Hedging in nonlinear models of illiquid financial markets

vorgelegt von
Dipl.-Math. Nadim Sah
aus Berlin

Von der Fakultät II - Mathematik und Naturwissenschaften
der Technischen Universität Berlin
zur Erlangung des akademischen Grades

Doktor der Naturwissenschaften
– Dr. rer. nat. –

genehmigte Dissertation

Vorsitzender: Prof. Dr. Fredi Tröltzsch
Berichter: Prof. Dr. Rüdiger Frey
Prof. Dr. Peter Bank

Tag der wissenschaftlichen Aussprache: 19. September 2014

Berlin 2014
D 83

To my grandmother.

Zusammenfassung

Der Großteil der gängigen mathematischen Modelle für die Bewertung und Replikation von derivativen Finanzinstrumenten beruht auf der impliziten Annahme, dass der Preis einer gehandelten Position linear in der Positionsgröße wächst. Dies ist zum Beispiel im wohlbekannten Black-Scholes Modell der Fall. Während diese Annahme für kleine Handelsvolumina gerechtfertigt sein mag, sind darauf beruhende Modelle unpassend in Situationen, in welchen die Transaktionsgröße signifikant im Verhältnis zur Gesamtgröße des Marktes ist. In der vorliegenden Dissertation studieren wir ein nichtlineares Modell für die Bewertung und Replikation von Derivaten, welches den Preiseinfluss derartig großer Transaktionen berücksichtigt.

Wir betrachten ein Setting, in welchem ein Market Maker mit einem großen Investor zu einem Preis handelt, der es dem Market Maker erlaubt, seinen Erwartungsnutzen beizubehalten. Dies ist der sogenannte *Marktindifferenzpreis*. In diesem Modell untersuchen wir die Zulässigkeit von Handelsstrategien und zeigen die Abwesenheit von Arbitragemöglichkeiten. Überdies charakterisieren wir die Menge der erreichbaren Contingent Claims und leiten asymptotische Approximationen für Hedginstrategien in diesem illiquiden Markt her. Während der erste Teil der Dissertation sich mit Power-Nutzenfunktionen befasst, ist der zweite Teil der Untersuchung von exponentiellen Nutzenfunktionen gewidmet.

Die vorliegende Arbeit erweitert das Marktindifferenzpreismodell für einen großen Investor von Bank und Kramkov [11, 14], in welchem die Autoren einen Nutzenindifferenzansatz verfolgen, um den Preiseinfluss von großen Transaktionen an Finanzmärkten zu beschreiben. Wir erweitern dieses Modell, indem wir die *Bewertung und Replikation von Contingent Claims* untersuchen, die in [11, 14] nicht thematisiert wurde. Überdies ergänzen wir den ursprünglichen Modellrahmen um zwei Aspekte: Zum einen betrachten wir im ersten Teil dieser Dissertation *Power-Nutzenfunktionen*, welche hyperbolische absolute Risikoaversion (HARA) besitzen, anstatt uns auf Nutzenfunktionen mit beschränkter absoluter Risikoaversion zu begrenzen, welche in [11, 14] betrachtet wurden. Zum anderen modellieren wir die Auszahlung des gehandelten Wertpapieres als Endwert einer *geometrischen Brown'schen Bewegung*, welche in der Analyse von Bank und Kramkov ausgeschlossen wurde, da nicht all ihre exponentiellen Momente endlich sind.

Auf mathematischer Ebene ist das Herzstück unseres Modells eine hoch nichtlineare stochastische Differentialgleichung (SDE), welche die Handelsdynamik bestimmt. Wir formulieren sowohl die Zulässigkeit von Handelsstrategien als auch die Bewertung und die Replikation von Derivaten als Fragen über die Existenz eines Kontrollprozesses für diese SDE, welcher die Existenz von starken Lösungen mit bestimmten Endbedingungen garantiert. Im vorliegenden Rahmen, in welchem das gehandelte Wertpapier mit Hilfe einer geometrischen Brown'schen Bewegung beschrieben wird, beantworten wir diese Fragen für Power-Nutzenfunktionen im ersten und für exponentielle Nutzenfunktionen im zweiten Teil dieser Dissertation.

Unsere Untersuchung der Erreichbarkeit von Contingent Claims führt zu neuartigen

und überraschend subtilen Fragen über die Eigenschaften der Lognormalverteilung. Während wir in der Lage sind ein Resultat über das asymptotische Verhalten der Laplacetransformierten der Lognormalverteilung zu zeigen, formulieren wir zwei verwandte Monotonieaussagen lediglich als Vermutungen. Obwohl die intuitive Korrektheit dieser Vermutungen von numerischen Experimenten untermauert wird, ist ein analytischer Beweis bislang nicht erbracht, so dass einige unserer Resultate bedingt auf diese Vermutungen formuliert werden müssen. Das anscheinend lückenhafte Verständnis der Eigenschaften dieser vielfach genutzten Verteilung ist überraschend und die betreffenden Resultate und offenen Fragen könnten von unabhängigem Interesse sein.

Summary

The majority of mathematical models concerned with the pricing and replication of derivatives in financial markets relies on the implicit assumption that the price for a traded position is linear in the position size, as is the case, for instance, in the famous Black-Scholes model. While suitable for small transactions, these models are inadequate if transaction sizes are large enough to be significant in comparison to the size of the market as a whole. In this thesis we study a nonlinear framework for the pricing and replication of derivatives which incorporates the price impact of such large transactions.

We consider a setting in which a market maker trades with a large investor at a price that allows him to preserve his level of expected utility, the so-called market indifference price. In this setting we investigate the admissibility of trading strategies and show the absence of arbitrage. We further characterise the set of attainable contingent claims and we derive asymptotic expansions for hedging strategies in this illiquid market. While the first part of the thesis establishes these results for power utility functions, its second part is dedicated to the case of exponential utilities.

The work at hand extends the model for a large investor trading at market indifference prices proposed by Bank and Kramkov in [11, 14], in which the authors follow a utility indifference pricing approach to study the price impact of large transactions in a financial market. We extend this model by investigating the *pricing and replication of contingent claims* in this setting which was not addressed in [11, 14]. Moreover, we extend the original framework in two other ways: Firstly, throughout the first part of this thesis, we consider *power utility functions* with hyperbolic absolute risk aversion (HARA) rather than utility functions with bounded absolute risk aversion which were studied in [11, 14]. Secondly, we model the payoff of the marketed security using *geometric Brownian motion*, which was excluded from the analysis by Bank and Kramkov due to its failure to satisfy a finite exponential moments condition.

On a mathematical level, the heart of our model is a highly nonlinear SDE which determines the dynamics of trading in our setting. The admissibility of trading strategies as well as the pricing and hedging of derivatives are formulated as questions about the existence of a control process for this SDE which ensures the existence of strong solutions with certain terminal conditions. In the present framework, where the traded security is modeled using geometric Brownian motion, these questions are answered for both power- and exponential utility functions in the first and second part of this thesis, respectively.

Our investigation of the attainability of contingent claims leads to novel and surprisingly delicate questions about the lognormal distribution. While we are able to prove a result concerning the asymptotic behaviour of its Laplace transform, two related monotonicity assertions are merely stated as conjectures. Even though these conjectures are strongly supported by numerical evidence and by intuition, an analytical proof is yet outstanding and several of our results must be stated conditional on the validity of these conjectures. The apparent lack of understanding of the properties

of this widely used distribution is quite surprising and the related results and open questions may be of interest in their own regard.

Contents

Introduction 1

I Power utility functions 11

Introduction to Part I 13

1 Admissible strategies 15
- 1.1 Model setup 15
- 1.2 Single transaction setting 17
- 1.3 Some auxiliary regularity results for conditional expectations 22
- 1.4 Simple strategies 25
- 1.5 Continuous-time strategies 28

2 Absence of arbitrage 39
- 2.1 Profit and loss of the large investor 39
- 2.2 Proof of absence of arbitrage 40

3 Hedging and replication of options 45
- 3.1 Acceptable and attainable contingent claims 45
- 3.2 A criterion for replicability 48
- 3.3 Hedging a call option 50
- 3.4 A more tractable condition for replicability of path independent claims 52
- 3.5 A claim which is acceptable but not admissible 54

4 Asymptotic analysis of price processes and hedging strategies 57
- 4.1 Marginal prices, liquidity corrections and the marginal indifference pricing measure 57
- 4.2 Asymptotic analysis of the replicating strategy 61
- 4.3 The influence of market depth on the approximations to the hedging position 75

5 Comparative statics for hedge ratio corrections 77
- 5.1 Positivity, unimodality and term structure 78

5.2 Variability with respect to the market maker's initial endowment and risk aversion . . . 79
5.3 Variability with respect to the parameters defining the process S . . . 80

6 Logarithmic utilities **85**

II Exponential utility functions 87

Introduction to Part II **89**

7 Admissible strategies **93**
7.1 Model setup . . . 93
7.2 Single transaction setting . . . 94
7.3 Simple strategies . . . 96
7.4 Continuous-time strategies . . . 97
7.5 Absence of arbitrage . . . 102

8 The wealth process of the large investor **105**
8.1 Wealth dynamics . . . 105
8.2 Some observations regarding the large investor's wealth process . . . 107

9 Hedging and replication of options **111**
9.1 Acceptable and attainable contingent claims . . . 111
9.2 Replication via the market maker's utility process . . . 113
9.3 Replication via the large investor's wealth process . . . 118
9.4 Hedging call- and put options . . . 120
9.5 A BSDE view of the hedging problem . . . 121

10 Asymptotic analysis of price processes and replicating strategies **123**
10.1 Marginal prices for contingent claims and for the traded security . . . 124
10.2 Liquidity correction for hedge ratios . . . 126

11 Comparative statics for hedge ratio corrections **135**
11.1 Positivity, unimodality and term structure . . . 137
11.2 Variability with respect to the market maker's initial endowment and risk aversion . . . 138
11.3 Variability with respect to the parameters defining the process S . . . 139

12 Some questions regarding the lognormal distribution **143**
12.1 A limit theorem for the lognormal Laplace transform . . . 143
12.2 Discussion of the monotonicity conjecture . . . 154

Index of Notation **157**

Bibliography **159**

Introduction

Current methods for the pricing and replication of derivatives in financial markets predominantly rely on the use of models in which the price of a traded position increases linearly in the position size. In particular, the current derivative pricing paradigm is dominated by the Black-Scholes model and its conceptual progeny. While this undoubtedly powerful theory is suitable to describe pricing and replication in liquid markets, a different approach is needed when we encounter market frictions as a result of large transactions which are not easily absorbed by the market. Such orders may possess a nonlinear impact on the price of the traded asset and hence on the prices and replicating strategies for derivatives written on it. It is therefore desirable to develop models which allow us to understand the shape and magnitude of such (il)liquidity effects and which, at the same time, retain a high degree of mathematical tractability.

One such model was recently proposed by Bank and Kramkov in [11, 14], where the authors follow a utility indifference pricing approach to study the price impact of large transactions in a financial market. The work at hand extends this model in various ways and investigates the *pricing and replication of contingent claims* in this setting. Before we discuss the focus and the contribution of this thesis in greater detail, let us briefly outline its motivation and background.

Background

The rigorous mathematical treatment of derivative pricing was initiated in Bachelier's famous thesis in 1900 after which the theory remained widely disregarded for almost seven decades. In 1965 Samuelson [47] expanded and refined Bachelier's work and introduced geometric Brownian motion as a mathematical model for the random development of asset prices in financial markets. Building on these ideas, in the 1970s and early 1980s, the investigation of derivative pricing resurfaced and gained critical mass with the seminal papers by Black and Scholes [17], Merton [40], Harrison and Kreps [30] and Harrison and Pliska [31] which established the modern theory of derivative pricing, including the famous Black-Scholes model, the idea of arbitrage free pricing and its most important insight, the fundamental theorem of asset pricing.

Since then, financial market participants have relied increasingly on mathematical modelling rather than pursuing the old-fashioned, intuitive and non-formalised way of trading and asset management. This mathematisation of finance introduced an abundance of new financial products to the markets which derived their value from the value of traditional assets and which could be fairly priced and replicated with the help of those new methods.

Most of the models which have been in use and under scrutiny since then have been generalised and extended in many important ways to capture empirically observed market effects that had formerly not been accounted for, such as e.g. stochastic volatility, stochastic interest rates and transaction costs. However, the vast majority of derivative pricing models, much in the spirit of Bachelier's original idea, specify the price processes of assets in an exogenous manner, usually via a semimartingale that allows for the use of Itô's calculus. In particular, this implies the assumption that asset prices do not depend on the actions of individual market participants. These models, for the sake of analytic tractability, neglect the fundamental economic insight that prices in a free market are formed via an equilibrium of supply and demand. This neglect is justified as long as these models operate under the assumption of a "liquid market" or, equivalently, of a "sufficiently small investor" whose actions are negligible when compared to the market as a whole.

A different approach is needed when this is not the case. If a market participant trades positions which are significant in relation to the size of the market, his actions have an impact on the price of the traded asset. Moreover, this impact propagates to influence the prices of financial instruments which derive their value from the price of this asset and, consequently, to the respective hedging strategies for those instruments.

The term *(il)liquidity*, under which such effects are usually subsumed, is ubiquitous in the financial market debate today, especially since the global financial crisis of 2008. The frequency with which this term is used and the importance unanimously assigned to it, however, are in stark contrast with our degree of understanding and our ability to model it. Most of the liquidity models which have been proposed over the last two decades account for the price impact of sizeable transactions by exogenously specifying a mathematically convenient functional relationship between the size of a traded position and the price per share, usually referred to as a reaction function, supply curve or demand pressure. By design, these exogenous solutions neglect the fact that liquidity effects arise endogenously from the dynamics of supply and demand. Moreover, in most of these models it is not clear how the proposed price impact translates to replicating strategies for derivatives written on illiquid assets.

Let us briefly mention some of the various approaches to modelling market illiquidity which have been pursued in the past. For a comprehensive and detailed survey of liquidity models we refer to Gökay, Roch and Soner [28].

One frequently taken approach is to consider the situation where a large investor has to sell a high volume of shares over a small period of time. This is done in Bertsimas and Lo [15], Almgren [5], Almgren and Chriss [6, 7] and Schied and Schöneborn [48]. While in [15] the authors seek to minimise the large investor's expected execution cost over all possible liquidation strategies, the model proposed in [5] and [6, 7] additionally takes into account the large investor's risk aversion. In [48], the authors generalise the setting of [5] and find the liquidation strategy which maximises the large investor's expected utility from terminal wealth.

In the optimal execution models of Obizhaeva and Wang [41] and Alfonsi, Fruth and Schied [1, 2] the large investor is, again, confronted with the problem of optimally liquidating a sizeable position of a risky asset within a given time. However, rather than modelling the price process directly, the authors model supply and demand in the form of a *limit order book* which is depleted by the large investor's orders and, thereafter, slowly recovers. This results in a resilient temporary price impact.

A different approach is taken by Cvitanić and Ma [22] and Cuoco and Cvitanić [21], where it is assumed that the drift and the volatility of the price process of a risky asset depend exogenously on the large investor's trading strategy.

Jarrow [33, 34] introduces a *reaction function* which explicitly describes the price impact for the marketed security as a function of the large investor's current holdings. Frey [25] extends the setting of Jarrow to continuous time and studies an illiquid market model in which option prices are obtained as solutions to a nonlinear partial differential equation. Frey and Stremme [26], Platen and Schweizer [43], Papanicolaou and Sircar [42] and Bank and Baum [10] use different instances of reaction functions to model the feedback effects of hedging strategies on asset prices.

Çetin, Jarrow and Protter [18] investigate temporary liquidity effects by specifying a stochastic field of supply curves which model the impact of the total order size on the price per share. Gökay and Soner [29] study a discrete-time version of this supply curve model and show that, in the limit, one recovers the setting of [18].

Roch [35] follows a limit order book approach in which he assumes that the permanent price impact is given by a stochastic process. Jarrow, Protter and Roch [36] use ideas from this setup to construct a liquidity-based model for asset price bubbles by comparing the (exogenously specified) fundamental price process for a traded security to the asset's market price, which is endogenously determined by trading activity.

The possibility of creating and exploiting arbitrage opportunities through liquidity-based price manipulations is investigated in Jarrow [33, 34], Gatheral [27], Huberman and Stanzl [32], Alfonsi and Schied [3] and Alfonsi, Schied and Slynko [4].

While these models provide valuable contributions to understanding market liquidity, they are not free of shortcomings. In the models of Cvitanic and Ma [22] and Cuoco and Cvitanic [21], for instance, the price process of the illiquid asset is not affected even by big jumps in the investor's trading strategy. The model of Çetin, Jarrow and Protter [18] only captures temporary liquidity effects which completely disappear for absolutely continuous strategies. Moreover, option prices in this setting are insensitive to liquidity effects. Finally, most of these models specify the price impact exogenously and therefore fail to comply with the fundamental economic principle of price formation by supply and demand.

In their recent papers [11, 14] Bank and Kramkov introduce a model for the price impact of a large investor which is based on utility indifference pricing. This model, rather than specifying an exogenous functional dependence of prices on demand pressure, derives the price impact endogenously from economic equilibrium considerations. A large investor trades with risk averse market makers at prices which allow the market makers to preserve their pre-transaction levels of expected utility. In this setting, Bank and Kramkov derive liquidity corrections for asset prices and describe the post-transaction allocation of wealth among the market makers. While [11] introduces the model for the single-period case, [14] establishes the continuous-time framework.

The benefit of the new approach by Bank and Kramkov consists in bridging the gap between two asset pricing paradigms, namely the semimartingale approach of financial mathematics on the one hand and the qualitative explanation of price formation provided by economic equilibrium theory on the other. The model incorporates the undeniably reasonable principle that prices are formed through an equilibrium of supply and demand while, at the same time, it allows for the use of the plentiful toolbox of stochastic analysis. As a result, the model possesses a high degree of mathematical tractability while it adheres to fundamental economic principles.

In the setting of Bank and Kramkov, Said [46] investigates the replication of options in the special case where the market makers possess exponential utility functions and where the value at maturity of the traded security is given by the terminal value of a Brownian motion. He shows that in this case the market is complete and he derives explicit representations for the replicating strategy and for the wealth process of the large investor. The use of Brownian motion for the value at maturity of the traded asset suggests the interpretation of [46] as an *illiquid Bachelier model*.

The thesis at hand extends the ideas of [11, 14] and [46] in three ways: Firstly, throughout the first part of this work, we consider *power utility functions* which display hyperbolic absolute risk aversion (HARA) rather than utility functions with bounded- or constant absolute risk aversion as were the object of study in [11, 14] and [46]. Secondly, we specify the underlying security as the terminal value of a *geometric Brownian motion*. This case was excluded from the previous three investigations due to the fact that geometric Brownian motion fails to satisfy the finite exponential moments condition imposed therein. Lastly, rather than analysing the price impact for the traded asset itself, the focus of this work is to investigate the replication of contingent claims as well as the liquidity effects on hedging strategies, neither of which are addressed in [11, 14]. With the same reasoning with which we can view [46] as an illiquid Bachelier model, we can consider the model presented in this thesis to be an *illiquid Black-Scholes model.*

Results

In Part I we establish the model and conduct an analysis of hedging and replication in the case where the payoff at maturity of the traded security is modeled using *geometric Brownian motion* and where the market maker's preferences are given by a *power utility function*. For this class of utility functions, the model displays very different characteristics from those in [11, 14]. Most prominently, we observe the emergence of position limits for trades which are the result of the market maker's unwillingness to assume net short positions in the traded security and in cash. As the upper trade size bound is stochastic, this phenomenon increases the difficulty of the characterisation of admissible trading strategies. In the continuous-time setting, for instance, it is generally not true that a locally bounded predictable process is an admissible trading strategy.

The *admissibility of trading strategies* is investigated in three consecutive steps: For a single trade, for simple strategies and, lastly, for continuous-time strategies. In all of these cases, admissible strategies are characterised with the help of the so-called *utility indifference principle*, which formalises the preservation of the market maker's level of expected utility. In the single-period case and for simple trading strategies this characterisation amounts to providing (stochastic) position limits for admissible trade sizes. In the case of continuous-time strategies, we formulate a nonlinear stochastic differential equation for the market maker's process of expected utility which governs the price formation and which implicitly determines the set of admissible trading strategies. Due to the dynamic domain restrictions which we encounter as a consequence of the aforementioned position limits, the question of existence and uniqueness of solutions to this SDE is very delicate: It turns out that the admissibility of a trading strategy depends on the interplay of the strategy itself and the utility level it induces. As a result, a simple "buy today and sell tomorrow" strategy may not be admissible. This is a consequence of the fact that the market maker may not have sufficient funds to enable the large investor to liquidate his position and it can be seen as a very severe form of illiquidity, culminating in a *market breakdown*.

We move on to show that the market maker's utility process is a (true) martingale. In particular, this implies the absence of arbitrage in our model. Doubling- and suicide strategies, which usually have to be excluded by imposing additional assumptions, are not admissible in our model as they violate the dynamic position limits. The fact that the market maker's utility process is a (true) martingale in the case of power utility functions constitutes a real difference to the settings of Part II of this thesis and of [14], where we merely obtain the submartingale property.

Our understanding of the market maker's utility process is used for the characterisation of the set of attainable contingent claims. For general contingent claims we show that a claim is attainable if and only if the integrand in the Itô representation of the utility process induced by the claim remains within certain bounds. In the special case of a path independent contingent claim, whose payoff depends solely on the value at maturity of the marketed security, this condition can be simplified to a boundedness condition on the growth of the claim's payoff function, where the bounds depend on the market maker's initial endowment. In particular, we show the attainability of (limited positions) of call- and put options for suitable initial endowments of the market maker. We further find that, if the market maker were to trade the contingent claim itself rather than the underlying security, the claim's market indifference price would be equal to the (illiquid) replication price of the claim.

The replicating strategy in our setting is specified in a highly implicit manner and there seems to be no hope to obtain a closed form solution. We therefore resort to deriving *asymptotic expansions for the replicating position* for a small number of contingent claims. We find that the *first order approximation* Δ can be viewed as a Black-Scholes delta in the sense that it is the sensitivity with respect to changes in the underlying of the claim's expected payoff under the *marginal indifference pricing measure* $\mathbb{Q}$. In our setting, this measure plays the role of the equivalent martingale measure as the marginal price processes of attainable contingent claims and of the marketed security are martingales with respect to $\mathbb{Q}$. We show that the first order approximation Δ can be interpreted as the (liquid) replicating strategy of a small investor who is trading at marginal prices and whose order sizes are negligible with respect to the overall market size.

For a large investor, whose trading activity causes a price impact, the *second order approximation* Λ becomes significant. It can be viewed as the *liquidity correction for the replicating strategy* and can be expressed as a linear combination of hedge ratios of auxiliary claims, i.e. as the sensitivities with respect to changes in the underlying of the (risk aversion corrected) expected payoffs of auxiliary claims which are composed of the claim itself and the traded security. This representation of the second order approximation provides a "liquid recipe for an illiquid hedge" in the sense that a portfolio manager who faces the task of replicating an option in our illiquid market can be advised to execute several parallel delta hedges which, together, hedge his liquidity risk.

We proceed to show that the liquidity correction Λ to the replicating strategy is inversely proportional to a linear scaling of the market maker's initial endowment in which the scaling parameter can be interpreted as the *market depth*. As further analytical statements about Λ are elusive, we present comparative statics for a call option which reveal that, in this case, Λ is positive (meaning that higher positions have to be taken when hedging) and unimodal with a maximum near the money. Moreover, we find that, for a call option, Λ reacts to changes in the model parameters as one would expect: It is increasing in the market maker's risk aversion, in the volatility of the underlying and in the remaining time to maturity.

Note that the asymptotic expansions are established only for the case where the market maker's risk aversion parameter satisfies $0 < a < 1$. This is due to the fact that, for the validity of the expansions, we need to assert that the term

$$k_1(p) \triangleq p \frac{\mathbb{E}[(p\psi + 1)^{-a}\psi]}{\mathbb{E}[(p\psi + 1)^{-a}]}$$

satisfies $k_1'(p) > 0$ for $p \geq 0$, where ψ is a lognormally distributed random variable. While the proof of this fact is straightforward for $0 < a < 1$, we are not yet able to provide an analytical proof for $a > 1$ although numerical evidence as well as intuition strongly support the validity of the statement also in this case. The assertion for $a > 1$ is stated towards the end of the first part of this thesis as Conjecture 1.

In Part II we revisit the "illiquid Black-Scholes model" from Part I which uses geometric Brownian motion to model the payoff at maturity of the underlying, albeit for a different class of utility functions. While in Part I we consider power utility functions which display hyperbolic absolute risk aversion (HARA), Part II investigates the model for exponential utility functions with constant absolute risk aversion (CARA). As before, we establish the *admissibility of trading strategies* for the three consecutive cases of single-period trading, simple trading strategies and, finally, continuous-time strategies. The methodology, as in Part I, relies on the *utility indifference principle*, which formalises the preservation of the market maker's expected utility from terminal wealth.

The most prominent difference to Part I is the fact that, for exponential utilities, we merely have a (deterministic) lower position limit on admissible trades as opposed to the upper and lower trade bounds which we encountered previously. Informally speaking, the power utility case is best described as *"the market maker will never take a short position, neither in stocks nor in cash"*, while the analog statement for the case of exponential utility functions reads *"the market maker will take short positions in cash but not in stocks"*.

The stochastic differential equation by which we define admissible trading strategies in the continuous-time case was observed to be nonlinear and highly unapproachable in Part I due to its dynamic domain restrictions. In Part II, thanks to the cash invariance of exponential utilities, it turns out to be a linear SDE. It is then easy to obtain the existence and uniqueness of solutions to this SDE for square-integrable predictable processes which are bounded from below by the market maker's initial stock position. Such processes, hence, are admissible trading strategies in the exponential utility setting.

We proceed to investigate the large investor's wealth process upon pursuit of an admissible trading strategy which, contrary to the HARA case, is well defined for exponential utility functions. In contrast to the vast majority of traditional models, the large investor's wealth process does not play a central role in our analysis on a technical level. In fact, it is more convenient to track the market maker's process of expected utility instead. The absence of arbitrage, as in Part I, is then obtained by showing that the market maker's utility process is a submartingale. However, contrary to the case of power utilities, it is not clear whether it is also a (true) martingale.

The subsequent analysis of attainability of contingent claims, again, relies heavily on the market maker's utility process. We show that a contingent claim can be replicated if and only if the integrand in the Itô representation of the utility process induced by the claim is non-negative. For path independent claims with Lipschitz-continuous payoff functions, this condition is equivalent to an easily verifiable condition on the claim's payoff function in relation to the market maker's initial endowment. This condition, in turn, can be used to show that (limited positions of) call- and put options are attainable. Moreover, as was the case in Part I, the replication price of an attainable claim coincides with the market indifference price which the claim would possess if the market maker were to trade it.

A significant complication of the analyis of attainable contingent claims for exponential utility functions arises from the need to show that $k_2(\theta) \to \infty$ for $\theta \to \infty$, where

$$k_2(\theta) \triangleq \theta \frac{\mathbb{E}[e^{-\theta\psi}\psi]}{\mathbb{E}[e^{-\theta\psi}]}$$

for a lognormally distributed random variable ψ. Although it is well known that $\mathbb{E}[e^{-\theta\psi}\psi]/\mathbb{E}[e^{-\theta\psi}] \to 0$ for $\theta \to \infty$, the rate of this convergence is not clear. We devote to the surprisingly lengthy proof of this result a separate section in which we also show why traditional methods of coping with limits of this kind such as Abelian and Tauberian theorems are not applicable. The result may be of interest in its own regard as it extends our knowledge of the not very well understood Laplace transform of the lognormal distribution; see for instance the very recent investigation by Asmussen, Jensen and Rojas-Nandayapa [8].

The question of the replicating strategy for an attainable contingent claim, as in Part I, is answered by providing *asymptotic expansions for the replicating position* for a small number of claims. The first and second order approximations to the replicating strategy admit representations which are structurally the same as those in the case of power utilities. However, we can use the constant absolute risk aversion of exponential utilities to obtain an alternative form for the second order approximation Λ which is of conceptual interest.

It was mentioned above that in the case of power utilities the extendibility of the expansions to the regime $a > 1$ is conditional on the validity of Conjecture 1. For exponential utility functions we face a similar situation: The validity of the expansions and all related results is conditional on the fact that $k_2'(\theta) > 0$ for $\theta \geq 0$. Again, this statement is supported by numerical evidence and intuition and it is furthermore consistent with the observations that $k_2(0) = 0$ and that $k_2(\theta) \to \infty$ for $\theta \to \infty$. However, an analytical proof of this assertion is outstanding and we state it in the second part of this thesis as Conjecture 2.

Conditional on the validity of this conjecture, we proceed to show that the interpretation of Δ and Λ as a small investor's replicating position and a large investor's liquidity correction, respectively, which we gave in Part I, remains valid for exponential utilities.

As in the case of power utilities, the liquidity correction Λ for the replicating strategy is hard to come by analytically and we resort once again to giving comparative statics. We find that, as for HARA utilities, Λ reacts to changes in the model parameters as one would expect: It is increasing in the market maker's risk aversion, in the remaining time to maturity and in the diffusion coefficient of the geometric Brownian motion that models the terminal price of the traded security. The exponential case differs from the case of power utilities insomuch as the cash invariance of exponential utilities renders the market maker's initial cash position irrelevant. In particular, we *do not* have the convenient scaling property with respect to the market depth that we obtained in Part I. Similarly to the case of power utilities, Λ decreases in the initial stock position of the market maker.

Acknowledgements

I would like to thank my advisor Prof. Dr. Peter Bank for his continuous support and his patient guidance during the last years. His intuition and his profound knowledge of financial mathematics have been a wellspring of helpful ideas. Without his support this thesis could not have been written. I would also like to thank Prof. Dr. Michael Scheutzow who has been guiding me academically ever since I first set foot in a university. He willingly took on the role of my mentor during the time of my doctorate and he provided me with valuable and insightful advice. I am very grateful to Prof. David Taylor who readily and warmly welcomed me as my academic host in Cape Town and, in the course of my stay, became a friend. I would like to thank Prof. Dr. Rüdiger Frey for kindly accepting the role of co-examiner for my thesis. Moreover, I would like to thank Prof. Dr. Walter Schachermayer and Prof. Dr. Martin Schweizer for asking helpful questions early on during a presentation of my work at the Kruger Park in 2011. I am indebted to Florence Siwak and Nadja Wisniewski for firmly holding together the framework within which we all operate and for smoothing the way in all administrative matters. Lastly, I thank the former graduate school IRTG Stochastic Models of Complex Processes, the Berlin Mathematical School, the Technische Universität Berlin – my alma mater – and the DAAD for providing me with the opportunity to attain the education which I am grateful to have received.

Part I

Power utility functions

Introduction to Part I

In the first part of this thesis we investigate the price impact of large transactions in an equilibrium pricing model with utility functions that display *hyperbolic absolute risk aversion* (HARA). A market maker trades with a large investor at a price that allows him to preserve his expected utility, the so-called *market indifference price*. This market indifference price is nonlinear in the size of the traded position and we interpret this nonlinearity as the price impact or liquidity effect caused by the size of the transaction. In this illiquid market we investigate the replication of contingent claims and we compute asymptotic approximations for (illiquid) hedge ratios.

The utility indifference approach to modeling the price impact of large transactions is due to Bank and Kramkov [11, 14] where it is established for utility functions with bounded absolute risk aversion and for traded securities with finite exponential moments. Said [46] treats the special case of exponential utility functions, where the traded security is modeled via Brownian motion. This suggests the interpretation of the latter setting as an "illiquid Bachelier model".

The work at hand extends the ideas of [11], [14] and [46] to a different class of utility functions and to a different marketed security: We consider *power utility functions* with hyperbolic absolute risk aversion and we model the traded security using *geometric Brownian motion*, which was not accommodated in the aforementioned works due to its failure to satisfy the finite exponential moment condition requested therein. In addition to these modeling differences, this work possesses a different focus from the previous analyses insofar as it is mainly concerned with the replication of contingent claims which has so far only been investigated in the special setting of [46].

A determining characteristic of our analysis when compared to those of [11], [14] and [46] is the emergence of trade size bounds for admissible strategies. A market maker whose preferences are modelled by a HARA utility function will never assume net short positions, neither in the traded security nor in cash. In a single-period setting this gives rise to a lower- and an upper bound on admissible trade sizes and it merely means that the market maker will not accommodate any trade exceeding these bounds. This phenomenon becomes more relevant when we extend our model to dynamic trading: There, the large investor can find himself in a situation where the upper bound on trade sizes is such that he cannot liquidate his asset position and that he becomes "trapped" in a long position of the traded security.

Chapter Overview

The first part of this thesis is structured as follows.

Chapter 1 establishes the notion of *admissible trading strategies*. This is done in three consecutive steps: First in a single-period setting, then for simple strategies (where only finitely many trades occur) and finally for continuous-time strategies. Our model is set up in such a way that the trade dynamics are guided by a *utility indifference principle* which formalises the idea of preservation of expected utility as the principle of price formation. A trading strategy is then defined to be admissible if it is such that the utility indifference principle can be adhered to.

Chapter 2 is dedicated to the proof of the absence of arbitrage in our model. This is accomplished by showing that the market maker's utility process is a (true) martingale. Moreover, we show that – contrary to most conventional models – there exists no well-defined profit and loss process for the large investor.

Chapter 3 establishes the setup for the replication of contingent claims in our model. We show that, if a claim is attainable, its replication price is equal to its market indifference price and we furthermore provide a necessary and sufficient condition for the attainability of claims. For path independent claims this condition reduces to an easily verifiable condition on the claim's payoff function in relation to the market maker's initial endowment. In particular, we prove that call- and put options are attainable under simple assumptions on the market maker's initial endowment.

Chapter 4 contains an asymptotic analysis of hedge ratios in our illiquid market. As there is no hope to obtain an explicit representation of the replicating position for an attainable claim, we consider small positions of claims and derive first- and second order approximations for the replicating position. While the first order approximation resembles the Black-Scholes delta, the second order approximation can be seen as a *liquidity correction for hedge ratios* in our model.

In **Chapter 5** we conduct a numerical investigation of the shape and magnitude of the liquidity correction term for the hedging position that was established in the previous chapter.

Chapter 6 briefly highlights the particularities of logarithmic utility functions in our model and explains why they are not included in our analysis.

Chapter 1

Admissible strategies

In this chapter we establish the notion of *admissibility* of trading strategies in our model which is achieved in three consecutive steps: We will first give a definition of *admissible transaction sizes* in a single transaction setting, where a trading strategy consists of the mere choice of the number of shares in a single transaction. We will then extend this notion to *admissible simple strategies*, where finitely many trades occur. Finally, we will express the trade dynamics for continuous-time strategies via an SDE for the market maker's process of expected utility which will be the main tool in defining *admissible continuous-time strategies.*

The way in which we establish these three notions of admissibility follows the same underlying idea: After stating a *utility indifference principle* which determines the trade dynamics in the respective setting, we will define the set of admissible trading strategies as the set of strategies for which the utility indifference principle can be adhered to. In the single-transaction setting and for simple strategies we will then provide transaction size bounds which characterize the respective sets of admissible strategies.

1.1 Model setup

Let us first introduce the main assumptions which define the setting of our model. These assumptions will concern the market maker's utility function and his initial endowment as well as the distribution of the traded security.

The randomness in our model is treated in the conventional way: We consider a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leq t\leq T}, \mathbb{P})$ satisfying the usual conditions and denote by $\mathbf{L}^0(\mathbb{R})$ the metric space of equivalence classes of all real-valued random variables differing on a set of measure zero endowed with the topology of convergence in probability. We write $\mathbf{L}^p(\mathbb{R})$, $p \geq 1$ for the Banach-space of p-integrable random variables. T is a finite maturity and $\mathcal{F}_T = \mathcal{F}$. Furthermore for a σ-algebra $\mathcal{A} \subseteq \mathcal{F}$ and a set $A \subseteq \mathbb{R}$ we denote by $\mathbf{L}^0(\mathcal{A}, A)$ and $\mathbf{L}^p(\mathcal{A}, A)$, $p \geq 1$, the respective subsets of $\mathbf{L}^0(\mathbb{R})$ and $\mathbf{L}^p(\mathbb{R})$ consisting of all $\mathcal{A}$-measurable random variables with values in A.

A single market maker quotes prices for a marketed security $\psi \in \mathbf{L}^0(\mathbb{R})$, where ψ is understood to be the cash payoff at maturity. The market maker's preferences are modeled by a utility function u for terminal wealth which in the following will satisfy

Assumption 1.1. The market maker's utility function $u : (0, \infty) \to \mathbb{R}$ has the form

$$u(x) = \frac{x^{1-a}}{1-a}, \text{ for some } a > 0,\ a \neq 1.$$

We set $u(0) \triangleq 0$ for $0 < a < 1$, $u(0) \triangleq -\infty$ for $a > 1$ and $u(x) \triangleq -\infty$ for $x < 0$.

Note that an agent who possesses such a utility function u has hyperbolic absolute risk aversion R, i.e.

$$R(x) \triangleq -\frac{u''(x)}{u'(x)} = \frac{a}{x}, \quad x > 0.$$

Moreover, u is a strictly concave, strictly increasing, continuously differentiable function on $(0, \infty)$.

We will further make

Assumption 1.2. The marketed security ψ is given by the value at maturity $\psi = S_T$ of a geometric Brownian motion $(S_t)_{0 \leq t \leq T}$ which is governed by the stochastic differential equation

$$dS_t = S_t(\mu dt + \sigma dW_t), \tag{1.1}$$

where $\mu \in \mathbb{R}$, $\sigma \in (0, \infty)$, $S_0 \in (0, \infty)$ and $(W_t)_{0 \leq t \leq T}$ is standard Brownian motion adapted to $(\mathcal{F}_t)$.

In the following we will denote the analytic solution of (1.1) with $S_0 = 1$ by

$$\mathcal{E}_t \triangleq e^{(\mu - \frac{\sigma^2}{2})t + \sigma W_t}, \quad t \geq 0. \tag{1.2}$$

The market maker further possesses an initial endowment $\Sigma_0 \in \mathbf{L}^0(\mathbb{R})$ which we will assume to satisfy

Assumption 1.3. The market maker's initial endowment Σ_0 is given as a combination of a position $p \geq 0$ in the traded security ψ and a cash amount $z \geq 0$, i.e.

$$\Sigma_0 = p\psi + z, \quad p, z \in \mathcal{D} \triangleq \mathbb{R}_+^2 \setminus \{(0,0)\}.$$

For the introduction of market indifference prices in our model we require that the market maker's initial level of expected utility is finite, i.e. that the market maker's initial endowment Σ_0 satisfies the integrability condition

$$\mathbb{E}[|u(\Sigma_0)|] < \infty. \tag{1.3}$$

It is for this reason that we demand in Assumption 1.3 that at least one of p and z be *strictly* greater than zero as otherwise the market maker's initial level of expected utility would be $\mathbb{E}[u(\Sigma_0)] = -\infty$. Assumptions 1.1, 1.2 and 1.3 together provide a setting in which (1.3) holds. Note that throughout this entire thesis we will assume that interest rates are zero.

1.2 Single transaction setting

A single transaction is realised by the passing of $q \in \mathbb{R}$ shares as well as a complementing cash amount $x \in \mathbb{R}$ from the large investor to the market maker. The market maker's total endowment changes from $\Sigma_0 = p\psi + z$ to the *post transaction endowment* Σ_1 given by

$$\Sigma_1 = \Sigma_0 + q\psi + x.$$

The cash amount x is the price for the transaction q and is quoted by the market maker in such a way that it allows him to preserve his expected utility, i.e. x satisfies the *utility indifference principle*

$$\mathbb{E}[u(\Sigma_0)] = \mathbb{E}[u(\Sigma_0 + q\psi + x)]. \tag{1.4}$$

For a discussion of the economic reasoning behind this principle of price formation see [11]. We will call x the *market indifference price* for a transaction of q shares. Note that we will view the transactions q and x of securities and cash from the point of view of the market maker; thus positive quantities of these variables denote an addition to the market maker's position, negative values a subtraction therefrom.

Definition 1.4. A transaction size $q \in \mathbb{R}$ is called *admissible* if there exists a cash amount x such that (1.4) is satisfied.

The following theorem provides a necessary and sufficient condition on the number of shares in a trade q for the existence and uniqueness of the market indifference price.

Theorem 1.5. *Let Assumptions 1.1, 1.2 and 1.3 hold. Then a market indifference price $x \in \mathbb{R}$ exists if and only if the transaction size q satisfies*

$$0 \le p + q \le \left(\frac{\mathbb{E}[(p\psi + z)^{1-a}]}{\mathbb{E}[\psi^{1-a}]} \right)^{\frac{1}{1-a}}. \tag{1.5}$$

Furthermore, if a market indifference price x exists, it is unique.

The "only if" statement in this theorem is proved by the following lemma.

Lemma 1.6. *Let Assumptions 1.1, 1.2 and 1.3 hold. If there exists a cash amount $x \in \mathbb{R}$ such that* (1.4) *holds then*

(i) $x + z \ge 0$

(ii) $p + q \geq 0$

(iii) $p + q \leq \left(\frac{\mathbb{E}[(p\psi+z)^{1-a}]}{\mathbb{E}[\psi^{1-a}]}\right)^{\frac{1}{1-a}}$.

Proof. *(ii)* Assume that $(p+q) < 0$. We will show that (1.4) does not possess a real solution. We have

$$\begin{aligned}
\mathbb{E}[u(\Sigma_1)] &= \mathbb{E}[u((p+q)\psi + x + z)] \\
&= \mathbb{E}[u((p+q)\psi + x + z)\mathbb{1}_{\{(p+q)\psi < -(x+z)\}}] \\
&\quad + \mathbb{E}[u((p+q)\psi + x + z)\mathbb{1}_{\{(p+q)\psi \geq -(x+z)\}}].
\end{aligned}$$

In the first term, $(p+q)\psi + x + z < 0$ and thus $u((p+q)\psi + x + z) = -\infty$. Taking into account that ψ is lognormally distributed and since $(p+q) < 0$, for any choice of $x \in \mathbb{R}$ we have $\mathbb{P}((p+q)\psi < -(x+z)) > 0$ and it follows that the first term is $-\infty$ while the second term is finite. By (1.3) we know that $\mathbb{E}[|u(\Sigma_0)|] < \infty$, so (1.4) cannot be satisfied.

(i) Note that if $x + z < 0$ then with positive probability $(p+q)\psi + x + z < 0$ since ψ is lognormally distributed. Using this observation the result follows analogously to the proof of *(ii)*.

(iii) Let $x, q \in \mathbb{R}$ satisfy (1.4). Then

$$\begin{aligned}
\mathbb{E}[u(\Sigma_0)] &= \mathbb{E}[u(\Sigma_0 + x + q\psi)] \\
&= \frac{1}{1-a}\mathbb{E}\big[\big((p+q)\psi + x + z\big)^{1-a}\big] \\
&\geq \frac{1}{1-a}(p+q)^{1-a}\mathbb{E}[\psi^{1-a}]
\end{aligned}$$

where the last inequality holds due to *(i)*. As $\Sigma_0 = p\psi + z$, it follows that

$$\frac{1}{1-a}\frac{\mathbb{E}[(p\psi + z)^{1-a}]}{\mathbb{E}[\psi^{1-a}]} \geq \frac{1}{1-a}(p+q)^{1-a}$$

which implies

$$\left(\frac{\mathbb{E}[(p\psi + z)^{1-a}]}{\mathbb{E}[\psi^{1-a}]}\right)^{\frac{1}{1-a}} \geq p + q.$$

□

The three statements of Lemma 1.6 possess intuitive economic interpretations: Equation *(i)* ensures that the market maker will never spend more cash than he possesses, i.e. that he will never assume a short cash position, while *(ii)* states that the market maker will never assume a short post-transaction position in the security ψ.

Equation (*iii*) constitutes an *upper* bound on the market maker's post-transaction position in ψ: The maximum transaction size that the market maker is able to accomodate is given by

$$\bar{q}(p,z) \triangleq \left(\frac{\mathbb{E}[(p\psi + z)^{1-a}]}{\mathbb{E}[\psi^{1-a}]} \right)^{\frac{1}{1-a}} - p. \tag{1.6}$$

We formalise this observation in the following corollary to Theorem 1.5.

Corollary 1.7. *Under the assumptions of Theorem 1.5 a transaction size* $q \in \mathbb{R}$ *is admissible if and only if* $q \in [-p, \bar{q}(p,z)]$.

Proof. The statement is an immediate consequence of Theorem 1.5. □

The existence of these bounds on admissible trade sizes is a defining characteristic of the model at hand and consitutes one of the main differences to the setting of [11], [14] and [46].

Note that the prices quoted by the market maker for a transaction of size q do not explode to zero or infinity when q approaches the boundary of the set of admissible trade sizes $[-p, \bar{q}(p,z)]$. Rather, $-p$ and $\bar{q}(p,z)$ constitute a minimum and a maximum transaction size, respectively, which both possess a (finite) market indifference price and can therefore be accommodated. For any transaction size $q \notin [-p, \bar{q}(p,z)]$, however, the market maker will not quote an indifference price: The bounds in Theorem 1.5 are sharp in the sense that for $q < -p$ and for $q > \bar{q}(p,z)$ equation (1.4) possesses no solution, i.e. the indifference principle cannot be satisfied. Transactions of these order sizes can therefore not be realised in this model.

In a single period setting these bounds on admissible trade sizes merely mean that the market maker will not accept arbitrarily large orders. Later on, in the case of dynamic trading, this issue will become more relevant: Then, it is possible that an earlier transaction cannot be unwound at a later date, i.e. that the large investor gets "trapped" in a long stock position which he cannot sell back to the market maker as the resulting position would violate the position limits.

The quantity $\bar{q}(p,z)$ possesses a simple economic interpretation: Letting $x(q)$ denote the market indifference price for an order of size q, $\bar{q}(p,z)$ is the (unique) number of shares which satisfies

$$x(\bar{q}(p,z)) = -z,$$

i.e. $\bar{q}(p,z)$ is the number of shares which, when transferred from the large investor to the market maker, must be complemented by the market maker's entire initial cash z. Hence, the existence of an upper bound on admissible transaction sizes is a direct consequence of the market maker's inability to assume a short cash position as a result of his HARA preferences. The highest transaction size that he will accommodate is precisely that which he can afford to pay for without borrowing additional funds.

Remark 1.8. Note that the upper limit $p+\bar{q}(p,z)$ on the market maker's total post-transaction position in the risky asset can also be expressed in terms of the market maker's initial utility level $u_0 \triangleq \mathbb{E}[u(\Sigma_0)]$ as

$$p+\bar{q}(p,z) = \left(\frac{u_0}{\mathbb{E}[u(\psi)]}\right)^{\frac{1}{1-a}}.$$

Let us now prove Theorem 1.5.

Proof of Theorem 1.5. It suffices to show the existence of an indifference price, i.e. of $x \in \mathbb{R}$ which satisfies (1.4); its uniqueness then follows directly from the fact that $u(y)$ is strictly increasing for $y > 0$.

If an indifference price x exists, (1.5) follows directly from Lemma 1.6. To prove the other direction, let us assume that (1.5) holds. Define $h : [-z,\infty) \to \mathbb{R}\cup\{-\infty\}$ by

$$h(x) \triangleq \mathbb{E}[u((p+q)\psi + x + z)] - \mathbb{E}[u(\Sigma_0)].$$

Then $h(x) = 0$ if and only if x solves (1.4). By Assumption 1.1,

$$h(x) = \frac{1}{1-a}\left(\mathbb{E}[((p+q)\psi + x + z)^{1-a}] - \mathbb{E}[\Sigma_0^{1-a}]\right).$$

In the special case where $p+q=0$ we can see that for $0<a<1$

$$h(-z) = -\frac{1}{1-a}\mathbb{E}[\Sigma_0^{1-a}] < 0$$

and for $a>1$

$$h(-z) = \lim_{x\downarrow -z} h(x) = -\infty < 0.$$

If $p+q>0$ then from inequality (1.5) we know that

$$\frac{1}{1-a}(p+q)^{1-a} \leq \frac{1}{1-a}\frac{\mathbb{E}[(p\psi+z)^{1-a}]}{\mathbb{E}[\psi^{1-a}]}$$

which, using the definition of $\Sigma_0 = p\psi + z$, implies

$$\begin{aligned} h(-z) &= \frac{1}{1-a}\left((p+q)^{1-a}\mathbb{E}[\psi^{1-a}] - \mathbb{E}[\Sigma_0^{1-a}]\right) \\ &\leq \frac{1}{1-a}\left(\mathbb{E}[(p\psi+z)^{1-a}] - \mathbb{E}[\Sigma_0^{1-a}]\right) \;=\; 0. \end{aligned}$$

Furthermore,

$$\lim_{x\to\infty} \mathbb{E}[((p+q)\psi + x + z)^{1-a}] = \begin{cases} \infty, & 0<a<1, \\ 0, & a>1, \end{cases}$$

and

$$\mathbb{E}[\Sigma_0^{1-a}] > 0.$$

For x sufficiently large

$$\operatorname{sign}(1-a) = \operatorname{sign}(\mathbb{E}[((p+q)\psi + x + z)^{1-a}] - \mathbb{E}[\Sigma_0^{1-a}])$$

from which it follows that

$$\lim_{x\to\infty} h(x) = \lim_{x\to\infty} \frac{1}{1-a}\left(\mathbb{E}[((p+q)\psi + x + z)^{1-a}] - \mathbb{E}[\Sigma_0^{1-a}]\right) > 0.$$

In the special case where $p + q = 0$, h is continuous on $(-z, \infty)$. Otherwise h is continuous on its entire domain. In either case the intermediate value theorem implies the existence of a zero for h. □

We conclude the section with the following proposition which collects the properties of the market indifference price.

Proposition 1.9. *Let Assumptions 1.1, 1.2 and 1.3 hold and let*

$$x(\cdot) : [-p, \bar{q}(p, z)] \to \mathbb{R}$$

denote the function which maps every admissible trade size q to its market indifference price $x(q)$. Then $x(\cdot)$ is strictly decreasing, strictly convex, positive for $-p \le q < 0$, negative for $0 < q \le \bar{q}(p, z)$ and $x(0) = 0$. Moreover, $x(\cdot)$ is twice continuously differentiable on $(-p, \bar{q}(p, z))$, the first and second order derivatives are given by

$$\partial_q x(q) = -\frac{\mathbb{E}[u'(\Sigma_0 + q\psi + x(q))\psi]}{\mathbb{E}[u'(\Sigma_0 + q\psi + x(q))]} \tag{1.7}$$

and

$$\partial_q^2 x(q) = -\frac{\mathbb{E}[u''(\Sigma_0 + q\psi + x(q))(\psi + \partial_q x(q))^2]}{\mathbb{E}[u'(\Sigma_0 + q\psi + x(q))]} \tag{1.8}$$

and both derivatives can be continuously extended to the boundary at $q = -p$ and $q = \bar{q}(p, z)$.

Proof. By definition $x(q)$ solves (1.4) so that the differentiability assertions about $x(\cdot)$ follow by the implicit function theorem. We differentiate both sides of (1.4) with respect to q to obtain

$$\mathbb{E}[u'(\Sigma_0 + q\psi + x(q))(\psi + \partial_q x(q))] = 0$$

and we differentiate both sides once more to obtain

$$\mathbb{E}[u''(\Sigma_0 + q\psi + x(q))(\psi + \partial_q x(q))^2] + \mathbb{E}[u'(\Sigma_0 + q\psi + x(q))\partial_q^2 x(q)] = 0.$$

Rearranging these equations yields the desired terms for the first- and second order derivatives. With regard to Assumptions 1.1 and 1.2 it follows that, firstly, $\partial_q x(q) < 0$ which implies that $x(\cdot)$ is strictly decreasing and that, secondly, $\partial_q^2 x(q) > 0$ which implies that $x(\cdot)$ is strictly convex. The assertions about the regions where x is positive and negative, respectively, are clear from (1.4) together with the strict monotonicity of u and Assumption 1.2. Lastly, by dominated convergence the identities (1.7) and (1.8) can be continuously extended to $q = -p$ and $q = \bar{q}(p, z)$. □

1.3 Some auxiliary regularity results for conditional expectations

Before we proceed to introduce the trade dynamics in continuous time let us state some preliminary technical results.

The following lemma ensures that we have sufficiently "nice" versions of certain processes which we will consider throughout this chapter.

Lemma 1.10. *Let Assumption 1.2 hold and let $f : (0, \infty) \to \mathbb{R}$ be a continuous function such that $|f|$ is monotonic and that for all $(p, z) \in \mathcal{D} \triangleq \mathbb{R}_+^2 \setminus \{(0,0)\}$ the integrability condition*

$$\mathbb{E}[|f(p\psi + z)|] < \infty$$

is satisfied. Consider the random field $K : \mathcal{D} \times [0, T] \to \mathbf{L}^1(\mathbb{R})$ defined by

$$K_t(p, z) := \mathbb{E}[f(p\psi + z)|\mathcal{F}_t]. \tag{1.9}$$

Then there exists a version of K such that

(i) for fixed $\omega \in \Omega$ and $t \in [0, T]$ the map

$$K_t(\cdot, \cdot)(\omega) : \mathcal{D} \to \mathbb{R}, \quad (p, z) \mapsto K_t(p, z)(\omega)$$

is continuous,

(ii) for fixed $(p, z) \in \mathcal{D}$ the martingale

$$\left(K_t(p, z)\right)_{0 \le t \le T}$$

is continuous.

Proof. (ii) is clear by the martingale representation theorem.

(i) Denote by $\mathring{\mathcal{D}}$ and $\partial\mathcal{D}$ the interior and boundary of $\mathcal{D}$, respectively. On $\mathring{\mathcal{D}}$, (i) is a direct consequence of Lemma C.1 in [13]. It remains to show that the extension of the map $K_t(\cdot, \cdot)(\omega)$ to the set

$$\mathcal{B} \triangleq \{(p, z) \in [0, \infty) \times [0, \infty) | p = 0 \text{ or } z = 0;\ p + z > 0\} = \partial\mathcal{D} \setminus \{(0,0)\}$$

is continuous. Let $(y_n)_{n\geq 1} = (p_n, z_n)_{n\geq 1}$ be a sequence in $\mathring{\mathcal{D}}$ converging to a point $y \in \mathcal{B}$. Then y is given either by $y = (0, z)$ for some $z > 0$ or by $y = (p, 0)$ for some $p > 0$. In either case, there exist $(\hat{p}, \hat{z}), (\tilde{p}, \tilde{z}) \in \mathcal{D}$ and an index $N \geq 1$ such that for all $n \geq N$, $\hat{p} \geq p_n \geq \tilde{p}$ and $\hat{z} \geq z_n \geq \tilde{z}$. If $|f|$ is increasing then

$$|f(\hat{p}\psi + \hat{z})| \geq |f(p_n\psi + z_n)|,$$

if $|f|$ is decreasing then

$$|f(\tilde{p}\psi + \tilde{z})| \geq |f(p_n\psi + z_n)|.$$

Thus, by the dominated convergence theorem, for $y = (0, z)$ we have

$$\lim_{n\to\infty} K_t(p_n, z_n)(\omega) = \lim_{n\to\infty} \mathbb{E}[f(p_n\psi + z_n)|\mathcal{F}_t](\omega) = f(z) = K_t(0, z)(\omega)$$

and for $y = (p, 0)$

$$\lim_{n\to\infty} K_t(p_n, z_n)(\omega) = \mathbb{E}[f(p\psi)|\mathcal{F}_t](\omega) = K_t(p, 0)(\omega),$$

which proves the continuity of $K_t(\cdot, \cdot)(\omega)$ on $\mathcal{D}$.

□

For the remainder of this chapter we will always consider the "nice" versions of Lemma 1.10 for stochastic fields of the form (1.9).

The next lemma will allow us to exchange the order of integration and differentiation when needed. It is a corollary of Lemma C.1 in [13].

Lemma 1.11. *Let N be an open subset of $\mathbb{R}$ and let $\eta : N \to \mathbf{L}^0(\mathbb{R})$ denote a random field such that for almost all $\omega \in \Omega$ the map*

$$\eta(\cdot)(\omega) : N \to \mathbb{R}, \quad x \mapsto \eta(x)(\omega)$$

is continuously differentiable and such that for any compact set $C \subset N$

$$\mathbb{E}\left[\sup_{x\in C} |\eta(x)|\right] + \mathbb{E}\left[\sup_{x\in C} |\eta'(x)|\right] < \infty.$$

Then there exists a version of the stochastic process

$$K_t(x) \triangleq \mathbb{E}[\eta(x)|\mathcal{F}_t], \quad 0 \leq t \leq T,\ x \in N,$$

which has continuous sample paths and which is such that, for any $(t, \omega) \in [0, T] \times \Omega$, the map

$$x \mapsto \mathbb{E}[\eta(x)|\mathcal{F}_t](\omega)$$

is continuously differentiable with derivative

$$\mathbb{E}[\eta'(x)|\mathcal{F}_t](\omega).$$

Proof. The statement follows directly from Lemma C.1 in [13]. □

The last technical lemma of this section ensures that we can differentiate $\mathcal{F}_t$-conditional expectations with respect to the value at time t of the process $(S_t)_{0\le t\le T}$ from Assumption 1.2.

Lemma 1.12. *Under Assumption 1.2 let $f:(0,\infty)\to\mathbb{R}$ be differentiable almost everywhere (resp. twice differentiable almost everywhere) and such that it satisfies the integrability condition*

$$\mathbb{E}[|f(\psi)|]+\mathbb{E}[|f'(\psi)\psi|]<\infty$$

(resp.

$$\mathbb{E}[|f(\psi)|]+\mathbb{E}[|f'(\psi)\psi|]+\mathbb{E}[|f''(\psi)\psi^2|]<\infty).$$

Let further

$$h(t,s)\triangleq\mathbb{E}_{t,s}[f(\psi)]=\mathbb{E}[f(\psi)|S_t=s]=\mathbb{E}[f(s\mathcal{E}_{T-t})], \tag{1.10}$$

so that $h(t,S_t)=\mathbb{E}[f(\psi)|\mathcal{F}_t]$. Then $h\in\mathcal{C}^{1,1}([0,T)\times(0,\infty),\mathbb{R})$ (resp. $h\in\mathcal{C}^{1,2}([0,T)\times(0,\infty),\mathbb{R})$),

$$\partial_s h(t,s)=\mathbb{E}[f'(s\mathcal{E}_{T-t})\mathcal{E}_{T-t}],$$

$$\left(\partial_s^2 h(t,s)=\mathbb{E}[f''(s\mathcal{E}_{T-t})\mathcal{E}_{T-t}^2]\right),$$

and

$$\partial_t h(t,s)=\mathbb{E}[f'(s\mathcal{E}_{T-t})\mathcal{E}_{T-t}Z_{T-t}],$$

where

$$Z_{T-t}=-\frac{\log(\mathcal{E}_{T-t})}{T-t}.$$

Proof. By Assumption 1.2,

$$h(t,s)=\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}f\left(se^{\sqrt{T-t}\sigma r+(T-t)(\mu-\frac{\sigma^2}{2})}\right)e^{\frac{-r^2}{2}}\mathrm{d}r.$$

By an application of Lebesgue's theorem, straightforward computations imply the differentiability of h and yield the desired forms for its derivatives. □

Note that under the assumptions of Lemma 1.12 one can even obtain the much stronger result that $h\in\mathcal{C}^{1,\infty}$. However, the reduced statement above suffices for our purposes.

The integrability condition in Lemma 1.12 is clearly satisfied if f is such that $\mathbb{E}[|f(\psi)|]<\infty$ and f has bounded derivative. Throughout this work, we will often use the notation

$$\partial_s\mathbb{E}_{t,s}[f(\psi)]|_{s=S_t}\triangleq\partial_s h(t,s)|_{s=S_t},$$

where h is given by (1.10). Note that, as a consequence of Lemma 1.12, for functions f satisfying the conditions therein, we have

$$\partial_s \mathbb{E}_{t,s}[f(\psi)]|_{s=S_t} = \frac{1}{S_t}\mathbb{E}[f'(\psi)\psi|\mathcal{F}_t]. \tag{1.11}$$

1.4 Simple strategies

We will now extend the model to a continuous-time setting which accommodates successive trading between the market maker and the large investor. The single-period case discussed above can then be viewed as the special case where the market maker and the large investor enter into a single trade at time zero and make no further transactions until maturity $T = 1$.

Trading between the market maker and the large investor is modelled by a predictable process $(Q_t)_{0\leq t\leq T}$, the *trading strategy* of the large investor, and a predictable process $(X_t)_{0\leq t\leq T}$, the *cash balance process*, which complements the trading. Q_t and X_t stand for the cumulative amount of stocks and cash, respectively, that have been transferred from the large investor to the market maker up to time t. The initial endowment of the market maker, as in the single-period case, is denoted by Σ_0.

We will first consider simple trading strategies where trading occurs only finitely often. We formalise this concept by giving

Definition 1.13. A *simple trading strategy* is a process $(Q_t)_{0\leq t\leq T}$ with

$$Q_t = \sum_{k=1}^{n} \theta_k \mathbb{1}_{(\tau_{k-1},\tau_k]}(t),\ 0 \leq t \leq T, \tag{1.12}$$

where $0 = \tau_0 \leq \cdots \leq \tau_n = T$ are stopping times and $\theta_k \in \mathbf{L}^0(\mathcal{F}_{\tau_{k-1}}, \mathbb{R})$.

It is reasonable to assume that, if it exists at all, the cash balance process $(X_t)_{0\leq t\leq T}$ which complements Q in terms of expected utility is of the same form and can be written as

$$X_t = \sum_{k=1}^{n} \xi_k \mathbb{1}_{(\tau_{k-1},\tau_k]}(t),\ 0 \leq t \leq T, \tag{1.13}$$

with $\xi_k \in \mathbf{L}^0(\mathcal{F}_{\tau_{k-1}}, \mathbb{R})$. Before any trading takes place, the market maker possesses the initial endowment Σ_0 and at any time after that, if X exists, his endowment is given by

$$\Sigma_t \triangleq \Sigma_0 + Q_t\psi + X_t.$$

For $k \in \{1,\ldots,n\}$, let

$$\Sigma_k \triangleq \Sigma_0 + \theta_k\psi + \xi_k,$$

then $\Sigma_k = \Sigma_{\tau_k}$. The principle that underlies the trading between the market maker and the large investor and by which we define the admissibility of trading strategies is yet again the preservation of expected utility. In the case of simple strategies, we can state the *utility indifference principle* as follows: For all $k \in \{1, \ldots, n\}$

$$\mathbb{E}[u(\Sigma_k)|\mathcal{F}_{\tau_{k-1}}] = \mathbb{E}[u(\Sigma_{k-1})|\mathcal{F}_{\tau_{k-1}}]. \tag{1.14}$$

Definition 1.14. Let Q be a simple trading strategy. We call Q *admissible* if there exists a complementing cash balance process X of the form (1.13) such that (1.14) possesses a solution ξ_k for each $k \in \{1, \ldots, n\}$.

Note that so far we are demanding that X be of the form (1.13). We will later show that, whenever it exists, the complementing cash balance process of a simple strategy is unique and that it must indeed be of that form. In order to make this statement and its demonstration rigorous we will have to introduce some additional notation which we shall do in the next section.

Let us now give a more tractable characterisation of admissible simple strategies. We state and prove the following preparatory lemma which can be viewed as a conditional version of Theorem 1.5.

Lemma 1.15. *Let Assumptions 1.1, 1.2 and 1.3 hold. Let $\mathcal{G}$ denote a sub-sigma-algebra of $\mathcal{F}_T$ and let $\theta, \pi, \zeta \in \mathbf{L}^0(\mathcal{G}, \mathbb{R})$ such that $\pi \geq 0$, $\zeta \geq 0$ and $\pi + \zeta > 0$. Then the following statements are equivalent:*

(i) There exists a unique random variable $\xi \in \mathbf{L}^0(\mathcal{G}, \mathbb{R})$ such that

$$\mathbb{E}[u((\pi + \theta)\psi + \zeta + \xi)|\mathcal{G}] = \mathbb{E}[u(\pi\psi + \zeta)|\mathcal{G}].$$

(ii) We have

$$-\pi \leq \theta \leq \bar{q}_{\mathcal{G}}(\pi, \zeta),$$

where

$$\bar{q}_{\mathcal{G}}(\pi, \zeta) \triangleq \left(\frac{\mathbb{E}[(\pi\psi + \zeta)^{1-a}|\mathcal{G}]}{\mathbb{E}[\psi^{1-a}|\mathcal{G}]} \right)^{\frac{1}{1-a}} - \pi.$$

Proof. The result follows by a conditional version of the argument given in the proof of Theorem 1.5. □

The following proposition is the extension of Corollary 1.7 to a multi-period setting.

Proposition 1.16. *Let Assumptions 1.1, 1.2 and 1.3 hold. Let Q be a simple trading strategy and for $k \in \{1, \ldots, n\}$ let $\Delta\theta_k \triangleq \theta_k - \theta_{k-1}$. Then Q is admissible if and only if almost surely*

$$-(p + \theta_{k-1}) \leq \Delta\theta_k \leq \bar{q}_{\mathcal{F}_{\tau_{k-1}}}(p + \theta_{k-1}, z + \xi_{k-1})$$

for all $k \in \{1, \ldots, n\}$.

Proof. Let $k \in \{1, \ldots, n\}$ and let further $\Delta\xi_k \triangleq \xi_k - \xi_{k-1}$. If Q is admissible then, by (1.14),

$$\mathbb{E}[u(\Sigma_{k-1} + \Delta\theta_k + \Delta\xi_k)|\mathcal{F}_{\tau_{k-1}}] = \mathbb{E}[u(\Sigma_{k-1})|\mathcal{F}_{\tau_{k-1}}].$$

By Lemma 1.15 this is equivalent to

$$-(p + \theta_{k-1}) \leq \Delta\theta_k \leq \bar{q}_{\mathcal{F}_{\tau_{k-1}}}(p + \theta_{k-1}, z + \xi_{k-1}).$$

□

Remark 1.17. We can alternatively express the statement of Proposition 1.16 in terms of the upper bound on the total stock position of the market maker. If, in the setting of Lemma 1.15, we define

$$\tilde{q}_{\mathcal{G}}(\pi, \zeta) \triangleq \left(\frac{\mathbb{E}[(\pi\psi + \zeta)^{1-a}|\mathcal{G}]}{\mathbb{E}[\psi^{1-a}|\mathcal{G}]}\right)^{\frac{1}{1-a}} = \bar{q}_{\mathcal{G}}(\pi, \zeta) + \pi$$

then the condition in Proposition 1.16 becomes

$$0 \leq \theta_k + p \leq \tilde{q}_{\mathcal{F}_{\tau_{k-1}}}(p + \theta_{k-1}, z + \xi_{k-1}).$$

We can see that even in the case of simple strategies the set of admissible trading strategies is given in a very implicit manner. In particular, it is not true that all simple trading strategies are admissible and it is not easy to check whether a given strategy is admissible or not.

To illustrate this, consider the following example. Let the market maker's initial endowment be given by $\Sigma_0 = p\psi + z$ and consider the simple trading strategy Q defined, at any time $t \in [0, T]$, by

$$Q_t \triangleq -p \cdot \mathbb{1}_{(0, \frac{T}{2}]}(t) + 0 \cdot \mathbb{1}_{(\frac{T}{2}, T]}(t),$$

i.e. in the notation of (1.12) $\theta_0 = -p$, $\theta_1 = 0$, $\tau_0 = 0$ and $\tau_1 = T/2$. Q is the strategy where the large investor purchases all the stocks from the market maker at time zero and seeks to sell them back to him at time $T/2$. Let $x(-p)$ denote the indifference price for the initial transaction of $-p$ shares. Then, for $t \in (0, \frac{T}{2}]$,

$$\tilde{q}_{\mathcal{F}_t}(p + \theta_t, z + \xi_t) = \tilde{q}_{\mathcal{F}_t}(0, z + x(-p)) = \frac{z + x(-p)}{\mathbb{E}[\psi^{1-a}|\mathcal{F}_t]^{\frac{1}{1-a}}}.$$

By Assumption 1.2, with positive probability we have

$$\frac{z + x(-p)}{\mathbb{E}[\psi^{1-a}|\mathcal{F}_{\frac{T}{2}}]^{\frac{1}{1-a}}} < p$$

in which case $\theta_1 + p = p > \tilde{q}_{\mathcal{F}_{\frac{T}{2}}}(0, z + x(-p))$. This means that the large investor cannot sell back his entire position p to the market maker and, in view of Remark 1.17, it implies that Q is not admissible.

This example can be generalised to show that any simple strategy with deterministic coefficients θ_k which is not decreasing is *not admissible.*

1.5 Continuous-time strategies

We will from now on assume that the trading strategy $(Q_t)_{0\leq t\leq T}$ is a (general) predictable process. The market maker's endowment at time $t = 0$ is yet again denoted by Σ_0 and at any time t thereafter by

$$\Sigma_t \triangleq \Sigma_0 + Q_t\psi + X_t.$$

It turns out that the indifference principle in continuous time is best stated by keeping track of the market maker's level of expected utility rather than of the complementing cash process X. Before we introduce the process that serves this very purpose, we define the set

$$\mathcal{D}_{z,p} \triangleq [-z,\infty) \times [-p,\infty) \setminus \{(-z,-p)\}$$

which is a shifted version of the equally named set in Lemma 1.10. The set $\mathcal{D}$ will ensure in future definitions that the market maker's utility level does not reach minus infinity.

In order to describe the time-evolution of the level of expected utility of the market maker we introduce the *static process of indirect utility* $F : \mathcal{D} \times [0,T] \to \mathbf{L}^0(\mathbb{R})$ defined by

$$F(x,q,t) \triangleq \mathbb{E}[u(\Sigma_0 + q\psi + x)|\mathcal{F}_t] = \mathbb{E}[u((p+q)\psi + x + z)|\mathcal{F}_t], \tag{1.15}$$

where the second equality holds if we impose Assumption 1.3, in which case z denotes the market maker's initial cash and p his initial security position. In the following, we will always consider a version of F which is nice in the sense of Lemma 1.10. Note that, in view of Lemma 1.10 (ii), the process $(F(x,q,t))_{0\leq t\leq T}$ is a continuous martingale. Furthermore, F possesses dynamics which are given in the following lemma.

Lemma 1.18. *Under Assumptions 1.1, 1.2 and 1.3 let F be defined as in* (1.15). *Then F admits the representation*

$$F(x,q,t) = F(x,q,0) + \int_0^t \tilde{g}(x,q,s)dW_s, \tag{1.16}$$

where the stochastic field $\tilde{g} : \mathcal{D} \times [0,T] \to \mathbf{L}^0(\mathbb{R})$ is given by

$$\tilde{g}(x,q,t) \triangleq \sigma S_t \partial_s h_{x,q}(t,S_t) = \sigma S_t \partial_s \mathbb{E}_{t,s}[u(\Sigma_0 + q\psi + x)]|_{s=S_t}$$

and the smooth function $h_{x,q} : [0,T] \times \mathbb{R}_+ \to \mathbb{R}$ *is given by*

$$h_{x,q}(t,s) \triangleq \mathbb{E}[u((p+q)s\mathcal{E}_{T-t} + x + z)].$$

Proof. By the Markov property of S,

$$F(x,q,t) = \mathbb{E}_{t,s}[u((p+q)S_{T-t} + x + z)]|_{s=S_t} = h_{x,q}(t,S_t).$$

By Lemma 1.12 we know that $h_{x,q}$ is continuously differentiable in t and twice continuously differentiable in s. An application of Itô's formula to $h_{x,q}$ yields

$$\begin{aligned} dF(x,q,t) &= \big(\partial_t h_{x,q}(t,S_t) + \mu S_t \partial_s h_{x,q}(t,S_t) + \tfrac{1}{2}\sigma^2 S_t^2 \partial_s^2 h_{x,q}(t,S_t)\big)\,dt \\ &\quad + \sigma S_t \partial_s h_{x,q}(t,S_t) dW_t. \end{aligned}$$

Since F is a martingale, the drift term vanishes and we obtain

$$dF(x,q,t) = \tilde{g}(x,q,t)dW_t.$$

□

The process $(F(x,q,t))_{0\le t\le T}$ models the dynamics of the market maker's indirect utility level after a transaction (x,q) and it will serve as the main ingredient when stating the indifference principle in continuous time. Before we come to that, let us introduce some additional notation. We define the *range of utilities* $\tilde{\mathcal{U}} \subseteq \mathbb{R}$ by

$$\tilde{\mathcal{U}} \triangleq u\big((0,\infty)\big) = \{\mathbf{u} \in \mathbb{R} \mid \exists r \in (0,\infty) \text{ s.t. } u(r) = \mathbf{u}\} = \begin{cases} (0,\infty), & 0 < a < 1, \\ (-\infty,0), & a > 1, \end{cases}$$

and the (random) *set of compatible pairs of utility levels and transaction sizes* by

$$\mathcal{A}(t,\omega) \triangleq \{(\mathbf{u},q) \mid \mathbf{u} \in \tilde{\mathcal{U}}, -p \le q \le \bar{q}(\mathbf{u},t,\omega)\}, \tag{1.17}$$

where the random field $\bar{q} : \tilde{\mathcal{U}} \times [0,T] \to \mathbf{L}^0((-p,\infty))$ is given by

$$\bar{q}(\mathbf{u},t) \triangleq \left(\frac{\mathbf{u}}{\mathbb{E}[u(\psi)|\mathcal{F}_t]}\right)^{\frac{1}{1-a}} - p. \tag{1.18}$$

The set $\mathcal{A}(t,\omega)$ is a crucial building block in our setup. It constitutes the unique set of pairs of utility levels and transaction sizes for which our version of market indifference pricing is technically possible. We will briefly discuss some properties and different characterisations of $\mathcal{A}(t,\omega)$ below.

Note that the expression (1.18) for the maximum transaction size $\bar{q}(\mathbf{u},t)$ is consistent with the ones in previous sections in the sense that for $\mathbf{u} = \mathbb{E}[u(\Sigma_0)]$

$$\bar{q}(\mathbf{u}, 0) = \bar{q}(p, z)$$

where $\bar{q}(p, z)$ is as in (1.6) and that for $\mathbf{u} = \mathbb{E}[u(\Sigma_0)|\mathcal{F}_t]$,

$$\bar{q}(\mathbf{u}, t) = \bar{q}_{\mathcal{F}_t}(p, z),$$

where $\bar{q}_{\mathcal{F}_t}(p, z)$ is as in Lemma 1.15 *(ii)*. In the above definition of $\mathcal{A}(t, \omega)$, fixing a utility level $\mathbf{u} \in \tilde{\mathcal{U}}$ implies a range of transaction sizes q for which $(\mathbf{u}, q) \in \mathcal{A}(t, \omega)$, i.e. for which $(\mathbf{u}, q)$ is a compatible transaction pair. We can define $\mathcal{A}(t, \omega)$ equivalently by

$$\mathcal{A}(t, \omega) \triangleq \{(\mathbf{u}, q) \in \tilde{\mathcal{U}} \times \mathbb{R} \mid q \geq -p, \mathbf{u} > u_{\min}(q, t, \omega)\}, \tag{1.19}$$

where the random field $u_{\min} : (-p, \infty) \times [0, T] \to \mathbf{L}^0(\tilde{\mathcal{U}})$ is given by

$$u_{\min}(q, t) \triangleq \mathbb{E}[u((p + q)\psi)|\mathcal{F}_t], \tag{1.20}$$

and $u_{\min}(-p, t) \triangleq 0$ if $0 < a < 1$; $u_{\min}(-p, t) \triangleq -\infty$ if $a > 1$. (The equivalence of (1.17) and (1.19) is a consequence of Lemma 1.19 below.) In this case, a fixed transaction size $q \geq -p$ implies a range of utilities $\mathbf{u}$ for which $(\mathbf{u}, q)$ is a compatible transaction pair.

The set $\mathcal{A}(t, \omega)$ is depicted in Figures 1.1 and 1.2 for $a = 2$ and $a = \frac{1}{2}$, respectively. We further define the (random) *set of compatible utility values* of the market maker, given a trade size q, by

$$\mathcal{U}(q, t, \omega) \triangleq \{\mathbf{u} \in \mathcal{U} \mid (\mathbf{u}, q) \in \mathcal{A}(t, \omega)\}$$

and the (random) *set of compatible transaction sizes*, given a level of indirect utility of the market maker $\mathbf{u}$, by

$$\mathcal{Q}(\mathbf{u}, t, \omega) \triangleq \{q \in \mathbb{R} \mid (\mathbf{u}, q) \in \mathcal{A}(t, \omega)\}.$$

Both sets can be expressed as intervals as follows:

$$\mathcal{U}(q, t, \omega) = \begin{cases} (u_{\min}(q, t, \omega), \infty) & 0 < a < 1, \\ (u_{\min}(q, t, \omega), 0) & a > 1 \end{cases}$$

and

$$\mathcal{Q}(\mathbf{u}, t, \omega) = [-p, \bar{q}(\mathbf{u}, t, \omega)]. \tag{1.21}$$

Let us now introduce the family of maps $(A_t)_{t \in [0,T]}$ which will serve as the main tool in formulating the indifference principle that determines the trade dynamics in

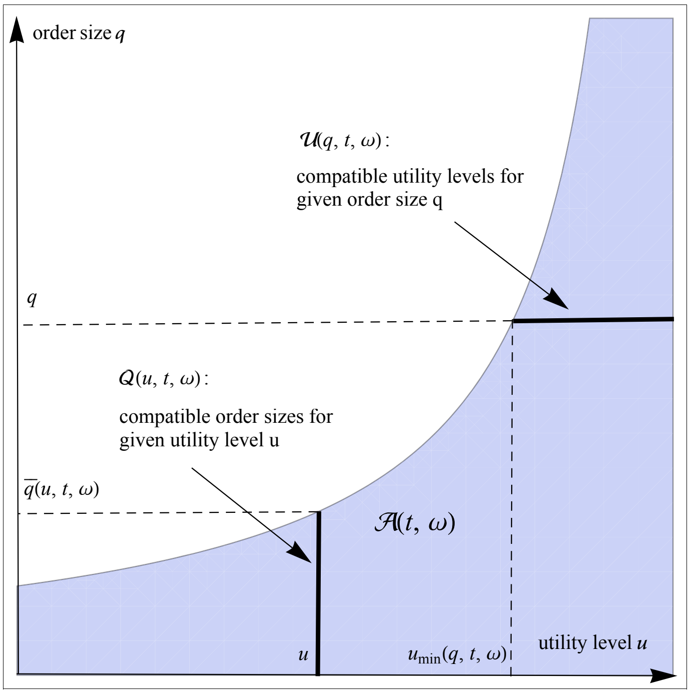

Figure 1.1: The set $\mathcal{A}(t, \omega)$ (blue) for $a > 1$

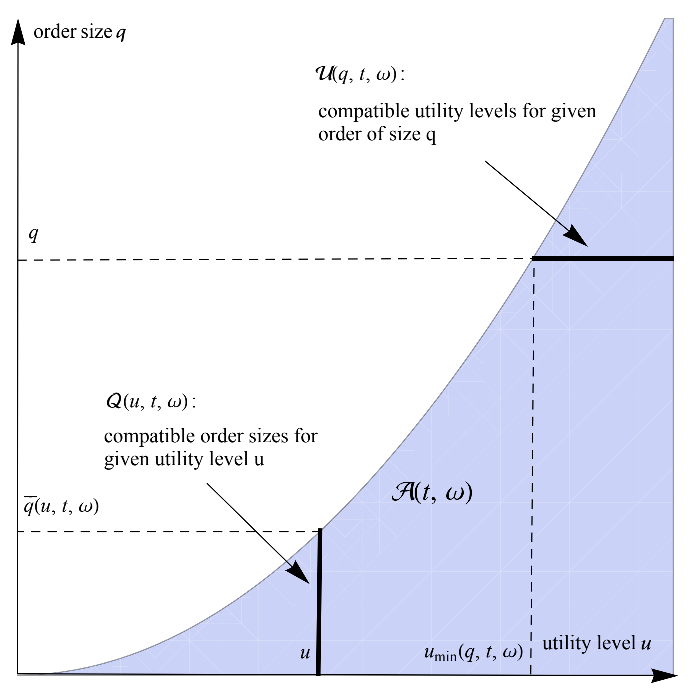

Figure 1.2: The set $\mathcal{A}(t, \omega)$ (blue) for $0 < a < 1$

continuous time. For each $t \in [0,T]$, $\omega \in \Omega$, we define the (random) map

$$A_t^\omega : \mathcal{A}(t,\omega) \to \mathbb{R}$$

by $A_t^\omega : (\mathbf{u}, q) \mapsto x$, where x solves

$$\mathbf{u} = F(x,q,t)(\omega). \tag{1.22}$$

Hence, for every pair $(\mathbf{u}, q) \in \mathcal{A}(t,\omega)$, the value of $A_t^\omega(\mathbf{u}, q)$ is given by the solution x to

$$\mathbf{u} = \mathbb{E}[u\big((p+q)\psi + z + x\big)|\mathcal{F}_t](\omega). \tag{1.23}$$

For notational convenience we will omit the superscript ω and merely write A_t. By Lemma 1.15 we know that (1.23) (and therefore (1.22)) possesses a unique solution x on the given domain.

Let us have a closer look at the family of maps (A_t). Fix $\omega \in \Omega$. At every time t, for a fixed transaction size q, there is a 1-to-1 correspondence between utility levels and cash amounts: Each utility level $\mathbf{u}$ corresponds to a unique cash amount x which complements the transaction of size q by preserving the level of expected utility. The map A_t reconstructs this very cash amount x from the utility level $\mathbf{u}$. Analogously, for a fixed utility level $\mathbf{u}$, there is a 1-to-1 correspondence between transaction sizes q and their complementing cash amounts x.

The following Lemma sheds light on the natural emergence of the quantities $u_{\min}$ and $\bar{q}$. While the former corresponds to the minimal utility level that the market maker must possess in order to be able to accomodate a transaction of size q, the latter corresponds to the maximum transaction size that the market maker can accomodate, given that he is currently at utility level $\mathbf{u}$. The two constraints (on possible utility levels and possible transaction sizes, respectively) are dual to each other in the sense that they both arise from the fact that the market maker will never assume a short position in cash.

Lemma 1.19. *Let Assumptions 1.1, 1.2 and 1.3 hold and let $\bar{q}(\mathbf{u}, t)$ and $u_{min}(q,t)$ be defined as in* (1.18) *and* (1.20)*, respectively. Then for $(\mathbf{u}, q) \in \mathcal{A}(t,\omega)$*

$$A_t(\mathbf{u}, q) = -z \;\; \textit{iff} \;\; \mathbf{u} = u_{min}(q,t) \;\; \textit{iff} \;\; q = \bar{q}(\mathbf{u}, t). \tag{1.24}$$

Proof. By definition of A_t we have $A_t(\mathbf{u}, q) = x$ if and only if

$$\mathbf{u} = \mathbb{E}[u((p+q)\psi + x + z)|\mathcal{F}_t]. \tag{1.25}$$

Substituting $\mathbf{u} = u_{\min}(q,t)$ into (1.25) implies $x = -z$. Conversely, for $x = -z$ in (1.25) we obtain $\mathbf{u} = u_{\min}(q,t)$. This proves the first equivalence. Considering $x = -z$ in (1.25) and rearranging it for q yields $\bar{q}(\mathbf{u}, t)$ which proves the second equivalence. $\square$

Both statements of Lemma 1.19 possess intuitive economic interpretations. The second equivalence means that $\mathbf{u}$ is the minimal utility level at which a transaction of size q can be accomodated if and only if q is the maximal transaction that can be accomodated when holding utility level $\mathbf{u}$. The first equivalence means that in this case the market maker has to assume a zero cash position to accomodate such a transaction. In this sense $u_{\min}(q,t)$ and $\bar{q}(\mathbf{u},t)$ describe the same phenomenon – namely the situation in which the market maker enters into a transaction which causes him to spend his entire cash – from different points of view. We define the level-set

$$\mathcal{A}^{-z}(t,\omega) \triangleq \{(\mathbf{u},q) \in \mathcal{A}(t,\omega) \mid A_t(\mathbf{u},q) = -z\},$$

i.e. $\mathcal{A}^{-z}(t,\omega)$ contains all compatible pairs $(\mathbf{u},q)$ of utility levels and transaction sizes which would cause the market maker to assume a zero cash position. By Lemma 1.19 it is clear that the sets $\{(u_{\min}(q,t,\omega),q)|q \in \mathbb{R}\}$ and $\{(\mathbf{u},\bar{q}(\mathbf{u},t,\omega))|\mathbf{u} \in \mathcal{U}\}$ are both identical to $\mathcal{A}^{-z}(t,\omega)$ and that they describe the "upper" boundary of the set of compatible transaction pairs $\mathcal{A}(t,\omega)$ in Figures 1.1 and 1.2.

We are now ready to describe the trade dynamics in the continuous-time setting by formulating the principle of preservation of expected utility for continuous-time strategies. This will be done in the form of an SDE for the market maker's utility process, following the approach of [14] and [12], where the analog SDE for utility functions with bounded absolute risk aversion is introduced and analysed.

We define the market maker's *dynamic process of indirect utility* $(U_t^Q)_{0\le t\le T}$ when the large investor is pursuing a trading strategy Q as the solution to the stochastic differential equation

$$U_t^Q = \mathbb{E}[u(\Sigma_0)] + \int_0^t F(A_s(U_s^Q,Q_s),Q_s;ds), \tag{1.26}$$

provided the solution exists and is uniquely determined. Of course this also means that the pair (U_t,Q_t) takes values in $\mathcal{A}_t$ for any $t \in [0,T]$ such that the right hand side of (1.26) makes sense. The nonlinear stochastic integral is to be understood in the sense of [39] Section 3.2. In differential form this equation reads

$$dU_t^Q = F(A_t(U_t^Q,Q_t),Q_t;dt) = g(U_t^Q,Q_t,t)dW_t, \tag{1.27}$$

where

$$g(\mathbf{u},q,t) \triangleq \tilde{g}(A_t(\mathbf{u},q),q,t) \tag{1.28}$$

and $\tilde{g}$ is defined as in Lemma 1.18. The nonlinear stochastic differential equation (1.26) formalises the principle of preservation of expected utility and can therefore be viewed as the *utility indifference principle* in continuous time.

The dynamic process of indirect utility is linked to the static process of indirect utility by the relationship

$$U_t^Q = F(X_t, Q_t, t),$$

where, at any time $t \in [0, T]$, the cash balance process X is given by

$$X_t = A_t(U_t^Q, Q_t).$$

In the following, we will refer to the market maker's *dynamic process of indirect utility* simply as his *process of indirect utility*.

Having established the utility indifference principle in continuous time, we are now ready to define admissibility for general predictable strategies.

Definition 1.20. We will call a predictable process $(Q_t)_{0 \le t \le T}$ an *admissible trading strategy* if it is such that (1.26) possesses a unique strong solution $(U_t^Q)_{0 \le t \le T}$. We will further call the (predictable) process $(X_t)_{0 \le t \le T}$ given by

$$X_t = X_t^Q \triangleq A_t(U_t^Q, Q_t) \tag{1.29}$$

for all $t \in [0, T]$ the *complementing cash-balance process* of the admissible trading strategy Q.

If Q is an admissible strategy then in particular this means that all terms in (1.26) are defined. In other words, for any admissible strategy Q, at any time $t \in [0, T]$, $(U_t^Q(\omega), Q_t(\omega))$ must be an allowed transaction pair, i.e.

$$(U_t^Q(\omega), Q_t(\omega)) \in \mathcal{A}(t, \omega)$$

for all $t \in [0, T]$. Equivalently we can say that at any time $t \in [0, T]$, the cumulative security position Q_t satisfies

$$Q_t(\omega) \in \mathcal{Q}(U_t^Q(\omega), t, \omega)$$

or, alternatively, at any time $t \in [0, T]$, the indirect utility level U_t^Q satisfies

$$U_t^Q(\omega) \in \mathcal{U}(Q_t(\omega), t, \omega).$$

Thus, an admissible trading strategy and the indirect utility process it induces are closely intertwined and cannot be investigated independently from one another.

These dynamic domain restrictions, of course, arise from the very same phenomenon that gave rise to the trade size bounds in the cases of a single transaction and of simple trading strategies: As the market maker will never assume a short position in the traded security, admissible strategies are bounded from below by $-p$; as he will never assume a short position in cash, admissible strategies are bounded from above by the dynamic upper trade size bound $\bar{q}(\mathbf{u}, t)$ introduced in (1.18).

These upper and lower bounds for admissible trading strategies constitute a crucial difference to the settings of [11], [14] and [46] and create significant technical obstacles. As a result of these dynamic domain restrictions, the question of existence of solutions to (1.26) is very delicate. In particular, it is *not true* that all locally bounded strategies are admissible. It is therefore not straightforward to determine whether a trading strategy Q is admissible or not and a tractable criterion which constitutes a sufficient condition for admissibility would be desirable. Fortunately, the question of uniqueness of solutions to (1.26) can be answered even in the absence of such a criterion as the following theorem shows.

Theorem 1.21. *Let Assumptions 1.1, 1.2 and 1.3 hold and let Q be a predictable process. Then solutions U^Q to* (1.26) *are unique up to indistinguishability, if they exist at all.*

Theorem 1.21 shows that we would not have needed to demand uniqueness in Definition 1.20 as any solution to (1.26) is unique. Before we proceed to prove the theorem, we will establish several lemmas.

Remark 1.22. For a simple strategy Q which is admissible in the sense of Definition 1.14 the SDE (1.26) possesses a strong solution U^Q. Thus, any simple admissible strategy is also admissible in the sense of Definition 1.20. Furthermore, Theorem 1.21 implies that the complementing cash balance process X of a simple strategy is unique (up to indistinguishability) and must therefore indeed possess the form (1.13) that we had previously imposed.

Let us now state and prove several lemmas concerning the important maps A_t and g.

Lemma 1.23. *Let Assumptions 1.1, 1.2 and 1.3 hold. Then for any $\omega \in \Omega$, $q \geq -p$ and $t \in [0, T]$ the map $\mathbf{u} \mapsto A_t^\omega(\mathbf{u}, q)$ of* (1.22) *is twice continuously differentiable, increasing and convex on its domain. Furthermore, we have*

$$\partial_{\mathbf{u}} A_t(\mathbf{u}, q) = \frac{1}{\mathbb{E}[u'\big(\Sigma_0 + q\psi + A_t(\mathbf{u}, q)\big)|\mathcal{F}_t]} \tag{1.30}$$

and

$$\partial^2_{\mathbf{u}} A_t(\mathbf{u}, q) = -\frac{\mathbb{E}[u''\big(\Sigma_0 + q\psi + A_t(\mathbf{u}, q)\big)|\mathcal{F}_t]}{\mathbb{E}[u'\big(\Sigma_0 + q\psi + A_t(\mathbf{u}, q)\big)|\mathcal{F}_t]^3}. \tag{1.31}$$

Proof. By definition of $A_t(\mathbf{u}, q)$,

$$\mathbf{u} = \mathbb{E}[u\big(\Sigma_0 + q\psi + A_t(\mathbf{u}, q)\big)|\mathcal{F}_t].$$

By Lemma 1.11 and the implicit function theorem $A_t(\cdot, q)$ is twice continuously differentiable. After differentiating both sides once and twice, respectively, with respect to $\mathbf{u}$ and rearranging the terms we obtain (1.30) and (1.31). Since $u' > 0$ it follows that $\partial_{\mathbf{u}} A_t(\mathbf{u}, q) > 0$, thus $A_t(\cdot, q)$ is increasing. Since $u'' < 0$ it follows that $\partial^2_{\mathbf{u}} A_t(\mathbf{u}, q) > 0$ which implies the convexity of $A_t(\cdot, q)$. □

Lemma 1.24. *Let Assumptions 1.1, 1.2 and 1.3 hold and let g be as in* (1.28)*. Then for any $\omega \in \Omega$, $q \geq -p$ and $t \in [0, T]$ the map*

$$g(\cdot, q, t, \omega) : \mathcal{U}(q, t, \omega) \to \mathbb{R}, \quad \mathbf{u} \mapsto g(\mathbf{u}, q, t, \omega)$$

is twice continuously differentiable, positive and strictly decreasing. Furthermore, we have

$$\partial_{\mathbf{u}} g(\mathbf{u}, q, t) = \sigma \mathbb{E}[u''\big(\Sigma_0 + q\psi + x\big)(p+q)\psi x' | \mathcal{F}_t]|_{x = A_t(\mathbf{u}, q), x' = \partial_{\mathbf{u}} A_t(\mathbf{u}, q)} \tag{1.32}$$

and

$$\begin{aligned} & \partial^2_{\mathbf{u}} g(\mathbf{u}, q, t) \\ = \ & \sigma \mathbb{E}[u'''\big(\Sigma_0 + q\psi + x\big)(p+q)\psi x'^2 + u''\big(\Sigma_0 + q\psi + x\big)(p+q)\psi x'' | \mathcal{F}_t] \\ & \quad |_{x = A_t(\mathbf{u}, q), x' = \partial_{\mathbf{u}} A_t(\mathbf{u}, q), x'' = \partial_{\mathbf{uu}} A_t(\mathbf{u}, q)}. \end{aligned}$$

Proof. By definition of g and by Lemma 1.12 we have

$$\begin{aligned} g(\mathbf{u}, q, t) &= \sigma S_t \partial_s \mathbb{E}_{t,s}[u\big(\Sigma_0 + q\psi + x\big)]|_{s = S_t, x = A_t(\mathbf{u}, q)} \\ &= \sigma \mathbb{E}_{t,s}[u'\big(\Sigma_0 + q\psi + x\big)(p+q)\psi]|_{s = S_t, x = A_t(\mathbf{u}, q)}. \end{aligned}$$

Since $u' > 0$ and $(p+q)\psi > 0$ it follows that g is positive. Differentiating this expression once and twice, respectively, with respect to $\mathbf{u}$ yields the desired terms for the first and second order derivatives. Since $u'' < 0$ and, by Lemma 1.23, $\partial_{\mathbf{u}} A_t(\mathbf{u}, q) > 0$ it follows that

$$\partial_{\mathbf{u}} g(\cdot, q, t) < 0$$

which implies that $g(\cdot, q, t)$ is strictly decreasing.

□

Lemma 1.25. *Let Assumptions 1.1, 1.2 and 1.3 hold and let g be as in* (1.28)*. Then for any $\omega \in \Omega$, $\mathbf{u} \in \tilde{\mathcal{U}}$ and $t \in [0, T]$ the map*

$$\partial_{\mathbf{u}} g(\mathbf{u}, \cdot, t, \omega) : \mathcal{Q}(\mathbf{u}, t, \omega) \to \mathbb{R}, \quad q \mapsto \partial_{\mathbf{u}} g(\mathbf{u}, q, t, \omega)$$

satisfies

$$0 = \partial_{\mathbf{u}} g(\mathbf{u}, -p, t, \omega) > \partial_{\mathbf{u}} g(\mathbf{u}, \cdot, t, \omega) > \partial_{\mathbf{u}} g(\mathbf{u}, \bar{q}(\mathbf{u}, t, \omega), t, \omega) = -\sigma a.$$

Proof. By (1.30) and (1.32) we have

$$\begin{aligned}
\partial_{\mathbf{u}} g(\mathbf{u},q,t) &= \sigma\mathbb{E}[u''\big(\Sigma_0+q\psi+x\big)(p+q)\psi x'|\mathcal{F}_t]|_{x=A_t(\mathbf{u},q),x'=\partial_{\mathbf{u}}A_t(\mathbf{u},q)} &(1.33)\\
&= \frac{\sigma\mathbb{E}[u''\big(\Sigma_0+q\psi+x\big)(\Sigma_0+q\psi+x-(x+z))|\mathcal{F}_t]}{\mathbb{E}[u'\big(\Sigma_0+q\psi+x\big)|\mathcal{F}_t]}|_{x=A_t(\mathbf{u},q)}\\
&= \frac{\sigma\big(-a\mathbb{E}[u'\big(\Sigma_0+q\psi+x\big)|\mathcal{F}_t]-\mathbb{E}[u''\big(\Sigma_0+q\psi+x\big)(x+z)|\mathcal{F}_t]\big)}{\mathbb{E}[u'\big(\Sigma_0+q\psi+x\big)|\mathcal{F}_t]}|_{x=A_t(\mathbf{u},q)}.\\
&= -\sigma\left(a+\frac{\mathbb{E}[u''\big(\Sigma_0+q\psi+x\big)(x+z)|\mathcal{F}_t]}{\mathbb{E}[u'\big(\Sigma_0+q\psi+x\big)|\mathcal{F}_t]}|_{x=A_t(\mathbf{u},q)}\right), &(1.34)
\end{aligned}$$

where we use the fact that, by Assumption 1.1, $xu''(x)=-au'(x)$ for all $x>0$. Since

$$\begin{aligned}
0 &\geq \frac{\mathbb{E}[u''\big(\Sigma_0+q\psi+x\big)(x+z)|\mathcal{F}_t]}{\mathbb{E}[u'\big(\Sigma_0+q\psi+x\big)|\mathcal{F}_t]}|_{x=A_t(\mathbf{u},q)}\\
&= -a\frac{\mathbb{E}[u'\big(\Sigma_0+q\psi+x\big)\frac{x+z}{(p+q)\psi+x+z}|\mathcal{F}_t]}{\mathbb{E}[u'\big(\Sigma_0+q\psi+x\big)|\mathcal{F}_t]}|_{x=A_t(\mathbf{u},q)}\\
&\geq -a,
\end{aligned}$$

it follows that

$$0 \geq \partial_{\mathbf{u}} g(\mathbf{u},q,t) \geq -a\sigma.$$

Furthermore, we can see from (1.33) that

$$\partial_{\mathbf{u}} g(\mathbf{u},-p,t)=0.$$

By Lemma 1.19 we have $A_t(\mathbf{u},\bar{q}(\mathbf{u},t))=-z$ and thus (1.34) implies

$$\partial_{\mathbf{u}} g(\mathbf{u},\bar{q}(\mathbf{u},t),t)=-\sigma a.$$

□

Given Lemma 1.25 the proof of uniqueness of solutions to (1.26) is now a straightforward adaptation of a classical Gronwall-argument:

Proof of Theorem 1.21. Let us assume that U and $\hat{U}$ are both strong solutions to (1.26) on $[0,T]$ and that $U_0=\hat{U}_0$. We define the family of stopping times

$$\tau_n \triangleq \inf\{t\in[0,T]:|U_t|>n\vee|\hat{U}_t|>n\},\ n\geq 1.$$

Then, using the notation of (1.27),

$$U_{t\wedge\tau_n}-\hat{U}_{t\wedge\tau_n}=\int_0^{t\wedge\tau_n}\big(g(U_s,Q_s,s)-g(\hat{U}_s,Q_s,s)\big)dW_s.$$

Applying the Itô isometry we obtain

$$
\begin{aligned}
\mathbb{E}[|U_{t\wedge\tau_n} - \hat{U}_{t\wedge\tau_n}|^2] &= \mathbb{E}\left[\left|\int_0^{t\wedge\tau_n} g(U_s, Q_s, s) - g(\hat{U}_s, Q_s, s) dW_s\right|^2\right] \\
&= \mathbb{E}\left[\int_0^{t\wedge\tau_n} \left|g(U_s, Q_s, s) - g(\hat{U}_s, Q_s, s)\right|^2 ds\right].
\end{aligned}
$$

By the mean value theorem we know that for some $\tilde{\mathbf{u}}_s \in [\min(U_s, \hat{U}_s), \max(U_s, \hat{U}_s)]$ we have

$$
\mathbb{E}\left[\int_0^{t\wedge\tau_n} \left|g(U_s, Q_s, s) - g(\hat{U}_s, Q_s, s)\right|^2 ds\right] = \mathbb{E}\left[\int_0^{t\wedge\tau_n} |\partial_{\mathbf{u}} g(\tilde{\mathbf{u}}_s, Q_s, s)|^2 |U_s - \hat{U}_s|^2 ds\right]
$$

and by Lemma 1.25

$$
|\partial_{\mathbf{u}} g(\tilde{\mathbf{u}}_s, Q_s, s)| \le \sigma a.
$$

It follows that

$$
\mathbb{E}\left[\int_0^{t\wedge\tau_n} |\partial_{\mathbf{u}} g(\tilde{\mathbf{u}}_s, Q_s, s)|^2 |U_s - \hat{U}_s|^2 ds\right] \le \sigma^2 a^2 \int_0^t \mathbb{E}\left[|U_{s\wedge\tau_n} - \hat{U}_{s\wedge\tau_n}|^2\right] ds.
$$

Hence,

$$
\mathbb{E}[|U_{t\wedge\tau_n} - \hat{U}_{t\wedge\tau_n}|^2] \le \sigma^2 a^2 \int_0^t \mathbb{E}\left[|U_{s\wedge\tau_n} - \hat{U}_{s\wedge\tau_n}|^2\right] ds.
$$

We apply Gronwall's inequality to

$$
f(t) \triangleq \mathbb{E}[|U_{t\wedge\tau_n} - \hat{U}_{t\wedge\tau_n}|^2]
$$

to obtain $f(t) \equiv 0$ which implies that U and $\hat{U}$ are modifications of one another. Since they are continuous it follows that they are indistinguishable. □

We will now proceed to show that our model is free of arbitrage.

Chapter 2

Absence of arbitrage

The question that arises naturally is under which assumptions our model is free of arbitrage. In this chapter we will consider the case of admissible trading strategies in continuous time and we will prove that under Assumptions 1.1, 1.2 and 1.3 the model is already free of arbitrage.

2.1 Profit and loss of the large investor

When the large investor pursues an admissible strategy Q the market maker's process of indirect utility U^Q is given as the unique solution to (1.26). The profit and loss of the large investor at maturity is then given by the $\mathcal{F}_T$-measurable real-valued random variable G^Q that satisfies

$$U_T^Q = u\big(\Sigma_0 - G^Q\big)$$

or, equivalently,

$$G^Q = -(Q_T\psi + X_T).$$

It would be a temptingly natural extension to define the P&L-process of the large investor $(G_t^Q)_{0\leq t\leq T}$ when following an admissible strategy Q as the unique solution at any time $t \in [0,T]$ to

$$U_t^Q = \mathbb{E}[u\big(\Sigma_0 - G_t^Q\big)|\mathcal{F}_t]. \tag{2.1}$$

However, this solution does not necessarily exist at all times $t < T$.

To illustrate this, we will give an example. Let $0 < \hat{p} < p$ and consider an admissible trading strategy Q for which the event $A \triangleq \{\Sigma_t = \hat{p}\psi\}$ occurs with positive probability. Then, on A, G_t^Q is given by the solution to

$$\mathbb{E}[u\big(\hat{p}\psi\big)|\mathcal{F}_t] = \mathbb{E}[u\big(p\psi + z - G_t^Q\big)|\mathcal{F}_t].$$

Note that for $G_t^Q \leq z$ we have

$$\mathbb{E}[u\big(p\psi + z - G_t^Q\big)|\mathcal{F}_t] \geq \mathbb{E}[u\big(p\psi\big)|\mathcal{F}_t] > \mathbb{E}[u\big(\hat{p}\psi\big)|\mathcal{F}_t]$$

and that for $G_t^Q > z$

$$\mathbb{E}[u(p\psi + z - G_t^Q)|\mathcal{F}_t] = -\infty.$$

Thus, there is no cash amount G_t^Q which solves (2.1). From an economic point of view, this situation corresponds to the case where, up to time t, the large investor pursued a trading strategy Q which allowed him to obtain all of the market maker's initial cash z as well as a part $p - \hat{p}$ of the market maker's initial shares. In this case, the market maker does not possess enough money to buy back the $p - \hat{p}$ shares so that the large investor cannot "cash in" his wealth before time T, when the traded security pays out. In this sense, it is possible that the large investor gets "trapped" in a long position of the traded security ψ. Note that, in this situation, the market maker fails to fulfill his mandate since he does no longer actually "make a market".

We will thus consider the large investor's profit and loss only at time $t = T$.

2.2 Proof of absence of arbitrage

In [14] the absence of arbitrage is deduced from the fact that the market maker's utility process is a submartingale. There, the submartingale property follows immediately from the upper boundedness of the market maker's utility function. While the same reasoning can be applied for the case $a > 1$ in our setting, the case $0 < a < 1$ necessitates a more involved argument which we present below.

We formalise the notion of an arbitrage as a riskless profit in the usual way and give the following definition:

Definition 2.1. We say that a predictable process Q is an *arbitrage* if $G^Q \geq 0$ a.s. and $\mathbb{P}(G^Q > 0) > 0$.

The following theorem ensures the absence of arbitrage in our model.

Theorem 2.2. *Let Assumptions 1.1, 1.2 and 1.3 hold and let Q be an admissible strategy. Then Q is not an arbitrage.*

Before we proceed to prove this theorem at the end of this section, we will establish several lemmas.

Lemma 2.3. *Let Assumption 1.1 hold. If for an admissible strategy Q the process of indirect utility U^Q of the market maker is a submartingale then Q is not an arbitrage.*

Proof. Assume that U^Q is a submartingale and assume that the large investor's P&L satisfies $G^Q \geq 0$ a.s. It is sufficient to show that this implies $\mathbb{P}(G^Q > 0) = 0$. Recall that G^Q is defined implicitly by

$$U_T^Q = u(\Sigma_0 - G^Q).$$

Since U^Q is a submartingale we have $\mathbb{E}[U_T^Q] \geq U_0^Q$ and hence

$$\mathbb{E}[u(\Sigma_0 - G^Q)] \geq \mathbb{E}[u(\Sigma_0)]. \tag{2.2}$$

As u is strictly increasing and $G^Q \geq 0$ a.s. this implies that $G^Q = 0$ a.s. and consequently $\mathbb{P}(G^Q > 0) = 0$. □

Remark 2.4. Under Assumptions 1.1, 1.2 and 1.3 let Q be an admissible strategy and let U^Q denote the market maker's indirect utility process when the large investor is pursuing Q. Then at all times $t \in [0, T]$

(i)
$$Q_t + p \geq 0$$

(ii)
$$A_t(U_t^Q, Q_t) + z \geq 0.$$

Lemma 2.5. *Let Assumptions 1.1, 1.2 and 1.3 hold. Let U^Q denote the market maker's indirect utility process when the large investor is pursuing an admissible trading strategy Q. Let u_{min} be as in (1.20). Then for all $t \in [0, T]$ and $\omega \in \Omega$*

$$U_t^Q(\omega) \geq u_{min}(Q_t(\omega), t, \omega).$$

Proof. By definition of $A_t(U_t^Q, Q_t)$ we have

$$U_t^Q = \mathbb{E}[u\big((p + Q_t)\psi + A_t(U_t^Q, Q_t) + z\big)|\mathcal{F}_t]$$

and from Remark 2.4 we know that

$$A_t(U_t^Q, Q_t) + z \geq 0.$$

Since u is increasing, this implies

$$\mathbb{E}[u\big((p + Q_t)\psi + A_t(U_t^Q, Q_t) + z\big)|\mathcal{F}_t] \geq \mathbb{E}[u\big((p + Q_t)\psi\big)|\mathcal{F}_t].$$

□

Lemma 2.6. *Let Assumptions 1.1, 1.2 and 1.3 hold. Then for $\omega \in \Omega$ and $t \in [0, T]$ we have*

$$\inf_{q \in (-p, \infty)} u_{min}(q, t, \omega) = \begin{cases} 0, & 0 < a < 1, \\ -\infty, & a > 1, \end{cases}$$

and this infimum is not attained on $(-p, \infty) \times [0, T] \times \Omega$.

Proof. The result follows immediately from

$$\lim_{q\downarrow -p} u_{\min}(q,t,\omega) = \lim_{q\downarrow -p} \mathbb{E}[u\big((p+q)\psi\big)|\mathcal{F}_t](\omega) = \begin{cases} 0, & 0<a<1, \\ -\infty, & a>1. \end{cases}$$

□

Lemma 2.7. *Let Assumptions 1.1, 1.2 and 1.3 hold. Let further $b=(q,t,\omega)\in(-p,\infty)\times[0,T]\times\Omega$. Then*

$$\frac{g(u_{min}(b),q,t,\omega)}{u_{min}(b)} \equiv \sigma(1-a).$$

In particular, the right hand side does not depend on b.

Proof. By definition of g and of $u_{\min}$, application of Lemmas 1.12 and 1.19 and by Assumption 1.1 we have

$$\begin{aligned} \frac{g(u_{\min}(b),q,t,\omega)}{u_{\min}(b)} &= \sigma\frac{\mathbb{E}[u'((p+q)\psi+z+A_t(u_{\min}(b),q)\big)(p+q)\psi|\mathcal{F}_t](\omega)}{\mathbb{E}[u\big((p+q)\psi\big)|\mathcal{F}_t](\omega)} \\ &= \sigma\frac{\mathbb{E}[u'((p+q)\psi\big)(p+q)\psi|\mathcal{F}_t](\omega)}{\mathbb{E}[u\big((p+q)\psi\big)|\mathcal{F}_t](\omega)} \\ &= \sigma\frac{\mathbb{E}[(p+q)^{1-a}\psi^{1-a}|\mathcal{F}_t](\omega)}{(1-a)^{-1}\mathbb{E}[(p+q)^{1-a}\psi^{1-a}|\mathcal{F}_t](\omega)} = \sigma(1-a). \end{aligned}$$

□

Lemma 2.8. *Let Assumptions 1.1, 1.2 and 1.3 hold. Let further Q be an admissible trading strategy and let U^Q denote the associated utility process. Then, for all $t\in[0,T]$,*

$$\left|\frac{g(U_t^Q,Q_t,t)}{U_t^Q}\right| \le |\sigma(1-a)|.$$

Proof. We have

$$g(U_t^Q,Q_t,t) = \sigma\mathbb{E}[u'(\Sigma_0+Q_t\psi+A_t(U_t^Q,Q_t))(p+Q_t)\psi|\mathcal{F}_t]$$

and

$$U_t^Q = \mathbb{E}[u(\Sigma_0+Q_t\psi+A_t(U_t^Q,Q_t))|\mathcal{F}_t].$$

Since $u'(x)\cdot x = (1-a)u(x)$ for all $x>0$ we have

$$\begin{aligned} g(U_t^Q,Q_t,t) &= \sigma\mathbb{E}[u'(\Sigma_0+Q_t\psi+A_t(U_t^Q,Q_t))\times \\ &\qquad \times\big((p+Q_t)\psi+z+A_t(U_t^Q,Q_t)-z-A_t(U_t^Q,Q_t)\big)|\mathcal{F}_t] \\ &= \sigma(1-a)\mathbb{E}[u(\Sigma_0+Q_t\psi+A_t(U_t^Q,Q_t))|\mathcal{F}_t] \\ &\qquad -\sigma\mathbb{E}[u'(\Sigma_0+Q_t\psi+A_t(U_t^Q,Q_t))\big(z+A_t(U_t^Q,Q_t)\big)|\mathcal{F}_t]. \end{aligned}$$

We know that $u' > 0$ and since by Remark 2.4 $z + A_t(U_t^Q, Q_t) \geq 0$, it follows that

$$\begin{aligned}&\sigma(1-a)U_t^Q - g(U_t^Q, Q_t, t)\\ = \;&\sigma\mathbb{E}[u'(\Sigma_0 + Q_t\psi + A_t(U_t^Q, Q_t))\big(z + A_t(U_t^Q, Q_t)\big)|\mathcal{F}_t] \geq 0.\end{aligned}$$

Thus, for $0 < a < 1$, we have

$$0 \leq \frac{g(U_t^Q, Q_t, t)}{U_t^Q} \leq \sigma(1-a)$$

and for $a > 1$

$$0 \geq \frac{g(U_t^Q, Q_t, t)}{U_t^Q} \geq \sigma(1-a).$$

□

Proposition 2.9. *Let Assumptions 1.1, 1.2 and 1.3 hold and let Q be an admissible trading strategy. Then the market maker's process of indirect utility U^Q is a (true) martingale.*

Proof. From (1.27) we know that the dynamics of the process U^Q can be expressed as

$$dU_t^Q = g(U_t^Q, Q_t, t)dW_t, \quad 0 \leq t \leq T,$$

where

$$g(U_t^Q, Q_t, t) = \sigma\mathbb{E}[u'\big(\Sigma_0 + Q_t\psi + A_t(U_t^Q, Q_t)\big)(p + Q_t)\psi|\mathcal{F}_t].$$

Thus, U^Q is a local martingale. We recall the Novikov condition which implies that in order to show that U^Q is a (true) martingale it is sufficient to show that

$$\mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^T \left|\frac{g(U_t^Q, Q_t, t)}{U_t^Q}\right|^2 dt\right)\right] < \infty.$$

By Lemma 2.8 at all times $t \in [0, T]$

$$\left|\frac{g(U_t^Q, Q_t, t)}{U_t^Q}\right| \leq |\sigma(1-a)|.$$

Thus

$$\begin{aligned}\int_0^T \left|\frac{g(U_t^Q, Q_t, t)}{U_t^Q}\right|^2 dt &\leq \int_0^T \sigma^2(1-a)^2 dt\\ &= \sigma^2(1-a)^2 T < \infty,\end{aligned}$$

which finishes the proof. □

We immediately obtain the absence of arbitrage in our model:

Proof of Theorem 2.2. By Proposition 2.9 U^Q is a (true) martingale and thus a submartingale. The result then follows by Lemma 2.3. $\square$

We already noted at the beginning of this chapter that in the regime $a > 1$ the absence of arbitrage can be obtained immediately: There, the submartingale property of U^Q is a direct consequence of the fact that U^Q is a local martingale which is bounded from above (since the market maker's utility function u is bounded from above). However, we included the case $a > 1$ in Proposition 2.9 as we will later on make use of the fact that U^Q is a true martingale.

We will now turn our attention to the replication of contingent claims.

Chapter 3

Hedging and replication of options

In this chapter we will examine the replicability of a contingent claim $H \in \mathbf{L}^0(\mathbb{R})$ which is not traded in our financial market. Starting from an initial cash amount v_{rep}, the large investor seeks to attain the position H at time T by dynamically trading the marketed security ψ with the market maker. Two questions arise naturally in this setup: Firstly, which claims can be replicated in our model, i.e. what is the set of attainable contingent claims? And, secondly, how is this replication accomplished, i.e. what is the replicating strategy for a given claim H?

In what follows, we will first derive a necessary and sufficient condition for a claim H to be attainable in the sense that there exists an admissible strategy Q which ensures that the large investor's wealth at time T is given by H. It turns out that, if a claim is attainable, its replication price is equal to its market indifference price.

Once the question of attainability has been addressed, we will turn our attention to the specific case of a call option and give an explicit criterion, namely a condition on the market maker's initial endowment, which constitutes a necessary and sufficient condition for the replicability of a call option in our model. We then proceed to give a necessary and sufficient condition for the attainability of path independent claims of the form $H = f(\psi)$. This easily verifiable condition is stated in terms of the claim's payoff function in relation to the market maker's initial endowment. Lastly, we will provide an example of a claim which possesses an indifference price but which is not attainable in our setting.

3.1 Acceptable and attainable contingent claims

If a claim $H \in \mathbf{L}^0(\mathbb{R})$ was traded in our financial market, we could apply the utility indifference principle to determine the cash amount v that complements the trading of H. Denoting the market maker's initial endowment in the usual way by

$$\Sigma_0 = p\psi + z,$$

we can see that if

$$\mathbb{E}[u\big(\Sigma_0 - H + v_t\big)|\mathcal{F}_t] = \mathbb{E}[u(\Sigma_0)|\mathcal{F}_t]$$

for some $v_t \in \mathbf{L}^0(\mathcal{F}_t, \mathbb{R})$ then v_t is the *market indifference price* of H at time t.

In what follows, we will consider contingent claims $H \in \mathbf{L}^0(\mathbb{R})$ which possess a market indifference price at all times. More precisely, we will consider claims which belong to the set $\mathcal{H}$ defined as follows.

Definition 3.1. We define the *set of acceptable contingent claims*

$$\begin{aligned}\mathcal{H} \quad \triangleq \quad & \{H \in \mathbf{L}^0(\mathbb{R}) \mid \forall t \in [0,T]\ \exists v_t \in \mathbf{L}^0(\mathcal{F}_t, \mathbb{R}) \text{ s.t.} \\ & \qquad \mathbb{E}[u(\Sigma_0 - H + v_t)|\mathcal{F}_t] = \mathbb{E}[u(\Sigma_0)|\mathcal{F}_t]\}.\end{aligned}$$

We will call a contingent claim $H \in \mathbf{L}^0(\mathbb{R})$ *acceptable*, if it belongs to the set of acceptable contingent claims.

Proposition 3.8 and Remark 3.9 below show that, under weak assumptions on the market maker's initial endowment, $\mathcal{H}$ contains e.g. call and put options.

On various occasions it will be necessary to restrict our analysis to path independent claims of the form $H = f(\psi) \in \mathcal{H}$. We therefore introduce the set

$$\mathcal{H}' \triangleq \{H \in \mathcal{H} \mid H = f(\psi) \text{ for some } f \in \mathcal{R}\} \subset \mathcal{H},$$

where $\mathcal{R}$ denotes the class of Lipschitz continuous functions on $\mathbb{R}$. Note that functions belonging to $\mathcal{R}$, in particular, satisfy the integrability condition of Lemma 1.12.

We proceed to define *attainability* of claims in our setting.

Definition 3.2. Let G^Q be defined as in Section 2.1. We will call a contingent claim $H \in \mathcal{H}$ *attainable* if there exists an admissible trading strategy $(Q_t^H)_{0\le t\le T}$ such that almost surely

$$v_{rep} + G^{Q^H} = H \tag{3.1}$$

for some cash amount $v_{rep} \in \mathbb{R}$. In that case, we will call v_{rep} a *replication price* of H and Q^H a *replicating strategy.*

The question that arises naturally is under what conditions an acceptable claim H is attainable and whether or not the replication price v_{rep} of H coincides with its utility indifference price at time zero, v_0. Proposition 3.3 below sheds light on this question.

Before we state and prove it, we introduce the process of indirect utility of the market maker when selling H at time $t = 0$ at price v_0. Denoting this process by $(U_t^H)_{0\le t\le T}$ we define

$$U_t^H \triangleq \mathbb{E}[u(\Sigma_0 - H + v_0)|\mathcal{F}_t]. \tag{3.2}$$

Note that this process is merely an auxiliary process in the sense that the market maker does not actually trade H. However, it is important to keep track of U^H as it will be the large investor's aim to follow an admissible trading strategy Q such that U^Q coincides with U^H. The following proposition makes this idea rigorous.

Proposition 3.3. *Let Assumptions 1.1, 1.2 and 1.3 hold. Consider a contingent claim $H \in \mathcal{H}$ and let v_0 denote the utility indifference price of H at time zero. Then*

(i) If there exists an admissible trading strategy $(Q_t)_{0\leq t\leq T}$ such that almost surely

$$U_t^H = U_t^Q \quad \forall t \in [0,T],$$

then $v_0 + G^Q = H$. In particular, this means that H is attainable with replicating strategy $Q^H = Q$ and replication price $v_{rep} = v_0$.

(ii) If H is attainable then $v_{rep} = v_0$ and almost surely

$$U_t^H = U_t^{Q^H} \ \forall t \in [0,T].$$

Proof. (i) Note that

$$U_T^Q = u\big(\Sigma_0 - G^Q\big)$$

and

$$U_T^H = \mathbb{E}[u(\Sigma_0 - H + v_0)|\mathcal{F}_T] = u(\Sigma_0 - H + v_0).$$

Since $U_T^Q = U_T^H$ and since u is strictly increasing and thus 1-to-1, we have

$$-G^Q = -H + v_0.$$

(ii) If H is attainable then there is an admissible trading strategy Q^H such that

$$v_{rep} + G^{Q^H} = H.$$

Then by definition of G we have

$$Q_T^H \psi + X_T = -G^{Q^H} = v_{rep} - H,$$

which, due to the monotonicity of u, implies that $U_T^{Q^H} = U_T^H$. Since U^H is a martingale and, by Proposition 2.9, U^{Q^H} is a martingale as well, it follows that

$$U_t^H = \mathbb{E}[U_T^H|\mathcal{F}_t] = \mathbb{E}[U_T^{Q^H}|\mathcal{F}_t] = U_t^{Q^H} \ \forall t \in [0,T].$$

Moreover, the equality of v_0 and v_{rep} follows from the monotonicity of u and the fact that

$$\mathbb{E}[u(\Sigma_0 + v_0 - H)] = \mathbb{E}[u(\Sigma_0)] = U_0^H = \mathbb{E}[U_T^H] = \mathbb{E}[u(\Sigma_0 + v_{rep} - H)]. \qquad \square$$

3.2 A criterion for replicability

In this section we will establish a criterion which constitutes a necessary and sufficient condition for a claim $H \in \mathcal{H}$ to be attainable with replication price $v_{rep} = v_0$. We begin with the following observation.

Remark 3.4. Let Assumptions 1.1, 1.2 and 1.3 hold and let $H \in \mathcal{H}$. Let further $(U_t^H)_{0\le t\le T}$ be defined as in (3.2). Then by the martingale representation theorem the process U^H, at any time $t \in [0,T]$, can be expressed as

$$U_t^H = U_0^H + \int_0^t i_s^H dW_s \tag{3.3}$$

for some predictable process $(i_t^H)_{0\le t\le T}$ with

$$\int_0^T (i_t^H)^2 dt < \infty.$$

If $H = f(\psi) \in \mathcal{H}'$ then, by Itô's formula and in view of (1.11), i_t^H is given by

$$\begin{aligned} i_t^H &= \sigma S_t \partial_s \mathbb{E}_{t,s}[u(\Sigma_0 - H + v_0)]|_{s=S_t} \\ &= \sigma \mathbb{E}[u'(\Sigma_0 - f(\psi) + v_0)(p - f'(\psi))\psi|\mathcal{F}_t]. \end{aligned} \tag{3.4}$$

From Proposition 3.3 we know that a claim H can be replicated with replication price $v_{rep} = v_0$ if and only if there exists an admissible trading strategy (Q_t^H) which ensures that almost surely

$$U_t^{Q^H} = U_t^H \ \forall t \in [0,T].$$

This is the idea that we will exploit in order to prove the following proposition which provides a necessary and sufficient condition for the attainability of an acceptable claim H.

Proposition 3.5. *Let Assumptions 1.1, 1.2 and 1.3 hold and let $H \in \mathcal{H}$. Let g be as in* (1.28) *and let U_t^H and i_t^H be as in Remark 3.4. Then H is attainable with replication price $v_{rep} = v_0$ if and only if there exists an admissible trading strategy Q^H such that almost surely for all $t \in [0,T]$*

$$i_t^H = g(U_t^H, Q^H, t). \tag{3.5}$$

Moreover, such a strategy Q^H exists if and only if for all $t \in [0,T]$

$$i_t^H \in [0, \sigma(1-a)U_t^H]. \tag{3.6}$$

Before we prove Proposition 3.5, we state and prove the following preparatory lemma which gives the range of g from (1.28) when g is considered as a function of the transaction size q.

Lemma 3.6. *Let Assumptions 1.1, 1.2 and 1.3 hold and let g be as in* (1.28). *Then for $\omega \in \Omega$, $\mathbf{u} \in \tilde{\mathcal{U}}$ and $t \in [0,T]$ the range of the map $g(\mathbf{u}, \cdot, t, \omega) : [-p, \bar{q}(\mathbf{u}, t, \omega)] \to \mathbb{R}$, $q \mapsto g(\mathbf{u}, q, t, \omega)$ is given by*

$$\text{im } g(\mathbf{u}, \cdot, t, \omega) = [0, \sigma(1-a)\mathbf{u}]. \tag{3.7}$$

Proof. By definition of g and Lemma 1.12 we have

$$\begin{aligned} g(\mathbf{u}, q, t) &= \sigma S_t \partial_s \mathbb{E}_{t,s}[u(\Sigma_0 + q\psi + x)]|_{s=S_t, x=A_t(\mathbf{u},q)} \\ &= \sigma \mathbb{E}[u'(\Sigma_0 + q\psi + x)(p+q)\psi | \mathcal{F}_t]|_{x=A_t(\mathbf{u},q)}. \end{aligned}$$

Thus, using the fact that by Assumption 1.1 $xu'(x) = (1-a)u(x)$ for all $x > 0$,

$$\begin{aligned} g(\mathbf{u}, q, t) &= \sigma \mathbb{E}[u'(\Sigma_0 + q\psi + x)(p+q)\psi | \mathcal{F}_t]|_{x=A_t(\mathbf{u},q)} \\ &= \sigma \mathbb{E}[u'(\Sigma_0 + q\psi + x)((p+q)\psi + x + z - (x+z)) | \mathcal{F}_t]|_{x=A_t(\mathbf{u},q)} \\ &= \sigma(1-a)\mathbb{E}[u(\Sigma_0 + q\psi + x) | \mathcal{F}_t]|_{x=A_t(\mathbf{u},q)} \\ &\quad - \sigma \mathbb{E}[u'(\Sigma_0 + q\psi + x)(x+z) | \mathcal{F}_t]|_{x=A_t(\mathbf{u},q)}. \end{aligned} \tag{3.8}$$

From the first line it is clear that g is non-negative. The second term in (3.8) is non-negative as well, which implies that the first term of (3.8) dominates g. Note that due to the definition of $A_t(\mathbf{u}, q)$

$$\mathbf{u} = \mathbb{E}[u(\Sigma_0 + q\psi + A_t(\mathbf{u}, q)) | \mathcal{F}_t].$$

Thus, the first term of (3.8) is equal to $\sigma(1-a)\mathbf{u}$. Let us consider the second term. For $q = -p$, the second term simplifies to

$$-\sigma \mathbb{E}[u'(\Sigma_0 + q\psi + x)(x+z) | \mathcal{F}_t]|_{x=A_t(\mathbf{u},q)} = -\sigma(A_t(\mathbf{u}, q) + z)^{1-a}$$

so that (3.8) is equal to zero. By Lemma 1.19 we know that $q = \bar{q}(\mathbf{u}, t)$ if and only if $A_t(\mathbf{u}, q) = -z$ which implies that for $q = \bar{q}(\mathbf{u}, t)$ the second term of (3.8) is equal to zero. The continuity of $q \mapsto g(\mathbf{u}, q, t)$ then implies (3.7). □

Proof of Proposition 3.5. By Proposition 3.3 we know that the claim H is attainable with replicating strategy Q^H and replication price $v_{rep} = v_0$ if and only if

$$U_t^{Q^H} = U_t^H \quad \forall t \in [0,T].$$

Due to (1.27) and to Remark 3.4 we know that this is the case if and only if

$$i_t^H = g(U_t^H, Q_t^H, t) \quad \mathbb{P} \otimes \mathrm{d}t - \text{a.e.},$$

which proves the first assertion. By Lemma 3.6 the range of the map $g(U_t^H, \cdot, t)$: $[-p, \bar{q}(U_t^H, t)] \to \mathbb{R}$ is given by $[0, \sigma(1-a)U_t^H]$, which proves the second assertion.

□

Remark 3.7. Note that for path independent claims $H = f(\psi) \in \mathcal{H}'$ condition (3.6) in Proposition 3.5 can be restated as a *bounded elasticity condition.* To this end, let

$$u^H(t,s) \triangleq \mathbb{E}_{t,s}[u(\Sigma_0 - H + v_0)]$$

so that $U_t^H = u^H(t, S_t)$ for all $t \in [0,T]$ and that further, by Itô's formula, $i_t^H = \sigma S_t \partial_s u^H(t, S_t)$ for all $t \in [0,T]$. Then (3.6) can be rewritten as

$$0 \leq \frac{s}{1-a} \frac{\partial_s u^H(t,s)}{u^H(t,s)} \leq 1 \quad \forall s > 0 \, \forall t \in [0,T].$$

However, throughout this chapter we prefer to use (3.6) for the sake of simplicity of computations.

Having established a necessary and sufficient condition for the attainability of an acceptable claim H, we will now proceed to look at the specific case of a call option.

3.3 Hedging a call option

Let us consider H to be a call option on the marketed security ψ with strike price $K \in \mathbb{R}_+$, i.e.

$$H = (\psi - K)^+.$$

We are interested in whether or not it is possible to hedge H, i.e. whether or not it is possible to find an admissible trading strategy Q^H which replicates H as described in the previous section. We answer this question by stating

Proposition 3.8. *Let Assumptions 1.1, 1.2 and 1.3 hold and let $H = (\psi - K)^+$ be a call option. Then ξH is attainable for any $\xi \in [0,p]$, where p denotes the market maker's initial position in the risky asset ψ.*

Proof. From Remark 3.4 we know that the dynamics of $U^{\xi H}$ can be expressed as

$$dU_t^{\xi H} = i_t^{\xi H} dW_t, \quad 0 \leq t \leq T,$$

for an appropriate $i_t^{\xi H}$. Furthermore, Proposition 3.5 tells us that ξH is attainable if and only if almost surely

$$i_t^{\xi H} \in [0, \sigma(1-a)U_t^{\xi H}] \quad \forall t \in [0,T].$$

Hence it is necessary and sufficient to show that for all $t \in [0,T]$ and $0 \leq \xi \leq p$

$$\sigma(1-a)U_t^{\xi H} - i_t^{\xi H} \geq 0 \tag{3.9}$$

and

$$i_t^{\xi H} \geq 0. \tag{3.10}$$

Let us first show (3.9). We know from Remark 3.4 that

$$i_t^{\xi H} = \sigma S_t \partial_s \mathbb{E}_{t,s}[u\big(\Sigma_0 - \xi(\psi - K)^+ + v_0\big)]|_{s=S_t}$$

which, after a straightforward calculation and with regard to Lemma 1.12, leaves us with

$$\begin{aligned} i_t^{\xi H} &= \sigma \mathbb{E}[u'\big((p-\xi)s\mathcal{E}_{T-t} + z + v_0 + K\big)(p-\xi)s\mathcal{E}_{T-t}\mathbb{1}_{\{s\mathcal{E}_{T-t}\geq K\}}]|_{s=S_t} \\ &\quad + \sigma \mathbb{E}[u'\big(ps\mathcal{E}_{T-t} + z + v\big)ps\mathcal{E}_{T-t}\mathbb{1}_{\{s\mathcal{E}_{T-t}<K\}}]|_{s=S_t}. \end{aligned}$$

Let $m \triangleq z + v_0 + K$. Since

$$\begin{aligned} U_t^{\xi H} &= \mathbb{E}[u((p-\xi)s\mathcal{E}_{T-t} + m)\mathbb{1}_{\{s\mathcal{E}_{T-t}\geq K\}}]|_{s=S_t} \\ &\quad + \mathbb{E}[u(ps\mathcal{E}_{T-t} + z + v)\mathbb{1}_{\{s\mathcal{E}_{T-t}<K\}}]|_{s=S_t} \end{aligned}$$

we can see that

$$\begin{aligned} &\sigma(1-a)U_t^{\xi H} - i_t^{\xi H} \\ &= \sigma\mathbb{E}\big[\big\{u((p-\xi)s\mathcal{E}_{T-t} + m)(1-a) \\ &\qquad -u'((p-\xi)s\mathcal{E}_{T-t} + m)(p-\xi)s\mathcal{E}_{T-t}\big\}\mathbb{1}_{\{s\mathcal{E}_{T-t}\geq K\}}\big]|_{s=S_t} \\ &\quad + \sigma\mathbb{E}\big[\big\{u(ps\mathcal{E}_{T-t} + z + v)(1-a) \\ &\qquad -u'(ps\mathcal{E}_{T-t} + z + v)ps\mathcal{E}_{T-t}\big\}\mathbb{1}_{\{s\mathcal{E}_{T-t}<K\}}\big]|_{s=S_t} \\ &= \sigma\mathbb{E}\big[\big\{((p-\xi)s\mathcal{E}_{T-t} + m)^{1-a} \\ &\qquad -((p-\xi)s\mathcal{E}_{T-t} + m)^{-a}\left\{(p-\xi)s\mathcal{E}_{T-t} + (m-m)\right\}\big\}\mathbb{1}_{\{s\mathcal{E}_{T-t}\geq K\}}\big]|_{s=S_t} \\ &\quad + \sigma\mathbb{E}\big[\big\{(ps\mathcal{E}_{T-t} + z + v)^{1-a} \\ &\qquad -(ps\mathcal{E}_{T-t} + z + v)^{-a}\left\{ps\mathcal{E}_{T-t}(z + v - (z+v))\right\}\big\}\mathbb{1}_{\{s\mathcal{E}_{T-t}<K\}}\big]|_{s=S_t} \\ &= \sigma\mathbb{E}[m((p-\xi)s\mathcal{E}_{T-t} + m)^{-a}\mathbb{1}_{\{s\mathcal{E}_{T-t}\geq K\}}]|_{s=S_t} \\ &\quad + \sigma\mathbb{E}[(z+v)(ps\mathcal{E}_{T-t} + z + v)^{-a}\mathbb{1}_{\{s\mathcal{E}_{T-t}<K\}}]|_{s=S_t} \\ &> 0, \end{aligned}$$

where the last inequality follows from the fact that $\xi \leq p$. This proves (3.9). Furthermore, (3.10) holds as, by Remark 3.4,

$$i_t^{\xi H} = \sigma \mathbb{E}[u'(\Sigma_0 - f(\psi) + v_0)(p - f'(\psi))\psi | \mathcal{F}_t] \geq 0.$$

Here, the last inequality is due to the fact that $u' > 0$ and $f'(\psi) = \xi \mathbb{1}_{\{\psi \geq K\}} \leq p$. □

From Proposition 3.8 it follows in particular, for $\xi = 1$, that a call option H is attainable if the market maker possesses an initial endowment Σ_0 with $p \geq 1$. This result illustrates the desire of the market maker to cover himself for the worst case loss: For large terminal values of the risky asset ψ the market maker will only be fully covered for the payout of the call option if he holds at least one unit of ψ. Since the market maker possesses a power utility function, he will never accept the risk of being left with negative terminal wealth and he must therefore have an initial endowment of at least one underlying security ψ in order to allow the short selling of H. Hence, a call option H which is attainable in our model is a so-called *covered call.*

Remark 3.9. Note that an analog result is true for put options $H = (K - \psi)^+$. In this case, the attainability of ξH is ensured by the condition $0 \leq \xi K \leq v_0 + z$ rather than by $0 \leq \xi \leq p$. Here, v_0 denotes the market indifference price for H at time zero while z, as usual, denotes the market maker's initial cash position.

3.4 A more tractable condition for replicability of path independent claims

The following proposition provides a sufficient condition for the replicability of path independent contingent claims $H = f(\psi)$ whose payoff at maturity only depends on the value at maturity of the underlying.

Proposition 3.10. *Let Assumptions 1.1, 1.2 and 1.3 hold. Consider a contingent claim $H = f(\psi) \in \mathcal{H}'$ and let v_0 denote the utility indifference price of H at time zero. Then H is attainable if and only if for almost all $x > 0$ the payoff function f satisfies*

$$p \geq f'(x) \geq \frac{f(x) - (z + v_0)}{x}, \tag{3.11}$$

where p and z denote the market maker's initial position in the marketed security ψ and in cash, respectively.

Proof. By Proposition 3.5 we know that H is replicable if and only if

$$\sigma(1-a)U_t^H - i_t^H \geq 0 \quad \forall t \in [0, T] \tag{3.12}$$

and

$$i_t^H \geq 0 \quad \forall t \in [0, T]. \tag{3.13}$$

By (3.4), in order for (3.13) to hold, it suffices that

$$p - f'(x) \geq 0 \quad \forall x > 0$$

which is true due to the first inequality in (3.11). Let us now show (3.12). For the sake of brevity in the ensuing computation let

$$\eta \triangleq \Sigma_0 - f(\psi) + v_0.$$

By Remark 3.4, using Assumption 1.1,

$$\begin{aligned} i_t^H &= \sigma \mathbb{E}_{t,s}[u'(\eta)(p\psi - \psi f'(\psi))]|_{s=S_t} \\ &= \sigma \mathbb{E}_{t,s}[u'(\eta)(\eta - (\psi f'(\psi) + z + v_0 - f(\psi))]|_{s=S_t} \\ &= \sigma(1-a)\mathbb{E}_{t,s}[u(\eta)]|_{s=S_t} - \sigma \mathbb{E}_{t,s}[u'(\eta)(\psi f'(\psi) + z + v_0 - f(\psi))]|_{s=S_t} \\ &= \sigma(1-a)U_t^H - \sigma \mathbb{E}_{t,s}[u'(\eta)(\psi f'(\psi) + z + v_0 - f(\psi))]|_{s=S_t}. \end{aligned}$$

Thus,

$$\begin{aligned} &\sigma(1-a)U_t^H - i_t^H \\ &= \sigma \mathbb{E}_{t,s}[u'(\Sigma_0 - f(\psi) + v_0)(\psi f'(\psi) + z + v_0 - f(\psi))]|_{s=S_t}. \end{aligned}$$

As $u' > 0$ and $\psi > 0$, for the last line to be non-negative it is necessary and sufficient that for almost all $x > 0$

$$x f'(x) + z + v_0 - f(x) \geq 0.$$

□

Note that the attainability of calls and puts could alternatively have been obtained as a corollary of Proposition 3.5: For a call option $H = f(\psi)$ with $f(x) = (x - K)^+$, for almost all $x > 0$, we have

$$f'(x) = \mathbb{1}_{\{x \geq K\}} \leq 1 \tag{3.14}$$

and

$$\begin{aligned} x f'(x) + z + v_0 - f(x) &= x\mathbb{1}_{\{x \geq K\}} + z + v_0 - (x - K)\mathbb{1}_{\{x \geq K\}} \\ &= K\mathbb{1}_{\{x \geq K\}} + z + v_0 \\ &> 0, \end{aligned}$$

so that, in view of (3.14), a call option is attainable if and only if $p \geq 1$.

Similarly, for a put option $H = f(x)$ with $f(x) = (K - x)^+$, for almost all $x > 0$, we have

$$f'(x) = -\mathbb{1}_{\{K \geq x\}} \leq 0$$

and

$$\begin{aligned} xf'(x) + z + v_0 - f(x) &= -x\mathbb{1}_{\{K\geq x\}} + z + v_0 - (K-x)\mathbb{1}_{\{K\geq x\}} \\ &= z + v_0 - K\mathbb{1}_{\{K\geq x\}}, \end{aligned}$$

so that a put option is attainable if and only if $z + v_0 \geq K$.

3.5 A claim which is acceptable but not admissible

The following example shows that there are claims which possess an indifference price but which cannot be replicated in our model.

Let the market maker's initial endowment be given by $\Sigma_0 = \psi$, let $a = 2$ and choose μ and σ in (1.1) in such a way that

$$\hat{\mu} \triangleq \left(\mu - \frac{\sigma^2}{2}\right) T > 0.$$

Consider the claim $H = f(\psi)$ with $f : \mathbb{R}_+ \to \mathbb{R}$ given by

$$f(x) = px - \frac{1}{x},$$

where p denotes the market maker's initial stock position; i.e. here, $p = 1$. We will now show that H is *acceptable* but *not attainable.*

H **is acceptable.** The indifference price v_0 of H is given as the solution to

$$\mathbb{E}[u(\Sigma_0 - H + v_0)] = \mathbb{E}[u(\Sigma_0)],$$

which here simplifies to

$$\mathbb{E}[u(\psi^{-1} + v_0)] = \mathbb{E}[u(\psi)]. \tag{3.15}$$

As $a = 2$, it follows that $u(x) = -x^{-1}$ so that the left hand side of (3.15), for $v_0 = 0$, reduces to

$$\mathbb{E}[u(\psi^{-1})] = -\mathbb{E}[\psi] = -e^{\hat{\mu}+\frac{\sigma^2}{2}}$$

while the right hand side is given by

$$\mathbb{E}[u(\psi)] = -\mathbb{E}[\psi^{-1}] = -e^{-\hat{\mu}+\frac{\sigma^2}{2}}.$$

Since $\hat{\mu} > 0$ it follows that

$$\mathbb{E}[u(\psi^{-1})] < \mathbb{E}[u(\psi)].$$

On the other hand, for $v_0 \to \infty$ the left hand side of (3.15) converges to 0. The mean value theorem then implies the existence of a solution v_0 to (3.15) which, in turn, implies that $H \in \mathcal{H}$.

***H* is not attainable.** Note that

$$f'(x) = p + \frac{1}{x^2} > p \quad \forall x > 0$$

so that the claim H does not satisfy the condition presented in Proposition 3.10. It follows that H is not attainable. Another way to see this is to note that

$$p - f'(\psi) = -\frac{1}{\psi^2} < 0.$$

In view of (3.4) this implies that $i_t^H < 0$ and, by Proposition 3.5, it follows that H is not attainable.

This concludes our investigation of the attainability of contingent claims. We will now turn our attention to the analysis of replicating strategies.

Chapter 4

Asymptotic analysis of price processes and hedging strategies

In the previous chapter we derived conditions which ensure that a contingent claim $H \in \mathbf{L}^0(\mathbb{R})$ can be replicated and we proved that call- and put options are attainable under easily verifiable conditions on the market maker's initial endowment. In this chapter we address the question of *how* this replication is accomplished and investigate a claim's replicating strategy Q^H.

Note that Q^H is specified only implicitly by the arguments in the previous chapter and that no closed form solution seems to exist, not even for simple products like the call option. We will therefore resort to an asymptotic analysis in which we look at the replication of *small positions* εH of an attainable claim. This enables us to compute first and second order approximations to the replicating position $Q_t^{\varepsilon H}$. We find that the first order approximation can be viewed as a Black-Scholes delta while the second order approximation, which can be interpreted as the *liquidity correction* for the replicating strategy, is given as a linear combination of risk-adjusted hedge ratios of auxiliary claims in some frictionless model.

Before we begin our examination of hedging strategies, we briefly investigate marginal prices and liquidity corrections for prices in our model which we will need for the ensuing asymptotic analysis.

4.1 Marginal prices, liquidity corrections and the marginal indifference pricing measure

In this section we will determine the prices that occur when small quantities of assets and contingent claims are traded. To this end, we compute the marginal price processes of the traded security ψ and of contingent claims H, i.e. we investigate the market indifference prices of $\varepsilon\psi$ and εH for $\varepsilon \downarrow 0$. We then introduce the so-called *marginal indifference pricing measure* $\mathbb{Q}$ under which marginal price processes are martingales and which will enable us to express marginal prices as expected payoffs.

Note that due to the nonlinear nature of our financial market it is not evident that the attainability of a claim H implies the attainability of smaller positions εH, $\varepsilon < 1$. It is for this reason that we introduce the set

$$\mathcal{H}'' \triangleq \{H \in \mathcal{H}' \mid \varepsilon H \text{ is attainable } \forall \varepsilon \in [0,1]\}$$

By Proposition 3.8 and Remark 3.9 it is clear that call- and put options belong to the set $\mathcal{H}''$ for a suitable initial endowment of the market maker. Moreover, it is easy to verify that any claim of the form $H = f(\psi)$ with f satisfying the conditions of Proposition 3.10 belongs to $\mathcal{H}''$.

Let now $H \in \mathcal{H}''$ be a contingent claim. Then the utility indifference price of a number of claims εH at time $t \in [0,T]$ is given by the unique cash amount $v_t^\varepsilon(H)$ which solves

$$\mathbb{E}[u(\Sigma_0)|\mathcal{F}_t] = \mathbb{E}[u\big(\Sigma_0 - \varepsilon H + v_t^\varepsilon(H)\big)|\mathcal{F}_t]. \tag{4.1}$$

Note that $v_t^0(H) = 0$ and that $v_0^1(H) = v_0$ is the market indifference price of H at time zero.

Since we cannot hope to obtain a closed-form representation for $v_t^\varepsilon(H)$, we examine the price of a small quantity of claims and do a Taylor expansion around zero, i.e. we write

$$v_t^\varepsilon(H) = \varepsilon \partial_\varepsilon v_t^\varepsilon|_{\varepsilon=0}(H) + \frac{\varepsilon^2}{2}\partial_\varepsilon^2 v_t^\varepsilon|_{\varepsilon=0}(H) + o(\varepsilon^2).$$

The first order approximation

$$\partial_\varepsilon v_t^\varepsilon|_{\varepsilon=0}(H)$$

is the *marginal utility indifference price* of H at time t while the second order approximation

$$\partial_\varepsilon^2 v_t^\varepsilon|_{\varepsilon=0}(H)$$

can be viewed as the (nonlinear) *liquidity correction* that arises in our model. For notational convenience we will merely write v_t^ε rather than $v_t^\varepsilon(H)$ from now on. The following proposition shows that the first and second order derivatives of v_t^ε and, consequently, the Taylor approximation exist.

Proposition 4.1. *Under Assumptions 1.1, 1.2 and 1.3 let $H \in \mathcal{H}''$. Then for any $\omega \in \Omega$ and $t \in [0,T]$ the function*

$$v_t^{(\cdot)}(\omega) : [0,1] \to \mathbb{R}, \quad \varepsilon \mapsto v_t^\varepsilon(\omega),$$

defined implicitly in (4.1)*, which maps a quantity ε of the contingent claim H to its market indifference price, is twice continuously differentiable. Furthermore, its derivatives can be extended continuously to $\varepsilon = 0$ and $\varepsilon = 1$ and in $\varepsilon = 0$ they are* *given by*

$$\partial_\varepsilon v_t^\varepsilon|_{\varepsilon=0} = \frac{\mathbb{E}[u'(\Sigma_0)H|\mathcal{F}_t]}{\mathbb{E}[u'(\Sigma_0)|\mathcal{F}_t]} \tag{4.2}$$

and

$$\partial_\varepsilon^2 v_t^\varepsilon|_{\varepsilon=0} = -\frac{\mathbb{E}[u''(\Sigma_0)(\partial_\varepsilon v_t^\varepsilon|_{\varepsilon=0} - H)^2|\mathcal{F}_t]}{\mathbb{E}[u'(\Sigma_0)|\mathcal{F}_t]}. \tag{4.3}$$

Proof. The indifference price v_t^ε of H at time $t \in [0,T]$ is defined implicitly via (4.1). Since $H \in \mathcal{H}''$, Lemma 1.11 together with the implicit function theorem implies the existence of the derivatives $\partial_\varepsilon v_t^\varepsilon$ and $\partial_\varepsilon^2 v_t^\varepsilon$ on $(0,1)$. Differentiating both sides of (4.1) with respect to ε yields

$$0 = \mathbb{E}[u'(\Sigma_0 - \varepsilon H + v_t^\varepsilon)(\partial_\varepsilon v_t^\varepsilon - H)|\mathcal{F}_t];$$

differentiating both sides of this equation once again with respect to ε yields

$$0 = \mathbb{E}[u''(\Sigma_0 - \varepsilon H + v_t^\varepsilon)(\partial_\varepsilon v_t^\varepsilon - H)^2 + u'(\Sigma_0 - \varepsilon H + v_t^\varepsilon)\partial_\varepsilon^2 v_t^\varepsilon|\mathcal{F}_t].$$

Rearranging both equalities with respect to $\partial_\varepsilon v_t^\varepsilon$ and $\partial_\varepsilon^2 v_t^\varepsilon$, respectively, yields

$$\partial_\varepsilon v_t^\varepsilon = \frac{\mathbb{E}[u'(\Sigma_0 - \varepsilon H + v_t^\varepsilon)H|\mathcal{F}_t]}{\mathbb{E}[u'(\Sigma_0 - \varepsilon H + v_t^\varepsilon)|\mathcal{F}_t]}$$

and

$$\partial_\varepsilon^2 v_t^\varepsilon = -\frac{\mathbb{E}[u''(\Sigma_0 - \varepsilon H + v_t^\varepsilon)(\partial_\varepsilon v_t^\varepsilon - H)^2|\mathcal{F}_t]}{\mathbb{E}[u'(\Sigma_0 - \varepsilon H + v_t^\varepsilon)|\mathcal{F}_t]}.$$

The dominated convergence theorem implies the existence of continuous extensions of these identities to the boundary points $\varepsilon = 0$ and $\varepsilon = 1$. The desired expressions (4.2) and (4.3) then follow for $\varepsilon = 0$. □

We will now turn our attention to the indifference price of a small position of the traded security $\varepsilon\psi$. Due to our convention of considering transfers of both the traded security and the complementing cash from the point of view of the market maker, we consider $H = -\psi$ in (4.1) which thus becomes

$$\mathbb{E}[u(\Sigma_0)|\mathcal{F}_t] = \mathbb{E}[u(\Sigma_0 + \varepsilon\psi + v_t^\varepsilon(-\psi))|\mathcal{F}_t]. \tag{4.4}$$

This results in a change of sign in the marginal price of ψ (when compared to the marginal price of H). In order to emphasize this difference, we will from now on denote the indifference price of a position of claims εH by v_t^ε while we will denote the indifference price of a position of the traded security $\varepsilon\psi$ by

$$x_t^\varepsilon \triangleq v_t^\varepsilon(-\psi). \tag{4.5}$$

Analog arguments to those in the proof of Proposition 4.1 yield

$$\partial_\varepsilon x_t^\varepsilon|_{\varepsilon=0} = -\frac{\mathbb{E}[u'(\Sigma_0)\psi|\mathcal{F}_t]}{\mathbb{E}[u'(\Sigma_0)|\mathcal{F}_t]} \tag{4.6}$$

and

$$\partial_\varepsilon^2 x_t^\varepsilon|_{\varepsilon=0} = -\frac{\mathbb{E}[u''(\Sigma_0)(\partial_\varepsilon x_t^\varepsilon|_{\varepsilon=0}+\psi)^2|\mathcal{F}_t]}{\mathbb{E}[u'(\Sigma_0)|\mathcal{F}_t]} \tag{4.7}$$

for the continuous extensions of the first and second order derivatives of x_t^ε to the point $\varepsilon = 0$. The form of marginal prices in our model suggests the following definition.

Definition 4.2. We define the *marginal indifference pricing measure* $\mathbb{Q}$ as the probability measure given by

$$\frac{d\mathbb{Q}}{d\mathbb{P}} \triangleq \frac{u'(\Sigma_0)}{\mathbb{E}[u'(\Sigma_0)]}. \tag{4.8}$$

Remark 4.3. When considered with respect to the marginal indifference pricing measure $\mathbb{Q}$, marginal prices assume the form

$$\partial_\varepsilon v_t^\varepsilon|_{\varepsilon=0} = \mathbb{E}^{\mathbb{Q}}[H|\mathcal{F}_t]$$

and

$$\partial_\varepsilon x_t^\varepsilon|_{\varepsilon=0} = -\mathbb{E}^{\mathbb{Q}}[\psi|\mathcal{F}_t].$$

In particular, the processes $(\partial_\varepsilon v_t^\varepsilon|_{\varepsilon=0})_{0\le t\le T}$ and $(\partial_\varepsilon x_t^\varepsilon|_{\varepsilon=0})_{0\le t\le T}$ are $\mathbb{Q}$-martingales. Moreover, the liquidity correction terms can be written as

$$\partial_\varepsilon^2 v_t^\varepsilon|_{\varepsilon=0} = \mathbb{E}^{\mathbb{Q}}[(\partial_\varepsilon v_t^\varepsilon|_{\varepsilon=0} - H)^2 R(\Sigma_0)|\mathcal{F}_t]$$

and

$$\partial_\varepsilon^2 x_t^\varepsilon|_{\varepsilon=0} = \mathbb{E}^{\mathbb{Q}}[(\partial_\varepsilon x_t^\varepsilon|_{\varepsilon=0} + \psi)^2 R(\Sigma_0)|\mathcal{F}_t],$$

where

$$R(\Sigma_0) = -\frac{u''(\Sigma_0)}{u'(\Sigma_0)}$$

denotes the market maker's risk aversion as introduced in Assumption 1.1.

The form of the second order terms $\partial_\varepsilon^2 v_t^\varepsilon|_{\varepsilon=0}$ and $\partial_\varepsilon^2 x_t^\varepsilon|_{\varepsilon=0}$ further motivates the introduction of the probability measure $\mathbb{R}$ defined by

$$\frac{d\mathbb{R}}{d\mathbb{P}} \triangleq \frac{u''(\Sigma_0)}{\mathbb{E}[u''(\Sigma_0)]}.$$

Note that the Radon-Nikodym densities of the measures $\mathbb{Q}$ and $\mathbb{R}$ with respect to one another are given by the market maker's (normalised) risk aversion $R(\Sigma_0)$ and his (normalised) *risk tolerance* $\tau(\Sigma_0) \triangleq R^{-1}(\Sigma_0)$, respectively; i.e.

$$\frac{d\mathbb{R}}{d\mathbb{Q}} = \frac{R(\Sigma_0)}{\mathbb{E}^{\mathbb{Q}}[R(\Sigma_0)]} \tag{4.9}$$

and

$$\frac{d\mathbb{Q}}{d\mathbb{R}} = \frac{R^{-1}(\Sigma_0)}{\mathbb{E}^{\mathbb{R}}[R^{-1}(\Sigma_0)]} = \frac{\tau(\Sigma_0)}{\mathbb{E}^{\mathbb{R}}[\tau(\Sigma_0)]}. \tag{4.10}$$

Under the measure $\mathbb{R}$, we have

$$\partial_\varepsilon^2 v_t^\varepsilon|_{\varepsilon=0} = \mathbb{E}^{\mathbb{R}}[(\partial_\varepsilon v_t^\varepsilon|_{\varepsilon=0} - H)^2|\mathcal{F}_t]\mathbb{E}^{\mathbb{Q}}[R(\Sigma_0)|\mathcal{F}_t]$$

and

$$\partial_\varepsilon^2 x_t^\varepsilon|_{\varepsilon=0} = \mathbb{E}^{\mathbb{R}}[(\partial_\varepsilon x_t^\varepsilon|_{\varepsilon=0} + \psi)^2|\mathcal{F}_t]\mathbb{E}^{\mathbb{Q}}[R(\Sigma_0)|\mathcal{F}_t].$$

The measure $\mathbb{R}$ will be used later on to express the second order approximation to the replicating position.

Remark 4.4. By definition of the measures $\mathbb{Q}$ and $\mathbb{R}$ we have

$$\mathbb{E}^{\mathbb{Q}}[R(\Sigma_0)K|\mathcal{F}_t] = -\frac{\mathbb{E}[u''(\Sigma_0)K|\mathcal{F}_t]}{\mathbb{E}[u'(\Sigma_0)|\mathcal{F}_t]} = \mathbb{E}^{\mathbb{R}}[K|\mathcal{F}_t]\ \mathbb{E}^{\mathbb{Q}}[R(\Sigma_0)|\mathcal{F}_t]$$

for all $K \in \mathbf{L}^0(\mathbb{R})$ for which these expectations exist.

Remark 4.5. Let $\sigma(\psi)$ denote the sigma algebra generated by $\psi = S_T$. Since $Z \triangleq d\mathbb{Q}/d\mathbb{P}$ is $\sigma(\psi)$-measurable and, consequently, for any bounded measurable $f : (0, \infty) \to \mathbb{R}$,

$$\mathbb{E}^{\mathbb{Q}}[f(\psi)|\mathcal{F}_t] = \mathbb{E}[Zf(\psi)|\mathcal{F}_t] = \mathbb{E}_{t,s}[Zf(\psi)]|_{s=S_t} = \mathbb{E}^{\mathbb{Q}}_{t,s}[f(\psi)]|_{s=S_t},$$

the process $(S_t)_{0\le t\le T}$ from Assumption 1.2 retains the Markov property under the marginal indifference pricing measure $\mathbb{Q}$. By the same reasoning S is Markov under the measure $\mathbb{R}$.

4.2 Asymptotic analysis of the replicating strategy

In this section we compute the first and second order approximations to the replicating strategy $(Q_t^{\varepsilon H})_{0\le t\le T}$ for a small quantity of claims εH, $\varepsilon > 0$. The large investor seeks to replicate εH by dynamically trading the marketed security ψ with the market maker. In view of Proposition 3.3, this is accomplished if the large investor follows an admissible trading strategy $Q^{\varepsilon H}$ such that almost surely

$$U_t^{\varepsilon H} = U_t^{Q^{\varepsilon H}} \quad \forall t \in [0, T].$$

This equation, however, defines the replicating strategy $Q^{\varepsilon H}$ in a strongly implicit manner and there seems to be no hope to find a closed form description of $Q^{\varepsilon H}$. We will therefore investigate $Q^{\varepsilon H}$ asymptotically by considering its Taylor expansion around zero, i.e.

$$Q_t^{\varepsilon H} = \varepsilon \cdot \partial_\varepsilon Q_t^{\varepsilon H}|_{\varepsilon=0} + \frac{\varepsilon^2}{2} \cdot \partial_\varepsilon^2 Q_t^{\varepsilon H}|_{\varepsilon=0} + o(\varepsilon^2).$$

The rest of this section will be devoted to calculating the first and second order approximations $\partial_\varepsilon Q_t^\varepsilon|_{\varepsilon=0}$ and $\partial_\varepsilon^2 Q_t^\varepsilon|_{\varepsilon=0}$. We first state several preparatory lemmas.

The following two lemmas establish the differentiability of the important maps A_t and g from (1.22) and (1.28) with respect to the transaction size q.

Lemma 4.6. *Let Assumptions 1.1, 1.2 and 1.3 hold. Let $\omega \in \Omega$, $t \in [0, T]$, $\mathbf{u} \in \tilde{\mathcal{U}}$ and let A_t^ω be as in* (1.22)*. Then the map*

$$A_t^\omega(\mathbf{u}, \cdot) : [-p, \bar{q}(\mathbf{u}, t, \omega)] \to \mathbb{R}, \quad q \mapsto A_t^\omega(\mathbf{u}, q)$$

is strictly decreasing, strictly convex and twice continuously differentiable on the interior of its domain. Furthermore, its derivatives are given by

$$\partial_q A_t(\mathbf{u}, q) = -\frac{\mathbb{E}[u'(\Sigma_0 + q\psi + A_t(\mathbf{u}, q))\psi|\mathcal{F}_t]}{\mathbb{E}[u'(\Sigma_0 + q\psi + A_t(\mathbf{u}, q))|\mathcal{F}_t]} \tag{4.11}$$

and

$$\partial_q^2 A_t(\mathbf{u}, q) = -\frac{\mathbb{E}[u''(\Sigma_0 + q\psi + A_t(\mathbf{u}, q))(\psi + \partial_q A_t(\mathbf{u}, q))^2|\mathcal{F}_t]}{\mathbb{E}[u'(\Sigma_0 + q\psi + A_t(\mathbf{u}, q))|\mathcal{F}_t]} \tag{4.12}$$

and both of them can be continuously extended to the boundary points $q = -p$ and $q = \bar{q}(\mathbf{u}, t)$.

Proof. The differentiability is a result of Lemma 1.11 together with the implicit function theorem. Implicit differentiation once and twice, respectively, in (1.23) yields the identities (4.11) and (4.12). By the dominated convergence theorem these identities can be continuously extended to the boundary points $q = -p$ and $q = \bar{q}(\mathbf{u}, t)$. Since $u' > 0$, it follows that $\partial_q A_t(\mathbf{u}, \cdot) < 0$ which implies that $A_t(\mathbf{u}, \cdot)$ is strictly decreasing. Finally, since $u'' < 0$, it follows that $\partial_q^2 A_t(\mathbf{u}, \cdot) > 0$ which implies that $A_t(\mathbf{u}, \cdot)$ is strictly convex. □

Lemma 4.7. *Let Assumptions 1.1, 1.2 and 1.3 hold. Let g be as in* (1.28)*. Then for any $\omega \in \Omega$, $\mathbf{u} \in \tilde{\mathcal{U}}$ and $t \in [0, T]$ the map*

$$g(\mathbf{u}, \cdot, t, \omega) : [-p, \bar{q}(\mathbf{u}, t, \omega)] \to [0, \sigma(1-a)\mathbf{u}], \quad q \mapsto g(\mathbf{u}, q, t, \omega)$$

is twice continuously differentiable on the interior of its domain. Furthermore, its derivatives can be extended continuously to the boundary points $-p$ and $\bar{q}(\mathbf{u}, t)$ where the first derivative takes the values

$$\partial_q g(\mathbf{u}, q, t)|_{q=-p} = \sigma\mathbb{E}[u'(A_t(\mathbf{u}, -p) + z)\psi|\mathcal{F}_t] = \sigma(A_t(\mathbf{u}, -p) + z)^{-a}\mathbb{E}[\psi|\mathcal{F}_t] \tag{4.13}$$

and

$$\partial_q g(\mathbf{u}, q, t)|_{q=\bar{q}(\mathbf{u},t)} = \sigma\mathbb{E}[u'((p+\bar{q}(\mathbf{u}, t))\psi)\psi|\mathcal{F}_t] = \sigma(p+\bar{q}(\mathbf{u}, t))^{-a}\mathbb{E}[\psi^{1-a}|\mathcal{F}_t]. \tag{4.14}$$

Moreover, for $0 < a < 1$, $g(\mathbf{u}, \cdot, t)$ *is strictly increasing.*

Proof. By definition

$$g(\mathbf{u}, q, t) = \tilde{g}(A_t(\mathbf{u}, q), q, t),$$

where $\tilde{g}$ and A_t are as in Lemma 1.18 and in (1.22), respectively. By Lemma 1.11 the function

$$\tilde{g}(x, q, t) = \sigma \mathbb{E}[u'(\Sigma_0 + q\psi + x)(p+q)\psi | \mathcal{F}_t]$$

is twice continuously differentiable in q. Moreover, by Lemma 4.6, $A_t(\mathbf{u}, q)$ is twice continuously differentiable in q which implies the differentiability assertions for $g(\mathbf{u}, \cdot, t)$. For notational convenience let

$$\eta(q) \triangleq (p+q)\psi + z + A_t(\mathbf{u}, q).$$

Then, using the fact that by Assumption 1.1 $xu''(x) = -au'(x)$,

$$\begin{aligned}
\frac{\partial_q g(\mathbf{u}, q, t)}{\sigma} &= \mathbb{E}[u''(\eta(q))(p+q)\psi(\psi + \partial_q A_t(\mathbf{u}, q)) + u'(\eta(q))\psi | \mathcal{F}_t] \\
= \quad & \mathbb{E}\left[u'(\eta(q)) \left\{ \frac{-a}{\eta(q)}(p+q)\psi(\psi + \partial_q A_t(\mathbf{u}, q)) + \psi \right\} \Big| \mathcal{F}_t \right] \\
= \quad & \mathbb{E}\left[u'(\eta(q)) \left\{ \frac{(1-a)(p+q)\psi + A_t(\mathbf{u}, q) + z - a(p+q)\partial_q A_t(\mathbf{u}, q)}{\eta(q)} \right\} \psi \Big| \mathcal{F}_t \right].
\end{aligned}$$

Since $A_t(\mathbf{u}, q) + z \geq 0$ and, by Lemma 4.6, $-\partial_q A_t(\mathbf{u}, q) > 0$ it follows that for $0 < a < 1$ the fraction in the last line is positive and hence $g(\mathbf{u}, \cdot, t)$ is strictly increasing. From a straightforward computation we obtain

$$\begin{aligned}
\frac{\partial_q^2 g(\mathbf{u}, q, t)}{\sigma} &= \mathbb{E}[u'''(\eta(q))(p+q)\psi(\psi + \partial_q A_t(\mathbf{u}, q))^2 \\
&\quad + u''(\eta(q))\psi(2\psi + \partial_q A_t(\mathbf{u}, q) + (p+q)\partial_q^2 A_t(\mathbf{u}, q)) | \mathcal{F}_t].
\end{aligned}$$

The dominated convergence theorem implies the existence of the desired continuous extensions of $\partial_q g$ and $\partial_q^2 g$ to the boundary points $q = -p$ and $q = \bar{q}(\mathbf{u}, t)$. □

The next lemma shows that the replicating position $Q_t^{\varepsilon H}$ for a small number of claims is differentiable with respect to ε and that, consequently, the Taylor approximation ansatz from the beginning of this section is justified.

Lemma 4.8. *Let Assumptions 1.1, 1.2 and 1.3 hold and let* $0 < a < 1$. *Let further* $H = f(\psi) \in \mathcal{H}''$ *and let* $(Q_t^{\varepsilon H})_{0 \leq t \leq T}$ *denote the replicating strategy for* εH, $\varepsilon \in [0, 1]$. *Then there exists a version of* $Q^{\varepsilon H}$ *such that for any* $(t, \omega) \in [0, T] \times \Omega$ *the map*

$$\varepsilon \mapsto Q_t^{\varepsilon H}(\omega)$$

is twice continuously differentiable. Furthermore, the first and second order derivatives can be continuously extended to the boundary points $\varepsilon = 0$ *and* $\varepsilon = 1$.

Proof. We saw in Proposition 3.5 that, at any time $t \in [0,T]$, $Q_t^{\varepsilon H}$ is the replicating position for εH if and only if $Q_t^{\varepsilon H}$ satisfies

$$i_t^{\varepsilon H} = g(U_t^{\varepsilon H}, Q_t^{\varepsilon H}, t),$$

where

$$i_t^{\varepsilon H} = \sigma \mathbb{E}[u'(\Sigma_0 - \varepsilon f(\psi) + v_0^\varepsilon)(p - \varepsilon f'(\psi))\psi | \mathcal{F}_t]$$

is as in (3.4). By Proposition 4.1 we know that the map $\varepsilon \mapsto v_0^{\varepsilon H}$ is twice continuously differentiable which, together with Lemma 1.11, implies that there exists a version of $i^{\varepsilon H}$ such that for any $(t,\omega) \in [0,T] \times \Omega$ the map $\varepsilon \mapsto i_t^{\varepsilon H}(\omega)$ is twice continuously differentiable.

By Lemma 4.7, for fixed $\mathbf{u} \in \tilde{\mathcal{U}}$ and $t \in [0,T]$, the map $q \mapsto g(\mathbf{u}, q, t)$ is twice continuously differentiable as well as strictly increasing and therefore 1-to-1. Hence, the implicit function theorem implies that the map $\varepsilon \mapsto Q_t^{\varepsilon H}$ is twice continuously differentiable on $[0,1]$. □

The next preliminary result ensures the differentiability with respect to ε of the complementing cash amount $X_t^{Q^{\varepsilon H}}$ for the replicating position $Q_t^{\varepsilon H}$.

Lemma 4.9. *Let Assumptions 1.1, 1.2 and 1.3 hold, let $0 < a < 1$, let $H \in \mathcal{H}''$ and let $(Q_t^{\varepsilon H})_{0 \le t \le T}$ be the replicating strategy for εH, $\varepsilon \in [0,1]$. Let $(X_t^{Q^{\varepsilon H}})_{0 \le t \le T}$ denote its complementing cash balance process. Let v_t^ε and x_t^ε as defined in (4.1) and (4.5) denote the indifference prices of εH and $\varepsilon \psi$, respectively. Then there exists a version of $X^{Q^{\varepsilon H}}$ such that for any $(t,\omega) \in [0,T] \times \Omega$ the map*

$$\varepsilon \mapsto X_t^{\varepsilon H}(\omega)$$

is twice continuously differentiable on $(0,1)$. Furthermore, its first and second order derivatives can be continuously extended to the boundary points $\varepsilon = 0$ and $\varepsilon = 1$. In $\varepsilon = 0$, they are given by

$$\partial_\varepsilon X_t^{Q^{\varepsilon H}}|_{\varepsilon=0} = -\left(\partial_\varepsilon v_t^\varepsilon|_{\varepsilon=0} - \partial_\varepsilon v_0^\varepsilon|_{\varepsilon=0}\right) + \partial_\varepsilon x_t^\varepsilon|_{\varepsilon=0} \partial_\varepsilon Q_t^{\varepsilon H}|_{\varepsilon=0}$$

and

$$\begin{aligned} \partial_\varepsilon^2 X_t^{Q^{\varepsilon H}}|_{\varepsilon=0} &= \frac{\mathbb{E}[u''(\Sigma_0)\{(\partial_\varepsilon v_0^\varepsilon|_{\varepsilon=0} - H)^2 - (\partial_\varepsilon Q_t^{\varepsilon H}|_{\varepsilon=0}\psi + \partial_\varepsilon X_t^{Q^{\varepsilon H}}|_{\varepsilon=0})^2\}|\mathcal{F}_t]}{\mathbb{E}[u'(\Sigma_0)|\mathcal{F}_t]} \\ &\quad + \frac{\mathbb{E}[u'(\Sigma_0)\{\partial_\varepsilon^2 v_0^\varepsilon|_{\varepsilon=0} - \partial_\varepsilon^2 Q_t^{\varepsilon H}|_{\varepsilon=0}\psi\}|\mathcal{F}_t]}{\mathbb{E}[u'(\Sigma_0)|\mathcal{F}_t]}. \end{aligned}$$

Proof. The complementing cash position $X^{Q^{\varepsilon H}}$ is defined implicitly, at any time $t \in [0,T]$, by

$$\mathbb{E}[u(\Sigma_0 + Q_t^{\varepsilon H}\psi + X_t^{Q^{\varepsilon H}})|\mathcal{F}_t] = \mathbb{E}[u(\Sigma_0 - \varepsilon H + v_0^\varepsilon)|\mathcal{F}_t]. \tag{4.15}$$

Since, by Lemma 4.8, there exists a version of $Q^{\varepsilon H}$ for which the map $\varepsilon \mapsto Q_t^{\varepsilon H}(\omega)$ is twice continuously differentiable on $(0,1)$ for all $(t,\omega) \in [0,T] \times \Omega$, the implicit function theorem together with Lemma 1.11 implies that there also exists a version of $X^{Q^{\varepsilon H}}$ such that the map $\varepsilon \mapsto X_t^{\varepsilon H}(\omega)$ is twice continuously differentiable on $(0,1)$ for all $(t,\omega) \in [0,T] \times \Omega$. Differentiating both sides of (4.15) with respect to ε yields

$$\begin{aligned} &\mathbb{E}[u'(\Sigma_0 + Q_t^{\varepsilon H}\psi + X_t^{Q^{\varepsilon H}})(\partial_\varepsilon Q_t^{\varepsilon H}\psi + \partial_\varepsilon X_t^{Q^{\varepsilon H}})|\mathcal{F}_t] \\ &= \mathbb{E}[u'(\Sigma_0 - \varepsilon H + v_0^\varepsilon)(\partial_\varepsilon v_0^\varepsilon - H)|\mathcal{F}_t]. \end{aligned} \tag{4.16}$$

Differentiating both sides once again with respect to ε yields

$$\begin{aligned} &\mathbb{E}[u''(\Sigma_0 + Q_t^{\varepsilon H}\psi + X_t^{Q^{\varepsilon H}})(\partial_\varepsilon Q_t^{\varepsilon H}\psi + \partial_\varepsilon X_t^{Q^{\varepsilon H}})^2 \\ &\quad + u'(\Sigma_0 + Q_t^{\varepsilon H}\psi + X_t^{Q^{\varepsilon H}})(\partial_\varepsilon^2 Q_t^{\varepsilon H}\psi + \partial_\varepsilon^2 X_t^{Q^{\varepsilon H}})|\mathcal{F}_t] \\ &= \mathbb{E}[u''(\Sigma_0 - \varepsilon H + v_0^\varepsilon)(\partial_\varepsilon v_0^\varepsilon - H)^2 + u'(\Sigma_0 - \varepsilon H + v_0^\varepsilon)\partial_\varepsilon^2 v_0^\varepsilon|\mathcal{F}_t]. \end{aligned} \tag{4.17}$$

After rearranging (4.16) for $\partial_\varepsilon X_t^{Q^{\varepsilon H}}$ we obtain

$$\begin{aligned} \partial_\varepsilon X_t^{Q^{\varepsilon H}} &= \frac{\mathbb{E}[u'(\Sigma_0 - \varepsilon H + v_0^\varepsilon)(\partial_\varepsilon v_0^\varepsilon - H)|\mathcal{F}_t]}{\mathbb{E}[u'(\Sigma_0 + Q_t^{\varepsilon H}\psi + X_t^{Q^{\varepsilon H}})|\mathcal{F}_t]} \\ &\quad - \frac{\mathbb{E}[u'(\Sigma_0 + Q_t^{\varepsilon H}\psi + X_t^{Q^{\varepsilon H}})\partial_\varepsilon Q_t^{\varepsilon H}\psi|\mathcal{F}_t]}{\mathbb{E}[u'(\Sigma_0 + Q_t^{\varepsilon H}\psi + X_t^{Q^{\varepsilon H}})|\mathcal{F}_t]} \end{aligned}$$

and, rearranging (4.17) for $\partial_\varepsilon^2 X_t^{Q^{\varepsilon H}}$, we obtain

$$\begin{aligned} \partial_\varepsilon^2 X_t^{Q^{\varepsilon H}} &= \frac{\mathbb{E}[u''(\Sigma_0 - \varepsilon H + v_0^\varepsilon)(\partial_\varepsilon v_0^\varepsilon - H)^2 + u'(\Sigma_0 - \varepsilon H + v_0^\varepsilon)\partial_\varepsilon^2 v_0^\varepsilon|\mathcal{F}_t]}{\mathbb{E}[u'(\Sigma_0 + Q_t^{\varepsilon H}\psi + X_t^{Q^{\varepsilon H}})|\mathcal{F}_t]} \\ &\quad - \frac{\mathbb{E}[u''(\Sigma_0 + Q_t^{\varepsilon H}\psi + X_t^{Q^{\varepsilon H}})(\partial_\varepsilon Q_t^{\varepsilon H}\psi + \partial_\varepsilon X_t^{Q^{\varepsilon H}})^2|\mathcal{F}_t]}{\mathbb{E}[u'(\Sigma_0 + Q_t^{\varepsilon H}\psi + X_t^{Q^{\varepsilon H}})|\mathcal{F}_t]} \\ &\quad - \frac{\mathbb{E}[u'(\Sigma_0 + Q_t^{\varepsilon H}\psi + X_t^{Q^{\varepsilon H}})\partial_\varepsilon^2 Q_t^{\varepsilon H}\psi|\mathcal{F}_t]}{\mathbb{E}[u'(\Sigma_0 + Q_t^{\varepsilon H}\psi + X_t^{Q^{\varepsilon H}})|\mathcal{F}_t]}. \end{aligned}$$

The dominated convergence theorem together with Lemma 4.8 implies that these identities can be extended continuously to the boundary points $\varepsilon = 0$ and $\varepsilon = 1$. Evaluating both terms at $\varepsilon = 0$ and using the representation of $\partial_\varepsilon v_t^\varepsilon|_{\varepsilon=0}$ which is given in Proposition 4.1 we obtain the desired forms for $\partial_\varepsilon X_t^{Q^{\varepsilon H}}|_{\varepsilon=0}$ and $\partial_\varepsilon^2 X_t^{Q^{\varepsilon H}}|_{\varepsilon=0}$. □

The last preliminary result shows that the denominator in Theorem 4.12 below is strictly positive.

Lemma 4.10. *Let Assumptions 1.1, 1.2 and 1.3 hold and let* $0 < a < 1$. *Then the function*

$$h : [0,T] \times (0,\infty) \to (0,\infty), \quad (t,s) \mapsto \mathbb{E}_{t,s}^{\mathbb{Q}}[\psi]$$

is continuously differentiable with respect to s and $\partial_s h(t,s) > 0$.

Proof. By Lemma 1.12 h is continuously differentiable in s and, after a straightforward computation, we obtain

$$s\partial_s h(t,s) = \frac{\mathbb{E}_{t,s}[u'(\Sigma_0)\psi]}{\mathbb{E}_{t,s}[u'(\Sigma_0)]} - ap\left(\frac{\mathbb{E}_{t,s}[u''(\Sigma_0)\psi^2]}{\mathbb{E}_{t,s}[u'(\Sigma_0)]} - \frac{\mathbb{E}_{t,s}[u'(\Sigma_0)\psi]\mathbb{E}_{t,s}[u''(\Sigma_0)\psi]}{\mathbb{E}_{t,s}[u'(\Sigma_0)]^2}\right).$$

Using the fact that, by Assumption 1.1, $u''(x) = -au'(x)/x$ for all $x > 0$, we rearrange these terms to obtain

$$s\partial_s h(t,s) = \frac{1}{\mathbb{E}_{t,s}[u'(\Sigma_0)]}\mathbb{E}_{t,s}\left[u'(\Sigma_0)\psi\left\{\frac{(1-a)p\psi + z + ap\mathbb{E}^{\mathbb{Q}}_{t,s}[\psi]}{p\psi + z}\right\}\right].$$

As $0 < a < 1$ and since $\mathbb{E}^{\mathbb{Q}}_{t,s}[\psi] > 0$ this implies that $\partial_s h(t,s) > 0$. □

Remark 4.11. In order to show that $\partial_q g > 0$ in Lemma 4.7 and $\partial_s h > 0$ in Lemma 4.10, respectively, we assumed that the market maker's risk aversion parameter a satisfies $0 < a < 1$. This is due to the fact that, as can be seen by following straightforward computations, both these derivatives are strictly positive if and only if $k_1'(p) > 0$ for all $p \geq 0$, where the map $k_1 : [0,\infty) \to \mathbb{R}$ is defined by

$$k_1(p) \triangleq p\frac{\mathbb{E}[(p\psi+1)^{-a}\psi]}{\mathbb{E}[(p\psi+1)^{-a}]}.$$

While for $0 < a < 1$ this is easy to see, this seemingly simple assertion is surprisingly elusive for $a > 1$. Even though numerical experiments suggest its correctness, we have not yet found an analytical proof. The expansions of Theorem 4.12 below as well as the ensuing results related to these expansions are therefore stated only for the case $0 < a < 1$. The extendability of all these results to the regime $a > 1$ is conditional on the validity of this assertion: Once it is proven, all the current proofs for $0 < a < 1$ hold verbatim for the case $a > 1$. We state this missing link as Conjecture 1 below. The proof of this conjecture and, consequently, the analytic vindication of the expansions of Theorem 4.12 for the regime $a > 1$ is highly desirable.

Conjecture 1. *Let ψ be a lognormally distributed random variable and let $a > 1$. Consider the function $k_1 : [0,\infty) \to \mathbb{R}$ defined by*

$$k_1(p) \triangleq p\frac{\mathbb{E}[(p\psi+1)^{-a}\psi]}{\mathbb{E}[(p\psi+1)^{-a}]}.$$

Then

$$k_1'(p) > 0 \quad \forall p \geq 0. \tag{4.18}$$

We are now ready to state and prove the following theorem which constitutes one of the main results of this chapter. It provides some insight into the replication of claims in our illiquid market by giving the first and second order approximations to the replicating strategy $Q^{\varepsilon H}$ for a small position εH of claims.

While in a liquid model perfect (delta-) replication of a claim is achieved by dynamically rebalancing ones portfolio so that at any time ones position in the risky asset is equal to the delta of the replicated claim, in our setting it becomes necessary to additionally account for liquidity effects. Theorem 4.12 identifies $\tilde{\Delta}$ and $\tilde{\Lambda}$ in the Taylor approximation

$$Q_t^{\varepsilon H} = \varepsilon \tilde{\Delta}_t + \frac{\varepsilon^2}{2} \tilde{\Lambda}_t + o(\varepsilon^2).$$

The first order term $\tilde{\Delta}$ is a *liquid hedging strategy* (in a sense which is made precise by Proposition 4.13) while $\tilde{\Lambda}$ can be viewed as the *liquidity correction to the replicating position* that arises in our illiquid market.

We find that we can express both $\tilde{\Delta}$ and $\tilde{\Lambda}$, and thus the approximation of the replicating position in our (nonlinear) illiquid market, in terms of objects pertaining to a (linear) liquid market, namely one in which assets are valued according to the marginal indifference pricing measure $\mathbb{Q}$ and where the measure $\mathbb{Q}$ plays the role of the equivalent martingale measure.

In this sense, Theorem 4.12 provides a *liquid hedging recipe for an illiquid world*: It specifies how (approximate) replication is achieved in an illiquid market by decomposing the replicating strategy into a portfolio of objects whose liquid replication will "do the trick".

Theorem 4.12. *Let Assumptions 1.1, 1.2 and 1.3 hold, let $0 < a < 1$ and let $H \in \mathcal{H}''$. Let further $(Q_t^{\varepsilon H})_{0 \le t \le T}$ denote the replicating strategy for the attainable claim εH and let*

$$\begin{aligned} \hat{H} &\triangleq H - \mathbb{E}^{\mathbb{Q}}[H], \\ \hat{\psi} &\triangleq \psi - \mathbb{E}^{\mathbb{Q}}[\psi]. \end{aligned}$$

Then

(i) the first order approximation to $Q^{\varepsilon H}$ is given by

$$\partial_\varepsilon Q_t^{\varepsilon H}|_{\varepsilon=0} = -\frac{\partial_s \mathbb{E}_{t,s}^{\mathbb{Q}}[H]|_{s=S_t}}{\partial_s \mathbb{E}_{t,s}^{\mathbb{Q}}[\psi]|_{s=S_t}} =: \tilde{\Delta}_t, \quad 0 \le t \le T, \tag{4.19}$$

(ii) and the second order approximation to $Q^{\varepsilon H}$ is given by

$$\partial_\varepsilon^2 Q_t^{\varepsilon H}|_{\varepsilon=0} = \frac{\sum_{i=1}^4 c_i(t, S_t) \partial_s \mathbb{E}_{t,s}^{\mathbb{Q}}[R(\Sigma_0) K_i]|_{s=S_t}}{\partial_s \mathbb{E}_{t,s}^{\mathbb{Q}}[\psi]|_{s=S_t}} =: \tilde{\Lambda}_t, \quad 0 \le t \le T, \tag{4.20}$$

where the random variables K_i, $i \in \{1, \ldots, 4\}$ are given by

$$\begin{aligned} K_1 &= \hat{H}^2, \\ K_2 &= \hat{\psi}^2, \\ K_3 &= \hat{\psi}, \\ K_4 &= 1 \end{aligned}$$

and the coefficients c_i, $i \in \{1, \ldots, 4\}$ are given by

$$\begin{aligned} c_1(t, S_t) &\equiv -1, \\ c_2(t, S_t) &= \tilde{\Delta}_t^2, \\ c_3(t, S_t) &= -2\tilde{\Delta}_t(\tilde{\Delta}_t \mathbb{E}_t^{\mathbb{Q}}[\hat{\psi}] + \mathbb{E}_t^{\mathbb{Q}}[\hat{H}]), \\ c_4(t, S_t) &= (\tilde{\Delta}_t \mathbb{E}_t^{\mathbb{Q}}[\hat{\psi}] + \mathbb{E}_t^{\mathbb{Q}}[\hat{H}])^2. \end{aligned}$$

The first order term $\tilde{\Delta}$ reminds us in its appearance of the Black-Scholes delta as it is the sensitivity with respect to changes in the underlying of the expected payoff of the claim H under the probability measure $\mathbb{Q}$. Proposition 4.13 below will show that it can be interpreted as the hedging strategy of a small investor who is trading in our illiquid market.

The second order term $\tilde{\Lambda}$, which can be viewed as the liquidity correction to the replicating position, can be expressed as a linear combination of hedge ratios (i.e. sensitivities with respect to changes in the underlying of the risk-aversion-corrected expected payoffs) of certain claims K_i which are given as functions of both the underlying ψ and the claim H. Later on, in Chapter 5, we provide comparative statics which give an intuition of the shape, magnitude and term structure of $\tilde{\Lambda}$.

Proof. Let $(U_t^{\varepsilon H})_{0 \le t \le T}$ as in (3.2) denote the utility process of the market maker when selling εH at time zero and let $(U_t^{Q^{\varepsilon H}})_{0 \le t \le T}$ denote his utility processes when the large investor is pursuing the replicating strategy $Q^{\varepsilon H}$. Since $Q^{\varepsilon H}$ is a replicating strategy we know by Proposition 3.3 that

$$U_t^{\varepsilon H} = U_t^{Q^{\varepsilon H}} \quad \forall t \in [0, T]. \tag{4.21}$$

By (1.27) and Remark 3.4 the dynamics of these processes are given by

$$U_t^{\varepsilon H} = U_0^{\varepsilon H} + \int_0^t i_s^{\varepsilon H} dW_s \tag{4.22}$$

and

$$U_t^{Q^{\varepsilon H}} = U_0^{Q^{\varepsilon H}} + \int_0^t g(U_s^{Q^{\varepsilon H}}, Q_s^{\varepsilon H}, s) dW_s. \tag{4.23}$$

Since

$$U_0^{\varepsilon H} = U_0^{Q^{\varepsilon H}},$$

it follows from (4.21), (4.22) and (4.23) that $\mathbb{P}\otimes dt$-almost everywhere

$$i_t^{\varepsilon H} = g(U_t^{Q^{\varepsilon H}}, Q_t^{\varepsilon H}, t). \tag{4.24}$$

By Proposition 4.1 and Lemmas 4.8 and 4.9 the functions $\varepsilon \mapsto v_0^\varepsilon$, $\varepsilon \mapsto Q_t^{\varepsilon H}$ and $\varepsilon \mapsto X_t^{Q^{\varepsilon H}}$ are twice continuously differentiable with respect to ε on $(0,1)$ and their derivatives can be continuously extended to the boundary point $\varepsilon = 0$.

In order to compute the first and second order derivatives of the left hand side and right hand side of (4.24) with respect to ε, we introduce the auxiliary notation

$$\begin{aligned}
\tilde\eta_0 &\triangleq v_0^\varepsilon,\\
\tilde\eta_1 &\triangleq \partial_\varepsilon v_0^\varepsilon,\\
\tilde\eta_2 &\triangleq \partial_\varepsilon^2 v_0^\varepsilon,\\
\tilde\delta_0 &\triangleq (Q_t^{\varepsilon H}, X_t^{Q^{\varepsilon H}}),\\
\tilde\delta_1 &\triangleq (\partial_\varepsilon Q_t^{\varepsilon H}, \partial_\varepsilon X_t^{Q^{\varepsilon H}}),\\
\tilde\delta_2 &\triangleq (\partial_\varepsilon^2 Q_t^{\varepsilon H}, \partial_\varepsilon^2 X_t^{Q^{\varepsilon H}}).
\end{aligned}$$

Furthermore, we let

$$\begin{aligned}
\eta_i &\triangleq \tilde\eta_i|_{\varepsilon=0},\\
\delta_i &\triangleq \tilde\delta_i|_{\varepsilon=0}
\end{aligned}$$

for $i \in \{1,2\}$. Note that $\tilde\eta_0|_{\varepsilon=0} = 0$ and $\tilde\delta_0|_{\varepsilon=0} = (0,0)$. The first order derivatives are then given by

$$\partial_\varepsilon i_t^{\varepsilon H}(\omega) = \sigma S_t \partial_s \mathbb{E}_{t,s}\big[u'(\Sigma_0 - \varepsilon H + v)(v' - H)\big]|_{(s,v,v')=(S_t,\tilde\eta_0,\tilde\eta_1)} \tag{4.25}$$

and

$$\begin{aligned}
&\partial_\varepsilon g(U_t(Q^{\varepsilon H}), Q_t^{\varepsilon H}, t, \omega) \\
&= \sigma S_t \partial_s \mathbb{E}_{t,s}\big[u'(\Sigma_0 + q\psi + x)(q'\psi + x')\big]|_{(s,(q,x),(q',x'))=(S_t,\tilde\delta_0,\tilde\delta_1)}.
\end{aligned} \tag{4.26}$$

The second order derivatives are given by

$$\begin{aligned}
&\partial_\varepsilon^2 i_t^{\varepsilon H}(\omega) \\
&= \sigma S_t \partial_s \mathbb{E}_{t,s}\big[u''(\Sigma_0 - \varepsilon H + v)(v' - H)^2 \\
&\quad + u'(\Sigma_0 - \varepsilon H + v)v''\big]|_{(s,v,v',v'')=(S_t,\tilde\eta_0,\tilde\eta_1,\tilde\eta_2)}
\end{aligned} \tag{4.27}$$

and

$$
\begin{aligned}
&\partial_\varepsilon^2 g(U_t(Q^{\varepsilon H}), Q_t^{\varepsilon H}, t, \omega) \\
&= \sigma S_t \partial_s \mathbb{E}_{t,s}\big[u''(\Sigma_0 + q\psi + x)(q'\psi + x')^2 \\
&\quad + u'(\Sigma_0 + q\psi + x)(q''\psi + x'')\big]|_{(s,(q,x),(q',x'),(q'',x''))=(S_t,\tilde{\delta}_0,\tilde{\delta}_1,\tilde{\delta}_2)}.
\end{aligned} \tag{4.28}
$$

Let us now prove (i). We differentiate both sides of (4.24) with respect to ε and evaluate in $\varepsilon = 0$. Using (4.25) and (4.26) this yields

$$
\begin{aligned}
&\partial_s \mathbb{E}_{t,s}[u'(\Sigma_0)(v' - H)]|_{(s,v')=(S_t,\eta_1)} \\
&= \partial_s \mathbb{E}_{t,s}[u'(\Sigma_0)(q'\psi + x')]|_{(s,(q',x'))=(S_t,\delta_1)}.
\end{aligned} \tag{4.29}
$$

From Lemma 4.9 we know that

$$
\partial_\varepsilon X_t^{Q^{\varepsilon H}}|_{\varepsilon=0} = (\partial_\varepsilon v_0^\varepsilon|_{\varepsilon=0} - \partial_\varepsilon v_t^\varepsilon|_{\varepsilon=0}) + \partial_\varepsilon x_t^\varepsilon|_{\varepsilon=0}\partial_\varepsilon Q_t^{\varepsilon H}|_{\varepsilon=0},
$$

which we use to solve (4.29) for $q' = \partial_\varepsilon Q_t^{\varepsilon H}|_{\varepsilon=0}$. This leaves us with

$$
\partial_\varepsilon Q_t^{\varepsilon H}|_{\varepsilon=0} = \frac{\partial_s \mathbb{E}_{t,s}[u'(\Sigma_0)]|_{s=S_t}\partial_\varepsilon v_t^\varepsilon|_{\varepsilon=0} - \partial_s \mathbb{E}_{t,s}[u'(\Sigma_0)H]|_{s=S_t}}{\partial_s \mathbb{E}_{t,s}[u'(\Sigma_0)]|_{s=S_t}\partial_\varepsilon x_t^\varepsilon|_{\varepsilon=0} + \partial_s \mathbb{E}_{t,s}[u'(\Sigma_0)\psi]|_{s=S_t}}. \tag{4.30}
$$

Recall that, by definition of $\mathbb{Q}$, for a path independent contingent claim $K = k(\psi) \in \mathcal{H}'$ we have

$$
\mathbb{E}_{t,s}^{\mathbb{Q}}[K]|_{s=S_t} = \frac{\mathbb{E}_{t,s}[u'(\Sigma_0)K]}{\mathbb{E}_{t,s}[u'(\Sigma_0)]}|_{s=S_t}.
$$

Thus,

$$
\begin{aligned}
\partial_s \mathbb{E}_{t,s}^{\mathbb{Q}}[K]|_{s=S_t} &= \partial_s \left(\frac{\mathbb{E}_{t,s}[u'(\Sigma_0)K]}{\mathbb{E}_{t,s}[u'(\Sigma_0)]}\right)|_{s=S_t} \\
&= \frac{\partial_s \mathbb{E}_{t,s}[u'(\Sigma_0)K]\mathbb{E}_{t,s}[u'(\Sigma_0)] - \mathbb{E}_{t,s}[u'(\Sigma_0)K]\partial_s \mathbb{E}_{t,s}[u'(\Sigma_0)]}{\mathbb{E}_{t,s}[u'(\Sigma_0)]^2}|_{s=S_t}.
\end{aligned} \tag{4.31}
$$

Using this fact for $K = \psi$ and $K = H$ and recalling that, by Proposition 4.1 and (4.6),

$$
\partial_\varepsilon v_t^\varepsilon|_{\varepsilon=0} = \frac{\mathbb{E}_{t,s}[u'(\Sigma_0)H]}{\mathbb{E}_{t,s}[u'(\Sigma_0)]}|_{s=S_t}
$$

and

$$
\partial_\varepsilon x_t^\varepsilon|_{\varepsilon=0} = -\frac{\mathbb{E}_{t,s}[u'(\Sigma_0)\psi]}{\mathbb{E}_{t,s}[u'(\Sigma_0)]}|_{s=S_t},
$$

we obtain from (4.30) that

$$
\partial_\varepsilon Q_t^{\varepsilon H}|_{\varepsilon=0} = -\frac{\partial_s \mathbb{E}_{t,s}^{\mathbb{Q}}[H]|_{s=S_t}}{\partial_s \mathbb{E}_{t,s}^{\mathbb{Q}}[\psi]|_{s=S_t}}.
$$

We proceed to prove (ii). We differentiate both sides of (4.24) twice with respect to ε and evaluate in $\varepsilon = 0$. Using (4.27) and (4.28), this yields

$$\begin{aligned} & \partial_s \mathbb{E}_{t,s}[u''(\Sigma_0)(v'-H)^2 + u'(\Sigma_0)v'']|_{(s,v',v'')=(S_t,\eta_1,\eta_2)} \\ = \; & \partial_s \mathbb{E}_{t,s}[u''(\Sigma_0)(q'\psi + x')^2 + u'(\Sigma_0)(q''\psi + x'')]|_{(s,(q',x'),(q'',x''))=(S_t,\delta_1,\delta_2)}. \end{aligned} \tag{4.32}$$

From Lemma 4.9 we know that

$$\begin{aligned} \partial^2_\varepsilon X_t^{Q^{\varepsilon H}}|_{\varepsilon=0} \;=\; & \frac{\mathbb{E}[u''(\Sigma_0)\{(\partial_\varepsilon v_0^\varepsilon|_{\varepsilon=0} - H)^2 - (\partial_\varepsilon Q_t^{\varepsilon H}|_{\varepsilon=0}\psi + \partial_\varepsilon X_t^{Q^{\varepsilon H}}|_{\varepsilon=0})^2\}|\mathcal{F}_t]}{\mathbb{E}[u'(\Sigma_0)|\mathcal{F}_t]} \\ & + \frac{\mathbb{E}[u'(\Sigma_0)\{\partial^2_\varepsilon v_0^\varepsilon|_{\varepsilon=0} - \partial^2_\varepsilon Q_t^{\varepsilon H}|_{\varepsilon=0}\psi\}|\mathcal{F}_t]}{\mathbb{E}[u'(\Sigma_0)|\mathcal{F}_t]}, \end{aligned}$$

which we use to solve (4.32) for $q'' = \partial^2_\varepsilon Q_t^{\varepsilon H}|_{\varepsilon=0}$. This leaves us with

$$\begin{aligned} \partial^2_\varepsilon Q_t^{\varepsilon H}|_{\varepsilon=0} \;=\; & \Big(\partial_s \mathbb{E}_{t,s}[u'(\Sigma_0)\psi]\mathbb{E}_{t,s}[u'(\Sigma_0)] - \partial_s \mathbb{E}_{t,s}[u'(\Sigma_0)]\mathbb{E}_{t,s}[u'(\Sigma_0)\psi]\Big)^{-1}|_{s=S_t} \times \\ \times \; & \Big(\partial_s \mathbb{E}_{t,s}[u''(\Sigma_0)\{(v'-H)^2 - (q'\psi + x')^2\}]\mathbb{E}_{t,s}[u'(\Sigma_0)] \\ & - \mathbb{E}_{t,s}[u''(\Sigma_0)\{(v'-H)^2 - (q'\psi + x')^2\}]\partial_s \mathbb{E}_{t,s}[u'(\Sigma_0)]\Big)|_{(s,v',(q',x'))=(S_t,\eta_1,\delta_1)}. \end{aligned}$$

Considering $K = ((v'-H)^2 - (q'\psi + x')^2)\frac{u''(\Sigma_0)}{u'(\Sigma_0)}$ in (4.31) we can see that

$$\partial^2_\varepsilon Q_t^{\varepsilon H}|_{\varepsilon=0} = \frac{\partial_s \mathbb{E}^{\mathbb{Q}}_{t,s}[\{(v'-H)^2 - (q'\psi + x')^2\}\frac{u''(\Sigma_0)}{u'(\Sigma_0)}]|_{(s,v',(q',x'))=(S_t,\eta_1,\delta_1)}}{\partial_s \mathbb{E}^{\mathbb{Q}}_{t,s}[\psi]|_{s=S_t}}. \tag{4.33}$$

Note that

$$\frac{\partial_s \mathbb{E}^{\mathbb{Q}}_{t,s}[(v'-H)^2\frac{u''(\Sigma_0)}{u'(\Sigma_0)}]|_{(s,v')=(S_t,\eta_1)}}{\partial_s \mathbb{E}^{\mathbb{Q}}_{t,s}[\psi]|_{s=S_t}} = -\frac{\partial_s \mathbb{E}^{\mathbb{Q}}_{t,s}[R(\Sigma_0)\hat{H}^2]|_{s=S_t}}{\partial_s \mathbb{E}^{\mathbb{Q}}_{t,s}[\psi]|_{s=S_t}}.$$

Thus, it remains to show that

$$-\partial_s \mathbb{E}^{\mathbb{Q}}_{t,s}\left[(q'\psi + x')^2\frac{u''(\Sigma_0)}{u'(\Sigma_0)}\right]|_{(s,(q',x'))=(S_t,\delta_1)} = \sum_{i=2}^{4} c_i(t,S_t)\partial_s \mathbb{E}^{\mathbb{Q}}_{t,s}[R(\Sigma_0)K_i]|_{s=S_t}.$$

It is

$$
\begin{aligned}
& -\partial_s \mathbb{E}^{\mathbb{Q}}_{t,s}\left[(q'\psi + x')^2 \frac{u''(\Sigma_0)}{u'(\Sigma_0)}\right]|_{(s,(q',x'))=(S_t,\delta_1)} \\
&= -\partial_s \mathbb{E}^{\mathbb{Q}}_{t,s}\left[(q'\hat{\psi} + q'\mathbb{E}^{\mathbb{Q}}[\psi] + x')^2 \frac{u''(\Sigma_0)}{u'(\Sigma_0)}\right]|_{(s,(q',x'))=(S_t,\delta_1)} \\
&= \Big((q')^2 \partial_s \mathbb{E}^{\mathbb{Q}}_{t,s}[R(\Sigma_0)\hat{\psi}^2] + 2q'(x' + q'\mathbb{E}^{\mathbb{Q}}[\psi])\partial_s \mathbb{E}^{\mathbb{Q}}_{t,s}[R(\Sigma_0)\hat{\psi}] \\
&\quad + (x' + q'\mathbb{E}^{\mathbb{Q}}[\psi])^2 \partial_s \mathbb{E}^{\mathbb{Q}}_{t,s}[R(\Sigma_0)]\Big)|_{(s,(q',x'))=(S_t,\delta_1)}.
\end{aligned} \tag{4.34}
$$

By definition of Δ

$$\partial_\varepsilon Q^{\varepsilon H}_t|_{\varepsilon=0} = \Delta_t$$

and by Lemma 4.9 and Remark 4.3

$$
\begin{aligned}
\partial_\varepsilon X^{Q^{\varepsilon H}}_t|_{\varepsilon=0} &= (\partial_\varepsilon v^\varepsilon_0|_{\varepsilon=0} - \partial_\varepsilon v^\varepsilon_t|_{\varepsilon=0}) + \partial_\varepsilon x^\varepsilon_t|_{\varepsilon=0}\partial_\varepsilon Q^{\varepsilon H}_t|_{\varepsilon=0} \\
&= -\Delta_t \mathbb{E}^{\mathbb{Q}}_t[\psi] - \mathbb{E}^{\mathbb{Q}}_t[\hat{H}],
\end{aligned}
$$

so that

$$\partial_\varepsilon X^{Q^{\varepsilon H}}_t|_{\varepsilon=0} + \Delta_t \mathbb{E}^{\mathbb{Q}}[\psi] = -(\Delta_t \mathbb{E}^{\mathbb{Q}}_t[\hat{\psi}] + \mathbb{E}^{\mathbb{Q}}_t[\hat{H}]).$$

Substituting this term into (4.34) yields the desired result. □

Note that, with regard to Remark 4.4, we can alternatively write the second order term $\tilde{\Lambda}$ in the theorem above as

$$\tilde{\Lambda} = \frac{\sum_{i=1}^4 c_i(t,S_t)\partial_s(\mathbb{E}^{\mathbb{Q}}_{t,s}[R(\Sigma_0)]\mathbb{E}^{\mathbb{R}}_{t,s}[K_i])|_{s=S_t}}{\partial_s \mathbb{E}^{\mathbb{Q}}_{t,s}[\psi]|_{s=S_t}}. \tag{4.35}$$

The following proposition illustrates the relationship of the marginal price processes of a claim H and the underlying ψ. In particular, it shows that the first order approximation $\tilde{\Delta}$ to the replicating position for a large investor can be interpreted as the replicating strategy for a small investor trading in our illiquid financial market: An investor whose order sizes are negligible compared to the market maker's endowment will not move the market maker's price quotation with his trading activity. The prices that he observes and which he trades at are therefore the marginal prices of H and ψ.

Consider such a small investor who seeks to attain a terminal position $\hat{H} = H - \mathbb{E}^{\mathbb{Q}}[H]$ by dynamically trading the marketed security ψ. We can view $\hat{H}$ as the small investor's *replication target*. The proposition below shows that pursuing the trading strategy $\tilde{\Delta}$ replicates the marginal price process of H and, in particular, achieves the small investors replication target $\hat{H}$.

Proposition 4.13. *Let Assumptions 1.1, 1.2 and 1.3 hold and let* $0 < a < 1$. *Let* $H = f(\psi) \in \mathcal{H}''$ *and let* $\tilde{\Delta}$ *be as in Theorem 4.12. Let* $(\tilde{S}_t)_{0\le t\le T}$*, defined by*

$$\tilde{S}_t \triangleq -\mathbb{E}^{\mathbb{Q}}[\psi|\mathcal{F}_t]$$

and $(\pi_t)_{0\le t\le T}$*, defined by*

$$\pi_t \triangleq \mathbb{E}^{\mathbb{Q}}[H|\mathcal{F}_t]$$

denote the marginal price processes of underlying ψ *and claim* H *respectively. Then for all* $t \in [0,T]$

$$\pi_t = \pi_0 + \int_0^t \tilde{\Delta}_s d\tilde{S}_s.$$

In particular, for $t = T$*,*

$$H - \mathbb{E}^{\mathbb{Q}}[H] = \int_0^T \tilde{\Delta}_s d\tilde{S}_s.$$

Proof. Let

$$h_1(t,s) \triangleq -\frac{\mathbb{E}_{t,s}[u'(\Sigma_0)\psi]}{\mathbb{E}_{t,s}[u'(\Sigma_0)]}$$

and

$$h_2(t,s) \triangleq \frac{\mathbb{E}_{t,s}[u'(\Sigma_0)f(\psi)]}{\mathbb{E}_{t,s}[u'(\Sigma_0)]}.$$

By Remark 4.5 the process $(S_t)_{0\le t\le T}$ from Assumption 1.2, defining ψ via $\psi = S_T$, is Markov under $\mathbb{Q}$. Hence,

$$h_1(t,S_t) = -\frac{\mathbb{E}_{t,s}[u'(\Sigma_0)\psi]}{\mathbb{E}_{t,s}[u'(\Sigma_0)]}|_{s=S_t} = -\mathbb{E}^{\mathbb{Q}}[\psi|\mathcal{F}_t] = \tilde{S}_t$$

and

$$h_2(t,S_t) = \frac{\mathbb{E}_{t,s}[u'(\Sigma_0)H]}{\mathbb{E}_{t,s}[u'(\Sigma_0)]}|_{s=S_t} = \mathbb{E}^{\mathbb{Q}}[H|\mathcal{F}_t] = \pi_t.$$

Since $H \in \mathcal{H}''$, Lemma 1.12 implies that h_1 and h_2 are both differentiable with respect to s. The processes $\tilde{S}$ and π are $\mathbb{Q}$-martingales and their respective dynamics can be expressed via Itô's formula by

$$d\tilde{S}_t = \sigma S_t \partial_s h_1(t,S_t)dW_t = -\sigma\partial_s \mathbb{E}^{\mathbb{Q}}_{t,s}[\psi]|_{s=S_t}dW_t \tag{4.36}$$

and

$$d\pi_t = \sigma S_t \partial_s h_2(t,S_t)dW_t = \sigma\partial_s \mathbb{E}^{\mathbb{Q}}_{t,s}[H]|_{s=S_t}dW_t. \tag{4.37}$$

Rearranging (4.36) yields

$$dW_t = \frac{d\tilde{S}_t}{\sigma S_t \partial_s h_1(t,S_t)}$$

which, substituted into (4.37), leaves us with

$$d\pi_t = \frac{\partial_s h_2(t,S_t)}{\partial_s h_1(t,S_t)}d\tilde{S}_t = \tilde{\Delta}_t d\tilde{S}_t.$$

□

The following proposition provides the complementing cash balance process for the small investor's replicating strategy $\tilde{\Delta}$.

Proposition 4.14. *Let Assumptions 1.1, 1.2 and 1.3 hold and let* $0 < a < 1$. *Let* $H = f(\psi) \in \mathcal{H}''$, *let the trading strategy* $(\tilde{\Delta}_t)_{0\le t\le T}$ *be as in Theorem 4.12 and let* $(X_t^{\tilde{\Delta}})_{0\le t\le T}$ *denote its complementing cash balance process. Then*

$$X_t^{\tilde{\Delta}} = -\mathbb{E}^{\mathbb{Q}}[\hat{H}|\mathcal{F}_t] - \tilde{\Delta}_t \mathbb{E}^{\mathbb{Q}}[\psi|\mathcal{F}_t], \quad 0 \le t \le T.$$

Proof. By Lemma 4.9, the complementing cash balance process for the trading strategy $\tilde{\Delta} = \partial_\varepsilon Q^{\varepsilon H}|_{\varepsilon=0}$, at any time $t \in [0,T]$, is given by

$$\partial_\varepsilon X_t^{Q^{\varepsilon H}}|_{\varepsilon=0} = -(\partial_\varepsilon v_t^\varepsilon|_{\varepsilon=0} - \partial_\varepsilon v_0^\varepsilon|_{\varepsilon=0}) + \partial_\varepsilon x_t^\varepsilon|_{\varepsilon=0}\partial_\varepsilon Q_t^{\varepsilon H}|_{\varepsilon=0}.$$

The result follows with the representations $\partial_\varepsilon v_t^\varepsilon|_{\varepsilon=0} = \mathbb{E}^{\mathbb{Q}}[H|\mathcal{F}_t]$ and $\partial_\varepsilon x_t^\varepsilon|_{\varepsilon=0} = -\mathbb{E}^{\mathbb{Q}}[\psi|\mathcal{F}_t]$ from Remark 4.3. □

The interpretation of $\tilde{\Delta}$ as a small investor's replicating strategy allows for an alternative interpretation of $\tilde{\Lambda}$, the second order term in Theorem 4.12. In the notation of that theorem and of Proposition 4.14 let

$$\xi(t) \triangleq \tilde{\Delta}_t\psi + X_t^{\tilde{\Delta}} \in \mathbf{L}^0(\mathcal{F}_T, \mathbb{R}).$$

The quantity $\xi(t)$ can be viewed as a small investor's net replicating position for H, composed of the stock position $\tilde{\Delta}_t\psi$ and its complementing cash amount $X_t^{\tilde{\Delta}}$. Then

$$\tilde{\Lambda}_t = \frac{\frac{\partial}{\partial s}\left\{-\mathbb{E}_{t,s}^{\mathbb{Q}}[R(\Sigma_0)]\left(\mathbb{E}_{t,s}^{\mathbb{R}}[\hat{H}^2] - \mathbb{E}_{t,s}^{\mathbb{R}}[(q'\psi + x')^2]\right)\right\}\Big|_{s=S_t, q'=\tilde{\Delta}_t, x'=X_t^{\tilde{\Delta}}}}{\partial_s \mathbb{E}_{t,s}^{\mathbb{Q}}[\psi]|_{s=S_t}},$$

which we write, in a slight abuse of notation, more concisely as

$$\tilde{\Lambda}_t = \frac{\frac{\partial}{\partial s}\left\{-\mathbb{E}_{t,s}^{\mathbb{Q}}[R(\Sigma_0)]\left(\mathbb{E}_{t,s}^{\mathbb{R}}[\hat{H}^2] - \mathbb{E}_{t,s}^{\mathbb{R}}[\xi^2(t)]\right)\right\}|_{s=S_t}}{\partial_s \mathbb{E}_{t,s}^{\mathbb{Q}}[\psi]|_{s=S_t}}. \tag{4.38}$$

This form for the second order term $\tilde{\Lambda}$ is obtained immediately from (4.33). It suggests the following heuristic interpretation: $\tilde{\Lambda}$ is the sensitivity with respect to changes in the underlying of the difference of the second moments of the net replicating position and the replication target. In other words, $\tilde{\Lambda}$ measures how changes in the underlying ψ influence the difference of the second moments of the target position and the net replicating position.

The following remark shows why – for power utilities – it is difficult to make this interpretation more rigorous by formulating it in terms of the variances of $\hat{H}$ and $\xi(t)$.

Remark 4.15. By definition of the variance and the measures $\mathbb{Q}$ and $\mathbb{R}$, for all $t \in [0,T]$, $s > 0$,

$$\mathrm{Var}_{t,s}^{\mathbb{R}}[\hat{H}] - \mathrm{Var}_{t,s}^{\mathbb{R}}[\xi(t)] = \mathbb{E}_{t,s}^{\mathbb{R}}[\hat{H}^2] - \mathbb{E}_{t,s}^{\mathbb{R}}[\xi^2(t)] - \left(\mathbb{E}_{t,s}^{\mathbb{R}}[\hat{H}]^2 - \mathbb{E}_{t,s}^{\mathbb{R}}[\xi(t)]^2\right).$$

In view of Remark 4.4, this equation can be rewritten as

$$\mathbb{E}^{\mathbb{R}}_{t,s}[\hat{H}^2] - \mathbb{E}^{\mathbb{R}}_{t,s}[\xi^2(t)] = \mathrm{Var}^{\mathbb{R}}_{t,s}[\hat{H}] - \mathrm{Var}^{\mathbb{R}}_{t,s}[\xi(t)] + \frac{\mathbb{E}^{\mathbb{Q}}_{t,s}[R(\Sigma_0)\hat{H}]^2 - \mathbb{E}^{\mathbb{Q}}_{t,s}[R(\Sigma_0)\xi(t)]^2}{\mathbb{E}^{\mathbb{Q}}_{t,s}[R(\Sigma_0)]^2}.$$

Moreover, by definition of ξ and by Proposition 4.14, we have $\xi(t) = \tilde{\Delta}_t\psi - (\tilde{\Delta}_t E^{\mathbb{Q}}_t[\psi] + \mathbb{E}^{\mathbb{Q}}_t[\hat{H}])$ and, consequently, taking the $\mathcal{F}_t$-conditional expectation with respect to $\mathbb{Q}$ on both sides,

$$\mathbb{E}^{\mathbb{Q}}_t[\xi(t)] = -\mathbb{E}^{\mathbb{Q}}_t[\hat{H}]. \tag{4.39}$$

We will see in Chapter 2 that, formally, the same calculations hold for exponential utility functions where the market maker's absolute risk aversion $R(\Sigma_0)$ is constant. In that case, due to (4.39), the term

$$\mathbb{E}^{\mathbb{Q}}_{t,s}[R(\Sigma_0)\hat{H}]^2 - \mathbb{E}^{\mathbb{Q}}_{t,s}[R(\Sigma_0)\xi(t)]^2$$

in the expression above vanishes and the difference of the second moments of $\hat{H}$ and $\xi(t)$ in (4.38) coincides with the difference of their variances under $\mathbb{R}$.

4.3 The influence of market depth on the approximations to the hedging position

In this section we will consider initial endowments of the form

$$\Sigma_0(d) \triangleq d(p\psi + z), \quad d > 0,$$

where $\Sigma_0 = p\psi + z$ satisfies Assumption 1.3. Clearly, $\Sigma_0(d)$ also satisfies Assumption 1.3. The scaling parameter d determines the size of orders which can be accommodated by the market maker and we will hence refer to it as the *market depth*. The following proposition describes the relationship between the market depth and the first and second order approximations to the hedging position from Theorem 4.12.

We observe that the first order approximation, corresponding to the replicating position of a small investor (as highlighted in Proposition 4.13), does not depend on the market depth, while the second order approximation, which is the liquidity correction to the large investor's replicating position, is inversely proportional to d.

Proposition 4.16. *Let Assumptions 1.1 and 1.2 hold and let* $0 < a < 1$*. Let further* $\tilde{\Delta}_t(d)$ *and* $\tilde{\Lambda}_t(d)$ *be the quantities in* (4.19) *and* (4.20)*, computed with respect to the initial endowment* $\Sigma_0(d) = d(p\psi + z)$*,* $d > 0$*. Then, at any time* $t \in [0, T]$*,*

$$\tilde{\Delta}_t(d) = \tilde{\Delta}_t(1) \tag{4.40}$$

and

$$\tilde{\Lambda}_t(d) = \frac{1}{d}\tilde{\Lambda}_t(1). \tag{4.41}$$

Proof. Let $\mathbb{Q}(d)$ denote the probability measure defined by

$$\frac{d\mathbb{Q}(d)}{d\mathbb{P}} \triangleq \frac{u'(d\Sigma_0)}{\mathbb{E}[u'(d\Sigma_0)]},$$

By Assumption 1.1 we have

$$\frac{d\mathbb{Q}(d)}{d\mathbb{P}} = \frac{u'(\Sigma_0(d))}{\mathbb{E}[u'(\Sigma_0(d))]} = \frac{u'(\Sigma_0)}{\mathbb{E}[u'(\Sigma_0)]} = \frac{d\mathbb{Q}(1)}{d\mathbb{P}}, \tag{4.42}$$

so that for all $d > 0$ the measures $\mathbb{Q}(d)$ and $\mathbb{Q} = \mathbb{Q}(1)$ coincide. Hence,

$$\tilde{\Delta}_t(d) = -\frac{\partial_s \mathbb{E}_{t,s}^{\mathbb{Q}(d)}[H]|_{s=S_t}}{\partial_s \mathbb{E}_{t,s}^{\mathbb{Q}(d)}[\psi]|_{s=S_t}} = -\frac{\partial_s \mathbb{E}_{t,s}^{\mathbb{Q}(1)}[H]|_{s=S_t}}{\partial_s \mathbb{E}_{t,s}^{\mathbb{Q}(1)}[\psi]|_{s=S_t}} = \tilde{\Delta}_t(1)$$

which proves (4.40). By Assumption 1.1,

$$R(\Sigma_0(d)) = \frac{a}{d(p\psi + z)} = \frac{1}{d} R(\Sigma_0)$$

and, by (4.40) and (4.42), the coefficients $c_i(t, S_t)$ in (4.20) do not depend on d. Using these observations together with (4.42) we deduce that

$$\tilde{\Lambda}_t(d) = \frac{1}{d} \frac{\sum_{i=1}^4 c_i(t, S_t) \partial_s \mathbb{E}_{t,s}^{\mathbb{Q}(1)}[R(\Sigma_0) K_i]|_{s=S_t}}{\partial_s \mathbb{E}_{t,s}^{\mathbb{Q}(1)}[\psi]|_{s=S_t}} = \frac{1}{d} \tilde{\Lambda}_t(1).$$

□

By the same reasoning by which we interpret d as the market depth, we can interpret its inverse

$$\rho \triangleq \frac{1}{d}$$

as the *relative size of the large investor* in comparison to the market maker. For $d \to \infty$ the size of the large investor ρ vanishes in comparison to the market maker and, in the limit, the large investor becomes the small investor from Proposition 4.13. Proposition 4.16 implies that

$$\frac{\tilde{\Lambda}_t(d)}{\tilde{\Delta}_t} = \frac{1}{d} \frac{\tilde{\Lambda}_t(1)}{\tilde{\Delta}_t} = \rho \frac{\tilde{\Lambda}_t(1)}{\tilde{\Delta}_t}.$$

Hence, depending on the market depth d (respectively on the size of the large investor ρ), the relative size of the liquidity correction $\tilde{\Lambda}$ in relation to the small investor's replicating position $\tilde{\Delta}$ can be of any order of magnitude. In other words, the liquidity correction $\tilde{\Lambda}$ for the hedge ratio can become vast if the large investor is large and it can become negligible if the large investor is small.

Chapter 5

Comparative statics for hedge ratio corrections

In the asymptotic analysis of the previous chapter we presented the approximation

$$Q_t^{\varepsilon H} = \varepsilon \tilde{\Delta}_t + \frac{\varepsilon^2}{2} \tilde{\Lambda}_t + o(\varepsilon^2)$$

to an investor's replicating position $Q_t^{\varepsilon H}$ for a claim εH. Recall that by our convention positions in both the marketed security and in cash are denoted from the point of view of the market maker. Throughout this chapter it is convenient to consider the quantities

$$\Delta \triangleq -\tilde{\Delta} \quad \text{and} \quad \Lambda \triangleq -\tilde{\Lambda}$$

which shift the point of view to that of the large investor. Note further that all those numerical experiments in this chapter for which $a > 1$ are to be viewed conditional on the validity of Conjecture 1 in Chapter 4.

We saw that the first order term Δ can be interpreted as the replicating strategy for H which a small investor, who is trading at marginal prices, would pursue in our market. As the small investor's trading activity does not have a price impact, he finds himself in a "liquid world" and his hedge Δ can be viewed as "liquid replication". In Figure 5.1 we depict Δ_t for a call option $H = (\psi - K)_+$ for various times to maturity and we observe that, unsurprisingly, its shape and term structure resemble that of the delta in the classical Black-Scholes model.

Let us now consider a large investor whose trading activity has a price impact as described in our model. Such an agent will have to account for liquidity effects by adjusting his replicating position in accordance with the second order term Λ, the *liquidity correction for the hedge ratio.* The term Λ constitutes the difference between the "liquid replicating position" of a small investor and the "illiquid replicating position" of a large investor and can be seen as an indicator of the (il)liquidity of the market.

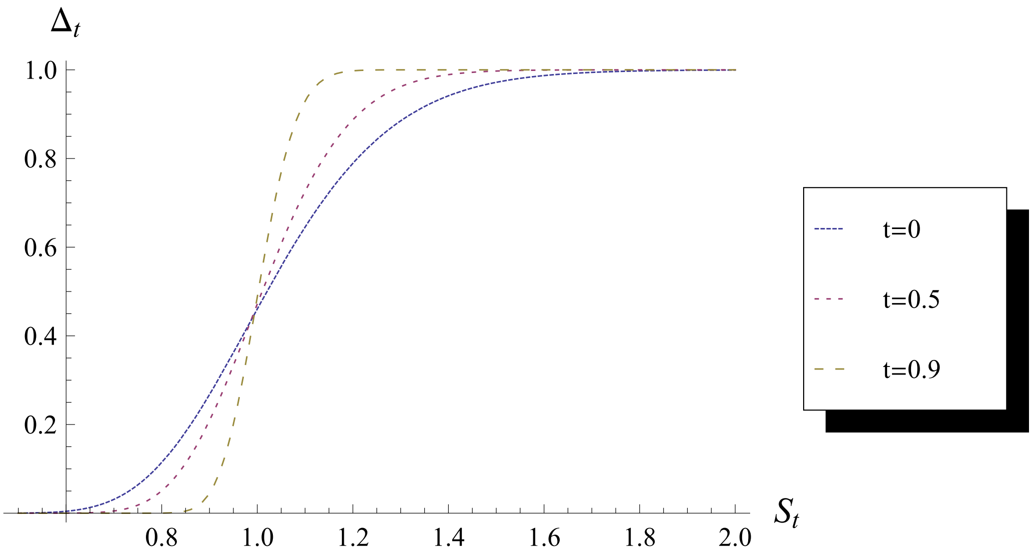

Figure 5.1: Term structure of Δ for a call option with strike $K = 1$; model parameters: $a = 2$, $p = 1$, $z = 1$, $\sigma = 0.2$, $\mu = 0$, $T = 1$

In what follows, we will present numerical results which give an idea of the shape and the order of magnitude of the liquidity correction Λ for a call option $H = (S_T - K)_+$ as well as of its dependence on changes in the parameters defining our model. We will investigate the term structure of Λ, i.e. its behaviour upon approaching maturity, as well as its reaction with respect to changes in the market maker's initial endowment p and z, his risk aversion a and the parameters μ and σ defining the price evolution of the marketed security.

5.1 Positivity, unimodality and term structure

Figure 5.2 displays the term Λ_t for a call option in dependence on S_t (from Assumption 1.2) for an exemplary non-degenerate set of parameters at various times $t \in [0, T]$. We observe that for the call option Λ is strictly positive. This means that, in order to account for liquidity effects, the large investor needs to assume a position in the underlying which is *strictly greater* than the replicating position Δ_t of the small investor. Moreover, we can see that Λ is *unimodal* with a unique global maximum near the money. This means that, as intuition suggests, the liquidity effect is stronger when prices are close to the money and weaker when prices are far away from the money.

Moreover, we can see that Λ decays as we approach maturity. This time evolution of the liquidity correction is, again, in line with what one would expect: As the remaining time – and therefore the degree of uncertainty – decreases, the liquidity correction term decreases in magnitude.

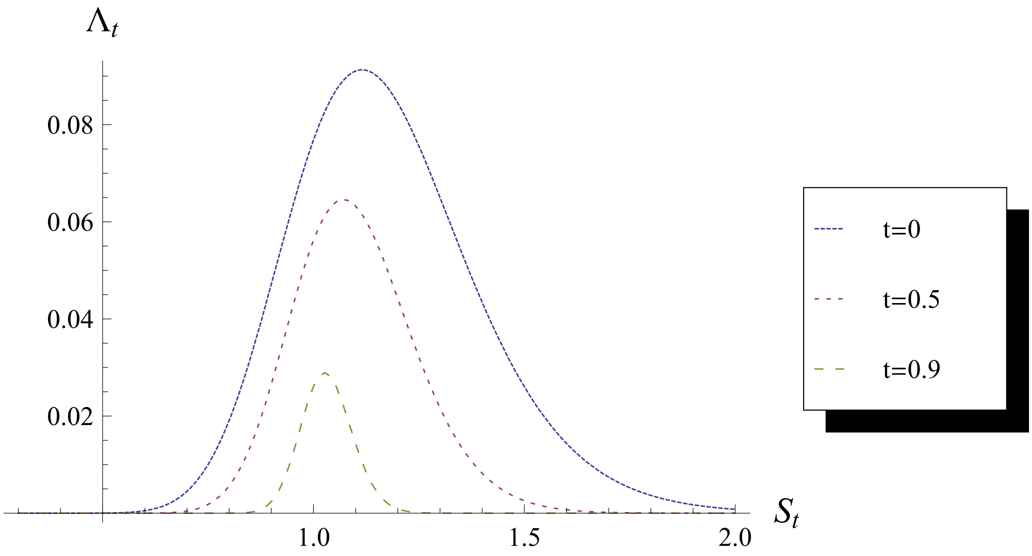

Figure 5.2: Term structure of Λ for a call option with strike $K = 1$; model parameters: $a = 2$, $p = 1$, $z = 1$, $\sigma = 0.2$, $\mu = 0$, $T = 1$

5.2 Variability with respect to the market maker's initial endowment and risk aversion

In order to highlight the relation between the market maker's initial endowment $\Sigma_0 = p\psi + z$ and the liquidity correction term Λ it is convenient to consider the reparametrisation of Σ_0 that is defined by $p = db$ and $z = d(1 - b)$, where $d > 0$ and $b \in [0, 1]$. Rearranging these relations for d and b yields

$$
\begin{aligned}
d &= p + z \in (0, \infty), \\
b &= 1 - \frac{z}{p + z} \in [0, 1].
\end{aligned}
$$

With respect to these coordinates, the initial endowment $\Sigma_0 = p\psi + z$ is given as

$$\Sigma_0 = d(b\psi + (1 - b)).$$

Here, $d > 0$ is the *market depth* from Section 4.3 and $b \in [0, 1]$ is the *portfolio balance parameter* which indicates the fraction of the portfolio that is invested in securities, while $1 - b$ corresponds to the fraction that is invested in cash.

As we have already understood the (inversely proportional) dependence of Λ on the market depth d in Proposition 4.16 it suffices to consider the case $d = 1$ and to investigate the variability of Λ with respect to b. Let us denote by $\Lambda(b)$ the liquidity correction term Λ computed with respect to the initial endowment $\Sigma_0(b) = b\psi + (1 - b)$. The dependence of $\Lambda(b)$ on b is visualised in Figure 5.3. We can see that the maximal liquidity correction $\max_{S_t > 0} \Lambda(b)$ attains a global minimum for some $b \in (0, 1)$ and that

$$\max_{S_t>0} \Lambda(0.01) > \max_{S_t>0} \Lambda(0.99),$$

which means that the liquidity correction is higher when the market maker possesses an initial endowment composed almost exclusively of cash than if he possesses one almost exclusively composed of shares.

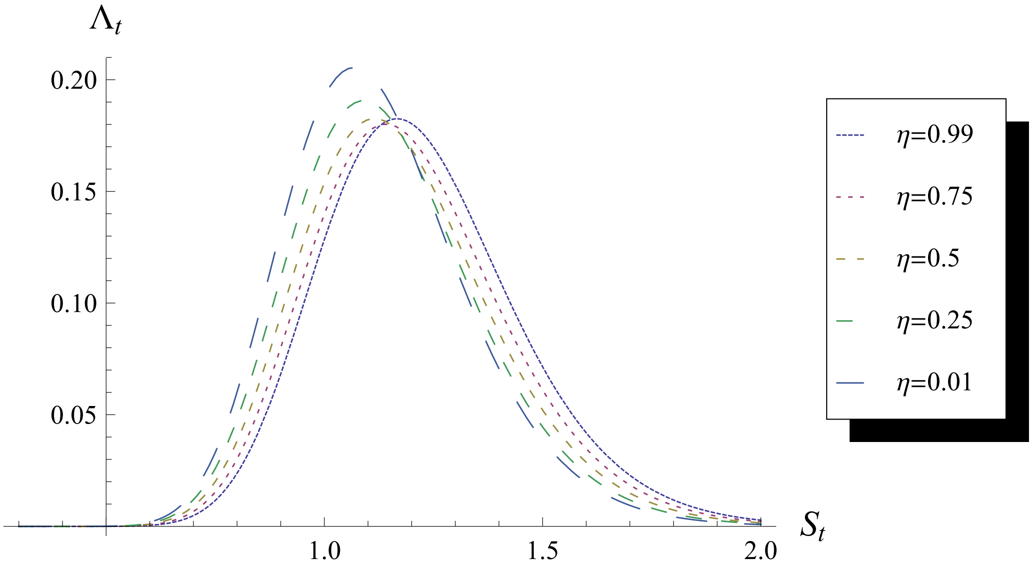

Figure 5.3: Liquidity correction Λ for various values of the portfolio balance parameter b; model parameters: $d = 1$, $\sigma = 0.2$, $\mu = 0$, $t = 0$, $T = 1$

Let us now consider the relation between the market maker's risk aversion a and the liquidity correction Λ. As is to be expected, a higher risk aversion necessitates a greater liquidity correction. Figure 5.4 illustrates this. It is not clear whether the exact relationship between the risk aversion a and the size of the liquidity correction Λ can be expressed in closed form. However, numerical experiments suggest that the dependence is roughly (but not precisely) linear, i.e. that $\Lambda_t(a) \approx a\Lambda_t(1)$, where $\Lambda(a)$ denotes the liquidity correction term Λ computed for a market maker with risk aversion a.

5.3 Variability with respect to the parameters defining the process S

Recall the geometric Brownian motion $(S_t)_{0\leq t\leq T}$ from Assumption 1.2 which models the payoff at maturity of the marketed security $\psi = S_T$. In this section we will investigate the dependence of Λ on the drift coefficient μ and the diffusion coefficient σ of S.

Contrary to the Black-Scholes model, the drift term μ has an influence on prices

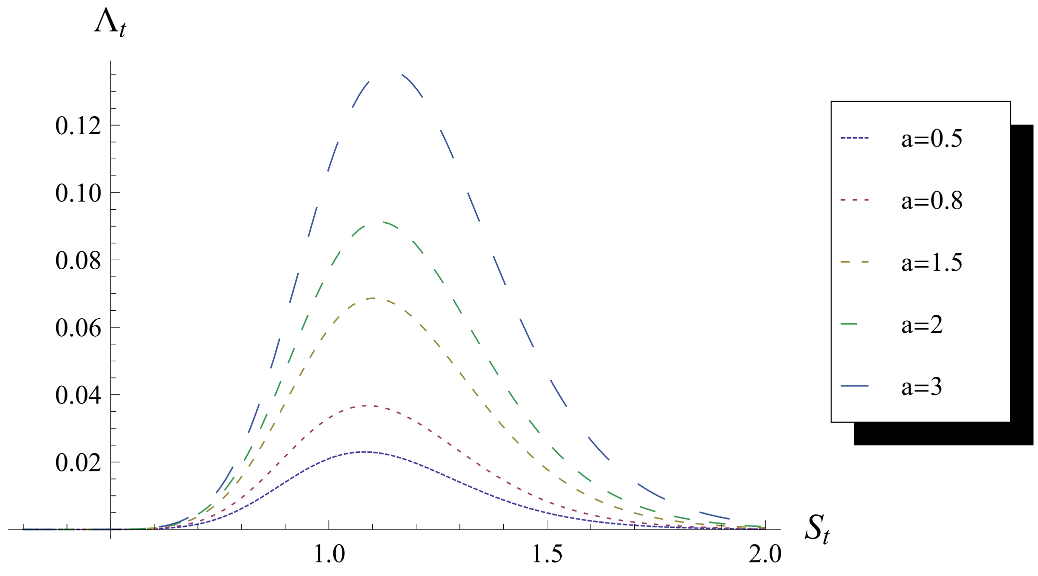

Figure 5.4: Liquidity correction Λ for various values of the market maker's risk aversions a; model parameters: $p = 1$, $z = 1$, $\sigma = 0.2$, $\mu = 0$, $t = 0$, $T = 1$

and on hedge ratios appearing in our model. In a Black-Scholes world μ plays no role as perfect replication renders the actual price development of the underlying irrelevant for the pricing of a contingent claim written on it. The irrelevance of μ in the Black-Scholes model is therefore a direct consequence of the agent's ability to hedge his exposure to changes in the price of the underlying. In our model, the market maker does *not* hedge. He merely takes the positions requested by the large investor and may therefore be fully exposed to the terminal value of ψ. It is natural that this exposure causes the market maker to incorporate his beliefs about the development of the process S into his price quote.

The effect of μ is visualised in Figure 5.5. We can see that the magnitude of Λ is not affected by changes in μ. Moreover, we observe that, in comparison to $\mu = 0$, Λ becomes "narrower" as μ grows large and that it becomes "wider" as μ becomes small (i.e. as $|\mu|$ becomes large for $\mu < 0$). This effect is simply due to exponential scaling. Note that, disregarding this exponential scaling effect, μ does not actually change the magnitude or shape of Λ (and Δ) but merely translates them along the S-axis. It can thus be viewed as a simple "discounting" applied by the market maker according to his believes about the future development of the process S. In other words, changes of μ have the sole effect of shifting the location of "at the money".

The relation between the liquidity correction term Λ and the diffusion coefficient σ is visualised in Figure 5.6. As is to be expected, the liquidity correction Λ is increasing in σ. Moreover, we observe that with increasing σ the location of the maximum of Λ tends to the right. This effect is a result of the $\frac{1}{2}\sigma^2$-term in (1.2). It can be interpreted as a discounting by the market maker in a similar sense to the discounting with respect

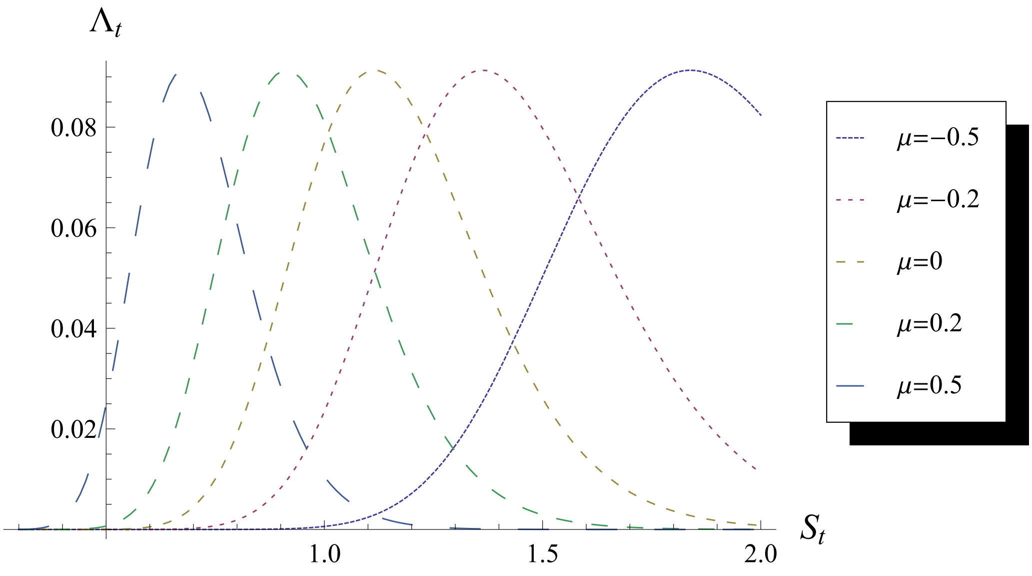

Figure 5.5: Liquidity correction Λ for various values of the drift coefficient μ; model parameters: $a = 2$, $p = 1$, $z = 1$, $\sigma = 0.2$, $t = 0$, $T = 1$

to μ explained above.

Summarising the observations of these numerical experiments we conclude that the liquidity correction to the replicating position Λ for a large investor who is hedging a call option in our illiquid market is coherent with our expectations: It is strictly positive, it attains a unique maximum near the money and it depends on the model parameters in an intuitive way. In particular, it is increasing with respect to parameters which reflect a higher degree of risk, i.e. in the market maker's risk aversion a, in the diffusion coefficient σ and in the remaining time to maturity $T - t$. Moreover, we saw that Λ is smallest when the market maker possesses an endowment which is balanced between stocks and cash and that, furthermore, Λ is inversely proportional to the *market depth*, which in our setting is a linear scaling parameter for the market maker's initial endowment.

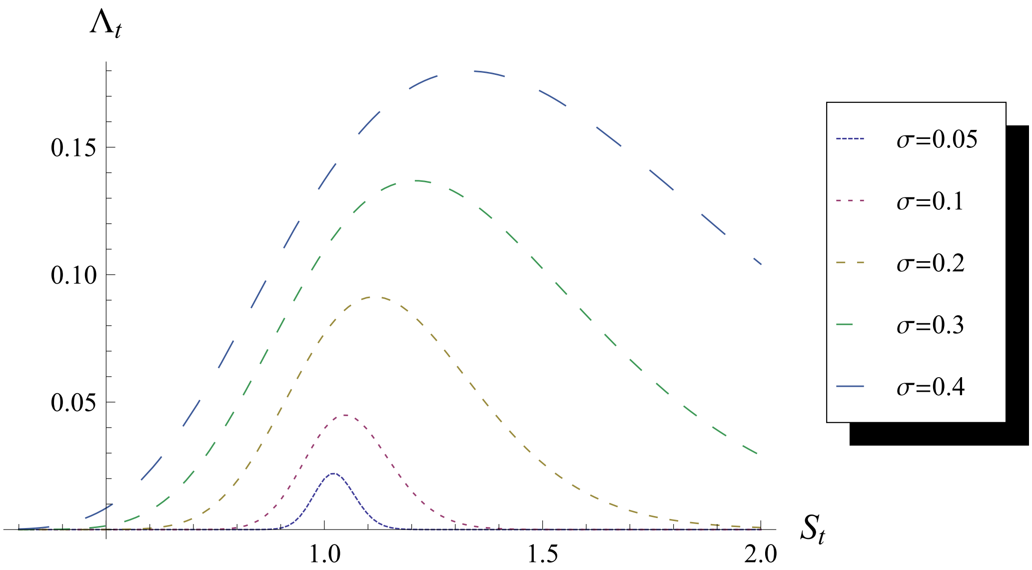

Figure 5.6: Liquidity correction Λ for various values of the diffusion coefficient σ; model parameters: $a = 2$, $p = 1$, $z = 1$, $\mu = 0$, $t = 0$, $T = 1$

Chapter 6

Logarithmic utilities

Throughout Part I of this thesis we restricted our analysis to the cases where the market maker's risk aversion coefficient a satisfied either $0 < a < 1$ or $a > 1$, deliberately leaving out the case $a = 1$ which corresponds to a logarithmic utility function. We did this for several reasons.

The terms and formulas throughout Part I are, in most cases, identical for the regimes $0 < a < 1$ and $a > 1$ but differ for $a = 1$. A logarithmic utility function therefore necessitates different arguments throughout the analysis and a significant amount of our proofs would need to be modified for the case $a = 1$. In some cases, it is not clear how our arguments need to be changed to accommodate the case $a = 1$.

In particular, our arguments do not prove that the market maker's utility process is a (true) martingale when $a = 1$ and not even that it is a submartingale. This leads to the possibility that the model is not free of arbitrage in this case, which, however, seems unlikely.

Another consequence of the absence of a proof for the martingale property of the market maker's utility process in the case $a = 1$ is that one direction of Proposition 3.3 does not necessarily hold. As a result we cannot say anymore whether for every attainable claim H there exists an admissible trading strategy Q such that $U_t^Q = U_t^H$.

A further complication in the case of logarithmic utilities is the fact that the logarithm takes both positive and negative values so that the market maker's expected utility may occasionally be zero. As we frequently consider quotients of expected utilities in our analysis, this issue would have to be addressed in order to avoid dividing by zero.

Some basic observations, however, are quickly made for logarithmic utility functions. First of all, we find that the existence of trade size bounds persists. In a single period setting, a trade size q is admissible, i.e. q can be complemented by a cash amount x in such a way that (1.4) is satisfied, if and only if

$$-p \leq q \leq \bar{q}(p, z),$$

where

$$\bar{q}(p,z) \triangleq \exp\left(\mathbb{E}\left[\log\left(\frac{\Sigma_0}{\psi}\right)\right]\right) - p.$$

As before, the upper bound on admissible trade sizes $\bar{q}(p,z)$ can be written in terms of the market maker's initial utility level $\mathbf{u}_0 \triangleq \mathbb{E}[\log(\Sigma_0)]$ as

$$\bar{q}(p,z) = \exp\left(\mathbf{u}_0 - \mathbb{E}[\log\psi]\right) - p,$$

which is consistent with Remark 1.8.

These observations render it likely that many of the results presented in Part I of this thesis possess an analogue in the case $a = 1$. A thorough analysis of our model for logarithmic utility functions would be a desirable extension.

Part II

Exponential utility functions

Introduction to Part II

In the second part of this thesis we will revisit the model from Part I, albeit for a different class of utility functions. Instead of power utilities with hyperbolic absolute risk aversion (HARA), we will consider exponential utility functions which display constant absolute risk aversion (CARA). As in Part I, we will model the value at maturity of the traded security using geometric Brownian motion.

In the context of our model, exponential utility functions have been used by Said [46], where an analysis of pricing and hedging is conducted for an illiquid *Bachelier model* in which the price process of the underlying is modeled by Brownian motion rather than geometric Brownian motion. The more general case of utility functions with bounded absolute risk aversion and weak assumptions on the distribution of the traded security is examined by Bank and Kramkov [11, 14]. However, the case of geometric Brownian motion is covered in none of the previous three analyses as GBM fails to satisfy the finite exponential moments condition required therein.

Moreover, the analysis at hand has a different focus from those previous works as it is mainly concerned with the *replication of contingent claims* under illiquidity which has not been addressed in [11] and [14].

Part II of this thesis can thus be seen as an extension of the ideas in [46], [11] and [14] to Black-Scholes-type price dynamics in the form of geometric Brownian motion with an emphasis on the question of replication of options.

When comparing the results in Part II to those in Part I, it turns out that for exponential utility functions some aspects of the analysis are significantly easier than for power utilities. In particular,

- we obtain a closed form expression for the market indifference price;
- there is no upper bound for admissible trading strategies;
- the absence of arbitrage can be obtained immediately as the market maker's utility function is bounded from above;
- we can define a (meaningful) wealth process for the large investor.

In some aspects, however, the exponential utility case proves to be more troublesome than the power utility case. In particular,

- the market maker's process of indirect utility is a local martingale but not necessarily a (true) martingale;
- the derivation of a sufficient condition for attainability of contingent claims is more involved;
- the validity of the asymptotic expansions for the hedging strategy is conditional on the yet unproven Conjecture 2.

Chapter Overview

In the second part of this thesis, we proceed as follows.

Chapter 7 is dedicated to the concept of *admissibility* of trading strategies. We first consider single-period trading which we then extend to the case of simple strategies where finitely many trades are allowed and, ultimately, to the case of continuous-time strategies. As in the power utility case the trade dynamics are guided by a *utility indifference principle* which, corresponding to the different types of strategies, comes in three different manifestations. A trading strategy is then said to be admissible if the respective utility indifference principle can be adhered to. We finish the chapter by showing that the model, without the need to impose further assumptions, is free of arbitrage.

In **Chapter 8** we introduce the large investor's *wealth process* when he is pursuing an admissible trading strategy. We then investigate its dynamics and highlight some of its properties. In contrast to the majority of classical models, the wealth process does not play a central role in our analysis.

Chapter 9 establishes the notion of *attainability* of claims. There, we provide a necessary and sufficient condition on the payoff function of a claim which guarantees that the claim can be replicated in our model. We achieve this aim by following two different approaches: Firstly, via replicating the market maker's utility process and, secondly, by tracking the large investor's wealth process. Finally, we prove the replicability of (limited quantities of) call- and put options in our model.

In **Chapter 10** we conduct an asymptotic analysis to derive liquidity corrections to replicating strategies in comparison to Black-Scholes-type models. To this end, we consider small positions of claims which allow us to express *liquidity corrections for hedge ratios* as a second order approximation to the replicating position. The results in this chapter as well as the numerical investigation of the subsequent chapter are conditional on the validity of a monotonicity assertion which we introduce as Conjecture 2. Even though this seemingly simple statement is strongly supported by numerical evidence and intuition, an analytical proof is yet oustanding.

Chapter 11 contains comparative statics for the liquidity correction to the hedge ratio that was established in the previous chapter. We give an idea of its shape and magnitude in the case of a call option and we provide numerical evidence which shows that the liquidity correction reacts to changes in the model parameters in such a way as one would expect.

In **Chapter 12** we address some questions related to the lognormal distribution. We first prove a *limit theorem for the Laplace transform of the lognormal distribution* which is needed to establish the notion of attainability of contingent claims in Chapter 9 and whose proof turns out to be surprisingly lengthy. We then briefly discuss Conjecture 2 whose validity is required for the validity of the asymptotic expansions of Chapter 10.

Chapter 7

Admissible strategies

In this section we will investigate the *admissibility* of trading strategies. We will introduce this notion in a single-period setting, then extend it to simple trading strategies and finally to continuous-time strategies.

7.1 Model setup

The general setup of this model as well as the trading dynamics therein are identical to those in Part I and we will restrict ourselves to highlighting the differences. Most prominently, Assumption 1.1 will be replaced by

Assumption 7.1. The market maker's utility function $u : \mathbb{R} \to \mathbb{R}$ has the form

$$u(x) = -\frac{1}{a}e^{-ax} \text{ for some } a > 0.$$

Note that an agent who possesses such a utility function u has constant absolute risk aversion R, i.e.

$$R(x) \triangleq -\frac{u''(x)}{u'(x)} = a.$$

Furthermore, note that u is a strictly concave, strictly increasing, twice continuously differentiable function on $\mathbb{R}$.

Throughout Part II we will continue to make use of Assumption 1.2 which we restate as

Assumption 7.2. The marketed contingent claim ψ is given by the value at maturity $\psi = S_T$ of a geometric Brownian motion $(S_t)_{0\leq t\leq T}$ which is governed by the stochastic differential equation

$$dS_t = S_t(\mu dt + \sigma dW_t), \tag{7.1}$$

where $S_0 > 0$ is constant and $(W_t)_{0\leq t\leq T}$ is standard Brownian motion adapted to $(\mathcal{F}_t)$.

We will again denote the analytic solution of (7.1) with $S_0 = 1$ by

$$\mathcal{E}_t \triangleq e^{(\mu - \frac{\sigma^2}{2})t + \sigma W_t}. \tag{7.2}$$

Moreover, we will impose

Assumption 7.3. The market maker's initial endowment Σ_0 is given as a combination of a position p in the traded security ψ and a cash amount z, i.e.

$$\Sigma_0 = p\psi + z \text{ for some } p \geq 0, z \in \mathbb{R}.$$

Note that Assumption 7.3 is slightly weaker than Assumption 1.3 in the power utility case since we only demand that $p \geq 0$ and $z \in \mathbb{R}$ rather than $p + z > 0$.

The assumption $p \geq 0$ is crucial to our analysis as otherwise, for $p < 0$, we have

$$\mathbb{E}[u(p\psi)] = -\frac{1}{a}\mathbb{E}[e^{-ap\psi}] = -\infty,$$

which makes it impossible to introduce a market indifference price for ψ as done further below. This problem with the exponential moments of ψ is the reason for which geometric Brownian motion was excluded from the analyses in [46], [11] and [14]. However, we will see that we can overcome this obstacle and conduct a meaningful investigation as long as we ensure that the market maker is left with a total long or neutral position in the traded asset ψ at all times. As in Part I, we will assume that interest rates are zero.

7.2 Single transaction setting

As in the power utility case, we will call a transaction size $q \in \mathbb{R}$ *admissible* if there is $x \in \mathbb{R}$ which solves the equation

$$\mathbb{E}[u(\Sigma_0)] = \mathbb{E}[u(\Sigma_0 + q\psi + x)].$$

Such an x will be called a *market indifference price* of the transaction q. Proposition 7.4 below is the exponential utility analog of Theorem 1.5. It turns out that the market indifference price x for an order of size $q \geq -p$ possesses an explicit representation and that, unlike in the power utility case, admissible trade sizes q are not bounded from above. The lack of an upper bound is due to the fact that, contrary to the case of power utilities, a market maker with constant absolute risk aversion will accept negative cash positions to allow for the purchase of further stocks. However, the lower bound $-p$ remains valid: If it is violated, i.e. for $p + q < 0$, we have $-a(p + q) > 0$ and consequently the market maker's expected utility explodes.

Proposition 7.4. *Under Assumptions 7.1, 7.2 and 7.3 a market indifference price $x \in \mathbb{R}$ exists if and only if the transaction size q satisfies*

$$p + q \geq 0. \tag{7.3}$$

In that case the market indifference price is given by

$$x = \frac{1}{a} \log \left(\frac{\mathbb{E}[u((p+q)\psi)]}{\mathbb{E}[u(p\psi)]} \right) = \frac{1}{a} \log \left(\frac{\mathbb{E}[e^{-a(p+q)\psi}]}{\mathbb{E}[e^{-ap\psi}]} \right). \tag{7.4}$$

Proof. The market indifference price x is given as the solution to

$$\mathbb{E}[u(\Sigma_0)] = \mathbb{E}[u(\Sigma_0 + q\psi + x)],$$

or, equivalently, to

$$-\frac{1}{a}\mathbb{E}\left[e^{-a(p\psi+z)}\right] = -\frac{1}{a}\mathbb{E}\left[e^{-a((p+q)\psi+z+x)}\right],$$

which after rearranging yields (7.4). As $a > 0$ and as ψ is the value at maturity of a geometric Brownian motion and thus lognormally distributed, $\mathbb{E}[e^{-a(p+q)\psi}]$ is finite if and only if $p + q \geq 0$. Hence, an indifference price $x \in \mathbb{R}$ exists if and only if $p + q \geq 0$. □

We conclude from Proposition 7.4 that the set of admissible trade sizes in the single-period case is given by the interval $[-p, \infty)$.

Proposition 7.5 below enlists the properties of the market indifference price x. In order to state and prove it, we introduce the *marginal indifference pricing measure* $\mathbb{Q}(r)$ of the market maker when he is holding $r \in [0, \infty)$ stocks defined by

$$\frac{d\mathbb{Q}(r)}{d\mathbb{P}} \triangleq \frac{u'(r\psi)}{\mathbb{E}[u'(r\psi)]} = \frac{e^{-ar\psi}}{\mathbb{E}[e^{-ar\psi}]}.$$

The measure $\mathbb{Q}(r)$ derives its name from the fact that, when the market maker holds r stocks, the marginal price of the traded asset ψ is given as its expected payoff under $\mathbb{Q}(r)$ as will be detailed in Section 10.1.

Proposition 7.5. *Let Assumptions 7.1, 7.2 and 7.3 hold and let*

$$x(\cdot) : [-p, \infty) \to \mathbb{R}, \quad q \mapsto x(q) \triangleq \frac{1}{a} \log \left(\frac{\mathbb{E}[e^{-a(p+q)\psi}]}{\mathbb{E}[e^{-ap\psi}]} \right)$$

denote the function which maps every admissible trade size q to its market indifference price $x(q)$. Then $x(\cdot)$ is strictly decreasing, convex, positive for $-p \leq q < 0$, negative for $q > 0$ and $x(0) = 0$. Moreover, $x(\cdot)$ is twice continuously differentiable on $(-p, \infty)$

and both derivatives can be extended continuously to the boundary point $q = -p$. Its first and second order derivatives are given by

$$\partial_q x(q) = -\frac{\mathbb{E}[e^{-a(p+q)\psi}\psi]}{\mathbb{E}[e^{-a(p+q)\psi}]} = -\mathbb{E}^{\mathbb{Q}(p+q)}[\psi] \tag{7.5}$$

and

$$\partial_q^2 x(q) = a\left(\frac{\mathbb{E}[e^{-a(p+q)\psi}\psi^2]}{\mathbb{E}[e^{-a(p+q)\psi}]} - \frac{\mathbb{E}[e^{-a(p+q)\psi}\psi]^2}{\mathbb{E}[e^{-a(p+q)\psi}]^2}\right) = a\,\mathrm{Var}^{\mathbb{Q}(p+q)}[\psi]. \tag{7.6}$$

Proof. The differentiability assertions about $x(\cdot)$ follow by dominated convergence; the terms for the first and second order derivatives are obtained by straightforward computations and by definition of the measure $\mathbb{Q}(\cdot)$. By Assumption 7.2 it is apparent from (7.5) that $\partial_q x(q) < 0$ from which it follows that $x(\cdot)$ is strictly decreasing. As the second order derivative can be expressed as a variance, it is clear that $\partial_q^2 x(q) > 0$, which implies the convexity of $x(\cdot)$. By the dominated convergence theorem both derivatives can be continuously extended to the boundary point $q = -p$. Finally, the assertions about $x(0)$ as well as the positivity and negativity of $x(\cdot)$ are clear from the definition of $x(\cdot)$ and the properties of the logarithm. $\square$

7.3 Simple strategies

As in the corresponding section in Part I, we consider simple strategies where trades occur at only finitely many points in time, i.e. we consider processes $(Q_t)_{0\leq t\leq T}$ of the form

$$Q_t = \sum_{k=1}^{n} \theta_k \mathbb{1}_{(\tau_{k-1},\tau_k]}(t),\ 0 \leq t \leq T, \tag{7.7}$$

where $0 = \tau_0 \leq \cdots \leq \tau_n = T$ are stopping times and $\theta_k \in \mathbf{L}^0(\mathcal{F}_{\tau_{k-1}}, \mathbb{R})$. It is reasonable to assume that the cash balance process $(X_t)_{0\leq t\leq T}$ which complements Q in terms of expected utility, if it exists at all, is of the same form and can be written as

$$X_t = \sum_{k=1}^{n} \xi_k \mathbb{1}_{(\tau_{k-1},\tau_k]}(t),\ 0 \leq t \leq T, \tag{7.8}$$

with $\xi_k \in \mathbf{L}^0(\mathcal{F}_{\tau_{k-1}}, \mathbb{R})$. As before, for $k \in \{1, \ldots, n\}$, we write

$$\Sigma_k \triangleq \Sigma_0 + \theta_k\psi + \xi_k$$

and we say that a simple trading strategy Q is *admissible* if there exists a complementing cash balance process X of the form (7.8) such that

$$\mathbb{E}[u(\Sigma_k)|\mathcal{F}_{\tau_{k-1}}] = \mathbb{E}[u(\Sigma_{k-1})|\mathcal{F}_{\tau_{k-1}}] \tag{7.9}$$

possesses a solution $\xi_k \in \mathbf{L}^0(\mathcal{F}_{\tau_{k-1}}, \mathbb{R})$ for all $k \in \{1, \ldots, n\}$. The following proposition characterises admissible simple strategies in terms of a lower trade bound. As in the single-period case treated in the previous section and in contrast to the case of power utilities, there is no upper bound for admissible simple strategies.

Proposition 7.6. *Let Assumptions 7.1, 7.2 and 7.3 hold and let Q be a simple trading strategy. Then Q is admissible if and only if*

$$\theta_k \geq -p \quad \forall k \in \{1, \ldots, N\}.$$

In that case, the process X of form (7.8) *defined by*

$$\xi_k = \frac{1}{a} \log \left(\frac{\mathbb{E}[u(\Sigma_0 + \theta_k \psi) | \mathcal{F}_{\tau_{k-1}}]}{\mathbb{E}[u(\Sigma_{k-1}) | \mathcal{F}_{\tau_{k-1}}]} \right), \quad k \in \{1, \ldots, N\},$$

complements Q in the sense of (7.9).

Proof. Solving

$$\mathbb{E}[u(\Sigma_k) | \mathcal{F}_{\tau_{k-1}}] = \mathbb{E}[u(\Sigma_{k-1}) | \mathcal{F}_{\tau_{k-1}}]$$

for the $\mathcal{F}_{\tau_{k-1}}$-measurable random variable ξ_k yields

$$\xi_k = \frac{1}{a} \log \left(\frac{\mathbb{E}[u(\Sigma_0 + \theta_k \psi) | \mathcal{F}_{\tau_{k-1}}]}{\mathbb{E}[u(\Sigma_{k-1}) | \mathcal{F}_{\tau_{k-1}}]} \right),$$

which proves the second assertion. Moreover, for $k = 1$ we obtain

$$\xi_1 = \frac{1}{a} \log \left(\frac{\mathbb{E}[u((p + \theta_1)\psi)]}{\mathbb{E}[u(p\psi)]} \right),$$

so that, by Proposition 7.4, $|\xi_1| < \infty$ if and only if $\theta_1 \geq -p$. The result then follows by induction. $\square$

We will see later on that if a process X complements an admissible simple strategy Q in the sense of (7.9) then that process is unique and therefore necessarily given by the process X in Proposition 7.6.

7.4 Continuous-time strategies

We will from now on assume the trading strategy $(Q_t)_{0 \leq t \leq T}$ to be a general predictable process. The market maker's endowment at time $t = 0$ is yet again denoted by Σ_0 and thereafter at any time t by

$$\Sigma_t \triangleq \Sigma_0 + Q_t \psi + X_t,$$

where the predictable process $(X_t)_{0 \leq t \leq T}$ is the complementing cash process for the strategy Q. As in the case of power utilities, it turns out that, in order to formulate a utility indifference principle for continuous-time strategies, it is advisable to track the market maker's utility process rather than the complementing cash process X. We introduce some additional notation for this purpose.

In order to describe the time-evolution of the level of expected utility of the market maker, we introduce the *static process of indirect utility* $F : \mathbb{R} \times [-p, \infty) \times [0, T] \to \mathbf{L}^0(\mathbb{R})$ defined by

$$F(x, q, t) \triangleq \mathbb{E}[u(\Sigma_0 + q\psi + x)|\mathcal{F}_t] = -\frac{1}{a}\mathbb{E}[e^{-a((p+q)\psi+x+z)}|\mathcal{F}_t], \tag{7.10}$$

where again z denotes the market maker's initial cash and p his initial security position. By construction $(F(x, q, t))_{0\leq t\leq T}$ is a martingale. In the following, we will always consider a version of F which is nice in the sense of Lemma 1.11. Similarly to the power utility case, F possesses dynamics which are given by the following lemma.

Lemma 7.7. *Under Assumptions 7.1, 7.2 and 7.3 let F be defined as in* (7.10)*. Then F possesses the representation*

$$F(x, q, t) = F(x, q, 0) + \int_0^t \tilde{g}(x, q, s)dW_s, \tag{7.11}$$

where the stochastic field $\tilde{g} : \mathbb{R} \times [-p, \infty) \times [0, T] \to \mathbf{L}^0(\mathbb{R})$ is given by

$$\tilde{g}(x, q, t) = \sigma S_t \partial_s h_{x,q}(t, S_t) = \sigma(p + q)e^{-a(x+z)}\mathbb{E}[e^{-a(p+q)\psi}\psi|\mathcal{F}_t]$$

and the function $h_{x,q} : [0, T] \times \mathbb{R}_+ \to \mathbb{R}$ is given by

$$h_{x,q}(t, s) = \mathbb{E}_{t,s}[u((p + q)\psi + x + z)] = -\frac{1}{a}e^{-a(x+z)}\mathbb{E}[e^{-a(p+q)s\mathcal{E}_{T-t}}]$$

Proof. The proof is analog to that of Lemma 1.18. □

Because

$$\mathbb{E}[e^{-a(p\psi+z)}|\mathcal{F}_t] = e^{-az}\mathbb{E}[e^{-ap\psi}|\mathcal{F}_t], \tag{7.12}$$

the preferences of agents with exponential utility functions display *cash invariance* in the sense that the solution x to

$$\mathbb{E}[u((p + q)\psi + z + x))|\mathcal{F}_t] = \mathbb{E}[u(p\psi + z)|\mathcal{F}_t]$$

is the same as the solution x to

$$\mathbb{E}[u((p + q)\psi + x))|\mathcal{F}_t] = \mathbb{E}[u(p\psi)|\mathcal{F}_t],$$

i.e. x does not depend on z. In the model at hand, this implies that the market maker's risk preferences, and therefore the utility indifference price x, do not depend on his current cash position. Moreover, note that

$$\tilde{g}(x, q, t) = e^{-ax}\tilde{g}(0, q, t). \tag{7.13}$$

In analogy to (1.23) in the power utility case, we define the family of maps A_t : $(-\infty, 0) \times [-p, \infty) \to \mathbf{L}^0(\mathbb{R})$, $t \in [0, T]$,

$$(\mathbf{u}, q) \mapsto -\frac{1}{a} \log \left(\frac{\mathbf{u}}{\mathbb{E}[u(\Sigma_0 + q\psi)|\mathcal{F}_t]} \right) =: x, \tag{7.14}$$

which, as their equivalent in the power utility case, reconstruct the complementing cash amount x for a given utility level $\mathbf{u}$ and transaction size $q \geq -p$ at time $t \in [0, T]$; i.e. $x = A_t(\mathbf{u}, q)$ solves

$$\mathbf{u} = \mathbb{E}[u(\Sigma_0 + q\psi + x)|\mathcal{F}_t].$$

These maps will serve as a tool in formulating the utility indifference principle in continuous time.

Without encountering the dynamic domain issues that we were facing in the power utility case, we define the market maker's *dynamic process of indirect utility* $(U_t^Q)_{0 \leq t \leq T}$ when the large investor is pursuing a trading strategy $Q \geq -p$ as the strong solution to

$$U_t^Q = \mathbb{E}[u(\Sigma_0)] + \int_0^t F(A_s(U_s^Q, Q_s), Q_s; ds), \tag{7.15}$$

provided the solution exists and is uniquely determined. The nonlinear stochastic integral, as before, is understood in the sense of [39] Section 3.2. The SDE (7.15) can be viewed as the indifference principle in continuous time. In differential form this equation reads

$$dU_t^Q = F(A_t(U_t^Q, Q_t), Q_t; dt) = g(U_t^Q, Q_t, t)dW_t, \tag{7.16}$$

where

$$g(\mathbf{u}, q, t) \triangleq \tilde{g}(A_t(\mathbf{u}, q), q, t) \tag{7.17}$$

and $\tilde{g}$ is defined as in Lemma 7.7. As in the power utility case, we will say that a predictable process Q is an *admissible trading strategy* if Q is such that the SDE (7.15) possesses a unique strong solution. However, unlike in the power utility case, it turns out that we can state an easily verifiable sufficient condition to ensure the admissibility of a trading strategy. This is done in Theorem 7.10 below.

The following proposition shows that (7.15) is really of a much simpler form, namely a linear SDE.

Proposition 7.8. *Let Assumptions 7.1, 7.2 and 7.3 hold. Then the SDE* (7.15) *can* *equivalently be written as*

$$\frac{dU_t^Q}{U_t^Q} = -a\sigma(p+Q_t)\mathbb{E}^{\mathbb{Q}(p+Q_t)}[\psi|\mathcal{F}_t]dW_t \tag{7.18}$$

with initial condition $U_0^Q = \mathbb{E}[u(\Sigma_0)]$.

Remark 7.9. The notation $\mathbb{E}^{\mathbb{Q}(p+Q_t)}[\psi|\mathcal{F}_t]$ is meant to denote the predictable process $(\mathbb{E}^{\mathbb{Q}(p+q)}[\psi|\mathcal{F}_t]|_{q=Q_t})_{0\leq t\leq T}$ given, at each time $t\in[0,T]$, by

$$\mathbb{E}^{\mathbb{Q}(p+q)}[\psi|\mathcal{F}_t]|_{q=Q_t} = \frac{\mathbb{E}[e^{-a(p+q)\psi}\psi|\mathcal{F}_t]|_{q=Q_t}}{\mathbb{E}[e^{-a(p+q)\psi}|\mathcal{F}_t]|_{q=Q_t}}.$$

By Lemma 1.11, there are smooth versions of these parametrised conditional expectations. In what follows, we will always consider these "nice" versions.

Proof of Proposition 7.8. Note first that, with regard to (7.13),

$$g(\mathbf{u},q,t) = e^{-aA_t(\mathbf{u},q)}\tilde{g}(0,q,t). \tag{7.19}$$

Using the definitions of A_t and F, this allows us to rewrite (7.15), resp. (7.16), as

$$U_t^Q = \mathbb{E}[u(\Sigma_0)] + \int_0^t U_s^Q \frac{\tilde{g}(0,Q_s,s)}{F(0,Q_s,s)}dW_s,$$

or, equivalently, as

$$\frac{dU_t^Q}{U_t^Q} = \frac{\tilde{g}(0,Q_t,t)}{F(0,Q_t,t)}dW_t.$$

Observe further that

$$\frac{\tilde{g}(0,q,t)}{F(0,q,t)} = -a\sigma(p+q)\mathbb{E}^{\mathbb{Q}(p+q)}[\psi|\mathcal{F}_t],$$

which finishes the proof. □

Note that, by the arguments in the proof of Proposition 7.8, it is apparent that the function g defined by (7.17) admits the representation

$$g(\mathbf{u},q,t) = -a\sigma\mathbf{u}(p+q)\mathbb{E}^{\mathbb{Q}(p+q)}[\psi|\mathcal{F}_t]. \tag{7.20}$$

Let us introduce some further notation. For two random variables α and β we denote the *$\mathcal{F}_t$-conditional covariance with respect to a probability measure* $\mathbb{L}$ by

$$\mathrm{Cov}_t^{\mathbb{L}}[\alpha,\beta] = \mathbb{E}_t^{\mathbb{L}}[\alpha\beta] - \mathbb{E}_t^{\mathbb{L}}[\alpha]\mathbb{E}_t^{\mathbb{L}}[\beta],$$

whenever it is defined, and the *$\mathcal{F}_t$-conditional variance with respect to a probability* *measure* $\mathbb{L}$ by

$$\operatorname{Var}_t^{\mathbb{L}}[\alpha] = \operatorname{Cov}_t^{\mathbb{L}}[\alpha, \alpha].$$

We will denote by $\operatorname{Cov}_{t,s}^{\mathbb{L}}$ and $\operatorname{Var}_{t,s}^{\mathbb{L}}$ the corresponding quantities with respect to the expectation $\mathbb{E}_{t,s}^{\mathbb{L}}$.

Having identified (7.15) as a linear SDE, we are now ready to prove the following theorem which shows that the set of admissible trading strategies includes all square integrable predictable processes which are bounded from below by $-p$.

Theorem 7.10. *Let Assumptions 7.1, 7.2 and 7.3 hold and let Q be a predictable process with $Q_t \geq -p$ for all $t \in [0, T]$ which satisfies*

$$\int_0^T Q_t^2 dt < \infty. \tag{7.21}$$

Then Q is an admissible trading strategy.

Proof. We have to show the existence and uniqueness of solutions to the SDE (7.15), which we saw can be expressed equivalently as (7.18). It is well known that a linear SDE of this form possesses a unique strong solution if

$$\int_0^T \left(-a\sigma(p + Q_t)\mathbb{E}^{\mathbb{Q}(p+Q_t)}[\psi|\mathcal{F}_t]\right)^2 dt < \infty. \tag{7.22}$$

For $q = -p$ we have

$$\mathbb{E}^{\mathbb{Q}(p+q)}[\psi|\mathcal{F}_t] = \mathbb{E}[\psi|\mathcal{F}_t]$$

while for $q > -p$, using Lemma 1.11, we find

$$\frac{\partial}{\partial q}\mathbb{E}^{\mathbb{Q}(p+q)}[\psi|\mathcal{F}_t] = -a \operatorname{Var}_t^{\mathbb{Q}(p+q)}[\psi] < 0,$$

so that $\mathbb{E}^{\mathbb{Q}(p+q)}[\psi|\mathcal{F}_t]$ is decreasing in q. Hence, for $q \in [-p, \infty)$ we have

$$0 \leq \mathbb{E}^{\mathbb{Q}(p+q)}[\psi|\mathcal{F}_t] \leq \mathbb{E}[\psi|\mathcal{F}_t].$$

Using Assumption 7.2 and (7.21), it follows that

$$\begin{aligned}\int_0^T (p + Q_t)^2\mathbb{E}^{\mathbb{Q}(p+Q_t)}[\psi|\mathcal{F}_t]^2 dt &\leq \int_0^T (p + Q_t)^2\mathbb{E}[\psi|\mathcal{F}_t]^2 dt \\ &\leq e^{2\mu T}\int_0^T (p + Q_t)^2 S_t^2 dt \quad < \infty.\end{aligned}$$

This implies (7.22) and finishes the proof. $\square$

Remark 7.11. As a direct consequence of Theorem 7.10 we know in particular that for a simple trading strategy Q of form (7.7) the complementing cash process X is unique and therefore, by Proposition 7.6, necessarily of form (7.8).

7.5 Absence of arbitrage

The absence of arbitrage in the case of exponential utilities is obtained significantly easier than in the power utility case. In [14] it was observed that the upper boundedness of the market maker's utility function implies the absence of arbitrage as in this case his utility process is a submartingale. We will show below that the same argument can be used in our setting.

As in the beginning of Section 2.1, let $G^Q \in \mathbf{L}^0(\mathcal{F}_T, \mathbb{R})$, defined by

$$U_T^Q = u\big(\Sigma_0 - G^Q\big),$$

denote the large investor's profit and loss at maturity T upon pursuit of an admissible trading strategy Q. As $U_T^Q = u(\Sigma_0 + Q_T\psi + X_T)$, G^Q can be equivalently expressed as

$$G^Q = -(Q_T\psi + X_T).$$

Furthermore, we define an arbitrage as in Definition 2.1.

Lemma 7.12. *Let Assumption 7.1 hold. If for an admissible strategy Q the process of indirect utility U^Q of the market maker is a submartingale then Q is not an arbitrage.*

Proof. Analog to the proof of Lemma 2.3. □

Proposition 7.13. *Let Assumptions 7.1, 7.2 and 7.3 hold and let Q be an admissible strategy. Then Q is not an arbitrage.*

Proof. Since the market maker's utility function u is bounded from above, so is the market maker's process of indirect utility U^Q. As U^Q is a local martingale, the boundedness implies that U^Q is a submartingale and thus Lemma 7.12 implies that Q is not an arbitrage. □

We conclude this section by briefly summarising what we have found. A predictable process $(Q_t)_{0\le t\le T}$ is called an admissible trading strategy in our setting if it is such that the SDE (7.15) possesses a unique strong solution U. Equation (7.15) serves as the indifference principle in the context of continuous-time strategies. For Q to be admissible it is sufficient that $Q_t \geq -p$ for all $t \in [0, T]$ and that further

$$\int_0^T Q_t^2 dt < \infty.$$

Moreover, if a strategy is admissible, it is not an arbitrage.

The set of admissible strategies constitutes one of the crucial differences between the exponential and power utility cases. While in the power utility case, the notion of admissibility is essentially explained by "borrowing cash is not acceptable and borrowing stocks is not acceptable", the analog idea in the case of exponential utility functions can be expressed as "borrowing cash is acceptable, borrowing stocks is not".

Chapter 8

The wealth process of the large investor

8.1 Wealth dynamics

While in the power utility case we found that there are admissible trading strategies which do not allow the definition of a wealth process at all times $t \in [0,T]$, the exponential case brings with it no such limitations. We define the *wealth process* of the large investor $(G_t^Q)_{0\le t\le T}$ when following an admissible trading strategy Q, at any time $t \in [0,T]$, as the (unique) $\mathcal{F}_t$-measurable solution G_t^Q to

$$U_t^Q = \mathbb{E}[u(\Sigma_0 - G_t^Q)|\mathcal{F}_t],$$

which can be rearranged to obtain

$$G_t^Q = \frac{1}{a}\log\left(\frac{U_t^Q}{\mathbb{E}[u(\Sigma_0)|\mathcal{F}_t]}\right).$$

Note that $G_T^Q = -(Q_T\psi + X_T)$ which is consistent with the definition of G^Q in the last section of the previous chapter and in the power utility case.

In order to describe the dynamics of the large investor's wealth process it is notationally convenient to introduce the family of functions $P_t(\cdot) : [0,\infty) \to [0,\infty)$ defined by

$$P_t(r) \triangleq r\mathbb{E}^{\mathbb{Q}(r)}[\psi|\mathcal{F}_t].$$

We can think of $P_t(r)$ as the marginal price of the total stock position r, i.e. as the money received if all stocks were sold off at their marginal (post-transaction) price $\mathbb{E}^{\mathbb{Q}(r)}[\psi]$ (see Section 10.1 for details on marginal prices). The following lemma describes the dynamics of the large investor's wealth process G^Q.

Lemma 8.1. *Let Assumptions 7.1, 7.2 and 7.3 hold and let Q be an admissible trading* *strategy. Then the dynamics of the large investor's wealth process G^Q are given by*

$$dG_t^Q = a\sigma^2 \left(P_t(p+Q_t)\big(P_t(p+Q_t) - P_t(p)\big) + \frac{1}{2}\big(P_t(p+Q_t) - P_t(p)\big)^2 \right) dt$$
$$- \sigma\big(P_t(p+Q_t) - P_t(p)\big) dW_t. \tag{8.1}$$

Proof. Recall the stochastic fields $\tilde{g}$ and F from Lemma 7.7 and g from (7.17). Define the auxiliary process $(\tilde{G}_t)_{0\le t\le T}$ by

$$\tilde{G}_t \triangleq \frac{\mathbb{E}[u(\Sigma_0)|\mathcal{F}_t]}{U_t^Q}$$

and let further $\tilde{g}(t) \triangleq \tilde{g}(0,0,t)$ and $F_t \triangleq F(0,0,t) = \mathbb{E}[u(\Sigma_0)|\mathcal{F}_t]$. Then by Itô's quotient rule we have

$$\frac{d\tilde{G}_t}{F_t/U_t^Q} = \left(\frac{\tilde{g}(t)}{F_t} - \frac{g(U_t^Q, Q_t, t)}{U_t^Q} \right) dW_t + \left(\frac{g(U_t^Q, Q_t, t)^2}{(U_t^Q)^2} - \frac{\tilde{g}(t) g(U_t^Q, Q_t, t)}{F_t U_t^Q} \right) dt.$$

An application of Itô's Lemma to $G^Q = -\frac{1}{a}\log(\tilde{G})$ then yields

$$\begin{aligned} dG_t^Q &= \frac{1}{a} \left(\left(\frac{g(U_t^Q, Q_t, t)}{U_t^Q} - \frac{\tilde{g}(t)}{F_t} \right) \frac{g(U_t^Q, Q_t, t)}{U_t^Q} + \frac{1}{2} \left(\frac{g(U_t^Q, Q_t, t)}{U_t^Q} - \frac{\tilde{g}(t)}{F_t} \right)^2 \right) dt \\ &\quad + \frac{1}{a} \left(\frac{g(U_t^Q, Q_t, t)}{U_t^Q} - \frac{\tilde{g}(t)}{F_t} \right) dW_t. \end{aligned}$$

Using the cash invariance of exponential utility functions we compute

$$\frac{\tilde{g}(t)}{F_t} = \frac{\sigma \mathbb{E}[e^{-ap\psi} p\psi|\mathcal{F}_t]}{-\frac{1}{a}\mathbb{E}[e^{-ap\psi}|\mathcal{F}_t]} = -a\sigma P_t(p) \tag{8.2}$$

and similarly

$$\frac{g(U_t^Q, Q_t, t)}{U_t^Q} = \frac{\sigma \mathbb{E}[e^{-a(p+Q_t)\psi}(p+Q_t)\psi|\mathcal{F}_t]}{-\frac{1}{a}\mathbb{E}[e^{-a(p+Q_t)\psi}|\mathcal{F}_t]} = -a\sigma P_t(p+Q_t). \tag{8.3}$$

Substituting these terms in the equation above yields

$$\begin{aligned} dG_t^Q &= a\sigma^2 \left(P_t(p+Q_t)\big(P_t(p+Q_t) - P_t(p)\big) + \frac{1}{2}\big(P_t(p+Q_t) - P_t(p)\big)^2 \right) dt \\ &\quad - \sigma\big(P_t(p+Q_t) - P_t(p)\big) dW_t. \end{aligned}$$

□

Using the notation $\hat{P}_t^Q \triangleq P_t(p+Q_t) - P_t(p)$, we can alternatively write the dynamics of the large investor's wealth process as

$$dG_t^Q = a\sigma^2 \left(P_t(p)\hat{P}_t^Q + \frac{3}{2}(\hat{P}_t^Q)^2 \right) dt - \sigma \hat{P}_t^Q dW_t. \tag{8.4}$$

We can see that in the case where, at a certain time $t \in [0, T)$, the large investor liquidates his entire stock position and decides to trade no more, i.e. where $Q_s = 0$ for all $s \in (t, T]$, we have $\hat{P}_t^Q \equiv 0$ so that the dynamics of his wealth process vanish and hence G_s^Q is constant for $s \in (t, T]$, as one would expect.

8.2 Some observations regarding the large investor's wealth process

In this section we present several identities related to the large investor's wealth process along with their economic interpretations. As these are not central to our analysis, the eager reader may skip this part and proceed directly to Chapter 9.

Denote by $(U_t^0)_{0 \le t \le T}$, $U_t^0 \triangleq \mathbb{E}[u(\Sigma_0)|\mathcal{F}_t] = F(0, 0, t)$, the market maker's utility process in the absence of trading and recall the family of maps $(A_t)_{t \in [0,T]}$ from (7.14).

Lemma 8.2. *Let Assumptions 7.1, 7.2 and 7.3 hold. Then the P&L-process of the large investor G^Q upon pursuit of an admissible trading strategy Q, at any time $t \in [0, T]$, is given by*

$$G_t^Q = -A_t(U_t^Q, Q_t) + A_t(U_t^0, Q_t). \tag{8.5}$$

Proof. We rearrange

$$\mathbb{E}[u(\Sigma_0 - G_t^Q)|\mathcal{F}_t] = \mathbb{E}[u(\Sigma_0 + Q_t\psi + X_t)|\mathcal{F}_t]$$

and use the fact that $X_t = A_t(U_t^Q, Q_t)$ to obtain

$$\begin{aligned} G_t^Q &= \frac{1}{a} \log \left(e^{-aA_t(U_t^Q, Q_t)} \frac{\mathbb{E}[u(\Sigma_0 + Q_t\psi)|\mathcal{F}_t]}{\mathbb{E}[u(\Sigma_0)|\mathcal{F}_t]} \right) \\ &= -A_t(U_t^Q, Q_t) - \frac{1}{a} \log \left(\frac{\mathbb{E}[u(\Sigma_0)|\mathcal{F}_t]}{\mathbb{E}[u(\Sigma_0 + Q_t\psi)|\mathcal{F}_t]} \right). \end{aligned}$$

The result then follows by the definition of

$$A_t(\mathbf{u}, q) = -\frac{1}{a} \log \left(\frac{\mathbf{u}}{\mathbb{E}[u(\Sigma_0 + q\psi)|\mathcal{F}_t]} \right).$$

□

Note that equivalent formulations of (8.5) are given by

$$G_t^Q = \frac{1}{a}\log\left(\frac{U_t^Q}{\mathbb{E}[u(\Sigma_0 + Q_t\psi)|\mathcal{F}_t]}\right) - \frac{1}{a}\log\left(\frac{U_t^0}{\mathbb{E}[u(\Sigma_0 + Q_t\psi)|\mathcal{F}_t]}\right)$$

and by

$$G_t^Q = \frac{1}{a}\log\left(\frac{F(X_t, Q_t, t)}{F(0, Q_t, t)}\right) - \frac{1}{a}\log\left(\frac{F(0,0,t)}{F(0, Q_t, t)}\right).$$

We can view (8.5) as a decomposition of the large investor's profit and loss into the *cash component* $-A_t(U_t^Q, Q_t)$ which is the cumulative cash amount which he received up to time t due to his trading with the market maker, and the *position value component* $A_t(U_t^0, Q_t) =: x$. This second component is the certainty equivalent x in the fictional trade

$$\mathbb{E}[u(\Sigma_0 + Q_t\psi + x)|\mathcal{F}_t] = \mathbb{E}[u(\Sigma_0)|\mathcal{F}_t];$$

i.e. it is the indifference price of a transaction of size Q_t at time t which the market maker would quote if, at that time, he would be holding the endowment Σ_0. Hence, from the large investor's point of view, (8.5) reads:

$$\text{wealth} \;=\; \text{cash received already} \;+\; \text{value of my stock position.}$$

Of course either or both of these amounts can be negative. The profit and loss of the large investor at time t is zero if and only if these amounts have the same absolute value but different signs, i.e. when the market indifference price of the large investor's stock position is equal to the amount that the large investor has paid so far – this case corresponds to the situation where $U_t^Q = U_t^0$.

Note that by rearranging

$$\mathbb{E}[u(\Sigma_0 - G_t^Q)|\mathcal{F}_t] = U_t^Q$$

we can alternatively express the large investor's wealth purely in terms of the market maker's indirect process of utility, namely as

$$G_t^Q = \frac{1}{a}\log\left(\frac{U_t^Q}{U_t^0}\right) = -A_t(U_t^Q, 0). \tag{8.6}$$

Here, we express the profit and loss up to time t as the *liquidation price* of the stock position Q_t; i.e. $-A_t(U_t^Q, 0)$ is the cash amount which the large investor retains upon closing his position in ψ at time t.

Remark 8.3. Lemma 8.2 together with (8.6) implies the identity

$$-A(U_t^Q, 0) = -A(U_t^Q, Q_t) + A(U_t^0, Q_t), \tag{8.7}$$

of which both sides are equal to G_t^Q by (8.5) and (8.6).

Chapter 9

Hedging and replication of options

In this chapter we investigate the attainability of a non-traded contingent claim $H \in \mathbf{L}^0(\mathbb{R})$. As in the power utility case, the large investor, starting from an initial cash amount v_{rep}, seeks to attain the position H at time T by trading the marketed security ψ with the market maker.

We approach the question of the replicability of H from two different angles: First, in Section 9.2, we use an argument which relies on tracking the market maker's utility process. Then, in Section 9.3, we present an alternative approach which is based on the large investor's wealth process.

We will see that we can guarantee the attainability of a path independent claim of the form $H = f(\psi)$ by imposing a simple condition on the claim's payoff function f. In particular, this allows us to prove the replicability of (limited positions of) call- and put options.

9.1 Acceptable and attainable contingent claims

As in the power utility case, we introduce the following set of eligible candidates for attainable claims.

Definition 9.1. For any time $t \in [0, T]$, we define the *set of acceptable contingent claims*

$$\mathcal{H}_t \triangleq \left\{ H \in \mathbf{L}^0(\mathbb{R}) | \exists v_t \in \mathbf{L}^0(\mathcal{F}_t, \mathbb{R}) \text{ s.t. } \mathbb{E}[u(\Sigma_0 - H + v_t)|\mathcal{F}_t] = \mathbb{E}[u(\Sigma_0)|\mathcal{F}_t] \right\}$$

and we say that a claim H is *acceptable at time* $t \in [0, T]$ if it belongs to the set $\mathcal{H}_t$.

The quantity v_t in the above definition can be viewed as the market indifference price of the claim H at time t. The following lemma gives a more easily verifiable condition for a claim H to belong to the set $\mathcal{H}_t$. Moreover, it shows that a claim which is acceptable at one time remains acceptable at any time thereafter. In particular, a claim which is acceptable at time zero remains acceptable at all times.

Lemma 9.2. *Let Assumptions 7.1, 7.2 and 7.3 hold and let $t_0 \in [0,T]$. Then $H \in \mathcal{H}_{t_0}$ if and only if*

$$\mathbb{E}^{\mathbb{Q}(p)}[e^{aH}|\mathcal{F}_{t_0}] < \infty.$$

Moreover, in that case, we have $H \in \mathcal{H}_t$ for all $t \in [t_0, T]$.

Proof. Let $H \in \mathcal{H}_{t_0}$. Rearranging the equation in the definition of $\mathcal{H}_{t_0}$ yields

$$v_{t_0} = \frac{1}{a}\log\left(\frac{\mathbb{E}[e^{-a(p\psi-H)}|\mathcal{F}_{t_0}]}{\mathbb{E}[e^{-ap\psi}|\mathcal{F}_{t_0}]}\right) = \frac{1}{a}\log\left(\mathbb{E}^{\mathbb{Q}(p)}[e^{aH}|\mathcal{F}_{t_0}]\right) \tag{9.1}$$

which implies that, firstly, $v_{t_0} > -\infty$ and that, secondly, $v_{t_0} < \infty$ if and only if $\mathbb{E}^{\mathbb{Q}(p)}[e^{aH}|\mathcal{F}_{t_0}] < \infty$. In that case,

$$\infty > \mathbb{E}^{\mathbb{Q}(p)}[e^{aH}|\mathcal{F}_{t_0}] = \mathbb{E}^{\mathbb{Q}(p)}\left[\mathbb{E}^{\mathbb{Q}(p)}[e^{aH}|\mathcal{F}_t]\big|\mathcal{F}_{t_0}\right] \quad \forall t \geq t_0,$$

which implies the second assertion. □

As of now, we will consider the set $\mathcal{H} \triangleq \mathcal{H}_0$ which, by the previous lemma, contains those claims which are acceptable at all times. Moreover, we will call a contingent claim $H \in \mathbf{L}^0(\mathbb{R})$ *acceptable*, if it belongs to the set $\mathcal{H}$. Note that the situation here differs from the case of power utilities where the acceptability of a claim at a time t_0 does not imply its acceptability at any time $t > t_0$ thereafter.

Recalling Definition 3.2 we will call a claim $H \in \mathcal{H}$ *attainable* if there is an admissible strategy $(Q^H_t)_{0\leq t\leq T}$ such that almost surely

$$v_{rep} + G_T^{Q^H} = H$$

for some cash amount $v_{rep} \in \mathbb{R}$, where G^{Q^H} denotes the large investor's wealth process which we introduced in the previous chapter. In this case, we refer to v_{rep} as a *replication price* of H and to Q^H as a *replicating strategy.*

In order to derive a tractable condition for the attainability of claims, we will restrict most of our analysis to those claims in $\mathcal{H}$ which are path independent with sufficiently integrable payoff functions. To this end, we introduce the set

$$\mathcal{H}' \triangleq \{H \in \mathcal{H} \mid H = f(\psi), f \in \mathcal{R}\},$$

where $\mathcal{R}$, as in Part I, denotes the class of Lipschitz continuous functions on $\mathbb{R}$.

In the following two sections we will introduce two different approaches to characterising the set of attainable claims belonging to $\mathcal{H}'$.

9.2 Replication via the market maker's utility process

In this section we will replicate an option via replicating the market maker's process of indirect utility. Proposition 9.3 will justify this approach by showing that any claim for which this can be done is attainable. In addition to the notion of *attainability* introduced above, we will call a claim $H \in \mathcal{H}$ *utility replicable* if there exists an admissible trading strategy $(Q_t^H)_{0\le t\le T}$ such that almost surely

$$U_t^H = U_t^{Q^H} \quad \forall t \in [0,T],$$

where $(U_t^H)_{0\le t\le T}$,

$$U_t^H \triangleq \mathbb{E}[u(\Sigma_0 - H + v_0)|\mathcal{F}_t], \quad 0 \le t \le T, \tag{9.2}$$

as in Part I, denotes the market maker's dynamic utility process induced by the claim H and v_0 denotes the market indifference price of H at time zero; i.e. v_0 is such that

$$\mathbb{E}[u(\Sigma_0 - H + v_0)] = \mathbb{E}[u(\Sigma_0)].$$

For power utilities the concepts of *attainability* and *utility replicability* coincide while for exponential utility functions we know merely that any utility replicable claim is attainable, as the following proposition shows. This, however, is sufficient for our purposes.

Proposition 9.3. *Let Assumptions 7.1, 7.2 and 7.3 hold. Consider a contingent claim $H \in \mathcal{H}$ and let v_0 denote its utility indifference price at time zero. If H is utility replicable with replicating strategy Q then*

$$v_0 + G_T^Q = H.$$

In particular this means that H is attainable with replicating strategy $Q^H = Q$ and replication price $v_{rep} = v_0$.

Proof. The proof is analog to the proof of part (i) of Proposition 3.3. □

Utility replicability constitutes a sufficient condition for the attainability of a claim $H \in \mathcal{H}$, albeit not a very tractable one. Theorem 9.5 below provides a characterisation of the set of utility replicable claims in $\mathcal{H}'$ by providing an easily verifiable criterion.

Before we proceed, observe that the market maker's dynamic utility process U^H induced by a claim $H \in \mathcal{H}'$, using Assumptions 7.2 and 7.3, can be written as

$$U_t^H = u^H(t,s)|_{s=S_t} \tag{9.3}$$

at any time $t \in [0, T]$, where the function $u^H \in \mathcal{C}^{1,2}$ is given by

$$u^H(t, s) = \mathbb{E}_{t,s}[u(p\psi + z - H + v_0)]. \tag{9.4}$$

By Assumption 7.2, this expression can be further simplified to

$$u^H(t, s) = \mathbb{E}[u(ps\mathcal{E}_{T-t} + z - f(s\mathcal{E}_{T-t}) + v_0)].$$

The following remark gives the dynamics of U^H which we will use in the proofs of Theorems 9.5 and 9.12. In order to state it, we recall the notation $\partial_s \mathbb{E}_{t,s}[f(\psi)]|_{s=S_t}$ introduced subsequent to Lemma 1.12.

Remark 9.4. Under Assumptions 7.1, 7.2 and 7.3 , by the martingale representation theorem, the process U^H, at any time $t \in [0, T]$, can be expressed as

$$U_t^H = U_0^H + \int_0^t i_s^H dW_s, \tag{9.5}$$

for some predictable process $(i_t^H)_{0 \le t \le T}$ with

$$\int_0^T (i_t^H)^2 dt < \infty.$$

Moreover, if $H = f(\psi) \in \mathcal{H}'$ is path independent with Lipschitz continuous f, an application of Itô's Lemma to $U_t^H = u^H(t, S_t)$ from (9.4) yields

$$i_t^H = \sigma S_t \partial_s u^H(t, s)|_{s=S_t} = \sigma S_t \partial_s \mathbb{E}_{t,s}[u(\Sigma_0 - H + v_0)]|_{s=S_t}. \tag{9.6}$$

As $f \in \mathcal{R}$, using Lemma 1.12, this expression can also be written as

$$i_t^H = \sigma \mathbb{E}[u'(\Sigma_0 - f(\psi) + v_0)(p - f'(\psi))\psi|\mathcal{F}_t]. \tag{9.7}$$

The following theorem provides the desired condition to ensure that a claim H is utility replicable and hence attainable.

Theorem 9.5. *Let Assumptions 7.1, 7.2 and 7.3 hold and let $H \in \mathcal{H}'$. Then $H = f(\psi)$ is utility replicable if and only if the function*

$$s \mapsto ps - f(s)$$

is increasing on $(0, \infty)$.

We will prove this theorem with the help of the following three lemmas.

Lemma 9.6. *Let ψ denote a lognormally distributed random variable. Then*

$$\lim_{\theta \to \infty} \left\{ \theta \frac{\mathbb{E}[e^{-\theta\psi}\psi]}{\mathbb{E}[e^{-\theta\psi}]} \right\} = \infty. \tag{9.8}$$

Even though it is well known that $\mathbb{E}[e^{-\theta\psi}\psi]/\mathbb{E}[e^{-\theta\psi}] \to 0$ for $\theta \to \infty$, the rate of this convergence is not clear. In particular, the proof of the seemingly simple assertion of Lemma 9.6 is surprisingly lengthy and therefore deferred to Section 12.1. There, we will also see that the statement of Lemma 9.6 is equivalent to

$$\lim_{\theta\to\infty}\left\{\theta\frac{\mathcal{L}_X(e^{\sigma^2}\theta)}{\mathcal{L}_X(\theta)}\right\} = \infty, \tag{9.9}$$

where $\mathcal{L}_X(r) \triangleq \mathbb{E}[e^{-rX}]$ denotes the Laplace transform of a lognormally distributed random variable $X = e^{\sigma N+\mu}$, $N \sim \mathcal{N}(0,1)$. The result, therefore, possesses an interest of its own as it sheds light on the properties of the not very well understood Laplace transform of the lognormal distribution.

The next lemma establishes the range of the function g from (7.17) by which we express the diffusion term in the dynamics of the market maker's utility process U^Q.

Lemma 9.7. *Let Assumptions 7.1, 7.2 and 7.3 hold. Let further* $\mathbf{u} < 0$*,* $t \in [0,T]$ *and let* g *be the function defined in* (7.17)*. Then the map* $g(\mathbf{u},\cdot,t) : [-p,\infty) \to \mathbb{R}$ *possesses the range*

$$g(\mathbf{u},[-p,\infty),t) = [0,\infty).$$

Proof. From (7.20) we know that g admits the representation

$$g(\mathbf{u},q,t) = -a\sigma\mathbf{u}(p+q)\mathbb{E}^{\mathbb{Q}(p+q)}[\psi|\mathcal{F}_t].$$

For $q = -p$, we have $g(\mathbf{u},q,t) = 0$. Moreover, since $-\mathbf{u} > 0$, for $q > -p$, we have $g(\mathbf{u},q,t) > 0$. By Lemma 9.6, using the definition of $\mathbb{Q}(\cdot)$ and denoting $\theta \triangleq a(p+q)$, we obtain

$$\lim_{q\to\infty}\left\{a(p+q)\mathbb{E}^{\mathbb{Q}(p+q)}[\psi|\mathcal{F}_t]\right\} = \lim_{\theta\to\infty}\left\{\theta\frac{\mathbb{E}[e^{-\theta s\mathcal{E}_{T-t}}s\mathcal{E}_{T-t}]}{\mathbb{E}[e^{-\theta s\mathcal{E}_{T-t}}]}|_{s=S_t}\right\} = \infty.$$

Since $g(\mathbf{u},\cdot,t)$ is continuous and non-negative on $[-p,\infty)$, it follows that its range is given by the interval $[0,\infty)$. $\square$

Lemma 9.8. *Let Assumptions 7.1, 7.2 and 7.3 hold. Let further* $H \in \mathcal{H}'$ *and let* u^H *be as in* (9.4)*. Then* H *is utility replicable if and only if the function* u^H *satisfies*

$$0 \le \partial_s u^H(t,s) < \infty \quad \forall t \in [0,T), s > 0. \tag{9.10}$$

Proof. Let $H \in \mathcal{H}$ be utility replicable. Then there exists an admissible strategy Q such that

$$U_t^Q = U_t^H \quad \forall t \in [0,T].$$

The dynamics of U^Q are given by (7.16) as

$$dU_t^Q = g(U_t^Q, Q_t, t)dW_t$$

and those of U^H are given in Remark 9.4 as

$$dU_t^H = \sigma S_t \partial_s u^H(t, S_t)dW_t.$$

It follows that u^H satisfies

$$\sigma S_t \partial_s u^H(t, S_t) = g(u^H(t, S_t), Q_t, t) \quad \mathbb{P} \otimes \mathrm{dt} - \text{a.e.}, \tag{9.11}$$

which implies that

$$\sigma s \partial_s u^H(t, s) = g(u^H(t, s), Q_t, t) \quad \forall t \in [0, T), s > 0.$$

As Q is admissible, we know that $Q_t \in [-p, \infty)$ for all $t \in [0, T]$. Hence, by Lemma 9.7,

$$0 \leq \partial_s u^H(t, s) < \infty \quad \forall t \in [0, T), s > 0.$$

Assume now that (9.10) holds. Since $g = g(\mathbf{u}, q, t)$ is continuous in q and, for any $t \in [0, T]$, $\mathbf{u} = U_t^H$, its range for varying q is given by $[0, \infty)$ (see Lemma 9.7), there exists a predictable process Q which solves the equation

$$g(U_t^H, Q_t, t) = \sigma S_t \partial_s u^H(t, S_t)$$

at any time $t \in [0, T)$. Hence, Q is an admissible trading strategy satisfying

$$dU_t^Q = dU_t^H \quad \forall t \in [0, T)$$

and consequently $U_t^Q = U_t^H$ for all $t \in [0, T]$, so that H is utility replicable. □

We proceed to prove Theorem 9.5.

Proof of Theorem 9.5. By Lemma 9.8 we know that $H = f(\psi)$ is utility replicable if and only if (9.10) holds. As f is differentiable almost everywhere with bounded derivative, by Lemma 1.12 we obtain

$$\begin{aligned} \partial_s u^H(t, s) &= \partial_s \mathbb{E}_{t,s}[u(p\psi - f(\psi) + z + v_0)] \\ &= \frac{1}{s}\mathbb{E}_{t,s}[u'(p\psi - f(\psi) + z + v_0)(p - f'(\psi))\psi]. \end{aligned}$$

Hence, as $u' > 0$ and since $|f'|$ is bounded, it is clear that (9.10) holds if the map $s \mapsto ps - f(s)$ is increasing on $(0, \infty)$.

Now assume, conversely, that (9.10) holds. As $(\partial_s u^H(t, S_t))_{0\le t\le T}$ is a continuous martingale, for $t \to T$, we have

$$\partial_s u^H(t, S_t) \xrightarrow{a.s.} \partial_s u^H(T, S_T)$$

and hence

$$\partial_s u^H(T, S_T) = u'(p\psi - f(\psi) + z + v_0)(p - f'(\psi)) \ge 0.$$

As $u' > 0$ it follows that $p - f'(s) \ge 0$ for almost all $s \in (0, \infty)$ which implies that the map $s \mapsto ps - f(s)$ is increasing on $(0, \infty)$. □

By similar arguments to those brought forward in Section 3.2, we can use our observations from Lemma 9.7 about the range of g to obtain a necessary and sufficient condition for the utility replicability of acceptable claims $H \in \mathcal{H}$ which are not necessarily path independent.

Proposition 9.9. *Let Assumptions 7.1, 7.2 and 7.3 hold and let $H \in \mathcal{H}$. Let further g be as in (7.17), U^H as in (9.2) and i^H as in Remark 9.4. Then H is utility replicable if and only if there exists an admissible trading strategy Q^H such that for all $t \in [0, T]$*

$$i_t^H = g(U_t^H, Q^H, t). \tag{9.12}$$

Moreover, such a strategy Q^H exists if and only if for all $t \in [0, T]$

$$0 \le i_t^H < \infty. \tag{9.13}$$

Proof. By definition the claim H is utility replicable with replicating strategy Q^H and replication price $v_{rep} = v_0$ if and only if

$$U_t^{Q^H} = U_t^H \quad \forall t \in [0, T]. \tag{9.14}$$

By Lemma 7.7 and Remark 9.4, (9.14) is equivalent to

$$i_t^H = g(U_t^H, Q_t^H, t) \quad \mathbb{P} \otimes \mathrm{d}t - \text{a.e.},$$

which proves the first assertion. By Lemma 9.7 the range of the map $g(U_t^H, \cdot, t) : [-p, \infty) \to \mathbb{R}$ is given by $[0, \infty)$, which proves the second assertion. □

We conclude this section by noting that, in view of Proposition 9.3, a utility replicable claim is, of course, attainable. Theorem 9.5 therefore provides a sufficient condition for the attainability of contingent claims $H \in \mathcal{H}'$; Proposition 9.9 provides one for contingent claims $H \in \mathcal{H}$.

9.3 Replication via the large investor's wealth process

In the previous section we derived a condition which is sufficient for the attainability of a path independent claim $H = f(\psi)$ by replicating the market maker's utility process induced by H. Alternatively, we can approach the replication of H by considering the large investor's wealth process G^Q.

This approach yields a criterion for replicability which is implied by Theorem 9.5 and therefore redundant from a mathematical point of view. However, this criterion is obtained without the detour of *utility replicability* and we include it in order to provide an additional perspective on replication, namely one in terms of the large investor's wealth dynamics rather than the market maker's utility dynamics.

For a claim $H \in \mathcal{H}$ we introduce the process $(G_t^H)_{0\le t\le T}$, defined at any time $t \in [0,T]$ as the (unique) $\mathcal{F}_t$-measurable solution to

$$U_t^H = \mathbb{E}[u(\Sigma_0 - G_t^H)|\mathcal{F}_t],$$

which, by definition of U^H, is given by

$$G_t^H = \frac{1}{a}\log\left(\frac{\mathbb{E}[u(\Sigma_0 + v_0 - H)|\mathcal{F}_t]}{\mathbb{E}[u(\Sigma_0)|\mathcal{F}_t]}\right). \tag{9.15}$$

Here, as in the definition of U^H, v_0 denotes the indifference price of H at time zero. The process G^H measures the wealth that the large investor would hold if he had purchased the claim H from the market maker at time zero in exchange for the cash amount v_0. The large investor's aim is to replicate G^H via trading only in the marketed security ψ; i.e. he needs to find an admissible trading strategy Q such that almost surely

$$G_T^H = G_T^Q.$$

To this end, let us investigate the dynamics of G^H.

Lemma 9.10. *Let Assumptions 7.1, 7.2 and 7.3 hold and let $H \in \mathcal{H}'$. Recall the notation i^H from Remark 9.4 and let $\tilde{g}_t \triangleq F(0,0,t)$. Then*

$$dG_t^H = \frac{1}{a}\left(\frac{i_t^H}{U_t^H}\left(\frac{i_t^H}{U_t^H} - \frac{\tilde{g}_t}{F_t}\right) + \frac{1}{2}\left(\frac{i_t^H}{U_t^H} - \frac{\tilde{g}_t}{F_t}\right)^2\right)dt + \frac{1}{a}\left(\frac{i_t^H}{U_t^H} - \frac{\tilde{g}_t}{F_t}\right)dW_t. \tag{9.16}$$

Proof. Define the auxiliary process $(\tilde{G}_t)_{0\le t\le T}$,

$$\tilde{G}_t^H \triangleq \frac{\mathbb{E}[u(\Sigma_0)|\mathcal{F}_t]}{\mathbb{E}[u(\Sigma_0 - H + v_0)|\mathcal{F}_t]}, \quad 0 \le t \le T.$$

By Itô's quotient rule we obtain

$$\frac{d\tilde{G}_t^H}{\tilde{G}_t} = \left(\frac{\tilde{g}_t}{F_t} - \frac{i_t^H}{U_t^H}\right) dW_t + \left(\frac{(i_t^H)^2}{(U_t^H)^2} - \frac{\tilde{g}_t i_t^H}{F_t U_t^H}\right) dt.$$

An application of Itô's formula to $G^H = -\frac{1}{a}\log(\tilde{G}^H)$ yields the desired result. □

For notational convenience we introduce the probability measure $\mathbb{H}$ associated with a claim $H \in \mathcal{H}'$, defined by

$$\frac{d\mathbb{H}}{d\mathbb{P}} \triangleq \frac{u'(p\psi - H)}{\mathbb{E}[u'(p\psi - H)]}.$$

Remark 9.11. If $H = f(\psi) \in \mathcal{H}'$ then, in view of Remark 9.4 and Lemma 1.11 and since the market maker's utility function u satisfies $u'(\cdot) = -au(\cdot)$, we have

$$\frac{i_t^H}{U_t^H} = -a\sigma\mathbb{E}^{\mathbb{H}}[p\psi - \psi f'(\psi)|\mathcal{F}_t] \quad \forall t \in [0,T]. \tag{9.17}$$

We are now ready to state and prove a condition which ensures that a claim $H \in \mathcal{H}'$ is attainable.

Theorem 9.12. *Let Assumptions 7.1, 7.2 and 7.3 hold and let $H = f(\psi) \in \mathcal{H}'$. If*

$$0 \ \leq \ \mathbb{E}^{\mathbb{H}}[p\psi - \psi f'(\psi)|\mathcal{F}_t] \ < \ \infty \quad \forall t \in [0,T] \tag{9.18}$$

then H is attainable.

Proof. Let (9.18) hold. At time zero, for any admissible trading strategy Q, we have

$$U_0^H = U_0^Q = \mathbb{E}[u(\Sigma_0)]$$

so that $G_0^H = G_0^Q$. We will now construct an admissible trading strategy Q such that

$$dG_t^H = dG_t^Q \quad \forall t \in [0,T] \tag{9.19}$$

and consequently $G_T^Q = G_T^H$.

Recall the predictable field $P_t(\cdot)$ introduced prior to Lemma 8.1. Observe that $P_t(0) = 0$ and that, by Lemma 9.6,

$$\lim_{q\to\infty} P_t(p+q) = \infty.$$

The continuity and non-negativity of the map $q \mapsto P_t(p+q)$ on $[-p,\infty)$ thus imply that its range is given by $[0,\infty)$. Hence, by (9.18), it follows that there exists a predictable process Q which satisfies

$$\mathbb{E}^{\mathbb{H}}[p\psi - \psi f'(\psi)|\mathcal{F}_t] = P_t(p + Q_t) \tag{9.20}$$

at all times $t \in [0, T]$.

Comparing the terms for dG^Q and dG^H given in (8.1) and (9.16), respectively, we find that for our choice of Q, in view of Remark 9.11, they are identical.

□

9.4 Hedging call- and put options

In this section we will show that call- and put options are utility replicable and therefore attainable in our model. For the sake of convenience we first state the following corollary to Theorem 9.5 which simplifies the ensuing proofs in this section.

Corollary 9.13. *Let Assumptions 7.1, 7.2 and 7.3 hold and let $H \in \mathcal{H}'$. Then a position $\xi \cdot H$, $\xi \in \mathbb{R}$, is utility replicable if and only if the function*

$$s \mapsto ps - \xi \cdot f(s)$$

is increasing on $(0, \infty)$.

Proof. The result follows immediately upon considering $\tilde{H} \triangleq \xi f(\psi)$ in Theorem 9.5. □

Using this result, we easily obtain the attainability of (limited positions of) call options and put options presented in the following two corollaries.

Corollary 9.14. *Let Assumptions 7.1, 7.2 and 7.3 hold and let $H = (\psi - K)_+$ denote a call option. Then a position ξH is attainable if and only if $\xi \leq p$.*

Proof. Consider the payoff function $f(s) = (s - K)_+$ of a call option. We can see that

$$s \mapsto ps - \xi f(s) = ((p - \xi)s + \xi K)\mathbb{1}_{\{s \geq K\}} + ps\mathbb{1}_{\{s < K\}}$$

is increasing on $(0, \infty)$ if and only if $\xi \leq p$. The result then follows from Corollary 9.13. □

Corollary 9.15. *Let Assumptions 7.1, 7.2 and 7.3 hold and let $H = (K - \psi)_+$ denote a put option. Then a position ξH is attainable if and only if $\xi \geq -p$.*

Proof. Consider the payoff function $f(s) = (K - s)_+$ of a put option. We can see that

$$s \mapsto ps - \xi f(s) = ((p + \xi)s - \xi K)\mathbb{1}_{\{s \leq K\}} + ps\mathbb{1}_{\{s > K\}}$$

is increasing on $(0, \infty)$ if and only if $\xi \geq -p$. The result then follows from Corollary 9.13. □

Note that for the attainability of a position ξH of calls or puts, respectively, the bounds stated in the two corollaries above are both sufficient and necessary: No position of calls or puts not satisfying these bounds can be replicated. This is immediately clear from the fact that positions violating these bounds are not *acceptable* (in the sense of Definition 9.1).

We observe that the number of call options which can be replicated in this model is bounded from above by the market maker's initial security position p while the number of put options that can be replicated is bounded from below by $-p$. In other words, the market maker's risk preferences imply that he would agree to write an arbitrarily large number of put options or buy an arbitrarily large number of call options while he would write a maximum of p call options or buy a maximum of p put options, respectively, were he to engage in the trading of options at all.

This phenomenon is due to the fact that in our model the market maker is willing to "borrow cash to buy stocks" while he will never "short stocks to obtain cash". The former behaviour, of course, is akin to the writing of a number of put options whose total liability lies beyond the coverage by his initial cash z, while the latter is akin to the writing of a number of call options which surpasses his initial stock position p.

Thus, the one-sided boundedness of attainable call- and put positions is a result of the market maker's refusal to take total short positions in the traded asset ψ and is therefore closely related to the one-sided boundedness of admissible trading strategies as detailed in Section 7.4.

9.5 A BSDE view of the hedging problem

Our analysis of the attainability of contingent claims was based on keeping track of the market maker's level of expected utility. Alternatively, it is possible to formulate the question of the attainability of a contingent claim $H \in \mathcal{H}$ as a BSDE problem. Recall the processes $(P_t(r))_{0\le t\le T}$ and $(\hat{P}_t^Q)_{0\le t\le T}$, defined by

$$P_t(r) \triangleq r\mathbb{E}^{\mathbb{Q}(r)}[\psi|\mathcal{F}_t]$$

and $\hat{P}_t^Q \triangleq P_t(p+Q_t) - P_t(p)$, respectively, which were introduced in Section 8.1. Let

$$Z_t \triangleq \sigma\hat{P}_t^Q, \quad 0 \le t \le T, \tag{9.21}$$

and

$$f(t, Z_t) \triangleq a\sigma^2(P_t(p)Z_t + \tfrac{3}{2}Z_t^2). \tag{9.22}$$

Then, in view of (8.4), the fact that Q is a hedge for H can be expressed as

$$Y_t = Y_T - \int_t^T f(s, Z_s)ds + \int_t^T Z_s dW_s, \quad Y_T = H. \tag{9.23}$$

This equation can be interpreted as a quadratic BSDE and the existence of a solution (Y, Z) to (9.23) can be obtained with e.g. the arguments in [38]. However, the existence of a solution to the BSDE is not the main concern: The actual problem consists in showing that, given a solution (Y, Z) to (9.23), there exists an admissible strategy Q which "induces" Z in the way of (9.21). Hence, the BSDE formulation of the problem of attainability is merely another way to express a question which we have answered above by keeping track of the market maker's utility process and using Itô's representation theorem.

This concludes our investigation of the attainability of contingent claims. The question that follows naturally and that we shall devote our attention to in the next chapter is *how* the replication of a claim is accomplished in our model.

Chapter 10

Asymptotic analysis of price processes and replicating strategies

In the previous chapter we derived conditions which guarantee that a path independent claim H is attainable. However, the corresponding replicating strategy Q^H was given only implicitly and it seems that no closed-form solution for Q^H exists; not even in the case where H is, e.g., a call option.

Hence, in order to investigate hedging strategies, we will conduct an asymptotic analysis where we consider small positions of an attainable contingent claim H in order to calculate first and second order approximations to the hedging strategy Q^H. We will see that the first order term, as in the power utility case, can be interpreted as the replicating strategy in a liquid setting while the second order term can again be viewed as a *liquidity correction* to the hedge ratio.

It turns out that the first and second order approximations are both structurally similar to their counterparts in Part I. However, in the case of exponential utilities, the market maker's absolute risk aversion

$$R(\Sigma_0) = -\frac{u''(\Sigma_0)}{u'(\Sigma_0)} = a$$

is constant, which, together with the cash invariance of exponential utilities, allows for significant simplifications of the expansions. In particular, we will see that the measures $\mathbb{Q}$ and $\mathbb{R}$, which were introduced in the power utility case, coincide in the exponential utility setting. This will allow us to express the second order approximation to the replicating position in a shorter and more intuitive form.

In Section 10.1 we briefly discuss marginal prices and liquidity premia for claims as well as for the traded security ψ as these will be needed for our asymptotic analysis of hedging strategies in Section 10.2.

Note that, as yet, the results in this chapter as well as the comparative statics in the following chapter are conditional on the validity of a monotonicity assertion related to the Laplace transform of the lognormal distribution which we state as Conjecture 2 further below. Although this conjecture is strongly supported by numerical evidence, an analytical proof has yet to be found.

10.1 Marginal prices for contingent claims and for the traded security

In this section we will determine the prices that occur when small quantities of assets and contingent claims are traded. We compute the first and second order approximations to the indifference prices of the traded security ψ and of acceptable contingent claims H. While it is well known that the first order approximation is the marginal price of the respective financial instrument, we will interpret the second order approximation as a (nonlinear) liquidity correction arising in our model.

Recall the marginal indifference pricing measure $\mathbb{Q}(\cdot)$ introduced in Section 7.4. This measure owes its name to the fact that the marginal prices of claims H and of the traded security ψ are given as their respective expected payoffs under the measure $\mathbb{Q}(p)$. As we will be dealing exclusively with $\mathbb{Q}(p)$ for the remainder of this chapter, we will omit the dependence on p and merely write $\mathbb{Q}$.

Throughout this chapter we will mostly consider contingent claims belonging to the set

$$\mathcal{H}'' \triangleq \{H \in \mathcal{H}' \mid \varepsilon H \text{ is utility replicable } \forall \varepsilon \in [0,1]\}.$$

As a consequence of Corollary 9.13 it is immediately clear that positions εH, $0 \leq \varepsilon \leq 1$, of utility replicable claims are utility replicable, so that we can write $\mathcal{H}''$ equivalently as

$$\mathcal{H}'' = \{H \in \mathcal{H}' \mid H \text{ is utility replicable}\}.$$

Let now $H \in \mathcal{H}$ be a contingent claim. Then the utility indifference price of a number of claims εH at time $t \in [0,T]$ is given by the unique cash amount $v_t^\varepsilon(H)$ which solves

$$\mathbb{E}[u(\Sigma_0)|\mathcal{F}_t] = \mathbb{E}[u\big(\Sigma_0 - \varepsilon H + v_t^\varepsilon(H)\big)|\mathcal{F}_t].$$

Solving this equation for $v_t^\varepsilon(H)$ yields the familiar expression

$$v_t^\varepsilon(H) = \frac{1}{a}\log\left(\frac{\mathbb{E}[u(\Sigma_0 - \varepsilon H)|\mathcal{F}_t]}{\mathbb{E}[u(\Sigma_0)|\mathcal{F}_t]}\right) = \frac{1}{a}\log\left(\frac{\mathbb{E}[e^{-a(p\psi - \varepsilon H)}|\mathcal{F}_t]}{\mathbb{E}[e^{-ap\psi}|\mathcal{F}_t]}\right). \tag{10.1}$$

Note that $v_t^0(H) = 0$ and that $v_0^1(H) = v_0$ is the market indifference price of H at time zero. For notational convenience we will merely write v_t^ε rather than $v_t^\varepsilon(H)$ from now on. The following proposition gives the first and second order derivatives of v_t^ε with respect to ε.

Proposition 10.1. *Under Assumptions 7.1, 7.2 and 7.3 let $H \in \mathcal{H}''$. Let further $(v_t^\varepsilon)_{0\le t\le T}$ denote the stochastic field defined by* (10.1)*. Then there exists a version of v^ε such that for any $\omega \in \Omega$, $t \in [0,T]$ the function*

$$\varepsilon \mapsto v_t^\varepsilon(\omega),$$

which maps a quantity ε of the contingent claim H to its market indifference price, is twice continuously differentiable on $(0,1)$*. Furthermore, its derivatives can be extended continuously to $\varepsilon = 0$ and $\varepsilon = 1$ and in $\varepsilon = 0$ they are given by*

$$\partial_\varepsilon v_t^\varepsilon|_{\varepsilon=0} = \frac{\mathbb{E}[u'(\Sigma_0)H|\mathcal{F}_t]}{\mathbb{E}[u'(\Sigma_0)|\mathcal{F}_t]} = \mathbb{E}_t^{\mathbb{Q}}[H] \tag{10.2}$$

and

$$\partial_\varepsilon^2 v_t^\varepsilon|_{\varepsilon=0} = -\frac{\mathbb{E}[u''(\Sigma_0)(\partial_\varepsilon v_t^\varepsilon|_{\varepsilon=0} - H)^2|\mathcal{F}_t]}{\mathbb{E}[u'(\Sigma_0)|\mathcal{F}_t]} = a\,\mathrm{Var}_t^{\mathbb{Q}}[H]. \tag{10.3}$$

Proof. Since $H \in \mathcal{H}''$, by Lemma 1.11 we can choose a version of v^ε which has twice continuously differentiable sample paths. Differentiating v_t^ε once and twice, respectively, with respect to ε yields

$$\partial_\varepsilon v_t^\varepsilon = \frac{\mathbb{E}[e^{-a(p\psi-\varepsilon H)}H|\mathcal{F}_t]}{\mathbb{E}[e^{-a(p\psi-\varepsilon H)}|\mathcal{F}_t]},$$

and

$$\partial_\varepsilon^2 v_t^\varepsilon = a\left(\frac{\mathbb{E}[e^{-a(p\psi-\varepsilon H)}H^2|\mathcal{F}_t]}{\mathbb{E}[e^{-a(p\psi-\varepsilon H)}|\mathcal{F}_t]} - \frac{\mathbb{E}[e^{-a(p\psi-\varepsilon H)}H|\mathcal{F}_t]^2}{\mathbb{E}[e^{-a(p\psi-\varepsilon H)}|\mathcal{F}_t]^2}\right).$$

The dominated convergence theorem implies the existence of continuous extensions of these identities to the boundary points $\varepsilon = 0$ and $\varepsilon = 1$. The desired expressions (10.2) and (10.3) then follow for $\varepsilon = 0$. □

The first order approximation $\partial_\varepsilon v_t^\varepsilon|_{\varepsilon=0}$ is the *marginal utility indifference price* of H at time t while the second order approximation $\partial_\varepsilon^2 v_t^\varepsilon|_{\varepsilon=0}$ can be viewed as the (nonlinear) *liquidity correction* that arises in our model.

We now turn our attention to the marginal price of the traded security ψ. In accordance with our convention of viewing transfers of both the traded security ψ and the complementing cash from the point of view of the market maker, we consider $H = -\psi$ in (10.1) which thus becomes

$$v_t^\varepsilon(-\psi) = \frac{1}{a}\log\left(\frac{\mathbb{E}[u(\Sigma_0+\varepsilon\psi)|\mathcal{F}_t]}{\mathbb{E}[u(\Sigma_0)|\mathcal{F}_t]}\right) = \frac{1}{a}\log\left(\frac{\mathbb{E}[e^{-a(p+\varepsilon)\psi)}|\mathcal{F}_t]}{\mathbb{E}[e^{-ap\psi}|\mathcal{F}_t]}\right). \tag{10.4}$$

In order to emphasize the difference between the marginal prices of H and ψ, we will denote the marginal price of the traded security ψ by

$$x_t^\varepsilon \triangleq v_t^\varepsilon(-\psi).$$

Analog arguments to those presented in the proof of Proposition 10.1 yield

$$\partial_\varepsilon x_t^\varepsilon|_{\varepsilon=0} = -\frac{\mathbb{E}[u'(\Sigma_0)\psi|\mathcal{F}_t]}{\mathbb{E}[u'(\Sigma_0)|\mathcal{F}_t]} = -\mathbb{E}_t^{\mathbb{Q}}[\psi] \tag{10.5}$$

and

$$\partial_\varepsilon^2 x_t^\varepsilon|_{\varepsilon=0} = -\frac{\mathbb{E}[u''(\Sigma_0)(\partial_\varepsilon x_t^\varepsilon|_{\varepsilon=0}+\psi)^2|\mathcal{F}_t]}{\mathbb{E}[u'(\Sigma_0)|\mathcal{F}_t]} = a\,\mathrm{Var}_t^{\mathbb{Q}}[\psi] \tag{10.6}$$

for the first and second order derivatives of x_t^ε. Again, the first order term x_t^ε is the marginal price of the traded security ψ and the second order term $\partial_\varepsilon^2 x_t^\varepsilon|_{\varepsilon=0}$ can be interpreted as a *liquidity correction* for the price of ψ.

The liquidity correction terms (10.3) and (10.6) both take the shape of a variance and are therefore positive. This means that the market maker will pay less for the acquisition, respectively charge more for the sale, of an additional ε shares than the marginal price suggests. The premium that he charges for taking on additional risk increases linearly in the variance of ψ (under the marginal indifference pricing measure $\mathbb{Q}$) and in his risk aversion a.

Remark 10.2. In Chapter 4 we introduced the measure $\mathbb{R}$ with Radon-Nikodym density $d\mathbb{R} \triangleq u''(\Sigma_0)/\mathbb{E}[u''(\Sigma_0)]\, d\mathbb{P}$ in order to express the analogs of the second order terms (10.3) and (10.6). For exponential utilities we have

$$\frac{d\mathbb{R}}{d\mathbb{P}} = \frac{u''(\Sigma_0)}{\mathbb{E}[u''(\Sigma_0)]} = \frac{u'(\Sigma_0)}{\mathbb{E}[u'(\Sigma_0)]} = \frac{d\mathbb{Q}}{d\mathbb{P}},$$

so that the measure $\mathbb{R}$ coincides with $\mathbb{Q}$.

Remark 10.3. By an analog argument to the one presented in Remark 4.5, the state process $(S_t)_{0\le t\le T}$ from Assumption 7.2 retains the Markov property under the marginal indifference pricing measure $\mathbb{Q}$.

10.2 Liquidity correction for hedge ratios

As in the power utility case, our aim is to compute the first and second order approximations to the replicating position $Q_t^{\varepsilon H}$ for a small position of claims $\varepsilon H \in \mathcal{H}''$. More precisely, we will compute the terms $\partial_\varepsilon Q_t^{\varepsilon H}|_{\varepsilon=0}$ and $\partial_\varepsilon^2 Q^{\varepsilon H}|_{\varepsilon=0}$ in the Taylor expansion

$$Q_t^{\varepsilon H} = \varepsilon \partial_\varepsilon Q_t^{\varepsilon H}|_{\varepsilon=0} + \frac{\varepsilon^2}{2}\partial_\varepsilon^2 Q^{\varepsilon H}|_{\varepsilon=0} + o(\varepsilon^2). \tag{10.7}$$

Theorem 10.8 below shows that the first and second order approximations $\partial_\varepsilon Q_t^{\varepsilon H}|_{\varepsilon=0}$ and $\partial_\varepsilon^2 Q^{\varepsilon H}|_{\varepsilon=0}$ are of a similar form to that in the power utility case.

Before we proceed to state and prove the theorem, we establish several preparatory lemmas. The first lemma shows that the important map g from (7.17) is sufficiently differentiable for our purposes and strictly increasing in the order size q. Its proof relies on the validity of the following conjecture.

Conjecture 2. *Let ψ be a lognormally distributed random variable. Let further*

$$k_2(\theta) \triangleq \theta \frac{\mathbb{E}[e^{-\theta\psi}\psi]}{\mathbb{E}[e^{-\theta\psi}]}.$$

Then

$$k_2'(\theta) > 0 \quad \forall \theta \geq 0.$$

Despite being strongly supported by numerical evidence, this conjecture has not yet been proven analytically. As a consequence, the validity of the expansions of Theorem 10.8 and of all related results is yet conditional on the validity of Conjecture 2. For a more detailed discussion of this conjecture see Section 12.2.

Lemma 10.4. *Let Assumptions 7.1, 7.2 and 7.3 hold and assume that Conjecture 2 is true. Let g be as in (7.17). Then for any $\omega \in \Omega$, $\mathbf{u} \in (-\infty, 0)$ and $t \in [0, T]$ the map*

$$g(\mathbf{u}, \cdot, t, \omega) : [-p, \infty) \to [0, \infty), \quad q \mapsto g(\mathbf{u}, q, t, \omega)$$

is twice continuously differentiable on the interior of its domain and its derivatives can be extended continuously to the boundary point $q = -p$, where the first order derivative takes the value

$$\partial_q g(\mathbf{u}, q, t, \omega)|_{q=-p} = -a\sigma\mathbf{u}\mathbb{E}[\psi|\mathcal{F}_t](\omega). \tag{10.8}$$

Moreover, $g(\mathbf{u}, \cdot, t)$ is strictly increasing.

Proof. By (7.17), (7.13) and (7.14) we have

$$g(\mathbf{u}, q, t) = \tilde{g}(A_t(\mathbf{u}, q), q, t) = e^{-aA_t(\mathbf{u},q)}\tilde{g}(0, q, t),$$

where

$$A_t(\mathbf{u}, q) = -\frac{1}{a}\log\left(\frac{\mathbf{u}}{\mathbb{E}[u(\Sigma_0 + q\psi)|\mathcal{F}_t]}\right).$$

By Lemma 1.11 the sample paths $A_t^\omega(\mathbf{u}, q)$ and $\tilde{g}(0, q, t, \omega)$ are twice continuously differentiable in q which implies that the map $g(\mathbf{u}, \cdot, t, \omega)$ is twice continuously differentiable. By the dominated convergence theorem the derivatives can be continuously extended to the boundary point $q = -p$. Using (7.20), we compute

$$\partial_q g(\mathbf{u}, q, t) = -a\sigma\mathbf{u}\left(\mathbb{E}_t^{\mathbb{Q}(p+q)}[\psi] - a(p+q)\operatorname{Var}_t^{\mathbb{Q}(p+q)}[\psi]\right),$$

which, for $q = -p$, yields (10.8). It is left to show that $g(\mathbf{u}, q, t)$ is strictly increasing in q. By (7.20) we know that g can be expressed as

$$g(\mathbf{u}, q, t) = -a\sigma\mathbf{u}(p+q)\mathbb{E}^{\mathbb{Q}(p+q)}[\psi|\mathcal{F}_t] = -a\sigma\mathbf{u}(p+q)\frac{\mathbb{E}[e^{-a(p+q)\psi}\psi|\mathcal{F}_t]}{\mathbb{E}[e^{-a(p+q)\psi}|\mathcal{F}_t]}.$$

Since $-\sigma\mathbf{u} > 0$, denoting $\theta \triangleq a(p+q)$, it follows that $g = g(\mathbf{u}, q, t)$ is strictly increasing in q if and only if

$$k_2(\theta) \triangleq \theta\frac{\mathbb{E}[e^{-\theta\psi}\psi|\mathcal{F}_t]}{\mathbb{E}[e^{-\theta\psi}|\mathcal{F}_t]}$$

is strictly increasing in θ. This statement, in turn, is implied by Conjecture 2. □

The next lemma shows that the replicating position $Q_t^{\varepsilon H}$ for a small number of claims is differentiable with respect to ε.

Lemma 10.5. *Let Assumptions 7.1, 7.2 and 7.3 hold and assume that Conjecture 2 is true. Let further $H = f(\psi) \in \mathcal{H}''$ and let $(Q_t^{\varepsilon H})_{0\le t\le T}$ denote the replicating strategy for εH. Then there exists a version of $Q^{\varepsilon H}$ such that for each $\omega \in \Omega$, $t \in [0,T]$ the function*

$$\varepsilon \mapsto Q_t^{\varepsilon H}(\omega)$$

is twice continuously differentiable on $(0,1)$. Furthermore, the first and second order derivatives can be continuously extended to the boundary point $\varepsilon = 0$.

Proof. We saw in Proposition 9.9 that, at any time $t \in [0,T]$, $Q_t^{\varepsilon H}$ is the utility-replicating position for εH if and only if $Q_t^{\varepsilon H}$ satisfies

$$i_t^{\varepsilon H} = g(U_t^{\varepsilon H}, Q_t^{\varepsilon H}, t),$$

where

$$i_t^{\varepsilon H} = \sigma\mathbb{E}[u'(\Sigma_0 - \varepsilon f(H) + v_0^{\varepsilon})(p - \varepsilon f'(\psi))\psi|\mathcal{F}_t]$$

is as in (9.7). By Proposition 10.1 we know that there exists a version of $v_0^{\varepsilon H}$ such that for any $\omega \in \Omega$ the map $\varepsilon \mapsto v_0^{\varepsilon H}(\omega)$ is twice continuously differentiable which, together with Lemma 1.11, implies that there exists a version of $i^{\varepsilon H}$ such that for any $\omega \in \Omega$, $t \in [0,T]$, the map $\varepsilon \mapsto i_t^{\varepsilon H}(\omega)$ is twice continuously differentiable.

For fixed $\mathbf{u} \in [0,\infty)$ and $t \in [0,T]$ Lemma 10.4 implies that the map $q \mapsto g(\mathbf{u}, q, t)$ is twice continuously differentiable as well as strictly increasing and therefore 1-to-1. Hence, there exists a version of $Q^{\varepsilon H}$ which is such that, by the implicit function theorem, the map $\varepsilon \mapsto Q_t^{\varepsilon H}(\omega)$ is twice continuously differentiable on $[0,1]$. □

The next preparatory lemma provides representations for the first and second order approximations to the complementing cash position $X_t^{Q^{\varepsilon H}}$ for the replicating position $Q_t^{\varepsilon H}$.

Lemma 10.6. *Let Assumptions 7.1, 7.2 and 7.3 hold, let $H \in \mathcal{H}''$ and assume that Conjecture 2 is true. Let $(Q_t^{\varepsilon H})_{0\le t\le T}$ be a replicating strategy for εH and let*

$(X_t^{Q^{\varepsilon H}})_{0\leq t\leq T}$ denote its complementing cash-balance process. Let further v_t^ε and x_t^ε, as defined in Section 10.1, denote the indifference prices of claim H and underlying ψ, respectively. Then there exists a version of $X^{Q^{\varepsilon H}}$ such that for any $\omega \in \Omega$, $t \in [0,T]$, the map

$$\varepsilon \mapsto X_t^{Q^{\varepsilon H}}(\omega)$$

is twice continuously differentiable on $(0,1)$. Furthermore, its first and second order derivatives can be continuously extended to the boundary points $\varepsilon = 0$ and $\varepsilon = 1$. In $\varepsilon = 0$, they are given by

$$\partial_\varepsilon X_t^{Q^{\varepsilon H}}|_{\varepsilon=0} \quad = \quad -\left(\partial_\varepsilon v_t^\varepsilon|_{\varepsilon=0} - \partial_\varepsilon v_0^\varepsilon|_{\varepsilon=0}\right) + \partial_\varepsilon x_t^\varepsilon|_{\varepsilon=0}\partial_\varepsilon Q_t^{\varepsilon H}|_{\varepsilon=0}$$

and

$$\begin{aligned}\partial_\varepsilon^2 X_t^{Q^{\varepsilon H}}|_{\varepsilon=0} \quad = \quad & \frac{\mathbb{E}[u''(\Sigma_0)\{(\partial_\varepsilon v_0^\varepsilon|_{\varepsilon=0} - H)^2 - (\partial_\varepsilon Q_t^{\varepsilon H}|_{\varepsilon=0}\psi + \partial_\varepsilon X_t^{Q^{\varepsilon H}})^2\}|\mathcal{F}_t]}{\mathbb{E}[u'(\Sigma_0)|\mathcal{F}_t]} \\ & + \frac{\mathbb{E}[u'(\Sigma_0)\{\partial_\varepsilon^2 v_0^\varepsilon|_{\varepsilon=0} - \partial_\varepsilon^2 Q_t^{\varepsilon H}|_{\varepsilon=0}\psi\}|\mathcal{F}_t]}{\mathbb{E}[u'(\Sigma_0)|\mathcal{F}_t]}.\end{aligned}$$

Proof. By definition of $X^{Q^{\varepsilon H}}$ we have

$$\mathbb{E}[u(\Sigma_0 + Q_t^{\varepsilon H}\psi + X_t^{Q^{\varepsilon H}})|\mathcal{F}_t] = \mathbb{E}[u(\Sigma_0 - \varepsilon H + v_0^\varepsilon)|\mathcal{F}_t]. \tag{10.9}$$

By Lemma 10.5, there exists a version of $Q^{\varepsilon H}$ such that for any $\omega \in \Omega$, $t \in [0,T]$ the map $\varepsilon \mapsto Q_t^{\varepsilon H}(\omega)$ is twice continuously differentiable on $(0,1)$. The implicit function theorem together with Lemma 1.11 thus implies the existence of a version of $X^{Q^{\varepsilon H}}$ such that, for any $\omega \in \Omega$, $t \in [0,T]$, the map $\varepsilon \mapsto X_t^{Q^{\varepsilon H}}(\omega)$ is twice continuously differentiable on $(0,1)$. The existence of continuous extensions of the derivatives to the boundary points $\varepsilon = 0$ and $\varepsilon = 1$ as well as the desired terms for the first and second order approximations are obtained by analog arguments and computations to those in the proof of Lemma 4.9. $\square$

The last preliminary lemma shows that the denominator for the expansions in Theorem 10.8 is strictly positive.

Lemma 10.7. *Let Assumptions 7.1, 7.2, 7.3 hold and assume that Conjecture 2 is true. Then the function*

$$h : [0,T] \times (0,\infty) \to (0,\infty), \quad (t,s) \mapsto \mathbb{E}_{t,s}^{\mathbb{Q}}[\psi]$$

is continuously differentiable with respect to s and $\partial_s h(t,s) > 0$.

Proof. Upon explicit computation of $\partial_s h(t,s)$ one can see that $\partial_s h(t,s) > 0$ if and only if Conjecture 2 holds. $\square$

We are now ready to state the following theorem which constitutes one of the main results of the second part of this thesis. Just like its analog in Part I, the theorem identifies the terms $\tilde{\Delta} \triangleq \partial_\varepsilon Q^{\varepsilon H}|_{\varepsilon=0}$ and $\tilde{\Lambda} \triangleq \partial^2_\varepsilon Q^{\varepsilon H}|_{\varepsilon=0}$ in the Taylor approximation

$$Q_t^{\varepsilon H} = \varepsilon \tilde{\Delta}_t + \frac{\varepsilon^2}{2}\tilde{\Lambda}_t + o(\varepsilon^2).$$

While we find that these terms possess representations which are structurally the same as in the case of power utilities, we will show later on that the second order approximation $\tilde{\Lambda}$ can be further simplified for exponential utility functions.

Theorem 10.8. *Let Assumptions 7.1, 7.2 and 7.3 hold and assume that Conjecture 2 is true. Let further $H \in \mathcal{H}''$ and let $(Q_t^{\varepsilon H})_{0\le t\le T}$ be the replicating strategy for εH. Moreover, let*

$$\begin{aligned} \hat{H} &\triangleq H - \mathbb{E}^{\mathbb{Q}}[H], \\ \hat{\psi} &\triangleq \psi - \mathbb{E}^{\mathbb{Q}}[\psi]. \end{aligned}$$

Then

(i) the first order approximation to $Q_t^{\varepsilon H}$ is given by

$$\partial_\varepsilon Q_t^{\varepsilon H}|_{\varepsilon=0} = -\frac{\partial_s \mathbb{E}^{\mathbb{Q}}_{t,s}[H]|_{s=S_t}}{\partial_s \mathbb{E}^{\mathbb{Q}}_{t,s}[\psi]|_{s=S_t}} =: \tilde{\Delta}_t, \quad 0 \le t \le T, \tag{10.10}$$

(ii) and the second order approximation to $Q_t^{\varepsilon H}$ is given by

$$\partial^2_\varepsilon Q_t^{\varepsilon H}|_{\varepsilon=0} = \frac{a\sum_{i=1}^3 c_i(t,S_t)\partial_s \mathbb{E}^{\mathbb{Q}}_{t,s}[K_i]|_{s=S_t}}{\partial_s \mathbb{E}^{\mathbb{Q}}_{t,s}[\psi]|_{s=S_t}} := \tilde{\Lambda}_t, \quad 0 \le t \le T, \tag{10.11}$$

where the random variables K_i, $i \in \{1,\dots,4\}$ are given by

$$\begin{aligned} K_1 &= \hat{H}^2, \\ K_2 &= \hat{\psi}^2, \\ K_3 &= \hat{\psi} \end{aligned}$$

and the coefficients c_i, $i \in \{1,\dots,4\}$ are given by

$$\begin{aligned} c_1(t,S_t) &\equiv -1, \\ c_2(t,S_t) &= \tilde{\Delta}_t^2, \\ c_3(t,S_t) &= -2\tilde{\Delta}_t(\tilde{\Delta}_t\mathbb{E}^{\mathbb{Q}}_t[\hat{\psi}] + \mathbb{E}^{\mathbb{Q}}_t[\hat{H}]). \end{aligned}$$

Proof. Having established Lemmas 10.5, 10.6, and 10.7 the proof is analog to that of

Theorem 4.12. □

The benefit of Theorem 10.8, as of its analog in the power utility case, consists in providing a "liquid-world answer" to the question of replication under illiquidity. The first order approximation (10.10) is akin to the Black-Scholes delta and can be computed with similar means while the second order approximation (10.11) is expressed as a linear combination of hedge ratios of auxiliary claims which, again, can be computed using the same methods as for the Black-Scholes delta.

This provides a "liquid recipe for illiquid replication" in the sense that a portfolio manager who finds himself presented with the task to replicate the claim εH in our model could (roughly) be advised to act as follows: "Delta-hedge the claim εH with respect to the marginal indifference pricing measure $\mathbb{Q}$ rather than the risk neutral measure and, additionally, delta-hedge a position of size ε^2 of these auxiliary claims which will offset your liquidity risk."

In Chapter 4, we saw that the first order approximation $\tilde{\Delta}$ can be interpreted as the replicating strategy of a small investor who trades the marketed security ψ at its marginal price. The same interpretation is valid in the exponential utility setting, as the following proposition shows.

Proposition 10.9. *Let Assumptions 7.1, 7.2 and 7.3 hold, let $H \in \mathcal{H}''$ and assume that Conjecture 2 is true. Let further $(\pi_t)_{0\le t\le T}$, defined by*

$$\pi_t \triangleq \mathbb{E}^{\mathbb{Q}}[H|\mathcal{F}_t]$$

and $(\tilde{S}_t)_{0\le t\le T}$, defined by

$$\tilde{S}_t \triangleq \mathbb{E}^{\mathbb{Q}}[\psi|\mathcal{F}_t]$$

denote the marginal price processes of the claim H and the marketed security ψ. Then

$$\pi_t = \pi_0 + \int_0^t \tilde{\Delta}_s d\tilde{S}_s.$$

In particular, for $t = T$,

$$H = \pi_0 + \int_0^T \tilde{\Delta}_s d\tilde{S}_s.$$

Proof. The proof is analog to that of Proposition 4.13. □

The small investor's replicating strategy $\tilde{\Delta}$ is complemented by the cash balance process which is given by the following proposition.

Proposition 10.10. *Let Assumptions 7.1, 7.2 and 7.3 hold, let $H = f(\psi) \in \mathcal{H}''$ and assume that Conjecture 2 is true. Let further $(\tilde{\Delta})_{0\le t\le T}$ be as in Theorem 10.8 and $\partial_\varepsilon X^{Q^{\varepsilon H}}|_{\varepsilon=0}$ as in Lemma 10.6. Then the complementing cash balance process*

$(X_t^{\tilde{\Delta}})_{0\le t\le T}$, *defined by*

$$X_t^{\tilde{\Delta}} \triangleq \partial_\varepsilon X_t^{Q^{\varepsilon H}}|_{\varepsilon=0}, \quad 0 \le t \le T,$$

is given by

$$X_t^{\tilde{\Delta}} = -\mathbb{E}^{\mathbb{Q}}[\hat{H}|\mathcal{F}_t] - \tilde{\Delta}_t \mathbb{E}^{\mathbb{Q}}[\psi|\mathcal{F}_t], \quad 0 \le t \le T.$$

Proof. The proof is analog to the proof of Proposition 4.14. □

In the exponential utility setting, the liquidity correction term $\tilde{\Lambda}$ can be expressed in a form which allows for a more intuitive interpretation than was the case in Part I. The following proposition shows that $\tilde{\Lambda}$ is the sensitivity with respect to changes in the underlying of the difference of variances of the claim H and a small investor's net replicating position for H under the measure $\mathbb{Q}$.

Proposition 10.11. *Let Assumptions 7.1, 7.2 and 7.3 hold, let $H \in \mathcal{H}''$ and assume that Conjecture 2 is true. Then*

$$\tilde{\Lambda}_t = \frac{\frac{\partial}{\partial s}\{\mathrm{Var}_{t,s}^{\mathbb{Q}}[H] - \tilde{\Delta}_t^2 \mathrm{Var}_{t,s}^{\mathbb{Q}}[\psi]\}|_{s=S_t}}{\partial_s \mathbb{E}_{t,s}^{\mathbb{Q}}[\psi]|_{s=S_t}}, \quad 0 \le t \le T. \tag{10.12}$$

Proof. Let

$$\xi(t) \triangleq \tilde{\Delta}_t \psi + X_t^{\tilde{\Delta}}.$$

We will prove that

$$\tilde{\Lambda}_t = \frac{\frac{\partial}{\partial s}\{\mathrm{Var}_{t,s}^{\mathbb{Q}}[\hat{H}] - \mathrm{Var}_{t,s}^{\mathbb{Q}}[q'\psi + x']\}|_{s=S_t, q'=\tilde{\Delta}_t, x'=X_t^{\tilde{\Delta}}}}{\partial_s \mathbb{E}_{t,s}^{\mathbb{Q}}[\psi]|_{s=S_t}},$$

which, in a slight abuse of notation, we write as

$$\tilde{\Lambda}_t = \frac{\frac{\partial}{\partial s}\{\mathrm{Var}_{t,s}^{\mathbb{Q}}[H] - \mathrm{Var}_{t,s}^{\mathbb{Q}}[\xi(t)]\}|_{s=S_t}}{\partial_s \mathbb{E}_{t,s}^{\mathbb{Q}}[\psi]|_{s=S_t}}. \tag{10.13}$$

As $\mathrm{Var}_{t,s}^{\mathbb{Q}}[\hat{H}] = \mathrm{Var}_{t,s}^{\mathbb{Q}}[H]$ and $\mathrm{Var}_{t,s}^{\mathbb{Q}}[\xi(t)] = \tilde{\Delta}_t^2 \mathrm{Var}_{t,s}^{\mathbb{Q}}[\psi]$, (10.13) then implies (10.12).

Analog computations to those leading up to (4.33) in the proof of Theorem 4.12 yield

$$\tilde{\Lambda}_t = \frac{\frac{\partial}{\partial s}\left\{-\mathbb{E}_{t,s}^{\mathbb{Q}}[R(\Sigma_0)]\left(\mathbb{E}_{t,s}^{\mathbb{R}}[\hat{H}^2] - \mathbb{E}_{t,s}^{\mathbb{R}}[\xi^2(t)]\right)\right\}|_{s=S_t}}{\partial_s \mathbb{E}_{t,s}^{\mathbb{Q}}[\psi]|_{s=S_t}}.$$

Note that, for exponential utility functions, $\mathbb{Q} = \mathbb{R}$. Moreover, for all $t \in [0, T]$, $s > 0$,

$$\mathrm{Var}_{t,s}^{\mathbb{Q}}[\hat{H}] - \mathrm{Var}_{t,s}^{\mathbb{Q}}[\xi(t)] = \mathbb{E}_{t,s}^{\mathbb{Q}}[\hat{H}^2] - \mathbb{E}_{t,s}^{\mathbb{Q}}[\xi^2(t)] - (\mathbb{E}_{t,s}^{\mathbb{Q}}[\hat{H}]^2 - \mathbb{E}_{t,s}^{\mathbb{Q}}[\xi(t)]^2).$$

By Lemma 10.6, we have $\xi(t) = \tilde{\Delta}_t\psi - (\tilde{\Delta}_t E_t^{\mathbb{Q}}[\psi] + \mathbb{E}_t^{\mathbb{Q}}[\hat{H}])$ and, consequently,

$$\mathbb{E}_t^{\mathbb{Q}}[\hat{H}] = -\mathbb{E}_t^{\mathbb{Q}}[\xi(t)].$$

This implies that

$$\mathbb{E}_{t,s}^{\mathbb{Q}}[\hat{H}]^2 - \mathbb{E}_{t,s}^{\mathbb{Q}}[\xi(t)]^2 = 0$$

and thus

$$\mathrm{Var}_{t,s}^{\mathbb{Q}}[\hat{H}] - \mathrm{Var}_{t,s}^{\mathbb{Q}}[\xi(t)] = \mathbb{E}_{t,s}^{\mathbb{Q}}[\hat{H}^2] - \mathbb{E}_{t,s}^{\mathbb{Q}}[\xi^2(t)].$$

□

In the case of exponential utility functions the term (10.10) for the first order approximation $\tilde{\Delta}$ admits an equivalent representation which we present in Proposition 10.12 below. Before we state and prove this proposition, note that for any $K \in \mathcal{H}''$ we have

$$\begin{aligned}\partial_s \mathbb{E}_{t,s}^{\mathbb{Q}}[K]|_{s=S_t} &= \partial_s \left(\frac{\mathbb{E}_{t,s}[u'(\Sigma_0)K]}{\mathbb{E}_{t,s}[u'(\Sigma_0)]} \right) |_{s=S_t} \\ &= \frac{\partial_s \mathbb{E}_{t,s}[u'(\Sigma_0)K]\mathbb{E}_{t,s}[u'(\Sigma_0)] - \mathbb{E}_{t,s}[u'(\Sigma_0)K]\partial_s \mathbb{E}_{t,s}[u'(\Sigma_0)]}{\mathbb{E}_{t,s}[u'(\Sigma_0)]^2}|_{s=S_t}.\end{aligned} \tag{10.14}$$

Proposition 10.12. *Under the assumptions of Theorem 10.8 let $\tilde{\Delta}$ be as in* (10.10)*. Assume further that $H = f(\psi) \in \mathcal{H}''$. Then*

$$\tilde{\Delta}_t = -\frac{\mathbb{E}_t^{\mathbb{Q}}[f'(\psi)\psi] - ap\,\mathrm{Cov}_t^{\mathbb{Q}}[H,\psi]}{\mathbb{E}_t^{\mathbb{Q}}[\psi] - ap\,\mathrm{Var}_t^{\mathbb{Q}}[\psi]}.$$

Proof. By (10.14) we have

$$\partial_s \mathbb{E}_{t,s}^{\mathbb{Q}}[H]|_{s=S_t} = \frac{\partial_s \mathbb{E}_{t,s}[u'(\Sigma_0)H]\mathbb{E}_{t,s}[u'(\Sigma_0)] - \mathbb{E}_{t,s}[u'(\Sigma_0)H]\partial_s \mathbb{E}_{t,s}[u'(\Sigma_0)]}{\mathbb{E}_{t,s}[u'(\Sigma_0)]^2}|_{s=S_t}.$$

Furthermore, we have

$$\partial_s f(s\mathcal{E}_{T-t}) = f'(s\mathcal{E}_{T-t})\mathcal{E}_{T-t}$$

and, by Assumption 7.1,

$$\partial_s u'(ps\mathcal{E}_{T-t} + z) = -ap\mathcal{E}_{T-t}u'(ps\mathcal{E}_{T-t} + z).$$

Thus, we can compute

$$
\begin{aligned}
& \partial_s \mathbb{E}^{\mathbb{Q}}_{t,s}[H]|_{s=S_t} \\
= \quad & \frac{\mathbb{E}_{t,s}[u'(\Sigma_0)(\frac{-ap}{s}H\psi + \frac{1}{s}f'(\psi)\psi)]}{\mathbb{E}_{t,s}[u'(\Sigma_0)]}|_{s=S_t} - \frac{\mathbb{E}_{t,s}[u'(\Sigma_0)H]\mathbb{E}_{t,s}[u'(\Sigma_0)(\frac{-ap}{s}\psi)]}{\mathbb{E}_{t,s}[u'(\Sigma_0)]^2}|_{s=S_t} \\
= \quad & \frac{1}{s}\frac{\mathbb{E}_{t,s}[u'(\Sigma_0)f'(\psi)\psi]}{\mathbb{E}_{t,s}[u'(\Sigma_0)]}|_{s=S_t} \\
& \quad - \frac{ap}{s}\left(\frac{\mathbb{E}_{t,s}[u'(\Sigma_0)H\psi]}{\mathbb{E}_{t,s}[u'(\Sigma_0)]} - \frac{\mathbb{E}_{t,s}[u'(\Sigma_0)H]}{\mathbb{E}_{t,s}[u'(\Sigma_0)]}\frac{\mathbb{E}_{t,s}[u'(\Sigma_0)(\psi)]}{\mathbb{E}_{t,s}[u'(\Sigma_0)]}\right)|_{s=S_t} \\
= \quad & \frac{1}{s}\left(\mathbb{E}^{\mathbb{Q}}_{t,s}[f'(\psi)\psi] - ap\,\mathrm{Cov}^{\mathbb{Q}}_{t,s}[H,\psi]\right)|_{s=S_t}.
\end{aligned}
$$

Note that this computation, in particular, holds for $H = \psi$, i.e. $f(x) = x$, in which case the last line can be further simplified to

$$
\partial_s \mathbb{E}^{\mathbb{Q}}_{t,s}[H]|_{s=S_t} = \frac{1}{s}\left(\mathbb{E}^{\mathbb{Q}}_{t,s}[\psi] - ap\,\mathrm{Var}^{\mathbb{Q}}_{t,s}[\psi]\right)|_{s=S_t}.
$$

The result then follows by applying this computation to both the numerator and denominator in

$$
\tilde{\Delta}_t = -\frac{\partial_s \mathbb{E}^{\mathbb{Q}}_{t,s}[H]|_{s=S_t}}{\partial_s \mathbb{E}^{\mathbb{Q}}_{t,s}[\psi]|_{s=S_t}}.
$$

□

Although we have established two different representations of the liquidity correction term for the hedge ratio $\tilde{\Lambda}$, it is not easy to investigate the term analytically. We will therefore resort to comparative statics in order to obtain an idea of the effect and magnitude of $\tilde{\Lambda}$. This is the subject of the next chapter.

Chapter 11

Comparative statics for hedge ratio corrections

In the previous chapter we found the approximation to the replicating position

$$Q_t^{\varepsilon H} = \varepsilon\tilde{\Delta}_t + \frac{\varepsilon^2}{2}\tilde{\Lambda}_t + o(\varepsilon^2), \quad 0 \le t \le T,$$

for a small number of claims εH. We saw further that $\tilde{\Delta}$ can be interpreted as a small investor's hedging strategy for H, while the second order approximation $\tilde{\Lambda}$ can be seen as the liquidity correction for the replicating position of a large investor. Throughout this chapter we will investigate the properties of $\tilde{\Lambda}$ numerically. We will deviate from our initial convention of viewing positions of both the traded security and cash from the point of view of the market maker. Instead, we will adopt the perspective of the large investor by considering the quantities $\Delta \triangleq -\tilde{\Delta}$ and $\Lambda \triangleq -\tilde{\Lambda}$.

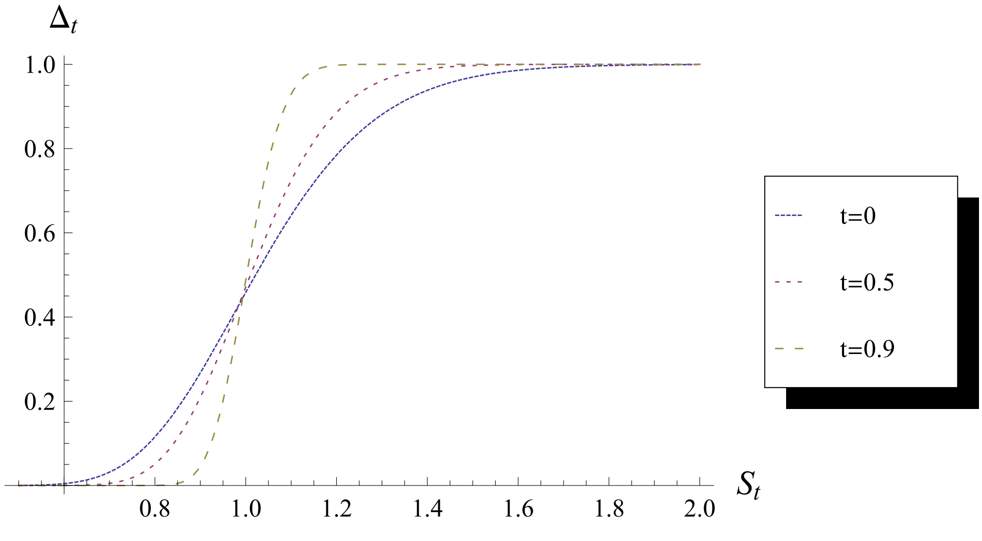

Figure 11.1: Term structure of Δ for a call option with strike $K = 1$; model parameters: $a = 1$, $p = 1$, $\sigma = 0.2$, $\mu = 0$, $T = 1$

Similarly to the power utility case we find that, for a call option, the first order approximation Δ resembles the Black-Scholes delta. It is depicted for various times to maturity in Figure 11.1.

As in the corresponding chapter in Part I, the quantitiy of interest throughout this chapter is the liquidity correction for the replicating position Λ. In what follows, we will investigate the shape and magnitude of Λ for a call option $H = (\psi - K)_+$. In particular, we will consider its term structure and its variability with respect to changes in the parameters defining our model, namely the market maker's risk aversion and initial endowment as well as the parameters defining the process S of Assumption 7.2.

Remark 11.1. Recall that in the case of power utilities we found that Δ was unaffected by changes in the size of the market maker's initial endowment. For exponential utility functions, this is *not* the case, as we can see in Figure 11.2. As the market maker's inventory p increases, the small investor's hedging position Δ flattens out and "moves to the right", i.e. the location of "the money" is shifted upwards.

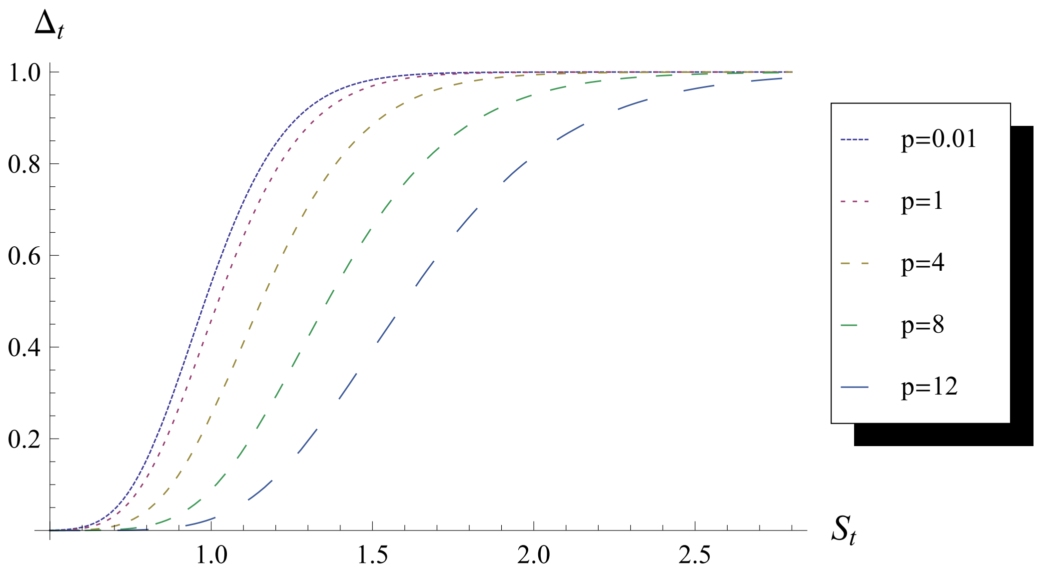

Figure 11.2: Liquidity correction term Λ for a call option with strike $K = 1$ for various values of the market maker's initial stock position p; model parameters: $a = 1$, $\sigma = 0.2$, $\mu = 0$, $t = 0$, $T = 1$

This difference between the cases of exponential utilities and power utilities is due to the fact that, for exponential utility functions, marginal prices display *cash invariance* while for power utility functions, they display *scale invariance*: Denoting by $\mathbb{Q}^{\exp}(p, z)$ the measure $\mathbb{Q}$ with respect to the market maker's initial endowment $\Sigma_0 = p\psi + z$ in the exponential utility case and by $\mathbb{Q}^{\text{pow}}(p, z)$ its analog in the power utility case, we have

$$\mathbb{E}^{\mathbb{Q}^{\exp}(p,z)}[\psi] = \mathbb{E}^{\mathbb{Q}^{\exp}(p,0)}[\psi]$$

as a result of the cash invariance of exponential utilities, while we have

$$\mathbb{E}^{\mathbb{Q}^{\text{pow}}(dp,dz)}[\psi] = \mathbb{E}^{\mathbb{Q}^{\text{pow}}(p,z)}[\psi], \quad \forall d > 0,$$

by Proposition 4.16. Hence, in the case of power utilities, Δ is not affected by a linear scaling $d > 0$ of the market maker's endowment while, for exponential utilities, Δ is merely invariant with respect to changes in the market maker's initial cash position z but not with respect to changes in his initial stock position p.

Recall that the validity of the expansions Δ and Λ and consequently of this entire chapter is conditional on the validity of Conjecture 2 from Chapter 10.

11.1 Positivity, unimodality and term structure

For a call option, the liquidity correction to the replicating strategy Λ is visualised in Figure 11.3 for different times to maturity. We can see that it has a similar shape to the one observed for power utility functions. Most notably, it is strictly positive, meaning that the large investor has to assume a replicating position which is strictly greater than the replicating position in a liquid market. Moreover, we can see that Λ is unimodal with a unique global maximum near the money and that the liquidity effect becomes negligible far away from the money. This is in accordance with what one would expect: The uncertainty is greatest near the money, while the replicating position deep in the money and deep out of the money is much less volatile. Upon approaching maturity, the liquidity correction term Λ vanishes. This, too, is natural, as the "remaining uncertainty" decreases with diminishing time to maturity.

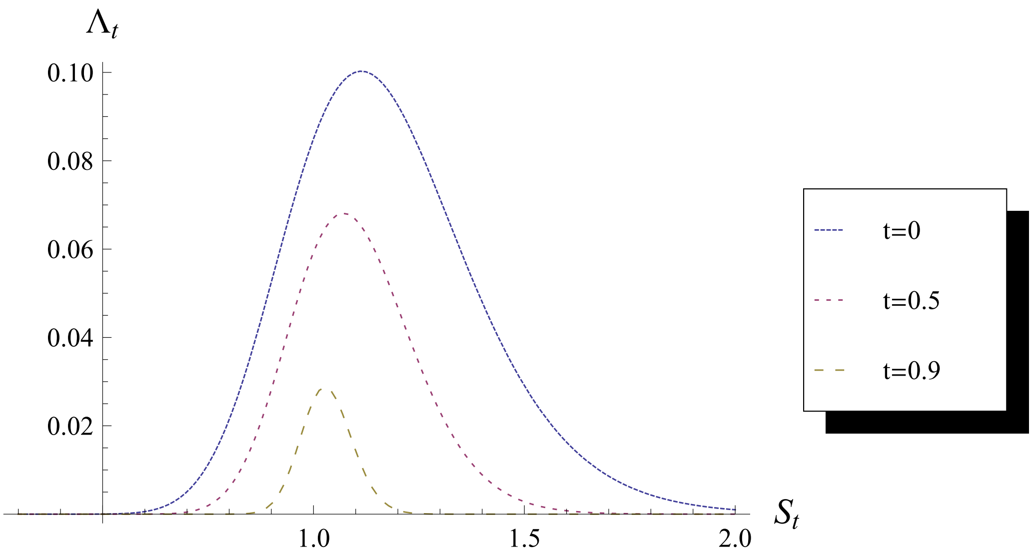

Figure 11.3: Term structure of the liquidity correction term Λ for a call option with strike $K = 1$; model parameters: $a = 1$, $p = 1$, $\mu = 0$, $t = 0$, $T = 1$

11.2 Variability with respect to the market maker's initial endowment and risk aversion

Note that, as a result of the cash invariance of exponential utility functions, the liquidity correction term Λ does not depend on z. This is clear from the fact that the measure $\mathbb{Q}$ in the definition of Λ in (10.11) does not depend on z.

As is to be expected, Λ is decreasing in the market maker's stock position p: When the market maker's initial endowment is large, the size of the large investor is less significant in relation to the total size of the market. His price impact decreases, and so does the liquidity correction term Λ; see Figure 11.4. Moreover, we can see that, as for Δ, an increase in the market maker's initial stock position p shifts the location of "the money" upwards.

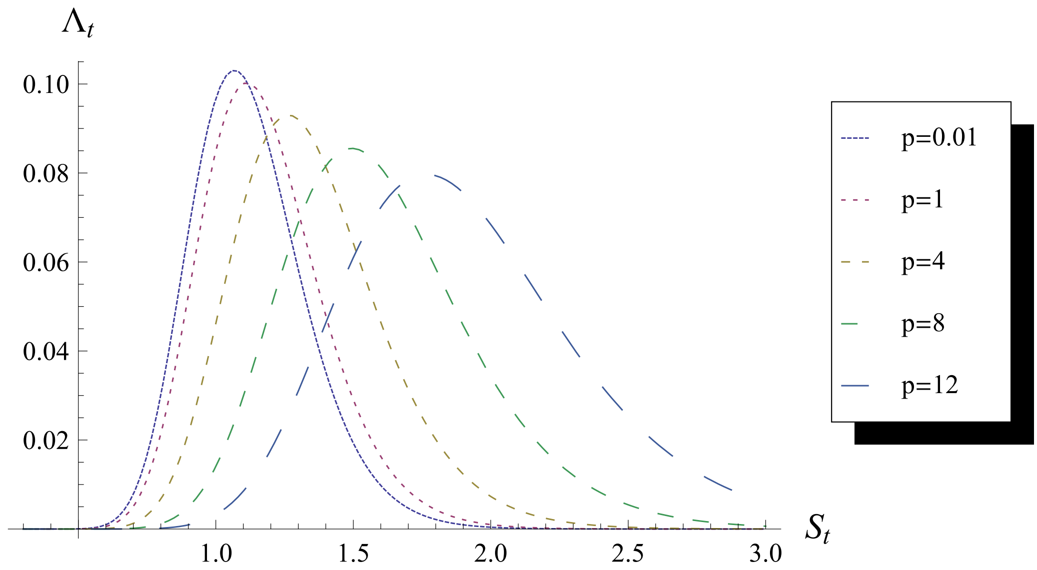

Figure 11.4: Liquidity correction term Λ for a call option with strike $K = 1$ for various values of the market maker's initial stock position p; model parameters: $a = 1$, $\sigma = 0.2$, $\mu = 0$, $t = 0$, $T = 1$

For power utilities, we saw that Λ was inversely proportional to a scaling parameter for the market maker's initial endowment. It thus displayed a self-similarity in the sense that the liquidity correction pertaining to a smaller initial endowment could be obtained by multiplying the liquidity correction for a larger initial endowment by the scaling parameter which expressed the quotient of the two endowment sizes. Hence, the shape of Λ was solely affected by the *balance* between stocks and cash in the market maker's initial endowment. Contrary to that, in the case of exponential utilities, the market maker's initial cash is irrelevant and the shape and magnitude of Λ are determined solely by the market maker's initial stock position p, as noted above.

The reaction of Λ with respect to changes in the market maker's risk aversion parameter a is, again, in line with what we would expect: As in the power utility case, a higher risk aversion induces a higher liquidity correction for the replicating position. This is illustrated in Figure 11.5. We observe that, similarly to the power utility case,

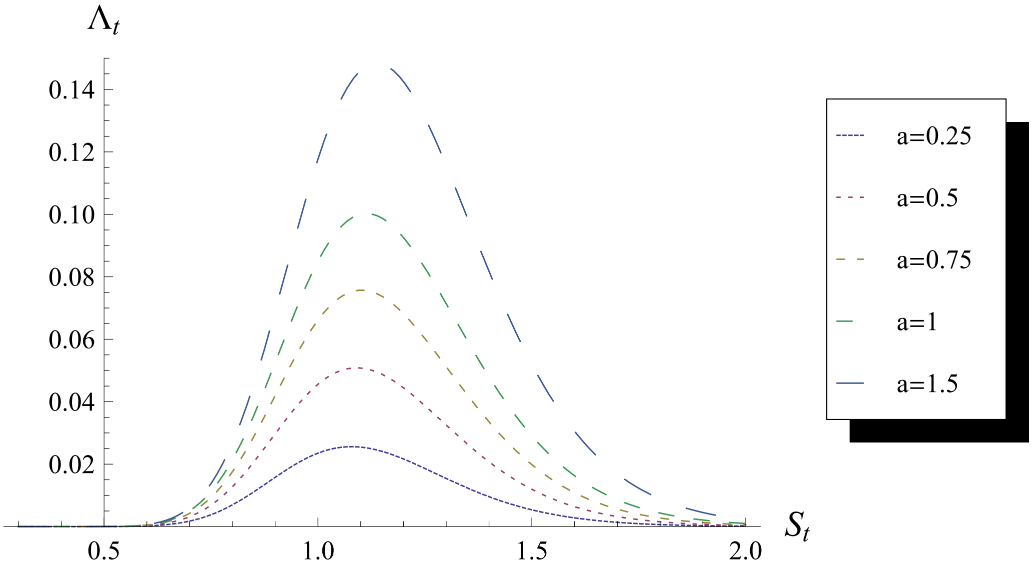

Figure 11.5: Liquidity correction term Λ for a call option with strike $K = 1$ for various values of the market maker's risk aversion a; model parameters: $p = 1$, $\sigma = 0.2$, $\mu = 0$, $t = 0$, $T = 1$

this relationship is roughly (but not precisely) linear for the maximal value of Λ, i.e. that

$$\Lambda(a) \approx a\Lambda(1),$$

where $\Lambda(a)$ denotes the liquidity correction term Λ as a function of the market maker's risk aversion a. In fact, such a "roughly linear relationship" is already hinted at by the multiplicative factor a in (10.11).

11.3 Variability with respect to the parameters defining the process S

The reaction of Λ to changes in the coefficients μ and σ of Assumption 7.2 is the same as in the power utility case: An increase in the volatility σ leads to an increase of the liquidity correction Λ, which is illustrated in Figure 11.6. This means that, as we would expect, the liquidity correction is higher when the risk associated with the marketed security is higher.

The changes in Λ with respect to changes in the drift coefficient μ can be understood as a discounting applied by the market maker according to his believes about the future evolution of S as was already explained in the corresponding Section 5.3 of Part

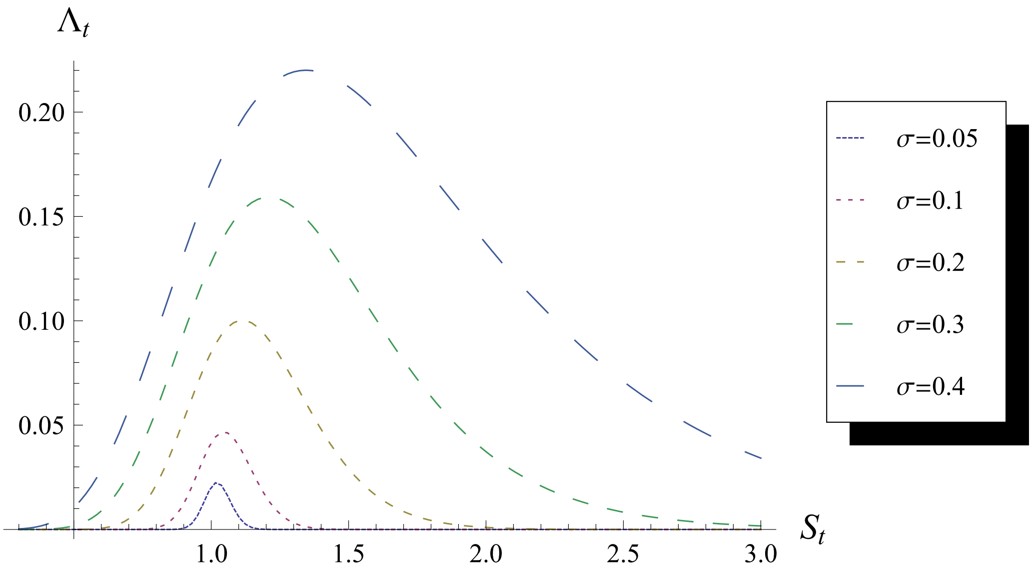

Figure 11.6: Liquidity correction term Λ for a call option with strike $K = 1$ for various values of the diffusion coefficient σ; model parameters: $a = 1$, $p = 1$, $\mu = 0$, $t = 0$, $T = 1$

I. Variations in μ merely move the location of "the money", while the magnitude of the liquidity correction Λ remains unaffected; see Figure 11.7. The changing "width" of Λ is due to the same exponential scaling effect that we already observed in Section 5.3.

Summarising our observations, we can see that the shape, magnitude and behaviour of the liquidity correction for the replicating position Λ for exponential utilities are similar to those in the case of power utilities. In particular, Λ is strictly positive, unimodal and maximal near the money. It is decreasing with decreasing time to maturity and it is increasing in the market maker's risk aversion a and in the diffusion coefficient σ which corresponds to a higher degree of risk associated with the marketed security. The most notable difference between the exponential- and power utility cases is the reaction of Λ with respect to changes in the market maker's initial endowment: For power utilities, Λ is inversely proportional to a linear scaling of the market maker's initial endowment and the influence of the market maker's initial endowment on the shape of Λ depends solely on the balance between the market maker's initial security and stock positions. For exponential utilities, the cash term is irrelevant and the influence of the market maker's initial endowment on the shape of Λ is determined only by his initial stock position.

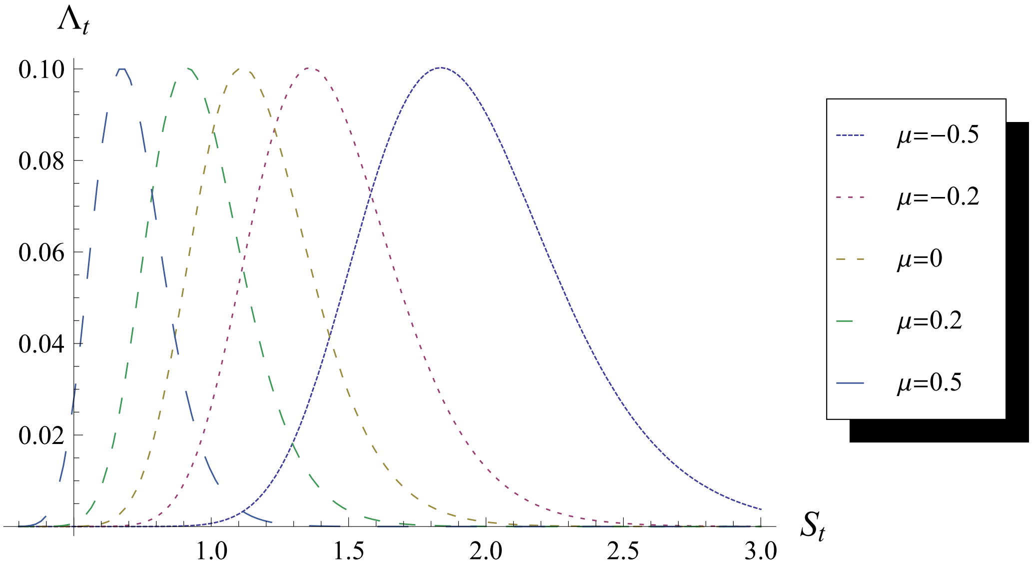

Figure 11.7: Liquidity correction term Λ for a call option with strike $K = 1$ for various values of the drift coefficient μ; model parameters: $a = 1$, $p = 1$, $\sigma = 0.2$, $t = 0$, $T = 1$

Chapter 12

Some questions regarding the lognormal distribution

In this chapter we address some questions related to the lognormal distribution which arise during our investigation of the replication of contingent claims in the exponential utility setting. To the best of our knowledge, the answers to these questions are not available in the literature. This apparent lack of understanding of the properties of the lognormal distribution is quite surprising and the questions and results in this chapter may be of interest in their own regard.

The term whose behaviour we seek to understand throughout this chapter is

$$k_2(\theta) \triangleq \theta \frac{\mathbb{E}[e^{-\theta\psi}\psi]}{\mathbb{E}[e^{-\theta\psi}]}, \quad \theta \geq 0.$$

In Section 12.1 we show that $k_2(\theta) \to \infty$ for $\theta \to \infty$, which is, as we will see, an asymptotic result for the Laplace transform of the lognormal distribution. Subsequently, in Section 12.2, we state as a conjecture the fact that $k_2'(\theta) > 0$. Even though this statement is intuitively convincing as well as strongly supported by numerical evidence, we have not yet been able to find an analytical proof.

12.1 A limit theorem for the lognormal Laplace transform

In this Section we will prove that

$$\lim_{\theta\to\infty} \left\{ \theta \frac{\mathbb{E}[e^{-\theta\psi}\psi]}{\mathbb{E}[e^{-\theta\psi}]} \right\} = \infty, \tag{12.1}$$

which we stated earlier as Lemma 9.6. It is somewhat surprising that the proof of this result requires such considerable effort. Although it is well known that the fraction $\mathbb{E}[e^{-\theta\psi}\psi]/\mathbb{E}[e^{-\theta\psi}]$ converges to zero for $\theta \to \infty$, the rate of convergence is not obvious. In fact, Tauberian and Abelian theorems as can be found e.g. in [16] and [24], which usually constitute powerful tools for investigating limiting behaviour of this kind, are not applicable. This is a consequence of the fact that the cumulative distribution function of the lognormal distribution

$$F(t) = \int_0^t \frac{1}{r\sqrt{2\pi}\sigma} e^{-\frac{(\log r-\mu)^2}{2\sigma^2}} dr$$

is not regularly varying. Recall that a function f is called regularly varying at zero if there exists $\rho \in (0, \infty)$ such that

$$\lim_{t \to 0+} \frac{f(tx)}{f(t)} = x^\rho \quad \forall x > 0.$$

For the sake of simplicity, let us assume that F is the lognormal distribution function induced by a standard normal with $\mu = 0$ and $\sigma = 1$. As $F(t) \to 0$ for $t \downarrow 0$, we can apply L'Hôpital's rule to obtain

$$\lim_{t \to 0+} \frac{F(tx)}{F(t)} = \lim_{t \to 0+} \frac{1}{x} e^{-\frac{1}{2}\left(2 \log t \log x + (\log x)^2\right)} = \begin{cases} \infty, & x > 1, \\ 1, & x = 1, \\ 0, & 0 < x < 1. \end{cases}$$

In particular, F is not regularly varying at zero and, consequently, the conditions of e.g. Karamata's, Feller's and de Bruijn's Tauberian theorems are not satisfied. Note that an analog argument for $t \to \infty$ shows that F is not regularly varying at infinity either.

Before we prove (12.1), let us briefly outline our strategy which relies on several auxiliary results. First, we will present an approximation $\tilde{\mathcal{L}}$ to the Laplace transform $\mathcal{L}$ of the lognormal distribution which is introduced in [8]. We will then show that we can, without loss of generality, consider $\psi = e^{\sigma N}$ in (12.1), where $N \sim \mathcal{N}(0, 1)$, $\sigma > 0$. Using this simplifying assumption, we proceed to show that the quotient $\mathcal{L}/\tilde{\mathcal{L}}$ of the Laplace transform and its approximation is bounded by constants above and below, so that, in particular, we can control the asymptotics of $\mathcal{L}$ with those of $\tilde{\mathcal{L}}$. As $\tilde{\mathcal{L}}$ is expressed most conveniently in terms of the Lambert W function, we will establish several of this function's properties which we will subsequently use to show a convergence result for $\tilde{\mathcal{L}}$. Finally, we will use Girsanov's theorem to express the terms in (12.1) as a quotient of Laplace transforms and combine the auxiliary results to show the desired convergence (12.1).

Note that the idea for the approximation $\tilde{\mathcal{L}}$ which we will use to prove (12.1) as well as the expression of $\tilde{\mathcal{L}}$ in terms of the Lambert W function are due to [8]. Among other things, the authors show that $\mathcal{L}(\theta) = \tilde{\mathcal{L}}(\theta)(1 + O(\log(\theta)^{-1}))$, so that, in particular, $\mathcal{L}$ and $\tilde{\mathcal{L}}$ are asymptotically equivalent. This very recent result could have been used in our proof of (12.1) instead of Lemma 12.2 below. However, it was not available at the time at which our proof was completed and we took a different approach from that in [8], obtaining – in the form of Lemma 12.2 – a somewhat different result.

Before we establish the auxiliary lemmas for the proof of (12.1) let us introduce some additional notation. Let $\psi = e^{\sigma N+\mu}$, $N \sim \mathcal{N}(0,1)$, $\mu \in \mathbb{R}$, $\sigma > 0$, be a lognormally distributed random variable and denote its Laplace transform by

$$\mathcal{L}_\psi(\theta) \triangleq \mathbb{E}[e^{-\theta\psi}] = \int_0^\infty \frac{1}{x\sqrt{2\pi}\sigma} \exp\left(-\theta x - \frac{(\log x - \mu)^2}{2\sigma^2}\right) dx.$$

For $k = 0, 1, 2, \ldots$, consider

$$\begin{aligned} \mathbb{E}[\psi^k e^{-\theta\psi}] &= \int_0^\infty \frac{x^{k-1}}{\sigma\sqrt{2\pi}} \exp\left(-\theta x - \frac{(\log x - \mu)^2}{2\sigma^2}\right) dx \\ &= \int_{-\infty}^\infty \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\theta e^y + ky - \frac{(y-\mu)^2}{2\sigma^2}\right) dy, \end{aligned}$$

where we used the substitution $y = \log(x)$. Laplace's method for approximating integrals of the form

$$\int_a^b e^{Mf(x)}\, dx$$

for some twice differentiable f, large M and a, b possibly infinite, now suggests to replace the expression

$$-\theta e^y + ky - \frac{(y-\mu)^2}{2\sigma^2} \tag{12.2}$$

by a Taylor approximation of second order around the value ρ_k which maximises (12.2), i.e. by

$$-\theta e^{\rho_k}\left(1 + (y-\rho_k) + \frac{(y-\rho_k)^2}{2}\right) + ky - \frac{(y-\mu)^2}{2\sigma^2},$$

where the maximising constant ρ_k is obtained to be

$$\rho_k = -L(\theta\sigma^2 e^{k\sigma^2+\mu}) + k\sigma^2 + \mu.$$

Here, $L : [-e^{-1}, \infty) \to \mathbb{R}$ is the Lambert W function (for real numbers), defined implicitly for any $z \in [-e^{-1}, \infty)$ by

$$z = e^{L(z)} L(z).$$

The resulting Gaussian integral

$$\tilde{\mathcal{L}}_\psi(\theta) \triangleq \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^\infty \exp\left(-\theta e^{\rho_k}\left(1 + (y-\rho_k) + \frac{(y-\rho_k)^2}{2}\right) + ky - \frac{(y-\mu)^2}{2\sigma^2}\right) dy$$

can be calculated explicitly and we obtain

$$\begin{aligned}\tilde{\mathcal{L}}_\psi(\theta) &= \frac{1}{\sqrt{L(\theta\sigma^2 e^{k\sigma^2+\mu})+1}} \times \\ &\quad \times \exp\left(-\frac{L(\theta\sigma^2 e^{k\sigma^2+\mu})^2 + 2L(\theta\sigma^2 e^{k\sigma^2+\mu}) - 2k\sigma^2\mu - k^2\sigma^4}{2\sigma^2}\right).\end{aligned}$$

For $k = 0$ this term simplifies to

$$\tilde{\mathcal{L}}_\psi(\theta) = \frac{1}{\sqrt{L(\theta\sigma^2 e^{\mu})+1}} \exp\left(-\frac{L(\theta\sigma^2 e^{\mu})^2 + 2L(\theta\sigma^2 e^{\mu})}{2\sigma^2}\right)$$

and for $\mu = 0$ to

$$\tilde{\mathcal{L}}_\psi(\theta) = \frac{1}{\sqrt{L(\theta\sigma^2)+1}} \exp\left(-\frac{L(\theta\sigma^2)^2 + 2L(\theta\sigma^2)}{2\sigma^2}\right). \tag{12.3}$$

The first preliminary lemma simplifies ensuing computations by allowing us to assume without loss of generality that $\mu = 0$.

Lemma 12.1. *Let $N \sim \mathcal{N}(0,1)$, $\sigma > 0$ and $\mu \in \mathbb{R}$. Then*

$$\lim_{\theta\to\infty} \left\{ \theta \frac{\mathbb{E}[e^{-\theta\psi}\psi]}{\mathbb{E}[e^{-\theta\psi}]} \right\} = \infty$$

for $\psi = e^{\sigma N + \mu}$, if and only if

$$\lim_{\theta\to\infty} \left\{ \theta \frac{\mathbb{E}[e^{-\theta\psi'}\psi']}{\mathbb{E}[e^{-\theta\psi'}]} \right\} = \infty$$

for $\psi' = e^{\sigma N}$.

Proof. Observe that for $\theta' \triangleq \theta e^{\mu}$ we have

$$\theta \frac{\mathbb{E}[e^{-\theta\psi}\psi]}{\mathbb{E}[e^{-\theta\psi}]} = \theta \frac{\mathbb{E}[e^{-\theta e^{\mu+\sigma N}} e^{\mu+\sigma N}]}{\mathbb{E}[e^{-\theta e^{\mu+\sigma N}}]} = \theta' \frac{\mathbb{E}[e^{-\theta' e^{\sigma N}} e^{\sigma N}]}{\mathbb{E}[e^{-\theta' e^{\sigma N}}]}.$$

□

The next lemma shows that the approximation $\tilde{\mathcal{L}}_\psi$ to $\mathcal{L}_\psi$ is sufficiently good for our purposes.

Lemma 12.2. *Let $N \sim \mathcal{N}(0,1)$, $\sigma > 0$, $\psi = e^{\sigma N}$ and let $\theta_0 > 0$. Then, there exists a constant $C(\sigma, \theta_0) > 0$ such that for all $\theta \geq \theta_0$*

$$\frac{1}{2} \leq \frac{\mathcal{L}_\psi(\theta)}{\tilde{\mathcal{L}}_\psi(\theta)} \leq C(\sigma, \theta_0).$$

Moreover, $C(\sigma, \theta_0)$ is decreasing in θ_0.

Proof. Let

$$f_\theta(y) \triangleq \frac{\sqrt{L(\theta\sigma^2)+1}}{\sqrt{2\pi}\sigma} \exp\left(-\frac{y^2}{2\sigma^2} - \theta e^y + \frac{L(\theta\sigma^2)^2 + 2L(\theta\sigma^2)}{2\sigma^2}\right)$$

and note that, by (12.3),

$$\frac{\mathcal{L}_\psi(\theta)}{\tilde{\mathcal{L}}_\psi(\theta)} = \int_{-\infty}^{\infty} f_\theta(y)dy.$$

It is our aim to show that

$$\frac{1}{2} \leq \int_{-\infty}^{\infty} f_\theta(y)dy \leq C(\sigma, \theta_0) \quad \forall\theta \geq \theta_0.$$

Consider the density of an $\mathcal{N}(-L(\theta\sigma^2), \frac{\sigma^2}{L(\theta\sigma^2)+1})$-distributed random variable

$$g_\theta(y) \triangleq \frac{\sqrt{L(\theta\sigma^2)+1}}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(y+L(\theta\sigma^2))^2}{2\sigma^2}\big(L(\theta\sigma^2)+1\big)\right).$$

Note that both g_θ and f_θ attain their unique global maxima in $\rho_0 = -L(\theta\sigma^2)$ and that

$$g_\theta(\rho_0) = f_\theta(\rho_0) = \frac{\sqrt{L(\theta\sigma^2)+1}}{\sqrt{2\pi}\sigma}.$$

Note further that, in what follows, we will often use the identity

$$xe^{-L(x)+r} = L(x)e^r \quad \forall x > 0\ \forall r \in \mathbb{R},$$

which is implied directly by the definition of L.

Lower bound. We compute

$$\frac{f_\theta}{g_\theta} = \exp\left(-\frac{y^2}{2\sigma^2} - \theta e^y + \frac{L^2(\theta\sigma^2) + L(\theta\sigma^2)}{2\sigma^2} + \frac{(y+L(\theta\sigma^2))^2}{2\sigma^2}(L(\theta\sigma^2)+1)\right).$$

Using the substitution $y = -L(\theta\sigma^2) + r$ we obtain

$$\frac{f_\theta}{g_\theta} = \exp\left(\frac{L(\theta\sigma^2)}{\sigma^2}(1 + r + \frac{r^2}{2} - e^r)\right).$$

By differentiating the exponent twice with respect to r we find that

$$(1 + r + \frac{r^2}{2} - e^r) > 0 \quad \text{iff} \quad 1 - e^r > 0 \quad \text{iff} \quad r < 0. \tag{12.4}$$

The case $r = 0$ corresponds to $y = -L(\theta\sigma^2)$, the value at which f_θ and g_θ attain their common global maximum. Observe that (12.4) implies

$$f_\theta(y) > g_\theta(y) \quad \forall y < -L(\theta\sigma^2),$$

which, in turn, implies the lower bound

$$\int_{-\infty}^{\infty} f_\theta(y)dy > \int_{-\infty}^{-L(\theta)} f_\theta(y)dy > \int_{-\infty}^{-L(\theta)} g_\theta(y)dy = \frac{1}{2},$$

where we made use of the fact that g_θ is symmetric about $-L(\theta\sigma^2)$ and its integral over the real line is normalised to one.

Upper bound. We consider the change of variable $y = -L(\theta\sigma^2) - r$ which yields

$$\int_{-\infty}^{\infty} f_\theta(y)dy = \int_{-\infty}^{\infty} \frac{\sqrt{L(\theta\sigma^2)+1}}{\sqrt{2\pi}\sigma} \exp\left(-\frac{r^2}{2\sigma^2} + \frac{L(\theta\sigma^2)}{\sigma^2}(1 - r - e^{-r})\right) dr.$$

Let us denote this new integrand by

$$\tilde{f}_\theta(r) \triangleq \frac{\sqrt{L(\theta\sigma^2)+1}}{\sqrt{2\pi}\sigma} \exp\left(-\frac{r^2}{2\sigma^2} + \frac{L(\theta\sigma^2)}{\sigma^2}(1 - r - e^{-r})\right).$$

$\tilde{f}_\theta$ attains its maximum in $r = 0$. Let $f_\theta^r, f_\theta^l \in \mathcal{C}^\infty([0,\infty))$, defined by

$$f_\theta^r : r \mapsto \tilde{f}_\theta(r),$$

and

$$f_\theta^l : r \mapsto \tilde{f}_\theta(-r),$$

denote the integrand's branches to the right and to the left of $r = 0$, respectively, where the latter is reflected to the positive half line. We find

$$\frac{f_\theta^r(r)}{f_\theta^l(r)} = \exp\left(\frac{L(\theta\sigma^2)}{\sigma^2}(-2r - e^{-r} + e^r)\right) = \exp\left(\frac{2L(\theta\sigma^2)}{\sigma^2}(\sinh(r) - r)\right),$$

from which it is apparent that, for $r > 0$,

$$f_\theta^r(r) > f_\theta^l(r)$$

and, consequently,

$$\int_{-\infty}^{\infty} \tilde{f}_\theta(r)dr < 2\int_0^{\infty} \tilde{f}_\theta(r)dr.$$

It remains to show that there is a constant $C(\sigma, \theta_0) > 0$, decreasing in θ_0, such that

$$2\int_0^{\infty} \tilde{f}_\theta(r)dr \le C(\sigma, \theta_0) \quad \forall \theta \ge \theta_0.$$

To this end, we will split the integral at $r = 1$. Let $\vartheta \triangleq \sqrt{L(\theta\sigma^2)+1}$. Then

$$\begin{aligned}\int_1^\infty \tilde{f}_\theta(r)dr &= \int_1^\infty \frac{\sqrt{L(\theta\sigma^2)+1}}{\sqrt{2\pi}\sigma}\exp\left(-\frac{r^2}{2\sigma^2}+\frac{L(\theta\sigma^2)}{\sigma^2}(1-r-e^{-r})\right)dr \\ &= \int_1^\infty \frac{\vartheta}{\sqrt{2\pi}\sigma}\exp\left(-\frac{r^2}{2\sigma^2}+\frac{\vartheta^2-1}{\sigma^2}(1-r-e^{-r})\right)dr \\ &< \int_1^\infty \frac{\vartheta}{\sqrt{2\pi}\sigma}\exp\left(-\frac{(r+\vartheta^2-1)^2}{2\sigma^2}+\frac{(\vartheta^2-1)^2}{2\sigma^2}+\frac{\vartheta^2-1}{\sigma^2}\right)dr,\end{aligned}$$

where the last inequality is due to neglecting the $-e^{-r}$-term. Hence,

$$\begin{aligned}&\int_1^\infty \frac{\vartheta}{\sqrt{2\pi}\sigma}\exp\left(-\frac{(r+\vartheta^2-1)^2}{2\sigma^2}+\frac{(\vartheta^2-1)^2}{2\sigma^2}+\frac{\vartheta^2-1}{\sigma^2}\right)dr \\ \leq\;& \int_1^\infty \frac{r+(\vartheta^2-1)}{1+(\vartheta^2-1)}\frac{\vartheta}{\sqrt{2\pi}\sigma}\exp\left(-\frac{(r+\vartheta^2-1)^2}{2\sigma^2}+\frac{(\vartheta^2-1)^2}{2\sigma^2}+\frac{\vartheta^2-1}{\sigma^2}\right)dr \\ =\;& \frac{\sigma}{\sqrt{2\pi e^{\sigma^{-2}}}\vartheta}.\end{aligned}$$

By definition, $\vartheta \geq 1$ so that

$$\frac{\sigma}{\sqrt{2\pi e^{\sigma^{-2}}}\vartheta} \leq \frac{\sigma}{\sqrt{2\pi e^{\sigma^{-2}}}} < \frac{\sigma}{\sqrt{2\pi}}.$$

Observe now that

$$\begin{aligned}\int_0^1 \tilde{f}_\theta(r)dr &= \int_0^1 \frac{\sqrt{L(\theta\sigma^2)+1}}{\sqrt{2\pi}\sigma}\exp\left(-\frac{r^2}{2\sigma^2}+\frac{L(\theta\sigma^2)}{\sigma^2}(1-r-e^{-r})\right)dr \\ &< \int_0^1 \frac{\vartheta}{\sqrt{2\pi}\sigma}\exp\left(\frac{\vartheta^2-1}{\sigma^2}(1-r-e^{-r})\right)dr,\end{aligned}$$

where the inequality is due to neglecting the $-\frac{r^2}{2\sigma^2}$-term. Choose $c > 0$ in such a way that

$$-cr^2 \geq 1-r-e^{-r} \quad \forall r \in [0,1].$$

Then

$$\int_0^1 \frac{\vartheta}{\sqrt{2\pi}\sigma}\exp\left(\frac{\vartheta^2-1}{\sigma^2}(1-r-e^{-r})\right)dr \leq \int_0^1 \frac{\vartheta}{\sqrt{2\pi}\sigma}\exp\left(-r^2c\frac{\vartheta^2-1}{\sigma^2}\right)dr.$$

Since

$$\int_{-\infty}^\infty \frac{\vartheta}{\sqrt{2\pi}\sigma}\exp\left(-r^2c\frac{\vartheta^2-1}{\sigma^2}\right)dr = \frac{1}{\sqrt{2c}}\frac{\vartheta}{\sqrt{\vartheta^2-1}} = \frac{1}{\sqrt{2c}}\sqrt{\frac{L(\theta\sigma^2)+1}{L(\theta\sigma^2)}},$$

we obtain

$$\int_0^1 \frac{\vartheta}{\sqrt{2\pi}\sigma}\exp\left(-r^2c\frac{\vartheta^2-1}{\sigma^2}\right)dr < \frac{1}{\sqrt{2c}}\sqrt{\frac{L(\theta\sigma^2)+1}{L(\theta\sigma^2)}}.$$

Hence, for $\theta \geq \theta_0$,

$$\int_0^\infty \tilde{f}_\theta(r)dr = \int_0^1 \tilde{f}_\theta(r)dr + \int_1^\infty \tilde{f}_\theta(r)dr < \frac{1}{\sqrt{2c}}\sqrt{\frac{L(\theta_0\sigma^2)+1}{L(\theta_0\sigma^2)}} + \frac{\sigma}{\sqrt{2\pi}}$$

so that

$$C(\sigma,\theta_0) \triangleq 2\left(\frac{1}{\sqrt{2c}}\sqrt{\frac{L(\theta_0\sigma^2)+1}{L(\theta_0\sigma^2)}} + \frac{\sigma}{\sqrt{2\pi}}\right)$$

satisfies the lemma's assertion. As $L(0) = 0$ and L is increasing, it follows that $C(\sigma,\cdot)$ is decreasing in θ_0. □

The third preliminary assertion highlights several properties of the Lambert W function.

Lemma 12.3. *Let L denote the Lambert W function and let $\lambda > 1$. Then*

(i)

$$\frac{L(\lambda\theta)}{L(\theta)} = \lambda e^{L(\theta)-L(\lambda\theta)},$$

(ii)

$$L(\theta) - L(\lambda\theta) = \log\left(\frac{L(\lambda\theta)}{L(\theta)}\right) - \log(\lambda),$$

(iii)

$$\frac{L(\lambda\theta)}{L(\theta)} \downarrow 1, \quad \theta \to \infty,$$

(iv)

$$L(\theta) - L(\lambda\theta) \downarrow -\log(\lambda), \quad \theta \to \infty.$$

Proof. From the definition of L it is apparent that

$$\frac{L(\lambda\theta)}{L(\theta)} = \frac{\lambda\theta e^{-L(\lambda\theta)}}{\theta e^{-L(\theta)}},$$

from which (i) and (ii) are obtained by simple algebraic transformations. Note that

$$\frac{\partial}{\partial\theta}\log(L(\theta)) = \frac{L'(\theta)}{L(\theta)}.$$

Using this observation, we compute

$$\log\left(\frac{L(\lambda\theta)}{L(\theta)}\right) = \log L(\lambda\theta) - \log L(\theta) = \int_\theta^{\lambda\theta}\frac{L'(x)}{L(x)}dx = \int_1^\lambda \frac{1}{y(1+L(\theta y))}dy,$$

where the last equality is due to the change of variables $x = \theta y$ and the fact that

$$L'(x) = \frac{L(x)}{x(1+L(x))}.$$

The last integral above converges to zero for $\theta \to \infty$ and it follows that

$$\lim_{\theta\to\infty} \log\left(\frac{L(\lambda\theta)}{L(\theta)}\right) = 0$$

which, as $L(\lambda\theta) > L(\theta)$, implies (iii). Finally, (iv) is a direct consequence of (ii) and (iii). $\square$

The fourth and last preliminary result shows that the approximation $\tilde{\mathcal{L}}_\psi$ displays the desired convergence.

Lemma 12.4. *Let $N \sim \mathcal{N}(0,1)$, $\sigma > 0$ and $\psi = e^{\sigma N}$. Then*

$$\lim_{\theta\to\infty} \left\{\theta \frac{\tilde{\mathcal{L}}_\psi(e^{\sigma^2}\theta)}{\tilde{\mathcal{L}}_\psi(\theta)}\right\} = \infty.$$

Proof. Using the representation (12.3) we compute

$$\begin{aligned}
\theta \frac{\tilde{\mathcal{L}}_\psi(e^{\sigma^2}\theta)}{\tilde{\mathcal{L}}_\psi(\theta)} &= \theta\sqrt{\frac{L(\theta\sigma^2)+1}{L(e^{\sigma^2}\theta\sigma^2)+1}} \times \\
&\quad \times \exp\left(-\frac{L(e^{\sigma^2}\theta\sigma^2)^2 + 2L(e^{\sigma^2}\theta\sigma^2)}{2\sigma^2} + \frac{L(\theta\sigma^2)^2 + 2L(\theta\sigma^2)}{2\sigma^2}\right).
\end{aligned}$$

Note that

$$\begin{aligned}
& \exp\left(-\frac{L(e^{\sigma^2}\theta\sigma^2)^2 + 2L(e^{\sigma^2}\theta\sigma^2)}{2\sigma^2} + \frac{L(\theta\sigma^2)^2 + 2L(\theta\sigma^2)}{2\sigma^2}\right) \\
= \;& \exp\left(\frac{1}{2\sigma^2}\big(L(\theta\sigma^2) - L(e^{\sigma^2}\theta\sigma^2)\big)\big(L(\theta\sigma^2) + L(e^{\sigma^2}\theta\sigma^2)\big)\right) \times \\
& \times \exp\left(\frac{1}{\sigma^2}\big(L(\theta\sigma^2) - L(e^{\sigma^2}\theta\sigma^2)\big)\right).
\end{aligned}$$

By Lemma 12.3 (iv) and the fact that $L(e^{\sigma^2}\theta\sigma^2) > L(\theta\sigma^2)$, we have

$$0 > L(\theta\sigma^2) - L(e^{\sigma^2}\theta\sigma^2) > -\log(e^{\sigma^2}) = -\sigma^2$$

which implies that

$$\begin{aligned}
& \theta \exp\left(\frac{1}{2\sigma^2}\big(L(\theta\sigma^2) - L(e^{\sigma^2}\theta\sigma^2)\big)\big(L(\theta\sigma^2) + L(e^{\sigma^2}\theta\sigma^2)\big)\right) \\
\geq \;& \theta \exp\left(-\frac{1}{2}\big(L(\theta\sigma^2) + L(e^{\sigma^2}\theta\sigma^2)\big)\right).
\end{aligned}$$

Moreover, as $L(e^{\sigma^2}\theta\sigma^2) > L(\theta\sigma^2) > 0$, it follows that

$$\theta \exp\left(-\frac{1}{2}\big(L(\theta\sigma^2) + L(e^{\sigma^2}\theta\sigma^2)\big)\right) \geq \theta \exp\left(-L(e^{\sigma^2}\theta\sigma^2)\right) = \frac{L(e^{\sigma^2}\theta\sigma^2)}{e^{\sigma^2}\sigma^2},$$

where the last equality follows from the definition of L. Finally, by Lemma 12.3 (*iv*) we know that

$$\exp\left(\frac{1}{\sigma^2}\big(L(\theta\sigma^2) - L(e^{\sigma^2}\theta\sigma^2)\big)\right) \to \frac{1}{e}, \qquad \theta \to \infty,$$

and by Lemma 12.3 (*iii*) that

$$\sqrt{\frac{L(\theta\sigma^2)+1}{L(e^{\sigma^2}\theta\sigma^2)+1}} \to 1, \qquad \theta \to \infty,$$

so that, combining these observations, we obtain

$$\lim_{\theta\to\infty}\left\{\theta\frac{\tilde{\mathcal{L}}_\psi(e\theta)}{\tilde{\mathcal{L}}_\psi(\theta)}\right\} \geq \lim_{\theta\to\infty}\sqrt{\frac{L(\theta\sigma^2)+1}{L(e^{\sigma^2}\theta\sigma^2)+1}}\frac{L(e^{\sigma^2}\theta\sigma^2)}{e^{\sigma^2}\sigma^2}e^{\frac{L(\theta\sigma^2)-L(e^{\sigma^2}\theta\sigma^2)}{\sigma^2}} = \infty.$$

□

We are now ready to assemble the pieces and prove (12.1).

Proof of Lemma 9.6. It is our objective to show that

$$\lim_{\theta\to\infty}\left\{\theta\frac{\mathbb{E}[e^{-\theta\psi}\psi]}{\mathbb{E}[e^{-\theta\psi}]}\right\} = \infty \tag{12.5}$$

for any lognormally distributed random variable ψ. Lemma 12.1 allows us to assume without loss of generality that ψ is given as $\psi = e^{\sigma N}$, where $N \sim \mathcal{N}(0,1)$, $\sigma > 0$. Note that, by the Girsanov Theorem, a change of measure yields

$$\mathbb{E}[e^{-\theta\psi}\psi] = e^{\frac{1}{2}\sigma^2}\mathbb{E}[e^{-\theta e^{\sigma N}}e^{\sigma N - \frac{1}{2}\sigma^2}] = e^{\frac{1}{2}\sigma^2}\mathbb{E}^{\mathbb{L}}[e^{-\theta e^{\sigma\tilde{N}+\sigma^2}}],$$

where the Radon-Nikodym derivative of $\mathbb{L}$ is given by

$$\frac{d\mathbb{L}}{d\mathbb{P}} = e^{\sigma N - \frac{1}{2}\sigma^2}$$

and the random variable $\tilde{N} \overset{\mathbb{L}}{\sim} \mathcal{N}(0,1)$ is normally distributed with respect to the measure $\mathbb{L}$. Note that

$$\mathbb{E}^{\mathbb{L}}[e^{-\theta e^{\sigma\tilde{N}+\sigma^2}}] = \mathbb{E}^{\mathbb{L}}[e^{-\theta e^{\sigma\tilde{N}}e^{\sigma^2}}] = \mathcal{L}_\psi(e^{\sigma^2}\theta),$$

where $\mathcal{L}_\psi$ denotes the Laplace transform of ψ. Hence, the fraction in (12.5) can be expressed as

$$\frac{\mathbb{E}[e^{-\theta\psi}\psi]}{\mathbb{E}[e^{-\theta\psi}]} = \frac{e^{\frac{1}{2}\sigma^2}\mathcal{L}_\psi(e^{\sigma^2}\theta)}{\mathcal{L}_\psi(\theta)}.$$

Let $C(\sigma,1)$ denote the constant in Lemma 12.2 for $\theta_0 = 1$. Then, for $\theta \geq 1$, we have

$$\theta\frac{e^{\frac{1}{2}\sigma^2}\mathcal{L}_\psi(e^{\sigma^2}\theta)}{\mathcal{L}_\psi(\theta)} = \theta\frac{e^{\frac{1}{2}\sigma^2}\tilde{\mathcal{L}}_\psi(e^{\sigma^2}\theta)}{\tilde{\mathcal{L}}_\psi(\theta)}\frac{\mathcal{L}_\psi(e^{\sigma^2}\theta)/\tilde{\mathcal{L}}_\psi(e^{\sigma^2}\theta)}{\mathcal{L}_\psi(\theta)/\tilde{\mathcal{L}}_\psi(\theta)} \geq \frac{e^{\frac{1}{2}\sigma^2}}{2C(\sigma,1)}\theta\frac{\tilde{\mathcal{L}}_\psi(e^{\sigma^2}\theta)}{\tilde{\mathcal{L}}_\psi(\theta)},$$

where the last inequality is a consequence of Lemma 12.2. An application of Lemma 12.4 concludes the proof. □

Note that the statement of Lemma 9.6, alternatively, can be expressed in terms of the elasticity of the Laplace transform of the lognormal distribution: As the following lemma shows, the fraction of Laplace transforms of interest throughout this section can equivalently be written as

$$\frac{\mathbb{E}[e^{-\theta\psi}\psi]}{\mathbb{E}[e^{-\theta\psi}]} = \frac{\mathcal{L}_\psi(e^{\sigma^2}\theta)}{\mathcal{L}_\psi(\theta)}e^{\frac{1}{2}\sigma^2} = -\frac{\partial_\theta\mathcal{L}_\psi(\theta)}{\mathcal{L}_\psi(\theta)}. \tag{12.6}$$

For the first equality, see the proof of Lemma 9.6; the second equality is implied by the lemma below.

Lemma 12.5. *Let $\psi = e^{\sigma N}$, where $N \sim \mathcal{N}(0,1)$, $\sigma > 0$. Then*

$$\frac{\partial^n}{\partial\theta^n}\mathcal{L}_\psi(\theta) = (-1)^n e^{\frac{n^2}{2}\sigma^2}\mathcal{L}(e^{n\sigma^2}\theta).$$

Proof. We will prove by induction that

$$\frac{\partial^n}{\partial\theta^n}\mathcal{L}_\psi(\theta) = (-1)^n\prod_{k=1}^{n} e^{\frac{2k-1}{2}\sigma^2}\mathcal{L}(e^{n\sigma^2}\theta).$$

This will imply the desired result as $\sum_{k=1}^{n+1}(2k-1) = n^2$. Note first that

$$\partial_\theta\mathcal{L}_\psi(\theta) = -\mathbb{E}[e^{-\theta\psi}\psi] = -e^{\frac{1}{2}\sigma^2}\mathbb{E}[e^{-\theta e^{\sigma N}}e^{\sigma N - \frac{1}{2}\sigma^2}].$$

Similarly to the argument in the proof of Lemma 9.6, an application of Girsanov's Theorem yields

$$\mathbb{E}[e^{-\theta e^{\sigma N}}e^{\sigma N-\frac{1}{2}\sigma^2}] = \mathbb{E}^{\mathbb{L}}[e^{-\theta e^{\sigma\tilde{N}+\sigma^2}}] = \mathcal{L}_\psi(e^{\sigma^2}\theta),$$

where $\frac{d\mathbb{L}}{d\mathbb{P}} = e^{\sigma N-\frac{1}{2}\sigma^2}$ and $\tilde{N} \overset{\mathbb{L}}{\sim} \mathcal{N}(0,1)$. This implies the correctness of the formula for $n=1$. Using a similar argument for $\partial_\theta\mathcal{L}_\psi(e^{n\sigma^2}\theta)$, we compute

$$\frac{\partial}{\partial\theta}\left((-1)^n\prod_{k=1}^{n} e^{\frac{2k-1}{2}\sigma^2}\mathcal{L}_\psi\big(e^{n\sigma^2}\theta\big)\right) = (-1)^{n+1}\prod_{k=1}^{n+1} e^{\frac{2k-1}{2}\sigma^2}\mathcal{L}_\psi\big(e^{(n+1)\sigma^2}\theta\big),$$

which verifies the validity of the induction step and hence the result. $\square$

12.2 Discussion of the monotonicity conjecture

The asymptotic expansions of Chapter 10 and all related results as well as the comparative statics of Chapter 11 were derived conditional on the validity of the following conjecture.

Conjecture 2. *Let ψ be a lognormally distributed random variable. Let further*

$$k_2(\theta) \triangleq \theta \frac{\mathbb{E}[e^{-\theta\psi}\psi]}{\mathbb{E}[e^{-\theta\psi}]}.$$

Then

$$k_2'(\theta) > 0 \quad \forall \theta \geq 0.$$

Even though this assertion is strongly supported by numerical evidence and intuition, an analytic proof eludes us as yet. Let us consider the statement of Conjecture 2 from an economic point of view. Recalling the marginal indifference pricing measure $\mathbb{Q}$, we can interpret the quantity $\mathbb{E}[e^{-\theta\psi}\psi]/\mathbb{E}[e^{-\theta\psi}]$ as the *marginal price of ψ when the market maker is already holding θ units of ψ.* Conjecture 2 then states the economically very reasonable fact that the mark-to-market value of a position of θ shares (i.e. the "marginal price for a total position of size θ") is strictly increasing in θ for all $\theta \geq 0$.

This intuition is reinforced by numerical evidence: Figure 12.1 displays the map $\theta \mapsto k_2(\theta)$ for varying values of $\sigma > 0$ when considering $\psi = e^{\sigma N}$, $N \sim \mathcal{N}(0, 1)$. We

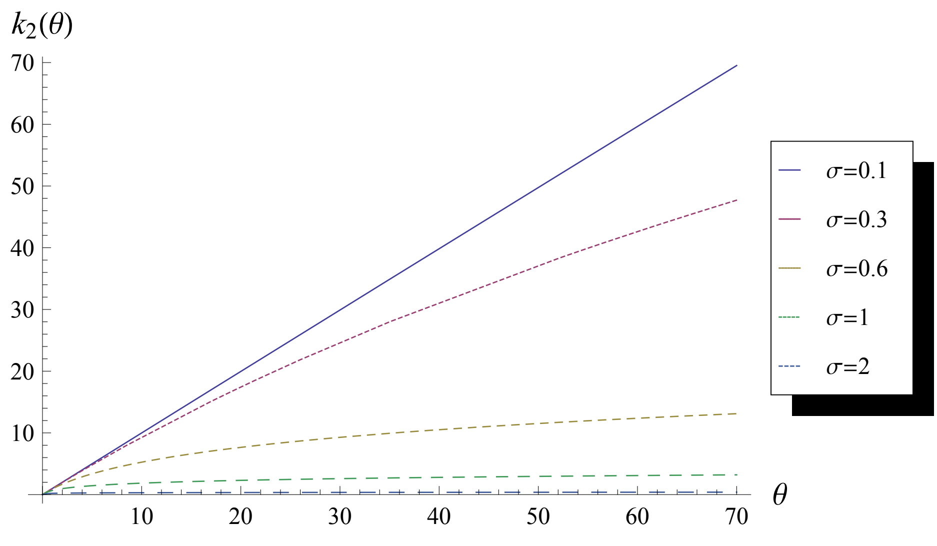

Figure 12.1: The map $\theta \mapsto k_2(\theta)$ vor varying values of σ

can see that k_2 appears to be strictly increasing in θ. Moreover, we observe that the rate at which k_2 increases depends strongly on σ: As σ becomes large, k_2 becomes almost "flat" (but, of course, still converges to ∞; see Lemma 9.6). In fact, this behaviour is to be expected. In the previous section we saw that, using Girsanov's theorem, k_2 can equivalently be expressed as

$$k_2(\theta) = \theta \frac{e^{\frac{1}{2}\sigma^2}\mathcal{L}_\psi(e^{\sigma^2}\theta)}{\mathcal{L}_\psi(\theta)}.$$

In particular, when σ is close to zero, we have

$$\frac{\mathcal{L}_\psi(e^{\sigma^2}\theta)}{\mathcal{L}_\psi(\theta)} \approx 1,$$

so that in this case k_2 is "almost linear" in θ. If, on the other hand, σ becomes large, the discrepancy between the arguments of the numerator and denominator of the fraction of Laplace transforms grows at the rapid rate of e^{σ^2} and k_2 is "pushed down" severely.

Note that Conjecture 2 seems to be related to Conjecture 1 from Part I, albeit not in a very tangible way. Recalling the notation

$$k_1(p) = p\frac{\mathbb{E}[(p\psi+1)^{-a}\psi]}{\mathbb{E}[(p\psi+1)^{-a}]}$$

of Conjecture 1 and slightly modifying the notation k_2 of Conjecture 2 as

$$\tilde{k}_2(ap) \triangleq ap\frac{\mathbb{E}[e^{-ap\psi}\psi]}{\mathbb{E}[e^{-ap\psi}]}$$

we can see from the power series representation of the exponential function that

$$\tilde{k}_2(ap) = ap\frac{\mathbb{E}[(1+p\psi+R(p,\psi))^{-a}\psi]}{\mathbb{E}[(1+p\psi+R(p,\psi))^{-a}]},$$

where

$$R(p,\psi) \triangleq \sum_{k=2}^{\infty} \frac{(p\psi)^k}{k!}.$$

In particular, $R(p,\psi)$ becomes insignificant for $p \to 0$, so that, for small p,

$$k_1(p) \approx \frac{\tilde{k}_2(ap)}{a}.$$

However, it is not clear whether the validity of either conjecture implies the validity of the other.

Index of Notation

$\tilde{\Delta}$	first order approximation to replicating position	pp. 67 and 130
$\tilde{\Lambda}$	second order approximation to replicating position	pp. 67 and 130
ψ	marketed security	pp. 16 and 93
Σ_0	market maker's initial endowment	pp. 16 and 94
a	market maker's risk aversion parameter	pp. 16 and 93
A_t	cash-recovering map	pp. 32 and 99
$\mathcal{A}_t$	set of compatible pairs of utility levels and transaction sizes	p. 29
b	balance parameter in reparametrisation of Σ_0	p. 79
d	market depth paramter in reparametrisation of Σ_0	p. 79
$\mathcal{E}$	analytic solution of SDE defining the GBM S	p. 16
F	static process of indirect utility	pp. 28 and 98
$\tilde{g}$	Itô integrand of F	pp. 28 and 28
g	Itô integrand of U^Q	pp. 33 and 99
G^Q	P&L (process) of the large investor	pp. 39 and 105
H	contingent claim written on underlying ψ	pp. 45 and 111
$\mathcal{H}$	set of acceptable contingent claims	pp. 46 and 111
$\mathcal{H}'$	set of claims in $\mathcal{H}$ which are path independent with Lipschitz continuous payoff functions	pp. 46 and 112
$\mathcal{H}''$	set of claims in $\mathcal{H}'$ of which small positions are attainable	pp. 58 and 124
i_t	Itô integrand of U^H	pp. 48 and 114
p	market maker's initial security position	pp. 16 and 94
$\mathbb{P}$	real world measure	p. 15
$\bar{q}$	maximal allowed transaction size	pp. 19, 26 and 29
$\mathbb{Q}$	marginal indifference pricing measure	pp. 60 and 95
Q	trading strategy of the large investor	pp. 25, 28, 96 and 97
$\mathcal{Q}_{\mathbf{u},t}$	set of compatible transaction sizes	p. 30
$\mathbb{R}$	real line *or* probability measure	p. 60 and 123
$\mathcal{R}$	class of Lipschitz continuous functions	pp. 46 and 112
R	market maker's absolute risk aversion	pp. 16 and 93
S	GBM which models payoff of ψ at maturity	pp. 16 and 93
T	maturity	p. 15
u	market maker's utility function	pp. 16 and 93
$U = U^Q$	dynamic process of indirect utility	pp. 33 and 99
$\mathcal{U}_{q,t}$	set of compatible utility levels	p. 30
$\tilde{\mathcal{U}}$	range of utilities	p. 29
v_{rep}	replication price of contingent claim H	pp. 46 and 112
v_t^ε	market indifference price of εH at time t	pp. 58 and 124
x_t^ε	market indifference price of $\varepsilon\psi$ at time t	pp. 59 and 126
$X = X^Q$	complementing cash process for a trading strategy Q	pp. 25, 34, 96 and 97
z	market maker's initial cash position	pp. 16 and 94

Bibliography

[1] A. Alfonsi, A. Fruth, and A. Schied. Constrained portfolio liquidation in a limit order book model. *Banach Center Publ.*, 83:9–25, 2008.

[2] A. Alfonsi, A. Fruth, and A. Schied. Optimal execution strategies in limit order books with general shape functions. *Quantitative Finance*, 10(2):143–157, 2010.

[3] A. Alfonsi and A. Schied. Optimal trade execution and absence of price manipulations in limit order book models. *SIAM Journal on Financial Mathematics*, 3:490–522, 2010.

[4] A. Alfonsi, A. Schied, and A. Slynko. Order book resilience, price manipulation, and the positive portfolio problem. *SIAM Journal on Financial Mathematics*, 3:511–533, 1965.

[5] A. Almgren. Optimal execution with nonlinear impact functions and trading-enhanced risk. *Applied Mathematical Finance*, 10(1):1–18, 2003.

[6] A. Almgren and N. Chriss. Value under liquidation. *Risk*, 12(12):61–63, 1999.

[7] A. Almgren and N. Chriss. Optimal execution of portfolio transactions. *Journal of Risk*, 3:5–39, 2000.

[8] S. Asmussen, J. L. Jensen, and L. Rojas-Nandayapa. On the laplace transform of the lognormal distribution. *Thiele Center preprint*, 2013. http://math.au.dk/publs?publid=994.

[9] Louis Bachelier. Théorie de la spéculation. *Annales scientifiques de l'E.N.S.*, 3(17):21–86, 1900.

[10] P. Bank and D. Baum. Hedging and portfolio optimization in financial markets with a large trader. *Mathematical finance*, 14:1–18, 2004.

[11] P. Bank and D. Kramkov. A model for a large investor trading at market indifference prices. i: single-period case. *preprint*, 2013. http://arxiv.org/abs/1110.3224.

[12] P. Bank and D. Kramkov. On a stochastic differential equation arising in a price impact model. *Stochastic Processes and their Applications*, 123(3):1160–1175, 2013.

[13] P. Bank and D. Kramkov. The stochastic field of aggregate utilities and its saddle conjugate. *preprint*, 2013. http://arxiv.org/abs/1310.7280.

[14] P. Bank and D. Kramkov. A model for a large investor trading at market indifference prices. ii: continuous-time case. *preprint*, 2014. http://arxiv.org/abs/1110.3229.

[15] D. Bertsimas and A. Lo. Optimal control of execution costs. *Journal of Financial Markets*, 1:1–50, 1998.

[16] N. H. Bingham, C. M. Goldie, and J. L. Teugels. *Regular Variation.* Cambridge University Press, 1989.

[17] Fischer Black and Myron Scholes. The pricing of options and corporate liabilities. *The Journal of Political Economy*, 81(3):637–654, 1973.

[18] U. Çetin, R. Jarrow, and P. Protter. Liquidity risk and arbitrage pricing theory. *Finance and Stochastics*, 8:311–341, 2004.

[19] U. Çetin and L.C.G. Rogers. Modelling liquidity effects in discrete time. *Mathematical Finance*, 17:15–29, 2007.

[20] U. Çetin, M. Soner, and N. Touzi. Option hedging for small investors under liquidity costs. *Finance and Stochastics*, 14(3):317–341, 2010.

[21] Domenico Cuoco and Jakša Cvitanić. Optimal consumption choices for a large investor. *Journal of Economic Dynamics and Control*, 22:401–436, 1998.

[22] Jakša Cvitanić and Jin Ma. Hedging options for a large investor and forward-backward sde's. *Annals of Applied Probability*, 6:370–398, 1996.

[23] F. Delbaen and W. Schachermayer. A general version of the fundamental theorem of asset pricing. *Math. Ann.*, 300(3):463–520, 1994.

[24] W. Feller. *An introduction to probability theory and its applications.* Wiley, 1971.

[25] Rüdiger Frey. Perfect option hedging for a large trader. *Finance and Stochastics*, 2:115–141, 1998.

[26] Rüdiger Frey and Alexander Stremme. Market volatility and feedback effects from dynamic hedging. *Mathematical Finance*, 7:351–374, 1997.

[27] Jim Gatheral. No-dynamic-arbitrage and market impact. *Quantitative Finance*, 10(7):749–759, 2010.

[28] Selim Gökay, Alexandre F. Roch, and Mete H. Soner. Liquidity models in continuous and discrete time. In Giulia Di Nunno and Bernt Øksendal, editors, *Advanced Mathematical Methods for Finance*, pages 333–365. Springer Berlin Heidelberg, 2011.

[29] Selim Gökay and Mete H. Soner. Hedging in an illiquid binomial market. *Nonlinear Analysis: Real World Applications*, 16:1–16, 2014.

[30] J. Michael Harrison and David M. Kreps. Martingales and arbitrage in multiperiod securities markets. *Journal of Economic Theory*, 20(3):381–408, 1979.

[31] J. Michael Harrison and Stanley R. Pliska. Martingales and stochastic integrals in the theory of continuous trading. *Stochastic Processes and their Applications*, 11(3):215–260, 1981.

[32] Gur Huberman and Werner Stanzl. Quasi-arbitrage and price manipulation. *Econometrica*, 72(4):1247–1275, 2004.

[33] Robert A. Jarrow. Market manipulation, bubbles, corners and short squeezes. *Journal of Financial and Quantitative Analysis*, 27(3):311–336, 1992.

[34] Robert A. Jarrow. Derivative securities markets, market manipulation and option pricing theory. *Journal of Financial and Quantitative Analysis*, 29(2):241–261, 1994.

[35] Robert A. Jarrow. Liquidity risk, price impacts and the replication problem. *Finance and Stochastics*, 15:399, 2011.

[36] Robert A. Jarrow, Philip Protter, and Alexandre F. Roch. A liquidity-based model for asset price bubbles. *Quantitative Finance*, 12(9):1339–1349, 2012.

[37] I. Karatzas and S.E. Shreve. *Brownian Motion and Stochastic Calculus*. Springer, 1991.

[38] M. Kobylanski. Backward stochastic differential equations with quadratic growth. *The Annals of Probability*, 28(2):558–602, 2000.

[39] H. Kunita. *Stochastic flows and stochastic differential equations*. Cambridge University Press, 1990.

[40] Robert C. Merton. Theory of rational option pricing. *The Bell Journal of Economic and Management Science*, 4(1):141–183, 1973.

[41] Anna Obizhaeva and Jiang Wang. Optimal trading strategy and supply/demand dynamics. *NBER Working Paper No 11444*, 2005.

[42] George Papanicolaou and Ronnie Sircar. General black-scholes models accounting for increased market volatility from hedging strategies. *Applied Mathematical Finance*, 5:45–82, 1998.

[43] Eckhard Platen and Martin Schweizer. On feedback effects from hedging derivatives. *Mathematical Finance*, 8:67–84, 1998.

[44] D. Revuz and M. Yor. *Continuous Martingales and Brownian Motion*. Springer, 2007.

[45] R.T. Rockafellar. *Convex Analysis*. Number 28 in Princeton Mathematical Series. Princeton University Press, Princeton, N.J., 1970.

[46] Jeffrey Said. *Nonlinear Stochastic Models of Liquidity Effects in Financial Markets*. PhD thesis, Columbia University, New York, 2007.

[47] Paul A. Samuelson. Proof that properly anticipated prices fluctuate randomly. *Industrial Management Review*, 6(2):41, 1965.

[48] Alexander Schied and Torsten Schöneborn. Risk aversion and the dynamics of optimal liquidation strategies in illiquid markets. *Finance and Stochastics*, 13(2):181–204, 2009.