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Three dimensional affine hyperspheres generated by 2-dimensional partial differential equations

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Abstract

It is well known that locally strongly convex affine hyperspheres can be determined as solutions of differential equations of Monge-Ampere type. In this paper we study in particular the 3-dimensional case and we assume that the hypersphere admits a Killing vector field (with respect to the affine metric) whose integral curves are geodesics with respect to both the induced affine connection and the Levi Civita connection of the affine metric. We show that besides the already known examples, such hyperspheres can be constructed starting from the 2-dimensional Laplace equation, the 2-dimensional sine-Gordon equation or the 2-dimensional cosh-Gordon equation.

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1 Introduction

In this paper we study nondegenerate affine hypersurfaces M^n into $\mathbb{R}^{n+1}$, equipped with its standard affine connection D. It is well known that on such a hypersurface there exists a canonical transversal vector field ξ, which is called the affine normal. With respect to this tranversal vector field one can decompose

$$D_X Y = \nabla_X Y + h(X,Y)\xi, \tag{1}$$

thus introducing the affine metric h and the induced affine connection ∇. The Pick-Berwald theorem states that ∇ coincides with the Levi Civita connection $\widehat{\nabla}$ of the affine metric h if and only if M is immersed as a nondegenerate quadric. The difference tensor K is introduced by

$$K_X Y = \nabla_X Y - \widehat{\nabla}_X Y \tag{2}$$

Deriving the affine normal, we introduce the affine shape operator S by

$$D_X \xi = -SX \tag{3}$$

Here, we will restrict ourselves to the case that the affine shape operator S is a multiple of the identity, i.e. $S = HI$. This means that all affine normals are parallel or pass through a fixed point. In case that the metric is positive definite one distinguishes the following classes of affine hyperspheres:

(i) elliptic affine hyperspheres, i.e. all affine normals pass through a fixed point and $H > 0$,

(ii) hyperbolic affine hyperspheres, i.e. all affine normals pass through a fixed point and $H < 0$,

(iii) parabolic affine hyperspheres, i.e. all the affine normals are parallel ($H = 0$).

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Due to the work of amongst others Calabi [2], Pogorelov [9], Cheng and Yau [3], Sasaki [10] and Li [7], positive definite affine hyperspheres which are complete with respect to the affine metric h are now well understood. In particular, the only complete elliptic or parabolic positive definite affine hyperspheres are respectively the ellipsoid and the paraboloid.

However, in the local case, one is far from obtaining a classification. The reason for this is that affine hyperspheres reduce to the study of the Monge-Ampere equations. This can be seen as follows.

If M is a parabolic affine hypersphere, then by an affine transformation, we may assume that the affine normal ξ is given by $(0, \ldots, 0, 1)$. It now follows that the immersion ϕ can be locally written as a graph

$$\phi(x_1, \ldots, x_n) = (x_1, \ldots, x_n, f(x_1, \ldots, x_n)). \tag{4}$$

It is then straightforward to check that the condition that M is a parabolic positive definite affine sphere is equivalent with the condition that f is a locally strongly convex function satisfying

$$\det(\tfrac{\partial^2 f}{\partial x_i \partial x_j}) = 1. \tag{5}$$

In case that the dimension is two, it already follows from the work of Blaschke that a 2-dimensional positive definite parabolic affine hypersphere can be locally represented by

$$(\tfrac{1}{2}(z - \bar{G}), \tfrac{1}{8}(z\bar{z} - G\bar{G}) + \tfrac{1}{2}Re(\int G) - \tfrac{1}{4}Re(G(z)z))$$

where G is a holomorphic function.

In the case of elliptic and hyperbolic positive definite affine hyperspheres one can obtain the determining Monge-Ampere equation as follows. We again write the immersion ϕ locally as

$$x_{n+1} = f(x_1, \ldots, x_n), \tag{6}$$

and we consider the Legendre transformation by

$$u_i = \tfrac{\partial f}{\partial x_i}. \tag{7}$$

It is then easy to check that this is a local diffeomorphism and that the function u defined by

$$u = \sum_{i=1}^{n} x_i u_i - f, \tag{8}$$

satisfies

$$\det(\tfrac{\partial^2 u}{\partial u_i \partial u_j}) = (Hu)^{-n-2}, \tag{9}$$

and that ϕ can be represented in terms of u_i by

$$\phi(u_1, \ldots, u_n) = (\tfrac{\partial u}{\partial u_1}, \ldots, \tfrac{\partial u}{\partial u_n}, -u + \sum_{i=1}^{n} u_i \tfrac{\partial u}{\partial u_i}). \tag{10}$$

Whereas in dimension 2, the techniques in order to obtain explicit examples are well developed, in higher dimensions there only exist few non trivial examples. For that reason, we will restrict ourselves to the case that M is 3-dimensional and admits some suitable symmetry which will allow us to reduce the classification problem to a lower dimensional one. The symmetry condition which we will impose is that the hypersurface M admits a Killing vector field whose integral curves are geodesics with respect to both the induced connection and the Levi Civita connection. Perhaps most surprising is that under those assumptions solutions can be obtained from either the Laplace equation, the sine-Gordon equation or the cosh-Gordon equation. Since these are well studied partial differential equations, we obtain in particular also solutions of the corresponding Monge Ampere equations.

The paper is organized as follows. In Section 2, we introduce the examples, starting from the above mentioned partial differential equations and in Section 3, we then characterize these examples by the property of admitting a Killing vector field. In particular, we prove the following theorems:

Theorem 1 *Let $\phi : M^3 \to \mathbb{R}^4$ be a positive definite parabolic affine sphere. Suppose that M^3 admits a Killing vector field whose integral curves are geodesics with respect to both the induced connection and the Levi Civita connection of the affine metric. Then there exists an open dense subset U of M such that for each point p of U, there exists a neighborhood V of p such that $\phi_{|V}$ is an open part of a quadric or $\phi_{|V}$ is congruent to one of the immersions described in Example 1 or 3.*

Theorem 2 *Let $\phi : M^3 \to \mathbb{R}^4$ be a positive definite elliptic affine sphere. Suppose that M^3 admits a Killing vector field whose integral curves are geodesics with respect to both the induced connection and the Levi Civita connection of the affine metric. Then there exists an open dense subset U of M such that for each point p of U, there exists a neighborhood V of p such that $\phi_{|V}$ is an open part of a quadric or $\phi_{|V}$ is congruent to one of the immersions described in Example 2 or 5.*

Theorem 3 *Let $\phi : M^3 \to \mathbb{R}^4$ be a positive definite hyperbolic affine sphere. Suppose that M^3 admits a Killing vector field whose integral curves are geodesics with respect to both the induced connection and the Levi Civita connection of the affine metric. Then there exists an open dense subset U of M such that for each point p of U, there exists a neighborhood V of p such that $\phi_{|V}$ is an open part of a quadric, an open part of $xyzw = 1$ or $\phi_{|V}$ is congruent to one of the immersions described in Example 4.*

2 Examples of affine hyperspheres admitting a Killing vector field

In this section, we introduce some examples of affine hyperspheres admitting a Killing vector field whose integral curves are geodesics with respect to both the induced connection and the Levi Civita connection of the affine metric h. The following two classes of examples are known, see [8] and [5].

Example 1 Let $\phi : M^2 \to \mathbb{R}^3$ be a positive definite parabolic affine sphere with constant affine normal vector field $\xi = (0, 0, 1)$. Writing it as a graph in the z-coordinate, we have

$$\phi(x_1, x_2) = (x_1, x_2, f(x_1, x_2)),$$

where f is a solution of

$$\det\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right) = 1.$$

It is then straightforward to compute that also

$$\psi(x_1, x_2, x_3) = (x_1, x_2, x_3, f(x_1, x_2) + \tfrac{1}{2}x_3^2),$$

is a positive definite parabolic affine sphere.

Example 2 In [5] it is shown how a positive definite 3-dimensional elliptic affine sphere for which

$$\{Z | K(X, Z) = 0 \quad \text{for all vector fields } X\} \tag{11}$$

spans a 1-dimensional distribution can be constructed from a surface immersed in the pseudo-sphere S^5_3 with the following properties

(i) the induced metric is positive definite

(ii) the immersion is minimal (vanishing mean curvature vector)

(iii) the image of $\alpha(v, v)$, where α denotes the second fundamental form of the immersion and v is a unit tangent vector describes a circle in each normal plane.

It is also shown there that such a surface is characterized by the following system of differential equations: Let $(A, \sigma) \colon N \to \mathbb{R}^2$, $(u, v) \mapsto (A(u, v), \sigma(u, v))$ be positive functions such that $(\ln\sigma)_{uu} + (\ln\sigma)_{vv} = -\sigma^2(1 + 2\sigma^{-6}A^2)$ and either

(i) $(\ln A)_{uu} + (\ln A)_{vv} = \sigma^2(\frac{1}{A^2} - 1)$ or

(ii) $(\ln A)_{uu} + (\ln A)_{vv} = -\sigma^2$.

Then in the first case there is a minimal linearly full immersion $g\colon N \to S^5_3$ and in the second case a linearly full minimal immersion $g\colon N \to S^4_2$ whose ellipses of curvature are non-degenerate circles and whose curvature is given by $k = 1 + 2A^2\sigma^{-6} > 1$.

From the formulas of [5] it follows that the unit vector field which spans the 1-dimensional distribution is a Killing vector field if and only if the corresponding surface is contained in a totally geodesic $S^4_2(1)$, i.e. if A and σ satisfy (ii).

The next classes of examples are new and show how to use the solutions of the 2-dimensional sine-Gordon equation, the 2-dimensional cosh-Gordon equation or the 2-dimensional Laplace equation in order to obtain examples of 3-dimensional positive definite affine hyperspheres in $\mathbb{R}^4$.

Example 3 Let $D \subset \mathbb{R}^2$ be a simply connected domain and let $f : D \to \mathbb{R}$ be a positive solution of the Laplace equation $f_{xx} + f_{yy} + 8 = 0$. Then, denoting by x and y the coordinates on $D \subset \mathbb{R}^2$ and by z the coordinate on $\mathbb{R}$, we define a metric h on $D \times \mathbb{R}$ by

$$h(\tfrac{\partial}{\partial x}, \tfrac{\partial}{\partial x}) = f + \tfrac{1}{16} f_y^2, \qquad h(\tfrac{\partial}{\partial y}, \tfrac{\partial}{\partial x}) = -\tfrac{1}{16} f_x f_y, \tag{12a}$$
$$h(\tfrac{\partial}{\partial y}, \tfrac{\partial}{\partial y}) = f + \tfrac{1}{16} f_x^2, \qquad h(\tfrac{\partial}{\partial x}, \tfrac{\partial}{\partial z}) = \tfrac{1}{4} f_y, \tag{12b}$$
$$h(\tfrac{\partial}{\partial y}, \tfrac{\partial}{\partial z}) = -\tfrac{1}{4} f_x, \qquad h(\tfrac{\partial}{\partial z}, \tfrac{\partial}{\partial z}) = 1, \tag{12c}$$

where f_x (resp. f_y) denotes the partial derivative of f with respect to x (resp. y) and a tensor K by

$$K(\tfrac{\partial}{\partial x}, \tfrac{\partial}{\partial x}) = \tfrac{f_x}{2f}\tfrac{\partial}{\partial y} + (1 + \tfrac{f_x^2}{8f})\tfrac{\partial}{\partial z}, \qquad K(\tfrac{\partial}{\partial y}, \tfrac{\partial}{\partial y}) = \tfrac{f_y}{2f}\tfrac{\partial}{\partial x} - (1 + \tfrac{f_y^2}{8f})\tfrac{\partial}{\partial z}, \tag{13a}$$
$$K(\tfrac{\partial}{\partial y}, \tfrac{\partial}{\partial x}) = \tfrac{f_x}{4f}\tfrac{\partial}{\partial x} + \tfrac{f_y}{4f}\tfrac{\partial}{\partial y}, \qquad K(\tfrac{\partial}{\partial x}, \tfrac{\partial}{\partial z}) = \tfrac{1}{f}(\tfrac{\partial}{\partial x} - \tfrac{1}{4} f_y \tfrac{\partial}{\partial z}) \tag{13b}$$
$$K(\tfrac{\partial}{\partial y}, \tfrac{\partial}{\partial z}) = -\tfrac{1}{f}(\tfrac{\partial}{\partial y} + \tfrac{1}{4} f_x \tfrac{\partial}{\partial z}), \qquad K(\tfrac{\partial}{\partial z}, \tfrac{\partial}{\partial z}) = 0, \tag{13c}$$

A straightforward computation shows that the Levi Civita connection $\widehat{\nabla}$ of this metric is given by

$$\widehat{\nabla}_{\frac{\partial}{\partial x}}\tfrac{\partial}{\partial x} = \tfrac{f_x}{2f}\tfrac{\partial}{\partial x} - (\tfrac{1}{8}\tfrac{1}{f} f_x f_y - \tfrac{1}{4} f_{xy})\tfrac{\partial}{\partial z}, \tag{14a}$$
$$\widehat{\nabla}_{\frac{\partial}{\partial y}}\tfrac{\partial}{\partial y} = \tfrac{f_y}{2f}\tfrac{\partial}{\partial y} + (\tfrac{1}{8}\tfrac{1}{f} f_x f_y - \tfrac{1}{4} f_{xy})\tfrac{\partial}{\partial z}, \tag{14b}$$
$$\widehat{\nabla}_{\frac{\partial}{\partial y}}\tfrac{\partial}{\partial x} = \tfrac{f_y}{4f}\tfrac{\partial}{\partial x} + \tfrac{f_x}{4f}\tfrac{\partial}{\partial y} + (-\tfrac{1}{4} f_{xx} - 1 + \tfrac{1}{16}\tfrac{f_x^2 - f_y^2}{f})\tfrac{\partial}{\partial z}, \tag{14c}$$
$$\widehat{\nabla}_{\frac{\partial}{\partial x}}\tfrac{\partial}{\partial z} = \tfrac{1}{f}(\tfrac{\partial}{\partial y} + \tfrac{1}{4} f_x \tfrac{\partial}{\partial z}), \tag{14d}$$
$$\widehat{\nabla}_{\frac{\partial}{\partial y}}\tfrac{\partial}{\partial z} = \tfrac{1}{f}(-\tfrac{\partial}{\partial x} + \tfrac{1}{4} f_y \tfrac{\partial}{\partial z}), \tag{14e}$$
$$\widehat{\nabla}_{\frac{\partial}{\partial z}}\tfrac{\partial}{\partial z} = 0, \tag{14f}$$

and that h and the connection $\nabla = \widehat{\nabla} + K$ satisfy the following properties

$$R(X,Y)Z = 0,$$
$$(\nabla_X h)(Y,Z) = (\nabla_Y h)(X,Z),$$
$$\nabla\omega_h = 0,$$

for all vector fields X, Y, Z on M.

Hence applying the basic existence theorem, with affine shape operator $S = 0$, see for example [4], we then obtain that there exists an immersion of $D \to \mathbb{R}$ as an improper affine sphere in $\mathbb{R}^4$ with h as induced metric and ∇ as induced affine connection. By construction, $\frac{\partial}{\partial z}$ is a Killing vector field whose integral curves are geodesics with respect to both connections.

Example 4 Let $D \subset \mathbb{R}^2$ be a simply connected domain and let $f : D \to \mathbb{R}$ be a positive solution of the Sine-Gordon equation $f_{xx} + f_{yy} - 8\sin f = 0$. Then, denoting by x and y the coordinates on $D \subset \mathbb{R}^2$ and by z the coordinate on $\mathbb{R}$, we define a metric on $D \times \mathbb{R}$ by

$$h(\tfrac{\partial}{\partial x}, \tfrac{\partial}{\partial x}) = \cos f + \tfrac{1}{16} f_y^2, \qquad h(\tfrac{\partial}{\partial y}, \tfrac{\partial}{\partial x}) = -\tfrac{1}{16} f_x f_y, \tag{15a}$$

$$h(\tfrac{\partial}{\partial y}, \tfrac{\partial}{\partial y}) = \cos f + \tfrac{1}{16} f_x^2, \qquad h(\tfrac{\partial}{\partial x}, \tfrac{\partial}{\partial z}) = -\tfrac{1}{4} f_y, \tag{15b}$$

$$h(\tfrac{\partial}{\partial y}, \tfrac{\partial}{\partial z}) = \tfrac{1}{4} f_x, \qquad h(\tfrac{\partial}{\partial z}, \tfrac{\partial}{\partial z}) = 1, \tag{15c}$$

where f_x (resp. f_y) denotes the partial derivative of f with respect to x (resp. y) and a tensor T by

$$K(\tfrac{\partial}{\partial x}, \tfrac{\partial}{\partial x}) = -\tfrac{f_x}{2\cos f} \tfrac{\partial}{\partial y} + (1 + \tfrac{f_x^2}{8\cos f}) \tfrac{\partial}{\partial z}, \qquad K(\tfrac{\partial}{\partial y}, \tfrac{\partial}{\partial y}) = -\tfrac{f_y}{2\cos f} \tfrac{\partial}{\partial x} - (1 + \tfrac{f_y^2}{8\cos f}) \tfrac{\partial}{\partial z}, \tag{16a}$$

$$K(\tfrac{\partial}{\partial y}, \tfrac{\partial}{\partial x}) = -\tfrac{f_x}{4\cos f} \tfrac{\partial}{\partial x} - \tfrac{f_y}{4\cos f} \tfrac{\partial}{\partial y}, \qquad K(\tfrac{\partial}{\partial x}, \tfrac{\partial}{\partial z}) = \tfrac{1}{\cos f} (\tfrac{\partial}{\partial x} + \tfrac{1}{4} f_y \tfrac{\partial}{\partial z}) \tag{16b}$$

$$K(\tfrac{\partial}{\partial y}, \tfrac{\partial}{\partial z}) = -\tfrac{1}{\cos f} (\tfrac{\partial}{\partial y} - \tfrac{1}{4} f_x \tfrac{\partial}{\partial z}), \qquad K(\tfrac{\partial}{\partial z}, \tfrac{\partial}{\partial z}) = 0, \tag{16c}$$

A straightforward computation shows that the Levi Civita connection $\widehat{\nabla}$ of this metric is given by

$$\widehat{\nabla}_{\frac{\partial}{\partial x}} \tfrac{\partial}{\partial x} = -\tfrac{f_x \sin f}{2\cos f} \tfrac{\partial}{\partial x} - (\tfrac{1}{8} \tfrac{\sin f}{\cos f} f_x f_y + \tfrac{1}{4} f_{xy}) \tfrac{\partial}{\partial z}, \tag{17a}$$

$$\widehat{\nabla}_{\frac{\partial}{\partial y}} \tfrac{\partial}{\partial y} = -\tfrac{f_y \sin f}{2\cos f} \tfrac{\partial}{\partial y} + (\tfrac{1}{8} \tfrac{\sin f}{\cos f} f_x f_y + \tfrac{1}{4} f_{xy}) \tfrac{\partial}{\partial z}, \tag{17b}$$

$$\widehat{\nabla}_{\frac{\partial}{\partial y}} \tfrac{\partial}{\partial x} = -\tfrac{f_y \sin f}{4\cos f} \tfrac{\partial}{\partial x} - \tfrac{f_x \sin f}{4\cos f} \tfrac{\partial}{\partial y} + (\tfrac{1}{4} f_{xx} - \sin f + \tfrac{1}{16}(f_x^2 - f_y^2) \tfrac{\sin f}{cos f}) \tfrac{\partial}{\partial z}, \tag{17c}$$

$$\widehat{\nabla}_{\frac{\partial}{\partial x}} \tfrac{\partial}{\partial z} = \tfrac{\sin f}{\cos f} (\tfrac{\partial}{\partial y} - \tfrac{1}{4} f_x \tfrac{\partial}{\partial z}), \tag{17d}$$

$$\widehat{\nabla}_{\frac{\partial}{\partial y}} \tfrac{\partial}{\partial z} = -\tfrac{\sin f}{\cos f} (\tfrac{\partial}{\partial x} + \tfrac{1}{4} f_y \tfrac{\partial}{\partial z}), \tag{17e}$$

$$\widehat{\nabla}_{\frac{\partial}{\partial z}} \tfrac{\partial}{\partial z} = 0, \tag{17f}$$

and that h and the connection $\nabla = \widehat{\nabla} + K$ satisfy the following properties

$$R(X,Y)Z = -h(Y,Z)X + h(X,Z)Y,$$
$$(\nabla_X h)(Y,Z) = (\nabla_Y h)(X,Z),$$
$$\nabla \omega_h = 0,$$

for all vector fields X, Y, Z on M.

Hence applying the basic existence theorem, with affine shape operator $S = -I$, see for example [4], we then obtain that there exists an immersion of $D \to \mathbb{R}$ as an hyperbolic affine sphere in $\mathbb{R}^4$ with h as induced metric and ∇ as induced affine connection. By construction, $\frac{\partial}{\partial z}$ is a Killing vector field whose integral curves are geodesics with respect to both connections.

Example 5 Let $D \subset \mathbb{R}^2$ be a simply connected domain and let $f : D \to \mathbb{R}$ be a positive solution of the Cosh-Gordon equation $f_{xx} + f_{yy} + 8\cosh f = 0$. Then, denoting by x and y the coordinates on $D \subset \mathbb{R}^2$ and by z the coordinate on $\mathbb{R}$, we define a metric on $D \times \mathbb{R}$ by

$$h(\tfrac{\partial}{\partial x}, \tfrac{\partial}{\partial x}) = \sinh f + \tfrac{1}{16} f_y^2, \qquad h(\tfrac{\partial}{\partial y}, \tfrac{\partial}{\partial x}) = -\tfrac{1}{16} f_x f_y, \tag{18a}$$

$$h(\tfrac{\partial}{\partial y}, \tfrac{\partial}{\partial y}) = \sinh f + \tfrac{1}{16} f_x^2, \qquad h(\tfrac{\partial}{\partial x}, \tfrac{\partial}{\partial z}) = \tfrac{1}{4} f_y, \tag{18b}$$

$$h(\tfrac{\partial}{\partial y}, \tfrac{\partial}{\partial z}) = -\tfrac{1}{4} f_x, \qquad h(\tfrac{\partial}{\partial z}, \tfrac{\partial}{\partial z}) = 1, \tag{18c}$$

where f_x (resp. f_y) denotes the partial derivative of f with respect to x (resp. y) and a tensor T by

$$K(\tfrac{\partial}{\partial x}, \tfrac{\partial}{\partial x}) = \tfrac{f_x}{2\sinh f} \tfrac{\partial}{\partial y} + (1 + \tfrac{f_x^2}{8\sinh f}) \tfrac{\partial}{\partial z}, \qquad K(\tfrac{\partial}{\partial y}, \tfrac{\partial}{\partial y}) = \tfrac{f_y}{2\sinh f} \tfrac{\partial}{\partial x} - (1 + \tfrac{f_y^2}{8\sinh f}) \tfrac{\partial}{\partial z}, \tag{19a}$$

$$K(\tfrac{\partial}{\partial y}, \tfrac{\partial}{\partial x}) = \tfrac{f_x}{4\sinh f} \tfrac{\partial}{\partial x} + \tfrac{f_y}{4\sinh f} \tfrac{\partial}{\partial y}, \qquad K(\tfrac{\partial}{\partial x}, \tfrac{\partial}{\partial z}) = \tfrac{1}{\sinh f} (\tfrac{\partial}{\partial x} - \tfrac{1}{4} f_y \tfrac{\partial}{\partial z}) \tag{19b}$$

$$K(\tfrac{\partial}{\partial y}, \tfrac{\partial}{\partial z}) = -\tfrac{1}{\sinh f} (\tfrac{\partial}{\partial y} + \tfrac{1}{4} f_x \tfrac{\partial}{\partial z}), \qquad K(\tfrac{\partial}{\partial z}, \tfrac{\partial}{\partial z}) = 0, \tag{19c}$$

A straightforward computation shows that the Levi Civita connection $\widehat{\nabla}$ of this metric is given by

$$\widehat{\nabla}_{\frac{\partial}{\partial x}}\frac{\partial}{\partial x} = \frac{f_x \cosh f}{2\sinh f}\frac{\partial}{\partial x} - \left(\frac{1}{8}\frac{\cosh f}{\sinh f} f_x f_y - \frac{1}{4} f_{xy}\right)\frac{\partial}{\partial z}, \tag{20a}$$

$$\widehat{\nabla}_{\frac{\partial}{\partial y}}\frac{\partial}{\partial y} = \frac{f_y \cosh f}{2\sinh f}\frac{\partial}{\partial y} + \left(\frac{1}{8}\frac{\cosh f}{\sinh f} f_x f_y - \frac{1}{4} f_{xy}\right)\frac{\partial}{\partial z}, \tag{20b}$$

$$\widehat{\nabla}_{\frac{\partial}{\partial y}}\frac{\partial}{\partial x} = \frac{f_y \cosh f}{4\sinh f}\frac{\partial}{\partial x} + \frac{f_x \cosh f}{4\sinh f}\frac{\partial}{\partial y} + \left(-\frac{1}{4} f_{xx} - \cosh f + \frac{1}{16}(f_x^2 - f_y^2)\frac{\cosh f}{sinh f}\right)\frac{\partial}{\partial z}, \tag{20c}$$

$$\widehat{\nabla}_{\frac{\partial}{\partial x}}\frac{\partial}{\partial z} = \frac{\cosh f}{\sinh f}\left(\frac{\partial}{\partial y} + \frac{1}{4} f_x \frac{\partial}{\partial z}\right), \tag{20d}$$

$$\widehat{\nabla}_{\frac{\partial}{\partial y}}\frac{\partial}{\partial z} = -\frac{\cosh f}{\sinh f}\left(\frac{\partial}{\partial x} - \frac{1}{4} f_y \frac{\partial}{\partial z}\right), \tag{20e}$$

$$\widehat{\nabla}_{\frac{\partial}{\partial z}}\frac{\partial}{\partial z} = 0, \tag{20f}$$

and that h and the connection $\nabla = \widehat{\nabla} + K$ satisfy the following properties

$$R(X,Y)Z = h(Y,Z)X - h(X,Z)Y,$$
$$(\nabla_X h)(Y,Z) = (\nabla_Y h)(X,Z),$$
$$\nabla\omega_h = 0,$$

for all vector fields X, Y, Z on M.

Hence applying the basic existence theorem, with affine shape operator $S = I$, see for example [4], we then obtain that there exists an immersion of $D \to \mathbb{R}$ as an elliptic affine sphere in $\mathbb{R}^4$ with h as induced metric and ∇ as induced affine connection. By construction, $\frac{\partial}{\partial z}$ is a Killing vector field whose integral curves are geodesics with respect to both connections.

In the next section, we then show that any affine hyperspheres which admit such a Killing vector field is either a quadric or it can be locally obtained as one of the previous examples.

3 Proof of the Main Theorem

We identify M with its image in $\mathbb{R}^4$. It is well known that an affine sphere with mean curvature H is determined by the following system of equations:

(i) $\widehat{R}(X,Y)Z = H(h(Y,Z)X - h(X,Z)Y) - [K_X, K_Y]Z$,

(ii) $(\widehat{\nabla}_X K)(Y,Z)$ is symmetric in X, Y and Z,

(iii) $h(K(X,Y),Z)$ is symmetric in X, Y and Z,

(iv) $\operatorname{trace} K_X = 0$ for any vector field X.

Let us now assume that M is 3-dimensional, positive definite and satisfies the assumptions of the Main Theorem. By applying an homothety, we may assume that $H \in \{-1, 0, 1\}$. We denote the Killing vector field by E_3. Since its integral curves with respect to both the Levi Civita connection and the induced connection are geodesics, we obtain that

$$\nabla_{E_3} E_3 = \widehat{\nabla}_{E_3} E_3 = K(E_3, E_3) = 0. \tag{21}$$

Clearly, we also can assume that E_3 has constant length 1 with respect to the affine metric h. Since $K_{E_3}E_3 = 0$, E_3 is an eigenvector of K_{E_3}. Denote by $U_1 = \{p \in M | K_{E_3} \neq 0\}$, by $U_2 = \{p \in M \backslash \bar{U}_1 | K \neq 0\}$ and $U_3 = M \setminus (\bar{U}_1 \cup \bar{U}_2)$. Then $U_1 \cup U_2 \cup U_3$ is an open dense subset of M.

Clearly on U_3 the difference tensor vanishes identically and hence by the Berwald Theorem, U_3 is affine congruent to an open part of a quadric. Next assume that $p \in U_2$. Then $K_{E_3} = 0$. Hence it follows that $K(E_3, X) = 0$, for any tangent vector field X. From [1] and [5] it now follows that, in the case that M is a proper affine hypersphere, M realizes the equality in the affine version of Chen's inequality.

Applying now the apolarity condition, together with the assumption that E_3 is a Killing vector field and the fact that $h(K(X,Y),Z)$ is totally symmetric it follows there exists vector fields E_1 and E_2 such that $\{E_1,E_2,E_3\}$ is an orthonormal basis, such that the second fundamental form h is given by

$$\begin{aligned}
&K(E_1,E_1) = -K(E_2,E_2) = \lambda E_1,\\
&K(E_1,E_2) = -\lambda E_2,\\
&K(E_1,E_3) = K(E_2,E_3) = K(E_3,E_3) = 0,
\end{aligned}$$

and such that the connection is given by

$$\begin{aligned}
&\widehat{\nabla}_{E_3}E_3 = 0, && \widehat{\nabla}_{E_1}E_3 = -bE_2, && \widehat{\nabla}_{E_1}E_1 = cE_2,\\
&\widehat{\nabla}_{E_3}E_1 = aE_2, && \widehat{\nabla}_{E_2}E_3 = bE_1, && \widehat{\nabla}_{E_2}E_2 = dE_1,\\
&\widehat{\nabla}_{E_3}E_2 = -aE_1, && \widehat{\nabla}_{E_1}E_2 = -cE_1 + bE_3, && \widehat{\nabla}_{E_2}E_1 = -dE_2 - bE_3.
\end{aligned}$$

Since $(\widehat{\nabla}_{E_1}K)(E_3,E_1) = (\widehat{\nabla}_{E_3}K)(E_1,E_1)$, we first deduce that $a = -\frac{1}{3}b$. It then follows from the Gauss equation that

$$\begin{aligned}
H &= h(\widehat{R}(E_1,E_3)E_3,E_1)\\
&= h(-\widehat{\nabla}_{E_3}(-bE_2) - \widehat{\nabla}_{-\frac{2}{3}bE_2}E_3, E_1)\\
&= \tfrac{1}{3}b^2 + \tfrac{2}{3}b^2.
\end{aligned}$$

Hence if $H = -1$, we obtain a contradiction. In the case that $H = 0$, it follows that $b = 0$. In that case the distribution spanned by E_1 and E_2 is integrable. Denote by N_1 an integral manifold of this distribution and denote the immersion of M into $\mathbb{R}^4$ by ϕ. It now follows immediately from the above formulas that $\phi|_{N_1}$ lies in a 3-dimensional subspace $\mathbb{R}^3$ of $\mathbb{R}^4$ and defines an improper affine sphere with constant transversal vector field ξ in that space. We also immediately see that $E_3|_{N_1}$ is a constant vector field along this immersion. We now introduce coordinates x,y,z such that ∂_x and ∂_y span N_1 and such that $\partial_z = E_3$. Since $D_{E_3}E_3 = -\xi$, it then follows that

$$\phi(x,y,z) = \phi|_{N_1}(x,y) + (z-z_0)E_3|_{N_1} - 1/2(z-z_0)^2\xi. \tag{22}$$

Applying now an affine transformation, together with a translation in the z-coordinate we see that ϕ is as obtained in Example 1.

Next, we consider the case that $H = 1$. In that case, if necessary by replacing E_2 by $-E_2$, we have that $b = 1$. Since M is now an elliptic affine sphere which realizes the equality in Chen's inequality we can apply the classification results of [5]. Taking into account that $b = 1$ and $h(\widehat{\nabla}_{E_1}E_3,E_1) = 0$, it follows that M can be obtained as described in Example 2.

Finally, we assume that $p \in U_1$. In this case, we can choose differentiable vector fields E_1 and E_2 such that

$$K(E_1,E_3) = \lambda E_1, \tag{23}$$
$$K(E_2,E_3) = -\lambda E_2, \tag{24}$$

where λ is a nonzero function. Since E_3 is a Killing vector field, it follows that we can write

$$\widehat{\nabla}_{E_1}E_3 = \mu E_2, \tag{25}$$
$$\widehat{\nabla}_{E_2}E_3 = -\mu E_1. \tag{26}$$

We now use the Gauss equation. Since

$$\begin{aligned}
\widehat{R}(E_1,E_3)E_3 &= HE_1 - K_{E_1}K_{E_3}E_3 + K_{E_3}K_{E_1}E_3\\
&= HE_1 + \lambda K_{E_3}E_1\\
&= (H+\lambda^2)E_1,
\end{aligned}$$

and

$$\begin{aligned}
\widehat{R}(E_1,E_3)E_3 =& \widehat{\nabla}_{E_1}\widehat{\nabla}_{E_3}E_3 - \widehat{\nabla}_{E_3}\widehat{\nabla}_{E_1}E_3 - \widehat{\nabla}_{\widehat{\nabla}_{E_1}E_3-\widehat{\nabla}_{E_3}E_1}E_3\\
=& - E_3(\mu)E_2 - \mu\widehat{\nabla}_{E_3}E_2 + \mu^2 E_1 - h(\widehat{\nabla}_{E_3}E_1,E_2)\mu E_1\\
=& - E_3(\mu)E_2 + \mu^2 E_1,
\end{aligned}$$

it follows that $E_3(\mu)=0$ and $\mu^2-\lambda^2=H$. By interchanging E_1 and E_2 if necessary, or replacing E_2 by $-E_2$, we may assume that $\lambda>0$ and $\mu\geq 0$, and therefore we can put

$$\lambda = \tfrac{1}{\sinh f}, \tag{27}$$
$$\mu = \tfrac{\cosh f}{\sinh f}, \tag{28}$$

if $H=1$, i.e. M is an elliptic affine sphere;

$$\lambda = \tfrac{1}{\cos f}, \tag{29}$$
$$\mu = \tfrac{\sin f}{\cos f}, \tag{30}$$

if $H=-1$, i.e. M is an hyperbolic affine sphere and in case that M is a parabolic affine sphere

$$\lambda = \mu = \tfrac{1}{f}, \tag{31}$$

for some local function f. Since μ did not depend on E_3 it follows from the previous equations that $E_3(f)=E_3(\lambda)=0$.

Next, we use that for an affine sphere K is a Codazzi tensor with respect to the Levi Civita connection of the affine metric $\widehat{\nabla}$. Therefore, since

$$\begin{aligned}
(\widehat{\nabla}_{E_1}K)(E_3,E_3) &= -2K(\widehat{\nabla}_{E_1}E_3,E_3) = 2\mu\lambda E_2,\\
(\widehat{\nabla}_{E_3}K)(E_1,E_3) &= E_3(\lambda)E_1 + \lambda\widehat{\nabla}_{E_3}E_1 - K(\widehat{\nabla}_{E_3}E_1,E_3) - K(E_1,\widehat{\nabla}_{E_3}E_3)\\
&= E_3(\lambda)E_1 + 2\lambda h(\widehat{\nabla}_{E_3}E_1,E_2)E_2,
\end{aligned}$$

we deduce that $h(\nabla_{E_3}E_1,E_2)=\mu$. It now follows that we can write

$$\widehat{\nabla}_{E_3}E_3 = 0 \qquad \widehat{\nabla}_{E_1}E_3 = \mu E_2 \qquad \widehat{\nabla}_{E_1}E_1 = cE_2 \tag{32}$$
$$\widehat{\nabla}_{E_3}E_1 = \mu E_2 \qquad \widehat{\nabla}_{E_2}E_3 = -\mu E_1 \qquad \widehat{\nabla}_{E_2}E_2 = dE_1 \tag{33}$$
$$\widehat{\nabla}_{E_3}E_2 = -\mu E_1 \qquad \widehat{\nabla}_{E_1}E_2 = -cE_1 - \mu E_3 \qquad \widehat{\nabla}_{E_2}E_1 = -dE_2 + \mu E_3 \tag{34}$$

From $(\widehat{\nabla}_{E_2}K)(E_1,E_3) = (\widehat{\nabla}_{E_1}K)(E_2,E_3)$ we deduce that

$$E_2(\lambda) = 2c\lambda, \tag{35}$$
$$E_1(\lambda) = 2d\lambda, \tag{36}$$
$$E_3(\lambda) = 0 \tag{37}$$

We now deal with the case that $\mu=0$ on an open set. Since $\mu^2-\lambda^2=H$, this can only happen if $\lambda=1$ and $H=-1$. Thus M is an hyperbolic affine sphere. It follows now from the differential equations for λ that $c=d=0$. This implies that the Levi Civita connection $\widehat{\nabla}$ of M is flat. It is now well known, see [11] or [6] that a flat hyperbolic affine sphere with nonvanishing difference tensor is congruent with an open part of $xyzw=1$. This completes the proof in this case.

Therefore, we may now assume that μ is different on the neighborhood of the point that we are considering. Since $\mu^2-\lambda^2=H$ this implies that the function μ satisfies the following system of differential equations:

$$\begin{aligned}
E_1(\mu) &= 2d\tfrac{\lambda^2}{\mu},\\
E_2(\mu) &= 2c\tfrac{\lambda^2}{\mu},\\
E_3(\mu) &= 0.
\end{aligned}$$

We know that

$$[E_1, E_3] = 0, \tag{38}$$
$$[E_2, E_3] = 0, \tag{39}$$
$$[E_1, E_2] = -cE_1 + dE_2 - 2\mu E_3. \tag{40}$$

Therefore, it follows computing the integrability conditions for (35), (36) and (37) that $E_3(c) = E_3(d) = 0$ and that $E_1(c) = E_2(d)$.

In the next step, we use the Gauss equation. Since on one hand we have that

$$\begin{aligned}
\widehat{R}(E_3, E_1)E_1 &= HE_3 - [K_{E_3}, K_{E_1}]E_1 \\
&= HE_3 - K_{E_3}K_{E_1}E_1 + \lambda K_{E_1}E_1 \\
&= (H + \lambda^2)E_3 + 2h(K(E_1, E_1), E_2)\lambda E_2, \\
\widehat{R}(E_3, E_2)E_2 &= HE_3 - [K_{E_3}, K_{E_2}]E_2 \\
&= HE_3 - K_{E_3}K_{E_2}E_2 - \lambda K_{E_2}E_2 \\
&= (H + \lambda^2)E_3 - 2h(K(E_2, E_2), E_1)\lambda E_1,
\end{aligned}$$

and on the other hand we have that

$$\begin{aligned}
\widehat{R}(E_3, E_1)E_1 &= \widehat{\nabla}_{E_3}(cE_2) - \widehat{\nabla}_{E_1}(\mu E_2) \\
&= -E_1(\mu)E_2 + \mu^2 E_3, \\
\widehat{R}(E_3, E_2)E_2 &= \widehat{\nabla}_{E_3}(dE_1) - \widehat{\nabla}_{E_2}(-\mu E_1) \\
&= E_2(\mu)E_1 + \mu^2 E_3,
\end{aligned}$$

we deduce that

$$h(K(E_1, E_1), E_2) = -\tfrac{E_1(\mu)}{2\lambda} = -d\tfrac{\lambda}{\mu}, \tag{41}$$
$$h(K(E_2, E_2), E_1) = -\tfrac{E_2(\mu)}{2\lambda} = -c\tfrac{\lambda}{\mu}. \tag{42}$$

Using now (41), (42), the apolarity condition and the total symmetry of $h(K(X,Y),Z)$ we obtain that

$$\begin{aligned}
K(E_1, E_1) &= c\tfrac{\lambda}{\mu}E_1 - d\tfrac{\lambda}{\mu}E_2 + \lambda E_3, \\
K(E_2, E_2) &= -c\tfrac{\lambda}{\mu}E_1 + d\tfrac{\lambda}{\mu}E_2 - \lambda E_3, \\
K(E_1, E_2) &= -d\tfrac{\lambda}{\mu}E_1 - c\tfrac{\lambda}{\mu}E_2.
\end{aligned}$$

Finally, it follows from

$$\begin{aligned}
h(\widehat{R}(E_1, E_2)E_2, E_1) &= H - h(K(E_2, E_2), K(E_1, E_1)) + h(K(E_1, E_2), K(E_1, E_2)) \\
&= H + \lambda^2 + 2(c^2 + d^2)\tfrac{\lambda^2}{\mu^2},
\end{aligned}$$

and

$$\begin{aligned}
h(\widehat{R}(E_1, E_2)E_2, E_1) &= h(\widehat{\nabla}_{E_1}(dE_1) - \widehat{\nabla}_{E_2}(-cE_1 - \mu E_3) - \widehat{\nabla}_{-cE_1 + dE_2 - 2\mu E_3}E_2, E_1) \\
&= E_1(d) + E_2(c) - \mu^2 - c^2 - d^2 - 2\mu^2,
\end{aligned}$$

that

$$E_1(d) + E_2(c) - 4\mu^2 - (c^2 + d^2)(1 + 2\tfrac{\lambda^2}{\mu^2}) = 0. \tag{43}$$

We now introduce vector fields X, Y and Z by

$$X = (\lambda)^{-\frac{1}{2}}(E_1 - \tfrac{1}{2}\tfrac{1}{\mu}cE_3), \tag{44}$$
$$Y = (\lambda)^{-\frac{1}{2}}(E_2 + \tfrac{1}{2}\tfrac{1}{\mu}dE_3), \tag{45}$$
$$Z = E_3. \tag{46}$$

Then, it follows from (43) that

$$\begin{aligned}
[X, Z] &= 0, \\
[Y, Z] &= 0, \\
[X, Y] &= \lambda^{-1}[E_1, E_2] - \tfrac{1}{2}\lambda^{-2}(E_1(\lambda)E_2 - E_2(\lambda)E_1) \\
&\quad + \tfrac{1}{2}\lambda^{-\frac{1}{2}}(E_1(\tfrac{\lambda^{-\frac{1}{2}}}{\mu}d) + E_2(\tfrac{\lambda^{-\frac{1}{2}}}{\mu}c))E_3 \\
&= (-2\tfrac{\mu}{\lambda} + \tfrac{1}{2}\tfrac{1}{\lambda\mu}(E_1(d) + E_2(c) - (c^2 + d^2)(1 + 2\tfrac{\lambda^2}{\mu^2}))E_3 \\
&= 0.
\end{aligned}$$

Hence there exist local coordinates x, y and z on M^3 such that $X = \frac{\partial}{\partial x}$, $Y = \frac{\partial}{\partial y}$ and $Z = \frac{\partial}{\partial z}$.

We now consider different cases depending on the fact that M is an hyperbolic, an elliptic or a parabolic affine sphere. First, we deal with the case that $H = 0$. In that case we had that $\lambda = \mu = f^{-1}$. From (35) and (36) it now follows that

$$\begin{aligned}
c &= \tfrac{1}{2}\tfrac{E_2(\lambda)}{\lambda} = -\tfrac{1}{2}\tfrac{E_2(f)}{f} = -\tfrac{1}{2}f^{-\frac{3}{2}}f_y, \\
d &= \tfrac{1}{2}\tfrac{E_1(\lambda)}{\lambda} = -\tfrac{1}{2}\tfrac{E_1(f)}{f} = -\tfrac{1}{2}f^{-\frac{3}{2}}f_x,
\end{aligned}$$

and hence (44), (45) and (46) can be rewritten as

$$\tfrac{\partial}{\partial x} = f^{\frac{1}{2}}E_1 + \tfrac{1}{4}f_y E_3, \tag{47a}$$

$$\tfrac{\partial}{\partial y} = f^{\frac{1}{2}}E_2 - \tfrac{1}{4}f_x E_3, \tag{47b}$$

$$\tfrac{\partial}{\partial z} = E_3. \tag{47c}$$

From (47) it now follows that the metric with respect to the coordinates x, y and z is given by (12). Moreover, inverting the above formulas, we get that

$$\begin{aligned}
E_1 &= f^{-\frac{1}{2}}(\tfrac{\partial}{\partial x} - \tfrac{1}{4}f_y\tfrac{\partial}{\partial z}), \\
E_2 &= f^{-\frac{1}{2}}(\tfrac{\partial}{\partial y} + \tfrac{1}{4}f_x\tfrac{\partial}{\partial z}), \\
E_3 &= \tfrac{\partial}{\partial z}.
\end{aligned}$$

Using the above formulas to rewrite the equation (43) as a differential equation in our coordinates, we see that f depends only on x and y and it is a solution of

$$f_{xx} + f_{yy} + 8 = 0.$$

Next, we consider the case that $H = -1$, i.e. M is an hyperbolic affine sphere. In this case, we had that $\lambda = \frac{1}{\cos f}$ and $\mu = \frac{\sin f}{\cos f}$. From (35) and (36) it now follows that

$$\begin{aligned}
c &= \tfrac{1}{2}\tfrac{E_2(\lambda)}{\lambda} = \tfrac{1}{2}\tfrac{\sin f E_2(f)}{\cos f} = \tfrac{1}{2}\tfrac{\sin f}{\cos^{\frac{3}{2}} f}f_y, \\
d &= \tfrac{1}{2}\tfrac{E_1(\lambda)}{\lambda} = \tfrac{1}{2}\tfrac{\sin f E_1(f)}{\cos f} = \tfrac{1}{2}\tfrac{\sin f}{\cos f^{\frac{3}{2}}}f_x,
\end{aligned}$$

and hence (44), (45) and (46) can be rewritten as

$$\tfrac{\partial}{\partial x} = (\cos f)^{\frac{1}{2}}E_1 - \tfrac{1}{4}f_y E_3, \tag{48a}$$

$$\tfrac{\partial}{\partial y} = (\cos f)^{\frac{1}{2}}E_2 + \tfrac{1}{4}f_x E_3, \tag{48b}$$

$$\tfrac{\partial}{\partial z} = E_3. \tag{48c}$$

From (48) it now follows that the metric with respect to the coordinates x, y and z is given by (15). Moreover, inverting the above formulas, we get that

$$
\begin{aligned}
E_1 &= (\cos f)^{-\frac{1}{2}}(\tfrac{\partial}{\partial x} + \tfrac{1}{4} f_y \tfrac{\partial}{\partial z}),\\
E_2 &= (\cos f)^{-\frac{1}{2}}(\tfrac{\partial}{\partial y} - \tfrac{1}{4} f_x \tfrac{\partial}{\partial z}),\\
E_3 &= \tfrac{\partial}{\partial z}.
\end{aligned}
$$

Using the above formulas to rewrite the equation (43) as a differential equation in our coordinates, we see that f depends only on x and y and it is a solution of

$$f_{xx} + f_{yy} - 8 \sin f = 0.$$

Finally, we consider the case that $H = 1$, i.e. M is an elliptic affine sphere. In this case, we had that $\lambda = \frac{1}{\sinh f}$ and $\mu = \frac{\cosh f}{\sinh f}$. From (35) and (36) it now follows that

$$
\begin{aligned}
c &= \tfrac{1}{2}\frac{E_2(\lambda)}{\lambda} = -\tfrac{1}{2}\frac{\cosh f\, E_2(f)}{\sinh f} = -\tfrac{1}{2}\frac{\cosh f}{\sinh^{\frac{3}{2}} f} f_y,\\
d &= \tfrac{1}{2}\frac{E_1(\lambda)}{\lambda} = -\tfrac{1}{2}\frac{\cosh f\, E_1(f)}{\sinh f} = -\tfrac{1}{2}\frac{\cosh f}{\sinh f^{\frac{3}{2}}} f_x,
\end{aligned}
$$

and hence (44), (45) and (46) can be rewritten as

$$\tfrac{\partial}{\partial x} = (\sinh f)^{\frac{1}{2}} E_1 + \tfrac{1}{4} f_y E_3, \tag{49a}$$

$$\tfrac{\partial}{\partial y} = (\sinh f)^{\frac{1}{2}} E_2 - \tfrac{1}{4} f_x E_3, \tag{49b}$$

$$\tfrac{\partial}{\partial z} = E_3. \tag{49c}$$

From (49) it now follows that the metric with respect to the coordinates x, y and z is given by (18). Moreover, inverting the above formulas, we get that

$$
\begin{aligned}
E_1 &= (\sinh f)^{-\frac{1}{2}}(\tfrac{\partial}{\partial x} - \tfrac{1}{4} f_y \tfrac{\partial}{\partial z}),\\
E_2 &= (\sinh f)^{-\frac{1}{2}}(\tfrac{\partial}{\partial y} + \tfrac{1}{4} f_x \tfrac{\partial}{\partial z}),\\
E_3 &= \tfrac{\partial}{\partial z}.
\end{aligned}
$$

Using the above formulas to rewrite the equation (43) as a differential equation in our coordinates, we see that f depends only on x and y and it is a solution of

$$f_{xx} + f_{yy} + 8 \cosh f = 0.$$

In order to complete the proof, we see that using the above formulas we can obtain the expression of the difference tensor K terms of our coordinates. It is easy to see that they correspond with those described in Example 3, 4 or 5, depending on whether M is an elliptic, an hyperbolic or a parabolic affine sphere. Therefore, applying the existence and uniqueness theorem of affine differential geometry we get that in a neighborhood of p, M^3 is affine congruent with an affine hypersphere as constructed in Example 3, 4 or 5. This completes the proof of the Main Theorem.

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