When do Projections Commute?

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Necessary and sufficient conditions for commutativity of two projections in Hilbert space are given through properties of so-called conditional connectives which are derived from the conditional probability operator PQP. This approach unifies most of the known proofs, provides a few new criteria, and permits certain suggestive interpretations for compound properties of quantum-mechanical systems.

1. Introduction

Commutativity of two projections P and Q in a complex Hilbert space H plays an important rôle in the mathematical formulation and physical interpretation of quantum-mechanical systems. $PQ = QP$ is interpreted as "commensurability" of the properties represented by P and Q. This means: On a quantum-mechanical system in a given state, measurement of P and Q can be made simultaneously or, a measurement of first P and then Q affects any state φ in the same way as does a measurement first of Q and then of P:

$$\langle \varphi, QP\varphi \rangle = \langle \varphi, PQ\varphi \rangle .$$

Mathematically speaking, this identity is equivalent to the fact that PQ is a projection onto the meet of P and Q, which in turn means physically that $PQ\ (= QP)$ is again a "compound" property of the system.

On the other hand, the meet (which by abuse of language we write $P \wedge Q$) of P and Q is uniquely defined even for non-commuting projections. In this case, PQ is not a projection and a fortiori $PQ \neq QP$. PQ is not even an observable (hermitian operator) in H and hence is not interpreted in quantum mechanics. The interpretation of $P \wedge Q$, however, has been controversial (see Jammer's book [4], p. 353–361).

The purpose of this paper is to present a somewhat unified approach to commutativity proofs for two projections in Hilbert space. We shall derive necessary and sufficient conditions for commutativity from properties of the so-called "conditional probability operator" PQP (cf. Bub's discussion in [3], and the relevant literature quoted there). This observable leads to the introduction of derived connectives $P \sqcap Q$, $P \sqcup Q$, and the material quasi-implication $P \to Q$. These connectives allow a reasonable physical interpretation for the meet $P \wedge Q$ even for non-commuting projections. Most of the following material can be proved in the more general setting of quasimodular orthocomplemented lattices (cf. [9] and [10]). These criteria are rephrasings of known results in terms of the new connectives; only (3.17) below appears to be new.

2. Conditional Connectives

Let P and Q be projections in a complex Hilbert space H. Because of the one-to-one correspondence between projections and their ranges, we denote the range of P by P as well, so that

$$Px = x \quad \text{and} \quad x \in P$$

have the same meaning.

Let $E_0(PQP)$ denote the null-space of PQP:

$$E_0(PQP) = \{x \in H \mid PQPx = 0\} .$$

2.1. Definition

The "conditional" connective $P \sqcap Q$ is (the projection onto) the orthocomplement of $E_0(PQP)$:

$$P \sqcap Q = E_0{}^{\perp}(PQP) ,$$

read "P and then Q".

In other words, $P \sqcap Q$ is the projection onto the range of PQP.

It follows from $\langle x, PQPx \rangle = \| QPx \|^2$ that $PQPx = 0$ if $QPx = 0$. Now the following representation of $E_0(PQP) = E_0(QP)$ ensues:

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2.2. Theorem

For all projections P, Q in H

$$E_0(PQP) = P^\perp \vee (P \wedge Q^\perp),$$

where $\vee$ may be replaced by $+$.

Proof. $QPx=0$ is equivalent to $Px \in Q$, and this is certainly fulfilled for all x in the subspace on the right-hand side.

Conversely, if x is such that $Px \in Q$, then we see from $x=(I-P)x+Px$ and $I-P=P^\perp$ that x belongs also to the right-hand side of the equality above.

2.3. Corollary

$$P \sqcap Q = P \wedge (P^\perp \vee Q).$$

2.3 justifies our reading of $P \sqcap Q$ as "P and then Q"; for, if we interpret the right-hand side of (2.3) via "classical" connectives, we see that $P \sqcap Q$ is true iff P is true and it is true that Q follows "materially" from P. It is now clear how to define $P \sqcup Q$: "P or then Q", and a material "quasi-implication" or "conditional implication" $P \to Q$:

2.4. Definitions

For projections P, Q in H put

$$\begin{aligned} P \sqcup Q &:= (P^\perp \sqcap Q^\perp)^\perp \\ &= E_0(P^\perp Q^\perp P^\perp) \\ &= \{x \mid Q^\perp P^\perp x = 0\} \\ &= P \vee (P^\perp \wedge Q); \end{aligned}$$

$$\begin{aligned} P \to Q &:= P^\perp \sqcup Q \\ &= E_0(PQ^\perp P) = \{x \mid Q^\perp Px = 0\} \\ &= P^\perp \vee (P \wedge Q). \end{aligned}$$

$P \to Q$ can be read as follows: "$P \to Q$ is true iff either $P^\perp$ is true or the occurrence of the yes-outcome of P leaves the system in a state which makes true Q." (cf. [2], p. 378).

2.5. Corollary

For all projections P, Q in H (commuting or not), we have

(1) $P \wedge Q \leqq P \sqcap Q$,

(2) $P \vee Q \geqq P \sqcup Q$,

(3) $P \wedge Q = P \sqcap (P^\perp \sqcup Q) = Q \sqcap (Q^\perp \sqcup P)$,

(4) $P \vee Q = P \sqcup (P^\perp \sqcap Q) = Q \sqcup (Q^\perp \sqcap P)$,
note that $P \sqcup (P^\perp \sqcap Q) = P + (P^\perp \sqcap Q)$.

Proof. (1) and (2) are clear from (2.3) and (2.4). We prove (4): Since

$$P \leqq P \vee Q, \quad P \vee Q - P = (P \vee Q) \wedge P^\perp.$$

(3) follows from (4).

$P \wedge Q$ and $P \vee Q$ can also be expressed in terms of the spectral measure of the observable PQP:

2.6. Theorem

For all projections P, Q in H

$$P \wedge Q = E_1(PQP) = E_1(QPQ),$$

where E_1 is the respective projection onto the eigenspace with the eigenvalue 1.

Proof. $x \in E_1(PQP)$ iff $PQPx = x$. From $\|x\|^2 = \langle PQPx, x\rangle = \|QPx\|^2 \leqq \|Px\|^2 \leqq \|x\|^2$ we see that $Px = x$, i.e. $x \in P$, and also $QPx = x$, and together with $Px = x$, that $Qx = x$. The converse is evident.

2.5 (3) allows a suggestive reading of the meet $P \wedge Q$, whether P and Q commute or not:

$P \wedge Q$ iff P and then P quasi-implies Q, which is the same as Q and then Q quasi-implies P.

3. Commutativity of Projections

Abbreviate $P \sim Q$ for $PQ = QP$. Obviously,

$$\begin{aligned} &P \sim Q \Leftrightarrow Q \sim P \Leftrightarrow P \sim Q^\perp \\ &\quad \Leftrightarrow P^\perp \sim Q \Leftrightarrow P^\perp \sim Q^\perp. \end{aligned} \tag{3.1}$$

Main Theorem: $P \sim Q$ is equivalent to each of the following equalities or inequalities in (3.2) through (3.17). (We shall prove only sufficiency; the proof that $P \sim Q$ implies (3.2) through (3.17) is straightforward and will be omitted).

$$P \wedge Q = P \sqcap Q. \tag{3.2}$$

Proof: (3.2) is the same as

$$E_1(PQP) = E_0{}^\perp(PQP),$$

which means that PQP is the projection onto $P \wedge Q$. Because of $E_0(PQP) = E_0(QP)$, PQP and QP coincide on $(P \wedge Q)^\perp$. On the other hand, for every $x \in P \wedge Q$, $x = QPx$ and $x = PQPx$, so that PQP and QP are identical.

$$P \vee Q = P \sqcup Q. \tag{3.3}$$

Proof: (3.2) and (3.1).

$$P = P \wedge Q + P \wedge Q^\perp. \tag{3.4}$$

Proof: $P \wedge Q^\perp \leqq P$ implies $P \wedge Q = P - P \wedge Q^\perp = P \wedge (P \wedge Q^\perp)^\perp = P \wedge (P^\perp \vee Q) = P \sqcap Q$. Now apply (3.2).

(3.2) is often used to define the 2-place relation C of commensurability

$$(P, Q) \in C \Leftrightarrow P = P \wedge Q + P \wedge Q^\perp$$

in an orthocomplemented lattice L_0 (e.g. Mittelstaedt [9], p. 32). It is worth noting that C is symmetrical if and only if L_0 is quasimodular (some authors say "orthomodular" or "weakly modular"):

$$P \leqq Q \Rightarrow Q \sqcap P = P.$$

If this implication holds in L_0, (3.1) is true in L_0 (see Mittelstaedt [9], p. 30—34); the reverse implication $Q \sqcap P = P \Rightarrow P \leqq Q$ is always true. If equality of the antecedent is weakened to mere inclusion, we get that $P \sim Q$ is a consequence of

$$Q \sqcap P \leqq P. \tag{3.5}$$

Proof: $P \wedge Q \leqq Q \sqcap P$ from (2.5) (1); $Q \sqcap P \leqq Q$ together with (3.5) and (3.2) give $P \sim Q$.

$$P \leqq Q \sqcup P. \tag{3.6}$$

Proof: (3.5) and (3.1).

$$P \leqq (Q \rightarrow P). \tag{3.7}$$

Proof: $Q \rightarrow P = Q^\perp \sqcup P \geqq P$. Apply (3.6) and (3.1). (We remark that always $P \sim (P \rightarrow Q)$; cf. [9], p. 41, 2.40 (b)).

$$P \rightarrow (Q \rightarrow P) = H. \tag{3.8}$$

Proof: (3.8) is the same as saying

$$P^\perp \sqcup (Q^\perp \sqcup P) = H.$$

Because of (2.5) (2) we have also $H = P^\perp \vee (Q^\perp \sqcup P)$. But then $P \leqq H$ implies $P \leqq Q^\perp \sqcup P$, and (3.6) together with (3.1) yield $P \sim Q$.

$$P \sqcap Q = Q \sqcap P. \tag{3.9}$$

Proof: Using (3.2), we have to prove that (3.9) implies $P \sqcap Q = P \wedge Q$. We know that always $P \sqcap Q \geqq P \wedge Q$. On the other hand, if $x \in P \sqcap Q = Q \sqcap P$, then $x \in P$ and $x \in Q$ (from the representation (2.3)), i.e. $x \in P \wedge Q$.

$$P \sqcup Q = Q \sqcup P. \tag{3.10}$$

Proof: (3.9) and (3.1).

$$P \rightarrow Q = Q^\perp \rightarrow P^\perp. \tag{3.11}$$

Proof: (3.11) says $P^\perp \sqcup Q = Q \sqcup P^\perp$. Apply (3.10) and (3.1).

$$w_\varphi(P \vee Q) \leqq w_\varphi(P) + w_\varphi(Q) \tag{3.12}$$

for all states $\varphi \in H$, where $w_\varphi(R) := \langle \varphi, R\varphi \rangle$ for projections R in H.

Proof: $P \vee Q = P + (P^\perp \sqcap Q)$ from (2.5) (4), and $w_\varphi(P^\perp \sqcap Q) \leqq w_\varphi(Q)$ for all states φ if and only if $P^\perp \sqcap Q \leqq Q$. Using (3.5) and (3.1) gives $P \sim Q$.

(Cf. also Jauch's lemma, p. 117 of [5]).

$$P = Q \sqcap P + Q^\perp \sqcap P. \tag{3.13}$$

Proof:

$$\begin{aligned} Q \sqcap P + Q^\perp \sqcap P &= \{x \mid PQx = 0\}^\perp \\ &\quad + \{x \mid PQ^\perp x = 0\}^\perp \\ &= \{x \mid PQx = 0 \ \text{ and } \\ &\qquad Px - PQx = 0\}^\perp \\ &= \{x \mid \ Px = 0 \ \text{ and } \\ &\qquad PQx = 0\}^\perp \\ &= (P^\perp \wedge Q^\perp \sqcup P^\perp)^\perp \\ &= P \vee (Q \sqcap P) \\ &= P + P^\perp \sqcap (Q \sqcap P). \end{aligned}$$

Therefore, (3.13) is equivalent to $P^\perp \sqcap (Q \sqcap P) = 0$ or $H = P \sqcup (Q^\perp \sqcup P^\perp) = P^\perp \rightarrow (Q \rightarrow P^\perp)$, which by (3.8) and (3.1) implies $P \sim Q$.

The proof of (3.13) shows also that $P^\perp \sqcap (Q \sqcap P) = P^\perp \sqcap (Q^\perp \sqcap P) =: I(P, Q)$, which may be called the "interference term", and (3.13) is equivalent to

$$I(P, Q) = 0. \tag{3.14}$$

$$P = QPQ + Q^\perp PQ^\perp. \tag{3.15}$$

Proof: Due to (3.13), (3.15) is the same as

$$QPQ + Q^\perp PQ^\perp = Q \sqcap P + Q^\perp \sqcap P.$$

Since always $QPQ \leqq Q \sqcap P$ and $Q^\perp PQ^\perp \leqq Q^\perp \sqcap P$, the latter equality can only hold if $QPQ = Q \sqcap P$ and $Q^\perp PQ^\perp = Q^\perp \sqcap P$. But this means that QPQ is the projection onto $Q \wedge P$, and thus coincides with PQ as was shown in the proof of (3.2).

From $QPQ + Q^\perp PQ^\perp = P - (QPQ^\perp + Q^\perp PQ)$ it can be seen that (3.15) holds if and only if $J(P, Q) := QPQ^\perp + Q^\perp PQ$ is zero:

$$J(P, Q) = 0. \tag{3.16}$$

$J(P, Q)$ is the observable which defines Mittelstaedt's probability of interference ([8], p. 215):

$$w_\varphi^{\text{int}}(P, Q) = \langle \varphi, J(P, Q)\varphi \rangle,$$

which is zero if and only if ($J(P, Q)$ is hermitian!) the condition (3.16) holds, i.e. iff $P \sim Q$.

(3.4), (3.13) and (3.15) are saying that each of the following representations for P is equivalent to $P \sim Q$:

$$P = P \wedge Q + P \wedge Q^{\perp},$$
$$P = Q \sqcap P + Q^{\perp} \sqcap P,$$
$$P = QPQ + Q^{\perp}PQ^{\perp}.$$

As our final criterium we show that $P \sim Q$ is equivalent to

$$PQP = QPQ. \tag{3.17}$$

Proof: By (3.2), $P \sim Q$ iff $P \wedge Q = P \sqcap Q$, i.e. $E_1(PQP) = E_0(PQP)$. This equality is true iff PQP is a projection. But, using (3.17) twice, we get

$$\begin{aligned}(PQP)^2 &= PQPPQP = PQ(PQP)\\ &= PQ(QPQ) = P(QPQ)\\ &= P(PQP) = PQP,\end{aligned}$$

so that the hermitian operator PQP is idempotent, i.e. in fact a projection.

From the standpoint of physical interpretation, (3.17) is to be expected: PQP is the defining operator for the "joint" probability of P and (then) Q and determines the conditional probability of Q, given P. Considering this interpretation, and (3.16), for instance, which is equivalent to (3.17), it comes as no surprise that $PQP = QPQ$ should imply $P \sim Q$. Mathematically speaking, however, this implication seems curious: (3.17) means that for $PQ = QP$ it is sufficient that PQ has the same value for Px as QP has for Qx, for all $x \in H$. In other words, (3.17) permits an implication from the equality of positive self-adjoint operators PQP and QPQ to the equality of prima facie more general operators PQ and QP. Putting $A = PQ$, $A^* = QP$, (3.17) may be restated as: $A = A^*$ is equivalent to $AA^* = A^*A$, i.e. for $A = PQ$ self-adjointness and normality are the same.

For this reason it may be of interest to have a proof of (3.17) independent of (3.2) and of the representation of $P \sqcap Q$ and $P \wedge Q$ through the spectral measure of PQP. We shall do so now.

Proposition. For any two projections P and Q in a complex Hilbert space H, the commutativity relation $PQ = QP$ is equivalent to $PQP = QPQ$.

Again, we need only prove the non-trivial direction.

We need the following

Lemma: Let A and B be bounded linear operators in H such that

(1) $AB = BA$,

(2) $A^2 = B^2$, and

(3) $(A - B) = -(A - B)^*$.

Then (4) E commutes with any transformation that commutes with $A - B$, and (5) $A = (2E - I)B$, where E is the orthogonal projection onto the null-space $M = E_0(A - B)$ of $A - B$.

Proof of the Lemma: (Cf. [1], p. 424, Theorem 23.3; note that in our proof A and B need not be self-adjoint!). Suppose that C commutes with $A - B$. This implies $C(M) \subset M$. From

$$\begin{aligned}C(A - B) = (A - B)C \Rightarrow (A - B)^*C^*\\ = C^*(A - B)^*\end{aligned}$$

and (3), we have

$$(A - B)C^* = C^*(A - B),$$

which implies $C^*(M) \subset M$. Therefore C reduces M, i.e. $CE = EC$, proving (4).

From (1) and (2) we have

$$(A - B)(A + B) = A^2 - B^2 = 0,$$

i.e. (6)

$$E(A + B) = A + B.$$

For any vector $z \in H$ write $z = x + y$, where $x \in M$ and $y \in M^{\perp}$. It follows

$$E(A - B)z = E(A - B)x + E(A - B)y.$$

The first term on the right is zero, because $x \in M = E_0(A - B)$ and the second is zero because E commutes with $A - B$, according to (4).

Hence

(7) $E(A - B) = 0$.

Combining (6) and (7), gives

$$E(A + B) - E(A - B) = A + B$$

or

$$A = 2 \cdot EB - B = (2E - I)B,$$

which proves (5).

We wish to apply the Lemma for $A = PQ$, $B = QP$. Assumption (1) is the same as $PQP = QPQ$. Using this, and observing

$$(8) \quad (PQ)^2 = PQPQ = PPQP = PQP = QPQ = (QP)^2,$$

we note that (2) is fulfilled.

Moreover,

$$(PQ - QP)^* = QP - PQ = -(PQ - QP),$$

which is assumption (3) of our Lemma.

Proof of the Proposition: From

$$(PQP)^2 = PQPPQP = PQPQP = PQQPQ = (PQ)^2$$

and (8), we see that $PQP = QPQ$ must be a projection. We claim that PQP is the projection onto $P \wedge Q$. This may be seen from

$$PQP = (PQP)^2 = (PQ)^2,$$

i.e.

$$PQP = (PQ)^{2k}, \quad k \geqq 1,$$

and from the fact that the projection onto $P \wedge Q$ is given by the limit of $(PQ)^n$, $n \to \infty$.

We prove $PQ = QP$.

If $z \in P \wedge Q$, trivially $PQz - QPz = z - z = 0$. If $z \in P^\perp \vee Q^\perp$, write $z = \lim (x_n + y_n)$, where $x_n \in P^\perp$ and $y_n \in Q^\perp$. Using (5) of our Lemma gives

$$PQ = (2E - I)\,QP \quad \text{and}$$
$$QP = (2E - I)\,PQ,$$

where E is the orthogonal projection onto

$$E_0(PQ - QP) = E_0(QP - PQ)$$

(note that the assumptions of the Lemma are symmetrical in A, B).

Therefore, using continuity,

$$PQz = \lim_n (PQx_n + PQy_n) = \lim_n PQx_n = \lim_n (2E - I)\,QPx_n = 0,$$

similarly $QPx = 0$.

Hence, PQ and QP coincide on H.

Remark

It should be noted that the Proposition is also a special case of a rather deep theorem by Fuglede-Putnam, Rosenblum (cf. [11], p. 300, Theorem 12.16, where Rosenblum's proof is given):

Assume that A, B, T are bounded transformations on H, A and B are normal, and

$$AT = TB.$$

Then $A^*T = TB^*$.

Taking $A = PQ$, $B = QP$, $T = P$ yields our Proposition.

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[1] G. Bachmann and L. Narici, Functional Analysis, Academic Press, London 1966.

[2] E. G. Beltrametti and G. Cassinelli, On state transformations induced by yes-no experiments, in the context of quantum logic, J. Phil. Logic **6**, 369 (1977).

[3] J. Bub, Von Neumann's Projection Postulate as a Probability Conditionalization Rule in Quantum Mechanics, J. Phil. Logic **6**, 381 (1977).

[4] M. Jammer, The Philosophy of Quantum Mechanics, John Wiley, New York 1974.

[5] J. M. Jauch, Foundation of Quantum Mechanics, Addison Wesley, Reading, Mass., 1968.

[6] H. Kröger, Zwerch-Assoziativität und verbandsähnliche Algebren, S.B. Bayer. Akad. Wiss., München 1973.

[7] H. Kröger, Das Assoziativgesetz als Kommutativitätsaxiom in Boole'schen Zwerchverbänden, J. Reine Angew. Math. **285**, 53 (1974).

[8] P. Mittelstaedt, Philosophische Probleme der modernen Physik, BI Mannheim, 1976. Engl. Transl. as Philosophical Problems of Modern Physics, Reidel, Dordrecht 1976.

[9] P. Mittelstaedt, Quantum Logic, Reidel, Dordrecht 1978.

[10] C. Piron, Foundations of Quantum Physics, Reading, Mass., 1976.

[11] W. Rudin, Functional Analysis, McGraw-Hill, New York 1973.