytopes. In particular, it seems that the known linear inequalities are rather weak and there might be either stronger restrictions or more extremal examples of polytopes. In this respect, the question of offside polytopes might be interesting.

Another apparent problem concerns f-vector shapes. One instance of this is the question whether the f-vectors of polytopes are unimodal in general.

We investigate this together with three other related properties of f-vector shapes: convexity, logarithmic convexity and Bárány's property. Different results are known, depending on the dimension. We show that convexity does not hold in general for polytopes of each dimension at least 5, whereas unimodality holds for 5-polytopes. For 6-polytopes we can establish Bárány's property. This leaves logarithmic convexity in dimension 7 as the major open problem, and we conclude by discussing examples that might be interesting in this respect.

The last part of Chapter 6 is devoted to centrally-symmetric polytopes. Kalai [37] posed three conjectures, the first of them—the 3^d-conjecture—stating that every centrally-symmetric polytope has at least 3^d non-empty faces. We show this for $d = 4$, presenting a slightly different proof than in the joint paper with Raman Sanyal and Günter M. Ziegler [46].

Theorem 6.3.16. *The 3^d-conjecture is true for 4-dimensional polytopes.*

It turns out that even the stronger Conjecture B of Kalai is true in dimension 4, although his Conjecture C fails. Additionally, both stronger conjectures fail to hold for dimensions 5 and higher. Counterexamples are surprisingly easy to describe and some of them, the Hansen polytopes, appear to be an interesting topic for further research. In the effort to obtain a proof of the 3^d-conjecture in all dimensions we also establish the following partial result:

Proposition 6.3.19. *For every $d \leq 7$, every centrally-symmetric d-polytope with $2d + 2$ vertices has at least 3^d non-empty faces.*

One can complement the discussion by characterising the cases where the bound is attained with equality, which is done for both of the above results.

Finally, Chapter 7 is concerned with shellings. We give a neat proof of the fact that there are few very small 2-simple, 2-simplicial 4-polytopes:

Theorem 7.2.13. *Up to combinatorial equivalence, there is only one non-trivial 2-simple, 2-simplicial 4-polytope with at most 9 vertices, namely the polytope P_9 from the family described in Theorem 4.2.2*

The last part of the chapter describes an experimental backtracking algorithm that uses shellings to search for polytopes with desired properties. We derive some simple criteria for cutting the search tree and also remark on the implementation and the used data structures. The implemented program has been used to search for cubical polytopes, spheres with $g_2 < 0$, and 2-simple, 2-simplicial 4-polytopes. In the first two cases, no suitable candidates were found, leading to the conjecture that there exists no cubical 4-polytope with 34 vertices and the conclusion that if there exist 2-simple, 2-simplicial

3-spheres with negative g_2 their number of vertices cannot be too small. On the other hand, the program found two candidates for a 2-simple, 2-simplicial 4-polytope on 12 vertices. However, these are not known to be realisable as polytopes. Their combinatorial data is given on page 171.

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Chapter 1

Preliminaries

This chapter serves mainly as an overview over all definitions, examples and general known results, as well as a place to introduce the notation that we will use throughout this thesis.

There will be no proofs in this chapter. Proofs for the important results can be found in the comprehensive books by Ziegler [58], Grünbaum [29], Brøndsted [17] and Stanley [53]. These books also contain a more thorough introduction to most of the examples.

We assume the reader to be familiar with the concepts of linear algebra, such as affine and linear subspaces, linear combinations, dimension etc.

1.1 Polytopes

We denote the *convex hull* of a set $M \subseteq \mathbb{R}^d$ by

$$\mathsf{conv}\, M \;:=\; \left\{ \sum_{i=1}^{n} \lambda_i \mathbf{v}_i \;\middle|\; n \in \mathbb{N}, \mathbf{v}_1, \dots, \mathbf{v}_n \in M, \lambda_1, \dots, \lambda_n \geq 0, \sum_{i=1}^{n} \lambda_i = 1 \right\}$$

and the *affine hull* of $M \subseteq \mathbb{R}^d$ by

$$\mathsf{aff}\, M \;:=\; \left\{ \sum_{i=1}^{n} \lambda_i \mathbf{v}_i \;\middle|\; n \in \mathbb{N}, \mathbf{v}_1, \dots, \mathbf{v}_n \in M, \lambda_1, \dots, \lambda_n \in \mathbb{R}, \sum_{i=1}^{n} \lambda_i = 1 \right\}.$$

The basic definition is that of a convex polytope.

Definition 1.1.1 (Convex polytope)**.** A *(convex) polytope* is the convex hull $\mathsf{conv}\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ of finitely many points in $\mathbb{R}^d$.

We also allow $n = 0$ points, so in particular $\mathsf{conv}\, \emptyset = \emptyset$ is a convex polytope. It is possible to define non-convex polytopes, which we will not consider in this thesis. Therefore the adjective “convex” will usually be dropped.

Equivalently, polytopes can be defined as the intersection of linear halfspaces, provided it is bounded. This is a rather non-trivial statement (see Ziegler [58, Chapter 1]) and has a lot to do with the concept of polarity or duality.

Definition 1.1.2 (Polytope dimension)**.** The *dimension of a polytope* P is the dimension of its affine hull: $\operatorname{dim} P := \operatorname{dim} \operatorname{aff} P$. If $\operatorname{dim} P = d$, we also call P a *d-polytope.*

If the dimension of a polytope $P \subset \mathbb{R}^d$ coincides with the dimension of the ambient space, that is, $\operatorname{dim} P = d$, we call P *full-dimensional.* For questions concerning the combinatorial structure of polytopes, we can usually assume that the polytopes are full-dimensional.

Definition 1.1.3 (Faces of polytopes)**.** A set $F \subseteq P \subset \mathbb{R}^d$ is a *face* of P if there exists a linear functional that is maximised among all points in P exactly on F; that is, if there exists some $\mathbf{a} \in \mathbb{R}^d$ and $b \in \mathbb{R}$ such that

$$\mathbf{a}^\top \mathbf{x} \le b \quad \forall \mathbf{x} \in P\,, \quad \mathbf{a}^\top \mathbf{x} < b \quad \forall \mathbf{x} \in P \setminus F \quad \text{and} \quad \mathbf{a}^\top \mathbf{x} = b \quad \forall \mathbf{x} \in F.$$

Note that every polytope P has the faces $\emptyset$ and P, which are defined by the inequalities $\mathbf{0}^\top \mathbf{x} \le -1$ and $\mathbf{0}^\top \mathbf{x} \le 1$ for instance.

The *dimension of a face* F is $\operatorname{dim} F = \operatorname{dim} \operatorname{aff} F$, if $F \neq \emptyset$. Additionally, we define $\operatorname{dim} \emptyset := -1$. Faces of dimension 0, 1, $\operatorname{dim} P - 2$ and $\operatorname{dim} P - 1$ are called *vertices*, *edges*, *ridges* and *facets*, respectively, of P. *Proper* faces of P are all faces $F \neq P$. We denote by $\operatorname{vert} P$ the set of all vertices of P.

It is easy to show that a face F of a polytope P is the convex hull of all vertices of P that are contained in F. With this we get the following basic property.

Proposition 1.1.4. Every polytope has finitely many faces.

If P is a polytope and F a face of P then F is again a polytope and every face of F is also a face of P. If F' is another face of P then $F \cap F'$ is also a face of P. □

Definition 1.1.5 (Simple and simplicial)**.** A d-polytope P is *simple* if every vertex of P is contained in exactly d facets. P is *simplicial* if every facet of P contains exactly d vertices.

There are many equivalent definitions of simple and simplicial, for example in terms of duality or in terms of the face lattice. See [58, Proposition 2.16] for an exhaustive treatment of simplicity and simpliciality.

Definition 1.1.6 (Affine and projective equivalence)**.** Polytopes $P_1 \subset \mathbb{R}^{d_1}$ and $P_2 \subset \mathbb{R}^{d_2}$ are *affinely equivalent* if there is an affine map $f : \mathbb{R}^{d_1} \to \mathbb{R}^{d_2}$ that maps P_1 onto P_2 bijectively.

P_1 and P_2 are *projectively equivalent* if they are affinely equivalent to full-dimensional d-polytopes P_1' and P_2', respectively, and there exists a projective transformation of $\mathbb{R}^d$ that maps P_1' bijectively onto P_2'.

For basic notions of projective transformations see [58, Section 2.6], for instance, and the references given there.

Both these relations are indeed equivalence relations and affine equivalence implies projective equivalence, as every affine map implies the existence of a suitable projective map.

Example 1.1.7. Here are some examples of polytopes that frequently serve as "building blocks" to obtain interesting examples with special properties.

(1) A d-dimensional *simplex* is the convex hull of any $d+1$ affinely independent points. Since $d+1$ affinely independent points define an affine map uniquely, all d-dimensional simplices are affinely equivalent.
The standard d-simplex is a subset of $\mathbb{R}^{d+1}$, defined by

$$\Delta_d \;:=\; \mathsf{conv}\{\mathbf{e}_i \mid 1 \le i \le d+1\} \;=\; \Big\{\mathbf{x} \in \mathbb{R}^{d+1} \;\Big|\; x_i \ge 0\,\forall i, \sum_{i=1}^{d+1} x_i = 1\Big\}$$

where $\mathbf{e}_i$ is the i-th unit vector. A full-dimensional standardised version of the simplex would be $\mathsf{conv}\{\mathbf{0}, \mathbf{e}_1, \ldots, \mathbf{e}_d\} \subset \mathbb{R}^d$.

(2) A 2-dimensional polytope with n vertices is the regular *n-gon*

$$\pentagon_n \;:=\; \mathsf{conv}\left\{\left(\cos\tfrac{2k\pi}{n}, \sin\tfrac{2k\pi}{n}\right)^{\top} \;\middle|\; 1 \le k \le n\right\}.$$

(3) The d-dimensional standard *cube* is given by

$$C_d \;:=\; \mathsf{conv}(\{-1,+1\}^d) \;=\; \{\mathbf{x} \in \mathbb{R}^d \mid -1 \le x_i \le 1\,\forall i\} \;=\; [-1,1]^d.$$

(4) The d-dimensional standard *crosspolytope* is

$$C_d{}^{\Delta} \;:=\; \{\mathbf{x} \in \mathbb{R}^d \mid |x_1| + \ldots + |x_d| \le 1\} \;=\; \mathsf{conv}\{\pm\mathbf{e}_i \mid 1 \le i \le d\}.$$

(5) Given real numbers $t_1 < t_2 < \ldots < t_n$, the d-dimensional *cyclic polytope* on n vertices, $n > d \geq 2$ can be defined as

$$\mathcal{C}_d(n) \;:=\; \mathsf{conv}\{(t_i, t_i^2, t_i^3, \ldots, t_i^d)^\top \mid 1 \leq i \leq n\}.$$

The facial structure of $\mathcal{C}_d(n)$ turns out to be independent of the choice of the parameters $t_1, \ldots, t_n$, so combinatorially $\mathcal{C}_d(n)$ is well-defined. For details see [58, Chapter 0].

(6) A generalisation of the standard simplex is the *d-dimensional hypersimplex*

$$\Delta_d(k) \;:=\; \mathsf{conv}\left\{\mathbf{x} \in \mathbb{R}^{d+1} \;\middle|\; 0 \leq x_i \leq 1 \;\forall i,\; \sum_{i=1}^{d+1} x_i = k\right\},$$

defined for $0 \leq k \leq d+1$. We have $\Delta_d(0) = \{0\}$ and $\Delta_d(1) = \Delta_d$, and it is easy to see that $\Delta_d(k)$ is affinely equivalent to $\Delta_d(d+1-k)$.

By applying "recycling operations" [58, p. 9] it is possible to produce more examples. Some of these constructions are standard and sometimes they are combined to obtain new polytopes with interesting properties.

Definition 1.1.8 (Standard polytope constructions)**.** Let P and Q be polytopes with vertices $\mathbf{v}_1, \ldots, \mathbf{v}_n$ and $\mathbf{w}_1, \ldots, \mathbf{w}_m$, respectively. Without loss of generality assume that P and Q are full-dimensional with the origin in the interior.

The *product* of P and Q is defined by

$$P \times Q \;:=\; \mathsf{conv}\left\{\begin{pmatrix} \mathbf{v}_i \\ \mathbf{w}_j \end{pmatrix} \;\middle|\; 1 \leq i \leq n, 1 \leq j \leq m\right\}.$$

The *direct sum* of P and Q is the polytope

$$P \oplus Q \;:=\; \mathsf{conv}\left\{\begin{pmatrix} \mathbf{v}_1 \\ \mathbf{0} \end{pmatrix}, \ldots, \begin{pmatrix} \mathbf{v}_n \\ \mathbf{0} \end{pmatrix}, \begin{pmatrix} \mathbf{0} \\ \mathbf{w}_1 \end{pmatrix}, \ldots, \begin{pmatrix} \mathbf{0} \\ \mathbf{w}_k \end{pmatrix}\right\}.$$

The *(free) join* of P and Q is

$$P * Q \;:=\; \mathsf{conv}\left\{\begin{pmatrix} \mathbf{v}_1 \\ -1 \\ \mathbf{0} \end{pmatrix}, \ldots, \begin{pmatrix} \mathbf{v}_n \\ -1 \\ \mathbf{0} \end{pmatrix}, \begin{pmatrix} \mathbf{0} \\ 1 \\ \mathbf{w}_1 \end{pmatrix}, \ldots, \begin{pmatrix} \mathbf{0} \\ 1 \\ \mathbf{w}_m \end{pmatrix}\right\}.$$

In the special case $\mathsf{dim}\, Q = 1$ the product is called a *prism over P*,

$$\mathsf{prism}\, P \;:=\; P \times [-1, 1],$$

while the direct sum is called a *bipyramid over* P:

$$\mathsf{bipyr}\, P \ := \ P \oplus [-1,1].$$

If $\mathsf{dim}\, Q = 0$, the join specialises to a *pyramid over* P, denoted by

$$\mathsf{pyr}\, P \ := \ P * \{0\}.$$

Defined like this, the pyramid is not full-dimensional, but it is affinely equivalent to

$$\mathsf{conv}\left\{ \begin{pmatrix} \mathbf{v}_1 \\ -1 \end{pmatrix}, \dots, \begin{pmatrix} \mathbf{v}_n \\ -1 \end{pmatrix}, \begin{pmatrix} \mathbf{0} \\ 1 \end{pmatrix} \right\}.$$

The point $(\mathbf{0}^\top, 1)$ is the *apex* of the pyramid.

Alternative notations also seen in the literature are $\otimes$ for the product and $\veebar$ for the join of polytopes. The product of two polytopes coincides with the usual set-theoretic product of two sets.

It is quite obvious that $\mathsf{dim}(P \times Q) = \mathsf{dim}(P \oplus Q) = \mathsf{dim}\, P + \mathsf{dim}\, Q$ and $\mathsf{dim}(P * Q) = \mathsf{dim}\, P + \mathsf{dim}\, Q + 1$. Accordingly, $\mathsf{dim}\,\mathsf{prism}\, P = \mathsf{dim}\,\mathsf{bipyr}\, P = \mathsf{dim}\,\mathsf{pyr}\, P = \mathsf{dim}\, P + 1$.

The next construction is due to Danzer, described by Eckhoff [22], and also by Ziegler [58, Example 8.41].

Definition 1.1.9 (Connected sum)**.** Let $P \subset \mathbb{R}^d$ be a simplicial and $Q \subset \mathbb{R}^d$ a simple d-polytope. Choose a facet F of P and a vertex $\mathbf{v}$ of Q and let $\mathbf{a} \in \mathbb{R}^d$ define an inequality that "cuts off" $\mathbf{v}$ from Q, that is

$$\mathbf{a}^\top \mathbf{v} < b \quad \text{and} \quad \mathbf{a}^\top \mathbf{x} > b \ \text{ for all other vertices } \mathbf{x} \in Q.$$

Now let $\widehat{Q}$ be the "cut" polytope

$$\widehat{Q} \ := \ Q \cap \{\mathbf{x} \in \mathbb{R}^d \mid \mathbf{a}^\top \mathbf{x} \geq b\}$$

and $\widehat{F}$ the facet that originated from the cut:

$$\widehat{F} \ := \ \widehat{Q} \cap \{\mathbf{x} \in \mathbb{R}^d \mid \mathbf{a}^\top \mathbf{x} = b\}.$$

If f is a projective transformation on $\mathbb{R}^d$ such that $f(\widehat{F}) = F = P \cap f(\widehat{Q})$ and the set

$$P \# Q \ := \ P \cup f(\widehat{Q})$$

is convex then $P \# Q$ is again a polytope, a *connected sum of* P *and* Q.

Due to the simpliciality of P and the simplicity of Q such projective transformations always exist, so a connected sum can always be obtained under these prerequisites.

The connected sum can be viewed as a generalisation of the stacking operation, which is described in Chapter 2 together with the slightly less standard procedure of "pseudostacking".

The fundamental observation from Proposition 1.1.4 that every polytope has finitely many faces suggests the following important definition.

Definition 1.1.10 (f-vector)**.** Let P be a d-polytope. Denote by $f_i(P)$ the number of i-dimensional faces of P. The d-tuple

$$f(P) \ := \ (f_0(P), f_1(P), \ldots, f_{d-1}(P)) \ \in \ \mathbb{Z}^d$$

is called the *f-vector* of P.

We denote by $\mathcal{F}_d$ the set of all f-vectors of d-polytopes:

$$\mathcal{F}_d \ := \ \{f(P) \mid P \text{ is a } d\text{-polytope}\} \ \subset \ \mathbb{Z}^d$$

It is often convenient to extend the f-vector by the two entries $f_{-1}(P) = f_d(P) = 1$, which correspond to the empty face and the full-dimensional face, respectively.

The last definition immediately leads to the natural problem to explicitly determine $\mathcal{F}_d$: *Give necessary and sufficient conditions for an integer d-tuple to be the f-vector of a d-polytope.*

This problem is trivial for $d = 0$, obvious for $d = 1$, easy for $d = 2$ and the first interesting case $d = 3$ was solved by Steinitz [54] in 1906. In Section 3.3 we give a sketch of the proof. For $d \geq 4$ the problem remains open and is a major motivation for the topics covered in this thesis.

A complete characterisation of f-vectors of simplicial polytopes is given by the g-Theorem, conjectured by McMullen [41] and proved by Stanley [50] and Billera & Lee [13]. McMullen [42] gave another proof for the necessity part using different techniques which is especially interesting for the proof of the nonnegativity of the g-vector (cf. Section 1.4).

Example 1.1.11. The entries of the f-vector of a d-simplex are binomial coefficients:

$$f_j(\Delta_d) \ = \ \binom{d+1}{j+1} \qquad \text{for } -1 \leq j \leq d$$

since every choice of $j+1$ vertices spans a j-dimensional face of Δ_d.

Example 1.1.12. If P is a d-polytope with f-vector $f(P)$ then the k-faces of a pyramid over P are exactly the k-faces of P, plus all joins of $(k-1)$-faces of P with the apex of the pyramid. Hence the f-vector of $\mathsf{pyr}\,P$ is

$$\begin{aligned} f(\mathsf{pyr}\,P) &= (f_0(P)+1, f_1(P)+f_0(P), \ldots, f_{d-1}(P)+f_{d-2}(P), 1+f_{d-1}(P)) \\ &= (f(P),1)+(1,f(P)) \end{aligned}$$

To understand the structures of polytopes, visualisations are doubtless helpful. While it is easy to make pictures of up-to-3-dimensional polytopes, there is no obvious way to do this for higher dimensions. We will regularly use the following technique to at least visualise 4-dimensional polytopes—the definition works for arbitrary dimension, however.

Definition 1.1.13 (Polytopal complex)**.** A *(finite) polytopal complex* $\mathcal{C}$ is a (finite) set of polytopes in $\mathbb{R}^d$ such that for all $P, Q \in \mathcal{C}$ their intersection is again a polytope $P \cap Q \in \mathcal{C}$. The *dimension* of a polytopal complex is the maximal dimension of one of its members.

Clearly, the set of all proper faces of a polytope is a finite polytopal complex. By projecting all the proper faces of a d-polytope in a suitable way, one obtains a finite polytopal complex of dimension $d-1$ that captures the whole combinatorial information of the polytope.

Definition 1.1.14 (Schlegel diagram)**.** Let $P \subset \mathbb{R}^d$ be a d-polytope and F a facet of P. Furthermore, let p be a projective transformation and $\pi_{p(F)}$ the orthogonal projection of $\mathbb{R}^d$ onto the hyperplane $\mathsf{aff}(p(F))$. If p is chosen in such a way that the whole polytope P is mapped into the image of F, more precisely,

$$(\pi_{p(F)} \circ p)(P \setminus F) \;=\; \mathsf{int}\,\big(p(F)\big),$$

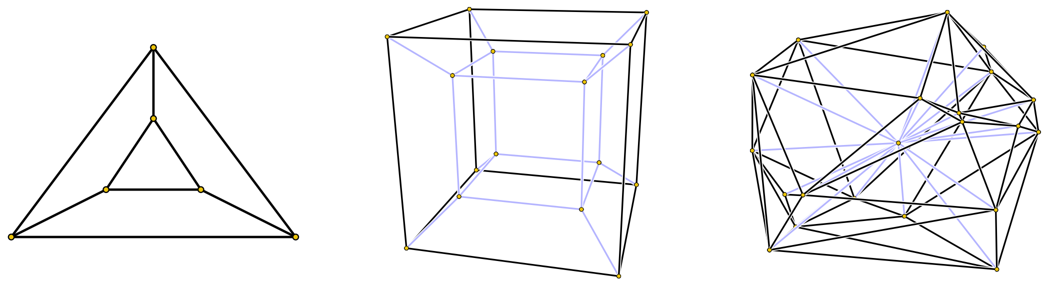
then the polytopal complex

$$\big\{(\pi_{p(F)} \circ p)(G) \mid G \text{ proper face of } P\big\}$$

is called a *Schlegel diagram* of P.

Figure 1.1 shows examples of Schlegel diagrams of some 3- and 4-dimensional polytopes.

An important issue in polytope theory is duality. In geometric terms this concept is described by the polar polytope, although we will usually refer to the more general, combinatorial definition introduced in Section 1.2.



(a) ...a prism Δ_2 (b) ...a 4-cube C_4 (c) ...some pyramid

Figure 1.1: Schlegel diagrams of...

Definition 1.1.15 (Polar polytope)**.** Let $P \subset \mathbb{R}^d$ be a d-polytope with vertices $\mathbf{v}_1, \ldots, \mathbf{v}_n \in \mathbb{R}^d$. Suppose $\mathbf{0} \in \mathbb{R}^d$ is contained in the interior of P. Then the *polar polytope of* P is defined by

$$P^\Delta := \{\mathbf{x} \in \mathbb{R}^d \mid \mathbf{x}^\top \mathbf{v}_i \leq 1 \text{ for } 1 \leq i \leq n\}.$$

Clearly, the restriction $\mathbf{0} \in \mathbb{R}^d$ is not a severe one, since we can translate every polytope such that this is satisfied. Also, it is immediate that the polar polytope again has $\mathbf{0}$ in its interior.

Proposition 1.1.16. For all polytopes P and Q with $\mathbf{0}$ in their interior we have:

(i) $\dim P = \dim P^\Delta$,

(ii) $P^{\Delta^\Delta} = P$,

(iii) $P \times Q = (P^\Delta \oplus Q^\Delta)^\Delta$,

(iv) $P * Q = (P^\Delta * Q^\Delta)^\Delta$.

The last topic in this section concerns face figures, a concept dual to faces as will become clear in the next section.

Definition 1.1.17 (Vertex figure)**.** Let $P \subset \mathbb{R}^d$ be a d-polytope and $\mathbf{v}$ a vertex of P. Let further $\mathbf{a} \in \mathbb{R}^d$ define a hyperplane that separates $\mathbf{v}$ from the other vertices of P, that is, for some $b \in \mathbb{R}$,

$$\mathbf{a}^\top \mathbf{v} < b \quad \text{and} \quad \mathbf{a}^\top \mathbf{w} > b \text{ for all other vertices } \mathbf{w} \in P.$$

Then the set $P/\mathbf{v} := P \cap \{\mathbf{x} \in \mathbb{R}^d \mid \mathbf{a}^\top \mathbf{x} = b\}$ defines a $(d-1)$-polytope, the *vertex figure* of $\mathbf{v}$ in P.

It is easy to see that every two vertex figures of the same vertex in the same polytope are projectively equivalent. Therefore it is justified to speak of *the* vertex figure.

The faces of $P/\mathbf{v}$ correspond bijectively to those faces of P that contain $\mathbf{v}$. For instance, the vertices of $P/\mathbf{v}$ represent the edges of P that end in $\mathbf{v}$. More general, every k-face of $P/\mathbf{v}$ represents a $(k+1)$-face of P containing $\mathbf{v}$ and vice versa, for $-1 \leq k \leq d-1$. In particular, the (-1)-face $\emptyset$ of $P/\mathbf{v}$ stands for the vertex $\mathbf{v}$ itself and the $(d-1)$-face $P/\mathbf{v}$ for the whole polytope P.

One consequence is that a polytope is simple if and only if every vertex figure is a simplex.

The construction of a vertex figure can be iterated to obtain a more general object.

Definition 1.1.18 (Face figure)**.** Let P be a polytope and F a face of P with $\mathsf{dim}\, F = k$ and $0 \leq k < \mathsf{dim}\, P$. The *face figure* P/F of F in P is

(I) the vertex figure $P/\mathbf{v}$ if $F = \mathbf{v}$ is a vertex, that is, if $\mathsf{dim}\, F = 0$, or

(II) the face figure P/F' of F' in $P/\mathbf{v}$, where $\mathbf{v}$ is an arbitrary vertex of F and F' the $(k-1)$-face of $P/\mathbf{v}$ that corresponds to the k-face F of P, if $\mathsf{dim}\, F > 0$.

Additionally, we define $P/\emptyset := P$ and $P/P := \emptyset$.

1.2 POSETS

Proposition 1.1.4 suggests to turn one's attention to the combinatorial structure of the set of faces of a polytope, since it essentially states that this set, together with inclusion, is a poset. In this section we recall the basic definitions and facts about posets that we will need later. A very compact overview can also be found in [58, Section 2.2]. For a more extensive coverage of the subject see the books of Stanley [53] and Aigner [3].

Definition 1.2.1 (Poset terminology)**.** A *poset* (or *partially ordered set*) is a finite set S together with a binary relation $\preceq$ satisfying

(i) *Reflexivity*: $x \preceq x$ for all $x \in S$,

(ii) *Antisymmetry*: if $x, y \in S$ with $x \preceq y$ and $y \preceq x$, then $x = y$,

(iii) *Transitivity*: $x \preceq y$ and $y \preceq z$ imply $x \preceq z$ for all $x, y, z \in S$.

We also frequently write $y \succeq x$ for $x \preceq y$, as well as $x \prec y$ (or $y \succ x$) if $x \preceq y$ and $x \neq y$. In situations of ambiguity the relation symbol is sometimes written as $\preceq_S$. A *minimal element* $\hat{0}$ in a poset S is an element such that $\hat{0} \preceq x$ for all $x \in S$. Accordingly, a *maximal element* $\hat{1}$ satisfies $\hat{1} \succeq x$ for all $x \in S$.

A *chain of length* n in a poset S is a set $C = \{x_1, \dots, x_{n+1}\} \subseteq S$ such that $x_1 \prec \dots \prec x_{n+1}$. We call a chain $C' \subseteq S$ a *subchain* of C if $C' \subseteq C$, and a *proper subchain* if additionally $C' \neq C$. A chain C is called *maximal* if S contains no chain such that C is a proper subchain of it.

A poset is *graded* if it has a minimal element $\hat{0}$ and for every $x \in S$ every maximal chain of the form $\hat{0} \prec \dots \prec x$ has the same length. In this case the length of such a chain is called the *rank of* x, denoted by $\mathsf{rank}\, x$. If a graded poset S also contains a maximal element then the *rank of* S is the rank of the maximal element. An *interval* $[x, y]$ in a poset S is the poset defined by all elements $w \in S$ with $x \preceq w \preceq y$, equipped with the induced order relation.

The *join of elements* $x, y \in S$, denoted by $x \vee y$, is the minimal element $z \in S$ with $z \succeq x$ and $z \succeq y$, provided it is unique. Accordingly, the *meet* $x \wedge y$ is the unique maximal element $z \in S$ with $z \preceq x$ and $z \preceq y$, if this exists. If S is a graded poset with $\mathsf{rank}\, S = r$ then the elements of rank 1 are the *atoms* and the elements of rank $r - 1$ are the *coatoms* of S.

Finally, the *dual poset* $S^\diamond$ is the poset on the same ground set S with the relation reversed:

$$x \preceq_{S^\diamond} y \;:\Longleftrightarrow\; y \preceq_S x.$$

Definition 1.2.2 (Lattice)**.** A *lattice* is a poset with a minimal and a maximal element, where all pairs of elements have a join (and a meet).

The set of all faces of P defines a graded lattice, which is atomic (that is, every element can be written as the join of atoms) and coatomic (every element can also be written as the meet of coatoms).

Definition 1.2.3 (Face lattice)**.** Let P be a polytope. The *face lattice* $L(P)$ of P is the graded lattice of all faces of P, together with the partial order induced by set inclusion.

We have $\mathsf{rank}\, L(P) = \mathsf{dim}\, P + 1$ and, more generally, $\mathsf{rank}\, F = \mathsf{dim}\, F + 1$ for all faces F of P. The minimal and maximal elements are the empty face $\emptyset$ and the whole polytope P, respectively. The atoms of $L(P)$ correspond to vertices, while coatoms correspond to facets of P. Furthermore, the meet of two elements of $F, F' \in L(P)$ is the face $F \cap F'$ and the join corresponds to the smallest face of P that contains both F and F'.

Dual posets correspond to polar polytopes: $L(P^\Delta) = L(P)^\diamond$. Furthermore, the intervals of the special form $[\emptyset, F]$ are exactly the face lattices $L(F)$ of faces F, while, due to duality, intervals of the form $[F, P]$ are the face lattices of the face figures P/F.

More general, an interval $[F, F']$ with faces $F \subseteq F'$ corresponds to the face figure F'/F of F in F', which is a polytope of dimension $\mathsf{dim}\, F' - \mathsf{dim}\, F - 1$. In particular, every interval of rank 2 is the face lattice of a 1-polytope and therefore contains exactly 4 elements in the form of a diamond.

The face lattice of a polytope captures the complete combinatorial information of its structure. Therefore it is natural to take it as a basis for the combinatorial description of a polytope.

Definition 1.2.4 (Combinatorial equivalence)**.** Two polytopes P and Q are *combinatorially equivalent*, denoted by $P \cong Q$, if their face lattices are isomorphic, that is, there exists a bijection $\phi : L(P) \to L(Q)$ such that $x \preceq_{L(P)} y \Leftrightarrow \phi(x) \preceq_{L(Q)} \phi(y)$ for all $x, y \in L(P)$.

In fact it suffices to claim the existence of a bijection that preserves the relations between the atoms and coatoms, since all faces are determined by the intersections of the vertex sets corresponding to the facets.

Since we are mainly interested in combinatorial properties, such as numbers of faces, we usually identify polytopes as geometric objects with their equivalence class with respect to combinatorial equivalence. In particular, we will regularly use notation for certain polytopes and constructions from the previous chapter, without reference to the concrete geometric definition.

EULERIAN POSETS

Face posets of polytopes are examples of an important class of posets, named after their most striking property, which in turn is named after Leonhard Euler (1707–1783). To define this, we introduce very briefly the Möbius function—this fits into a much broader theory in general, for which the book of Stanley [53] is a good reference.

Definition 1.2.5 (Möbius function)**.** Let L be a graded poset. The *Möbius function* $\mu_L : L \times L \to \mathbb{Z}$ of L is inductively defined by

$$\mu_L(x,y) \;:=\; \begin{cases} 0 & \text{if } x \not\preceq y \\ 1 & \text{if } x = y \\ -\sum\limits_{x \preceq w \prec y} \mu_L(x,w) & \text{if } x \prec y \end{cases}$$

Definition 1.2.6 (Eulerian poset)**.** A graded poset L with maximal and minimal element is called *Eulerian* if for all $x, y \in L$ with $x \preceq y$ we have

$$\mu_L(x, y) \;=\; (-1)^{\mathsf{rank}[x,y]}.$$

Proposition 1.2.7. If P is a polytope then $L(P)$ is an Eulerian lattice. $\square$

Using the *order complex* of a poset one can show that for certain *cell complexes* Γ the value $\mu_{L(\Gamma)}(\hat{0}, \hat{1})$ of the Möbius function of its face poset equals the *reduced Euler characteristic* of the underlying topological space $|\Gamma|$ (see, for instance, Stanley [53, Chapter 3.8] for details).

For polytopes this yields the following outstanding result, which is the general statement of Euler's polyhedral formula.

Theorem 1.2.8 (Euler's equation)**.** If P is a d-polytope and $(f_0, \ldots, f_{d-1})$ its f-vector, then

$$\sum_{i=0}^{d-1} (-1)^i f_i \;=\; 1 - (-1)^d.$$

$\square$

Euler's equation is the first deep result about the face numbers of polytopes. The theory of poset combinatorics, however, provides even more insight. Stanley [49] invented the following general concept of the h-vector.

Definition 1.2.9 (Generalised h-vector and toric g-vector)**.** Recursively define two polynomials $h(P, x)$ and $g(P, x)$, associated to an Eulerian poset P of rank $d + 1$, by

$$h(P, x) \;:=\; \begin{cases} 1 & \text{if } \mathsf{rank}\, P = 0 \\ \sum\limits_{\hat{0} \preceq F \prec P} g([\hat{0}, F], x)(x-1)^{d-\mathsf{rank}\, F} & \text{if } d \geq 0 \end{cases}$$

and

$$g(P, x) \;:=\; \begin{cases} 1 & \text{if } \mathsf{rank}\, P = 0 \\ \sum\limits_{i=0}^{m} (h_i(P) - h_{i-1}(P))x^i & \text{if } d \geq 0 \end{cases}$$

where $h(P, x) = \sum\limits_{i=0}^{d} h_i(P)\, x^i$ and we set $h_{-1}(P) = 0$ and $m = \lfloor d/2 \rfloor$.

The *h-vector* of P is the $(d+1)$-tuple of coefficients of the polynomial $h(P, x)$, and the *g-vector* is the $(\lfloor d/2 \rfloor + 1)$-tuple of coefficients of $g(P, x)$:

$$h(P) \;:=\; (h_0(P), \ldots, h_d(P)) \quad \text{and} \quad g(P) \;:=\; (g_0(P), \ldots, g_{\lfloor d/2 \rfloor}(P)).$$

We have $h_0(P) = 1$ for all P, as can be shown by induction using Euler's equation. Examples for this quite intricate definition, as well as the proof of the following result can be found in Stanley's book [53, Section 3.14].

Theorem 1.2.10 (Dehn-Sommerville equations)**.** If P is an Eulerian poset of rank $d+1$ then $h_i(P) = h_{d-i}(P)$ for $0 \le i \le d$. $\square$

If now P is a d-polytope, we can define its h-vector to be the h-vector of its face lattice, $h(P) := h(L(P))$. This is then a $(d+1)$-tuple of integers and accordingly we get the g-vector $g(P) := g(L(P))$.

For a simplicial d-polytope P the h-vector has a direct combinatorial interpretation, which makes it possible to reconstruct the f-vector from the h-vector. See [58, Section 8.3] for a thorough treatment of this topic.

Theorem 1.2.11. Let P be a simplicial d-polytope, $(h_0, \ldots, h_d)$ its h-vector, $(f_0, \ldots, f_{d-1})$ its f-vector and $f_{-1} = 1$. Then we have

$$\begin{aligned} h_k &= \sum_{i=0}^{k} (-1)^{k-i} \binom{d-i}{d-k} f_{i-1} \quad \text{for} \quad 0 \le k \le d \quad \text{and} \\ f_i &= \sum_{k=0}^{i+1} \binom{d-k}{i-k+1} h_k \quad \text{for} \quad 0 \le i \le d-1. \end{aligned}$$

$\square$

This shows that for simplicial polytopes the h-vector completely determines the f-vector and vice versa. Even more, due to the Dehn-Sommerville equations, the g-vector already contains all information about the f-vector. This culminates in the g-Theorem, conjectured by McMullen [41] and proved by Billera and Lee [13] and Stanley [50], which gives a complete characterisation of the f-vectors of simplicial polytopes.

The same problem for non-simplicial polytopes seems to be much harder and is a big open question in connection with combinatorics of polytopes. One reason for this is that in the general case the h-vector cannot be written in terms of the f-vector any more, but only in terms of the flag vector, which we define below.

For notational convenience we introduce the notation $[d] := \{0, \ldots, d-1\}$ for the set of all non-negative integers smaller than d. Note that in this thesis, in contrast to the common notation in books treating posets, such as Stanley [53], the elements of this set are shifted by one, due to the fact that we mainly have to deal with polytopes.

Definition 1.2.12 (Flag vector)**.** Let P be a graded poset of rank $d+1$. For a set $S = \{i_1, \ldots, i_\ell\} \subseteq [d]$ with $i_1 < \ldots < i_\ell$ let

$$f_S(P) \;:=\; \Big|\big\{\hat{0} \prec F_1 \prec \ldots \prec F_\ell \prec \hat{1} \mid F_j \in P,\, \mathsf{rank}\, F_j = i_j + 1\big\}\Big|.$$

The 2^d-tuple

$$\big(f_S(P)\big)_{S \subseteq [d]}$$

is the *flag vector* of P.

For a polytope P its flag vector is the flag vector of its face lattice. Hence for a polytope the index set S exactly prescribes the dimensions of the faces in the chains to be counted.

When dealing with a specific entry of the flag vector we usually leave out the set braces and the commas if there is no danger of confusion, writing only f_{02} instead of $f_{\{0,2\}}$, for instance. Moreover, the elements of the index set are always denoted in increasing order.

As in the case of the f-vector (see Definition 1.1.10) it is sometimes convenient to regard the index set S as a subset of $\{-1, 0, \ldots, d-1, d\}$. Then for d-polytopes P we have $f_S(P) = f_{S \setminus \{-1\}}(P) = f_{S \setminus \{d\}}(P)$, since there is only one empty face and one d-dimensional face.

It is immediate that the f-vectors can be seen as projections of the flag vectors:

$$f(P) \;=\; \big(f_S(P)\big)_{S \subseteq [d], |S| = 1}.$$

Also, $f_\emptyset(P) = 1$ for all polytopes P.

Example 1.2.13. The flag vector of a simplicial polytope P is already determined by its f-vector. A complete formula can be obtained recursively, using the f-vector of the simplex (see Example 1.1.11) and the fact that $f_S(P) = f_{S \setminus \{k\}}(\Delta_k) \cdot f_k(P)$, with $k = \mathsf{max}\, S$. This results in

$$f_{i_1, i_2, \ldots, i_k}(P) \;=\; \binom{i_2 + 1}{i_1 + 1} \cdots \binom{i_k + 1}{i_{k-1} + 1} \cdot f_{i_k}(P).$$

Example 1.2.14 (cf. Stenson [55, Lemma 8])**.** If P is a polytope then the flag vector of the pyramid $\mathsf{pyr}\, P$ can be expressed in terms of the flag vector of P:

$$f_S(\mathsf{pyr}\, P) \;=\; f_S(P) + \sum_{k \in S} f_{S_{<k} \cup (S_{\geq k} - 1)}(P).$$

Here, $S_{<k} := \{j \in S \mid j < k\}$, accordingly $S_{\geq k} := \{j \in S \mid j \geq k\}$, and $S - 1 := \{j - 1 \mid j \in S\}$. This in particular implies the formula for the f-vector of $\mathsf{pyr}\, P$, see Example 1.1.12.

The sum in the above formula will show up rather frequently in Chapter 2, so for the sake of shorter notation we introduce an abbreviation for it:

$$\operatorname*{apf}_S P \;:=\; \sum_{k\in S} f_{S_{<k}\cup(S_{\geq k}-1)}(P).$$

In plain words, $\mathsf{apf}_S P$ is the number of "additional pyramid flags", that is, flags with index set S of the pyramid $\mathsf{pyr}\, P$, that contain faces which themselves contain the apex of the pyramid. It is worth mentioning some special cases: For $\dim P = d$ we have

$$\begin{aligned}
\operatorname*{apf}_{\emptyset} P &= 0,\\
\operatorname*{apf}_{\{j\}} P &= f_{j-1}(P) \quad \text{for } 0 \leq j \leq d,\\
\operatorname*{apf}_{\{0,2\}} P &= f_{-1,1}(P) + f_{0,1}(P) \;=\; 3f_1(P) \quad \text{for } d \geq 2.
\end{aligned}$$

In particular, $\mathsf{apf}_{\{d\}} P$ is the number of facets of P and $\mathsf{apf}_{\{d+1\}} P = 1$. Note that the index set S might also contain -1 or $d+1$, in which case we have $\mathsf{apf}_{S\setminus\{-1\}} P = \mathsf{apf}_S P = \mathsf{apf}_{S\setminus\{d+1\}} P$.

Obviously, the flag vector carries much more information than the f-vector. Still, this information is highly redundant, in the sense that the entries of the flag vectors of Eulerian posets satisfy a number of relations. These "Generalized Dehn-Sommerville equations" were first obtained by Bayer and Billera and can be viewed as the analogue for flag vectors of Euler's equation for the f-vector.

Theorem 1.2.15 (Generalized Dehn-Sommerville equations [7])**.** Let P be an Eulerian poset of rank d and $S = \{i_1, \ldots, i_\ell\} \subseteq [d]$. Suppose we have $-1 = i_0 < i_1 < \ldots < i_\ell < i_{\ell+1} = d$ and $i_k < i_{k+1} - 1$ for some k. Then

$$\sum_{j=i_k+1}^{i_{k+1}-1} (-1)^{j-i_k-1} f_{S\cup\{j\}}(P) \;=\; \left(1-(-1)^{i_{k+1}-i_k-1}\right) f_S(P)$$

□

The proof of the equations is remarkably short and simply uses the fact that every interval of an Eulerian poset is again Eulerian und hence Euler's equation can be applied. Not surprisingly, the theorem comprises Euler's equation itself as the special case $S = \emptyset$.

The equations in fact completely describe the affine space that contains all flag vectors of Eulerian posets of a given dimension. We will get back to this in Chapter 3.

The Generalized Dehn-Sommerville equations imply that the complete flag vector of a polytope is already determined by a subset of its entries. A natural choice is described in the following.

Definition 1.2.16 (Sparse sets)**.** Let $d \geq 1$. The set Ψ_d is the set of all subsets of $\{0, \dots, d-2\}$ that do not contain two consecutive integers. The elements of Ψ_d are the *sparse subsets* of $[d]$.

The number of sparse subsets of $[d]$ is given by the $(d+1)$-st *Fibonacci number*, defined by $F_0 = F_1 = 1$ and $F_d = F_{d-1} + F_{d-2}$ for $d \geq 2$.

Theorem 1.2.17 (Bayer & Billera [7])**.** Let $d \geq 1$. For all $T \subseteq [d]$ there is a relation

$$f_T(P) \; = \; \sum_{S \in \Psi_d} c_S f_S(P)$$

with certain $c_S \in \mathbb{Z}$, valid for all Eulerian posets P of rank $d+1$. □

Definition 1.2.18 (Reduced flag vector)**.** Given an Eulerian poset P of rank $d+1$, we call $\mathbf{f}(P) := (f_S(P))_{S \in \Psi_d}$ the *reduced flag vector* of P.

If P is a polytope then $\mathbf{f}(P)$ is an integer vector of length $F_{\mathsf{dim}\, P}$. We denote by $\mathcal{F}\ell_d \subset \mathbb{Z}^{F_d}$ the set of reduced flag vectors of all d-polytopes. The reverse lexicographic order on $[d]$ induces a natural ordering of the components of the reduced flag vector. Stated explicitly, the reduced flag vector can be denoted as follows:

$$\mathbf{f} = (f_\emptyset,\ f_0,\ f_1,\ f_2, f_{02},\ f_3, f_{03}, f_{13},\ f_4, f_{04}, f_{14}, f_{24}, f_{024},\ f_5, f_{05}, \dots,\ f_{\tau,\dots,d-2})$$

where τ is 0 or 1 if d is even or odd, respectively. We will always use this convention in formal expressions and examples involving the reduced flag vector.

Casually, Theorem 1.2.17 states that knowing the entries of the flag vector of an Eulerian poset subscripted by the sparse subsets suffices to know the whole flag vector. In terms of linear algebra, the flag vectors of Eulerian posets of rank $d+1$ are all contained in an affine subspace of $\mathbb{R}^{2^d}$ of dimension $F_d - 1$, since $f_\emptyset(P) = 1$ for every Eulerian poset P.

1.3 THE cd-INDEX

The **cd**-index captures the same information as the flag vector in a very elegant form. It was invented by J. Fine and first published by Bayer and

Klapper [10]. Since then a wealth of results has been produced in this field by a number of people, notably Bayer, Billera, Ehrenborg, Readdy, and Stanley, see for instance [9], [11], [12], [24], [25], and [52].

Let P be an Eulerian poset of rank $d+1$. For a subset $S \subseteq [d]$ define $w_S := w_0 \cdot \dots \cdot w_{d-1}$, where

$$w_i \;:=\; \begin{cases} \mathbb{b} & \text{if } i \in S \\ \mathbb{a} - \mathbb{b} & \text{if } i \notin S \end{cases}$$

with non-commuting variables $\mathbb{a}$ and $\mathbb{b}$. Then the $\mathbb{ab}$-*index of* P is

$$\Psi(P) \;:=\; \sum_{S \subseteq [d]} f_S(P)\, w_S.$$

The $\mathbb{ab}$-index is a homogeneous polynomial of degree d in $\mathbb{a}$ and $\mathbb{b}$ with integer coefficients. Bayer and Klapper [10] proved that for Eulerian posets this polynomial can be written uniquely in terms of the two (also non-commuting) variables

$$\mathbb{c} \;:=\; \mathbb{a} + \mathbb{b} \quad \text{and} \quad \mathbb{d} \;:=\; \mathbb{ab} + \mathbb{ba}$$

and then becomes the $\mathbb{cd}$-*index of* P. With $\deg \mathbb{c} = 1$ and $\deg \mathbb{d} = 2$ the $\mathbb{cd}$-index of an Eulerian poset is then also a homogeneous polynomial of degree d.

The $\mathbb{cd}$-*index of a polytope* P is $\Psi(P) := \Psi(L(P))$, the $\mathbb{cd}$-index of its face lattice. It is not hard to see that the $\mathbb{cd}$-index of the polar polytope is obtained by simply reversing the order of the variables in each monomial. In particular, the $\mathbb{cd}$-index of self-dual polytopes exhibits a certain symmetry in this respect, as can be seen from the examples below.

Example 1.3.1. Here are the $\mathbb{cd}$-indices of a few polytopes of not too high dimension, computed with `polymake` [28], using Theorem 1.3.2 below.

$$\begin{aligned}
\Psi([0,1]) &= \mathbb{c} \\
\Psi(\pentagon_n) &= \mathbb{c}^2 + (n-2)\mathbb{d} \\
\Psi(\Delta_4) &= \mathbb{c}^4 + 3\mathbb{c}^2\mathbb{d} + 5\mathbb{cdc} + 3\mathbb{dc}^2 + 4\mathbb{d}^2 \\
\Psi(C_4) &= \mathbb{c}^4 + 6\mathbb{c}^2\mathbb{d} + 16\mathbb{cdc} + 14\mathbb{dc}^2 + 20\mathbb{d}^2 \\
\Psi({C_4}^\Delta) &= \mathbb{c}^4 + 14\mathbb{c}^2\mathbb{d} + 16\mathbb{cdc} + 6\mathbb{dc}^2 + 20\mathbb{d}^2 \\
\Psi(\pentagon_7 * \pentagon_7) &= \mathbb{c}^5 + 12\mathbb{c}^3\mathbb{d} + 49\mathbb{c}^2\mathbb{dc} + 49\mathbb{cdc}^2 + 84\mathbb{cd}^2 + 12\mathbb{dc}^3 \\
&\quad + 74\mathbb{dcd} + 84\mathbb{d}^2\mathbb{c}
\end{aligned}$$

It follows directly from the definition that the coefficients of the $\mathtt{cd}$-index are linear combinations of flag vector entries. While for some monomials their coefficients in the $\mathtt{cd}$-index of polytopes can be calculated easily, in general this is a complicated matter.

We adopt the following convenient notation from Ehrenborg [24] to express those coefficients: For two $\mathtt{cd}$-monomials u and v let

$$\langle u \mid v \rangle \;:=\; \delta_{u,v} \;=\; \begin{cases} 1 & \text{if } u = v \\ 0 & \text{otherwise} \end{cases}$$

This extends linearly to a bilinear form on the space of all $\mathtt{cd}$-polynomials.

It is now an easy observation that if P is a d-polytope then $\langle \mathtt{c}^d \mid \Psi(P)\rangle = 1$ and $\langle \mathtt{c}^{d-2}\mathtt{d} \mid \Psi(P)\rangle = f_{d-1}(P) - 2$, and dually $\langle \mathtt{d}\mathtt{c}^{d-2} \mid \Psi(P)\rangle = f_0(P) - 2$. In general, to obtain the expression of a coefficient in terms of the flag vector, a recipe can be given, as was done by Billera, Ehrenborg and Readdy, see [11, Section 7].

Theorem 1.3.2 (Billera & Ehrenborg [11, Proposition 7.1])**.** Given a $\mathtt{cd}$-monomial $u = \mathtt{c}^{n_1}\mathtt{d}\mathtt{c}^{n_2}\cdots\mathtt{c}^{n_p}\mathtt{d}\mathtt{c}^{n_{p+1}}$, for every polytope P,

$$\langle u \mid \Psi(P)\rangle \;=\; \sum_{i_1,\ldots,i_p} (-1)^{(m_1-i_1)+\ldots+(m_p-i_p)}\, k_{i_1,\ldots,i_p}(P)$$

where $m_0 := 0$ and $m_i := m_{i-1} + n_i + 2$ for $i \geq 1$, the sum is over all p-tuples $(i_1, \ldots, i_p)$ with $m_{j-1} \leq i_j \leq m_j - 2$ for $1 \leq j \leq p$, and the *sparse flag k-vector* is defined by

$$k_S(P) \;:=\; \sum_{T \subseteq S} (-2)^{|S|-|T|} f_T(P) \qquad \text{for } S \in \Psi_d$$

□

Theorem 1.3.2 in fact establishes a one-to-one correspondence between the reduced flag vectors and the $\mathtt{cd}$-indices of Eulerian posets. Consequently, the number of $\mathtt{cd}$-monomials of degree d is again the Fibonacci number F_d, as can easily be shown by induction, matching the size of the reduced flag vectors of the degree $d{+}1$ Eulerian posets. Also, both the reduced flag vector, as well as the $\mathtt{cd}$-index start with a constant term: $f_\emptyset(P) = \langle \mathtt{c}^d \mid \Psi(P)\rangle = 1$.

An important feature of the $\mathtt{cd}$-index of polytopes is the non-negativity of its coefficients, proved by Stanley [52]. In fact, even a stronger statement is true. For two $\mathtt{cd}$-polynomials Ψ_1 and Ψ_2 write $\Psi_1 \geq \Psi_2$ if every coefficient of the polynomial $\Psi_1 - \Psi_2$ is non-negative, that is, $\langle u \mid \Psi_1 - \Psi_2\rangle \geq 0$ for all $\mathtt{cd}$-monomials u. In other words, we read inequalities for polynomials coefficient-wise.

Theorem 1.3.3 (Billera & Ehrenborg [11, Theorem 5.3])**.** If P is a d-polytope then $\Psi(P) \geq \Psi(\Delta_d)$. In other words, $\langle u \mid \Psi(P)\rangle - \langle u \mid \Psi(\Delta_d)\rangle \geq 0$ for every $\mathbb{cd}$-monomial u. $\square$

Stanley's result then follows from the non-negativity of the $\mathbb{cd}$-index of the simplex. See, for instance, Purtill [45, Section 6], where he shows that the coefficients of the $\mathbb{cd}$-index of the simplex, the cube and the octahedron count certain classes of André permutations.

1.4 Linear inequalities

*Why do we care** about such a complicated technique as the $\mathbb{cd}$-index? One reason is that due to the inequalities from Theorem 1.3.3 and the fact that the coefficients can be written in terms of the flag vector they provide linear inequalities and therefore necessary conditions for the flag vectors of polytopes.

Although it looks as though we are dealing with affine inequalities here, since they usually have a non-zero right-hand-side, they are indeed linear inequalities in the reduced flag vectors, since the constant term can always be written as a multiple of $f_\emptyset = 1$.

Example 1.4.1. As remarked in Section 1.3, $\langle \mathbb{dc}^{d-2} \mid \Psi(P)\rangle = f_0(P) - 2$ for every d-polytope P. Together with $\langle \mathbb{dc}^{d-2} \mid \Psi(\Delta_d)\rangle = f_0(\Delta_d) - 2 = d - 1$, Theorem 1.3.3 implies $f_0(P) \geq d + 1$.

This is of course quite a trivial statement, but it shows how the machinery works. Since every flag vector entry denoted by a sparse set can be reformulated in terms of the $\mathbb{cd}$-index, it is possible to write every linear flag vector relation as a $\mathbb{cd}$-polynomial. Using Ehrenborg's notation, a linear functional L on the reduced flag vectors, evaluated on an Eulerian poset P, is then simply $\langle z \mid \Psi(P)\rangle$, where z is the $\mathbb{cd}$-polynomial corresponding to L.

Example 1.4.2. The $\mathbb{cd}$-polynomial $\mathbb{dc}^{d-2} - (d-1)\mathbb{c}^d$ encodes the functional $f_0 - (d+1)f_\emptyset$, cf. Example 1.4.1. For a slightly less obvious example consider

$$-2(d-2)\mathbb{c}^d + 2\mathbb{cdc}^{d-3} - (d-2)\mathbb{dc}^{d-2}$$

which translates into the innocently looking functional $2f_1 - df_0$.

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Note that both functionals in Example 1.4.2 are non-negative for all polytopes of suitable dimension. This is not at all obvious from the $\mathbb{cd}$-polynomials (at least not for the second example), but it can be derived using the techniques to be described in this section.

Before we focus on that, let us consider another source of linear inequalities, which is the g-vector of a polytope. Recall from Section 1.2 that the entries of the g-vector are simply the differences of the entries of the h-vector. Theorem 1.2.10 implies that only one half of the g-vector is in fact interesting and for simplicial polytopes P we have $g_k(P) \geq 0$ for $0 \leq k \leq \lfloor d/2 \rfloor$.

This follows from the proof of the necessity part of the g-Theorem by Stanley [50] and uses a considerable amount of algebraic geometry and topology. For non-simplicial polytopes the situation is even more complicated. For polytopes with rational coordinates Stanley [49] proved that the entries of the h-vector are the *Betti numbers* with respect to *intersection cohomology* of the *toric variety* associated with the polytope. It then follows from the *Hard Lefschetz Theorem* that $h_0 \leq \ldots \leq h_{\lfloor d/2 \rfloor}$.

Karu [38] provided a proof of the Hard Lefschetz Theorem for *projective fans*, thus extending the unimodality property of the h-vector to general (non-rational) polytopes.

Theorem 1.4.3 (Karu [38]). If P is a d-polytope then $g_i(P) \geq 0$ for all $i \in \{0, \ldots, \lfloor d/2 \rfloor\}$. □

The special cases $g_1(P) \geq 0$ and $g_2(P) \geq 0$ correspond to the fact that $2f_1 \geq df_0$ (cf. Example 1.4.2), respectively to the rigidity inequality

$$f_1(P) - d\, f_0(P) + f_{02}(P) - 3f_2(P) + \binom{d+1}{2} \geq 0,$$

due to Kalai [35] and Whiteley.

Since every entry of the g-vector can be seen as a linear combination of entries of the flag vector, the g-vector can again be stated in terms of a $\mathbb{cd}$-polynomial. An explicit formula was given by Bayer and Ehrenborg in [9] and [24]. For example, the above expression for $g_2(P)$ using the $\mathbb{cd}$-index reads $g_2(P) = \langle g_2^d \mid \Psi(P) \rangle$ with the $\mathbb{cd}$-polynomial

$$g_2^d \;=\; \frac{d(d-3)}{2}\mathbb{c}^d - \mathbb{c}^2\mathbb{d}\mathbb{c}^{d-4} - (d-3)\mathbb{d}\mathbb{c}^{d-2} + \mathbb{d}^2\mathbb{c}^{d-4}.$$

CONVOLUTION

It is possible to recycle known linear inequalities for the flag vector to obtain new ones in higher dimensions. One possibility to do this is convolution, introduced by Kalai [36].

If we denote the polytope dimension with a superscript at the flag vector, we can define the following formal multiplication for flag vector entries of d-, respectively e-dimensional polytopes:

$$(f_S^d * f_T^e) \;:=\; f_{S\cup\{d\}\cup(T+d+1)}^{d+e+1}$$

where $T+k := \{t+k \mid t \in T\}$ denotes the set T, shifted by k.

Now suppose we are given two linear functionals

$$L = \sum_{S\subseteq[d]} \alpha_S f_S^d \qquad \text{and} \qquad M = \sum_{T\subseteq[e]} \beta_T f_T^e$$

for flag vectors of d- and e-polytopes, respectively. Then we can define a new functional for flag vectors of $(d+e+1)$-dimensional polytopes by

$$L * M \;:=\; \sum_{S\subseteq[d],\,T\subseteq[e]} \alpha_S \beta_T (f_S^d * f_T^e).$$

If the functionals are non-negative for all polytopes, that is, if

$$L(P) \;:=\; \sum_{S\subseteq[d]} \alpha_S f_S^d(P) \;\geq\; 0 \qquad \text{and likewise} \qquad M(Q) \geq 0$$

for all d-polytopes P, respectively all e-polytopes Q, then also

$$(L * M)(R) \geq 0$$

for all $(d+e+1)$-dimensional polytopes R. The reason is that

$$f_{S\cup\{d\}\cup(T+d+1)}^{d+e+1}(R) \;=\; \sum_{\substack{F\ d\text{-face}\\ \text{of } R}} f_S^d(F) \cdot f_T^e(R/F)$$

and therefore

$$\begin{aligned}
(L * M)(R) \;&=\; \sum_{\substack{S\subseteq[d],\,T\subseteq[e]\\ F\ d\text{-face of } R}} \alpha_S f_S^d(F) \cdot \beta_T f_T^e(R/F) \\
&=\; \sum_{\substack{F\ d\text{-face}\\ \text{of } R}} \underbrace{\left(\sum_{S\subseteq[d]} \alpha_S f_S^d(F)\right)}_{\geq 0} \cdot \underbrace{\left(\sum_{T\subseteq[e]} \beta_T f_T^e(R/F)\right)}_{\geq 0}
\end{aligned}$$

Example 1.4.4. For $d \geq 1$, the elementary inequality $2f_1 \geq df_0$ arises as a convolution: Consider the two inequalities

$$f_\emptyset^0 \ \geq \ 0 \quad \text{and} \quad f_0^{d-1} - df_\emptyset \ \geq \ 0$$

for 0-, respectively $(d-1)$-dimensional polytopes. The convolution of the left-hand-side functionals is

$$f_{01}^d - df_0^d$$

which, using the Generalized Dehn-Sommerville equations (or simply the fact that every edge contains 2 vertices) to obtain $f_{01} = 2f_1$, yields the above inequality. Since $f_\emptyset^0$ is always equal to 1, the inequality is tight exactly for polytopes where all vertex figures attain equality in the second inequality, that is, if and only if the polytope is simple.

The convolution of two inequalities can also be expressed in terms of the $\mathtt{cd}$-index, see, for instance, Stenson [55]. The following quite compact formulation was given by Ehrenborg [24, Section 2]. If two homogeneous $\mathtt{cd}$-polynomials expressing two linear inequalities are given, then their convolution corresponds to the $\mathtt{cd}$-polynomial obtained by linear extension of the following multiplication rule, defined on the monomials: Let u, v be $\mathtt{cd}$-monomials and write $u = \tilde{u}\bar{u}$, $v = \bar{v}\tilde{v}$ with

$$\bar{u} \ := \ \begin{cases} \mathtt{c}\ , \text{ if } u \text{ ends with } \mathtt{c} \\ 1\ , \text{ if } u \text{ ends with } \mathtt{d} \end{cases} \quad \text{and} \quad \bar{v} \ := \ \begin{cases} \mathtt{c}\ , \text{ if } v \text{ starts with } \mathtt{c} \\ 1\ , \text{ if } v \text{ starts with } \mathtt{d} \end{cases}$$

Then $u * v$ is the polynomial $\tilde{u}p\tilde{v}$ with

$$p \ := \ \begin{cases} 2\mathtt{c} & \text{if } \{\bar{u}, \bar{v}\} = \{1\} \\ 2\mathtt{c}^2 + \mathtt{d} & \text{if } \{\bar{u}, \bar{v}\} = \{1, \mathtt{c}\} \\ 2\mathtt{c}^3 + \mathtt{dc} + \mathtt{cd} & \text{if } \{\bar{u}, \bar{v}\} = \{\mathtt{c}\} \end{cases}$$

Example 1.4.5. Let $p := 1$ and $q := \mathtt{dc}^{d-3} - (d-2)\mathtt{c}^{d-1}$, which encode linear inequalities for 0- and $(d-1)$-dimensional polytopes, respectively (cf. Example 1.4.2). Their convolution amounts to

$$\begin{aligned} p * q &= (1 * \mathtt{dc}^{d-3}) - (d-2)(1 * \mathtt{c}^{d-1}) \\ &= (1 \cdot 2\mathtt{c} \cdot \mathtt{dc}^{d-3}) - (d-2)(1 \cdot (2\mathtt{c}^2 + \mathtt{d}) \cdot \mathtt{c}^{d-2}) \\ &= 2\mathtt{cdc}^{d-3} - 2(d-2)\mathtt{c}^d - (d-2)\mathtt{dc}^{d-2} \end{aligned}$$

which explains the second polynomial in Example 1.4.2.

The question how the $\mathtt{cd}$-index and convolution are related was investigated by Stenson [55]. She showed that there are inequalities that arise from Theorem 1.3.3, but cannot be obtained via convolution. Likewise, there are convoluted g-vector inequalities that are not implied by the basic $\mathtt{cd}$-index inequalities.

LIFTING THE $\mathtt{cd}$-INDEX

Another way to get inequalities from lower-dimensional ones is provided by the following Lifting Theorem by Ehrenborg [24].

Theorem 1.4.6 (Ehrenborg [24, Theorem 3.1])**.** Let p be a $\mathtt{cd}$-polynomial that represents a non-negative linear inequality, that is,

$$\langle p \mid \Psi(P)\rangle \;\geq\; 0$$

for all polytopes P. Furthermore, let u and v be $\mathtt{cd}$-monomials such that u does not end in $\mathtt{c}$ and v does not begin with $\mathtt{c}$. Then

$$\langle u \cdot p \cdot v \mid \Psi(P)\rangle \geq 0$$

for all polytopes P. □

Combining Theorems 1.3.3 and 1.4.6 one obtains the following family of inequalities (see Ehrenborg [24, Theorem 3.7]):

$$\langle u \cdot p \cdot v \mid \Psi(P)\rangle \;\geq\; \langle p \mid \Psi(\Delta_k)\rangle \cdot \langle u \cdot \mathtt{c}^k \cdot v \mid \Psi(P)\rangle$$

for all $\mathtt{cd}$-monomials u, v and p and every polytope P, where k is the degree of p.

Note that one cannot lift with completely arbitrary monomials, but only with those that end, respectively begin in a compatible way. Furthermore, the Lifting Theorem in general yields inequalities that are not tight at the simplex, even if one starts with tight inequalities, as the following example demonstrates.

Example 1.4.7. Consider the $\mathtt{cd}$-polynomial describing the entry g_2 of the g-vector of 6-dimensional polytopes,

$$g_2^6 \;=\; 9\mathtt{c}^6 - \mathtt{c}^2\mathtt{d}\mathtt{c}^2 - 3\mathtt{d}\mathtt{c}^4 + \mathtt{d}^2\mathtt{c}^2$$

Lifting with the monomials 1 from the left and $\mathtt{d}$ from the right according to Theorem 1.4.6 gives $\langle g_2^6 \cdot \mathtt{d} \mid \Psi(P)\rangle \geq 0$ for all 8-polytopes P, while $\langle g_2^6 \cdot \mathtt{d} \mid \Psi(\Delta_8)\rangle = 3$. So the lifted inequality is not tight for Δ_8, although $g_2(\Delta_6) = 0$.

It is still an open question whether the theorem can be strengthened in this respect. This can also be seen as a non-symmetric analogue of Kalai's Conjecture C for centrally-symmetric polytopes. For details see the discussion at the end of Section 6.3.

Chapter 2

Inductive construction of polytopes

Since a polytope is the convex hull of its vertices, all possible properties of polytopes can theoretically be described in an inductive way: For a d-polytope choose out of its vertices $d+1$ affinely independent ones, which will yield a d-simplex $P^{(1)}$. Then for every further vertex v_k describe the polytope $P^{(k+1)} := \mathsf{conv}(P^{(k)} \cup \{v_k\})$ in terms of the polytope $P^{(k)}$, for $k \geq 1$.

This principle appears for instance in Grünbaum [29, Section 5.2] and was extended by Altshuler & Shemer [4]. Similar concepts were introduced by Shephard [48, Section 3] and by Edelsbrunner [23, Section 8.4.1]. Algorithmically, this describes the basic idea of the *beneath-and-beyond* algorithm to compute the convex hull of points (cf. Joswig [32] and de Berg et al. [20, Chapter 11]).

Unfortunately, for a complete combinatorial characterisation of polytopes this approach is rather unsuitable, due to the vast number of possibilities that have to be taken into account. However, there are special cases in which the situation is managable and we will discuss some of these construction steps here.

We describe stacking and pseudostacking only for full-dimensional polytopes. This presents no loss of generality, since one could easily extend the constructions to the general case. This would, however, entail a greater level of technicality in both notation and argumentation. So unless stated otherwise, for the rest of this chapter P always denotes a d-polytope in $\mathbb{R}^d$.

2.1 Stacking

Definition 2.1.1 (Beneath and beyond)**.** Let P be a d-polytope and F a facet of P, defined by the linear inequality $\mathbf{a}^\top \mathbf{x} \leq b$. A point $\mathbf{p} \in \mathbb{R}^d$ is *beyond* F if $\mathbf{a}^\top \mathbf{p} > b$ and *beneath* F if $\mathbf{a}^\top \mathbf{p} < b$.

The interior of a polytope is therefore the set of all points that are beneath all of the polytope's facets.

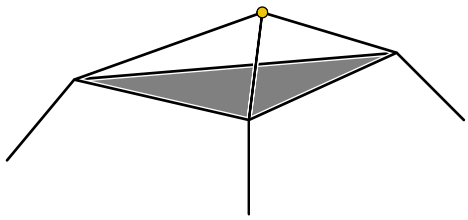


Figure 2.1: Stacking beyond the shaded triangle

Definition 2.1.2 (Stacking)**.** Let P be a polytope and F a facet of P. Furthermore, let $\mathbf{v}$ be a point beyond F and beneath all other facets of P. Then the polytope P' that arises from P by *stacking* beyond the facet F is $P' := \mathsf{conv}(P \cup \{\mathbf{v}\})$.

It is clear that a point $\mathbf{v}$ as in Definition 2.1.2 always exists. For instance take

$$\mathbf{v} \ := \ \mathbf{b}_F + \varepsilon\,(\mathbf{b}_F - \mathbf{b}_P)$$

for small enough $\varepsilon > 0$, where $\mathbf{b}_F$ is the vertex-barycenter of the facet F and $\mathbf{b}_P$ the barycenter of the polytope P.

Furthermore, for the combinatorial properties of P' it is irrelevant which point exactly is chosen—any point beyond F and beneath all other facets will yield the same combinatorial type of polytope.

The dual operation to "stacking beyond a facet" is "cutting off a vertex", which adds a new facet as well as all faces of it.

Casually, stacking beyond a facet means to "glue" a pyramid over the facet on the polytope such that the result is convex again and all faces, except for the selected facet itself, survive the operation. With this in mind it is easy to see how the flag vector changes when the stacking operation is applied.

Proposition 2.1.3. Let P be a d-polytope and F a facet of P. Suppose P' arises out of P by stacking beyond F. Then for the flag vector of P' we have

$$f_S(P') \ = \ \begin{cases} f_S(P) + \mathsf{apf}_S\, F & \text{if } d-1 \notin S \\ f_S(P) + \mathsf{apf}_S\, F - f_S(F) & \text{if } d-1 \in S \end{cases}$$

Proof. Let $S \subseteq [d]$. If $d-1 \notin S$ then every flag of faces that is counted by $f_S(P')$ is either a flag in P or one in $\mathsf{pyr}\, F$. However, flags of faces of F are counted by both $f_S(P)$ and $f_S(\mathsf{pyr}\, F)$, but occur only once in P'. Hence,

$$f_S(P') \ = \ f_S(P) + f_S(\mathsf{pyr}\, F) - f_S(F).$$

Using the formula for the flag vector of a pyramid in Example 1.2.14, the last term cancels out and we get the first part of the claim.

If $d-1 \in S$, then the same holds, except that those flags of both P and $\mathsf{pyr}\, F$ that contain F are not at all present in P'. The number of these flags is exactly $f_{S\setminus\{d-1\}}(F)$, so we get

$$f_S(P') \;=\; f_S(P) + f_S(\mathsf{pyr}\, F) - 2f_{S\setminus\{d-1\}}(F).$$

Since $\mathsf{dim}\, F = d-1$ and therefore $f_{S\setminus\{d-1\}}(F) = f_S(F)$, the rest of the assertion follows again from the pyramid formula. □

By considering the special case $S = \{j\}$, the proposition immediately yields the following formula for the f-vector.

Corollary 2.1.4. For P, F and P' as in Proposition 2.1.3 we have

$$\begin{aligned} f_j(P') &= f_j(P) + f_{j-1}(F) \qquad \text{for } 0 \le j \le d-2, \text{ and} \\ f_{d-1}(P') &= f_{d-1}(P) + f_{d-2}(F) - 1. \end{aligned}$$

□

Classically, stacking is regarded with respect to simplex facets. In this case, the above formula can be stated more explicitly, using the formula in Example 1.1.11.

Corollary 2.1.5. Let P, F and P' be as in Proposition 2.1.3 and F a $(d-1)$-simplex. Then

$$\begin{aligned} f_j(P') &= f_j(P) + \binom{d}{j} \qquad \text{for } 0 \le j \le d-2, \text{ and} \\ f_{d-1}(P') &= f_{d-1}(P) + d - 1. \end{aligned}$$

□

Special classical examples are stacked polytopes, defined as follows.

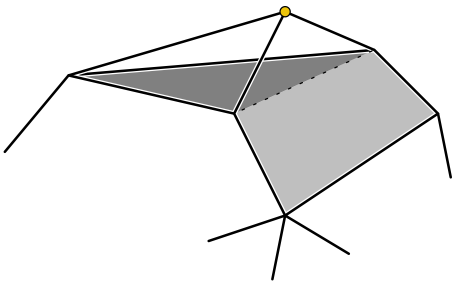
Definition 2.1.6 (Stacked polytope)**.** A d-dimensional *stacked polytope* on n vertices is the result of $n-d-1$ consecutive stacking operations, initially applied to a d-simplex.

It follows immediately that all stacked polytopes are simplicial. To calculate the flag vector, one therefore only has to consider the f-vector (see Example 1.2.13).

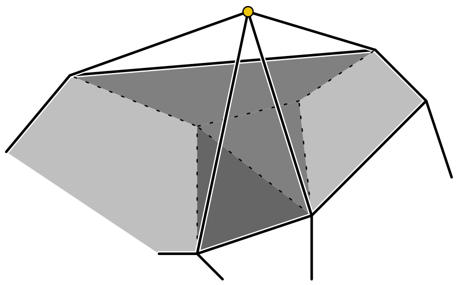
Proposition 2.1.7. If P is a d-dimensional stacked polytope on n vertices, then

$$\begin{aligned} f_j(P) &= \binom{d+1}{j+1} + (n-d-1)\binom{d}{j} \qquad \text{for } 0 \le j \le d-2, \text{ and} \\ f_{d-1}(P) &= d + 1 + (n-d-1)(d-1) \;=\; (n-d)(d-1) + 2. \end{aligned}$$

□



(a) Pseudostacking into one adjacent facet (light grey)



(b) Pseudostacking beyond one adjacent facet and into two others

Figure 2.2: Pseudostacking beyond a triangle

2.2 Pseudostacking

Pseudostacking can on the one hand be seen as a generalisation of stacking, on the other hand also as a special—and therefore more easily treatable—case of the general situation investigated by Grünbaum [29, Section 5.2] and Altshuler & Shemer [4].

We follow more or less the notation already used in the joint paper with A. Paffenholz [43], but confine ourselves to dimensions at least 3. The 2-dimensional case is usually trivial and would complicate statements considerably.

Definition 2.2.1 (Pseudostacking)**.** Let P be a d-polytope and F a facet of P. Denote by $\mathsf{adj}\, F$ the facets of P that are adjacent to F, that is, the facets F' of P such that $F \cap F'$ is a ridge of P.

For two disjoint subsets $\mathcal{F}, \mathcal{N} \subseteq \mathsf{adj}\, F$ we call a point $\mathbf{v} \in \mathbb{R}^d$ a *pseudostacking point* with respect to F, $\mathcal{F}$ and $\mathcal{N}$ if $\mathbf{v}$ is beyond F and all facets in $\mathcal{N}$, lies in the affine hull of every facet in $\mathcal{F}$ and beneath all remaining facets of P.

The polytope

$$\mathsf{PS}^F_{\mathcal{F},\mathcal{N}}(P) \;:=\; \mathsf{conv}(P \cup \{\mathbf{v}\})$$

is then obtained by *pseudostacking* P beyond the facet F with respect to $\mathcal{F}$ and $\mathcal{N}$.

The usual stacking operation is the special case $\mathcal{F} = \mathcal{N} = \emptyset$.

Already with this cautious generalisation, quite a lot of things can happen. Figure 2.2 shows two examples: On the left the case that $\mathcal{F}$ contains exactly one element and $\mathcal{N} = \emptyset$. On the right we have an example where $|\mathcal{F}| = 2$

and $|\mathcal{N}| = 1$; note that the number of vertices decreases. Moreover, another facet of P vanishes, although it is not contained in $\mathcal{F}$ nor $\mathcal{N}$, not even in $\mathsf{adj}\, F$.

In contrast to the usual stacking, it is possible that a pseudostacking point does not exist at all. In terms of the notation of Altshuler & Shemer [4]: the pair $\mathcal{N} \mid \mathcal{F}$ might not be *coverable* in P. In the following section we study some situations for which the existence of a pseudostacking vertex—that is, the coverability of the pair $\mathcal{N} \mid \mathcal{F}$—can be established and investigate the combinatorics and the flag vector of the polytope that arises from the pseudostacking operation.

The general facial structure of pseudostacked polytopes can be read off from the following theorem of Grünbaum, which in fact covers an even more general situation. We quote the version of Altshuler & Shemer [4, Section 2], which remedies a slight error in the original formulation.

Theorem 2.2.2 (Grünbaum [29, Theorem 5.2.1]). Let $P \subset \mathbb{R}^d$ be a d-polytope and $\mathbf{v} \in \mathbb{R}^d \setminus P$. Let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ be the partition of the facets of P such that $\mathbf{v}$ lies in the affine hull of every $A \in \mathcal{A}$, beyond every $B \in \mathcal{B}$ and beneath every $C \in \mathcal{C}$. Define three types of sets G:

(A) G is a face of a member of $\mathcal{C}$.

(B) $G = \mathsf{conv}(F \cup \{\mathbf{v}\})$, where F is the intersection of a subset of $\mathcal{A}$ (or, equivalently, F is a face of P and $\mathbf{v} \in \mathsf{aff}\, F$), with $\bigcap \emptyset := P$.

(C) $G = \mathsf{conv}(F \cup \{\mathbf{v}\})$, where F is a face of a member of $\mathcal{B}$ and also a face of a member of $\mathcal{C}$.

Then the sets of types (A), (B) and (C) are faces of $P' := \mathsf{conv}(P \cup \{\mathbf{v}\})$, and each face of P' is of exactly one of those types. $\square$

We will make extensive use of this theorem to work out the flag vectors of pseudostacked polytopes.

General considerations

The most modest modification of stacking is to choose the point in the affine hull of only one adjacent facet. This is always possible.

Proposition 2.2.3. Let P be a d-polytope, $d \geq 3$, and F a facet of P. Choose an arbitrary facet $F' \in \mathsf{adj}\, F$ and define $\mathcal{F} := \{F'\}$ and $\mathcal{N} := \emptyset$, Then a pseudostacking point with respect to F, $\mathcal{F}$ and $\mathcal{N}$ exists and for the flag vector of the polytope $\mathsf{PS}^F_{\mathcal{F},\mathcal{N}}(P)$ we have

$$f_S(\mathsf{PS}^F_{\mathcal{F},\mathcal{N}}(P)) \;=\; f_S(P) + \underset{S}{\mathsf{apf}}\, F - \Xi_S$$

for $S \subseteq [d]$, with

$$\Xi_S \;=\; \begin{cases} 0 & \text{if } \{d-2, d-1\} \cap S = \emptyset \\ f_{S_{<d-2}}(R) & \text{if } \{d-2, d-1\} \cap S = \{d-2\} \\ f_S(F) + f_{S_{<d-2}}(R) & \text{if } \{d-2, d-1\} \cap S = \{d-1\} \\ f_S(F) + 2f_{S_{<d-2}}(R) & \text{if } \{d-2, d-1\} \subseteq S \end{cases}$$

where $R := F \cap F'$ is the ridge between F and F'.

For a sketch of the situation see Figure 2.2(a), where the facet F' is shaded light grey and the ridge R is the dotted edge.

Proof. It is easy to show that a pseudostacking point $\mathbf{v}$ exists: Consider the polyhedron defined by the reversed inequality for the facet F and the facet-defining inequalities for all other facets of P. This polyhedron is non-empty and since F' intersects F in a ridge of P, the inequality corresponding to F' also defines a facet of the polyhedron. Any point in the relative interior of this facet is a valid choice for $\mathbf{v}$.

All facets of P are again facets of $\mathsf{PS}^F_{\mathcal{F},\mathcal{N}}(P)$, except for F which vanishes in the interior of $\mathsf{PS}^F_{\mathcal{F},\mathcal{N}}(P)$, and F' which is replaced by the new facet $\widehat{F'} := \mathsf{conv}(F' \cup \{\mathbf{v}\})$, that is F' stacked beyond R.

Let $S \subseteq [d]$ be given. Similar to the situation in Proposition 2.1.3, the flags of $\mathsf{PS}^F_{\mathcal{F},\mathcal{N}}(P)$ are exactly the flags in P, plus those in $\mathsf{pyr}\, F = F * \mathbf{v}$, except for

(A) one instance of each flag of proper faces of F (they are counted twice in $f_S(P) + f_S(\mathsf{pyr}\, F)$),

(B) all flags in both P and $\mathsf{pyr}\, F$ that contain F (which is not a face of $\mathsf{PS}^F_{\mathcal{F},\mathcal{N}}(P)$ any more),

(C) all flags in $\mathsf{pyr}\, F$ that contain $R * \mathbf{v}$ and apart from that only faces in R (they are also counted twice, see the remark below),

(D) all flags in both P and $\mathsf{pyr}\,F$ that contain R (which also vanishes in $\mathsf{PS}^F_{\mathcal{F},\mathcal{N}}(P)$).

Note that the flags of type (C) contain $R * \mathbf{v}$, which is, strictly speaking, not a face of $\mathsf{PS}^F_{\mathcal{F},\mathcal{N}}(P)$; however, it is merged with F' into the facet $\widehat{F'}$ and hence these flags are in one-to-one correspondence to the flags containing $\widehat{F'}$ and faces of R. So in general we have

$$f_S(\mathsf{PS}^F_{\mathcal{F},\mathcal{N}}(P)) \;=\; f_S(P) + f_S(\mathsf{pyr}\,F) - x_S$$

and it remains to determine the number x_S of flags that are overcounted. With the formula for $f_S(\mathsf{pyr}\,F)$ from Example 1.2.14 and the definition of $\mathsf{apf}_S\,F$ we then obtain the assertion with $\Xi_S = x_S - f_S(F)$.

For $d-1, d-2 \notin S$ we only have to subtract the flags in (A) that we counted twice and their number is exactly $f_{S\setminus\{d-1\}} = f_S(F)$. Therefore $x_S = f_S(F)$ and this yields $\Xi_S = 0$.

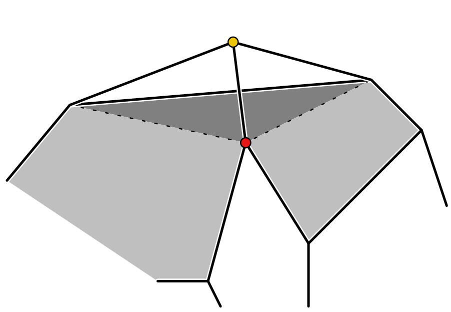
If $d-1 \notin S$ and $d-2 \in S$, then as before we subtract the twice counted flags, plus those flags of type (D) that are not already covered by type (A), that is, all flags of faces of R in $\mathsf{pyr}\,F$. Since there are $f_{S_{<d-2}}(R)$ of these, we get $x_S = f_S(F) + f_{S_{<d-2}}(R)$, which implies $\Xi_S = f_{S_{<d-2}}(R)$.

Now suppose $d-1 \in S$, but $d-2 \notin S$. Then the number of flags of type (B) has to be subtracted, which amounts to $2f_S(F)$. Additionally, the flags in (C) do not exist in $\mathsf{PS}^F_{\mathcal{F},\mathcal{N}}(P)$ and their number is exactly $f_{S_{<d-2}}(R)$. Hence $x_S = 2f_S(F) + f_{S_{<d-2}}(R)$ and therefore $\Xi_S = f_S(F) + f_{S_{<d-2}}(R)$.

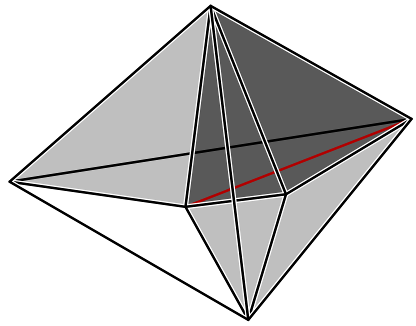
Finally, suppose S contains both $d-1$ and $d-2$. Then again all flags of type (B) are not present in $\mathsf{PS}^F_{\mathcal{F},\mathcal{N}}(P)$. Also the flags in (D) vanish, but half of them (those that contain both R and F) have already been scheduled for subtraction, since they are also listed under (A). The remaining ones (those that contain R and F', respectively $R * \mathbf{v}$) add up to $2f_{S_{<d-2}}(R)$. Summing up, $x_S = 2f_S(F) + 2f_{S_{<d-2}}(R)$, so $\Xi_S = f_S(F) + 2f_{S_{<d-2}}(R)$, which finishes the proof. □

In particular, Proposition 2.2.3 implies that $f_0(\mathsf{PS}^F_{\mathcal{F},\mathcal{N}}(P)) = f_0(P) + 1$, that is, the number of vertices really increases. In a more general situation this might not be true, and it can even decrease, as one can see from the example in Figure 2.2(b).

Also, the combinatorial description of the resulting polytope might be much more complicated, since it depends not only on the sets $\mathcal{F}$ and $\mathcal{N}$, but also on other facets that might "squeeze in" between those facets we are considering. If this is the case then at least no harm is done to those faces that are not directly involved.



(a) The red vertex is contained in an uninvolved facet



(b) The red edge is contained in the nonshaded facet

Figure 2.3: Nonsimple facet sets (light grey) adjacent to a (dark grey) facet

Lemma 2.2.4. Let P be a polytope, F a facet of P and $\mathcal{F}$ and $\mathcal{N}$ disjoint subsets of $\mathsf{adj}\, F$. Assume that a pseudostacking point with respect to F, $\mathcal{F}$ and $\mathcal{N}$ exists. Then a face of F is again a face of $\mathsf{PS}^F_{\mathcal{F},\mathcal{N}}(P)$ if and only if it is contained in at least one facet which is not in $\mathcal{F} \cup \mathcal{N}$.

Proof. Follows directly from Grünbaum's Theorem 2.2.2. □

It is desirable to have the condition in Lemma 2.2.4 for all faces of dimension up to $d-3$, since this gives a bit more control over the combinatorics and the resulting flag vector. Under certain prerequisites this situation can be ensured.

Definition 2.2.5 (Nonsimple facet set)**.** Let P be a polytope and F a facet of P. A subset $\mathcal{F}$ of $\mathsf{adj}\, F$ is called *nonsimple* if for all facets $F_1, F_2 \in \mathcal{F}$ the face $G := F_1 \cap F_2 \cap F$ of F is $(d-3)$-dimensional and contained in some other facet $F' \notin \mathcal{F}$.

Figure 2.3 shows examples of nonsimple facet sets. In the light of Definition 2.2.5, Lemma 2.2.4 states that if we choose $\mathcal{F}, \mathcal{N} \subseteq \mathsf{adj}\, F$ such that $\mathcal{F} \cup \mathcal{N}$ is nonsimple then the only lower-dimensional faces that completely vanish in $\mathsf{PS}^F_{\mathcal{F},\mathcal{N}}(P)$ are the ridges between F and the facets in $\mathcal{F} \cup \mathcal{N}$.

Proposition 2.2.6. Let P be a d-polytope and F a facet of P. Choose a nonsimple set $\mathcal{F} = \{F_1, \dots, F_k\} \subseteq \mathsf{adj}\, F$. Set $\mathcal{N} := \emptyset$ and assume that there exists a pseudostacking point with respect to F, $\mathcal{F}$ and $\mathcal{N}$. Then for $S \subseteq [d]$,

$$f_S(\mathsf{PS}^F_{\mathcal{F},\mathcal{N}}(P)) \;=\; f_S(P) + \underset{S}{\mathsf{apf}}\, F - \Xi_S$$

with

$$\Xi_S \;=\; \begin{cases} 0 & \text{if } \{d-2,d-1\} \cap S = \emptyset \\ \sum_{i=1}^{k} f_{S_{<d-2}}(R_i) & \text{if } \{d-2,d-1\} \cap S = \{d-2\} \\ f_S(F) + \sum_{i=1}^{k} f_{S_{<d-2}}(R_i) & \text{if } \{d-2,d-1\} \cap S = \{d-1\} \\ f_S(F) + 2\sum_{i=1}^{k} f_{S_{<d-2}}(R_i) & \text{if } \{d-2,d-1\} \subseteq S \end{cases}$$

where $R_i := F \cap F_i$ is the ridge between F and F_i.

Proof. The same arguments as in the proof of Proposition 2.2.3 hold here, except that the overcounted flags have to be subtracted for every facet in $\mathcal{F}$. Since $\mathcal{F}$ is nonsimple, all faces of dimension at most $d-3$ survive the pseudostacking, including the proper faces of the vanishing ridges R_i. Hence, the numbers that have to be subtracted are independent of each other. $\square$

We also have to investigate less restricted situations. This means that we have to generalise the above setting in some respect, which we will do in the rest of this section. Proofs for the following results that are analogous to the one of Proposition 2.2.3 are still possible in principle, however, they tend to get quite messy. Therefore we give another approach which uses more the combinatorics of the face poset than the geometric view.

Lemma 2.2.7. Let P be a d-polytope and $S \subseteq [d]$. Let $m := \max S$. Then

$$f_S(P) \;=\; \sum_{\substack{F\ m\text{-face} \\ \text{of } P}} f_S(F) \;=\; \sum_{\substack{F\ m\text{-face} \\ \text{of } P}} f_{S_{<m}}(F).$$

In particular,

$$f_S(P) \;=\; \begin{cases} \sum\limits_{\substack{F \text{ facet} \\ \text{of } P}} f_S(F) & \text{if } d-1 \in S \\ \frac{1}{2} \sum\limits_{\substack{F \text{ facet} \\ \text{of } P}} f_S(F) & \text{if } \{d-2,d-1\} \cap S = \{d-2\} \end{cases}$$

Proof. Clearly, for $S = \{i_1, \ldots, i_\ell, m\}$ with $i_1 < \ldots < i_\ell < m$, the family

$$\Big\{ \{\emptyset \subset F_1 \subset \ldots \subset F_\ell \subset F_m \subset P \mid F_j \text{ face of } F_m\,,\ \dim F_j = i_j\} \;\Big|\; F_m\ m\text{-face of } P \Big\}$$

is a partition of the set of face flags

$$\big\{\emptyset \subset F_1 \subset \ldots \subset F_\ell \subset F_m \subset P \mid F_j, F_m \text{ faces of } P, \dim F_j = i_j, \dim F_m = m\big\}$$

whose cardinality determines $f_S(P)$. The rest of the first assertion follows from the fact that $f_S(F) = f_{S_{<\dim F}}(F)$, see the remark on page 21.

The second assertion is a direct consequence of the first. For the third observe that, for $\max S = d-2$ we have $f_{S\cup\{d-1\}}(P) = 2f_S(P)$, since every $(d-2)$-face of P is contained in exactly two facets of P, or, alternatively, by the Generalized Dehn-Sommerville equations, Theorem 1.2.15. ☐

Lemma 2.2.8. Let F be a $(d-1)$-polytope and $S \subseteq [d]$.

(a) If $d-1 \in S$ then

$$\sum_{\substack{R \text{ facet} \\ \text{of } F}} \Big(f_{S_{<d-1}}(R) + \underset{S_{<d-1}}{\mathsf{apf}}\ R \Big) \ = \ \underset{S}{\mathsf{apf}}\, F.$$

(b) If $\{d-2, d-1\} \cap S = \{d-2\}$, then

$$\sum_{\substack{R \text{ facet} \\ \text{of } F}} \Big(f_{S_{<d-1}}(R) + \underset{S_{<d-1}}{\mathsf{apf}}\ R \Big) \ = \ f_S(F) + 2\,\underset{S}{\mathsf{apf}}\, F.$$

Proof. If $d-1 \in S$ then by Lemma 2.2.7 we can write

$$\begin{aligned} f_S(\mathsf{pyr}\, F) \ &= \ \sum_{\substack{F' \text{ facet} \\ \text{of } \mathsf{pyr}\, F}} f_{S_{<d-1}}(F') \ = \ f_{S_{<d-1}}(F) + \sum_{\substack{R \text{ facet} \\ \text{of } F}} f_{S_{<d-1}}(\mathsf{pyr}\, R) \\ &= \ f_{S_{<d-1}}(F) + \sum_{\substack{R \text{ facet} \\ \text{of } F}} \Big(f_{S_{<d-1}}(R) + \underset{S_{<d-1}}{\mathsf{apf}}\ R \Big). \end{aligned}$$

On the other hand, since $\dim F = d-1$,

$$f_S(\mathsf{pyr}\, F) \ = \ f_S(F) + \underset{S}{\mathsf{apf}}\, F \ = \ f_{S_{<d-1}}(F) + \underset{S}{\mathsf{apf}}\, F$$

which implies (a).

If $d-1 \notin S$, but $d-2 \in S$, then the analogous calculation using Lemma 2.2.7 yields

$$f_S(\mathsf{pyr}\, F) \ = \ \frac{1}{2} f_{S_{<d-1}}(F) + \frac{1}{2} \sum_{\substack{R \text{ facet} \\ \text{of } F}} \Big(f_{S_{<d-1}}(R) + \underset{S_{<d-1}}{\mathsf{apf}}\ R \Big)$$

and this implies

$$\begin{aligned}\sum_{\substack{R \text{ facet}\\ \text{of } F}} \Big(f_{S_{<d-1}}(R) + \operatorname*{apf}_{S_{<d-1}} R \Big) &= 2 f_S(\mathsf{pyr}\, F) - f_{S_{<d-1}}(F) \\ &= f_S(F) + 2 \operatorname*{apf}_{S} F.\end{aligned}$$

□

Lemma 2.2.9. Let P be a d-polytope and F a facet of P. Choose disjoint subsets $\mathcal{F}, \mathcal{N} \subseteq \mathsf{adj}\, F$ such that the set $\mathcal{F} \cup \mathcal{N}$ is nonsimple. Assume that there is a pseudostacking point $\mathbf{v}$ with respect to $F, \mathcal{F}$ and $\mathcal{N}$. Then for $S \subseteq [d]$,

$$f_S(\mathsf{PS}^F_{\mathcal{F},\mathcal{N}}(P)) \;=\; f_S(P) + \operatorname*{apf}_{S} F + \sum_{F' \in \mathcal{N}} \operatorname*{apf}_{S} F' - \sum_{F' \in \mathcal{N}} \operatorname*{apf}_{S_{<d-1}} (F' \cap F) - \Xi_S$$

with

$$\Xi_S = \begin{cases} 0 & \text{if } \{d-2, d-1\} \cap S = \emptyset \\ \sum_{F' \in \mathcal{F} \cup \mathcal{N}} f_{S_{<d-2}}(F' \cap F) & \text{if } \{d-2, d-1\} \cap S = \{d-2\} \\ \Gamma & \text{if } \{d-2, d-1\} \cap S = \{d-1\} \\ \Gamma + \sum_{F' \in \mathcal{F}} f_{S_{<d-2}}(F' \cap F) & \text{if } \{d-2, d-1\} \subseteq S \end{cases}$$

and

$$\begin{aligned}\Gamma \;=\; & f_S(F) + \sum_{F' \in \mathcal{F} \cup \mathcal{N}} f_{S_{<d-2}}(F' \cap F) \\ & + \sum_{F' \in \mathcal{N}} \Big(f_S(F') + f_{S_{<d-2}}(F' \cap F) + \operatorname*{apf}_{S_{<d-1}} (F' \cap F) \Big).\end{aligned}$$

Proof. By Grünbaum's Theorem 2.2.2, the facets of $\mathsf{PS}^F_{\mathcal{F},\mathcal{N}}(P)$ can be partitioned into three types:

(A) facets of P, not in $\mathcal{F} \cup \mathcal{N} \cup \{F\}$,

(B) facets of the form $\mathsf{conv}(F' \cup \{\mathbf{v}\})$ for all facets $F' \in \mathcal{F}$,

(C) pyramids $R * \mathbf{v}$ for all ridges R of P that satisfy the following conditions: R is a facet of some facet $F' \in \{F\} \cup \mathcal{N}$, but not in $\{F \cap F' \mid F' = \mathcal{F} \cup \mathcal{N}\}$.

Hence, if $d-1 \in S$, then by Lemma 2.2.7 we get

$$f_S(\mathsf{PS}^F_{\mathcal{F},\mathcal{N}}(P)) \;=\; \sum_{\substack{F' \text{ of} \\ \text{type } (\mathsf{A})}} f_{S_{<d-1}}(F') + \sum_{\substack{F' \text{ of} \\ \text{type } (\mathsf{B})}} f_{S_{<d-1}}(F') + \sum_{\substack{F' \text{ of} \\ \text{type } (\mathsf{C})}} f_{S_{<d-1}}(F')$$

where the sums are over all facets F' of the respective types. The first term simply equals

$$\sum_{\substack{F' \text{ of} \\ \text{type } (\mathsf{A})}} f_{S_{<d-1}}(F') \;=\; \sum_{F' \notin \mathcal{F} \cup \mathcal{N} \cup \{F\}} f_{S_{<d-1}}(F').$$

In the second term, the sum is over all facets of type (B) and by nonsimplicity, each of these facets is the original facet F', stacked beyond the ridge $F' \cap F$. Hence, by Proposition 2.1.3

$$\sum_{\substack{F' \text{ of} \\ \text{type } (\mathsf{B})}} f_{S_{<d-1}}(F') \;=\; \sum_{F' \in \mathcal{F}} \Big(f_{S_{<d-1}}(F') + \underset{S_{<d-1}}{\mathsf{apf}}\,(F' \cap F) - \delta_S f_{S_{<d-1}}(F' \cap F) \Big)$$

where $\delta_S := 0$ if $d-2 \notin S$ and $\delta_S := 1$ if $d-2 \in S$.

The third term can be rewritten into a sum over all ridges satisfying the conditions in (C), that is, all ridges of F and all of every facet $F' \in \mathcal{N}$, less those that connect F to the facets in $\mathcal{F}$ as well as those that are between F and the facets in $\mathcal{N}$—note that the latter were counted twice before.

$$\begin{aligned}
\sum_{\substack{F' \text{ of} \\ \text{type } (\mathsf{C})}} f_{S_{<d-1}}(F') \;&=\; \sum_{\substack{R \text{ ridge} \\ \text{as in } (\mathsf{C})}} f_{S_{<d-1}}(\mathsf{pyr}\, R) \\
&= \sum_{\substack{R \text{ facet} \\ \text{of } F}} f_{S_{<d-1}}(\mathsf{pyr}\, R) + \sum_{F' \in \mathcal{N}} \sum_{\substack{R \text{ facet} \\ \text{of } F'}} f_{S_{<d-1}}(\mathsf{pyr}\, R) \\
&\qquad - \sum_{F' \in \mathcal{F}} f_{S_{<d-1}}(\mathsf{pyr}\, R) - \sum_{F' \in \mathcal{N}} 2 f_{S_{<d-1}}(\mathsf{pyr}\, R) \\
&= \sum_{\substack{R \text{ facet} \\ \text{of } F}} \Big(f_{S_{<d-1}}(R) + \underset{S_{<d-1}}{\mathsf{apf}}\; R \Big) + \sum_{F' \in \mathcal{N}} \sum_{\substack{R \text{ facet} \\ \text{of } F'}} \Big(f_{S_{<d-1}}(R) + \underset{S_{<d-1}}{\mathsf{apf}}\; R \Big) \\
&\qquad - \sum_{F' \in \mathcal{F}} \Big(f_{S_{<d-1}}(F' \cap F) + \underset{S_{<d-1}}{\mathsf{apf}}\,(F' \cap F) \Big) \\
&\qquad - \sum_{F' \in \mathcal{N}} 2 \Big(f_{S_{<d-1}}(F' \cap F) + \underset{S_{<d-1}}{\mathsf{apf}}\,(F' \cap F) \Big)
\end{aligned}$$

$$
\begin{aligned}
&= \operatorname*{apf}_S F + \sum_{F' \in \mathcal{N}} \operatorname*{apf}_S F' - \sum_{F' \in \mathcal{F}} \Big(f_{S_{<d-1}}(F' \cap F) + \operatorname*{apf}_{S_{<d-1}} (F' \cap F) \Big) \\
&\qquad - \sum_{F' \in \mathcal{N}} 2 \Big(f_{S_{<d-1}}(F' \cap F) + \operatorname*{apf}_{S_{<d-1}} (F' \cap F) \Big)
\end{aligned}
$$

For the last equation we have used Lemma 2.2.8. Gathering all the terms and reordering the summands, we obtain:

$$
\begin{aligned}
&f_S(\mathsf{PS}^F_{\mathcal{F},\mathcal{N}}(P)) = \\
&\quad \sum_{F' \notin \mathcal{N} \cup \{F\}} f_{S_{<d-1}}(F') + \operatorname*{apf}_S F + \sum_{F' \in \mathcal{N}} \operatorname*{apf}_S F' - \sum_{F' \in \mathcal{N}} \operatorname*{apf}_{S_{<d-1}} (F' \cap F) \\
&\qquad - \sum_{F' \in \mathcal{F} \cup \mathcal{N}} f_{S_{<d-1}}(F' \cap F) - \sum_{F' \in \mathcal{N}} \Big(f_{S_{<d-1}}(F' \cap F) + \operatorname*{apf}_{S_{<d-1}} (F' \cap F) \Big) \\
&\qquad - \delta_S \sum_{F' \in \mathcal{F}} f_{S_{<d-1}}(F' \cap F).
\end{aligned}
$$

From Lemma 2.2.7 we get

$$
\sum_{F' \notin \mathcal{N} \cup \{F\}} f_{S_{<d-1}}(F') = f_S(P) - \sum_{F' \in \mathcal{N}} f_{S_{<d-1}}(F') - f_{S_{<d-1}}(F)
$$

and this yields

$$
\begin{aligned}
f_S(\mathsf{PS}^F_{\mathcal{F},\mathcal{N}}(P)) &= f_S(P) + \operatorname*{apf}_S F + \sum_{F' \in \mathcal{N}} \operatorname*{apf}_S F' - \sum_{F' \in \mathcal{N}} \operatorname*{apf}_{S_{<d-1}} (F' \cap F) \\
&\qquad - \Gamma - \delta_S \sum_{F' \in \mathcal{F}} f_{S_{<d-1}}(F' \cap F),
\end{aligned}
$$

with

$$
\begin{aligned}
\Gamma &= f_S(F) + \sum_{F' \in \mathcal{F} \cup \mathcal{N}} f_{S_{<d-2}}(F' \cap F) \\
&\qquad + \sum_{F' \in \mathcal{N}} \Big(f_S(F') + f_{S_{<d-2}}(F' \cap F) + \operatorname*{apf}_{S_{<d-1}} (F' \cap F) \Big)
\end{aligned}
$$

as claimed.

If $d-1 \notin S$, but $d-2 \in S$, we can basically do the same calculations, but have to take care of minor changes due to the differences in the Lemmas. From Lemma 2.2.7 we have

$$
f_S(\mathsf{PS}^F_{\mathcal{F},\mathcal{N}}(P)) = \frac{1}{2} \Bigg(\sum_{\substack{F' \text{ of} \\ \text{type (A)}}} f_{S_{<d-1}}(F') + \sum_{\substack{F' \text{ of} \\ \text{type (B)}}} f_{S_{<d-1}}(F') + \sum_{\substack{F' \text{ of} \\ \text{type (C)}}} f_{S_{<d-1}}(F') \Bigg)
$$

The first sum yields the same expression as before, as well as the second, with $\delta_S = 1$ by assumption. The third sum now rewrites as

$$
\begin{aligned}
\sum_{\substack{F' \text{ of} \\ \text{type (C)}}} f_{S_{<d-1}}(F') &= \\
&= \sum_{\substack{R \text{ facet} \\ \text{of } F}} \Big(f_{S_{<d-1}}(R) + \underset{S_{<d-1}}{\mathsf{apf}}\ R \Big) + \sum_{F' \in \mathcal{N}} \sum_{\substack{R \text{ facet} \\ \text{of } F'}} \Big(f_{S_{<d-1}}(R) + \underset{S_{<d-1}}{\mathsf{apf}}\ R \Big) \\
&\quad - \sum_{F' \in \mathcal{F}} \Big(f_{S_{<d-1}}(F' \cap F) + \underset{S_{<d-1}}{\mathsf{apf}}\ (F' \cap F) \Big) \\
&\quad - \sum_{F' \in \mathcal{N}} 2 \Big(f_{S_{<d-1}}(F' \cap F) + \underset{S_{<d-1}}{\mathsf{apf}}\ (F' \cap F) \Big) \\
&= f_S(F) + 2 \underset{S}{\mathsf{apf}}\, F + \sum_{F' \in \mathcal{N}} \Big(f_S(F') + 2 \underset{S}{\mathsf{apf}}\, F' \Big) \\
&\quad - \sum_{F' \in \mathcal{F}} \Big(f_{S_{<d-1}}(F' \cap F) + \underset{S_{<d-1}}{\mathsf{apf}}\ (F' \cap F) \Big) \\
&\quad - \sum_{F' \in \mathcal{N}} 2 \Big(f_{S_{<d-1}}(F' \cap F) + \underset{S_{<d-1}}{\mathsf{apf}}\ (F' \cap F) \Big)
\end{aligned}
$$

by Lemma 2.2.8. Summing up and using Lemma 2.2.7 again,

$$
\begin{aligned}
f_S(\mathsf{PS}^F_{\mathcal{F},\mathcal{N}}(P)) &= \frac{1}{2} \sum_{F' \notin \mathcal{N} \cup \{F\}} f_{S_{<d-1}}(F') + \frac{1}{2} \sum_{F' \in \mathcal{N}} f_S(F') + \frac{1}{2} f_S(F) \\
&\quad + \underset{S}{\mathsf{apf}}\, F + \sum_{F' \in \mathcal{N}} \underset{S}{\mathsf{apf}}\, F' - \sum_{F' \in \mathcal{N}} \underset{S_{<d-1}}{\mathsf{apf}}\ (F' \cap F) \\
&\quad - \sum_{F' \in \mathcal{F}} f_{S_{<d-1}}(F' \cap F) - \sum_{F' \in \mathcal{N}} f_{S_{<d-1}}(F' \cap F) \\
&= f_S(P) + \underset{S}{\mathsf{apf}}\, F + \sum_{F' \in \mathcal{N}} \underset{S}{\mathsf{apf}}\, F' - \sum_{F' \in \mathcal{N}} \underset{S_{<d-1}}{\mathsf{apf}}\ (F' \cap F) \\
&\quad - \sum_{F' \in \mathcal{F} \cup \mathcal{N}} f_{S_{<d-1}}(F' \cap F).
\end{aligned}
$$

Finally, if $m := \max S < d-2$ then all faces of dimension m of P are also faces of $\mathsf{PS}^F_{\mathcal{F},\mathcal{N}}(P)$, as can be seen from Grünbaums Theorem 2.2.2. Therefore, by Lemma 2.2.7 we can sum up over all m-faces of P, plus all m-faces of $\mathsf{pyr}\, F'$ for all $F' \in \mathcal{N}$ and have to subtract the term for all m-faces that are counted twice. Here we imagine that $\mathsf{PS}^F_{\mathcal{F},\mathcal{N}}(P)$ arises from P by "gluing" pyramids on

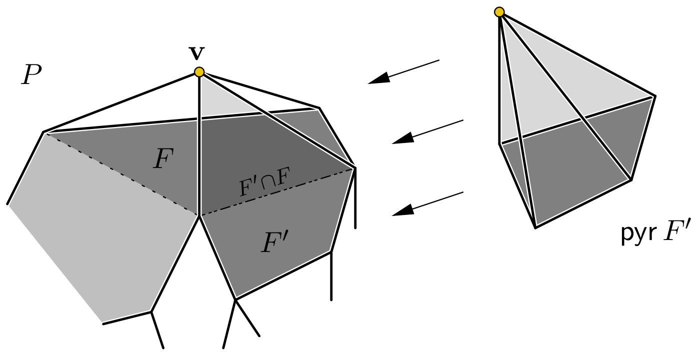


Figure 2.4: Proof of Lemma 2.2.9 if $\max S < d-2$

F and all facets F' in $\mathcal{N}$, each of them along the pyramid facet $(F' \cap F) * \mathbf{v}$; see Figure 2.4 for an illustration. This leads to

$$\begin{aligned}
&f_S(\mathsf{PS}^F_{\mathcal{F},\mathcal{N}}(P)) \\
&\quad = \quad f_S(P) + f_S(\mathsf{pyr}\, F) + \sum_{F' \in \mathcal{N}} f_S(\mathsf{pyr}\, F') \\
&\qquad\qquad - f_S(F) - \sum_{F' \in \mathcal{N}} \Big(f_S(F') + f_S(\mathsf{pyr}(F' \cap F)) - f_S(F' \cap F) \Big) \\
&\quad = \quad f_S(P) + f_S(F) + \underset{S}{\mathsf{apf}}\, F + \sum_{F' \in \mathcal{N}} \Big(f_S(F') + \underset{S}{\mathsf{apf}}\, F' \Big) \\
&\qquad\qquad - f_S(F) - \sum_{F' \in \mathcal{N}} \Big(f_S(F') + f_S(F' \cap F) + \underset{S}{\mathsf{apf}}(F' \cap F) - f_S(F' \cap F) \Big) \\
&\quad = \quad f_S(P) + \underset{S}{\mathsf{apf}}\, F + \sum_{F' \in \mathcal{N}} \underset{S}{\mathsf{apf}}\, F' - \sum_{F' \in \mathcal{N}} \underset{S_{<d-1}}{\mathsf{apf}}\, (F' \cap F).
\end{aligned}$$

□

The next case is similar to the previous one, except that the nonsimplicity gets lost to some extend; see Figure 2.5 for a 3-dimensional sketch. We call a face F of a polytope P a *simple face* if the face figure P/F is a simplex. In particular, a simple face of dimension $d-k$ of a d-polytope P is contained in exactly k facets of P. We now allow one carefully chosen simple face among the faces of the otherwise nonsimple set $\mathcal{F} \cup \mathcal{N}$.

Proposition 2.2.10. Let P be a d-polytope and F a facet of P. Furthermore, let $\mathcal{F}$ and $\mathcal{N}$ be non-empty, disjoint subsets of $\mathsf{adj}\, F$ with $F_0 \in \mathcal{N}$ and $F_1 \in \mathcal{F}$. Suppose the set $\mathcal{F} \cup \mathcal{N}$ is nonsimple except that the $(d-3)$-face

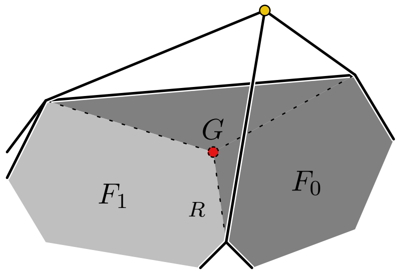


Figure 2.5: The situation in Proposition 2.2.10

$G := F \cap F_0 \cap F_1$ is a simple face of P. Then, if there exists a pseudostacking point with respect to F, $\mathcal{F}$ and $\mathcal{N}$,

$$f_S(\mathsf{PS}^F_{\mathcal{F},\mathcal{N}}(P)) \;=\; f_S(P) + \operatorname*{\mathsf{apf}}_S F + \sum_{F' \in \mathcal{N}} \operatorname*{\mathsf{apf}}_S F' - \sum_{F' \in \mathcal{N}} \operatorname*{\mathsf{apf}}_{S_{<d-1}} (F' \cap F) - \Xi_S$$

with

$$\Xi_S \;=\; \begin{cases} 0 & \text{if } \mathsf{max}\, S < d-3 \\ f_{S_{<d-3}}(G) & \text{if } \mathsf{max}\, S = d-3 \\ \Gamma + f_{S_{<d-3}}(G) & \text{if } \mathsf{max}\, S = d-2 \\ \Gamma + \tilde{\Gamma}_S + f_{S_{<d-1}}(F) + \sum\limits_{F' \in \mathcal{N}} \Big(f_{S_{<d-1}}(F') & \\ \qquad + f_{S_{<d-1}}(F' \cap F) + \operatorname*{\mathsf{apf}}\limits_{S_{<d-1}} (F' \cap F) \Big) & \text{if } \mathsf{max}\, S = d-1 \end{cases}$$

where

$$\Gamma \;=\; \operatorname*{\mathsf{apf}}_{S_{<d-2}} G + f_{S_{<d-1}}(R) + \sum_{F' \in \mathcal{F} \cup \mathcal{N}} f_{S_{<d-1}}(F' \cap F)$$

and

$$\tilde{\Gamma}_S \;=\; \begin{cases} 0 & \text{if } \mathsf{max}\, S_{<d-1} < d-3 \\ f_{S_{<d-3}}(G) & \text{if } \mathsf{max}\, S_{<d-1} = d-3 \\ 2 f_{S_{<d-3}}(G) + \operatorname*{\mathsf{apf}}\limits_{S_{<d-2}} G & \\ \quad + f_{S_{<d-1}}(R) + \sum\limits_{F' \in \mathcal{F}} f_{S_{<d-1}}(F' \cap F) & \text{if } \mathsf{max}\, S_{<d-1} = d-2 \end{cases}$$

with $R := F_0 \cap F_1$.

Proof. By assumption and Grünbaum's Theorem 2.2.2, G is the only $(d-3)$-face of P that is not a face of $\mathsf{PS}^F_{\mathcal{F},\mathcal{N}}(P)$. Additionally, all proper faces of

G are again faces of $\mathsf{PS}^F_{\mathcal{F},\mathcal{N}}(P)$, since they are contained in facets not in $\mathcal{F} \cup \mathcal{N} \cup \{F\}$. Therefore, if $\max S < d-3$ nothing changes, compared to the proof of Lemma 2.2.9 and if $\max S = d-3$ we only have to subtract a term $f_{S_{<d-3}}(G)$.

Now suppose $d-1 \in S$. We again split the term for $f_S(\mathsf{PS}^F_{\mathcal{F},\mathcal{N}}(P))$ into three sums over the respective facet types. Nothing changes for the first sum. For the second observe that the facet F_1 of P is responsible for a facet $\tilde{F}_1$ of $\mathsf{PS}^F_{\mathcal{F},\mathcal{N}}(P)$ of type (B) and we have

$$\tilde{F}_1 \;=\; \mathsf{conv}(F_1 \cup \{\mathbf{v}\}) \;\cong\; \mathsf{PS}^{F_1\cap F}_{\emptyset,\{R\}}(F_1)$$

combinatorially, since F_1 intersects F_0 in the ridge R. Therefore, we can use Lemma 2.2.9 itself to calculate $f_{S_{<d-1}}(F_1)$ and get

$$\begin{aligned}
&\sum_{\substack{F' \text{ of}\\ \text{type (B)}}} f_{S_{<d-1}}(F')\\
&= \sum_{F'\in\mathcal{F}\setminus\{F_1\}} \Big(f_{S_{<d-1}}(F') + \operatorname*{\mathsf{apf}}_{S_{<d-1}}(F'\cap F) - \delta_S f_{S_{<d-1}}(F'\cap F)\Big)\\
&\qquad + f_{S_{<d-1}}(F_1) + \operatorname*{\mathsf{apf}}_{S_{<d-1}}(F_1\cap F) + \operatorname*{\mathsf{apf}}_{S_{<d-1}} R - \operatorname*{\mathsf{apf}}_{S_{<d-2}} G\\
&\qquad - \begin{cases} 0 & \text{if } d-3, d-2 \notin S\\ f_{S_{<d-3}}(G) & \text{if } d-3 \in S, d-2 \notin S\\ f_{S_{<d-1}}(F_1\cap F) + 2f_{S_{<d-3}}(G) & \\ \qquad + f_{S_{<d-1}}(R) + \operatorname*{\mathsf{apf}}_{S_{<d-2}} G & \text{if } d-2 \in S\end{cases}\\
&= \sum_{F'\in\mathcal{F}} \Big(f_{S_{<d-1}}(F') + \operatorname*{\mathsf{apf}}_{S_{<d-1}}(F'\cap F)\Big) + \operatorname*{\mathsf{apf}}_{S_{<d-1}} R - \operatorname*{\mathsf{apf}}_{S_{<d-2}} G - \tilde{\Gamma}_S
\end{aligned}$$

with $\tilde{\Gamma}_S$ as stated.

In the third sum the additional term $f_{S_{<d-1}}(\mathsf{pyr}\, R)$ has to be subtracted from the expression in the previous proof, since R is not a face of $\mathsf{PS}^F_{\mathcal{F},\mathcal{N}}(P)$ either. This yields

$$\begin{aligned}
&\sum_{\substack{F' \text{ of}\\ \text{type (C)}}} f_{S_{<d-1}}(F') \;=\\
&\quad \operatorname*{\mathsf{apf}}_{S} F + \sum_{F'\in\mathcal{N}} \operatorname*{\mathsf{apf}}_{S} F' - \sum_{F'\in\mathcal{F}} \Big(f_{S_{<d-1}}(F'\cap F) + \operatorname*{\mathsf{apf}}_{S_{<d-1}}(F'\cap F)\Big)\\
&\quad - \sum_{F'\in\mathcal{N}} 2\Big(f_{S_{<d-1}}(F'\cap F) + \operatorname*{\mathsf{apf}}_{S_{<d-1}}(F'\cap F)\Big) - \Big(f_{S_{<d-1}}(R) + \operatorname*{\mathsf{apf}}_{S_{<d-1}} R\Big)
\end{aligned}$$

Adding up all three sums gives the asserted expression.

The last open case is that $d-2 \in S$ and $d-1 \notin S$. Then the second sum evaluates to

$$\begin{aligned}
&\sum_{\substack{F' \text{ of} \\ \text{type (B)}}} f_{S_{<d-1}}(F') \\
&= \sum_{F' \in \mathcal{F} \setminus \{F_1\}} \Big(f_{S_{<d-1}}(F') + \underset{S_{<d-1}}{\mathsf{apf}}\,(F' \cap F) - f_{S_{<d-1}}(F' \cap F) \Big) \\
&\quad + f_{S_{<d-1}}(F_1) + \underset{S_{<d-1}}{\mathsf{apf}}\,(F_1 \cap F) + \underset{S_{<d-1}}{\mathsf{apf}}\,R - \underset{S_{<d-2}}{\mathsf{apf}}\,G \\
&\quad - f_{S_{<d-1}}(F_1 \cap F) - f_{S_{<d-3}}(G) - f_{S_{<d-1}}(R) - f_{S_{<d-3}}(G) - \underset{S_{<d-2}}{\mathsf{apf}}\,G \\
&= \sum_{F' \in \mathcal{F}} \Big(f_{S_{<d-1}}(F') + \underset{S_{<d-1}}{\mathsf{apf}}\,(F' \cap F) - f_{S_{<d-1}}(F' \cap F) \Big) \\
&\quad + \underset{S_{<d-1}}{\mathsf{apf}}\,R - 2 f_{S_{<d-3}}(G) - 2\,\underset{S_{<d-2}}{\mathsf{apf}}\,G - f_{S_{<d-1}}(R)
\end{aligned}$$

and the last sum, as in the previous proof, with $f_{S_{<d-1}}(\mathsf{pyr}\, R)$ subtracted:

$$\begin{aligned}
\sum_{\substack{F' \text{ of} \\ \text{type (C)}}} f_{S_{<d-1}}(F') \;&=\; f_S(F) + 2\,\underset{S}{\mathsf{apf}}\,F + \sum_{F' \in \mathcal{N}} \Big(f_S(F') + 2\,\underset{S}{\mathsf{apf}}\,F' \Big) \\
&\quad - \sum_{F' \in \mathcal{F}} \Big(f_{S_{<d-1}}(F' \cap F) + \underset{S_{<d-1}}{\mathsf{apf}}\,(F' \cap F) \Big) \\
&\quad - \sum_{F' \in \mathcal{N}} 2 \Big(f_{S_{<d-1}}(F' \cap F) + \underset{S_{<d-1}}{\mathsf{apf}}\,(F' \cap F) \Big) - \Big(f_{S_{<d-1}}(R) + \underset{S_{<d-1}}{\mathsf{apf}}\,R \Big).
\end{aligned}$$

Finally,

$$\begin{aligned}
f_S(\mathsf{PS}^F_{\mathcal{F},\mathcal{N}}(P)) \;&=\; \frac{1}{2} \sum_{F' \notin \mathcal{N} \cup \{F\}} f_{S_{<d-1}}(F') + \frac{1}{2} \sum_{F' \in \mathcal{N}} f_{S_{<d-1}}(F') + \frac{1}{2} f_{S_{<d-1}}(F) \\
&\quad + \underset{S}{\mathsf{apf}}\,F + \sum_{F' \in \mathcal{N}} \underset{S}{\mathsf{apf}}\,F' - \sum_{F' \in \mathcal{F}} f_{S_{<d-1}}(F' \cap F) - \underset{S_{<d-2}}{\mathsf{apf}}\,G - f_{S_{<d-3}}(G) \\
&\quad - \sum_{F' \in \mathcal{N}} \Big(f_{S_{<d-1}}(F' \cap F) + \underset{S_{<d-1}}{\mathsf{apf}}\,(F' \cap F) \Big) - f_{S_{<d-1}}(R) \\
&= f_S(P) + \underset{S}{\mathsf{apf}}\,F + \sum_{F' \in \mathcal{N}} \underset{S}{\mathsf{apf}}\,F' - \sum_{F' \in \mathcal{N}} \underset{S_{<d-1}}{\mathsf{apf}}\,(F' \cap F) - \Gamma - f_{S_{<d-3}}(G).
\end{aligned}$$

□

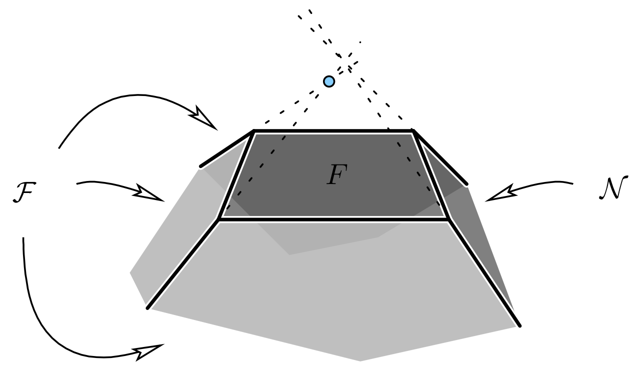


Figure 2.6: A pseudostacking point cannot be obtained

For the last few results we had to assume that a pseudostacking point exists. This cannot always be guaranteed in general. Figure 2.2 illustrates an example where this is not possible and cannot be made possible by a projective transformation either: Here, $\mathcal{F}$ consists of 3 facets whose defining hyperplanes already fix a unique candidate for a pseudostacking point, which is, however, beneath the facet in $\mathcal{N}$.

Therefore, in the following we consider pseudostacking beyond a *simplex facet*, that is, a facet which is combinatorially equivalent to a $(d-1)$-simplex. We show that in this case it is virtually always possible to find a pseudostacking point, almost independent on how $\mathcal{F}$ and $\mathcal{N}$ are chosen.

Lemma 2.2.11. Let $P \subset \mathbb{R}^d$ be a d-polytope and F a simplex facet of P. Choose two disjoint sets $\mathcal{F}, \mathcal{N} \subset \mathsf{adj}\, F$ such that $\mathcal{F} \cup \mathcal{N} \neq \mathsf{adj}\, F$. Then there exists a pseudostacking point with respect to F, $\mathcal{F}$ and $\mathcal{N}$.

Proof. For a facet F' of P let $H_{F'} := \mathsf{aff}\, F'$ and $H^+_{F'}$ and $H^-_{F'}$ be the open halfspaces of points beneath F' and beyond F', respectively. Furthermore, denote by $\mathbf{n}_{F'}$ the inward pointing normal vector to the facet F', that is

$$H_{F'} \;=\; \{\mathbf{v} \mid \mathbf{n}_{F'} \cdot \mathbf{v} = a_{F'}\} \qquad \text{and} \qquad H^-_{F'} \;=\; \{\mathbf{v} \mid \mathbf{n}_{F'} \cdot \mathbf{v} < a_{F'}\}.$$

Since F is a simplex, the intersection of all facets F' in $\tilde{\mathcal{F}} := \mathcal{F} \cup \mathcal{N}$ is non-empty and therefore the subspace

$$U_{\tilde{\mathcal{F}}} \;:=\; \bigcap_{F' \in \tilde{\mathcal{F}}} H_{F'}$$

of $\mathbb{R}^d$ is of dimension at least 1. (Here we have used that $\tilde{\mathcal{F}} \neq \mathsf{adj}\, F$.) In particular, we can choose a point $\tilde{\mathbf{v}} \in \mathsf{relint}(H^-_F \cap U_{\tilde{\mathcal{F}}})$.

Now let $\mathbf{m}_{F'}$ be the vector $\mathbf{n}_{F'}$, projected to the subspace $U_{\tilde{\mathcal{F}}\setminus\{F'\}}$ and consider

$$\mathbf{v} \;:=\; \tilde{\mathbf{v}} + \varepsilon \sum_{F'\in\mathcal{N}} \mathbf{m}_{F'}$$

for $\varepsilon > 0$. Since every $\mathbf{m}_{F'}$ is contained in $U_{\mathcal{F}}$, the point $\mathbf{v}$ is again in the affine hull of every facet $F' \in \mathcal{F}$. Additionally, $\mathbf{v}$ is beyond every $F' \in \mathcal{N}$ and therefore is a pseudostacking point with respect to F, $\mathcal{F}$ and $\mathcal{N}$. $\square$

The condition $\mathcal{F} \cup \mathcal{N} \neq \mathsf{adj}\, F$ in Lemma 2.2.11 can also be dropped if the facet F is *in bounded position* (see [43]), which basically means that the region $H_F^- \cap \bigcap_{F'\in\mathsf{adj}\,F} H_{F'}^+$ is bounded. This can always be achieved by applying a projective transformation if there is at least one facet F' of P with $F' \notin \mathsf{adj}\, F$.

PSEUDOSTACKING 4-POLYTOPES

In the following chapters, we will often be interested in 4-polytopes. For this case, the above results can be substantially specialised. We only give the sparse entries of the flag vectors, from which the whole flag vectors can be reconstructed, if needed.

Proposition 2.2.12. Let P be a 4-polytope and F, F', $\mathcal{F}$ and $\mathcal{N}$ as in Proposition 2.2.3. Then

$$\begin{aligned}
f_0(\mathsf{PS}^F_{\mathcal{F},\mathcal{N}}(P)) &= f_0(P) + 1,\\
f_1(\mathsf{PS}^F_{\mathcal{F},\mathcal{N}}(P)) &= f_1(P) + f_0(F),\\
f_2(\mathsf{PS}^F_{\mathcal{F},\mathcal{N}}(P)) &= f_2(P) + f_1(F) - 1,\\
f_{02}(\mathsf{PS}^F_{\mathcal{F},\mathcal{N}}(P)) &= f_{02}(P) + 3f_1(F) - f_0(F \cap F').
\end{aligned}$$

Proof. Apply Proposition 2.2.3 and use that $\mathsf{apf}_{\{0,2\}}\, F = 3f_1(F)$ (see Section 1.2) and $f_\emptyset(F \cap F') = 1$. $\square$

Corollary 2.2.13. Let P be a 4-polytope, F a simplex facet of P, and $F' \in \mathsf{adj}\, F$. Then

$$(f_0, f_1, f_2, f_{02})(\mathsf{PS}^F_{\{F'\},\emptyset}(P)) \;=\; (f_0, f_1, f_2, f_{02})(P) + (1, 4, 5, 15).$$

$\square$

We close this chapter with different 4-dimensional versions of the remaining results of the previous sections. These will all be used in Section 4.2 to construct a family of elementary 2-simple, 2-simplicial 4-polytopes.

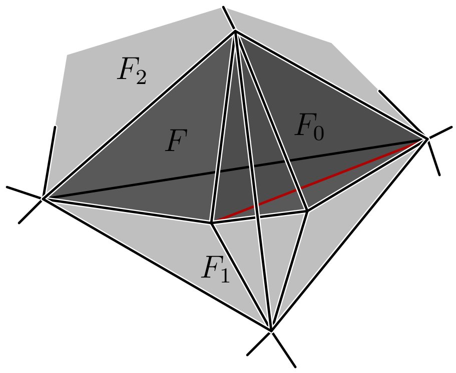


Figure 2.7: The red edge is contained in F, F_0 and F_1, but in no other facet (F_1 is the flat triangular bipyramid at the bottom)

Proposition 2.2.14. Let P be a 4-polytope, F a simplex facet of P, and $\mathcal{F} \subset \mathsf{adj}\, F$ a nonsimple set of two facets. Then

$$(f_0, f_1, f_2, f_{02})(\mathsf{PS}^F_{\mathcal{F},\emptyset}(P)) \;=\; (f_0, f_1, f_2, f_{02})(P) + (1, 4, 4, 12).$$

Proof. A pseudostacking point exists due to Lemma 2.2.11 and the formula for the flag vector follows directly from Proposition 2.2.6, with $F \cong \Delta_3$ and $R_i \cong \Delta_2$ for all i. □

Proposition 2.2.15. Let P be a 4-polytope, F simplex facet of P, and $\mathcal{F} \subset \mathsf{adj}\, F$ a nonsimple set of three facets. Then

$$(f_0, f_1, f_2, f_{02})(\mathsf{PS}^F_{\mathcal{F},\emptyset}(P)) \;=\; (f_0, f_1, f_2, f_{02})(P) + (1, 4, 3, 9).$$

Proof. Analogous to the previous proof. □

The last result of this chapter applies Proposition 2.2.10 to a very special case. Figure 2.7 shows a sketch of a part of a Schlegel diagram that illustrates the setting.

Proposition 2.2.16. Let P be a 4-polytope, F and F_0 simplex facets of P with $F_0 \in \mathsf{adj}\, F$, and $F_1, F_2 \in \mathsf{adj}\, F \setminus \{F_0\}$ such that the $(d-3)$-face $F \cap F_0 \cap F_1$ is a simple face of P, but each of the faces $F \cap F_0 \cap F_2$ and $F \cap F_1 \cap F_2$ is not.

Then with $\mathcal{F} = \{F_1, F_2\}$ and $\mathcal{N} = \{F_0\}$,

$$(f_0, f_1, f_2, f_{02})(\mathsf{PS}^F_{\mathcal{F},\mathcal{N}}(P)) \;=\; (f_0, f_1, f_2, f_{02})(P) + (1, 4, 4, 12).$$

Proof. Apply Lemma 2.2.11 and Proposition 2.2.10 and use $f(F) = f(F_0) = f(\Delta_3)$ and $f(F \cap F_0) = f(\Delta_2)$. □

Chapter 3

f- and flag vectors

In this chapter we aim at a first closer examination of f-vectors and flag vectors of polytopes. First we mention two different polytope bases, that is, sets of polytopes whose flag vectors span the affine space defined by the Generalized Dehn-Sommerville equations (Theorem 1.2.15).

We then examine linear inequalities for the flag vectors of polytopes. The techniques decribed in Section 1.4 give rise to plenty of inequalities, where for the vast majority of them questions like necessity or tightness are open. These topics will be considered for low dimensions in later chapters.

However, these linear restrictions can be used to visualise the space inhabited by the polytope flag vectors. Here we describe this method in general and later use it for polytopes of not too high dimension to obtain visualisations that lead to new conjectures and questions. Although we mainly deal with flag vectors in this chapter, one can specialise the statements and get analogous results for the f-vectors.

In the last section we illustrate the given visualisation method by considering 3-dimensional polytopes and review Steinitz' characterisation of their f-vectors. Additionally, we characterise the f-vectors of centrally-symmetric 3-polytopes.

3.1 Polytope bases

We will see in this section that the flag vectors of Eulerian posets—indeed, even of polytopes—affinely span the subspace defined by Theorem 1.2.15. This implies that

$$\operatorname{\mathsf{dim\,aff}} \mathcal{F}_d = d - 1 \qquad \text{and} \qquad \operatorname{\mathsf{dim\,aff}} \mathcal{F\ell}_d = F_d - 1.$$

To show this one has to find a *basis of polytopes* of size F_d, that is, a set Ω of d-polytopes such that $|\Omega| = F_d$ and $\mathsf{aff}\{\mathbf{f}(P) \mid P \in \Omega\} = \mathsf{aff}\, \mathcal{F\ell}_d$. Two such bases are discussed in the next two subsections.

THE BASIS OF BAYER AND BILLERA

The basis of polytopes described by Bayer and Billera [7] can be obtained by recursively applying pyramiding and bipyramiding operations.

In contrast to the basis of Kalai that we describe later in this section, we will not use the Bayer-Billera basis again. Nevertheless it is interesting in its own right due to an additional property not mentioned in the original proof in [7]: the flag vectors of its polytopes span the affine integer lattice of the corresponding dimension.

Definition 3.1.1 (Bayer-Billera basis)**.** For $d \geq 1$ define $\Omega_d := \Omega_d^{\mathsf{pyr}} \cup \Omega_d^{\mathsf{bipyr}}$, where

$$\Omega_d^{\mathsf{pyr}} := \begin{cases} \{\Delta_1\} & \text{if } d = 1 \\ \{\mathsf{pyr}\, P \mid P \in \Omega_{d-1}\} & \text{if } d > 1 \end{cases}$$

and

$$\Omega_d^{\mathsf{bipyr}} := \begin{cases} \emptyset & \text{if } d = 1 \\ \{\mathsf{bipyr}\, P \mid P \in \Omega_{d-1}^{\mathsf{pyr}}\} & \text{if } d > 1 \end{cases}$$

Furthermore, let $\widetilde{\Omega}_d := \Omega_d \setminus \{\Delta_d\}$.

Ω_d is the set of all d-polytopes that arise from iteratively building pyramids or bipyramids over the 1-simplex, without ever constructing two bipyramids consecutively. It is easy to show that $|\Omega_d| = |\Psi_d| = F_d$.

Definition 3.1.2 (Relative flag vector)**.** For $d \geq 1$ let $\widetilde{\Psi}_d := \Psi_d \setminus \{\emptyset\}$. The *relative reduced flag vector* of a d-polytope P is the vector

$$\widetilde{\mathbf{f}}(P) := (\mathbf{f}(P) - \mathbf{f}(\Delta_d))|_{\widetilde{\Psi}_d}$$

where we consider only the components corresponding to the non-empty sparse sets.

Note that $f_\emptyset(P) - f_\emptyset(\Delta_d) = 0$ for all polytopes P, so no information is lost by restricting the relative flag vector to the $F_d - 1$ essential components.

Proposition 3.1.3. The system

$$\left\{\widetilde{\mathbf{f}}(P) \;\middle|\; P \in \widetilde{\Omega}_d\right\}$$

of $F_d - 1$ reduced relative flag vectors spans $\mathbb{Z}^{F_d - 1}$.

Proof (cf. Bayer and Billera [7]). Let $\widetilde{L}_d$ be the $(F_d - 1) \times (F_d - 1)$ matrix, rows indexed by $\widetilde{\Omega}_d$, columns indexed by $\widetilde{\Psi}_d$, representing the vectors in question, that is,

$$\widetilde{L}_d = \left(\widetilde{f}_S(P)\right)_{P \in \widetilde{\Omega}_d, S \in \widetilde{\Psi}_d}$$

We show that $\det \widetilde{L}_d = 1$. Define the $F_d \times F_d$ matrix

$$L_d \;=\; \left(\begin{array}{c|ccc} 1 & d+1 & \frac{d(d+1)}{2} & \cdots \\ \hline 0 & & & \\ \vdots & & \widetilde{L}_d & \\ 0 & & & \end{array}\right)$$

that contains the reduced flag vector of the simplex in the first row. Obviously $\det \widetilde{L}_d = \det L_d$. A row of L_d indexed by $P \in \Omega_d$, $P \neq \Delta_d$ contains the relative flag vector of the polytope P, so adding the first row to every other yields the matrix

$$K_d \;=\; \left(f_S(P)\right)_{P \in \Omega_d, S \in \Psi_d}$$

and $\det L_d = \det K_d$. Therefore it suffices to show that $\det K_d = 1$.

For $d \leq 3$ one can easily verify that $\det K_d = 1$ holds. Bayer and Billera show in [7, Lemma 2.4] that $K_d = A_d T_d$ where A_d is a certain $F_d \times F_d$ matrix and

$$T_d \;=\; \begin{pmatrix} 1 & 0 & & & & \\ 0 & 1 & & & & \\ & & K_1 & & & \\ & & & K_2 & & \\ & & & & \ddots & \\ & & & & & K_{d-2} \end{pmatrix}$$

By induction, $\det T_d = \det K_1 \cdot \cdots \cdot \det K_{d-2} = 1$. Furthermore, Bayer and Billera show that $\det A_d = 1$ for $d \leq 3$ and A_d can be constructed from an $(F_d \times F_d)$-matrix

$$E \;:=\; \left(\begin{array}{c|c} A_{d-1} & * \\ \hline 0 & I \end{array}\right)$$

by performing row and column operations that do not change the determinant. Therefore we have $\det A_d = \det E = 1$ by induction, which implies $\det K_d = \det A_d \cdot \det T_d = 1$. $\square$

THE BASIS OF KALAI

Kalai [36] gave a different basis of polytopes. Its members are constructed from sums and joins of simplices whose dimensions comply with certain rules. We give the basic definitions and the main result, on which we build in Chapter 5.

Definition 3.1.4 (Admissible sequences)**.** An *admissible sequence* is a $(2k)$-tuple $(m_1, \ell_1; \ldots; m_k, \ell_k)$ of non-negative integers, $k \geq 1$, such that

(i) $m_i \geq \ell_i \geq 1$ for $i \in \{1, \ldots, k-1\}$ and

(ii) $m_k \geq \ell_k \geq 0$.

We call $\mathsf{dim}\, b := m_1 + \ell_1 + \ldots + m_k + \ell_k + k - 1$ the *dimension* of the admissible sequence b and denote by $\mathbf{B}_d$ the set of all admissible sequences of dimension d.

For example, the sequences $(3, 0)$ and $(2, 1)$ are admissible sequences of dimension 3, while $(1, 1; 1, 0)$ and $(2, 1; 0, 0)$ are admissible sequences of dimension 4. The tuples $(4, 2; 1, 0; 1, 1)$ and $(1, 3; 3, 2; 1, 1)$ are no admissible sequences, and also $(3, 1; 1, 1; 0, 1)$ is not.

It is not hard to show that the number of admissible sequences is $|\mathbf{B}_d| = F_d$, see [36, Claim 2.1].

Definition 3.1.5 (Kalai basis)**.** Given an admissible sequence

$$b \;=\; (m_1, \ell_1; \ldots; m_k, \ell_k)\ , \quad \mathsf{dim}\, b = d,$$

define the polytope

$$P[b] \;:=\; ((\ldots(\Delta_{m_1} \oplus \Delta_{\ell_1}) * \Delta_{m_2}) \oplus \Delta_{\ell_2}) * \ldots * \Delta_{m_k}) \oplus \Delta_{\ell_k}$$

Let $\Theta_d := \{P[b] \mid b \in \mathbf{B}_d\}$ be the set of all polytopes defined by admissible sequences of dimension d.

Definition 3.1.5 to some extend justifies the postulations in Definition 3.1.4: It is obvious that $\mathsf{dim}\, P[b] = \mathsf{dim}\, b$ for an admissible sequence b. Furthermore, the sum of a polytope with a 0-dimensional simplex yields the same polytope, so ℓ_i should be greater than 0 for all indices i except possibly the last.

Θ_d always contains the d-dimensional simplex $\Delta_d = P[(d, 0)]$. Another easily recognisable element of Θ_d for $d \geq 2$ is $P[(d-1, 1)] = \Delta_{d-1} \oplus \Delta_1 = \mathsf{bipyr}\, \Delta_{d-1}$.

Additionally, to every admissible sequence a linear functional for flag vectors of d-polytopes can be associated. From Section 1.4 we know that every entry of the g-vector can be expressed as a linear combination of entries of the flag vector. Accordingly, we denote by g_ℓ^d, for $d \geq 2\ell$, the linear flag vector functional related to the ℓ-th entry of the g-vector of d-polytopes.

Now, given an admissible sequence $b = (m_1, \ell_1; \ldots; m_k, \ell_k)$ of dimension d, associate to b the linear functional

$$G[b] \ := \ g_{\ell_1}^{m_1+\ell_1} * \ldots * g_{\ell_k}^{m_k+\ell_k}$$

where $*$ denotes convolution (see Section 1.4).

To show that the system Θ_d defines a polytope basis, Kalai proved that the functionals $G[b]$ applied to the polytopes $P[b]$ yield 0 sufficiently often. More precisely, define the following partial order on $\mathbf{B}_d$:

$$\begin{aligned} b = (m_1, \ell_1, \ldots, m_k, \ell_k) \ \preccurlyeq \ b' = (m'_1, \ell'_1, \ldots, m'_{k'}, \ell'_{k'}) \quad &:\Longleftrightarrow \\ \sum_{i=1}^{j}(m_i + \ell_i) \ \leq \ \sum_{i=1}^{j}(m'_i + \ell'_i) \text{ for all } j \in \{1, \ldots, \mathsf{min}\{k, k'\}\} &\text{ or} \\ k = k', \text{ and } m_i + \ell_i = m'_i + \ell'_i \text{ and } \ell_i \geq \ell'_i \text{ for all } i \in \{1, \ldots, k\}& \end{aligned}$$

With this partial order, the sequence $(d, 0) \in \mathbf{B}_d$ is always maximal, and $(1, 1; \ldots; 1, 1; 1, 0)$, respectively $(1, 1; \ldots; 1, 1; 1, 1)$, depending on the parity of d, is minimal. Then the set Θ_d is a polytope basis as a consequence of the following theorem.

Theorem 3.1.6 (Kalai [36, Theorem 4.1])**.** $G[b](P[b]) = 1$ for every $b \in \mathbf{B}_d$, and $G[c](P[b]) = 0$ for all $b, c \in \mathbf{B}_d$ with $b \not\preccurlyeq c$.

Important parts of the proof will be referred to in Chapter 5, where the polytopes in Θ_d are used to show tightness of certain flag vector inequalities.

3.2 LINEAR INEQUALITIES AND VISUALISATION

The results in the previous section, together with the Generalized Dehn-Sommerville equations fully describe the affine hull of all flag vectors of d-polytopes. The next step in understanding flag vectors would be to determine the "region" within this affine hull that contains all flag vectors. More explicitly, we are looking for the *convex* hull

$$\mathcal{F\ell C}_d \ := \ \mathsf{conv}\, \mathcal{F\ell}_d\,, \qquad \text{respectively} \qquad \mathcal{FC}_d \ := \ \mathsf{conv}\, \mathcal{F}_d.$$

This task seems to be much more difficult than the previous and indeed we can currently only give linear inequalities describing a generally larger set containing the closure of this convex hull.

An efficient way to encode a linear inequality for flag vectors of d-polytopes is by stating the (dual) vector $(\alpha_S)_{S\in\Psi_d} \in \mathbb{R}^{F_d}$ of coefficients on the left hand side. With the convention that the sense of the inequality is always "$\geq$", a linear inequality is completely described by such a vector α_S.

Example 3.2.1. Consider the valid inequalities

$$\begin{aligned} 2f_0(P) - d\,f_1(P) &\leq 0 \\ 2f_{d-1}(P) - d\,f_{d-2}(P) &\leq 0 \\ g_2(P) \;=\; f_1(P) - d\,f_0(P) + f_{02}(P) - 3f_2(P) + \binom{d+1}{2} &\geq 0 \end{aligned}$$

They yield the following vectors $(\alpha_S)_{S\in\Psi_d} \in \mathbb{R}^{F_d}$:

($\alpha_\emptyset$,	α_0 ,	α_1 ,	α_2 ,	α_{02} ,	$\ldots$,	α_{d-2} ,	$\ldots$)
(0 ,	-2 ,	d ,	0 ,	0 ,	$\ldots$,	0 ,	$\ldots$)
($2\,(1-(-1)^d)$,	$2\,(-1)^d$,	$-2\,(-1)^d$,	$2\,(-1)^d$,	0 ,	$\ldots$,	$2-d$,	$\ldots$)
($\binom{d+1}{2}$,	$-d$,	1 ,	-3 ,	1 ,	$\ldots$,	0 ,	$\ldots$)

Note that the second inequality had to be rewritten using the Euler equation in order to express it in terms of sparse sets.

Given a vector $\alpha \in \mathbb{R}^{F_d}$ and a d-polytope P we can evaluate the linear functional corresponding to α on the flag vector of P:

$$\alpha(P) \;:=\; \sum_{S\in\Psi_d} \alpha_S f_S(P) \;=\; \alpha \cdot \mathbf{f}(P).$$

Then the statement that the inequality represented by α is valid for all flag vectors of d-polytopes can be reformulated as

$$\alpha(P) \;\geq\; 0 \qquad \text{for all } d\text{-polytopes } P.$$

All the examples mentioned above are linear inequalities that are tight at the flag vector $\mathbf{f}(\Delta_d)$ of the d-simplex. Inequalities of this type help us to visualise f- and flag-vectors, provided the dimension is not too high. This was first done by Ziegler [59] for f-vectors of 4-polytopes. In the rest of this section we describe this method in general to apply it later.

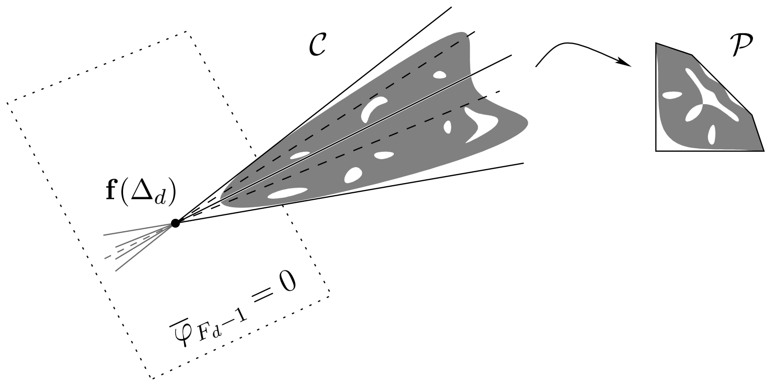


Figure 3.1: Projective coordinates for a cone of flag vectors.

Suppose we are given a finite set of linear inequalities, represented by vectors $\alpha^{(1)}, \ldots, \alpha^{(m)}$, that are tight at $\mathbf{f}(\Delta_d)$, that is, $\alpha^{(\ell)}(\Delta_d) = 0$ for all ℓ. We further assume that $\alpha^{(1)}|_{\widetilde{\Psi}_d}, \ldots, \alpha^{(m)}|_{\widetilde{\Psi}_d}$ span the linear space $\mathbb{R}^{F_d-1}$ and therefore the set

$$\mathcal{C} \;:=\; \{\mathbf{f} \mid \alpha^{(\ell)} \cdot \mathbf{f} \geq 0 \text{ for all } \ell\} \cap \{\mathbf{f} \mid f_\emptyset = 1\}$$

is a polyhedral cone with apex $\mathbf{f}(\Delta_d)$.

We now introduce—in two steps—coordinates such that $\mathcal{C}$ is mapped to a polyhedron $\mathcal{P}$ of dimension $F_d - 2$, making it possible to visualise the region defined by the inequalities if d is not too large. Geometrically, this amounts to considering a suitable cut through the cone $\mathcal{C}$. See Figure 3.1 for an illustration.

As first step we consider a linear transformation, given by an invertible matrix

$$T \;=\; (\tau_{k,S})_{\substack{0 \leq k \leq F_d-1 \\ S \in \Psi_d}}$$

that satisfies the following conditions (where τ_k is the k-th row of T):

(i) $\tau_0 = (1, 0, \ldots, 0)$,

(ii) $T \cdot \mathbf{f}(\Delta_d) = (1, 0, \ldots, 0)^\top$,

(iii) $\{\mathbf{f} \mid \tau_{F_d-1} \cdot \mathbf{f} = 0\} \cap \mathcal{C} = \{\mathbf{f}(\Delta_d)\}$.

In other words, T allows us to express any (potential) flag vector $\mathbf{f} \in \mathcal{C}$ in terms of different coordinates $\overline{\varphi} = (\overline{\varphi_0}, \ldots, \overline{\varphi}_{F_d-1})$, given by

$$\overline{\varphi} \;=\; T \cdot \mathbf{f} \tag{3.1}$$

where $\overline{\varphi}_0 = f_\emptyset = 1$ and the last coordinate $\overline{\varphi}_{F_d-1} \neq 0$ if $\mathbf{f} \in \mathcal{C} \setminus \{\mathbf{f}(\Delta_d)\}$.

In the second step we "mod out" the last coordinate by considering the map

$$\Phi : \left(\overline{\varphi}_0, \ldots, \overline{\varphi}_{F_d-1}\right) \mapsto (\varphi_0, \ldots, \varphi_{F_d-2})$$

with

$$\varphi_k \;:=\; \frac{\overline{\varphi}_k}{\overline{\varphi}_{F_d-1}} \qquad \text{for } 0 \leq k \leq F_d - 2 \tag{3.2}$$

and define

$$\mathcal{P} \;:=\; \Phi(T \cdot \mathcal{C}).$$

From the properties of the two maps and the prerequisites on the inequalities we get the following statement.

Proposition 3.2.2. Under the conditions (i)–(iii) above, the cone $\mathcal{C}$ maps to a polyhedron $\mathcal{P} \subseteq \left\{(\varphi_0, \ldots, \varphi_{F_d-2})^\top \in \mathbb{R}^{F_d-1} \mid \varphi_0 = 1\right\}$ with $\operatorname{dim} \mathcal{P} = F_d - 2$.

In the space $\left\{(\varphi_1, \ldots, \varphi_{F_d-2})^\top \mid \varphi_1, \ldots, \varphi_{F_d-2} \in \mathbb{R}\right\} \cong \mathbb{R}^{F_d-2}$, the polyhedron $\mathcal{P}$ is defined by the inequalities

$$\sum_{k=1}^{F_d-2} (\alpha^{(\ell)} T^{-1})_k \, \varphi_k \;\geq\; -(\alpha^{(\ell)} T^{-1})_{F_d-1}$$

for $1 \leq \ell \leq m$.

Proof. Due to condition (i) the set $\{\mathbf{f} \mid f_\emptyset = 1\}$ is mapped to the affine subspace $\left\{(\varphi_0, \ldots, \varphi_{F_d-2})^\top \in \mathbb{R}^{F_d-1} \mid \varphi_0 = 1\right\}$.

By (3.1) we have $\mathbf{f} = T^{-1}\overline{\varphi}$, therefore the points in $\mathcal{C}$ are exactly those with $\overline{\varphi}$ satisfying $\alpha^{(\ell)} T^{-1} \overline{\varphi} \geq 0$ for all ℓ and $\overline{\varphi}_0 = 1$. By (3.2) and condition (iii) this is for all $\mathbf{f} \in \mathcal{C} \setminus \{\mathbf{f}(\Delta_d)\}$ equivalent to

$$\sum_{k=0}^{F_d-2} (\alpha^{(\ell)} T^{-1})_k \, \varphi_k \;+\; (\alpha^{(\ell)} T^{-1})_{F_d-1} \;\geq\; 0$$

for $1 \leq \ell \leq m$. We show that $(\alpha^{(\ell)} T^{-1})_0 = 0$ for all ℓ, which implies the assertion. Assume that $(\alpha^{(\ell)} T^{-1})_0 \neq 0$ for some ℓ. By condition (ii) we get

$$\alpha^{(\ell)} \mathbf{f}(\Delta_d) \;=\; \alpha^{(\ell)} T^{-1} \cdot (1, 0, \ldots, 0)^\top \;=\; (\alpha^{(\ell)} T^{-1})_0 \;\neq\; 0,$$

in contradiction to $\alpha^{(\ell)}(\Delta_d) = 0$ for all ℓ. $\square$

Hence every vertex of $\mathcal{P}$ corresponds to an extreme ray of the cone $\mathcal{C}$ and facets of $\mathcal{P}$ correspond to facets of $\mathcal{C}$.

We will in later chapters apply this procedure to linear inequalities that hold for all flag vectors of polytopes. Then our main concern will be the question if extreme rays or facets of $\mathcal{C}$ are also rays or facets, respectively, of the set $\mathcal{F\ell C}_d$.

Definition 3.2.3 (Facet-defining inequality)**.** Let α be an inequality for flag vectors of d-polytopes. We say that α is *facet-defining* for $\mathcal{F\ell C}_d$ if

$$\mathsf{dim\,aff}\left(\mathcal{F\ell C}_d \cap \{\mathbf{f} \mid \alpha \cdot \mathbf{f} = 0\}\right) \;=\; \mathsf{dim}\,\mathcal{F\ell C}_d - 1 \;=\; F_d - 2$$

that is, there are $F_d - 1$ polytopes such that their flag vectors are affinely independent and satisfy $\alpha \cdot \mathbf{f} = 0$.

However, $\mathcal{F\ell C}_d$ is defined as the convex hull over infinitely many points. Therefore chances are that it is not a polyhedron and in general it is not even closed (see Bayer [6]). Hence there might be inequalities that are not facet-defining, but cannot be improved either.

Definition 3.2.4 (Asymptotic faces and inequalities)**.** A face (vertex/edge/ray/facet) F of $\mathcal{C}$ is called an *asymptotic face (vertex/edge/ray/facet)* of $\mathcal{F\ell C}_d$ if

$$\mathsf{dim\,aff}\left(\overline{\mathcal{F\ell C}_d} \cap F\right) \;=\; \mathsf{dim}\,F.$$

Here $\overline{\mathcal{F\ell C}_d}$ denotes the closure of $\mathcal{F\ell C}_d$.

A linear inequality is *asymptotically facet-defining* or an *asymptotic inequality* if it defines an asymptotic facet of $\mathcal{F\ell C}_d$, that is, an asymptotic face of dimension $F_d - 2$.

To show that a face is an asymptotic face one can use the transformation described above and apply it to the flag vectors of suitable families of polytopes. In this way asymptotic flag vectors are obtained.

Definition 3.2.5 (Asymptotic flag vector)**.** A point $\mathbf{f} \in \mathbb{R}^{F_d}$ is an *asymptotic flag vector* if there is a family $\mathcal{F}$ of polytopes such that the image point $\Phi(T\cdot\mathbf{f})$ is a limit point of the set $\{\Phi(T \cdot \mathbf{f}(P)) \mid P \in \mathcal{F}\}$.

Then F is an asymptotic face if it contains enough points that are asymptotic flag vectors of polytope families or genuine flag vectors of polytopes that span $\mathsf{aff}\,F$. With this it is not hard to see that asymptotic inequalities that are not facet-defining really exist; see Chapters 4 and 5.

3.3 The 3-dimensional case

To illustrate the method we have described in the last section we apply it to the well-known case of 3-dimensional polytopes. Since $\Psi_3 = \{\emptyset, \{0\}, \{1\}\}$, the characterisation of flag vectors of 3-polytopes is contained in the characterisation of f-vectors, which in turn was done by Steinitz [54] in 1906.

Theorem 3.3.1 (Steinitz [54])**.** An integer vector (f_0, f_1, f_2) is the f-vector of a 3-polytope if and only if the following conditions hold:

(i) $f_0 - f_1 + f_2 = 2$,

(ii) $f_2 \leq 2f_0 - 4$ and $f_0 \leq 2f_2 - 4$.

The inequalities in (ii) are satisfied with equality for simplicial and simple 3-polytopes, respectively.

Sketch of proof. Necessity of the conditions can be derived from the fact that the graphs of 3-polytopes are planar. Proofs of Euler's formula and the relations between the number of vertices, edges and faces of planar graphs can be found in every basic book on graph theory; see [21] or [34] for instance.

To show sufficiency, one has to prove that every integer point contained in the 2-dimensional cone $\mathcal{C}$ defined by the equation and the two inequalities is the f-vector of some 3-polytope. This can be done by giving constructions that "reach" a new point $f' \in \mathcal{C}$ from an old one $f \in \mathcal{C}$, provided f is already established as the f-vector of some polytope. By taking the symmetry into account, which is the consequence of duality of polytopes, it is sufficient to consider one operation and two f-vectors to start with.

The operation is the usual stacking over a triangular facet, which establishes the point $f' = f + (1, 3, 2)$. The initial f-vectors are that of a tetrahedron, $(4, 6, 4)$, and that of a pyramid over a quadrangle, $(5, 8, 5)$. Starting with these two points, every integer point in $\mathcal{C}$ can be reached by suitable application of stacking and dualisation. $\square$

Figure 3.2 gives an illustration of the cone $\mathcal{C}$ and the integer points in it.

Reformulated in terms of the reduced flag vector Steinitz' theorem reads as follows.

Theorem 3.3.2. An integer vector $\mathbf{f} = (f_\emptyset, f_0, f_1)$ is the reduced flag vector of a 3-polytope if and only if $f_\emptyset = 1$ and $2f_1 \geq 3f_0 \geq f_1 + 6$.

The two inequalities are tight exactly for simple and simplicial polytopes, respectively.

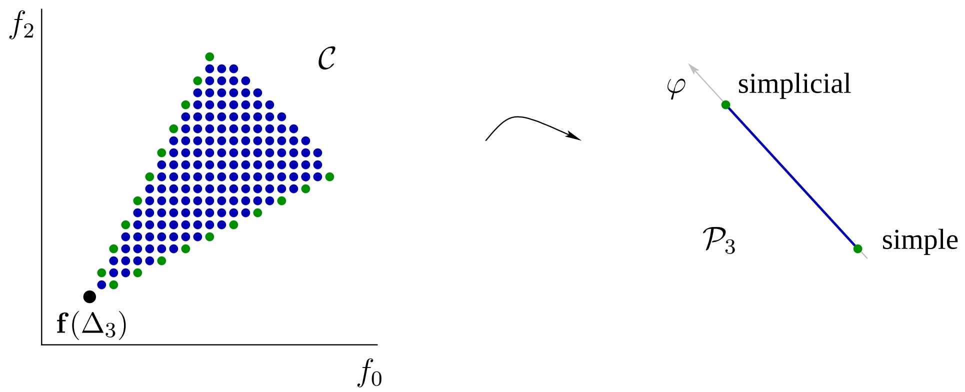


Figure 3.2: Cone of flag vectors of 3-polytopes

In terms of the previous section, the flag vectors of 3-polytopes are exactly the integer vectors of the cone $\mathcal{C}_3$, defined by the inequalities given by

$$\begin{aligned} \alpha^{(1)} &= (0, -3, 2) \\ \alpha^{(2)} &= (-6, 3, -1) \end{aligned}$$

As the transformation matrix we choose

$$T = \begin{pmatrix} 1 & 0 & 0 \\ -2 & -1 & 1 \\ -4 & 1 & 0 \end{pmatrix}$$

which is obviously regular and has the property that $\overline{\varphi}_2 = T \cdot \mathbf{f} = f_0 - 4f_\emptyset = 0$ occurs only if $\mathbf{f}$ is the flag vector of a 3-polytope with 4 vertices, which can then only be a 3-simplex. Additionally, $T \cdot \mathbf{f}(\Delta_3) = T \cdot (1, 4, 6)^\top = (1, 0, 0)^\top$.

After performing the second stage and applying Proposition 3.2.2 we arrive at the 1-dimensional polytope $\mathcal{P}_3 = \{\varphi \in \mathbb{R} \mid 1/2 \leq \varphi \leq 2\}$, see Figure 3.2.

The two vertices of $\mathcal{P}_3$ coming from the two rays—which are also facets—of $\mathcal{C}_3$ correspond to simple and simplicial polytopes, respectively. In particular, $\mathcal{C}_3$ has no asymptotic faces that are not also genuine faces.

Due to Steinitz' theorem we know that in the 3-dimensional case $\mathcal{C}_3$ equals $\mathcal{F}\ell\mathcal{C}_3$. Even more, every integer point in $\mathcal{F}\ell\mathcal{C}_3$ is a flag vector of a 3-polytope. Both statements are false already for dimension 4. Even considering the closure $\overline{\mathcal{F}\ell\mathcal{C}_d}$, it is an open question if this in general defines a cone at all, cf. Conjecture 6.3.26.

Centrally-symmetric 3-polytopes

Centrally-symmetric polytopes are an interesting special class of polytopes, due to their outstanding geometric feature. It turns out that in dimension 3 we can easily identify the flag vectors of centrally-symmetric polytopes among those of general polytopes.

Without loss of generality we can restrict ourselves to central symmetry with respect to the origin.

Definition 3.3.3 (Centrally-symmetric)**.** A polytope $P \subset \mathbb{R}^d$ is *centrally symmetric* if $x \in P$ implies $-x \in P$ for all $x \in \mathbb{R}^d$.

Before we state the characterisation of flag vectors of centrally-symmetric 3-polytopes, we gather some general observations that will be helpful and whose easy proofs we omit. The less obvious implication of part (b) is proved in Lemma 6.3.10.

Proposition 3.3.4. Let P be a centrally-symmetric d-polytope.

(a) All entries of the flag vector $\mathbf{f}(P)$ are even, except for $f_\emptyset(P) = 1$.

(b) $f_0(P) \geq 2d$ and equality holds if and only if P is affinely equivalent to the d-dimensional crosspolytope.

(c) The polar polytope P^Δ is again centrally-symmetric.

Proposition 3.3.4 already contains a characterisation of the possible flag vectors of centrally-symmetric 2-polytopes.

Lemma 3.3.5. Let P be a centrally-symmetric d-polytope, $d \geq 2$. Then

$$f_0(P) + f_{d-1}(P) \;\geq\; \begin{cases} 8 & \text{for } \; d = 2 \\ 14 & \text{for } \; d = 3 \\ 4d + 4 & \text{for } \; d \geq 4 \end{cases}$$

The bound is tight for $d = 2$ and $d = 3$, satisfied with equality exactly for the cube and the crosspolytope of the respective dimension.

Proof. Suppose $f_0(P) = 2d$. Then by Proposition 3.3.4 (b) P is a d-crosspolytope, therefore $f_{d-1}(P) = 2^d$ and

$$f_0(P) + f_{d-1}(P) \; \begin{cases} = \; 8 & \text{for } \; d = 2 \\ = \; 14 & \text{for } \; d = 3 \\ > \; 4d + 4 & \text{for } \; d \geq 4 \end{cases}$$

d	**Polytope**	f_0	f_{d-1}	$f_0 + f_{d-1}$	**Bound**
4	$\mathsf{bipyr}\, C_3$	10	12	22	20
5	$\Delta_5(3)$	12	20	32	24
	$H(G_4)$	16	16	32	
6	$H(G_5)$	24	24	48	28
7	$\mathsf{prism}(C_3 \oplus C_3)$	32	38	70	32

Table 3.1: Centrally-symmetric polytopes nearest to the bound in Lemma 3.3.5

Now suppose $f_{d-1}(P) = 2d$. Then by duality and Proposition 3.3.4, P is a d-cube, $f_0(P) = 2^d$ and the same conclusion as above holds.

Therefore we are left with the case that $f_0(P) \geq 2d+2$ and $f_{d-1}(P) \geq 2d+2$, but this immediately implies that $f_0(P) + f_{d-1}(P) \geq 4d + 4$, as claimed. $\square$

The bound for $d \geq 4$ in Lemma 3.3.5 is not tight. An enumeration approach (cf. Section 6.3) shows that for $d = 4$ the correct bound is 22 instead of 20, and is attained at $\mathsf{bipyr}\, C_3$.

For $5 \leq d \leq 7$ the results of the computer search in fact imply that the bound is at least $4d + 8$, but it is probably much larger in general. Table 3.1 shows polytopes that are interesting in this respect. Currently known to be nearest to the bound for $d = 5$ are the hypersimplex $\Delta_5(3)$ and the Hansen polytope of the path on four vertices. They both achieve the value 32, thereby also disproving the extended 3^d-conjecture—see Section 6.3 and [46] for details, as well as the paper of Hansen [31].

Theorem 3.3.6 (f-vectors of centrally-symmetric 3-polytopes)**.** An integer vector (f_0, f_1, f_2) is the f-vector of a centrally-symmetric 3-polytope if and only if the following conditions hold:

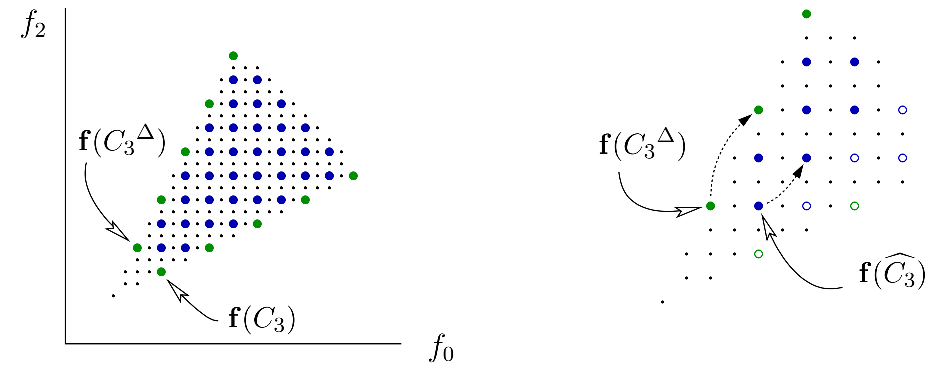
(i) f_0, f_1, f_2 are even,

(ii) $f_0 - f_1 + f_2 = 2$,

(iii) $f_2 \leq 2f_0 - 4$ and $f_0 \leq 2f_2 - 4$,

(iv) $f_0 + f_2 \geq 14$.

The inequalities in (iii) are tight exactly for simple and simplicial polytopes, respectively, and the inequality (iv) is tight only for the octahedron and the cube.



(a) Black dots mark lattice points that are flag vectors of polytopes, but not of centrally-symmetric ones

(b) Closeup of the cone showing the constructions needed in the proof of Theorem 3.3.6

Figure 3.3: Flag vectors of centrally-symmetric 3-polytopes

Proof. Necessity of the conditions as well as the tightness assertions follow from Theorem 3.3.1, Proposition 3.3.4 (a) and Lemma 3.3.5.

It remains to show that every vector satisfying the conditions (i) to (iv) is the flag vector of a centrally-symmetric 3-polytope. This is done in an analogous way as in the proof of Steinitz' Theorem. The illustration in Figure 3.3(a) might help to get an overview over what has to be done.

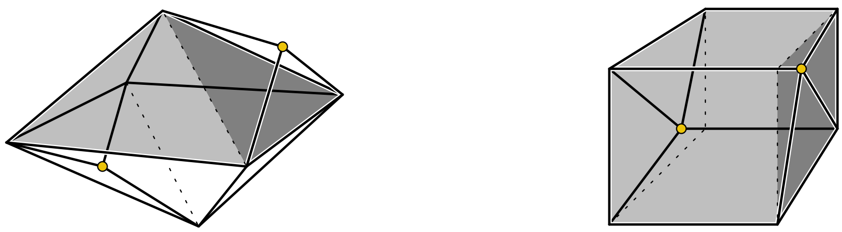
Let A be the polyhedron defined by the conditions (ii) to (iv) in the theorem. As before, we use two constructions to "reach" all desired points in the upper half of A, that is, all points in A with $f_2 \geq f_0$ and even entries; see Figure 3.3(b). The first construction is the usual stacking operation, applied in a centrally-symmetric way to two opposite triangular facets. By Proposition 2.1.4 this results in the f-vector $f + 2 \cdot (1,3,2) = f + (2,6,4)$ when applied to a polytope with f-vector f.

The second construction amounts to symmetrically pseudostack two opposite triangular facets. In the notation of Section 2.2, given a centrally-symmetric polytope P with a triangular facet F, choose an adjacent facet $F' \in \mathsf{adj}\, F$, and let $\mathcal{F} := \{F'\}$. Then there are symmetric facets $-F$ and $-F'$, and with $\mathcal{F}' := \{-F'\}$ the polytope

$$\mathsf{PS}^{-F}_{\mathcal{F}',\emptyset}(\mathsf{PS}^{F}_{\mathcal{F},\emptyset}(P))$$

exists by Proposition 2.2.3, is centrally-symmetric again, and the f-vector increases by $2 \cdot (1,2,1) = (2,4,2)$. See Figure 3.4(a).

Both constructions require a centrally-symmetric polytope with a triangular facet. Therefore we still have to show that all polytopes onto which we want



(a) To obtain $f + (2, 4, 2)$ (b) To obtain f-vector $(8, 14, 8)$

Figure 3.4: Pseudostacking to prove Theorem 3.3.6

to perform the described steps, have such a facet. Both operations create new simplex facets and therefore every resulting polytope can again be used as input to the constructions.

The octahedron ${C_3}^{\Delta}$ is simplicial and therefore clearly contains triangular facets. Now let $\widehat{C_3}$ be the polytope obtained by symmetrically pseudostacking a cube C_3 such that each added vertex is contained in the affine hull of an edge, see Figure 3.4(b). Using pseudostacking terminology,

$$\widehat{C_3} := \mathsf{PS}^{-F}_{\{-F_1,-F_2\},\emptyset}(\mathsf{PS}^{F}_{\{F_1,F_2\},\emptyset}(C_3))$$

with an arbitrary facet F of C_3 and two intersecting facets $F_1, F_2 \in \mathsf{adj}\, F$. Section 2.2 contains no statement that this can be done in general, since the vertex $F \cap F_1 \cap F_2$ is not nonsimple and F is not a simplex—in this case, however, it is obvious that a pseudostacking point exists, and also how the f-vector changes. The two facets F and $-F$ are broken into two triangles each, while the rest of the cube remains combinatorially unchanged. Hence $f(\widehat{C_3}) = (8, 14, 8)$ and in particular $\widehat{C_3}$ contains triangles.

Taking either ${C_3}^{\Delta}$ or $\widehat{C_3}$ as a starting point, we can now for any given vector f in $A \cap \{(f_0, f_1, f_2) \mid f_2 \geq f_0\}$ with even entries obtain a centrally-symmetric polytope P with $f(P) = f$. See Figure 3.3(b), where the two operations applied to the two starting polytopes are indicated by dotted lines.

Finally, all points in A with $f_0 \geq f_2$ are also f-vectors of centrally-symmetric polytopes, by taking duals of the above constructed polytopes. □

For sake of completeness we again rewrite the above theorem in terms of the reduced flag vector.

Theorem 3.3.7. An integer vector $\mathbf{f} = (f_\emptyset, f_0, f_1)$ is the reduced flag vector of a centrally-symmetric 3-polytope if and only if the following conditions hold:

(i) $f_\emptyset = 1$,

(ii) f_0 and f_1 are even,

(iii) $2f_1 \geq 3f_0 \geq f_1 + 6 \geq 18$.

The first two inequalities are tight exactly for simple and simplicial polytopes, respectively, and the last one is tight only for the octahedron and the cube.

We conclude this section by mentioning that one could also focus on other subclasses of 3-polytopes. A concise account of the region spanned by flag vectors of cubical polytopes and zonotopes, for instance, has been provided by Stenson [56].

Chapter 4

The cone of flag vectors of 4-polytopes

The first visualisation of the flag vector cone of 4-polytopes was given by Bayer [6]. Using the method described in Section 3.2 we provide a new version of the picture and explain classes of extremal polytopes that occur.

We further prove that one of the rays of the cone completely belongs to $\mathcal{F}\ell\mathcal{C}_4$. This new result is also published in [43]. Similarly, we give a construction showing that one of the facets is an asymptotic facet of $\mathcal{F}\ell\mathcal{C}_4$.

With these results we can conlude that none of the inequalities that define the cone are redundant, that is, the cone is in this sense a close approximation of $\mathcal{F}\ell\mathcal{C}_4$. Additionally, we state the closest further inequalities that the currently known examples would allow.

4.1 Visualisation and known facts

Bayer [6] gave a list of all known linear inequalities valid for all flag vectors of 4-polytopes. Despite some effort, no new such inequalities have been proved up to this day, so the following list that describes the cone $\mathcal{C}_4$ is the closest approximation to $\mathcal{F}\ell\mathcal{C}_4$ that we currently have.

Theorem 4.1.1 (Bayer [6, Theorem 1])**.** The reduced flag vector $\mathbf{f}$ of every 4-polytope satisfies the six inequalities

$$\begin{aligned}
f_0 - 5f_\emptyset &\geq 0,\\
6f_1 - 6f_0 - f_{02} &\geq 0,\\
f_2 - f_1 + f_0 - 5f_\emptyset &\geq 0,\\
f_{02} - 3f_1 &\geq 0,\\
f_{02} - 3f_2 &\geq 0,\\
f_{02} - 3f_2 + f_1 - 4f_0 + 10f_\emptyset &\geq 0.
\end{aligned}$$

□

We now apply the visualisation recipe from Section 3.2 to obtain a projective version $\mathcal{P}_4$ of $\mathcal{C}_4$. For this we use the transformation matrix

$$T = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -5 & 1 & 0 & 0 & 0 \\ 0 & -6 & 6 & 0 & -1 \\ -5 & 1 & -1 & 1 & 0 \\ -20 & 2 & -2 & 0 & 1 \end{pmatrix}$$

Again, $T \cdot \mathbf{f}(\Delta_4) = (1, 0, 0, 0, 0)^\top$, and $-20 + 2f_0 - 2f_1 + f_{02} = f_{03} - 20 = 0$ is equivalent to a 4-polytope having exactly five facets with four vertices each, which can only be a 4-simplex. Also, $\det T = 4$, hence it meets all the prerequisites from Section 3.2. After transformation to the coordinates

$$\begin{aligned} \varphi_1 &= \frac{f_0 - 5}{f_{03} - 20} \\ \varphi_2 &= \frac{6f_1 - 6f_0 - f_{02}}{f_{03} - 20} = \frac{2(f_1 + f_2) - 2(f_0 + f_3) - f_{03}}{f_{03} - 20} \\ \varphi_3 &= \frac{f_0 - f_1 + f_2 - 5}{f_{03} - 20} = \frac{f_3 - 5}{f_{03} - 20} \end{aligned}$$

we get the polyhedron $\mathcal{P}_4$ defined by the inequalities

$$\varphi_1 \geq 0 \tag{4.1}$$
$$\varphi_2 \geq 0 \tag{4.2}$$
$$\varphi_3 \geq 0 \tag{4.3}$$
$$12\varphi_1 + \varphi_2 \leq 3 \tag{4.4}$$
$$\varphi_2 + 12\varphi_3 \leq 3 \tag{4.5}$$
$$3\varphi_1 + 3\varphi_3 \leq 1 \tag{4.6}$$

For an illustration of $\mathcal{P}_4$ see Figure 4.1. The vertices of $\mathcal{P}_4$ are highlighted and named in accordance with the sketch of Bayer [6].

The symmetry of $\mathcal{P}_4$ reflects duality of polytopes: Let

$$\varphi = (\varphi_1, \varphi_2, \varphi_3) = \Phi(T \cdot \mathbf{f}(P))$$

for a polytope P, then $(\varphi_3, \varphi_2, \varphi_1) = \Phi(T \cdot \mathbf{f}(P^\Delta))$. In other words, the involution $p : (\varphi_1, \varphi_2, \varphi_3) \mapsto (\varphi_3, \varphi_2, \varphi_1)$ maps the projective image of the flag vector of a polytope to that of its polar. Therefore $\mathcal{P}_4$ (and also $\mathcal{C}_4$ for an analogous mapping) are invariant under p. Additionally, for all self-dual

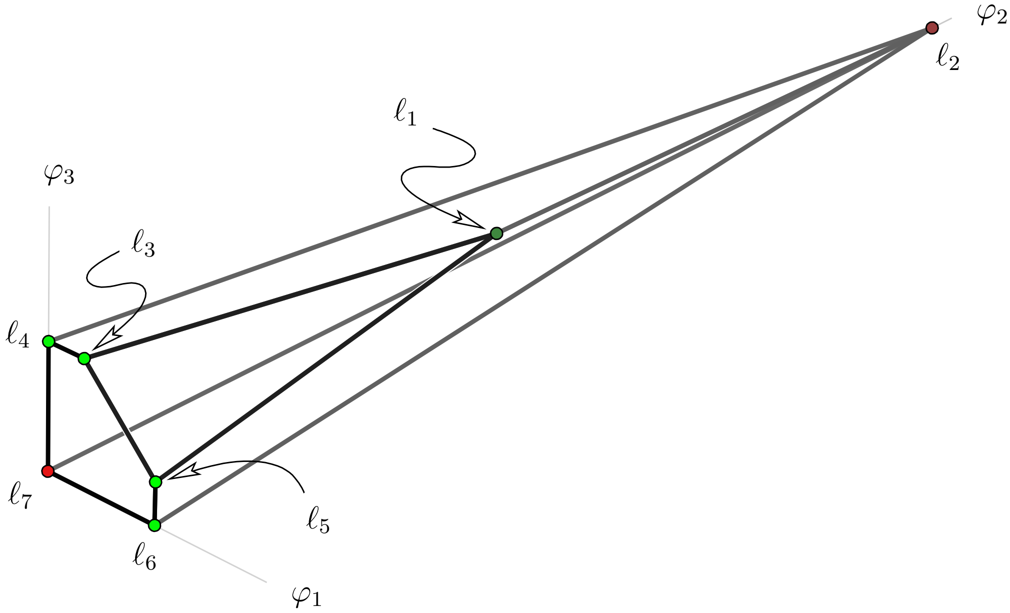


Figure 4.1: Projective cone of flag vectors of 4-polytopes

polytopes φ is contained in the set of fixed points of p, which is the plane given by $\varphi_1 = \varphi_3$.

This has the consequence that in the following we only have to investigate "one half" of $\mathcal{P}_4$ and get the corresponding statements about the "other half" by duality.

Each facet of $\mathcal{P}_4$ represents an inequality, which can be interpreted as a special property that 4-polytopes can have.

Definition 4.1.2 (Extremal properties of 4-polytopes)**.** A 4-dimensional polytope P is called

(a) *elementary* if $g_2(P) = 0$,

(b) 2-*simplicial* if every 2-face of P is a triangle (equivalently, every facet of P is simplicial),

(c) 2-*simple* if every edge of P is contained in 3 facets (equivalently, P^Δ is 2-simplicial),

(d) *center-boolean* if every facet of P is simple (equivalently, the vertex figure P/v of every vertex v of P is simplical).

Proposition 4.1.3. Let P be a 4-polytope and $\mathbf{f} = \mathbf{f}(P)$ its flag vector.

(a) P is elementary if and only if $f_{02} - 3f_2 + f_1 - 4f_0 + 10 = 0$, that is, (4.6) holds with equality.

(b) P is 2-simplicial if and only if $f_{02} - 3f_2 = 0$, that is, (4.5) holds with equality.

(c) P is 2-simple if and only if $f_{02} - 3f_1 = 0$, that is, (4.4) holds with equality.

(d) P is center-boolean if and only if $6f_1 - 6f_0 - f_{02} = 0$, that is, (4.2) holds with equality.

Proof. (a) follows directly from the definition of elementary, by the formula for g_2, cf. Sections 1.4 and 3.2.

(b) P is 2-simplicial if and only if every 2-face contains exactly 3 vertices, that is

$$f_{02} \;=\; \sum_{\text{2-faces } F} f_0(F) \;=\; \sum_{\text{2-faces } F} 3 \;=\; 3f_2.$$

(c) P is 2-simple if and only if every edge is contained in exactly 3 facets, that is $f_{13} = 3f_1$. By the Generalized Dehn-Sommerville equations we get

$$f_{02} \;=\; f_{03} - 2f_0 + 2f_1 \;=\; f_{13} - 2f_0 + 2f_1 - 2f_2 + 2f_3 \;=\; f_{13}.$$

(d) P is center-boolean if and only if every facet F is a simple 3-polytope, that is $f_0(F) = 2f_2(F) - 4$. Summing up over all facets we get

$$f_{03} \;=\; 2f_{23} - 4f_3 \;=\; 4f_2 - 4f_3$$

and writing this in terms of f_{02} as before and using Euler's equation

$$f_{02} \;=\; 2f_1 - 2f_0 + 4f_2 - 4f_3 \;=\; 6f_1 - 6f_2.$$

□

Another way to interpret $g_2(P)$ is by way of the *complexity* of a polytope

$$C \;:=\; \frac{f_{03} - 20}{f_0 + f_3 - 10}$$

as introduced by Ziegler [59]. By definition, we have $\varphi_1 + \varphi_3 = 1/C$ and inequality (4.6) states that the complexity of every polytope is at least 3. So the polytopes with flag vectors on the facet $\langle \ell_1, \ell_3, \ell_5 \rangle$ are exactly those with minimal complexity.

Table 4.1 summarises the depiction of the above properties. Note that the two facets $\langle \ell_2, \ell_4, \ell_7 \rangle$ and $\langle \ell_2, \ell_6, \ell_7 \rangle$ cannot contain any flag vectors of polytopes. To see this assume $P \neq \Delta_4$ is a 4-polytope with $\varphi = \Phi(T \cdot \mathbf{f}(P))$ and $\varphi_1 = 0$ or $\varphi_3 = 0$. Then $f_0 = 5$ or $f_3 = 5$, but this both implies that P is a simplex. Accordingly, the edge of $\mathcal{P}_4$ on the φ_2-axis can only represent an asymptotic face. If there really are asymptotic flag vectors on the respective 2-face of $\mathcal{FlC}_4$ is an open problem and is equivalent to the question if there exist polytopes of arbitrarily high *fatness*

$$F \;:=\; \frac{f_1 + f_2 - 20}{f_0 + f_3 - 10}$$

cf. Section 6.1 and Ziegler [59] for details.

In the following we describe extremal polytopes to show that all of the six inequalities for $\mathcal{C}_4$ also define facets of $\mathcal{FlC}_4$, some of them asymptotic. Most of these examples are well-known, others are relatively new and some constructions—see Sections 4.2 and 4.3—have not been described before.

To show that an inequality is (possibly asymptotically) facet-defining we have to find 3 affinely independent (possibly asymptotic) flag vectors of polytopes with the respective property. For this it is useful to give polytopes that feature more than one of these properties, resulting in flag vectors on edges or even vertices of $\mathcal{P}_4$, that is, rays of $\mathcal{C}_4$.

Such polytopes can indeed be found and are listed in Table 4.2, together with the projective coordinates of the rays of $\mathcal{C}_4$. The given examples show that the rays marked by the green dots in Figure 4.1 are rays of $\mathcal{FlC}_4$, either asymptotic, like ℓ_4 and ℓ_6, or exact, like ℓ_1, ℓ_3 and ℓ_5 which contain flag vectors of polytopes with arbitrarily many vertices.

Facet	**Property**	**Associated flag vector inequality**
$\langle \ell_1, \ell_3, \ell_5 \rangle$	elementary	$g_2 \;=\; f_{02} - 3f_2 + f_1 - 4f_0 + 10 \;\geq\; 0$
$\langle \ell_1, \ell_2, \ell_3, \ell_4 \rangle$	2-simplicial	$f_{02} - 3f_2 \;\geq\; 0$
$\langle \ell_1, \ell_2, \ell_5, \ell_6 \rangle$	2-simple	$f_{02} - 3f_1 \;\geq\; 0$
$\langle \ell_3, \ell_4, \ell_5, \ell_6, \ell_7 \rangle$	center-boolean	$6f_1 - 6f_0 - f_{02} \;\geq\; 0$
$\langle \ell_2, \ell_4, \ell_7 \rangle$	few vertices	$f_0 - 5 \;\geq\; 0$
$\langle \ell_2, \ell_6, \ell_7 \rangle$	few facets	$f_3 - 5 \;\geq\; 0$

Table 4.1: Properties associated to the facets of $\mathcal{P}_4$

Since ℓ_4 and ℓ_6 are asymptotic rays, we have so far only shown that the involved inequalities (4.4), (4.5) and (4.2) are asymptotically facet-defining. However, we can give other examples away from the rays that span the desired facets.

All simplicial polytopes have flag vectors resulting in points on the edge $\langle \ell_3, \ell_4 \rangle$. This is due to the fact that all facets of a simplicial polytope P are simplices and hence both simplicial and simple; that is, P is both 2-simplicial and center-boolean. Examples of simplicial polytopes away from ℓ_3 and ℓ_4 include the 4-dimensional crosspolytope, as well as direct sums of 2-polytopes. The latter are especially interesting, since

$$\mathbf{f}(\pentagon_n \oplus \pentagon_m) \;\to\; \ell_4 \quad \text{for } n = m \to \infty$$

and

$$\mathbf{f}(\pentagon_n \oplus \pentagon_3) \;\to\; \ell_3 \quad \text{for } n \to \infty.$$

So in the projective picture their flag vectors somehow "interpolate" between those of cyclic and stacked polytopes.

This settles the facet-defining property for the three mentioned inequalities and implies the following (intermediate) result about $\mathcal{C}_4$.

Proposition 4.1.4. The facets

$$\langle \ell_1, \ell_3, \ell_5 \rangle\ , \quad \langle \ell_1, \ell_2, \ell_3, \ell_4 \rangle\ , \quad \langle \ell_1, \ell_2, \ell_5, \ell_6 \rangle \quad \text{and} \quad \langle \ell_3, \ell_4, \ell_5, \ell_6, \ell_7 \rangle$$

of $\mathcal{C}_4$ represent facet-defining inequalities for $\mathcal{F}\ell\mathcal{C}_4$. □

It remains to show that the two hyperplanes spanned by $\langle \ell_2, \ell_4, \ell_7 \rangle$ and $\langle \ell_2, \ell_6, \ell_7 \rangle$ are indeed asymptotically facet-defining, where we again—due to polytope duality—have to consider only one of them. The ray ℓ_4 already contains an asymptotic flag vector, that of cyclic polytopes. It turns out

Ray	$(\varphi_1, \varphi_2, \varphi_3)$	**Polytope properties**	**Examples**
ℓ_1	$(\frac{1}{6}, 1, \frac{1}{6})$	elementary 2-simple, 2-simplicial	see Section 4.2
ℓ_2	$(0, 3, 0)$	"fat" 2-simple, 2-simplicial	unknown
ℓ_7	$(0, 0, 0)$	"fat" center-boolean	unknown
ℓ_4	$(0, 0, \frac{1}{4})$	simplicial with few vertices	cyclic polytopes
ℓ_3	$(\frac{1}{12}, 0, \frac{1}{4})$	elementary simplicial	stacked polytopes
ℓ_5	$(\frac{1}{4}, 0, \frac{1}{12})$	elementary simple	truncated polytopes
ℓ_6	$(\frac{1}{4}, 0, 0)$	simple with few facets	dual cyclic polytopes

Table 4.2: Known polytopes on or close to the rays of $\mathcal{C}_4$

that we can find asymptotic flag vectors for both faces, $\langle \ell_4, \ell_7 \rangle$ and $\langle \ell_2, \ell_4 \rangle$, away from ℓ_2.

The polytopes in a family with an asymptotic flag vector for $\langle \ell_4, \ell_7 \rangle$ are asymptotically center-boolean, which means that nearly all facets are simple, and the number of vertices is less than the number of facets by order of magnitude. Polytopes that accomplish this and are asymptotically not simplicial are for instance projected products of polygons (see Ziegler [60]) or the neighbourly cubical polytopes obtained by Joswig & Ziegler in [33]. Also, Thilo Rörig constructed "Armadillotopes", polytopes with large simple facets, but few vertices, that yield the same asymptotic flag vector as the neighbourly cubical polytopes.

Flag vectors of polytopes in the interior of $\langle \ell_2, \ell_4 \rangle$ had up to now not been explicitly described. Such polytopes can be obtained by breaking facets of neighbourly cubical polytopes, which is the result of Section 4.3. This finally shows the following statement complementing Proposition 4.1.4.

Proposition 4.1.5. The facets $\langle \ell_2, \ell_4, \ell_7 \rangle$ and $\langle \ell_2, \ell_6, \ell_7 \rangle$ of $\mathcal{C}_4$ are asymptotically facet-defining for $\mathcal{F}\ell\mathcal{C}_4$. $\square$

The conclusion is that all of Bayer's inequalities are indeed necessary.

Theorem 4.1.6. None of the six inequalities for flag vectors of polytopes in Theorem 4.1.1 is redundant or can be improved—all of them are facet-defining, at least asymptotically. $\square$

However, this does not mean that $\mathcal{F}\ell\mathcal{C}_4$ is completely described. As mentioned, the quest for polytopes with arbitrarily high fatness is still under way. The "fattest" examples currently known are projected products of polygons and their duals with $F = 9-\varepsilon$ (see Ziegler [60]), and polytopes that arise from the E-construction (see Paffenholz and Ziegler [44]). In Figure 4.2 the asymptotic flag vectors of these are represented by the points $P = (0, 0, \frac{1}{16})$ and $E = (\frac{1}{8}, \frac{3}{2}, \frac{1}{8})$, respectively.

Together with the asymptotic flag vectors of the broken neighbourly cubical polytopes—the point $N = (0, \frac{1}{4}, \frac{11}{48})$—and their duals, this gives rise to three more inequalities which would be satisfied with equality asymptotically by the respective examples. They are indicated as shaded planes in Figure 4.2.

Conjecture 4.1.7. The reduced flag vectors of 4-polytopes satisfy the following linear inequalities:

$$\begin{aligned} 42f_0 - 26f_1 + 16f_2 - f_{02} &\geq 140, \\ 594f_0 - 234f_1 + 48f_2 + 29f_{02} &\geq 1980, \\ 594f_0 - 546f_1 + 360f_2 + 29f_{02} &\geq 1980. \end{aligned}$$

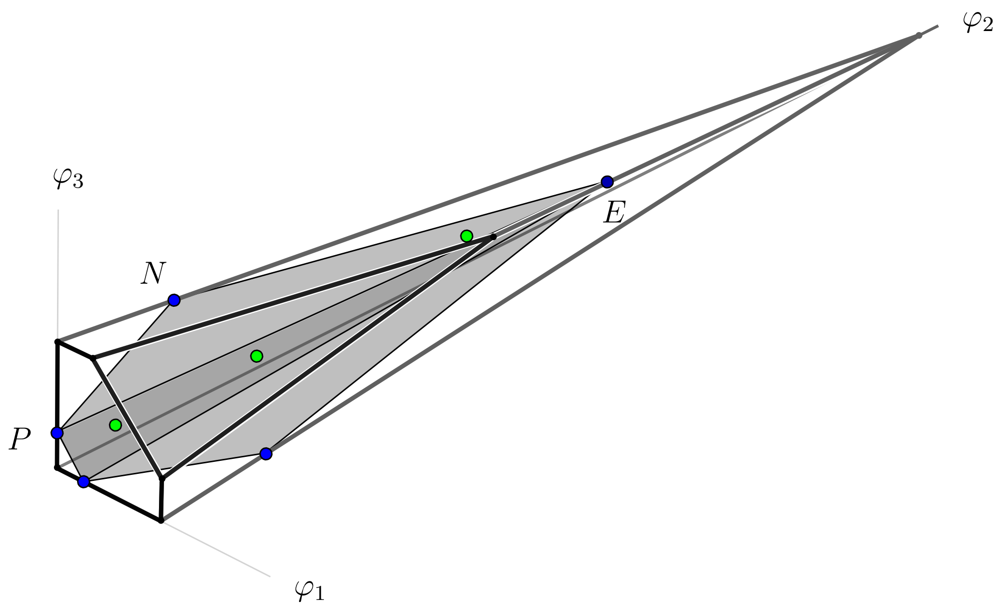


Figure 4.2: More special flag vectors and conjectured inequalities

While the last two of the inequalities above seem a bit too strange to be capable of a sensible interpretation, the first one is quite interesting. Since it is self-dual, it can be written more symmetrically:

$$20f_0 - 4f_1 - 4f_2 + 20f_3 + f_{03} \;\geq\; 140$$

and in terms of fatness and complexity:

$$4F - C \;\leq\; 20.$$

In particular, together with $C \leq 2F - 2$ (see Ziegler [59]) this would imply that $F \leq 9$ and therefore solve the fatness problem for the f-vectors of 4-polytopes.

Concluding this overview, we remark that there are also examples of polytopes with flag vectors in the relative interior of facets of $\mathcal{C}_4$, indicated by green dots in Figure 4.2:

- The Kalai basis polytope $P[(1,1;1,0)] = \mathsf{pyr}^2\, C_2$ as well as the multiplexes (see Bayer [8]) are elementary and give rise to points in the interior of facet $\langle \ell_1, \ell_3, \ell_5 \rangle$.

- Center-boolean polytopes having none of the other special properties in Proposition 4.1.3 can be obtained by taking connected sums (see Section 1.1) of *non*-elementary simplicial polytopes with their duals. Examples for these include direct sums of polygons, as mentioned above.
- Finally, 2-simplicial polytopes that are not 2-simple can be constructed from 2-simple, 2-simplicial polytopes that contain at least one simplex facet. Performing only the first three pseudostacking steps described in Section 4.2 destroys 2-simplicity while keeping 2-simpliciality.

4.2 THE RAY ℓ_1

The aim of this section is to prove that the ray ℓ_1 of $\mathcal{C}_4$ is indeed a ray of $\mathcal{F}\ell\mathcal{C}_4$. This result first appeared in [43]. Before, only two examples of polytopes were explicitly known to have flag vectors on ℓ_1, namely the simplex and the hypersimplex $\Delta_4(2)$. In the following we will construct arbitrarily large polytopes with flag vectors on ℓ_1.

We do this in a similar way as in the proofs of Theorems 3.3.1 and 3.3.6: we describe an operation that produces a new polytope from a smaller one with the desired effect on the flag vector, that is, the change in the flag vector matches exactly the direction of the ray ℓ_1.

The construction that has all the desired properties consists of four pseudostacking steps and we refer to the results from Section 2.2 to prove the claims. Figures 4.3 and 4.4 illustrate the construction. For the sake of a shorter notation we use the abbreviation

$$[p_1, \dots, p_n] \ := \ \mathsf{conv}\{p_1, \dots, p_n\}$$

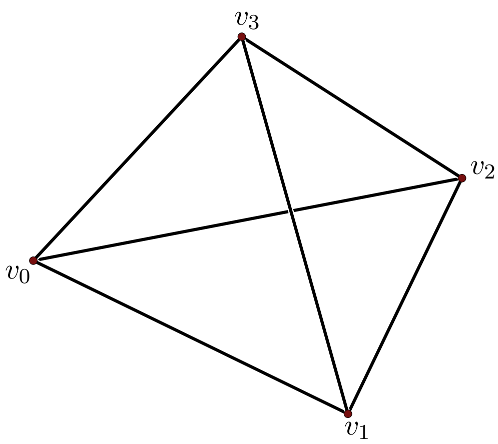
for points $p_1, \dots, p_n \in \mathbb{R}^4$ in the following. Furthermore, given a facet F of a polytope P and a ridge $R = [p_1, \dots, p_n]$ contained in F, then we denote by $F(p_1, \dots, p_n)$ the unique facet F' of P with the property that $F' \cap F = R$.

THE CONSTRUCTION

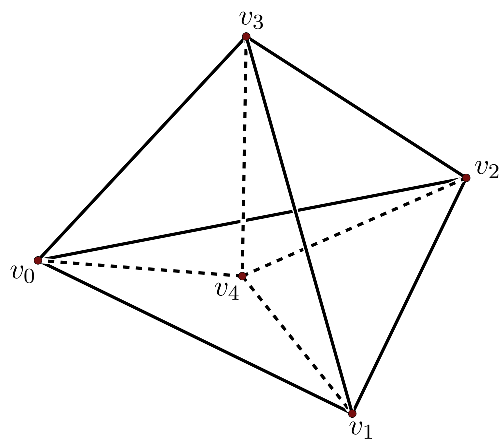
Let P be an elementary 2-simple, 2-simplicial 4-polytope that contains a simplex facet F. Denote the vertices of F by v_0, v_1, v_2, v_3, see Figure 4.3(a).

Step (i): Let $F^{(\mathrm{i})} := F$ and $F' := F(v_0, v_1, v_3)$. Then by Corollary 2.2.13 the polytope $P^{(\mathrm{i})} := \mathsf{PS}^{F^{(\mathrm{i})}}_{\{F'\},\emptyset}(P)$ exists and

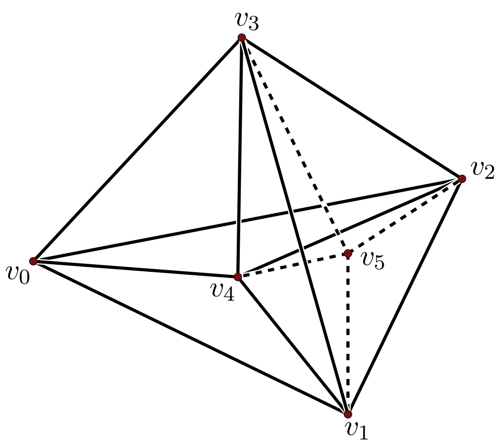
$$\mathbf{f}(P^{(\mathrm{i})}) \ = \ \mathbf{f}(P) + (0, 1, 4, 5, 15).$$



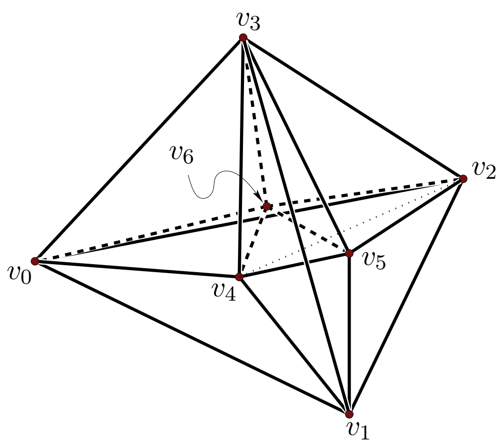
(a) Part of the Schlegel diagram of P with the labelling of the vertices of the facet F



(b) Part of the Schlegel diagram of $P^{(\mathrm{i})}$ with the labelling inherited from P



(c) Part of the Schlegel diagram of $P^{(\mathrm{ii})}$ with the bipyramid facet $[v_0, v_1, v_2, v_4, v_5]$



(d) Part of the Schlegel diagram of $P^{(\mathrm{iii})}$; the (simple) edge $[v_2, v_4]$ vanishes

Figure 4.3: Constructing $\mathcal{I}(P)$ from P—first three steps

We denote the pseudostacking point in this step by v_4; see Figure 4.3(b).

Step (ii): Let $F^{(\mathrm{ii})} := [v_1, v_2, v_3, v_4]$, which is a simplex facet of $P^{(\mathrm{i})}$ by Grünbaum's Theorem 2.2.2, and $F_1 := [v_0, v_1, v_2, v_4]$ and $F_2 := F^{(\mathrm{ii})}(v_1, v_2, v_3)$. The edge $[v_1, v_2] = F^{(\mathrm{ii})} \cap F_1 \cap F_2$ is contained in at least one other facet of $P^{(\mathrm{i})}$ (namely, $F_1(v_0, v_1, v_2)$), hence the set $\{F_1, F_2\}$ is nonsimple.
Then by Proposition 2.2.14 the polytope $P^{(\mathrm{ii})} := \mathsf{PS}^{F^{(\mathrm{ii})}}_{\{F_1,F_2\},\emptyset}(P^{(\mathrm{i})})$ exists

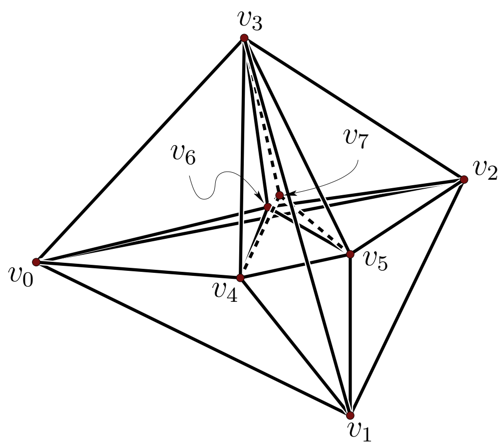


Figure 4.4: Last step of the construction, showing a part of the Schlegel diagram of $\mathcal{I}(P) = P^{(\text{iv})}$

and

$$\mathbf{f}(P^{(\text{ii})}) \;=\; \mathbf{f}(P^{(\text{i})}) + (0,1,4,4,12) \;=\; \mathbf{f}(P) + (0,2,8,9,27).$$

Denote the pseudostacking point by v_5.

Step (iii): Let $F^{(\text{iii})} := [v_0, v_2, v_3, v_4]$, which is a simplex facet of $P^{(\text{ii})}$, again by Grünbaum's Theorem 2.2.2, $F_0 := [v_2, v_3, v_4, v_5]$, $F_1 = [v_0, v_1, v_2, v_4, v_5]$ and $F_2 := F^{(\text{iii})}(v_0, v_2, v_3)$. The two edges $[v_0, v_2]$ and $[v_2, v_3]$ are again contained in uninvolved facets and the edge $[v_2, v_4]$ is simple. Thus the prerequisites of Proposition 2.2.16 are satisfied and we get the polytope $P^{(\text{iii})} := \mathsf{PS}^{F^{(\text{iii})}}_{\{F_1,F_2\},\{F_0\}}(P^{(\text{ii})})$ with

$$\mathbf{f}(P^{(\text{iii})}) \;=\; \mathbf{f}(P^{(\text{ii})}) + (0,1,4,4,12) \;=\; \mathbf{f}(P) + (0,3,12,13,39).$$

In this step, the vertex v_6 is added and the simple edge $[v_2, v_4]$ of $P^{(\text{ii})}$ vanishes, see Figure 4.3(d).

Step (iv): For the last step (see Figure 4.4) let $F^{(\text{iv})} := [v_3, v_4, v_5, v_6]$ and $F_1 := [v_0, v_3, v_4, v_6]$, $F_2 := [v_1, v_3, v_4, v_5]$ and $F_3 := [v_2, v_3, v_5, v_6]$. The edges $[v_3, v_4]$, $[v_3, v_5]$ and $[v_3, v_6]$ are again contained in other facets, so Proposition 2.2.15 applies and yields the polytope $P^{(\text{iv})} := \mathsf{PS}^{F^{(\text{iv})}}_{\{F_1,F_2,F_3\},\emptyset}(P^{(\text{iii})})$ with

$$\mathbf{f}(P^{(\text{iv})}) \;=\; \mathbf{f}(P^{(\text{iii})}) + (0,1,4,3,9) \;=\; \mathbf{f}(P) + (0,4,16,16,48).$$

Eventually, define $\mathcal{I}(P) := P^{(\mathrm{iv})}$ as the outcome of the construction, applied to P.

Proposition 4.2.1. If P is an elementary, 2-simple, 2-simplicial 4-polytope that contains a simplex facet, then $\mathcal{I}(P)$ is again elementary, 2-simple, 2-simplicial and contains a simplex facet.

Proof. In Step (iv), the facet $F^{(\mathrm{iv})}(v_4, v_5, v_6)$ is adjacent to $F^{(\mathrm{iv})}$ and the pseudostacking point v_7 is beneath it. Hence, by Grünbaum's Theorem 2.2.2 $P^{(\mathrm{iv})} = \mathcal{I}(P)$ contains a pyramid over the ridge $[v_4, v_5, v_6]$ as facet, which is the simplex $[v_4, v_5, v_6, v_7]$.

2-simpliciality can be read off from the f-vector, by Proposition 4.1.3:

$$f_{02}(\mathcal{I}(P)) \;=\; f_{02}(P) + 48 \;=\; 3f_2(P) + 3 \cdot 16 \;=\; 3f_2(\mathcal{I}(P))$$

and accordingly 2-simplicity:

$$f_{02}(\mathcal{I}(P)) \;=\; f_{02}(P) + 48 \;=\; 3f_1(P) + 3 \cdot 16 \;=\; 3f_1(\mathcal{I}(P)).$$

Finally, $\mathcal{I}(P)$ is elementary, since

$$\begin{aligned} g_2(\mathcal{I}(P)) &= f_1(\mathcal{I}(P)) - 4f_0(\mathcal{I}(P)) + 10 \\ &= (f_1(P) + 16) - 4(f_0(P) + 4) + 10 \\ &= f_1(P) - 4f_0(P) + 10 \\ &= 0. \end{aligned}$$

Here we have already used 2-simpliciality of both, P and $\mathcal{I}(P)$. □

The last thing that remains to do is to find a polytope P to apply Proposition 4.2.1 to. But this is easy—the simplex has all the desired properties.

Theorem 4.2.2. For every integer $k \geq 1$ there exists an elementary 2-simple, 2-simplicial 4-polytope P_{4k+1} on $4k + 1$ vertices.

Proof. For $k = 1$ take $P_5 := \Delta_4$, which is clearly elementary, 2-simple and 2-simplicial. The existence of P_{4k+1} for $k > 1$ then follows by induction using Proposition 4.2.1. □

Corollary 4.2.3. The ray ℓ_1 is contained in $\mathcal{FlC}_4$, the convex hull of flag vectors of 4-polytopes. □

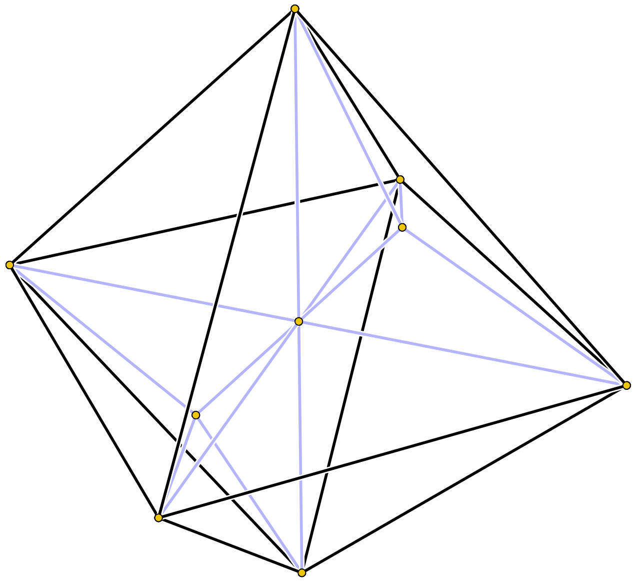


Figure 4.5: Symmetric Schlegel diagram of P_9.

It turns out that the polytope $P_9 := \mathcal{I}(\Delta_4)$ is the unique smallest 2-simple, 2-simplicial 4-polytope except for the simplex itself; see Section 7.2 for a proof. A particularly symmetric realisation of P_9 is given by

$$\operatorname{conv}\left\{\begin{pmatrix}3\\0\\0\\0\end{pmatrix},\begin{pmatrix}0\\3\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\\3\\0\end{pmatrix},\begin{pmatrix}1\\1\\1\\1\end{pmatrix},\begin{pmatrix}-3\\0\\0\\0\end{pmatrix},\begin{pmatrix}0\\-3\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\\-3\\0\end{pmatrix},\begin{pmatrix}-1\\-1\\-1\\1\end{pmatrix},\begin{pmatrix}0\\0\\0\\\frac{3}{2}\end{pmatrix}\right\}$$

see Figure 4.5.

In [43], examples of elementary 2-simple, 2-simplicial 4-polytopes are constructed for every integer $n \geq 9$ of vertices, except for 12. It is still open, whether even a non-elementary 2-simple, 2-simplicial 4-polytope on 12 vertices exists. Two examples of combinatorial spheres are given in Section 7.3. No geometric realisation of either of these examples is known to the author. Also, both fail to be elementary, with $g_2 = 1$ and $g_2 = 2$, respectively.

4.3 Polytopes with asymptotically few vertices

In this section we describe how to obtain 4-polytopes that are interesting primarily for two reasons: first, they have asymptotically fewer vertices than facets, that is, their flag vectors approach the facet $\langle \ell_2, \ell_4, \ell_7 \rangle$ of the flag vector cone and second, they are not center-boolean, that is, their flag vectors stay away from the facet $\langle \ell_3, \ell_4, \ell_5, \ell_6, \ell_7 \rangle$.

Together with the cyclic polytopes and projected products of polygons these *broken neighbourly cubical polytopes* establish the asymptotic facet-defining property of $\langle \ell_2, \ell_4, \ell_7 \rangle$. Taking polarity into account, this completes the proof of Proposition 4.1.5.

The examples are constructed by pushing a carefully chosen subset of vertices of neighbourly cubical polytopes, as defined by Joswig and Ziegler [33], thus modifying their facets in an appropriate way. It turns out that the resulting polytopes are additionally 2-simplicial, hence their flag vectors actually approach the interior of the face $\langle \ell_2, \ell_4 \rangle$ of the flag vector cone $\mathcal{C}_4$.

We first give a general description of how to break facets of polytopes before we recall the statements about neighbourly cubical polytopes that we are going to use. Eventually, we describe the concrete approach and verify the claims about the flag vectors.

Definition 4.3.1 (Pushing)**.** Let $P \subset \mathbb{R}^d$ be a d-polytope and $\mathbf{v} \in \mathsf{vert}\, P$ a vertex of P. Then P' arises from P by *pushing* $\mathbf{v}$ if

$$P' \;=\; \mathsf{conv}\,(\mathsf{vert}\, P \setminus \{\mathbf{v}\} \cup \{\mathbf{v}'\})$$

where $\mathbf{v}'$ is a point in P such that the intersection of the half-open segment $]\mathbf{v}, \mathbf{v}']$ with every hyperplane spanned by points in $\mathsf{vert}\, P$ is empty.

This definition appears for example in a paper of Klee [39]. He continues with a lower bound for the number of faces of P' and also proves that every face of P' that contains $\mathbf{v}'$ is in fact a pyramid with $\mathbf{v}'$ as its apex [39, Theorem 2.4]. We will use this fact, and also some of the statements in Klee's proof, for the proof of the following proposition.

We call a vertex $\mathbf{v}$ of a d-polytope Q a *simple vertex in* Q if the vertex figure $Q/\mathbf{v}$ is a $(d-1)$-simplex; equivalently, $\mathbf{v}$ has exactly d neighbours in Q, that is, the set

$$N_Q(\mathbf{v}) \;:=\; \{\mathbf{w} \mid \mathbf{w} \in \mathsf{vert}\, Q \text{ and } [\mathbf{v}, \mathbf{w}] \text{ edge of } Q\}$$

has exactly d elements.

Proposition 4.3.2. Let $P \subset \mathbb{R}^d$ be a d-polytope and $\mathbf{v} \in \mathsf{vert}\, P$. Suppose $\mathbf{v}$ is a simple vertex in all facets of P containing it.

If P' arises from P by pushing $\mathbf{v}$ then each facet of P' belongs to exactly one of the following three types:

(A) facets F of P with $\mathbf{v} \notin F$,

(B) facets of the form $\mathsf{conv}(N_F(\mathbf{v}) \cup \{\mathbf{v}'\}) \cong \Delta_{d-1}$ for all facets F of P with $\mathbf{v} \in F$,

(C) facets of the form $\mathsf{conv}(\mathsf{vert}\, F \setminus \{\mathbf{v}\})$ for all facets F of P with $\mathbf{v} \in F$ and $F \ncong \Delta_{d-1}$.

Proof. Let $\mathbf{v}'$ be the pushed vertex. Obviously, facets F of P with $\mathbf{v} \notin F$ are again facets of P' with $\mathbf{v}' \notin F$ and vice versa.

Now let F be a facet of P with $\mathbf{v} \in F$ and $h := \mathsf{aff}\, F$ the hyperplane defining F. Since $\mathbf{v}$ is a simple vertex in F, it has $d-1$ neighbouring vertices $\mathbf{v}_1, \dots, \mathbf{v}_{d-1} \in F$. If F contains further vertices, that is, $F \ncong \Delta_{d-1}$, then it follows from Klee's proof that h defines a facet $\mathsf{conv}(\mathsf{vert}\, F \setminus \{\mathbf{v}\})$ of P' (see [39, p. 709], statement (c)), which is of type (C). By the same statement (c) of Klee the set $\mathsf{conv}(\mathbf{v}', \mathbf{v}_1, \dots, \mathbf{v}_{d-1})$ is also a facet of P', and it is clearly a $(d-1)$-simplex.

Conversely, let F' be a facet of P' and suppose $\mathbf{v}' \in F'$. We show that in this case F' is a facet of type (B) and for this we have to find a facet of P that contains $\mathbf{v}$, in which the neighbours of $\mathbf{v}$ are exactly those of $\mathbf{v}'$ in F'.

By Klee's theorem [39, Theorem 2.4], F' is a pyramid with $\mathbf{v}'$ at its apex. Hence, if $\mathbf{v}_1, \dots, \mathbf{v}_n$ are the vertices in $\mathsf{vert}\, F' \setminus \{\mathbf{v}'\}$, they are all connected to $\mathbf{v}'$ by an edge, $\mathsf{aff}\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is a $(d-2)$-dimensional affine space, and $R := \mathsf{conv}\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is a ridge of P'.

This implies that there is a facet F'' of P' such that $R = F' \cap F''$ and a hyperplane h'' defining F'' as a facet of P'. All vertices of P' are contained in the same closed halfspace, defined by h'', and $\mathbf{v}$ is also in the same half-space: otherwise the segment $]\mathbf{v}, \mathbf{v}']$ would intersect h'', in contradiction to the definition of pushing. Since the vertices of P coincide with the vertices of P'—up to $\mathbf{v}$ and $\mathbf{v}'$—we get that h'' also defines a facet F of P.

If $\mathbf{v} \notin F$, that is, $F'' = F$, then R is also a ridge of P and we have a unique facet $\tilde{F}$ of P such that $F \cap \tilde{F} = R$. This facet has $\mathbf{v}$ as a vertex, since otherwise it would also be a facet of P', by Proposition 4.3.2, and then there were three different facets of P' containing the ridge R. For the same reason, none of the other vertices of P can be a vertex of $\tilde{F}$, so $\tilde{F}$ is a pyramid over R with $\mathbf{v}$ at the apex, since $\mathbf{v} \notin \mathsf{aff}\, R$ (otherwise $\mathbf{v}$ would be in h'' and therefore also in F). But $\mathbf{v}$ is a simple vertex in $\tilde{F}$, which implies that $n = d-1$ and $F' \cong \Delta_{d-1}$. Hence F' is of type (B), coming from the facet $\tilde{F} \cong \Delta_{d-1}$.

If $\mathbf{v} \in F$ then $\mathbf{v}$ is obviously also in h'' and F turns out to be F'', pseudostacked beyond R. To prove this, we show that in h''

(i) $\mathbf{v}$ is beneath or in the affine hull of every facet R' of F'', except for R, and

(ii) $\mathbf{v}$ is beyond R.

Let $R' \neq R$ be another facet of F''. Then R' is a ridge of P' and hence there exists some facet F''' such that $F''' \cap F'' = R'$. This facet does in fact not contain $\mathbf{v}'$, since otherwise $\mathbf{v}'$ had more neighbours in F'' than only $\mathbf{v}_1, \dots, \mathbf{v}_n$, by Klee's theorem [39, Theorem 2.4]. Let $h''' := \mathsf{aff}\, F'''$. Then all vertices of P' that are not in $\mathsf{vert}\, F'''$ (this includes $\mathbf{v}'$) are in the same open halfspace with respect to h'''. Since the segment $]\mathbf{v}, \mathbf{v}']$ does not intersect h''', the point $\mathbf{v}$ also lies in this halfspace, possibly only in its closure. Since $h''' \cap h'' = \mathsf{aff}\, R'$, this shows (i).

For (ii) let $\mathbf{w} \in \mathsf{vert}\, P' \setminus (\mathsf{vert}\, F'' \cup \{\mathbf{v}'\})$ and consider the hyperplane $h := \mathsf{aff}(R \cup \{\mathbf{w}\})$. Then F'' and $\mathbf{v}'$ are in different halfspaces defined by h, since $\mathbf{w}$ is in neither of the two facets incident to R. Again by the definition of pushing, $\mathbf{v}$ is in the same halfspace as $\mathbf{v}'$, that is, beyond or in the affine hull of R as a facet of F'' in h''. In fact, $\mathbf{v} \notin \mathsf{aff}\, R$, since otherwise it would also be in R, by (i).

Now Theorem 2.2.2 implies that for each vertex $\mathbf{v}_i$ there has to be a facet R' of F'' with $\mathbf{v}_i \in R'$ such that $\mathbf{v}$ is indeed beneath R' in h''. Again by Theorem 2.2.2 we get that $\mathsf{conv}\{\mathbf{v}_i, \mathbf{v}\}$ is a face of F (of type (C)), in other words, $\mathbf{v}_i$ is a neighbour of $\mathbf{v}$ in F. Since $\mathbf{v}_1, \dots, \mathbf{v}_n$ span the $(d-2)$-dimensional space $\mathsf{aff}\, R$, we have $n \geq d-1$, and since $\mathbf{v}$ is a simple vertex in F, we get $n = d-1$. As above, $F' \cong \Delta_{d-1}$ is of type (B), coming from the facet F.

Finally, consider the case that $\mathbf{v}' \notin F'$ and let $h := \mathsf{aff}\, F'$ be the hyperplane such that $F' = P' \cap h$. Then every vertex of P is in the same closed halfspace defined by h, since the vertices of P are the vertices of P', except that $\mathbf{v}'$ is replaced by $\mathbf{v}$. In fact, would $\mathbf{v}$ be in the open halfspace that does not contain the remaining vertices of P' then the segment $]\mathbf{v}, \mathbf{v}']$ would intersect the hyperplane h, in contradiction to the definition of pushing.

If $\mathbf{v} \in h$ then F' is of type (C), otherwise F' is of type (A). This finishes the proof. $\square$

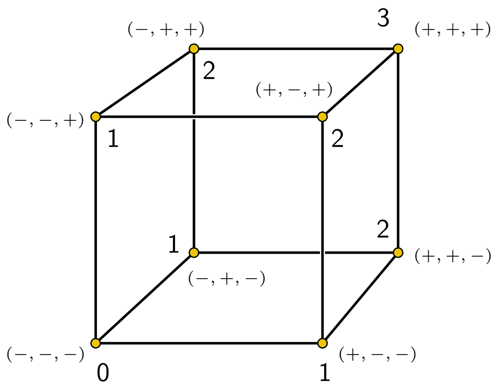


Figure 4.6: Sign vectors and heights of vertices in a 3-cube

Next we introduce the polytopes to which we want to apply the above pushing operation. The following theorem is the main result of [33], in fact in a more general statement than is needed for our construction.

Theorem 4.3.3 (Joswig & Ziegler [33, Theorem 16])**.** *For every $n \geq d \geq 2r+2$ there exists a combinatorial n-cube $C_n \subset \mathbb{R}^n$ and a linear projection map $\pi : \mathbb{R}^n \to \mathbb{R}^d$ such that $C_d^n := \pi(C_n)$ is a cubical d-polytope, and π induces an isomorphism between the r-skeleton of C_d^n and that of C_n.*

The combinatorics of these *neighbourly cubical polytopes* can be described more explicitly, which leads to a formula for the number of facets of C_d^n (see [33, Corollary 19]). We only need the 4-dimensional case, for which Theorem 4.3.3 provides us with cubical 4-polytopes on 2^n vertices, which have the graph of the n-dimensional cube. Their flag vectors are given by

$$(f_0, f_1, f_2, f_3, f_{03})(C_d^n) \;=\; 2^{n-2} \cdot \big(4, 2n, 3(n-2), n-2, 8(n-2)\big).$$

Let us shortly review a standard notation to describe the combinatorics of the cube. The vertices of the n-cube correspond to the n-tuples in $\{-,+\}^n$. More precisely, every vertex of a combinatorial n-cube C_n can be associated via a combinatorial equivalence to a vertex of the standard cube $[-1,1]^n$, which is an n-dimensional vector in $\{-1,1\}^n$. Replacing -1 by $-$ and $+1$ by $+$, we get a unique *sign vector* in $\{-,+\}^n$ for every vertex of C_n, see Figure 4.6. Having done this, we define the *height* $\mathsf{ht}\, v$ of a vertex v of C_n as the number of $+$ in its combinatorial description.

We can even describe every face of C_n in a similar way. Choose k out of the n positions in an n-tuple and fix a sign for all the remaining $n-k$ positions. Now all vertices of C_n whose sign vectors coincide at the $n-k$ fixed positions and are filled up with either sign at the chosen k positions span a k-face F of C_n. Hence, F can be associated to the vector in $\{-,+,0\}^n$ that has the common signs of its vertices at the $n-k$ fixed positions and 0 at the chosen k positions. Conversely, every k-face can by uniquely described in this way.

Proposition 4.3.4. Every face F of an n-dimensional cube can be associated uniquely to a sign vector $\sigma(F) \in \{-,+,0\}^n$. The dimension of F equals the number of 0's in $\sigma(F)$, and the sign vectors of faces $G \subseteq F$ are exactly those that are obtained from $\sigma(F)$ by replacing 0-entries with either $+$ or $-$. $\square$

For instance, in Figure 4.6, the bottommost edge corresponds to the vector $(0,-,-)$, and the front quadrangle to $(0,-,0)$.

The faces of the neighbourly cubical polytopes inherit this combinatorial description from the cubes they are projected from. By Theorem 4.3.3, for $d \geq 2$ all vertices are retained by the projection and hence for every k-face F of C_d^n there is a k-face of the original n-cube, spanned by the corresponding vertices, that is projected to F. Accordingly, a height can be assigned to every vertex of C_d^n, via its sign vector.

In particular, in the 4-dimensional case, every facet F of a neighbourly cubical polytope C_4^n is equipped with a sign vector in $\{-,+,0\}^n$ that has exactly three 0-entries. Furthermore, the vertices of F are the ones that correspond to sign vectors obtained from the sign vector of F by replacing the 0's with $-$ or $+$. From this it is clear that the following statement holds.

Proposition 4.3.5. For every facet F of C_d^n the heights of its vertices are four consecutive integer numbers $h, h+1, h+2, h+3$, with some $h \in \{0,\dots,n-3\}$. Furthermore, F comprises exactly one vertex of each height h and $h+3$, and three of height $h+1$ and $h+2$, respectively. $\square$

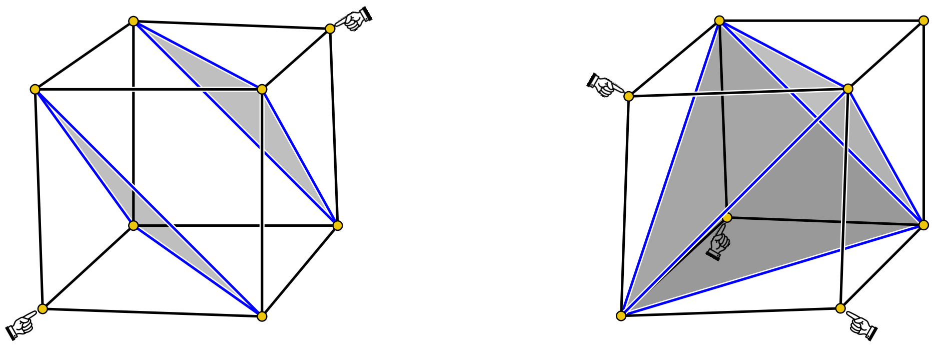
Compare Figure 4.6, where $h = 0$. Proposition 4.3.5 turns out to be the crucial observation for the facet breaking strategy, which we describe in the following section.

BREAKING FACETS OF NEIGHBOURLY CUBICAL POLYTOPES

The strategy is as follows: push every vertex v of C_d^n with height

$$\mathsf{ht}\, v \equiv 0 \mod 3.$$

We call the polytope obtained in this way a *broken neighbourly cubical polytope*, denoted by $\check{C}_d^n$. The last observation of the previous section, together with Proposition 4.3.2 allows us to describe the facets of $\check{C}_d^n$.



(a) An octahedron and two simplices arising from an $\mathcal{O}$-facet

(b) A bipyramid and three simplices arising from a $\mathcal{B}$-facet

Figure 4.7: Broken 3-cubes—the blue edges, together with the shaded triangle 2-faces arise from pushing in the indicated way

Consider a facet F of C_d^n and let $h := \mathsf{min}\,\{\mathsf{ht}\,v \mid v \in F,\ v \text{ vertex of } C_d^n\}$. If $h \equiv 0 \mod 3$ then by Proposition 4.3.5 two opposite vertices of F are pushed and Proposition 4.3.2 implies that the facet breaks up into an octahedron and two simplices; see Figure 4.7(a). In this case we call F an *$\mathcal{O}$-facet*.

If, on the other hand, $h \equiv 1 \mod 3$ or $h \equiv 2 \mod 3$, then three vertices of F are pushed which are all neighbours of some other vertex in F. Proposition 4.3.2 now implies that F is replaced by a bipyramid over a triangle and three simplices in $\check{C}_d^n$. Then F is called a *$\mathcal{B}$-facet*, as illustrated in Figure 4.7(b).

Proposition 4.3.6. Let $n \geq 4$. Then for the flag vector of the 4-dimensional broken neighbourly cubical polytope $\check{C}_4^n$ we have

$$(f_0, f_1, f_2, f_3;\, f_{03})(\check{C}_4^n) \;=$$
$$\big(2^n, 2^{n-2}(5n-6), 9(n-2)\cdot 2^{n-2} - o, 2^n(n-2) - o;\ 17(n-2)\cdot 2^{n-2} - 3o\big)$$

with $0 \leq o \leq f_3(C_4^n) = 2^{n-2}(n-2)$.

Furthermore, $\check{C}_4^n$ is 2-simplicial.

Proof. Let o be the number of $\mathcal{O}$-facets of C_4^n. The pushing operation does not change the number of vertices of a polytope, so $f_0(\check{C}_4^n) = f_0(C_4^n) = 2^n$.

Clearly, every facet of C_4^n is either an $\mathcal{O}$-facet or a $\mathcal{B}$-facet. Furthermore, every $\mathcal{O}$-facet gives rise to 3, every $\mathcal{B}$-facet to 4 facets of $\check{C}_4^n$. Hence

$$f_3(\check{C}_4^n) \;=\; 3o + 4\,(f_3(C_4^n) - o) \;=\; 2^n(n-2) - o.$$

The number of 2-faces can be computed in the following way: for every $\mathcal{O}$-facet 2 new triangles are created and 6 quadrilaterals break into 2 triangles each. The triangles that arise from the broken quadrilaterals, however, are counted twice, since they also break apart in the neighbouring facets. Therefore, every $\mathcal{O}$-facet accounts for $2 + \frac{1}{2} \cdot 6 = 5$ new 2-faces in $\check{C}_4^n$. Similarly, every $\mathcal{B}$-facet creates $3 + \frac{1}{2} \cdot 6 = 6$ new 2-faces. In total, we get

$$\begin{aligned} f_2(\check{C}_4^n) &= f_2(C_4^n) + 5o + 6\,(f_3(C_4^n) - o) \\ &= 3\,(n-2) \cdot 2^{n-2} + 6\,(n-2) \cdot 2^{n-2} - o \;=\; 9\,(n-2) \cdot 2^{n-2} - o. \end{aligned}$$

The number of edges can then be computed from Euler's equation.

Finally, for every $\mathcal{O}$-facet in C_4^n the broken $\check{C}_4^n$ contains 2 facets with 4 vertices each and 1 facet with 6 vertices. Analogously, for every $\mathcal{B}$-facet we have 3 facets with 4 vertices each and 1 facet with 5 vertices in $\check{C}_4^n$. This leads to

$$f_{03}(\check{C}_4^n) \;=\; o(2 \cdot 4 + 6) + (f_3(C_4^n) - o)(3 \cdot 4 + 5) \;=\; 17\,(n-2) \cdot 2^{n-2} - 3o.$$

2-simpliciality follows from the fact that every quadrilateral of C_4^n is broken into triangles and all new 2-faces are also triangles. □

Theorem 4.3.7. The family of broken neighbourly cubical polytopes $\check{C}_4^n$ gives rise to an asymptotic flag vector in the interior of the face $\langle \ell_2, \ell_4 \rangle$ of $\mathcal{C}_4$.

Proof. Using the transformation in Section 4.1, we obtain from Proposition 4.3.6:

$$\begin{aligned} \varphi_1 &= \frac{f_0 - 5}{f_{03} - 20} = \frac{2^n - 5}{17(n-2) \cdot 2^{n-2} - 3o - 20} \\ &\leq \frac{2^n - 5}{14(n-2) \cdot 2^{n-2} - 20} \overset{n\to\infty}{\longrightarrow} 0 \\ \varphi_2 &= \frac{2(f_1 + f_2) - 2(f_0 + f_3) - f_{03}}{f_{03} - 20} \\ &= \frac{2(2^{n-2}(14n - 24) - o) - 2(2^n(n-1) - o) - (17(n-2) \cdot 2^{n-2} - 3o)}{17(n-2) \cdot 2^{n-2} - 3o - 20} \\ &= \frac{3 \cdot 2^{n-2}(n-2) + 3o}{17 \cdot 2^{n-2}(n-2) - 3o - 20} \geq \frac{3 \cdot 2^{n-2}(n-2)}{17 \cdot 2^{n-2}(n-2)} \overset{n\to\infty}{\longrightarrow} \frac{3}{17} > 0 \end{aligned}$$

Here we have used both bounds for o from Proposition 4.3.6. The first limit shows that we get arbitrarily close to $\langle \ell_2, \ell_4, \ell_7 \rangle$ and the second that the asymptotic flag vector has positive distance from the face $\langle \ell_3, \ell_4, \ell_5, \ell_6, \ell_7 \rangle$. Furthermore, 2-simpliciality of $\check{C}_4^n$ ensures that all flag vectors, including the asymptotic, lie on the face $\langle \ell_1, \ell_2, \ell_3, \ell_4 \rangle$. □

Using the combinatorial description of neighbourly cubical polytopes by Sanyal and Ziegler [47] one can obtain explicit formulas for the numbers of $\mathcal{O}$-facets and $\mathcal{B}$-facets, respectively. A precise calculation then yields an asymptotic flag vector of $(f_0, f_1, f_2, f_{02}) = (0, 5, 26/3, 26)$ for the $\check{C}_4^n$ family. From this one gets the point $N = (0, 1/4, 11/48)$ in Figure 4.2.

A promising way to improve the above results (and thereby disprove Conjecture 4.1.7) might be to apply the pushing operation to break the facets of projected products of polygons (see Ziegler [60]) instead of neighbourly cubical polytopes. Their facet structure, however, is more complicated and it seems much harder to provide a pushing strategy that works.

Chapter 5

Kalai's and Braden's sequences

What has been done in the last chapter for dimension 4, namely trying to approximate the convex hull of flag vectors, $\mathcal{F}\ell\mathcal{C}_d$, by the polyhedron given by all linear inequalities we know, can in principle be done for arbitrary dimension. However, we have to deal with some difficulties.

Firstly, the dimension of the object we want to understand is too high to make useful visualisations of it. Furthermore, the number of inequalities "grows rapidly", as Ehrenborg remarks [24, p. 217]. On top of that, starting with dimension 6, the known inequalities do not longer determine a cone, that is, we cannot apply the projective transformation method to capture the complete information available.

Therefore we focus on selected inequalities which seem promising due to their special structure, and try to show that they are facet-defining for $\mathcal{F}\ell\mathcal{C}_d$. Braden (unpublished [16]) conjectured this for a certain set of inequalities that arise from convolutions of g-vector entries. He worked this out for dimension up to 6, and we will present a proof here. Furthermore, we show that some of these inequalities are asymptotically facet-defining in general, and cannot be more than that.

5.1 Admissible sequences and associated polytopes

Recall from Section 3.1 the basis polytopes defined by Kalai [36]: for a $(2k)$-tuple $b = (m_1, \ell_1; \ldots; m_k, \ell_k) \in \mathbb{N}^{2k}$ with $k \geq 1$ we define the polytope

$$P[b] := \begin{cases} \Delta_{m_1} \oplus \Delta_{\ell_1} & \text{if } k = 1 \\ \left(P[\hat{b}] * \Delta_{m_k}\right) \oplus \Delta_{\ell_k} & \text{if } k > 1 \end{cases}$$

where $\hat{b} = (m_1, \ell_1; \ldots; m_{k-1}, \ell_{k-1})$ if $k > 1$. We call b admissible, if

- for all i we have $m_i \geq \ell_i$, and
- for all i, except possibly the last, $\ell_i \geq 1$.

The number of admissible sequences of dimension d is the Fibonacci number F_d. This coincides with the affine dimension of the space of flag vectors of d-polytopes, see Section 1.2.

We now define a different set of sequences, which eventually yields the same polytopes. These sequences were proposed by Tom Braden [16].

Definition 5.1.1 (Braden's sequences)**.** Let $k \geq 1$ be an integer number. A $(2k)$-tuple $b = (m_1, \ell_1; \ldots; m_k, \ell_k) \in \mathbb{N}^{2k}$ is a *Braden sequence*, if it satisfies the following properties:

(i) If $\ell_i > 0$ for some $i \in \{1, \ldots, k\}$ then $m_i = \ell_i$.

(ii) If $\ell_i = 0$ for some $i \in \{1, \ldots, k-1\}$ then $\ell_{i+1} > 0$.

As before, the *dimension* of b is $\mathsf{dim}\, b := \left(\sum_{i=1}^{k} m_i + \ell_i\right) + k - 1$, and the *length* of b is k. We denote by $\mathbf{T}_d$ the set of all Braden sequences of dimension d.

We then have a one-to-one correspondence between the elements of $\mathbf{B}_d$ and those of $\mathbf{T}_d$.

Proposition 5.1.2. There exists a bijective mapping $\phi : \mathbf{B}_d \to \mathbf{T}_d$.

Proof. To define the mapping ϕ let $b = (m_1, \ell_1; \ldots; m_k, \ell_k) \in \mathbf{B}_d$. Then $\phi(b)$ is the result of the following transformation: for every $i \in \{1, \ldots, k\}$ with $m_i \neq \ell_i > 0$ replace

$$\ldots; m_i, \ell_i; \ldots \qquad \text{by the two pairs} \qquad \ldots; m_i - \ell_i - 1, 0; \ell_i, \ell_i; \ldots$$

Since $b \in \mathbf{B}_d$, we have $m_i > \ell_i$, hence $m_i - \ell_i - 1 \geq 0$. Clearly, the length of $\phi(b)$ is at least the length of b.

That $\phi(b)$ is in $\mathbf{T}_d$ is easy to see: (i) is satisfied since every pair that violated $m_i = \ell_i$ was replaced by two correct ones, and (ii) holds since the replacements create no two consecutive pairs with $\ell_i = 0$ and the only possible zero entry in the admissible sequence b could have been the last one.

Obviously, the transformation is reversible: If $b = (m_1, \ell_1; \ldots; m_k, \ell_k) \in \mathbf{T}_d$, then $m_i \geq \ell_i$ for every $i \in \{1, \ldots, k\}$ by definition. If $\ell_i = 0$ for some $i \in \{1, \ldots, k-1\}$ then $\ell_{i+1} > 0$, and we can replace the two pairs

$$\ldots; m_i, 0; m_{i+1}, \ell_{i+1}; \ldots \qquad \text{by the one pair} \qquad \ldots; m_i + m_{i+1} + 1, \ell_{i+1}; \ldots$$

This yields an admissible sequence and the two mappings are inverse to each other. $\square$

dim.	admissible	$\leftrightarrow$ Braden	dim.	admissible	$\leftrightarrow$ Braden
4	$(4,0)$	$\leftrightarrow (4,0)$	6	$(6,0)$	$\leftrightarrow (6,0)$
	$(3,1)$	$\leftrightarrow (1,0;1,1)$		$(5,1)$	$\leftrightarrow (3,0;1,1)$
	$(2,2)$	$\leftrightarrow (2,2)$		$(4,2)$	$\leftrightarrow (1,0;2,2)$
	$(2,1;0,0)$	$\leftrightarrow (0,0;1,1;0,0)$		$(3,3)$	$\leftrightarrow (3,3)$
	$(1,1;1,0)$	$\leftrightarrow (1,1;1,0)$		$(4,1;0,0)$	$\leftrightarrow (2,0;1,1;0,0)$
5	$(5,0)$	$\leftrightarrow (5,0)$		$(3,2;0,0)$	$\leftrightarrow (0,0;2,2;0,0)$
	$(4,1)$	$\leftrightarrow (2,0;1,1)$		$(3,1;1,0)$	$\leftrightarrow (1,0;1,1;1,0)$
	$(3,2)$	$\leftrightarrow (0,0;2,2)$		$(2,2;1,0)$	$\leftrightarrow (2,2;1,0)$
	$(3,1;0,0)$	$\leftrightarrow (1,0;1,1;0,0)$		$(2,1;2,0)$	$\leftrightarrow (0,0;1,1;2,0)$
	$(2,2;0,0)$	$\leftrightarrow (2,2;0,0)$		$(2,1;1,1)$	$\leftrightarrow (0,0;1,1;1,1)$
	$(2,1;1,0)$	$\leftrightarrow (0,0;1,1;1,0)$		$(1,1;3,0)$	$\leftrightarrow (1,1;3,0)$
	$(1,1;2,0)$	$\leftrightarrow (1,1;2,0)$		$(1,1;2,1)$	$\leftrightarrow (1,1;0,0;1,1)$
	$(1,1;1,1)$	$\leftrightarrow (1,1;1,1)$		$(1,1;1,1;0,0)$	$\leftrightarrow (1,1;1,1;0,0)$

Table 5.1: Admissible and Braden sequences of dimensions 4 to 6

The partial order on the set B_d described in Section 3.1 induces, via ϕ, a partial order on the set T_d: for $b, b' \in \mathsf{T}_d$ define

$$b \preccurlyeq b' \iff \phi^{-1}(b) \preccurlyeq \phi^{-1}(b')$$

Also, the number of elements in T_d coincides with the number of admissible sequences.

Corollary 5.1.3. The number of Braden sequences of dimension d is F_d. $\square$

Table 5.1 lists the Braden sequences in T_4, T_5 and T_6 together with their counterparts in B_4, B_5 and B_6, in decreasing order.

Though the sequences themselves are different, every Braden sequence yields the same polytope as its counterpart in B_d.

Proposition 5.1.4. For all $b \in \mathsf{T}_d$ we have $P[b] \cong P[\phi^{-1}(b)]$.

Proof. Let $b = (m_1, \ell_1; \ldots; m_k, \ell_k) \in \mathsf{T}_d$ and suppose $b \neq \phi^{-1}(b)$. Then $\ell_i = 0$ for some $i < k$. Let i be the smallest such index. Then by definition of ϕ, the sequence $\phi^{-1}(b)$ contains the pair

$$\ldots; m_i + m_{i+1} + 1, \ell_{i+1}; \ldots \quad \text{instead of} \quad \ldots; m_i, 0; m_{i+1}, \ell_{i+1}; \ldots$$

in b. The construction of the polytope $P[b]$ involves the operations

$$P[(m_1, \ell_1; \ldots; m_{i-1}, \ell_{i-1})] * \Delta_{m_i} \oplus \Delta_0 * \Delta_{m_{i+1}} \oplus \Delta_{\ell_{i+1}}$$

(applied from left to right). By definition of direct sum and join (see Definition 1.1.8), we have $P \oplus \Delta_0 \cong P$ and $P * \Delta_n * \Delta_m \cong P * (\Delta_n * \Delta_m) \cong P * \Delta_{n+m+1}$ for every polytope P, hence

$$\begin{aligned} P[(m_1, \ell_1; \ldots; m_{i-1}, \ell_{i-1})] * \Delta_{m_i} \oplus \Delta_0 * \Delta_{m_{i+1}} \oplus \Delta_{\ell_{i+1}} \\ \cong\ & P[(m_1, \ell_1; \ldots; m_{i-1}, \ell_{i-1})] * \Delta_{m_i} * \Delta_{m_{i+1}} \oplus \Delta_{\ell_{i+1}} \\ \cong\ & P[(m_1, \ell_1; \ldots; m_{i-1}, \ell_{i-1})] * \Delta_{m_i + m_{i+1} + 1} \oplus \Delta_{\ell_{i+1}} \\ \cong\ & P[(m_1, \ell_1; \ldots; m_{i-1}, \ell_{i-1}; m_i + m_{i+1} + 1; \ell_{i+1})] \end{aligned}$$

Since i was the smallest index where b differs from $\phi^{-1}(b)$, this is the polytope constructed in the first i steps from $\phi^{-1}(b)$. Iterating this argument proves the assertion. □

5.2 DEFINING FACETS

Every admissible or Braden sequence gives rise to a flag vector functional

$$G[b] \;=\; g_{\ell_1}^{m_1+\ell_1} * \ldots * g_{\ell_k}^{m_k+\ell_k}$$

cf. Section 3.1. We have $G[b](P) \geq 0$ for all b and all polytopes P of the correct dimension $\mathsf{dim}\, P = \mathsf{dim}\, b$, since the g-vector is non-negative for polytopes, and convolution preserves non-negativity of linear functionals (see Section 1.4).

Kalai's Theorem 3.1.6 implies that the inequality given by the minimal admissible sequence b_{min} defines a facet of $\mathcal{F\ell C}_d$: $G[b_{\mathsf{min}}](P[b]) = 0$ for all admissible sequences $b \neq b_{\mathsf{min}}$, that is, there are $F_d - 1$ many affinely independent flag vectors of polytopes satisfying $G[b_{\mathsf{min}}]$ with equality, which then span a facet of $\mathcal{F\ell C}_d$.

Note that affine independence follows from the theorem itself: The mapping $\mathbf{f}(P[b]) \mapsto (G[c](P[b]))_{c \in \mathbf{B}_d}$ for $b \in \mathbf{B}_d$, defines a linear map on the space spanned by the vectors $\{\mathbf{f}(P[b]) \mid b \in \mathbf{B}_d\}$. By Kalai's Theorem this map has full rank and therefore the vectors in this set are linearly independent and span the F_d-dimensional linear space of flag vectors. In particular, the affine span $\mathsf{aff}\{\mathbf{f}(P[b]) \mid b \in \mathbf{B}_d\}$ is mapped to an $(F_d - 1)$-dimensional affine space.

In this section we show that the result about the facet-defining sequence carries over to inequalities defined by Braden sequences, and try to extend it. The desired result would be the following conjecture.

Conjecture 5.2.1 (Braden [16])**.** All $F_d - 1$ non-trivial inequalities represented by $G[b]$, with $b \in \mathbf{T}_d \setminus \{(d, 0)\}$, are (possibly asymptotically) facet-defining for $\mathcal{F\ell C}_d$.

We prove the conjecture for $d \leq 6$, following the sketch of proof given by Braden. The next section considers special parts of the conjecture for general d.

To keep the calculations reasonably clear, we introduce some more notation. Most of it is taken over from Kalai's paper [36]. For an admissible or Braden sequence $b = (m_1, \ell_1; \ldots; m_k, \ell_k)$ define $s(b) := (m_1 + \ell_1, \ldots, m_k + \ell_k)$ and $\ell(b) := (\ell_1, \ldots, \ell_k)$. If P is a d-polytope and $s = (s_1, \ldots, s_k)$ such that $\big(\sum_{i=1}^{k} s_i\big) + k - 1 = d$, we call a chain $\emptyset = F_0 \subset F_1 \subset \ldots \subset F_{k-1} \subset F_k = P$ of faces of P an *s-chain*, if $\mathsf{dim}\, F_i/F_{i-1} = s_i$ for $1 \leq i \leq k$. We denote by $\mathcal{C}(s, P)$ the set of all s-chains of the polytope P.

Braden sequences can be "turned round": for $b = (m_1, \ell_1; \ldots; m_k, \ell_k) \in \mathsf{T}_d$ define $b^* = (m_k, \ell_k; \ldots; m_1, \ell_1)$, the *dual sequence*. This turns out to be quite useful for our purposes.

Proposition 5.2.2. Let $b \in \mathsf{T}_d$. Then $b^* \in \mathsf{T}_d$ and for any d-polytope P the convolution $G[b]$ and that of the dual sequence, $G[b^*]$, satisfy

$$G[b](P) = G[b^*](P^\Delta).$$

Proof. Let $b = (m_1, \ell_1; \ldots; m_k, \ell_k) \in \mathsf{T}_d$. Obviously, b^* also satisfies the conditions in Definition 5.1.1. Furthermore

$$G[b^*](P^\Delta) \;=\; \sum_{\substack{F_0 \subset \ldots \subset F_k \\ \text{in } \mathcal{C}(s(b^*), P^\Delta)}} \prod_{i=1}^{k} g_{\ell_i}^{s_i}(F_i/F_{i-1})$$

If $\ell_i = 0$ for some i then $g_{\ell_i}^{s_i}(F_i/F_{i-1}) = 1 = g_{\ell_i}^{s_i}((F_i/F_{i-1})^\Delta)$. For $\ell_i \geq 1$ we have $s_i = m_i + \ell_i = 2\ell_i$ by 5.1.1.(i). Hence $g_{\ell_i}^{s_i}(F_i/F_{i-1}) = g_{\ell_i}^{s_i}((F_i/F_{i-1})^\Delta)$, since the highest entry of the g-vector of an even-dimensional polytope coincides with the one of the polar polytope (see, for instance, Bayer and Ehrenborg [9, Theorem 5.1]).

Denoting with F^Δ the face in P^Δ corresponding to the face F of P, we have $(F_i/F_{i-1})^\Delta \cong {F_{i-1}}^\Delta/{F_i}^\Delta$. Every chain $F_0 \subset \ldots \subset F_k$ in $\mathcal{C}(s(b^*), P^\Delta)$ corresponds uniquely to the chain ${F_k}^\Delta \subset \ldots \subset {F_0}^\Delta$ in $\mathcal{C}(s(b), P)$ and vice versa. Putting everything together, we get

$$\sum_{\substack{F_0 \subset \ldots \subset F_k \\ \text{in } \mathcal{C}(s(b^*), P^\Delta)}} \prod_{i=1}^{k} g_{\ell_i}^{s_i}(F_i/F_{i-1}) \;=\; \sum_{\substack{{F_k}^\Delta \subset \ldots \subset {F_0}^\Delta \\ \text{in } \mathcal{C}(s(b), P)}} \prod_{i=1}^{k} g_{\ell_i}^{s_i}({F_{i-1}}^\Delta/{F_i}^\Delta) \;=\; G[b](P).$$

$\square$

To verify Conjecture 5.2.1 we have to find—for every non-trivial Braden sequence b—enough polytopes whose flag vectors give linearly independent points in $\mathcal{F}\ell\mathcal{C}_d$ and yield 0 when evaluated on $G[b]$. Here, "enough" means $F_d - 2$, since the d-simplex already provides one suitable flag vector for every $b \neq (d, 0)$. As in Kalai's paper, affine independence can be proved using the values $G[c](P[b])$, $b, c \in \mathbf{T}_d$ themselves.

Some inequalities will be only asymptotically facet-defining. For these we have to include parametrised families of polytopes P_n and consider the values $G[b](P_n)$ by order of magnitude, with respect to the parameter $n \to \infty$.

To compute $G[b](P)$ for a given polytope P, one has to sum a certain product of g-vector entries, over all $s(b)$-chains of P. Under some circumstances such a chain contributes nothing to the sum, namely, if one of the factors $g^{s_i}_{\ell_i}(F_i/F_{i-1})$ is 0. This happens, for instance, if the quotient polytope F_i/F_{i-1} is a simplex and $\ell_i > 0$.

Definition 5.2.3 (Degenerate)**.** Given a sequence $b = (m_1, \ell_1; \ldots; m_k, \ell_k)$, an $s(b)$-chain $\emptyset = F_0 \subset F_1 \subset \ldots \subset F_{k-1} \subset F_k = P$ is *degenerate* if F_i/F_{i-1} is an s_i-simplex for some $i \in \{1, \ldots, k\}$ with $s_i = m_i + \ell_i$ and $\ell_i > 0$.

For admissible sequences this notion of degeneracy is slightly less restrictive than the one in Kalai's paper [36, Section 4]. In other words, an admissible sequence that is degenerate due to Kalai's definition is also degenerate in the sense of Definition 5.2.3. The formulation given here is the natural generalisation that also works for Braden sequences.

In terms of chains of faces, passing from a Braden sequence to the corresponding admissible sequence amounts to taking subchains (see Section 1.2).

Lemma 5.2.4. Let $b = (m_1, l_1; \ldots; m_k, \ell_k) \in \mathbf{T}_d$ and let $\mathcal{F}$ be an $s(b)$-chain $\emptyset = F_0 \subset \ldots \subset F_k = P$ in the face lattice of the polytope P. Then there is a unique $s(\phi^{-1}(b))$-subchain $\mathcal{F}'$ of $\mathcal{F}$.

If $\mathcal{F}'$ is degenerate then $\mathcal{F}$ is degenerate and

$$\prod_{i=1}^{k} g^{s_i}_{\ell_i}(F_i/F_{i-1}) \;=\; 0.$$

Proof. Let $s(b) = (s_1, \ldots, s_k)$. Then for every $i < k$ with $\ell_i = 0$ the two entries s_i, s_{i+1} in $s(b)$ are replaced by the entry $s_i + s_{i+1} + 1$ in $s(\phi^{-1}(b))$. Therefore removing every face F_i where $i < k$ with $\ell_i = 0$ from the chain $\mathcal{F}$ uniquely defines an $s(\phi^{-1}(b))$-subchain $\mathcal{F}'$ of $\mathcal{F}$.

If $\mathcal{F}'$ is degenerate, then it contains two consecutive faces $F \subset \tilde{F}$, whose quotient is a simplex of dimension at least 2. There are two possibilities:

(I) F and $\tilde{F}$ are also faces in the chain $\mathcal{F}$, that is, $F = F_{i-1}$ and $\tilde{F} = F_i$ for some $i \leq k$. Since $\mathcal{F}'$ was degenerate, we have $\ell_i > 0$ and the chain $\mathcal{F}$ is also degenerate.

(II) Otherwise, exactly one face F_i in $\mathcal{F}$ is missing in $\mathcal{F}'$ between the faces $F = F_{i-1}$ and $\tilde{F} = F_{i+1}$, where $1 \leq i \leq k-1$. Since $\ell_i = \ell_{i+1} = 0$ is not allowed for sequences in $\mathbf{T}_d$, one of s_i or s_{i+1} has to be at least 2. That is, either F_i/F_{i-1} or F_{i+1}/F_i is of dimension at least 2, and both of these quotients are simplices, since they are quotients in the simplex $\tilde{F}/F$. Hence $\mathcal{F}$ is degenerate.

□

The following theorem is the main statement of this section and the counterpart of Kalai's Theorem 3.1.6 for Braden sequences.

Theorem 5.2.5. Let $b \in \mathbf{T}_d$ and $c \in \mathbf{T}_d$ with $b \not\preccurlyeq c$. Then

(a) $G[b](P[b]) > 0$ and

(b) $G[c](P[b]) = 0$.

Proof. We prove this by adapting the proof of [36, Theorem 4.1.], using Proposition 5.1.4 and Lemma 5.2.4. Let $b, c \in \mathbf{T}_d$ with $b \not\preccurlyeq c$ and for notation $b' := \phi^{-1}(b)$, $c' := \phi^{-1}(c)$. Then $b' \not\preccurlyeq c'$ and $P[b] = P[b']$ by Proposition 5.1.4.

We first show (b). As in Kalai's proof, one of the following two cases applies:

(I) $s(b') \not\leq s(c')$. Then Kalai shows (see [36, Proposition 4.2]) that every $s(c')$-chain in $P[b']$ is degenerate, and by Lemma 5.2.4 also every $s(c)$-chain in $P[b]$ is degenerate, so $G[c](P[b]) = 0$.

(II) $s(b') = s(c')$ and $\ell_\eta(c') > \ell_\eta(b')$ for some $\eta \leq k$. Then, again by [36, Proposition 4.2], every $s(c')$-chain in $P[b']$ is degenerate, with one exception—the *central chain*

$$\emptyset \subset P[(m_1(b'), \ell_1(b'))] \subset P[(m_1(b'), \ell_1(b'); m_2(b'), \ell_2(b'))] \subset \ldots \subset P[b']$$

Therefore we only have to care about $s(c)$-chains, of which the central chain is a subchain. We show that the factor $g^{s_\eta(c')}_{\ell_\eta(c')}$ occurs in $G[c]$, and, applied to the appropriate quotient polytope in each such chain, equals 0.

Comparing c' and c there are again two possibilites: Either the pair $m_\eta(c'), \ell_\eta(c')$ in c' is also contained in c (which means $m_\eta(c') = \ell_\eta(c')$), or it is replaced by the two successive pairs $d_\eta, 0; \ell_\eta(c'), \ell_\eta(c')$ in c, where $d_\eta := m_\eta(c') - \ell_\eta(c') - 1$. In the first case the factor

$$g^{2\ell_\eta(c')}_{\ell_\eta(c')}\left(\Delta_{m_\eta(b')} \oplus \Delta_{\ell_\eta(b')}\right)$$

occurs in every summand corresponding to a subchain of the central chain, and since $\ell_\eta(b') < \ell_\eta(c') < m_\eta(b')$, this factor equals 0.

Otherwise, the part $g^{s_\eta(c')}_{\ell_\eta(c')}$ in $G[c']$ is replaced by $g^{d_\eta}_0 * g^{2\ell_\eta(c')}_{\ell_\eta(c')}$ in $G[c]$, and every summand contains a factor

$$g^{2\ell_\eta(c')}_{\ell_\eta(c')}\left((\Delta_{m_\eta(b')} \oplus \Delta_{\ell_\eta(b')})/F\right)$$

where F is a face of $\Delta_{m_\eta(b')} \oplus \Delta_{\ell_\eta(b')}$ of dimension d_η. Then the quotient $(\Delta_{m_\eta(b')} \oplus \Delta_{\ell_\eta(b')})/F$ is combinatorially equivalent to a sum $\Delta_{m_\eta(b')-x} \oplus \Delta_{\ell_\eta(b')-y}$ with $x, y \geq 0$. Since $\ell_\eta(c') > \ell_\eta(b') \geq \ell_\eta(b') - y$, again $g^{2\ell_\eta(c')}_{\ell_\eta(c')}(\Delta_{m_\eta(b')-x} \oplus \Delta_{\ell_\eta(b')-y}) = 0$.

This proves (b). For the proof of (a), again by [36, Proposition 4.2], we have to investigate only the contribution of those chains that contain the central chain as a subchain. Consider any pair $m_\eta(b'), \ell_\eta(b')$. As before, two cases are possible.

If $m_\eta(b') = \ell_\eta(b')$, the pair survives the mapping ϕ and in $G[b]$ the evaluation of any chain of interest includes the factor $g^{s_\eta(b')}_{\ell_\eta(b')}(\Delta_{m_\eta(b')} \oplus \Delta_{\ell_\eta(b')}) = 1$.

Otherwise, as above, the pair splits up under ϕ into $d_\eta, 0; \ell_\eta(b'), \ell_\eta(b')$, with $d_\eta := m_\eta(b') - \ell_\eta(b') - 1$, and the product over any $s(b)$-chain contains a factor

$$g^{d_\eta}_0(F) \cdot g^{2\ell_\eta(b')}_{\ell_\eta(b')}\left((\Delta_{m_\eta(b')} \oplus \Delta_{\ell_\eta(b')})/F\right)$$

where F is a face of $\Delta_{m_\eta(b')} \oplus \Delta_{\ell_\eta(b')}$ of dimension d_η. Since $\Delta_{m_\eta(b')} \oplus \Delta_{\ell_\eta(b')}$ is combinatorially equivalent to $(\Delta_{d_\eta} * \Delta_{\ell_\eta(b')}) \oplus \Delta_{\ell_\eta(b')}$, at least one of these d_η-faces F is a simplex. Then $(\Delta_{m_\eta(b')} \oplus \Delta_{\ell_\eta(b')})/F \cong \Delta_{\ell_\eta(b')} \oplus \Delta_{\ell_\eta(b')}$. Thus there is at least one chain, such that

$$\underbrace{g^{d_\eta}_0(\Delta_{d_\eta})}_{=1} \cdot \underbrace{g^{2\ell_\eta(b')}_{\ell_\eta(b')}(\Delta_{\ell_\eta(b')} \oplus \Delta_{\ell_\eta(b')})}_{=1} = 1$$

is contained in the product.

Altogether we can find at least one chain of faces in $P[b]$, such that the product $\prod_{i=1}^{k} g_{\ell_i(b)}^{s_i(b)}(F_i/F_{i-1}) = 1$, and therefore

$$G[b](P[b]) \;=\; \sum_{\substack{F_0 \subset \ldots \subset F_k \\ \text{in } \mathcal{C}(s(b),P[b])}} \prod_{i=1}^{k} g_{\ell_i(b)}^{s_i(b)}(F_i/F_{i-1}) \;\geq\; 1.$$

□

Corollary 5.2.6. Let b_{min} be the smallest (by partial order) Braden sequence of dimension d. Then the inequality represented by $G[b_{\mathsf{min}}]$ is tight and facet-defining for $\mathcal{F\ell C}_d$, as is the inequality given by $G[b^*_{\mathsf{min}}]$.

Proof. By Theorem 5.2.5(b) the inequality $G[b_{\mathsf{min}}] \geq 0$ is satisfied with equality by the flag vectors of every $P[b]$ with $b \in \mathbf{T}_d$, $b \not\preccurlyeq b_{\mathsf{min}}$, which are exactly $F_d - 1$ many. Affine independence of these flag vectors follows from Theorem 5.2.5(a).

Proposition 5.2.2 implies that the inequality $G[b^*_{\mathsf{min}}]$ is also facet-defining, by considering the polar polytopes $P[b]^{\Delta}$. □

Dimensions 4 to 6

For moderate dimensions* one can arrange the values $G[b](P)$ in a relatively concise way and try to find more polytopes to prove Conjecture 5.2.1. We build up a matrix of size $n \times (F_d - 1)$, indexing the rows by the polytopes involved and the columns by the non-trivial Braden sequences, ordered according to a total order extending the given partial order.

Then for every column we have to find $F_d - 2$ rows containing a 0, which means that the flag vectors of the specified polytopes lie on the hyperplane defined by the specified convolution. Furthermore, there has to be an additional row such that the resulting submatrix of size $(F_d - 1) \times (F_d - 1)$ has full rank. This ensures that the flag vectors of the chosen polytopes are affinely independent. Note that we leave out the simplex $P[(d, 0)]$, which gives equality for all non-trivial inequalities, thus already giving us one polytope for free.

By Theorem 5.2.5 the last column of this matrix already contains $F_d - 2$ zeros. More generally, ordering the polytopes $P[b]$ with $b \in \mathbf{T}_d$ consistently to the inequalities shows a triangular $(F_d - 1) \times (F_d - 1)$-matrix with positive

*The "definition" of *moderate dimension* happens to be essentially the same as in Chapter 6

	$(1,0;1,1)$	$(2,2)$	$(0,0;1,1;0,0)$	$(1,1;1,0)$
$P[(1,0;1,1)]$	6	0	0	0
$P[(2,2)]$	9	1	0	0
$P[(0,0;1,1;0,0)]$	3	0	3	0
$P[(1,1;1,0)]$	1	0	2	1
C_4	0	2	0	24

Table 5.2: Facet-defining Braden sequences in dimension 4

values on the diagonal. Adding more polytopes to treat the other inequalities unfortunately gives a less clear structure, so regularity of the respective submatrix has to be verified explicitly.

In dimension 4 we have $F_4 - 1 = 4$ inequalities and need 3 polytopes for each of them. Table 5.2 summarises the data. The sequence $(1,1;1,0)$ is smallest, so its inequality, as well as the one arising from the dual sequence, $(1,0;1,1)$, are facet-defining by Corollary 5.2.6.

Furthermore, the inequality of $(2,2)$ is facet-defining, already by considering Kalai's basis polytopes. For the last remaining inequality, corresponding to the sequence $(0,0;1,1;0,0)$, we just need one other polytope. The 4-cube gives a suitable flag vector.

In dimension 5 we have to deal with $F_5 - 1 = 7$ sequences. The smallest sequence $(1,1;1,1)$ is dual to itself, therefore no other inequality is proved to be facet-defining by Corollary 5.2.6. In addition to that, Kalai's basis polytopes show that the inequality arising from $(2,2;0,0)$ is facet-defining. By Proposition 5.2.2, this also settles the dual sequence $(0,0;2,2)$. For the inequality corresponding to $(0,0;1,1;1,0)$ and its dual the 5-cube, the prism over the 4-dimensional crosspolytope and four of the basis polytopes suffice.

There remain the inequalities from the sequence $(1,1;2,0)$ and its dual. As we will prove in Section 5.3, these inequalities are only asymptotically facet-defining. This can be seen by taking as a parametrised family of polytopes the join of two n-gons, $\pentagon_n * \pentagon_n$. The value $G[(1,1;2,0)](\pentagon_n * \pentagon_n)$ grows asymptotically less than that of other considered convolutions. More precisely, let

$$\tilde{G}[b](\pentagon_n * \pentagon_n) \; := \; \frac{G[b](\pentagon_n * \pentagon_n)}{f_0(\pentagon_n * \pentagon_n)} \; = \; \frac{G[b](\pentagon_n * \pentagon_n)}{n^2}$$

Then $\tilde{G}[(1,1;2,0)](\pentagon_n * \pentagon_n) \to 0$ and $\tilde{G}[(1,1;1,1)](\pentagon_n * \pentagon_n) \to 2$ as $n \to \infty$. This shows that the asymptotic flag vector of the family is in the hyperplane defined by $(1,1;2,0)$, and regularity holds with the limits $\lim\limits_{n\to\infty} \tilde{G}[b](\pentagon_n * \pentagon_n)$,

	(2,0;1,1)	(0,0;2,2)	(1,0;1,1;0,0)	(2,2;0,0)	(0,0;1,1;1,0)	(1,1;2,0)	(1,1;1,1)
$P[(2,1;1,1)]$	10	0	0	0	0	0	0
$P[(0,0;2,2)]$	18	4	0	0	0	0	0
$P[(1,0;1,1;0,0)]$	6	0	6	0	0	0	0
$P[(2,2;0,0)]$	9	1	9	1	0	0	0
$P[(0,0;1,1;1,0)]$	3	0	6	0	3	0	0
$P[(1,1;2,0)]$	1	0	3	0	3	1	0
$P[(1,1;1,1)]$	15	2	4	0	4	1	1
C_5	0	0	0	20	0	80	0
$\mathsf{prism}\, {C_4}^{\Delta}$	24	0	48	4	0	24	24
$⬠_n * ⬠_n$	$\to 0$	0	$\to 2$	0	$\to 2$	$\to 0$	$\to 2$

Table 5.3: (Asymptotically) facet-defining Braden sequences in dimension 5

hence the inequality is asymptotically facet-defining. See Table 5.3 for the summary.

Finally consider the situation for $d = 6$. We have $F_6 = 13$, so there are 12 inequalities and 12 basis polytopes. Again, by Corollary 5.2.6, the smallest inequality $(1,1;1,1;0,0)$ and also its dual $(0,0;1,1;1,1)$ are facet-defining. To show that in fact all inequalities are facet-defining, several additional polytopes are needed, as well as parametrised families, cf. Section 5.3. Apart from the standard constructions and examples from Section 1.1, we use the following construction suggested by Braden [16].

Definition 5.2.7 (Capped prism)**.** Let P be a polytope. The *capped prism* $\mathsf{F}(P)$ over P is the polytope that arises as $\mathsf{prism}\, P$, stacked over the two respective facets $P \times \{-1\}$ and $P \times \{1\}$.

In other words, $\mathsf{F}(P) := \mathsf{conv}(\mathsf{prism}\, P \cup \{\mathbf{p}_-, \mathbf{p}_+\})$, where $\mathbf{p}_-$ and $\mathbf{p}_+$ are points beyond the facets $P \times \{-1\}$, respectively $P \times \{1\}$.

All used polytopes are listed in Table 5.4. The inequalities $(3,3)$, $(2,2;1,0)$ and $(1,1;1,1;0,0)$ are obviously (by use of Kalai's polytopes and Corollary 5.2.6, respectively) facet-defining. Proposition 5.2.2 then implies that also $(1,0;2,2)$ and $(0,0;1,1;1,1)$ are facet-defining. Furthermore, inequality $(1,1;0,0;1,1)$ can be seen to be facet-defining by replacing $P[(1,1;2,1)]$ by $\Delta_2 \oplus \Delta_1 * \Delta_1 \oplus \Delta_1$. Similarly, $(0,0;2,2;0,0)$ is done by replacing $P[(3,2;0,0)]$

		(3,0;1,1)	(1,0;2,2)	(3,3)	(2,0;1,1;0,0)	(0,0;2,2;0,0)	(1,0;1,1;1,0)	(2,2;1,0)	(0,0;1,1;2,0)	(0,0;1,1;1,1)	(1,1;3,0)	(1,1;0,0;1,1)	(1,1;1,1;0,0)
$(*)$	$P[(5,1)]$	15	0	0	0	0	0	0	0	0	0	0	0
$(*)$	$P[(4,2)]$	30	10	0	0	0	0	0	0	0	0	0	0
$(*)$	$P[(3,3)]$	36	16	1	0	0	0	0	0	0	0	0	0
$(**)$	$P[(4,1;0,0)]$	10	0	0	10	0	0	0	0	0	0	0	0
$(**)$	$P[(3,2;0,0)]$	18	4	0	18	4	0	0	0	0	0	0	0
	$P[(3,1;1,0)]$	6	0	0	12	0	6	0	0	0	0	0	0
	$P[(2,2;1,0)]$	9	1	0	18	2	9	1	0	0	0	0	0
	$P[(2,1;2,0)]$	3	0	0	9	0	9	0	3	0	0	0	0
$(**)$	$P[(2,1;1,1)]$	28	7	0	12	0	12	0	3	3	0	0	0
	$P[(1,1;3,0)]$	1	0	0	4	0	6	0	4	0	1	0	0
	$P[(1,1;2,1)]$	21	3	0	6	0	9	0	5	3	1	3	0
	$P[(1,1;1,1;0,0)]$	15	2	0	19	2	8	0	5	1	1	1	1
	$\pentagon_n * \pentagon_n * \Delta_0$	$\to 0$	0	0	$\to 2$	0	$\to 4$	0	$\to 2$	$\to 2$	$\to 0$	$\to 2$	$\to 2$
	$\pentagon_n * \pentagon_n \oplus \Delta_1$	$\to 1$	$\to 2$	$\to 1$	$\to 4$	0	$\to 6$	0	$\to 2$	$\to 6$	$\to 0$	$\to 6$	0
	$\Delta_2 \oplus \Delta_1 * \Delta_1 \oplus \Delta_1$	0	0	0	3	0	12	7	12	0	28	0	3
	$\Delta_1 \oplus \Delta_1 * \Delta_2 \oplus \Delta_1$	1	0	0	5	0	9	3	6	0	21	3	3
$(*)$	C_6	0	0	5	0	0	0	120	0	0	240	0	0
$(**)$	$\Delta_6(3)$	0	0	0	0	140	0	0	630	0	0	0	0
$(**)$	$\mathsf{prism}\, {C_5}^{\Delta}$	80	20	15	160	40	0	0	0	0	40	240	0
$(*)$	$\mathsf{F}(\Delta_4 \oplus \Delta_1)$	80	20	0	0	0	0	0	0	0	20	30	0
$(**)$	$\mathsf{F}(\Delta_3 \oplus \Delta_1) * \Delta_0$	50	8	0	50	8	0	0	14	6	14	6	6
$(*)$	$\mathsf{F}(\Delta_3 \oplus \Delta_1) \oplus \Delta_1$	164	68	5	0	0	0	0	28	12	14	60	0
$(***)$	$\Delta_4 \times \Delta_1 * \Delta_0$	0	0	0	0	0	0	0	10	0	10	0	0
$(***)$	$\Delta_3 \times \Delta_2 * \Delta_0$	0	0	0	0	4	0	4	18	0	18	0	0
$(***)$	$\Delta_3 \times \Delta_1 * \Delta_1$	0	0	0	0	0	6	0	12	0	6	0	0
$(***)$	$\Delta_2 \times \Delta_2 * \Delta_1$	0	1	0	0	2	9	1	18	0	9	0	0
$(***)$	$\mathsf{prism}^2\, {C_4}^{\Delta}$	24	0	13	96	0	96	40	0	0	104	144	96
$(***)$	$\pentagon_n * \mathsf{pyr}\,\mathsf{prism}\,\Delta_2$	0	0	$\to 0$	$\to 0$	0	$\to 3$	$\to 7$	$\to 3$	0	$\to 25$	0	$\to 3$

Table 5.4: Non-trivial Braden sequences in dimension 6

and $P[(2,2;1,0)]$ by $\Delta_1 \oplus \Delta_1 * \Delta_2 \oplus \Delta_1$ and the 6-cube. The last (strictly) facet-defining inequality is $(1,0;1,1;1,0)$, which can be seen by considering all polytopes marked with $(*)$ and $(**)$.

The remaining sequences yield asymptotically facet-defining inequalities. For $(2,0;1,1;0,0)$ one can take all polytopes marked with $(*)$ and $(***)$, which includes a family of joins involving n-gons. Again, by Proposition 5.2.2 this also settles the inequality given by $(0,0;1,1;2,0)$. Finally, $(1,1;3,0)$ and the dual sequence $(3,0;1,1)$ are asymptotically facet-defining by using the two families $\pentagon_n * \pentagon_n * \Delta_0$ and $\pentagon_n * \pentagon_n \oplus \Delta_1$ instead of $P[(1,1;2,1)]$ and $P[(1,1;1,1;0,0)]$, and Proposition 5.2.2. This last argument is an example of the more general principle discussed in the next section, see Proposition 5.3.5.

Summarising, we have the following result.

Proposition 5.2.8. Conjecture 5.2.1 is true for $d \leq 6$. In other words, for $d \leq 6$ all inequalities given by convolutions $G[b]$ with $b \in \mathbf{T}_d \setminus \{(d,0)\}$, are facet-defining or asymptotically facet-defining for $\mathcal{FlC}_d$.

5.3 Asymptotic facets

In dimension 5 we needed an infinite family of polytopes, the join of n-gons, to prove that the sequence $(1,1;2,0)$ defines a facet of $\mathcal{FlC}_5$. Similarly, in dimension 6 we needed two polytope families for the sequence $(1,1;3,0)$. The question arises, whether this is necessary or if we may be able to find enough individual polytopes with linearly independent flag vectors in the hyperplane.

It turns out that this is not possible. The situation is similar to that in Chapter 4, where the inequalities $f_0 \geq 5$ and $f_3 \geq 5$ turn out to be asymptotically facet-defining, without containing any other flag vector than that of the simplex. The difference is that for the inequalities considered here several tight flag vectors of polytopes can be found, however, not enough linearly independent ones.

In this section we show that there are lots of sequences that can produce only asymptotically facet-defining inequalities. Furthermore, for some of these sequences in arbitrary high dimension we show that they are indeed asymptotically facet-defining.

Lemma 5.3.1. Let w_1, respectively w_2, be linear functionals that are non-negative on all flag vectors of d_1-, respectively d_2-polytopes, $d_1, d_2 \geq 0$. Let

$$G_1 := w_1 * g_0^e\ , \quad G_2 := g_0^e * w_2 \quad \text{and} \quad G_3 := w_1 * g_0^e * w_2$$

with $e \geq 0$, and suppose for a polytope P of dimension $d_1 + e + 1$, $e + d_2 + 1$ or $d_1 + e + d_2 + 2$ we have $G_i(P) = 0$, with

$$i = \begin{cases} 1 & \text{if } \dim P = d_1 + e + 1 \\ 2 & \text{if } \dim P = e + d_2 + 1 \\ 3 & \text{if } \dim P = d_1 + e + d_2 + 2 \end{cases}$$

Then for every linear functional w on flag vectors of e-polytopes the convolution

$$G'_i := \begin{cases} w_1 * w & \text{if } i = 1 \\ w * w_2 & \text{if } i = 2 \\ w_1 * w * w_2 & \text{if } i = 3 \end{cases}$$

also evaluates to 0 on P, that is: $G'_i(P) = 0$.

Proof. We only show the case $i = 3$ explicitly, the other cases being exactly the same, with w_1, respectively w_2, ignored. By definition of the convolution,

$$G_3(P) = \sum_{\substack{F_1 \subset F_2 \text{ faces of } P \\ \text{with } \dim F_1 = d_1, \\ \dim F_2 = d_1 + e + 1}} w_1(F_1) \cdot g_0^e(F_2/F_1) \cdot w_2(P/F_2).$$

F_1, F_2/F_1 and P/F_2 are again polytopes, so we have

$$w_1(F_1) \geq 0\,, \; g_0^e(F_2/F_1) = 1\,, \text{ and } \; w_2(P/F_2) \geq 0$$

for all occuring F_1 and F_2. Since $G_3(P) = 0$, every term in the sum must be 0, and therefore for every chain $F_1 \subset F_2$ either $w_1(F_1) = 0$ or $w_2(P/F_2) = 0$. This implies

$$G'_3(P) = \sum_{\substack{F_1 \subset F_2 \text{ faces of } P \\ \text{with } \dim F_1 = d_1, \\ \dim F_2 = d_1 + e + 1}} \underbrace{w_1(F_1) \cdot w(F_2/F_1) \cdot w_2(P/F_2)}_{=0} = 0.$$

□

Applying this lemma to the special case of Braden sequences immediately gives the following result.

Proposition 5.3.2. Consider a Braden sequence b of the form

$$b = (m_1, \ell_1; \ldots; m_{\mu-1}, \ell_{\mu-1}; m_\mu, 0; m_{\mu+1}, \ell_{\mu+1}; \ldots; m_k, \ell_k)$$

with $1 \leq \mu \leq k$. Suppose there is a polytope P with $G[b](P) = 0$. Then for every Braden sequence

$$b' = (m_1, \ell_1; \ldots; m_{\mu-1}, \ell_{\mu-1};\; \tilde{b}\;; m_{\mu+1}, \ell_{\mu+1}; \ldots; m_k, \ell_k)$$

with any $\tilde{b} \in \mathbf{T}_{m_\mu}$, also $G[b'](P) = 0$. □

Corollary 5.3.3. Let b be a Braden sequence of dimension $d \geq 5$ of the form as in Proposition 5.3.2, where $k \geq 2$ and $m_\mu \geq 2$, and

$$\Bigl(\sum_{i<\mu} m_i + \ell_i\Bigr) + \mu - 2 \;\geq\; 2 \quad \text{or} \quad \Bigl(\sum_{i>\mu} m_i + \ell_i\Bigr) + k - \mu - 1 \;\geq\; 2.$$

Then the affine dimension of the set

$$\{\mathbf{f}(P) \mid P \; d\text{-polytope with } G[b](P) = 0\}$$

of all polytope flag vectors in the hyperplane defined by $G[b]$ is at most $F_d - 3$. In other words, $G[b]$ can only be asymptotically facet-defining for $\mathcal{F}\ell\mathcal{C}_d$.

Proof. The hypotheses guarantee the existence of a Braden sequence b' different from b, for which the conclusion of Proposition 5.3.2 holds: For instance, let

$$\tilde{b} \;:=\; \begin{cases} (\frac{m_\mu}{2}, \frac{m_\mu}{2}), & \text{for } m_\mu \text{ even} \\ (0,0; \frac{m_\mu-1}{2}, \frac{m_\mu-1}{2}), & \text{for } m_\mu \text{ odd and } (\sum_{i<\mu} m_i + \ell_i) + \mu - 2 \geq 2 \\ (\frac{m_\mu-1}{2}, \frac{m_\mu-1}{2}; 0, 0), & \text{for } m_\mu \text{ odd and } (\sum_{i<\mu} m_i + \ell_i) + \mu - 2 \not\geq 2 \end{cases}$$

then $b' = (m_1, \ell_1; \ldots; m_{\mu-1}, \ell_{\mu-1};\ \tilde{b}\ ; m_{\mu+1}, \ell_{\mu+1}; \ldots; m_k, \ell_k) \in \mathsf{T}_d$ and $b' \neq b$. Now for every d-polytope P with $G[b](P) = 0$, by Proposition 5.3.2 also $G[b'](P) = 0$. This means that the above set of flag vectors is contained in the intersection of the two hyperplanes defined by $G[b]$ and $G[b']$. By Theorem 5.2.5 these hyperplanes are linearly independent, hence their intersection has strictly smaller dimension. ☐

As a result of the hypotheses of Corollary 5.3.3, the first dimension, where sequences of this form occur is 5. The first example is the sequence $(1,1;2,0)$, for which we had to use joins of n-gons in Section 5.2. This situation can be generalised to show that in each dimension $d \equiv 2 \bmod 3$ the penultimate sequence is in fact asymptotically facet-defining.

Denote the j-fold join of n-gons by

$$\pentagon_n^{*j} \;:=\; \underbrace{\pentagon_n * \ldots * \pentagon_n}_{j \text{ times}}.$$

Proposition 5.3.4. Let $d = 3k-1$ with $k \geq 2$. Then the inequality represented by the Braden sequence of length k,

$$(\underbrace{1,1;\ldots;1,1}_{k-1 \text{ pairs}};2,0)$$

is asymptotically facet-defining for $\mathcal{F}\ell\mathcal{C}_d$.

Proof. Consider the set

$$B \;:=\; \big\{P[b] \mid b \in \mathbf{T}_d \setminus \{(d,0),(1,1;\ldots;1,1;1,1)\}\big\}.$$

The flag vectors of the polytopes in B are linearly independent by Theorem 5.2.5. We first show that adding the "asymptotic" flag vector of the family $\pentagon_n^{*k}$ does not destroy linear independence. For this we claim that evaluating the convolutions corresponding to the last two sequences on $\pentagon_n^{*k}$ yields

$$\begin{aligned}\left(g_1^2 * \ldots * g_1^2 * g_0^2\right)\left(\pentagon_n^{*k}\right) &= k!\,(n-3)^{k-1},\\ \left(g_1^2 * \ldots * g_1^2 * g_1^2\right)\left(\pentagon_n^{*k}\right) &= k!\,(n-3)^{k}.\end{aligned}$$

We prove both equalities by induction: For $k=2$ we have

$$\left(g_1^2 * g_0^2\right)\left(\pentagon_n^{*2}\right) \;=\; \sum_{\substack{\text{2-faces } F\\ \text{of } \pentagon_n^{*2}}} g_1^2(F)\, g_0^2(\pentagon_n^{*2}/F)$$

The 2-faces of $\pentagon_n^{*2}$ are triangles and two n-gons. For F being a triangle we have $g_1^2(F)=0$, for F an n-gon, $g_1^2(F)=n-3$. Hence,

$$\left(g_1^2 * g_0^2\right)\left(\pentagon_n^{*2}\right) \;=\; \sum_{\substack{n\text{-gon faces } F\\ \text{of } \pentagon_n^{*2}}} \underbrace{g_1^2(F)}_{=n-3}\, \underbrace{g_0^2(\pentagon_n^{*2}/F)}_{=1} \;=\; 2(n-3).$$

For the other convolution,

$$\left(g_1^2 * g_1^2\right)\left(\pentagon_n^{*2}\right) \;=\; \sum_{\substack{\text{2-faces } F\\ \text{of } \pentagon_n^{*2}}} g_1^2(F)\, g_1^2(\pentagon_n^{*2}/F).$$

Again only the two n-gon faces have to be considered for F and then $\pentagon_n^{*2}/F$ is also an n-gon. Therefore

$$\left(g_1^2 * g_1^2\right)\left(\pentagon_n^{*2}\right) \;=\; \sum_{\substack{n\text{-gon faces } F\\ \text{of } \pentagon_n^{*2}}} \underbrace{g_1^2(F)}_{=n-3}\, \underbrace{g_1^2(\pentagon_n^{*2}/F)}_{=n-3} \;=\; 2(n-3)(n-3).$$

Now let $k>2$. Then

$$\left(g_1^2 * \ldots * g_1^2 * g_0^2\right)\left(\pentagon_n^{*k}\right) \;=\; \sum_{\substack{\text{2-faces } F\\ \text{of } \pentagon_n^{*k}}} g_1^2(F)\cdot\left(g_1^2 * \ldots * g_1^2 * g_0^2\right)\left(\pentagon_n^{*k}/F\right).$$

As before, only the n-gon faces F are interesting, but now there are k of them in $\pentagon_n^{*k}$. Moreover, if F is an n-gon, then $\pentagon_n^{*k}/F$ is combinatorially equivalent to $\pentagon_n^{*(k-1)}$, so

$$\begin{aligned}
\left(g_1^2 * \ldots * g_1^2 * g_0^2\right)\left(\pentagon_n^{*k}\right) &= k(n-3)\cdot\left(g_1^2 * \ldots * g_1^2 * g_0^2\right)\left(\pentagon_n^{*(k-1)}\right)\\
&= k\,(n-3)\cdot(k-1)!\,(n-3)^{k-2}\\
&= k!\,(n-3)^{k-1}
\end{aligned}$$

by induction. Finally, the analogous argument yields

$$\begin{aligned}
\left(g_1^2 * \ldots * g_1^2 * g_1^2\right)\left(\pentagon_n^{*k}\right) &= \sum_{\substack{\text{2-faces } F\\ \text{of } \pentagon_n^{*k}}} g_1^2(F)\cdot\left(g_1^2 * \ldots * g_1^2 * g_1^2\right)\left(\pentagon_n^{*k}/F\right)\\
&= k(n-3)\cdot\left(g_1^2 * \ldots * g_1^2 * g_1^2\right)\left(\pentagon_n^{*(k-1)}\right)\\
&= k\,(n-3)\cdot(k-1)!\,(n-3)^{k-1}\\
&= k!\,(n-3)^{k}.
\end{aligned}$$

Now dividing the obtained expressions by $f_0(\pentagon_n^{*k}) = (n-3)^k$, letting $n \to \infty$ and writing all polytopes and convolutions into a matrix as described in Section 5.2 gives

	$(d-3,0;1,1)$	$\cdots$	$(1,1;\ldots;1,1;2,0)$	$(1,1;\ldots;1,1;1,1)$
$P[(d-3,0;1,1)]$	>0			
$\vdots$		$\ddots$		0
$P[(1,1;\ldots;1,1;2,0)]$	$*$		>0	
$\pentagon_n^{*k}$			$\to 0$	$\to k!$

Obviously, linear independence still holds.

Furthermore, $G[(1,1;\ldots;1,1;2,0)](P) = 0$ for all $P \neq P[(1,1;\ldots;1,1;2,0)]$ in B and for the family $\pentagon_n^{*k}$ we have

$$G[(1,1;\ldots;1,1;2,0)](\pentagon_n^{*k})/(n-3)^k \xrightarrow{n\to\infty} 0$$

From this can be concluded that the inequality $G[(1,1;\ldots;1,1;2,0)] \geq 0$ is asymptotically facet-defining. $\square$

For dimensions $d \equiv 0 \mod 3$ a similar argument can be given for the antepenultimate sequence.

Proposition 5.3.5. Let $d = 3k$ with $k \geq 2$. Then the inequality represented by the Braden sequence

$$(\underbrace{1,1;\ldots;1,1}_{k-1 \text{ pairs}};3,0)$$

is asymptotically facet-defining for $\mathcal{F}\ell\mathcal{C}_d$.

Proof. The proof is very similar to the previous one, so we will carry out only the details that are different. This time let

$$B := \big\{P[b] \mid b \in \mathbf{T}_d \setminus \{(d,0),(1,1;\ldots;1,1;0,0;1,1),(1,1;\ldots;1,1;0,0)\}\big\}.$$

We replace the two excluded polytopes by the two infinite families $\mathsf{pyr}\,\pentagon_n^{*k}$ and $\mathsf{bipyr}\,\pentagon_n^{*k}$ and claim that

$$\begin{aligned}
\big(g_1^2 * \ldots * g_1^2 * g_0^3\big)\,\big(\mathsf{pyr}\,\pentagon_n^{*k}\big) &= k!\,(n-3)^{k-1} && (5.1)\\
\big(g_1^2 * \ldots * g_1^2 * g_0^0 * g_1^2\big)\,\big(\mathsf{pyr}\,\pentagon_n^{*k}\big) &= k!\,(n-3)^{k} && (5.2)\\
\big(g_1^2 * \ldots * g_1^2 * g_1^2 * g_0^0\big)\,\big(\mathsf{pyr}\,\pentagon_n^{*k}\big) &= k!\,(n-3)^{k} && (5.3)
\end{aligned}$$

and

$$\begin{aligned}
\big(g_1^2 * \ldots * g_1^2 * g_0^3\big)\,\big(\mathsf{bipyr}\,\pentagon_n^{*k}\big) &= k!\,(n-3)^{k-1} && (5.4)\\
\big(g_1^2 * \ldots * g_1^2 * g_0^0 * g_1^2\big)\,\big(\mathsf{bipyr}\,\pentagon_n^{*k}\big) &= 3k!\,(n-3)^{k-1}(n-2) && (5.5)\\
\big(g_1^2 * \ldots * g_1^2 * g_1^2 * g_0^0\big)\,\big(\mathsf{bipyr}\,\pentagon_n^{*k}\big) &= 0 && (5.6)
\end{aligned}$$

To prove (5.1), (5.2) and (5.3) the same arguments as in the previous proof work. Here the induction step relies on the observation that for any 2-face F of $\mathsf{pyr}\,\pentagon_n^{*k}$ that is not a triangle (that is, it is an n-gon), the quotient polytope $(\mathsf{pyr}\,\pentagon_n^{*k})/F$ is equivalent to $\mathsf{pyr}\,\pentagon_n^{*(k-1)}$.

We still have to establish (5.4), (5.5) and (5.6). For shorter notation let $P_j := \mathsf{bipyr}\,\pentagon_n^{*j}$. Again, the only 2-faces of P_k that are not triangles are n-gons and P_k has k of them, which are all already contained in $\pentagon_n^{*k}$. Therefore, the quotient polytope P_k/F of an n-gon face F is again P_{k-1} and (5.4) follows by induction as before.

For the proof of (5.5) let $k = 2$ and consider $P_2 = \mathsf{bipyr}(\pentagon_n * \pentagon_n)$. Then

$$\big(g_1^2 * g_0^0 * g_1^2\big)\,(P_2) = \sum_{\substack{F \subset F' \text{ faces of } P_2\\ \text{with } \dim F = 2,\\ \dim F' = 3}} g_1^2(F)\,g_0^0(F'/F)\,g_1^2(P_2/F')$$

Any n-gon F is contained in n different n-gon pyramids, which were already 3-faces of $\pentagon_n * \pentagon_n$, and additionally in two n-gon pyramids, which arise as pyramids over F by the bipyramid operation. For F' being of the first kind we have $P/F' \cong \pentagon_4$, since F' is contained in four different facets of P_2, namely, the pyramids with the two apices, each over the two facets of $\pentagon_n * \pentagon_n$ containing F'. For the second kind of 3-face we have $P/F' \cong \pentagon_n$, since the facets containing F' are now exactly the pyramids with respect to the given apex over those facets of $\pentagon_n * \pentagon_n$ that contain F—and there are precisely n of those. Putting it all together,

$$\begin{aligned}
\big(g_1^2 * g_0^0 * g_1^2\big)\,(P_2) &= 2\,(n-3)\Big(\sum_{\substack{\text{3-faces } F' \\ \text{of 1st kind}}} g_1^2(\underbrace{P_2/F'}_{\pentagon_4}) + \sum_{\substack{\text{3-faces } F' \\ \text{of 2nd kind}}} g_1^2(\underbrace{P_2/F'}_{\pentagon_n})\Big) \\
&= 2\,(n-3)\big(n \cdot 1 + 2 \cdot (n-3)\big) \\
&= 2\,(n-3) \cdot 3\,(n-2).
\end{aligned}$$

Now (5.5) follows again by induction, using the fact that $P_k/F \cong P_{k-1}$ for every n-gon 2-face F of P_k.

Finally, (5.6) can be proved be considering the facets of P_k. For every $k \geq 2$,

$$\big(g_1^2 * \ldots * g_1^2 * g_1^2 * g_0^0\big)\,(P_k) \;=\; \sum_{F \text{ facet of } P_k} (g_1^2 * \ldots * g_1^2 * g_1^2)(F)\; g_0^0(P_k/F).$$

If F is a facet of P_k then it is a pyramid over one of the facets of $\pentagon_n^{*k}$, which are themselves of the form $\pentagon_n * \ldots * \Delta_1 * \ldots * \pentagon_n$. In other words, F is combinatorially equivalent to $\pentagon_n^{*(k-1)} * \Delta_2 \cong \mathsf{pyr}^2\, \pentagon_n^{*(k-1)}$. Therefore, (5.6) follows from the assertion that

$$\big(g_1^2 * \ldots * g_1^2\big)\,\big(\mathsf{pyr}^3\, \pentagon_n^{*(k-1)}\big) \;=\; 0.$$

This, in turn, can be shown by induction, using $(\mathsf{pyr}^3\, \pentagon_n^{*j})/\pentagon_n \cong \mathsf{pyr}^3\, \pentagon_n^{*(j-1)}$ for $j \geq 1$.

Now the proposition is eventually proved by again dividing by $(n-3)^k$ and letting $n \to \infty$. We get the following matrix:

	$(d-3,0;1,1)$	$\cdots$	$(1,1;\ldots;1,1;3,0)$	$(1,1;\ldots;1,1;0,0;1,1)$	$(1,1;\ldots;1,1;1,1;0,0)$
$P[(d-3,0;1,1)]$	>0				
$\vdots$		$\ddots$		0	
$P[(1,1;\ldots;1,1;3,0)]$			>0		
$\mathsf{pyr}\,\pentagon_n^{*k}$	$*$		$\to 0$	$\to k!$	$\to k!$
$\mathsf{bipyr}\,\pentagon_n^{*k}$			$\to 0$	$\to 3k!$	0

Again, linear independence is obvious and the hyperplane represented by the sequence $(1,1;\ldots;1,1;3,0)$ contains the flag vectors of F_d-4 polytopes out of the set B, plus the asymptotic flag vectors of the two families $\mathsf{pyr}\,\pentagon_n^{*k}$ and $\mathsf{bipyr}\,\pentagon_n^{*k}$. □

QUESTIONS

The situation in low dimensions suggests some questions for arbitrary d.

Conjecture 5.3.6. For even d the inequality given by $G[(\frac{d}{2},\frac{d}{2})]$ is facet-defining, as is for odd d the one given by $G[(\frac{d-1}{2},\frac{d-1}{2};0,0)]$.

Additionally, for even d the convolution $G[(\frac{d}{2}-1,\frac{d}{2}-1;1,0)]$ determines a facet of $\mathcal{F\ell C}_d$.

This might even be provable by only using the basis polytopes $P[b]$, $b\in\mathbf{T}_d$.

Conjecture 5.3.7. If $G[b_1]$ and $G[b_2]$ define facets of $\mathcal{F\ell C}_{\mathsf{dim}\,b_1}$ and $\mathcal{F\ell C}_{\mathsf{dim}\,b_2}$, then $G[(b_1;b_2)]$ is facet-defining for $\mathcal{F\ell C}_{\mathsf{dim}\,b_1+\mathsf{dim}\,b_2+1}$, provided $(b_1;b_2)\in\mathbf{T}_d$.

Chapter 6

f-vectors of moderate-dimensional polytopes

In this chapter we will be concerned with f-vectors. Since the flag vector of a polytope encodes a superset of the information in the f-vector, we can use results about the flag vector to conclude properties of the f-vector.

The first topic is the application of our visualisation method from Section 3.2 to f-vectors of moderate-dimensional polytopes. Here "moderate" means 4 or 5, where the 4-dimensional case is just the executive summary of Ziegler's presentation in [59]. In the 5-dimensional case a number of interesting observations can be made.

A major question is that of unimodality of f-vectors. It is known that for high-dimensional polytopes f-vectors are not unimodal in general. However, the situation is not so clear in moderate dimensions, and there are some related properties that are worth studying. Some of them can be proved, others disproved, for polytopes for moderate dimensions, where "moderate" now extends to dimension 6, partly even 7.

Finally, we come back to centrally-symmetric polytopes, which we already discussed for dimension 3 in Section 3.3. We examine the 3^d-conjecture by Kalai [37] and prove it for 4-polytopes. Some partial results towards a proof for higher dimensions are mentioned, as well as related conjectures and a number of counterexamples to them. Some of these, notably the Hansen polytopes, reveal quite interesting properties and might be worth being studied further.

6.1 Visualisation

The methods to visualise the space of flag vectors that we developed in Section 3.2 can naturally be applied to f-vectors. This yields 2- and 3-dimensional representations of the space of f-vectors of 4- and 5-polytopes, respectively. In either case we use all currently known linear inequalities for these f-vectors.

DIMENSION 4

The visualisation for 4-polytopes was first done by Ziegler in [59], where he discussed essentially the representation in Figure 6.1. It can be obtained from the inequalities

$$\begin{aligned} f_0 - 5 &\geq 0 \\ f_3 - 5 &\geq 0 \\ f_1 - 2f_0 &\geq 0 \\ f_3 - 2f_2 &\geq 0 \\ 2f_1 - 5f_0 - 5f_3 + 2f_2 + 10 &\geq 0 \end{aligned}$$

considering the transformation

$$T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -5 & 1 & 0 & 0 \\ -5 & 1 & -1 & 1 \\ -20 & 0 & 1 & 1 \end{pmatrix}$$

which leads to the projective coordinates

$$\varphi_1 = \frac{f_0 - 5}{f_1 + f_2 - 10} \quad , \qquad \varphi_2 = \frac{f_3 - 5}{f_1 + f_2 - 10}$$

We get a polytope $\widetilde{\mathcal{P}}_4$ in (φ_1, φ_2)-space described by

$$\begin{aligned} \varphi_1 &\geq 0 \\ \varphi_2 &\geq 0 \\ 1 - 3\varphi_1 - \varphi_2 &\geq 0 \\ 1 - \varphi_1 - 3\varphi_2 &\geq 0 \\ 2 - 5\varphi_1 - 5\varphi_2 &\geq 0 \end{aligned}$$

as a projective representation of our approximation for $\mathcal{FC}_4$, the convex hull of f-vectors of 4-polytopes. $\widetilde{\mathcal{P}}_4$ is in fact a 5-gon, see Figure 6.1, and it is symmetric with respect to the axis $\{\varphi_1 = \varphi_2\}$, which contains points coming from self-dual polytopes.

As in the case of flag vectors of 4-polytopes in Section 4.1 the five facets of $\widetilde{\mathcal{P}}_4$ correspond to special properties of 4-polytopes. The first two inequalities

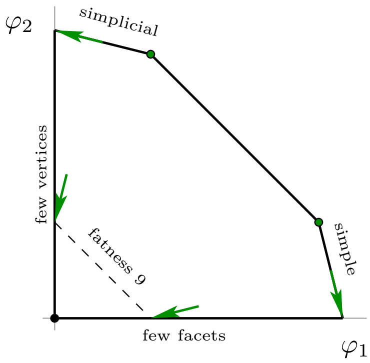


Figure 6.1: $\widetilde{\mathcal{P}}_4$ as in [59]

can be satisfied with equality only by the simplex, and therefore they can at most be asymptotic facets. This they are, and polytope families which show this are for instance the cyclic polytopes and projected products of polygons, indicated by the arrows in Figure 6.1.

Cyclic polytopes, together with stacked polytopes and duality, also show that the third and fourth inequality, where equality characterise simple and simplicial polytopes, respectively, define facets of $\mathcal{FC}_4$. With these two examples—and their duals—we can already conclude that all vertices of $\widetilde{\mathcal{P}}_4$, except for the point $(0, 0)$, represent rays (asymptotic in the case of cyclic polytopes) of $\mathcal{FC}_4$, and also that the last inequality is facet-defining for $\mathcal{FC}_4$.

This last inequality is in fact the most interesting one. Rewritten in terms of the fatness $F = (f_1 + f_2 - 20)/(f_0 + f_3 - 10)$, it states that $F \geq 5/2$ for all 4-polytopes. In fact, polytopes with the same fatness F yield points in $\widetilde{\mathcal{P}}_4$ on lines $\{\varphi_1 + \varphi_2 = 1/F\}$, parallel to the facet defined by the last inequality. In this respect, this inequality gives a lower bound for the fatness of 4-polytopes.

Two questions arise immediately and are unsolved by now. First, is there a corresponding upper bound, and second, is the lower bound also true for more general objects than polytopes?

Problem 6.1.1. Find a polytope family with arbitrarily high fatness or prove that fatness is bounded for 4-polytopes.

The current record holders with respect to fatness are the projected products of polygons (Ziegler [60]) with $F = 9 - \varepsilon$ with arbitrarily small $\varepsilon > 0$.

Problem 6.1.2. Is it true that the inequality $2f_1 - 5f_0 - 5f_3 + 2f_2 + 10 \geq 0$ holds for all regular 3-spheres having the intersection property?

Part of Section 7.3 is connected to this problem. At the time being, however, no examples of 3-spheres violating this inequality are known.

Dimension 5

These are all the currently known linear inequalities for f-vectors of 5-polytopes, stated in terms of the reduced f-vector:

$$
\begin{aligned}
f_0 - 6f_\emptyset &\geq 0 &(6.1)\\
f_3 - f_2 + f_1 - f_0 - 4f_\emptyset &\geq 0 &(6.2)\\
2f_1 - 5f_0 &\geq 0 &(6.3)\\
5f_0 - 5f_1 + 5f_2 - 3f_3 - 10f_\emptyset &\geq 0 &(6.4)\\
3f_2 - 2f_1 - 2f_3 &\geq 0 &(6.5)
\end{aligned}
$$

All these inequalities are tight for the f-vector of the 5-simplex. Inequality (6.5) is due to Kalai [36, Theorem 7.1].

We get as an approximation of $\mathcal{FC}_5$ a 4-dimensional cone $\mathcal{C}_5$ in $\mathbb{R}^5$ with its apex at the point $f(\Delta_5) = (1, 6, 15, 20, 15)$. To transform this cone into a 3-dimensional polytope $\mathcal{P}_5$, we consider the transformation matrix

$$
T = \begin{pmatrix}
1 & 0 & 0 & 0 & 0\\
-6 & 1 & 0 & 0 & 0\\
0 & 0 & 1 & 0 & -1\\
-4 & -1 & 1 & -1 & 1\\
-20 & 0 & 0 & 1 & 0
\end{pmatrix}
$$

We have $\det T = -2$ and $T \cdot f(\Delta_5) = (1, 0, 0, 0, 0)^\top$ and $f_2(P) - 20 = 0$ if and only if $P \cong \Delta_5$, so all the necessary prerequisites are satisfied. Passing to projective coordinates yields

$$
\varphi_1 = \frac{f_0 - 6}{f_2 - 20}\,, \quad \varphi_2 = \frac{f_1 - f_3}{f_2 - 20}\,, \quad \varphi_3 = \frac{f_3 - f_2 + f_1 - f_0 - 4}{f_2 - 20} = \frac{f_4 - 6}{f_2 - 20}
$$

Our original inequalities (6.1) to (6.5) translate into inequalities for $\varphi_0, \varphi_2, \varphi_3$, defining the polytope $\mathcal{P}_5$ (see Figure 6.2):

$$
\begin{aligned}
(6.1) &\iff \varphi_1 \geq 0\\
(6.2) &\iff \varphi_3 \geq 0\\
(6.3) &\iff 1 - 4\varphi_1 + \varphi_2 + \varphi_3 \geq 0\\
(6.4) &\iff 1 + \varphi_1 - \varphi_2 - 4\varphi_3 \geq 0\\
(6.5) &\iff 1 - 2\varphi_1 - 2\varphi_3 \geq 0
\end{aligned}
$$

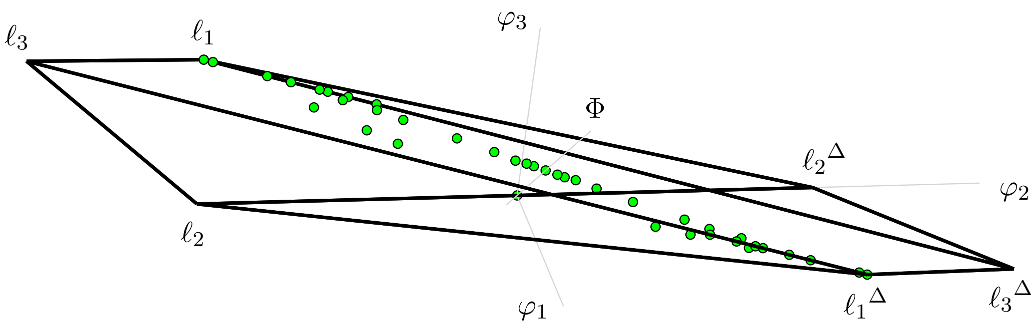


Figure 6.2: $\mathcal{P}_5$ with f-vectors on the boundary (green)

In the following we gather some properties of the polytope $\mathcal{P}_5$ as well as consequences for the f-vectors of 5-polytopes. The vertices of the geometric realisation are

$$\ell_1 = \begin{pmatrix} 0 \\ -1 \\ 1/2 \end{pmatrix}, \quad \ell_2 = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}, \quad \ell_3 = \begin{pmatrix} 0 \\ -3/2 \\ 1/2 \end{pmatrix},$$
$$\ell_1{}^\Delta = \begin{pmatrix} 1/2 \\ 1 \\ 0 \end{pmatrix}, \quad \ell_2{}^\Delta = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \ell_3{}^\Delta = \begin{pmatrix} 1/2 \\ 3/2 \\ 0 \end{pmatrix}$$

and as in the case of flag vectors of 4-polytopes in Chapter 4 they correspond to rays of $\mathcal{C}_4$, which (by slight abuse of notation) we also denote by $\ell_1, \ldots, \ell_3{}^\Delta$.

$\mathcal{P}_5$ is symmetric with respect to the axis

$$\Phi \ := \ \{(\varphi_1, \varphi_2, \varphi_3) \in \mathbb{R}^3 \mid \varphi_1 = \varphi_3, \varphi_2 = 0\}$$

and dualisation corresponds to exchanging φ_1 and φ_3 and reversing the sign of φ_2. This justifies the above notation for the "dual" rays and also implies that f-vectors of self-dual 5-polytopes appear on Φ.

For simplicial polytopes equality holds in (6.4) and (6.5) by the Dehn-Sommerville equations (see Theorem 1.2.10 and cf. Kalai [36, Section 7]). Hence the corresponding f-vectors can be found in the intersection of the related facets $\langle \ell_1, \ell_2{}^\Delta, \ell_3{}^\Delta \rangle$ and $\langle \ell_1, \ell_3, \ell_1{}^\Delta, \ell_3{}^\Delta \rangle$. Additionally, the inequalities obtained from the Lower Bound Theorem by Barnette [5] define a smaller polytope within $\mathcal{P}_5$ (cf. Figure 6.3), which therefore also contains all f-vectors of simplicial polytopes.

This forces any point coming from the f-vector of a simplicial polytope to lie on the line segment between the point $(1/10, -1/2, 2/5)^\top$ and the vertex ℓ_1. By duality, an analogous statement holds for simple polytopes.

Again, the facets described by (6.1) and (6.2) contain only asymptotic f-vectors. Examples are given by the following families:

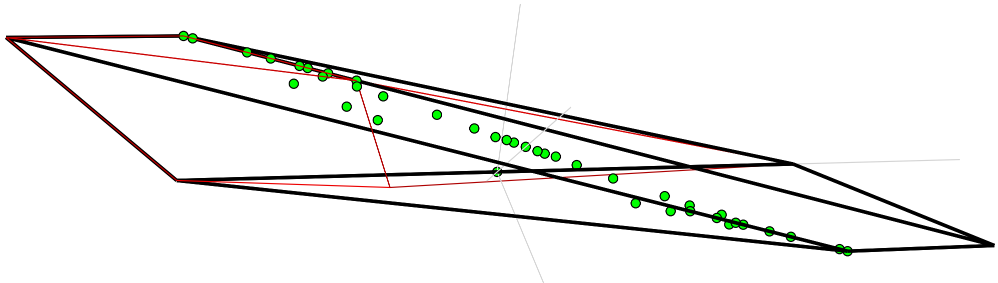


Figure 6.3: $\mathcal{P}_5$ with region of simplicial polytopes by the Lower Bound Theorem (red)

- joins of n-gons, $\pentagon_n * \pentagon_n$,
- neighbourly cubical 5-polytopes,
- prisms and pyramids over cyclic 4-polytopes,
- cyclic 5-polytopes.

The asymptotic flag vector arising from cyclic 5-polytopes is in fact on ℓ_1 and those of joins of n-gons are in both of the facets (6.1) and (6.2), that is, on the face $\langle \ell_2, {\ell_2}^\Delta \rangle$.

The only facet of $\mathcal{C}_5$ which actually contains f-vectors in its relative interior is the one corresponding to Kalai's inequality (6.5). It is obviously self-dual and polytopes P satisfying it with equality are exactly those with the following two properties (see Kalai [36, Theorem 7.1]:

- P is 2-*hypersimplectic*, that is, $(g_1^2 * g_1^2)(P) = 0$. Geometrically, this means that for every 2-face F of P we have $F \cong \Delta_2$ or $P/F \cong \Delta_2$.
- P is *center-boolean*, that is, every proper face of P is simple (cf. Definition 4.1.2 for 4-polytopes); in Kalai's terminology, P is *semisimplectic*.

Examples include simplicial and simple polytopes, as remarked earlier, as well as connected sums of simplicial and simple polytopes. A particularly interesting example is $\Delta_5 \# \Delta_5$, whose f-vector is in the symmetry center of the facet $\langle \ell_1, \ell_3, {\ell_1}^\Delta, {\ell_3}^\Delta \rangle$.

The fact that simplicial polytopes have f-vectors on Kalai's facet implies that the facet $\langle \ell_1, {\ell_2}^\Delta, {\ell_3}^\Delta \rangle$ corresponding to inequality (6.4) can at best contain asymptotic f-vectors. If it really does, that is, if it defines an asymptotic

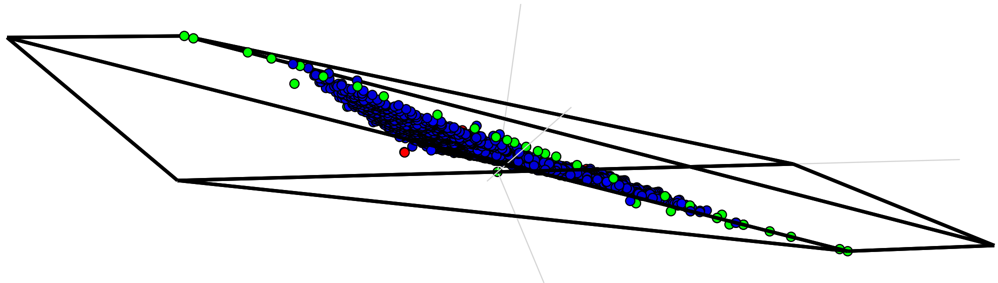


Figure 6.4: $\mathcal{P}_5$ with f-vectors of 0/1-polytopes (blue) and offside points (red; only one dually offside point is well visible)

facet of $\mathcal{FC}_5$, is in fact an open question. To solve this, it would be enough to find an asymptotic f-vector on $\langle \ell_1, {\ell_2}^{\Delta}, {\ell_3}^{\Delta} \rangle$, away from the face $\langle \ell_1, {\ell_3}^{\Delta} \rangle$, for which we already know f-vectors that span it.

Problem 6.1.3. Find a family P_n of polytopes with f-vectors $f^{(n)} := f(P_n)$ and corresponding $\varphi_1^{(n)}, \varphi_2^{(n)}, \varphi_3^{(n)}$ such that

$$1 + \varphi_1^{(n)} - \varphi_2^{(n)} - 4\varphi_3^{(n)} \stackrel{n\to\infty}{\longrightarrow} 0\,, \qquad \text{but} \qquad 1 - 2\varphi_1^{(n)} - 2\varphi_3^{(n)} \geq \varepsilon \;\text{ for all } n$$

for some $\varepsilon > 0$.

Reformulated in terms of the f-vector, the desired properties are

$$2f_3^{(n)} - 5f_4^{(n)} \stackrel{n\to\infty}{\longrightarrow} 0\,, \qquad \text{and} \qquad 3f_2^{(n)} - 2f_1^{(n)} - 2f_3^{(n)} \geq \varepsilon \;\text{ for all } n,$$

that is, the polytopes are "asymptotically closer to being simplicial than to being 2-hypersimplectic and center-boolean".

Problem 6.1.3 concerns "extremal" polytopes in the sense that their f-vectors are close to one of the facets. Another, somehow weaker, concept of extremality reveals itself if one tries to examine the region of $\mathcal{C}_5$ that contains the f-vectors of those polytopes having none of the facet-defining properties.

A possible source for many non-simplicial and non-simple polytopes are 0/1-polytopes. Figure 6.4 shows points in $\mathcal{P}_5$ arising from all possible 0/1-polytopes, as well as some other constructions that yield non-simplicial and non-simple polytopes. The 0/1-polytopes were obtained using the classification of Aichholzer [2] and the accompanying database.

What is remarkable is that our usual examples of 5-polytopes seem to fill only a very restricted subset of $\mathcal{P}_5$. More precisely, one could be tempted to conjecture that all f-vectors of 5-polytopes in the $(\varphi_1, \varphi_2, \varphi_3)$-space lie on one side of the hyperplane defined by the origin and the face $\langle \ell_1, {\ell_3}^\Delta \rangle$, and also, by symmetry, the one defined by $(0,0,0)$ and $\langle {\ell_1}^\Delta, \ell_3 \rangle$. This yields the inequalities

$$3\varphi_1 - \varphi_2 - 2\varphi_3 \geq 0 \quad \text{and} \quad -2\varphi_1 + \varphi_2 + 3\varphi_3 \geq 0$$

which can again be retranslated into two dual inequalities for the f-vectors.

Definition 6.1.4. A 5-polytope P is called *offside* if

$$5f_0(P) - 3f_1(P) + 2f_2(P) - f_3(P) \ < \ 10$$

and *dually offside* if P^Δ is offside.

So the above conjecture would be equivalent to the statement that there are no offside 5-polytopes. This, however, is not true—there are 5-polytopes that violate one of the two inequalities. The following examples are known to the author and are indicated in Figure 6.4 as red dots:

- a single offside 0/1-polytope with 18 vertices,
- the 2-simplicial, 2-simple polytopes $E_2(Q_n^5)$ for $n \geq 1$ (see Paffenholz and Ziegler [44]) are dually offside,
- glued 5-cubes with stackings performed on it; more precisely: Let $C_{n,k}$ be a polytope obtained by gluing together n 5-cubes and after that stacking k of the resulting facets ($k \leq 8n+2$)—then for example $C_{2,16}$, $C_{2,17}$ and $C_{2,18}$ are offside. In general, for all $n > 1$, the "maximally stacked" polytope $C_{n,8n+2}$ is offside.

The natural question to ask is whether these examples are really special or there are actually a lot of offside polytopes. It might also be interesting to know "how far" offside a 5-polytope can be.

Problem 6.1.5. Characterise and/or find more examples of offside or dually offside polytopes. Is the value $5f_0 - 3f_1 + 2f_2 - f_3$ (or its dual) bounded from below for 5-polytopes?

Obviously, Problems 6.1.3 and 6.1.5 are related: if one finds a family as in Problem 6.1.3 this also gives examples of offside polytopes.

6.2 UNIMODALITY AND f-VECTOR SHAPES

The general motivation for this section is to understand the possible "shapes" of polytope f-vectors. Particularly interesting might be the question whether f-vectors of polytopes are unimodal, that is, the numbers $f_k(P)$ increase up to some k and then decrease again. This is known to be false in general; for an extensive treatment of this problem see Ziegler's book [58] and the paper of Eckhoff [22].

Besides unimodality there are a number of related properties concerning the shape of the f-vector $f(P) = (f_0, \dots, f_{d-1})$ of a d-polytope P that are worth investigating:

(C) convexity: $f_k \geq (f_{k-1} + f_{k+1})/2$ for all $k \in \{1, \dots, d-2\}$

(L) logarithmic convexity: $f_k^2 \geq f_{k-1}f_{k+1}$ for all $k \in \{1, \dots, d-2\}$

(U) unimodality: $f_0 \leq \dots \leq f_k \geq \dots \geq f_{d-1}$ for some $k \in \{0, \dots, d-1\}$

(B) Bárány's property: $f_k \geq \mathsf{min}\{f_0, f_{d-1}\}$ for all $k \in \{0, \dots, d-1\}$

It is easy to see that each property implies the next one:

$$(\mathsf{C}) \;\Rightarrow\; (\mathsf{L}) \;\Rightarrow\; (\mathsf{U}) \;\Rightarrow\; (\mathsf{B}).$$

As remarked, unimodality is false in general for $d \geq 8$, and counterexamples are, for instance, given by connected sums of cyclic polytopes with their duals; see [58, Example 8.41] and [22]. This implies that (C) and (L) are also false for every $d \geq 8$. On the other hand, (U)—and therefore also (B)—is (rather trivially) true for all $d \leq 4$.

For simplicial polytopes of arbitrary dimension a weaker version of unimodality was proved by Björner. It states that up to $f_{\lfloor d/2 \rfloor}$ the f-vector entries increase and decrease again, starting with $f_{\lfloor 3(d-1)/4 \rfloor}$; see [58, Theorem 8.39]. With this, (U) follows for simplicial—and therefore also for simple—polytopes of dimensions $d \leq 8$ and $d = 10$.

Similarly, convexity is trivially true up to $d \leq 3$, and for $d = 4$ it follows easily from $f_0 \geq 5$ and $f_2 \geq 2f_3$ together with Euler's equation and duality. Therefore all four properties are true for $d \leq 4$.

It remains to investigate all properties in dimensions 5 to 7, as well as Bárány's property for arbitrary dimenson. We show that (C) is false for $d \geq 5$, while (U) can be proved for $d = 5$. For dimension 6 we show (B), while for dimension 7 the same line of argumentation gives not enough information to prove (B). We conclude with some calculations that suggest that in fact logarithmic convexity (L) might be true for $d \leq 7$.

DIMENSIONS 5 TO 7

That unimodality is true for 5-dimensional polytopes was also observed by Eckhoff [22].

Proposition 6.2.1. Unimodality (U) holds for f-vectors of polytopes of dimension $d \le 5$.

Proof. Let P be a 5-polytope and $f(P) = (f_0, f_1, f_2, f_3, f_4)$ its f-vector. Obviously, $5f_0 \le 2f_1$ and $5f_4 \le 2f_3$, therefore $f_0 < f_1$ and $f_3 > f_4$. Kalai's inequality (6.5) in Section 6.1 implies $3f_2 \ge 2f_1 + 2f_3$, hence

$$f_2 \;\ge\; \frac{2}{3}(f_1 + f_3) \;>\; \frac{f_1 + f_3}{2}.$$

In particular, $f_2 > f_1$ or $f_2 > f_3$. Therefore $f(P)$ is unimodal. □

This implies that also (B) holds for 5-polytopes. On the other hand, convexity is false in general, already in dimension 5.

Proposition 6.2.2. Convexity (C) fails to hold for $d \ge 5$, that is, the f-vectors of d-polytopes are not convex in general.

Proof. For dimension 5 the f-vector of the cyclic polytope with n vertices is given by

$$f(\mathcal{C}_5(n)) = \Big(n,\ \frac{n(n-1)}{2},\ 2(n^2 - 6n + 10),\ \frac{5(n-3)(n-4)}{2},\ (n-3)(n-4)\Big)$$

(cf. [58, Chapter 8]), which implies

$$f_1 \;=\; \frac{n^2 - n}{2} \;<\; \frac{2n^2 - 11n + 20}{2} \;=\; \frac{f_0 + f_2}{2}$$

for $n \ge 8$; see Figure 6.5.

For $d \ge 6$, cyclic d-polytopes are 3-neighbourly, therefore $f_1 = \binom{f_0}{2}$ and $f_2 = \binom{f_0}{3}$ with $f_0 \ge d + 1$. We conclude that

$$f_0 + f_2 - 2f_1 \;=\; \frac{1}{6}\, f_0(f_0 - 2)(f_0 - 7) \;>\; 0$$

for cyclic d-polytopes with $f_0 \ge \max\{d + 1, 8\}$. Thus for $d \ge 7$ already the d-simplex is a counterexample for (C). □

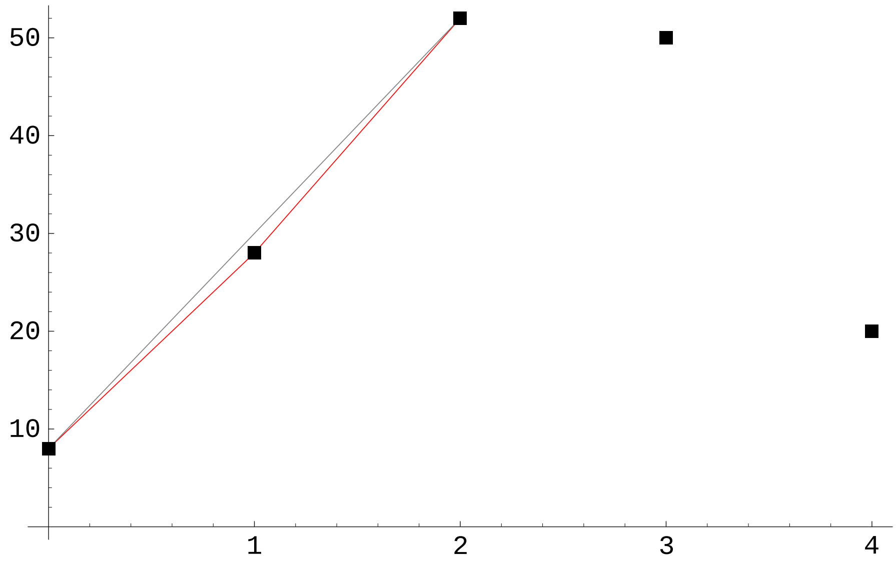


Figure 6.5: (non-convex) f-vector of $\mathcal{C}_5(8)$

Hence for $d = 5$ the only remaining open question is whether logarithmic convexity holds.

If we want to prove unimodality for f-vectors of 6-polytopes, we can use some trivial facts, such as $f_0 < f_1$ and $f_4 > f_5$. From this, (U) would simply follow from the statement

$$f_1(P) \;\leq\; f_2(P) \tag{6.6}$$

for all 6-polytopes P or, equivalently by duality, from $f_3(P) \geq f_4(P)$. However, this does not follow from the commonly known linear inequalities—we only have a weaker statement.

Proposition 6.2.3. Let $(f_0, \ldots, f_5)$ be the f-vector of a 6-polytope. Then

$$f_2 \;\geq\; \frac{2}{3} f_1 + 21.$$

Proof. We claim that the following inequalities hold:

$$f_1 - 3f_0 \;\geq\; 0 \tag{6.7}$$

$$f_0 - f_1 + f_2 - 21 \;\geq\; 0 \tag{6.8}$$

The assertion then follows by multiplying (6.8) by 3 and adding (6.7).

Inequality (6.7) is the familiar statement that every vertex is in at least 6 edges. For the proof of (6.8) we use Ehrenborg's Lifting Theorem 1.4.6, which implies that $\langle \mathbb{c}^2\mathbb{d}\mathbb{c}^2 - 19\mathbb{c}^6 \mid \Psi(P)\rangle \geq 0$. Expressing the $\mathbb{c}\mathbb{d}$-word $\mathbb{c}^2\mathbb{d}\mathbb{c}^2$ in terms of the flag vector of the 6-polytope P by applying Theorem 1.3.2 yields

$$\langle \mathbb{c}^2\mathbb{d}\mathbb{c}^2 \mid \Psi(P)\rangle \;=\; \sum_{i=0}^{2}(-1)^{4-i}k_i \;=\; k_0 - k_1 + k_2.$$

For the sparse flag k-vector we have

$$k_i \;=\; \sum_{T\subseteq\{i\}} (-2)^{1-|T|} f_T \;=\; -2f_\emptyset + f_i \;=\; f_i - 2$$

and therefore $\langle \mathbb{c}^2\mathbb{d}\mathbb{c}^2 \mid \Psi(P)\rangle = f_0 - f_1 + f_2 - 2$. The trivial $\mathbb{c}\mathbb{d}$-word $\mathbb{c}^6$ translates into $f_\emptyset = 1$, hence

$$\langle \mathbb{c}^2\mathbb{d}\mathbb{c}^2 - 19\mathbb{c}^6 \mid \Psi(P)\rangle \;=\; f_0 - f_1 + f_2 - 21.$$

□

Corollary 6.2.4. The f-vectors of 6-polytopes satisfy Bárány's property (B).

Proof. Let $(f_0, \ldots, f_5)$ be the f-vector of a 6-dimensional polytope. Clearly, $f_1 \geq 3f_0 > f_0$, thus by Proposition 6.2.3

$$f_2 \;\geq\; \frac{2}{3}f_1 + 21 \;\geq\; 2f_0 + 21 \;>\; f_0.$$

Dually, we have $f_3 > f_5$ and $f_4 > f_5$. □

As the desired inequality (6.6) for unimodality does not follow from the known linear inequalities, one can find potential flag vectors that satisfies all these, but not (6.6). An example for a family of vectors is

$$\begin{aligned}\mathbf{f}^{(\ell)} \;&=\; (f_0, f_1, f_2, f_3, f_4;\\ &\qquad f_{02}, f_{03}, f_{04}, f_{13}, f_{14},\\ &\qquad f_{24};\; f_{024})\\ &=\; (22+\ell, 111+3\ell, 110+2\ell, 35+4\ell, 21+6\ell;\\ &\qquad 780+15\ell, 1340+50\ell, 1080+51\ell, 2010+90\ell, 2160+132\ell,\\ &\qquad 1260+114\ell;\; 6480+396\ell)\end{aligned}$$

for $\ell \geq 0$. It is not clear whether a polytope P with $\mathbf{f}(P) = \mathbf{f}^{(\ell)}$ exists. For the number of facets of such a polytope we get $f_5(P) = 7 + 2\ell$ by Euler's equation.

A similar statement to the one in Proposition 6.2.3 holds for 7-polytopes. However, this is not enough to prove even Bárány's property (B) completely, since we have no condition for f_3.

Proposition 6.2.5. Let $(f_0, \dots, f_6)$ be the f-vector of a 7-polytope. Then

$$f_2 \;\geq\; \frac{5}{7}\, f_1 + 36.$$

Proof. As before, we consider two valid inequalities which together imply the assertion:

$$\begin{aligned} 2f_1 - 7f_0 &\;\geq\; 0 & (6.9)\\ f_0 - f_1 + f_2 - 36 &\;\geq\; 0 & (6.10) \end{aligned}$$

Again, (6.9) is obvious. The nonnegativity of $\langle \mathbb{c}^2\mathbb{d}\mathbb{c}^3 - 34\mathbb{c}^7 \mid \Psi(P)\rangle$, due to Theorem 1.4.6, gives inequality (6.10) by the same calculation as in the proof of Proposition 6.2.3 (the additional $\mathbb{c}$ at the end makes no difference):

$$\langle \mathbb{c}^2\mathbb{d}\mathbb{c}^3 \mid \Psi(P)\rangle \;=\; f_0 - f_1 + f_2 - 2.$$

Together with $\mathbb{c}^7$, which again represents $f_\emptyset$, we get

$$\langle \mathbb{c}^2\mathbb{d}\mathbb{c}^3 - 34\mathbb{c}^7 \mid \Psi(P)\rangle \;=\; f_0 - f_1 + f_2 - 36.$$

□

Corollary 6.2.6. If $(f_0, \dots, f_6)$ is the f-vector of a 7-polytope then

$$f_k \;\geq\; \mathsf{min}\{f_0, f_6\} \qquad \text{for } k = 0, 1, 2, 4, 5, 6$$

with strict inequality for $k \neq 0, 6$.

Proof. We have $f_1 \geq \frac{7}{2} f_0 > f_0$, and Proposition 6.2.5 implies

$$f_2 \;\geq\; \frac{5}{7}\, f_1 + 36 \;\geq\; \frac{5}{2}\, f_0 + 36 \;>\; f_0.$$

$f_4 > f_6$ and $f_5 > f_6$ follow by duality. □

Again, one can find vectors satisfying all known linear inequalities, but violating both $f_3 \geq f_0$ and $f_3 \geq f_6$; take, for instance, the potential flag vector

$$\begin{aligned} \mathbf{f} \;&=\; (f_0, f_1, f_2, f_3, f_4, f_5;\\ &\qquad f_{02}, f_{03}, f_{04}, f_{05}, f_{13}, f_{14}, f_{15}, f_{24}, f_{25}, f_{35};\\ &\qquad f_{024}, f_{025}, f_{035}, f_{135})\\ &=\; (134, 469, 371, 70, 371, 469;\\ &\qquad 2814, 6580, 10360, 8484, 9870, 20720, 21210, 13790, 20720, 9870;\\ &\qquad 62160, 84840, 84840, 127260). \end{aligned}$$

Dimension	≤ 4	5	6	7	≥ 8
(C)	✔	✘	✘	✘	✘
(L)	✔	?	?	?	✘
(U)	✔	✔	?	?	✘
(B)	✔	✔	✔	?	?

Table 6.1: Properties of f-vector shapes of polytopes

A polytope with this flag vector would obviously be a counterexample for (B). From Euler's equation, we get $f_6 = 134$. Note that $\mathbf{f}$ is symmetric, that is, it could be the flag vector of a self-dual polytope.

The results from this section can be summarised as shown in Table 6.1. A ✔, respectively ✘, in the table indicates that the property in question holds, respectively does not hold, for all polytopes of the given dimension, whereas the question marks illustrate the cases that are still open.

Logarithmically convex examples

Another approach to solutions to the open questions is to try and find counterexamples. Most promising may be connected sums of cyclic polytopes, since this construction yields counterexamples for unimodality in dimension 8.

The effect of this construction on the f-vector essentially amounts to adding the f-vectors of the involved polytopes (see for instance [58, pp. 274f]): If P is a simplicial and Q a simple d-polytope, $d \geq 3$, then the f-vector of $P\#Q$ is given by

$$f_i(P\#Q) = \begin{cases} f_i(P) + f_i(Q) & \text{for } 1 \leq i \leq d-2 \\ f_i(P) + f_i(Q) - 1 & \text{for } i = 0 \text{ or } i = d-1 \end{cases}$$

In particular, the f-vector of the connected sum of a d-polytope P with its polar is symmetric:

$$\begin{aligned} f_i(P\#P^\Delta) &= \begin{cases} f_i(P) + f_{d-1-i}(P) & \text{for } 1 \leq i \leq d-2 \\ f_i(P) + f_{d-1-i}(P) - 1 & \text{for } i = 0 \text{ or } i = d-1 \end{cases} \\ &= f_{d-1-i}(P\#P^\Delta) \end{aligned}$$

A straightforward calculation now shows that the connected sums of cyclic 7-polytopes with their duals are still logarithmically convex, although they are asymptotically close to violating (L).

Proposition 6.2.7. For all $n \geq 8$, the f-vector of $P_7^n := \mathcal{C}_7(n)\#\mathcal{C}_7(n)^\Delta$ is logarithmically convex and

$$\frac{f_3(P_7^n)^2}{f_2(P_7^n)f_4(P_7^n)} \xrightarrow{n\to\infty} 1$$

Proof. The f-vector of the cyclic 7-polytope on n vertices is given by

$$\begin{aligned} f(\mathcal{C}_7(n)) \quad = \quad & \big(n,\ \tfrac{n(n-1)}{2},\ \tfrac{n(n-1)(n-2)}{6},\ \tfrac{5(n-4)(n^2-8n+21)}{6}, \\ & \tfrac{(n-4)(3n^2-31n+84)}{2},\ \tfrac{7(n-4)(n-5)(n-6)}{6},\ \tfrac{(n-4)(n-5)(n-6)}{3}\big) \end{aligned}$$

(cf. [58, Chapter 8]). From this we obtain for $f(P_7^n) = (f_0(n), \dots, f_6(n))$:

$$\begin{aligned} f_0(n) &= \tfrac{(n-3)(n^2-12n+41)}{3} &,\quad f_1(n) &= \tfrac{7n^3-102n^2+515n-840}{6}, \\ f_2(n) &= \tfrac{5n^3-66n^2+313n-504}{3} &,\quad f_3(n) &= \tfrac{5(n-4)(n^2-8n+21)}{3} \end{aligned}$$

By symmetry of $f(P_7^n)$, these entries suffice to verify logarithmic convexity. We get

$$\begin{aligned} \frac{f_1(n)^2}{f_0(n)f_2(n)} &= \frac{(7n^3-102n^2+515n-840)^2}{4(n-3)(n^2-12n+41)(5n^3-66n^2+313n-504)} > 1\,, \\ \frac{f_2(n)^2}{f_1(n)f_3(n)} &= \frac{2(5n^3-66n^2+313n-504)^2}{5(n-4)(n^2-8n+21)(7n^3-102n^2+515n-840)} > 1\,, \\ \frac{f_3(n)^2}{f_2(n)f_4(n)} &= \frac{25(n-4)^2(n^2-8n+21)^2}{(5n^3-66n^2+313n-504)^2} > 1 \end{aligned}$$

for $n \geq 8$. Since the leading coefficients of the polynomials in the numerator and the denominator of the last fraction are equal,

$$\frac{f_3(P_7^n)^2}{f_2(P_7^n)f_4(P_7^n)} \xrightarrow{n\to\infty} 1.$$

□

With this in mind, the following conjecture seems plausible.

Conjecture 6.2.8. (L) holds for all d-polytopes of dimension $d \leq 7$.

The affirmative answer to this conjecture would settle the given open questions for moderate dimensions, leaving only Bárány's property for $d \geq 8$ as the last open problem.

6.3 Centrally-symmetric polytopes

As in Section 3.3 we call a polytope P *centrally-symmetric* (or *cs* for short) if $P = -P$, that is, it is centrally-symmetric with respect to the origin. A conjecture of Kalai [37, Conjecture A] states that the total number of faces of centrally-symmetric polytopes is minimised precisely by the iterated sums and products of 1-polytopes, which were studied by Hanner [30].

Definition 6.3.1 (Hanner polytope)**.** A centrally-symmetric d-polytope P is a *Hanner polytope* if $d = 0$ or $P = [-1, +1]$ or $P = P_1 \times P_2$ or $P = P_1 \oplus P_2$ for two Hanner polytopes P_1 and P_2.

Conjecture 6.3.2 (Kalai [37, Conjecture A])**.** Let P be a centrally-symmetric d-polytope with f-vector $f(P) = (f_0, \dots, f_{d-1})$. Then the number of non-empty faces of P is at least 3^d, that is,

$$\sum_{i=0}^{d} f_i \;\geq\; 3^d$$

Furthermore, equality holds if and only if P is a Hanner polytope.

Note that the sum also counts the face P itself, so taking only the proper faces into account the conjectured inequality is equivalent to

$$\sum_{i=0}^{d-1} f_i \;\geq\; 3^d - 1$$

Conjecture 6.3.2 is trivially true for $0 \leq d \leq 2$ and can easily be shown by elementary methods for $d = 3$. Stanley [51] proved it for simplicial (and therefore also for simple) polytopes for arbitrary d. Furthermore, one direction of the equivalence can easily be shown by induction.

Proposition 6.3.3. Let P be a d-dimensional Hanner polytope. Then

$$\sum_{i=0}^{d-1} f_i(P) \;=\; 3^d - 1.$$

□

In this section we prove Conjecture 6.3.2 for 4-dimensional centrally-symmetric polytopes, see Theorem 6.3.16. This result also appeared in [46], with a partly different proof.

Again, we more or less explicitly make use of the basic facts in Proposition 3.3.4. In addition, we have that the total number of faces of the polar polytope P^Δ equals that of P itself, that is,

$$\sum_{i=0}^{d-1} f_i(P) \;=\; \sum_{i=0}^{d-1} f_i(P^\Delta)$$

and $f_{0,d-1}(P) = f_{0,d-1}(P^\Delta)$. (This actually holds for arbitrary polytopes, not only for centrally-symmetric ones.)

For the proof of Theorem 6.3.16, the main work goes into the examination of a centrally-symmetric 4-polytope having 10 vertices. To this end, and also to establish the basis for the computer enumeration leading to Proposition 6.3.19, we study polytopes that arise as the convex hull of the regular crosspolytope together with a symmetric pair of new vertices.

This comprises a more detailed analysis of Grünbaum's Theorem 2.2.2. More precisely, we have to understand how the resulting polytope changes if we perturb the new vertex in a controlled way. Roughly spoken, "the more in general position" the added vertex is, the more faces the resulting polytope will have. This can be viewed as a generalisation of the well known fact that the vertices of a nonsimplicial polytope can always be perturbed to obtain a simplicial polytope with more faces. Here however, the situation is more subtle.

Facet-hyperplane arrangements of polytopes

Given a d-polytope $P \subset \mathbb{R}^d$, we denote by $\mathcal{A}(P)$ the hyperplane arrangement induced by the facet-defining hyperplanes of P. Adding a new point $\mathbf{p} \in \mathbb{R}^d$ gives us a polytope $P' := \mathsf{conv}(P \cup \{\mathbf{p}\})$, as in Chapter 2. The point $\mathbf{p}$ will be located in the relative interior of some face $\mathcal{F}$ of the arrangement $\mathcal{A}(P)$, and again the combinatorics of P' does not depend on the exact position of $\mathbf{p}$ within $\mathsf{relint}\,\mathcal{F}$. However, the combinatorial structure does change if $\mathbf{p}$ is contained in an arrangement face of dimension less than d and perturbed freely within $\mathbb{R}^d$.

To describe the situation combinatorially, we associate shadings to the facets of P, with respect to the point $\mathbf{p}$. Similar concepts were introduced for instance by Shephard [48, Section 3], and also by Edelsbrunner [23, Section 8.4], who associates (mixed) colours to the faces. Without loss of generality we can always assume that the considered polytope is full-dimensional.

Definition 6.3.4 (Shadings of facets)**.** A facet F of a full-dimensional polytope P is called

$$\left\{\begin{array}{c} \textit{black} \\ \textit{grey} \\ \textit{white} \end{array}\right\} \text{ with respect to } \mathbf{p} \text{ if } \mathbf{p} \text{ is } \left\{\begin{array}{l} \text{beneath } F \\ \text{on the hyperplane } \mathsf{aff}\, F \\ \text{beyond } F \end{array}\right.$$

To illustrate the terminology imagine the point $\mathbf{p}$ to be a source of light which illuminates the white facets, leaves black facets in the dark and sheds some twilight on the grey facets. The special cases considered in Chapter 2 are those where all facets are black, except for one, which is white (stacking), and, more generally, one facet F is white, all facets that do not intersect F are black and the remaining ones can be black, grey or white (pseudostacking).

Grünbaum's Theorem 2.2.2 tells us that black facets of P will again be facets of P', grey facets expand to new facets of P' with the same facet-defining hyperplane, whereas white facets vanish in the interior of P'.

We can further categorise all faces of P by recording the shadings of the facets they are contained in. Table 6.2 lists the possible types of faces of P with respect to $\mathbf{p}$ and what happens to them in P' by Theorem 2.2.2. Note that faces F of P with $\mathbf{p} \in \mathsf{aff}\, F$, that is, faces of type **g** are replaced by $\mathsf{conv}(F \cup \{\mathbf{p}\})$ in P'.

Vertices of polytopes are special faces, in the sense that they have no non-empty facets themselves. Due to this reason we have to take special care of them.

Definition 6.3.5. Let $P \subset \mathbb{R}^d$ be a d-polytope and v a vertex of P. We say that a point $\mathbf{p} \in \mathbb{R}^d \setminus P$ *swallows* v if v is not a vertex of $\mathsf{conv}(P \cup \{\mathbf{p}\})$.

Furthermore, a point $\mathbf{p} \in \mathbb{R}^d$ is *non-swallowing* (with respect to P) if $\mathbf{p} \notin P$ and $\mathbf{p}$ does not swallow any vertex of P; in other words, if we have $\mathsf{vert}(\mathsf{conv}(P \cup \{\mathbf{p}\})) = \mathsf{vert}(P) \cup \{\mathbf{p}\}$.

type of F	shadings of facets of P containing F	status in P'
b	only black	again face of P'
bg	black and grey	again face of P'
bw	black and white	again face of P'
g	only grey	again face of P', as $\mathsf{conv}(F \cup \{\mathbf{p}\})$
gw	grey and white	vanishes in $\partial P'$
w	only white	vanishes in $\mathsf{int}\, P'$
bgw	black, grey and white	again face of P'

Table 6.2: Possible types of faces with respect to $\mathbf{p}$

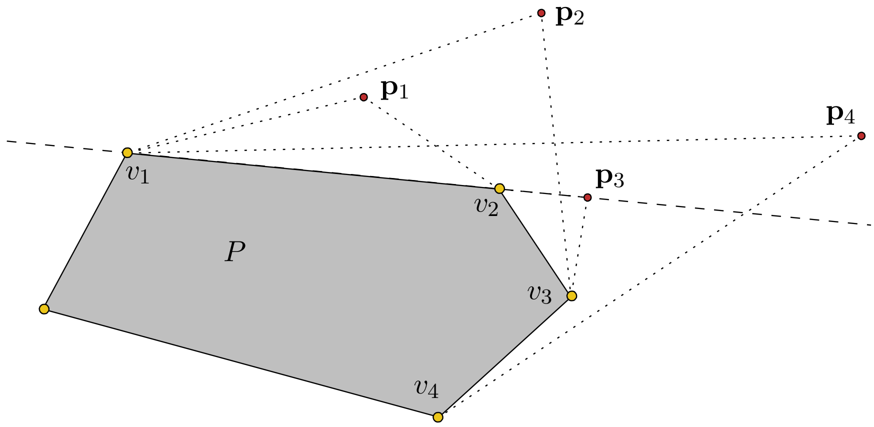


Figure 6.6: Swallowing vertices of P: $\mathbf{p}_1$ is non-swallowing, while $\mathbf{p}_2$ and $\mathbf{p}_3$ both swallow the vertex v_2; the point $\mathbf{p}_4$ swallows two vertices

See Figure 6.6 for an illustration.

Lemma 6.3.6. Let $P \in \mathbb{R}^d$ be a d-polytope, $\mathbf{p} \in \mathbb{R}^d \setminus P$ and v a vertex of P. Then $\mathbf{p}$ swallows v if and only if there is no facet-defining hyperplane h of P containing v such that $\mathbf{p}$ is beneath h, in other words, v is of type **w** or **gw** with respect to $\mathbf{p}$.

Proof. If the vertex v were a **g**-face of P with respect to $\mathbf{p}$, then $v = \mathbf{p}$, in contradiction to $\mathbf{p} \notin P$. Therefore $\mathbf{p}$ swallows v if and only if v is of type **gw** or **w**, as can be seen from Table 6.2. This happens if and only if every hyperplane of $\mathcal{A}(P)$ that contains v comes from a grey or white facet, in other words, $\mathbf{p}$ is on it or beyond the corresponding facet. □

In addition to the surviving faces of P, the new polytope P' has the following new faces:

- the vertex $\mathbf{p}$ itself, if $\mathbf{p} \notin P$,
- pyramids $\mathsf{conv}(F \cup \{\mathbf{p}\})$ for every **bw**-face F of P,
- pyramids $\mathsf{conv}(F \cup \{\mathbf{p}\})$ for every **bgw**-face F of P.

We call the faces of the latter types **bw**-pyramids and **bgw**-pyramids, respectively.

As noted by Shephard [48], every polytope has at least one black and one white facet, provided $\mathbf{p} \notin P$. Furthermore, if F is a k-dimensional face of type **g**, then the point $\mathbf{p}$ is contained in the affine hull of F, and we can deduce the type of a face $G \subseteq F$ "within P" from its type "within F".

Lemma 6.3.7. Let $P \subset \mathbb{R}^d$ be a d-polytope and $\mathbf{p} \in \mathbb{R}^d \setminus P$. Let F be a k-dimensional face of P of type $\mathbf{g}$ with respect to $\mathbf{p}$, for $k \geq 1$.

(a) F contains at least one $(k-1)$-dimensional face of type **bg** and at least one $(k-1)$-dimensional face of type **gw**.

(b) If F contains only one **bg**-face G of dimension $k-1$ then every vertex v of F that is not contained in G has type **gw**.

Proof. (a) Since F is grey, we have $\mathbf{p} \in (\mathsf{aff}\, F) \setminus F$. The face F is a full-dimensional polytope in $\mathsf{aff}\, F$, so we can find a facet G_1 of F such that $\mathbf{p}$ is beyond G_1 in $\mathsf{aff}\, F$. Let F' be a facet of P containing G_1. If F' also contains F, it is grey, otherwise $\mathsf{aff}\, F' \cap \mathsf{aff}\, F = \mathsf{aff}\, G_1$, and therefore $\mathbf{p}$ is beyond F', which means that F' is a white facet. This shows that G_1 is a **gw**-face of P. Analogously, there is a facet G_2 of F such that $\mathbf{p}$ is beneath G_2 in $\mathsf{aff}\, F$, and by the same argument G_2 is then a **bg**-face of P.

(b) Consider a vertex $v \in F \setminus G$ and let F' be some facet of P containing v. If F' contains F then it is grey. Otherwise $h := \mathsf{aff}\, F \cap \mathsf{aff}\, F'$ is a hyperplane in $\mathsf{aff}\, F$, since it contains v and is therefore not empty. Then $G' := h \cap F$ is a face of F, not contained in G. Let $G_1, \ldots, G_k$ be the facets of F that contain G'. Then h is a positive combination of the respective hyperplanes that define $G_1, \ldots, G_k$. Since p is beyond or on every one of these hyperplanes by hypothesis, it is also beyond or on h, and therefore also beyond or on $\mathsf{aff}\, F'$. This shows that F' is white or grey and hence v is of type **gw**. $\square$

We now come to the main tool we want to apply in the next subsection. This basically states that perturbing the added point into some lower-dimensional face of the facet-hyperplane arrangement yields a polytope with smaller or equal total number of faces. We also describe some situations, in which the number of faces decreases strictly.

Theorem 6.3.8. Let $P \subset \mathbb{R}^d$ be a d-polytope and $\mathcal{F}$ and $\mathcal{G}$ faces of the arrangement $\mathcal{A}(P)$ with $\mathcal{G} \subset \mathcal{F} \not\subseteq P$ and $0 \leq \mathsf{dim}\, \mathcal{G} = \mathsf{dim}\, \mathcal{F} - 1$. Furthermore, let $\mathbf{p}_{\mathcal{G}} \in \mathsf{relint}\, \mathcal{G}$ and $\mathbf{p}_{\mathcal{F}} \in \mathsf{relint}\, \mathcal{F}$, with $\mathbf{p}_{\mathcal{F}}$ non-swallowing with respect to P, and $P_X := \mathsf{conv}(P \cup \{\mathbf{p}_X\})$, where $X \in \{\mathcal{G}, \mathcal{F}\}$. Then

$$\sum_{i=0}^{d-1} f_i(P_{\mathcal{G}}) \ \leq \ \sum_{i=0}^{d-1} f_i(P_{\mathcal{F}})$$

Strict inequality holds in each of the following cases:

(I) $\mathcal{G} \subset P$.

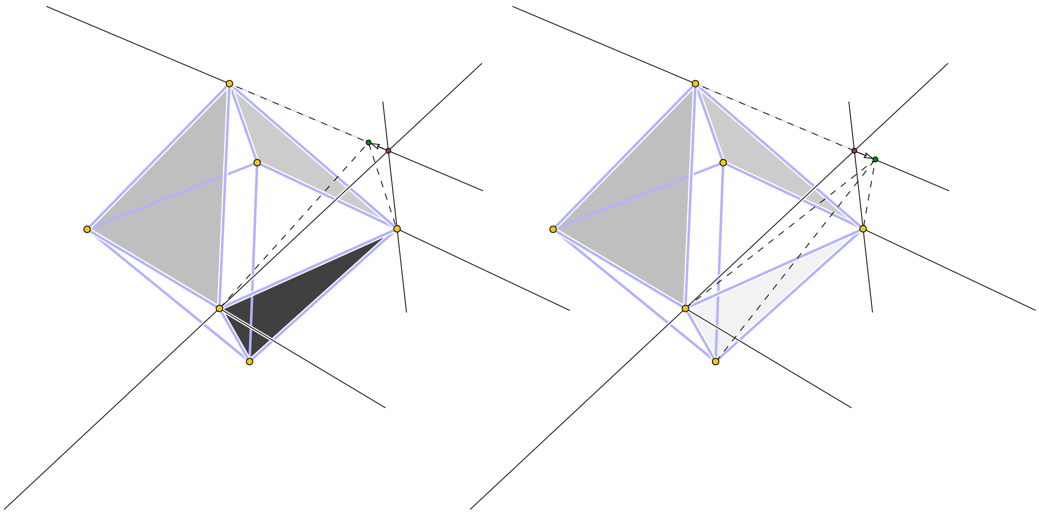


Figure 6.7: Pushing (left) and pulling (right) a point (red). A formerly grey facet becomes black, respectively white. Broken lines indicate new faces of P' that were not present in P.

(ii) The point $\mathbf{p}_{\mathcal{G}}$ is *pushed* with respect to some hyperplane, that is, there is a facet of P that is grey with respect to $\mathbf{p}_{\mathcal{G}}$, but black with respect to $\mathbf{p}_{\mathcal{F}}$.

(iii) The point $\mathbf{p}_{\mathcal{G}}$ is *pulled* with respect to some hyperplane, that is, some facet F of P is grey with respect to $\mathbf{p}_{\mathcal{G}}$ and white with respect to $\mathbf{p}_{\mathcal{F}}$, and there are at least two ridges in F of type **bg** with respect to $\mathbf{p}_{\mathcal{G}}$.

Figure 6.7 shows examples of pushing and pulling a point in $\mathcal{A}({C_3}^{\Delta})$.

Proof. We compare the numbers of faces of $P_{\mathcal{G}}$ and $P_{\mathcal{F}}$ by a close examination of the types of faces of P with respect to $\mathbf{p}_{\mathcal{G}}$, respectively $\mathbf{p}_{\mathcal{F}}$. Since $\mathcal{G}$ is a face of $\mathcal{F}$, every hyperplane of $\mathcal{A}(P)$ that contains $\mathbf{p}_{\mathcal{F}}$ also contains $\mathbf{p}_{\mathcal{G}}$. At least one hyperplane, however, contains $\mathbf{p}_{\mathcal{G}}$, but not $\mathbf{p}_{\mathcal{F}}$, since $\operatorname{dim} \mathcal{G} < \operatorname{dim} \mathcal{F}$. In terms of shadings of facets of P this means that every facet that is black or white with respect to $\mathbf{p}_{\mathcal{G}}$ has the same shading with respect to $\mathbf{p}_{\mathcal{F}}$, while there is at least one grey facet with respect to $\mathbf{p}_{\mathcal{G}}$ that is either black or white with respect to $\mathbf{p}_{\mathcal{F}}$.

We therefore distinguish further between those facets of P that are grey with respect to $\mathbf{p}_{\mathcal{G}}$, but white, respectively black, with respect to $\mathbf{p}_{\mathcal{F}}$ and assign the types $\mathbf{g}^+$, respectively $\mathbf{g}^-$ to these. Then P may contain faces of types $\mathbf{bg}^-$, $\mathbf{gg}^+$, $\mathbf{bgg}^-\mathbf{w}$, etc. Table 6.3 describes all faces of P that yield faces of $P_{\mathcal{G}}$, but vanish in $P_{\mathcal{F}}$ or vice versa.

Additionally, there are different new faces of $P_{\mathcal{G}}$, respectively $P_{\mathcal{F}}$, which are listed in Table 6.4. The listing shows that each new face of $P_{\mathcal{G}}$ corresponds to some new face of $P_{\mathcal{F}}$, whereas some new faces of $P_{\mathcal{F}}$ have no counterparts in $P_{\mathcal{G}}$.

The asserted inequality now follows from Lemma 6.3.7(a), since to every face of $P_{\mathcal{G}}$ we can associate at least one face of $P_{\mathcal{F}}$. To the faces described in the starred lines in Table 6.3, which are present in $P_{\mathcal{G}}$ but not in $P_{\mathcal{F}}$, we associate the faces described in the correspondingly marked lines in Table 6.4—this is possible by Lemma 6.3.7(a). Note that the prerequisite of the Lemma, namely that $\mathsf{dim}\, F \geq 1$, is met, since $\mathbf{p}_{\mathcal{F}}$ is non-swallowing with respect to P by hypothesis, that is, there is no 0-dimensional face of P that is of type $\mathsf{\textbf{w}}$ or $\mathsf{\textbf{gw}}$ with respect to $\mathbf{p}_{\mathcal{F}}$, by Lemma 6.3.6.

It remains to prove that in the cases mentioned in the theorem $P_{\mathcal{G}}$ has strictly less faces than $P_{\mathcal{F}}$.

(I) If $\mathcal{G} \subset P$ then $\mathbf{p}_{\mathcal{G}} \in P$ and hence this point does not create a new face in $P_{\mathcal{G}}$, while $\mathbf{p}_{\mathcal{F}}$ does in $P_{\mathcal{F}}$.

(II) If F is a $\mathsf{\textbf{g}}^{-}$-facet of P with respect to $\mathbf{p}_{\mathcal{G}}$ then by Lemma 6.3.7(a) there is at least one $\mathsf{\textbf{g}}^{-}\mathsf{\textbf{w}}$-ridge which yields a face in $P_{\mathcal{F}}$, as well as a new $\mathsf{\textbf{bw}}$-pyramid in $P_{\mathcal{F}}$, as can be seen from Tables 6.3 and 6.4—both these faces have no counterpart in $P_{\mathcal{G}}$. Hence $P_{\mathcal{G}}$ has at least two faces less than $P_{\mathcal{F}}$.

(III) If F is a $\mathsf{\textbf{g}}^{+}$-facet of P with respect to $\mathbf{p}_{\mathcal{G}}$ then, by Table 6.3, F yields a facet in $P_{\mathcal{G}}$, but not in $P_{\mathcal{F}}$. However, if there are at least two ridges in F of type $\mathsf{\textbf{bg}}^{+}$ then $P_{\mathcal{F}}$ has at least two $\mathsf{\textbf{bw}}$-pyramids that have no counterpart in $P_{\mathcal{G}}$, see Table 6.4, compensating for the "lost" facet F.

□

The special case we need for understanding centrally-symmetric polytopes with $2d+2$ vertices is the adding of two symmetric points to the d-dimensional

	type of face F of P		status of face F	
	w.r.t. $\mathbf{p}_{\mathcal{G}}$	w.r.t. $\mathbf{p}_{\mathcal{F}}$	in $P_{\mathcal{G}}$	in $P_{\mathcal{F}}$
(∗)	$\mathsf{\textbf{g}}^{+}$	$\mathsf{\textbf{w}}$	$\mathsf{conv}(F \cup \{\mathbf{p}_{\mathcal{G}}\})$	—
(∗∗)	$\mathsf{\textbf{gg}}^{+}$	$\mathsf{\textbf{gw}}$	$\mathsf{conv}(F \cup \{\mathbf{p}_{\mathcal{G}}\})$	—
	$\mathsf{\textbf{g}}^{-}\mathsf{\textbf{w}}$	$\mathsf{\textbf{bw}}$	—	F (again a face)

Table 6.3: Faces of P yielding different faces of $P_{\mathcal{G}}$, respectively $P_{\mathcal{F}}$.

new faces of $P_{\mathcal{G}}$	corr. to	new faces of $P_{\mathcal{F}}$
$\mathbf{p}_{\mathcal{G}}$, if $\mathcal{G} \not\subset P$	$\longleftrightarrow$	$\mathbf{p}_{\mathcal{F}}$
bw-pyramids $\mathsf{conv}(F \cup \{\mathbf{p}_{\mathcal{G}}\})$	$\longleftrightarrow$	bw-pyramids $\mathsf{conv}(F \cup \{\mathbf{p}_{\mathcal{F}}\})$
—	$(*)$	bw-pyramids over faces of type bg^+ w.r.t. $\mathbf{p}_{\mathcal{G}}$
—	—	bw-pyramids over faces of type $\mathsf{g}^-\mathsf{w}$ w.r.t. $\mathbf{p}_{\mathcal{G}}$
$\mathsf{bg}^{\pm}\mathsf{w}$-pyramids $\mathsf{conv}(F \cup \{\mathbf{p}_{\mathcal{G}}\})$	$\longleftrightarrow$	bw-pyramids $\mathsf{conv}(F \cup \{\mathbf{p}_{\mathcal{F}}\})$
bgw-pyramids $\mathsf{conv}(F \cup \{\mathbf{p}_{\mathcal{G}}\})$	$\longleftrightarrow$	bgw-pyramids $\mathsf{conv}(F \cup \{\mathbf{p}_{\mathcal{F}}\})$
—	$(**)$	bgw-pyramids over faces of type bgg^+ w.r.t. $\mathbf{p}_{\mathcal{G}}$
—	—	bgw-pyramids over faces of type $\mathsf{gg}^-\mathsf{w}$ w.r.t. $\mathbf{p}_{\mathcal{G}}$

Table 6.4: New faces of $P_{\mathcal{G}}$, respectively $P_{\mathcal{F}}$

crosspolytope. We can assume that the new points do not swallow any vertices of the crosspolytope, otherwise the result will have only $2d$ vertices. We call an arrangement face $\mathcal{F}$ of $\mathcal{A}(P)$ with $\dim \mathcal{F} > 0$ *non-swallowing* if some point in $\mathsf{relint}\,\mathcal{F}$ (equivalently every point in $\mathsf{relint}\,\mathcal{F}$) is non-swallowing with respect to P.

Lemma 6.3.9. Let $\mathcal{F}$ be a non-swallowing face of the arrangement $\mathcal{A}({C_d}^{\Delta})$ with $0 < \dim \mathcal{F} \leq d$, $\mathbf{p} \in \mathsf{relint}\,\mathcal{F}$ and $P := \mathsf{conv}({C_d}^{\Delta} \cup \{\mathbf{p}\})$.

(a) If $\mathcal{F}$ has a non-swallowing vertex $\mathbf{p}'$ then for $P' := \mathsf{conv}({C_d}^{\Delta} \cup \{\mathbf{p}'\})$ we have $\mathsf{vert}\,P' = \mathsf{vert}\,{C_d}^{\Delta} \cup \{\mathbf{p}\}$ and

$$\sum_{i=0}^{d-1} f_i(P') \;<\; \sum_{i=0}^{d-1} f_i(P)$$

(b) If $\mathcal{F}$ has no non-swallowing vertices then $\sum\limits_{i=0}^{d-1} f_i(P) > 3^d - 1$.

Proof. (a) Since $\mathbf{p}'$ is non-swallowing, it is a vertex of P', as are all vertices of ${C_d}^{\Delta}$. Furthermore, by induction on the dimension of $\mathcal{F}$, using Theorem 6.3.8, we have

$$\sum_{i=0}^{d-1} f_i(P') \;\leq\; \sum_{i=0}^{d-1} f_i(P).$$

Assume that equality holds. Then in each step of the induction no pushing is allowed and pulling is allowed only with respect to some facet-hyperplane for which the corresponding facet contains only one **bg**-ridge. The facet is necessarily grey with respect to the pulled point and since it has only one **bg**-ridge, by Lemma 6.3.7(b) there is a vertex v_F of F of type **gw**. Then v_F vanishes in the resulting polytope (see Table 6.2), and therefore also in P, in contradiction to $\mathbf{p} \in \mathcal{F}$ being non-swallowing.

(b) Let $\mathbf{v}$ be a vertex of $\mathcal{F}$. Then either it is a vertex of ${C_d}^\Delta$ or it swallows a vertex of ${C_d}^\Delta$. In both cases, $P'' := \mathsf{conv}({C_d}^\Delta \cup \{\mathbf{v}\})$ is combinatorially equivalent to ${C_d}^\Delta$ and hence $\sum_{i=0}^{d-1} f_i(P'') = 3^d - 1$.

Since $\mathcal{F}$ is non-swallowing, $\mathbf{p}$ is beneath at least one facet-hyperplane for every vertex of ${C_d}^\Delta$, by Lemma 6.3.6. Hence, to get from the swallowing point $\mathbf{v}$ to $\mathbf{p}$, one has to push at least once, and therefore, by Theorem 6.3.8, the total number of faces of P is strictly larger than that of P''. $\square$

Concluding this subsection, we state the two lemmas that we will use in the end, first for the proof of the 4-dimensional case, later for general d. The first one is independent of the previous considerations and also gives a proof of Proposition 3.3.4(b).

Lemma 6.3.10. Let $P \subset \mathbb{R}^d$ be a centrally-symmetric d-polytope with $2d$ vertices. Then P is affinely equivalent to ${C_d}^\Delta$.

Proof. Denote the vertices of P by $v_1, \ldots, v_d, -v_1, \ldots, -v_d$. Since P is full-dimensional, $v_1, \ldots, v_d$ are linearly independent. Therefore there is a (unique) non-singular, linear transformation that maps the standard unit normals e_i to v_i for $1 \leq i \leq d$. Hence P is the image under an affine transformation of the d-crosspolytope ${C_d}^\Delta$. $\square$

Lemma 6.3.11. Let $P \subset \mathbb{R}^d$ be a centrally-symmetric d-polytope with $2d+2$ vertices.

(a) There exists a point $\mathbf{p} \notin {C_d}^\Delta$ such that P is combinatorially equivalent to $\mathsf{conv}({C_d}^\Delta \cup \{\mathbf{p}, -\mathbf{p}\})$.

(b) If the point $\mathbf{p}$ in (a) is contained in at most $d-1$ hyperplanes of $\mathcal{A}({C_d}^\Delta)$, then P has strictly more than $3^d - 1$ proper faces, or there exists a centrally-symmetric polytope $P' = \mathsf{conv}({C_d}^\Delta \cup \{\mathbf{p}', -\mathbf{p}'\})$ with $2d + 2$ vertices such that $\mathbf{p}'$ is a point contained in d hyperplanes of $\mathcal{A}({C_d}^\Delta)$ and P has strictly more faces than P'.

Proof. (a) As in the previous proof choose d vertices $v_1, \ldots, v_d$ of P that are linearly independent. Then $\mathsf{conv}\{\pm v_1, \ldots, \pm v_d\}$ is again affinely equivalent to

${C_d}^{\Delta}$, and P itself is the image under the corresponding affine transformation of the polytope $\mathsf{conv}({C_d}^{\Delta} \cup \{\pm\mathbf{p}\})$, where $\mathbf{p} \in \mathbb{R}^d$ is a non-swallowing vertex with respect to ${C_d}^{\Delta}$.

(b) If $\mathbf{p}$ is in less than d hyperplanes of $\mathcal{A}({C_d}^{\Delta})$ then $\mathbf{p} \in \mathsf{relint}\,\mathcal{F}$ for some arrangement face $\mathcal{F}$ of dimension at least 1. Additionally, since $\mathbf{p}$ is non-swallowing, $\mathcal{F}$ is non-swallowing with respect to ${C_d}^{\Delta}$. (b) now follows from Lemma 6.3.9 by taking as $\mathbf{p}'$ a non-swallowing vertex of $\mathcal{F}$ if one exists or applying Part (b) of the Lemma, if not. □

Dimension 4

The proof of Conjecture 6.3.2 for dimension 4 is done in several steps. First, we consider polytopes with relatively small number of vertices or facets and enumerate in an elementary way the possibilities that can occur. If both the numbers of vertices and facets is large enough—which means, at least 12—then the statement follows from certain linear inequalities, which partly are more sophisticated.

For the rest of this subsection let $P \subset \mathbb{R}^4$ denote a centrally-symmetric 4-polytope with f-vector $f(P) = (f_0, f_1, f_2, f_3)$. For convenience we set

$$s := \sum_{i=0}^{3} f_i.$$

Then for the 4-dimensional case Conjecture 6.3.2 states that $s \geq 80$.

Lemma 6.3.12. If $f_0 = 8$ then P is affinely equivalent to the 4-dimensional crosspolytope and $s = 80$.

Proof. This is Lemma 6.3.10 for $d = 4$. Since the crosspolytope is a Hanner polytope, we get $s = 3^4 - 1 = 80$, by Proposition 6.3.3. □

Lemma 6.3.13. If $f_0 = 10$ then $s \geq 80$. Equality holds if and only if P is combinatorially equivalent to the bipyramid over the 3-dimensional cube, $\mathsf{bipyr}\,C_3$.

Proof. By Lemma 6.3.11(a), P is combinatorially equivalent to a polytope $\mathsf{conv}({C_4}^{\Delta} \cup \{\mathbf{v}, -\mathbf{v}\})$ for some point $\mathbf{v} \notin {C_4}^{\Delta}$. We enumerate all possibilities—combinatorially and up to symmetry—for choosing the last vertex $\mathbf{v}$ in the relative interior of some face of the facet-hyperplane arrangement $\mathcal{A}({C_4}^{\Delta})$.

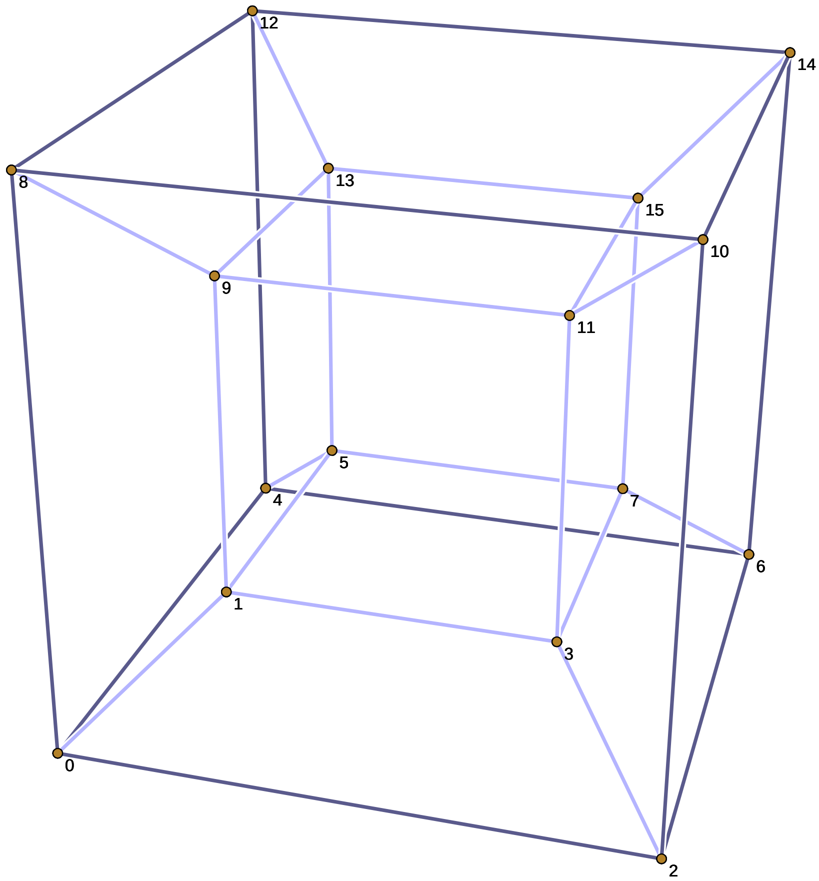


Figure 6.8: Labelling of the vertices of C_4.

By part (b) of Lemma 6.3.11 it suffices to consider only the vertices of the hyperplane arrangement.

Dualising the situation, P^Δ can be viewed as the image under an affine transformation of a polytope obtained by chopping parts off the cube C_4 with two symmetric hyperplanes $h, -h$. These hyperplanes can be chosen in such a way that all facet-defining hyperplanes for C_4 remain facet-defining for the resulting polytope, that is, no facet of C_4 "completely vanishes" and—by dualising the result obtained from Lemma 6.3.11—h passes through 4 vertices of C_4.

Therefore it remains to show that for all possible choices of the hyperplane h we have $s \geq 80$, with equality only if P^Δ is equivalent to $\mathsf{prism}\, C_3{}^\Delta$. To keep track of the different cases, we refer to the Schlegel diagram of C_4 in Figure 6.8 and denote the vertices by the numbers in the picture. Also, we identify faces of C_4 with the set of their vertices.

First observe that the symmetry of C_4 allows to pick one fixed vertex out of the 4 that are supposed to be contained in h, say vertex $\mathtt{0}$. The task is then the following: first, find all possible choices of 3 out of the remaining vertices $\mathtt{1}$ to $\mathtt{15}$ that yield 4-tuples $(v_1 = \mathtt{0}, v_2, v_3, v_4)$ with the properties (i) and (ii) below; and second, in each case examine the respective polytope that arises from intersecting C_4 with two symmetric halfspaces that are defined by the respective vertices. More precisely, the halfspaces contain the origin and are bounded by h and $-h$, respectively, where h is the hyperplane defined by the given vertices.

The vertex tuples have to satisfy the following conditions:

(i) No two opposite vertices are contained in the 4-tuple, that is, $v_i \neq -v_j$ for $1 \leq i < j \leq 4$; using the numbering in Figure 6.8 this says that $v_i + v_j \neq 15$ for $1 \leq i < j \leq 4$. (Otherwise this yields a hyperplane which cuts through the origin and therefore a lower-dimensional polytope.)

(ii) Not all 4 vertices lie on the same facet of C_4. (Otherwise the hyperplane h defines this facet and we get C_4 again after cutting.)

Again we have to consider these tuples only up to symmetry of the cube.

Case 1: Three of the four vertices lie in one facet, w.l.o.g. the vertices v_1, v_2 and v_3, in the facet $F = \{\mathtt{0}, \mathtt{1}, \mathtt{2}, \mathtt{3}, \mathtt{4}, \mathtt{5}, \mathtt{6}, \mathtt{7}\}$. Then by (ii), we have to choose v_4 from the set $\{\mathtt{8}, \mathtt{9}, \mathtt{10}, \mathtt{11}, \mathtt{12}, \mathtt{13}, \mathtt{14}, \mathtt{15}\}$.
Case 1.1: v_1, v_2, v_3 even lie in one ridge, w.l.o.g. in the ridge $\{\mathtt{0}, \mathtt{1}, \mathtt{2}, \mathtt{3}\}$.
Then w.l.o.g. we can take $v_2 = \mathtt{1}$ and $v_3 = \mathtt{2}$. By (i), $v_4 \in \{\mathtt{15}, \mathtt{14}, \mathtt{13}\}$ rules out, and by (ii) we cannot choose $v_4 \in \{\mathtt{8}, \mathtt{9}, \mathtt{10}, \mathtt{11}\}$, therefore $v_4 = \mathtt{12}$. Then the hyperplane $h = \mathsf{aff}\{\mathtt{0}, \mathtt{1}, \mathtt{2}, \mathtt{12}\}$ contains the origin, so cutting C_4 with h and $-h$ yields a lower dimensional polytope.

Case 1.2: v_1, v_2, v_3 are not all contained in some ridge.
Case 1.2.1: Two of the three vertices are connected by an edge, w.l.o.g. the vertices $v_1 = \mathtt{0}$ and $v_2 = \mathtt{1}$.
Then the third vertex in F cannot be $\mathtt{2}$, $\mathtt{3}$, $\mathtt{4}$ or $\mathtt{5}$, otherwise three vertices would lie in a ridge; hence v_3 has to be $\mathtt{6}$ or $\mathtt{7}$. For the same reason v_4 cannot be $\mathtt{8}$ or $\mathtt{9}$, so $v_4 \in \{\mathtt{12}, \mathtt{13}, \mathtt{11}, \mathtt{10}\}$, by (i). All these choices of v_3 and v_4 then yield equivalent polytopes by symmetry, namely prisms over a 3-dimensional crosspolytope.
Case 1.2.2: There are no edges between the three vertices.
Then $v_2, v_3 \in \{\mathtt{3}, \mathtt{5}, \mathtt{6}\}$ and w.l.o.g. we take $v_2 = \mathtt{3}$ and $v_3 = \mathtt{5}$. Respecting property (i) and the fact that we are not in Case 1.2.1 leaves only the vertices

`9` and `14` for the last vertex v_4. Both choices yield equivalent polytopes, which have f-vector $(14, 36, 32, 10)$ (see the proof of Theorem 6.3.25 for the computation of the f-vector) and therefore $s = 92$.

Case 2: No three of the four vertices lie in the same facet.
Case 2.1: One of v_2, v_3, v_4, say v_2, lies in a ridge with `0`, w.l.o.g. in $\{0, 1, 4, 5\}$.
Case 2.1.1: v_2 even lies on some edge with `0`, w.l.o.g. $v_2 = 1$.
Then the only choice for v_3 and v_4, taking into account the hypothesis of Case 2, are `14` and `15`, but this 4-tuple obviously violates (i).
Case 2.1.2: v_2 is not connected to `0` by an edge.
Then $v_2 = 5$ and for v_3 and v_4 we can choose `10`,`11`,`14` or `15` without violating the hypothesis of Case 2. Vertices `15` and `10` are forbidden by (i), so w.l.o.g. $v_3 = 11$ and $v_4 = 14$. But then $h = \mathsf{aff}\{0, 5, 11, 14\}$ again contains the origin, as in Case 1.1.

Case 2.2: none of v_2, v_3, v_4 lies in a ridge with `0`.
Then, respecting (i), $v_2, v_3, v_4 \in \{7, 11, 13, 14\}$, and for every possible choice there is a facet containing all three of them, in contradiction to the hypothesis of Case 2. □

The next ingredient for the proof of Conjecture 6.3.2 for dimension 4 is a special case of a theorem by A'Campo-Neuen [1, Theorem 1.2]. Note that she stated the theorem for rational polytopes, nevertheless it also holds for non-rational polytopes by Karu's proof of the Hard Lefschetz Theorem [38].

Proposition 6.3.14. For every centrally-symmetric 4-polytope P we have $f_{03}(P) \geq 3f_0(P) + 3f_3(P) - 8$.

Proof. By [1, Theorem 1.2], we have $h_2(P) - 6 \geq h_1(P) - 4$, that is, $g_2(P) \geq 2$. Expressing the g-vector in terms of the flag vector (see Chapter 1) implies the assertion. □

A different proof of Proposition 6.3.14 appears in [46]. It proceeds by adapting arguments of Kalai [35] using rigidity, to the centrally-symmetric situation.

Lemma 6.3.15. If $f_0 \geq 12$ and $f_3 \geq 12$ then $s > 80$.

Proof. We first prove that $s \geq 80$. By Proposition 6.3.14, the following inequalities hold for the flag vector of P:

$$f_{03} \geq 3f_0 + 3f_3 - 8 \tag{6.11}$$
$$f_{03} \leq 4f_1 - 4f_0 \tag{6.12}$$
$$f_{03} \leq 4f_2 - 4f_3 \tag{6.13}$$

Inequalities (6.12) and (6.13) are basically the same statement, by Euler's equation, and arise from Bayer's inequality $6f_1 - 6f_0 - f_{02} \geq 0$ (see Theorem 4.1.1) by applying the Generalized Dehn-Sommerville equations. Combining (6.11) and (6.12), we get

$$4f_1 \;\geq\; 7f_0 + 3f_3 - 8 \;\geq\; 7 \cdot 12 + 3 \cdot 12 - 8 \;=\; 112$$

and therefore $f_1 \geq 28$. Analogously, by (6.11) and (6.13), $f_2 \geq 28$. Hence,

$$s \;=\; f_0 + f_1 + f_2 + f_3 \;\geq\; 12 + 28 + 28 + 12 \;=\; 80.$$

Now suppose $s = 80$. Then P has the f-vector $(12, 28, 28, 12)$. By (6.11), $f_{03}(P) \geq 3 \cdot 12 + 3 \cdot 12 - 8 = 64$ and by (6.12), $f_{03}(P) \leq 4 \cdot 28 - 4 \cdot 12 = 64$, so $f_{03}(P) = 64$. In particular, in (6.12) and (6.13) equality holds, which means that P is *center-boolean*, that is, all its facets are simple (cf. Proposition 4.1.3). Since P has only 12 vertices, any of its facets can have at most 6 vertices, by central symmetry. Therefore, any facet of P is either a tetrahedron or a prism over a triangle.

Let t and p be the number of facets of P that are tetrahedra and prisms, respectively. Then we have

$$\begin{aligned} f_3 &= \;\; t + \;\; p = 12 \\ f_{03} &= 4t + 6p = 64 \end{aligned}$$

which implies that $p = 8$ and $t = 4$.

The polar polytope P^Δ is again centrally-symmetric, center-boolean and has the same flag vector, so all the above calculations also hold for P^Δ. This means, by duality, that P has 4 vertices of degree 4 and 8 vertices of degree 5, and, in particular, no vertex of degree higher than 5.

Now consider some prism facet P_0 of P, see Figure 6.9. It contains three quadrilateral ridges that connect it to three different facets P_1, P_2, P_3, also of prism type. Further, let R be one of the two triangular ridges and F the facet of P such that $R = P_0 \cap F$. Then F must be a tetrahedron—otherwise F was the opposite facet of P_i for some $i \in \{0, 1, 2, 3\}$, but it intersects each of the P_i in at least an edge.

Finally, let v be the vertex in F not contained in R. It is clearly not a vertex of P_0, so it has to be a vertex of $P_7 := -P_0$, since P_0 already contains half of all the vertices of P. Hence there are three edges from v to other vertices in P_7. Adding to the already known three edges to the vertices of $R \subset P_0$, the vertex v has at least six incident edges, and therefore degree larger than 5, a contradiction. □

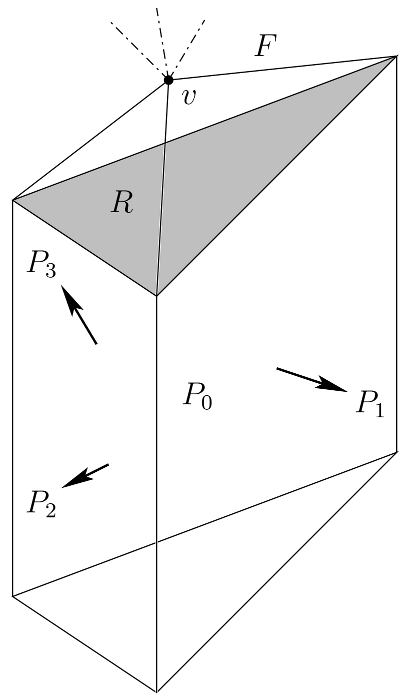


Figure 6.9: A prism facet P_0 of P

Now the main result of this section, the proof that Kalai's Conjecture A is true in the 4-dimensional case, is a direct consequence of the three lemmas above.

Theorem 6.3.16. Conjecture 6.3.2 is true for $d = 4$. In other words, every centrally-symmetric 4-polytope has at least 3^4 non-empty faces, with equality if and only if it is combinatorially equivalent to a 4-dimensional Hanner polytope.

Proof. If $f_0 = 8$ then, by Lemma 6.3.12, P is the 4-dimensional crosspolytope and $s = 80$. If $f_3 = 8$ then, by duality and Lemma 6.3.12, P is the 4-cube and also $s = 80$.

If $f_0 = 10$ then, by Lemma 6.3.13, $s \geq 80$ with equality if and only if P is a bipyramid over a 3-dimensional cube. Dually, if $f_3 = 10$ then Lemma 6.3.13 implies that $s \geq 80$ with equality if and only if P is dual to a bipyramid over a 3-cube, that is, P is a prism over a 3-dimensional crosspolytope.

Finally, if $f_0 \geq 12$ and $f_3 \geq 12$ then, by Lemma 6.3.15, $s > 80$. $\square$

HIGHER DIMENSIONS AND RELATED CONJECTURES

The proof above can not easily be generalised to higher dimensions. Although we can still use linear inequalities to get analogues to Lemma 6.3.15, for $d > 4$ the condition on f_0 and f_{d-1} must be much stronger to get the desired bound for the sum of face numbers, as the following computation shows.

Proposition 6.3.17. Let P be d-polytope with f-vector $(f_0, \ldots, f_{d-1})$ and $s_d := \sum_{i=0}^{d-1} f_i$.

(a) If $d = 5$ and $f_0 + f_4 \geq 42$, then $s_5 \geq 242 = 3^5 - 1$.

(b) If $d = 6$ and $f_0 + f_5 \geq 116$, then $s_6 > 728 = 3^6 - 1$.

Proof. (a) From Euler's equation and Kalai's inequality (see (6.5) in Section 6.1) we get $f_1 + f_3 \geq 3f_0 + 3f_4 - 6$, which implies

$$\begin{aligned} s_5 &= (f_0 + f_2 + f_4) + (f_1 + f_3) \;=\; (f_1 + f_3 + 2) + (f_1 + f_3) \\ &\geq 6f_0 + 6f_4 - 10 \;\geq\; 6 \cdot 42 - 10 \;=\; 242. \end{aligned}$$

(b) For $d = 6$ we use the inequalities $f_1 \geq 3f_0$ and $f_2 \geq \frac{2}{3}f_1 + 21$ (see Proposition 6.2.3) and their dual counterparts. Then

$$\begin{aligned} s_6 &= (f_0 + f_1 + f_2) + (f_3 + f_4 + f_5) \;\geq\; (6f_0 + 21) + (6f_5 + 21) \\ &= 6 \cdot 116 + 42 \;=\; 738 \;>\; 728. \end{aligned}$$

□

In other words, if a 5-polytope, respectively 6-polytope, violates Kalai's conjecture then the number of vertices and facets cannot be too large. Hence, in principle it suffices to consider "small" polytopes. However, the gap that has to be bridged by the enumeration grows exponentially in d, which makes this approach quite useless.

Similarly, Lemma 6.3.11 can in principle be used to examine the centrally-symmetric d-polytopes with $2d + 2$ vertices. One could either hope for a higher-dimensional version of Lemma 6.3.13 or a counterexample to Conjecture 6.3.2. The following special case is in the scope of this approach.

Conjecture 6.3.18. Let P be a centrally-symmetric d-polytope with $2d+2$ vertices. Then the number of proper faces of P is at least $3^d - 1$ with equality if and only if P is a $(d-3)$-fold bipyramid over a 3-cube.

A computer-based enumeration using an adapted version of Edelsbrunner's algorithm for constructing hyperplane arrangements (see [23, Section 7]) gave the following result.

Proposition 6.3.19. Conjecture 6.3.18 is true for $d \leq 7$.

Instead of special cases of Kalai's conjecture, one could also try to prove stronger statements. In [37], Kalai gave other conjectures, which would imply Conjecture 6.3.2.

Conjecture 6.3.20 (Kalai [37, Conjecture B])**.** For every centrally-symmetric d-polytope P there exists a d-dimensional Hanner polytope H such that $f_i(H) \leq f_i(P)$ for all $i \in \{0, \ldots, d-1\}$.

Obviously, Conjecture 6.3.20 implies Conjecture 6.3.2, so to prove the latter, one could prove the more general one instead. For dimension 4 this was done in [46]. However, for $d \geq 5$, Conjecture 6.3.20 fails, and counterexamples can be constructed from the *central hypersimplices*

$$\tilde{\Delta}(k) \;:=\; \{\mathbf{x} \in C_{2k-1} \mid -1 \leq x_1 + \ldots + x_{2k-1} \leq 1\}.$$

The terminology is justified by the fact that $\tilde{\Delta}(k)$ is affinely equivalent to $\Delta_{2k-1}(k)$. This can be seen using the affine transformation

$$\phi : \mathbb{R}^{2k} \to \mathbb{R}^{2k}\,,\; \mathbf{x} \mapsto 2\mathbf{x} - \mathbf{1}$$

If $\tilde{\mathbf{x}} = \phi(\mathbf{x})$ then $0 \leq x_i \leq 1$ implies $-1 \leq \tilde{x}_i \leq 1$ and $x_1 + \ldots + x_{2k} = k$ implies $\tilde{x}_1 + \ldots + \tilde{x}_{2k} = 0$. Therefore,

$$\begin{aligned} \phi(\Delta_{2k-1}(k)) &= \phi(\{\mathbf{x} \in [0,1]^{2k} \mid x_1 + \ldots + x_{2k} = k\}) \\ &= \{\mathbf{x} \in C_{2k} \mid x_1 + \ldots + x_{2k} = 0\}. \end{aligned}$$

Furthermore, if $\mathbf{x} \in C_{2k}$ then $-1 \leq x_{2k} \leq 1$ and $x_1 + \ldots + x_{2k} = 0$ implies $-1 \leq x_1 + \ldots + x_{2k-1} \leq 1$. Conversely, if we have an $\mathbf{x}' \in C_{2k-1}$ with $-1 \leq x'_1 + \ldots + x'_{2k-1} \leq 1$ then by defining $x'_{2k} := -(x'_1 + \ldots + x'_{2k-1}) \in [-1,1]$, we get $\mathbf{x} := (\mathbf{x}', x'_{2k})$ with $x_1 + \ldots + x_{2k} = 0$. That is, $\tilde{\Delta}(k)$ is the bijective image of $\phi(\Delta_{2k-1}(k))$ under the projection to the first $2k-1$ coordinates.

Proposition 6.3.21. The f-vector of the hypersimplex is given by

$$f_i(\Delta_d(k)) \;=\; \begin{cases} \binom{d+1}{k} & \text{if } i = 0 \\ \binom{d+1}{i+1} \sum\limits_{p=1}^{i} \binom{d-i}{k-p} & \text{if } 1 \leq i \leq d-1 \end{cases}$$

In particular, for the central hypersimplex of dimension $d = 2k-1$ we have

$$f_0(\tilde{\Delta}(k)) \;=\; \binom{2k}{k} \qquad \text{and} \qquad f_{d-1}(\tilde{\Delta}(k)) \;=\; 4k.$$

Proof. Let $h := \{\mathbf{x} \in \mathbb{R}^{d+1} \mid \sum_{i=1}^{d+1} x_i = k\}$. The hypersimplex $\Delta_d(k)$ is obtained by cutting the 0/1-cube $C := [0,1]^{d+1}$ with the hyperplane h.

Let F be a face of C with sign vector $\sigma(F) \in \{-,+,0\}^{d+1}$ (see Proposition 4.3.4), and suppose $\sigma(F)$ has p pluses and m minuses. h intersects the relative interior of F if and only if F has two vertices on different sides of h. The vertex $\mathbf{v}$ of F with the smallest possible value for the sum of its entries has a sign vector that can be obtained from $\sigma(F)$ by replacing all 0's with $-$'s, and then $\sum v_i = p$. Similarly, replacing all 0's in $\sigma(F)$ with $+$'s gives the vertex $\mathbf{w}$ of F with largest possible sum $\sum v_i = d+1-m$. Hence, $h \cap \mathsf{relint}\, F \neq \emptyset$ if and only if $p \leq k-1$ and $d+1-m \geq k+1$, that is, $m \leq d-k$.

Furthermore, if $h \cap \mathsf{relint}\, F \neq \emptyset$ then the number of 0's in $\sigma(F)$ equals

$$d+1-(p+m) \;\geq\; d+1-(k-1+d-k) \;\geq\; 2$$

and therefore, by Proposition 4.3.4 we get $\mathsf{dim}\, F \geq 2$. In other words, no edge of $[0,1]^{d+1}$ is intersected in its interior by h and hence the vertices of the hypersimplex are exactly those vertices of $[0,1]^{d+1}$ that lie on h. These are the 0/1-vectors with exactly k entries equal to 1, and their number is

$$f_0(\Delta_d(k)) \;=\; \binom{d+1}{k}$$

Similarly, no edge of $[0,1]^{d+1}$ can be completely contained in h, since at least one of the two vertices is not on h, and consequently no face of higher dimension is completely contained in h. Therefore, the i-dimensional faces of $\Delta_d(k)$ (for $1 \leq i \leq d-1$) are in one-to-one correspondence to those $(i+1)$-dimensional faces of the cube that intersect h in their interior. They in turn are given by the $(d+1)$-tuples that meet the conditions above and have exactly $i+1$ zeros, that is, of the remaining $d-i$ entries at most $d-k$ are minuses and at most $k-1$ are pluses. The first of these condition states that there are at least $k+1$ entries that are either pluses or zeros, that is, if we have $i+1$ zeros, there are at least $k-i$ pluses.

Now given a fixed number $i+1$ of zeros, there are $\binom{d+1}{i+1}$ ways to distribute them to the entries of the $(d+1)$-tuple. Additionally, for an admissible number p of pluses, with $k-i \le p \le k-1$, there are $\binom{d-i}{p}$ ways to put them into the remaining free entries. The number of i-faces of $\Delta_d(k)$, for $1 \le i \le d-1$, is therefore

$$f_i(\Delta_d(k)) \;=\; \binom{d+1}{i+1} \sum_{p=k-i}^{k-1} \binom{d-i}{p} \;=\; \binom{d+1}{i+1} \sum_{p=1}^{i} \binom{d-i}{k-p}$$

Since $\tilde{\Delta}(k)$ is combinatorially equivalent to $\Delta_{2k-1}(k)$, we immediately get the asserted number of vertices, and the number of facets is

$$f_{d-1}(\tilde{\Delta}(k)) \;=\; \binom{d+1}{d} \sum_{p=1}^{d-1} \binom{d-(d-1)}{k-p} \;=\; (d+1) \sum_{p=1}^{2k-2} \binom{1}{k-p}$$

For $k \ge 2$, the only non-zero terms in the sum are the ones where $k-p=0$ and $k-p=1$, that is, for $p=k$ and $p=k-1$, and we get

$$f_{d-1}(\tilde{\Delta}(k)) \;=\; 2k\left(\binom{1}{1} + \binom{1}{0}\right) \;=\; 4k.$$

□

Since the d-dimensional central hypersimplices have $2d+2$ facets, we only have to investigate the Hanner polytopes with relatively few facets to establish them as counterexamples to Conjecture 6.3.20.

Lemma 6.3.22. Let H be a d-dimensional Hanner polytope, $d \ge 3$. Then $f_{d-1}(H) \ge 2d$ and

(a) if $f_{d-1}(H) = 2d$ then $H = C_d$,

(b) if $f_{d-1}(H) = 2d+2$ then $H = C_{d-3} \times {C_3}^{\Delta}$.

Proof. The bound on the number of facets and (a) follow from Proposition 3.3.4, respectively Lemma 6.3.10, by duality, since H is centrally-symmetric.

(b) is true for $d=3$, since there are only two Hanner polytopes in this case, the cube and the crosspolytope. Now assume that $d \ge 4$. Then H is the sum or the product of two Hanner polytopes H' and H'' of dimensions i and $d-i$, respectively, with $1 \le i \le d/2$. If $H = H' \oplus H''$, then, by induction on d,

$$f_{d-1}(H) \;=\; f_{i-1}(H') \cdot f_{d-i-1}(H'') \;\ge\; 2i \cdot 2(d-i) \;\ge\; 4(d-1) \;\ge\; 2d+4$$

where for the last two inequalities we used $1 \leq i \leq d/2$ and $d \geq 4$, respectively.

Therefore, if $f_{d-1}(H) = 2d+2$, we can assume that $H = H' \times H''$, and then $f_{d-1}(H) = f_{i-1}(H') + f_{d-i-1}(H'')$. The condition in (b) is satisfied if and only if it is satisfied for one of the polytopes H' or H'', which then necessarily has dimension at least 3, and for the other we have the condition from (a). Since the product of polytopes is (combinatorially) a commutative operation, we can, by induction, assume that $H' = C_i$ and $H'' = C_{d-i-3} \times {C_3}^{\Delta}$, where $d - i \geq 3$. Since the product is also associative, we conclude that

$$H \;=\; C_i \times C_{d-i-3} \times {C_3}^{\Delta} \;=\; C_{d-3} \times {C_3}^{\Delta}.$$

□

This now implies that in odd dimensions a Hanner polytope with not more facets than the central hypersimplex has in fact many more vertices. For even dimensions the counterexamples are given by prisms over hypersimplices.

Theorem 6.3.23. For $d \geq 5$, Conjecture 6.3.20 is false. Counterexamples are given by the central hypersimplices $\tilde{\Delta}(k)$ for odd dimension $d = 2k-1$ and $\mathsf{prism}\,\tilde{\Delta}(k)$ for even dimension $d = 2k$, for $k \geq 3$.

Proof. Let $d = 2k-1 \geq 5$, and suppose H is a d-dimensional Hanner polytope with $f_i(H) \leq f_i(\tilde{\Delta}(k))$ for all $i \in \{0, \ldots, d-1\}$. Since, by Proposition 6.3.21, the hypersimplex has $2d+2$ facets, it follows from Lemma 6.3.22 that H is either C_{2k-1} or $C_{2k-4} \times {C_3}^{\Delta}$. We have

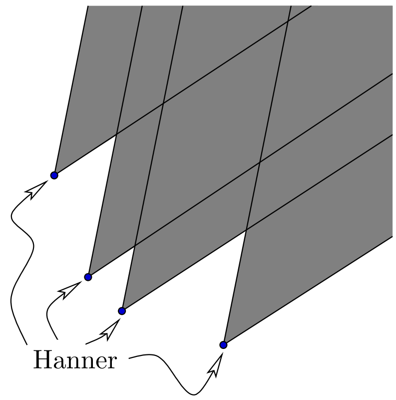
$$\begin{aligned} f_0(C_{2k-1}) \;&=\; 2^{2k-1} \;=\; 4 \cdot 2^{2k-3} \text{ and} \\ f_0(C_{2k-4} \times {C_3}^{\Delta}) \;&=\; f_0(C_{2k-4}) \cdot f_0({C_3}^{\Delta}) \;=\; 2^{2k-4} \cdot 6 \;=\; 3 \cdot 2^{2k-3} \end{aligned}$$

so in either case, $f_0(H) \geq 3 \cdot 2^{2k-3} > \binom{2k}{k} = f_0(\tilde{\Delta}(k))$, where the strict inequality holds for $k \geq 3$.

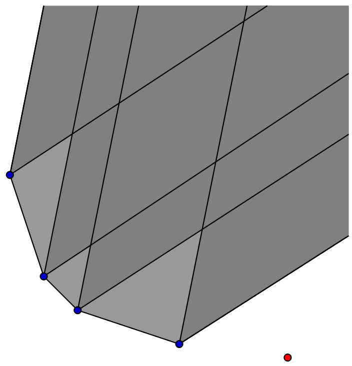
For even dimension $d = 2k$ consider $\mathsf{prism}\,\tilde{\Delta}(k) = [-1,1] \times \tilde{\Delta}(k)$, which has $2(2k-1)+2+2 = 2d+2$ facets. Again by Lemma 6.3.22, every Hanner polytope H with componentwise smaller f-vector is of the form $[-1,1] \times H'$, with some $(d-1)$-dimensional Hanner polytope H' and the result follows from the odd-dimensional case. □

Kalai also stated a third, even stronger conjecture. Recall the notation of linear functionals $\alpha \in \mathbb{R}^{F_d}$ on flag vectors of d-polytopes from Section 3.2. Such a functional is non-negative for all d-polytopes P, if

$$\alpha(P) \;:=\; \sum_{S \in \Psi_d} \alpha_S f_S(P) \;\geq\; 0.$$



(a) Flag vectors of the Hanner polytopes and their cones



(b) Flag vector of a counterexample for Conjecture 6.3.24 (red)

Figure 6.10: Geometric interpretation of Kalai's Conjecture C and Conjecture 6.3.24

Kalai's Conjecture C in [37] then claims that for all centrally-symmetric d-polytopes P there exists a Hanner polytope H such that for all linear functionals α that are non-negative for all d-polytopes we have $\alpha(H) \leq \alpha(P)$.

There is a nice geometric interpretation for this statement, see Figure 6.10(a). For a Hanner polytope H the inequalities $\alpha \cdot \mathbf{f} \geq \alpha(H)$ for all α as above define a cone $\mathcal{C}_H$ with apex $\mathbf{f}(H)$. The conjecture states that every flag vector of a centrally-symmetric polytope is contained in one of the cones $\mathcal{C}_H$ for some Hanner polytope H. So, to find counterexamples to the conjecture one has to give a polytope with flag vector outside the union of the cones $\mathcal{C}_H$.

We consider the slightly weaker statement given below and show that this is false in general, already for dimension 4. In the geometric interpretation, we give a counterexample with flag vector not only outside the union of the cones, but separated from it by a linear function, that is, even outside the convex hull of this union, see Figure 6.10(b). Obviously, this implies that Kalai's Conjecture C does not hold for 4-dimensional centrally-symmetric polytopes.

Conjecture 6.3.24. Let $\alpha \in \mathbb{R}^{F_d}$ be a flag vector functional, which is non-negative for all d-polytopes. Then for every centrally-symmetric d-dimensional polytope P there exists a d-dimensional Hanner polytope H such that $\alpha(H) \leq \alpha(P)$.

Theorem 6.3.25. Conjecture 6.3.24 is false for $d = 4$. Counterexamples are given, for instance, by the polytope

$$P_4 \ := \ C_4 \cap \{\mathbf{x} \in \mathbb{R}^4 \mid -2 \leq x_1 + x_2 + x_3 + x_4 \leq 2\}$$

or the 24-cell.

Proof. Consider the flag vector functional

$$\frac{1}{2}(f_{02} - 3f_2) + \frac{1}{2}(f_{02} - 3f_1)$$

which we denote by α, that is, $\alpha := (0, 0, -3/2, -3/2, 1) \in \mathbb{R}^{F_d}$. We have $\alpha(P) \geq 0$ for all 4-polytopes P and $\alpha(P) = 0$ if and only if P is 2-simple, 2-simplicial (see Proposition 4.1.3).

The polytope P_4, as stated in the theorem, is obtained from the 4-cube by "chopping off" the two vertices $-\mathbf{1}$ and $\mathbf{1}$ with hyperplanes that pass through the respective neighbouring vertices. To compute the flag vector of P_4 first note that all vertices of C_4 are again vertices of P_4, except for $-\mathbf{1}$ and $\mathbf{1}$, and no new vertices show up, since the cutting hyperplanes do not intersect any edge of C_4 in its relative interior. Therefore $f_0(P_4) = f_0(C_4) - 2 = 14$. Furthermore, all facet-defining hyperplanes for C_4 are again facet-defining (that is, no facet of C_4 is completely cut away), and there are two new facets, defined by the cutting hyperplanes. Hence $f_3(P_4) = f_3(C_4) + 2 = 10$. The number of edges of P_4 is the number of edges of C_4, subtracted those that are cut away, and added those that arise as the intersection of 2-faces of the cube with the cutting hyperplanes. We cut away the four incident edges to each of the cube vertices $-\mathbf{1}$ and $\mathbf{1}$, and create one new edge for every 2-face that contains one of these vertices. Therefore,

$$f_1(P_4) \ = \ f_0(C_4) - 2 \cdot 4 + 2 \cdot 6 \ = \ 32 - 8 + 12 \ = \ 36$$

The number of 2-faces of P_4 can now be obtained by Euler's equation:

$$f_2(P_4) \ = \ f_3(P_4) + f_1(P_4) - f_0(P_4) \ = \ 10 + 36 - 14 \ = \ 32$$

Finally, to obtain $f_{02}(P_4)$, note that of the 32 2-faces there are $2 \cdot 4 = 8$ triangles that arise as faces of the newly created simplex facets, $2 \cdot 6 = 12$ more triangles originate from cutting quadrilaterals of the cube and the remaining 12 are quadrilaterals of the cube that are not intersected by the cutting hyperplanes. Therefore,

$$f_{02}(P_4) \ = \ 3 \cdot (8 + 12) + 4 \cdot 12 \ = \ 108$$

	(f_0, f_1, f_2, f_3)	f_{02}	α
P_4	$(14, 36, 32, 10)$	108	6
24-cell	$(24, 96, 96, 24)$	288	0
C_4	$(16, 32, 24, 8)$	96	12
${C_4}^\Delta$	$(8, 24, 32, 16)$	96	12
bipyr C_3	$(10, 28, 30, 12)$	96	9
prism ${C_3}^\Delta$	$(12, 30, 28, 10)$	96	9

Table 6.5: 4-dimensional Hanner polytopes and their f-vectors

and we get $\alpha(P_4) = (108 - 3 \cdot 32 + 108 - 3 \cdot 36)/2 = 6$.

The facets of the 24-cell are all octahedra, therefore it is 2-simplicial. It is also self-dual, which implies 2-simplicity. Hence $\alpha(\text{24-cell}) = 0$. (See, for example, Coxeter [19, Section 8.2] for a discussion of the combinatorial structure of the 24-cell.)

Compare that to the 4-dimensional Hanner polytopes, which are listed in Table 6.5. The value of α is at least 9 for each of the four 4-dimensional Hanner polytopes, and this implies the assertion. □

The polytope P_4 is in fact one of the examples that have to be considered during the proof of Lemma 6.3.13. There we have seen that it does not yield a counterexample for Conjecture 6.3.2, although it does for the stronger Conjecture 6.3.24.

It might be worth considering a non-symmetric version of Conjecture 6.3.24. The problem is that there is no obvious general analogue to the Hanner polytopes as "minimal polytopes", except possibly the simplex. With this, the conjecture reads as follows.

Conjecture 6.3.26. Let $\alpha \in \mathbb{R}^{F_d}$ be a flag vector functional, which is non-negative for all d-polytopes. Then for every d-dimensional polytope P we have $\alpha(\Delta_d) \leq \alpha(P)$.

A proof of this conjecture would imply that the inequalities obtained by Ehrenborg's Lifting Theorem 1.4.6, which are not sharp for the simplex, can be improved. On the other hand, a counterexample to the conjecture might arise from a lifted inequality.

The geometric interpretation of Kalai's Conjecture C can also be carried over to Conjecture 6.3.20, by considering only the very special linear functionals f_i, $0 \leq i \leq d-1$, instead of all α. Then for dimension 5, the central hypersimplex

(a) Path on 4 vertices (b) Bull graph

Figure 6.11: Graphs as input for the Hansen construction

$\tilde{\Delta}(3)$, is also a counterexample to a weaker version of Conjecture 6.3.20. There are in fact more counterexamples with very interesting properties. They arise from a construction given by Hansen [31], which we describe shortly.

Consider a simple graph G (that is, without loops and multiple edges) on the vertex set $\{1, \ldots, n\}$. An *independent set* is a subset $I \subseteq \{1, \ldots, n\}$ such that the subgraph of G induced by the vertices in I has no edges. For an independent set I define the characteristic vector $\chi^I \in \{0,1\}^n$ by

$$\chi_i^I := \begin{cases} 0 & \text{if } i \notin I \\ 1 & \text{if } i \in I \end{cases}$$

The Hansen polytope of G is

$$H(G) := \mathsf{conv}\Big\{ \pm \begin{pmatrix} 1 \\ \chi^I \end{pmatrix} \in \mathbb{R}^{n+1} \;\Big|\; I \subseteq \{1, \ldots, n\} \text{ independent set of } G \Big\}$$

From the definition it is clear that the Hansen polytopes are centrally-symmetric. Hansen [31] gave a combinatorial description of these polytopes for perfect graphs G, using the cliques of G. From this, one sees that $H(G)$ is self-dual if G is self-complementary. Unfortunately, it turns out that it is quite difficult to read off the f-vector from the combinatorics. An easy observation is that $f_0(H(G))$ is twice the number of independent sets of G and $f_n(H(G))$, the number of facets, equals twice the number of cliques of G, but beyond that not much is known in general.

However, computations for small examples show that the Hansen polytopes might be interesting in different respects. Consider as graphs G_4, the path on 4 vertices, and G_5, the "bull graph" on 5 vertices, see Figure 6.11. Both are self-complementary and perfect. Then $H(G_4)$ and $H(G_5)$ are also strong counterexamples to Conjecture 6.3.20 in dimensions 5 and 6, respectively.

Additionally, the total number of their faces is much closer to 3^d than for any other non-Hanner polytope. This phenomenon seems to continue with examples in growing dimension, as computations by Ragnar Freij and Matthias Henze [27] suggest. Therefore, Hansen polytopes of suitable graphs might be good candidates for counterexamples to the 3^d-conjecture for larger d.

Even more surprising, the very same examples seem to be close to violating the *Mahler conjecture*, one version of which states that the product of the volumes of a polytope and its polar is minimised by the centrally-symmetric cube. See Tao [57] for an overview. It would be interesting to investigate the connection of this open problem to the 3^d-conjecture.

Finally, we mention that $H(G_4)$ and $H(G_5)$ are also the polytopes closest to the bound given in Lemma 3.3.5. This suggests the question, if there is an improved bound which is tight for certain Hansen polytopes.

Chapter 7

Shelling 4-polytopes

By describing a shelling we combinatorially build up a discrete structure—in our case a polytope, or, more generally, a regular sphere—by successively adding the facets, according to certain rules.

Shellings provide an efficient way to formulate proofs using invariants that are valid during the building procedure. An outstanding example might be the Upper Bound Theorem which can be proved using shellings (see [58, Section 8.4]).

In this chapter we consider only 4-dimensional polytopes and we use some rather simple observations concerning shellings to aim at the construction of polytopes with special properties. In Section 7.3 we describe an algorithmic way to do this. An implementation of this framework exists and, for instance, led the way to the construction of the polytopes described in Section 4.2. The first interesting member of this family is also the smallest possible non-trivial 2-simple, 2-simplicial 4-polytope, which we prove in Section 7.2.

7.1 Shelling polytopes

There are different variants in the literature of which rules exactly should be postulated to define a shelling. The following definition coincides with the one in Ziegler's book [58, Chapter 8] for the special case of polytopes.

Definition 7.1.1 (Shelling of a polytope)**.** A *shelling* of a polytope P is a total ordering $F_1, \ldots, F_m$ of the facets of P such that either $\mathsf{dim}\, P \leq 1$ or the following condition is satisfied: For every $j \in \{2, \ldots, m\}$ the intersection

$$T_j \ := \ F_j \cap \bigcup_{i=1}^{j-1} F_i$$

is non-empty and there is a shelling $G_1, \ldots, G_k$ of the facet F_j such that $T_j = G_1 \cup \ldots \cup G_\ell$ for some $\ell \leq k$.

It can easily be shown that every polytope has a shelling. The first, very intriguing proof was given by Brugesser and Mani [18], and describes a *line shelling*. This concept is actually strong enough to guarantee the existence of shellings with additional useful properties. See [58, Section 8.2] for proofs.

Theorem 7.1.2 (Brugesser & Mani [18])**.** Let P be a polytope. Given any two facets F and F' of P, there exists a shelling of P that starts with F and ends with F'.

Similarly, given any face G of P, there exists a shelling of P that starts with exactly those facets that contain G.

Furthermore, given any vertex v and a facet F containing v, there exists a shelling of P that starts with F and shells all the other facets containing v next.

Finally, reversing the order of the facets in a shelling of P again gives a shelling of P. □

We are solely concerned about shellings of 4-dimensional polytopes. This basically amounts to "gluing together" 3-polytopes along their 2-faces, such that the obtained polyhedral complex is topologically a 3-ball during the whole process, only becoming a 3-sphere—the boundary of the shelled 4-polytope—in the last step.

7.2 Small 2-simple, 2-simplicial 4-polytopes

A 4-polytope is 2-simple, 2-simplicial if every edge is contained in 3 facets and every 2-face is a triangle (see Definition 4.1.2). These polytopes prove to be rather interesting in different respects, as can be seen from Chapter 4.

Trivially, the 4-simplex is a 2-simple, 2-simplicial 4-polytope. Eppstein, Kuperberg and Ziegler [26] were the first to construct infinitely many 2-simple, 2-simplicial 4-polytopes. Rational realisations were given by Paffenholz and Ziegler [44], the smallest of which is the hypersimplex, having 10 vertices and 30 edges.

An obvious question is whether there exists a smaller example and if so, how many combinatorially different ones there are. In this section we show that there is—up to combinatorial equivalence—only one non-trivial 2-simple, 2-simplicial 4-polytope with at most 9 vertices, namely the polytope P_9 described in Theorem 4.2.2.

As discussed in Section 4.2, 2-simplicity and 2-simpliciality can be read off from the flag vector: P is 2-simplicial if and only if $f_{02}(P) = 3f_2(P)$, and it is 2-simple if and only if $f_{13}(P) = 3f_1(P)$. Additionally, we make frequent use of the following simple facts.

Lemma 7.2.1. Let P be a 2-simple, 2-simplicial 4-polytope.

(a) P^Δ is also 2-simple, 2-simplicial.

(b) The f-vector of P is symmetric: $f_0(P) = f_3(P)$ and $f_1(P) = f_2(P)$.

(c) For the flag vector we have $f_{03}(P) = 2f_0(P) + f_1(P)$.

Proof. (a) is obvious, 2-simplicity and 2-simpliciality being dual properties to each other.

(b) By 2-simpliciality, every 2-face has 3 edges, so $f_{12} = 3f_2$. On the other hand, every edge is contained in 3 different 2-faces, by 2-simplicity, therefore $f_{12} = 3f_1$. We immediately conclude that $f_1 = f_2$ and by Euler's equation $f_0 = f_1 - f_2 + f_3 = f_3$.

(c) By 2-simpliciality, $f_{02} = 3f_2$. By the Generalized Dehn-Sommerville equations (Theorem 1.2.15), we have $f_{01} - f_{02} + f_{03} = 2f_0$. Combining these with (b),

$$f_{03} \;=\; 2f_0 + f_{02} - f_{01} \;=\; 2f_0 + 3f_2 - 2f_1 \;=\; 2f_0 + 3f_1 - 2f_1 \;=\; 2f_0 + f_1.$$

□

Lemma 7.2.2. Let P be a 2-simple, 2-simplicial 4-polytope. If P contains a facet with k vertices, then $f_0(P) \geq 2k - 3$.

Proof. Let F be a facet of P with k vertices. Then F is a simplicial 3-polytope and contains $2k - 4$ triangles, which are ridges of P. Therefore, P has at least $2k - 4$ facets in addition to F. From Lemma 7.2.1(b) we get $f_0(P) = f_3(P) \geq 2k - 4 + 1 = 2k - 3$. □

Lemma 7.2.3. Let P be a 4-polytope. Suppose v is a vertex of P and F a facet containing v such that v has degree at least k within F, that is, v is contained in at least k edges of F. Then v^Δ is a facet of P^Δ with at least $k + 1$ vertices.

Proof. The vertex v having degree k within F translates to the facet v^Δ having at least k different 2-faces that all contain the vertex F^Δ. In other words, v^Δ is a 3-polytope that contains a vertex of degree k and therefore has at least $k + 1$ vertices. □

LESS THAN 9 VERTICES

We first prove that there cannot be a non-trivial 2-simple, 2-simplicial 4-polytope with less than 9 vertices. The following lemma is a special case of the well-known fact that every k-simplicial, h-simple d-polytope is a simplex if $k + h \geq d + 1$ (see, for instance, Grünbaum [29, Exercise 4.8.12])—note that simplicial is equivalent to 3-simplicial.

Lemma 7.2.4. Every simplicial, 2-simple 4-dimensional polytope is combinatorially equivalent to the 4-simplex. $\square$

Proposition 7.2.5. There is no 2-simple, 2-simplicial 4-polytope with less than 9 vertices, except for the simplex.

Proof. Let P be a 2-simple, 2-simplicial 4-polytope with at most 8 vertices, and not combinatorially equivalent to the 4-simplex. By Lemma 7.2.4, P contains at least one facet F which has 5 or more vertices. On the other hand, F cannot contain 6 or more vertices by Lemma 7.2.2.

Therefore, F has exactly 5 vertices. Since it is simplicial, it is a bipyramid over a triangle. Let v be a vertex of F that is incident to 4 different triangles $R_1, \ldots, R_4$ in F. Then P has facets $F_1, \ldots, F_4$ such that $F_i \cap F = R_i$, and v is contained in at least these 5 facets.

If v was in fact contained in more than 5 facets, then P^Δ would be a 2-simple, 2-simplicial 4-polytope with a facet v^Δ containing at least 5 vertices, so $f_0(P^\Delta) \geq 9$ by Lemma 7.2.2. Since $f_0(P^\Delta) = f_3(P)$, this contradicts Lemma 7.2.1(b). Hence, the facet v^Δ can only be a bipyramid over a triangle, since it has to be simplicial. Back in the primal setting, this means that the vertex figure P/v is a triangular prism.

Figure 7.1 attempts a sketch of the local situation around v. Here, dashed edges belong to the facet F, as do the patterned triangles $R_1, \ldots, R_4$ which are partly hidden. Shaded triangles are 2-faces of the facets $F_1, \ldots, F_4$. Equally coloured regions and vertices are identified.

Since v was an arbitrary vertex of degree 4 in F, the above also holds for each of the other two. Now consider a shelling of P where F and all its neighbouring facets are shelled first. Such a shelling exists, because except for F and its 6 neighbours there is at most one other facet of P left. This can be taken as the first facet of a shelling, and then the reverse shelling is the one we look at. Considering the complex after the last neighbour of F was added leaves two possibilities, up to symmetry. The sketches are to be read in the same fashion as before.

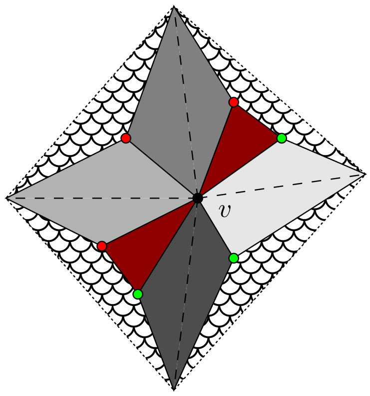


Figure 7.1: Combinatorics of the vertex figure P/v

(i) All vertex figures at the equator of F are "equally directed", see Figure 7.2. Then the two yellow circled vertices are both contained in two different facets, hence they are in the 2-face that arises as their intersection. By 2-simpliciality, this 2-face must be a triangle, but there is no edge between the two vertices.

(ii) Otherwise one of the three vertex figures is "directed contrary" to the other two and the situation looks like in Figure 7.3. Here we have an

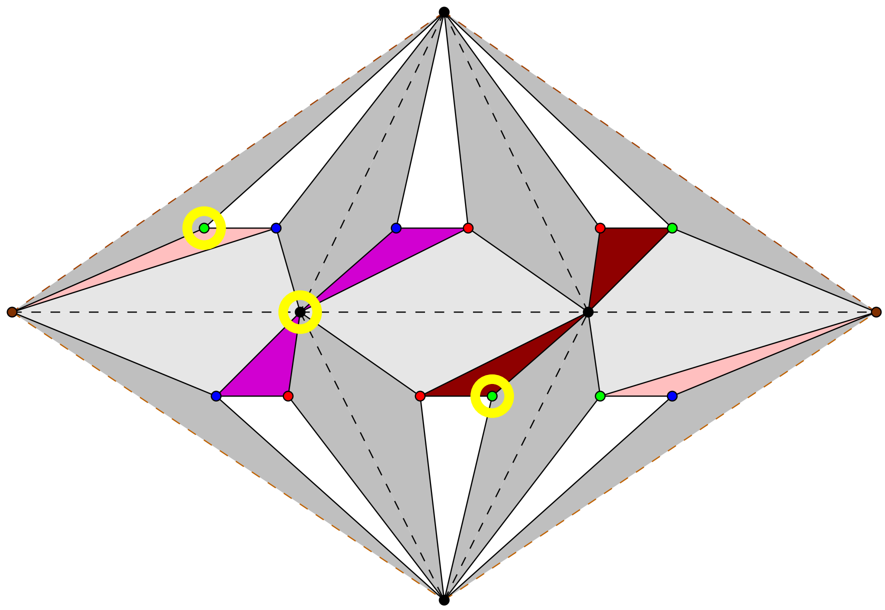


Figure 7.2: Case (i) in the proof of Proposition 7.2.5

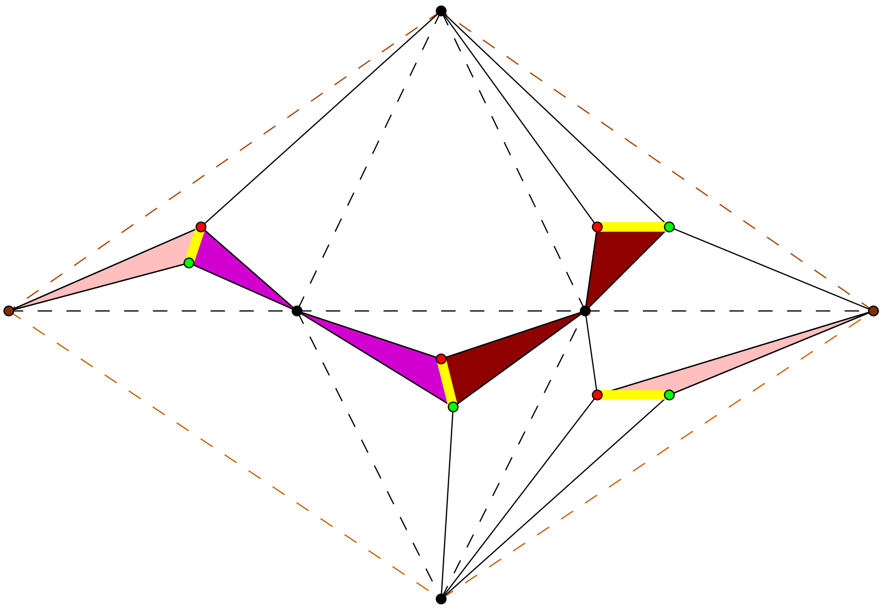


Figure 7.3: Case (II) in the proof of Proposition 7.2.5

edge—coloured yellow in the picture—that is contained in 4 different facets, contradicting 2-simplicity of P.

□

EXACTLY 9 VERTICES

There is one example of a 2-simple, 2-simplicial 4-polytope with exactly 9 vertices we know from Chapter 4, which was denoted by P_9. It is self-dual with f-vector $(9, 26, 26, 9)$ and features as facets one octahedron, two simplices and six triangular bipyramids.

For the rest of this section let P be any 2-simple, 2-simplicial 4-polytope with exactly 9 vertices. We want to show that P is combinatorially equivalent to P_9.

First of all, observe that by Lemma 7.2.2 there cannot be a facet with 7 or more vertices in P. Additionally, due to 2-simpliciality, the only possible types of facets with 6 vertices are the octahedron and the twice stacked simplex. We call these two types *large facets*.

Lemma 7.2.6. P does not contain 4 or more large facets.

Proof. If P contains at least 4 large facets, then $f_{03}(P) \geq 4 \cdot 6 + 5 \cdot 4 = 44$. Counting missing edges in the large facets gives $f_1(P) \leq \binom{9}{2} - 4 \cdot 3 = 24$, and applying Lemma 7.2.1 yields $f_{03}(P) \leq 2 \cdot 9 + 24 = 42$, a contradiction. □

Lemma 7.2.7. *P does not contain 3 or more large facets.*

Proof. Suppose, P has 3 large facets. By Lemma 7.2.6 there are no more large facets. We show that additionally P has at least 2 facets which are triangular bipyramids.

First of all, any two of the large facets must have precisely 3 vertices in common: If two of them had at most 2 vertices in common, P would have at least $2 + 2 \cdot 4 = 10$ vertices. On the other hand, two facets intersect in a ridge, which, by 2-simpliciality, contains at most 3 vertices.

No vertex can be contained in all three of the large facets—otherwise there would again be $6 + 3 + 1 = 10$ vertices. Altogether, we get that every vertex of P is incident to exactly two large facets.

Every vertex of degree at least 4 within a facet must be contained in two other facets in which it also has degree at least 4, by 2-simplicity. Consider a vertex that has degree at least 4 in one of the large facets. There have to be two other facets in which it also has degree at least 4, but only one of them can be another large facet. The only candidate for the other one is a triangular bipyramid, which provides 3 vertices with degree 4.

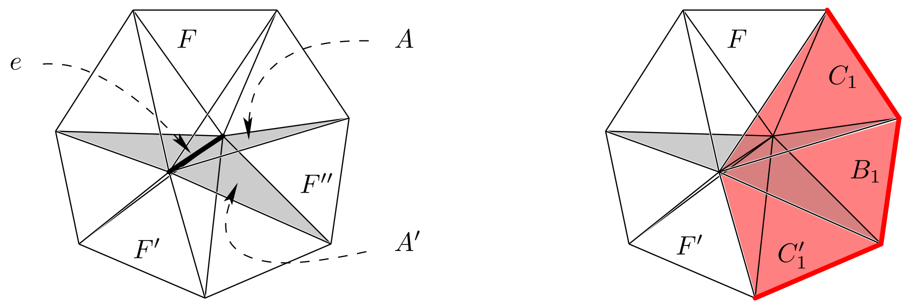
Now we count the number of such vertices. If one of the large facets is an octahedron then there are 6 vertices of degree 4 in that. Otherwise there are at least 4 vertices in each of the 3 large facets, but we might have counted each one twice. Hence there are at least $3 \cdot 4/2 = 6$ vertices in P that have degree at least 4 in some large facet. Each of these vertices has to be in a triangular bipyramid, so in either case there have to be at least $6/3 = 2$ such bipyramids.

Having established this, the same calculation as before yields a contradiction: $f_{03} \geq 3 \cdot 6 + 2 \cdot 5 + 4 \cdot 4 = 44$, but $f_{03} \leq 2 \cdot 9 + \binom{9}{2} - 3 \cdot 3 - 2 = 43$. □

Lemma 7.2.8. *P does not contain 2 or more large facets.*

Proof. Suppose P has two large facets F and F'. As before, F and F' share a ridge of P, otherwise there would be too many vertices. Now there are 3 cases.

(I) F and F' are both twice stacked simplices.



(a) The twice stacked simplices F and F'

(b) Three edges (red) in $F_1 \cap F_2$, but not in a ridge

Figure 7.4: Case (I) in the proof of Lemma 7.2.8

Then both vertices v_1 and v_2 of degree 5 within F must coincide with the vertices of degree 5 within F'—otherwise, by Lemma 7.2.3, P^Δ had more than 2 facets with 6 vertices, in contradiction to Lemma 7.2.7.

Let e be the edge between v_1 and v_2, see Figure 7.4(a). Then by 2-simplicity, e can be in only one other facet F'' and $F'' \cap (F \cup F')$ consists of two triangles $A \subset F$ and $A' \subset F'$ with $A \cap A' = e$. Additionally, F'' cannot have more vertices than the ones in $A \cup A'$, since P has only 9 vertices. Therefore, F'' is a simplex.

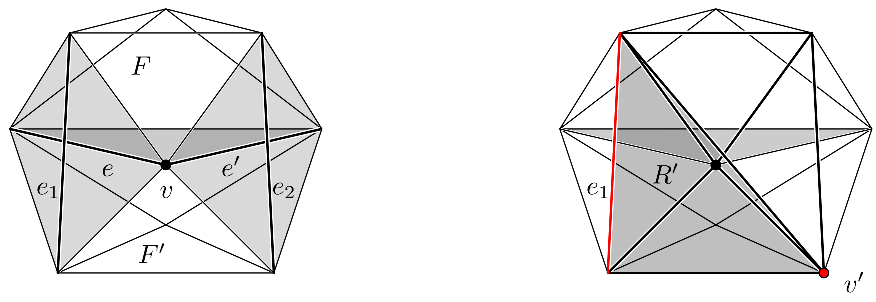
Now let $B_1 \neq A, A'$ be one of the remaining triangle faces of F'', and F_1 the facet adjacent to F'' with $B_1 = F_1 \cap F''$, see Figure 7.4(b). Then by 2-simplicity, F_1 contains 2-faces C_1 and C'_1, belonging to F and F', respectively, that both intersect with B_1. The same holds for analogous 2-faces B_2, C_2, C'_2 and another facet F_2.

Since C_1 and C_2, as well as C'_1 and C'_2, intersect in F, respectively F', we now have 3 edges of P that are contained in two different facets, F_1 and F_2, but not in a common ridge, a contradiction.

(II) F and F' are both octahedra.

Let R be the ridge between F and F', v a vertex in R and e an edge of R, incident to v. Like in Case (I), e is contained in a simplex S_1, intersecting both F and F' in ridges. The same is true for the other edge e' in R that is incident to v, so we get another simplex facet S_2, containing e'. Denote by e_1 and e_2 the edges in S_1 and S_2, respectively, that do not intersect e, respectively e', see Figure 7.5(a).

We get that v belongs to one more facet F_1, in which it has degree 4,



(a) The two octahedra F and F' and two simplices

(b) The ridge R' in the facets F_1 (bold edges) and F_2 (not drawn)

Figure 7.5: Case (ii) in the proof of Lemma 7.2.8

and which shares ridges with both simplices, S_1 and S_2. F_1 must be a bipyramid over a triangle by Lemma 7.2.7.

Since the same holds for the other vertices of R, the simplex S_1 is also adjacent to another triangular bipyramid F_2, which also contains the edge e_1. By 2-simplicity, e_1 cannot be contained in any other facet, therefore also F_1 and F_2 intersect in a ridge R'. This ridge contains e_1 and an additional vertex v', either in F or in F'. The same ridge, however, as a 2-face of F_2 contains a different vertex from either F or F', a contradiction. See Figure 7.5(b).

(iii) F is an octahedron and F' is a twice stacked simplex.

Then there is a vertex v with degree 5 within F'. Since v has degree 4 in F, there are at least 7 incident edges to v. Dually, P^Δ has a facet v^Δ with 7 or more 2-faces. Since v^Δ is a simplicial 3-polytope, it has in fact 8 or more 2-faces and therefore v lies on at least one more edge, leading to another vertex v'.

v' can neither be contained in F nor in F'. Both of these facets have 6 vertices, 3 of them in common. Therefore, P must again have at least $2 \cdot 6 - 3 + 1 = 10$ vertices.

□

Lemma 7.2.9. *P does not contain a twice stacked simplex as facet.*

Proof. By Lemma 7.2.3, every vertex v of degree 5 within a facet yields a facet v^Δ of P^Δ with at least 6 vertices. Since a twice stacked simplex has two vertices of degree 5, P^Δ would contain at least 2 facets with 6 vertices, in contradiction to Lemma 7.2.8. □

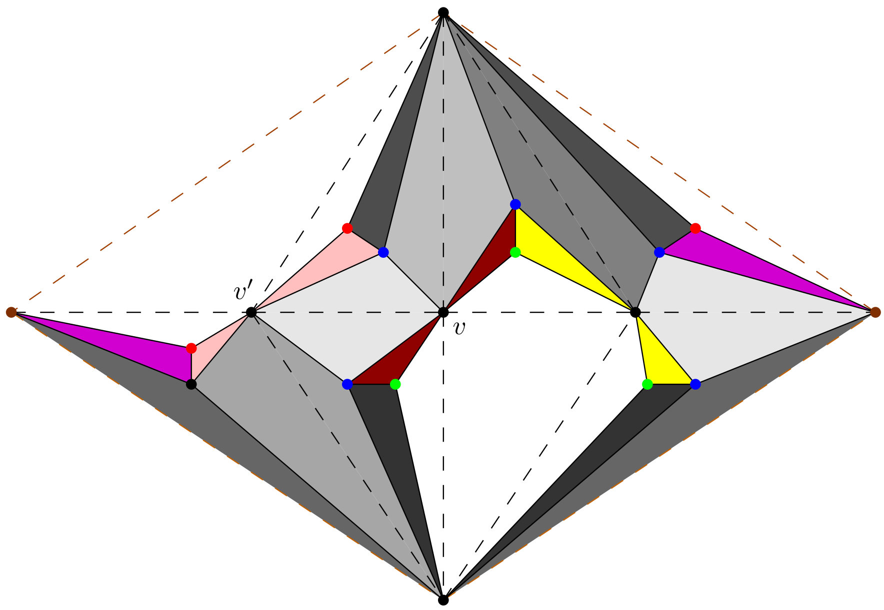


Figure 7.6: Case (I) in the proof of Lemma 7.2.10

Lemma 7.2.10. If P contains an octahedron, then it is combinatorially equivalent to P_9.

Proof. Let F be an octahedral facet of P and v an arbitrary vertex of F. We again consider the facet v^Δ of P^Δ. v^Δ cannot be a facet with 7 or more vertices by Lemma 7.2.2, it cannot be a twice stacked simplex by Lemma 7.2.9 and it cannot be a simplex, because in contains at least 5 vertices by Lemma 7.2.3.

Suppose v^Δ is an octahedron. Then v is contained in some facet F', which is different from F and from all four facets that are adjacent to F via a ridge containing v. By Lemma 7.2.1(b) there are only 8 facets different from F, and they all share a ridge with the octahedron F. Hence, F' and F would share a ridge that does not contain v, in addition to the vertex v. This is a contradiction.

Therefore, v^Δ is a bipyramid over a triangle and P/v is a triangular prism, like in the proof of Proposition 7.2.5. This holds for all vertices of F, and by Lemmas 7.2.8 and 7.2.9 all other facets are simplices or triangular bipyramids. Choosing a neighbouring vertex v', this leaves two possibilities, up to symmetry, for the combinatorics of F and its adjacent facets:

(I) The vertex figures at v and v' are "equally oriented", see Figure 7.6. Then the remaining two facets, which are not indicated in the sketch,

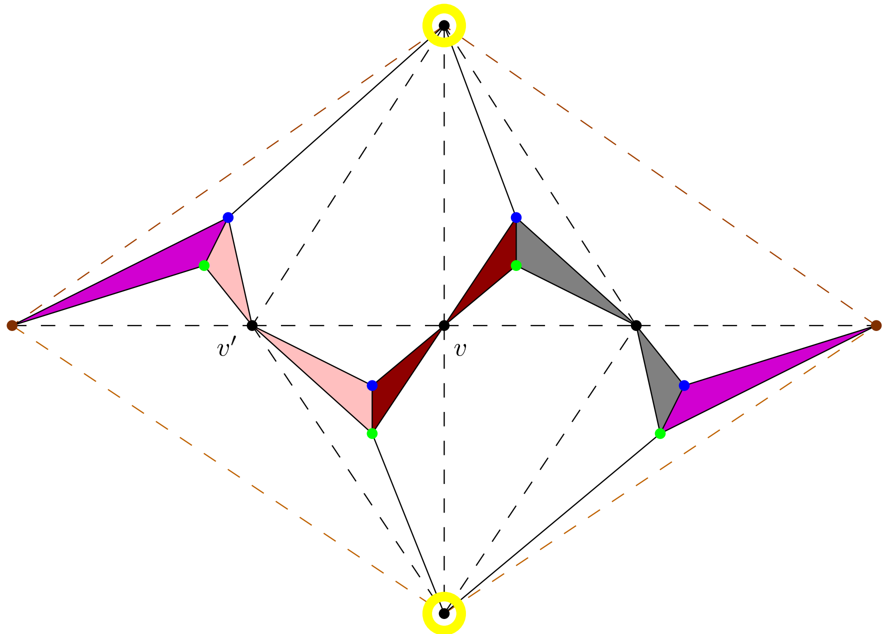


Figure 7.7: Case (II) in the proof of Lemma 7.2.10

have to be simplices, and the combinatorics arising is equivalent to that of P_9.

(II) Otherwise there is only space for 4 bipyramids and the remaining facets are simplices again, see Figure 7.7. Then the two yellow circled vertices cannot have the required prism vertex figure and we get a contradiction. □

Lemma 7.2.11. *P or P^{Δ} contains an octahedron.*

Proof. Suppose, P as well as P^{Δ}, contains only facets with at most 5 vertices. Let F be a facet of P which is a triangular bipyramid and v a vertex of degree 4 within F. Then by the assumption and Lemma 7.2.3, v^{Δ} is again a bipyramid over a triangle. Now the same argument as in the proof of Proposition 7.2.5 leads to a contradiction.

Therefore, P or P^{Δ} contains a facet with 6 vertices, which can only be an octahedron by Lemma 7.2.9. □

Proposition 7.2.12. *P is combinatorially equivalent to P_9.*

Proof. By Lemma 7.2.11, P or its polar contains an octahedron, hence by Lemma 7.2.10, P or P^{Δ} is combinatorially equivalent to P_9. The claim now follows from the self-duality of P_9. □

Concluding, we get the main result of this section.

Theorem 7.2.13. P_9 is the only non-trivial 2-simple, 2-simplicial 4-polytope with $f_0 \leq 9$, up to combinatorial equivalence. □

7.3 A computer-based approach

In the previous section we used shellings to prove that certain situations cannot occur under certain circumstances. Shellings can also be used to find interesting examples of polytopes. The basic idea is to enumerate all possible shelling sequences of a possibly existing polytope which has required properties. We give a depth-first search method which adds one facet in each iteration, thus enumerating the shellings of polytopes with as many facets as the maximal search depth.

In principle this yields two possible outcomes. Either we cannot find a shelling, which means that such a polytope does not exist, or we get the complete combinatorial information about a desired polytope. The algorithm described here uses no geometric information, which means that in the second case we get the combinatorics of a 3-sphere. This still has to be realised as a polytope if we insist on finding such. In general, we will therefore rather use the terminology "sphere" instead of "polytope" in this section.

The aimed type of polytope obviously restricts the possible shelling sequences. For instance, for searching 2-simplicial spheres we would only have to use simplicial polytopes as facets. Also, Theorem 7.1.2, as well as some rather simple observations, provide means to speed up the enumeration. Still, the algorithm is somehow a brute-force method which can probably be optimised considerably.

General algorithm

The program starts with an empty polyhedral complex or, optionally, with a given starting configuration. The shelling itself is then done by a recursive function, triggered by the main procedure.

The concrete initialisation depends on the given *shelling type*, that is, the properties the found spheres are supposed to have. Actually implemented shelling types undertake the search for 2-simple, 2-simplicial spheres, cubical spheres, and spheres with $g_2 < 0$, but a lot more are imaginable. The general structure of the initialisation routine is relatively fixed. It creates a space to collect the found spheres in and returns the main data structure, the *algorithm structure*.

Algorithm *Main program*
Input: shelling type t, options o
Output: collection of found 3-spheres
$s \leftarrow initialise\_algorithm(t, o)$
$b \leftarrow$ empty collection of 3-spheres
$shell\_recursive(s, b)$
return b

The algorithm structure s is the central object. It contains all essential data and functionality, tailored to the aimed shelling type, that is, for every shelling type there has to be an adapted version of the algorithm structure. Before we explain it in detail, we have a closer look at a single shelling step.

Each call to the recursive function *shell_recursive* corresponds to a shelling step. Here a list of possibilities for extending the current polyhedral complex is assembled and afterwards successively executed, unless

- the current structure represents a completed sphere—in which case a subroutine is called that tests for previous isomorphic results and saves it if there are none; or
- for some reason backtracking comes into play and the current branch of the search tree is cut—for example because the maximal search depth has been reached or there are any obstacles that prohibit the current complex from becoming a valid sphere with the prescribed properties.

Algorithm *shell_recursive*
Input: algorithm structure s, collection b of already found spheres
Output: —
if $s.finished()$
  then $b.add(s)$
    return
if $s.cut\_branch()$
  then return
$s.calc\_extension\_possibilities()$
while $s.is\_extendable()$
  do $s' = s.extend()$
    $shell\_recursive(s', b)$

Everything that is called in this recursive method—except for the function $b.add$—is part of the algorithm structure s, which we examine closer next.

Algorithm structure

The algorithm structure comprises all information as well as all routines that are necessary to execute all possible shellings. In the general version it contains the following parts:

- A structure to describe the current state of the shelling. This contains on the one hand the boundary surface of the current 3-ball, represented by two dual planar graphs with additional information on the vertices, edges and faces, and on the other hand the complete vertex-facet incidences of the complex.
- Information about the current and maximal search depth.
- A flag that can be set at various stages of a shelling step and indicates if there has been an obstruction to further extending the complex to a valid sphere of the given type.
- A list, together with an iterator, of possibilities to further extend the complex.

The important routines that decide on the backtracking, calculate possible further extensions and actually extend the current complex are mostly part of the specialised versions, since they largely depend on the shelling type. In the following we give an overview over the parts that all versions have in common, and discuss some more details about the specialisations in the concluding sections.

Recognising when the algorithm has arrived at a valid result, is rather easy: The subroutine *s.finished* simply decides whether the boundary surface of the current complex is empty or not.

The decision to backtrack, exercised in the method *s.cut_branch*, in its most general form depends exclusively on the current search depth, equally the current number of facets: If this is larger than or equals the maximal number of facets allowed, the recursion is stopped. There are, however, some criteria to recognise earlier that a branch of the search tree has to be cut off. This leads to some improvements of the general method, reducing the runtime of the program, that are discussed in the next subsection.

The second half of the recursive function concerns the creation and application of further shelling steps. After setting up a list of possible extensions in *s.calc_extension_possibilities*, the loop simply traverses this list until the end. In each iteration a copy of the algorithm structure is created, with the

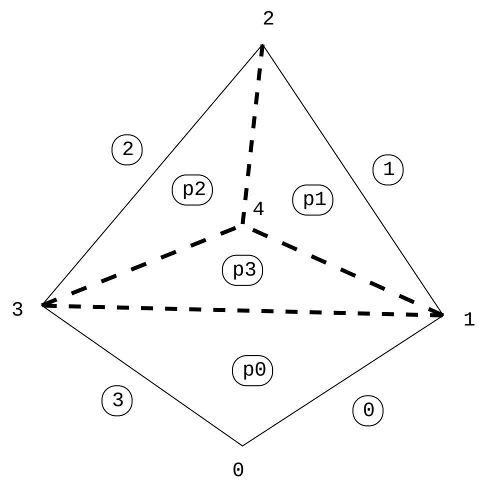


Figure 7.8: Possible extension surface—the dashed edges are about to vanish from the surface, as well as the vertex `4`

respective extension applied and the shelling function is called recursively with this copy as a parameter.

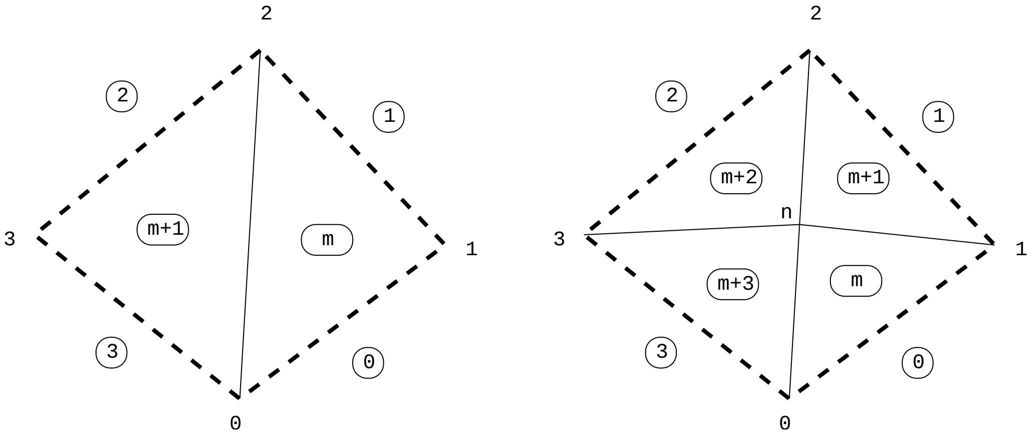
The calculation of possible extensions, that is, next shelling steps, is the main effort of this part, and again, the details differ considerably for the different shelling types. The overall principle is to some extent given by the used data structures that describe the boundary surface of the ball.

This surface determines, which facets can possibly be used in a next shelling step, and how. Precisely, we have to match parts of the surface with parts of the boundaries of the allowed facets to obtain the possible intersections of the next facet in a shelling with the current polyhedral complex. We call these 2-dimensional subcomplexes *extension surfaces*. They have to be either topological 2-balls or the complete surface, which is topologically a 2-sphere, in the last shelling step. Figure 7.8 shows an example of a possible extension surface, which can be used to glue different new facets, see Figure 7.9.

Therefore, the procedure *s.calc_extension_possibilities* first identifies all such subcomplexes as subgraphs of the dual graph contained in the boundary surface description. Then for each suitable extension surface it exhibits all possibilities to "glue" a facet onto this subcomplex, and records the changes the boundary undergoes in each case, to be applied later by the function *s.extend*.

The information to encode these changes consists of

- a list of vertices that have to be added to the primal surface graph;



(a) Extension with a triangular bipyramid (b) Extension with a twice stacked simplex

Figure 7.9: Result of extending the surface part in Figure 7.8 with different facets—the continuous lines represent new edges of the surface, the dashed ones are now contained in one more facet

- a list of vertices that have to be added to the dual surface graph, that is, faces that are new to the boundary surface;
- a list of edges that are new to the primal surface graph;
- a list of edges that already existed in the primal surface graph, but now change their status, since they are contained in one more facet.

Note that both the new as well as the changing edges contain the information needed to keep the duality between the two graphs consistent. See Figure 7.9 for examples.

Additionally, the vertex-facet-incidences of the complete complex have to be updated. This amounts to simply adding a new facet containing all vertices in the extension surface and all new vertices. The data necessary for this can also be read off from the information above.

Improving runtime

As remarked, it is advisable to make the search tree as small as possible while ensuring that the algorithm does not miss anything. Apart from some simple observations about shellings, Theorem 7.1.2 provides us with possibilities for doing this.

For example, it suffices to consider shellings that start with all facets around a given vertex. This fact can be applied on two ways. Either we declare some arbitrary vertex of the first facet—the one with index 0 for instance—to be the start vertex, and then we can skip any extension possibility that does not involve this vertex, until it disappears from the surface.

The other possibility is to declare the start vertex to be one of the vertices of the potentially resulting sphere with maximal degree. This is a bit more tricky, since initially we do not know the vertices with highest degree. They are determined in the first step in which a vertex disappears from the surface. We then record the highest degree of the now interior vertices and in the following prohibit any extension that would create vertices with higher degree.

However, this strategy cannot be combined with the following one, since the vertex with the highest degree is not necessarily contained in a specific facet.

Additionally, we can restrict the search to shellings starting with a chosen facet. This means that facets that have been used to start earlier shellings need not be considered any more, because any polytope that would contain such a facet would have been found before. This reduces the number of extension possibilities immensely, once the program completed searching through the first main branches of the search tree.

By default, the extension possibilities are applied in an undefined order. It can make sense to sort the list after it is created, according to the size of the surface parts that lie beneath the respective extension possibilities, ascendingly or descendingly.

This actually does not speed up the overall runtime. In practise, however, it seems that it could lead to resulting spheres found earlier. An explanation may be that the smaller the extension surfaces used in the first steps, the larger the overall surface of the complex—which is soon likely to be too large to be completed within the remaining steps.

Some conditions can be given that imply that a complex has too large a surface in this respect. They can be expressed using the number of vertices and edges (following ideas of Arnold Waßmer).

Definition 7.3.1 (Vertex and edge hunger)**.** Let $F_1, \dots, F_k$ be a shelling sequence of a polytope P and $1 \leq i \leq k$. Let G be an arbitrary face of the polyhedral complex $\mathcal{C}_i$ defined by the facets $F_1, \dots, F_i$.

The *facet degree* of G is

$$\mathsf{fdeg}_i(G) \ := \ |\{F_j \,|\, 1 \leq j \leq i, G \subseteq F_j\}|.$$

The *vertex hunger of a vertex* v of $\mathcal{C}_i$ at step i is

$$\mathsf{VH}_i(v) := \begin{cases} 0 & \text{if } v \text{ is not on } \partial\mathcal{C}_i \\ 1 & \text{if } v \text{ is on } \partial\mathcal{C}_i \text{ and } \mathsf{fdeg}_i(v) \geq 3 \\ 4 - \mathsf{fdeg}_i(v) & \text{otherwise} \end{cases}$$

The *edge hunger of an edge* e of $\mathcal{C}_i$ at step i is

$$\mathsf{EH}_i(e) := \begin{cases} 0 & \text{if } e \text{ is not on } \partial\mathcal{C}_i \\ 1 & \text{if } e \text{ is on } \partial\mathcal{C}_i \text{ and } \mathsf{fdeg}_i(e) \geq 2 \\ 3 - \mathsf{fdeg}_i(e) & \text{otherwise} \end{cases}$$

The *vertex hunger*, respectively *edge hunger, of the complex* $\mathcal{C}_i$ at step i is

$$\mathsf{VH}_i := \sum_{v \text{ vertex of } \mathcal{C}_i} \mathsf{VH}_i(v)\ , \qquad \text{respectively} \qquad \mathsf{EH}_i := \sum_{e \text{ edge of } \mathcal{C}_i} \mathsf{EH}_i(e).$$

So the vertex, respectively edge hunger, at step i gives a lower bound on the number of facets that occur after step i and contain the given vertex, respectively edge. Using this we can then give upper bounds on the vertex and edge hunger, depending on the maximal total number of facets (the maximal recursion depth) and the current number of facets (the current recursion depth), see the concluding subsections. If such a bound is exceeded, the shelling cannot be completed with less than the given total number of facets, and can therefore be stopped early.

Notes on the implementation

The implementation of the described shelling algorithm comprises some more features, such as interfaces for debugging and printing log messages to monitor the progress of the shelling. Furthermore, the initialisation module can read an option file which contains detailed information about the possible extensions, maximal search depth and optional settings as described above. This also includes the possibility of providing a skip list, making it possible to continue an interrupted shelling process at roughly the point where it stopped.

The program is completely implemented in `C++`, using the Standard Template Library, as well as data structures from `polymake` [28]. For locating extension surfaces, that is, suitable subgraphs of the dual surface graph, as well as isomorphism testing of the found spheres, the functionality of `nauty` [40] is embedded.

2-SIMPLE, 2-SIMPLICIAL 4-POLYTOPES

One possible instance of a shelling type is to look for 2-simple, 2-simplicial spheres. In this case we can further specialise some parts of the algorithm, notably the computation of extension surfaces.

The boundary surface is now a simplicial 2-dimensional complex. The enumeration of all extension surfaces can be done with another recursive search on the dual surface graph. We start at some arbitrary triangle Δ of the boundary surface and search for a neighbouring triangle that is separated from Δ by an edge of degree 2. Note that every such edge, together with the neighbouring triangle has to be contained in a facet that is to be glued onto Δ, to ensure 2-simplicity. If a neighbour Δ' is found, it is added to the extension surface, receives a mark, to prevent it from being found from another direction in the further progress, and the search is continued recursively on Δ'.

The extension surface is completed if there are no more unmarked neighbouring triangles left. We still have to take some care about the validity of the result. For example, edges of degree 1 may not lie inside the extension surface. If such a thing occurs, then there is no chance of completing the current shelling. Similarly, there may be no identified vertices on the boundary of the extension surface, since this would change its topological type to something else than a 2-ball. Finally, there might be vertices which are not connected by an edge in the extension surface, but both contained in some earlier facet—these would both also be contained in the new facet, a situation which is not allowed for a polyhedral complex.

As remarked after Definition 7.3.1, we can also give bounds on the vertex and edge hunger that allow the algorithm to earlier recognise the necessity for backtracking.

Proposition 7.3.2. Let P be a 2-simple, 2-simplicial 3-sphere with k facets having at most α vertices each. Then for every shelling and $1 \leq i < k$ we have $\mathsf{VH}_i \leq \alpha(k-i)$ and $\mathsf{EH}_i \leq (3\alpha-6)(k-i)$. Furthermore, the number of 2-faces of $\partial\mathcal{C}_i$ is at most $(2\alpha-6)(k-i)+2$.

Proof. After shelling step i there are $k-i$ more facets to come, which are simplicial 3-polytopes, and therefore each have at most α vertices, $3\alpha-6$ edges and $2\alpha-4$ triangles. From this we immediately get the bounds on the vertex and edge hunger.

Furthermore, in every shelling step after i, except for the last one, at most $2\alpha-4$ triangles vanish from the surface boundary and at least 1 turns up new, so in each of $k - i - 1$ shelling steps the number of triangles on the surface decreases by at most $2\alpha - 6$. Adding the last facet of the shelling, at most $2\alpha - 4$ triangles vanish. In total, after step i the surface cannot have had more than $(2\alpha - 6)(k - i - 1) + 2\alpha - 4 = (2\alpha - 6)(k - i) + 2$ triangles. ☐

Using these specialisations, the program enumerated all 2-simple, 2-simplicial 4-polytopes with up to 11 vertices, and found several instances on 13 and 14 vertices. Among them were the first two members of the family described in Section 4.2, as well as the other small examples described in [43, Section 4].

We know that there are 2-simple, 2-simplicial 4-polytopes on n vertices for $n \in \{5, 9, 10, 11\}$ and all $n \geq 13$ (see [43]), but none for $6 \leq n \leq 8$ (see Section 7.2). It is not known, however, if there exists an example on 12 vertices.

Conjecture 7.3.3. There exists a 2-simple, 2-simplicial 4-polytope on 12 vertices.

The program discovered two 2-simple, 2-simplicial combinatorial 3-spheres on 12 vertices with f-vectors $(12, 40, 40, 12)$ and $(12, 39, 39, 12)$, respectively. Their vertex-facet incidences are given in Table 7.1. Realising one (or both) of these as polytopes would solve Conjecture 7.3.3 to the affirmative.

CUBICAL 4-POLYTOPES

Cubical 4-polytopes were extensively studied in several papers by Blind and Blind. Among other results, they showed in [14] that all even-dimensional cubical polytopes have an even number of vertices and classified the cubical d-polytopes with up to 2^{d+1} vertices, for $d \geq 4$ (see Blind and Blind [15]).

For $d = 4$ they use a shelling argument which is very similar to our algorithmic approach. In this case one gets a characterisation of all cubical 4-polytopes with at most 32 vertices. The first gap in vertex numbers is at 34.

Conjecture 7.3.4. There is no cubical 4-polytope with 34 vertices.

In the following we describe how to specialise the general algorithm to search for cubical 3-spheres. Most procedures for the cubical case are very similar to those for the 2-simple, 2-simplicial case. The difference is mainly in obtaining the extension surfaces. There are obviously only very few possible types, since there is only one type of facet. On the other hand, different possible extension

$\{v_0, v_1, v_2, v_3\}$	$\{v_0, v_1, v_3, v_4, v_6, v_8, v_{11}\}$
$\{v_0, v_2, v_3, v_4, v_5, v_6, v_7\}$	$\{v_0, v_2, v_3, v_6, v_7\}$
$\{v_0, v_1, v_3, v_4, v_8, v_9\}$	$\{v_0, v_2, v_4, v_5, v_7, v_8, v_9\}$
$\{v_0, v_1, v_2, v_6, v_9, v_{10}\}$	$\{v_0, v_4, v_6, v_7\}$
$\{v_0, v_4, v_7, v_8\}$	$\{v_1, v_2, v_3, v_5, v_6, v_9, v_{10}\}$
$\{v_0, v_5, v_6, v_{10}\}$	$\{v_1, v_2, v_4, v_6, v_7\}$
$\{v_0, v_5, v_7, v_8, v_9, v_{10}\}$	$\{v_0, v_1, v_5, v_8, v_{10}\}$
$\{v_1, v_2, v_3, v_4, v_{10}, v_{11}\}$	$\{v_1, v_2, v_4, v_8, v_{10}\}$
$\{v_2, v_5, v_6, v_8, v_{10}, v_{11}\}$	$\{v_0, v_1, v_5, v_9, v_{11}\}$
$\{v_1, v_8, v_9, v_{10}, v_{11}\}$	$\{v_0, v_2, v_3, v_9, v_{11}\}$
$\{v_1, v_4, v_5, v_7, v_8, v_{11}\}$	$\{v_2, v_5, v_8, v_{10}\}$
$\{v_2, v_4, v_5, v_{11}\}$	$\{v_1, v_3, v_9, v_{11}\}$

Table 7.1: Vertex-sets of the facets of two 2-simple, 2-simplicial 3-spheres on 12 vertices

surfaces might now intersect, in contrast to the 2-simple, 2-simplicial case, since edges can have degree larger than 2. We therefore have no simple method to compute extension surfaces and have to enumerate them manually.

Additionally, we can use some nice properties of cubical polytopes. For example, the vertex set of cubical polytopes of even dimension can be coloured in such a way that adjacent vertices get different colours and the number of vertices of the one colour equals that of the other, see Blind and Blind [14]. This means that during the shelling newly added vertices are equipped with a suitable colour. If this is not possible in some step or we get too many vertices of one colour, then the shelling will not yield a valid cubical polytope.

Furthermore, when searching for polytopes with at most 34 vertices, several of the statements used by Blind and Blind in [15] can also be applied in our case, including bounds on the vertex degrees, the simpliciality of the vertex figures, and the fact that without loss of generality we can start the shelling with one of two special complexes.

Similarly to the 2-simple, 2-simplicial case, there are also bounds on the vertex and edge hunger.

Proposition 7.3.5. Let P be a cubical 4-polytope on k facets, and a shelling given. Then for $1 \leq i < k$ we have $\mathsf{VH}_i \leq 8(k-i)$ and $\mathsf{EH}_i \leq 12(k-i)$. Furthermore, the number of 2-faces of $\partial\mathcal{C}_i$ is at most $4(k-i)+2$.

Proof. This is proved in the same way as Proposition 7.3.2, using the fact that all facets are 3-cubes with 8 vertices, 12 edges and 6 quadrilaterals as 2-faces. □

Runs of the program with the two start complexes yielded no cubical combinatorial 3-sphere with 34 vertices. While we do not count this as a proof, computer-based proofs being a very special subject anyway, it might still be an indication that Conjecture 7.3.4 could be correct.

SPHERES WITH $g_2 < 0$

For every polytope P we have $g_2(P) \geq 0$, see Theorem 1.4.3. This is still unproved for general spheres. In particular, it is an open question whether there are 3-spheres with $g_2 < 0$.

One possible application of the shelling algorithm might therefore be the quest for examples of spheres with negative g_2. We hereby restrict the search to 2-simple, 2-simplicial spheres, in which case we can use everything said in the corresponding section above. Additionally, we have the following statement about the vertex hunger.

Proposition 7.3.6. Suppose P is a 2-simple, 2-simplicial 3-sphere with $g_2(P) < 0$ and $F_1, \ldots, F_k$ a shelling sequence of P. Then for $1 \leq i < k$,

$$\sum_{j=1}^{i} f_0(F_j) + \mathsf{VH}_i \;<\; 6k - 10.$$

Proof. If P is 2-simple, 2-simplicial then $g_2(P) = f_1(P) - 4f_0(P) + 10$ (cf. Section 1.4). With Lemma 7.2.1 we get

$$g_2(P) \;=\; 2f_0(P) + f_1(P) - 6f_0(P) + 10 \;=\; f_{03}(P) - 6f_3(P) + 10 \;<\; 0$$

The vertex-facet incidences can be counted separately for the facets before step i and those after:

$$f_{03}(P) \;=\; \sum_{j=1}^{i} f_0(F_j) + \sum_{j=i+1}^{k} f_0(F_j)$$

Now the vertex hunger at step i counts the incidences of vertices that are on the surface in step i with the facets F_j, $i < j \leq k$, therefore

$$\mathsf{VH}_i \;\leq\; \sum_{j=i+1}^{k} f_0(F_j)$$

This implies

$$\sum_{j=1}^{i} f_0(F_j) + \mathsf{VH}_i \;\leq\; f_{03}(P) \;<\; 6f_3(P) - 10 \;=\; 6k - 10$$

□

The search for 3-spheres with up to 25 facets and $g_2 < 0$ did not result in any examples up to now. On the other hand, such examples might be quite large, so one might have to increase search depth considerably, resulting in very long computation times.

Bibliography

[1] A. A'Campo-Neuen, *On generalized h-vectors of rational polytopes with a symmetry of prime order*, Discrete Comput. Geom. **22**, no. 2 (1999), pp. 259–268.

[2] O. Aichholzer, *Extremal properties of* $0/1$*-polytopes of dimension* 5, in Polytopes—combinatorics and computation (Oberwolfach, 1997), DMV Sem. 29, Birkhäuser, Basel, 2000, pp. 111–130.

[3] M. Aigner, *Combinatorial theory*, Classics in Mathematics, Springer-Verlag, Berlin, 1997. Reprint of the 1979 original.

[4] A. Altshuler and I. Shemer, *Construction theorems for polytopes*, Israel J. Math. **47**, no. 2-3 (1984), pp. 99–110.

[5] D. Barnette, *A proof of the lower bound conjecture for convex polytopes*, Pacific J. Math. **46** (1973), pp. 349–354.

[6] M. M. Bayer, *The extended f-vectors of* 4*-polytopes*, J. Combin. Theory Ser. A **44**, no. 1 (1987), pp. 141–151.

[7] M. M. Bayer and L. J. Billera, *Generalized Dehn-Sommerville relations for polytopes, spheres and Eulerian partially ordered sets*, Invent. Math. **79**, no. 1 (1985), pp. 143–157.

[8] M. M. Bayer, A. M. Bruening, and J. D. Stewart, *A combinatorial study of multiplexes and ordinary polytopes*, Discrete Comput. Geom. **27**, no. 1 (2002), pp. 49–63. Geometric combinatorics (San Francisco, CA/Davis, CA, 2000).

[9] M. M. Bayer and R. Ehrenborg, *The toric h-vectors of partially ordered sets*, Trans. Amer. Math. Soc. **352**, no. 10 (2000), pp. 4515–4531 (electronic).

[10] M. M. Bayer and A. Klapper, *A new index for polytopes*, Discrete Comput. Geom. **6**, no. 1 (1991), pp. 33–47.

[11] L. J. Billera and R. Ehrenborg, *Monotonicity of the cd-index for polytopes*, Math. Z. **233**, no. 3 (2000), pp. 421–441.

[12] L. J. Billera, R. Ehrenborg, and M. Readdy, *The cd-index of zonotopes and arrangements*, in Mathematical essays in honor of Gian-Carlo Rota

(Cambridge, MA, 1996), Progr. Math. 161, Birkhäuser Boston, Boston, MA, 1998, pp. 23–40.

[13] L. J. Billera and C. W. Lee, *A proof of the sufficiency of McMullen's conditions for f-vectors of simplicial convex polytopes*, J. Combin. Theory Ser. A **31**, no. 3 (1981), pp. 237–255.

[14] G. Blind and R. Blind, *Gaps in the numbers of vertices of cubical polytopes. I*, Discrete Comput. Geom. **11**, no. 3 (1994), pp. 351–356.

[15] G. Blind and R. Blind, *Cubical* 4*-polytopes with few vertices*, Geom. Dedicata **66**, no. 2 (1997), pp. 223–231.

[16] T. Braden. personal communication and handwritten notes, 2004.

[17] A. Brøndsted, *An introduction to convex polytopes*, Graduate Texts in Mathematics 90, Springer-Verlag, New York, 1983.

[18] H. Bruggesser and P. Mani, *Shellable decompositions of cells and spheres*, Math. Scand. **29** (1971), pp. 197–205 (1972).

[19] H. S. M. Coxeter, *Regular polytopes*, Dover Publications Inc., New York, third ed., 1973.

[20] M. de Berg, M. van Kreveld, M. Overmars, and O. Schwarzkopf, *Computational geometry*, Springer-Verlag, Berlin, revised ed., 2000. Algorithms and applications.

[21] R. Diestel, *Graph theory*, Graduate Texts in Mathematics 173, Springer-Verlag, Berlin, third ed., 2005.

[22] J. Eckhoff, *Combinatorial properties of f-vectors of convex polytopes*, Normat **54**, no. 4 (2006), pp. 146–159.

[23] H. Edelsbrunner, *Algorithms in Combinatorial Geometry*, EATCS Monographs on Theoretical Computer Science 10, Springer-Verlag, Berlin, 1987.

[24] R. Ehrenborg, *Lifting inequalities for polytopes*, Adv. Math. **193**, no. 1 (2005), pp. 205–222.

[25] R. Ehrenborg and M. Readdy, *Coproducts and the cd-index*, J. Algebraic Combin. **8**, no. 3 (1998), pp. 273–299.

[26] D. Eppstein, G. Kuperberg, and G. M. Ziegler, *Fat 4-polytopes and fatter 3-spheres*, in Discrete geometry, Monogr. Textbooks Pure Appl. Math. 253, Dekker, New York, 2003, pp. 239–265.

[27] R. Freij and M. Henze, *Independence complexes of perfect graphs.* Preprint, April 2009, 16 pages.

[28] E. GAWRILOW AND M. JOSWIG, Polymake: *A framework for analyzing convex polytopes*, in Polytopes — Combinatorics and Computation, G. Kalai and G. M. Ziegler, eds., DMV Seminar 29, Birkhäuser-Verlag, Basel, 2000, pp. 43–73.

[29] B. GRÜNBAUM, *Convex polytopes*, Graduate Texts in Mathematics 221, Springer-Verlag, New York, second ed., 2003. Prepared and with a preface by Volker Kaibel, Victor Klee and Günter M. Ziegler.

[30] O. HANNER, *Intersections of translates of convex bodies*, Math. Scand. **4** (1956), pp. 67–89.

[31] A. B. HANSEN, *On a certain class of polytopes associated with independence systems*, Math. Scand. **41**, no. 2 (1977), pp. 225–241.

[32] M. JOSWIG, *Beneath-and-beyond revisited*, in Algebra, geometry, and software systems, Springer, Berlin, 2003, pp. 1–21.

[33] M. JOSWIG AND G. M. ZIEGLER, *Neighborly cubical polytopes*, Discrete Comput. Geom. **24**, no. 2-3 (2000), pp. 325–344. The Branko Grünbaum birthday issue.

[34] D. JUNGNICKEL, *Graphs, networks and algorithms*, Algorithms and Computation in Mathematics 5, Springer, Berlin, third ed., 2008.

[35] G. KALAI, *Rigidity and the lower bound theorem. I*, Invent. Math. **88**, no. 1 (1987), pp. 125–151.

[36] G. KALAI, *A new basis of polytopes*, J. Combin. Theory Ser. A **49**, no. 2 (1988), pp. 191–209.

[37] G. KALAI, *The number of faces of centrally-symmetric polytopes*, Graphs Combin. **5** (1989), pp. 389–391.

[38] K. KARU, *Hard Lefschetz theorem for nonrational polytopes*, Invent. Math. **157**, no. 2 (2004), pp. 419–447.

[39] V. KLEE, *On the number of vertices of a convex polytope*, Canad. J. Math. **16** (1964), pp. 701–720.

[40] B. MCKAY, nauty *homepage:* http://cs.anu.edu.au/~bdm/nauty/.

[41] P. MCMULLEN, *The numbers of faces of simplicial polytopes*, Israel J. Math. **9** (1971), pp. 559–570.

[42] P. MCMULLEN, *On simple polytopes*, Invent. Math. **113**, no. 2 (1993), pp. 419–444.

[43] A. PAFFENHOLZ AND A. WERNER, *Constructions for 4-polytopes and the cone of flag vectors*, in Algebraic and geometric combinatorics, Contemp. Math. 423, Amer. Math. Soc., Providence, RI, 2006, pp. 283–303.

[44] A. PAFFENHOLZ AND G. M. ZIEGLER, *The E_t-construction for lattices, spheres and polytopes*, Discrete Comput. Geom. **32**, no. 4 (2004), pp. 601–621.

[45] M. PURTILL, *André permutations, lexicographic shellability and the cd-index of a convex polytope*, Trans. Amer. Math. Soc. **338**, no. 1 (1993), pp. 77–104.

[46] R. SANYAL, A. WERNER, AND G. M. ZIEGLER, *On Kalai's conjectures about centrally symmetric polytopes*, Discrete Comput. Geometry **41** (2009), pp. 183–198. published online September 3, 2008; `http://arxiv.org/abs/0708.3661[arxiv.org]`.

[47] R. SANYAL AND G. M. ZIEGLER, *Construction and analysis of projected deformed products.* Preprint `http://arxiv.org/abs/0710.2162`.

[48] G. C. SHEPHARD, *Sections and projections of convex polytopes*, Mathematika **19** (1972), pp. 144–162.

[49] R. STANLEY, *Generalized H-vectors, intersection cohomology of toric varieties, and related results*, in Commutative algebra and combinatorics (Kyoto, 1985), Adv. Stud. Pure Math. 11, North-Holland, Amsterdam, 1987, pp. 187–213.

[50] R. P. STANLEY, *The number of faces of a simplicial convex polytope*, Adv. in Math. **35**, no. 3 (1980), pp. 236–238.

[51] R. P. STANLEY, *On the number of faces of centrally-symmetric simplicial polytopes*, Graphs Combin. **3**, no. 1 (1987), pp. 55–66.

[52] R. P. STANLEY, *Flag f-vectors and the cd-index*, Math. Z. **216**, no. 3 (1994), pp. 483–499.

[53] R. P. STANLEY, *Enumerative combinatorics. Vol. 1*, Cambridge Studies in Advanced Mathematics 49, Cambridge University Press, Cambridge, 1997. With a foreword by Gian-Carlo Rota, Corrected reprint of the 1986 original.

[54] E. STEINITZ, *Über die Eulerschen Polyederrelationen*, Archiv für Mathematik und Physik **11** (1906), pp. 86–88.

[55] C. STENSON, *Relationships among flag f-vector inequalities for polytopes*, Discrete Comput. Geom. **31**, no. 2 (2004), pp. 257–273.

[56] C. STENSON, *Families of tight inequalities for polytopes*, Discrete Comput. Geom. **34**, no. 3 (2005), pp. 507–521.

[57] T. TAO, *Open question: the Mahler conjecture on convex bodies*. Blog page started March 8, 2007, http://terrytao.wordpress.com/2007/03/08/open-problem-the-mahler-conjecture-on-convex-bodies/.

[58] G. M. ZIEGLER, *Lectures on polytopes*, Graduate Texts in Mathematics 152, Springer-Verlag, New York, 1995.

[59] G. M. ZIEGLER, *Face numbers of 4-polytopes and 3-spheres*, in Proceedings of the International Congress of Mathematicians, Vol. III (Beijing, 2002), Beijing, 2002, Higher Ed. Press, pp. 625–634.

[60] G. M. ZIEGLER, *Projected products of polygons*, Electron. Res. Announc. Amer. Math. Soc. **10** (2004), pp. 122–134 (electronic).

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Symbol Index

$*$	convolution of linear functionals (p. 27)
$*$	join of polytopes (p. 10)
$[d]$	the set $\{0, \ldots, d-1\}$ (p. 19)
$[x, y]$	interval between elements x and y in a poset (p. 16)
Ω_d	Bayer-Billera basis in dimension d (p. 54)
Θ_d	Kalai basis in dimension d (p. 56)
$\alpha(P)$	linear functional α evaluated on the flag vector of P (p. 58)
$\langle . \mid . \rangle$	bilinear form on $\mathbb{cd}$-monomials (p. 24)
$\Psi(P)$	$\mathbb{cd}$-index of P (p. 23)
$\tilde{\Delta}(k)$	central hypersimplex of dimension $2k-1$ (p. 142)
χ_I	characteristic vector of the independent set I (p. 149)
$\cong$	combinatorial equivalence (p. 17)
$\#$	connected sum of polytopes (p. 11)
b^*	dual sequence to b (p. 95)
$S^\diamond$	dual poset of S (p. 16)
$\Delta_d(k)$	d-dimensional hypersimplex with parameter k (p. 10)
$\preccurlyeq$	partial order on $\mathbf{B}_d$ (p. 57)
$\preccurlyeq$	partial order on $\mathbf{T}_d$ (p. 93)
μ_L	Möbius function of the poset L (p. 17)
$\pentagon_n$	n-gon (p. 9)
ϕ	bijective mapping between $\mathbf{B}_d$ and $\mathbf{T}_d$ (p. 92)
P^Δ	polar polytope of P (p. 14)

$\times$	product of polytopes (p. 10)
$\oplus$	sum of polytopes (p. 10)
Δ_d	d-dimensional simplex (p. 9)
Ψ_d	sparse subsets of $[d]$ (p. 22)
$\vee$	join of poset elements (p. 16)
$\wedge$	meet of poset elements (p. 16)
$\hat{0}$	minimal element of a poset (p. 16)
$\hat{1}$	maximal element of a poset (p. 16)
$\mathsf{aff}\, M$	affine hull of a set M (p. 7)
$\mathcal{A}(P)$	hyperplane arrangement defined by all facet hyperplanes of P (p. 127)
$\mathsf{apf}_S\, P$	additional pyramid flags (p. 21)
$\mathsf{bipyr}\, P$	bipyramid over P (p. 11)
$\mathbf{B}_d$	set of admissible sequences of dimension d (p. 56)
$\mathcal{C}(s, P)$	set of all s-chains of P (p. 95)
$\mathsf{conv}\, M$	convex hull of a set M (p. 7)
$C_d{}^{\Delta}$	d-dimensional crosspolytope (p. 9)
C_d	d-dimensional cube (p. 9)
$\mathcal{C}_d(n)$	cyclic d-polytope on n vertices (p. 10)
C_d^n	neighbourly cubical d-polytope on 2^n vertices (p. 85)
$\check{C}_d^n$	broken neighbourly cubical d-polytope on 2^n vertices (p. 86)
$\mathsf{dim}\, P$	dimension of a polytope P (p. 8)
EH_i	edge hunger of a complex in shelling step i (p. 168)
$\mathsf{EH}_i(e)$	edge hunger of edge e in shelling step i (p. 168)
$\mathbf{e}_i$	i-th unit vector (p. 9)
$\mathsf{F}(P)$	capped prism over P (p. 101)
$\mathcal{FC}_d$	convex hull of $\mathcal{F}_d$ (p. 57)

$\mathsf{fdeg}_i(G)$	facet degree of G in shelling step i (p. 167)
F_d	d-th Fibonacci number (p. 22)
$\mathbf{f}(P)$	reduced flag vector of P (p. 22)
$\mathcal{F}\ell\mathcal{C}_d$	convex hull of $\mathcal{F}\ell_d$ (p. 57)
$\mathcal{F}\ell_d$	set of all flag vectors of d-polytopes (p. 22)
$\mathcal{F}_d$	set of all f-vectors of d-polytopes (p. 12)
$\widetilde{\mathbf{f}}(P)$	relative reduced flag vector of P (p. 54)
$f(P)$	f-vector of P (p. 12)
$f_i(P)$	number of i-dimensional faces of P (p. 12)
$f_S(P)$	entry of the flag vector of P corresponding to the set S (p. 20)
$g(P)$	(toric) g-vector of P (p. 18)
$G[b]$	g-vector convolution associated to sequence b (p. 57)
$\mathsf{ht}\, v$	height of a vertex of C_n (p. 85)
$H(G)$	Hansen polytope of the graph G (p. 149)
$h(P)$	(generalised) h-vector of P (p. 18)
$\ell(b)$	lower index sequence of b (p. 95)
ℓ_i	i-th ray of the flag vector cone of 4-polytopes (p. 71)
$L(P)$	face lattice of the polytope P (p. 16)
$\mathsf{prism}\, P$	prism over P (p. 10)
$\mathsf{pyr}\, P$	pyramid over P (p. 11)
$\mathsf{PS}^F_{\mathcal{F},\mathcal{N}}(P)$	pseudostacked polytope (p. 34)
$P[b]$	basis polytope corresponding to sequence b (p. 56)
$\mathsf{rank}\, x$	rank of the element x in a poset (p. 16)
$s(b)$	sum sequence of b (p. 95)
$\mathbf{T}_d$	set of all Braden sequencee of dimension d (p. 92)
$\mathsf{vert}\, P$	set of vertices of a polytope P (p. 8)
VH_i	vertex hunger of a complex in shelling step i (p. 168)
$\mathsf{VH}_i(v)$	vertex hunger of vertex v in shelling step i (p. 168)