# Online Cooperative Cost Sharing*

Janina Brenner[1] and Guido Schäfer[2]

[1] Institute of Mathematics, Technical University Berlin, Germany
brenner@math.tu-berlin.de
[2] Algorithms, Combinatorics and Optimization, CWI Amsterdam, The Netherlands
g.schaefer@cwi.nl

**Abstract.** The problem of sharing the cost of a common infrastructure among a set of strategic and cooperating players has been the subject of intensive research in recent years. However, most of these studies consider cooperative cost sharing games in an *offline* setting, i.e., the mechanism knows all players and their respective input data in advance. In this paper, we consider cooperative cost sharing games in an *online* setting: Upon the arrival of a new player, the mechanism has to take instantaneous and irreversible decisions without any knowledge about players that arrive in the future. We propose an online model for general demand cost sharing games and give a perfect characterization of both weakly group-strategyproof and group-strategyproof online cost sharing mechanisms for this model. Moreover, we present a simple method to derive incremental online cost sharing mechanisms from online algorithms such that the competitive ratio is preserved. Based on our general results, we develop online cost sharing mechanisms for several binary demand and general demand cost sharing games.

## 1 Introduction

The pivotal point in mechanism design is to achieve a global objective even though part of the input information is owned by selfish players. In cost sharing, the aim is to share the cost of a common service in a fair manner while the players' valuations for the service are private information. Based on the declared bids of the players, a cost sharing mechanism has to determine a service allocation and distribute the incurred cost among the served players. In many cost sharing games, the common service is represented by a combinatorial optimization problem like minimum Steiner tree, machine scheduling, etc., which defines a cost for every possible service allocation. We consider cooperative cost sharing games, i.e., players may form coalitions to coordinate their bidding strategies.

During the last decade, there has been substantial research on *binary demand* cost sharing games, where a service allocation determines simply whether or not a player is served. In this paper, we consider the *general demand* setting in which players require not only one but several *levels* of service and the mechanism has

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to determine which service level is granted to each player and at what price. We assume that players are concerned about the *quantity* of service levels they obtain, e.g., the number of distinct connections to a source, executions of their job, etc. Moreover, once a player's request for a certain service level was refused, she will not be granted a higher level. This general demand cost sharing model has recently been investigated quite intensively; see [1, 3, 9, 10].

To the best of our knowledge, all previous studies of cooperative cost sharing games considered *offline* settings, where the entire input instance is known in advance. That is, the mechanism has complete information about the input data associated with every player (bids for different service levels and other relevant player characteristics) that can be taken into account in order to determine the allocation and payment scheme. However, several natural cost sharing games inherently bear an *online* characteristic in the sense that players arrive over time and reveal their input data only at their arrival. In such settings, the mechanism needs to take instantaneous and irreversible decisions with respect to the assigned service level and payment of the player without any knowledge about players that arrive in the future.

Problems in which the input data is revealed gradually and irreversible decisions have to be taken without further knowledge of future requests are the subject of *online computation* [2]. The standard yardstick to assess the quality of an online algorithm is by means of its *competitive ratio*, i.e., the worst case ratio of the cost of the solution produced by the online algorithm compared to the cost of an optimal offline algorithm that knows the entire input data in advance.

*Our Contributions.* The main contributions of this paper are as follows:

1. We propose the first online model for general demand cost sharing games: In its most general form, every player arrives several times to request an additional service level. Upon the arrival of a player, the online mechanism immediately determines a price for her new request. We require that at each point of time, the sum of the collected payments approximates the cost of the (optimal offline) solution for the current allocation.
2. We give a perfect characterization of both weakly group-strategyproof and group-strategyproof (formal definitions are given below) online mechanisms for the online general demand cost sharing game: We show that online cost sharing mechanisms are automatically weakly group-strategyproof for online binary demand games. In the general demand case, this is true if the marginal costs of the underlying cost function are increasing. Moreover, we prove necessary and sufficient conditions for group-strategyproofness of online cost sharing mechanisms.
3. We present a simple yet effective method to derive online cost sharing mechanisms from competitive online algorithms: Given a $\rho$-competitive algorithm for the underlying problem, we show that the induced *incremental* online mechanism is $\rho$-budget balanced at all times. Together with the above characterization, this enables us to derive online mechanisms for several binary

demand and general demand cost sharing games for network design and scheduling problems. For example, we obtain an $O(\log^2 |V|)$-budget balanced group-strategyproof online mechanism for the online binary demand Steiner forest cost sharing game, where $V$ denotes the set of vertices of the underlying graph.

*Related Work.* Moulin [10] introduced incremental cost sharing mechanisms in the offline setting. He studies fully budget balanced cost sharing mechanisms in two polar cases with respect to the underlying cost function. He states that if the cost function is supermodular and marginal costs are increasing, essentially only incremental mechanisms can be group-strategyproof. On the other hand, if the cost function is submodular and marginal costs are decreasing, all cross-monotonic cost sharing methods for *binary demand* games yield group-strategyproof mechanisms, but almost only sequential stand alone mechanisms (a particularly simple subclass of incremental mechanisms) can be group-strategyproof for *general demand* cost sharing games.

We complement Moulin's results by studying the *submodular* case with *increasing* marginal costs: We define incremental mechanisms slightly differently in that they accept requests in the borderline cases in which a player's bid equals the offered price. This small difference allows us to guarantee the strong notion of group-strategyproofness for all incremental mechanisms in this case while achieving weak group-strategyproofness for the whole class of games with increasing marginal cost functions.

Independently of our work, Juarez studied group-strategyproofness of binary demand cost sharing games in an unpublished work [8].

## 2 Online General Demand Cost Sharing Games

We first review *offline general demand cost sharing games* as studied in [1, 3, 9, 10]. Let $U$ be a set of players that are interested in a common service. In a *general demand* cost sharing game, every player has valuations for a finite number of service levels, i.e. the maximum service level requested is bounded by a constant $L \in \mathbb{N}$. Let $(i, l)$ denote player $i$'s request for service level $l$. Each player $i \in U$ has a *valuation* vector $v_i \in \mathbb{R}_+^L$, where $v_{i,l}$ denotes how much more (additive) player $i$ likes service level $l$ compared to service level $l-1$. The valuation vectors are private information, i.e. $v_i$ is known to $i$ only. Additionally, each player $i$ announces a *bid* vector $b_i \in \mathbb{R}_+^L$. $b_{i,l}$ represents the maximum price player $i$ is willing to pay for service level $l$ (in addition to service level $l-1$).

An *allocation* of goods or service to the set of players $U$ is denoted by a vector $\mathbf{x} \in \mathbb{N}_0^U$, where $x_i \in \mathbb{N}_0$ indicates the level of service that player $i$ obtains; here $x_i = 0$ represents that $i$ does not receive any good or service. Note that as a characteristic of this model, only subsequent service levels can be allocated to a player (i.e. if a player obtains service level $l$, then she also obtains service levels $1, \ldots, l-1$). We denote by $\mathbf{e}_i \in \mathbb{N}_0^U$ the $i$th unit vector.

The *servicing cost* of an allocation $\mathbf{x} \in \mathbb{N}_0^U$ is given by a cost function $C : \mathbb{N}_0^U \to \mathbb{R}_+$. We assume that $C$ is non-decreasing in every component and

$C(\mathbf{0}) = 0$ for the all-zero allocation $\mathbf{0}$. In the examples we study, the common service is represented by a combinatorial optimization problem like e.g. Steiner tree, machine scheduling, etc. In these cases, we define $C(\mathbf{x})$ as the cost of an offline optimal solution to the underlying optimization problem.

A general demand *cost sharing mechanism* solicits the bid vectors $b_i$ from all players $i \in U$, and computes a service allocation $\mathbf{x} \in \mathbb{N}_0^U$ and (non-negative) payments $\phi_{i,l} \in \mathbb{R}$ for every player $i \in U$ and service level $l \leq L$. We assume that the mechanism complies with the following standard assumptions:

1. *Individual rationality*: A player is charged only for service levels that she receives, and for any service level, her payment is at most her bid, i.e. for all $i, l$: $\phi_{i,l} = 0$ if $x_i < l$ and $\phi_{i,l} \leq b_{i,l}$ if $x_i \geq l$.
2. *No positive transfer*: A player is not paid for receiving service, i.e. $\phi_{i,l} \geq 0$ for all $i, l$.

For notational convenience, we define $v_{i,0} = \phi_{i,0} = 0$ for all players $i \in U$.

Let $\bar{C}(\mathbf{x})$ denote the cost of the actually computed solution for allocation $\mathbf{x}$. A cost sharing mechanism is *$\beta$-budget balanced* if the total payment obtained from all players deviates by at most a factor $\beta \geq 1$ from the total cost, i.e.

$$\bar{C}(\mathbf{x}) \leq \sum_{i \in U} \sum_{l=1}^{L} \phi_{i,l} \leq \beta \cdot C(\mathbf{x}).$$

If $\beta = 1$, we simply call the cost sharing mechanism budget balanced.

We assume that players act strategically and every player's goal is to maximize her own utility. The *utility* of player $i$ is defined as

$$u_i(\mathbf{x}, \phi) := \sum_{l=1}^{x_i} (v_{i,l} - \phi_{i,l}).$$

Since the outcome $(\mathbf{x}, \phi)$ computed by the mechanism solely depends on the bids $\mathbf{b}$ of the players (and not on their true valuations), a player may have an incentive to declare a bid vector $b_i$ that differs from her true valuation vector $v_i$. We say that a mechanism is *strategyproof* if bidding truthfully is a dominant strategy for every player. That is, for every player $i \in U$ and every two bid vectors $\mathbf{b}, \mathbf{b}'$ with $b_i = v_i$ and $b_j = b'_j$ for all $j \neq i$, we have $u_i(\mathbf{x}, \phi) \geq u_i(\mathbf{x}', \phi')$, where $(\mathbf{x}, \phi)$ and $(\mathbf{x}', \phi')$ are the solutions output by the mechanism for bid vectors $\mathbf{b}$ and $\mathbf{b}'$, respectively. Note that in our model, a player cannot lie about the characteristics or arrival times of her requests.

In *cooperative* mechanism design, it is assumed that players can form coalitions in order to coordinate their bids. A mechanism is *group-strategyproof* if no coordinated bidding of a coalition $S \subseteq U$ can ever strictly increase the utility of some player in $S$ without strictly decreasing the utility of another player in $S$. More formally, for every coalition $S \subseteq U$ and every two bid vectors $\mathbf{b}, \mathbf{b}'$ with $b_i = v_i$ for every $i \in S$ and $b_i = b'_i$ for every $i \notin S$, if there is some $i \in S$ with $u_i(\mathbf{x}', \phi') > u_i(\mathbf{x}, \phi)$ then there is some $j \in S$ with $u_i(\mathbf{x}', \phi') < u_i(\mathbf{x}, \phi)$.

A mechanism is *weakly group-strategyproof* if no coordinated bidding can ever strictly increase the utility of *every* player in a coalition. That is, for every coalition $S \subseteq U$ and every two bid vectors $\mathbf{b}, \mathbf{b}'$ with $b_i = v_i$ for every $i \in S$ and $b_i = b'_i$ for every $i \notin S$, there is some $i \in S$ with $u_i(\mathbf{x}', \phi') \leq u_i(\mathbf{x}, \phi)$. Intuitively, weak group-strategyproofness suffices if we assume that players adopt a slightly more conservative attitude with respect to their willingness of joining a coalition: While in the group-strategyproof setting a player will participate in a coalition even if her utility is not affected, she only participates if she is strictly better off in the weakly group-strategyproof setting.

We now extend general demand cost sharing games to an *online* scenario [2]. Many cost sharing games studied in the literature are derived from combinatorial optimization problems. We take this as a motivation to define *online cost sharing games* very generally with respect to the varying online characteristics inherited from different online optimization problems.

In our model, an online mechanism must immediately fix the payment for a requested service at the point of time when it is revealed, without any knowledge about future requests. In line with the offline model, we assume that an online mechanism will never accept any further requests from a player that has previously been rejected. When the cost sharing game is derived from a combinatorial optimization problem, the mechanism has to maintain a (possibly suboptimal) feasible solution for the current service allocation. We allow the online solution to be modified as in the underlying online optimization problem.

Following Borodin et al. [2], we describe an online list model as a basic example. Here, service requests $(i, l)$ arrive according to an online list. (Note that for certain problems like online scheduling, jobs may have release dates which are then treated as arrival times of the respective requests.) Upon arrival, the player reveals the characteristic of her new request (the input information for the underlying combinatorial optimization problem) and her bid $b_{i,l}$. Each time a new request arrives, the respective player is offered an additional level of service for a price $p$ that may depend on previous inputs and decisions only. If her bid $b_{i,l}$ is larger or equal to this price, the request is accepted and added to the current allocation. Otherwise, the request is rejected and all further appearances of player $i$ are implicitly deleted from the online list (formally, we may define $p = \infty$ for all subsequent requests of player $i$). Algorithm 1 gives a more formal description.

Let $\mathbf{x}^t$ denote the current allocation after processing request $t \in T = \{1, 2, \ldots\}$. Let $\bar{C}(\mathbf{x}^t)$ denote the cost of the actually computed solution for $\mathbf{x}^t$. We call an online cost-sharing mechanism *$\beta$-budget balanced at all times* for some $\beta \geq 1$ if for all requests $t \in T$ :

$$\bar{C}(\mathbf{x}^t) \leq \sum_{i \in U} \sum_{l=1}^{x_i^t} \phi_{i,l} \leq \beta \cdot C(\mathbf{x}^t).$$

The conditions of individual rationality and no positive transfer as well as the different forms of incentive compatibility transfer in a straightforward way.

**Algorithm 1**: Online general demand cost sharing mechanism.

**Input**: online cost sharing game
**Output**: allocation vector $\mathbf{x} = (x_i)_{i \in U}$, payment vector $\phi = (\phi_{i,l})_{i \in U, l \leq L}$

Initialize $\mathbf{x}^0 = \mathbf{0}$
**forall** *requests* $t \in T$ **do**
  Read out input data and bid $b_{i,l}$ of newly arrived request $t =: (i, l)$.
  Determine payment $p$ for new request.
  **if** $b_{i,l} \geq p$ **then** set $\mathbf{x}^t = \mathbf{x}^{t-1} + \mathbf{e}_i$ and $\phi_{i,l} = p$
  **else** set $\mathbf{x}^t = \mathbf{x}^{t-1}$ and $\phi_{i,l} = 0$, and ignore all further appearances of player $i$.
**end**
Output allocation vector $\mathbf{x}$ and payments $\phi$

# 3 Incentive Compatibility

The following characterizations hold for all online mechanisms in our framework. Note that the requirements for group-strategyproofness highly depend on the fact that requests are accepted if the announced bid is equal to the offered price.

## 3.1 Strategyproofness

To achieve strategyproofness, we need to bound the increase in marginal valuations of individual players. As expressed by Fact 1 below, this is essential to prevent players from overbidding for some level to obtain positive utility for higher levels. In previous works on general demand cost sharing [1, 9], players' valuations were assumed to be non-increasing. However, we can slightly relax this condition by introducing a positive factor $\lambda$:

**Definition 1.** *A valuation vector $v_i \in \mathbb{R}^L$ is* $\lambda$-decreasing *if for all $1 < l \leq L$,*

$$v_{i,l} \leq \lambda \cdot v_{i,l-1}.$$

Given $\lambda$-decreasing valuations for all players, an online mechanism is guaranteed to be weakly group-strategyproof if and only if the induced cost shares grow faster than the valuations (the proof is given in Section 3.2):

**Definition 2.** *A cost sharing mechanism has* $\lambda$-increasing prices *if for every bid vector $\mathbf{b}$ and player $i \in U$, the price for any service level $1 < l \leq L$ is at least $\lambda$ times the price for the previous service level, i.e.*

$$\phi_{i,l}(\mathbf{b}) \geq \lambda \cdot \phi_{i,l-1}(\mathbf{b}).$$

We would like to remark that the above conditions can be further generalized by letting $\lambda$ vary for every player (and/or level) or by adding constant terms to the right hand sides. However, the following fact emphasizes that a set of conditions similar to the above are indeed necessary to achieve strategyproofness.

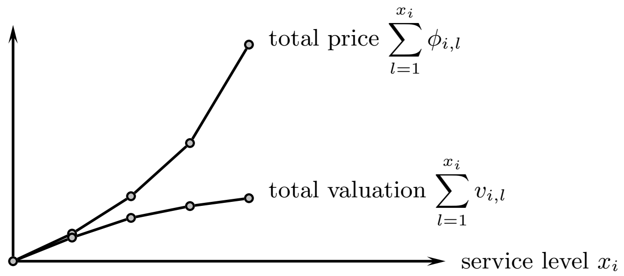


**Fig. 1.** An example for $\lambda$-decreasing valuations and $\lambda$-increasing prices with $\lambda = 1$

**Fact 1** *A general demand online mechanism is not strategyproof if cost shares do not increase by more than valuations per service level.*

*Proof.* We assume for simplicity that there is only one player. Further, assume that for some service level $l$, $\phi_l(\mathbf{b}) < \lambda \cdot \phi_{l-1}(\mathbf{b})$, say the difference is $\lambda$. By individual rationality, we know that $x(\mathbf{b}) \geq l-1$. Since the mechanism is online, $\phi_l(\mathbf{b})$ does not depend on $b_l$ and we can assume that $b_l > \phi_l(\mathbf{b})$.

We define the player's valuations as $v_{l-1} = \phi_{l-1}(\mathbf{b}) - \epsilon$, $v_l = \lambda \cdot v_{l-1}$ and $v_k = \phi_k(\mathbf{b})$ for $k < l-1$. Thus, the valuation increases by a factor $\lambda$ when going from level $l-1$ to level $l$, whereas the prices increase by less than a factor $\lambda$.

With this valuation vector, the player obtains positive utility in the run on $\mathbf{b}$: $u(\mathbf{b}) = 0 + u_{l-1}(\mathbf{b}) + u_l(\mathbf{b}) = -\epsilon + \lambda(\phi_{l-1}(\mathbf{b}) - \epsilon) - (\lambda\phi_{l-1}(\mathbf{b}) - \delta) = \delta - (\lambda+1)\epsilon > 0$ for sufficiently small $\epsilon$. On the other hand, she gets zero utility if she bids truthfully, hence the mechanism is not strategyproof. The same argumentation can be pursued with additive instead of multiplicative increase. $\square$

### 3.2 Weak Group-Strategyproofness

We now prove that in fact, under the above conditions, every online mechanism is weakly group-strategyproof.

**Theorem 1.** *If valuations are $\lambda$-decreasing, a general demand online cost sharing mechanism with $\lambda$-increasing prices is weakly group-strategyproof.*

*Proof.* Fix a coalition $S \subseteq U$ and a bid vector $\mathbf{b}$ with $b_i = v_i$ for all $i \in S$. Assume for contradiction that all members of the coalition can strictly increase their utilities by changing their bids to $\mathbf{b}'$ (while $b_i = b'_i$ for all $i \notin S$). Let $(i,l)$ be the first request for which the mechanism makes different decisions in the runs on $\mathbf{b}$ and $\mathbf{b}'$. We have $\phi_{i,l}(\mathbf{b}) = \phi_{i,l}(\mathbf{b}')$ since all previous decisions of the mechanism were equal in both runs. There are two possible cases:

1. $v_{i,l} < \phi_{i,l} \leq b'_{i,l}$. Because of $\lambda$-decreasing valuations and $\lambda$-increasing prices, we have $\ldots \leq \lambda^{-2} v_{i,l+2} \leq \lambda^{-1} v_{i,l+1} \leq v_{i,l} < \phi_{i,l}(\mathbf{b}') \leq \lambda^{-1}\phi_{i,l+1}(\mathbf{b}') \leq \lambda^{-2}\phi_{i,l+2}(\mathbf{b}') \leq \ldots$, and hence player $i$ has negative utility for service levels $l$ and higher in the run on $\mathbf{b}'$, whereas the utility for each level is non-negative when bidding truthfully.

2. $b'_{i,l} < \phi_{i,l} \leq v_{i,l}$. Then, player $i$ obtains only $l-1$ levels of service in the run on $\mathbf{b}'$, whereas she gets additional utility by accepting level $l$ in the run on $\mathbf{b}$.

Consequently, player $i$ gets less or equal utility in the run on $\mathbf{b}'$, a contradiction to the assumption. □

Remark that for binary demand cost sharing games, both Definitions 1 and 2 are always fulfilled since there is only one level of service. Hence, binary demand online cost sharing mechanisms are always weakly group-strategyproof.

### 3.3 Group-Strategyproofness

In order to ensure the stronger notion of group-strategyproofness, we need to prevent that *dropping out*, like in the second case of the proof of Theorem 1, can help subsequent players. Towards this end, we need the following property of cross-monotonicity, which is equivalent to Moulin's submodular costs condition and generalizes the well-known notion of cross-monotonicity for binary demand cost sharing games [10].

Consider a fixed instance of an online cost sharing game and let $\phi_{i,l}(\mathbf{b})$ denote the price that player $i$ is offered for service level $l$ when $\mathbf{b}$ is the bid vector input to the mechanism. Throughout this section, we assume $\lambda$-decreasing valuations and $\lambda$-increasing prices.

**Definition 3.** *An online mechanism is* cross-monotonic *if for every player $i \in U$ and service level $l$, the offered price does not decrease when a subset of requests are accepted in previous iterations, i.e.*

$$\phi_{i,l}(\mathbf{b}') \geq \phi_{i,l}(\mathbf{b})$$

*for all bid vectors $\mathbf{b}, \mathbf{b}'$ such that $x^{t-1}(\mathbf{b}') \leq x^{t-1}(\mathbf{b})$, where $(i,l)$ is request $t$.*

This condition is a sufficient for an online cost sharing mechanism to be group-strategyproof. The main proof ideas are the following: First, dropping out can never help others since it only increases cost shares of subsequent bidders. Second, the first member of a coalition who overbids for an additional level of service can only decrease her utility by doing this, since prices increase more than valuations in terms of service levels.

**Theorem 2.** *If valuations are $\lambda$-decreasing, a general demand online cost sharing mechanism with $\lambda$-increasing prices is group-strategyproof if it is cross-monotonic.*

*Proof.* Fix a coalition $S \subseteq U$ and a bid vector $\mathbf{b}$ with $b_i = v_i$ for all $i \in S$. Assume that every member of the coalition increases or maintains her utility when the coalition changes their bids to $\mathbf{b}'$ (while $b_i = b'_i$ for all $i \notin S$).

We first prove that $\mathbf{x}^t(\mathbf{b}') \leq \mathbf{x}^t(\mathbf{b})$ for all $t \in T$. Assume for contradiction that there is a request which is accepted in the run on $\mathbf{b}'$ but not in the run on $\mathbf{b}$. Let $(i,l)$ be the earliest such request, say request $t$. That is, $\mathbf{x}^\tau(\mathbf{b}') \leq \mathbf{x}^\tau(\mathbf{b})$

for all $\tau < t$. By cross-monotonicity, we have $\phi_{i,l}(\mathbf{b}') \geq \phi_{i,l}(\mathbf{b})$. Since players outside the coalition submit the same bids in both runs, player $i$ must be a member of the coalition to gain service in the run on $\mathbf{b}'$. But then, $\phi_{i,l}(\mathbf{b}') \geq \phi_{i,l}(\mathbf{b}) > b_{i,l} = v_{i,l}$ and hence by $\lambda$-decreasing valuations and $\lambda$-increasing prices, player $i$ has negative utility for service levels $l$ and higher in the run on $\mathbf{b}'$. Since $\mathbf{x}^\tau(\mathbf{b}') \leq \mathbf{x}^\tau(\mathbf{b})$ for all $\tau < t$, by cross-monotonicity $\phi_{i,k}(\mathbf{b}') \geq \phi_{i,k}(\mathbf{b})$ for all $k < l$ as well, and therefore $u_i(\mathbf{b}') < u_i(\mathbf{b})$, a contradiction to the first assumption.

We can conclude that $\mathbf{x}^t(\mathbf{b}') \leq \mathbf{x}^t(\mathbf{b})$ for all $t \in T$. Hence, $\phi_{i,l}(\mathbf{b}') \geq \phi_{i,l}(\mathbf{b})$ for all $i, l$ by cross-monotonicity. This means that

$$u_i(\mathbf{b}') = \sum_{l=1}^{x_i(\mathbf{b}')} (v_{i,l} - \phi_{i,l}(\mathbf{b}')) \leq \sum_{l=1}^{x_i(\mathbf{b})} (v_{i,l} - \phi_{i,l}(\mathbf{b})) = u_i(\mathbf{b})$$

for all $i$ and $l$, hence we obtain group-strategyproofness. □

We prove next that Theorem 2 actually holds in an if and only if fashion, even in the binary demand case.

**Theorem 3.** *An binary demand online mechanism is not group-strategyproof if it is not cross-monotonic.*

*Proof.* Consider an online mechanism that is not cross-monotonic; let $L = 1$. That is, there are bid vectors $\mathbf{b}, \mathbf{b}'$ with $\mathbf{x}^{t-1}(\mathbf{b}') \leq \mathbf{x}^{t-1}(\mathbf{b})$ and $\phi_i(\mathbf{b}') < \phi_i(\mathbf{b})$ for some player $i$. For simplicity, assume that $i$ is the last player in the online instance. Since the mechanism is online, $\phi_i(\mathbf{b}')$ does not depend on $b'_i$, so we can assume that $b'_i = \phi_i(\mathbf{b})$. We will define valuations such that there is a coalition $S$ which has an incentive to misreport their valuations.

Define $S := \{j \in U \mid b_j \neq b'_j\} \cup \{i\}$. Assume that all $j \in U \setminus S$ bid $b_j = b'_j$. Now, define $v_j := \phi_j(\mathbf{b})$ for all $j \in S$. Observe that if all players in $S$ bid truthfully, the outcome of the mechanism is the same as for bid vector $\mathbf{b}$. Now, if the coalition changes their bids to $\mathbf{b}'$, some players $j \in S \setminus \{i\}$ lose service but all retain their previous utility of zero. Meanwhile, player $i$ increases her utility from zero to $\phi_i(\mathbf{b}) - \phi_i(\mathbf{b}') > 0$. Hence, the mechanism is not group-strategyproof. □

## 4 Incremental Online Mechanisms

In this section, we describe how to turn competitive online algorithms into *incremental* online cost sharing mechanisms. Given a combinatorial optimization problem $\mathcal{P}$ and a $\rho$-competitive online algorithm ALG for this problem, we define an online mechanism for the corresponding cost sharing game which is $\rho$-budget balanced at all times. This mechanism is weakly group-strategyproof if the algorithm's marginal costs are increasing, which is gratuitous in the binary demand case.

Let ALG be a $\rho$-competitive algorithm for an online combinatorial optimization problem $\mathcal{P}$. Consider an instance $\mathcal{I}$ of $\mathcal{P}$. The incremental online mechanism induced by ALG works as follows: Requests arrive according to $\mathcal{I}$. Each time a new request arrives, we simulate ALG on the online instance induced by the requests that have previously been accepted plus the new item. The price $p$ for the additional service level is set to be the incremental cost caused by the update in the competitive algorithm. We call an online algorithm ALG *cross-monotonic* if the induced incremental online mechanism is cross-monotonic.

It is straightforward to see that the budget balance factor of the incremental online mechanism is inherited from the competitive ratio of the input algorithm:

**Lemma 1.** *The incremental online mechanism is $\rho$-budget balanced at all times.*

*Proof.* In every iteration $t$ of the mechanism, we have $\sum_{i \in U} \sum_{l=1}^{x_i^t} \phi_{i,l} = \bar{C}(\mathbf{x}^t)$, since every accepted player pays exactly the incremental cost for adding her to the current set of served players. Since ALG is a $\rho$-competitive algorithm, we obtain

$$\bar{C}(\mathbf{x}^t) = \sum_{i \in U} \sum_{l=1}^{x_i^t} \phi_{i,l} \leq \rho \cdot C(\mathbf{x}^t),$$

which proves $\rho$-budget balance at all times. $\square$

### 4.1 Binary Demand Examples

We now apply our framework to competitive online algorithms for several combinatorial optimization problems. In this section, we give examples for *binary demand* cost sharing games, i.e. the maximum service level is $L = 1$ and every player has only one request.

**Online Scheduling.** Consider the parallel machine scheduling problem with the objective of minimizing the makespan. Any list scheduling algorithm has an approximation factor of at most 2 for this problem. Hence, the online algorithm that adds each arriving job to the machine with the currently least load is 2-competitive. Unfortunately, it is not cross-monotonic as deleting jobs can cause higher or lower completion times for subsequent jobs. Nonetheless, our framework leads to a 2-budget balanced, weakly group-strategyproof online mechanism. Note that in this scenario, jobs do not have release dates and so the online order is not coupled with scheduling time.

**Corollary 1.** *There is a 2-budget balanced weakly group-strategyproof incremental online mechanism for the minimum makespan scheduling problem on parallel machines $P||C_{\max}$.*

**Online Steiner Tree and Forest.** Given an undirected graph $G$ with edge costs, connection requests arrive online. In the Steiner forest problem, each request consists of a pair of vertices $s_i, t_i$; in the Steiner tree problem, all requests have one vertex in common, i.e. $s_i = s_j$ for all $i, j \in U$. The goal is to select a minimum cost set of edges such that each vertex pair is connected by a path. Let $n$ denote the number of players or commodities.

The online greedy Steiner tree algorithm picks the shortest path to the current tree each time a new commodity arrives. It has a competitive ratio of $\log n$, while the competitive ratio of any online algorithm is shown to be at least $1/2 \log n$ [7]. Hence, our framework gives a weakly group-strategyproof $\Theta(\log n)$-budget balanced online cost sharing mechanism for the Steiner tree problem, which is asymptotically best possible. The greedy algorithm for the online Steiner forest problem achieves an approximation ratio of $O(\log^2 n)$.

**Corollary 2.** *There is an $O(\log^2 n)$-budget balanced weakly group-strategyproof incremental online mechanism for the Steiner forest game. This mechanism is $(\log n)$-budget balanced for the Steiner tree game.*

Unfortunately, the greedy algorithm is not cross-monotonic, as dropping out of players can provoke others to switch their paths, which in turn can have arbitrary effects on the costs incurred by subsequent players. This issue can be overcome if paths are unambiguous; e.g. if $G = (V, E)$ is a forest, the above mechanisms are group-strategyproof. Pushing this observation even further, we obtain an $O(\log |V|)$-budget balanced group-strategyproof mechanism for the Steiner forest game if the underlying graph is known in advance: We use the *oblivious* online Steiner forest algorithm proposed by Gupta et al. [5] which connects every arriving vertex pair by a predefined path which depends on the underlying graph only. We remark that the online notion is here somewhat perturbed as $G$ needs to be known beforehand.

**Corollary 3.** *There is an $O(\log^2 |V|)$-budget balanced group-strategyproof incremental online mechanism for the Steiner forest game, where $V$ is the vertex set of the underlying graph.*

### 4.2 General Demand Examples

In this section, we exploit the whole range of our framework by deriving incremental mechanisms for *general demand* cost sharing games. In the first example, players are assumed to arrive only once with the complete list of their requests, while in the second example, the arrival sequence is mixed, i.e. players can takes turns announcing additional requests.

**Online Preemptive Scheduling.** A common problem in preemptive scheduling is the parallel machine setting in which each job has a release date. The cost of a solution is given by the sum of all completion times. The single machine case is solved optimally by the *shortest remaining processing time* (SRPT) algorithm [11]. SRPT is a 2-approximation algorithm for the parallel machine case [4]. In the corresponding scheduling game, we treat the release date of a job as its arrival time in the cost sharing framework. The cost $\bar{C}$ of an allocation $\mathbf{x}$ is the total cost of the SRPT schedule for $\mathbf{x}$. Whenever a job arrives, the SRPT solution is updated by adding the new job, and the resulting increase in total cost is set to be the cost share for this request.

In our model, each player may request multiple executions of their job. E.g. consider a student who asks a copy shop to print and bind several copies of his master's thesis, or a joinery is asked to produce a few of the same individual piece of furniture. In these scenarios, it is very natural to assume that the marginal valuation for each additional copy is decreasing, i.e. $v_{i,l} \geq v_{i,l+1}$ for all $i, l$. We assume that each player arrives only once with all her requests. In scheduling terms, each player owns several jobs which all have the same release date and processing time. Before subsequent players arrive, the SRPT algorithm schedules all of player $i$'s jobs subsequently, hence each of them delays the same number of jobs, and later copies have larger completion times. Therefore, the general demand incremental online mechanism induced by SRPT has increasing marginal prices, and we obtain:

**Corollary 4.** *There is a 2-budget balanced weakly group-strategyproof general demand incremental online mechanism for the preemptive scheduling problem with release dates* $P|r_i, pmtn|\sum C_i$*. This mechanism is budget balanced in the single machine case.*

**Online Multicommodity Routing.** In an online multicommodity routing problem, we are given a directed graph with monotonically increasing cost functions on each arc. Commodities arrive online and request routing of $l$ units of capacity from some vertex to another. We assume that the routing is splittable in integer units. The greedy algorithm which routes each unit of flow separately in an optimal way is $(3 + 2\sqrt{2})$-competitive for this problem [6]. It is clear that marginal costs are increasing, since the cost functions on each arc grow with increasing traffic. This is true even when players arrive in a mixed order and request to route additional units between their source-destination pair. However, this is a congestion-type game (the more players in the game, the higher the costs per request), and so we cannot expect group-strategyproofness.

**Corollary 5.** *There is a* $(3 + 2\sqrt{2})$*-budget balanced weakly group-strategyproof incremental online mechanism for the online multicommodity routing problem in which each player arrives multiple times.*

## 5 Conclusion

We characterized strategyproofness, weak group-strategyproofness and group-strategyproofness of mechanisms in a new framework for online general demand cost sharing games. Quite surprisingly, weak group-strategyproofness comes for free for all binary demand problems; in the general demand case cost shares for subsequent service levels must increase faster than valuations to achieve this property. In both cases, online mechanisms are group-strategyproof if and only if *dropping out* cannot help subsequent players. Consequently, we cannot expect incremental cost sharing mechanisms for problems with congestion effects like e.g. scheduling games to be group-strategyproof, whereas this seems easier for network design problems. It would be interesting to see more applications to our framework, possibly also when the online mechanisms are not directly derived from competitive algorithms as in our incremental mechanisms.

We consider this work as a very natural and general starting point to exploit the possibilities and limits of cooperative cost sharing in different online contexts. Our model restricts feasible allocations to a continuous sequence of accepts for each player, starting with their first request. This feature of the model enhances truthfulness as it prevents players from underbidding to reject some service request and then to obtain it later for a cheaper price. One interesting line of research would therefore be to allow for more general mechanisms which might accept further requests of players even after a request has been rejected.

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