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AT-free, coAT-free Graphs and AT-free Posets

(Extended Abstract)

by

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1 Introduction

An asteroidal triple of a graph is an independent triple of vertices such that any pair of those vertices is joined by a path that does not contain any of the vertices in the neighborhood of the third vertex. Graphs not containing an asteroidal triple are called AT-free graphs.

In this paper we consider both structural and algorithmic properties of two subclasses of AT-free graphs. In the first part we examine AT-free, coAT-free graphs, i.e., neither the graph nor its complement contains an AT. For this family of graphs we study the number of minimal separators and show that this number is bounded by a polynomial in the number of vertices of the graph. As a corollary we get polynomial time algorithms for a variety of problems, when restricted to AT-free, coAT-free graphs.

The second part of the paper is devoted to a somewhat more restricted family graphs, or rather a family of posets, the AT-free posets. This family is defined to be the class of posets with an AT-free comparability graph. AT-free posets seem to be a bit awkward at first, but nevertheless, there is a strong motivation to study them: The class of permutation graphs, which are characterized as the graphs that are both comparability and cocomparability, and even more the corresponding posets enjoy a lot of interest since the posets whose comparability graph is a permutation graph are exactly the posets of dimension less than or equal to two. Hence, it is a reasonable question to ask, whether the nice properties of permutation graphs are preserved when generalizing from cocomparability graphs to AT-free graphs, or in other words, when we require the complement of a comparability graph to be coAT-free instead of being a comparability graph. In this second section we will look at this kind of poset a little closer and establish different structural properties of AT-free posets. In particular we study the relationship between dominating pair vertices and extremal elements. Furthermore we introduce an ordering for posets that visualizes the overall structure of AT-free posets.

2 Minimal Separators in AT-Free and coAT-Free Graphs

The common definition of an efficient algorithm is usually done by requiring the running time of the algorithm in question to be bounded by a polynomial in the size of the input. For problems that are known to be hard, several weaker requirements have been introduced. One of those weakenings is to use, besides the size of the input, an additional, possibly exponential parameter to bound the running time. A parameter that turned out to be useful especially for certain completion problems, as treewidth and minimum fill-in, is the number of minimal separators of the given graph. Recall, that in a connected graph $G = (V, E)$ a set of vertices $S \subseteq V$ is called a minimal separator of G if there are vertices $a, b \in V$ such that a and b are in different connected components of $G \setminus S$ and no subset of S separates a and b. Although, already for very restricted families of graphs, as for example cobipartite graphs, the number of minimal separators can be exponential in the number of vertices of the graph, there are several graph classes where this value can be bounded by a polynomial. For instance chordal graphs, permutation graphs and weakly triangulated graphs share this property.

As it turns out, for graphs which are both AT-free and coAT-free a similar result can be shown to be true as well. By observing that all graphs with $\alpha(G) \leq 2$ and $\omega(G) \leq 2$ have at most 5 vertices and by applying that a separator S of G is minimal if and only of there are at least two full components in $G \setminus S$ we can prove the following theorem (the proofs of lemmata and theorems are omitted in this extended abstract because of space restrictions).

Theorem 2.1 *Let G be an AT-free and coAT-free graph. Then the number of minimal separators of G is polynomial in the size of G.*

Kloks et al. [KKS97, KMW98] designed efficient algorithms for the problems treewidth, pathwidth, minimum fill-in, minimum interval graph completion and vertex ranking when the input is restricted to AT-free graphs where the number of minimal separators in the graph is polynomially bounded by the number of its vertices.

Theorem 2.2 ([KKS97, KMW98]) *There are $O(n^5 R + n^3 R^3)$ time algorithms computing treewidth, pathwidth, minimum fill-in, minimum interval graph completion, and vertex ranking of a given AT-free graph G, where R is the number of minimal separators of G.*

Using Theorem 2.1, we get the following corollary.

Corollary 2.3 *There are polynomial time algorithms computing treewidth, pathwidth, minimum fill-in, minimum interval graph completion and vertex ranking of a given AT-free and coAT-free graph.*

As Kloks showed [Klo97], the number of minimal separators in a weakly chordal graph is bounded by $O(\overline{m})$ (see also [BT99]). It seems to be an interesting question, whether this property is valid also for graphs containing larger induced cycles. As the following observation shows, this is not possible.

Observation 2.4 *Let $\mathcal{G}$ be the family of graphs, that do neither contain an induced cycle with more than four vertices nor an induced complement of a cycle with more than six vertices. Then the number of minimal separators of graphs in $\mathcal{G}$ cannot be bounded polynomially.*

This observation suggests that the result of Theorem 2.1 is hardly extendible to a larger graph class since the graphs constructed in the proof of Observation 2.4 are AT-free and their complements do not contain any induced cycle with more than six vertices. However, there is a generalization, by using the asteroidal number of the given graph and of its complement. The asteroidal number of a graph G, $\mathrm{an}(\overline{G})$, is defined to be the maximum cardinality of an independent set of vertices S, such that for each $s \in S$ all vertices of $S \setminus \{s\}$ are contained in the same connected component of $G \setminus \mathrm{N}[s]$. For proving this generalization we use a similar method as for the proof of Theorem 2.1.

Theorem 2.5 *Let G be a graph such that there are constants $k, l \in \mathbb{N}$ with $\mathrm{an}(G) < k$ and $\mathrm{an}(\overline{G}) < l$, then the number of minimal separators of G is polynomial in the size of G.*

As a corollary of this theorem we get a bound on the number of minimal separators of a given graph. To the best of our knowledge, no upper bound for this parameter has been known before.

Corollary 2.6 *The number of minimal separators of a graph G is bounded by $O(n^c)$, where $c = 2(\mathrm{r}(\mathrm{an}(\overline{G})+1, \mathrm{an}(G)) - 1)$ and $\mathrm{r}(\cdot,\cdot)$ is the Ramsey number.*

For AT-free graphs we know that the number of minimal separators can be quite large. By Corollary 2.6, we can give a bound on this number using the asteroidal number of the complementary graph.

Corollary 2.7 *An AT-free graph G has no more than $O(n^{2\,\mathrm{an}(\overline{G})})$ minimal separators.*

Finally, we can draw a corollary stating that for AT-free graphs which are complements of graphs with bounded asteroidal number the above mentioned problems can also be solved in polynomial time.

Corollary 2.8 *There are $O(n^{6\,\mathrm{an}(\overline{G})+3})$ time algorithms for computing treewidth, pathwidth, minimum fill-in, minimum interval graph completion, and the vertex ranking number of a given AT-free graph.*

REMARK. Determining the asteroidal number is an NP-complete problem, as shown by Kloks et al. [KKM97]. Even worse, when the problem is restricted to coAT-free graphs it remains NP-complete as the reduction used in [KKM97] is done on planar, 3-regular, triangle-free graphs and, of course, every triangle-free graph is coAT-free.

Note, however, that for applying the above algorithms it is not necessary to know how large the asteroidal number of the graph and its complement is. Only for giving a bound on the running time of the algorithm this information is needed.

3 AT-Free Posets

As in graph theory an important topic is the investigation of graph classes which have some interesting property, there is a similar interest in order theory to investigate posets which have certain properties (e.g. interval orders, cycle-free posets or planar posets). Since we are interested in asteroidal triples, we study the class of *AT-free posets*. An *AT-free poset* is a poset with an AT-free comparability graph.

For reasons of simplicity we will use the concepts *poset* and *graph* interchangeably. So if we consider an AT-free poset G we usually examine the underlying AT-free comparability graph together with an arbitrary but fixed transitive orientation. We do not restrict ourself to the edges of the Hasse diagram but instead always consider the transitive edges too.

A dominating pair of a graph is a pair of vertices, such that any path connecting those two vertices is a dominating set of the graph. A vertex x is called a dominating pair vertex if there is a vertex y such that (x, y) is a dominating pair. Corneil et al.[COS97] proved that every AT-free graph contains a dominating pair. Even stronger, for AT-free graphs with diameter at least 4 they showed that there exist two nonempty, disjoint sets of vertices X, Y, such that a pair (x, y) of vertices is a dominating pair if and only if $x \in X, y \in Y$. Consequently, dominating pair vertices form some kind of polar points in an AT-free graph and visualize nicely what is meant by the notion of linear structure of AT-free graphs. When working with partially ordered sets an equally important notion as dominating pair vertices for AT-free graphs is that of an extremal element, i.e., an element that is either a minimal or a maximal element. This leads naturally to the question, whether there is any relationship between these concepts and whether there are dominating pairs that consist of minimal and/or maximal elements in an AT-free poset. At the beginning we consider the restricted case of posets of diameter 2.

Lemma 3.1 *Let G be a connected AT-free poset with* $\operatorname{diam}(G) = 2$. *Then there is a dominating pair (a, b) in G such that a is a maximal and b is a minimal element of G.*

To get a similar result about dominating pairs in AT-free posets in general, we make use of a properties of LBFS orderings in AT-free graphs (see [COS99] for details on two-sweep LBFS).

Lemma 3.2 *Let G be an AT-free poset. If (x, y) is a dominating pair of G, found by a two-sweep LBFS and neither x nor y is an extremal element, then there is a minimal element m comparable to y, such that (x, m) is a dominating pair of G.*

The corresponding algorithmic result is the following.

Lemma 3.3 *Let G be a connected AT-free poset and let (x, y) be a dominating pair of G found by a two-sweep LBFS. There is an $O(n + m)$ algorithm that finds a minimal vertex m such that (x, m) forms a dominating pair of G.*

Using a result of Corneil et al.[COS97] on dominating pair vertices in AT-free graphs of large diameter we get the general result.

Theorem 3.4 *Let G be a connected AT-free poset. Then there is a dominating pair (x, y), such that x is minimal and y is maximal.*

To study the overall structure of AT-free posets we now define the concept of a zigzag ordering. A *zigzag ordering* of a poset is a partition of the vertices into layers $L_1, L_2, \dots, L_k$ such that

(i) There are only edges inside a layer or between consecutive layers.
(ii) All edges between two consecutive layers are oriented in the same direction.
(iii) If $k > 1$ then every vertex of L_1 has at least one neighbor in L_2.
(iv) There is a unique element x that is minimal or maximal for L_1; for $i > 1$ every vertex in layer L_i has at least one neighbor in L_{i-1}.

Consequently, the layers alternate between being source and sink layers, when considering only edges between the layers. It is not hard to show that every connected poset has a zigzag ordering. The special structure of AT-free graphs allows to consider a more restricted zigzag ordering. We call a zigzag ordering an *admissible* zigzag ordering if there is no $\mathbf{2} + \mathbf{2}$ consisting of edges between consecutive layers, where a $\mathbf{2} + \mathbf{2}$ is an induced suborder consisting of two disjoint pairs of comparable elements which share no further comparabilities.

Lemma 3.5 *Every connected AT-free poset G has an admissible zigzag ordering.*

Theorem 3.6 *For every bipartite poset G which has an admissible zigzag ordering, both the interval dimension and the semiorder dimension of G is at most 2.*

A simple corollary of Lemma 3.5 and Theorem 3.6 is of course, that every bipartite AT-free poset has interval dimension less than or equal to two. This can also be proved by the fact that every bipartite AT-free graph is a bipartite permutation graph.

AT-free posets are generalizations of posets of dimension less than or equal to two. It is quite obvious that this generalization does not maintain the dimension: From Gallai [Gal67] it follows that every AT-free poset with dimension larger than two has to contain the complement a graph out of four infinite families as induced subgraph. It seems quite natural to ask what properties AT-free posets have with respect to their interval dimension. As shown by Lekkerkerker and Boland [LB62], there is a close relationship between AT-free graphs and interval orders—though they considered the complementary graph. Theorem 3.6 suggests that there could be a similarly close relationship for AT-free posets and posets of interval dimension less than or equal to two. Of course, not all posets of interval dimension one are AT-free posets. More precisely, an interval order is AT-free if and only if the corresponding comparability graph does not contain any of three special graphs as induced subgraph. Hence, all we can hope for is to show that the interval dimension of AT-free posets is bounded, maybe

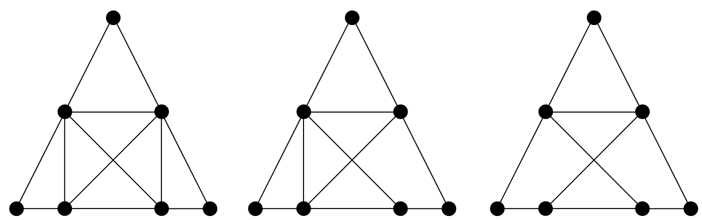

Figure 1: Forbidden subgraphs for AT-free interval orders.

by a constant. Unfortunately, we could not prove any such bound yet, but we make the following conjecture.

Conjecture 3.7 *The interval dimension of AT-free posets is not greater than two. In other words, every coAT-free cocomparability graph is a trapezoid graph .*

Finally we would like to mention an observation concerning a closure operation that maintains AT-freeness. Möhring showed that every minimal triangulation of an AT-free graph is AT-free again[Möh96]. Hence, the "minimal chordal closure" does not destroy the property to be AT-free. As the next result shows, also the transitive closure does maintain the property of a given graph to be AT-free; thus also the comparability graph of an AT-free Hasse-diagram is AT-free.

Theorem 3.8 *Let $G = (V, E)$ be an AT-free graph with an acyclic orientation F. Then the underlying graph G' of the transitive closure of F is AT-free again.*

Remark. There are other graph modifying operations, that maintain the property of a graph to be AT-free as well. One example is the contraction of an arbitrary connected induced subgraph.

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