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Representing polyhedra by few polynomial inequalities

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Abstract

This work presents results in the field of real algebraic geometry as well as in the theory on polyhedra. The main result is that every d-dimensional polyhedron can be described by at most $2d$ polynomial inequalities and, moreover, an explicit construction for these polynomials is provided.

It is also shown that for any d-dimensional pointed polyhedral cone there is a description using $(2d-2)$ polynomial inequalities, and that for bounded polyhedra there is a representation involving $2d-1$ polynomial inequalities. A construction for the necessary polynomials is provided for both cases.

In each case, the number of polynomials constructed is close to the lower bound: To represent a d-dimensional polyhedron containing a vertex, at least d polynomial inequalities are needed.

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> "Without friends no one would choose to
> live, though he had all other goods."
> Aristotle

Thank You.

Contents

1. Introduction

The central result of this thesis is that any Polyhedron in $\mathbb{R}^d$ can be described using $2d$ polynomial inequalities, and that these can moreover be constructed.

A deep result in semi-algebraic geometry by Bröcker and Scheiderer ([Brö91], [Sch89]), states that any closed semi-algebraic set of dimension d can be described by at most $d(d+1)/2$ polynomial inequalities. Surprisingly, the number of polynomials necessary is completely independent of the geometric or algebraic complexity of the considered set. Unfortunately, all known proofs of this result are non-constructive.

A class of semi-algebraic sets crucial to optimization are polyhedra. This thesis presents a constructive proof for the theorem of Bröcker and Scheiderer in the case of polyhedra, providing an appropriate algorithm. Moreover in this case the upper bound on the number of polynomials necessary is also reduced from $d(d+1)/2$ to $2d$.

1.1 Background and motivation

The power of linear programming, one of today's most important optimization techniques, is - to a large extent - based on deep insights into the interplay between the geometry and the algebraic description of polyhedra.

A linear program can be interpreted as the task to optimize a linear objective function restricted to a polyhedron. Consequently all known algorithms solving linear programs incorporate some geometric properties of polyhedra, and their successful implementations use special analytic or algebraic representations of polyhedra: The simplex-algorithm is based on an efficient matrix-representation of vertices, inner-point-methods use analytic centers or the central path in polyhedra.

A special challenge in combinatorial optimization is the complexity of linear programs, arising from combinatorial problems. Often the number of facets of the involved polyhedra is exponential in their dimension, and with this the number of linear inequalities used to describe them. There are techniques that deal with this efficiently – the ellipsoid-method in theory and in praxis the simplex-algorithm together with cutting plane techniques. Nevertheless, there is still potential for improvement here, subsequently leading to an interest in fundamentally new procedures.

In principle, the result of Bröcker and Scheiderer offers the possibility to describe any d dimensional polyhedron by $d(d+1)/2$ polynomial inequalities, even though its number of facets might be exponentially in d. Unfortunately, all known proofs of the corresponding result are non-constructive. In contrast, this thesis presents an algorithm resulting in a description using $2d$ polynomial inequalities which describe a given d-dimensional polyhedron.

Thus combinatorial optimization problems could be reformulated as the task to maximize a linear objective function over a system of *few* polynomial inequalities. The use of methods of nonlinear optimization could then lead to efficient numeric treatment of the resulting polynomials.

Related Work

The ideas used in the construction presented in this thesis, evolve from earlier approaches for the description of *special* polyhedra by means of polynomial inequalities:

In [Brö91], Example 2.10, or in [ABR96], Example 4.7, a description for a regular convex n-gon in $\mathbb{R}^2$ using two polynomials is given. This result was generalized to arbitrary convex polygons by vom Hofe in [Hof91] using three polynomial inequalities, which was improved by Bernig showing in [Ber98] that any convex polygon may be described using two polynomial inequalities. Grötschel and Henk give a constructive description in [GH03] of simple convex polytopes of any dimension d involving $O(d^d)$ polynomials.

The presented approach is the first to examine polyhedra without additional restrictions, the results have been published in advance in [BGH04]. A refined version of the corresponding approach is presented in this work.

1.2 Main Theorem

In this paper we will show the following:

Theorem 1.2.1 (Main Theorem). *Any d-dimensional convex polyhedron can be described using at most $2d$ polynomial inequalities.*
These polynomials can be constructed using basic techniques such as linear optimization.

The **explicit construction** for the necessary polynomials is provided in Chapter 3, leading to Algorithms 3.6.1 and 3.6.4 (pg. 61 and 65).

From the viewpoint of Real Algebraic Geometry the given thesis presents a constructive proof for a stricter version of the results of BRÖCKER and SCHEIDERER on a subclass of basic closed semi-algebraic sets in $\mathbb{R}^d$, specifically those defined by linear polynomials.

A corollary from the presented construction is that the the interior of a given d-dimensional polyhedron can be described using $2d$ *strict* polynomial inequalities. Rephrased, for any open semi-algebraic set in $\mathbb{R}^d$ defined by linear polynomials, this work provides a description using $2d$-many polynomial inequalities.

1.3 This Thesis is organized as follows

The main ambition of this work is to prove the Main Theorem and therefore to present a construction for the necessary polynomials. This is done in three steps.

The first chapter introduces the used notation and reduces the proof to the case of polyhedral cones. Chapter 2 reviews the combinatorial and geometrical concepts hidden in the actual construction. This is followed by the main work of this thesis, the algebraic proof for the Main Theorem: Chapter 3 presents an algorithm for the calculation of the polynomials describing a given input-polyhedron which is studied on an Example.

The last two chapters examine the construction presented in the first three chapters. In Chapter 4 other possible approaches are presented and discussed. The final chapter concludes this thesis by a review on the given construction, the algorithms and resulting implications.

The first part of this Thesis is important to the conception of the whole work: The geometric interplay between the objects created is a key to the understanding of the overall algebraic construction.

1.3.1 The concept

In anticipation of the following definitions the concept of this work can be sketched as follows:

The main principle is to replace linear polynomials used in the description of a given polyhedron P by their product. Unavoidably this results in a set containing additional points, these are then removed by introducing new polynomials to the resulting semi-algebraic set. Special properties of the introduced polynomials ensure that they again can be replaced by their products – but without changing the corresponding set.

In brief the following holds: For each face F of P two polynomials $\mathfrak{p}_F$ and $\mathfrak{q}_F$ are constructed, such that the semi-algebraic set defined by all these polynomials is P. For each dimension $k = 0, \ldots, d$ all polynomials $\mathfrak{p}_F$ can be replaced by their product without changing the set. The same holds for the polynomials $\mathfrak{q}_F$. This results in $2d$ polynomials describing P. The key to this is the geometric property of each semi-algebraic set defined by two polynomials $\mathfrak{p}_F, \mathfrak{q}_F$.

1.4 Notation and Definitions

1.4.1 Semi-algebraic sets

In the following we denote basic closed semi-algebraic sets (algebraic varieties) based on polynomials $\mathfrak{f}_1, \ldots, \mathfrak{f}_m \in \mathbb{R}^n[X]$ in the following way:

$$\begin{aligned}
\mathcal{S}^{\geq}(\mathfrak{f}_1, \ldots, \mathfrak{f}_m) &:= \{x \in \mathbb{R}^n \,:\, \mathfrak{f}_1(x) \geq 0, \ldots, \mathfrak{f}_m \geq 0\}, \\
\mathcal{S}^{>}(\mathfrak{f}_1, \ldots, \mathfrak{f}_m) &:= \{x \in \mathbb{R}^n \,:\, \mathfrak{f}_1(x) > 0, \ldots, \mathfrak{f}_m > 0\}, \quad \text{and} \\
\mathcal{S}^{=}(\mathfrak{f}_1, \ldots, \mathfrak{f}_m) &:= \{x \in \mathbb{R}^n \,:\, \mathfrak{f}_1(x) = 0, \ldots, \mathfrak{f}_m = 0\} \quad \text{(a variety)}.
\end{aligned}$$

The *algebraic variety* $\mathcal{S}^{=}(\mathfrak{f}_1, \ldots, \mathfrak{f}_m)$ is called the *Zariski-closure* of the first two semi-algebraic sets. If a polynomial $\mathfrak{f}$ is non-positive in some point $x \in \mathbb{R}^d$, this can be expressed by $-\mathfrak{f}(x) \geq 0$, and so the corresponding semi-algebraic set of all such x is $\mathcal{S}^{\geq}(-\mathfrak{f})$. The central idea of this work is to replace polynomials in a semi-algebraic set $\mathcal{S}^{\geq}(\mathfrak{f}_1, \ldots, \mathfrak{f}_m)$ by their product, meaning the set $\mathcal{S}^{\geq}(\mathfrak{f}_1 * \ldots * \mathfrak{f}_m)$ is created, where $\mathfrak{f} * \mathfrak{g}$ denotes the product of the polynomials $\mathfrak{f}$ and $\mathfrak{g}$.

1.4.2 Closed cones

This work uses special properties of closed cones:

Definition 1.4.1. *A cone $\mathcal{C} \subseteq \mathbb{R}^d$ is a set for which the following holds: With $y \in \mathcal{C}$ one has $\lambda y \in \mathcal{C}$ for every $\lambda \geq 0$. $\mathcal{C}$ is called* pointed *if it contains no linear space as a subset.*

In this work *closed* is used in the topological sense: A cone $\mathcal{C}$ is closed, when its complement is open, i.e., for each $x \notin \mathcal{C}$ there is an ε-ball wrapping x which is also not contained in $\mathcal{C}$.

We will be working both with cones which are semi-algebraic sets – called *semi-algebraic cones*, as well as with *polyhedral* cones, these are cones, which are polyhedra:

1.4.3 Convex polyhedra and their faces

For a broad characterization of polyhedra we would like to refer the interested reader to [Zie98]. The only difference of this work to the general usage of terminology is the fact that here polyhedra are defined by linear polynomials which are *positive* on the set.

Convex polyhedra

A subset $H \subseteq \mathbb{R}^d$ is called *half-space*, if there is a non-zero vector $a \in \mathbb{R}^d$ and some $b \in \mathbb{R}$ such that

$$H = \{x \in \mathbb{R}^d \,:\, a \cdot x \geq b_i\}.$$

The corresponding Zariski-closure $\{x \in \mathbb{R}^d \,:\, a \cdot x = b_i\}$ is called a hyper-plane. A **convex polyhedron** is the intersection of finitely many half-spaces

$$P = \{x \in \mathbb{R}^d \,:\, a_1 \cdot x + b_1 \geq 0, \ldots, a_m \cdot x + b_m \geq 0\},$$

it is a semi-algebraic set defined on linear polynomials, where $a_i \in \mathbb{R}^d$ and $b_i \in \mathbb{R}$ for $i = 1, \ldots, m$. This is the *half-space- or $\mathcal{H}$-representation* of P. Opposed to this, P can also be described as the Minkowski-sum of convex combinations of vertices, a cone and a linear space:

$$P = \mathrm{conv}(V) + \mathrm{cone}(W) + \mathrm{lin}(U).$$

This is the *$\mathcal{V}$-representation* of P, where $\mathrm{conv}(V)$ is the convex combination of a finite set of vectors V and $\mathrm{cone}(W)$ is the conic hull of a finite set of vectors W.

The lineality-set

The lineality set of a polyhedron $P \subseteq \mathbb{R}^d$ is the maximal linear space V, such that the Minkowski-sum of both results in the original set, i.e., such that $P = P + V$ holds. A *bounded* polyhedron is called a ***polytope***.

Faces of polyhedra and support-vectors

The procedure outlined in this work makes use of the combinatorics of faces of polyhedra. Let $P \subseteq \mathbb{R}^d$ be a polyhedron. A set $F \subseteq P$ is called a ***face*** of P, if there is a linear polynomial $a \cdot x + b$ with $a \in \mathbb{R}^d$ and $b \in \mathbb{R}$ such that

$$F = \{x \in \mathbb{R}^d \,:\, a \cdot x + b = 0\} \quad \text{and} \quad P \subseteq \{x \in \mathbb{R}^d \,:\, a \cdot x + b \geq 0\}$$

The corresponding hyper-plane $\{x \in \mathbb{R}^d \,:\, a \cdot x + b = 0\}$ is called a ***supporting*** hyper-plane of F, the corresponding vector a a ***support-vector*** of F. The whole is rephrased by stating that the linear inequality $a \cdot x + b \geq 0$ is ***incident*** to F. The experienced reader should note that the *support-vectors used in this work point **into*** the polyhedron. This concept of defining faces will be generalized leading to the definition of support-polynomials.

By $[F] \subseteq \{1, \ldots, m\}$ we denote the set of all indices of inequalities $a_i \cdot x + b_i \geq 0$ incident to F, i.e., for each $i \in [F]$ and each $x \in F$ one has $a_i \cdot x - b_i = 0$.

The ***dimension*** $\dim(F)$ of F is the minimum of dimensions of affine spaces containing F. The face F is called ***proper***, if both $F \neq P$ and $F \neq \emptyset$ hold. Both sets, P itself and $\emptyset$ are faces of P, one defines $\dim(\emptyset) := -1$. With $\mathscr{F}(P)$ we denote the set of all *proper* faces of P, with $\mathscr{F}_k(P)$ the set of all k-faces, faces of dimension k. A face of dimension $d-1$ is called a ***facet*** of P.

Polyhedral cones are given in irredundant $\mathcal{H}$-description

A cone C that is a polyhedron is called *polyhedral cone.* The $\mathcal{H}$-representation of such sets has the form $C = \{x \in \mathbb{R}^d \,:\, (a_1 \cdot x) \geq 0, \ldots, (a_m \cdot x) \geq 0\}$ (cf. [Zie98]). This representation is called ***irredundant*** if each polynomial $(a_i \cdot x) \geq 0$ is incident to a *different* facet $F_i \in \mathscr{F}_{d-1}(C)$, i.e., $F_i \neq F_j$ holds for pairs $i \neq j$. In the following we assume that the description of any given polyhedral cone is irredundant.

1.4.4 This work uses support-polynomials which form cones

The polynomials used in this work have two important properties. The first links them to a face of the given polyhedron, which helps exploiting the combinatorial properties of faces of polyhedra. This causes the d in the $2d$. The second is of "geometrical nature" and ensures the 2 in $2d$.

All used polynomials are support-polynomials

Definition 1.4.2. *Given a face $F \in \mathscr{F}(P)$, any (not necessary linear) polynomial $\mathfrak{f}_F$ fulfilling*

$$\text{i) } P \subseteq \mathcal{S}^{\geq}(\mathfrak{f}_F) \qquad \textit{and} \qquad \text{ii) } F = P \cap \mathcal{S}^{=}(\mathfrak{f}_F), \tag{1.1}$$

is called **support-polynomial** *of the face F. This is rephrased by stating that the polynomial $\mathfrak{f}_F$* **defines** *the face F.*

Remark 1.4.3. *Any face $F \in \mathscr{F}(P)$, can be obtained from P using a support-polynomial $\mathfrak{f}_F$, by observing $F = \mathcal{S}^{\geq}(-\mathfrak{f}_F) \cap P$.*

This seemingly unimportant property is the basis for the main idea of this work. It is the reason why *all* polynomials used in this construction are support-polynomials for some face of a polyhedron.

All used semi-algebraic sets are cones

The semi-algebraic sets used in our construction are cones, which approximate polyhedral cones. They are defined using polynomials which are either linear, or sums of squares. Both properties ensure that the resulting semi-algebraic sets are cones:

Lemma 1.4.4. *Let $\mathfrak{f}$ be a sum of squares, i.e.,*

$$\mathfrak{f}(x) = \sum_{i=1}^{m} \lambda_i \mathfrak{l}(x)^2$$

holds for some polynomials $\mathfrak{l}_i$, and some $\lambda_i \in \mathbb{R}$, then both sets $\mathcal{S}^{\geq}(\mathfrak{f})$ and $\mathcal{S}^{\geq}(-\mathfrak{f})$ are cones.

The reason is that for each $y \in \mathcal{S}^{\geq}(\mathfrak{f})$ and $\mu \in \mathbb{R}$ one has $\mathfrak{f}(\mu y) = \mu^2 \mathfrak{f}(y) \geq 0$. The corresponding holds for $\mathcal{S}^{\geq}(-\mathfrak{f})$. An example for this is the cone $\{x \in \mathbb{R}^3 : (x_1)^2 - (x_2)^2 - (x_3)^2 \geq 0\}$.

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2. The combinatorics and geometry behind the construction

2.1 Introduction

This chapter is an introduction to the combinatorial and geometric principles implicitly used in the algorithm presented in Chapter 3.

The proof of the Main Theorem is divided into three parts: In the first section of this chapter, the proof is reduced to the special case of polyhedral cones, making use of homogenization. The second section motivates, that the necessary polynomials can be constructed, making a short journey through algebra, combinatorics and the geometry of the corresponding semi-algebraic sets. Finally, in the the next chapter, the $2d-2$ polynomials describing a polyhedral cone will be calculated in detail. The second section of this chapter is for motivation only, the information given there is only implicitly used in the resulting construction. Thus if uninterested in the geometric and algebraic background, the reader may stick to the third chapter.

The basic idea behind the construction in this work is to replace polynomials which appear in a semi-algebraic set by their product and some additional polynomials: Given a pointed polyhedral cone C in semi-algebraic description, $C = \{x \in \mathbb{R}^d : \mathfrak{p}_1(x) \geq 0, \ldots, \mathfrak{p}_m(x) \geq 0\}$, the polynomials used are multiplied into a single polynomial $\mathfrak{P} = \mathfrak{p}_1 * \mathfrak{p}_2 * \ldots * \mathfrak{p}_m$. Here $*$ is used to indicate the product of real numbers as well as the product of polynomials. In C all $\mathfrak{p}_i$ are positive, so their product $\mathfrak{P}$ is, too. But the resulting semi-algebraic set $\mathcal{S}^{\geq}(\mathfrak{P})$ contains various other polyhedra besides C. These will be removed by adding new polynomial inequalities to the description $\mathcal{S}^{\geq}(\mathfrak{P})$. Each polynomial used thereby corresponds to some face of the cone C, allowing to exploit the combinatorics of the faces of C. Grouping the new polynomials by dimension of the corresponding face, each group of polynomials is replaced by their product. Due to other special "geometric" properties of the used polynomials, the unavoidably resulting undesired sets "cancel out".

The task of this chapter is to motivate these properties, which will be used implicitly in the explicit construction in Chapter 3.

2.2 Reduction to polyhedral cones

The following theorem shows, that it suffices to prove the Main Theorem for cones. This simplifies the construction, polynomials will be explicitly constructed only for the case that the given polyhedron is a polyhedral cone.

Theorem 2.2.1. *Let $C \subseteq \mathbb{R}^d$ be a d-dimensional polyhedral cone. Then one can construct $(2d-2)$ polynomials $\mathfrak{p}_i \in \mathbb{R}[x]$, $1 \leq i \leq 2d-2$, such that $C = \mathcal{S}^{\geq}(\mathfrak{p}_1, \ldots, \mathfrak{p}_{2d-2})$.*

The general case – that of polyhedra – can now be derived through homogenization: To prove the Main Theorem 1.2.1, any given polyhedron P is homogenized (cf. [Zie98]) resulting in a polyhedral cone $C := \{\lambda(x, 1) \in \mathbb{R}^{d+1} : x \in P, \lambda \geq 0\}$. According to Theorem 2.2.1, C can be described using $2(d+1)-2 = 2d$ polynomials $\mathfrak{p}_i(x) \in \mathbb{R}[X]^{d+1}$. Thus restricting both C and all $\mathfrak{p}_i$ to the hyper-plane induced by $x_{d+1} = 1$, one obtains a description of P:

$$P = \{x \in \mathbb{R}^d : \mathfrak{p}_1(x, 1) \geq 0, \ldots, \mathfrak{p}_{2d}(x, 1) \geq 0\}.$$

In case P is a polytope, one of the $\mathfrak{p}_i$ is redundant in this description for P, as we will see later. This is a consequence of the special construction behind Theorem 2.2.1.

2.3 General Concept

In the following, let $C := \{x \in \mathbb{R}^d : a_1 \cdot x \geq 0, \ldots, a_m \cdot x \geq 0\}$ always be a *pointed* polyhedral cone, given in irredundant description. This means, each polynomial $a_i \cdot x$ is support-polynomial for a *different* facet $F_i \in \mathscr{F}_{d\text{-}1}(C)$, i.e., $F_i \neq F_j$ holds for $i \neq j$.

2.3.1 Replacing polynomials by their product

In this work the following approach is followed:
To reduce the number of polynomials used in the description of a given polyhedral cone $C := \{x \in \mathbb{R}^d : a_1 \cdot x \geq 0, \ldots, a_m \cdot x \geq 0\}$, the given

linear polynomials are replaced by their product

$$\mathfrak{P}(x) := \prod_{i=1}^{m} (a_i \cdot x).$$

In the following this will be referred to as "multiplying polynomials in a description of C". The resulting semi-algebraic set $\mathcal{S}^{\geq}(\mathfrak{P})$, based on a single polynomial inequality, is the union of C and all sets where an even number of the given inequalities are inverted to $a_i \cdot x \leq 0$ (see Figure 2.4). In order to retrieve C from $\mathcal{S}^{\geq}(\mathfrak{P})$ new polynomials are added to the description, i.e., $C = \mathcal{S}^{\geq}(\mathfrak{P}, \mathfrak{f}_1, \ldots, \mathfrak{f}_n)$ holds for some polynomials $\mathfrak{f}_1, \ldots, \mathfrak{f}_n \in \mathbb{R}[X]^d$. This can be understood as intersecting $\mathcal{S}^{\geq}(\mathfrak{P})$ with new semi-algebraic sets $\mathcal{S}^{\geq}(\mathfrak{f}_i)$:

$$\mathcal{S}^{\geq}(\mathfrak{P}, \mathfrak{f}_1, \ldots, \mathfrak{f}_n) = \mathcal{S}^{\geq}(\mathfrak{P}) \cap \bigcap_{i=1,\ldots,n} \mathcal{S}^{\geq}(\mathfrak{f}_i).$$

In the following this is used to link this procedure to the *geometry* of such sets.

Repetitive multiplication

The mentioned description $C = \mathcal{S}^{\geq}(\mathfrak{P}, \mathfrak{f}_1, \ldots, \mathfrak{f}_n)$ involving $\mathfrak{P}$, might use more than the initially given m linear polynomials $(a_1 \cdot x), \ldots, (a_m \cdot x)$. To again reduce the number of polynomials used in the description, the newly introduced polynomials $\mathfrak{f}_i$ are replaced by their product, as done above. The resulting set $\mathcal{S}^{\geq}(\mathfrak{P}, \prod_{i=1}^{n} \mathfrak{f}_i)$ contains C, and again new polynomials must be introduced in order to retrieve C. The cycle of introducing new polynomials and multiplying them, comes to a halt after d steps due to special "combinatorial" properties of the constructed polynomials. This is examined more detailed in the following, starting with a sketch in the following Subsection 2.3.2.

Other approaches

Aside of the approach of multiplying all linear polynomials describing C, it is also possible to split the given linear polynomials describing C into separate groups and to build the products in groups. Such approaches, based on coloring theorems, lead to fewer polynomial inequalities only for very special polyhedra. This will be one of the subjects discussed in Chapter 4.

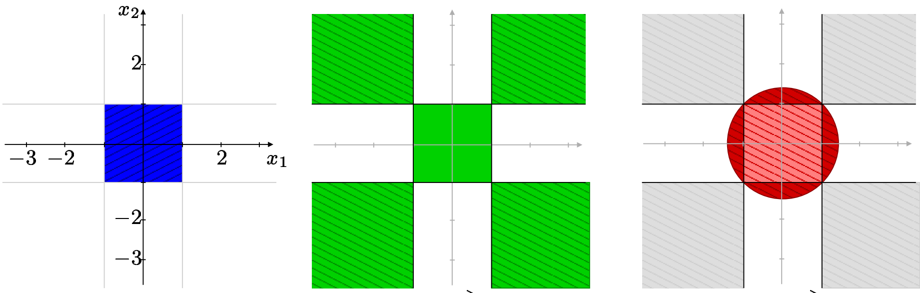
Example

The unit-square $W = \{ x \in \mathbb{R}^2 : |x_i| \leq 1, \ i = 1, 2 \}$ can be described as $W = \mathcal{S}^{\geq}(\mathfrak{f}, \mathfrak{g})$, where $\mathfrak{f}(x) := (1 - (x_1)^2) \ * \ (1 - (x_2)^2)$ and $\mathfrak{g}(x) :=$

$2-(x_1)^2-(x_2)^2$ (see Figure 2.1). The polynomial $\mathfrak{f}$ is the product of four linear polynomials used in the following description

$$W = \{x \in \mathbb{R}^2 \;:\; 1+x_i \geq 0,\;\; 1-x_i \geq 0,\;\; i=1,2\}$$

Figure 2.1 visualizes the replacement of linear polynomials by their



i) The unit-square $1+x_i \geq 0,\; 1-x_i \geq 0,\; i=1,2$

ii) The set $\mathcal{S}^{\geq}(\mathfrak{f})$ $\mathfrak{f}(x) = (1-(x_1)^2)*(1-(x_2)^2)$

iii) The set $\mathcal{S}^{\geq}(\mathfrak{g})$ $\mathfrak{g}(x) = 2-(x_1)^2-(x_2)^2$

Fig. 2.1: Replacing polynomials by their product and adding a correction-polynomial

products: Figure 2.1 i) shows the set W. The semi-algebraic set $\mathcal{S}^{\geq}(\mathfrak{f})$ defined by the product of all all linear inequalities defining W is displayed in ii). Finally, iii) shows the disc $\{x \in \mathbb{R}^2 \;:\; (x_1)^2+(x_1)^2 \leq 2\}$, which contains W, but contains no point of $\mathcal{S}^{\geq}(\mathfrak{f}) \setminus W$.

In the example, the undesired part of $\mathcal{S}^{\geq}(\mathfrak{f})$, namely $\mathcal{S}^{\geq}(\mathfrak{f}) \setminus W$ is connected to W in its vertices. Moreover, the the polynomial $\mathfrak{g}$ zeroes in these four points, so the boundary $\{x \in \mathbb{R}^2 \;:\; 2-(x_1)^2-(x_2)^2 = 0\}$ of the correcting set $\mathcal{S}^{\geq}(\mathfrak{g})$ passes through these vertices. These properties are important for the intended procedure and are generalized in the next section.

2.3.2 Exploiting the combinatorics of faces of polyhedra

In this work we use the following combinatorial invariant: In a description of C, a set of ***at most one*** support-polynomial $\mathfrak{f}_F$ for each k-face $F \in \mathscr{F}_k(C)$ can be replaced by their product $\mathfrak{F}_k := \prod_{F \in \mathscr{F}_k(C)} \mathfrak{f}_F$, when introducing appropriate support-polynomials for faces of dimension at most $k-1$ to the description (cf. Lemma 2.4.1).

A combinatorial view on the algorithm

This principle leads to the following iterative process:

- Each of the linear polynomials $a_i \cdot x$ in an irredundant $\mathcal{H}$-description $C = \{x \in \mathbb{R}^d : a_1 \cdot x \geq 0, \ldots, a_m \cdot x \geq 0\}$ is a support-polynomial for a different $d-1$ dimensional face of C. Therefore all $(a_i \cdot x)$ can be replaced by their product, when introducing ***several*** appropriate support-polynomials for each face of dimension at most $d-2$.

- The resulting support-polynomials for all $(d-2)$-faces again can be replaced by ***several*** products and some support-polynomials for faces of dimension at most $d-3$, and so on.

In each step of this iterative process, the dimension of faces for which support-polynomials are added gets lower. So it must come to a halt after d steps, because there is only one face in C of dimension 1, namely the vertex.

This approach has been followed in [GH03], leading to $O(d^d)$ polynomial inequalities. The reason why this principle *alone* does not lead to $O(d)$ many polynomials is that for a face F two different support-polynomials may never end up in the same product. If at any point of the iteration above, the description of C contains n support-polynomials for *the same* face, this results in at least n products in a following step. So the *introduction* of new support-polynomials in each step is a problem.

The "2" in the $2d$ polynomials used in a description of C

Instead of introducing new polynomials in each step of the iteration, here the polynomials added in the very first step of the iteration are "reused" in all following steps. Due to special geometric properties of the sets moreover two polynomials suffice for each face:

In the very first step two support-polynomials $\mathfrak{p}_F, \mathfrak{q}_F$ are introduced for each face F. The polynomials are constructed such, that when replacing $\mathfrak{p}_F, \mathfrak{q}_F$ for k-faces by their products $\mathfrak{P}_k := \prod_{F \in \mathscr{F}_k(C)} \mathfrak{p}_F$ and $\mathfrak{Q}_k := \prod_{F \in \mathscr{F}_k(C)} \mathfrak{q}_F$, the errors caused by this are undone by all $\mathfrak{p}_F, \mathfrak{q}_F$ with $\dim(F) < k$.

2.3.3 A sketch of the algorithm

In this construction, each polynomial used is a support polynomial for some face of the polyhedral cone $C = \{x \in \mathbb{R}^d : a_1 \cdot x \geq 0, \ldots, a_m \cdot x \geq 0\}$. As a start, in the given *irredundant* description, each linear polynomial $a_i \cdot x$ is the support-polynomial of some face $F_i \in \mathscr{F}_{d-1}(C)$, a facet of C. In each step of the following iteration, support-polynomials of k-faces are replaced by their products. The errors resulting from this are undone by two support-polynomials $\mathfrak{p}_F, \mathfrak{q}_F$ of faces of dimension at most $k-1$.

As a start, one can replace the polynomials $(a_i \cdot x)$, by their product $\mathfrak{P}_{d-1}(x) := (a_1 \cdot x) * \ldots (a_n \cdot x)$, when introducing *two* special polynomials $\mathfrak{p}_F, \mathfrak{q}_F$ for each face $F \in \mathscr{F}(C)$ with $\dim(F) \le d-2$:

$$C \;=\; \mathcal{S}^{\geq}(\mathfrak{P}_{d-1}) \cap \bigcap_{k=0}^{d-2} \bigcap_{F \in \mathscr{F}_k(C)} \mathcal{S}^{\geq}(\mathfrak{p}_F,\, \mathfrak{q}_F). \tag{2.1}$$

The reason why for each face two polynomials suffice, lies in the geometry of all sets $\mathcal{S}^{\geq}(\mathfrak{p}_F, \mathfrak{q}_F)$, examined later. Now successively for $k = d-2, \ldots, 1$ the polynomials $\mathfrak{p}_F$ and $\mathfrak{q}_F$ for k-faces $F \in \mathscr{F}_k(C)$ will be separately replaced by their products

$$\mathfrak{P}_k := \prod_{F \in \mathscr{F}_k(C)} \mathfrak{p}_F \qquad \text{and} \quad \mathfrak{Q}_k := \prod_{F \in \mathscr{F}_k(C)} \mathfrak{q}_F.$$

When replacing the polynomials $\mathfrak{p}_F$ and $\mathfrak{q}_F$ for faces of dimension $d-2$ by their products, without any additional insights the resulting set contains the set in (2.1), one obtains

$$C \;\subseteq\; \mathcal{S}^{\geq}(\mathfrak{P}_{d-1}, \mathfrak{P}_{d-2},\, \mathfrak{Q}_{d-2}) \;\cap\; \bigcap_{k=0}^{d-2} \bigcap_{F \in \mathscr{F}_k(C)} \mathcal{S}^{\geq}(\mathfrak{p}_F,\, \mathfrak{q}_F). \tag{2.2}$$

Now any point in $\mathcal{S}^{\geq}(\mathfrak{P}_{d-2},\, \mathfrak{Q}_{d-2})$ not in $\bigcap_{F \in \mathscr{F}_{d-1}(C)} \mathcal{S}^{\geq}(\mathfrak{p}_F,\, \mathfrak{q}_F)$ can be removed using support-polynomials for faces of dimension at most $d-3$. In the following section we motivate, that the remaining polynomials $\mathfrak{p}_F, \mathfrak{q}_F$ can be constructed such that they do so – in addition to their original task. This results in

$$C \;=\; \mathcal{S}^{\geq}(\mathfrak{P}_{d-1}, \mathfrak{P}_{d-2},\, \mathfrak{Q}_{d-2}) \;\cap\; \bigcap_{k=0}^{d-2} \bigcap_{F \in \mathscr{F}_k(C)} \mathcal{S}^{\geq}(\mathfrak{p}_F,\, \mathfrak{q}_F).$$

Due to the same reasons, repetitive multiplication results in a description of C using $2d-1$ polynomials after d steps:

$$C = \mathcal{S}^{\geq}(\mathfrak{P}_{d-1}, \mathfrak{P}_{d-2},\, \mathfrak{Q}_{d-2}, \ldots, \mathfrak{P}_1,\, \mathfrak{Q}_1) \cap \bigcap_{k=0}^{0} \bigcap_{F \in \mathscr{F}_k(C)} \mathcal{S}^{\geq}(\mathfrak{p}_F,\, \mathfrak{q}_F).$$

The vertex v is the only 0-dimensional face of the pointed cone C, leading to

$$\mathcal{S}^{\geq}(\mathfrak{p}_v, \mathfrak{q}_v) = \bigcap_{k=0}^{0} \bigcap_{F \in \mathscr{F}_k(C)} \mathcal{S}^{\geq}(\mathfrak{p}_F,\, \mathfrak{q}_F).$$

As shown later, one has $\mathfrak{Q}_{d-2} = \mathfrak{P}_{d-2}$, such that one polynomial in the above is obsolete, leading to $2d-2$ polynomials.

Reusing support-polynomials

The most important property in the algorithm sketched above is, that for each face one has to construct only two polynomials $\mathfrak{p}_F, \mathfrak{q}_F$. This is due to the special geometry of the corresponding sets $\mathcal{S}^{\geq}(\mathfrak{p}_F, \mathfrak{q}_F)$.

The central *geometric* idea of this work is, to choose all $\mathfrak{p}_F, \mathfrak{q}_F$ such that each set $\mathcal{S}^{\geq}(\mathfrak{p}_F, \mathfrak{q}_F)$ is a *closed cone* approximating C. With this, according to Lemma 2.4.2, when replacing the polynomials $\mathfrak{p}_F, \mathfrak{q}_F$ by their products $\mathfrak{P}_k, \mathfrak{Q}_k$, the resulting additional sets

$$\mathcal{S}^{\geq}(\mathfrak{P}_k, \mathfrak{Q}_k) \setminus \bigcap_{F \in \mathscr{F}_k(C)} \mathcal{S}^{\geq}(\mathfrak{p}_F, \mathfrak{q}_F)$$

are contained in *closed cones* $\mathscr{C}_k^G$, each intersecting C in a face G of dimension at most $k-1$. Viewed from the perspective of such a face G, there are appropriate polynomials $\mathfrak{p}_G, \mathfrak{q}_G$ such that no set $\mathscr{C}_k^G \setminus C$ is contained in the closed cone $\mathcal{S}^{\geq}(\mathfrak{p}_G, \mathfrak{q}_G)$.

For example viewed from the perspective of the vertex v, there are some *closed cones* $\mathscr{C}_{d-1}^v, \ldots, \mathscr{C}_1^v$ attached to v (see Figure 2.2). Since

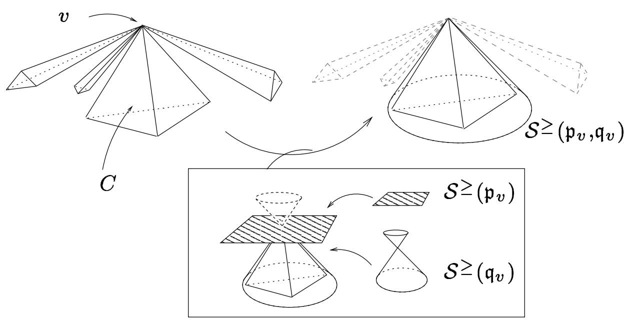


Fig. 2.2: An approximation $\mathcal{S}^{\geq}(\mathfrak{p}_v, \mathfrak{q}_v)$ removes several closed cones *simultaneously* from a polyhedral cone C.

all $\mathscr{C}_i^v$ are closed, there is some cone $\mathcal{S}^{\geq}(\mathfrak{p}_v, \mathfrak{q}_v)$ containing C and not containing any point in $\mathscr{C}_n^i \setminus C$, see Figure 2.2. The set $\mathcal{S}^{\geq}(\mathfrak{p}_v, \mathfrak{q}_v)$ is a semi-algebraic cone approximating C. Here the polynomial $\mathfrak{p}_v$ is linear, and $\mathfrak{q}_v$ is a sum of squares. An example for such a pointed cone $\mathcal{S}^{\geq}(\mathfrak{p}_v, \mathfrak{q}_v)$ is the well known cone $\{x \in \mathbb{R}^3 \ : \ x_3 \geq 0, \, (x_3)^2 - (x_1)^2 - (x_2)^2 \geq 0\}$.

Accordingly, for each other k-face F, the set $\mathcal{S}^{\geq}(\mathfrak{p}_F, \mathfrak{q}_F)$ is also an approximation to C, which removes closed cones $\mathscr{C}_{d-1}^F, \ldots, \mathscr{C}_{k+1}^F$ from C. Roughly speaking, by improving how good $\mathcal{S}^{\geq}(\mathfrak{p}_F, \mathfrak{q}_F)$ approximates C, one can adjust its capability in removing closed cones from C (see Figure 2.3).

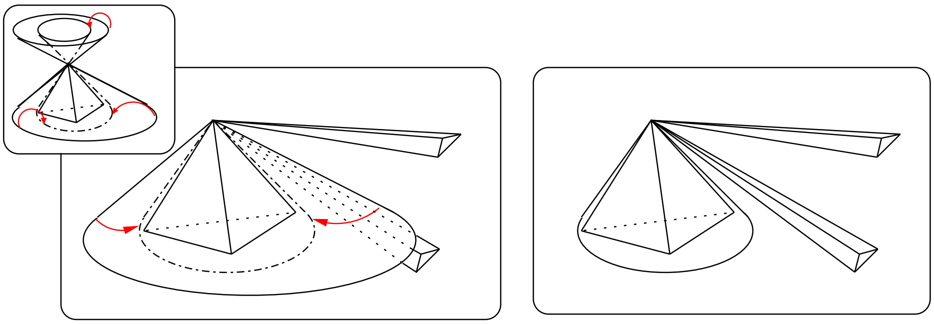


Fig. 2.3: Improving the quality of approximation, $\mathcal{S}^{\geq}(\mathfrak{p},\mathfrak{q})$ can remove more closed cones from C.

This way restrictions are "handed down" in dimension, fixing some n-face G the restrictions for the polynomials $\mathfrak{p}_G, \mathfrak{q}_G$, are imposed from $\mathfrak{p}_F, \mathfrak{q}_F$ for faces of dimension at least $n+1$. Roughly speaking, the lower the dimension of G, the better $\mathcal{S}^{\geq}(\mathfrak{p}_G, \mathfrak{q}_G)$ must approximate C. This is examined closer in the following sections.

Example

As an example examine the replacement of all $(a_i \cdot x)$ by their product $\mathfrak{P}_{d-1}(x) := (a_1 \cdot x) * \ldots (a_n \cdot x)$. The resulting set $\mathcal{S}^{\geq}(\mathfrak{P}_{d-1})$ contains – besides C – all sets where an even number of inequalities are inverted, such as for example the set

$$\mathscr{C}_{d-1} := \{x \in \mathbb{R}^d \ : \ (a_1 \cdot x) \leq 0, (a_2 \cdot x) \leq 0, \ a_3 \cdot x \geq 0, \ldots, a_n \cdot x \geq 0\}.$$

Here we call these sets *undesired*. Just as C, these undesired sets are closed, polyhedral cones, and more precisely the set $\mathcal{S}^{\geq}(\mathfrak{P}) \setminus C$ is contained in the union of all such sets (cf. Lemma 2.4.2). Each undesired set intersects C in a face $F \in \mathscr{F}(C)$ of dimension at most $d-2$, for example

$$\begin{aligned} C \cap \mathscr{C}_{d-1} & \\ = \quad & \{x \in \mathbb{R}^d : \ (a_1 \cdot x) \geq 0, (a_2 \cdot x) \geq 0, \ a_3 \cdot x \geq 0, \ldots, a_n \cdot x \geq 0\} \\ \cap\, & \{x \in \mathbb{R}^d : \ (a_1 \cdot x) \leq 0, (a_2 \cdot x) \leq 0, \ a_3 \cdot x \geq 0, \ldots, a_n \cdot x \geq 0\} \\ = \quad & \{x \in \mathbb{R}^d : \ (a_1 \cdot x) = 0, (a_2 \cdot x) = 0, \ a_3 \cdot x \geq 0, \ldots, a_n \cdot x \geq 0\} \end{aligned}$$

holds, where the last set is a face of C. In each undesired set $\mathscr{C}$ at least two polynomials are negative. Thus at least $k=2$ polynomials $a_i \cdot x$ zero in the intersection $F = C \cap \mathscr{C}$, making F a face of dimension $d-k = d-2$ or less.

2.4 Using the combinatorial structure of polyhedra

When replacing the polynomials in the description of a semi-algebraic set $\mathcal{A} = \mathcal{S}^{\geq}(\mathfrak{f}_1, \ldots, \mathfrak{f}_n)$ by their product, in general, the resulting set $\mathcal{B} = \mathcal{S}^{\geq}(\mathfrak{f}_1 * \ldots * \mathfrak{f}_n)$ contains additional points. In order to reobtain $\mathcal{A}$ from $\mathcal{B}$, the *artifact* $\mathcal{A} \setminus \mathcal{B}$ is removed by adding more polynomial inequalities to the description.

As roughly introduced in the previous section, it is possible to replace support-polynomials for k-faces of C by their product when introducing appropriate support-polynomials for faces of dimension at most $k-1$. In this section this is first proved, then this is examined in its geometrical meaning. Please note that although the corresponding proof uses the *sum* of polynomials, this work uses *other* polynomials. The approach of using the sums of polynomials has been examined in [GH03], leading to a description of simplicial polyhedra using $O(d^d)$ polynomials.

2.4.1 The combinatorial invariant

In this work we use the following combinatorial invariant:

Lemma 2.4.1. *In a description of C, a set of* one *support-polynomial for each N-face can be replaced by their product and additional support-polynomials $\mathfrak{p}_H$ for faces $H \in \mathscr{F}(C)$ with* $\dim(H) < N$.

To prove this we examine the sets “gained” by replacing polynomials with their product. Roughly, the reduction of dimension then follows since in each such set two polynomials are negative. Lemma 2.4.1 is a direct consequence of

Lemma 2.4.2 (Main Lemma 1). *Let $N \in \{1, \ldots, d-1\}$ and for each N-face $F \in \mathscr{F}_N(C)$ let $\mathfrak{f}_F$ be some support-polynomial. Replacing the polynomials defining $\mathcal{A}_N := \bigcap_{F \in \mathscr{F}_N(C)} \mathcal{S}^{\geq}(\mathfrak{f}_F)$ by their product $\mathfrak{F}_N := \prod_{F \in \mathscr{F}_N(C)} \mathfrak{f}_F$ one obtains*

$$\mathcal{A}_N \subseteq \mathcal{S}^{\geq}(\mathfrak{F}_N) \subseteq \mathcal{A}_N \cup \bigcup_{\substack{F,G \in \mathscr{F}_N(C) \\ F \neq G}} \mathcal{S}^{\geq}(-\mathfrak{f}_F, -\mathfrak{f}_G) \setminus \mathcal{S}^{=}(\mathfrak{f}_F, \mathfrak{f}_G). \quad (2.3)$$

Moreover for each pair $F, G \in \mathscr{F}(C)$ with $F \neq G$ there is a support-polynomial $\mathfrak{p}_{F \cap G}$ for the corresponding face $F \cap G$ removing the additional sets:

$$\mathcal{A}_N = \mathcal{S}^{\geq}(\mathfrak{F}_N) \cap \bigcap_{\substack{F,G \in \mathscr{F}_N(C) \\ F \neq G}} \mathcal{S}^{\geq}(\mathfrak{p}_{F \cap G}). \quad (2.4)$$

Proof. The first inclusion in (2.3) holds, since for any $y \in \mathcal{A}_N$ all $\mathfrak{f}_F$ are positive, leading to $\mathfrak{F}(y) \geq 0$. For any point $x \in \mathcal{S}^{\geq}(\mathfrak{F}_N) \setminus \mathcal{A}_N$, due to $x \notin \mathcal{A}_N$ there is at least one $\mathfrak{f}_F$ with $\mathfrak{f}_F(x) < 0$. In addition there is at least one $\mathfrak{f}_G$ with $F \neq G$ and $\mathfrak{f}_G(x) \leq 0$, otherwise $\prod_{F \in \mathscr{F}_N(C)} \mathfrak{f}_F(x) \geq 0$ would not hold. So $-\mathfrak{f}_F(x) \geq 0$ and $-\mathfrak{f}_G(x) \geq 0$ hold, this together with $\mathfrak{f}_F(x) \neq 0$ lead to $x \in \mathcal{S}^{\geq}(-\mathfrak{f}_F, -\mathfrak{f}_G) \setminus \mathcal{S}^{=}(\mathfrak{f}_F, \mathfrak{f}_G)$, proving (2.3). Moreover $\mathfrak{p}_{F \cap G} := (\mathfrak{f}_F + \mathfrak{f}_G)/2$ is a support-polynomial for the corresponding face and $\mathfrak{p}_{F \cap G}(x) < 0$ holds, implying $x \notin \mathcal{S}^{\geq}(\mathfrak{f}_{F \cap G})$. Since x is chosen arbitrarily this proves (2.4). □

With the used combinatorial invariant Lemma 2.4.1 follows immediately from Lemma 2.4.2 since if $F \neq G$ and $\dim(F) = \dim(G) = N$ hold, then $\dim(F \cap G) < N$ follows.

Replacing polynomials results in additional *artifacts*

In this section, the additional sets resulting from replacing polynomials by their product in a semi-algebraic set is called artifact:

Definition 2.4.3. *For a semi-algebraic set $\mathcal{A} := \mathcal{S}^{\geq}(\mathfrak{f}_1, \ldots, \mathfrak{f}_n)$ the set*

$$\operatorname{artifact}(\mathfrak{f}_1, \ldots, \mathfrak{f}_n) := \mathcal{S}^{\geq}(\mathfrak{f}_1 * \ldots * \mathfrak{f}_n) \setminus \mathcal{S}^{\geq}(\mathfrak{f}_1, \ldots, \mathfrak{f}_n)$$

is called the artifact of $\mathfrak{f}_1, \ldots, \mathfrak{f}_n$*. If clear which polynomials are meant, this is abbreviated to* "the artifact of $\mathcal{A}$"*.*

For example Figure 2.1 ii), (page 22) depicts the artifact of the unit square W. For two polynomials $\mathfrak{f}_1, \mathfrak{f}_2$ the artifact is the set where both are non-positive *and* at least one is negative: If the product $\mathfrak{f}_1 * \mathfrak{f}_2$ is positive, then either both polynomials are positive or both are negative, leading to $\mathcal{S}^{\geq}(\mathfrak{f}_1 * \mathfrak{f}_2) = \mathcal{S}^{\geq}(-\mathfrak{f}_1, -\mathfrak{f}_2) \cup \mathcal{S}^{\geq}(\mathfrak{f}_1, \mathfrak{f}_2)$. This results in

$$\begin{aligned} \operatorname{artifact}(\mathfrak{f}_1, \mathfrak{f}_2) &= \mathcal{S}^{\geq}(\mathfrak{f}_1 * \mathfrak{f}_2) \setminus \mathcal{S}^{\geq}(\mathfrak{f}_1, \mathfrak{f}_2) \\ &= \mathcal{S}^{\geq}(-\mathfrak{f}_1, -\mathfrak{f}_2) \setminus \mathcal{S}^{=}(\mathfrak{f}_1, \mathfrak{f}_2). \end{aligned}$$

According to Lemma 2.4.2 (2.3) any additional point resulting from replacing polynomials with their product, is contained in a set of this form, i.e.,

$$\mathcal{S}^{\geq}(\mathfrak{f}_1 * \ldots * \mathfrak{f}_n) \setminus \mathcal{S}^{\geq}(\mathfrak{f}_1, \ldots, \mathfrak{f}_n) \subseteq \bigcup_{1 \leq i < j \leq n} \mathcal{S}^{\geq}(-\mathfrak{f}_i, -\mathfrak{f}_j) \setminus \mathcal{S}^{=}(\mathfrak{f}_i, \mathfrak{f}_j).$$

With Definition 2.4.3 this is rephrased by

$$\operatorname{artifact}(\mathfrak{f}_1, \ldots, \mathfrak{f}_n) \subseteq \bigcup_{1 \leq i < j \leq n} \operatorname{artifact}(\mathfrak{f}_1, \mathfrak{f}_2).$$

So when multiplying polynomials used in a description of C, one needs to remove the artifact of each pair of the corresponding polynomials from C.

This can be done using support-polynomials, for faces of *lower* dimension.

2.4.2 Motivating the combinatorial invariant

To interpret the combinatorial invariant stated in Lemma 2.4.1 *geometrically*, a slight shift in the perception of (2.4) is necessary: Given a description $C = \mathcal{S}^{\geq}(\mathfrak{f}_1, \ldots, \mathfrak{f}_n)$, the set C is reobtained from $\mathcal{A} := \mathcal{S}^{\geq}(\mathfrak{f}_1 * \ldots * \mathfrak{f}_n)$ by intersection with an appropriate set $\mathcal{B}$. But although the task of $\mathcal{B}$ is to "remove parts **of** $\mathcal{A}$ resulting in C", from the perspective of C, this is reinterpreted as "$\mathcal{B}$ cuts off something which was attached to C":

Definition 2.4.4. *We say the set $\mathcal{B}$* **cuts off** *some set $\mathcal{D}$ from C, if it literally does, i.e., $C = (C \cup \mathcal{D}) \cap \mathcal{B}$ holds. This can be rephrased by $C \subseteq \mathcal{B}$ and $(\mathcal{D} \setminus C) \cap \mathcal{B} = \emptyset$.*

With this shift of perception, reobtaining C from $\mathcal{A}$ by intersection with $\mathcal{B}$, reduces to three insights:

1. A set $\mathcal{B}$ can only fulfill $C = \mathcal{A} \cap \mathcal{B}$, if $C \subseteq \mathcal{B}$ and $(\mathcal{A} \setminus C) \cap \mathcal{B} = \emptyset$ hold. Thus the task of reobtaining C from $\mathcal{A}$ is equivalent to finding some $\mathcal{B}$ which cuts off $\mathcal{A} \setminus C$ from C.

2. Due to a relaxation of (2.3), $\mathcal{A} \setminus C$ is contained in the union of all sets of the form $\mathcal{S}^{\geq}(-\mathfrak{f}_i, -\mathfrak{f}_j)$. So the previous task can be reduced to *cutting off* all such sets from C.

3. According to (2.5), for support-polynomials $\mathfrak{f}_F$ and $\mathfrak{f}_G$ for faces F and G respectively, $\mathcal{S}^{\geq}(-\mathfrak{f}_F, -\mathfrak{f}_G)$ intersects C in $F \cap G$, and can be cut off from C using a set $\mathcal{S}^{\geq}(\mathfrak{f}_{F\cap G})$, where $\mathfrak{f}_{F\cap G}$ is a support-polynomial for the corresponding face $F \cap G$.

The following motivates the last insight, the corresponding proof is part of the proof of Lemma 2.4.1. A visualization is found in Figure 2.4, page 30.

Additional points are removed from C using support polynomials

So although the final goal is to remove parts of $\mathcal{A}$, this is done finding polynomial inequalities which cut off some set $\mathcal{S}^{\geq}(-\mathfrak{f}_F, -\mathfrak{f}_G)$ from C, for a pair of faces $F \neq G$. This can be done using a support-polynomial for the corresponding face $F \cap G$. The geometric intuition for this is that the set $\mathcal{S}^{\geq}(-\mathfrak{f}_F, -\mathfrak{f}_G)$ intersects C in $F \cap G$: The set C is contained in $\mathcal{S}^{\geq}(\mathfrak{f}_F, \mathfrak{f}_G)$ and by by definition $F = C \cap \mathcal{S}^{=}(\mathfrak{f}_F)$ holds, implying

$$\begin{aligned} C \cap \mathcal{S}^{\geq}(-\mathfrak{f}_F, -\mathfrak{f}_G) &= C \cap \mathcal{S}^{\geq}(\mathfrak{f}_F, \mathfrak{f}_G) \;\cap\; \mathcal{S}^{\geq}(-\mathfrak{f}_F, -\mathfrak{f}_G) \\ &= C \cap \mathcal{S}^{=}(\mathfrak{f}_F, \mathfrak{f}_G) \\ &= F \cap G. \end{aligned} \tag{2.5}$$

Thus $\mathcal{S}^{\geq}(-\mathfrak{f}_F, -\mathfrak{f}_G)$ can be separated from C using a support-polynomial $\mathfrak{f}_{F\cap G}$ for this face as done in Figure 2.4.

So as a whole this motivates, that a set of *one* support-polynomial $\mathfrak{f}_F$ per k-face $F \in \mathscr{F}_k(C)$ used in a description of C, can be replaced by their product and support-polynomials for faces of dimension at most $k-1$.

Example

For example given the cone $\mathcal{S}^{\geq}(\mathfrak{f}_1, \ldots, \mathfrak{f}_4)$ in Figure 2.4, the set $\mathcal{A} := \mathcal{S}^{\geq}(\mathfrak{f}_1 * \ldots * \mathfrak{f}_4)$ contains also the set $\mathcal{S}^{\geq}(-\mathfrak{f}_1, \ldots, -\mathfrak{f}_4)$ where all inequalities $\mathfrak{f}_i(x) \geq 0$ are inverted to $-\mathfrak{f}_i(x) \geq 0$. This set is contained in

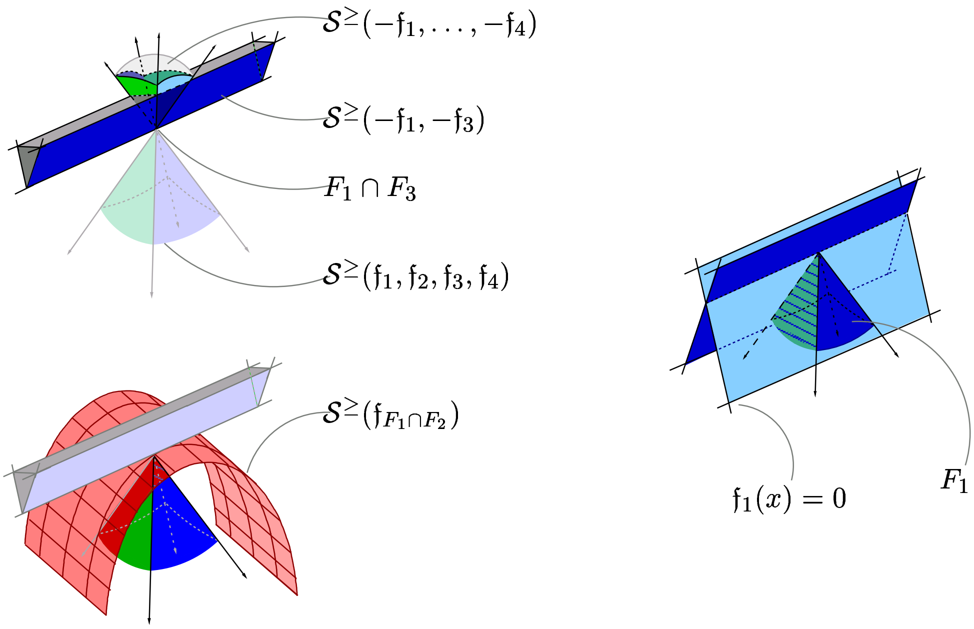


Fig. 2.4: Each undesired point in $\mathcal{S}^{\geq}(\mathfrak{f}_1 * \ldots * \mathfrak{f}_n)$ is contained in some set $\mathcal{S}^{\geq}(-\mathfrak{f}_i, -\mathfrak{f}_j)$.

$\mathcal{S}^{\geq}(-\mathfrak{f}_1, -\mathfrak{f}_3)$, which intersects C in the face $F_1 \cap F_3$ and thus can be removed from this set using an appropriate support-polynomial $\mathfrak{f}_{F_1 \cap F_3}$ for

$F_1 \cap F_3$. So the set $\mathcal{S}^{\geq}(\mathfrak{f}_1 * \ldots * \mathfrak{f}_4, \mathfrak{f}_{F_1 \cap F_3})$ is a more precise approximation to C than $\mathcal{A}$.

2.4.3 A combinatorial view on the algorithm

To replace a set of one support-polynomial $\mathfrak{f}_F$ per face $F \in \mathscr{F}_N(C)$ by their product, one needs to introduce support polynomials for the faces $F \cap G$, which remove a set

$$\text{artifact}(\mathfrak{f}_F, \mathfrak{f}_G) = \mathcal{S}^{\geq}(-\mathfrak{f}_F, -\mathfrak{f}_G) \setminus \mathcal{S}^{=}(\mathfrak{f}_F, \mathfrak{f}_G)$$

from C. Viewed from the perspective of a face H with $\dim(H) \ll N$, there can be "quite a bunch" of sets to remove, namely all $\text{artifact}(\mathfrak{f}_F, \mathfrak{f}_G)$ with $F, G \in \mathscr{F}_N(C)$ and $F \cap G = H$. For example choosing H as the single vertex $v \in C$, all $N = 1$ dimensional faces of C pairwise intersect only in v.

A straight-forward solution for this is to construct *one* polynomial $\mathfrak{f}_{F,G}$ for *each* such pair, resulting in "many" support-polynomials for the face H. The problem with "many" support-polynomials for one face is, that two of these must end up in a different product, when replacing polynomials by their products. For example the set $\mathcal{S}^{\geq}(\mathfrak{f}_H * \mathfrak{f}_H) = \mathbb{R}^d$ can not be removed from C using a support-polynomial for a lower dimensional face.

A sketch of the algorithm

In this paper, exactly two polynomials $\mathfrak{p}_H, \mathfrak{q}_H$ are constructed for each face H. The resulting set $\mathcal{S}^{\geq}(\mathfrak{p}_H, \mathfrak{q}_H)$ approximates C. Roughly speaking it "mimics" the geometry of C close to the face H. This way the set $\mathcal{S}^{\geq}(\mathfrak{p}_H, \mathfrak{q}_H)$ is able to remove as "many" sets $\mathcal{S}^{\geq}(-\mathfrak{f}_F, -\mathfrak{f}_G)$ as necessary, just by improving the quality of its approximation. For the vertex of C this is visualized in the Figures 2.2 and 2.3 on page 25 and 26 respectively. The motivation for this property and additional restrictions on all $\mathfrak{p}_F, \mathfrak{q}_F$ are given in the next section.

Algorithm 2.4.5. A combinatorial sketch of the algorithm

Input:	*An irredundant description* $C = \{x \in \mathbb{R}^d \, : \, a_1 \cdot x \geq 0, \ldots, a_m \cdot x \geq 0\}$.
Output:	*A set of* $2d$ *polynomials* $\mathfrak{P}_i, \mathfrak{Q}_i$ $(i = 1 \ldots, d)$, *such that* $C = \mathcal{S}^{\geq}(\mathfrak{P}_1, \mathfrak{Q}_1, \ldots, \mathfrak{P}_d, \mathfrak{Q}_d)$.
Init:	*Set* $k := d-1$ *and for each* $i = 1, \ldots, m$ *set* $\mathfrak{p}_{F_i}(x) := \mathfrak{q}_{F_i}(x) := a_i \cdot x$ *(redundantly), where* $F_i \in \mathscr{F}_N(C)$ *is the facet incident to* $a_i \cdot x = 0$. *Set* $\mathfrak{P}_{d-1}(x) = \mathfrak{Q}_{d-1}(x) = \prod_{F \in \mathscr{F}_{d-1}(C)} \mathfrak{p}_F$
Step 1:	*Set* $k := k-1$.
Step 2:	*For each* $H \in \mathscr{F}_k(C)$, *construct support-polynomials* $\mathfrak{p}_H, \mathfrak{q}_H$ *such that: For each pair of faces* $F \neq G$ *with* $F \cap G = H$, *the set* $\mathcal{S}^{\geq}(\mathfrak{p}_H, \mathfrak{q}_H)$ *removes both* $\mathrm{artifact}(-\mathfrak{p}_F, -\mathfrak{p}_G)$ *and* $\mathrm{artifact}(-\mathfrak{q}_F, -\mathfrak{q}_G)$ *from* C.
Step 3:	*Set* $\mathfrak{P}_k := \prod_{F \in \mathscr{F}_N(C)} \mathfrak{p}_F$ *and* $\mathfrak{Q}_k := \prod_{F \in \mathscr{F}_N(C)} \mathfrak{q}_F$
Step 4:	*If* $k > -1$, *go to Step 1, else end.*

The algorithm comes to a halt, when k reaches -1: There is only one face of dimension 0, namely the vertex $F = \mathbf{0}$ of C, leading to $\mathfrak{P}_0 = \mathfrak{p}_v$ and $\mathfrak{Q}_0 = \mathfrak{q}_v$. So no polynomials are multiplied. Here a brief sketch of the proof for the correctness of the corresponding Algorithm, Main Algorithm, on page 61, is given. For a complete proof, see there.

Sketch of the proof for the algorithm
The idea behind the above algorithm is an iterative argument: Any point in some set $\mathcal{S}^{\geq}(\mathfrak{P}_k, \mathfrak{Q}_k)$ which is not in C, is contained in some set of the form $\mathrm{artifact}(-\mathfrak{p}_F, -\mathfrak{p}_G)$ or $\mathrm{artifact}(-\mathfrak{q}_F, -\mathfrak{q}_G)$ with $\dim(F) = \dim(G) = k$ (cf. Lemma 2.4.2). But such sets are removed from C using $\mathcal{S}^{\geq}(\mathfrak{p}_{F \cap G}, \mathfrak{q}_{F \cap G})$, where $\dim(F \cap G) < k$. This way there is some minimal dimension n for which $x \in \mathcal{S}^{\geq}(\mathfrak{P}_n, \mathfrak{Q}_n)$ holds, leading to $x \notin \bigcap_{i=0}^{d-1} \mathcal{S}^{\geq}(\mathfrak{P}_i, \mathfrak{Q}_i)$. ☐

The above can work only, since in each step the restrictions on the polynomials $\mathfrak{p}_F, \mathfrak{q}_F$ with $\dim(F) = k$ are imposed by the properties of polynomials $\mathfrak{p}_F, \mathfrak{q}_F$ with $\dim(F) \geq k$. In each step the added polynomials must remove more sets, than the polynomials added before did. So the polynomials $\mathfrak{p}_F, \mathfrak{q}_F$ must be "controllable" in their ability of removing sets from C. The sets $\mathcal{S}^{\geq}(\mathfrak{p}_F, \mathfrak{q}_F)$ are approximations to the cone C, and the mentioned improvement is done by improving "how good" $\mathcal{S}^{\geq}(\mathfrak{p}_F, \mathfrak{q}_F)$ approximates C. So the overall process works due to the *special geometry* of the sets involved, which is examined in the following section.

2.5 Geometry: using 2 support-polynomials per face

Where the last section motivated, that a polyhedral cone $C = \{x \in \mathbb{R}^d : a_1 \cdot x \geq 0, \dots, a_m \cdot x \geq 0\}$ can be described using a number of polynomials linear in d, this section motivates that $2d$ suffice.

In the following, for each $i = 1, \dots, m$ let $F_i \in \mathscr{F}_{d-1}(C)$ be the facet induced by $a_i \cdot x = 0$, and to simplify notation, (redundantly) define $\mathfrak{p}_{F_i}(x) := \mathfrak{q}_{F_i}(x) := a_i \cdot x$. By definition, for any support-polynomial $\mathfrak{f}_F$ of some face $F \in \mathscr{F}(C)$, one obtains $C \subseteq \mathcal{S}^{\geq}(\mathfrak{f}_F)$. So, with the above definition of $\mathfrak{p}_{F_i}$ and $\mathfrak{q}_{F_i}$ and *any* choice of support-polynomials $\mathfrak{p}_F, \mathfrak{q}_F$ for each *other* face $F \in \mathscr{F}(C)$ with $\dim(F) \leq d-2$ one obtains

$$C = \bigcap_{k=0}^{d-1} \bigcap_{F \in \mathscr{F}_k(C)} \mathcal{S}^{\geq}(\mathfrak{p}_F,\, \mathfrak{q}_F). \tag{2.6}$$

The task of this section is to motivate the properties, that must hold in order to multiply polynomials in (2.6), without changing the set. More precisely, defining $\mathfrak{P}_k := \prod_{F \in \mathscr{F}_k(C)} \mathfrak{p}_F$ and $\mathfrak{Q}_k := \prod_{F \in \mathscr{F}_k(C)} \mathfrak{q}_F$, we motivate that for special polynomials $\mathfrak{p}_F, \mathfrak{q}_F$ the following holds:

$$C = \bigcap_{k=0}^{d-1} \mathcal{S}^{\geq}(\mathfrak{P}_k,\, \mathfrak{Q}_k). \tag{2.7}$$

This is due to the special geometric properties of the corresponding sets $\mathcal{S}^{\geq}(\mathfrak{p}_F, \mathfrak{q}_F)$, stated in (2.8). This geometric invariant basically expresses, that there are two polynomials, which separate C from the union of several semi-algebraic sets. In general, the separation of semi-algebraic sets using polynomials is a non trivial task (cf. [AAB99]). Therefore this is motivated after the corresponding proof.

2.5.1 The geometric invariant

Two polynomials per face suffice

In review of Algorithm 2.4.5, the following Lemma 2.5.1 proves that there are polynomials $\mathfrak{p}_F, \mathfrak{q}_F$ such that replacing these polynomials by their products in the description (2.6) of C results in (2.7), a description of C involving $2d$ polynomials.

Algorithm 2.4.5 implies the following: In the description (2.6) of C, the polynomials $\mathfrak{p}_F$ for all k-faces can be replaced by their product $\mathfrak{P}_k$ without loss, if for each pair $F, G \in \mathscr{F}_k(C)$ the corresponding artifact$(\mathfrak{p}_F, \mathfrak{p}_G)$ is removed from the resulting set. The same holds regarding all $\mathfrak{q}_F$. We claim that there are *two adequate* polynomials $\mathfrak{p}_H$ and $\mathfrak{q}_H$, such that the set $\mathcal{S}^{\geq}(\mathfrak{p}_H, \mathfrak{q}_H)$ removes all artifacts "belonging" to the corresponding face $H = F \cap G$. From the perspective of the face H this reads as follows:

Lemma 2.5.1 (Main Lemma 2). *All* $\mathfrak{p}_H, \mathfrak{q}_H$ *with* $0 \leq \dim(H) \leq d-2$ *in (2.6) can be constructed such, that for any two different faces* $F, G \in \mathscr{F}_k(C)$ *of equal dimension* $k > \dim(H)$ *intersecting in* $H = F \cap G$ *one obtains*

$$\begin{array}{l} \mathrm{artifact}(\mathfrak{p}_F, \mathfrak{p}_G) \cap \mathcal{S}^{\geq}(\mathfrak{p}_H, \mathfrak{q}_H) = \emptyset \quad \textit{and} \\ \mathrm{artifact}(\mathfrak{q}_F, \mathfrak{q}_G) \cap \mathcal{S}^{\geq}(\mathfrak{p}_H, \mathfrak{q}_H) = \emptyset. \end{array} \tag{2.8}$$

The center of the following proof is the construction done in the next chapter. Since there the Main Theorem is explicitly proved, the following proof is meant to be a motivation for the "cryptic" restrictions (2.9) on the polynomials constructed in the following chapter.

Proof. Corollary 3.5.4 in Chapter 3 proves that all polynomials $\mathfrak{p}_F, \mathfrak{q}_F$ with $0 \leq \dim(F) \leq d-2$ can be explicitly constructed such that

$$\mathcal{S}^{\geq}(\mathfrak{p}_{F\cap G}, \mathfrak{q}_{F\cap G}) \subseteq \mathcal{S}^{>}(\mathfrak{p}_F, \mathfrak{q}_F) \cup \mathcal{S}^{>}(\mathfrak{q}_F, \mathfrak{q}_G) \cup \mathcal{S}^{=}(\mathfrak{p}_F, \mathfrak{q}_F, \mathfrak{p}_G, \mathfrak{q}_G) \tag{2.9}$$

holds for any pair F, G with $\dim(F) = \dim(G)$. Now we prove, that (2.9) essentially is equivalent to the claim in (2.8):

Assume $H \in \mathscr{F}(C)$ with $0 \leq \dim(H) \leq d-2$, and choose two $F, G \in \mathscr{F}_N(C)$ with $N > \dim(H)$ such that $H = F \cap G$. Then (2.8) essentially claims that $\mathcal{S}^{\geq}(\mathfrak{p}_H, \mathfrak{q}_G)$ is contained in the complement of both artifact$(\mathfrak{q}_F, \mathfrak{q}_G)$ and artifact$(\mathfrak{p}_F, \mathfrak{q}_F)$. For each point x in the complement of artifact$(\mathfrak{p}_F, \mathfrak{q}_F)$, either $x \in \mathcal{S}^{=}(\mathfrak{p}_F, \mathfrak{p}_G)$ holds, or one of both polynomials is *strictly* positive in x:

$$\left(\ \mathcal{S}^{\geq}(-\mathfrak{p}_F, -\mathfrak{p}_G) \setminus \mathcal{S}^{=}(\mathfrak{p}_F, \mathfrak{p}_G) \ \right)^c \quad = \quad \mathcal{S}^{>}(\mathfrak{p}_F) \cup \mathcal{S}^{>}(\mathfrak{p}_G) \cup \mathcal{S}^{=}(\mathfrak{p}_F, \mathfrak{p}_G)$$

(For a visualization of the complement see Figure 2.7). The corresponding holds for artifact$(\mathfrak{q}_F, \mathfrak{q}_G)$, with (2.9) leading to

$$\begin{array}{lllll}
\mathcal{S}^{\geq}(\mathfrak{p}_H, \mathfrak{q}_H) \subseteq & & & & \\
\subseteq & \mathcal{S}^{>}(\mathfrak{p}_F, \mathfrak{q}_F) & \cup\ \mathcal{S}^{>}(\mathfrak{q}_F, \mathfrak{q}_G) & \cup\ \mathcal{S}^{=}(\mathfrak{p}_F, \mathfrak{q}_F, \mathfrak{p}_G, \mathfrak{q}_G) & \\
\subseteq & [\ \mathcal{S}^{>}(\mathfrak{p}_F) & \cup\ \mathcal{S}^{>}(\mathfrak{p}_G) & \cup\ \mathcal{S}^{=}(\mathfrak{p}_F, \mathfrak{p}_G) &] \\
\cap & [\ \mathcal{S}^{>}(\mathfrak{q}_F) & \cup\ \mathcal{S}^{>}(\mathfrak{q}_G) & \cup\ \mathcal{S}^{=}(\mathfrak{q}_F, \mathfrak{q}_G) &] \\
= & \multicolumn{4}{l}{[\mathrm{artifact}(\mathfrak{p}_F, \mathfrak{p}_G)]^c \ \cap\ [\mathrm{artifact}(\mathfrak{q}_F, \mathfrak{q}_G)]^c.}
\end{array}$$

This proves Lemma 2.5.1 since with the polynomials constructed in the next chapter, (2.8) holds. $\square$

In review of Algorithm 2.4.5, Lemma 2.5.1 proves that there are polynomials $\mathfrak{p}_F, \mathfrak{q}_F$ such that replacing these polynomials by their products in the description (2.6) of C results in (2.7), a description of C involving $2d$ polynomials.

A geometric view on the algorithm

The main message of Lemma 2.5.1 is the following: One can construct support-polynomials $\mathfrak{p}_F, \mathfrak{q}_F$ for faces of dimension $\dim(F) = 0, \dots d-2$, such that defining

$$\mathfrak{P}_k := \prod_{F \in \mathscr{F}_k(C)} \mathfrak{p}_F \qquad \text{and} \qquad \mathfrak{Q}_k := \prod_{F \in \mathscr{F}_k(C)} \mathfrak{q}_F.$$

leads to

$$\begin{aligned}
C &= \bigcap_{k=0}^{d-1} \bigcap_{F \in \mathscr{F}_k(C)} \mathcal{S}^{\geq}(\mathfrak{p}_F,\, \mathfrak{q}_F) \\
&= \bigcap_{k=d-1}^{d-1} \mathcal{S}^{\geq}(\mathfrak{P}_k,\, \mathfrak{Q}_k) \ \cap\ \bigcap_{k=0}^{d-2} \bigcap_{F \in \mathscr{F}_k(C)} \mathcal{S}^{\geq}(\mathfrak{p}_F,\, \mathfrak{q}_F) \\
&= \bigcap_{k=d-2}^{d-1} \mathcal{S}^{\geq}(\mathfrak{P}_k,\, \mathfrak{Q}_k) \ \cap\ \bigcap_{k=0}^{d-3} \bigcap_{F \in \mathscr{F}_k(C)} \mathcal{S}^{\geq}(\mathfrak{p}_F,\, \mathfrak{q}_F) \qquad (2.10)\\
&\ \ \vdots \\
&= \bigcap_{k=1}^{d-1} \mathcal{S}^{\geq}(\mathfrak{P}_k,\, \mathfrak{Q}_k) \ \cap\ \bigcap_{k=0}^{0} \bigcap_{F \in \mathscr{F}_k(C)} \mathcal{S}^{\geq}(\mathfrak{p}_F,\, \mathfrak{q}_F).
\end{aligned}$$

This results in a description of C using $2d$ polynomials, since there is exactly one face of C with dimension 0, leading to $\bigcap_{H \in \mathscr{F}_0(C)} \mathcal{S}^{\geq}(\mathfrak{p}_H,\, \mathfrak{q}_H) = \mathcal{S}^{\geq}(\mathfrak{p}_0,\, \mathfrak{q}_0)$. Two of the polynomials, namely $\mathfrak{Q}_{d-1}$ and $\mathfrak{Q}_{d-2}$ are redundant in such a description. This is shown in the following chapter, leading to a description of C using $2d-2$ polynomials.

The idea behind (2.10) is, that any point in some set $\mathcal{S}^{\geq}(\mathfrak{P}_k, \mathfrak{Q}_k)$ not contained in $\bigcap_{F \in \mathscr{F}_k(C)} \mathcal{S}^{\geq}(\mathfrak{p}_F, \mathfrak{q}_F)$, is part of some artifact of the form $\mathrm{artifact}(\mathfrak{p}_F, \mathfrak{p}_G)$ or $\mathrm{artifact}(\mathfrak{q}_F, \mathfrak{q}_G)$ with $\dim(F) = \dim(G) = k$. But such sets are removed from C using $\mathcal{S}^{\geq}(\mathfrak{p}_{F \cap G}, \mathfrak{q}_{F \cap G})$, so in each line of (2.10), equality holds.

The reason why Lemma 2.5.1 works, is the geometry of the sets $\mathcal{S}^{\geq}(\mathfrak{p}_F, \mathfrak{q}_F)$, which we examine now, before they are constructed in the next chapter.

2.5.2 Motivation for the geometric invariant

Separating semi-algebraic sets is non-trivial

The geometric invariant stated in Lemma 2.5.1 basically expresses, that there are two polynomials, which separate the basic closed semi-algebraic set C from the union of several semi-algebraic sets. This insight is non trivial. In fact, the separation of semi-algebraic sets has attracted attention during the last years (cf. [AAB99], [Brö88], and [ABR96])

The construction used in this work leads to one of the common examples of two semi-algebraic sets which can not be separated using a single polynomial inequality. However, the separation is possible with the help of square roots or rational functions $\mathfrak{p}(x)/\mathfrak{q}(x)$, due to the Minkowski separation theorem (see [ABR96]). Implicitly, we construct a rational function $\mathfrak{p}(x)/\mathfrak{q}(x)$ for separation, but instead of using the inequality $\mathfrak{q}(x)/\mathfrak{p}(x) \geq 0$, two inequalities $\mathfrak{p}(x) \geq 0$ and $\mathfrak{q}(x) \geq 0$ are used. This is depicted in Figure 2.6.

All polynomials constructed define closed cones

Lemma 2.5.1 holds due to special properties of the polynomials $\mathfrak{p}_F, \mathfrak{q}_F$ *constructed* later. To obtain a continuous argumentation, such properties should be valid for all $\mathfrak{p}_F$ and $\mathfrak{q}_F$. So one can only impose properties, that the linear polynomials defining C posses: Each such linear polynomial $\mathfrak{f}_F$ is support-polynomial for a facet F and

i) $\mathcal{S}^{\geq}(-\mathfrak{f}_F)$ is a cone, which is topologically closed,
ii) $\mathcal{S}^{\geq}(-\mathfrak{f}_F)$ has lineality-set $\mathrm{lin}(F)$, meaning $\quad$ (2.11)
$\mathcal{S}^{\geq}(-\mathfrak{f}_F) = \mathcal{S}^{\geq}(-\mathfrak{f}_F) + \mathrm{lin}(F)$.

These properties are also valid for each $\mathcal{S}^{\geq}(-\mathfrak{p}_F)$ and $\mathcal{S}^{\geq}(-\mathfrak{q}_F)$ we will construct. Here we would like to *motivate*, that these special *geometric* properties are the main reason why claim 2.5.1 works *geometrically*.

Motivating Lemma 2.5.1 by iteration

We motivate Lemma 2.5.1 by showing that if all polynomials $\mathfrak{p}_F, \mathfrak{q}_F$ have the properties in (2.11), then (2.8) can hold. This is done using an iteration: As a start by definition all polynomials $\mathfrak{p}_F, \mathfrak{q}_F$ with $\dim(F) = d-1$ have the properties in (2.11).

Fix $0 \leq N \leq d-2$ and assume all $\mathfrak{p}_F, \mathfrak{q}_F$ with $\dim(F) > N$ have the properties in (2.11). Let $H \in \mathscr{F}_N(C)$ be a face of dimension N, then there are $\mathfrak{p}_H, \mathfrak{q}_H$ having properties in (2.11), such that (2.8) holds. To motivate this the geometry of the set to be removed by $\mathcal{S}^{\geq}(\mathfrak{p}_H, \mathfrak{q}_H)$, is examined:

Reviewing (2.8) – the essence of Lemma 2.5.1 – from the perspective of the face H, the set $\mathcal{S}^{\geq}(\mathfrak{p}_H, \mathfrak{q}_H)$ has to remove a "hole bunch" of sets from C: For any pair of faces $F \neq G$ of equal dimension with $F \cap G = H$, one has to ensure $\mathcal{S}^{\geq}(\mathfrak{p}_H, \mathfrak{q}_H)$ separates the appropriate artifacts from C. The restrictions on F and G imply $\dim(F) = \dim(G) > \dim(H)$, and so the union of all the corresponding artifacts can be written as

$$\mathcal{A}_H \;:=\; \bigcup_{\dim(H)<k\leq d-1} \; \bigcup_{\substack{H=F\cap G\\ F,G\in\mathscr{F}_k(C)}} \; [\,\mathrm{artifact}(\mathfrak{p}_F, \mathfrak{p}_G) \cup \mathrm{artifact}(\mathfrak{q}_F, \mathfrak{q}_G)\,]$$

By definition, for any support-polynomial $\mathfrak{f}_F$ of a face $F \in \mathscr{F}(C)$, $\mathfrak{f}_F$ zeroes in F, and thus $\mathrm{lin}(F) \subseteq \mathcal{S}^{=}(\mathfrak{f}_F)$ holds. So the artifact of two support-polynomials $\mathfrak{f}_F, \mathfrak{f}_G$ is contained in

$$\begin{aligned} \mathrm{artifact}(\mathfrak{f}_F, \mathfrak{f}_G) &= \mathcal{S}^{\geq}(-\mathfrak{f}_F, -\mathfrak{f}_G) \setminus \mathcal{S}^{=}(\mathfrak{f}_F, \mathfrak{f}_G) \\ &\subseteq \mathcal{S}^{\geq}(-\mathfrak{f}_F, -\mathfrak{f}_G) \setminus \mathrm{lin}(F \cap G). \end{aligned}$$

Here the first line follows from the definition for an artifact. This leads to $\mathcal{A}_H \subseteq \mathcal{B}_H \setminus \mathrm{lin}(H)$, defining

$$\mathcal{B}_H \;:=\; \bigcup_{\dim(H)<N\leq d-1} \; \bigcup_{\substack{H=F\cap G\\ F,G\in\mathscr{F}_N(C)}} \; [\,\mathcal{S}^{\geq}(-\mathfrak{p}_F, -\mathfrak{p}_G) \cup \mathcal{S}^{\geq}(-\mathfrak{q}_F, -\mathfrak{q}_G)\,].$$

So to remove $\mathcal{A}_H$ from C, it suffices to remove $\mathcal{B}_H \setminus \mathrm{lin}(H)$ from C. To this end we examine the geometric properties of $\mathcal{B}_H$, namely those in (2.12). These properties allow for a semi-algebraic cone $\mathcal{S}^{\geq}(\mathfrak{p}_H, \mathfrak{q}_H)$ removing $\mathcal{B}_H \setminus \mathrm{lin}(H)$ from C– see Figure 2.5.

Due to the properties in (2.11), each set $\mathcal{S}^{\geq}(-\mathfrak{p}_F, -\mathfrak{p}_G)$ in the definition of $\mathcal{B}_H$ is a closed cone – see (2.11) *i*) – with lineality-set $F \cap G = H$ – see (2.11) *ii*). Moreover each set $\mathcal{S}^{\geq}(-\mathfrak{p}_F, -\mathfrak{p}_G)$ intersects C in

$F \cap G = H$ – see (2.5) on page 30. So as a finite union of such sets, $\mathcal{B}_H$ has the following three properties:

$$\begin{array}{lllr} \text{a)} & \mathcal{B}_H \text{ is a (closed) cone} & \text{due to (2.11) } i), & \\ \text{b)} & \mathcal{B}_H = \mathcal{B}_H + \operatorname{lin}(H) & \text{due to (2.11) } ii) \text{ and,} & (2.12) \\ \text{c)} & \mathcal{B}_H \cap C = H & \text{due to (2.5).} & \end{array}$$

For a visualization see Figure 2.5 ii). Property *b)* ensures that the

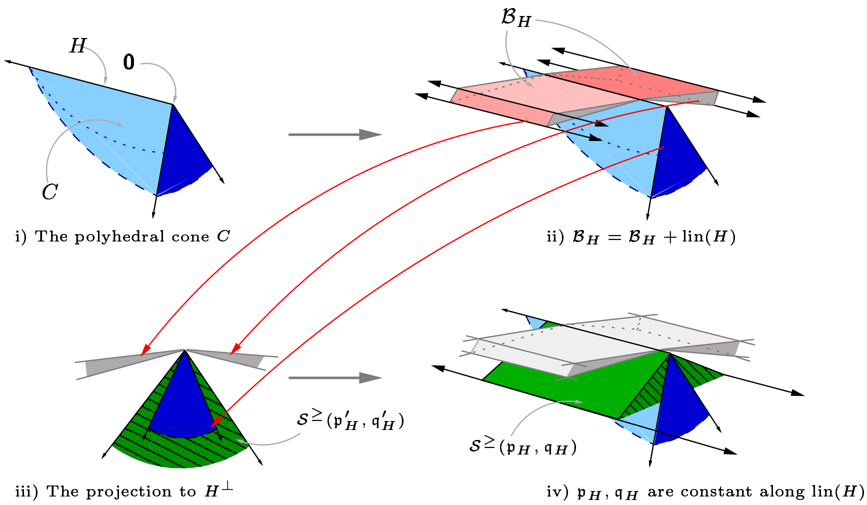


Fig. 2.5: $\mathcal{S}^{\geq}(\mathfrak{p}_H, \mathfrak{q}_H)$ removes $\mathcal{B}_H$ from C, where $\mathcal{B}_H = \mathcal{B}_H + \operatorname{lin}(H)$ and $\mathcal{B}_H \cap C = H$.

geometry of $\mathcal{B}_H$ is determined by its projection to $H^{\perp}$: Let π_H be the orthogonal projection of $\mathbb{R}^d$ onto $H^{\perp}$, then one has $\mathcal{B}_H = \pi_H(\mathcal{B}_H) + \operatorname{lin}(H)$. Both the projections of C and $\mathcal{B}_H$ are closed cones, intersecting in $\pi_H(C) \cap \pi_H(\mathcal{B}_H) = \pi_H(H) = \{\mathbf{0}\}$ – see Figure 2.5 iii).

So the task of removing $\mathcal{B}_H \setminus \operatorname{lin}(H)$ from C, boils down to removing $\pi_H(\mathcal{B}_H) \setminus \{\mathbf{0}\}$ from the pointed polyhedral cone $\pi_H(C)$. Here $\pi_H(\mathcal{B}_H)$ is the finite union of some pointed, closed cones. This can be done using a semi-algebraic cone $\mathcal{S}^{\geq}(\mathfrak{p}'_H\mathfrak{q}'_H)$, which is an "appropriate" approximation to C, see Figure 2.6. Here $\mathfrak{p}'_H$ is a linear polynomial, such that $\mathcal{S}^{\geq}(\mathfrak{p}'_H)$ is a supporting-half-space of H. While $\mathfrak{q}'_H$ is a polynomial of the form $(\mathfrak{p}'_H)^{2k} - \sum_{i=1}^{m} (\mathfrak{l}'_i)^{2k}$ with some linear polynomials $\mathfrak{l}'_i$. An example for such a pointed cone $\mathcal{S}^{\geq}(\mathfrak{p}'_H, \mathfrak{q}'_H)$ is the well known cone $\{x \in \mathbb{R}^3 \,:\, x_3 \geq 0,\, (x_3)^2 - (x_1)^2 - (x_2)^2 \geq 0\}$.

We define $\mathfrak{p}_H(x) := \mathfrak{p}'_H(\pi(x))$ and $\mathfrak{q}_H := \mathfrak{q}'_H(\pi(x))$, which means that $\mathfrak{p}_H(x + y) = \mathfrak{p}_H(x)$ holds for all $y \in \operatorname{lin}(F)$, and similarly for $\mathfrak{q}_H$. So both resulting polynomials are constant in direction of $\operatorname{lin}(H)$, leading to $\mathcal{S}^{\geq}(\mathfrak{p}_H, \mathfrak{q}_H) = \mathcal{S}^{\geq}(\mathfrak{p}_H, \mathfrak{q}_H) + \operatorname{lin}(H)$. Moreover the resulting set $\mathcal{S}^{\geq}(\mathfrak{p}_H, \mathfrak{q}_H)$ removes $\mathcal{B}_H \setminus \operatorname{lin}(H)$ from C (see Figure 2.5 *iv*).

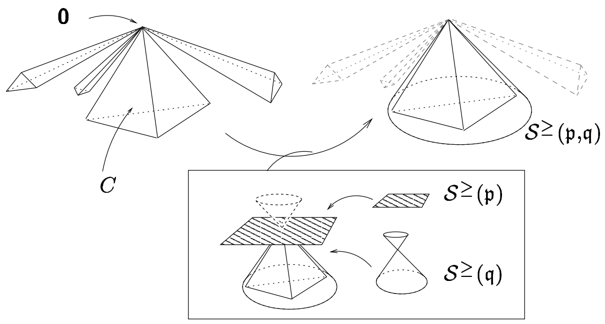


Fig. 2.6: An approximation $\mathcal{S}^{\geq}(\mathfrak{p},\mathfrak{q})$ removes several closed cones *simultaneously* from a polyhedral cone C.

Aside from this the polynomials have the appropriate properties (2.11): By definition of $\mathfrak{p}_H, \mathfrak{q}_H$ are independent of $\mathrm{lin}(F)$ leading to property (2.11) *ii*). Since $\mathfrak{p}'_H$ is linear and since $\mathfrak{q}_H$ is the sum of polynomials of *even* degree, both $\mathcal{S}^{\geq}(\mathfrak{p}'_H)$ and $\mathfrak{q}'_H$ are closed cones – which is property (2.11) *i*). Thus separating $\mathcal{B}_H$ from C can be done using two polynomials $\mathfrak{p}_H$, $\mathfrak{q}_H$ with the desired properties (2.11).

2.5.3 Example: Where to put correction sets

Figure 2.7 shows a brief example of placing a "correction set" into the complement of an artifact. It visualizes $\mathcal{S}^{\geq}(\mathfrak{f}_1,\mathfrak{f}_2) = \mathcal{S}^{\geq}(\mathfrak{f}_1 * \mathfrak{f}_2, \mathfrak{f}_3)$. The sign of each polynomial is indicated by '$-$' and '$+$' in each area. For simplicity, assume that $\mathfrak{f}_i(x) > 0$ holds for each x in the topological interior of each set $\mathcal{S}^{\geq}(\mathfrak{f}_i)$.

Image 1 shows the set $\mathcal{S}^{\geq}(\mathfrak{f}_1 * \mathfrak{f}_2)$ (green), this set contains both $\mathcal{S}^{\geq}(\mathfrak{f}_1,\mathfrak{f}_2)$ and $\mathcal{S}^{\geq}(-\mathfrak{f}_1,-\mathfrak{f}_2)$. The intersection of these sets, $\mathcal{S}^{\geq}(-\mathfrak{f}_1,-\mathfrak{f}_2)\cap \mathcal{S}^{\geq}(\mathfrak{f}_1,\mathfrak{f}_2) = \mathcal{S}^{=}(\mathfrak{f}_1,\mathfrak{f}_2)$, belongs to the originally given set $\mathcal{S}^{\geq}(\mathfrak{f}_1,\mathfrak{f}_2)$ and so the undesired part of $\mathcal{S}^{\geq}(\mathfrak{f}_1 * \mathfrak{f}_2)$ has the form

$$\mathcal{S}^{\geq}(\mathfrak{f}_1 * \mathfrak{f}_2) \setminus \mathcal{S}^{\geq}(\mathfrak{f}_1,\mathfrak{f}_2) \quad = \quad \mathcal{S}^{\geq}(-\mathfrak{f}_1,-\mathfrak{f}_2) \setminus \mathcal{S}^{=}(\mathfrak{f}_1,\mathfrak{f}_2).$$

Image 2 shows the removal of this undesired part of $\mathcal{S}^{\geq}(\mathfrak{f}_1 * \mathfrak{f}_2)$, called artifact, through $\mathcal{S}^{\geq}(\mathfrak{f}_3)$ (orange). The "correction-set" $\mathcal{S}^{\geq}(\mathfrak{f}_3)$ is placed in the complement of $\mathcal{S}^{\geq}(-\mathfrak{f}_1,-\mathfrak{f}_2)\setminus\mathcal{S}^{=}(\mathfrak{f}_1,\mathfrak{f}_2)$. For each point x of this complement, either $x \in \mathcal{S}^{=}(\mathfrak{f}_1,\mathfrak{f}_2)$ holds, or one of both polynomials is *strictly* positive in x:

$$\left(\ \mathcal{S}^{\geq}(-\mathfrak{f}_1,-\mathfrak{f}_2) \setminus \mathcal{S}^{=}(\mathfrak{f}_1,\mathfrak{f}_2)\ \right)^c \quad = \quad \mathcal{S}^{>}(\mathfrak{f}_1) \cup \mathcal{S}^{>}(\mathfrak{f}_2) \cup \mathcal{S}^{=}(\mathfrak{f}_1,\mathfrak{f}_2).$$

Sets of this form are part of the central proof. In Image 2, $\mathcal{S}^{>}(\mathfrak{f}_1)$ and $\mathcal{S}^{>}(\mathfrak{f}_2)$ are easy to find, they are the interior of the two discs $\mathcal{S}^{\geq}(\mathfrak{f}_1)$ and $\mathcal{S}^{\geq}(\mathfrak{f}_2)$, while $\mathcal{S}^{=}(\mathfrak{f}_1,\mathfrak{f}_2)$ are the two indicated points where their

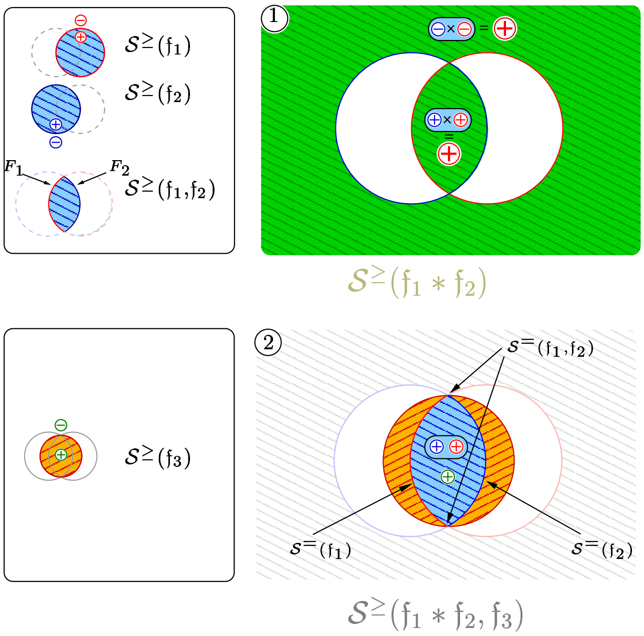


Fig. 2.7: Two steps: 1st multiplying polynomials, 2nd adding a correction-polynomial

outer hull intersect. The set $\mathcal{S}^{\geq}(\mathfrak{f}_3)$ is mostly contained in the first, the border of $\mathcal{S}^{\geq}(\mathfrak{f}_3)$ neatly passes through the latter.

2.6 Summary

Using semi-algebraic cones which approximate C, it is possible to describe C using $2d$ polynomial inequalities. The d in $2d$ results from replacing support-polynomials for faces of equal dimension by their product. The changes in the resulting set can be undone using 2 support-polynomials for lower dimensional faces, leading to the 2 in $2d$. This is due to two invariants, namely:

- Support-polynomials for faces of dimension k can be replaced by their product when having appropriate support-polynomials $k-1$ in the description.
- Closed cones which intersect C in F can be removed using a set $\mathcal{S}^{\geq}(\mathfrak{p}_F, \mathfrak{q}_F)$.

When replacing polynomials by products the inevitably resulting additional points are contained in sets $\mathcal{S}^{\geq}(-\mathfrak{p}_F, -\mathfrak{p}_G)$ or $\mathcal{S}^{\geq}(-\mathfrak{q}_F, -\mathfrak{q}_G)$. These are closed cones that can be removed from the resulting set using appropriate approximations to C, built on support-polynomials for faces of lower dimensions. So more or less each set $\mathcal{S}^{\geq}(\mathfrak{p}_F, \mathfrak{q}_G)$ mimics the geometry of C in F.

The Main Lemmata 2.4.2 and 2.5.1 state these invariants. Lemma 2.5.1 is proven by anticipating the following construction using (2.9) which is the central line (3.17) of Lemma 3.5.4.

3. Construction

3.1 Introduction

The task of this chapter is to prove the Main Theorem, by constructing a semi-algebraic description for d-dimensional polyhedra using $2d$ polynomial inequalities. As proved in the first chapter, this can be reduced to constructing a semi-algebraic description for polyhedral cones involving $2d-2$ polynomials. Concerning this, the previous chapter motivates that the key to constructing the corresponding polynomials is the construction of a controlled approximation for polyhedral cones.

Therefore the first part of this chapter introduces a construction for an approximation of a polyhedral cone. With the polynomials constructed, the Main Theorem then is proven.

3.1.1 The constructed polynomials

In the following let $C = \{x \in \mathbb{R}^d \,:\, a_i \cdot x \geq 0,\ i = 1, \ldots, m\}$ be a d-dimensional, pointed, polyhedral cone, given in irredundant description. The previous chapter motivated the following: If a special semi-algebraic cone $\mathcal{S}^{\geq}(\mathfrak{p}_F, \mathfrak{q}_F)$ is constructed for each proper face $F \in \mathscr{F}(C)$, the resulting polynomials can be used in a description of C and then replaced by their products. Roughly speaking, when replacing polynomials by their products in a semi-algebraic description of C, the introduction of $\mathfrak{p}_F, \mathfrak{q}_F$ ensures that C does not change close to F. To this end it is crucial to control how good $\mathcal{S}^{\geq}(\mathfrak{p}_F, \mathfrak{q}_F)$ approximates C.

Each semi-algebraic cone $\mathcal{S}^{\geq}(\mathfrak{p}_F, \mathfrak{q}_F)$ is constructed as an approximation to a relaxation of C, the *face cone* $C_{[F]}$ (cf. definition 3.2.2). This polyhedral cone inherits all relevant geometric information about the face F, passing it on to its approximation $\mathcal{S}^{\geq}(\mathfrak{p}_F, \mathfrak{q}_F)$.

The approximation itself is based on an an idea of Minkowski (cf. [Min03]) for the approximation of bounded polytopes. Therefore in the following several statements for bounded polytopes are used. The sets appearing in the construction are mostly unbound, and so boundedness is introduced by intersection with hyper-planes. Moreover, the construction uses both the $\mathcal{H}$- and the $\mathcal{V}$-description of the given face-cone. Therefore each set and constant involved is defined for both descriptions.

3.2 Polyhedra used approximating C

3.2.1 Reduction to bounded sets by vertex figures

The vertex figure

The approximation used here evolves from an approximation for polytopes given by Minkowski. A good way to bring in polytopes when working with cones is to use vertex-figures: The intersection of a pointed polyhedral cone C with an appropriate hyperplane results in a bounded polytope - a vertex figure P of C. To obtain such a vertex-figure a support-vector u of the apex of the appropriate polyhedral cone is used, namely $P := C \cap \{x \in \mathbb{R}^d \ : \ (u \cdot x) = 1\}$.

Measuring approximation quality using a linear function

The set C is the infinite union of parallel vertex-figures, e.g. one has $C = \cup_{\lambda \geq 0} \lambda P$. In each such slice $u \cdot x$ is constant, $u \cdot x = \lambda$ holds for any $x \in \lambda P$. So to obtain an approximation to C, from an approximation to P, will involve the usage of $u \cdot x$. If one achieves to approximate P "ε-good" with a semi-algebraic set, then each set λP can be approximated "$\lambda\varepsilon$-good" with a similar set. This leads to the (rough) insight, that for an approximation to C, the measurement $u \cdot x$ with $x \in C$ is a criterion of measuring the quality of approximation.

3.2.2 Support-vectors

The approach followed in this work involves constructing an approximation for each proper face of C. Therefore a support-vector u_F will be needed for each face $F \in \mathscr{F}(C)$. To ease notation, we fix some arbitrary choice of support-vectors.

Definition 3.2.1. *For each proper face $F \in \mathscr{F}(C)$ choose the support-vector $u_F := \sum_{i \in \mathscr{F}(C)} a_i$.*

This choice is made purely in desire to ease notation. Without fixing the support-vectors, virtually every object in the following would have an additional index. *The construction works with any other choice of* *support-vectors, no additional properties of this special choice are used in the following.*

3.2.3 The set to approximate: The face cone

A face cone is a relaxation of C

In the following let $F \in \mathscr{F}(C)$ be a proper face of C, *proper* meaning that both $F \neq C$ and $F \neq \emptyset$ hold. For F, a polyhedral cone called the face-cone is defined, which is a relaxation of C, which represents the geometrical properties of F.

Definition 3.2.2. *For a proper face $F \in \mathscr{F}(C)$ let $[F]$ denote the set of all indices of linear inequalities incident to F:*

$$[F] := \{i \in \{1, \ldots, m\} \ : \ (a_i \cdot x) = 0 \textit{ for all } x \in F\}.$$

The face-cone $C_{[F]}$ *of F, is defined by all inequalities incident to F:*

$$C_{[F]} := \{x \in \mathbb{R}^d \ : \ a_i \cdot x \geq 0 \ , \ i \in [F]\}.$$

Corollary 3.2.3. *The vectors a_i, $i \in [F]$ span $F^\perp$.*

The set $F^\perp$ is the set of all vectors in $\mathbb{R}^d$ perpendicular to F. By definition, for each $i \in [F]$, one has $a_i \in F^\perp$, since $a_i \cdot x = 0$ holds for all $x \in F$. Defining $k := \dim(F)$, one finds that there is a subset of $d - k$ linearly independent vectors in $\{a_i \ : \ i \in [F]\}$. Since $\dim(F^\perp) = d - k$ holds, this implies that the vectors in $\{a_i \ : \ i \in [F]\}$ span $F^\perp$.

The face cone $C_{[F]}$ inherits geometric and combinatorial properties of F

For any face F the face-cone $C_{[F]}$ contains C, so $C_{[F]}$ can be understood as an approximation to C, which inherits the "important" geometric information of the face F. For example, for the single vertex v of C one has $C_{[v]} = C$, since any inequality defining C is incident to v. Moreover, face-cones preserve the combinatorial properties of faces:

If $G \subset F$ holds for some face $G \in \mathscr{F}(C)$, then this implies $C_{[G]} \subset C_{[F]}$. This holds since $C_{[G]}$ is defined using a subset of the inequalities defining $C_{[F]}$; the inverse inclusion $[F] \subset [G]$ holds for the index-set of incident inequalities.

The $\mathcal{V}$-representation of a face cone

In the general case $C_{[F]}$ is not pointed, since it contains $\text{lin}(F)$, but since the dimension of $F^\perp$ equals the rank of all a_i, $i \in [F]$, the cone $C_{[F]} \cap F^\perp$ is pointed. Later, for each face-cone of C, an approximation by a semi-algebraic set will be constructed. The following helps reducing the corresponding construction to approximating pointed cones:

Lemma 3.2.4. *For each face $F \in \mathscr{F}(C)$ one has*

$$C_{[F]} = C + \text{lin}(F) = (C_{[F]} \cap F^\perp) + \text{lin}(F),$$

where $C_{[F]} \cap F^\perp$ is pointed.

Proof. The inclusion $C_{[F]} \supseteq C + \text{lin}(F)$ must hold, since for any triple $i \in [F]$, $y \in C$ and $w \in \text{lin}(F)$ one has $a_i \cdot y + w = a_u \cdot y \geq 0$ and thus $y + w \in C_{[F]}$.

The reverse inclusion $C_{[F]} \supseteq C + \text{lin}(F)$ follows from two facts: First of all one has $C_{[F]} = C_{[F]} + \text{lin}(F)$, since $a_i \cdot x = 0$ holds for all $x \in \text{lin}(F)$ and $i \in [F]$. Moreover, as proven below, the projection of C to $F^\perp$ is $C_{[F]} \cap F^\perp$. These two facts imply, that for every $x \in C_{[F]}$ there is a $y \in \text{lin}(F)$ such that $x - y \in C$, implying $x \in C + \text{lin}(F)$.

Using the Fourier-Motzkin-Projection Method, it is now shown that the projection of C to $F^\perp$ is $C_{[F]} \cap F^\perp$:

Assume that $F = \text{cone}\{v_1, \ldots, v_n\}$ holds for a set of vectors v_i. Starting with C, each k-th step of the Fourier-Motzkin-Algorithm projects the previously constructed cone to $v_k^\perp$. This way C is projected to $F^\perp$ in n steps. In the first step of the algorithm the set of all a_i is sorted into groups, depending on the sign of $a_i \cdot v_1$. Since $v_1 \in C$ holds, this sign is never negative, and due to the properties of the algorithm the projection of C to $v_1^\perp$ is $\{x \in \mathbb{R}^d \,:\, a_i \cdot x \geq 0\,,\, i \in [\text{cone}(v_1)]\} \cap v_1^\perp$. Here $[\text{cone}(v_1)]$ is the index set of all inequalities incident to the one-dimensional-face $\text{cone}(v_1)$ of C. Each following projection step follows the same rules leading to the set

$$\{x \in \mathbb{R}^d \,:\, a_i \cdot x \geq 0\,,\, i \in [\text{cone}(v_1, \ldots, v_k)]\} \cap \text{lin}\{v_1, \ldots, v_k\}^\perp$$

after k steps. This set is exactly $C_{[F]} \cap F^\perp$ for $k = n$.

The cone $C_{[F]} \cap F^\perp$ is pointed, since $\text{lin}(F)$ is a face of $C_{[F]}$; therefore $\text{lin}(F) \cap F^\perp = \{\mathbf{0}\}$ is a *vertex* of $C_{[F]} \cap F^\perp$, which thus is pointed. $\square$

Corollary 3.2.5 ($\mathcal{V}$-representation). *Let $V \in \mathbb{R}^d$ be a finite set such that $C = \text{cone}(V)$. Then for each face $F \in \mathscr{F}(C)$ one has*

$$C_{[F]} = \text{cone}\{v \in V \,:\, v \notin F\} + \text{lin}(F).$$

3.3 Principles of approximating cones

In the following we will construct a semi-algebraic set, which approximates a given polyhedral cone. The approximation itself will be a cone and will have the same lineality-space as the given polyhedral cone. Roughly speaking, the quality of approximation will be measured by determining how far it is for each point in C to the outside of the approximation. This is measured using an auxiliary polyhedron Δ which has the same lineality set as the given polyhedral cone C. For each point in C, the approximation provides some space of the form $x + \varepsilon(u \cdot x)\Delta$, where u is a support-vector for the apex of C.

At first, the general idea used in this approximation is examined:

3.3.1 Approximating a bounded polytope

Approximation of an interval

The overall construction of approximating cones is based on an approximation to polytopes by Minkowski presented in [Min03]. Since the approximation is the central element of this thesis, we would like to give a brief introduction to how these approximations work. The technique Minkowski introduced, makes use of approximating intervals by a single polynomial inequality. The main idea is to replace absolute values by a quadratic functions in inequalities, e.g. $|x| \leq 1$ is equivalent to $x^2 \leq 1$.

So any Interval $I = [m - r, m + r] \subset \mathbb{R}$, with midpoint m and radius $r \geq 0$, can be described using a single polynomial of even degree, i.e., $I = \{y \in \mathbb{R} \ : \ ((y - m)/r)^{2k} \leq 1\}$, with $k \in \mathbb{N}$. From this description an approximation to I can be derived, namely the set $\{y \in \mathbb{R} \ : \ ((y - m)/r)^{2k} \leq \mathbf{2}\}$. The last polynomial inequality transforms to $|y - m| \leq r \sqrt[2k]{2}$, where $\sqrt[2k]{2}$ converges to 1 as k goes to infinity.

Approximating polytopes, the intersection of "multi-dimensional intervals"

Replacing y above by a linear polynomial $a \cdot x$, the set enclosed between two parallel hyperplanes in $\mathbb{R}^d$ can be described or approximated using a single polynomial inequality. Every bounded polytope $P = \{x \in \mathbb{R}^d \ : \ a_i \cdot x \geq b_i, \ i = 1, \dots, n\}$ can be understood as the intersection of such sets: Using the support-function $h(a_i, P) := \max\{a_i \cdot x \ : \ x \in P\}$ one has

$$P = \{x \in \mathbb{R}^d \ : \ b_i \leq a_i \cdot x \leq h(a_i, P), \ i = 1, \dots, n\}.$$

So defining m_i, r_i such that $[b_i, h(a_i,P)] = [m_i - r_i, m_i + r_i]$ by setting

$$r_i := \frac{h(a_i,P) - b_i}{2} \quad \text{and} \quad m_i := \frac{h(a_i,P) + b_i}{2}$$

for each $i \in \{1, \ldots, n\}$, one obtains a description and and an approximation of P using monomes of even degree by observing

$$\begin{array}{rllll} P & = & \{x \in \mathbb{R}^d : & |a_i \cdot x - m_i| \;\; /r_i & \leq 1, \;\; i \in \{1, \ldots, n\}\} \\ & = & \{x \in \mathbb{R}^d : & (a_i \cdot x - m_i)^{2k}/r_i^{2k} & \leq 1, \;\; i \in \{1, \ldots, n\}\} \\ & \subseteq & \{x \in \mathbb{R}^d : & \sum_{i=1}^n (a_i \cdot x - m_i)^{2k}/r_i^{2k} & \leq n \quad \} =: \mathcal{P}_k \end{array}$$

All summands in the definition of $\mathcal{P}_k$ are positive, so for $x \in \mathcal{P}_k$ the single polynomial inequality "simultaneously" implies $(a_i \cdot x - m_i)^{2k} \leq r_i^{2k}\, n$ for each i, equivalent to $|a_i \cdot x - m_i| \leq r_i \sqrt[2k]{n}$. Therefore $\mathcal{P}_k$ approximates P arbitrarily good, when $k \in \mathbb{N}$ is chosen large. For a given $\varepsilon > 0$ choose $k \in \mathbb{N}$ such that $\sqrt[2k]{n} \leq 1 + \varepsilon$ holds, then the above leads to the following set, bounding $\mathcal{P}_k$:

$$\begin{array}{rlllll} \mathcal{P}_k & \subseteq & \{x \in \mathbb{R}^d : & (a_i \cdot x - m_i)^{2k} & \leq r_i^{2k} n, & i \in \{1, \ldots, n\}\} \\ & \subseteq & \{x \in \mathbb{R}^d : & |a_i \cdot x - m_i| & \leq r_i\,(1+\varepsilon), & i \in \{1, \ldots, n\}\} \\ & \subseteq & \{x \in \mathbb{R}^d : & a_i \cdot x & \geq b_i - \varepsilon r_i, & i \in \{1, \ldots, n\}\} \end{array}$$

Here the last inclusion derives from $b_i = m_i - r_i$. *Geometrically*, the last set bounding $\mathcal{P}_k$ is P "inflated", i.e., the half-spaces defining P have each been "moved away" by εr_i from P. With this bounding, one finds that with k going to infinity – and thus ε going to zero – $\mathcal{P}_k$ converges to P, since $b_i - \varepsilon r_i$ goes to b_i.

The remaining question is "how good" $\mathcal{P}_k$ approximates P. This can be answered using the auxiliary polytope $\Delta := \{x \in \mathbb{R}^d \,:\, |a_i \cdot x| \leq 2r_i\}$. Utilizing special properties of the barycenter of P, one can show that $\mathcal{P}_k$ is bounded by the Minkowski-sum

$$\mathcal{P}_k \subseteq P + \varepsilon\, d\, \Delta.$$

So how close $\mathcal{P}_k$ really is to P depends on the geometric properties of Δ, which are examined later on.

3.3.2 The "radius" of a face-cone is a linear function

The difficulty that arises when approximating polyhedral cones, is that here "r_i" is a linear function, not a constant. For P which is bounded, the relative width $2r_i = h(a_i,P) - h(-a_i,P)$ is bounded for any direction $a_i \in \mathbb{R}^d$. This is not true in general for a pointed polyhedral cone, but it holds true for any of its vertex-figures.

Calculating the "radius" of $C_{[F]}$ using a vertex-figure P^F

In the following let $C = \{x \in \mathbb{R}^d \,:\, a_i \cdot x \geq 0\}$ be a pointed polyhedral cone. Further for each face $F \in \mathscr{F}(C)$, let u_F be a fixed support-vector of F. To obtain a measure of relative width in each direction a_i, define the following:

Definition 3.3.1. *For any proper face $F \in \mathscr{F}(C)$ with support-vector u_F let P^F be the following vertex-figure of $C_{[F]}$:*

$$P^F := \{x \in C_{[F]} \,:\, u_F \cdot x = 1\}.$$

For every index $i \in [F]$, define $r_i^F := max\{(a_i \cdot x)/2 \,:\, x \in P^F\}$, where $2r_i^F$ is the relative width of P^F in direction of a_i.

Note that although the vertex-figure P^F not necessarily is bounded, r_i^F is finite, see Lemma 3.3.3, below. With the value of r_i^F the cone $C_{[F]}$ can be rewritten:

Lemma 3.3.2. *For any proper face $F \in \mathscr{F}(C)$, the cone $C_{[F]}$ can be rewritten as*

$$\begin{aligned} C_{[F]} &= \{x \in \mathbb{R}^d \,:\, 0 \leq (a_i \cdot x) \leq 2r_i^F(u_F \cdot x),\ i \in [F]\} && (3.1)\\ &= \{x \in \mathbb{R}^d \,:\, |a_i \cdot x - r_i^F(u_F \cdot x)| \leq r_i^F(u_F \cdot x),\ i \in [F]\} && (3.2)\end{aligned}$$

Thus for any $x \in C_{[F]}$ and $i \in [F]$ one finds $(a_i \cdot x) \leq 2r_i^F(u_F \cdot x)$.

Proof. Let $F \in \mathscr{F}(C)$ be any proper face of C. By definition, one has $C_{[F]} = \{x \in \mathbb{R}^d \,:\, 0 \leq (a_i \cdot x),\ i \in [F]\}$, so to prove (3.1) it suffices to prove the inclusion "$\subseteq$". To this end, we choose $j \in [F]$ and $y \in C_{[F]}$ arbitrarily, and prove $a_j \cdot y \leq 2r_j^F(u_F \cdot y)$:

Since u_F is a support-vector for a face of $C_{[F]}$, one has $u_F \cdot y \geq 0$. If $u_F \cdot y > 0$ holds, then this implies $(a_j \cdot y) \leq 2r_j^F(u_F \cdot y)$ by the definition of r_j^F, since $y/(u_F \cdot y) \in P^F$. If $u_F \cdot y = 0$ holds, then one has $y \in \operatorname{lin}(F)$, due to $y \in C_{[F]}$. Thus $a_j \cdot y = 2r_j^F(u_F \cdot y) = 0$ holds, by the definition of $[F]$. So in both cases y is part of the set of the right-hand-side of (3.1), proving the examined inclusion.

The set in (3.2) can be obtained from (3.1) by direct calculation, observing that for each $i \in [F]$ the value $r_i^F(u_F \cdot x) \geq 0$ is positive. Viewing r_i^F as both radius *as well* as the midpoint of the interval $[0, 2r_i^F(u_F \cdot x)]$, this calculation follows the previous discussion on intervals. ☐

To calculate r_i^F, observe the following:

Lemma 3.3.3 ($\mathcal{V}$-description). *For every pair of a proper face $F \in \mathscr{F}(C)$ and index $i \in [F]$, the value r_i^F is finite, since for any $\mathcal{V}$-description $C = \mathrm{conv}(V)$ one obtains*

$$r_i^F = \max\left\{ \frac{(a_i \cdot v)}{2(u_F \cdot v)} \;:\; v \in V,\;\; a_i \cdot v \neq 0 \right\}.$$

Proof. Let $F \in \mathscr{F}(C)$ be a proper face and choose some $i \in [F]$. The set P^F is the vertex figure of $C_{[F]}$, which can be rewritten as $C^F = \mathit{cone}\{v \in V \;: v \notin F\} + \mathrm{lin}(F)$ due to Corollary 3.2.5. For any $v \in V$, the restriction $v \in F$ is equivalent to $u_F \cdot v = 0$, implying the $\mathcal{V}$-description

$$P^F = \mathrm{conv}\left\{ \frac{v}{(u_F \cdot v)} \;: v \in V,\; u_F \cdot v \neq 0 \right\} + \mathrm{lin}(F).$$

Now the definition of $[F]$ leads to $\{v \in V \;:\; a_i \cdot v \neq 0\} \subset \{v \in V \;: u_F \cdot v \neq 0\}$. Moreover, $a_i \cdot v \geq 0$ holds for all $v \in V$, and there is at least one $v \in V$ with $a_i \cdot v > 0$. Otherwise C is not full-dimensional. This leads to

$$\begin{aligned} r_i^F &:= \max \;\; \{(a_i \cdot x)/2 \;:\; x \in P^F\} \\ &= \max \;\; \left\{ \tfrac{(a_i \cdot v)}{2(u_F \cdot v)} \;:\; v \in V,\; a_i \cdot v \neq 0 \right\}, \end{aligned}$$

proving the lemma. ☐

3.3.3 The auxiliary polyhedron Δ^F

The following polyhedron is an auxiliary object, it is used measuring the quality of approximation to face-cones. Though never explicitly appearing in the actual construction, it is used in the following proofs and so its properties have to be examined to some extent.

Definition 3.3.4. *For any proper face $F \in \mathscr{F}(C)$ of C let Δ^F be the following polyhedron:*

$$\Delta^F := \{x \in \mathbb{R}^d \;:\; |a_i \cdot x| \leq 2r_i^F,\; i \in [F]\}.$$

These polyhedra are unbounded in the general case, each containing $\mathrm{lin}(F)$, but their intersection with $F^\perp$ is bounded. In case F is a facet, the set Δ^F is the intersection of two parallel half-spaces. This must hold, since in this case there is only one possible support-vector $u_F = a_j$, where j is the only element of $[F]$. So one obtains $r_j^F = 1/2 = \max\{(a_i \cdot x)/2 \;:\; x \in P^F\}$, leading to $\Delta^F = \{x \in \mathbb{R}^d \;:\; |a_j \cdot x| \leq 1\}$.

The auxiliary polyhedron Δ^F is bounded in direction of u_F

Since $a_i \cdot x = 0$ holds for every $i \in [F]$ and $x \in \mathrm{lin}(F)$, the lineality-set of Δ^F contains $\mathrm{lin}(F)$, meaning that one has $\Delta^F = (\Delta^F \cap F^\perp) + \mathrm{lin}(F)$. Here the set $\Delta^F \cap F^\perp$ is bounded:

Lemma 3.3.5. *For each proper face $F \in \mathscr{F}(C)$ the set $\Delta^F \cap F^\perp$ is bounded and moreover for each $v \in \Delta^F$ one has*

$$|u_F \cdot v| \leq (d+1). \tag{3.3}$$

Proof. Let $F \in \mathscr{F}(C)$ be a proper face of C. Then $\Delta^F \cap F^\perp$ is bounded due to the following:

Any ray in $F^\perp$ is of the form $\mathrm{cone}(y)$, where $y \in F^\perp$. If such a ray is contained in Δ^F then $|a_i \cdot (\lambda y)| \leq 2r_i^F$ must hold for any $\lambda \geq 0$ and all $i \in [F]$, implying $a_i \cdot y = 0$ for all $i \in [F]$. The vectors a_i with $i \in [F]$ span $F^\perp$ (see Corollary 3.2.3) and thus the above implies $y \in (F^\perp)^\perp = \mathrm{lin}(F)$, leading to $y = \mathbf{0}$ due to $y \in F^\perp$. This shows that $\Delta^F \cap F^\perp$ contains the trivial ray $\{\mathbf{0}\}$ only and thus is bounded.

To prove (3.3), first restrict to the case where F is the vertex of C. The special in this case is that $C_{[F]} = C$ is pointed and thus its vertex-figure P^F is bounded. Let s be the barycenter of the *polytope* P^F and $i \in [F]$ an arbitrary index. Then with [BF34] and $h(a_i, P^F) = 2r_i^F$ and $h(-a_i, P^F) = 0$ one obtains

$$\frac{1}{d+1} 2r_i^F \leq \; a_i \cdot s \leq \frac{d}{d+1} 2r_i^F.$$

Thus $C_{[F]}$ translated by $-(d+1)s$ must contain Δ^F:

$$\begin{array}{rcll} \Delta^F & \subseteq & \{x \in \mathbb{R}^d \; : \; a_i \cdot x \geq -2r_i, & i \in [F]\} \\ & \subseteq & \{x \in \mathbb{R}^d \; : \; a_i \cdot x \geq -(d+1)(a_i \cdot s), & i \in [F]\} \\ & = & C_{[F]} - (d+1)s. & \end{array}$$

Since $u_F \cdot y \geq 0$ holds for any $y \in C_{[F]}$, the above implies that for any $x \in \Delta^F$ one has $u_F \cdot x \geq -(d+1)(u_F \cdot s) = -(d+1)$. The last derives from $u_F \cdot s = 1$, which is due to $s \in P^F$. Since Δ^F is symmetric to the origin, this implies $u_F \cdot x \leq (d+1)$ and thus shows $|u_F \cdot x| \leq d+1$.

Now if F is not the vertex of C, then one uses the above in the space $F^\perp$. One has $C_{[F]} = (C_{[F]} \cap F^\perp) + \mathrm{lin}(F)$ and $\Delta^F = (\Delta_F \cap F^\perp) + \mathrm{lin}(F)$. Here $C_{[F]} \cap F^\perp$ is pointed and so the same argumentation as above leads to $|u_F \cdot x| \leq d+1$ for any $x \in \Delta^F \cap F^\perp$. Recalling that $u_F \cdot v = 0$ holds for any $v \in \mathrm{lin}(F)$, as a whole this leads to $|u_F \cdot x| \leq d+1$ for any $x \in \Delta^F$. □

The auxiliary polyhedron Δ^F contains $P^F - P^F$

The following relates Δ^F to the vertex figure P^F. Note that thereby the Minkowski-sum $P^F - P^F$ is in general *not* just $\{\mathbf{0}\}$.

Lemma 3.3.6. *For a proper face $F \in \mathscr{F}(C)$, one finds $P^F - P^F \subseteq \Delta^F$.*

Proof. Let $i \in [F]$ be an arbitrary index of a proper face F. Each $x \in P^F$ fulfills $0 \le a_i \cdot x \le 2r_i^F$ and thus each $y \in P^F + (-P^F)$ must fulfill $|a_i \cdot y| \le 2r_i^F$, implying $y \in \Delta^F$. $\square$

3.4 Central Construction

The following is the construction of an approximation to a face-cone, whose outer hull is close (3.5), but not too close (3.4) to $C_{[F]}$:

Lemma 3.4.1. *Let $F \in \mathscr{F}(C)$ be a proper face and $0 < \varepsilon \le 1/2$. Then there is a polynomial $\mathfrak{q}_F^\varepsilon$ such that defining $\mathfrak{p}_F(x) := u_F \cdot x$ one obtains*

$$\begin{aligned} \{x + \varepsilon\omega\,(u_F \cdot x)\Delta^F \;:\; x \in C_{[F]}\} &\subseteq \mathcal{S}^{>}(\mathfrak{p}_F, \mathfrak{q}_F^\varepsilon) \cup \mathcal{S}^{=}(\mathfrak{p}_F, \mathfrak{q}_F^\varepsilon) \qquad (3.4) \\ \{x + \;\varepsilon\,(u_F \cdot x)\Delta^F \;:\; x \in C_{[F]}\} &\supseteq \mathcal{S}^{\ge}(\mathfrak{p}_F, \mathfrak{q}_F^\varepsilon), \qquad (3.5) \end{aligned}$$

where $\omega = 2^{-4}(d+1)^{-2}$. Moreover, one has $\mathcal{S}^{=}(\mathfrak{p}_F, \mathfrak{q}_F^\varepsilon) = \operatorname{lin}(F)$.

Proof. Let $F \in \mathscr{F}(C)$ be a proper face of C and define $n := |[F]|$ as the amount of indices in $[F]$. Set $\bar{\varepsilon} := \varepsilon/(2d+2)$. With these definitions define

$$\mathfrak{q}_F^\varepsilon(x) \;:=\; (u_F \cdot x)^{2k} - \sum_{i \in [F]} \frac{1}{n} \left(\frac{(a_i \cdot x) - r_i^F (u_F \cdot x)}{(1+\bar{\varepsilon}) r_i^F} \right)^{2k}, \qquad (3.6)$$

where $k := \left\lceil \frac{1}{2} \frac{\ln(n)}{\ln(1+\bar{\varepsilon})} \right\rceil$. Both the proofs for (3.4) and (3.5) fall into two cases, each depending on if the considered point y fulfills $u_F \cdot y = 0$ or $u_F \cdot y > 0$. In the latter case, the proof will use a vertex-figure of $C_{[F]}$ in order to exploit properties of bounded polyhedra.

Proof for (3.4):
Choose an arbitrary point y in the set on the left-hand side in (3.4): To this end, let x, v be arbitrary points with $x \in C_{[F]}$ and $v \in \Delta^F$, and with $\widetilde{\varepsilon} := \omega\varepsilon$ define $y := x + (u_F \cdot x)\widetilde{\varepsilon} v$. The definition of $\widetilde{\varepsilon}$ leads to $0 < \widetilde{\varepsilon} < \varepsilon/(d+1)$, which together with $|u_F \cdot v| \le (d+1)$ implies $|\widetilde{\varepsilon}(u_F \cdot v)| \le \varepsilon/2 \le 1/2$, leading to the estimation

$$2(u_F \cdot x) \ge (u_F \cdot y) = (u_F \cdot x)(1 + \widetilde{\varepsilon}(u_F \cdot v)) \;\ge (u_F \cdot x)/2 \qquad (3.7)$$

This shows that $u_F \cdot y \geq 0$ must hold and more over, $u_F \cdot y = 0$ holds if and only if $u_F \cdot x = 0$.

So assume $u_F \cdot y = 0$, which implies $u_F \cdot x = 0$. Together with $x \in C_{[F]}$ this leads to $x \in \text{lin}(F)$, implying that $a_i \cdot x = 0$ holds for each $i \in [F]$. From this $a_i \cdot y = 0$ is derived by observing $a_i \cdot y = a_i \cdot x + (u_F \cdot x)\widetilde{\varepsilon}(a_i \cdot w) = 0$. With $u_F \cdot y = 0$ and $a_i \cdot y = 0$, direct insertion leads to $\mathfrak{q}_F^\varepsilon(y) = 0$. Thus $y \in \mathcal{S}^=(\mathfrak{p}_F, \mathfrak{q}_F^\varepsilon)$ holds, proving (3.4) in this case.

Now assume $u_F \cdot y > 0$, then one has

$$\mathfrak{q}_F^\varepsilon(y) = (u_F \cdot y)^{2k} \left[1 - \sum_{i \in [F]} \frac{1}{n} \left(\frac{(a_i \cdot y)/(u_F \cdot y) - r_i^F}{(1 + \overline{\varepsilon})r_i^F} \right)^{2k} \right], \tag{3.8}$$

which implies that to prove $q_F(y) > 0$, it suffices to show $(a_i \cdot y)/(u_F \cdot y) - r_i^F < (1 + \overline{\varepsilon})r_i^F$ for each $i \in [F]$. So let $i \in [F]$ be an arbitrary index. The core task is to bounded $a_i \cdot y$ relative to $u_F \cdot y$. To this end define $\overline{x} := x\,(u_F \cdot y)/(u_F \cdot x)$, where $u_F \cdot x \neq 0$ holds since with $u_F \cdot y > 0$ one has $u_F \cdot x > 0$. In addition of $\overline{x}$ fulfilling $u_F \cdot y = u_F \cdot \overline{x}$, it is a positive multiple of x and thus a point in the cone $C_{[F]}$. Moreover the value of $\overline{\varepsilon}$ is chosen such that $y \in \overline{x} + \varepsilon\,(u_F \cdot y)\Delta^F$ must hold: Direct calculation leads to the first line of

$$\begin{array}{rclcccccc} |a_i \cdot (y - \overline{x})| & = & |\ (u_F \cdot x) & \widetilde{\varepsilon} & (a_i \cdot w) & - & (u_F \cdot w) & \widetilde{\varepsilon} & (a_i \cdot x)| \\ & \leq & 2(u_F \cdot y) & \widetilde{\varepsilon} & 2r_i^F & + & (d+1) & \widetilde{\varepsilon} & 4r_i^F(u_F \cdot y) \\ & < & \overline{\varepsilon} r_i^F (u_F \cdot y). & & & & & & \end{array}$$

The second inequality follows from $0 \leq a_i \cdot x \leq 2r_i^F(u_F \cdot x)$, $0 < u_F \cdot x \leq 2u_F \cdot y$ (cf. (3.7)) and $|u_F \cdot w| \leq (d+1)$ together with the triangle inequality. The last inequality uses $d > 0$ leading to $4(d+2)\widetilde{\varepsilon} < 8(d+1)\widetilde{\varepsilon} = \overline{\varepsilon}$. The above implies

$$\begin{array}{rcccc} |(a_i \cdot y) - r_i^F(u_F \cdot y)| & \leq & |(a_i \cdot \overline{x}) - r_i^F(u_F \cdot y)| & + & |a_i \cdot (y - \overline{x})| \\ & < & r_i^F(u_F \cdot y) & + & \overline{\varepsilon} r_i^F (u_F \cdot y). \end{array}$$

Here in the second line $0 \leq (a_i \cdot \overline{x}) \leq 2r_i^F(a_i \cdot \overline{x})$ is used, which holds due to $\overline{x} \in C_{[F]}$. So for the index i one obtains

$$\left(\frac{(a_i \cdot y)/(u_F \cdot y) - r_i^F}{(1 + \overline{\varepsilon})r_i^F} \right)^{2k} < 1 \tag{3.9}$$

Since $i \in C_{[F]}$ was chosen arbitrarily, each summand in (3.8) corresponding to the fraction in (3.9) is strictly smaller than $1/n = 1/|[F]|$. So the above proves that $\mathfrak{q}_F^\varepsilon(y) > 0$ must hold, this shows $y \in \mathcal{S}^>(\mathfrak{p}, \mathfrak{q}_F^\varepsilon)$, finally proving (3.4).

Proof for (3.5):
Choose an arbitrary point $y \in \mathbb{R}^d$ fulfilling $\mathfrak{p}(y) \geq 0$ and $\mathfrak{q}_F^\varepsilon(y) \geq 0$. With $\mathfrak{p}(y) = u_F \cdot y \geq 0$ there are two cases to examine:

First assume that $u_F \cdot y = 0$ holds. Direct insertion leads to

$$\mathfrak{q}_F^\varepsilon(y) = -\sum_{i\in[F]} \frac{1}{n}\left(\frac{(a_i \cdot y)}{(1+\varepsilon/2)r_i^F}\right)^{2k} \geq 0,$$

implying that $a_i \cdot y = 0$ holds for each $i \in [F]$. Thus y is a point in $\operatorname{lin}(F) \subset C_{[F]}$, which together with $\mathbf{0} \in \Delta^F$ proves that y is contained in the set on the left-hand side of (3.5).

Now assume $u_F \cdot y > 0$ holds. To prove the inclusion in (3.5), it suffices to restrict to the case $u_F \cdot y = 1$. This is due to the fact, that if $\overline{y} := y/(u_F \cdot y)$ is part of the left-hand side of (3.5), y is too:

First of all, $\overline{y} \in \mathcal{S}^{\geq}(\mathfrak{p}, \mathfrak{q}_F^\varepsilon)$ must hold, since one has $\mathfrak{p}(\overline{y}) = \mathfrak{p}(y)/(u_F \cdot y) \geq 0$ and $\mathfrak{q}_F^\varepsilon(\overline{y}) = \mathfrak{q}_F^\varepsilon(y)/(u_F \cdot y)^{2k} \geq 0$. Moreover, if $\overline{y}$ is in the left hand side of (3.5), i.e. there are $\overline{x} \in C_{[F]}$ and $w \in \Delta^F$ such that $\overline{y} = \overline{x} + \varepsilon(u_F \cdot \overline{x})w$, then $y = x + \varepsilon(u_F \cdot x)w$ holds for $x = (u_F \cdot y)\overline{x} \in C_{[F]}$. Here one has $x \in C_{[F]}$, since x is a positive multiple of $\overline{x}$.

So assume w.l.o.g., that $u_F \cdot y = 1$, and let s be the barycenter of the *bounded* polytope $P^F \cap F^\perp$. The next paragraph proves

$$y - s \in (1+\varepsilon)(P^F - s), \tag{3.10}$$

which with $s \in P^F$ and $P^F - P^F \subseteq \Delta^F$ leads to

$$y \in (1+\varepsilon)P^F - \varepsilon\, s \subseteq P^F + \varepsilon(P^F - P^F) \subseteq P^F + \varepsilon\Delta^F.$$

This proves the lemma by showing that there is $x \in P^F \subset C_{[F]}$ and $w \in \Delta^F$ such that $y = x + \varepsilon w$. So it suffices to show (3.10), which is done by finding an estimation on the ratio of $a_i \cdot y$ and the values of $a_i \cdot x$ for any $x \in P^F - s$. To this end special properties of s are used:

From $u_F \cdot y = 1$ one derives directly $u_F \cdot (y - s) = 0$ and obtains

$$\mathfrak{q}_F^\varepsilon(y) = 1 - \sum_{i\in[F]} \frac{1}{n}\left(\frac{(a_i \cdot y) - r_i^F}{(1+\overline{\varepsilon})r_i^F}\right)^{2k} \geq 0,$$

by direct insertion. This inequality implies, that each negative summand may not exceed 1, more precise

$$|(a_i \cdot y) - r_i^F| \leq (1+\overline{\varepsilon})r_i^F \sqrt[2k]{n} \tag{3.11}$$

must hold for each $i \in [F]$. Here, by choice of k, one has $\sqrt[2k]{n} \leq (1+\bar{\varepsilon})$ and so $\sqrt[2k]{n}(1+\bar{\varepsilon}) \leq (1+4\bar{\varepsilon})$ holds due to $\bar{\varepsilon} \leq 1$. Using this, (3.11) can be relaxed to the statement

$$-b_i - 4\bar{\varepsilon}\, r_i^F \leq (a_i \cdot y - s) \leq (2+4\bar{\varepsilon}) r_i^F - b_i, \tag{3.12}$$

where $b_i := a_i \cdot s$ for each $i \in [F]$. Using the last definition one obtains

$$P^F - s = \{x \in \mathbb{R}^d \ : \ -b_i \leq (a_i \cdot x) \leq 2r_i^F - b_i, \ \ i \in [F]\,, \ \ (u_F \cdot x) = 0\},$$

by additionally observing that $(u_F \cdot x) = 0$ must hold for any $x \in P^F - s$ due to $(u_F \cdot s) = 1$. So defining

$$\lambda := \max_{i \in [F]} \ \max \left\{ \frac{-b_i - 4\bar{\varepsilon}\, r_i^F}{-b_i} \ \ , \ \ \frac{2r_i^F - b_i + 4\bar{\varepsilon}\, r_i^F}{2r_i^F - b_i} \right\},$$

one has $(y-s) \in \lambda\,(P^F - s)$. From [BF34] one obtains for each $i \in [F]$ the estimations

$$\frac{1}{d+1} 2r_i^F \ \leq b_i \leq \ \frac{d}{d+1} 2r_i^F,$$

which lead to

$$\left.\begin{array}{rcl} \frac{-b_i - 4\bar{\varepsilon}\, r_i^F}{-b_i} & = & 1 + 4\bar{\varepsilon}\, \frac{r_i^F}{b_i} \\ \frac{2r_i^F - b_i + 4\bar{\varepsilon} r_i^F}{2r_i^F - b_i} & = & 1 + 4\bar{\varepsilon}\, \frac{r_i^F}{2r_i^F - b_i} \end{array}\right\} \leq 1 + 4\bar{\varepsilon}\, \frac{d+1}{2} = 1 + \varepsilon.$$

This proves $\lambda \leq 1 + \varepsilon$, leading to $y - s \in (1+\varepsilon)(P^F - s)$, proving the lemma – using the argumentation above.

□

3.5 Proof of the Main Theorem

3.5.1 Replacing polynomials with their product

Polynomials can be replaced if approximations fit into each other

The main objective of the presented proof strategy is to replace polynomials $\mathfrak{q}_{F,\varepsilon}$ and $\mathfrak{p}_F$ for faces $F \in \mathscr{F}_k(C)$ of equal dimension $k \in \{0,\dots,d-1\}$ by their products. For a special choice of the parameters ε, the arising $2d$ polynomials $\mathfrak{P}_k$ and $\mathfrak{Q}_k$, give a complete description of the cone C. Roughly speaking, to prove this one has to study, for two k-faces F and G, how to “fit” $\mathcal{S}^{\geq}(\mathfrak{p}_{F\cap G}, \mathfrak{q}^{\varepsilon}_{F\cap G})$ into the union of $\mathcal{S}^{\geq}(\mathfrak{p}_F, \mathfrak{q}^{\varepsilon}_F)$ and $\mathcal{S}^{\geq}(\mathfrak{p}_G, \mathfrak{q}^{\varepsilon}_G)$. More precise, $\mathcal{S}^{\geq}(\mathfrak{p}_{F\cap G}, \mathfrak{q}^{\varepsilon}_{F\cap G})$ must be fitted into the complement of the corresponding artifacts, this is the geometric invariant stated in Lemma 2.5.1.

Approximations are compared using points in the approximated sets

This is done using the sets introduced in Lemma 3.3.2. Using the insights presented there, we find that $\mathcal{S}^{\geq}(\mathfrak{p}_{F\cap G}, \mathfrak{q}^{\varepsilon}_{F\cap G})$ is wrapped by $\{x + (u_{F\cap G}\cdot x)\,\varepsilon\,\Delta^{F\cap G}\ : x \in C_{[F\cap G]}\}$. Given some $x \in C_{[F\cap G]}$, the set $\mathcal{S}^{\geq}(\mathfrak{p}_F, \mathfrak{q}^{\varepsilon}_F)$ “provides” some space $x + \omega\varepsilon\ (u_F\cdot x)\ \Delta^F$ in its interior, the similar holds for $\mathcal{S}^{\geq}(\mathfrak{p}_G, \mathfrak{q}^{\varepsilon}_G)$.

So to prove $\mathcal{S}^{\geq}(\mathfrak{p}_{F\cap G}, \mathfrak{q}^{\varepsilon}_{F\cap G})$ is contained in the two other sets, first it is proven that that $M\Delta^{F\cap G} \subseteq \Delta^F$ and $M\Delta^{F\cap G} \subseteq \Delta^G$ hold for some constant M. Then, for each $x \in C_{[F\cap G]}$ it is shown that $u_{F\cap G}\cdot x$ is related linearly to the maximum $\max\{u_F\cdot x, u_G\cdot x\}$ involving the same constant M. This holds since x can not be close to F as well as G without being close to $F\cap G$, and the three linear functions $(u_{F\cap G}\cdot x)$, $(u_F\cdot x)$, and $(u_G\cdot x)$ represent these distances.

3.5.2 Fitting auxiliary sets into each other

The central constant M

To calculate the constant M it is necessary to know how close a point $y_F \in F$ can be to some other face G, measured relative to $u_{F\cap G}\cdot y_F$. This is expressed by the constant $r_{\min}$:

Definition 3.5.1. *As above, let $C = \mathrm{cone}(V)$ be the* irredundant $\mathcal{V}$-*description of the pointed, polyhedral cone C. For each face $F \in \mathscr{F}(C)$*

let u_F be the chosen support-vector. Then one defines $M := r_{\min}/r_{\max}$ by setting

$$\begin{aligned} r_{\min} &:= \min_{F\in\mathscr{F}(C)} \min_{i\in[F]} \quad \min\left\{ \tfrac{(a_i\cdot v)}{2(u_F\cdot v)} \ :\ v\in V,\ \ a_i\cdot v\neq 0 \right\}, \\ r_{\max} &:= \max_{F\in\mathscr{F}(C)} \max_{i\in[F]} \quad \max\left\{ \tfrac{(a_i\cdot v)}{2(u_F\cdot v)} \ :\ v\in V,\ \ a_i\cdot v\neq 0 \right\}. \end{aligned}$$

Please note that both constants depend on the choice of support-vectors u_F for each face $F\in\mathscr{F}(C)$. Moreover $r_{\max}$ is just the maximum of all possible values of r_i^F (see proof below) while $r_{\min}$ is deliberately chosen to be *lesser* than the minimum of all these values.

Corollary 3.5.2. *For any pair of a face $F\in\mathscr{F}(C)$ and index $i\in[F]$, one obtains $0<r_{\min}\le r_i^F\le r_{\max}$ and thus $0<M\le 1$.*

Proof. Choose some $F\in\mathscr{F}(C)$ and $i\in[F]$, then $r_{\min}\le r_i^F\le r_{\max}$ is implied by the definition of r_i^F in Lemma 3.3.3. To prove $0<r_{\min}$, observe that there is some $v\in V$ with $a_i\cdot v\neq 0$, otherwise C would not be full-dimensional. Moreover for any such v one has $u_F\cdot v\neq 0$ since by definition of $[F]$, the polynomial $a_i\cdot x$ zeroes in F, implying $v\notin F$ and thus $u_F\cdot v\neq 0$. This leads to $a_i\cdot v>0$ implying $r_{\min}>0$. $\square$

Comparing polytopes Δ^F and support-polynomials $(u_F\cdot x)$

Lemma 3.5.3. *For every pair F,G of k-faces of C and for every $x\in C_{[F\cap G]}$ one finds*

$$\begin{aligned} (M/2)\,(u_{F\cap G}\cdot x) &\le \max\{(u_F\cdot x),(u_G\cdot x)\} \qquad (3.13)\\ M\,\Delta^{F\cap G} &\subseteq \Delta^F. \qquad (3.14) \end{aligned}$$

Proof. The lemma holds trivially, if one has $F=G$, due to $M\le 1$. So assume $F\neq G$ and choose $x\in C_{[F\cap G]}$ arbitrarily.

Proof for (3.14): One has $F\cap G\subseteq F$, leading to the inverse inclusion $[F]\subseteq[F\cap G]$ for the indices of incident inequalities. So for $i\in[F]$ both r_i^F and $r_i^{F\cap G}$ are defined and one has $M\,r_i^{F\cap G}\le r_{\min}\le r_i^F$. Thus by definition one has

$$\begin{aligned} M\Delta^{F\cap G} &= \{x\in\mathbb{R}^d\ :\ |a_i\cdot x|\le 2Mr_i^{F\cap G},\quad i\in[F\cap G]\} \\ &\subseteq \{x\in\mathbb{R}^d\ :\ |a_i\cdot x|\le 2r_i^F,\quad i\in[F]\} \qquad =\Delta^F. \end{aligned}$$

Proof for (3.13): Here it suffices to show

$$M\,(u_{F\cap G}\cdot x)\ \le\ (u_F+u_G)\cdot x, \qquad (3.15)$$

when observing $(u_F + u_G) \cdot x \leq 2 \max\{u_F \cdot x, u_G \cdot x\}$. To this end assume the *irredundant* $\mathcal{V}$-description of C is given by $C = \operatorname{cone}(V)$, where $V \subset \mathbb{R}^d$ is a finite set. Defining $V^*_{F\cap G} := \{v \in V \,:\, v \notin (F \cap G)\}$, one has $C_{[F\cap G]} = \operatorname{cone}(V^*_{F\cap G}) + \operatorname{lin}(F\cap G)$, due to Lemma 3.2.4. So there are $y \in \operatorname{cone}(V^*_{F\cap G})$ and $w \in \operatorname{lin}(F \cap G)$ with $x = y + w$. It suffices to prove (3.15) separately for both y and w, due to the linearity of the involved scalar-products:

Both, $u_{F\cap G}$ and $(u_F + u_G)$, are support-vectors for the face $F \cap G \in \mathscr{F}(C)$, implying $u_{F\cap G} \cdot w = (u_F + u_G) \cdot w = 0$. So for w (3.15) holds trivially.

The vector y is the conic combination $y = \sum_{v \in V^*_{F\cap G}} \lambda_v v$ for some $\lambda_v \geq 0$. Here each v involved is part of $C \setminus (F \cap G)$ and thus by definition of support-vectors $u_{F\cap G} \cdot v > 0$ must hold for each. This leads to a proof for (3.15), by observing

$$\begin{array}{rcl} M(u_{F\cap G} \cdot y) & \leq & \sum_{v \in V^*_{F\cap G}} \lambda_v \left(\frac{(u_F + u_G)\cdot v}{(u_{F\cap G} \cdot v)} \right) (u_{F\cap G} \cdot v) \\ & = & (u_F + u_G) \cdot y. \end{array} \tag{3.16}$$

Here the inequality holds since each v can not be in both F and G:

Let $v \in V^*_{(F\cap G)}$, then one has $v \notin F \cap G$, so at least one of $v \notin F$ or $v \notin G$ must hold. Assuming w.l.o.g. $v \notin F$, one finds an index $i \in [F] \subset [F \cap G]$ with $a_i \cdot v > 0$. Using the definitions of $r_{\max}$ and $r_{\min}$ one obtains:

$$\frac{(u_F + u_G) \cdot v}{(u_{F\cap G} \cdot v)} = \frac{(u_F \cdot v)}{(a_i \cdot v)} \frac{(a_i \cdot v)}{(u_{F\cap G} \cdot v)} \geq \frac{1}{r_{\max}} r_{\min} = M,$$

which proves (3.16) since $v \in V^*_{(F\cap G)}$ was chosen arbitrarily. So subsequently (3.15) (and thus the lemma) is proven. □

3.5.3 Fitting approximations into each other

A corollary of Lemma 3.5.3 is that one can explicitly calculate ε_k, $0 \leq k \leq d-1$, such that given $F, G \in \mathscr{F}_k(C)$ and $l := \dim(F \cap G)$ the inclusion (3.17) holds. To understand this "cryptic" property, notice the following: The meaning of (3.17), is that $\mathcal{S}^{\geq}(\mathfrak{p}_H, \mathfrak{q}_H^{\varepsilon_l})$ is contained in the complements of both $\operatorname{artifact}(\mathfrak{p}_F, \mathfrak{p}_G)$ and $\operatorname{artifact}(\mathfrak{q}_F^{\varepsilon_k}, \mathfrak{q}_F^{\varepsilon_k})$. This insight is the central part of the proof for Main Lemma 2 (see (2.9) page 34), and Main Lemma 2 directly leads to a description of C using $2d$ polynomials.

Corollary 3.5.4. *For each $k \in \{0, \ldots, d-1\}$ let*

$$\varepsilon_k := \left(\frac{M^2}{2} \frac{1}{2^4(d+1)} \right)^{d-1-k},$$

then for any pair of two different k-faces $F, G \in \mathscr{F}_k(C)$, $k \in \{1, \dots, d-1\}$, with intersection $H := F \cap G$ and $l := \dim(H)$ one has

$$\begin{aligned} \mathcal{S}^{\geq}(\mathfrak{p}_H, \mathfrak{q}_H^{\varepsilon_l}) \quad \subseteq \quad & \mathcal{S}^{>}(\mathfrak{p}_F, \mathfrak{q}_F^{\varepsilon_k}) \cup \mathcal{S}^{>}(\mathfrak{p}_G, \mathfrak{q}_G^{\varepsilon_k}) \\ \cup \quad & \mathcal{S}^{=}(\mathfrak{p}_F, \mathfrak{q}_F^{\varepsilon_k},\ \mathfrak{p}_G, \mathfrak{q}_G^{\varepsilon_k}). \end{aligned} \tag{3.17}$$

Proof. One has $0 < \varepsilon_0 < \ldots < \varepsilon_{d-1} = 1$, so each ε_k meets the restriction $0 < \varepsilon_k \leq 1$ for Lemma 3.4.1, which states $\mathcal{S}^{\geq}(\mathfrak{p}_H, \mathfrak{q}_H^{\varepsilon_l}) \subseteq \{x + \varepsilon_l\,(u_H \cdot x)\Delta^H \ :\ x \in C_{[H]}\}$. Thus choosing an arbitrary $y \in C_{[H]}$, it suffices to show

$$\begin{aligned} y + \varepsilon_l\,(u_H \cdot y)\Delta^H \quad \subseteq \quad & \mathcal{S}^{>}(\mathfrak{p}_F, \mathfrak{q}_F^{\varepsilon_k}) \cup \mathcal{S}^{>}(\mathfrak{p}_G, \mathfrak{q}_G^{\varepsilon_k}) \\ \cup \quad & \mathcal{S}^{=}(\mathfrak{p}_F, \mathfrak{q}_F^{\varepsilon_k},\ \mathfrak{p}_G, \mathfrak{q}_G^{\varepsilon_k}). \end{aligned}$$

in order to prove (3.17). To this end distinguish the cases $(u_H \cdot y) = 0$ and $(u_H \cdot y) > 0$:

In case $u_H \cdot y = 0$ holds, one has $y \in \operatorname{lin}(H) = \operatorname{lin}(F \cap G)$ due to $y \in C_{[H]}$, implying

$$y + \varepsilon_l\,(u_H \cdot y)\Delta^H = \{y\} \subseteq \operatorname{lin}(F \cap G) = \mathcal{S}^{=}(\mathfrak{p}_F, \mathfrak{q}_F^{\varepsilon_k},\ \mathfrak{p}_G, \mathfrak{q}_G^{\varepsilon_k}).$$

The last equation is a result of Lemma 3.4.1, which states $\operatorname{lin}(F) = \mathcal{S}^{=}(\mathfrak{p}_F, \mathfrak{q}_F^{\varepsilon_k})$, accordingly this holds also for G.

Now assume $u_H \cdot y > 0$ holds, and w.l.o.g. assume that $u_F \cdot y \geq u_G \cdot y$ holds. In addition one infers $u_F \cdot y \geq 0$ from $y \in C_{[F]}$. With this, Lemma 3.5.3 leads to the first two of the following inclusions:

$$\begin{aligned} y + \varepsilon_l\,(u_H \cdot y)\Delta^H \quad & \subseteq \quad y + \ \tfrac{\mathbf{2}}{\mathbf{M}} \ \ \varepsilon_l(u_{\mathbf{F}} \cdot y)\Delta^H \\ & \subseteq \quad y + \ \tfrac{2}{M^{\mathbf{2}}} \ \ \varepsilon_l(u_F \cdot y)\Delta^{\mathbf{F}} \\ & \subseteq \quad y + \ \omega \ \ \varepsilon_k(u_F \cdot y)\Delta^F \\ & \subseteq \quad \mathcal{S}^{>}(\mathfrak{p}_F, \mathfrak{q}_F^{\varepsilon_k}) \cup \operatorname{lin}(F). \end{aligned} \tag{3.18}$$

The last inclusion is a direct consequence of (3.5) in Lemma 3.4.1, the inclusion before follows from the value $\omega = \frac{1}{2^4(d+1)}$ defined there, i.e. one has

$$\frac{2}{M^2}\varepsilon_l \leq \frac{2}{M^2}\varepsilon_{k+1} = \frac{1}{2^4(d+1)}\varepsilon_k = \omega\ \varepsilon_k.$$

What is left to show in order to prove the lemma, is that in (3.18) the set $\operatorname{lin}(F)$ is obsolete. To this end we show that for each $v \in \omega\varepsilon_k\ \Delta^F$ one has $u_F \cdot (y + (u_F \cdot y)v) > 0$ and thus $(y + (u_F \cdot y)\ v) \notin \operatorname{lin}(F)$. One has $|u_F \cdot v| \leq \omega\varepsilon_k(d+1) = \varepsilon_k/2^4 < 1/2$, implying

$$u_F \cdot (y + (u_F \cdot y)v) = (u_F \cdot y)(1 + (u_F \cdot v)) > (u_F \cdot y)/2 > 0.$$

So since by assumption $u_H \cdot y > 0$ holds, the above and (3.18) lead to

$$y + \varepsilon_l\,(u_H \cdot y)\Delta^H \quad \subseteq \mathcal{S}^{>}(\mathfrak{p}_F, \mathfrak{q}_F^{\varepsilon_k}).$$

This finally proves the lemma. □

Since every $(d-2)$-face H of C is given by the intersection of two uniquely determined facets F and G of C we may even set

$$\varepsilon_{d-2} := 1/2, \quad \mathfrak{q}_H^{\varepsilon_{d-2}}(x) := \mathfrak{p}_H(x) = u_H \cdot x \tag{3.19}$$

without violating the validity of Corollary 3.5.4. Now we come to the definition of the polynomials, which give us a representation of an n-dimensional pointed polyhedral cone.

3.5.4 Proof of the Main Theorem

Definition 3.5.5. *Let ε_k, $0 \le k \le d-1$, be chosen according to Corollary 3.5.4 and* (3.19). *For $F \in \mathscr{F}_k(C)$, let $\mathfrak{p}_F, \mathfrak{q}_F^{\varepsilon_k} \in \mathbb{R}[X]$ be given as in Lemma 3.4.1. Then, for $k = 0, \dots, n-1$, let*

$$\mathfrak{P}_k(x) := \prod_{F \in \mathscr{F}_k(C)} \mathfrak{p}_F(x) \quad \text{and} \quad \mathfrak{Q}_k(x) := \prod_{F \in \mathscr{F}_k(C)} \mathfrak{q}_F^{\varepsilon_k}(x).$$

Proof of the Main Theorem. First we show that

$$C = \left\{ x \in \mathbb{R}^d : \ \mathfrak{P}_k(x) \ge 0,\ \mathfrak{Q}_k(x) \ge 0,\ k = 0, \dots, d-1 \right\}.$$

The inclusion $\subseteq$ is obvious. So let $y \notin C$, but suppose that y satisfies all the polynomial inequalities. Since $y \notin C$ one of the facet defining inequalities has to be violated, i.e., there exists an $(d-1)$-face F with $\mathfrak{p}_F(y) < 0$. Hence we may define $l \in \{0, \dots, d-1\}$ as the minimum number (index) for which one of the factors in the polynomials $\mathfrak{P}_l(x)$ or $\mathfrak{Q}_l(x)$ is violated. Since both, $\mathfrak{P}_0(x)$ and $\mathfrak{Q}_0(x)$, consist only of one such polynomial factor we have $l \in \{1, \dots, d-1\}$.

Let $F \in \mathscr{F}_l(C)$ such that $\mathfrak{p}_F(y) < 0$ or $\mathfrak{q}_F^{\varepsilon_l}(y) < 0$. Since $\mathfrak{P}_l(y) \ge 0$ and $\mathfrak{Q}_l(y) \ge 0$ there must exist a $G \in \mathscr{F}_l(C)$ with $\mathfrak{p}_G(y) \le 0$ (in the case that $\mathfrak{p}_F(y) < 0$) or with $\mathfrak{q}_G^{\varepsilon_l}(y) \le 0$ (if $\mathfrak{q}_F^{\varepsilon_l}(y) < 0$). Thus we know that y is neither contained in $\mathcal{S}^{>}(\mathfrak{p}_F, \mathfrak{q}_F^{\varepsilon_l})$ nor in $\mathcal{S}^{>}(\mathfrak{p}_G, \mathfrak{q}_G^{\varepsilon_l})$ nor in the linear space $\mathcal{S}^{=}(\mathfrak{p}_F, \mathfrak{q}_F^{\varepsilon_l}, \mathfrak{p}_G, \mathfrak{q}_G^{\varepsilon_l}) = \mathrm{lin}(F) \cap \mathrm{lin}(G)$. By the choice of $\varepsilon_{\dim(F\cap G)}$ and Corollary 3.5.4, however, those points y are cut off by the cone $\mathcal{S}^{\ge}(\mathfrak{p}_{F\cap G}, \mathfrak{q}_{F\cap G}^{\varepsilon_{\dim(F\cap G)}})$. Thus we must have

$$y \notin \mathcal{S}^{\ge}(\mathfrak{p}_{F\cap G}, \mathfrak{q}_{F\cap G}^{\varepsilon_{\dim(F\cap G)}}),$$

contradicting the minimum property of l. Finally, we observe that $\mathfrak{P}_{d-1} = \mathfrak{Q}_{d-1}$ and $\mathfrak{P}_{d-2} = \mathfrak{Q}_{d-2}$ hold, and hence one obtains only $2d-2$ polynomials. □

3.6 Algorithm

Algorithm 3.6.1. *Main Algorithm*

Input A polyhedral cone C given in double-description by

$$\begin{aligned} C &= \mathrm{cone}(V) \\ &= \{x \in \mathbb{R}^d \, : \, (a_i \cdot x) \geq 0, \ i = 1, \dots, m\}. \end{aligned}$$

Output Polynomials $\mathfrak{P}_k, \mathfrak{Q}_k$ with $k = 0, \dots, d-1$ such that

$$C = \left\{x \in \mathbb{R}^d : \ \mathfrak{P}_k(x) \geq 0, \, \mathfrak{Q}_k(x) \geq 0, \ \ k = 0, \dots, d-1\right\}.$$

Step 1 For each $F \in \mathscr{F}(C)$ calculate $[F] \subseteq \{1, \dots, m\}$, the index-set of all linear inequalities incident to F and fix a support-vector u_F.

Step 2 Calculate r_i^F and $\bar{r}_i^F$ for each pair $F \in \mathscr{F}(C)$, $i \in [F]$ as well as r_{min}, r_{max}:

$$\begin{aligned} \bar{r}_i^F &:= \min \left\{ \tfrac{(a_i \cdot v)}{2(u_F \cdot v)} \ : \ v \in V, \ (a_i \cdot v) \neq 0 \right\}, \\ r_i^F &:= \max \left\{ \tfrac{(a_i \cdot v)}{2(u_F \cdot v)} \ : \ v \in V, \ (a_i \cdot v) \neq 0 \right\}. \end{aligned}$$

$$\begin{aligned} r_{\min} &:= \min\nolimits_{F \in \mathscr{F}(C)} \ \min\nolimits_{i \in [F]} \ \bar{r}_i^F, \\ r_{\max} &:= \max\nolimits_{F \in \mathscr{F}(C)} \ \max\nolimits_{i \in [F]} \ r_i^F. \end{aligned}$$

Step 3 Construction of the support-polynomials $\mathfrak{p}_F, \mathfrak{q}_F$:
For each $F \in \mathscr{F}(C)$ set $\mathfrak{p}_F(x) := (u_F \cdot x)$ and
– if $\dim(F) \geq d-2$ set $\mathfrak{q}_F = \mathfrak{p}_F$,
– if $\dim(F) < d-2$ set

$$\bar{\varepsilon}_F := \frac{1}{2(d+1)} \left(\frac{r_{\min}^2}{r_{\max}^2} \frac{1}{2^5(d+1)} \right)^{d-1-\dim(F)},$$

$$n_F := \left\lceil \frac{\ln(|[F]|)}{2\ln(1+\bar{\varepsilon}_F)} \right\rceil \text{ and}$$

$$\mathfrak{q}_F(x) := (u_F \cdot x)^{2n_F} - \sum_{i \in [F]} \frac{1}{|[F]|} \left(\frac{(a_i \cdot x) - r_i^F (u_F \cdot x)}{r_i^F (1 + \bar{\varepsilon}_F)} \right)^{2n_F}.$$

Step 4 For each $k = 0, \dots, d-1$ set

$$\begin{aligned} \mathfrak{P}_k(x) &:= \textstyle\prod_{F \in \mathscr{F}_k(C)} \mathfrak{p}_F(x) \text{ and} \\ \mathfrak{Q}_k(x) &:= \textstyle\prod_{F \in \mathscr{F}_k(C)} \mathfrak{q}_F(x). \end{aligned}$$

Proof of correctness of the Main Algorithm. The polynomials $\mathfrak{P}_k$ and $\mathfrak{Q}_k$ constructed in correspond to the polynomials in Definition 3.5.5. According to the proof of the Main Theorem on page 60, these polynomials describe C, i.e.,

$$C = \mathcal{S}^{\geq}(\mathfrak{P}_{d-1}, \mathfrak{P}_{d-2}, \mathfrak{P}_{d-3}, \mathfrak{Q}_{d-3}, \ldots, \mathfrak{P}_0, \mathfrak{Q}_0)$$

holds. □

3.6.1 Discussion

Complexity

The complexity of the Algorithm 3.6.1 in its given form is not polynomial in the input-size: In general, the number of faces $F \in \mathscr{F}(C)$ is exponential in the input-size of all a_i and all v used in the initial descriptions. Since one has to calculate a polynomial for each face $F \in \mathscr{F}(C)$, in the worst case the running time is not polynomially bounded in the input-size. This general obstruction can not be overcome.

The general intention is to use a semi-algebraic description for polyhedra arising in combinatorial optimization. For such sets mostly the facet-defining inequalities are known, implying knowledge on the corresponding support-vectors. This gives rise to the following question: Assume C is given in $\mathcal{H}$-representation together with a support-vector for each face. Is it possible to calculate the core values $r_{\min}, r_{\max}$, and r_i^F more efficiently than traversing all extremals v of C?

The calculation of each r_i^F and $r_{\max}$ can be reduced to solving an LP of the form $\max(a_i \cdot x)$ s.t. $x \in \{x \in \mathbb{R}^d \,:\, a_i \cdot x \geq 0,\ i \in [F] \text{ and } (u_F \cdot x) = 1\}$. Each such operation can be solved in polynomial-time in the given input-size, see [GLS88].

Unfortunately, it is not clear, how to calculate $r_{\min}$ more efficiently in such a setting. This value reflects how close any $v \in V$ can be to F, if $v \notin F$ holds. More precise it depends on the minimum angle between two rays $\operatorname{cone}(v)$ and $\operatorname{cone}(w)$ which are contained in different faces. For the given construction calculating a lower-bound for $r_{\min}$ would suffice, but unfortunately it is not clear how to calculate such a bound, without traversing all extremals of C.

Preliminaries

The given cone C does not need to be pointed, in order to apply Algorithm 3.6.1. The presented construction is restricted to pointed polyhedral cones to simplify the proofs involved. In fact a polyhedral cone in $\mathbb{R}^d$ with an n-dimensional apex, i.e. each face of it has at least dimension n, can be described using at most $2(d-n)$ polynomials.

The initially given $\mathcal{H}$- and $\mathcal{V}$-representations of C need not be irredundant. But to keep the resulting exponents as low as possible, one should use at least an irredundant $\mathcal{V}$-representation: The exponent $2n_F$ of each polynomial $\mathfrak{q}_F$ is greater than $1/\ln(1+r_{\min}^2)$. Where the value of $r_{\min}$ becomes arbitrarily close to 0, when adding a single redundant vector $v \in C$ to V, if v is close to some face F.

3.6.2 Implications for semi-algebraic geometry

The following corollaries offer the explicit construction of polynomials describing special semi-algebraic sets. The number of polynomials involved is relatively far from the proven *lower bounds*, namely $d(d+1)/2$ polynomials for closed and d polynomials for open semi-algebraic sets (cf. [BCR98] p. 122 and p. 259). But since the latter bounds are obtained non-constructive, the following corollaries are interesting because this work provides a construction for the corresponding polynomials:

Corollary 3.6.2. *Any d-dimensional basic open semi-algebraic set $\mathcal{A} \subseteq \mathbb{R}^d$ defined by linear polynomials can be described using $2d$ strict polynomial inequalities. The corresponding polynomials can be constructed.*

Proof. Let

$$\mathcal{P} = \{x \in \mathbb{R}^d \ : \ a_1 \cdot x + b_1 > 0, \dots, a_m \cdot x + b_m > 0\}$$

be a basic open semi-algebraic set, defined by strict linear inequalities. Then the corresponding polyhedron $P = \{x \in \mathbb{R}^d \ : \ a_1 \cdot x + b_1 \geq 0, \dots, a_m \cdot x + b_m \geq 0\}$ can be described using $2d$ polynomials $P = \mathcal{S}^{\geq}(\mathfrak{P}_d, \mathfrak{P}_{d-1}, \mathfrak{P}_{d-2}, \mathfrak{Q}_{d-2}, \dots, \mathfrak{P}_0, \mathfrak{Q}_0)$. In Algorithm 3.6.1, these polynomials are constructed such, that each polynomial $\mathfrak{P}_k$ and $\mathfrak{Q}_k$ vanishes on all k-faces of P. Any point $x \in \mathcal{P}$ is contained in P and not contained in any of its faces, implying that

$$\mathcal{P} = \mathcal{S}^{>}(\mathfrak{P}_d, \mathfrak{P}_{d-1}, \mathfrak{P}_{d-2}, \mathfrak{Q}_{d-2}, \dots, \mathfrak{P}_0, \mathfrak{Q}_0)$$

holds. □

The following corollary is an observation of Claus Scheiderer:

Corollary 3.6.3. *Any d-dimensional basic closed semi-algebraic set $\mathcal{A} = \mathcal{S}^{\geq}(\mathfrak{f}_1, \dots, \mathfrak{f}_m) \subseteq \mathbb{R}^d$ defined by polynomials $\mathfrak{f}_i$ of degree at most n, can be described using $2\binom{n+d}{n} - 2$ strict polynomial inequalities. The corresponding polynomials can be constructed.*

Proof. Each inequality $\mathfrak{f}_i(x) \geq 0$ can be understood as a linear inequality in the space of monomes: Let $m_1, \dots, m_N \in \mathbb{R}[X]$ be the monomes such that $\mathfrak{f}_i = \sum_{j=1}^{N} \lambda_{ij} m_j$ holds for each $\mathfrak{f}_i$ and some coefficients $\lambda_{ij} \in \mathbb{R}$. Fixing these coefficients, one defines the polyhedron

$$P := \left\{ y \in \mathbb{R}^N \ : \ \sum_{j=1}^{N} \lambda_{1j} y \geq 0, \dots, \sum_{j=1}^{N} \lambda_{mj} y \geq 0 \right\}.$$

This set can be described using $2N$ polynomials $\mathfrak{P}_i$, which moreover can be constructed, one obtains $P = \mathcal{S}^{\geq}(\mathfrak{P}_1, \dots \mathfrak{P}_{2N})$. The set $\mathcal{A}$ is the pre-image of P under the polynomial mapping $\mathfrak{p} : \mathbb{R}^d \to \mathbb{R}^N$, $x \mapsto (m_1(x), \dots, m_N(x))$, and so one obtains $\mathcal{A} = \{x \in \mathbb{R}^d \ : \ \mathfrak{P}_1(\mathfrak{p}(x)) \geq 0, \dots, \mathfrak{P}_{2N}(\mathfrak{p}(x)) \geq 0\}$. This proves the corollary, since the number of possible monomes is bounded by $N \leq \binom{n+d}{n} - 1$. $\square$

3.6.3 A description for bounded polytopes

Based on Algorithm 3.6.1, any polyhedron in $\mathbb{R}^d$ can be described using $2d$ polynomials as stretched out in the first chapter. For a *bounded* polytope P, the corresponding algorithm possesses a simple form, leading to a semi-algebraic description of P involving $2d-1$ polynomials. This results from the following: The semi-algebraic description of P is obtained from running Algorithm 3.6.1 for the homogenization $C \subset \mathbb{R}^{d+1}$ of P. The cone C has a single vertex $v = \mathbf{0}$, and a possible support-polynomial $\mathfrak{p}_v$ for v is $x_{d+1} \geq 0$. The description for P is obtained from the description of C by fixing $x_{d+1} = 1$ in all polynomials. This way the polynomial $\mathfrak{P}_0(x, 1) = \mathfrak{p}_v(x, 1) = 1$ becomes obsolete for the description of P.

The following algorithm calculates such a description. The seemingly only change is the replacement of the homogeneous linear polynomials $a_i \cdot x$ and $u_F \cdot x$ by linear polynomials of the form $a_i \cdot x + b_i$ or $a_F \cdot x + b_F$.

Algorithm 3.6.4. *Algorithm for bounded polytopes*

Input A bounded polytope P given in double-description by

$$\begin{aligned} C &= \operatorname{conv}(V) \\ &= \{x \in \mathbb{R}^d \,:\, (a_i \cdot x) + b_i \geq 0,\ i = 1, \ldots, m\}. \end{aligned}$$

Output Polynomials $\mathfrak{F}_{d-1}, \mathfrak{F}_{d-2}, \mathfrak{G}_{-1}$ and $\mathfrak{F}_k, \mathfrak{G}_k$ with $k = 0, \ldots, d-3$ such that

$$C = \mathcal{S}^{\geq} \{\mathfrak{F}_{d-1}, \mathfrak{F}_{d-2},\ \mathfrak{F}_{d-3}, \mathfrak{G}_{d-3}, \ldots, \mathfrak{F}_0, \mathfrak{G}_0,\ \mathfrak{G}_{-1}\}.$$

Init For $i = 1, \ldots, m$, set $\mathfrak{l}_i(x) := (a_i \cdot x) + b_i$.

Step 1 For each $F \in \mathscr{F}(C)$ calculate $[F] \subseteq \{1, \ldots, m\}$ and fix a linear support-polynomial $\mathfrak{f}_F$. Set $\mathfrak{f}_\emptyset := 1$ and $[\emptyset] := \{1, \ldots, m\}$.

Step 2 For each $k = 0, \ldots, d-1$ set

$$\mathfrak{F}_k(x) \quad := \quad \prod_{F \in \mathscr{F}_k(C)} \mathfrak{f}_F(x) \text{ and}$$

Step 3 For each pair $F \in \mathscr{F}(C) \cup \{\emptyset\}$, $i \in [F]$ calculate

$$\begin{aligned} \underline{R}_i^F &:= \min \left\{ \frac{\mathfrak{l}_i(v)}{2\mathfrak{f}_F(v)} \;:\; v \in V,\ \mathfrak{l}_i(v) \neq 0 \right\}, \\ R_i^F &:= \max \left\{ \frac{\mathfrak{l}_i(v)}{2\mathfrak{f}_F(v)} \;:\; v \in V,\ \mathfrak{l}_i(v) \neq 0 \right\}. \end{aligned}$$

$$\begin{aligned} R_{\min} &:= \min_{F \in \mathscr{F}(C)} \ \min_{i \in [F]} \ \underline{R}_i^F, \\ R_{\max} &:= \max_{F \in \mathscr{F}(C)} \ \max_{i \in [F]} \ R_i^F. \end{aligned}$$

Step 4 For each $F \in \mathscr{F}(C) \cup \{\emptyset\}$ with $\dim(F) < d-2$ set

$$\widetilde{\varepsilon}_F \quad := \quad \frac{1}{2(d+1)} \left(\frac{R_{\min}^2}{R_{\max}^2} \frac{1}{2^5(d+1)} \right)^{d-1-\dim(F)},$$

$$\widetilde{n}_F \quad := \quad \left\lceil \frac{\ln(|[F]|)}{2\ln(1+\widetilde{\varepsilon})} \right\rceil \text{ and}$$

$$\mathfrak{g}_F(x) \quad := \quad \mathfrak{f}_F(x)^{2\widetilde{n}_F} - \sum_{i \in [F]} \frac{1}{|[F]|} \left(\frac{\mathfrak{l}_i(x) - R_i^F \mathfrak{f}_F(x)}{R_i^F(1+\widetilde{\varepsilon}_F)} \right)^{2\widetilde{n}_F}.$$

Step 5 For each $k = -1, \ldots, d-1$ set

$$\mathfrak{G}_k(x) \quad := \quad \prod_{F \in \mathscr{F}_k(C)} \mathfrak{g}_F(x).$$

Proof of correctness of Algorithm 3.6.4

Proof. The proof of correctness for Algorithm 3.6.4 comes through Algorithm 3.6.1 (the Main Algorithm) with the help of homogenization. To simplify notation, in the following (x, λ) with $x \in \mathbb{R}^d$ and $\lambda \in \mathbb{R}$ is used to represent a vector $(x^T, \lambda)^T \in \mathbb{R}^{d+1}$. The polytope P can be reobtained from its homogenization $C := \{(\lambda\, x, \lambda) \in \mathbb{R}^{d+1} \ : \ x \in P,\ \lambda \geq 0\}$ by fixing $x_{d+1} = 1$ in any semi-algebraic description of C:

Taking the lift of dimension into account when indexing, the Main Algorithm yields a description

$$C = \mathcal{S}^{\geq}(\mathfrak{P}_d, \mathfrak{P}_{d-1}, \mathfrak{P}_{d-2}, \mathfrak{Q}_{d-2}, \ldots, \mathfrak{P}_0, \mathfrak{Q}_0),$$

from which one derives

$$P = \{x \in \mathbb{R}^d \ : \ \mathfrak{P}_d(x,1) \geq 0 \ldots, \mathfrak{P}_0(x,1) \geq 0, \mathfrak{Q}_0(x,1) \geq 0\}. \tag{3.20}$$

So the task is to compare the polynomials $\mathfrak{P}_k, \mathfrak{Q}_k$ calculated in Algorithm 3.6.1 for C with the polynomials constructed in the Main Algorithm 3.6.4, i.e., one must show that $\mathfrak{F}_{k-1}(x) = \mathfrak{P}_k(x,1)$ and $\mathfrak{G}_{k-1}(x) = \mathfrak{Q}_k(x,1)$ hold, and that the polynomial $\mathfrak{P}_0(x,1)$ is obsolete in (3.20).

The homogenization $C := \{(\lambda\, x, \lambda) \in \mathbb{R}^{d+1} \ : \ x \in P,\ \lambda \geq 0\}$ of P is a pointed cone since P is bounded. Setting $\overline{a}_i = (a_i, b_i)$ for $i = 1, \ldots, m$ and $\overline{V} := \{(v,1) \ : \ v \in V\}$ its $\mathcal{H}$- and $\mathcal{V}$-representation are given by (cf. [Zie98]):

$$\begin{aligned} C &= \{y \in \mathbb{R}^{d+1} \ : \ y_{d+1} \geq 0, \overline{a}_i \cdot x \geq 0, \ldots, \overline{a}_m \cdot x \geq 0\} && (3.21) \\ &= \operatorname{cone}(\overline{V}). && (3.22) \end{aligned}$$

The values calculated in Algorithm 3.6.4 are the same calculated in the Main Algorithm: Let $w \in \overline{V}$ and $v \in V$ such that $w = (v,1)$, then one has $\overline{a}_i \cdot w = a_i \cdot v + b_i = \mathfrak{l}_i(v)$. Moreover each face $\overline{F} \in \mathscr{F}(C)$ is the homogenization of a face $F \in \mathscr{F}(P)$, except for the vertex $w \in C$. Let $\mathfrak{f}_F(x) = a_F \cdot x + b_F$ be the chosen linear support-polynomial for some face $F \in \mathscr{F}(P)$, then $\overline{u}_{\overline{F}} := (a_F, b_F)$ is a support-vector for the corresponding face $\overline{F} \in \mathscr{F}(C)$. For compatibility one defines $\overline{\emptyset} := w$ and chooses $u_w := (\mathbf{0}, 1)$ as a support-vector for the vertex $w \in C$ With these choices on $\overline{u}_{\overline{F}}$, the values calculated in Algorithm 3.6.4 result in the values calculated in the Main Algorithm:

$$\begin{aligned} R_i^F &= \max\left\{ \tfrac{\mathfrak{l}_i(v)}{2\mathfrak{p}_F(v)} \ : \ v \in V,\ \mathfrak{l}_i(v) \neq 0 \right\} \\ &= \max\left\{ \tfrac{\overline{a}_i \cdot w}{2\overline{u}_{\overline{F}} \cdot w} \ : \ w \in W,\ \overline{a}_i \cdot w \neq 0 \right\} = r_i^{\overline{F}}, \end{aligned}$$

$$\begin{aligned} \mathfrak{f}_F(x) &= \overline{u}_{\overline{F}} \cdot (x,1) \quad \text{and} \\ \mathfrak{l}_i(x) &= \overline{a}_i \cdot (x,1). \end{aligned}$$

Accordingly $R_{\max} = r_{\max}$ and $R_{\min} = r_{\min}$ hold, Moreover, the lift of dimensions $\dim(F) + 1 = \dim(\overline{F})$ and $\dim(P) + 1 = \dim(C) = d + 1$ cancel out, in the definitions of $\widetilde{\varepsilon}_F = \overline{\varepsilon}_{\overline{F}}$ and $\widetilde{n}_F = n_{\overline{F}}$. Here the values with tilde appear in Algorithm 3.6.4 and the others in the Main Algorithm.

All in all this leads to $\mathfrak{g}_F(x) = \mathfrak{q}_{\overline{F}}(x, 1)$, such that for the polynomials $\mathfrak{P}_k, \mathfrak{Q}_k$ calculated in Algorithm 3.6.1 one obtains:

$$\mathfrak{F}_{k-1}(x) = \mathfrak{P}_k(x, 1) \quad \text{and} \mathfrak{G}_{k-1}(x) = \mathfrak{Q}_k(x, 1),$$

leading to $P = \mathcal{S}^{\geq}(\mathfrak{F}_{d-1}, \mathfrak{F}_{d-2}, \mathfrak{F}_{d-3}, \mathfrak{G}_{d-3}, \ldots, \mathfrak{F}_{-1}, \mathfrak{G}_{-1})$. Here the polynomial $\mathfrak{F}_{-1}(x) = \mathfrak{p}_w(x, 1) = \overline{u}_w \cdot (x, 1) = 1$ is obsolete, which finally proves the algorithm. $\square$

3.6.4 Example

The description of the d-cube $D_d := \{x \in \mathbb{R}^d : -1 \leq x_i \leq 1\}$ already uses $2d$ polynomial inequalities. These can easily be reduced to d polynomial inequalities of degree 2, namely $D_d = \{x \in \mathbb{R}^d : (x_1)^2 \leq 1, \ldots, (x_d)^2 \leq 1\}$. The construction stretched out in Algorithm 3.6.4, results in more polynomials of higher degree: A direct calculation for every k-face $F \in \mathscr{F}_k(C_d)$, leads to the following exponent $2n_F$ of $\mathfrak{f}_F$:

$$2n_F = \left\lceil \ln(d-k) \left(\ln \left(1 + \frac{1}{2(d+1)(32\, d^2(d+1))^{d-1-k}} \right) \right)^{-1} \right\rceil.$$

The degree $2n_v$ of a polynomial $\mathfrak{f}_v$ constructed in Algorithm 3.6.4 for some vertex $v \in C_d$ is given in Table 3.1. Recalling that every cube C_d

$d =$	2	3	4	5	6	7	8	9	10	15	20
$n_v =$	2^{11}	2^{24}	2^{38}	2^{54}	2^{70}	2^{87}	2^{105}	2^{123}	2^{142}	2^{242}	2^{350}

Tab. 3.1: The exponent n_v of a single polynomial $\mathfrak{f}_v$

has 2^d vertices, these exponents have to be taken to the power of 2^d to obtain the degree of the resulting product $\mathfrak{G}_0$. In $\mathbb{R}^3$ the resulting polynomial $\mathfrak{G}_0$ has a degree of 2^{192}. This obstructs the direct usage of Algorithm 3.6.4 to present an example. But in fact, with reduced exponents $2n_F$, the polynomials constructed in Algorithm 3.6.4, lead to a description of $C_3 = \mathcal{S}^{\geq}(\mathfrak{F}_2, \mathfrak{F}_1, \mathfrak{F}_0, \mathfrak{G}_0, \mathfrak{G}_{-1})$. The corresponding polynomials are presented on the following page in order to demonstrate the resulting degrees. On the following page, Figure 3.1 presents the geometry of the corresponding semi-algebraic sets.

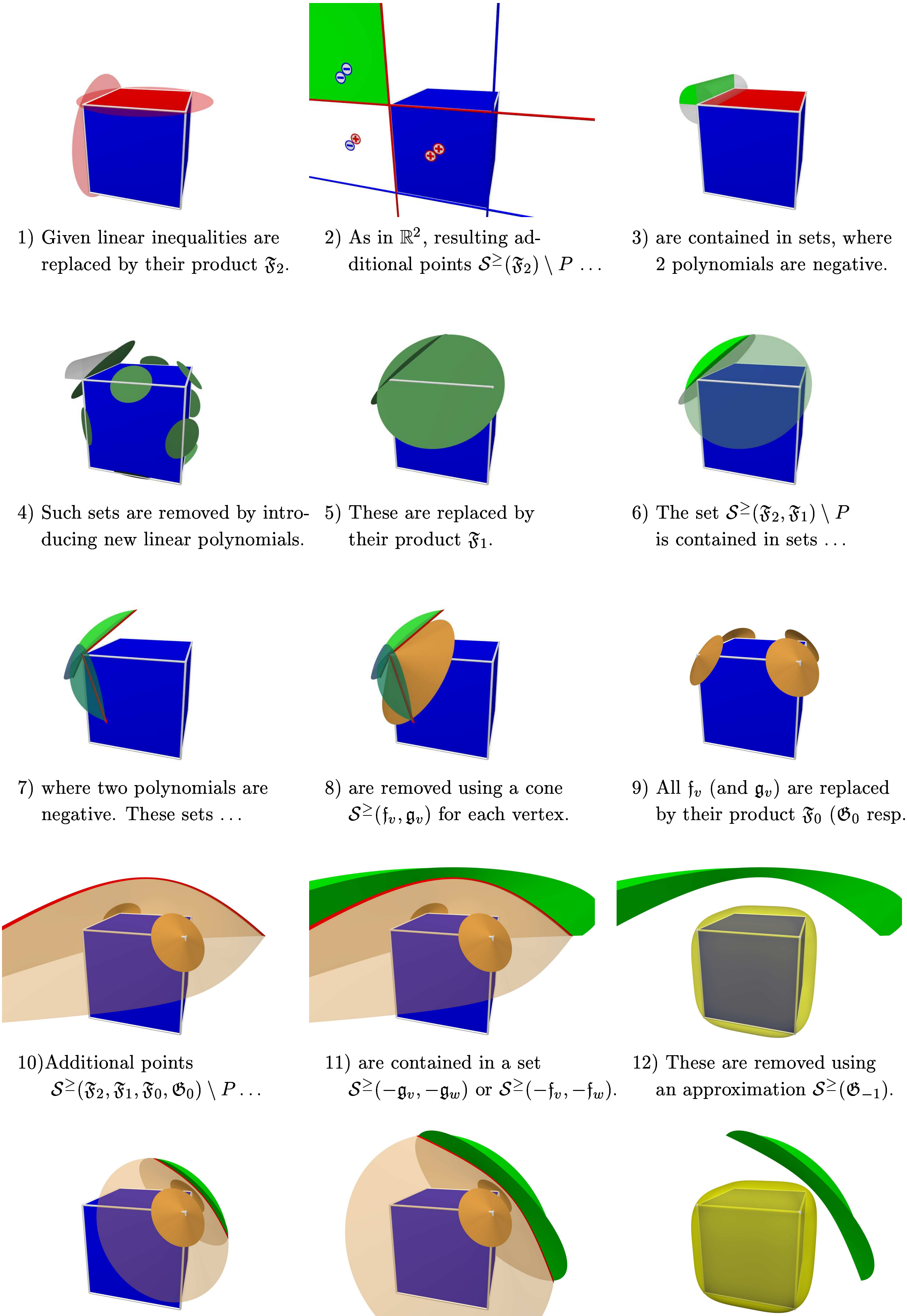
$$\mathfrak{F}_2(x) := -x^2y^2z^2 + x^2y^2 + x^2z^2 - x^2 + y^2z^2 - y^2 - z^2 + 1$$

$$\mathfrak{F}_1(x) := x^8y^4 - 2\,x^8y^2z^2 - 8\,x^8y^2 + x^8z^4 - 8\,x^8z^2 + 16\,x^8 - 2\,x^6y^6 + 2\,x^6y^4z^2 + 2\,x^6y^2z^4 + 64\,x^6y^2z^2 + 96\,x^6y^2 - 2\,x^6z^6 + 96\,x^6z^2 - 256\,x^6 + x^4y^8 + 2\,x^4y^6z^2 - 6\,x^4y^4z^4 - 48\,x^4y^4z^2 + 48\,x^4y^4 + 2\,x^4y^2z^6 - 48\,x^4y^2z^4 - 256\,x^4y^2z^2 - 640\,x^4y^2 + x^4z^8 + 48\,x^4z^4 - 640\,x^4z^2 + 1536\,x^4 - 2\,x^2y^8z^2 - 8\,x^2y^8 + 2\,x^2y^6z^4 + 64\,x^2y^6z^2 + 96\,x^2y^6 + 2\,x^2y^4z^6 - 48\,x^2y^4z^4 - 256\,x^2y^4z^2 - 640\,x^2y^4 - 2\,x^2y^2z^8 + 64\,x^2y^2z^6 - 256\,x^2y^2z^4 + 512\,x^2y^2z^2 + 2560\,x^2y^2 - 8\,x^2z^8 + 96\,x^2z^6 - 640\,x^2z^4 + 2560\,x^2z^2 - 4096\,x^2 + y^8z^4 - 8\,y^8z^2 + 16\,y^8 - 2\,y^6z^6 + 96\,y^6z^2 - 256\,y^6 + y^4z^8 + 48\,y^4z^4 - 640\,y^4z^2 + 1536\,y^4 - 8\,y^2z^8 + 96\,y^2z^6 - 640\,y^2z^4 + 2560\,y^2z^2 - 4096\,y^2 + 16\,z^8 - 256\,z^6 + 1536\,z^4 - 4096\,z^2 + 4096$$

$$\mathfrak{F}_0(x) := x^8 - 4\,x^6y^2 - 4\,x^6z^2 - 36\,x^6 + 6\,x^4y^4 + 4\,x^4y^2z^2 + 36\,x^4y^2 + 6\,x^4z^4 + 36\,x^4z^2 + 486\,x^4 - 4\,x^2y^6 + 4\,x^2y^4z^2 + 36\,x^2y^4 + 4\,x^2y^2z^4 - 360\,x^2y^2z^2 + 324\,x^2y^2 - 4\,x^2z^6 + 36\,x^2z^4 + 324\,x^2z^2 - 2916\,x^2 + y^8 - 4\,y^6z^2 - 36\,y^6 + 6\,y^4z^4 + 36\,y^4z^2 + 486\,y^4 - 4\,y^2z^6 + 36\,y^2z^4 + 324\,y^2z^2 - 2916\,y^2 + z^8 - 36\,z^6 + 486\,z^4 - 2916\,z^2 + 6561$$

$$\begin{aligned}\mathfrak{G}_0(x) := {} & (11\,x^2 - 16\,xy^2 - 32\,xyz - 58\,xy - 16\,xz^2 - 58\,xz - 78\,x - 4\,y^4 - 16\,y^3z - 56\,y^3 - 24\,y^2z^2 - 168\,y^2z - 289\,y^2 - 16\,yz^3 - 168\,yz^2 - 578\,yz - 678\,y - 4\,z^4 - 56\,z^3 - 289\,z^2 - 678\,z - 657) \\ & (11\,x^2 - 16\,xy^2 + 32\,xyz - 58\,xy - 16\,xz^2 + 58\,xz - 78\,x - 4\,y^4 + 16\,y^3z - 56\,y^3 - 24\,y^2z^2 + 168\,y^2z - 289\,y^2 + 16\,yz^3 - 168\,yz^2 + 578\,yz - 678\,y - 4\,z^4 + 56\,z^3 - 289\,z^2 + 678\,z - 657) \\ & (11\,x^2 - 16\,xy^2 + 32\,xyz + 58\,xy - 16\,xz^2 - 58\,xz - 78\,x - 4\,y^4 + 16\,y^3z + 56\,y^3 - 24\,y^2z^2 - 168\,y^2z - 289\,y^2 + 16\,yz^3 + 168\,yz^2 + 578\,yz + 678\,y - 4\,z^4 - 56\,z^3 - 289\,z^2 - 678\,z - 657) \\ & (11\,x^2 - 16\,xy^2 - 32\,xyz + 58\,xy - 16\,xz^2 + 58\,xz - 78\,x - 4\,y^4 - 16\,y^3z + 56\,y^3 - 24\,y^2z^2 + 168\,y^2z - 289\,y^2 - 16\,yz^3 + 168\,yz^2 - 578\,yz + 678\,y - 4\,z^4 + 56\,z^3 - 289\,z^2 + 678\,z - 657) \\ & (11\,x^2 + 16\,xy^2 - 32\,xyz + 58\,xy + 16\,xz^2 - 58\,xz + 78\,x - 4\,y^4 + 16\,y^3z - 56\,y^3 - 24\,y^2z^2 + 168\,y^2z - 289\,y^2 + 16\,yz^3 - 168\,yz^2 + 578\,yz - 678\,y - 4\,z^4 + 56\,z^3 - 289\,z^2 + 678\,z - 657) \\ & (11\,x^2 + 16\,xy^2 + 32\,xyz + 58\,xy + 16\,xz^2 + 58\,xz + 78\,x - 4\,y^4 - 16\,y^3z - 56\,y^3 - 24\,y^2z^2 - 168\,y^2z - 289\,y^2 - 16\,yz^3 - 168\,yz^2 - 578\,yz - 678\,y - 4\,z^4 - 56\,z^3 - 289\,z^2 - 678\,z - 657) \\ & (11\,x^2 + 16\,xy^2 + 32\,xyz - 58\,xy + 16\,xz^2 - 58\,xz + 78\,x - 4\,y^4 - 16\,y^3z + 56\,y^3 - 24\,y^2z^2 + 168\,y^2z - 289\,y^2 - 16\,yz^3 + 168\,yz^2 - 578\,yz + 678\,y - 4\,z^4 + 56\,z^3 - 289\,z^2 + 678\,z - 657) \\ & (11\,x^2 + 16\,xy^2 - 32\,xyz - 58\,xy + 16\,xz^2 + 58\,xz + 78\,x - 4\,y^4 + 16\,y^3z + 56\,y^3 - 24\,y^2z^2 - 168\,y^2z - 289\,y^2 + 16\,yz^3 + 168\,yz^2 + 578\,yz + 678\,y - 4\,z^4 - 56\,z^3 - 289\,z^2 - 678\,z - 657)\end{aligned}$$

$$\mathfrak{G}_{-1}(x) := -x^8 - y^8 - z^8 + 3 + 1/12230590464$$



1) Given linear inequalities are replaced by their product $\mathfrak{F}_2$.

2) As in $\mathbb{R}^2$, resulting additional points $\mathcal{S}^{\geq}(\mathfrak{F}_2) \setminus P \dots$

3) are contained in sets, where 2 polynomials are negative.

4) Such sets are removed by introducing new linear polynomials.

5) These are replaced by their product $\mathfrak{F}_1$.

6) The set $\mathcal{S}^{\geq}(\mathfrak{F}_2, \mathfrak{F}_1) \setminus P$ is contained in sets $\dots$

7) where two polynomials are negative. These sets $\dots$

8) are removed using a cone $\mathcal{S}^{\geq}(\mathfrak{f}_v, \mathfrak{g}_v)$ for each vertex.

9) All $\mathfrak{f}_v$ (and $\mathfrak{g}_v$) are replaced by their product $\mathfrak{F}_0$ ($\mathfrak{G}_0$ resp.).

10) Additional points $\mathcal{S}^{\geq}(\mathfrak{F}_2, \mathfrak{F}_1, \mathfrak{F}_0, \mathfrak{G}_0) \setminus P \dots$

11) are contained in a set $\mathcal{S}^{\geq}(-\mathfrak{g}_v, -\mathfrak{g}_w)$ or $\mathcal{S}^{\geq}(-\mathfrak{f}_v, -\mathfrak{f}_w)$.

12) These are removed using an approximation $\mathcal{S}^{\geq}(\mathfrak{G}_{-1})$.

Fig. 3.1: The unit-cube in $\mathbb{R}^3$ is described using 3 polynomials.

4. Other approaches

This chapter reviews other possible approaches that can be followed in order to obtain a description for polyhedra involving few polynomials. In fact, there are *exactly* two basic principles that can be followed to do so. This is due to algebraic principles "hidden" behind the presented construction. So opposed to the presented construction there is the possibility to follow a generally different principle. In addition, it may be possible to refine the construction introduced in this thesis.

As a whole, there are exactly two points where one can stray from the procedure presented in this work: Either right at the beginning, or after the first step. The resulting two approaches and their capabilities are demonstrated in the following by two examples. While refining the given construction may lead to to a description of d-dimensional polyhedra in d polynomial inequalities, following the other approach in general can not.

4.1 Review on Construction and Principles used

4.1.1 Algebraic background

There are three underlying principles of semi-algebraic geometry behind this work: Let $P = \{x \in \mathbb{R}^d : (a_1 \cdot x) + b_1 \geq 0, \dots (a_m \cdot x) + b_m \geq 0\}$ be a d-dimensional polyhedron with an additional semi-algebraic description $P = \mathcal{S}^{\geq}(\mathfrak{f}_1, \dots, \mathfrak{f}_n)$. Then the following holds:

1. Given any proper face $F \in \mathscr{F}(P)$, then at least $d - \dim(F)$ polynomials $\mathfrak{f}_i$ vanish in F. This is a consequence of Proposition 2.1 [GH03], a work of M. Henk and M. Grötschel.

2. Any polynomial $(a_i \cdot x) + b_i$ is a factor of at least one of the polynomials $\mathfrak{f}_i$. This is due to Proposition 2.1 in [GH03].

3. Any polynomial $\mathfrak{f}$ strictly positive on P, can be written as the conic combination of products of the linear polynomials defining

P: There are $\lambda_i \in \mathbb{R}$ with $\lambda_i \geq 0$ and $N, n_{ij} \in \mathbb{N} \cup \{0\}$ such that

$$\mathfrak{f}(x) = \sum_{i=1}^{N} \lambda_i \prod_{j=1}^{m} (a_j \cdot x + b_j)^{n_{ij}}$$

holds. This result is due Handelmann (cf. [Han88]) a constructive version is Theorem 2 in [RP01], a work of B. Reznick and V. Powers.

How this work meets the principles

Given a polyhedron P, the construction used in this work meets principle 1), by defining a polynomial for each dimension $k = 0, \ldots, d-1$, which vanishes simultaneously on all faces $F \in \mathscr{F}_k(P)$: Each polynomial $\mathfrak{p}_F$ for a k-face $F \in \mathscr{F}_k(C)$ vanishes on F, so their product $\mathfrak{P}_k$ vanishes on all k-faces. The same holds for the corresponding $\mathfrak{q}_F$ and $\mathfrak{Q}_k$. So for each face $F \in \mathscr{F}_k(C)$, from the polynomials used in in the description

$$P = \mathcal{S}^{\geq}(\mathfrak{P}_{d-1}, \mathfrak{P}_{d-2}, \mathfrak{P}_{d-3}, \mathfrak{Q}_{d-3}, \ldots, \mathfrak{P}_0, \mathfrak{Q}_0),$$

at least $d - k$ vanish on F, namely $\mathfrak{P}_{d-1}, \ldots, \mathfrak{P}_k$. In addition, the approximating polynomials $\mathfrak{q}_F$ are constructed in close observation of principle 3).

4.2 Further research and unused techniques

Two possible concepts have not been used in this work: The first is to split the given linear polynomials defining a polyhedron into groups and then replace each group by its product. The second is the use of conic combination of polynomials positive on the polyhedron. Both concepts are explained in the following by means of toy examples.

Possible other approaches

One of the remaining questions is, if in general there is a description for a d-dimensional polyhedron P using only d polynomials – this is the lower bound if P contains a vertex, due to principle 1). So the question is, at which point one can stray from the given construction. Due to the principles given above, there are only *two* general approaches of describing polyhedra by polynomials. As a consequence of principle 2), any algorithm constructing a semi-algebraic description of P must output products of the linear polynomials defining P. So all such algorithms can be split into two classes:

(I) Such where the product of all $a_i \cdot x + b_i$ is part of the output and

(II) others. At some point these must split all linear polynomials into some (not necessarily disjoint) groups and build the product of each group.

Following Approach (I) may lead to a description of a d dimensional polyhedron in d polynomial inequalities in the general case, following Approach (II) can not. Approach (II) can be followed using colorings of facets, or more general using coverings of stable sets, explained below. For *special* d dimensional polyhedra, this way one can find a description using less than $2d$ polynomials. Following approach (I), one may end up with d polynomial inequalities describing P, if one finds an appropriate polynomial $\mathfrak{F}_k$ for each $k = d-2, \ldots, 1$ vanishing on all faces of the corresponding dimension $F \in \mathscr{F}_k(P)$.

Both ideas are presented in more detail in the following using two examples.

4.2.1 Grouping polynomials

Grouping leads to fewer polynomials in special examples

The effectiveness of replacing groups of polynomials by their product is easily demonstrated using the cube

$$\{x \in \mathbb{R}^d \,:\, 1 + x_i \geq 0,\ 1 - x_i \geq 0,\ i = 1, \ldots d\}$$

which can be represented as $\{x \in \mathbb{R}^d \,:\, 1-(x_i)^2 \geq 0,\ i = 1, \ldots d\}$. Here each pair of linear polynomials $1 + x_i$ and $1 - x_i$ can be replaced by their product $1 - (x_i)^2$ without changing the set. So in some cases it is better to first group the linear polynomials defining a polyhedron and then replace each group by its product. Such grouping can be done by coloring the corresponding facets, this is explained using an example in the following. After this the general drawback for this approach in $\mathbb{R}^d$ is motivated.

How to group

Let

$$P = \{x \in \mathbb{R}^d \,:\, \mathfrak{l}_F(x) \geq 0,\ F \in \mathscr{F}_{d-1}(C)\}$$

be a polyhedron, defined by linear polynomials $\mathfrak{l}_F(x) \in \mathbb{R}[X]$, each incident to a different facet $F \in \mathscr{F}_{d-1}(P)$. When replacing a group of the linear polynomials by their product in this description, the unavoidably resulting additional points are contained in sets of the form $\mathcal{S}^{\geq}(-\mathfrak{l}_F, -\mathfrak{l}_G)$, which intersect P in $F \cap G$. It is desirable to reduce the dimension of such faces, this can be done covering the facets with stable sets, i.e., the polynomials $\mathfrak{l}_F$ are grouped such that

- each $\mathfrak{l}_F$ is contained at least in one group,
- and for two polynomials $\mathfrak{l}_F$ and $\mathfrak{l}_G$ in one group, $F \cap G$ is a face of dimension at most $d - N$.

Here N can be chosen as desired, in the general case resulting in more groups, the higher N is chosen. In general this approach can not result in a description involving d polynomials, the counterexample is the d-dimensional tetrahedron. Each pair of its $d+1$ facets intersect in a ridge, leading to at least $d+1$ polynomials resulting from any such grouping.

In $\mathbb{R}^3$ coloring leads to fewer polynomials in some examples

The facets of a 3-dimensional polyhedron $P \subset \mathbb{R}^3$ can be grouped into 4 groups such that two facets in the same group intersect at most in a vertex. This is due to the four color theorem (cf. [AH77],[RSST97]). Using such a grouping, the corresponding polynomials $\mathfrak{l}_F$ for each group are replaced by their product in the $\mathcal{H}$-description of P. Each additional set $\mathcal{S}^{\geq}(-\mathfrak{l}_F, -\mathfrak{l}_G)$ resulting from this, intersects P in $F \cap G$, which is either a vertex or the face $\emptyset$.

In the following example, the facets of the cross-polytope are split into two groups using coloring. Each group of corresponding polynomials $\mathfrak{l}_F$ is replaced by its product. The additional sets resulting from this are attached to vertices of the cross-polytope. These are removed from P using a single polynomial inequality, involving a polynomial zeroing in all vertices. In general, using 4 colors to color the facets of a polyhedron in $\mathbb{R}^3$, this concept leads to a description of P using at least 5 polynomials.

Example

The set $P := \{x \in \mathbb{R}^3 \ : \ 1 - a \cdot x \geq 0, \ a \in \{-1,1\}^3\}$ is called the cross-polytope, depicted in Figure 4.2.1. Splitting the vectors a into two groups

$$A^+ := \{a \in \{-1,1\}^3 : \prod_{i=1}^{3} a_i = 1\}$$

and $A^- = \{-1,1\}^3 \setminus A^+$, the set P can be described using three polynomials, i.e., one obtains $P = \mathcal{S}^{\geq}(\mathfrak{F}_1^+, \mathfrak{F}_1^-, \mathfrak{F}_0)$ for

$$\begin{array}{rcl}\mathfrak{F}_1^+(x) & := & \prod_{a\in A^+}(1 - a\cdot x)\\ \mathfrak{F}_1^-(x) & := & \prod_{a\in A^-}(1 - a\cdot x)\\ \mathfrak{F}_0\,(x) & := & 1-(x_1)^2-(x_2)^2-(x_3)^2.\end{array}$$

The splitting up in this case was done such that the facets corresponding to two vectors in A^+ intersect only in a vertex of P. In Figure 4.2.1 the facets corresponding to A^+ are blue, those corresponding to A^- are green. The proof for the description is obtained by looking at the set

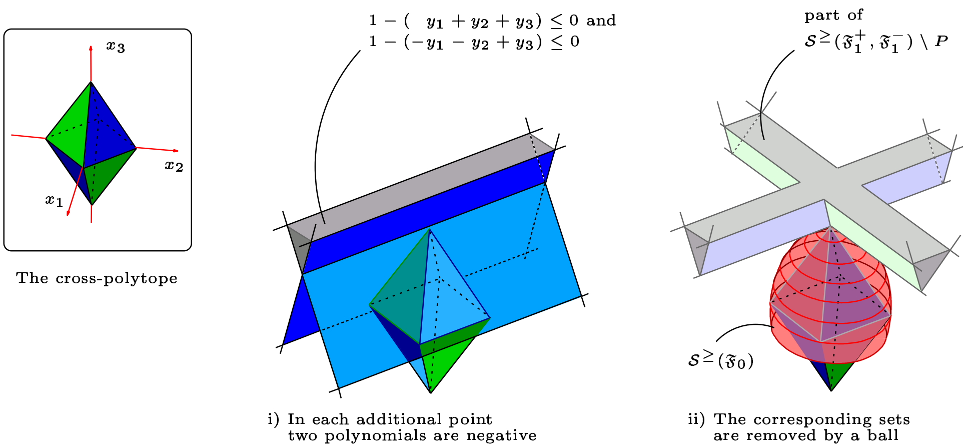


Fig. 4.1: The cross-polytope: Replacing groups of linear polynomials by their product.

$$P^+ := \{x \in \mathbb{R}^3 \ : \ 1 - a\cdot x \geq 0,\ a \in A^+\}$$

and the accordingly defined P^-.

Replacing the linear polynomials defining P^+ by their product leads to $\mathcal{S}^{\geq}(\mathfrak{F}_1^+)$. In any resulting additional point $y \in P^+ \setminus \mathcal{S}^{\geq}(\mathfrak{F}_1^+)$ two of the corresponding linear polynomials are negative, one of both strictly. Due to symmetry one can assume w.l.o.g. $1-(y_1+y_2+y_3) < 0$ and $1-(-y_1-y_2+y_3) \leq 0$ implying $1 < y_3$ and thus $\mathfrak{F}_0(y) < 0$. Figure 4.2.1 i) shows the corresponding set, the union of two such sets is presented in Figure 4.2.1 i). The same holds for the set $P^- \setminus \mathcal{S}^{\geq}(\mathfrak{F}_1^-)$. Since $P = P^+ \cap P^-$ holds, any point $\mathcal{S}^{\geq}(\mathfrak{F}_1^+, \mathfrak{F}_1^-) \setminus P$ is not contained in $\mathcal{S}^{\geq}(\mathfrak{F}_0)$. Geometrically this is presented in Figure 4.2.1 ii), depicting two sets in which two polynomials are negative. These are removed from P using the ball $\mathcal{S}^{\geq}(\mathfrak{F}_0)$.

The drawback and advantages of this approach

A general advantage of this approach is to result in polynomials of lower degree than the approach presented in this work. But aiming for a description for a d-dimensional polyhedron in d polynomial inequalities this approach must fail in $\mathbb{R}^3$ already. In general the coloring leads to 4 groups of polynomials and thus to 4 products already. Secondly, for $d > 3$ there are highly neighborly polytopes, whose facets can in general not be grouped into *few* groups such that no two facets in the same group intersect in a low-dimensional face.

4.2.2 Constructing one polynomial for all k-Faces

Principle 1), one of the three principles stated on page 71 implies the following: Given a d-dimensional polyhedron P in a semi algebraic-description $P = \mathcal{S}^{\geq}(\mathfrak{f}_1, \ldots, \mathfrak{f}_n)$, then in any proper face $F \in \mathscr{F}(P)$ at least $d - \dim(F)$ polynomials $\mathfrak{f}_i$ vanish. Aiming for a description of P involving d polynomials $\mathfrak{F}_{d-1}, \ldots, \mathfrak{F}_0$, a possible way to meet these restrictions is is the following:

First one replaces all linear polynomials in a $\mathcal{H}$-description of P by their product $\mathfrak{F}_{d-1}$. Then one has to find "correction" polynomials $\mathfrak{F}_k$ with $k = d-2, \ldots, 0$, undoing the errors caused. In observation of the above, each polynomial $\mathfrak{F}_k$ should vanish on all faces of the corresponding dimension k, namely all $F \in \mathscr{F}_k(P)$.

The polynomials $\mathfrak{P}_k$ and $\mathfrak{Q}_k$ constructed in Algorithm 3.6.1 have these properties. This results from the properties of their factors $\mathfrak{p}_F$ (and $\mathfrak{q}_F$ resp.) which zero on a face F and are positive on P. The question is now, if there are other ways to construct appropriate polynomials aside of using products of one polynomial for each face. In the following example, for each dimension there is such a polynomial.

4.2.3 Using surfaces with many double points

Let $P \subset \mathbb{R}^3$ be the tetrahedron defined by the four vertices

$$\begin{array}{rclcl} v_1 & := & (\ \ 1, 1, \ \ 1), & & v_2 := (\ \ 1, -1, -1) \\ v_3 & := & (-1, 1, -1) & & v_4 := (-1, -1, \ \ 1). \end{array}$$

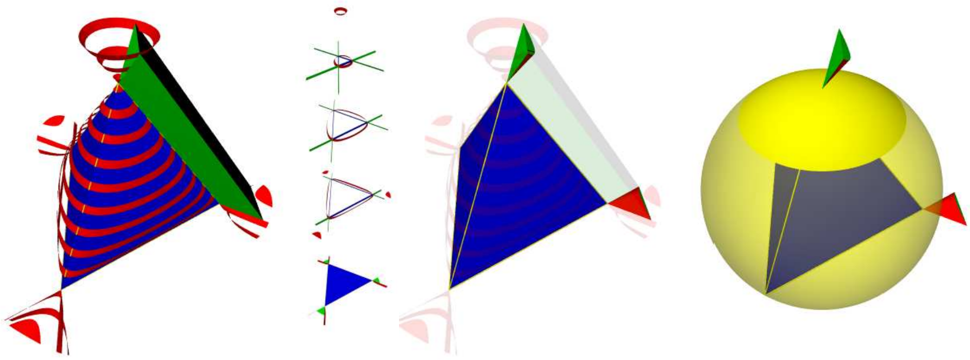
This polytope can be described using three polynomials $\mathfrak{F}_2, \mathfrak{F}_1$, and $\mathfrak{F}_0$, i.e., $P = \mathcal{S}^{\geq}(\mathfrak{F}_2, \mathfrak{F}_1, \mathfrak{F}_0)$ holds. These are constructed such, that each $\mathfrak{F}_k$ zeroes in all k-faces and is strictly positive on the rest of P. Roughly speaking each $\mathfrak{F}_k$ is support-polynomial for all k-faces. The corresponding surfaces induced by each $\mathfrak{F}_1(x) = 0$ and $\mathfrak{F}_2(x) = 0$ are shown in Figure 4.2. The $\mathcal{H}$-description of P is

$$P = \{x \in \mathbb{R}^d : \quad 1 + x_1 + x_2 + x_3 \geq 0, \qquad 1 + x_1 - x_2 - x_3 \geq 0, \\ 1 - x_1 + x_2 - x_3 \geq 0, \qquad 1 - x_1 - x_2 + x_3 \geq 0\}.$$

The product of these polynomials is $\mathfrak{F}_2$, it vanishes on all facets. The polynomial

$$\mathfrak{F}_1(x) := 1 - {x_1}^2 - {x_2}^2 - {x_3}^2 + 2x_1x_2x_3$$

is positive on P and vanishes in all edges of P. The corresponding surface $\mathcal{S}^=(\mathfrak{F}_1)$ is called *Cayley cubic* (see Figure 4.3). It has four



i) The set $\mathcal{S}^{\geq}(\mathfrak{F}_2)$ contains P and additional sets.

ii) The parts of additional sets contained in $\mathcal{S}^{\geq}(\mathfrak{F}_1)$...

iii) ... are removed by $\mathcal{S}^{\geq}(\mathfrak{F}_0)$

Fig. 4.2: Any point in $\mathcal{S}^{\geq}(\mathfrak{F}_2) \setminus P$ is not contained in $\mathcal{S}^{\geq}(\mathfrak{F}_1, \mathfrak{F}_0)$

double-points, exactly in the vertices of P. The polynomial $\mathfrak{F}_1$ zeroes in all edges of P: For example any point in the edge (v_1, v_2) is of the form $y_\lambda = (1, \lambda, \lambda)$ with $\lambda \in [-1, 1]$, leading to $\mathfrak{F}_1(y_\lambda) = 1-1-2\lambda^2+2\lambda^2 = 0$. Moreover for any point $x \in P$, one has $\mathfrak{F}_1(x) \geq 0$: For the linear-combination $y_\lambda = \sum_{i=1}^4 \lambda_i v_i$ with $\lambda_i \in [0, 1]$ and $\sum_{i=1}^4 \lambda_i = 1$, a longer calculation leads to

$$\mathfrak{F}_1(y_\lambda) = \lambda_1\lambda_2\lambda_3 + \lambda_1\lambda_2\lambda_4 + \lambda_1\lambda_3\lambda_4 + \lambda_2\lambda_3\lambda_4 \geq 0.$$

Finally a polynomial that vanishes on the vertices is used, namely $\mathfrak{F}_0 := 3 - (x_1)^2 - (x_2)^2 - (x_3)^2$. The corresponding set $\mathcal{S}^{\geq}(\mathfrak{F}_0)$ can be found in Figure 4.2 iii).

The geometric idea

The proof for $P = \mathcal{S}^{\geq}(\mathfrak{F}_2, \mathfrak{F}_1, \mathfrak{F}_3)$ is slightly long and tedious, therefore here only the geometric insights are presented. Since the polynomial $\mathfrak{F}_2$ is the product of all linear polynomials defining P, the set $\mathcal{S}^{\geq}(\mathfrak{F}_2)$ contains P and several other points. In each such additional point two of

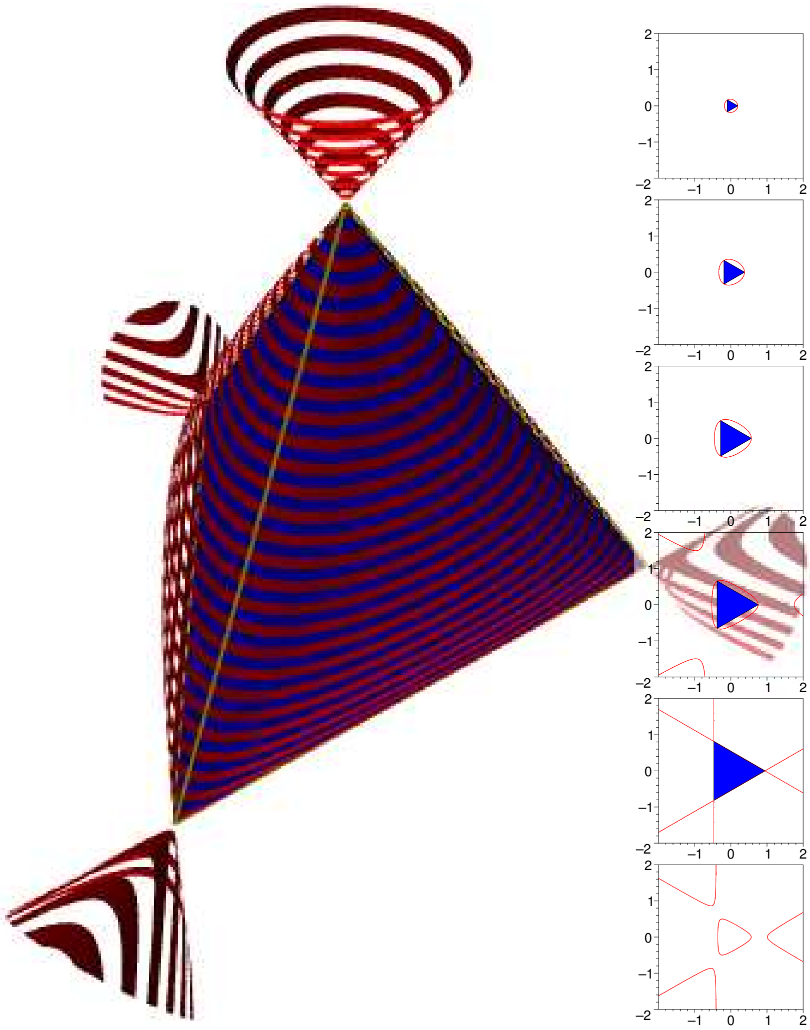


Fig. 4.3: The Cayley cubic passes neatly through the edges of a tetrahedron

the given polynomials are negative, one of the corresponding sets is depicted in Figure 4.2 i). The idea is now to understand the set $\mathcal{S}^{\geq}(\mathfrak{F}_1, \mathfrak{F}_2)$ as a correction-tool removing these additional points from $\mathcal{S}^{\geq}(\mathfrak{F}_2)$: Figure 4.2 i) depicts the set $\mathcal{S}^{\geq}(\mathfrak{F}_1)$ in red, it “cuts additional sets off the edges” of P, leaving sets that are attached only to vertices of P, as shown in Figure 4.2 ii). These remaining sets are then removed using $\mathcal{S}^{\geq}(\mathfrak{F}_0)$, as can be seen in Figure iii). So all in all $P = \mathcal{S}^{\geq}(\mathfrak{F}_2, \mathfrak{F}_1, \mathfrak{F}_0)$ holds.

A polynomial such as $\mathfrak{F}_1$ zeroing on all edges $e \in P$ should have double points in the vertices of P, therefore to generalize the sketched approach it seems worthy to examine such sets.

5. Conclusions and Outlook

5.1 Results

5.1.1 Main Result

The central result of this thesis is that any Polyhedron in $\mathbb{R}^d$ can be described using $2d$ polynomial inequalities, and that these can moreover be constructed.

Using fundamental techniques from linear algebra, together with homogenization, Algorithm 3.6.1 results in a description using $2d$ polynomial inequalities for any d dimensional polyhedron. Algorithm 3.6.4 yields a semi-algebraic description involving $2d-1$ polynomials for every d-dimensional polytope.

5.1.2 Implications for Semi-Algebraic Geometry

The results [Brö91] and [Sch89] of Bröcker and Scheiderer show that any d-dimensional semi-algebraic set can be represented by at most $d(d+1)/2$ polynomial inequalities. Thus for the case of d-dimensional polyhedra, this work additionally lowers this upper bound from quadratic in d to $2d$. This is close to the lower bound d for pointed polyhedra in d dimensions.

The construction behind Algorithm 3.6.1 leads to the insight, that the interior of a d dimensional polyhedron can be described using $2d$ strict inequalities. Reformulated, any basic *open* semi-algebraic set in $\mathbb{R}^d$ defined by strict *linear* inequalities, can be described using $2d$ strict polynomial inequalities (see Corollary 3.6.2).

From this and the main result Scheiderer immediately observed the following: For any basic closed semi-algebraic set $\mathcal{A} \subseteq \mathbb{R}^d$ defined by polynomials of degree at most n, one can construct a description involving at most $2\binom{n+d}{n} - 2$ polynomial inequalities. This is done using the presented construction in the space of monomes as shown in Corollary 3.6.3. The same holds for any d-dimensional basic *open* semi-algebraic set. In both cases the number of polynomials are relatively far from the proven *lower bounds* for the general case, namely $d(d+1)/2$

polynomials for closed and d polynomials for open semi-algebraic sets (cf. [BCR98] p. 122 and p. 259). But to this time there are no constructive proofs known fro these bounds. In contrast, this work provides the corresponding polynomials.

Chapter 4 suggests an approach, which could lead to the description of d dimensional polytopes by means of d polynomial inequalities. The example given there presents a description for a polyhedron $P \subset \mathbb{R}^3$ involving $d = 3$ polynomials. In addition the total degree of the used polynomials is bounded from above by the dimension and the number of facets of P. A generalization of this would clearly improve the application of the resulting description.

5.1.3 Implications for Optimization

This work was initially motivated by the desire to follow new strategies in the attack of combinatorial optimization problems. A common way of solving these, is to represent the corresponding feasible set by a polyhedron and maximize a linear function on this set. The aim is, to exploit special properties of the corresponding polyhedra when representing them by polynomial inequalities, and then apply methods from nonlinear optimization.

The class of polynomials used in the presented approach theoretically meet the requirements: The polynomials constructed in the presented algorithm are based on sums of squares. For these there is both a rich theoretical background (cf. [ML], [Las97]) and applicable algorithms (cf. [Las01]) exist.

But a general disadvantage of the description of polyhedra using *few* polynomials is the high degree of the appropriate polynomials, obstructing their numerical treatment: If a d dimensional polyhedron with m facets is described by $2d$ polynomial inequalities, then at least one of the used polynomials has a degree of $m/(2d)$ or higher (cf. [GH03]). Since the number of facets m is independent of the dimension n, the degree of the resulting polynomials can obstruct their numeric treatment.

In brief, in a semi-algebraic description of a polyhedron involving few polynomials, only the degree can reflect the geometrical complexity of the set. Consequently, the degree of the polynomials resulting from the presented construction depends on geometric properties of the corresponding polyhedron.

However there is hope, to obtain a balance between the number and the degree of the used polynomials in the future. Accordingly, a goal of further research would be a description of polyhedra, where – *relatively to the number of facets* – few polynomials are needed.

5.2 The principles and invariants used

The given construction of a semi-algebraic description for a polyhedron P follows a trail through combinatorics and geometry to algebra, ending with a set of consistent polynomials positive on P:

For each face F of P two polynomials $\mathfrak{p}_F, \mathfrak{q}_F$ are constructed. All together these polynomials form a semi-algebraic description of P. Moreover, in this description all $\mathfrak{p}_F$ for faces F of dimension k can be replaced with their product, without changing the set. The same holds for all $\mathfrak{q}_F$. This results from three insights, the last two are of iterative nature:

- When replacing all $\mathfrak{p}_F$ with $\dim(F) = k$ by their product in a semi-algebraic description of P, the resulting set contains additional points. The algebraic observation is, that in each such point at least two polynomials are negative. Accordingly this holds for all $\mathfrak{q}_F$.

- The following geometric detection is, that for k-faces F and G the corresponding sets $\mathcal{S}^{\geq}(-\mathfrak{p}_F, -\mathfrak{p}_G)$ and $\mathcal{S}^{\geq}(-\mathfrak{q}_F, -\mathfrak{q}_G)$ are closed cones intersecting P in a face of lower dimension, namely $F \cap G$.

- The final insight is of geometrical nature: A collection of several closed cones can be simultaneously removed from $F \cap G$ using a semi-algebraic cone $\mathcal{S}^{\geq}(\mathfrak{p}_{F\cap G}, \mathfrak{q}_{F\cap G})$, where $\mathfrak{p}_{F\cap G}$ and $\mathfrak{q}_{F\cap G}$ are support-polynomials, having the properties leading to the previous point.

So when replacing all $\mathfrak{p}_F, \mathfrak{q}_F$ for faces of equal dimension k by their products $\mathfrak{P}_k$ and $\mathfrak{Q}_k$, the polynomials $\mathfrak{p}_F, \mathfrak{q}_F$ for faces of dimension at most $k-1$ undo the resulting errors. This allows to do so for each dimension, resulting in a description of P involving $2d$ polynomials.

5.3 The algorithm

The presented algorithm works in theory, but it is not of practical use. Some of the obstacles responsible for this can not be overcome.

5.3.1 Complexity

Both presented algorithms, the Main Algorithm 3.6.1 and Algorithm 3.6.4 are not polynomial in the input-size.

The most costly step in both algorithms is the construction of a polynomial for each face of the given polyhedron. There is no general bound for the number of *all* faces of a polyhedron, which is polynomial in the number of all its facets and all its vertices. So even if the polyhedron is given in both its $\mathcal{H}$ and $\mathcal{V}$-description, this step can not be done in polynomial time. This is an inevitable consequence of the construction scheme and can not be overcome without changing the construction. In contrast to this, Section 4.2.2 indicates that there could be a different approach which overcomes this drawback.

5.3.2 Numerical stability

Algorithm 3.6.1 constructs a pair of polynomials for each face of a given polyhedral cone. It is absolutely crucial for the given construction that the semi-algebraic set defined by each such pair is a cone. The problem arising from this is that slight rounding on the "wrong" coefficients results in sets without a singularity. This makes the approach unstable in its numerical handling, when calculating the corresponding polynomials straight forward. But this can be overcome by the right choice of for the free parameters of the algorithm, namely by choosing an appropriate support-vector for each face.

5.3.3 Degree of resulting polynomials

The polynomials calculated for the Example in Section 3.6.4 show, that in some cases it is possible to lower the degree of the polynomials resulting from Algorithm 3.6.1. But in general, the high degree of the output-polynomials can not be overcome, since these are due to the principle of construction: The algorithm creates a polynomial for each k-face then all these are multiplied. Using this principle to describe a d-dimensional polyhedron P by polynomials, at least one of the resulting polynomials has degree $|\mathscr{F}(C)|/d$ or higher, where $|\mathscr{F}(C)|$ is the number of all faces of C.

5.4 Remaining open problems

It is still an open question if in general there is a semi-algebraic description for a d dimensional polyhedron involving at most d polynomial inequalities. To examine this, it should be fruitful to examine conic combinations of polynomials positive on P rather than their products. In $\mathbb{R}^3$ candidates for such polynomials are those that result in varieties with many double points, just as in the example in Section 4.2.2.

Another question is, whether for special classes of polyhedra the given construction can be simplified. Relating this to combinatorial optimization, another question arises: For the construction presented in this work, it is necessary to characterize all faces of a given polyhedron. For several polyhedra resulting from combinatorial optimization problems, there exists a complete characterization of their facets. So the question arises, if there is a generic description for the corresponding lower-dimensional faces in some cases.

But the practically most relevant task is to find a balance between the *amount* of polynomials used in the description of P and their *degree.* The hope is here that these two values level out in a description using *relatively* few polynomials of *relatively* low degree. The search for the corresponding construction would be a rewarding challenge.

Index

Bibliography

[AAB99] F. Aquistapace, C. Andradas, and F. Broglia, *Separation of semi-algebraic sets*, Journal of the American Mathematical Society **12** (1999), no. 3, 703–728.

[ABR96] C Andradas, L. Bröcker, and J.M Ruiz, *Constructible sets in real geometry*, Springer-Verlag, Berlin, 1996.

[AH77] K. Appel and W. Haken, *Every planar map is four colorable (part i.)*, Illinois J. Math. **21** (1977), 429–490.

[BCR98] J. Bochnak, M. Coste, and M.-F. Roy, *Real Algebraic Geometry*, Springer-Verlag, New York, 1998.

[Ber98] A. Bernig, *Constructions for the Theorem of Bröcker and Scheiderer*, Master's thesis, Universität Dortmund, 1998.

[BF34] T. Bonnesen and W. Fenchel, *Theorie der konvexen Körper*, Springer-Verlag, Berlin, 1934.

[BGH04] Hartwig Bosse, Martin Grötschel, and Matrin Henk, *Polynomial inequalities representing polyhedra*, Mathematical Programming, published online s10107-004-0563-2 (2004).

[Brö88] L. Bröcker, *On the separation of basic semialgebraic sets by polynomials*, Manuscripta Math. **60** (1988), no. 4, 497–508.

[Brö91] ———, *On basic semi-algebraic sets*, Expo Math. **9** (1991), 289–334.

[GH03] Martin Grötschel and Matrin Henk, *The representation of polyhedra by polynomial inequalities*, Descrete and Computational Geometry **29** (2003), no. 4, 485–504.

[GLS88] Martin Grötschel, László Lovász, and Alexander Schrijver, *Geometric algorithms and combinatorial optimization*, Springer, Heidelberg, 1988.

[Han88] D. Handelman, *Representing polynomials by positive linear functions on compact convex polyhedra*, Pacific Journal of Mathematics **132** (1988), 35–62.

[Hof91] G.vom Hofe, *Beschreibung von ebenen konvexen n-Ecken durch höchstens drei algebraische Ungleichungen*, Master's thesis, Universität Dortmund, 1991.

[Las97] J.P. Lasserre, *Semidefinite programming vs. lp relaxations for polynomial programming*, Mathematics of Operations Research **27** (1997), 347–366.

[Las01] ______, *Global optimization with polynomials and the problems of moments*, SIAM Journal on Optimization **11** (2001), 798–817.

[Min03] Herrman Minkowski, *Volumen und Oberfläche*, Math. Ann. **57** (1903), 447–495.

[ML] Franz Rendl Monique Laurent, *Semidefinite programming and integer programming*, To appear in the Handbook on Discrete Optimization edited by K.Aardal, G. Nemhauser and R. Weismantel.

[RP01] B. Reznick and V. Powers, *A new bound for Pòlya's Theorem with applications to polynomials positive on polyhedra*, J. Pure Appl. Alg. **164** (2001), 221–229.

[RSST97] N. Robertson, D. P. Sanders, P. D. Seymour, and R. Thomas, *The four colour theorem*, J. Combin. Theory **B. 70** (1997), 2–44.

[Sch89] C. Scheiderer, *Stability index of real varieties*, Inventiones Math **97** (1989), no. 3, 467–483.

[Zie98] Günter M. Ziegler, *Lectures on Polytopes*, Springer, New York, 1998.