# Laughlin's wave function, plasma analogies and the fractional quantum Hall effect on infinite cylinders

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Von der Fakultät II - Mathematik und Naturwissenschaften
der Technischen Universität Berlin
zur Erlangung des akademischen Grades
Doktor der Naturwissenschaften
- Dr. rer. nat. -

genehmigte Dissertation

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Tag der wissenschaftlichen Aussprache: 22. Juni 2007

Berlin 2007

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# Zusammenfassung

Gegenstand dieser Dissertation ist die Untersuchung von Laughlins Wellenfunktion auf einem Zylinder. Die Ergebnisse sind sowohl für die Theorie des fraktionalen Quanten-Hall-Effekts als auch für die klassische statische Mechanik geladener Teilchen in einem neutralisierenden Hintergrund von Interesse.

Wir zeigen, dass Laughlins Wellenfunktion als "Quanten-Polymer-System" dargestellt werden kann. Die $L^2$-Norm ist die Zustandssumme eines Polymersystems mit translationsinvarianter Aktivität. Die Aktivität kann zu einer stabilen Aktivität reskaliert werden. Das Polymersystem steht wiederum in engem Zusammenhang mit einem Erneuerungsprozess. Unter der Voraussetzung einer endlichen mittleren Wartezeit in diesem Prozess zeigen wir, dass Laughlins Zustand einen wohldefinierten thermodynamischen Limes besitzt. Dabei lassen wir die Länge des Zylinders bei konstanter Dichte und konstantem Radius gegen unendlich gehen. Der Grenzzustand ist invariant und mischend bzgl. bestimmter Translationen längs der Zylinderachse. Bei Füllfaktor $1/p$ ist die Periode gerade das $p$-fache der Periode des gefüllten Landaubandes. Für hinreichend dünne Zylinder zeigen wir, dass die Wartezeit des Erneuerungsprozesses tatsächlich endlich ist und die soeben erwähnte Periode auch die kleinste Periode ist.

Weiter definieren wir modifizierte Zylinder- und Torusfunktionen, die ebenfalls in Verbindung zu Polymersystemen stehen. Die Definition verwendet Funktionen mit kompaktem Träger. Bei hinreichend kleinem Träger entspricht die Zylinderfunktion einem Monomer-Dimer-System, während die Torusfunktion einem Monomer-Dimer-System auf einem Ring, möglicherweise noch mit einem zusätzlichen langen Polymer, entspricht. Dieses Monomer-Dimer-System ist explizit lösbar; Torus- und Zylinder- Monomer-Dimer- Funktionen sind im Limes langer Zylinder äquivalent.

Wir gehen auf die Bedeutung unserer Ergebnisse im Sinne von Laughlin's Plasmaanalogie ein. Wir zeigen, dass im Limes dünner Zylinder unsere Ergebnisse bzgl. der Normierungskonstanten bzw. freier Energien und bzgl. der Einteilchendichten konsistent mit bekannten Ergebnissen über eindimensionale Plasmasysteme sind. Ferner zeigen wir, dass unsere Schranken über Normierungskonstanten gut zu Asymptotiken der freien Energien breiter halbperiodischer Streifen passen.

Laughlins Funktion bei Füllfaktor 1/3 ist der exakte Grundzustand eines geeigneten Hamiltonoperators. Wir weisen nach, dass Grundzustände dieses Hamiltonoperators mit unendlich vielen Teilchen bei Füllfaktor 1/3 notwendigerweise eine Symmetriebrechung aufweisen, wenn es eine Spektrallücke über dem Grundzustand gibt.

Schließlich betrachten wir ein einfaches Modell für Ladungstransport auf Zylindern und untersuchen den Zusammenhang zwischen der Periodizität der Einteilchendichte und fraktionalem Ladungstransport.

# Abstract

We investigate Laughlin's wave function on a cylinder. The results are of interest for the fractional quantum Hall effect and for the classical statistical mechanics of charged particles moving in a neutralizing background.

We show that Laughlin's cylinder function is related to a "quantum polymer" system. The $L^2$-norm squared is a polymer partition function with translationally invariant activity. The activity can be rescaled to a stable activity. The polymer system is related to a renewal process. We show that if the renewal process has finite mean, Laughlin's state on the cylinder has a well-defined thermodynamic limit as the cylinder gets infinitely long, the radius being kept fixed. The limiting state is invariant and mixing with respect to shifts along the axis. At filling factor $1/p$, the period is $p$ times the period of the filled Landau band. On sufficiently thin cylinders, we show that the associated renewal process has indeed finite mean and the period mentioned above is the smallest period.

We define a class of modified torus and cylinder Laughlin-type wave functions. These are still associated with polymer systems. The definition uses functions of compact support. If the support is small, the cylinder function is associated with a monomer-dimer system, and the torus function with a monomer-dimer system on a ring with possibly one additional long polymer. The monomer-dimer case is solvable, and monomer-dimer cylinder and torus functions are equivalent in the limit of long cylinders.

We interpret our result in view of Laughlin's plasma analogy. We show that in the limit of thin cylinders, our results on normalization / free energies and the one-particle density are consistent with existing results on one-dimensional systems. We show that our bounds on normalization constants are consistent with large strip asymptotics of the free energy of a jellium system on a semiperiodic strip.

Laughlin's function at filling factor 1/3 is the ground state of a suitable Hamiltonian. We show that, on the cylinder, gapped infinite volume ground states of this Hamiltonian necessarily display a kind of translational symmetry breaking.

Finally, we look at a simple model of bulk charge transport on the cylinder and examine the relationship between our symmetry breaking result and fractional charge transport.

# Contents

# Introduction

The classical Hall effect, named after its discoverer E. Hall, is a well-known electrodynamic phenomenon: when a current $I_H$ flows through a thin sheet of conducting material in the form of a bar, subject to a magnetic field perpendicular to the sheet, a potential difference $V_H$ between opposite sides of the bar appears. According to classical electrodynamics, the Hall conductance $\sigma_H = I_H/V_H$ is a linear function of the ratio $n/B$ of the areal charge carrier density and magnetic field strength.

Hundred years after the discovery of the Hall effect, K. von Klitzing and his collaborators [KDP80] observed experimentally that at low temperatures, the Hall conductance is not a linear function of $n/B$; instead, it has plateaus where the conductance equals an integer multiple of a fundamental unit. This is the *integer quantum Hall effect* (IQHE). Later, plateaus at fractional multiples were found [TSG82], giving rise to the *fractional quantum Hall effect* (FQHE).

Since their discovery in the early 1980s, the integer and fractional quantum Hall effects have spurred a wealth of research among experimental, theoretical and mathematical physicists. A large body of knowledge is now general consensus and can be found in one of the many books and reviews on the quantum Hall effect, see e.g. [PG87, Yos98, DDPR04]. A number of mathematical results are available (a good review can be found in [BvESB94]). Some questions are still controversial while others have well-accepted, but yet unproved answers. Meanwhile, the theory of the quantum Hall effect has reverberated on other fields, including quantum computation [Day05], models for Mott insulators [LL04], spin systems [KL87] and rotating Bose condensates [WGS98].

The integer quantum Hall effect is in general explained in terms of independent electrons (see however [SG04]). In contrast, electron interactions are of utmost importance for the understanding of the fractional effect. Consequently, the FQHE led to an intense research on the nature of the ground state of the electron gas in FQHE samples. Early numerical computations confronted theoreticians with an interesting phenomenon: the degeneracy of the ground state depends on the geometry of the sample. On rectangles with quasiperiodic boundary conditions - i.e., tori - a degeneracy, associated to a translational symmetry breaking, was observed, whereas explicitly isotropic geometries like the disk and sphere had non-degenerate ground states (see [Hal85] and the references therein). The degeneracy of FQHE ground states on tori was initially dismissed as physically irrelevant [Hal85]. Later, the very fact that degeneracy depends on geometry has been considered as an intrinsic feature of FQHE systems, in close connection with the notion of topological order now popular among physicists [WN90]. From a more mathematical point of view however, it is not very suprising that boundary conditions affect the finite volume degeneracy, and the interesting question is whether infinite volume ground states are degenerate or not.

Today, it is widely accepted that electrons at FQHE fractions form an "incompressible quantum fluid" whose bulk properties are well described by Laughlin's wave function [Lau83].

The wave function has been initially proposed to describe electrons on a disk, but since then has been adapted to the cylinder [Tho84], torus [HR85b] and sphere [Hal83]. Laughlin invoked a plasma analogy to conclude that his disk function describes a homogeneous electron gas. While the existence of the thermodynamic limit of the free energy for Coulomb systems has been proved a while ago [LL72], analytical results pertaining to correlation functions are more sparse, see the review by Brydges and Martin [BM99]. The homogeneity of the plasma related to Laughlin's function follows from numerical results; proofs are to our knowledge still missing.

The present work is devoted to the study of Laughlin's wave function on a cylinder. This choice is motivated by the use of the cylinder geometry in another famous contribution by Laughlin [Lau81], which has now come to be known as *Laughlin's argument*. Let us briefly sketch the main mathematical problem underlying this thesis. Consider the function $\Psi_N$ of $N$ complex variables $z_j = x_j + iy_j$ defined through

$$\Psi_N(z_1, .., z_N) := \prod_{1 \le j < k \le N} (e^{z_j/R} - e^{z_k/R})^p e^{-\frac{1}{2}\sum_{j=1}^N x_j^2},$$

where $R$ is a positive number and $p$ an odd integer. The function $\Psi_N$ has the period $2\pi R$ in each $y_j$. Thus we may consider that it describes a gas of $N$ electrons moving on a cylinder of radius $R$. The position of each particle is specified by a complex number $z = x + iy$; $x$ is the coordinate along the cylinder axis, $y$ is an angular variable. $\Psi_N$ is Laughlin's cylinder wave function. The density of the gas at position $z$ is

$$\rho_N(z) = \frac{N}{||\Psi_N||_{L^2}^2} \int_{(\mathbb{R}\times[0,2\pi R])^{N-1}} |\Psi_N(z, x_2 + iy_2, .., x_N + iy_N)|^2 dx_2\, dy_2 .. dx_N\, dy_N.$$

One can show that the density is a function of the coordinate $x$ alone and decays exponentially outside $[0, pN/R]$. Thus Laughlin's cylinder function describes a gas of electrons living on a finite cylinder. The quantity $\nu = 1/p$ is related to the average density and is called the filling factor. At filling factor $1/p = 1$ (referred to as the filled Landau level), the density can be computed explicitly and is a sum of equally weighted Gaussians centered in integer multiples of $1/R$. The following conjecture pertaining to $p > 1$ is at the heart of the present work:

> In the limit $N \to \infty$, well in the middle of the finite cylinder, the electronic density approaches a function of the coordinate along the cylinder axis that is periodic with minimal period $p/R$.

We will be interested not only in the one-particle density but also in other correlation functions. We will look at the problem both in the FQHE context and from the point of view of classical Coulomb systems.

The question of non-trivial periodicity on the cylinder is motivated by the combination of the contributions of Laughlin to the integer [Lau81] and the fractional [Lau83] quantum Hall effect. In [Lau81], Laughlin uses a the cylinder geometry and gauge invariance arguments to explain the IQHE. It has been suggested [TW84] that a ground state degeneracy is required on cylinders in order to reconcile Laughlin's argument with fractionally quantized Hall conductances; the simplest picture is a $p$-fold degeneracy associated to a $p$-fold translational symmetry breaking, as conjectured for Laughlin's cylinder state. This rather heuristic connection between ground state degeneracy is rigorously established for systems on tori. In the Chern number approach, fractional quantization of the Hall conductance implies ground

state degeneracy [KS90].

Our results are backed by a number of existing works. Rezayi and Haldane [RH94] investigated the dependence of the cylinder density $\rho_N$ on the cylinder radius. They showed that if the numbers of particles is finite and fixed, there is a well pronounced periodicity as the radius $R$ gets small. Similar results have been obtained numerically for the Laughlin-type wave functions on tori [SFL$^+$05, LL04]. Let us mention here that the thin torus picture has also been used for investigations on more complicated filling fractions, see e.g. [BK06].

Laughlin connected his wave function to the theory of classical Coulomb systems by observing that the modulus squared of the disk function is the Boltzmann weight of a classical one-component plasma. This plasma analogy can be transferred to the cylinder geometry. The symmetry breaking phenomenon is well-known at coupling constant $\Gamma = 2$, corresponding to filled Landau level functions (i.e., Laughlin functions with filling factor $1/p = 1$) [CFS83]. Translational symmetry breaking has been proved to occur for one-dimensional systems [Kun74, BL75]. Based on the $\Gamma = 2$ and one-dimensional results, [ŠWK04] have investigated numerically jellium tubes at coupling constants $\Gamma = 2p$. Unlike in [RH94], they vary the number of particles as well as the radius, and find that a periodicity appears in the limit of long jellium strips; again, the amplitude of oscillations is large when the strip is thin. The periodicity of the charge density in jellium tubes is conjectured also in [AGL01].

In view of the plasma analogy, results on the $L^2$ norm of Laughlin's wave function give information on free energies. The asymptotic behavior of the jellium strip free energy as the strip gets large has been examined by Forrester [For91], who also studied the plasma analogy on tori [For06], motivated by the question of universal finite-size corrections of Coulomb free energies (see also [TT04]).

Laughlin's wave function is essentially the power of a Vandermonde determinant. It can be expanded in Slater determinants of lowest Landau level basis functions; the expansion coefficients are closely related to the expansion coefficients of powers of Vandermonde determinants, written as a sum over monomials. Di Francesco et al. [FGIL94] and Dunne [Dun93] investigate Laughlin's disk wave function from this point of view. The relation with the FQHE has also spurred combinatorial work on the Vandermonde determinant [STW94, KTW01].

The main point behind the proof of our results is a product rule for expansion coefficients of Vandermonde determinants established in [FGIL94]. We use this product rule to represent Laughlin's wave function as a *quantum polymer system*: we choose this denomination because of the analogy to the *quantum dimer* model, proposed by Rokhsar and Kivelson [RK88] as an idealization of resonating valence bond states.

The thesis is organized as follows:

Chapter 1 gives a synthetic presentation of the Laughlin-type wave functions and some background material on particles in magnetic fields and Laughlin's plasma analogy.

In Chapter 2, we prove the main results of our thesis: For sufficiently small cylinder radius, the thermodynamic limit of Laughlin's cylinder state exists and defines a periodic, mixing state with minimal period $p/R$. The key ingredient is the representation of Laughlin's wave function as a quantum polymer. The $L^2$-norm is a one-dimensional polymer partition function. It satisfies a recurrence relation known in stochastics as a renewal equation. If the associated renewal process has finite mean interarrival time, the (non-quantum) polymer system has a well-defined thermodynamic limit, closely related to the thermodynamic limit

of Laughlin's state.

We define a class of solvable models by replacing Gaussians with functions of compact support. The cylinder and torus functions can still be represented as polymer systems. When the support of the compact functions is small, the system is a pure monomer system. As the support gets larger, the system becomes a monomer-dimer system and then - without intermediary steps - a system where polymers of arbitrary length have non-vanishing activity. The monomer and monomer-dimer cases define solvable models. When the cylinder system is a monomer-dimer system, the modified torus function is a monomer-dimer system on a ring, with possibly one long additional polymer covering the whole ring, and we can prove the equivalence of cylinder and torus wave functions in this setting.

We translate our results into the jellium picture. This allows an interpretation of results on the asymptotics of normalization constants in terms of free energies, and also sheds some light on the renewal equation.

In the last section, we reexamine the results in the light of the characterization of Laughlin's wave function at filling factor 1/3 as an exact ground state of truncated interactions [Hal83, PT85, TK85]. We adapt an argument by Koma [Kom04] that shows that the symmetry breaking is to be expected if the truncated interaction has a gapped ground state. This result should be understood as a consistency check of different assumptions on FQHE systems: the FQHE ground state at simple filling fractions is expected to be gapped and incompressible. If the truncated interaction with Laughlin's wave function as ground state reproduces this feature, symmetry breaking necessarily follows.

In Chapter 3, we examine the charge transport on cylinders (and tori). We recall Laughlin's argument and the Chern number approach. We present the Chern number approach in a way that allows a simple comparison with the cylinder setting and prove that the adiabatic curvatures in absence of background interactions can be expressed in terms of the one-particle density. We examine a simple model of charge transport on cylinders, closely following the approach to the IQHE in [Ric00], Chapter 1.

**Acknowledgements** At this point I would like to express my gratitude to Prof. R. Seiler and Prof. E. H. Lieb for their continous support and their invaluable help. This thesis would not have been possible without their contribution. I thank Prof. H. Schulz-Baldes for useful discussions on quantum Hall systems and for the numerous references he pointed out to me, and R. Siegmund-Schultze for his help on renewal processes. I would also like to thank Prof. Y. Avron for bringing [Osh00] to my attention and Prof. D. Brydges for pointing out [TT04]. Furthermore, I am deeply indebted to colleagues, friends and family for their support, patience and encouragement.

# Chapter 1

# Laughlin's wave function

The object of this thesis is the investigation of Laughlin's wave function on a cylinder. The Laughlin wave function is essentially a power of a Vandermonde polynomial; it is a very simple function, and in principle we could write down the function and the integrals defining the reduced correlation matrices, and formulate in a few lines the problem we wish to solve as sketched in the Introduction. We take a different approach and give some background information, in more detail than strictly necessary for the proofs of the main results of this thesis. The connection to the fractional quantum Hall effect is explained in Chapter 3.

Laughlin's wave function was introduced as an approximate ground state for electrons in a magnetic field. We start by recalling some basic facts on particles in a magnetic field (Section 1.1). Then we give the definition of Laughlin-type wave function in different geometries (Section 1.2). The Laughlin wave functions are indexed by odd integers $p$, related to the average density of the electron gas they describe. The function pertaining to $p = 1$ corresponds to the filled Landau level; on a cylinder, it describes an electron gas with a periodicity along the cylinder axis (Section 1.3). The Laughlin functions can be characterized as exact ground states of truncated interactions, which we present in detail in Section 1.4. The existence of gaps, of truncated interactions as well as "real" Hamiltonians, is related to the notion of incompressibility. Since the connection between these notions is often left implicit, we explain this precisely in Section 1.5. Finally, we present the plasma analogy relating the modulus squared of the wave functions considered to Boltzmann weights of classical one-component plasma (Section 1.6).

Most of the material presented in this chapter summarizes well-known facts. The presentation given here differs from the usual presentation by the simultaneous treatment of different geometries and the addition of some details and proofs generally omitted in the physics literature.

## 1.1 Particles in a magnetic field

The Laughlin wave functions are considered as good approximations to the ground state of a two-dimensional, spin-polarized interacting electron gas in a magnetic field. Before we introduce the wave functions, let us summarize some facts about the quantum mechanics of charged particles in a magnetic field.

In the following, the same symbol is used for functions and their multiplication operators; the notation will not distinguish between a formal differential operator and the closed or self-adjoint operators arising with suitable domain of definition. We do not specify the precise nature of commutators.

Consider an electron of $m$ and charge $-e < 0$ moving in the $x, y$-plane. A magnetic field $B = B_z \mathbf{e}_z$ is applied perpendicular to the plane. Let $A = (A_x, A_y) : \mathbb{R}^2 \to \mathbb{R}^2$ be a differentiable function such that $\partial_x A_y - \partial_y A_x = B_z$. $A$ is the vector potential generated by the magnetic field. The motion of the electron is described by the *Landau Hamiltonian*

$$H = \frac{1}{2m}(p + eA)^2 = \frac{1}{2m}\big((-i\hbar\partial_x + eA_x)^2 + (-i\hbar\partial_y + eA_y)^2\big). \tag{1.1}$$

With a suitable domain of definition, $H$ defines a self-adjoint operator with discrete spectrum

$$\sigma(H) = \{(n + \frac{1}{2})\hbar\omega \mid n \in \mathbb{N}_0\}, \quad \omega := \frac{e|B_z|}{m}. \tag{1.2}$$

Note that $H$ does not include any spin contribution: the electrons are assumed to have all their spins aligned (spin-polarized gas). The quantity $\omega$ is called *cyclotron frequency*. Other useful quantities are the *flux quantum* $\phi_0$ and the *magnetic length* defined through

$$\phi_0 := \frac{h}{e} = \frac{2\pi\hbar}{e}, \qquad l := \sqrt{\frac{\hbar}{e|B_z|}}.$$

The magnetic flux through a disk of radius $l$ equals the flux quantum: $|B_z|\, 2\pi l^2 = \phi_0$. The Hamiltonian commutes with the *center operators*

$$c_x := x + \frac{1}{m\omega}(p_y + eA_y), \quad c_y := y - \frac{1}{m\omega}(p_x + eA_x)$$

and the *magnetic translations*

$$t(a) = e^{i(a_1 c_y - a_2 c_x)/l^2}, \quad a = (a_1, a_2) \in \mathbb{R}^2.$$

The magnetic translations act like usual translations except for a phase factor: $(t(a)\psi)(z) = e^{i\gamma(z;a)}\psi(z - a)$. Here and in the following, we identify $\mathbb{C}$ and $\mathbb{R}^2$ in the usual way: $z = x + iy \equiv (x, y)$. The magnetic translations satisfy

$$t(a + b) = e^{-i(a\wedge b)/2l^2} t(a)t(b), \quad a \wedge b = a_1 b_2 - a_2 b_1$$

whence the commutation relation $t(a)t(b) = e^{i(a\wedge b)/l^2} t(b)t(a)$. As a consequence, two magnetic translations $t(a)$ and $t(b)$ commute if and only if the area of the parallelogram spanned by $a$ and $b$ is an integer multiple of $2\pi l^2$. Let $b := \frac{1}{l\sqrt{2}}(c_x - ic_y)$. Then $[b, b^*] = 1$. The *lowest Landau level* $\mathcal{N}(H - \hbar\omega/2)$ has a complete orthonormal set $\Omega_m$, $m \in \mathbb{N}_0$ such that

$$(c_x^2 + c_y^2)\Omega_m = (2b^*b + 1)l^2\Omega_m = (2m + 1)l^2\Omega_m. \tag{1.3}$$

The center operators are closely related to the coordinates $x, y$: for example, for all $\phi$ in the lowest Landau level, the following identities hold:

$$(\phi, x\phi) = (\phi, c_x\phi), \qquad (\phi, (x^2 + y^2)\phi) = (\phi, (c_x^2 + c_y^2 + l^2)\phi),$$
$$(\phi, y\phi) = (\phi, c_y\phi) \qquad (\phi, e^{i(q_1 x + q_2 y)}\phi) = (\phi, e^{i(q_1 c_x + q_2 c_y)} e^{-q^2 l^2/4}\phi).$$

There are different possible choices for the vector potential $A$, giving rise to different, but unitarily equivalent Hamiltonians. Two gauges are particularly useful: the *symmetric gauge* and the *Landau gauge*. We give some details on these gauges. In the following, we suppose that the magnetic field points downwards: $B_z < 0$. This leads to the expression of lowest Landau level wave functions in terms of holomorphic functions. The opposite choice $B_z > 0$ would lead to antiholomorphic functions.

**Symmetric gauge.** In the symmetric gauge $A = (-B_z y/2, B_z x/2)$, the lowest Landau level is

$$\mathcal{N}(H - \hbar\omega/2) = \{\psi \in L^2(\mathbb{R}^2) \mid \psi(z) = f(z)e^{-|z|^2/4l^2} \text{for some holomorphic } f\}$$

and the set of functions $(\Omega_m)_{m\in\mathbb{N}_0}$ defined through

$$\Omega_m(z) := \frac{1}{\sqrt{2\pi l^2 m!(2l^2)^m}} z^m e^{-|z|^2/4l^2}. \tag{1.4}$$

is a complete orthonormal set of the lowest Landau level fulfilling (1.3). The magnetic translations are

$$\big(t(a)\psi\big)(z) = e^{\frac{i}{2l^2}a\wedge z}\psi(z-a).$$

**Landau gauge.** In the Landau gauge $A = (0, B_z x)$,

$$\mathcal{N}(H - \hbar\omega/2) = \{\psi \in L^2(\mathbb{R}^2) \mid \psi(z) = f(z)e^{-x^2/2l^2} \text{for some holomorphic } f\}.$$

The magnetic translations become

$$\big(t(a)\psi\big)(z) = e^{i(a_1 y - \frac{1}{2}a_1 a_2)/l^2}\psi(z-a).$$

For $\phi \in \mathcal{S}(\mathbb{R})$, let

$$(V\phi)(x,y) := \frac{1}{\sqrt{2\pi\sqrt{\pi}}} \int_{-\infty}^{\infty} \phi(k) e^{iky/l} e^{-(x-kl)^2/2l^2} dk.$$

Note that $(V\phi)(x,\cdot)$ is related to the Fourier transform of $k \mapsto \phi(k)e^{-(x-kl)^2/2l^2}$. $V$ extends to a unitary map from $L^2(\mathbb{R})$ onto the lowest Landau level. Moreover,

$$VkV^{-1} = c_x/l, \quad Vi\frac{d}{dk}V^{-1} = c_y/l$$

and we can set $\Omega_m := VF_m$ where $F_m$ are the Hermite functions. The set of functions $(\Omega_m)$ then forms a complete orthonormal system of eigenfunctions of $c_x^2 + c_y^2$.

The previous considerations apply to infinite samples. In practice, we wish to model finite samples. The most popular geometries are the disk, cylinder or semiperiodic strip, and torus, which we now describe.

**Disk.** In principle, the motion of an electron in a disk $B_R = \{z \in \mathbb{C} \equiv \mathbb{R}^2 \mid |z| \le R\}$ should be modeled by a Hamiltonian in $L^2(B_R)$ with suitable boundary conditions. However, it is more common to look at the Hamiltonian $H$ in $L^2(\mathbb{R}^2)$ restricted to a spectral subspace of $c_x^2 + c_y^2$. The corresponding lowest Landau level is simply $\text{span}\{\Omega_m \mid 0 \le m \le M\}$ where $M \in \mathbb{N}$ is the largest integer such that $(2M+1)l^2 \le R^2$, and $\Omega_m$ are as in (1.3).

**Infinite cylinder.** Let $R > 0$. The expression (1.1) with periodic boundary conditions $t(2\pi R\mathbf{e}_y)\psi = \psi$ defines a self-adjoint operator in $L^2(\mathbb{R} \times [0, 2\pi R])$ with spectrum as in (1.2). The semiperiodic strip can be seen as a cylinder of radius $R$, the $y$-coordinate plays the role of an angular variable. In the Landau gauge, the lowest Landau level has the complete orthonormal set $\psi_k, k \in \mathbb{Z}$ with

$$\psi_k(z) := \frac{1}{\sqrt{2\pi R l\sqrt{\pi}}} e^{iky/R} e^{-(x-k\gamma l)^2/2l^2}$$

where $\gamma := l/R$. They are eigenfunctions of the center operator $c_x$. The eigenvalue gives the center of the Gaussian part of $\psi$: $c_x\psi_k = k\gamma l\psi_k$. The magnetic translations act as follows:

$$t(n\gamma l\mathbf{e}_x)\psi_k = e^{in\gamma c_y/l}\psi_k = \psi_{k+n}, \quad t(a\mathbf{e}_y)\psi_k = e^{-iac_x/l^2}\psi_k = e^{-ika/R}\psi_k$$

for all $n \in \mathbb{Z}$ and $a \in \mathbb{R}$. The Hamiltonian commutes with all translations in the $y$-direction, but only with a discrete set of translation in the $x$-direction: $t(a\mathbf{e}_x)$ preserves the boundary condition $t(2\pi R\mathbf{e}_y)\psi = \psi$ if and only if $a \cdot 2\pi R \in 2\pi l^2\mathbb{Z}$.

**Torus.** Let $L > 0$. We wish to define a Hamiltonian $H$ in $L^2([0, L] \times [0, 2\pi R])$ through (1.1) and the boundary conditions

$$t(L\mathbf{e}_x)\psi = \psi, \quad t(2\pi R\mathbf{e}_y)\psi = \psi.$$

This is possible if $t(L\mathbf{e}_x)$ and $t(2\pi R\mathbf{e}_y)$ commute, i.e., $L2\pi R = N_f \cdot 2\pi l^2$ for some $N_f \in \mathbb{N}$. In this case, the flux through the surface is an integer multiple of the flux quantum $\phi_0$: $|B_z| \cdot 2\pi RL = N_f\phi_0$, whence the denomination *number of fluxes* of $N_f$. The resulting Hamiltonian has the same spectrum as in (1.2). Each eigenvalue is $N_f$-fold degenerate. In the Landau gauge, the lowest Landau level consists of functions $\psi(x, y) = f(z)e^{-x^2/2l^2}$ where $f$ is holomorphic and satisfies

$$f(z + 2\pi R) = f(z), \quad f(z + L) = e^{N_f z/R + L^2/2l^2} f(z). \tag{1.5}$$

The lowest Landau level has the orthonormal basis $\tilde{\psi}_k$, $k \in \mathbb{Z}/N_f\mathbb{Z}$, where

$$\tilde{\psi}_k = \sum_{r=-\infty}^{\infty} \psi_{k+rN_f}.$$

The Hamiltonian commutes with magnetic translations $t(a\mathbf{e}_x + b\mathbf{e}_y)$ for vectors $(a, b)$ in the lattice $\gamma l\mathbb{Z} \times \frac{2\pi R}{N_f}\mathbb{Z}$, and

$$t(\gamma l\mathbf{e}_x)\tilde{\psi}_k = e^{i\gamma c_y/l}\tilde{\psi}_k = \tilde{\psi}_{k+1}, \quad t(\frac{2\pi R}{N_f}\mathbf{e}_y)\tilde{\psi}_k = e^{-i2\pi\gamma c_x/N_f l}\tilde{\psi}_k = e^{-i2\pi k/N_f}\tilde{\psi}_k$$

**Finite cylinder.** Let $L > 0$, $R > 0$ and $a, b \in \mathbb{R}$, $a < b$. In the spirit of the disk approach described above, the lowest Landau level on a cylinder is often modeled by a restriction to the spectral subspace of $c_x$ where $a \le c_x \le b$, i.e. the considered space is

$$\operatorname{span}\{\psi_k \mid a \le k\gamma l \le b\} \subset L^2(\mathbb{R} \times [0, 2\pi R]). \tag{1.6}$$

A more satisfactory approach uses the *spectral* or *chiral boundary conditions* introduced in [AANS98]. These are described in more detail in Chapter 3. Let us only mention the following: the expression (1.1) together with spectral boundary conditions defines a Hamiltonian in $L^2([a, b] \times [0, 2\pi R])$ with a spectrum of the form

$$\sigma(H) = \{e_n(kl^2/R) \mid n \in \mathbb{N}_0, k \in \mathbb{Z}\}$$

where $e_0(\rho) < .. < e_n(\rho) < ...$ are the eigenvalues of a norm-continuous family of operators $h(\rho)$ in $L^2([a,b])$. The "ground state curve" $e_0(\cdot)$ satisfies $e_0(\rho) \geq \hbar\omega/2$ with equality if and only if $a \leq \rho \leq b$. Thus the ground state energy of $H$ is $\hbar\omega/2$. The ground state has the orthogonal basis $\psi_k$ where $k \in \mathbb{Z}$ is such that $a \leq k\gamma l \leq b$ and the functions $\psi_k$ are considered as elements of $L^2([a,b] \times [0, 2\pi R])$. Thus we recover essentially the space (1.6).

## 1.2 Laughlin-type wave functions

The previous section describes particles moving in a two-dimensional world subject to magnetic field, but otherwise free. The one-particle Hamiltonians have a discrete spectrum and explicitly known eigenfunctions. The next step is to ask for the ground state of a gas of electrons that interact between themselves and with a neutralizing background. We are looking for the ground state of a Hamiltonian of the type

$$H_N = \sum_{j=1}^{N} \frac{1}{2m}(p_j + eA(z_j))^2 + \sum_{j=1}^{N} W(z_j) + \sum_{1\leq j<k\leq N} V(z_j - z_k).$$

$V$ is essentially a $1/r$-Coulomb interaction, adapted to the geometry (torus, cylinder..). In the limit of large magnetic fields $|B_z| \to \infty$, the gap $\hbar\omega \propto |B_z|$ separating the lowest Landau level from the excited states of the one-particle Hamiltonian becomes large. Thus as an approximation, one can look for ground states of $\Pi^{\otimes N} H_N \Pi^{\otimes N}$ where $\Pi$ is the projection onto the lowest Landau level. In the lowest Landau level, the kinetic energy gives a constant $N\hbar\omega/2$, and we are left with the Hamiltonian

$$\Pi^{\otimes N}\big(\sum_{j=1}^{N} W(z_j) + \sum_{1\leq j<k\leq N} V(z_j - z_k)\big)\Pi^{\otimes N}.$$

Laughlin [Lau83] proposed a lowest Landau level wave function in the disk geometry that is commonly accepted as a good approximation to the ground state, as far as bulk properties are concerned. Later, Thouless [Tho84] adapted Laughlin's wave function to the cylinder geometry, and Haldane and Rezayi defined torus Laughlin-type wave functions [HR85b]. There is also an analogue for a spherical geometry, defined by Haldane [Hal83], but we shall not use this wave function here. Let us mention however that the spherical geometry is advantageous for numerical computations [HR85a, FOC86].

**Definition 1.1.** *Let $l, R, L > 0$ and $N, p \in \mathbb{N}$ and $2\pi RL = pN2\pi l^2$. Set $\gamma := l/R$. Let*

$$\Psi_N^D(z_1, .., z_N) := \frac{1}{\sqrt{N!}} \prod_{N\geq j>k\geq 1} (z_j - z_k)^p e^{-\sum_{j=1}^N |z_j|^2/4l^2}$$

$$\Psi_N^C(z_1, .., z_N) := \frac{1}{\sqrt{N!}} \prod_{N\geq j>k\geq 1} (e^{z_j/R} - e^{z_k/R})^p e^{-\sum_{j=1}^N x_j^2/2l^2} \frac{e^{-\frac{1}{2}p^2\gamma^2\sum_{j=0}^{N-1} j^2}}{(2\pi Rl\sqrt{\pi})^{N/2}}$$

$$\Psi_N^T(z_1, .., z_N) := F^{cm}(\sum_{j=1}^{N} z_j) \prod_{1\leq j<k\leq N} \theta_1(\frac{i(z_j - z_k)}{2R}|i\frac{L}{2\pi R})^p e^{-\frac{1}{2l^2}\sum_{j=1}^N x_j^2}$$

*where $\theta_1$ is Jacobi's odd elliptic theta function and $F$ is in the $p$-dimensional subspace of holomorphic functions such that*

$$F(Z + i2\pi R) = (-1)^{p(N-1)} F(Z), \quad F(Z + L) = (-1)^{p(N-1)} e^{pZ/R} e^{pL/(2R)} F(Z). \tag{1.7}$$

*These functions will be called disk, cylinder and torus Laughlin wave functions.*

For $N = 1$, the products $\prod_{1\le j<k\le N}$ are empty. We follow the convention that the product over an empty set is 1. The multiplicative factor in the definition of $\Psi_N^C$ is chosen so that the normalization constant $C_N = ||\Psi_N^C||^2$ has properties derived in the next chapter. We refer the reader to [AS72] for a list of properties and to [FK01] for a modern account on theta functions. Let us recall some facts: let $\tau \in \mathbb{C}$ with $\Im\tau > 0$. Jacobi's odd elliptic theta function is defined as

$$\theta_1(z \mid \tau) = -i \sum_{n=-\infty}^{\infty} (-1)^n e^{i\pi\tau(n-1/2)^2} e^{2i(n-1/2)z}.$$

As a function of $z$, it is holomorphic, doubly quasiperiodic (see (1.8) below) with periods 1 and $\tau$, and

$$\theta_1(z \mid \tau) = 0 \;\Leftrightarrow\; \exists m, n \in \mathbb{Z} :\; z = m + n\tau.$$

The zeroes of $\theta_1(\cdot \mid \tau)$ are simple. The fact that the holomorphic solutions of (1.7) form a $p$-dimensional subspace follows from the theory of theta functions ([FK01], Chapter 7).

The wave functions of the previous definition are symmetric if $p$ is even, and antisymmetric if $p$ is odd. They lie in the disk, cylinder and torus lowest Landau level introduced in the previous section:

**Lemma 1.2.** *1.* $\Psi_N^D \in \otimes^N \mathrm{span}\{\Omega_m \mid 0 \le m \le p(N-1)\}$*, the lowest Landau level on a disk with radius* $((2p(N-1)+1))^{1/2} l$ *(symmetric gauge).*

*2.* $\Psi_N^C \in \otimes^N \mathrm{span}\{\psi_k \mid 0 \le k \le p(N-1)\}$*, the spectral subspace* $0 \le c_x \le p(N-1)\gamma l$ *in the lowest Landau level on a cylinder (Landau gauge).*

*3.* $\Psi_N \restriction ([a,b] \times [0, 2\pi R])^N$ *is made up of wave functions in the ground state of the Landau Hamiltonian in* $L^2([a,b] \times [0, 2\pi R])$ *with spectral boundary conditions, provided* $a \le 0 < p(N-1)\gamma l \le b$ *(Landau gauge).*

*4.* $\Psi_N^T \in \otimes^N \mathrm{span}\{\tilde{\psi}_k \mid k \in \mathbb{Z}/pN\mathbb{Z}\}$*, the lowest Landau level on the torus* $\mathbb{R}/pN\gamma l\mathbb{Z} \times \mathbb{R}/2\pi R\mathbb{Z}$ *(Landau gauge).*

*Proof. 1.* Let

$$V_N(z_1, .., z_N) := \det\big(z_j^{k-1}\big)_{1\le j,k\le N} = \prod_{N\ge j>k\ge 1} (z_j - z_k)$$

be the Vandermonde determinant. Its $p$-th power can be expanded in monomials

$$V_N(z_1, .., z_N)^p = \sum_{m_1,..,m_N} b_N(m_1, .., m_N) z_1^{m_1} .. z_N^{m_N}.$$

The Vandermonde determinant is a polynomial of maximal degree $N-1$ in each variable, thus $V_N^p$ is a polynomial of maximal degree $p(N-1)$ in each variable and only $0 \le m_1, .., m_N \le p(N-1)$ contributes to the sum above. The disk Laughlin function is $V_N^p$ multiplied by a Gaussian factor. The basis functions $\Omega_m$ are monomials $z^m$ times the Gaussian factor $e^{-|z|^2/4l^2}$. Thus $\Psi_N^D$ is a sum of tensor products of functions $\Omega_m$ with $0 \le m \le p(N-1)$.

*2.* and *3.* The cylinder function is $V_N(e^{z_1/R},..,e^{z_N/R})^p$ times a Gaussian factor. Using $\psi_k(z) = (2\pi Rl\sqrt{\pi})^{1/2} e^{kz/R} e^{-k^2\gamma^2/2} e^{-x^2/2l^2}$, we get

$$\begin{aligned}\Psi_N^C(z_1,..,z_N) &= \frac{1}{\sqrt{N!}}\big(V_N(e^{z_1/R},..,e^{z_M/R})\big)^p e^{-\sum_{j=1}^N x_j^2/2l^2} \frac{e^{-p^2\gamma^2\sum_{j=0}^{N-1} j^2}}{(2\pi Rl\sqrt{\pi})^{N/2}} \\ &= \frac{1}{\sqrt{N!}} \sum_{0\le m_1,..,m_N\le p(N-1)} a_N(m_1,..,m_N)\psi_{m_1}\otimes..\otimes\psi_{m_N}\end{aligned}$$

where

$$a_N(m_1,..,m_N) := b_N(m_1,..,m_N) e^{\gamma^2\sum_{j=1}^N (m_j^2 - p^2(j-1)^2)/2}.$$

This proves *2.*. The functions $\psi_k$ are in the ground state of the Hamiltonian with spectral boundary conditions in $L^2([a,b]\times[0,2\pi R])$ if $a \le k\gamma l \le b$, whence also *3.*

*4.* The $N$-fold tensor product of the torus Landau level is the space of functions of the form $f(z_1,..,z_N)\exp(-\sum_{j=1}^N x_j^2/2l^2)$ where $f$ is holomorphic and satisfies the quasiperiodicity condition (1.5) in each complex variable $z_j$, $N_f = pN$. Theta functions are holomorphic, thus it remains to prove that

$$f(z_1,..,z_N) := F(\sum_{j=1}^N z_j) \prod_{1\le j<k\le N} \theta_1(\frac{i(z_j-z_k)}{2R}|i\frac{L}{2\pi R})^p e^{-\frac{1}{2l^2}\sum_{j=1}^N x_j^2}$$

is quasiperiodic. This is checked using (1.7) and

$$\theta_1(z+\pi\mid\tau) = -\theta_1(z\mid\tau) = \theta_1(-z\mid\tau), \quad \theta_1(z+\pi\tau\mid\tau) = -e^{-2iz-i\pi\tau}\theta_1(z\mid\tau). \tag{1.8}$$

□

**Filling factor.** In view of the previous lemma, the Laughlin wave functions describe $N$-particle states living on samples of size essentially $pN\cdot 2\pi l^2$. This leads to an average density $1/(p\cdot 2\pi l^2)$ (see also Corollary 2.15). The one-particle lowest Landau level of e.g. the torus space has dimension $pN$. Therefore the case $p=1$ is called the *filled Landau level*. It corresponds to an average density $1/(2\pi l^2)$ (see also the next section). The ratio $1/p$ of the average density and the filled Landau level density is called the *filling factor*.

## 1.3 Filled Landau level

The filled Landau level function ($p=1$), e.g., on the cylinder, is

$$\Psi_N^C = \psi_0\wedge\psi_1\wedge..\wedge\psi_{N-1}.$$

The one-particle density and other correlation functions have simple expressions. The $n$-point reduced density matrix associated to an $N$-particle wave function $\Psi_N\in L^2(\Omega^N)$ is

$$\rho_N(z_1',..,z_n';z_1,..,z_n) = \frac{N!}{(N-n)!}\int_{\Omega^{N-n}} \overline{\Psi_N(z_1,..,z_n,w_{n+1},..,w_N)} \Psi_N(z_1',..,z_n',w_{l+1},..,w_N)dw/||\Psi_N||^2. \tag{1.9}$$

For $\Psi_N^C$ as above, it can be written as

$$\rho_N(z'_1,..,z'_n;z_1,..,z_n) = n! \sum_{0\le m_1<..<m_n\le N-1} \psi_{m_1}\wedge..\wedge\psi_{m_l}(z'_1,..,z'_n)\overline{\psi_{m_1}\wedge..\wedge\psi_{m_l}(z_1,..,z_n)}.$$

If we shift the function on the cylinder so that the origin is in the middle of the cylinder, the sum ranges from $-M$ to $N-1-M$ where $M$ is an integer close to $N/2$. Taking then the limit $N\to\infty$, we get the expressions for the filled Landau level reduced matrices

$$\rho(z'_1,..,z'_n;z_1,..,z_n) = n! \sum_{\substack{m_1,..,m_n\in\mathbb{Z}:\\ m_1<..<m_n}} \psi_{m_1}\wedge..\wedge\psi_{m_l}(z'_1,..,z'_n)\overline{\psi_{m_1}\wedge..\wedge\psi_{m_l}(z_1,..,z_n)}.$$

The one-particle density is

$$\rho(z) := \rho(z;z) = \sum_{m=-\infty}^{\infty} |\psi_m|^2(z) = \frac{1}{2\pi Rl\sqrt{\pi}} \sum_{m\in\mathbb{Z}} e^{-(x-ml^2/R)^2/l^2}.$$

Using Poisson's summation formula, this can be rewritten as

$$\rho(z) = \frac{1}{2\pi l^2}(1+2\sum_{n=1}^{\infty} e^{-n^2\pi^2R^2/l^2}\cos(2\pi nxR/l^2)).$$

In particular, the one-particle density is independent of the coordinate $y$ around the cylinder and is periodic in the direction of the cylinder axis, of period $l^2/R=\gamma l$. The amplitude of the oscillations is large when the radius of the cylinder is small. The average of the one-particle density is $1/(2\pi l^2)$, which is the value of the constant one-particle density of the filled Landau level on the infinite plane:

$$\rho_{\text{plane}}(z) = \sum_{n=0}^{\infty} |\Omega_n|^2(z) = \frac{1}{2\pi l^2}\sum_{n=1}^{\infty}\frac{|z|^{2n}}{n!(2l^2)^n}e^{-|z|^2/2l^2} = \frac{1}{2\pi l^2}.$$

Note that as the radius $R$ goes to infinity at fixed magnetic length $l$, we recover $\rho_{\text{plane}}$ from the cylinder density $\rho$.

The aim of the next chapter is to prove a similar statement for the cylinder function at general filling factor $1/p$, namely, that the one-particle density is a periodic function of the coordinate along the axis. The period is just $p$ times the filled Landau level period $\gamma l$. The amplitude of the oscillations is large when the radius of the cylinder is small. The average density is $1/(p\cdot 2\pi l^2)$.

## 1.4 Truncated interactions

Laughlin's wave functions are widely accepted as good approximations to the ground state of the "true" lowest Landau level projected Hamiltonian $\Pi^{\otimes N}H_N\Pi^{\otimes N}$. An additional interesting feature is that Laughlin's state is the *exact* ground state of a modified Hamiltonian, as observed in [Hal83, PT85, TK85].

This observation is interesting for several reasons. First, it allows an indirect approach to the question of how far away Laughlin's state is from the exact ground state: in the spirit of numerical backward error analysis, we can compare Hamiltonians instead of ground states. It

has been suggested to treat the difference between the lowest Landau level projected Coulomb interaction and the truncated interaction as a small perturbation [HR85a].
Second, we can consider the truncated interaction as a toy Hamiltonian for the FQHE and ask whether it reproduces features such as incompressibility and existence of gaps. The modified Hamiltonian has been used also as a toy Hamiltonian for investigations on Mott insulators [LL04]. Finally, the characterization of the Laughlin state as an exact ground state will be used in section 2.4 to investigate the equivalence of the cylinder and torus functions.

There are two a priori distinct definitions of truncated interactions. Haldane (see his contribution in [PG87]) defines the interaction in terms of projections onto eigenfunctions of the relative angular momentum, where as [TK85, PT85] characterize Laughlin's wave function as a ground state of projected $\Delta^j\delta$-interactions. Both approaches give the same result, as we will see in this section.

The Laughlin wave functions at filling factor $1/p$ are characterized by the order of their zeroes: as two particles get close $z_j - z_k \to 0$, the wave function is $O((z_j - z_k)^p)$. This makes Laughlin's wave function an exact ground state of $\Delta^j\delta$-interactions [TK85, PT85] projected to the lowest Landau level. These lowest Landau level projected interactions define bounded operators. In the lowest Landau level on infinite samples (or in the disk geometry), the bounded operators are projections on eigenfunctions of the separation of the particles, measured through the center operators.

The definition of the operator in the disk geometry follows Haldane's presentation in [PG87]. For the cylinder, the operators are presented in a slightly different way in [RH94]. The second quantized expressions for the torus and cylinder that can be found in [SFL$^+$05] and [LL04] will be recovered from our definition in Corollary 1.7.

Let $\mathcal{H} \subset L^2(\mathbb{R}^2)$ be the lowest Landau level in the symmetric gauge. The two-particle space $\mathcal{H} \otimes \mathcal{H}$ has a complete orthonormal set

$$\psi_{n,m}(z_1, z_2) = c_{n,m}(z_1 + z_2)^n (z_1 - z_2)^m e^{-(|z_1|^2+|z_2|^2)/4l^2}, \quad m, n \in \mathbb{N}_0$$

(with $c_{n,m}$ suitable normalization constants). Let

$$\mathcal{M} = \frac{1}{4l^2}\big((c_x^{(1)} - c_x^{(2)})^2 + (c_y^{(1)} - c_y^{(2)})^2\big) - \frac{1}{2}.$$

Here, we use the notation $c_x^{(1)} = c_x \otimes \mathbf{1}$, $c_x^{(2)} = \mathbf{1} \otimes c_x$ and similarly for $c_y^{(1)}, c_y^{(2)}$. Then $\mathcal{M}\psi_{n,m} = m\psi_{n,m}$. Thus $\sigma(\mathcal{M}) = \mathbb{N}_0$. The operator $\mathcal{M}$ is sometimes called *relative angular momentum*.

**Definition 1.3.** *Let $\psi_n$, $\tilde{\psi}_n$ be the cylinder and torus lowest Landau level basis functions, and $\gamma = l/R$. For $m \in \mathbb{N}_0$, let $F_m$ be the $m$-th Hermite function (eigenfunction of $-\partial_k^2 + k^2$) and let $P_m$, $P_m^C$, $P_m^T$ be the following two-particle operators acting respectively in the lowest Landau level on the plane, cylinder and torus:*

- *$P_m = P_m^D$ is the projection on the eigenspace $\mathcal{N}(\mathcal{M} - m)$ of $\mathcal{M}$.*
- *$P_m^C$ is the bounded operator determined through*

$$(\psi_{k_1} \otimes \psi_{k_2}, P_m^C \psi_{n_1} \otimes \psi_{n_2}) = \gamma\sqrt{2}\delta_{k_1+k_2,n_1+n_2} F_m\Big(\frac{(k_1 - k_2)\gamma}{\sqrt{2}}\Big) F_m\Big(\Big(\frac{n_1 - n_2)\gamma}{\sqrt{2}}\Big). \quad (1.10)$$

- $P_m^T$ *is the unique bounded operator such that*

$$(\tilde{\psi}_{k_1} \otimes \tilde{\psi}_{k_2}, P_m^T \tilde{\psi}_{n_1} \otimes \tilde{\psi}_{n_2})$$
$$= \gamma\sqrt{2} \sum_{r\in\mathbb{Z}} \delta_{k_1+k_2, n_1+n_2+rN_f} \sum_{\substack{s,t\in\mathbb{Z}: \\ s-t=r \mod 2}} F_m(\frac{(k_1-k_2+sN_f)\gamma}{\sqrt{2}}) F_m(\frac{(n_1-n_2+tN_f)\gamma}{\sqrt{2}}). \tag{1.11}$$

*Remark.* $P_m$ is bounded because it is a projection. The boundedness of (1.11) follows from the fact that the lowest Landau level on a torus is finite-dimensional. The decay properties of the Hermite functions can be used to show that (1.10) defines a bounded operator of norm

$$||P_m^C|| \leq \sqrt{2}\gamma \sum_{k\in\mathbb{Z}} F_m(\frac{k\gamma}{\sqrt{2}})^2;$$

see also (1.19) below.

The operators $P_m^R$, $R = D, C, T$ are associated to the projection of the delta function interaction

$$\frac{4\pi l^2}{j!} e^{|z_1-z_2|^2/4l^2} l^{2j} \Delta_{z_1-z_2}^j \delta(z_1 - z_2)$$

to the lowest Landau level. More precisely:

**Lemma 1.4.** *Let* $\Delta_{z_1-z_2} := (\frac{\partial}{\partial z_1} - \frac{\partial}{\partial z_2})(\frac{\partial}{\partial \bar{z}_1} - \frac{\partial}{\partial \bar{z}_2})$ *be the Laplacian with respect to* $z_1 - z_2$. *Let* $\mathcal{H}_D$ *be the lowest Landau level in the symmetric gauge,* $\mathcal{H}_C$ *the lowest Landau level on the cylinder and torus. Let* $D = \mathbb{R}^2$, $C = \mathbb{R} \times [0, 2\pi R]$, $T = [0, L] \times [0, 2\pi R]$. *Then for* $R \in \{D, C, T\}$ *and* $f, g \in \mathcal{H}_R$*:*

$$(f, P_j^R g) = \frac{4\pi l^2}{j!} \int_R \Big( l^{2j} \Delta_{z_1-z_2}^j (e^{|z_1-z_2|^2/4l^2} \bar{f} g) \Big)(z, z) dxdy. \tag{1.12}$$

*Proof.* Assume without loss of generality $l = 1$. For $z_1, z_2 \in \mathbb{C}$ let $Z := (z_1 + z_2)/\sqrt{2}$, $z := (z_1 - z_2)/\sqrt{2}$.

We start with the symmetric gauge. First, remark that the right-hand side of (1.12) defines a bounded sesquilinear form. To see this, consider first

$$\phi(z) = f(z)e^{-|z|^4/2}, \quad \psi(z) = g(z)e^{-|z|^2/4}$$

two functions in the lowest Landau level. Thus $f$ and $g$ are holomorphic. By a classical result on Bergman spaces, for $r > 0$ and $m \in \mathbb{N}_0$ there exists a constant $c_m > 0$ such that

$$|\frac{\partial^m f}{\partial z^m}(0)| \leq c_m \int_{|z|<r} |f(z)|^2 dxdy \leq c_m e^{r^2/4} ||\phi||.$$

and similarly for $g, \psi$. Thus

$$|\big(\Delta^j \bar{\phi}\psi e^{|z|^2/2}\big)(0)| = |\big((4\frac{\partial^2}{\partial z \partial \bar{z}})^j \bar{f} g\big)(0)| \leq 4^j e^{r^2/2} c_j^2 ||\phi||\, ||\psi||.$$

The boundedness of the right-hand side of (1.12) can be inferred from this. Since the left-hand side is bounded too, it is enough to check (1.12) for functions $f, g$ in a suitable complete

orthonormal system. Let $\Omega_m$, $m \in \mathbb{N}_0$ be the functions defined in (1.4). Let $M, N, m, n \in \mathbb{N}_0$ and

$$f(z_1, z_2) := \Omega_M(Z)\Omega_m(z), \quad g(z_1, z_2) := \Omega_N(Z)\Omega_n(z). \tag{1.13}$$

Then

$$\begin{aligned}
&\big(\Delta^j_{z_1-z_2} e^{|z_1-z_2|^2/4}(\bar f g)\big)(w,w) \\
&\quad = (\bar\Omega_M\Omega_N)(\sqrt{2}w)\frac{1}{2\pi\sqrt{2^{m+n}m!n!}}\big(\Delta^j_{z_1-z_2}\bar z^m z^n\big)(0) \\
&\quad = (\bar\Omega_M\Omega_N)(\sqrt{2}w)\frac{1}{2\pi 2^{m+n}\sqrt{m!n!}})\big(\Delta^j_{z_1-z_2}(\overline{z_1-z_2})^m(z_1-z_2)^n\big)(0) \\
&\quad = (\bar\Omega_M\Omega_N)(\sqrt{2}w)\frac{1}{2\pi 2^{m+n}\sqrt{m!n!}}\delta_{m,j}\delta_{n,j}m!n!4^j \\
&= \quad \bar\Omega_M\Omega_N(\sqrt{2}w)\frac{j!}{2\pi}\delta_{m,j}\delta_{n,j}.
\end{aligned}$$

Since

$$\int_{\mathbb{R}^2}\bar\Omega_M\Omega_N(\sqrt{2}x,\sqrt{2}y)dxdy = \frac{1}{2}(\Omega_M,\Omega_N) = \frac{1}{2}\delta_{M,m}$$

we obtain

$$\int\big(\Delta^j_{z_1-z_2}e^{|z_1-z_2|^2/4}\bar f g)\big)(w,w) = \frac{j!}{4\pi}(f, P^D_j g).$$

Functions of the form (1.13) form a complete orthonormal set in the lowest Landau level, thus using the boundedness of $P^D_j$ we get the desired result.

Now let us turn to a cylinder of radius $R$. Let $\gamma = l/R = 1/R$. Again, it is enough to check (1.12) for functions in a complete orthonormal system. The lowest Landau level basis functions are $\psi^\gamma_n(z) = (\gamma/(2\pi\sqrt{\pi}))^{1/2}\exp(iny/R)\exp(-(x-n\gamma)^2/2)$, $n \in \mathbb{Z}$. A simple computation gives

$$\psi_{n_1}(z_1)\psi_{n_2}(z_2) = \sqrt{2}\psi^{\gamma/\sqrt{2}}_{n_1+n_2}(Z)\psi^{\gamma/\sqrt{2}}_{n_1-n_2}(z).$$

Hence for $f := \psi_{m_1}\otimes\psi_{m_2}$, $g := \psi_{n_1}\otimes\psi_{n_2}$

$$\begin{aligned}
&\big(\Delta^j_{z_1-z_2}e^{|z_1-z_2|^2/4}(\bar f g)\big)(w,w) \\
&\qquad = 2(\bar\psi^{\gamma/\sqrt{2}}_{m_1+m_2}\psi^{\gamma/\sqrt{2}}_{n_1+n_2})(\sqrt{2}w)\big(\frac{1}{2^j}\Delta^j e^{|z|^2/2}\bar\psi^{\gamma/\sqrt{2}}_{m_1-m_2}\psi^{\gamma/\sqrt{2}}_{n_1-n_2}\big)(0)
\end{aligned} \tag{1.14}$$

The Hermite polynomials $H_n(x)$ satisfy the following generating function identity:

$$e^{2xt-t^2} = \sum_{n=0}^\infty H_n(x)\frac{t^n}{n!}.$$

From this we deduce

$$\begin{aligned}
\bar\psi^\gamma_k(z)\psi^\gamma_n(z)e^{|z|^2/2} &= \frac{\gamma}{2\pi\sqrt{\pi}}e^{-\gamma^2(k^2+n^2)/2}e^{\gamma k\bar z}e^{\gamma nz}e^{-x^2}e^{(x^2+y^2)/2} \\
&= \frac{\gamma}{2\pi\sqrt{\pi}}e^{-\gamma^2(k^2+n^2)/2}e^{\gamma k\bar z-\bar z^2/4}e^{\gamma nz-z^2/4} \\
&= \frac{\gamma}{2\pi\sqrt{\pi}}e^{-\gamma^2(k^2+n^2)/2}\sum_{r,s=0}^\infty\frac{H_r(k\gamma)H_s(n\gamma)}{2^{r+s}r!s!}\bar z^r z^s.
\end{aligned}$$

Since $\Delta = 4\partial\bar\partial$, we get

$$\big(\Delta^j\overline{\psi_k^\gamma(z)}\psi_n^\gamma(z)e^{|z|^2/2}\big)(0) = \frac{\gamma}{2\pi\sqrt{\pi}}e^{-\gamma^2(k^2+n^2)/2}H_j(k\gamma)H_j(n\gamma) = \frac{2^j j!\gamma}{2\pi}F_j(k\gamma)F_j(n\gamma) \tag{1.15}$$

where $F_n(x) = (2^n n!\sqrt{\pi})^{-1/2}H_n(x)e^{-x^2/2}$ is the $n$-th Hermite function. Combining (1.14) and (1.15), we obtain

$$\big(\Delta^j_{z_1-z_2}e^{|z_1-z_2|^2/4}(\bar f g)\big)(w,w) = \frac{j!\gamma}{\pi\sqrt{2}}(\bar\psi^{\gamma/\sqrt{2}}_{m_1+m_2}\psi^{\gamma/\sqrt{2}}_{n_1+n_2})(\sqrt{2}w)\big(F_j(\frac{k\gamma}{\sqrt{2}})F_j(\frac{n\gamma}{\sqrt{2}})\big).$$

But now

$$\int_0^{2\pi/\gamma}dy\int_{-\infty}^{\infty}dx(\bar\psi^{\gamma/\sqrt{2}}_{m_1+m_2}\psi^{\gamma/\sqrt{2}}_{n_1+n_2})(\sqrt{2}x,\sqrt{2}y)$$
$$= \frac{1}{2}\int_0^{2\pi\sqrt{2}/\gamma}dy\int_{-\infty}^{\infty}dx(\bar\psi^{\gamma/\sqrt{2}}_{m_1+m_2}\psi^{\gamma/\sqrt{2}}_{n_1+n_2})(x,y) = \frac{1}{2}\delta_{m_1+m_2,n_1+n_2}.$$

Thus we get

$$\int\big(\Delta^j_{z_1-z_2}e^{|z_1-z_2|^2/4}(\bar f g)\big)(w,w) = \frac{j!\gamma}{2\pi\sqrt{2}}\delta_{m_1+m_2,n_1+n_2}F_j(\frac{k\gamma}{\sqrt{2}})F_j(\frac{n\gamma}{\sqrt{2}}) = \frac{j!}{4\pi}(f,P_j^C g).$$

Now let us turn to the torus. Let $T_\gamma = [0,N_f\gamma]\times[0,2\pi/\gamma]$, $N_f\in\mathbb{N}$. The lowest Landau level basis functions $\tilde\psi_m$ are related to the cylinder functions through $\tilde\psi_m = \sum_{k\in\mathbb{Z}}\psi_{m+kN_f}$. If $f,g$ are two smooth functions, let

$$q_j(f,g) := \int_T\big(\Delta^j_{z_1-z_2}e^{|z_1-z_2|^2/4}(\bar f g)\big)(w,w)dxdy.$$

Mimicking the reasoning done for the cylinder case, we get

$$q_j(\psi_{k_1}\otimes\psi_{k_2},\psi_{n_1}\otimes\psi_{n_2}) = \frac{j!\gamma}{2\pi\sqrt{2}}\delta_{K,N}\big(\frac{1}{\sqrt{\pi}}\int_0^{\sqrt{2}N_f\gamma}e^{-(x-K\gamma/\sqrt{2})^2}dx\big)F_j(\frac{k\gamma}{\sqrt{2}})F_j(\frac{n\gamma}{\sqrt{2}}) \tag{1.16}$$

where we used capital letters for sums of indices and small letters for differences ($K = k_1+k_2, k = k_1-k_2$). We are interested in

$$q_j(\tilde\psi_{k_1}\otimes\tilde\psi_{k_2},\tilde\psi_{n_1}\otimes\tilde\psi_{n_2}) = \sum_{s_1,s_2,t_1,t_2\in\mathbb{Z}}q_j(\psi_{k_1+s_1N_f}\otimes\psi_{k_2+s_2N_f},\psi_{n_1+t_1N_f}\otimes\psi_{n_2+t_2N_f}) \tag{1.17}$$

In view of (1.16), the sum vanishes unless $K+SN_f = N+TN_f$ for some $S,T\in\mathbb{Z}$. Suppose that $K-N = rN_f$ with $r\in\mathbb{Z}$. Note that

$$\sum_{s_1,s_2,t_1,t_2\in\mathbb{Z}}\delta_{s_1+s_2-t_1-t_2,r}a(s_1+s_2)b(s_1-s_2)b(t_1-t_2)$$
$$= \sum_{\substack{S,s\in\mathbb{Z}:\\ S=s \mod 2}}\ \sum_{\substack{T,t\in\mathbb{Z}:\\ T=t \mod 2}}\delta_{S-T,r}a(S)b(s)b(t)$$
$$= \sum_{S\in\mathbb{Z}}a(S)\sum_{s=S \mod 2}b(s)\sum_{t=S+r \mod 2}b(t)$$
$$= \sum_{\alpha=0}^{1}\big(\sum_{S\in\mathbb{Z}}a(2S+\alpha)\big)\big(\sum_{s\in\mathbb{Z}}b(2s+\alpha)\big)\big(\sum_{t\in\mathbb{Z}}b(2t+r+\alpha)\big)$$

Combining this with (1.16) and (1.17), we get

$$q_j(f,g) = \frac{j!\gamma}{2\pi\sqrt{2}} \sum_{\substack{s,t\in\mathbb{Z}:\\ s-t=r \mod 2}} F_j(\frac{(k+sN_f)\gamma}{\sqrt{2}})F_j(\frac{(n+tN_f)\gamma}{\sqrt{2}}). \qquad \square$$

**Lemma 1.5.** *The operators $P_j^R$ are bounded, self-adjoint and non-negative. Let $m \in \mathbb{N}_0$ and $V_0, V_1, .., V_m > 0$. Let $\psi$ be in the disk, cylinder, or torus lowest Landau level depending on $R \in \{D, C, T\}$. Then*

$$\psi \in \mathcal{N}(\sum_{j=0}^{m} V_j P_j^R) \;\Leftrightarrow\; \forall Z:\, \psi(Z+z, Z-z) \underset{z\to 0}{=} O(|z|^{m+1}). \tag{1.18}$$

*Proof.* The operators are bounded by definition (see also the remark following Definition 1.3), the self-adjointness is obvious for $P_m^C$. For $P_m^C$ and $P_m^T$ the symmetry follows from $F_j(-x) = (-1)^j F_j(x)$ or from (1.12). As a projection, $P_m^D$ is obviously non-negative. To see that $P_m^C$ is positive, let

$$U : \mathcal{H}_C \otimes \mathcal{H}_C \to l^2(\mathbb{Z}^2) \equiv l^2(\mathbb{Z}) \otimes l^2(\mathbb{Z}^2), \quad \psi_{n_1} \otimes \psi_{n_2} \mapsto (\delta_{n_1+n_2,m}\delta_{n_1-n_2,n})_{m,n\in\mathbb{Z}}.$$

$U$ is only a partial isometry since $n_1 + n_2 = n_1 - n_2 \mod 2$. Then

$$UP_j^C = \big(\mathbf{1} \otimes (\gamma\sqrt{2}|F_j^\gamma\rangle\langle F_j^\gamma|)\big)U, \quad F_j^\gamma := (F_j(\frac{n\gamma}{\sqrt{2}}))_{n\in\mathbb{Z}}. \tag{1.19}$$

This implies the positivity of $P_j^C$. To treat the torus case, let $\hat{\delta}_{n,s}$ take the value 1 if $n = s \mod 2N_f$ and 0 else, and let $\hat{e}_n = (\hat{\delta}_{n,s})_{s\in\mathbb{Z}}$. Define

$$\tilde{U} : \quad \mathcal{H}_T \otimes \mathcal{H}_T \to l^2((\mathbb{Z}/2N_f\mathbb{Z})^2) \equiv l^2(\mathbb{Z}/2N_f\mathbb{Z})^{\otimes 2} \tag{1.20}$$

$$\tilde{\psi}_{n_1} \otimes \tilde{\psi}_{n_2} \mapsto \frac{1}{\sqrt{2}}(\hat{e}_{n_1+n_2} \otimes \hat{e}_{n_1-n_2} + \hat{e}_{n_1+n_2+N_f} \otimes \hat{e}_{n_1-n_2+N_f}). \tag{1.21}$$

Again, $\tilde{U}$ is a partial isometry. Let

$$G_j := (\sum_{k\in\mathbb{Z}} F_j(\frac{(n+2kN_f)\gamma}{\sqrt{2}}))_{n\in\mathbb{Z}}, \quad H_j := (G_j(n+N_f))_{n\in\mathbb{Z}}.$$

Then

$$\tilde{U}P_j^T = \gamma\sqrt{2}\big(\mathbf{1} \otimes (|G_j\rangle\langle G_j| + |H_j\rangle\langle H_j|)\big)\tilde{U},$$

whence the positivity of $P_j^T$. Thus the operators $P_j^R$ are positive and $\psi \in \mathcal{N}(\sum_{j=0}^m V_j P_j)$ if and only if $(\psi, P_j^R\psi) = 0$ for $j = 0, 1.., m$. Let $\psi(z_1, z_2)$ be in the lowest Landau level of two particles and let $l = 1$. Write $\psi(z_1, z_2) = f(\frac{z_1+z_2}{2}, z_1 - z_2)e^{-|z_1-z_2|^2/8}$ so that

$$(\psi, P_j^R\psi) = \frac{4\pi}{j!}\int_R (\Delta_z^j|f|^2)(Z, 0)dXdY$$

where the Laplacian $\Delta_z$ acts on the second variable of $f(Z, z)$. Note that $\psi(Z+z, Z-z) = O(|z|^{m+1})$ if and only if $f(Z, z) = O(|z|^{m+1})$. Since functions in the lowest Landau level are holomorphic up to a factor of the type $\exp(-ax^2 - by^2)$, we can write for sufficiently small $z$

$$f(Z, z) = \sum_{m,n\in\mathbb{N}_0} a_{m,n}(Z)z^m\bar{z}^n, \quad |f(Z, z)|^2 = \sum_{m,n\in\mathbb{N}_0} b_{m,n}(Z)z^m\bar{z}^n$$

and we get

$$(\psi, P_j^R \psi) = 4^{j+1} j! \pi \int_R b_{j,j}(Z) dX dY.$$

We proceed by induction on $m$. Suppose first $(\psi, P_0^R \psi) = 0$. Since $b_{0,0}(Z) = |f(Z,0)|^2$ we get $f(Z,0) = 0$ for all $Z$ and thus $f(Z,z) = O(|z|)$. Now suppose $(\psi, P_j^R \psi) = 0$ for $j = 0, .., m+1$ and by induction $f(Z,z) = O(|z|^{m+1})$. Then

$$f(Z,z) = \sum_{k=0}^{m+1-k} a_{k,m+1-k} z^k \bar{z}^{m+1-k} + O(|z|^{m+2}), \quad b_{m+1,m+1}(Z) = \sum_{k=0}^{m+1} |a_{k,m+1-k}|^2.$$

Thus $(\psi, P_{m+1}^R \psi) = 0$ implies $a_{k,m+1-k} = 0$ for $k = 0, .., m+1$. □

Now we can characterize the Laughlin-type wave functions as ground states of suitable Hamiltonians.

**Proposition 1.6.** *Let $p \in \mathbb{N}$, $V_0, .., V_{p-1} > 0$ and $N \in \mathbb{N}$, $N \geq 2$. Define the Hilbert spaces*

$$\mathcal{H}_N^D := \otimes^N \text{span}\{\Omega_j \mid 0 \leq j \leq p(N-1)\}$$
$$\mathcal{H}_N^C := \otimes^N \text{span}\{\psi_j \mid 0 \leq j \leq p(N-1)\}$$
$$\mathcal{H}_N^T := \otimes^N \text{span}\{\tilde{\psi}_j \mid j \in \mathbb{Z}/N_f\mathbb{Z}\}.$$

*Let $\Pi_N^D$ be the projection onto $\mathcal{H}_N^D$ in $\otimes^N \text{span}\{\Omega_n \mid n \in \mathbb{N}_0\}$, and $\Pi_N^C$ the projection onto $\mathcal{H}_N^C$ in $\otimes^N \text{span}\{\psi_j \mid j \in \mathbb{Z}\}$. Let $\Pi_N^T$ be the identity in $\mathcal{B}(\mathcal{H}_N^T)$. For $R \in \{D, C, T\}$ let $H_N^R$ be the $N$-particle Hamiltonian in $\mathcal{B}(\mathcal{H}_N^R)$ defined as*

$$H_N^R = \Pi_N^R \sum_{1 \leq j < k \leq N} (V_0 P_0^{R,jk} + .. + V_{p-1} P_{p-1}^{R,jk}) \, \Pi_N^R$$

*where $P^{jk}$ refers to $P$ acting on the $j$-th and $k$-th variables. Then $H_N^R$, $H_N^C$ have unique ground states $\Psi_N^D$, $\Psi_N^C$. $H_N^T$ has a $p$-fold degenerate ground state, the space of $\Psi_N^T$ functions.*

*Proof.* By Lemma 1.5, $H_N^R$ is positive and a function $\Psi \in \mathcal{H}_N^R$ is in the null space of $H_N^R$ if and only if for any $j < k$, $\Psi(z_1, .., z_N) = O(|z_j - z_k|^p)$ as $|z_j - z_k|$ goes to zero, the sum $z_j + z_k$ and the variables $z_n, n \neq j, k$ being fixed. From this we see that the Laughlin-type wave functions are indeed ground states, and it remains to show that these are the only ones.

The argument for the disk is simple (see Haldane's article in [PG87]): let $\Psi \in \mathcal{H}_N^D$. Then there is a a polynomial of degree $\leq p(N-1)$ in each variable such that

$$\Psi(z_1, .., z_N) = P(z_1, .., z_N) e^{-\sum_j |z_j|^2/4l^2}.$$

If $\Psi(z_1, .., z_N) = O(|z_j - z_k|^p)$ as $z_j - z_k \to 0$, $P$ is a multiple of $(z_j - z_k)^p$. Thus if $\Psi \in \mathcal{N}(H_N^D)$, there is a polynomial $Q$ so that

$$P(z_1, .., z_N) = Q(z_1, .., z_N) \prod_{N \geq j > k \geq 1} (z_j - z_k)^p.$$

Since $P$ has maximal degree $p(N-1)$ in each variable, $Q$ is a constant and $\Psi$ is a multiple of $\Psi_N^D$. The argument for the cylinder is similar: note that if $\Psi$ is in $\mathcal{H}_N^C$, there is a polynomial $P$ of degree $\leq p(n-1)$ in each variable such that

$$\Psi(z_1, .., z_N) = P_N(e^{z_1/R}, .., e^{z_N/R}) e^{-\frac{1}{2l^2} \sum_{j=1}^N x_j^2}.$$

If $\Psi \in \mathcal{H}_N^T$, there is a holomorphic function $f$ with the quasiperiodicity

$$f(z_1,..,z_j+2\pi Ri,..,z_N) = f(z_1,..,z_N),$$
$$f(z_1,..,z_j+L,..,z_N) = e^{N_f z_j/R} e^{N_f^2 l^2/2R^2} f(z_1,..,z_N)$$

in each variable $z_j$ such that $\Psi(z) = f(z_1,..,z_N) e^{-\frac{1}{2l^2}\sum_{j=1}^N x_j^2}$. If $\Psi \in \mathcal{N}(H_N^T)$, there is a holomorphic function $g$ such that

$$f(z_1,..,z_N) = g(z_1,..,z_N) \prod_{1\le j<k\le N} \theta_1(i\frac{z_j-z_k}{2R} \mid i\frac{L}{2\pi R})^p. \tag{1.22}$$

The function $g$ satisfies for each $j \in \{1,..,N\}$

$$g(z_1,.,z_j+i2\pi R,..,z_N) = (-1)^{p(N-1)} g(z_1,..,z_N)$$
$$g(z_1,.,z_j+L,..,z_N) = (-1)^{p(N-1)} e^{p\sum_{k=1}^N z_k/R} e^{pL/2R} g(z_1,..,z_N). \tag{1.23}$$

Define a function $G$ through $g(z_1,..,z_N) = G(\sum_{j=1}^N z_j, z_2-z_1,..,z_N-z_1)$; equivalently,

$$G(Z,z_2',..,z_N') = g(z_1, z_1+z_2',..,z_1+z_N'), \quad z_1 := \frac{1}{N}(Z - \sum_{j=2}^N z_j').$$

$G$ is holomorphic. It follows from (1.23) that for $b_1,..,b_N \in \mathbb{Z}$,

$$g(z_1+b_1L,..,z_N+b_NL) = (-1)^{p(N-1)B} e^{pBZ/R} e^{LB^2/2R} g(z_1,..,z_N),\ B = \sum_{j=1}^N b_j.$$

This can be used to show that $G$ is doubly periodic in $z_j'$ with periods $iN2\pi R$ and $NL$ in each variable $z_j'$. Since it is holomorphic, this means that $G(Z,z_2',..,z_N') = F(Z)$ for some holomorphic function $F$. Thus $g(z_1,..,z_N) = F(z_1+..+z_N)$. Together with (1.22) and (1.23), this proves that $\Psi$ is a torus Laughlin wave function $\Psi_N^T$ as in Definition 1.1. □

To summarize, we have shown that the lowest Landau level projections of $\Delta^j\delta$-interactions are the bounded operators defined in the beginning of this section, The Laughlin-type wave functions can be characterized as the ground states of Hamiltonians defined with these truncated interactions. This holds for even $p$ as well as for odd $p$ and requires no assumption on the parity of the wave function. Some simplifications are possible when we do assume something on the parity. We exemplify this for Laughlin's 1/3-function and antisymmetric functions.

**Second quantized version.** If we restrict to antisymmetric functions, we can express the various Hamiltonians used in terms of fermionic creation and annihilation operators. Let us recall their definition. For $\Omega \subset \mathbb{R}^2$ and $\mathcal{H} \subset L^2(\Omega)$, let $\mathcal{F} = \bigoplus_{N=0}^\infty \wedge^N \mathcal{H}$ be the associated fermionic Fock space. For $r,s \in \mathbb{N}$, the wedge product of $f \in \wedge^r(\mathcal{H}) \subset L^2(\Omega^r)$ and $g \in \wedge^s\mathcal{H}$ is the element of $\wedge^{r+s}\mathcal{H}$ determined through

$$(f\wedge g)(z_1,..,z_{r+s}) = \sqrt{\frac{(r+s)!}{r!s!}} A_{r+s}(f\otimes g)(z_1,..,z_{r+s})$$
$$= \frac{1}{\sqrt{r!s!(r+s)!}} \sum_{\pi\in\mathcal{S}_{r+s}} \mathrm{sgn}(\pi) f(z_{\pi(1)},..,z_{\pi(r)}) g(z_{\pi(r+1)},..,z_{\pi(r+s)}).$$

where $A_n$ is the antisymmetrisation operator in $L^2(\Omega^n) \equiv \mathcal{H}^{\otimes n}$. The wedge product of $f \in \wedge^0\mathcal{H} \equiv \mathbb{C}$ with $g \in \wedge^s\mathcal{H}$ is $f \cdot g$. The map thus defined extends uniquely to a bounded bilinear map $\wedge : \mathcal{F} \times \mathcal{F} \to \mathcal{F}$. For $f \in \mathcal{H}$, the creation operator $c^*(f) \in \mathcal{B}(\mathcal{H})$ is the operator $\psi \mapsto f \wedge \psi$, and the annihilation operator $c(f)$ is its adjoint. The truncated interaction has a simple expression [LL04, SFL$^+$05]:

**Corollary 1.7.** *Let $p = 3$, $N \in \mathbb{N}\backslash\{1\}$ and $R > 0$. Let $\psi_n$, $\tilde{\psi}_n$ be the cylinder and torus lowest Landau level basis functions, and $c_n = c(\psi_n)$, $\tilde{c}_n = c(\tilde{\psi}_n)$. Let $f(n) = ne^{-n^2\gamma^2/4}$, $\tilde{f}(n) = \sum_{k\in\mathbb{Z}} f(n + 6kN)$. The $N$-particle Laughlin states at filling factor $1/3$ on the cylinder and torus can be characterized as the ground states of*

$$\sum_{0\le k_1,k_2,n_1,n_2\le 3N-3} \delta_{k_1+k_2,n_1+n_2} f(k_1-k_2)f(n_1-n_2)c^*_{k_2}c^*_{k_1}c_{n_1}c_{n_2}, \tag{1.24}$$

*and*

$$\sum_{\substack{R,x,y\in\mathbb{Z}/6N\mathbb{Z}\\ R=x=y \mod 2}} \tilde{f}(x)\tilde{f}(y)\tilde{c}^*_{\frac{R-x}{2}}\tilde{c}^*_{\frac{R+x}{2}}\tilde{c}_{\frac{R+y}{2}}\tilde{c}_{\frac{R-y}{2}} \tag{1.25}$$

*acting in $\wedge^N\mathrm{span}\{\psi_n \mid 0 \le n \le 3N-3\}$ resp. $\wedge^N\{\tilde{\psi}_n \mid n \in \mathbb{Z}/(3N)\mathbb{Z}\}$.*

*Sketch of proof.* (1.24) is obviously a second-quantized version of $P_1^C$ projected to $\wedge^2\mathrm{span}\{\psi_k \mid 0 \le k \le 3N-3\}$, up to a positive multiplicative constant. The projection of $P_2^C$ and $P_0^C$ to antisymmetric functions vanish. Applying Proposition 1.6 with $p = 3$, we see that $\Psi_n^L$ is the unique ground state of (1.24).
The argument for the torus is similar. The computation needed to check that (1.25) is the second-quantized version of $P_1^T$ uses essentially the following: if $(R, x) \in (\mathbb{Z}/6N\mathbb{Z})^2$ is such that $R$ and $x$ are both even or odd (this does not depend on the choice of representants because $6N$ is even), $((R + x)/2, (R - x)/2)$ is well-defined as an element of $(\mathbb{Z}/3N\mathbb{Z})^2$. Furthermore if $(k_1, k_2) \in (\mathbb{Z}/3N\mathbb{Z})^2$,

$$\begin{cases} \frac{R+x}{2} = k_1 \mod 3N \\ \frac{R-x}{2} = k_2 \mod 3N \end{cases} \Leftrightarrow \begin{cases} k_1 + k_2 = R \mod 6N, \\ k_1 - k_2 = x \mod 6N \end{cases} \text{ or } \begin{cases} k_1 + k_2 = R + 3N \mod 6N, \\ k_1 - k_2 = x + 3N \mod 6N. \end{cases}$$

□

**Comparison of truncated interaction and projected Coulomb interactions.** Let $V(|z_1 - z_2|)$ be a rotationally invariant interaction and $\Pi$ the projection on the lowest Landau level $\mathcal{H}$ of in the plane. Then $\Pi^{\otimes 2}V(|z_1 - z_2|)\Pi^{\otimes 2}$ can be expanded in terms of the projections $P_m$:

$$\Pi^{\otimes 2}V(|z_1 - z_2|)\Pi^{\otimes 2} = \sum_{m=0}^{\infty} V_m P_m.$$

The numbers $V_m$, $m \in \mathbb{N}_0$ are called pseudopotential parameters (see Haldane's contribution in [PG87]). For the Coulomb interaction $V(z) = 1/|z_1 - z_2|$, they can be computed as

$$V_m = \frac{\Gamma(m + \frac{1}{2})}{2m!}.$$

The truncated interactions $\sum_{j=0}^{p-1} V_m P_m$ can be viewed as the short-range part of $\Pi V \Pi$. This raises the question how large the error done by cutting off the long-range part is. Here, one should be aware that *two* questions show up:

1. Is $\sum_{j=1}^{p-1} V_m P_m$ a good approximation to $\Pi^{\otimes 2} V \Pi^{\otimes 2}$?
2. Is the truncated many-body Hamiltonian from Proposition 1.6 that has Laughlin's function as its ground state, a good approximation to the lowest Landau level projected Hamiltonian

$$\Pi^{\otimes N} \sum_{1 \le j < k \le N} V(z_j - z_k) \Pi^{\otimes N} + \Pi^{\otimes N} \sum_{j=1}^{N} W(z_j) \Pi^{\otimes N} \ ?$$

The comparison of truncated and "true" lowest-Landau level projected Hamiltonians has been performed by [HR85a] in a spherical geometry. On the sphere, the background potential $W$ gives only a constant contribution to the total Hamiltonian and the distinction between the two questions blurs. [HR85a] found that the truncated Hamiltonian is a good approximation to the lowest Landau level projected Hamiltonian. In samples with boundaries, the situation is quite different: the truncated interaction is in general not a good approximation to the true interaction.

Consider for example two particles in a disk. Let $\Pi_3$ be the projection in $\wedge^2 L^2(\mathbb{R}^2)$ onto $\mathrm{span}\{\Omega_m \wedge \Omega_n \mid 0 \le m, n \le 3\}$, the Hilbert space of two fermions in the lowest Landau level with center $c_x^2 + c_y^2 \le 3$. The ground state of the truncated interaction $\Pi_3 P_1 \Pi_3$ is $(z_1 - z_2)^3 e^{-(|z_1|^2 + |z_2|^2)/4l^2}$. The ground state of $\Pi_3 V \Pi_3$, on the other hand, is $\Omega_2 \wedge \Omega_3$, as can be shown by a simple computation. This reflects the fact that excess charge in Coulomb systems accumulates at the boundary; this feature of Coulomb interaction is due to its long range part and thus not captured by a truncated interaction.

Thus in samples with boundaries, the answer to the first question is in general negative. However, this does not exclude that truncated Hamiltonian and the lowest Landau level Hamiltonian including the contribution from a neutralizing background have similar ground states.

It is interesting to examine the way the physics literature treats samples with boundaries. For example, Laughlin [Lau83] compares his wave function to various "true" ground states: these are obtained as ground states of Hamiltonians without background potential, projected to the space of lowest Landau level many-body functions with total angular momentum $pN(N-1)/2$. The restriction of angular momentum is justified on grounds that this is the angular momentum of Laughlin's function at filling factor $1/p$. A similar approach is used by Dev and Jain [DJ91], and Girvin and Jach [GJ83] who even *define* the filling factor through the total angular momentum (equation (9)). The conclusion of the numerical investigations cited is that "true" ground states are close to the Laughlin state, but it seems to us that the treatment or rather neglection of the background as well as the restriction to a given angular momentum sector require more justification. Thus we feel that the numerical comparisons of exact and Laughlin states in the spherical geometry [HR85a, FOC86] are much more convincing.

## 1.5 Incompressibility and gaps

The aim of this section is to discuss the notion of a gap above the ground state and its relation to incompressibility. We do this mainly because we will use gap conditions in Section 2.4 and Chapter 3, and we wish to clarify the relation between the two in general *different* gaps that

are used. However, apart from making things (hopefully) a little bit clearer, the following considerations are not necessary for the understanding of the remainder of the thesis.

In the ground state of a quantum Hall system at certain densities, the electrons form an incompressible gas (see below). This is justified by numerical results on the ground state energy per particle and by the existence of a gap above the ground state. The relation between gaps and incompressibility is often only sketched (see however [Yos98]); we give some details here. As soon as one refers to systems of interacting particles, there are several natural notions of gaps; this is implicit when one talks of the gappedness of different excitations and raises the question which gap leads to incompressibility of the system.

**Incompressible fluid at zero temperature.** Consider a system of $N$ particles moving in a domain of volume $V$, modeled by a Hamiltonian with ground state energy $E(N,V)$. Suppose that the limit $u(n) := \lim_{N\to\infty} E(N,V)/N$ taken along suitable sequences of domains such that $N \to \infty$, $N/V \to n$, exists. The limit $u$ is the energy per particle at density $n$ and temperature 0. Numerical computations [YHL83] suggest that for quantum Hall systems, $u$ is a piecewise differentiable function of $n$ with a "cusp" at certain densities: for example, $du/dn$ is discontinuous at $n_* = (3 \cdot 2\pi l^2)^{-1}$. The discontinuity of $du/dn$ is the manifestation of *incompressibility*. A system is usually called incompressible if its volume is independent of the pressure. Let us briefly explain the connection to the cusp in $u(n)$ and other characterizations, e.g. discontinuity of the chemical potential (see also [Yos98]).

The bulk thermodynamics of a system with one species of particles is described by a set of intensive parameters among which, at zero temperature, "only one is independent": suppose that in a neighborhood of a certain density $n_*$, the bulk thermodynamics is described by a continuous curve $J \ni s \mapsto (u,n,p,\mu)(s)$, where $J$ is an open interval and $u,n,p,\mu$ are the energy per particle, density, pressure and chemical potential. We interpret the projections of the curve to the $(n,u)$ and $(n,p)$ planes as the graphs of (possibly multi-valued) functions $u(n)$, $p(n)$. Suppose that the energy per particle $u$ is a continuously differentiable (single-valued) function of the density $n$ except at $n = n_*$ where it has different left and right derivatives. Then by the usual thermodynamic laws, for $n \neq n_*$,

$$p(n) = n^2 \frac{du}{dn}$$

(this is just a different way of writing $du = -pdv$, with $v = 1/n$ the volume per particle). In particular, the pressure is a single-valued function of $n$ except possibly at $n = n_*$. Let

$$p_\pm := n_*^2 \lim_{n\to n_*\pm} \frac{du}{dn}.$$

We require that the system has a non-negative compressibility, i.e., the density increases when the pressure increases and vice versa. Then $p_- < p_+$ and the pressure is a continuous, increasing function of $n$ except at $n = n_*$ where it jumps from $p_-$ to $p_+$. In view of the continuity of $s \mapsto (n(s), p(s))$, it follows that at $n = n_*$, the pressure can take any value between $[p_-, p_+]$. Equivalently, we may say that the density is constant along the pressure interval $[p_-, p_+]$: the system is incompressible.

Now let us turn to the behavior of the chemical potential. At zero temperature, the chemical potential is

$$\mu = u + pv = u + p/n = \frac{d(nu)}{dn} \tag{1.26}$$

where the last identity holds only in points of differentiability of $u$. Let $\mu_\pm = u + p_\pm/n$. If $n = n_*$, the chemical potential can take any value in $[\mu_-, \mu_+]$; by a slight abuse of language, we may say that the chemical potential is a discontinuous function of the density. Equivalently, the density is constant along the chemical potential interval $[\mu_-, \mu_+]$.

To conclude, we have the following manifestations of incompressibility:

- $du/dn$ is discontinous at $n = n_*$
- the volume per particle $v$ is constant along a pressure interval $[p_-, p_+]$
- the chemical potential is discontinous at $n = n_*$
- the density is constant along a chemical potential interval $[\mu_-, \mu_+]$.

**Different energy gaps.** The incompressibility of a system, characterized through the discontinuity of the chemical potential, is related to the existence of gaps. For example, Girvin [Gir05] writes the discontinuity of the chemical potential at filling factor 1/3 as $\Delta\mu = 3(\Delta_+ + \Delta_-)$, where $\Delta_\pm$ represent quasielectron / quasihole excitation energies (p. 158). On the other hand, the existence of a gap is also invoked to justify the application of adiabatic theorems in the various quantum Hall effect arguments involving slow addition of a flux quantum. This gap however is in general different from the gaps related to incompressibility. For systems of interacting particles, two gaps can be considered: the gap at a fixed number of particles, and the gap of a grand-canonical Hamiltonian $H - \mu\hat{N}$. The discontinuity of the chemical potential is related to these quantities, but in generals differs from them.

Let $\mathcal{H}$ be the Hilbert space for the motion of one particle moving. For simplicity, suppose $\dim \mathcal{H} = d < \infty$. Let $\mathcal{F} := \bigoplus_{N=0}^{d} \wedge^N \mathcal{H}$ be the corresponding fermionic Fock space, $H_N \in \mathcal{B}(\wedge^N \mathcal{H})$ the Hamiltonian for $N$-particle systems.

*Gap at a fixed number of particles.* The first natural notion of a gap is simply the gap at a fixed number of particles: let $E_0(N) < E_1(N) < ..$ be the (possibly degenerate) eigenvalues of $H_N$. We shall refer to $E_1(N) - E_0(N)$ as the gap above the ground state of $H_N$ (we assume that $H_N$ is not a multiple of the identity). This gap is essentially the gap used in adiabatic theorems: typical quantum Hall effect gedanken experiments involve flux dependent Hamiltonians $H_N(\phi)$ and use a condition of the type $E_1(N, \phi) - E_0(N, \phi) \geq g_{min} > 0$ for some $\phi$-independent $g_{min}$, at a given number of particles.

*Discontinuity of the chemical potential.* Since incompressibility can be characterized in terms of the chemical potential, it is natural to turn to the grand-canonical ensemble. Let $\mu \in \mathbb{R}$ be a chemical potential, and

$$H - \mu\hat{N} := \bigoplus_{N=0}^{d} (H_N - \mu N P_N) \in \mathcal{B}(\mathcal{F}).$$

$P_N$ is the projection on $\wedge^N \mathcal{H}$, considered as a subspace of $\mathcal{F}$. At zero temperature, the expected number of particles is $N(\mu) = \mathrm{tr} P_\mu \hat{N}/\mathrm{tr} P_\mu$, where $P_\mu$ is the projection on the ground state of $H - \mu\hat{N}$. For each number of particles $N \in \mathbb{N}$, we define left and right chemical potentials $\mu_-(N) := E_0(N) - E_0(N-1)$ and $\mu_+(N) := E_0(N+1) - E_0(N)$ [1] We

[1] The definition of the chemical potentials $\mu_\pm(N)$ appears in the work [LW68] on Mott insulators; there, the nonvanishing of $\mu_+ - \mu_-$ is related to Mott insulating behavior.

assume that

$$\forall N \in \mathbb{N}: \quad \mu_-(N) < \mu_+(N), \tag{1.27}$$

i.e., $E_0(N)$ is a strictly convex function of $N$. For $\mu_-(N) < \mu < \mu_+(N)$, $P_\mu$ is the projection on the ground state of $H_N$, considered as a subspace of $\mathcal{F}$, so that

$$\mu_-(N) < \mu < \mu_+(N) \;\Rightarrow\; N(\mu) = N. \tag{1.28}$$

In order to apply one of the criteria of incompressibility, we have to take thermodynamic limits. We do this only at a formal level and restore the dependence of the volume in the notation by writing $\mu_\pm(N,V)$, $E_0(N,V)$ instead of $\mu_\pm(N)$, $E_0(N)$. For large $N$, $E_0(N,V) \approx Nu(N/V) = Nu(n)$ with $n = N/V$. Then

$$E_0(N+1,V) - E_0(N,V) \approx (N+1)u(n+\frac{1}{V}) - Nu(n) \approx u(n) + nu'_+(n) =: \mu_+(n)$$
$$E_0(N,V) - E_0(N-1,V) \approx Nu(n) - (N-1)u(n-\frac{1}{V}) \approx u(n) + nu'_-(n) =: \mu_-(n)$$

where we assume that $u$ has left and right derivatives $u'_-$ and $u'_+$. The previous relations are consistent with (1.26). In the thermodynamic limit, (1.28) becomes

$$\mu_-(n) \le \mu \le \mu_+(n) \;\Rightarrow\; n(\mu) = n.$$

The system is incompressible at a density $n$ if the limiting value $\mu_+(n) - \mu_-(n)$ of the difference $\mu_+(N,V) - \mu_-(N,V)$ as $N \to \infty$, $N/V \to n$ does not vanish. Thus incompressibility at zero temperature involves only the ground state energies $E_0(N,V)$, and the existence of gaps above the ground states of $H_N$ in $\wedge^N\mathcal{H}$ is irrelevant.

*Gap of $H - \mu\hat{N}$.* The discontinuity of the chemical potential has nothing to do with the gap of $H_N$ at fixed $N$. However, it is related to the gap of $H - \mu\hat{N}$ in $\mathcal{F}$. More precisely, suppose $\mu_-(N) < \mu < \mu_+(N)$ so that the ground state of $H - \mu\hat{N}$ coincides with the ground state of $H_N$. Then the gap

$$g(H - \mu\hat{N}) := \min \sigma(H - \mu\hat{N}) \backslash \{E_0(N) - \mu N\} - (E_0(N) - \mu N)$$

above the ground state of $H - \mu\hat{N}$ is bounded by the discontinuity of the chemical potential and the gap $E_1(N) - E_0(N)$ above the ground state of $H_N$ in $\mathcal{H}_N$:

$$g(H - \mu\hat{N}) \le \min(E_1(N) - E_0(N), \mu_+(N) - \mu_-(N)). \tag{1.29}$$

*Proof of (1.29).* Note that

$$\begin{aligned} g(H - \mu\hat{N}) &\le \min\{E_1(N) - E_0(N)\} \cup \{E_0(N') - E_0(N) - \mu(N' - N) \mid N' \neq N\} \\ &= \min\{E_1(N) - E_0(N), \mu_+(N) - \mu, \mu - \mu_-(N)\} \\ &\le \min\{E_1(N) - E_0(N), \mu_+(N) - \mu_-(N)\}. \end{aligned}$$

The second line is obtained by using (1.27) and writing for $N' \ge N+1$

$$\begin{aligned} E_0(N') - E_0(N) - \mu(N' - N) &= \sum_{k=1}^{N'-N} (E_0(N+k) - E_0(N+k-1) - \mu) \\ &\ge E_0(N+1) - E_0(N) - \mu. \end{aligned}$$

An analogous inequality holds for $N' \le N - 1$. □

To summarize, we have the following relationship between the discontinuity of the chemical potential and the two gaps: the discontinuity of the chemical potential has nothing to do with the existence of gaps at a fixed number of particles, but involves only ground state energies $E_0(N)$. In contrast, if $H - \mu\hat{N}$ has a gapped ground state, then the chemical potential is discontinous and there is a gap at fixed number of particles, and the gap of $H - \mu\hat{N}$ is bounded from above by these two quantities.

Finally, let us mention how the quantities are related for FQHE states: the FQHE state admits a curve of "collective excitations" depending on a wave vector $k$. The curve approaches the discontinuity of the potential at small wavelengths ($k$ large), but has a minimum smaller than $\Delta\mu$ ([HR85a, Gir05]). Thus the gap at a fixed number of particles is smaller than the discontinuity of the chemical potential.

*Remarks: 1. Independent particles.* For non-interacting particles, the distinctions made above are superfluous: the gaps and the discontinuity of the chemical potential equal the gap of the one-particle Hamiltonian.

*2. Different gaps for different excitations.* In solid state physics, it is common to talk of gaps for specific excitations. In this perspective, the gap at fixed number of particles $E_1(N) - E_0(N)$ is the gap for excitations that preserve the total number of particles. In contrast, the differences $E_0(N') - E_0(N)$ show up when we allow excitations that do change the number of particles. This aspect reappears in the $C^*$-algebraic definition of gaps for systems with an infinite number of particles: two algebras of operators are introduced (in Section 2.4, $\mathcal{A}^{U(1)}$ and $\mathcal{A}$), and there are two notions of ground states and gappedness.

*3. Gapless edge excitations.* At this point we stress that the incompressibility and gappedness of the FQHE ground state refer to bulk properties. It is well-known that on samples with boundaries, there are gapless excitations. This can be explained with the spectral boundary conditions (see p. 8 and p. 113). The Hamiltonian $H$ with spectral boundary conditions determines a splitting of the Hilbert space into invariant subspaces $\mathcal{H}_b$ and $\mathcal{H}_e$ corresponding to bulk and edge states. The gap of $H$ restricted to bulk states $\mathcal{H}_b$ is larger or equal to $\hbar\omega$. In contrast, the gap of $H$ on the whole space $L^2([a,b] \times [0, 2\pi R])$ is bounded by

$$\min\big\{e_0(kl^2/R) - \hbar\omega/2 \mid k \in \mathbb{Z},\ kl^2/R \notin [a,b]\big\},$$

where $e_0(\rho)$ is a continuous function with minimum $\hbar\omega/2$. In the limit $R \to \infty$, the magnetic length $l$ and the interval $[a,b]$ being fixed, the gap vanishes. Thus in the limit of large samples, the Hamiltonian has gapless excitations, associated to edge states.

## 1.6 Plasma analogies

Laughlin interpreted the modulus squared of his wave function as the Boltzmann weight of a two-dimensional one-component plasma. This analogy can be made in different geometries and provides useful intuition when it comes to the translational symmetry breaking. In this section, we shall describe the relevant plasma models, and give the Laughlin - plasma analogy in the various geometries used. In order to compare free energies, special care must be devoted to additive constants in the definition of energy functions.

We use the following "Coulomb potentials":

$$\begin{aligned} V_D(z) &= -\log|z|/R_0 && (z \in \mathbb{C}\backslash\{0\}),\\ V_C(z) &= -\log|\frac{2R}{R_0}\sinh\frac{z}{2R}| && (z \in \mathbb{C}\backslash\{2\pi Rki \mid k\in\mathbb{Z}\}),\\ V_T(z) &= -\log|\frac{2R}{R_0}\frac{\theta_1(\frac{iz}{2R}|\frac{iL}{2\pi R})}{\theta_1'(0\mid\frac{iL}{2\pi R})}| + \frac{x^2}{2RL}, && (z \in \mathbb{C}\backslash(L\mathbb{Z}+2\pi Ri\mathbb{Z})). \end{aligned}$$

The index $D, C, T$ refers to the geometry (disk, cylinder, torus). The cylinder has radius $R$, the torus is $\mathbb{R}/L\mathbb{Z}\times\mathbb{R}/2\pi R\mathbb{Z}$, and $R_0$ is a reference length scale. The notation $\theta_1'(z|\tau)$ refers to the derivative of $\theta_1$ with respect to the complex variable $z$. The potentials fulfill suitable Poisson equations, the cylinder and torus potentials have appropriate periodicity, and all potentials behave as $V_D$ for small $|z|$:

**Properties:** *The functions $V_D, V_C, V_T$ define tempered distributions $\phi\in\mathcal{S}(\mathbb{R}^2)\mapsto\int_{\mathbb{R}^2}V\phi$. Let $\delta_a:\phi\mapsto\phi(a)$. The following equations hold in the sense of distributions:*

$$-\Delta V_D = 2\pi\delta_0, \qquad -\Delta V_C = 2\pi\sum_{k\in\mathbb{Z}}\delta_{(0,2\pi Rk)} \qquad -\Delta V_T = 2\pi\big(\sum_{m,k\in\mathbb{Z}}\delta_{(mL,k2\pi R)} - \frac{1}{2\pi RL}\big).$$

*Furthermore, for $k, m\in\mathbb{Z}$, we have*

$$V_C(z+2\pi Ri) = V_C(z), \quad V_T(z+mL+k2\pi Ri) = V_T(z).$$

*and $V_{C,T} = V_D(z) + o(1)$ as $z\to 0$.*

Thus $V_D$ is a two-dimensional Coulomb potential. $V_D$ is the 2D potential adapted to a system confined to a strip $[0,L]\times[0,2\pi R]$ with periodic boundary conditions in the $y$-direction (i.e., a cylinder). Similarly, $V_T$ is a torus Coulomb potential, i.e., adapted to the rectangle $[0,L]\times[0,2\pi R]$ with periodic boundary conditions. $V_T$ is the potential created by a charge (the $\delta_0$-part in the Poisson equation), its images (the other Dirac distributions), and their neutralizing background (the constant term).

Now consider a system of $N$ particles of charge $q$ and positions $z_j = x_j + iy_j$ moving in a neutralizing background of charge density $-nq$ on a disk, a cylinder or a torus. For $R, L > 0$, let

$$D := \{z\in\mathbb{C}\mid |z|<R\}, \quad C := [-L/2, L/2]\times[0,2\pi R] =: T.$$

Now we can define the potential energy $U_\alpha, \alpha\in\{D,C,T\}$ through

$$U_\alpha(z_1,..,z_N) := \sum_{1\le j<k\le N} q^2V_\alpha(z_j-z_k) - nq^2\sum_{j=1}^N\int_\alpha V_\alpha(z_j-z)dz + \frac{1}{2}(nq)^2\int_{\alpha\times\alpha}V_\alpha(z-z')dzdz' \quad (1.30)$$

where $n = N/\pi R^2$ if $\alpha = D$ (disk) and $n = N/(2\pi RL)$ if $\alpha\in\{C,T\}$ (cylinder or torus). The

expression of the Boltzmann weights are:

$$e^{-\beta U_D} = \prod_{j<k} |\frac{z_j - z_k}{R_0}|^\Gamma e^{-\Gamma \frac{\pi n}{2} \sum_{j=1}^N r_j^2} e^{\Gamma \frac{N^2}{2}(\frac{3}{4} - \log|\frac{R}{R_0}|)} \tag{1.31}$$

$$e^{-\beta U_C} = \prod_{j<k} |2\sinh \frac{z_j - z_k}{2R}|^\Gamma e^{-\Gamma \pi n \sum_{j=1}^N x_j^2} e^{-\frac{\lambda^2}{12} N^3}, \text{ if } R = R_0 \tag{1.32}$$

$$e^{-\beta U_T} = \prod_{j<k} |\frac{2R}{R_0} \frac{\theta_1(\frac{iz}{2R}|\frac{iL}{2\pi R})}{\theta_1'(0 \mid \frac{iL}{2\pi R})}|^\Gamma e^{-\Gamma \pi n \sum_{j<k}(x_j - x_k)^2/N} e^{-N^3\lambda^2/12} e^{-\frac{N^2\Gamma}{2} \log \frac{R}{R_0}} \prod_{n=1}^\infty (1-u^{2n})^2. \tag{1.33}$$

where $\Gamma = \beta q^2$, $\lambda^2 = \Gamma/(4\pi n R^2)$ and $u := e^{-2\pi R^2/L}$. Details on the computation can be found in [FGIL94] for the disk energy $U_D$, [ŠWK04] for the cylinder and [For06] for the torus energy.

*Remark:* The reader may feel that we have done some overcounting in the definition of the torus energy, since the interaction $V_T$ already includes a background contribution. There is however, a standard approach to the definition of energy functions for long range interactions in periodic geometries (see [BST66] for the Coulomb interaction). In the following paragraph, we briefly sketch this approach, which allows not only to check that our definition of the torus energy $U_T$ coincides with the more standard definition, but also to give a meaning to a "periodic $1/r$ interaction" (used for quantum Hall samples on tori in [YHL83]).
To simplify matters, we replace first the logarithmic interaction with an interaction $V \in \mathcal{S}(\mathbb{R}^2)$. The potential created by a uniform background, charges located in $z_1, .., z_N \in [0, L_1] \times [0, L_2]$ and their images is

$$W(z; z_1, .., z_N) = \sum_{j=1}^N \sum_{m \in \Lambda} V(z - z_j - m) - \frac{N}{L_1 L_2} \int_{\mathbb{R}^2} V(z - z')dzdz'$$

where $\Lambda = L_1\mathbb{Z} \times L_2\mathbb{Z}$. Let $\mu := \sum_{j=1}^N \delta_{z_j} - \frac{N}{L_1 L_2}$ be the charge density in the unit cell $[0, L_1] \times [0, L_2]$. Then

$$U(z_1, .., z_N) := \frac{1}{2}(\int_{[0,L_1]\times[0,L_2]} W d\mu - NV(0))$$

represents the energy of the unit cell minus the self-interaction of the particles. The energy is in general reexpressed as [BST66]

$$U(z_1, .., z_N) = \sum_{j<m} \psi(z_j - z_m) + \frac{N}{2}(\psi(0) - V(0)) \tag{1.34}$$

where $\psi(r) = \frac{2\pi}{L_1 L_2} \sum_{k \in \Lambda' \setminus \{(0,0)\}} \hat{V}(k) e^{ik\cdot r}$, and $\Lambda'$ denotes the lattice dual to $\Lambda$. But $U$ can also be expressed by the formula (1.30) with integrations carried out on $[0, L_1] \times [0, L_2]$ and interaction potential

$$\tilde{V}(z) = \sum_{m \in \Lambda} V(z - m) - \sum_{m \in \Lambda \setminus \{0\}} V(m) = \psi(z) - \psi(0) + V(0)$$

so that $\tilde{V}(0) = V(0)$. In view of $V_T(z) = V_D(z) + o(1)$ as $|z| \to 0$, this is the analogue of the formula we used in order to define $U_T$. In contrast, (1.34) gives the more familiar expression

$$U(r_1, .., r_N) = \sum_{j<m} \sum_{k \in \mathbb{Z}^2 \setminus \{(0,0)\}} \frac{1}{k^2} e^{ik\cdot(r_j - r_m)} + \text{const.}$$

A similar expression is used in [YHL83] to define the energy function used for numerical computations for quantum Hall systems on a torus, except that $1/k^2$ is replaced with const$/|k|$, the two-dimensional Fourier transform of $1/|r|$, and the resulting Hamiltonian is projected to the lowest Landau level.

The plasma analogy relates the Boltzmann weights to the modulus squared of the Laughlin type wave functions:

**Proposition 1.8.** *Let $\Psi_N^D$, $\Psi_N^C$, $\Psi_N^D$ be the Laughlin wave functions at filling factor $1/p$ defined on p.9, and $e^{-\beta U_D}, e^{-\beta U_C}, e^{-\beta U_T}$ the Boltzmann weigths as in (1.31), (1.32) and (1.33). Suppose that $\Gamma = 2p$ and $n = (p \cdot 2\pi l^2)^{-1}$. Then*

$$|\Psi_N^D|^2 = e^{-\beta U_D} e^{-\Gamma \frac{N^2}{2}(\frac{3}{4} - \log|\frac{R}{R_0}|)} \tag{1.35}$$

$$|\Psi_N^C(z_1, .., z_N)|^2 = \frac{1}{N!} \frac{1}{(2\pi R l \sqrt{\pi})^N} e^{\frac{N\lambda^2}{12}} e^{-\beta U_C}(\{z_j - p(N-1)l^2/(2R)\}) \tag{1.36}$$

$$|\Psi_N^T(z_1, .., z_N)|^2 \propto |F^{cm}(\sum_{j=1}^N z_j)|^2 e^{\frac{1}{Nl^2}(\sum_{j=1}^N x_j)^2} e^{-\beta U_T}(z_1, .., z_N). \tag{1.37}$$

*where the multiplicative constant in (1.37) can be determined from (1.33).*

*Sketch of proof.* The first identity (1.35) is obvious. (1.36) uses the identities $e^s - e^t = 2 \exp \frac{s+t}{2} \sinh \frac{s-t}{2}$ and

$$\sum_{1 \le j < k \le N} (x_j + x_k) = \frac{N-1}{2} \sum_{j=1}^N x_j.$$

(1.37) is shown using

$$\sum_{1 \le j < k \le N} (x_j - x_k)^2 = N \sum_{j=1}^N x_j^2 - (\sum_{j=1}^N x_j)^2. \qquad \square$$

The plasma analogy was used by Laughlin [Lau81] in the disk geometry to transfer knowledge on plasma systems to the electron gas described by Laughlin's wave function. The existence of the thermodynamic limit of the free energy of the two-dimensional jellium system has been proved in various geometries [SM76]. There is a wealth of numerical results on two-dimensional jellium systems (see the references in [FGIL94]). They suggest that for coupling constants below $\Gamma \sim 140$, the plasma is translationally invariant, whereas it becomes crystalline above.

Semi-periodic strips should be seen as quasi one-dimensional systems rather than two-dimensional. The formation of Wigner crystals in one-dimensional jellium has been proved [Kun74, BL75]. Motivated by this and the exact results on plasmas at coupling $\Gamma = 2$ (corresponding to the filled Landau level), [ŠWK04] investigated numerically symmetry breaking on jellium strips.

It is interesting to observe that the very analogy invoked by Laughlin to show that his function describes a homogeneous gas suggests a non-trivial periodicity of the density on cylinders.

# Chapter 2

# Thermodynamic limits

The filled Landau level on a cylinder corresponds to an electron gas whose density is not uniform, but has a periodicity in the direction along the cylinder axis, as we have seen in Section 1.3. The main result of this thesis is that for Laughlin cylinder functions at lower filling factor $1/p$, a similar statement can be proved provided the cylinder radius is sufficiently small. The period of the one-particle density is $p$ times the period of the filled Landau level. Furthermore, for thin cylinders the thermodynamic limit of all correlation functions exists. The limiting state is mixing with respect to a suitable translation group. The proof of these results is the object of Section 2.1. It makes crucial use of the representation of Laughlin's wave function as a polymer system and a relationship to discrete renewal processes. The fundamental result is a statement on the asymptotics of normalization constants (Theorem 2.12) which allows to infer that on thin cylinders, the associated renewal process has finite mean interarrival time.

Interestingly, the relationship with polymer systems subsists when we consider modified Laughlin type wave functions. In Section 2.2, we define modified cylinder and torus wave functions and show that, for a certain range of parameters, they are associated with monomer-dimer systems, define solvable models, and monomer-dimer torus and cylinder wave functions are equivalent in a suitable limit.

In Section 2.3, we look at the problem from the point of view of classical jellium systems. We compare the jellium tube to one- and two-dimensional systems, and derive an inequality from Newton's electrostatic theorem. This inequality is related to a recurrence relation fulfilled by the normalization constant of Laughlin's wave function. The semiperiodic Coulomb interaction is translationally invariant with respect to translations along the cylinder axis by any real value. Therefore in the jellium tube picture, our results imply symmetry breaking for *all* values of the radius. On thin strips, we know that the period is $pl^2/R$, whereas on large strips, the period might be $l^2/R$, but in any case the one-particle density cannot be constant.

In Section 2.4, we examine infinite volume ground states, among which we find the limits of Laughlin's wave function, of truncated interactions. The main interest lies in the relationship between the existence of gaps and symmetry breaking.

## 2.1 Laughlin's cylinder wave function

In this section, we prove the main result of this thesis, namely that, at least for sufficiently thin cylinders, the thermodynamic limit of Laughlin's state at filling factor $1/p$ exists and defines a state with a translational period that is $p$ times the period of the filled Landau level. The key ingredient is the representation of Laughlin's cylinder function as a polymer system. The norm squared $C_N$ of Laughlin's function is the polymer partition function of a system of hard rods on the one-dimensional lattice $\mathbb{Z}$ with translationally invariant activity. It satisfies a recurrence relation known in stochastics as a renewal equation. When the associated renewal process has finite mean, the thermodynamic limit of the polymer system exists; Laughlin's function essentially inherits this feature from the polymer system, as well as clustering properties. The translational invariance of the limiting polymer state translates into periodicity of Laughlin's state. For sufficiently thin cylinders, the associated renewal process has indeed finite mean, and $pl^2/R$ is the smallest period of the thermodynamic limit of Laughlin's state.

This section is organized as follows:

- In the first subsection, we give some basic properties of Laughlin's wave function, and in particular its expansion into Slater determinants of lowest Landau level basis functions. The main result is that the expansion coefficients satisfy a product rule and that Laughlin's wave function can be represented as a polymer system. The product rule has been shown to hold for expansion coefficients of powers of Vandermonde determinants [FGIL94]. The novelty here is the observation that on cylinders, the product rule holds also for the expansion coefficients of Laughlin's function.

- In Subsection 2.1.2, we review well-known results on polymer systems and renewal processes that will be used later on. The main message is that systems of hard rods on a lattice with stable, translationally invariant activity are associated with discrete renewal processes. When the associated renewal process has finite mean, the system of rods has a nicely behaved thermodynamic limit, while processes with infinite mean give rise to complications.

- As a consequence, in order to make the polymer representation of Laughlin's wave function useful, we have to answer two questions: Is the activity of the associated system stable? Does the associated renewal process have a finite mean? These questions are answered in Subsection 2.1.3 by looking at the asymptotics of the $L^2$ norm squared of Laughlin's wave function. The answer to the first question is always positive. For the second question, we have a positive answer only for thin cylinders; the proof uses the polymer analogy and combines the well-known results presented in Subsection 2.1.2 with the fact that in the limit of large radii, all polymers except monomers have vanishing activity. What happens for large radii is open.

- In Subsection 2.1.4, we prove that under the assumption of a finite mean, Laughlin's state has a well-defined thermodynamic limit. The limiting state is periodic in the direction along the cylinder axis, and $pl^2/R$ is a period; furthermore, the limiting state is mixing with respect to shifts by multiples of $pl^2/R$. The results on one-dimensional systems of hard rods presented in Subsection 2.1.2 serve as a guiding intuition, but our results are quantum mechanical and thus go beyond Subsection 2.1.2.

- In the last subsection, we prove that on thin cylinders, the limiting state of Subsection 2.1.4 has $pl^2/R$ as its smallest period. Since Laughlin's state can be characterized as

the ground state of a Hamiltonian with period $l^2/R$ (see Section 2.4), this result shows that there is symmetry breaking.

### 2.1.1 Basic properties

Laughlin's wave function has a few number of basic, but important properties that we present in this subsection. The simplest are two symmetry properties (Lemma 2.1). The second observation is that the one-particle density is a sum of Gaussians centered on the lattice $l\gamma\mathbb{Z}$, and therefore, in the limit of infinite cylinders, the one-particle density cannot be constant: at best, it can have a period that is a multiple of $l^2/R = \gamma l$ (Lemma 2.2). Laughlin's wave function is a sum of products of lowest Landau level basis functions (see (2.1) below). The most important result of this subsection is that the expansion coefficients satisfy a product rule (Lemma 2.4) that allows the representation of Laughlin's function as a polymer system (Lemma 2.6). Later, we will give an estimate of the activities of the associated polymer system based on an auxiliary result proved in this subsection (Lemma 2.5).

The starting point is the intimate relation between Laughlin's wave function and powers of a Vandermonde determinant. Recall from the proof of Lemma 1.2 that

$$\Psi_N^C(z_1,..,z_N) = \frac{1}{\sqrt{N!}} \sum_{0\leq m_1,..,m_N\leq p(N-1)} a_N(m_1,..,m_N)\psi_{m_1}\otimes..\otimes\psi_{m_N} \tag{2.1}$$

$$a_N(m_1,..,m_N) := b_N(m_1,..,m_N)e^{\gamma^2\sum_{j=1}^N (m_j^2-p^2(j-1)^2)/2} \tag{2.2}$$

where $\psi_m(z) \propto e^{imy/R}e^{-(x-m\gamma l)^2/2l^2}$ are the normalized cylinder lowest Landau level basis functions and the coefficients $b_N$ are defined through the expansion of the $p$-th power of the Vandermonde determinant:

$$V_N(z_1,..,z_N)^p = \sum_{m_1,..,m_N} b_N(m_1,..,m_N)z_1^{m_1}..z_N^{m_N}.$$

Remember that we give empty products the value 1, so that

$$V_1(z)^p = \prod_{1\leq j<k\leq 1}(z_k-z_j)^p = 1 = b_1(0)$$

and

$$\Psi_1^C(z) = \frac{1}{\sqrt{2\pi Rl\sqrt{\pi}}}e^{-x^2/2l^2} = a_1(0)\psi_0(z), \quad a_1(0)=1. \tag{2.3}$$

Let us explain here how the geometry affects the expansion. As observed by Dunne [Dun93], the expansion coefficients of Laughlin's function as a sum of *normalized* lowest Landau level basis function is made up of *two* parts: in (2.2), we see that $a_N(m_1,..,m_N)$ is the product of $b_N(m_1,..,m_N)$, coming from the expansion of the polynomial $V_N^p$, and an exponential factor that accounts for the fact that $z\mapsto e^{z/R}e^{-x^2/2l^2}$ is not normalized. A similar expansion holds for the disk function. It suffices to replace the cylinder basis functions $\psi_k$ in (2.1) with the disk basis functions $\Omega_k(z) = z^k e^{-|z|^2/4l^2}(2\pi l^2 k!(2l^2)^k)^{-1/2}$ and to set

$$a_N^D(m_1,..,m_N) := b_N(m_1,..,m_N)(2\pi l^2)^{N/2}\Big(\prod_{j=1}^N m_j!(2l^2)^{m_j}\Big)^{1/2}. \tag{2.4}$$

A comparison of this formula with (2.2) shows how the geometry enters the scene. The difference in the contributions from normalization constants is responsible for the fact that

cylinder expansions coefficients satisfy the product rule of Lemma 2.4 while disk expansion coefficients do not.

The relationship of Laughlin's function to powers of a Vandermonde determinant has been exploited in [Dun93, FGIL94] and motivated combinatorial investigations [STW94, KTW01]. We use here two properties of the expansion coefficients $b_N(\{m_j\})$ proved in [Dun93, FGIL94]: the reversal invariance (2.10) and the product rule and admissibility conditions (Lemma 2.4 for the $b_N$'s instead of $a_N$'s).

Let us mention that although, to our knowledge, there is no closed expression giving $b_N(\{m_j\})$ as a function of the $m_j$'s, a number of formulas is available for specific $m_j$'s: for example,

$$b_N(0, p, .., pN-p) = 1,$$
$$b_N(1, p, .., p(N-2), pN-p-1) = (-1)^{N-1} p(p-1)^{N-2}, \tag{2.5}$$
$$b_N(s(N-1), s(N-1)+1, .., (s+1)(N-1)) = (-1)^{\frac{sN(N-1)}{2}} \frac{\big((s+1)N\big)!}{N!(s+1)!^N}, \quad p = 2s+1, \tag{2.6}$$

see [Dun93], Section 6 for the $p = 3$ case of the formulas and [FGIL94], Section 5, for a proof of the third identity. The equality (2.5) is proved in a way similar to (2.75) in the proof of Lemma 2.23 below. Similar formulas are verified for $N \leq 8$, $p$ odd, in [RH94]. Additional formulas can be found also in [KTW01].

Laughlin's cylinder function is an eigenvector of total $y$-momentum and of a rotation by 180°, which is defined as follows: let $(s_0\psi)(z) := \psi(-z)$ and for $a \in \mathbb{R}$,

$$(s_a\psi)(z) := (t(a\mathbf{e}_x)s_0 t(-a\mathbf{e}_x)\psi)(z) = e^{i2ay/l^2}\psi(2a - z).$$

The rotation $s_a$ preserves the boundary condition $\psi(z + i2\pi R) = \psi(z)$ if and only if $a \in \frac{\gamma l}{2}\mathbb{Z}$. Moreover,

$$\forall r \in \frac{1}{2}\mathbb{Z},\ \forall k \in \mathbb{Z}: \quad s_{r\gamma l}\psi_k = \psi_{2r-k}. \tag{2.7}$$

**Lemma 2.1.** *Laughlin's cylinder function is an eigenfunction of the total y-momentum and of an overall* 180° *rotation around the middle of the cylinder:*

$$\sum_{j=1}^{N} c_{x,j}\Psi_N^C = -il^2 \sum_{j=1}^{N} \frac{\partial}{\partial y_j}\Psi_N^C = \frac{pN(N-1)}{2}\gamma l \Psi_N^C \tag{2.8}$$
$$s_{p(N-1)\gamma l/2}^{\otimes N}\Psi_N^C = \Psi_N^C. \tag{2.9}$$

*Proof.* The translational invariance in the $y$-direction (2.8) follows from the fact that $V_N^p$ is a homogeneous polynomial of total degree $p(0 + 1 + .. + N - 1) = pN(N-1)/2$ so that $a_N(m_1, .., m_N)$ vanishes unless $m_1 + .. + m_N = pN(N-1)/2$. In [Dun93] (Section 6) and [FGIL94] (Section 4, Property 0) it is shown that

$$b_N(p(N-1) - m_N, ..., p(N-1) - m_1) = b_N(m_1, .., m_N). \tag{2.10}$$

This follows from

$$\prod_{j=1}^{N-1} z_j^{p(N-1)} V_N(\frac{1}{z_1}, .., \frac{1}{z_N})^p = (-1)^{pN(N-1)/2} V_N(z_1, .., z_N)^p$$

and the antisymmetry / symmetry $b_N(m_{\pi(1)},..,m_{\pi(N)}) = (\mathrm{sgn}(\pi))^p b_N(m_1,..,m_N)$; the permutation $1 \mapsto N, 2 \mapsto N-1$, etc., has the sign $(-1)^{\lfloor N/2 \rfloor} = (-1)^{N(N-1)/2}$ where $\lfloor N/2 \rfloor$ is the integer part of $N/2$. Combining (2.10) and (2.2), we obtain

$$a_N(p(N-1)-m_N,...,p(N-1)-m_1) = a_N(m_1,..,m_N). \tag{2.11}$$

The rotational invariance (2.9) then follows from (2.7) and (2.1). □

*Remark.* Following [Dun93], we will also say *reversal invariance* instead of rotational invariance.

The symmetries have simple consequences for the reduced $n$-point matrices which we now introduce. It is convenient to use the language of second quantization. We work out the details only for odd $p$. Most of the statements however, hold for even $p$ also; the most important technical difference comes from the unboundedness of bosonic creation and annihilation operators [BR79b].

Let $\mathcal{F} := \wedge\mathcal{H}$, $\mathcal{H} = L^2(\mathbb{R} \times [0, 2\pi R])$ be the fermionic Fock space over the infinite cylinder and $\mathcal{A}$ the anticommutation algebra over $\mathcal{H}$, i.e., the closed sub-algebra of $\mathcal{B}(\mathcal{F})$ generated by $\mathbf{1}_{\mathcal{B}(\mathcal{F})}$ and the creation and annihilation operators $c^*(f), c(f)$, $f \in \mathcal{H}$. Laughlin's wave function defines a state on $\mathcal{A}$ through

$$\langle a \rangle_{\Psi_N^C} = \langle \Psi_N^C, a\Psi_N^C \rangle / ||\Psi_N^C||^2.$$

If $\langle\cdot\rangle$ is a state on $\mathcal{A}$, the reduced $n$-point matrix ([BR79b], Section 6.3.3), if it exists, is the function $\rho^{(n)} : (\mathbb{R} \times [0, 2\pi R])^n \times (\mathbb{R} \times [0, 2\pi R])^n \to \mathbb{R}$ such that for all $f_1,..,f_n, g_1,..,g_n \in \mathcal{H}$,

$$\begin{aligned}&\langle c^*(f_1)..c^*(f_n)c(g_n)..c(g_1)\rangle \\ &\qquad = \int_{((\mathbb{R}\times[0,2\pi R])^{2n}} \overline{g_1(z_1')..g_n(z_n')}\rho^{(n)}(z_1',..,z_n';z_1,..,z_n)f_1(z_1)..f_n(z_n)dz_1'..dz_n.\end{aligned}$$

Put differently, it is the integral kernel of the operator $\rho^{(n)}$ in $\otimes^n\mathcal{H}$ with expectation values

$$\langle g_1 \otimes .. \otimes g_n, \rho^{(n)} f_1 \otimes .. \otimes f_n \rangle = \langle c^*(f_1)..c^*(f_n)c(g_n)..c(g_1)\rangle.$$

Using $(c(f)\Psi_N)(z_2,..,z_N) = \sqrt{N}\int \overline{f(z_1)}\Psi_N(z_1,..,z_N)dz_1$, one sees that the reduced density matrices of $\langle\cdot\rangle_{\Psi_N^C}$ are given by integrals of the type (1.9). Thus the definitions given in the section on the filled Landau level are consistent with the definitions given here. Let $c_k = c(\psi_k)$. The $r$-point reduced density matrix of $\Psi_N^C$ can also be expressed as

$$\begin{aligned}\rho_N^{(r)}(z_1',..,z_r';z_1,..,z_r) = r! \sum_{\substack{0\le n_1<..<n_r\le pN-p \\ 0\le m_1<..<m_r\le pN-p \\ n_1+..+n_r=m_1+..+m_r}} &\langle c_{m_1}^*..c_{m_r}^* c_{n_r}..c_{n_1}\rangle_{\Psi_N^C} \\ &\psi_{m_1}\wedge..\wedge\psi_{m_r}(z_1',..,z_r')\overline{\psi_{n_1}\wedge..\wedge\psi_{n_r}(z_1,..,z_r)}.\end{aligned}$$

The restriction to $m_1 + .. + m_r = n_1 + .. + n_r$ comes from

$$\rho_N^{(r)}(z_1' - ia, ..; .., z_r' - ia) = \rho_N^{(r)}(z_1, ..; ..z_r),$$

which is a consequence of the translational invariance (2.8). In particular, the one-particle density

$$\rho_N(z) := \rho_N^{(1)}(z;z) = \sum_{k=0}^{pN-p} \langle c_k^* c_k\rangle_{\Psi_N^C} |\psi_k(z)|^2 = \frac{1}{2\pi Rl\sqrt{\pi}} \sum_{k=0}^{pN-p} \langle c_k^* c_k\rangle_{\Psi_N^C} e^{-(x-k\gamma l)^2}$$

is a sum of Gaussians, independent of the coordinate $y$ around the cylinder. This will subsist in the thermodynamic limit. Thus instead of looking at the one-particle density, we can look at the sequence of occupation numbers $\langle c_k^* c_k \rangle$. The following lemma ensures a one-to-one relation between the periodicity of the one-particle density and of the sequence of occupation numbers.

**Lemma 2.2.** *Let $(n_k)_{k\in\mathbb{Z}}$ be a sequence of numbers in $[0,1]$ that is not identically zero and $\rho(x) := \sum_{k=-\infty}^{\infty} n_k |\psi_k(x)|^2$. Then any period of $\rho(\cdot)$ is a multiple of $l^2/R = \gamma l$. Moreover, for $p \in \mathbb{N}$, $\rho$ is periodic with period $p\gamma l$ if and only if $(n_k)$ is $p$-periodic. In this case the Fourier coefficients of $\rho$ can be expressed as*

$$\frac{1}{p\gamma l}\int_0^{p\gamma l} e^{-ikx}\rho(x)dx = \frac{1}{p}\frac{1}{2\pi l^2} e^{-\frac{\pi^2 k^2}{p^2\gamma^2}} \sum_{j=0}^{p-1} n_j e^{-i2\pi jk/p}. \tag{2.12}$$

*Proof.* $\rho = f * \mu$ is the convolution of a function $f$ and a measure $\mu$

$$f(x) := \frac{1}{2\pi R l\sqrt{\pi}} e^{-x^2/l^2}, \quad \mu := \sum_{k\in\mathbb{Z}} n_k \delta_{k\gamma l}.$$

Taking Fourier transforms, we get

$$\hat{\rho} = \sqrt{2\pi}\hat{f}\cdot\hat{\mu} = \frac{1}{2\pi R} e^{-(lx)^2/4}\hat{\mu} \tag{2.13}$$

which should be read in the distributional sense, i.e., $\forall \phi \in \mathcal{S}(\mathbb{R}) : \int_{-\infty}^{\infty} \rho\hat{\phi} = \sqrt{2\pi}\int_{-\infty}^{\infty} \widehat{\hat{f}\phi} d\mu$. Suppose $\mu$ and $\nu$ are tempered measures such that $f*\mu = f*\nu$. Then $\hat{f}\cdot\hat{\mu} = \hat{f}\cdot\hat{\nu}$. Since $\hat{f}$ is a continuous, strictly positive function this implies $\hat{\mu} = \hat{\nu}$ and the injectivity of the Fourier transform gives $\mu = \nu$. For $h \in \mathbb{R}$, let $\tau_h\mu$ be the shifted measure $\tau_h\mu(B) = \mu(B+h)$. Suppose $\rho = f*\mu$ is periodic with period $h$, then

$$(f*\mu)(x) = \rho(x) = \rho(x-h) = \int_{-\infty}^{\infty} f(x-h-y)d\mu(y) = (f*\tau_h)\mu(x)$$

and thus $\tau_{-h}\mu = \mu$. Thus the support of $\mu$ is invariant by a shift by $h$. Since the support is non-empty by the assumption $\sum_k n_k > 0$, and obviously $\mathrm{supp}\mu \subset \gamma l\mathbb{Z}$, $h$ has to be a multiple of $\gamma l$. Now let $p \in \mathbb{N}_0$. If $\rho$ is $p\gamma l$ periodic, so is $\mu$, whence $n_{k+p} = n_k$. Conversely, if $n_{k+p} = n_k$, $\rho$ is obviously $p\gamma l$-periodic. Let

$$a_k(\rho) := \frac{1}{p\gamma l}\int_0^{p\gamma l} e^{-i2\pi kx/(p\gamma l)}\rho(x)dx,$$
$$a_k(\mu) = \frac{1}{p\gamma l}\int_{[0,p\gamma l[} e^{-i2\pi kx/(p\gamma l)}d\mu = \frac{1}{p\gamma l}\sum_{j=0}^{p-1} n_j e^{-i2\pi kj/p}.$$

Then

$$\hat{\rho} = \sum_{k\in\mathbb{Z}} a_k(\rho)\sqrt{2\pi}\delta_{2\pi k/(p\gamma l)}, \quad \hat{\mu} = \sum_{k\in\mathbb{Z}} a_k(\mu)\sqrt{2\pi}\delta_{2\pi k/(p\gamma l)}.$$

(See [Sch73], p. 224–229 for an account on periodic distributions and their Fourier series.) Together with (2.13) we obtain (2.12). □

*Remarks. 1.* The previous lemma shows in particular that $\rho$ cannot be constant, since constant functions admit periods that are not multiples of $\gamma l$.

*2.* The mapping $(n_k) \mapsto \rho$ is actually injective.

*3.* If we set $p = 1$, we recover the result of Poisson's summation formula for the filled Landau level (see Chapter 1).

*4.* There is an analogue of (2.12) for finitely many particles:

$$\rho(x) = \sum_{k\in\mathbb{Z}} n_k |\psi_k(x)|^2, \quad \sum_{k\in\mathbb{Z}} n_k = N$$
$$\Rightarrow \frac{1}{Np\gamma l}\int_{-\infty}^{\infty} e^{-2\pi ikx/(p\gamma l)}\rho(x)dx = \frac{1}{p}\frac{1}{2\pi l^2}e^{-\frac{\pi^2 k^2}{p^2\gamma^2}}\frac{1}{N}\sum_{j\in\mathbb{Z}} n_j e^{-i2\pi jk/p}.$$

*5.* For even $p$, Laughlin's wave function describes bosons and the occupation numbers can be greater than 1. We can relax the condition $n_k \in [0,1]$ to $n_k \geq 0$ and the requirement that $(n_k)$ is polynomially bounded in $-\infty$ and $+\infty$ (i.e., it defines a tempered measure).

Di Francesco et al. proved a necessary condition for the non-vanishing of $b_N(m_1,..,m_N)$ ([FGIL94], Property 3) and a product rule for the expansion coefficients $b_N$ (Property 5). These properties also hold for the expansion coefficients $a_N$. Before we state them, we give some auxiliary definitions.

**Definition 2.3.** *Let $N \in \mathbb{N}$, $m = (m_1,..,m_N) \in \mathbb{Z}^N$. We say*

- *$m$ is* $N$-admissible *if $0 \leq m_1,..,m_N \leq p(N-1)$ and*

$$\forall k \in \{1,..,N\} : \sum_{j=1}^{k} m_j \geq \sum_{j=1}^{k} p(j-1)$$

  *with equality for $k = N$: $\sum_{j=1}^{N} m_j = pN(N-1)/2$.*

- *$k$ is a* renewal point *of $m$ if $k \in \{0,..,N\}$ and $\sum_{j=1}^{k}(m_j - p(j-1)) = 0$. $R(m)$ denotes the set of renewal points of $m$. If $m$ is $N$-admissible, $R(m)$ contains $0$ and $N$.*

- *$m$ is* reducible *if it has a renewal point in $\{1,....N-1\}$, and* irreducible *in the opposite case.*

The notion of admissibility is related to the majorization partial order ([HLP64], p.45): for $\alpha, \alpha' \in \mathbb{R}_+^N$, one writes $\alpha \prec \alpha'$ if the finite sequences $(\alpha_j)$ and $(\alpha'_j)$, rearranged in decreasing order ($\alpha_{\sigma(1)} \geq ..\alpha_{\sigma(N)}$, $\alpha'_{\tau(1)} \geq ..\alpha'_{\tau(N)}$), satisfy $\sum_{j=1}^{k}\alpha'_{\tau(j)} \leq \sum_{j=1}^{k}\alpha_{\sigma(j)}$ for all $k \in \{1,..,N\}$ with equality for $k = N$. Then

$$(m_{\pi(1)},..,m_{\pi(N)}) \text{ is N-admissible for all } \pi \in \mathcal{S}_N$$
$$\Leftrightarrow (m_1,..,m_N) \prec (0,p,2p,..,(N-1)p). \quad (2.14)$$

Furthermore, by a classical theorem ([HLP64], p.49) a sequence is majorated by the reference sequence $(0,..,(N-1)p)$ if and only if there is a doubly stochastic matrix $P$ such that

$$(m_1,..,m_N)^T = P(0,p,2p,..,(N-1)p)^T. \quad (2.15)$$

Renewal points of $m$ correspond to block-diagonal stochastic matrices: if $P$ is block-diagonal with an upper left $k \times k$ block, $k$ is a renewal point of $m$. By the Birkhoff-von Neumann theorem, the set of doubly stochastic matrices is the convex hull of permutation matrices. We will see that if $a_N(m_1, .., m_N) \neq 0$, $m$ is majorated by $(0, ..., p(N-1))$ and the stochastic matrix in (2.15) can be chosen as a simple average of $p$ permutation matrices: $P = \frac{1}{p}(P_{\pi_1} + .. + P_{\pi_p})$.

**Lemma 2.4.** *Let $(m_1, .., m_N) \in \mathbb{Z}^N$.*

1. *If $a_N(m_1, .., m_N) \neq 0$, then $(m_1, .., m_N)$ is $N$-admissible.*
2. *Suppose $m$ is reducible with a renewal point $k \in \{1, .., N-1\}$. Then*

$$a_N(m_1, .., m_N) = a_k(m_1, .., m_k) a_{N-k}(m_{k+1} - pk, .., m_N - pk). \tag{2.16}$$

*Moreover, if $m_1 \leq .. \leq m_N$, then $(m_1, .., m_N)$ is $N$-admissible if and only if $(m_1, .., m_k)$ is $k$-admissible and $(m_{k+1} - pk, .., m_N - pk)$ is $N-k$-admissible.*

Note that the product (2.16) is 0 if one of the sequences does not have the correct admissibility. Before we turn to the proof of the lemma, let us stress that the product rule (2.16) really uses the cylinder geometry: remember that the Laughlin function expansion coefficients are made up of two contributions. The first, $b_N(m_1, .., m_N)$, coming from the power of the Vandermonde polynomial, always satisfies the factorization rule [FGIL94], while the normalization contribution (see (2.2) and (2.4)) does so only in the cylinder geometry.

*Proof.* The lemma is based on similar results holding for the coefficients $b_N(m_1, .., m_N)$, proved in [FGIL94]. We give slightly different proofs. The proof of the equivalence of admissibilities follows [KTW01]. For simplicity suppose $p = 3$. The reasoning for other values of $p$ is strictly analogous.

*1.* Suppose $a_N(m_1, .., m_N) \neq 0$. Then by (2.2), we must have $b_N(m_1, .., m_N) \neq 0$. Writing

$$V_N(z_1, .., z_N)^p = (\sum_{\pi \in \mathcal{S}_N} \mathrm{sgn}\pi z_1^{\pi(1)-1} ... z_N^{\pi(N)-N})^p \tag{2.17}$$

$$= \sum_{\pi, \sigma, \tau \in \mathcal{S}_N} \mathrm{sgn}\pi\sigma\tau \prod_{j=1}^{N} z_j^{\pi(j)+\sigma(j)+\tau(j)-3} \tag{2.18}$$

we obtain

$$b_N(m_1, ..., m_N) = \sum_{\substack{\pi, \sigma, \tau \in \mathcal{S}_N : \\ \forall j:\ m_j = \pi(j)+\sigma(j)+\tau(j)-3}} \mathrm{sgn}\pi\sigma\tau. \tag{2.19}$$

Thus $b_N(m_1, .., m_N) \neq 0$ implies that there exist permutations $\pi, \sigma, \tau \in \mathcal{S}_N$ such that $m_j = \pi(j) + \sigma(j) + \tau(j) - 3$ for all $j \in \{1, .., N\}$. Since $\pi$ is in $\mathcal{S}_N$, for all $k \in \{1, .., N\}$,

$$\sum_{j=1}^{k} \pi(j) \geq \sum_{j=1}^{k} j \tag{2.20}$$

with equality if $k = N$. Similar inequalities hold for $\sigma$ and $\tau$. Adding them up, we obtain

$$\sum_{j=1}^{k} m_j = \sum_{j=1}^{k} (\pi(j) + \sigma(j) + \tau(j) - 3) \geq \sum_{j=1}^{k} (3j - 3), \tag{2.21}$$

with equality for $k = N$, whence *1.*

*2.* Now suppose $(m_1, .., m_N) \in \mathbb{Z}^N$ is such that $m_1 + .. + m_k = 3k(k-1)/2$ for some $k$ in $\{1, .., N-1\}$. Let $\pi, \sigma, \tau \in \mathcal{S}_N$ be such that $m_j = \pi(j) + \sigma(j) + \tau(j) - 3$ for all $j \in \{1, .., N\}$. Then there is equality in (2.21) and in (2.20) for $\pi, \sigma, \tau$. This implies that $\pi, \sigma$ and $\tau$ leave $\{1, .., k\}$ and $\{k+1, .., N\}$ invariant, Let $\pi' \in \mathcal{S}_k$ be the restriction of $\pi$ to $\{1, .., k\}$ and $\pi'' \in \mathcal{S}_{N-k}$ such that $\pi(k+j) = \pi''(j) + k$ for all $j \in \{1, .., N-k\}$. Define $\sigma', \tau', \sigma'', \tau''$ in a similar way. Then $m_{k+j} - pk = \pi''(j) + \sigma''(j) + \tau''(j)$ for all $j \in \{1, .., N-k\}$. Therefore (2.19) becomes

$$\begin{aligned} & b_N(m_1, .., m_N) \\ & \quad = \Big( \sum_{\substack{\pi', \sigma', \tau' \in \mathcal{S}_k : \\ \forall j:\ m_j = \pi'(j) + \sigma'(j) + \tau'(j) - 3}} \mathrm{sgn}\pi'\sigma'\tau' \Big) \Big( \sum_{\substack{\pi'', \sigma'', \tau'' \in \mathcal{S}_{N-k} : \\ \forall j:\ m_{k+j} - 3k = \pi''(j) + \sigma''(j) + \tau''(j) - 3}} \mathrm{sgn}\pi''\sigma''\tau'' \Big) \\ & \quad = b_k(m_1, .., m_k) b_{N-k}(m_{k+1} - 3k, .., m_N - 3k). \end{aligned}$$

Since

$$\begin{aligned} & \sum_{j=k+1}^{N} ((m_j - 3k)^2 - 3^2(j-k-1)^2) \\ & \quad = \sum_{j=k+1}^{N} (m_j^2 - 3^2(j-1)^2) - 6k \sum_{j=k+1}^{N} (m_j - 3(j-1)) = \sum_{j=k+1}^{N} (m_j^2 - 3^2(j-1)^2), \end{aligned}$$

the exponential factors satisfy a product rule too, whence

$$a_N(m_1, .., m_N) = a_k(m_1, .., m_k) a_{N-k}(m_{k+1} - 3k, .., m_N - 3k).$$

Finally, let $(m_1, .., m_N)$ be $N$-admissible and increasing $0 \le m_1, .., m_N \le pN - p$. We obtain immediately the $k$-admissibility of $(m_1, ..., m_k)$. It remains to see that $(m_{k+1} - pk, .., m_N - pk)$ is $N-k$-admissible. For $r \ge 1$, write

$$\begin{aligned} \sum_{j=1}^{r} (m_{k+j} - pk - p(j-1)) &= \sum_{j=k+1}^{k+r} (m_j - p(j-1)) \\ &= \sum_{j=1}^{k+r} (m_j - p(j-1)) \ge 0 \end{aligned}$$

with equality for $r = N$. If in addition $m_1 \le .. \le m_N$, the previous inequalities imply $0 \le m_{k+1} - pk \le .. \le m_N - pk \le p(N-k-1)$, so that $(m_{k+1} - pk, .., m_N - pk)$ is $N-k$-admissible. Conversely, if $(m_1, .., m_k)$ is $k$-admissible, $(m_{k+1} - pk, .., m_N - pk)$ is $N-k$-admissible, one can check that $(m_1, ., m_N)$ is $N$-admissible. □

Depending on the parity of $p$, the coefficients $a_N(m_1, .., m_N)$ are either symmetric or anti-symmetric functions of the $m_j$'s. Thus if $a_N(m_1, .., m_N) \neq 0$, any permutation of the $m_j$'s is $N$-admissible, i.e., $(m_1, .., m_N)$ is majorized by the reference vector $(0, p, .., p(N-1))$ and can be written as a doubly stochastic matrix times this vector. In the proof of the previous lemma, for $p = 3$, the only sequences that occur are of the form

$$m_j = \pi(j) + \sigma(j) + \tau(j) - 3, \quad j = 1, .., N.$$

Put differently,

$$(m_1, .., m_N)^T = \frac{1}{3}(P_\pi + P_\sigma + P_\tau)(0, 3, .., 3(N-1))^T.$$

Thus we can choose the stochastic matrix as a simple average of permutation matrices. This holds for all $p \in \mathbb{N}$ with the obvious modifications.

If $a_N(m_1, .., m_N) \neq 0$, the $N$-tupel $(m_j)$ is majorized by $(p(j-1))$. This means that the $m_j$'s are less spread out than $0, p, .., p(N-1)$. The following lemma quantifies this "less" for irreducible sequences by comparing their variances. This amounts to a comparison of sums of squares, since

$$\frac{1}{N^2}\sum_{j<k}(m_j - m_k)^2 = \frac{1}{N}\sum_{j=1}^N m_j^2 - (\frac{1}{N}\sum_{j=1}^N m_j)^2 = \frac{1}{N}\sum_{j=1}^N m_j^2 - (\frac{1}{N}\sum_{j=1}^N p(j-1))^2.$$

**Lemma 2.5.** *Let $m \in \mathbb{Z}^N$ be $N$-admissible, irreducible and increasing: $m_1 < .. < m_N$. Then*

$$\sum_{j=1}^N \big(m_j^2 - p^2(j-1)^2\big) \leq -(p+1)(N-1). \tag{2.22}$$

*If $m_1 \leq .. \leq m_N$, the upper bound is $-p(N-1)$.*

*Proof.* Let $0 < m_1 < .. < m_N \leq p(N-1)$ be $N$-admissible. Then

$$\forall k \in \{0, .., N\} : \; n_k := \sum_{j=1}^N \big(m_j - p(j-1)\big) \geq 0,$$

$n_0 = n_N = 0$, and $m_j = p(j-1) + n_j - n_{j-1}, \quad j = 1, .., N$. The monotonicity of $(m_j)$ translates into

$$n_{j+1} - 2n_j + n_{j-1} + p \geq 1, \quad j = 1, .., N. \tag{2.23}$$

A summation by parts gives

$$\sum_{j=1}^N (m_j^2 - p^2(j-1)^2) = -\sum_{j=1}^N n_j(p + n_{j+1} - 2n_j + n_{j-1}) \leq -(p+1)\sum_{j=1}^{N-1} n_j$$

The irreducibility of $(m_1, .., m_N)$ implies $n_j \geq 1$ for $j \in 1, .., N-1$, whence (2.22). If $m_1 \leq .. \leq m_N$, the lower bound in (2.23) is 0, and the summation by parts gives the upper bound $-p(N-1)$. □

The previous lemma will be important later on. It allows to prove that long polymers in an associated polymer system have small activity.

The factorization rule (2.16) leads to a representation of Laughlin's wave function in terms of "building blocks" that is reminiscent of polymer systems (see next section). Let $\Gamma := \{\{j, .., j+n-1\} \mid j \in \mathbb{Z}, n \in \mathbb{N}\}$. Elements of $\Gamma$ will be referred to as *rods* or *polymers*. For $X, Y \in \Gamma$, we write $X < Y$ if $x < y$ for all $x \in X, y \in Y$. For $N \in \mathbb{N}$, let $\mathcal{P}_N$ be the set of ordered partitions $(X_1, .., X_D)$ of $\{0, .., N-1\}$ into rods: $X_1 < .. < X_D$, $\{0, ..., N-1\} = X_1 \dot\cup .. \dot\cup X_D$. For $j \in \mathbb{Z}$, $N \in \mathbb{N}$ and odd $p$, define:

$$u_{\{j,..j+N-1\}} = \sum_{\substack{0 \leq m_1 \leq .. \leq m_N \leq pN-p \\ \text{irreducible}}} a_N(m_1, .., m_N)\psi_{m_1+pj} \wedge .. \wedge \psi_{m_N+pj}.$$

(The definition for even $p$ is similar: $\otimes_s$ instead of $\wedge$.) In view of (2.3), the monomer functions are simply

$$u_{\{j\}} = \psi_{pj}. \tag{2.24}$$

Thus for $X \in \Gamma$ with $N(X) \geq 2$,, $u_X$ is an antisymmetric function of $N(X) = |X|$ complex variables made up of Gaussians $\psi_k$ with $k \in pX$. Moreover, the magnetic translations act as

$$t(jp\gamma l\mathbf{e}_x)^{\otimes N(X)} u_X = u_{j+X}, \quad j \in \mathbb{Z}.$$

**Lemma 2.6.** *Let $p \in \mathbb{N}$ odd, $N \in \mathbb{N}$ and $\mathcal{P}_N$ the set of increasingly ordered partitions of $\{0, .., N-1\}$ as defined above. Let $I \subset \mathbb{R}$ and $\Omega = I \times [0, 2\pi R]$. Then*

$$\Psi_N = \sum_D \sum_{(X_1,..,X_D)\in\mathcal{P}_N} u_{X_1} \wedge .. \wedge u_{X_D} \tag{2.25}$$

$$||\Psi_N^C||^2_{L^2(\Omega^N)} = \sum_D \sum_{(X_1,..,X_D)\in\mathcal{P}_N} \Phi(X_1) \cdot .. \cdot \Phi(X_D) \tag{2.26}$$

*where $\Phi(X) := ||u_X||^2_{L^2(\Omega^{N(X)})}$.*

If $p$ is even, a formula similar to (2.25) holds with symmetric products $\otimes_s$ instead of wedge products; (2.26) holds unchanged.

*Proof.* We start with

$$\Psi_N^C = \sum_{0\leq m_1<..<m_N\leq p(N-1)} a_N(m_1, .., m_N)\psi_{m_1} \wedge .. \wedge \psi_{m_N}. \tag{2.27}$$

Only admissible sequences contribute to the sum. Let $m$ be an admissible sequence with renewal points $R(m) = \{r_0, .., r_k\}$, $0 = r_0 < r_1 < .. < r_k = N$. Then

$$a_N(m_1, .., m_N)\psi_{m_1} \wedge .. \wedge \psi_{m_N} = \left(a_{r_1}(m_1, ..., m_r)\psi_{m_1} \wedge .. \wedge \psi_{m_r}\right) \\ \wedge ... \wedge \left(a_{r_k-r_{k-1}}(m_{r_k+1} - pr_k, ..., m_r - pr_k)\psi_{m_{r_{k-1}+1}} \wedge .. \wedge \psi_{m_N}\right),$$

and the subsequences $(m_1, ..., m_{r_1})$,..,$(m_{r_k+1} - pr_k, ..., m_r - pr_k)$ are irreducible. When $R = \{r_0, .., r_k\}$ is kept fixed and we sum over sequences $m$ with renewal points $R(m) = R$, we obtain

$$u_{r_1} \wedge t(r_1p\gamma l)^{\otimes(r_2-r_1)}u_{r_2-r_1} \wedge .. \wedge t(r_{k-1}p\gamma l)^{\otimes(N-r_{k-1})}u_{N-r_{k-1}}. \tag{2.28}$$

Each strictly increasing sequence $0 = r_0 < r_1 .. < r_k = N$ defines a partition of $\{0, .., N-1\}$: the points are interpreted as starting points of rods, we get the partition $X_1 = \{0, .., r_1 - 1\}$, $X_2 = \{r_2, .., r_3 - 1\}$,.., $X_k = \{r_{k-1}, .., N-1\}$. The function (2.28) is nothing else but $u_{X_1} \wedge .. \wedge u_{X_k}$. Thus if in (2.27) we sum first over sequences $m$ with a given set of renewal points and then over all possible sets, identified with the corresponding partition, we obtain (2.25). For the normalization constant, note that

$$\begin{aligned} ||\Psi_N^C||^2_{L^2(\Omega^N)} &= \sum_{0\leq m_1<..<m_N\leq p(N-1)} |a_N(m_1, .., m_N)|^2 \prod_{j=1}^N ||\psi_{m_j}||^2_{L^2(\Omega)} \\ ||u_{\{j,..,j+N-1\}}||^2_{L^2(\Omega^N)} &= \sum_{\substack{0\leq m_1<..<m_N\leq p(N-1)\\ \text{irred.}}} |a_N(m_1, .., m_N)|^2 \prod_{j=1}^N ||\psi_{m_j}||^2_{L^2(\Omega)}. \end{aligned} \tag{2.29}$$

This follows from the orthogonality of the functions with respect to $y$-integration:

$$\{m_1,..,m_N\} \neq \{n_1,..,n_N\} \Rightarrow$$
$$\int_{[0,2\pi R]^N} \big(\overline{(\psi_{m_1} \wedge .. \wedge \psi_{m_N})}\psi_{n_1} \wedge .. \wedge \psi_{n_N}\big)(x_1+iy_1,..,x_N+iy_N)dy_1..dy_N = 0.$$

Summing in two steps, as for the proof of (2.25), gives (2.26). □

*Example.* It is instructive to work out by hand the polymer representation for two particles and $p = 3$. We have

$$\begin{aligned}
\Psi_2^C(z_1,z_2) &= \frac{1}{\sqrt{2}}(e^{z_2/R} - e^{z_1/R})^3 e^{-(x_1^2+x_2^2)/2l^2} \frac{e^{-\frac{1}{2}\gamma^2 3^2 \sum_{j=1}^2 (j-1)^2}}{2\pi R l\sqrt{\pi}} \\
&= \frac{1}{\sqrt{2}}\big((e^{3z_2/R} - e^{3z_1/R}) - 3(e^{2z_2/R}e^{z_1/R} - e^{z_2/R}e^{2z_1/R})\big)\frac{e^{-9\gamma^2/2}}{2\pi R l\sqrt{\pi}} \\
&= e^{\frac{1}{2}\gamma^2(0^2+3^2-9)}\psi_0 \wedge \psi_3 - 3e^{\frac{1}{2}\gamma^2(1^2+2^2-9)}\psi_1 \wedge \psi_2 \\
&= \psi_0 \wedge \psi_3 - 3e^{-2\gamma^2}\psi_1 \wedge \psi_2.
\end{aligned}$$

This can be rewritten as

$$\Psi_2^C = u_{\{0\}} \wedge u_{\{1\}} + u_{\{0,1\}}, \quad u_{\{0\}} = \psi_0,\ u_{\{1\}} = \psi_3,\ u_{\{0,1\}} = -3e^{-2\gamma^2}\psi_1 \wedge \psi_2.$$

The previous lemma gives the normalization constant as a polymer partition function. It turns out that the correlation functions are closely related to those of the associated polymer system, which we describe in the next section. Moreover, there is an analogy to the *dimer model* introduced by Rokhsar and Kivelson [RK88] as an idealization of resonating valence bond states (RVB). Consider a lattice of spin 1/2-particles. To each dimer covering of the lattice, we can associate a state where lattice sites in a dimer form a singlet. The sum of states corresponding to different dimer coverings is an RVB state. Different dimer coverings do not give rise to orthogonal functions. Nonetheless, [RK88] propose a model of hard core dimers on a lattice, where wave functions corresponding to different dimer coverings are *defined* as orthogonal. The representation obtained in the previous lemma is similar: Laughlin's wave function is a sum of contributions from different polymer coverings, and the contributions from different coverings are orthogonal. Thus we may call the Laughlin function a *quantum polymer system.*

### 2.1.2 Associated polymer system

By the results of the previous subsection, the $L^2$ norm squared $C_N$ of Laughlin's wave function is a polymer partition function. More precisely, it is the grand-canonical partition function for a system of hard rods on the one-dimensional lattice $\mathbb{Z}$. The normalization $C_N$ satisfies a recurrence relation known in stochastics as a *renewal equation.* In this subsection, we give a short overview of results on polymer systems, discrete renewal processes and thermodynamic limits of systems of hard rods on $\mathbb{Z}$. This will allow us to infer results on the asymptotics of $(C_N)$ and will serve as a useful guiding intuition for the thermodynamic limits of correlation functions in Laughlin's state.

We begin with a brief overview on systems of polymers on a lattice. We proceed with a short summary on discrete renewal processes, and then present results on the thermodynamic limits

of systems of hard rods. Let us summarize the main aspect of the hard rod system. The recurrence relation satisfied by the partition function is equivalent to the formal power series identity

$$1 + \sum_{n=1}^{\infty} C_n t^n = \frac{1}{1 - \sum_{n=1}^{\infty} \alpha_n t^n}.$$

The number $\alpha_n \geq 0$ is the activity of a rod of lenght $n$. We assume that the power series have a non-vanishing radius of convergence (this amounts to a stability assumption on the activity $(\alpha_n)$). Then one of the following conditions must hold:

1. There exists some $r > 0$ such that $\sum_n \alpha_n r^n = 1$ and $\sum_n n\alpha_n r^n < \infty$. (The renewal process with interarrival distribution $(r^n \alpha_n)$ is positive recurrent.)
2. There exists some $r > 0$ such that $\sum_n \alpha_n r^n = 1$, but $\sum_n n\alpha_n r^n = \infty$. (The renewal process with interarrival distribution $(r^n \alpha_n)$ is null recurrent.)
3. The radius of convergence $r$ of $\sum_n \alpha_n t^n$ satisfies $\sum_n r^n \alpha_n < 1$. (The renewal process with interarrival distribution $(r^n \alpha_n)$ is transient.)

In the first case, the polymer system has a well-behaved thermodynamic limit (see Proposition 2.9 below). In the second and third cases, one needs in general additional knowledge on the coefficients $(\alpha_n)$. Typically, in those cases for large volumes, the polymer system will tend to fill the whole volume with a few number of long rods ([Gia07]). For Laughlin's cylinder function, we will see that we are actually in the first case for sufficiently thin cylinders. What happens on thick cylinders is open.

The summary on lattice polymer systems follows [GK71]. An account on discrete renewal processes can be found e.g. [Fel62] in the sections on recurrent events. The recurrence relation for the one-dimensional system of hard rods is used in [IVZ06]. For monomer-dimer systems, it boils down to the two-step recurrence relation of [HL72], Section III. Generating functions and renewal theory have been applied to a large variety of biochemical models and the models are still an active field of research in probability theory [Lif64, BD77, Fis84, CGZ06, Gia07]. These are frequently termed polymer models. Let us stress however that there is a slight difference in the use of the word "polymer": in this thesis, we use "polymer" essentially as a synonym for "hard rod" and look at partitions of intervals into rods. Renewal points correspond to starting points of rods. In the biochemical models evoked above, one typically looks at the configuration of *one* single long polymer (a molecular chain, a double string of DNA..), possibly in more than one dimension. Renewal points are points where "something happens", e.g., the polymer chain hits a wall.

In a lot of models, the coefficients $(\alpha_n)$ are functions of some parameter. When the parameter is varied, it is possible to switch between one of the three cases above. This gives rise to a phase transition; depending on the model described, one finds among others, the denominations localization-delocalization transition, wetting transition, denaturation transitions. The intermediate case 2. corresponds to a critical regime.

**Lattice polymer systems**

We start with a brief overview following [GK71]. A *polymer* is a set $X \subset \mathbb{Z}^d$. The cardinality of $X$ is denoted $N(X)$ and may be viewed as a number of sites in $X$. *Monomers* are polymers with $N(X) = 1$, *dimers* correspond to $N(X) = 2$, $n$-mers to $N(X) = n$ ($n \in \mathbb{N}$). The *activity* is a map $\Phi : \Gamma \to [0, \infty[$. Allowed configurations of the system are subsets $\{X_1, .., X_D\}$ of $\Gamma$

that define a partition of $\Lambda$: $\Lambda = X_1 \dot\cup \cdot .. \dot\cup X_D$. The activity $\Phi$ defines a probability measure $P$ through

$$P_\Lambda(\{\{X_1, .., X_D\}\}; \Phi) = \Phi(X_1) \cdot .. \cdot \Phi(X_D)/Q_\Lambda[\Phi]$$

with the *polymer partition function*

$$Q_\Lambda[\Phi] := \sum_{\{X_1,..,X_D\}}{}' \Phi(X_1) \cdot .. \cdot \Phi(X_D). \tag{2.30}$$

The sum ranges over all partitions of $\Lambda$ into polymers. If the activity of monomers is $\Phi(\{j\}) = 1$, the monomers can be interpreted as empty sites and we can rewrite $Q_\Lambda$ as

$$Q_\Lambda[\Phi] = \sum_{\substack{X_1,..,X_D \subset \Lambda: \\ N(X_j) \geq 2}} \Phi(X_1) \cdot .. \Phi(X_D) e^{-\beta U(X_1,..,X_D)}$$

where $U(X_1, .., X_d)$ is 0 when $X_i \cap X_j =$ for all $i \neq j$, and $\infty$ else. Thus $Q_\Lambda$ is the grand-canonical partition function for a hard core interaction. The *pressure* and *correlation functions* are defined through

$$\begin{aligned} \beta p_\Lambda &:= \frac{1}{|\Lambda|} \log Q_\Lambda, \\ \rho_\Lambda(X_1, .., X_p) &:= \sum_{\substack{\{Y_1,..,Y_n\}: \\ X_1 \dot\cup .. X_D \dot\cup Y_1 \dot\cup .. Y_n = \Lambda}} \Phi(X_1) .. \Phi(X_p) \Phi(Y_1) \cdot .. \cdot \Phi(Y_n)/Q_\Lambda. \end{aligned}$$

The correlation functions give the probability of finding the polymers $X_1, .., X_p$. The sum rule

$$\forall x \in \Lambda : \sum_{X \ni x} \rho_\Lambda(X) = 1 \tag{2.31}$$

expresses the fact that every site of $\mathbb{Z}$ is covered by exactly one polymer. One can show that

$$Q_{\Lambda \dot\cup \Lambda'} \geq Q_\Lambda Q_{\Lambda'}. \tag{2.32}$$

For $\xi > 0$, $\xi^N \Phi$ denotes the rescaled activity

$$(\xi^N \Phi)(X) := \xi^{N(X)} \Phi(X).$$

The probability measure $P_\Lambda$ and the correlation functions $\rho_\Lambda$ are invariant with respect to such a rescaling, whereas

$$Q_\Lambda[\xi^N \Phi] = \xi^{|\Lambda|} Q_\Lambda[\Phi].$$

The activity is called *stable* if

$$\sum_{X \ni 0} \frac{\Phi(X)}{N(X)} < \infty$$

and *translationally invariant* if $\Phi(a + X) = \Phi(X)$ for all $a \in \mathbb{Z}^d$.

We will be interested in one-dimensional lattice systems where $\Phi(X) = 0$ if $X$ is not of the form $\{j, ..j + N(X) - 1\}$ for some $j \in \mathbb{Z}$. Such systems are closely related to discrete renewal processes, which we now briefly describe. Details can be found in the book by W. Feller [Fel62] in the sections on recurrent events.

**Discrete renewal processes**

A point process on $\mathbb{N}$ is a (discrete) renewal process if the waiting times between consecutive events are independent identically distributed random variables. More precisely, let $(\Omega, \mathcal{F}, P)$ be a probability space, $E := \mathcal{P}(\mathbb{N})$, and $\mathcal{E}_0$ the standard $\sigma$-algebra on $E$ [1]. Let $R : (\Omega, \mathcal{F}) \to (E, \mathcal{E}_0)$ be a random variable. We call $R$ a *renewal process* with *interarrival distribution* $(p_n)_{n\in\mathbb{N}\cup\{\infty\}}$ if there is a family $(T_n)_{n\in\mathbb{N}}$ of $\mathbb{N}\cup\{\infty\}$-valued, independent identically distributed random variables on $\Omega$ such that $P(T_j = n) = p_n$ for all $n \in \mathbb{N}\cup\{\infty\}$ and

$$R(\omega) = \{S_n(\omega) \mid n \in \mathbb{N},\ S_n(\omega) < \infty\}$$

for all $\omega \in \Omega$, where

$$S_n(\omega) = \sum_{j=1}^{n} T_j(\omega).$$

The elements of $R(\omega)$ are called *renewal points*, and the *renewal probability* is

$$u_n := P(\text{“}n \text{ is a renewal point”}) = P(n \in R) = P(\exists k \in \mathbb{N}_0 :\ S_k = n).$$

If $T_j$ takes the value $\infty$ with a non-zero probability, the process is called *transient*.

*Example: Bernoulli process.* Let $0 \le p, q \le 1$ with $p+q = 1$, and let $(X_n)_{n\in\mathbb{N}}$ be independent random variables such that $P(X_n = 0) = q$, $P(X_n = 1) = p$. Define $R$ as

$$R(\omega) := \{n \in \mathbb{N} \mid X_n(\omega) = 1\}.$$

Then $R$ is a renewal process with geometric interarrival distribution and constant renewal probability:

$$\forall n \in \mathbb{N} : \quad p_n = pq^{n-1}, \quad u_n = p = \frac{1}{\sum_k k p_k}.$$

More generally, the *renewal theorem* says that if $(p_n)$ is aperiodic (i.e. the greatest common divisor of $\{n \in \mathbb{N} \mid p_n > 0\}$ is 1)

$$\lim_{n\to\infty} u_n = 1/\mu, \quad \mu := \sum_{n\in\mathbb{N}\cup\{\infty\}} n p_n \tag{2.33}$$

with the convention $1/\infty = 0$ and $\infty \cdot 0 = 0$. The proof makes heavy use of the *renewal equation*

$$\forall n \ge 1 : \quad u_n = p_1 u_{n-1} + p_2 u_{n-2} + .. + p_{n-1} u_1 + b_n, \quad b_n := P(S_1 = n). \tag{2.34}$$

Here, $b_n = p_n$, but the equation holds also for renewal processes where $T_1 = S_1$ is allowed to have a distribution different from $(p_j)$. These are called *delayed* renewal processes. This is best seen by writing

$$\begin{aligned} u_n = P(\exists k : S_k = n) &= P(S_1 = n) + \sum_{k=2}^{n}\sum_{j=1}^{n-1} P(S_k = n \text{ and } T_k = j) \\ &= P(T_1 = n) + \sum_{j=1}^{n-1}\sum_{k=2}^{n} P(S_{k-1} = n-j) P(T_k = j) \\ &= b_n + \sum_{j=1}^{n-1} u_{n-j} p_j. \end{aligned}$$

[1] Subsets $A \subset \mathbb{N}$ are identified with their characteristic functions $\chi_A \in \{0,1\}^{\mathbb{N}}$, and on $\{0,1\}^{\mathbb{N}}$ we consider the $\sigma$-algebra generated by the cylinder sets.

This recurrence relation is equivalent to the formal power series identity

$$\sum_{n=1}^{\infty} u_n t^n = \frac{\sum_{n=1}^{\infty} b_n t^n}{1 - \sum_{n=1}^{\infty} p_n t^n}$$

or, if $b_n = p_n$ and with $u_0 := 1$

$$\sum_{n=0}^{\infty} u_n t^n = \frac{1}{1 - \sum_{n=1}^{\infty} p_n t^n}.$$

For a given interarrival distribution $(p_n)_{n\in\mathbb{N}\cup\{\infty\}}$, the problem "find a probability distribution $(b_n)_{n\in\mathbb{N}}$ on $\mathbb{N}$ so that the associated delayed renewal process has a constant renewal probability" has a solution if and only if $\mu = \sum_{n\in\mathbb{N}\cup\{\infty\}} np_n < \infty$. In this case the unique solution is $b_n = \sum_{k=n}^{\infty} p_k/\mu$ and the constant renewal probability is $1/\mu$.

If $\mu < \infty$, the delayed process with constant renewal probability can be extended to a stationary process on $\mathbb{Z}$ as follows: Let $S_0 : \Omega \to \{0, -1, -2, ..\}$, $S_1 : \Omega \to \mathbb{N}$ be two random variables with joint distribution

$$m \leq 0,\ 1 \leq n :\ P(S_0 = m \text{ and } S_1 = n) = p_{n-m}/\mu.$$

$S_0$ and $S_1$ represent the renewal points that are closest to the origin 0. Note that

$$P(S_1 = n) = \sum_{k=n}^{\infty} p_k/\mu = b_n, \quad P(S_1 - S_0 = n) = np_n/\mu.$$

Let $(T_n)_{n\in\mathbb{Z}\setminus\{1\}}$ be a family of independent identically distributed random variables with values in $\mathbb{N}$, $P(T_k = n) = p_n$. For $n \leq -1$, let $S_n := S_0 - \sum_{k=1}^{|n|} T_{-k}$, and for $n \geq 2$, $S_n := S_1 + \sum_{k=2}^{n} T_k$. We will call

$$R : \Omega \to \mathcal{P}(\mathbb{Z}), \quad \omega \mapsto \{S_n(\omega) \mid n \in \mathbb{Z}\}$$

a stationary renewal process on $\mathbb{Z}$. The renewal probability $u_n = P(n \in R(\omega))$ is constant: for all $n \in \mathbb{Z}$, $u_n = 1/\mu$. Moreover, the sequence $(S_n)_{n\geq 1}$ defines the delayed renewal process with constant renewal probability described above. If we look at $(S_n)_{n\geq 1}$ and condition on $S_0 = 0$, we recover the (non-delayed) renewal process on $\mathbb{N}$.

A subset of $\mathbb{Z}$ can also be described by the sequence of its points, arranged in increasing order and suitably numbered. Thus the renewal process $R$ on $\mathbb{Z}$ is described by the sequence $(S_n)_{n\in\mathbb{Z}}$, the numbering is chosen so that $S_0 \leq 0 < S_1$. Equivalently, we may describe the process through its waiting intervals

$$X_n = \{S_n, S_n + 1, ..., S_{n+1} - 1\}, \quad n \in \mathbb{Z}.$$

These define a partition of $\mathbb{Z}$. If we interpret the waiting intervals as polymers, the distribution of $R$ is a probability distribution on infinite polymer configurations; it turns out that this distribution arises as the thermodynamic limit of suitable finite volume polymer distributions $P_\Lambda$.

**Rods on a one-dimensional lattice**

Let us go back to the one-dimensional lattice polymer system. Suppose the activity is translationally invariant and supported by "rods":

$$\Phi(X) = \begin{cases} \alpha_{N(X)}, & \text{if } X = \{j, j+1, .., j+N(X)-1\} \text{ for some } j \in \mathbb{Z}, \\ 0, & \text{else} \end{cases}$$

for some sequence $(\alpha_n)$ of non-negative numbers with $\sum \alpha_n > 0$. For $N \in \mathbb{N}$, let

$$C_N = Q_{\{0,..,N-1\}}[\Phi].$$

Then (2.30) can be rewritten as

$$C_N = \sum_{n_1+..+n_D=N} \alpha_{n_1} \cdot .. \cdot \alpha_{n_D}.$$

Together with $C_0 := 1$, this implies the formal power series identity

$$\sum_{n=0}^{\infty} C_n t^n = (1 - \sum_{n=1}^{\infty} \alpha_n t^n)^{-1} \tag{2.35}$$

and the recurrence relation

$$C_n = \alpha_1 C_{n-1} + \alpha_2 C_{n-2} + .. + \alpha_{n-1} C_1 + \alpha_n \quad (n \geq 1). \tag{2.36}$$

The thermodynamic limit of the pressure exists and can be expressed in terms of $(\alpha_n)$.

**Lemma 2.7.** *Let* $r := \max\{t \geq 0 \mid \sum_{n=1}^{\infty} \alpha_n t^n \leq 1\}$. *Then with the convention* $-\log 0 := \infty$ *we have*

$$\beta p := \lim_{N\to\infty} \frac{1}{N} \log C_N = \sup \frac{1}{N} \log C_N = -\log r.$$

*Proof.* The supermultiplicativity (2.32) together with the translational invariance of the activity implies $C_{N+M} \geq C_N C_M$. This shows that $\lim \frac{1}{N} \log C_N = \sup \frac{1}{N} \log C_N$. The limit is $-\log r_C$, with $r_C$ the radius of convergence $r_C$ of $\sum C_n t^n$. It remains to show that $r = r_C$. Note first that (2.36) implies $C_n \geq \alpha_n$, thus $r_C \leq r_\alpha$, where $r_\alpha$ is the radius of convergence of $\sum \alpha_n t^n$. We continue by looking more closely at $r$. It follows from $\sum_n \alpha_n t^n = \infty$ if $t > r_\alpha$ that $r \leq r_\alpha$. If $r_\alpha = 0$, obviously $r = 0$. If $r_\alpha > 0$, there are two possibilities: either $r < r_\alpha$ and $\alpha(r) = 1$, $r > 0$, or $r = r_\alpha$, in which case Abel's theorem on power series with nonnegative coefficients implies

$$\alpha(r) = \alpha(r_\alpha) = \lim_{t \nearrow r_\alpha} \alpha(t) \leq 1.$$

Several cases can be distinguished:

1. $r_\alpha = r = 0$. Then by $r_C \leq r_\alpha$ we also have $r_C = 0$.
2. $r_\alpha > r > 0$. Then $\alpha(r) = 1$, (2.35) is an identity of functions on $[0, r[$, and $r_C \geq r$. Moreover
$$\sum_{n=0}^{\infty} r^n C_n = \lim_{t \nearrow r} \frac{1}{1 - \alpha(r)} = \infty$$
whence $r_C = r$.

3. $r_\alpha = r > 0$. Again, (2.35) is an identity of functions on $[0, r[$, $r_C \geq r = r_\alpha$. Since $r_C \leq r_\alpha$ because of $C_n \geq \alpha_n$, we have $r_C = r_\alpha = r$. □

*Remarks: 1.* The quantity $r$ can be characterized in a slightly more eloquent way as follows: the equation $\sum_{n=1}^{\infty} \alpha_n t^n = 1$, $t \in [0, \infty[$ has either no solution or a unique solution. In the first case, $r$ equals the radius of convergence $R$ of $\sum_n \alpha_n t^n$. In the second case, $r$ is the unique solution of the equation and lies necessarily in $]0, R]$.

*2. Generalized ensemble.* Lifson [Lif64] points out a relation between the "generating functions approach" and "Guggenheim's divergent generalized partition functions". The generating function $\sum C_n t^n$ can be given the meaning of a partition function. The partition function $C_V$ is the grand-canonical partition function for a system of rods on the lattice $\{0, .., V-1\}$ with a hard core interaction. Let $p > 0$ be a pressure, $\beta$ an inverse temperature. Then the generating function

$$C(e^{-\beta p}) = \sum_{V=0}^{\infty} e^{-\beta p V} C_V$$

can be thought of as a partition function too. It corresponds to the so-called *generalized ensemble*. More generally, let $Q(\beta, \{N_j\}, V)$ be the canonical partition function for a system of particles of different species $k = 1, ., K$, each present with $N_k$ particles, at temperature $\beta^{-1}$, in a volume $V$; then the generalized partition function would be

$$\Gamma(\beta, p_e, \{\mu_j\}) = \sum_{\{N_j\}} \sum_{V=0}^{\infty} e^{-\beta(pV + \sum_j \mu_j N_j)} Q(\beta, \{N_j\}, V) = \sum_{V=0}^{\infty} e^{-\beta p V} \Xi(\beta, V, \{\mu_j\}).$$

This partition functions differs from other partition functions (e.g. canonical, grand-canonical, isothermal-isobaric..): it is a function of intensive parameters only. There is no free extensive parameter. Thus the usual procedure of letting extensive parameters go to infinity for taking thermodynamic limits does not work. In the grand-canonical ensemble, the pressure is obtained through

$$\beta p = \lim_{V \to \infty} \frac{1}{V} \log \Xi(\beta, V, \{\mu_j\})$$

if the limit exists. For this value of $p$, the generalized partition function $\Gamma$ diverges. Thus the procedure that replaces thermodynamic limits in the generalized ensemble is to look for the point where $\Gamma$ diverges, whence the name *divergent partition function*. This is just what we do: in Lemma 2.7, we express the pressure in terms of the radius of convergence of $C(t)$, i.e., we look for the point where $C(t)$ starts to diverge.

The recurrence relation (2.36) is similar to the renewal equation (2.34). Let us make the connection to renewal processes more explicit. In the following, we suppose that the power series $\sum \alpha_n t^n$ has a non-vanishing radius of convergence; equivalently, $r > 0$ or $\beta p < \infty$. This is the case if and only if $\xi^N \Phi$ is stable for some $\xi > 0$, since

$$\sum_{X \ni 0} \frac{\xi^{N(X)} \Phi(X)}{N(X)} = \sum_{N=1}^{\infty} \xi^N \frac{\alpha_N}{N} \sum_{\substack{j \in \mathbb{Z}: \\ 0 \in \{j, .., j+N-1\}}} 1 = \sum_{N=1}^{\infty} \alpha_N \xi^N.$$

Let $r$ be defined as in Lemma 2.7 and $p_n := r^n \alpha_n$, $p_\infty := 1 - \sum p_n$. This defines a probability distribution on $\mathbb{N} \cup \{\infty\}$. Let $R_0$ be a renewal process on $\mathbb{N}$ with interarrival distribution $(p_n)$ and renewal probabilities $(u_n)_{n \in \mathbb{N}}$. Combining $p_n = r^n \alpha_n$, $C_0 = u_0 = 1$ and the recurrence relations (2.34), (2.36), we see that $u_n = r^n C_n$ for all $n \in \mathbb{N}_0$. The renewal theorem (2.33) gives us a more refined version of Lemma 2.7.

**Lemma 2.8.** *Let $r$ be defined as in Lemma 2.7. If $\{n \in \mathbb{N} \mid \alpha_n \neq 0\}$ has greatest common divisor 1,*

$$\lim_{n\to\infty} r^n C_n = \begin{cases} (\sum_{n=1}^{\infty} n\alpha_n r^n)^{-1}, & \text{if } \sum_{n=1}^{\infty} \alpha_n r^n = 1, \\ 0, & \text{if } \sum_{n=1}^{\infty} \alpha_n r^n < 1. \end{cases}$$

*Remark:* If $\sum_n \alpha_n r^n = 1$ and $\sum_n n r^n \alpha_n = \infty$, $r^n C_n \to 0$. If $\sum_n \alpha_n r^n < 1$, (2.35) implies the stronger statement

$$\sum_{n=1}^{\infty} r^n C_n = (1 - \sum_{n=1}^{\infty} r^n \alpha_n)^{-1}.$$

The renewal process associated to the polymer system is not only useful to derive something about the asymptotics of the partition functions, but also serves a better understanding of correlation functions and their thermodynamic limit.

The finite volume correlation functions can be described in terms of conditional probabilities of the renewal process. Let $0 < S_1 < .. < S_n < ..$ be the renewal points, and let

$$\mathcal{I}_0 := \{\{0, .., S_1 - 1\}, \{S_1, .., S_2 - 1\}, .., \{S_{n-1}, .., S_n - 1\}, ..\}$$

be the set of waiting intervals. Then if $X_i \neq X_j$ for $i \neq j$,

$$\rho_{\{0,..,N-1\}}(X_1, .., X_k) = P(X_1, .., X_k \in \mathcal{I}_0 \mid N \in R_0).$$

If the associated renewal process has finite mean, we can define a stationary renewal process $R$ on $\mathbb{Z}$. Let $\mathcal{I}$ be the collection of its waiting intervals. The previous relation becomes

$$\rho_{\{a,..,b-1\}}(X_1, .., X_k) = P(X_1, .., X_k \in \mathcal{I} \mid a, b \in R).$$

The thermodynamic limit $a \to -\infty$, $b \to \infty$ is given by the probability distribution of $R$, without any conditions.

**Proposition 2.9.** *Suppose $\sum_n \alpha_n r^n = 1$, $\mu := \sum_n n\alpha_n r^n < \infty$, and $\gcd\{n \mid \alpha_n \neq 0\} = 1$. Let $R$ be the associated stationary renewal process and $P$ the distribution of $R$. Let $\mathcal{I}$ be the set of waiting intervals ($\mathcal{I}$ is a random variable). Then for all rods $X_1, .., X_k$, the thermodynamic limit of the correlation functions for the polymer system with activities $\Phi(X) = \alpha_{N(X)}$ exists and equals*

$$\rho(X_1, .., X_k) := \lim_{\substack{a\to-\infty \\ b\to\infty}} \rho_{\{a,..,b-1\}}(X_1, .., X_k) = \begin{cases} P(\{X_1, .., X_k\} \subset \mathcal{I}), & \text{if } X_i \neq X_j \text{ for } i \neq j, \\ 0, & \text{else.} \end{cases}$$

*The correlation functions are translationally invariant and mixing: for all sets of rods $X_1, .., X_m$, $Y_1, .., Y_n$, we have*

$$\rho(X_1, .., X_m, Y_1 + k, .., Y_n + k) \underset{k\to\infty}{=} \rho(X_1, .., X_m)\rho(Y_1, .., Y_n) + O(|r^k C_k - \mu^{-1}|). \quad (2.37)$$

*Proof.* Let $X_1, .., X_k$ be some rods. If $X_i \cap X_j \neq \emptyset$ for some $i \neq j$, the probability of finding these rods as well as the probability that they are waiting intervals for the renewal process is 0. Thus we can suppose that they are non-overlapping and without loss of generality assume $X_1 < ... < X_k$, that is they are numbered from left to right. Write $X_k = \{s_k, s_k, .., s_k +$

$N(X_k)-1\} = \{s_k, .., e_k - 1\}$ and take $|a|, b$ large enough so that $X_1, .., X_k$ are contained in $\{a, .., b-1\}$. Then, the correlation function can be written as

$$\begin{aligned}
\rho_{\{a,..,b-1\}}(X_1, .., X_k) &= \sum_{\substack{Y_1,..,Y_D: \\ Y_1 \dot\cup .. \dot\cup X_k = \{a,..,b-1\}}} \alpha_{N(Y_1)} ... \alpha_{N(Y_D)} \alpha_{N(X_1)} ... \alpha_{N(X_k)} / Q_{\{a,..,b-1\}} \\
&= \alpha_{N(X_1)} . \alpha_{N(X_k)} . Q_{\{a,..,b=1\} \setminus (X_1 \cdot \cup .. \cdot \cup X_D\}} / Q_{\{a,..,b-1\}} \\
&= r^{b-a} \alpha_{N(X_1)} . \alpha_{N(X_k)} C_{s_1 - a} C_{s_2 - e_1} .. C_{b-e_k} / (r^{b-a} C_{b-a}) \\
&= u_{s_1 - a} p_{N(X_1)} .. p_{N(X_k)} u_{b-e_k} / u_{b-a} \\
&\underset{b-a \to \infty}{\longrightarrow} \mu^{-1} p_{N(X_1)} u_{s_2 - e_1} ... u_{b-e_k} \mu^{-1} / \mu^{-1} \\
&= \mu^{-1} p_{N(X_1)} u_{s_2 - e_1} ... p_{N(X_k)} \qquad (2.38) \\
&= P(X_1, .., X_k \in \mathcal{I}).
\end{aligned}$$

The expression (2.38) shows that $\rho$ is translationally invariant. For the clustering properties, let $X_1, .., X_m, Y_1, ., Y_n$ and denote as before their starting and endpoints with $s_1, .., s_m, e_1, .., e_m$, $s'_1, .., e'_n$. Then, for sufficiently large $k$,

$$\begin{aligned}
\rho(X_1, .., X_m, Y_1 + k, .., Y_n + k) &= \mu^{-1} p_{N(X_1)} u_{s_2 - e_1} ... p_{N(X_m)} u_{s'_1 + k - e_m} p_{N(X_1)} u_{s'_2 - e'_1} ... p_{N(Y_n)} \\
&= \rho(X_1, .., X_m)\, \mu u_{s'_1 + k - e_m}\, \rho(Y_1, .., Y_n)
\end{aligned}$$

whence (2.37). □

*Remarks: 1. Semi-infinite line.* The same reasoning shows that the limits of correlation functions $\rho_{\{0,..,N-1\}}$, $N \to \infty$, exist and can be described in terms of the renewal process on $\mathbb{N}$.

*2. Probability distribution on the length of rods.* The probability for finding a given rod $X = \{j, .., j + N(X) - 1\}$ is $\rho(X) = q r^{N(X)} \alpha_{N(X)}$. Let $k \in \mathbb{Z}$. The probability that $k$ is in a rod, or waiting interval, $X$ of length $N(X) = n$ is, in loose notation

$$P(k \in X,\ N(X) = n) = \sum_{\substack{X:\ k \in X, \\ N(X) = n}} \rho(X) = \sum_{\substack{j \in \mathbb{Z}: \\ j \le k \le j + n - 1}} q r^n \alpha_n = q n r^n \alpha_n = \mu^{-1} n p_n.$$

The sum over $n$ is 1: this is the sum rule $\sum_{X \ni 0} \rho(X) = 1$. Thus if $\mu = q^{-1}$ is finite, there is a natural probability distribution $(q n r^n \alpha_n)$ on the length of polymers. In the renewal theory picture, this distribution is called the *size-biased* distribution. The expected length of the polymer containing a given point (in the renewal setting, the expected lifetime or age at death) is

$$\sum_{n=1}^{\infty} q n^2 r^n \alpha_n = \sum_{n=1}^{\infty} \mu^{-1} n^2 p_n.$$

The lifetime expectation can be infinite although the process has finite mean $\mu$. This is related to the so-called *inspection paradox*.

*3. Rate of clustering.* In the previous proposition, the rate of clustering is determined by the speed of convergence of $u_n \to \mu^{-1}$, where $u_n = r^n C_n$ is the renewal probability. We have the following:

$$\sum_{n=1}^{\infty} |u_n - \mu^{-1}| < \infty \Leftrightarrow \sum_{n=1}^{\infty} n^2 p_n < \infty.$$

The implication "⇒" can be shown by taking the limit $t \nearrow 1$ in

$$\sum_{n=0}^{\infty}(u_n - \mu^{-1})t^n = \frac{1}{1-\sum_{n=1}^{\infty} p_n t^n} - \frac{1}{\sum_{n=1}^{\infty} n p_n (1-t)} = \frac{1}{1-p(t)} - \frac{1}{p'(1)(1-t)}.$$

The converse uses results on rates of convergence in renewal theory, see [Sto65]. A closer look at the generating functions also shows that if the radius $R$ of convergence of $\sum_n r^n \alpha_n$ is strictly larger than $r$, $u_n$ goes to $\mu^{-1}$ exponentially fast:

$$r^n C_n - q = u_n - \mu^{-1} = O((r/R)^N).$$

**$\mu = \infty$ as a close-packing limit.** The previous proposition is of no help when the renewal process has infinite mean $\mu$. The case $\mu = \infty$ should be thought of as a close-packing limit (in [Gia07], Section 1.2 it is shown that in this regime the average density of renewal points, i.e. the "contact density" in the picture used in the book, is typically 0). We illustrate this by an example. Let $\nu \in ]0,1[$, $c > 0$ and

$$\alpha_n := \frac{\delta}{S}\frac{1}{n^{1+\nu}}, \quad S := \sum_{n=1}^{\infty} n^{-1-\nu}.$$

Then $\alpha(t) = \sum_{n=1}^{\infty} \alpha_n t^n$ has radius of convergence 1. We look at sequences of volumes $\Lambda_N = \{-N/2, .., N/2-1\}$, $N$ even, and ask for the limiting behavior of the polymer correlation function

$$\rho_N(X) = \frac{C_{a+N/2}\alpha_{b-a}C_{N/2-b}}{C_N}, \quad X = \{a, .., b-1\}.$$

Depending on the value of the parameter $c$, we are in one of the three cases sketched in the beginning of this subsection:

1. When $\delta > 1$, the equation $\alpha(t) = 1$ has a solution $r < 1$ and we are in the case where the associated renewal process has finite mean, and the thermodynamic limit is described by Proposition 2.9. In particular, $\lim_{N\to\infty} \rho_N(X)$ exists and is strictly positive.

2. When $\delta = 1$, the solution to the equation $\alpha(t) = 1$ is $r = 1$, and $\sum_n n\alpha_n r^n = \sum_n n^{-\nu}\delta/S = \infty$. Then it is known ([Don97], Theorem B, [Gia07], Theorem A.7) that there exists a $c_\nu > 0$ such that

   $$C_n \underset{n\to\infty}{\sim} \frac{c_\nu}{n^{1-\nu}}.$$

   As a consequence

   $$\frac{C_{a+n/2}C_{n/2-b}}{C_n} \underset{n\to\infty}{\sim} \frac{c_\nu^2}{n^{1-\nu}}\big((\frac{1}{2}+\frac{a}{n})(\frac{1}{2}-\frac{b}{n})\big)^{\nu-1}$$

   and $\lim_{N\to\infty} \rho_N(X) = 0$ for every fixed $X$.

3. When $\delta < 1$, the equation $\alpha(t) = 1$ has no solution. It is known [Gia07], Theorem A.4, that

   $$u_n \underset{n\to\infty}{\sim} \frac{\alpha_n}{(1-\delta)^2} = \frac{\delta}{S(1-\delta)^2}\frac{1}{n^{1+\nu}}$$

   and we have again $\lim_N \rho_N(X) = 0$ for each fixed rod $X$.

Thus for this example, if the associated renewal process has infinite mean, each fixed rod $X$ has a probability that vanishes in the limit $N \to \infty$. Since on the other hand the sum rule

$$\sum_{Y:\ k\in Y} \rho_N(Y) = 1$$

(see (2.31)) must hold for all $N \in 2\mathbb{N}$ and $k \in \{-N/2, .., N/2-1\}$, we conclude that long intervals tend to be filled with long polymers.

Later, we will apply the results of this subsection to Laughlin's wave function. We can think of this function as a polymer system where each polymer has internal degrees of freedom. We prove in the next subsection that on thin cylinders the associated renewal process has finite mean. What happens for large radius is open. *If* it happens that the associated renewal process has infinite mean, the previous considerations tell us that the system is governed by long polymers, and any statement about the thermodynamic limit would require a closer look at the internal structure of long polymers.

### 2.1.3 Normalization

By Lemma 2.6, Laughlin's cylinder wave function is closely related to a system of hard rods on a one-dimensional lattice with activity

$$\Phi(X) = ||u_X||^2_{L^2\left((I\times[0,2\pi R])^{N(X)}\right)}.$$

The polymer representation is valid for any choice of interval $I \subset \mathbb{R}$. Here, we will choose $I = \mathbb{R}$. This choice leads to a translationally invariant activity. Indeed, we have seen that translation of a rod $X$ results in magnetic translation of the function:

$$u_{a+X} = t(a \cdot p\gamma l \mathbf{e}_x)^{\otimes N(X)} u_X.$$

Hence if we choose the infinite cylinder as domain of integration, $I = \mathbb{R}$, we obtain a translationally invariant activity and we can write

$$\begin{aligned} \Phi(X) &= ||u_X||^2_{L^2((\mathbb{R}\times[0,2\pi R])^{N(X)})} = \alpha_{N(X)} \\ \alpha_N &= \sum_{\substack{0\le m_1\le ..\le m_N\le p(N-1)\\ \text{irred.}}} |a_N(m_1, .., m_N)|^2. \end{aligned} \tag{2.39}$$

Let us recall that the one-particle density of Laughlin's $N$-particle function is a sum of Gaussians with centers inside $[0, pN\gamma l]$. Thus the one-particle density decays exponentially outside the finite cylinder $[0, pN\gamma l] \times [0, 2\pi R]$, even though we integrate on infinite cylinders $\mathbb{R} \times [0, 2\pi R]$. By a slight abuse of language, we shall refer to the limit $N \to \infty$ as the limit of infinite cylinders.

In view of the results summarized in the previous section, two questions on the associated polymer system naturally arise. The first question is whether the activity is stable or, more generally, whether it can be rescaled to a stable activity. On p.46, we have observed that the activity $(\alpha_n)$ can be rescaled to a stable activity if and only if the radius of convergence of $\sum_n \alpha_n t^n$ is strictly positive. From Lemma 2.6 we know that the $L^2$ norm squared of Laughlin's function, $C_N = ||\Psi_N^c||^2$ is the polymer partition function associated with the activity $(\alpha_n)$. It follows from the relation (2.35) that $\sum_n \alpha_n t^n$ has a positive radius of convergence if

and only if $\sum_n C_n t^n$ has. Thus the question of the stability of the associated activity boils down to the question whether the radius of convergence $r$ of $\sum_n C_n t^n$ is strictly positive. Lemma 2.10 below shows that $r > 0$ for all (finite) cylinder radii $R$.

The second question is whether the associated renewal process has a finite mean $\mu$. By Lemma 2.8, the normalization constant $(C_n)$ obeys the asymptotics $r^n C_n \to q$ (the aperiodicity assumption of Lemma 2.8) is fulfilled because in the system associated with Laughlin's function, monomers have activity $\alpha_1 = 1$). The limit $q$ is the inverse of the expected interarrival time of the renewal process, $q = \mu^{-1}$. Thus the question whether $\mu < \infty$ reduces to the question whether $\lim_n r^n C_n$ is strictly positive or not. This question is addressed in Theorem 2.12, where we prove that $q > 0$ on sufficiently thin cylinders. The key ingredient is an estimate on the radius dependence of the activity (Lemma 2.11).

The following lemma gives the asymptotics of the normalization constant. In particular, we prove that the radius of convergence of $\sum_n C_n r^n$ is strictly positive, for all values of the radius. Apart from the information $r > 0$, we do not use the bounds (2.40) below in the quantum Hall context. The bounds are however of interest in the plasma analogy, which is why we prove them.

**Lemma 2.10.** *For all $p \in \mathbb{N}$ and $\gamma = l/R > 0$, there exist $r > 0$ and $q \geq 0$ such that*

$$-\log r = \lim_{N\to\infty} \frac{1}{N} \log C_N = \sup_N \frac{1}{N} \log C_N, \quad q = \lim_{n\to\infty} C_n r^n.$$

*Moreover, $r = r_p(\gamma)$ satisfies the bound*

$$\Big(e^p \sum_{\substack{n_1,..,n_p\in\mathbb{Z}:\\ n_1+..+n_p=0}} e^{-\frac{\pi^2}{p\gamma^2}(n_1^2+..+n_p^2)}\Big)^{-1} \leq r_p(\gamma) p^{1-\frac{p}{2}} (\frac{e\gamma}{\sqrt{\pi}})^{1-p} \leq 1. \tag{2.40}$$

Before we turn to the proof, let us briefly look at $p = 1$ (filled Landau level). Laughlin's cylinder function at filling factor 1 is simply

$$\Psi_N^C = \psi_0 \wedge .. \wedge \psi_{N-1}.$$

It follows that $C_N = 1$ for all $N \in \mathbb{N}$, thus $r = 1$. This should be compared to the bound (2.40) which for $p = 1$ reads $e^{-1} \leq r \leq 1$. Thus for the filled Landau level, the lower bound is actually too small whereas the upper bound is exact.

*Proof of Lemma 2.10.* By Lemma 2.6 and the supermultiplicativity (2.32), we have $C_{N+M} \geq C_N C_M$. In particular, $\lim_{N\to\infty} \frac{1}{N} \log C_N = \sup_N \frac{1}{N} \log C_N$ and $r$ is well-defined (but possibly 0). Now let

$$\phi_n(z) := \frac{1}{(2\pi R l \sqrt{\pi})^{1/2p}} e^{iny/R} e^{-\frac{1}{2pl^2}(x-pn\gamma l)^2}.$$

Then

$$\Psi_N(z_1,..,z_N) = \frac{1}{\sqrt{N!}} (\det(\phi_{k-1}(z_j))_{1\leq j,k\leq N})^p =: \frac{1}{\sqrt{N!}} (\det A(z_1,..,z_N))^p$$

and $|\Psi_N|^2 = (\det A^* A)^p / N!$. Lower and upper bounds on $C_N$ are now obtained with Hadamard's and Hölder's inequalities. Using Hadamard's inequality

$$|\det A| \leq \prod_{j=1}^N (\sum_{k=1}^N |A_{jk}|^2)^{1/2}$$

we see

$$\begin{aligned} C_N &\le \frac{1}{N!}\Big(\int_{\mathbb{R}\times[0,2\pi R]} \big(\sum_{k=0}^{N-1} |\phi_k(z)|^2\big)^p dxdy\Big)^N \\ &= \frac{1}{N!}\Big(\int_{-\infty}^{\infty} \frac{1}{l\sqrt{\pi}} \big(\sum_{k=0}^{N-1} e^{-\frac{1}{pl^2}(x-pk\gamma l)^2}\big)^p dx\Big)^N \\ &= \frac{1}{N!}\Big(\int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi}} \big(\sum_{k=0}^{N-1} e^{-\frac{1}{p}(s-pk\gamma)^2}\big)^p ds\Big)^N \end{aligned}$$

By Poisson's summation formula,

$$\sum_{k=-\infty}^{\infty} e^{-\frac{1}{p}(x-kp\gamma)^2} = \frac{\sqrt{\pi}}{\gamma\sqrt{p}} \sum_{n=-\infty}^{\infty} e^{-\pi^2 n^2/(p\gamma^2)} e^{i2\pi nx/(p\gamma)} =: f(x).$$

Thus

$$\begin{aligned} \int_{-p\gamma/2}^{(N-1/2)p\gamma} &\frac{1}{\sqrt{\pi}} \big(\sum_{k=0}^{N-1} e^{-\frac{1}{p}(x-pk\gamma)^2}\big)^p dx \le \frac{1}{\sqrt{\pi}} N \int_0^{p\gamma} f(x)^p dx \\ &= N \frac{p\gamma}{\sqrt{\pi}} \Big(\frac{\sqrt{\pi}}{\gamma\sqrt{p}}\Big)^p \sum_{n_1+..+n_p=0} e^{-\frac{\pi^2}{p\gamma^2}(n_1^2+..+n_p^2)} \\ &= N p^{1-\frac{p}{2}} \big(\frac{\sqrt{\pi}}{\gamma}\big)^{p-1} \sum_{n_1+..+n_p=0} e^{-\frac{\pi^2}{p\gamma^2}(n_1^2+..+n_p^2)} \\ &=: N b(\gamma). \end{aligned}$$

The integral from $-\infty$ to $-p\gamma/2$ can be bounded from above by

$$\int_{-\infty}^{-p\gamma/2} \big(\sum_{k=0}^{\infty} e^{-(x-pk\gamma)^2/p}\big)^p dx =: c$$

which is an $N$-independent, finite number. For symmetry reasons this is also a bound to the integral from $(N-1/2)p\gamma$ to $\infty$. Thus we obtain

$$C_N \le \frac{1}{N!}(Nb(\gamma) + 2c)^N.$$

Using Stirling's formula, we obtain the upper bound $r \ge 1/(eb(\gamma)) > 0$. Now we turn to a lower bound for $C_N$. Let $I_N := [-p\gamma l/2, (N-1/2)p\gamma l] \times [0, 2\pi R]$. With Hölder's inequality $|\int_\Omega f| \le |\Omega|^{1-1/p} (\int f^p)^{1/p}$ , we get

$$\begin{aligned} C_N &\ge \frac{1}{N!} \int_{I_N^N} |\det A|^{2p} \ge \frac{1}{N!} \frac{1}{|I_N^N|^{p-1}} \big(\int_{I_N^N} |\det A|^2\big)^p \\ &= \frac{N!^{p-1}}{(Np\gamma l 2\pi R)^{p-1}} ||\phi_0 \wedge .. \wedge \phi_{N-1}||^{2p}_{L^2(I_N^N)} \\ &= \frac{N!^{p-1}}{(Np\gamma l 2\pi R)^{N(p-1)}} (2\pi R)^{N(p-1)} \prod_{k=0}^{N-1} \int_{-(k+\frac{1}{2})p\gamma}^{(N-k-1/2)p\gamma} e^{-s^2/p} ds \\ &= \Big(\frac{N!\sqrt{\pi}^N}{(Np\gamma)^N}\Big)^{p-1} \sqrt{p}^{pN} \prod_{k=0}^{N-1} \Big(1 - \frac{1}{2}\mathrm{erfc}((N-k-\frac{1}{2})\sqrt{p}\gamma) - \frac{1}{2}\mathrm{erfc}((k+\frac{1}{2})\sqrt{p}\gamma)\Big)^p \end{aligned}$$

where erfc$x := \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2} dt$ is the complementary error function. The decay properties of erfc can be used to show that the product $\prod_{k=0}^{N-1}$ stays bounded away from zero. Making use again of Stirling's formula, we get

$$r \leq \frac{(p\gamma)^{p-1} e^{p-1}}{\sqrt{\pi}^{p-1} \sqrt{p}^p} = (\frac{\gamma e}{\sqrt{\pi}})^{p-1} p^{\frac{p}{2}-1}.$$

The asymptotics $r^n C_n \to q \geq 0$ follows from Lemma 2.8. $1/q$ is the expected interarrival time of the associated renewal process. Note that the monomer activity of the system associated to Laughlin's function is $\alpha_1 = 1$, thus the aperiodicity condition $\gcd\{n \in \mathbb{N} \mid \alpha_n > 0\} = 1$ needed to apply Lemma 2.8 is fulfilled. □

It follows from the previous lemma that $r > 0$, and therefore $\sum_n \alpha_n t^n$ must have a non-zero radius of convergence. Put differently, the activity can be rescaled to a stable activity. It is still open whether the associated renewal process has finite mean $\mu < \infty$, or equivalently, if $q = \mu^{-1} > 0$.

*Remarks: 1.* The bounds on $r$ are interesting for small $\gamma$. They can be related to a conjecture on the relation between strip free energies and bulk free energies of Coulomb systems (see Section 2.3). We now turn to the large $\gamma$ behavior.

*2.* For $\gamma = 0$, the activity cannot be rescaled to a stable activity. Let us first define the activity for $\gamma = 0$. Note that for general $\gamma$, the activity and the normalization constants are polynomials of $e^{-\gamma^2}$ with integer coefficients:

$$\begin{aligned} C_N &= \sum_{0 \leq m_1 \leq .. \leq m_N \leq pN-p} |b_N(m_1, .., m_N)|^2 (e^{-\gamma^2})^{\sum_{j=1}^N (p^2(j-1)^2 - m_j^2)} \\ \alpha_N &= \sum_{\substack{0 \leq m_1 \leq .. \leq m_N \leq pN-p \\ \text{irred}}} |b_N(m_1, .., m_N)|^2 (e^{-\gamma^2})^{\sum_{j=1}^N (p^2(j-1)^2 - m_j^2)} \end{aligned} \tag{2.41}$$

where $b_N(m_1, .., m_N)$ are the expansion coefficients of the $p$-th power of the Vandermonde determinant. The $b_N$'s are integers since they are sums of signs of permutations, see p.37. These expressions are obtained by combining (2.29) and (2.2). Note that by Lemma 2.5, the sums in (2.41) contain only non-negative powers of $e^{-\gamma^2}$. For $\gamma = 0$, corresponding to the limit of infinite radii, we may define $C_N$ and $\alpha_N$ through (2.41). are well-defined for $\gamma = 0$, corresponding to the limit of infinite radii. The value of $C_N$ for $\gamma = 0$ is known: it is $C_N = (pN)!/(N!p!^N)$ see e.g. [FGIL94], Section 5, Property 6. It follows that for $p \geq 2$ and $\gamma = 0$, the series $\sum_n C_n t^n$ has a vanishing radius of convergence: $r = 0$.

Lemma 2.10 shows that for all (finite) values of the radius $R$, the activity of the polymer system associated with Laughlin's function can be rescaled to a stable activity. Now it remains to see whether the associated renewal process has finite mean. Before we turn to this question, let us have a closer look at the activity $(\alpha_n)$.

**Lemma 2.11.** *The monomer activity is $\alpha_1 = 1$. Furthermore, there exist integers $b_{N,m} \in \mathbb{N}_0$ such that*

$$\forall N \geq 1 : \ \alpha_N = \sum_{m=p(N-1)}^{\infty} b_{N,m} e^{-m\gamma^2}.$$

*In particular, as $\gamma \to \infty$, $N$-mers have activity $O(e^{-p(N-1)\gamma^2})$.*

*Proof.* Recall from (2.24) that the monomer functions are $u_{\{j\}} = \psi_{pj}$. Thus the monomer activity is $||u_{\{j\}}||^2 = 1$. By Lemma 2.5, $\sum_{j=1}^{N}(p^2(j-1)^2 - m_j^2) \geq p(N-1)$ for any increasing, $N$-admissible irreducible $(m_1, .., m_N)$. The second statement of the lemma now follows immediately from the formula (2.41). □

*Remark.* Rezayi and Haldane [RH94] have observed that for systems with a fixed, finite number of particles, Laughlin's wave function approaches the *Tao-Thouless state* [2] in the limit of vanishing cylinder radius:

$$\Psi_N^C \simeq \psi_0 \wedge \psi_p \wedge ... \wedge \psi_{p(N-1)} \text{ as } \gamma \to \infty.$$

Using the polymer representation, this can be rephrased as: in the limit of thin cylinders, the system is a pure monomer system. We will see that a similar statement holds for systems with infinitely many particles. This behavior is consistent with the fact that in the limit of thin cylinders, all polymers except monomers have vanishing activity, as shown in the previous lemma.

Now let us come back to the question whether $q = \lim_n r^n C_n > 0$, or equivalently, whether the associated renewal process has finite mean. Recall that this is the case if and only if $\sum_{n=1}^{\infty} \alpha_n r^n = 1$ and $\sum_{n=1}^{\infty} n\alpha_n r^n < \infty$. The crucial observation is that these two conditions are automatically fulfilled if $\sum_n \alpha_n t^n$ has a radius of convergence $R$ strictly larger than the radius of convergence $r$ of $\sum_n C_n t^n$. Thus we are going to compare domains of convergence. Furthermore, observe that the monomer case is rather trivial: for a monomer system,

$$\sum_{n=1}^{\infty} \alpha_n t^n = t, \quad \sum_{n=0}^{\infty} C_n t^n = \frac{1}{1-t}$$

whence $r = 1$ and the "series" $\sum_n t^n \alpha_n = t$ has an infinite radius of convergence.

Lemma 2.11 tells us that on thin cylinders, the system looks almost like a monomer system and the associated renewal process should have finite mean. This heuristic argument gives the main intuition behind the next theorem.

We wish to apply Lemma 2.11. Thus it is useful to keep track of the $\gamma$-dependence of the various quantities involved. By (2.41), the activity and the normalization constants are polynomials of $e^{-\gamma^2}$ with coefficients in $\mathbb{N}$ (see also the previous lemma). We make the $\gamma$-dependence more explicit in the notation by writing $\alpha_n(e^{-\gamma^2})$, $C_n(e^{-\gamma^2})$. The power series

$$C(t, e^{-\gamma^2}) := 1 + \sum_{n=1}^{\infty} C_n(e^{-\gamma^2}) t^n, \quad A(t, e^{-\gamma^2}) := t + \sum_{n=2}^{\infty} \alpha_n(e^{-\gamma^2})\, t^n$$

are actually power series of *two* variables, $t$ and $e^{-\gamma^2}$. They have non-negative integer coefficients. By the results of Section 2.1.2, $C$ and $A$ are related through

$$C(t, e^{-\gamma^2}) = \frac{1}{1 - A(t, e^{-\gamma^2})}.$$

[2] At filling factor $1/p$, the Tao-Thouless state is a simple Slater determinant where every $p$-th Landau orbital is occupied. It was proposed as a candidate FQHE ground state in [TT83] but later abandoned [Tho85].

The quantities $r = r(e^{-\gamma^2})$ and $R = R(e^{-\gamma^2})$ are $\gamma$-dependent too. They define curves that may be characterized through the boundaries of the domains of convergence of $C(t,u)$ and $A(t,u)$ in $\mathbb{R}^2_+$, see Figure 2.1. We know that $r(u) \le R(u)$ for all $u$. The case $u = 0$ corresponds to the infinitely thin cylinder. By Lemma 2.11, in this limit we have a pure monomer system. Consequently, $r(0) = 1$, $R(0) = \infty$ and $r(0) < R(0)$.

The strategy in the proof of the next theorem is to show the strict inequality $r(u) < R(u)$ subsists for sufficiently small $u$. This will imply that on thin cylinders the associated renewal process has finite mean, or equivalently, $r^n C_n \to q > 0$. Furthermore, we show some analyticity results.

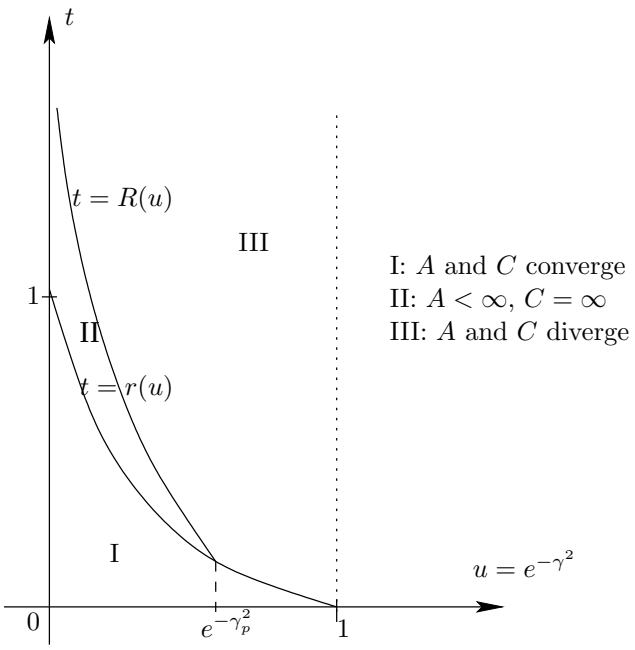


Figure 2.1: Domains of convergence of $A(t,u)$ and $C(t,u)$ for $p \ge 2$. The curve $r(u)$ delimits the domain of convergence of $A$, $R(u)$ the domain of convergence of $C$. Both series diverge when $u \ge 1$. We know that $r(u) = 1 + O(u)$ and $R(u) \ge \text{const} \cdot u^{-p}$ as $u \to 0$. When $u < e^{-\gamma^2}$, $r(u) < R(u)$. It is an open question whether the curves $r$ and $R$ touch for some $u_p = e^{-\gamma_p^2}$ strictly below 1.

**Theorem 2.12.** *Let $p \ge 2$ be fixed. Define the functions $C(t,u), A(t,u), r(u), R(u)$ as above. Let $r = r_p(\gamma)$, $q = q_p(\gamma)$ as in Lemma 2.10. The following holds:*

1. *There exists a $\gamma_p > 0$ such that for $u < e^{-\gamma_p^2}$, $r(u) < R(u)$.*
2. *The functions $]\gamma_p, \infty[ \ni \gamma \mapsto r_p(\gamma), q_p(\gamma)$ are analytic and strictly positive. As $\gamma \to \infty$,*

$$r_p(\gamma) = 1 + O(e^{-\gamma^2}), \quad q_p(\gamma) = 1 + O(e^{-\gamma^2}).$$

*Proof.* The main idea is to use Lemma 2.11 and look at what happens when $e^{-\gamma^2} \to 0$. Observe that $A(t,0) = t$. It follows that $R(0) = \infty$, $C(t,0) = 1/(1-t)$ and $r(0) = 1 < R(0)$.

Let

$$\mu(u) := \sum_{n=1}^{\infty} n\alpha_n(u)(r(u))^n = r(u)\frac{\partial A}{\partial t}(r(u), u). \tag{2.42}$$

We have $\mu(0) = 1$. The crucial obervation is that

$$\lim_{u\to 0} R(u) = \infty, \quad \forall u \in [0, 1[: \ 0 < r(u) \leq 1.$$

To see this, note first that $R(u) \geq r(u)$ and $r(u) > 0$ for $u = e^{-\gamma^2} \in ]0, 1[$ by Lemma 2.10. Next, let $0 < u < v < 1$; by Lemma 2.5 , there exist nonnegative numbers $b_{mn}$ so that

$$\alpha_n(u) = \sum_{m\geq p(n-1)} b_{mn}u^m \leq (\frac{u}{v})^{p(n-1)} \sum_{m\geq p(n-1)} b_{mn}v^m = (\frac{u}{v})^{p(n-1)}\alpha_n(v)$$

Since $R(u) = 1/(\limsup_n(\alpha_n(u))^{1/n})$, this shows

$$u^p R(u) \geq v^p R(v).$$

For $v < 1$, $R(v) > 0$, thus letting $u \to 0$ in the previous inequality, we obtain $\lim_{u\to 0} R(u) = \infty$. On the other hand, since $A(1, u) \geq 1 \geq A(r(u), u)$ for all $u \geq 0$, $r(u) \leq 1$ for all $u \geq 0$. Thus for sufficiently small $u$, $r(u) \leq 1 < R(u)$.

Let $\gamma_p > 0$ such that $r(u) < R(u)$ if $u < e^{-\gamma_p^2} =: u_p$. On $[0, u_p[$, the curve $r(u)$ is in the interior of the domain of convergence of $A(t, u)$, thus $\mu(u) < \infty$. Furthermore $r(u)$ is the solution of $A(r(u), u) = 1$. The power series $A(t, u)$ defines a holomorphic function of two variables, defined in

$$D_A = \{(z, w) \in \mathbb{C}^2 \mid |z| < R(|w|)\}.$$

We know $A(1, 0) = 1$ and $\partial_z A(1, 0) = 1 \neq 0$. Thus by an implicit function theorem, there is a neighborhood $U \subset \mathbb{C}^2$ of $(1, 0)$ such that for $(z, w) \in U$, the equation $A(z, w) = 1$ has a unique solution $z(w)$; furthermore, $z(w)$ is holomorphic. But for $w = u \in \mathbb{R}$, $z(u) = r(u)$, thus $r(u)$ is analytic in a neighborhood of 0. Let $u \in [0, e^{-\gamma_p^2}[$. We know that $(r(u), u)$ is in the interior of $D_A$, and $(\partial_z A)(r(u), u) \geq r(u) > 0$. Thus the same reasoning as before gives the analyticity of $r(\cdot)$ in a neighborhood of $u$. It follows that $r(u)$ is analytic on $[0, e^{-\gamma^2}[$. Around $u = 0$, it has a power series expansion

$$r(e^{-\gamma^2}) = 1 + \sum_n \beta_n \, (e^{-\gamma^2})^n$$

whence $r(e^{-\gamma^2}) = 1 + O(e^{-\gamma^2})$. The quantity $\mu(u)$ from (2.42) inherits the analyticity, and from $\mu(0) = 1$ we get $\mu(e^{-\gamma^2}) = 1 + O(e^{-\gamma^2})$. Since $q = \mu^{-1}$, this concludes the proof. □

*Remark.* The characterization of $r(u)$ and $R(u)$ as boundaries of domains of convergence gives access to convexity properties: for example, the set

$$\{(\log t, \log u) \mid t, u > 0, \ C(t, u) < \infty\}$$

is convex (see e.g. the book [Hör90]). It follows that the functions $\log u \mapsto \log r(u)$ and

$$]0, \infty[\ni \gamma^2 \mapsto -\log r_p(\gamma)$$

are convex.

### 2.1.4 Correlation functions and clustering

The normalization of Laughlin's wave function was expressed as a polymer partition function. For sufficiently thin cylinder (i.e. sufficiently large $\gamma = l/R$), the associated renewal process has finite mean. In Section 2.1.2, we showed that this implies the existence of the thermodynamic limit of correlation functions for the associated polymer system, the resulting state is translationally invariant and mixing.

Laughlin's wave function inherits the properties of the associated polymer system. The thermodynamic limit of all correlation functions exist (Theorem 2.13). Since $u_{j+X} = t(j \cdot p\gamma l)^{\otimes N(X)} u_X$ [3] , the translational invariance of the polymer system becomes invariance of Laughlin's state with respect to magnetic translations by multiples of $p\gamma l$; moreover Laughlin's state is reversal invariant (Theorem 2.14). We prove also that Laughlin's state is mixing with respect to the group of translations $t(n \cdot p\gamma l)$ $(n \in \mathbb{Z})$ (Theorem 2.16). The question whether $p\gamma l$ is the smallest period of the state is deferred to the next section.

In the following, the meaning of $\mathcal{A}$ depends on the parity of $p \in \mathbb{N}$: when $p$ is odd, $\mathcal{A}$ is the canonical anticommutation algebra over $\mathcal{H} = L^2(\mathbb{R} \times [0, 2\pi R]$, i.e. the $C^*$-algebra generated by $\mathbf{1}$ and the creation and annihilation operators $c^*(f), c(g)$, $f, g \in L^2(\mathbb{R} \times [0, 2\pi R])$. When $p$ is even, it is the canonical commutation relation algebra over $\mathcal{H}$, i.e. the algebra generated by the Weyl operators $W(f) = \exp(i\Phi(f))$ where $\Phi(f)$ is the closure of the operator $(c(f) + c^*(f))/\sqrt{2}$ defined in terms of the bosonic creation and annihilation operators, on the bosonic Fock space (see [BR79b], Section 5.2).

Details of proofs will be worked out only for odd $p$.

**Theorem 2.13.** *Suppose $r^n C_n \to q > 0$. Then there is a state $\langle\cdot\rangle$ on $\mathcal{A}$ such that for all sequences $(a_N)_N$ of integers with $a_N \to -\infty$ and $N + a_N \to \infty$, the state $\langle\cdot\rangle_N$ associated to $t(a_N p\gamma l)^{\otimes N}\Psi_N$ converges weakly to $\langle\cdot\rangle$: for all $a \in \mathcal{A}$,*

$$\langle a\rangle_N = \langle t(a_N p\gamma)^{\otimes N}\Psi_N, at(a_N p\gamma)^{\otimes N}\Psi_N\rangle/||\Psi_N||^2 \underset{N\to\infty}{\to} \langle a\rangle. \tag{2.43}$$

*Proof.* It is enough to prove the convergence (2.43) for operators $a = c^*(f_1)..c^*(f_n)c(g_1)..c(g_m)$ where $f_1, .., g_m$ are elements of a complete orthonormal system of $L^2(\mathbb{R} \times [0, 2\pi R])$. The expectation $\langle a\rangle_N$ vanishes if $m \neq n$. Since $c(f)\Psi_N = 0$ if $f$ is orthogonal to the lowest Landau level, it is enough to consider functions in the lowest Landau level. Thus we are left with the case $a = c^*_{\ell'_1}..c^*_{\ell'_r}c_{\ell_r}..c_{\ell_1}$ where again for $k \in \mathbb{Z}$. $c_k = c(\psi_k)$. These operators anticommute and we can take $\ell_1 < .. < \ell_r$ and similarly $\ell'_1 < .. < \ell'_r$.

*Strategy.* For $L = \{\ell_1 < .. < \ell_r\} \subset \mathbb{Z}$, let $c_L := c_{\ell_1}..c_{\ell_r}$ and $c^*_L := (c_L)^*$. Let $b_N := N + a_N$. We will see that $\langle c^*_{L'}c_L\rangle_N$ can be written as

$$\langle c^*_{L'}c_L\rangle_N = \sum_{n=1}^{N}\sum_{j=a_N}^{b_N-n} \frac{C_{j-a_N}C_{b_N-j-n}}{C_N} f_n(L' - pj, L - pj). \tag{2.44}$$

for a suitable family of functions $(f_n)_{n\in\mathbb{N}}$. We use the notation $L - pj = \{\ell_1, .., \ell_r\} - pj$ for the set $\{\ell_1 - pj, .., \ell_r - pj\}$. Let us anticipate and mention some of the properties of $(f_n)_{n\in\mathbb{N}}$:

- $\forall n \in \mathbb{N}\ \forall L \subset \mathbb{Z}:\ f_n(L, L) \geq 0.$

[3] For $a \in \mathbb{R}$, $t(a)$ is a shorthand for a translation in the $x$-direction: $t(a) = t(a\mathbf{e}_x)$.

- The functions have compact support: $f_n(L', L) \neq 0$ implies $L \cup L' \subset \{0, .., pn - p\}$.

For fixed $j$ and $n$, $C_{j-a_N} C_{b_N-j-n}/C_N \to qr^n$ due to $C_n r^n \to q > 0$. This suggests the following definition:

$$\langle c^*_{L'} c_L \rangle := \sum_{n=1}^{\infty} qr^n \sum_{j=-\infty}^{\infty} f_n(L' - pj, L - pj). \tag{2.45}$$

The theorem is proved in three steps:

1. Define $f_n$ and prove (2.44).
2. Show $\langle c^*_L c_L \rangle \leq 1$ and $\langle c^*_L c_L \rangle_N \to \langle c^*_L c_L \rangle$.
3. Prove $\sum_{j,n} qr^n |f_n(L' - pj, L - pj)| \leq \langle c^*_L c_L \rangle^{1/2} \langle c^*_{L'} c_{L'} \rangle^{1/2}$ and $\langle c^*_{L'} c_L \rangle_N \to \langle c^*_{L'} c_L \rangle$.

*1.* Let $L', L \subset \mathbb{Z}$ with $|L'| = |L| = r$. We start with the representation

$$\langle \Psi_N, c^*_{L'} c_L \Psi_N \rangle = \sum_{m', m} \overline{a_N(m')} a_N(m) \langle \psi_m, c^*_{L'} c_L \psi_m \rangle. \tag{2.46}$$

The sum ranges over increasing $N$-admissible sequences $m = (m_1, ., m_N)$, $m' = (m'_1, ., m'_N)$, and $a_N(m)$, $\psi_m$ are short-hands for $a_N(m_1, ., m_N)$, $\psi_{m_1} \wedge .. \wedge \psi_{m_N}$. Suppose $m$ and $m'$ have common renewal points $s, t$ such that $L \cup L' \subset \{ps, .., pt - p\}$. Then

$$\begin{aligned} \langle \psi_m, c^*_{L'} c_L \psi_m \rangle = \prod_{j \in \{1,..,s\} \cup \{t,..,N\}} & \delta_{m_j, m'_j} |a_s(m_1, .., m_s)|^2 \, |a_{N-t}(m_{t+1} - pt, .., m_N)|^2 \\ & \cdot \overline{a_{t-s}(m'_{s+1} - ps, .., m'_t - ps)} a_{t-s}(m_{s+1} - ps, .., m_t - ps) \\ & \cdot \langle \psi_{m'_{s+1}} \wedge .. \wedge \psi_{m'_t}, c^*_{L'} c_L \psi_{m_{s+1}} \wedge .. \wedge \psi_{m_t} \rangle. \end{aligned} \tag{2.47}$$

This motivates the following definitions: we call $m, m'$ $L \cup L'$-irreducible if $m$ and $m'$ have no common renewal points except 0 and $N$ that are below or above $L \cup L'$:

$$\begin{aligned} & k \in R(m) \cap R(m') \cap \{1, .., N-1\} \\ & \qquad \Rightarrow (L \cup L') \cap \{0, .., pk - p\} \neq \emptyset \text{ and } (L \cup L') \cap \{pk, .., pN - p\} \neq \emptyset. \end{aligned}$$

Similarly, we call a single sequence $m$ $L \cup L'$-irreducible if it has no renewal point except possibly 0 and $N$ below or above $L \cup L'$. The function $f_N(L', L)$ is defined by (2.46) except that the sum runs over increasing $N$-admissible pairs of sequences that are $L \cup L'$-irreducible. With this definition, combining (2.46) and (2.47) we obtain (2.44).

*2.* For $L = L'$, the definition of $f_n$ gives

$$f_N(L, L) = {\sum_m}' |a_N(m)|^2 \chi_{L \subset \{m_1, .., m_N\}}$$

where the sum ranges over increasing $N$-admissible sequences that are $L$-irreducible. In particular, $f_N(L, L) \geq 0$. Moreover, if $f_N(L, L) \neq 0$, there exists an $N$-admissible sequence $m$ such that $L \subset \{m_1, .., m_N\}$, whence $L \subset \{0, .., pN - p\}$.

Let $d \in \mathbb{N}$. From (2.44) and $f_n(L,L) \geq 0$ we get

$$\sum_{n=1}^{d} \sum_{j=a_N}^{b_N-j-n} \frac{C_{j-a_N} C_{b_N-j-n}}{C_N} f_n(L-pj, L-pj) \leq \langle c_L^* c_L \rangle_N \leq 1$$

If $f_n(L-pj, L-pj) \neq 0$ we must have $L - pj \subset \{0, .., pn-p\}$, thus only a finite number of $j$'s contribute to the sum and we can take the limit $N \to \infty$, which gives

$$\sum_{n=1}^{d} \sum_{j=-\infty}^{\infty} qr^n f_n(L-pj, L-pj) \leq 1.$$

Letting $d \to \infty$, we obtain the convergence of the sum defining $\langle c_L^* c_L \rangle$. The proof of $\langle c_L^* c_L \rangle_N \to \langle c_L^* c_L \rangle$ is completed by an $\epsilon/3$ argument. For $\epsilon > 0$, there exists an $m \in \mathbb{N}$ such that

$$\sum_{n=d+1}^{\infty} \sum_{j=-\infty}^{\infty} qr^n f_n(L-pj, L-pj) \leq \epsilon/3. \tag{2.48}$$

By Lemma 2.10, $0 < C_n r^n \leq 1$ and $C_n r^n \to q$. By the assumption $q > 0$, we get $\min_n r^n C_n =: c > 0$. Therefore

$$\frac{C_j C_{N-j-n}}{C_N} \leq \frac{C_{N-n}}{C_N} \leq \frac{r^{-(N-n)}}{c r^{-N}} = \frac{1}{c} r^n.$$

It follows that

$$\sum_{n=d+1}^{\infty} \sum_{j=a_N}^{b_N-n} \frac{C_{j-a_N} C_{b_N-j-n}}{C_N} f_n(L-pj, L-pj) \leq \frac{\epsilon}{3qc} \tag{2.49}$$

Finally, there is an $M \in \mathbb{N}$ such that for $N \geq M$,

$$\Big| \sum_{n=1}^{d} \Big( \sum_{j=a_N}^{b_N-n} \frac{C_{j-a_N} C_{b_N-j-n}}{C_N} f_n(L-pj, L-pj) - \sum_{j=-\infty}^{\infty} qr^n f_n(L-pj, L-pj) \Big) \Big| \leq \epsilon/3. \tag{2.50}$$

Putting together (2.48), (2.49) and (2.50) we see that for $N \geq M$,

$$|\langle c_L^* c_L \rangle_N - \langle c_L^* c_L \rangle| \leq \frac{2 + (qc)^{-1}}{3} \epsilon$$

and we obtain the desired convergence.

*3.* Let $d \in \mathbb{N}$. As before, we have

$$\sum_{n=1}^{d} qr^n \sum_{j=-\infty}^{\infty} |f_n(\mathbf{l}' - pj, \mathbf{l} - pj)| = \lim_{N \to \infty} \sum_{n=1}^{m} \sum_{j=a_N}^{b_N-j-n} \frac{C_j C_{b_N-n-j}}{C_N} |f_n(\mathbf{l}' - pj, \mathbf{l} - pj)|$$

since due to $n \leq d$ only finitely many terms contribute to the sum. But now

$$\begin{aligned}
&\sum_{n=1}^{d}\sum_{j=a_N}^{b_N-n}\frac{C_{j-a_N}C_{b_N-n-j}}{C_N}|f_n(L'-pj,L-pj)| \\
&\qquad\leq \sum_{n=1}^{N}\sum_{j=a_N}^{b_N-n}\frac{C_{j-a_N}C_{b_N-n-j}}{C_N}|f_n(L'-pj,L-pj)| \\
&\qquad\leq \sum_{m,m'}|a_N(m')a_N(m)\langle\psi_{m'},c^*_{L'}c_L\psi_m\rangle|/C_N \\
&\qquad= \sum_{k\in\mathbb{Z}^{N-r}}|a_N(\ell'\cup k)a_N(\ell\cup k)|/C_N \\
&\qquad\leq \big(\sum_{k\in\mathbb{Z}^{N-r}}|a_N(\ell'\cup k)|^2\big)^{1/2}\big(\sum_{k\in\mathbb{Z}^{N-r}}|a_N(\ell\cup k)|^2\big)^{1/2}/C_N \\
&\qquad= \langle c^*_{L'}c_{L'}\rangle_N^{1/2}\langle c^*_Lc_L\rangle_N^{1/2}\leq 1.
\end{aligned}$$

Here $r = |L| = |L'|$ and $\ell, \ell' \in \mathbb{Z}^r$ are such that $L = \{\ell_1, ., \ell_r\}$, $L' = \{\ell'_1, .., \ell'_r\}$. The notation $k \cup \ell$ refers to the increasing sequence obtained by rearranging $k_1, .., k_{N-r}, \ell_1, .., \ell_r$. Letting first $d$ and then $N$ go to infinity, we get the absolute convergence of the sum defining $\langle c^*_{L'}c_L\rangle$ and the bound $|\langle c^*_{L'}c_L\rangle| \leq \langle c^*_{L'}c_{L'}\rangle^{1/2}\langle c^*_Lc_L\rangle^{1/2}$ (Cauchy-Schwarz for the state $\langle\cdot\rangle$). An argument strictly analogous to *2.* then shows $\langle c^*_{L'}c_L\rangle_N \to \langle c^*_{L'}c_L\rangle$. □

Now let us turn to the symmetries of Laughlin's state. A unitary map $U$ in $\mathcal{B}(\mathcal{H})$, $\mathcal{H} = L^2(\mathbb{R}\times[0,2\pi R])$ induces the unitary map $\Gamma(U) := \bigoplus_{N\in\mathbb{N}_0}U^{\otimes N}$ in $\mathcal{F}$ and a map

$$\tau_U : \mathcal{A}\to\mathcal{A},\quad \tau_U(a) := \Gamma(U)a\Gamma(U)^{-1}.$$

The map $\tau_U$ is the unique $C^*$-automorphism such that $\tau_U(c(f)) = c(Uf)$ for all $f \in \mathcal{H}$ (the Bogoliubov automorphism induced by $U$). This construction can be applied to the 180° rotation (or *reversal*) $s_0$, introduced at the beginning of the chapter, to the magnetic translation $t(\gamma l\mathbf{e}_x)$ in the $x$-direction and to the translations $t(a\mathbf{e}_y)$ ($a \in \mathbb{R}$) in the $y$-direction; note that in the Landau gauge, magnetic translations in the $y$ direction coincide with the usual translation $\psi \mapsto \psi(\cdot - ia)$. We will call the resulting morphisms of $\mathcal{A}$ $\tau_s$, $\tau_x$ and $\tau^a_y$.

**Theorem 2.14.** *Suppose $r^nC_n \to q > 0$ and let $\omega(\cdot) = \langle\cdot\rangle$ be the limiting state of the previous theorem. Then $\omega$ is invariant with respect to reversals, translations in the $x$ direction by multiples of $p\gamma l$, and arbitrary translations in the $y$ direction:*

$$\forall n\in\mathbb{Z},\ \forall a\in\mathbb{R}:\ \omega = \omega\circ\tau_s = \omega\circ\tau_x^{np} = \omega\circ\tau_y^a.$$

*Proof.* The invariance with respect to $y$-translations and reversal follow from Lemma 2.1: the $y$-translational invariance is a direct consequence of (2.8). For the reversal invariance, suppose $N$ is odd and let $a_N := -(N-1)/2$. By (2.9),

$$s_0t(a_Np\gamma l\mathbf{e}_x)^{\otimes N}\Psi_N = t(a_Np\gamma l\mathbf{e}_x)^{\otimes N}s_{p(N-1)\gamma l/2}\Psi_N = t(a_Np\gamma l\mathbf{e}_x)^{\otimes N}\Psi_N,$$

whence the reversal invariance of $\langle\cdot\rangle_N$. Taking limits along sequences of odd integers $N = (2M+1)\to\infty$, we obtain the reversal invariance of $\langle\cdot\rangle$. The invariance with respect to $\tau_x^p$

follows from (2.45):

$$\begin{aligned}\langle \tau_x^p(c^*_{L'}c_L)\rangle = \langle c^*_{L'+p}c_{L+p}\rangle &= \sum_{n=1}^{\infty} qr^n \sum_{j=-\infty}^{\infty} f_n(L'+p-pj, L+p-pj)\\ &= \sum_{n=1}^{\infty} qr^n \sum_{j=-\infty}^{\infty} f_n(L'-pj, L-pj)\\ &= \langle c^*_{L'}c_L\rangle.\end{aligned}$$

□

We were initially interested in the thermodynamic limit of the one-particle density. Using the representation

$$\rho_N(z) = \sum_{k=a_N}^{b_N} \langle c_k^* c_k\rangle_N |\psi_k(z)|^2 = \frac{1}{2\pi Rl\sqrt{\pi}} \sum_{k=a_N}^{b_N} \langle c_k^* c_k\rangle_N e^{-(x-k\gamma l)^2/l^2}$$

we obtain immediately that, under the assumptions of Theorem 2.13, $\rho_N(z) \to \rho(z)$, where

$$\rho(z) = \frac{1}{2\pi Rl\sqrt{\pi}} \sum_{k=-\infty}^{\infty} \langle c_k^* c_k\rangle e^{-(x-k\gamma l)^2/l^2}. \tag{2.51}$$

*Remark: Occupation numbers with even p.* Strictly speaking, the previous expression holds true only when $p$ is odd. When $p$ is even, the occupation number operator $\hat{n}_k$ is unbounded and thus $\hat{n}_k \notin \mathcal{A}$. The eigenprojections of $\hat{n}_k$ are in $\mathcal{A}$, thus we may write $\hat{n}_k = \sum_k n_k P_k$ and set $\langle \hat{n}_k\rangle = \sum_k n_k \langle P_k\rangle$. It is implicit in the proofs of the following theorems that the sums involved stay finite. Another procedure to give a meaning to $\langle \hat{n}_k\rangle$ is to check that the state $\omega = \langle\cdot\rangle$ is *regular* ([BR79b] p.24), in which case there is a natural definition of an operator $\hat{n}_{k,\omega}$ in the cyclic representation $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$ and one can set $\langle \hat{n}_k\rangle := \langle \Omega_\omega, \hat{n}_{k,\omega}\Omega_\omega\rangle$. In the following, we will keep on writing $\langle \hat{n}_k\rangle$ without explicitly requiring $p$ odd.

The one-particle density $\rho(z)$ is $p\gamma l$-periodic in the direction along the cylinder axis and has the correct average value:

**Corollary 2.15.** *Suppose $r^n C_n \to q > 0$ and let $\langle\cdot\rangle$ be the state of Theorem 2.13. The one-particle density $\rho$ associated to $\langle\cdot\rangle$ is independent of the coordinate $y$ around the axis and is periodic with period $p\gamma l$ in the direction along the axis. Moreover, it has average density $(p\cdot 2\pi l^2)^{-1}$:*

$$\frac{1}{p\gamma l}\int_0^{p\gamma l} \rho(x)dx = \frac{1}{2\pi l^2}\frac{1}{p}\sum_{k=0}^{p-1}\langle c_k^* c_k\rangle = \frac{1}{p}\frac{1}{2\pi l^2}.$$

*Proof.* The one-particle density $\rho(z)$ is given by (2.51) and obviously independent of $y = \Im z$. The periodicity of $\rho$ follows from the periodicity of $\langle\cdot\rangle$ proved in the previous theorem. By Lemma 2.2,

$$\frac{1}{p\gamma l}\int_0^{p\gamma l} \rho(x)dx = \frac{1}{2\pi l^2}\frac{1}{p}\sum_{k=0}^{p-1}\langle c_k^* c_k\rangle.$$

Let $n_k := \langle c_k^* c_k\rangle$. It remains to prove $n_0 + n_1 + .. + n_{p-1} = 1$. A close look at the definition of $f_n$ in the proof of Theorem 2.13 shows that $f_n(\{k\},\{k\})$ can be expressed in terms of the polymer functions $u_X$ from Lemma 2.6:

$$f_n(\{k\}-pj, \{k\}-pj) = \langle u_{\{j,..,j+n-1\}}, \hat{n}_k u_{\{j,..,j+n-1\}}\rangle.$$

The formula (2.45) can be rewritten in terms of the polymer correlation functions $\rho^P(X) = qr^{N(X)}\alpha_{N(X)}$ from Proposition 2.9 as

$$n_k = \langle \hat{n}_k \rangle = \sum_X \rho^P(X) \frac{\langle u_X, \hat{n}_k u_X \rangle}{||u_X||^2}. \tag{2.52}$$

This formula is of interest in itself: in a probabilistic language, the probability for finding a particle in the lattice site $k$ given that there is a given polymer $X$ covering the site is $\langle u_X, \hat{n}_k u_X \rangle / ||u_X||^2$. For our purpose it is more convenient to write it as

$$n_k = \sum_{n=1}^{\infty} qr^n \sum_{j=-\infty}^{\infty} \langle u_n, \hat{n}_{k-pj} u_n \rangle, \quad u_n = u_{\{0,..,n-1\}}.$$

Using $\sum_{k=-\infty}^{\infty} \langle u_n, \hat{n}_k u_n \rangle = n\alpha_n$ ($u_n$ is an $n$-particle wave function with norm $||u_n||^2 = \alpha_n$), it follows that

$$\begin{aligned} \sum_{k=0}^{p-1} n_k &= \sum_{n=1}^{\infty} qr^n \sum_{k=0}^{p-1} \sum_{j=-\infty}^{\infty} \langle u_n, \hat{n}_{k-pj} u_n \rangle = \sum_{n=1}^{\infty} qr^n \sum_{k=-\infty}^{\infty} \langle u_n, \hat{n}_k u_n \rangle \\ &= \sum_{n=1}^{\infty} qr^n n\alpha_n = 1 \end{aligned}$$

(recall $q = 1/(\sum_n nr^n\alpha_n)$). □

The formula (2.52) can be generalized: any expectation value of a product of occupation numbers $\hat{n}_k$ can be expressed in terms of the polymer correlation functions $\rho^P$ from Proposition 2.9 and the functions $u_X$ of Lemma 2.6. For example, the diagonal two-point correlation is

$$\langle \hat{n}_k \hat{n}_\ell \rangle = \sum_{\substack{X: \\ pX \supset \{k,\ell\}}} \rho^P(X) \frac{\langle u_X, \hat{n}_k \hat{n}_\ell u_X \rangle}{||u_X||^2} + \sum_{\substack{X,Y: \\ pX \ni k,\ pY \ni \ell}} \rho^P(X,Y) \frac{\langle u_X, \hat{n}_k u_X \rangle}{||u_X||^2} \frac{\langle u_Y, \hat{n}_\ell u_Y \rangle}{||u_Y||^2}. \tag{2.53}$$

This suggests that the clustering properties of $\rho^P$ transfer to the state $\langle \cdot \rangle$: if $k$ and $\ell$ are far apart, the probability that they are in the same polymer is small, thus in (2.53) the second summand dominates; but then recall $\rho^P(X,Y) - \rho^P(X)\rho^P(Y) = O(|u_{\mathrm{dist}\,(X,Y)} - \mu^{-1}|)$, where $u_n = r^n C_n \to \mu^{-1} = q > 0$. One can show that

$$\langle \hat{n}_k \hat{n}_\ell \rangle - \langle \hat{n}_k \rangle \langle \hat{n}_\ell \rangle \underset{|k-\ell| \to \infty}{\longrightarrow} 0.$$

The speed of convergence is essentially determined by $\sum_{n \geq |k-\ell|} nr^n\alpha_n$ (this gives a bound to the probability that $k$ and $\ell$ are in the same polymer) and $|u_{|k-\ell|} - \mu^{-1}|$. Similar statements hold for general correlation functions.

**Theorem 2.16.** *Suppose $r^n C_n \to q > 0$. Then the state $\omega(\cdot) = \langle \cdot \rangle$ of Theorem 2.13 is mixing with respect to the shifts $\tau_x^{np}, n \in \mathbb{Z}$:*

$$\forall a, b \in \mathcal{A} : \lim_{n \to \infty} \omega(a \tau_x^{pn}(b)) = \omega(a)\omega(b). \tag{2.54}$$

*Proof.* Again, it is enough to check (2.54) for operators $a = c_{L'}^* c_L$, $b = c_{K'}^* c_K$, where $L, L', K, K' \subset \mathbb{Z}$ and we agree $c_\emptyset = c_\emptyset^* = \mathbf{1}$. The main idea is to write a formula similar to (2.53). The correlation function can be written as a sum of two parts: one where

$L \cup L'$ and $K \cup K'$ are in a same compound block (this will be the sum $\sum_n g(..)$ defined below) this part will be bounded by the probability amplitude that a long polymer occurs. Another part of the sum consists of contributions where $L \cup L'$ and $K \cup K'$ are in different blocks; this converges to the product of the correlations as $L \cup L'$ and $K \cup K'$ are far away.

We start by treating some special cases. Recall that $\langle c^*_{L'} c_L \rangle \neq 0$ implies $|L| = |L'|$ and $\sum_{k \in L'} k = \sum_{k \in L} = k$ (this comes from the translational invariance $\omega \circ \tau^a_y = \omega$, $\tau^a_y(c_k) = e^{-ika/R} c_k$).

*1.* $|L| + |K| \neq |L'| + |K'|$. Then $\omega(a \tau^{np}_x(b)) = 0$ and we must have $|K| \neq |K'|$ or $|L| \neq |L'|$, whence $\omega(a)\omega(b) = 0$.

*2.* $|L| + |K| = |L'| + |K'|$ but $|K'| \neq |K|$. Then $\omega(a)\omega(b) = 0$. Suppose for example $|K'| > |K|$. Then

$$\sum_{k \in L' \cup (pn+K')} k - \sum_{k \in L \cup (pn+K)} k = \sum_{k \in L' \cup K} k - \sum_{k \in L \cup K} k + pn(|K'| - |K|) > 0$$

for sufficiently large $n$, whence $\omega(a \tau^{pn}_x(b)) = 0$ for sufficiently large $n$.

*3.* $|L| = |L'|$, $|K| = |K'|$, but $\sum_{k \in L \cup K} k \neq \sum_{k \in L' \cup K'} k$. Then

$$\sum_{k \in L' \cup (pn+K')} k - \sum_{k \in L \cup (pn+K)} k = \sum_{k \in L' \cup K'} k - \sum_{k \in L \cup K} k \neq 0$$

thus $\omega(a \tau^{np}_x(b))0$. Moreover $\sum_{k \in K} k \neq \sum_{k \in K'} k$ or $\sum_{k \in L} k \neq \sum_{k \in L} k$, therefore $\omega(a)\omega(b) = 0$.

We are left with the case $|L'| = |L|$, $|K'| = |K|$, $\sum_{k \in K \cup L} k = \sum_{K' \cup L'} k$. We will show that $\omega(ab) - \omega(a)\omega(b)$ is small when $d := \min K \cup K' - \max L \cup L'$ is large.

Suppose $m$, $m'$ are increasing $N$-admissible and have a common renewal point between $L \cup L'$ and $K \cup K'$, i.e.:

$$s \in R(m) \cap R(m') : \; L \cup L' \subset \{0, .., ps - p\} \text{ and } K \cup K' \subset \{ps, .., pN - p\}.$$

Then

$$\begin{aligned} &\overline{a_N(m')} a_N(m) \langle \psi_{m'}, ab \psi_m \rangle \\ &\quad = \overline{a_s(\{m'_j\}_{1 \leq j \leq s})} a_{N-s}(\{m_j\}_{1 \leq j \leq s}) \langle \psi_{m'_1} \wedge .. \wedge \psi_{m'_s}, a \psi_{m_1} \wedge .. \wedge \psi_{m_s} \rangle \\ &\cdot \overline{a_{N-s}(\{m'_j - ps\}_{j \geq s+1})} a_{N-s}(\{m_j - ps\}_{j \geq s+1}) \langle \psi_{m'_{s+1}} \wedge .. \wedge \psi_{m'_N}, b \psi_{m_{s+1}} \wedge .. \wedge \psi_{m_N} \rangle \end{aligned} \tag{2.55}$$

Thus if we define

$$g_n(L', L; K', K) := {\sum}' a_N(m') a_N(m) \langle \psi_{m'}, c^*_{L'} c_L c^*_{K'} c_K \psi_m \rangle$$

with a sum over $L \cup L' \cup K \cup K'$ - irreducible pairs of increasing $N$-admissible sequences $m, m'$ with no renewal point between $L \cup L'$ and $K \cup K'$ and the functions $f_n$ as in the proof of Theorem 2.13, we obtain

$$f_N(L' \cup K', L \cup K) = \sum_{0 < i \leq j < N} f_i(L', L) C_{j-i} f_{N-j}(K' - pj, K - pj) + g_N(L', L; K', K).$$

For $X = \{j, .., j + N - 1\}$, it is convenient to define

$$f_X(L', L) = f_N(L' - pj, L - pj), \quad g_X(L', L; K', K) = g_N(L' - pj, L - pj; K' - pj, K - pj).$$

Using the representation (2.45), we get $\langle ab\rangle = F + G$ where

$$F = \sum_{X<Y} qr^{N(X)+N(Y)} f_X(L', L)(C_{d(X,Y)} r^{d(X,Y)} - q) f_Y(K', K)$$
$$G = \sum_X q^{N(X)} g_X(L', L; K', K)$$

Here, if $X = \{s_X, .., s_X + N(X) - 1\}$ and $Y = \{s_Y, .., s_Y + N(Y) - 1\}$, $d(X, Y) := s_Y - (s_X + N(X))$. The previous expressions should be compared with

$$\langle a\rangle\langle b\rangle = \sum_{X,Y} q^2 r^{N(X)+N(Y)} f_X(L'; L) f_Y(K'; K).$$

We start by estimating $F - \langle a\rangle\langle b\rangle$. Write $X = \{s_X, .., e_X - 1\}$, $Y = \{s_Y, .., e_Y - 1\}$. The difference of $F$ and $\langle a\rangle\langle b\rangle$ is a sum over $X, Y$ which we split into two parts. Let $M \in \mathbb{N}$. The first part is a sum over $X, Y$ with $s_Y - e_X \geq M$. It is bounded by

$$\sup_{n\geq M} |1 - q^{-1} r^n C_n| \Big(\sum_X qr^{N(X)} |f_X(L', L)|\Big) \Big(\sum_Y qr^{N(Y)} |f_Y(K', K)|\Big)$$
$$\leq \sup_{n\geq M} |1 - q^{-1} r^n C_n| (\langle c^*_{L'} c_{L'}\rangle \langle c^*_L c_L\rangle \langle c^*_{K'} c_{K'}\rangle \langle c^*_K c_K\rangle)^{1/2} \leq \sup_{n\geq M} |1 - q^{-1} r^n C_n|.$$

The second part is a sum over $X, Y$ with $s_Y - e_X \leq M - 1$. Writing $d = (\min(K \cup K') - ps_Y) + p(s_Y - e_X) + (pe_X - \max(L \cup L'))$, we obtain

$$\min(K \cup K') - ps_Y \geq (d - pM)/2 \text{ or } pe_X - \max(L \cup L') \geq (d - pM)/2.$$

Using $C_n r^n \leq q$, we can bound give a bound on the second part of the sum as

$$2 \sum_{\substack{X: \\ pe_X - \max(L\cup L') \geq (d-pM)/2}} qr^{N(X)} |f_X(L', L)| + 2 \sum_{\substack{Y: \\ \min(K\cup K') - ps_Y \geq (d-pM)/2}} qr^{N(Y)} |f_Y(K', K)|.$$

If we go back to the definition of $f_X$ as a sum over $N$-admissible, $L \cup L'$ irreducible sequences $m', m$, we see that only $m$ sequences with no renewal point between $L \cup L'$ and $e_X$ contribute. By a procedure similar to step *3.* of the proof of Theorem 2.13, one can show that

$$\sum_{\substack{X: \\ pe_X - \max(L\cup L') \geq (d-pM)/2}} qr^{N(X)} |f_X(L', L)| \leq \sum_{n\geq (d-pM)/2p} nr^n \alpha_n$$

the right-hand side representing the probability in the polymer ensemble of Proposition 2.9 that a given point is in a polymer of length greater or equal to $(d - pM)/2p$. The sum over $Y$ can be estimated in a similar way. Thus we get

$$|F - \langle a\rangle\langle b\rangle| \leq \sup_{n\geq M} |1 - q^{-1} r^n C_n| + 4 \sum_{n\geq (d-pM)/2p} nr^n \alpha_n.$$

It remains to give a bound on $G$. We start by looking at the non-vanishing contributions to $g_n(L', L; K', K)$. Without loss of generality, we may assume $\sum_{k\in L'} k \geq \sum_{k\in L} k$ (otherwise,

use $\langle a^* \rangle = \overline{\langle a \rangle}$ to interchange $L'$ and $L$).

*Claim.* Suppose $m', m$ are increasing $N$-admissible sequences and $\sum_{k\in L'} k \geq \sum_{k\in L} k$. and $\langle \psi_m, c^*_{L'} c_L c^*_{K'} c_K \psi_{m'} \rangle \neq 0$. Then if $s$ is a renewal point of $m'$ such that $L \cup L' \subset \{0, .., ps-p\}$, $K \cup K' \subset \{ps, .., pN-p\}$, $s$ is also a renewal point of $m$.

*Proof of the claim.* Let $m', m, K, L, K', L', s$ be as described above, and $M = \{m_1, .., m_N\}$, similarly for $M'$. Then we must have $M' \backslash (K' \cup L') = M \backslash (K \cup L)$. Intersecting with $\{0, .., ps-p\}$ we get $M' \backslash L' \cap \{0, .., ps-p\} = M \backslash L \cap \{0, .., ps-p\}$. In particular $\sum_{j=1}^{s} (m'_j - m_j) = \sum_{k\in L'} k - \sum_{k\in L} k \geq 0$. Thus

$$0 = \sum_{j=1}^{s} (m'_j - p(j-1)) \geq \sum_{j=1}^{s} (m_j - p(j-1)) \geq 0.$$

The inequality on the right-hand side must be an equality, and $s$ is a renewal point of $m$.

As a consequence, if a pair $m', m$ contributes to $g_n(L', L; K', K)$ and $X'_1, .., X'_D$ is the partition of $\{0, .., N-1\}$ determined by the renewal points of $m'$, there is a rod $X'_j$ such that $pX'_j \cap (L' \cup L) \neq 0$ and $pX'_j \cap (K' \cup K) \neq 0$. In particular, $N(X'_j) \geq \min(K' \cup K) - \max(L \cup L') = d$. Thus in the spirit of step *3.* in the proof of Theorem 2.13, we have

$$\begin{aligned} &|\sum_{n=1}^{N} \sum_{j=0}^{N-n} \frac{C_j C_{N-n-j}}{C_N} g_n(L' - pj, L - pj; K' - pj, K - pj)| \\ &\quad \leq {\sum_{m,m'}}' \overline{a_N(m)} a_N(m') \langle \psi_{m'}, ab \psi_m \rangle| / C_N \\ &\quad \leq \langle c^*_K c_K c^*_L c_L \rangle^{1/2}_{\Psi_N} \big( {\sum_{m'}}' |a_N(m')|^2 \langle \psi_{m'}, c^*_{L'} c_{L'} c^*_{K'} c_{K'} \psi_{m'} \rangle \big)^{1/2} / C_N. \end{aligned}$$

where $\sum'$ is the sum over sequences $m'$ having no renewal point between $L \cup L'$ and $K' \cup K$. Taking the limit $N \to \infty$ after shifting the origin to the middle of the cylinder, we obtain a bound of $G$ in terms of the square root of the probability that $\max L \cup L'/p$ is in a polymer of length greater or equal to $p$. Finally, we obtain the bound

$$|\langle ab \rangle - \langle a \rangle \langle b \rangle| \leq \sup_{n \geq M} |1 - q^{-1} r^n C_n| + 4 \sum_{n \geq (d-pM)/2p} n r^n \alpha_n + \big( \sum_{n \geq d/p} n r^n \alpha_n \big)^{1/2} \qquad (2.56)$$

for any $M \in \mathbb{N}$. Choosing $M$ of the order of $d/(2p)$, we see that $|\langle ab \rangle - \langle a \rangle \langle b \rangle|$ goes to zero as $d \to \infty$. □

*Remark: Rate of convergence.* Suppose $\sum_n \alpha_n t^n$ has radius of convergence $R > r$. By Theorem 2.12, this is the case for sufficiently large $\gamma$. For $\Lambda \subset \mathbb{Z}$, $|\Lambda| < \infty$, let $\mathcal{A}_\Lambda$ be the algebra generated by $\mathbf{1}, c_k, c^*_m, k, m \in \Lambda$. If $a \in \mathcal{A}_\Lambda$, $b \in \mathcal{A}_{\Lambda'}$, then $\langle ab \rangle - \langle a \rangle \langle b \rangle$ goes to zero like $(r/R)^{\mathrm{dist}\,(\Lambda, \Lambda')/4p}$: there is exponential clustering.

### 2.1.5 Symmetry breaking

In Theorem 2.14, we proved that the state $\omega = \langle \cdot \rangle$ of Theorem 2.13 is invariant with respect to translations $\tau_x^{np}$ by multiples of $p\gamma l$. Thus $p\gamma l$ is a translational period. This leaves the question open whether $p\gamma l$ is the smallest period. We will prove that for sufficiently thin

cylinders, $p\gamma l$ is indeed the smallest period of $\omega$. Since $\omega$ is an infinite volume ground state of a Hamiltonian invariant with respect to translations by multiples of $\gamma l$ (see Section 2.4), this means that there is symmetry breaking.

In principle, it is possible that the state has $p\gamma l$ as its smallest period, but a $\gamma l$-periodic one-particle density. We will prove that for sufficiently thin cylinders, this does not happen and $p\gamma l$ is the smallest period of the one-particle density.

The proof of the non-trivial periodicity of the one-particle density uses the analyticity results of Theorem 2.10. They are consistent with numerical results by [ŠWK04] and related to numerical results for tori [SFL$^+$05]. This work looks at torus wave functions at filling factor 1/3 and gives numerical estimates of the order parameter

$$\mathcal{O} = \frac{1}{N}\sum_{j=1}^{3N} e^{i2\pi j/3}\langle c^*(\tilde{\psi}_j)c(\tilde{\psi}_j)\rangle$$

($\tilde{\psi}_k,\ k \in \mathbb{Z}/3N\mathbb{Z}$ are the torus lowest Landau level basis functions). The authors find that the curve $e^{-\gamma^2} \mapsto \mathcal{O}$ looks like $e^{-c^2/\gamma^2}$ for some constant $c > 0$. In particular, for large radius $R = l/\gamma$, $\mathcal{O}$ is very small *but non-zero*.

As a complement, we prove that the states $\omega$, $\omega \circ \tau_x$,..,$\omega \circ \tau_x^{p-1}$ are orthogonal if

$$\sum_{n=1}^{\infty} n^2 r^n \alpha_n < \infty. \tag{2.57}$$

We call the states $\omega_1, .., \omega_p$ on $\mathcal{A}$ *orthogonal* if for all positive functionals $\tilde{\omega}$ on $\mathcal{A}$, and all $k \in \{1, .., p\}$,

$$\left(\tilde{\omega} \leq \omega_k \text{ and } \tilde{\omega} \leq \sum_{j\neq k} \omega_j\right) \Rightarrow \tilde{\omega} = 0$$

or equivalently, if in the cyclic representation $(\pi_\omega, \mathcal{H}_\omega, \Omega_\omega)$ associated to $\omega := \omega_1 + ... + \omega_p$, there exist mutually orthogonal projections $P_1, .., P_p \in \pi_\omega(\mathcal{A})'$ such that

$$\omega_j(a) = \langle P_j\Omega_\omega, \pi_\omega(a)\Omega_\omega\rangle \quad (j \in \{1, .., p\}),\ a \in \mathcal{A}$$

(see [BR79a] Lemma 4.1.19, p.333). The analogous notion for probability measures is disjointness of supports.

For the proof of the orthogonality, it is enough to restrict to a commutative subalgebra and we can use a probabilistic picture. The proof then follows ideas of [AM80], who look at thermodynamic limits of a classical one-dimensional jellium systems. They notice a simple condition on the range of the electric field. This condition subsists in the thermodynamic limit and shows that the distribution of the electric field cannot be translationally invariant. [AM80] then use an ergodic theorem to prove that the electric field is a function of particle positions and conclude that the distribution of particle positions cannot be translationally invariant (see also p.71 for more details). Something similar can be done here. The second moment condition (2.57) ensures the integrability of an auxiliary random variable, needed for the ergodic theorem. A related approach is also used in [AGL01], who conjectured that their results could be applied to jellium tubes.

**Theorem 2.17.** *Suppose $C_n r^n \to q > 0$ and let $\omega = \langle\cdot\rangle$ be the state of Theorem 2.13.*

1. *Suppose that the second moment $\sum_n n^2 r^n \alpha_n$ is finite. Then the states $\omega \circ \tau_x^k$, $k = 0, ..., p-1$ are orthogonal. In particular, $\omega \circ \tau_x^k = \omega$ if and only if $k$ is an integer multiple of $p$.*

2. *Let $\gamma_p$ be as in Theorem 2.12. Then we have:*

   (a) *The second moment condition and the conclusions of 1. hold for all $\gamma > \gamma_p$.*

   (b) *Let $\rho$, $n_k = \omega(c_k^* c_k)$ be the one-particle density and occupation numbers associated to $\omega$. Then the order parameter*

$$\langle \mathcal{O}_p \rangle := \frac{1}{p\gamma l} \int_0^{p\gamma l} \rho(x) e^{-i2\pi x/(p\gamma l)} dx = \frac{1}{p} \frac{1}{2\pi l^2} e^{-\frac{\pi^2}{p^2\gamma^2}} \sum_{j=0}^{p-1} n_j e^{-i2\pi j/p}$$

   *is an analytic function of $\gamma \in ]\gamma_p, \infty[$, and $\langle \mathcal{O}_p \rangle = 1 + O(e^{-\gamma^2})$ as $\gamma \to \infty$. Hence on $]\gamma_p, \infty[$ $\langle \mathcal{O}_p \rangle$ cannot vanish except possibly on a discrete set, and for sufficiently large $\gamma$, $\langle \mathcal{O}_p \rangle > 0$.*

*Remark:* For $p = 3$, the reversal symmetry implies $n_1 = n_{-1} = n_2$ and the order parameter equals

$$\langle \mathcal{O}_3 \rangle = \frac{1}{3\gamma l} \int_0^{3\gamma l} \rho(x) e^{-i2\pi x/(3\gamma l)} dx = \frac{1}{3} \frac{1}{2\pi l^2} e^{-\frac{\pi^2 k^2}{9\gamma^2}} (n_0 - n_1).$$

Thus $\langle \mathcal{O}_3 \rangle$ is really a measure of the amplitude of oscillations.

Again, we give details of the proof only for odd $p$. When $p$ is even, the occupation numbers $\hat{n}_k$ are unbounded operators (thus $\hat{n}_k \notin \mathcal{A}$). We start with the simpler part of the proof, namely with the proof of *2.*

*Proof of 2.* Recall from Theorem 2.12 that for $\gamma > \gamma_p$, the series $\sum_n \alpha_n t^n$ has a radius of convergence $R$ strictly larger than the radius of convergence $r$ of $\sum_n C_n t^n$. It follows that for $\gamma > \gamma_p$, the second moment condition (2.57) holds. Now let us turn to the order parameter. The analyticity is essentially a consequence of the analyticity of $r_p(\gamma)$ and $q_p(\gamma)$ (see Theorem 2.12). For the asymptotics of the order parameter as $\gamma \to \infty$, remember that the natural variable is $u = e^{-\gamma^2}$, and $u = 0$ describes a pure monomer system. The asymptotics follows from an expansion around the point $u = 0$. We give more details now.

Let $\hat{n}_k = c_k^* c_k$. Remember

$$n_k = \sum_{n=1}^{\infty} q r^n \alpha_n \sum_{j \in \mathbb{Z}} \langle u_n, \hat{n}_{k-pj} u_n \rangle / ||u_n||^2.$$

For $n \in \mathbb{N}$, we have

$$\sum_{k=0}^{p-1} e^{-i2\pi k/p} \sum_{j=-\infty}^{\infty} \langle u_n, \hat{n}_{k-pj} u_n \rangle = p \sum_{k \in \mathbb{Z}} e^{-i2\pi k/p} \langle u_n, \hat{n}_k u_n \rangle = p \sum_{k=0}^{pn-p} e^{-i2\pi k/p} \langle u_n, \hat{n}_k u_n \rangle$$

Let $\langle \mathcal{O}_p \rangle' := \frac{1}{p} \sum_{k=0}^{p-1} \langle \hat{n}_k \rangle e^{-i2\pi k/p}$. Then

$$\langle \mathcal{O}_p \rangle' = \sum_{n=1}^{\infty} q n r^n \alpha_n \langle \mathcal{O}_p \rangle'_n, \quad \langle \mathcal{O}_p \rangle'_n := \frac{1}{pn} \langle u_n, \big( \sum_{k=0}^{pn-p} e^{-i2\pi k/p} \hat{n}_k \big) u_n \rangle / ||u_n||^2.$$

Recall that $(qnr^n\alpha_n)$ represents a probability distribution on the lengths of polymers. The previous formula expresses the fact that each polymer contributes a part to $\langle \mathcal{O}_p \rangle$, depending only on its length. Both $||u_n||^2 = \alpha_n$ and $\langle u_n \hat{n}_k u_n \rangle$ are polynomials of $e^{-\gamma^2}$, for example

$$\langle u_n \hat{n}_k u_n \rangle = \sum_{\substack{0 \le m_1 \le .. \le m_N \le p(N-1) \\ \text{irred.}}} \chi_{\{m_1,..,m_N\}}(k) |b_N(m_1,..,m_N)|^2 e^{\gamma^2 \sum_{j=1}^N (m_j^2 - p^2(j-1)^2)}$$

where $b_N(m_1,..,m_N)$ are integers. By Theorem (2.12), $r(e^{-\gamma^2})$ and $q$ are analytic functions of $e^{-\gamma^2} \in [0, e^{-\gamma_p^2}[$. Since $|\langle \mathcal{O}_p \rangle'_n| \le 1$, and $\alpha_n(e^{-\gamma^2}) \le \alpha_n(e^{-\gamma'^2})$ for $\gamma \ge \gamma'$, the sum $\sum_n nr^n \alpha_n \langle \mathcal{O}_p \rangle'_n$ is a uniformly convergent sum of analytic functions, hence it is analytic itself. In particular, it has a convergent power series expansion in $e^{-\gamma^2} = 0$ around $u = 0$. Since $\alpha_n(e^{-\gamma^2} = 0) = 0$ for $n \ge 1$, $r(0) = 1$ and $q(0) = 1$, we obtain $\langle \mathcal{O}_p \rangle'_n = 1 + O(e^{-\gamma^2})$. □

*Proof of 1.* The main ingredients of the proof of *1.* are a lemma that we will state and prove below and the observation that orthogonality of states needs only be checked on a subalgebra. In the problem at hand, there is a natural choice of subalgebra: it is the algebra $\mathcal{A}_0$ generated by the occupation numbers $\hat{n}_k$. The algebra $\mathcal{A}_0$ is abelian and thus can be treated using a probabilistic language. The associated probability space $\Omega$ describes point particles on a lattice. The state $\omega$ on $\mathcal{A}$ is associated with a probability measure $P$ on $\Omega$, and shifted states $\omega \circ \tau_x^k$ correspond to shifted measures $P_k$. In order to prove orthogonality of the states, we need to prove that the shifted measures are mutually singular. The singularity of the measures follows from the existence of a random variable $Q$ with the following property: there is a set of *pairwise different* values $q_0, .., q_{p-1}$ such that for each $k \in \{0, .., p-1\}$, the random variable $Q$ takes the value $q_k$ $P_k$-almost surely. This means that the measures $P_k$ have their supports in the disjoint preimages under $Q$ of $q_0, .., q_{p-1}$.

The existence of such a random variable $Q$ is the object of Lemma 2.18 below. The main work to be done is the proof of this lemma, which we postpone to p.71 where the reader will find also a description of the physical intuition behind this lemma.

Let $\mathcal{A}_0$ be the closed subalgebra of $\mathcal{A}$ generated by the occupation number operators $\hat{n}_k = c_k^* c_k, k \in \mathbb{Z}$ and the identity $\mathbf{1}$. We start by showing that it is enough to check orthogonality on $\mathcal{A}_0$. Let $\omega_n := \omega \circ \tau_x^{pn}$, and let $\tilde{\omega}$ be a positive functional on $\mathcal{A}$ with $\tilde{\omega}(a) \le \omega_0(a)$ and $\tilde{\omega}(a) \le (\omega_1(a) + ... + \omega_{p-1}(a))$ for all $a \in \mathcal{A}$. In particular, the inequalities hold for $a \in \mathcal{A}_0$. If the states $\omega_j|_{\mathcal{A}_0}$ are orthogonal, it follows that $\tilde{\omega}|_{\mathcal{A}_0} = 0$. Since $\mathbf{1} \in \mathcal{A}_0$, we have $\tilde{\omega}(\mathbf{1}) = 0$. By the positivity of $\tilde{\omega}$, for all $a \in \mathcal{A}$, $|\tilde{\omega}(a)| \le \tilde{\omega}(\mathbf{1})||a|| = 0$, whence $\tilde{\omega} = 0$.

Thus we need only show that the restrictions of $\omega_0, ., \omega_{p-1}$ to $\mathcal{A}_0$ are orthogonal. But $\mathcal{A}_0$ is a commutative algebra and we can use a probabilistic language. Let us briefly recall the construction of the probability space. The occupation number operator $\hat{n}_k$ gives the number of particles at lattice site $k$. The probability space $\Omega$ we will consider describes particles on a lattice. Each particle configuration $X \in \Omega$ is characterized uniquely by the sequence of occupation numbers $(n_k(X))_{k \in \mathbb{Z}}$, where $n_k(X)$ gives the number of particles at the lattice site $k$. For simplicity, suppose $p$ odd. Then Laughlin's function is antisymmetric and thus describes fermions. There is at most one particle per lattice site and we may take $\Omega = \mathcal{P}(\mathbb{Z}) \equiv \{0,1\}^{\mathbb{Z}}$, where we identify sets with their characteristic functions. We give $\{0,1\}$ the discrete topology and $\{0,1\}^{\mathbb{Z}}$ the product topology, and let $\mathcal{E}$ be the Borel-$\sigma$-algebra. The mapping $\hat{n}_k \mapsto n_k(\cdot)$ extends uniquely to a homomorphism $\phi$ from $\mathcal{A}_0$ to $C(\Omega)$, and there is a unique probability measure $P$ on $(\Omega, \mathcal{E})$ such that for all $a \in \mathcal{A}_0$, $\omega(a) = \int \phi(a) dP$.

The probability of cylinder sets can be specified as follows: for $K, A \subset \mathbb{Z}$,

$$P(\{X \mid X \cap K = A\}) = \omega\big(\prod_{k\in A} \hat{n}_k \prod_{k\in K\setminus A} (1-\hat{n}_k)\big). \tag{2.58}$$

Equivalently, for all $K = \{k_1, .., k_r\} \subset \mathbb{Z}$, $|K| = r < \infty$, and all $(\epsilon_1, .., \epsilon_r) \in \{0,1\}^r$,

$$P(\{X \mid n_{k_1}(X) = \epsilon_1, .., n_{k_r}(X) = \epsilon_r\}) = \omega\big(\prod_{\substack{k\in K\\ \epsilon_k=1}} \hat{n}_k \prod_{\substack{k\in K\\ \epsilon_k=0}} (1-\hat{n}_k)\big).$$

Shifted states area associated with shifted measures: let $\tau : \Omega \to \Omega$, $X \mapsto X+1$. Then the measures $P_j$ associated to the states $\omega_j = \omega \circ \tau_x^j$ are given by $P_j = P \circ \tau^j$. To conclude the proof, we need the following lemma:

**Lemma 2.18.** *There is a map $\mathcal{Q} : \Omega \to \mathbb{R}^{\mathbb{Z}}$, $X \mapsto (\mathcal{Q}_X(x))_{x\in\mathbb{Z}}$ such that:*

- *for all $X \in \Omega$, for all $x, s \in \mathbb{Z}$: $\mathcal{Q}_{X+s}(x) = \mathcal{Q}_X(x-s)$, i.e., $\mathcal{Q}$ is "covariant";*
- *for $P$-almost all $X \in \Omega$, for all $x \in \mathbb{Z}$: $\mathcal{Q}_X(x) \in -x/p + \mathbb{Z}$.*

The lemma will be proved a little later. For a given set of particles, $\mathcal{Q}_X(x)$ should be thought of as the total charge accumulated in $]-\infty, x[$, where the particles account for a charge 1 and a uniform background of density $-1/p$ is added. By Lemma 2.18, there exists a measurable subset $\mathcal{M}$ of $\Omega$ such that $P(\mathcal{M}) = 1$ and for all $X \in \mathcal{M}$ and $x \in \mathbb{Z}$: $\mathcal{Q}_X(x) \in -x/p + \mathbb{Z}$. We have

$$P(\mathcal{M}) = 1 \quad \text{and for all } X \in \mathcal{M} : \ \mathcal{Q}_X(0) \in \mathbb{Z}.$$

It follows that for $j \in \mathbb{Z}$, the shifted measures $P_j = P \circ \tau^j$ satisfy

$$P_j(\mathcal{M} - j) = 1 \quad \text{and for all } X \in \mathcal{M} + j : \ \mathcal{Q}_X(0) \in -j/p + \mathbb{Z}. \tag{2.59}$$

To see this, let $X \in \mathcal{M} - j$; we may write $X = Y - j$ with $y \in \mathcal{M}$ and

$$\mathcal{Q}_X(0) = \mathcal{Q}_{Y-j}(0) = \mathcal{Q}_Y(j) \in -j/p + \mathbb{Z}.$$

By (2.59), the measures $P = P_0, P_1, .., P_{p-1}$ are mutually singular.

As a consequence, the states $\omega_j$ on $\mathcal{A}_0$ are orthogonal. Indeed, let $\tilde{\omega}$ be a positive functional on $\mathcal{A}_0$ with $\tilde{\omega} \le \omega_0$ and $\tilde{\omega} \le (\omega_1 + .. + \omega_{p-1})$. Let $\mu$ be the measure (not necessarily a probability measure) on $(\Omega, \mathcal{F})$ associated to $\tilde{\omega}$ through a relation similar to (2.58). Then, for all $E \in \mathcal{F}$, $\mu(E) \le P_0(E)$ and $\mu(E) \le (P_1(E) + .. + P_{p-1}(E))$. Applying this to $E_j = \mathcal{M} + j$, we get $\mu(E_j) = 0$ for all $j \in \{0, .., p-1\}$, hence $\mu = 0$ and $\tilde{\omega} = 0$. □

The proof of Lemma 2.18 follows ideas of [AM80, AGL01]. We start with a summary of the ideas of these works that we shall use. In [AM80], Section 4, Aizenman and Martin consider a one-dimensional classical jellium system, consisting of $N$ particles of negative charge $-e$ moving in a neutralizing background of charge density $\rho e$ on a line segment $[-L/2, L/2] \subset \mathbb{R}$. For simplicity, we will assume $e = 1$. The one-dimensional Coulomb potential is $-|x|$. Each particle configuration is characterized by a set of particle positions $x_1, .., x_N$. The electric field at $x$ is a function of the particle positions: for $x \in [-L/2, L/2]$,

$$E(x; \{x_j\}) = -\sum_{j=1}^{N} \mathrm{sgn}(x - x_j) + 2\rho x. \tag{2.60}$$

(Actually, the field is not well-defined at the location of a particle, $x = x_j$. We shall not deal with this difficulty here.) Due to the overall neutrality of the system, the electric field vanishes outside the volume containing the charges:

$$E(-L/2; \{x_j\}) = 0 = E(L/2; \{x_j\}). \tag{2.61}$$

Furthermore, for $x \leq y$,

$$E(y; \{x_j\}) - E(x; \{x_j\}) = 2\rho(y - x) - 2 \sum_{x_j \in [x,y]} 1. \tag{2.62}$$

It follows that

$$E(x; \{x_j\}) = E(x; \{x_j\}) - E(-L/2; \{x_j\}) = 2\rho x + N - 2 \sum_{x_j \in [-L/2;x]} 1$$

Suppose we take $N$ even; then

$$E(x; \{x_j\}) \in 2\rho x + 2\mathbb{Z}. \tag{2.63}$$

This constraint will subsist in the limit $N \to \infty$, the density $\rho$ being kept fixed, and shows that the limiting distribution of the field $E$ cannot be translationally invariant. It can at best be periodic with period $1/\rho$. We would like to know whether the particle positions have a translationally invariant distribution. In the limit of infinite volume, the particle positions can be recovered from the electric field as the locations of its discontinuities. But a function of a periodic quantity may very well be translationally invariant, so this is of no help. In order to conclude that the particle positions have a non translationally invariant distribution, we need to know that the field is a function of the particle positions. This is of course the case for finite volumes, but for systems with infinitely many particles the formula (2.60) no longer makes sense. Nevertheless, the field *can* be recovered from the particle positions. This is shown in [AM80] using an ergodic theorem.

An abstract version of the argument previously described is considered in [AGL01], who look at point processes and at one-dimensional charge distributions. Let us sketch some aspects of [AGL01] for the special case of one species of particles moving on a lattice. Let $P$ be a translationally invariant probability distribution on the space of particle configurations $\mathcal{P}(\mathbb{Z}) \equiv \{0,1\}^{\mathbb{Z}}$. Let $\rho$ be the average number of particles at a given site. We might think of the particles as point charges with charge 1 moving in a neutralizing background of charge density $-\rho$. For a given configuration $X \subset \mathbb{Z}$ and $a, b \in \mathbb{Z}$ $(a < b)$, define

$$F_X(]a,b]) := |X \cap ]a,b]| - \rho(b - a). \tag{2.64}$$

$F_X(]a,b])$ is the difference between the number of particles in $]a,b]$ in the configuration $X$ and the average number of particles in $]a,b]$. It can also be considered as the total "charge" in $]a,b]$. [AGL01] prove that if the random variables $X \mapsto F_X(]a,b])$, $a, b \in \mathbb{Z}$, $a < b$ have variances uniformly bounded in $a, b$ (actually, [AGL01] give weaker conditions), there exists a *covariant antiderivative*, i.e., there exists a family of random variables $X \mapsto \mathcal{E}_X(a)$, $a \in \mathbb{Z}$, such that for $P$-almost all $X \subset \mathbb{Z}$ and all $a, b \in \mathbb{Z}$, $a < b$:

$$\mathcal{E}_{X+1}(a) = \mathcal{E}_X(a - 1), \quad F_X(]a,b]) = \mathcal{E}_X(b) - \mathcal{E}_X(a). \tag{2.65}$$

Combining the second identity with (2.64) gives an expression for $\mathcal{E}_X(b) - \mathcal{E}_X(a)$ analogous to (2.62). Thus $\mathcal{E}_X(a)$ can be thought of as the "field" in $a$, or simply as the total charge

accumulated in $]-\infty, a]$. Now the probability space can be partitioned into level sets of $X \mapsto \mathcal{E}_X(0)$ (modulo 1). For $\theta \in \mathbb{R}/\mathbb{Z}$, let

$$\mathcal{M}_\theta := \{X \in \mathcal{P}(\mathbb{Z}) \mid \mathcal{E}_X(0) = \theta \mod 1\}.$$

Using (2.65), one easily sees that the shift $T_k : X \mapsto X + k$ maps $\mathcal{M}_\theta$ onto $\mathcal{M}_{\theta+k\rho}$. If $\rho$ is not an integer, there are several sets $\mathcal{M}_\theta$, and [AGL01] prove that the probability measure $P$ decomposes into a mixture of mutually singular measures $P_\theta$, supported in $\mathcal{M}_\theta$. In the particular case that the density is rational and not integer: $\rho = p/q$, $p, q$ coprime and $q \geq 2$, we obtain a decomposition of the probability space into $q$ subsets invariant under shifts by multiples of $q$. In particular, the probability measure $P$ cannot be ergodic with respect to shifts by multiples of $q$: in this sense, there is symmetry breaking.

Now let us come back to Lemma 2.18. The idea of proof is a combination of ideas found in [AM80] and [AGL01]. From [AGL01] we take the idea to construct a generalized "electric field" or "total charge" random variable. This is the variable $\mathcal{Q}_X(x)$ of Lemma 2.18. For the construction of the random variable, we follow the idea [AM80] of looking at finite volumes first and then making a detour involving a larger probability space and an ergodic theorem. As we have seen in the proof of Theorem 2.17, the level sets of the field play an important role for the symmetry breaking, just as in [AGL01] and [AM80].

*Proof of Lemma 2.18.* The starting point is the following: the state $\omega$ is associated with a probability measure $P$ on $\Omega = \mathcal{P}(\mathbb{Z})$, where a particle configuration is characterized through the set $X \subset \mathbb{Z}$ of occupied lattice points. The measure $P$ is invariant with respect to shifts (on $\mathbb{Z}$) by multiples of $p$. By Corollary 2.15, the average density is

$$\frac{1}{p}\sum_{j=0}^{p-1}\langle \hat{n}_j \rangle = \frac{1}{p}.$$

We think of the particles as point charges of charge 1 moving in a neutralizing background of charge density $-1/p$. Each configuration $X \subset \mathbb{Z}$ defines a "charge distribution" $F$ on $\mathbb{Z}$ through

$$F_X(A) := |X \cap A| - \frac{1}{p}|A|, \quad A \subset \mathbb{Z}.$$

The previous considerations suggest to search for a covariant antiderivative of $F_X$, i.e. to search for a map $X \mapsto (\mathcal{Q}_X(x))_{x\in\mathbb{Z}}$ such that

$$\mathcal{Q}_{X+s}(x) = \mathcal{Q}_X(x-s), \quad F_X([x,y[) = \mathcal{Q}_X(y) - \mathcal{Q}_X(x)$$

and to show that the probability measure $P$ is supported in the set where $\mathcal{Q}_X(x) = -x/p \mod 1$. We will proceed as follows:

1. We start by looking at finite volumes. For systems with finitely many particles $N$, we can simply define $\mathcal{Q}_X^N(x)$ as the total charge accumulated in $]-\infty, x[$ (with a suitable background distribution). The field thus constructed is not covariant, but essentially satisfies the condition $\mathcal{Q}_X^N(x) = -x/p \mod 1$. We will see that finite volume configurations may be characterized through an auxiliary space $(\Omega'_N, \mathcal{F}'_N, P'_N)$; this serves as a guide for the infinite volume case.

2. As a second step, we construct an auxiliary probability space $(\Omega', \mathcal{F}', P')$ space from which the particle positions may be recovered, i.e., there is a random variable $M : \Omega' \to$

$\Omega = \mathcal{P}(\mathbb{Z})$ such that the measure $P$ is the image of $P'$ under $M$: $P = P' \circ M$. The approach in [AM80] suggests to take as larger probability space $\Omega'$ the configurations of the field, but we will see that in our setting, a different construction is more natural; the auxiliary space arises roughly as infinite volume limit of the space $\Omega'_N$ of 1.

3. We proceed by showing that there exists a field $\Omega' \ni \nu \mapsto (Q_\nu(x))_{x\in\mathbb{Z}}$, defined on the auxiliary space, that is an antiderivative of the particle positions in the sense that

$$F_{M(\nu)}([x, y[) = Q_\nu(y) - Q_\nu(x),$$

where $M : \Omega' \to \Omega$ reconstructs the particle locations from the auxiliary variable $\nu \in \Omega'$. We show that $Q$ has a covariance property, that $Q_\nu(x) = -x/p \mod 1$ $P'$-almost surely and that $\nu \mapsto Q_\nu(x)$ has finite expectation value; this is the place where the second moment condition (2.57) enters.

4. Just as in [AM80], we use an ergodic theorem for $Q_\nu(x)$ to show that the field $Q_\nu$ can be recovered from the particle locations, i.e., there is a map $\mathcal{Q} : \Omega \ni X \mapsto (\mathcal{Q}_X(x))_{x\in\mathbb{Z}}$ such that

$$\forall x \in \mathbb{Z} : \; Q_\nu(x) = \mathcal{Q}_{M(\nu)}(x) \quad \text{for } P' - \text{almost all } \nu \in \Omega'.$$

The map $\mathcal{Q}$, suitably defined, is the map of Lemma 2.18.

*1. Finite volumes.* For $N \in \mathbb{N}$, let $\langle\cdot\rangle_N$ be the state defined by $\Psi_N$ and $P_N$ the corresponding probability measure on $\Omega = \mathcal{P}(\mathbb{Z})$. Note that $X \subset \{0, .., pN - p\}$ and $|X| = N$ $P_N$-almost surely. More precisely, let $a_N(m_1, .., m_N)$ be the expansion coefficients of Laughlin's $N$-particle state, $C_N$ its $L^2$-norm squared, then the probability $P_N$ is defined as

$$P_N(\{m_1, .., m_N\}) = |a_N(m_1, .., m_N)|^2/C_N$$

where for $X \subset \mathbb{Z}$ we write $P(X)$ instead of $P(\{X\})$. Think of $P_N$ as describing $N$ particles located in $\{0, .., pN - 1\}$, with a uniform neutralizing background so that the total charge accumulated in $]-\infty, x[$ vanishes if $x \leq 0$ or $x \geq pN$. Let

$$\mathcal{Q}^N_X(x) := \begin{cases} |X \cap [0, x[| - \frac{1}{p}x, & \text{if } x \in \{0, .., pN\}, \\ 0, & \text{else.} \end{cases}$$

For $X \subset \{0, .., pN - 1\}$, $|X| = N$, we have

$$\mathcal{Q}^N_X(0) = \mathcal{Q}^N_X(pN) = 0.$$

This equation is the analogue of (2.61). If $x, y$ are in $\{0, .., pN\}$ and $x \leq y$, we have obviously

$$\mathcal{Q}^N_X(y) - \mathcal{Q}^N_X(x) = |X \cap [0, x[| - \frac{1}{p}x;$$

in this sense, $\mathcal{Q}^N_X$ is an antiderivative. Furthermore, it is obvious from the definition of $\mathcal{Q}^N$ that

$$\forall X \subset \mathbb{Z}, \; \forall x \in \{0, .., pN\} : \; \mathcal{Q}^N_X(x) = -x/p \mod 1.$$

This is the analogue of the constraint (2.63) on the electric field for one-dimensional jellium.

Now let us use the additional structure present here. First, note that if $X \subset \mathbb{Z}$ has a non-zero probability, $P_N(X) > 0$, there exists an increasing, $N$-admissible finite sequence $m = (m_1, .., m_N)$ such that $X = \{m_1, .., m_N\}$. Now, we claim that:

Let $m = (m_1, .., m_N)$ be increasing $m_1 \leq .. \leq m_N$ and $N$-admissible. Let $X = \{m_1, .., m_N\}$ with $P_N(X) > 0$. Suppose $k$ is a renewal point of $m$. Then $\mathcal{Q}_X^N(pk) = 0$ and for $pk \leq x \leq pN$,

$$\mathcal{Q}_X^N(x) = |X \cap [pk, x[\,| - (x - pk)/p.$$

Thus renewal points give zeroes of the field, and the field at $x$ depends only on the particle configuration up to the next renewal point.

To see why this holds, let $m$, $X$, $k$. $P_N(X) > 0$ implies that $(m_1, .., m_k)$ and $(m_{k+1} - pk, .., m_N - pk)$ are $k$ (resp. $N-k$)-admissible. In particular, $0 \leq m_1, .., m_k \leq pk - p$, and $pk \leq m_{k+1}, .., m_N \leq pN - p$, therefore if $x \geq pk$

$$\begin{aligned}\mathcal{Q}_X^N(x) &= \big(|X \cap [0, pk[\,| - k\big) + \big(|X \cap [pk, x[\,| - (x - pk)/p\big) \\ &= \big(|\{m_1, ., m_k\}| - k\big) + \big(|X \cap [pk, x[ - (x - pk)/p\big) \\ &= |X \cap [pk, x[ - (x - pk)/p,\end{aligned}$$

whence the claim.

*Definition of an auxiliary probability space: finite volumes.* We would like to use the observation on renewal points to define $\mathcal{Q}_X$ for infinite systems. However, defining renewal points for infinite sets $X \subset \mathbb{Z}$ seems even harder than defining a total charge, thus we have to make a little detour. Recall from the proof of Lemma 2.5 that $N$-admissible sequences can be characterized in terms of sequences $\nu_0, .., \nu_N$ with

$$\begin{gathered}m_j = p(j-1) + \nu_j - \nu_{j-1}, \quad \nu_0 = \nu_N = 0 \\ \nu_j \geq 0, \quad \nu_{j+1} - 2\nu_j + \nu_{j-1} + p \geq 1.\end{gathered}$$

Furthermore $\nu_k = 0$ if and only if $k$ is a renewal point of $m$. The probability measure $P_N$ defines a measure $P'_N$ on the space $\Omega'_N$ of $(\nu_j)$-sequences through

$$P'_N(\{(0, \nu_1, .., \nu_N, 0)\}) = P_N(\{(m_1, .., m_N)\}) = |a_N(m_1, .., m_N)|^2 / C_N.$$

The factorization rule from Lemma 2.4 leads to

$$\begin{aligned}&P'_N(\{0, \nu_1, .., \nu_{k-1}, 0, \nu_{k+2}, .., \nu_{N-1}, 0)\}) = \\ &\qquad C_k P'_k(\{(0, \nu_1, .., \nu_{k-1}, 0)\})\, C_{N-k} P'_{N-k}(\{(0, \nu_{k+1}, .., \nu_{N-1}, 0)\})/C_N,\end{aligned}$$

moreover $P'_N(\nu_k = 0) = C_k C_{N-k}/C_N$ for all $k \in \{0, .., N\}$. Thus for $k \to \infty$, $N - k \to \infty$, $P'_N(\nu_k = 0) \to \mu^{-1} = q$.

*2. Auxiliary probability space: infinite volume.* More generally, there is a natural probability measure on the space of infinite sequences. Let $\Omega' \subset \mathbb{N}_0^{\mathbb{Z}}$ be the set of sequences $(\nu_j)_{j \in \mathbb{Z}}$ such that

$$\forall j \in \mathbb{Z}: \quad \nu_{j+1} - 2\nu_j + \nu_{j-1} + p \geq 1$$

and the set $\{j \mid \nu_j = 0\}$ is bounded neither from below nor above. Define a probability measure $P'$ on $\Omega'$ by requiring that the zeroes of $(\nu_j)$ are distributed like a stationary renewal process with interarrival distribution $(p_n) = (r^n \alpha_n)$, constant renewal probability

$$P'(\nu_j = 0) = q = \mu^{-1} = 1/(\sum_n n r^n \alpha_n),$$

and for all $a \in \mathbb{Z}$, $N \in \mathbb{N}$, and $\epsilon_{a+1}, .., \epsilon_{a+N-1}$ in $\mathbb{N}_0$,

$$P'(\{(\nu_j) \mid \nu_a = 0, \nu_{a+1} = \epsilon_{a+1}, ..., \nu_{a+N-1} = \epsilon_{a+N-1}, \nu_{a+N} = 0\}) \\ = \mu^{-1} C_N r^N P'_N(\{(0, \epsilon_{a+1}, .., \epsilon_{a+N-1}, 0)\}).$$

The measure $P'$ is invariant and mixing with respect to shifts $(\nu_j) \mapsto (\nu_{j-1})$. The probability measure $P$ on $\Omega$ is recovered with

$$M : \Omega' \to \Omega, \quad (\nu_j) \mapsto \{p(j-1) + \nu_j - \nu_{j-1} \mid j \in \mathbb{Z}\}$$

as $P = P' \circ M$. Note that $M((\nu_{j-1})) = p + M((\nu_j))$.

*3. Definition of an auxiliary map Q.* For $\nu \in \Omega'$ and $x \in \mathbb{Z}$, let $j \in \mathbb{Z}$ be such that $\nu_j = 0$ and $pj \leq x$. Define $Q_\nu(x) := F_{M(\nu)}([pj, x[)$. This definition is motivated by the observation made in 1. that a renewal point arising through $\nu_j = 0$ give zeroes of the field $Q_\nu(pj) = 0$. Note that the definition does not depend on the choice of $j$: suppose $pj' < pj \leq x$ and $\nu_{j'} = \nu_j = 0$. Let $m_k := p(k-1) + \nu_{k+1} - \nu_k$. By the convexity property of $(\nu_k)_k$, $(m_k)$ is increasing, furthermore $\nu_k \geq 0$ implies

$$m_{j'} \leq p(j'-1), \quad pj' \leq m_{j'+1} \leq .. \leq m_j \leq p(j-1), \quad pj \leq m_{j+1}..$$

whence $M(\nu) \cap [pj', pj[ = \{m_{j'+1}, .., m_j\}$ and

$$F_{M(\nu)}([pj', x[) - F_{M(\nu)}([pj, x[) = F_{M(\nu)}([pj', pj[) = 0.$$

$Q_\nu$ has a series of properties:

1. $Q_\nu$ satisfies a constraint on its range: $Q_\nu(x) = -x/p \mod 1$ for all $\nu \in \Omega$
2. $Q_\nu$ is an antiderivative of the charge density $F$: if $x < y$, $Q_\nu(y) - Q_\nu(x) = F_{M(\nu)}([x, y[)$
3. $Q_\nu$ is covariant: $Q_{\tau\nu}(x) = Q_\nu(x - p)$, where $(\tau\nu)_j = \nu_{j-1}$
4. The field at $x$ has finite expectation value: $\int_{\Omega'} |Q_\nu(x)| dP'(\nu) < \infty$ for all $x \in \mathbb{Z}$.

Indeed, let $x \in \mathbb{Z}, j \in \mathbb{Z}$ with $\nu_j = 0$, $pj \leq x$, then

$$Q_\nu(x) = F_{M(\nu)}([pj, x[) = |M(\nu) \cap [pj, x[| - (x - pj)/p = -x/p \mod 1.$$

If $x < y$,

$$Q_\nu(y) - Q_\nu(x) = F_{M(\nu)}([pj, y[) - F_{M(\nu)}([pj, x[) = F_{M(\nu)}([x, y[).$$

Choose $j$ small enough so that $p(j+1) \leq x$ and let $\nu'_j := \nu_{j-1}$. Then $\nu'_{j+1} = 0$, and

$$Q_{\tau\nu}(x) = Q_{\nu'}(x) = F_{M(\tau\nu)}([p(j+1), x[) \\ = F_{p+M(\nu)}([pj + p, x[) = F_{M(\nu)}([pj, x - p[) = Q_\nu(x - p).$$

Finally, suppose $pj \leq x \leq p(j+n-1)$ and $\nu_j = \nu_{j+n} = 0$, $j \in \mathbb{Z}$, $n \in \mathbb{N}$. Then $M(\nu) \cap [pj, p(j+n) - p] = \{m_{j+1}, .., m_{j+n}\}$ and $|Q_\nu(x)| \leq n + (x - pj)/p \leq 2n$. For $\nu \in \Omega'$, let $n_\nu(x)$ be the smallest possible $n$. Then

$$\int |Q_\nu(x)| dP'(\nu) \leq 2 \int_{\Omega'} n_\nu(x) dP'(\nu) = 2 \sum_{n=1}^{\infty} n \cdot n r^n \alpha_n < \infty,$$

where we used that $P(n_\nu(x) = n) = nr^n\alpha_n$.

*4. Definition of $\mathcal{Q}_X(x)$.* We wish to define $\mathcal{Q}_X(x)$ such that $Q_\nu(x) = \mathcal{Q}_{M(\nu)}(x)$. Note that by $P = P' \circ M$, for $P$-almost all $X$, there is a $\nu \in \Omega'$ such that $X = M(\nu)$. We need to check however, that $Q_\nu(x)$ depends only on $M(\nu)$, i.e. $M(\nu) = M(\nu')$ implies $Q_\nu = Q_{\nu'}$. This is done by means of the ergodic theorem ($P'$ is mixing with respect to shifts). Let

$$\bar{Q} := \frac{1}{p}\int (Q_\nu(x) + Q_\nu(x+1) + .. + Q_\nu(x+p-1))dP'(\nu).$$

Then for $P'$-almost all $\nu \in \Omega'$,

$$\begin{aligned}\bar{Q} &= \lim_{k\to\infty}\frac{1}{pk}\sum_{r=0}^{k-1}\big(Q_{\tau^{-r}\nu}(x) + .. + Q_{\tau^{-r}\nu}(x+p-1)\big)\\ &= \lim_{k\to\infty}\frac{1}{pk}\sum_{r=0}^{k-1}\big(Q_\nu(x+pr) + .. + Q_\nu(x+pr+p-1)\big)\\ &= \lim_{k\to\infty}\frac{1}{pk}\sum_{y=x}^{x+pk-1} Q_\nu(y) = \lim_{k\to\infty}\frac{1}{pk}\sum_{y=x}^{x+pk-1}(Q_\nu(x) + F_{M(\nu)}([x,y[))\end{aligned}$$

It follows that for $P'$-almost all $\nu \in \Omega$,

$$Q_\nu(x) = \bar{Q} - \lim_{k\to\infty}\frac{1}{pk}\sum_{y=x}^{x+pk-1} F_{M(\nu)}([x,y[).$$

The right-hand side depends on $M(\nu)$ only, hence $Q_\nu(x) = \mathcal{Q}_{M(\nu)}(x)$ for $P'$-almost all $\nu$ with

$$\mathcal{Q}_X(x) := \begin{cases}\bar{Q} - \lim_{k\to\infty}\frac{1}{pk}\sum_{y=x}^{x+pk-1} F_X([x,y[), & \text{if the limit exists}\\ 0, & \text{else.}\end{cases}$$

If the limit $\mathcal{Q}_X(x)$ exists for some $x \in \mathbb{Z}$, it exists for all $y \in \mathbb{Z}$. Let $\mathcal{M}'$ be the set of $X \in \Omega$ such that the limit exists. Then

$$\forall X \in \mathcal{M}' :\ \mathcal{Q}_X(y) - \mathcal{Q}_X(x) = F_X([x,y[).$$

For $s, x \in \mathbb{Z}$,

$$\begin{aligned}-\frac{1}{pk}\sum_{y=x}^{x+pk-1} F_{X+s}([x,y[) = -\frac{1}{pk}\sum_{y=x}^{x+pk-1} F_{X+s}([x,y[) &= -\frac{1}{pk}\sum_{y=x}^{x+pk-1} F_X([x-s,y-s[)\\ &= -\frac{1}{pk}\sum_{y'=x-s}^{x+pk-1-s} F_X([x-s,y'[)\end{aligned}$$

hence if $X \in \mathcal{M}'$, then also $X + s \in \mathcal{M}'$ and $\mathcal{Q}_{X+s}(x) = \mathcal{Q}_X(x-s)$. This relation is also true if $X \notin \mathcal{M}'$, in which case it reads $0 = 0$. Thus

$$\forall X \in \Omega,\ \forall x, s \in \mathbb{Z} :\ \mathcal{Q}_{X+s}(x) = \mathcal{Q}_X(x-s).$$

Finally, $\mathcal{Q}_X$ inherits from $Q_\nu$ the condition on its range:

$$\mathcal{Q}_X(x) = -x/p \mod 1 \quad \text{for } P\text{-almost all } X. \qquad \square$$

Theorem 2.17 answers the question whether there is symmetry breaking in Laughlin's state only for thin cylinders. It leaves open what happens on large strips. In view of the numerical results mentioned at the beginning of this section, we believe that, at least for small $p$, the result applies to broad cylinders, but this still awaits a proof.

## 2.2 Solvable models

In this section, we define generalized Laughlin-type cylinder and torus wave functions. The starting point for the cylinder function is the representation as a power of a Slater determinant of Gaussians. The generalized function is obtained by allowing functions $f$ different from Gaussians. These functions are interesting mainly for two reasons.

The first reason is that for particular choices of $f$, the modified cylinder function defines a solvable model: if we choose $f$ as a function with sufficiently small, compact support, the modified cylinder function is associated with a monomer-system on $\mathbb{Z}$. The partition function of a monomer-dimer system on a linear chain satisfies a two-step recurrence relation (see e.g. [HL72], p.196). Consequently the $L^2$-norm squared of the modified function satisfies a two-step recurrence relation, which is of course explicitly solvable. The correlation functions can be computed explicitly as well. We look only at the one-particle density.

More generally, the modified cylinder functions retain a lot of the structure present for Laughlin's cylinder function, and most of the results of Section 2.1 have an analogue. In particular, the modified wave functions are associated with polymer systems. An important parameter in the description of Laughlin's wave function is the ratio $\gamma = l/R$ of magnetic length and cylinder radius. We have seen that as $\gamma \to \infty$, the system approaches a monomer system. A similar phenomenon appears here: the definition of modified functions involves not only the function $f$ but also its translates $f(\cdot - pn\gamma l), n \in \mathbb{Z}$. We choose $f$ as a function of compact support. Then we observe the following phenomenon:

1. When $\gamma$ is very large, $f(\cdot)$ and its neighbors $f(\cdot - n\gamma l)$ have non-overlapping supports, and the associated polymer system is a monomer system.

2. When $\gamma$ is such that only nearest neighbors $f(\cdot)$ and $f(\cdot \pm \gamma l)$ overlap, we obtain the monomer-dimer system mentioned above.

3. As soon as $\gamma$ is small enough so that next nearest neighbors $f(\cdot)$ and $f(\cdot - 2n\gamma l)$ overlap, the associated polymer system has polymers of non-vanishing activity of arbitrary length.

The second motivation is the comparison of cylinder and torus functions. Up to now, this chapter has only been concerned with cylinder functions, and our goal was to show that they display a kind of symmetry breaking. This kind of symmetry breaking is well-known for torus functions [HR85b, Hal85]. A torus can be seen as a cylinder with periodic boundary conditions. It is widely accepted that for small $p$ ($p \leq 7$), Laughlin's function describes an incompressible liquid. Thus boundary conditions should not affect the bulk behavior, and we expect that the cylinder and torus wave functions are equivalent in the limit of long cylinder / tori. We check this equivalence for the solvable monomer-dimer model: when the cylinder modified function is associated with a monomer-dimer system on a linear chain, a suitably defined modified torus function is associated with a monomer-dimer system on a ring, with possibly one additional long polymer covering the whole ring.

### 2.2.1 Generalized Laughlin wave functions

We start by giving the definition of generalized cylinder and torus functions $\Phi_N^C$, $\Phi_N^T$. In Proposition 2.20, we verify that the modified functions have periodicity properties that justify the denominations "cylinder" and "torus" functions. The definition of the modified functions involves a function $f : \mathbb{R} \to \mathbb{R}$. In Proposition 2.21, we check that we recover the Laughlin wave functions of Definition 1.1 if we choose $f$ as a suitable Gaussian.

Let $f : \mathbb{R} \to \mathbb{C}$ be a continuous even function ($f(x) = f(-x)$) such that $\int_{-\infty}^{\infty} |f|^{2p} = 1$. For $n \in \mathbb{N}$ and $N \in \mathbb{N}$, define

$$\phi_n(z) = \frac{1}{\sqrt[2p]{2\pi R}} e^{iny/R} f(x - pn\gamma l), \quad \tilde{\phi}_n(z) = \sum_{k=-\infty}^{\infty} \phi_{n+kN}. \tag{2.66}$$

**Definition 2.19.** *Let $f, \phi_n, \tilde{\phi}_n$ be as above. The generalized Laughlin cylinder and torus wave functions for $N$ electrons at filling factor $1/p$ are the functions*

$$\Phi_N^C(z_1, .., z_N) = \frac{1}{\sqrt{N!}} \Big( \det(\phi_{n-1}(z_j))_{1 \le n,j \le N} \Big)^p.$$
$$\Phi_N^T(z_1, .., z_N) = \frac{1}{\sqrt{N!}} \prod_{s=1}^{p} \det(e^{iX_s y_j/(pNl^2)} \tilde{\phi}_{n-1}(z_j - \frac{Z_s}{N}))_{1 \le n,j \le N}$$

*where $Z_1, .., Z_p \in \mathbb{C}$ satisfy $\sum_{s=1}^p X_s/(pN\gamma l) \in \mathbb{Z}$, $\sum_{s=1}^p Y_s/(2\pi R) \in \mathbb{Z}$.*

The conditions on $Z_1, .., Z_p$ are similar to conditions imposed on the zeroes of theta functions, see [FK01], Chapter 7. Let us remark that there is a small difference between the definitions of $\Phi_N^C$ and $\Phi_N^T$: the cylinder function is, up to a prefactor, the $p$-th power of a determinant, whereas the torus function is in general only a product of $p$ determinants.

Recall the expression of the magnetic translation in the $x$-direction:

$$(t(a\mathbf{e}_x)\psi)(z) = e^{iay/l^2} \psi(z - a).$$

If $\Psi$ is an $N$-particle wave function, we denote by $t_j(a)$, $j \in \{1, .., N\}$ the magnetic translation in the $j$-th variable. For example, $(t_1(a\mathbf{e}_y)\Psi)(z_1, .., z_N) = \Psi(z_1 - ia, z_2, .., z_N)$.

**Proposition 2.20.** *The functions $\Phi_N^C$ and $\Phi_N^T$ have the following periodicity:*

$$t_j(2\pi R\mathbf{e}_y)\Phi_N^T = \Phi_N^T, \quad t_j(2\pi R\mathbf{e}_y)\Phi_N^C = \Phi_N^C,$$
$$t_j(pN\gamma l\mathbf{e}_x)\Phi_N^T = \Phi_N^T$$

*for all $j \in \{1, .., N\}$.*

**Proposition 2.21.** *Let*

$$f(x) = \frac{1}{\sqrt[2p]{l\sqrt{\pi}}} e^{-\frac{1}{2pl^2}x^2}.$$

*Then the modified cylinder functions equal Laughlin's function $\Phi_N^C = \Psi_N^C$, and the functions $\Phi_N^T$ are torus Laughlin wave functions. Conversely, any torus Laughlin wave function $\Psi_N^T$ is of the form $\Psi_N^T = \lambda\Phi_N^T$ with $\lambda \in \mathbb{C}$ and $\Phi_N^T$ as in Definition 2.19.*

*Proof of Proposition 2.20.* The periodicity in the direction around the cylinder follows from $\phi_n(z+2\pi Ri)=\phi_n(z)$ and

$$e^{i\sum_{s=1}^p X_s 2\pi R/(pNl^2)} = e^{i2\pi\sum_{s=1}^p X_s/(pN\gamma l)} = 1.$$

For the quasiperiodicity of the torus wave function, we compute

$$\big(t_1(pN\gamma l)\Phi_N^T\big)(z_1,..,z_N) = e^{ipNy_1/R}\Phi_N^T(z_1-pN\gamma l,..,z_N).$$

Using

$$e^{iNy_1/R}\tilde\phi_0(z_1-pN\gamma l-\frac{Z_s}{N}) = e^{iY_s/R}\tilde\phi_0(z_1-\frac{Z_s}{N})$$

and $\sum_{s=1}^p Y_s\in 2\pi R\mathbb{Z}$, one can check $t_1(pN\gamma l)\Phi_N^T=\Phi_N^T$. The periodicity in the variables $z_2,..,z_N$ is shown in an analogous way. □

*Proof of Proposition 2.21: cylinder function.* The cylinder Laughlin wave function is

$$\begin{aligned}\Phi_N^C(z_1,..,z_N) &= \frac{1}{\sqrt{N!}}\prod_{N\geq j>k\geq 1}(e^{z_j/R}-e^{z_k/R})^p e^{-\sum_{j=1}^N x_j^2/2l^2}\frac{e^{-\frac12 p^2\gamma^2\sum_{j=0}^{N-1}j^2}}{(2\pi Rl\sqrt{\pi})^{N/2}}\\ &= \frac{1}{\sqrt{N!}}\frac{1}{(2\pi Rl\sqrt{\pi})^{N/2}}\Big(\det(e^{(n-1)z_j/R})_{1\leq n,j\leq N}\ e^{-\frac{1}{2pl^2}\sum_{j=1}^N x_j^2}e^{-\frac12 p\gamma^2\sum_{n=1}^N(n-1)^2}\Big)^p\\ &= \frac{1}{\sqrt{N!}}\frac{1}{(2\pi Rl\sqrt{\pi})^{N/2}}\Big(\det(e^{i(n-1)y_j/R}e^{-\frac{1}{2pl^2}(x_j-p(n-1)\gamma l)^2})_{1\leq j,n\leq N}\Big)^p\end{aligned}$$

This is of the form given in Definition 2.19 provided we choose $f(x)$ as a Gaussian with the correct multiplication constant.

*Torus functions.* We start by showing that any $\Phi_N^T$ function is a torus Laughlin wave function. Note first that

$$\Phi_N^T(z_1,..,z_N) = f(z_1,..,z_N)e^{-\frac{1}{2l^2}\sum_{j=1}^N x_j^2}$$

for some holomorphic function $f$. This is checked using

$$e^{iX_sy/R}\tilde\phi_n(z-\frac{Z_s}{N}) \propto \Big(e^{X_sz/(pNl^2)}\sum_{k=-\infty}^{\infty}e^{(n+kN)z/R-p\gamma^2(n+kN)^2/2}\Big)e^{-x^2/(2pl^2)}e^{-X_s^2/(2pl^2N^2)}.$$

Together with the quasiperiodicity of Proposition 2.20, this shows that $\Phi_N^T$ is in the torus lowest Landau level. On the other hand, $\Phi_N^T$ has a zero of order at least $p$ as two particles get close $(z_j-z_k)\to 0$. Thus by the results of Section 1.4 it is a torus Laughlin wave function.

The functions $\tilde\phi_n$ can be considered as the basis functions of the lowest Landau level on a torus $[0,pN\gamma l]\times[0,2\pi R]$ with a modified magnetic length $l'=\sqrt{p}l$. The corresponding filled Landau level wave functions are

$$G(z_1,..,z_N) = F(\sum_{j=1}^N z_j)\prod_{1\leq j<k\leq N}\theta_1(\frac{i(z_j-z_k)}{2R}\mid\frac{iL}{2\pi R})e^{-\frac{1}{2pl^2}\sum_{j=1}^N x_j^2}$$

where $L=pN\gamma l$ and $F$ is a holomorphic function satisfying

$$F(Z+i2\pi R)=(-1)^{N-1}F(Z),\quad F(Z+L)=(-1)^{N-1}e^{Z/R}e^{L/(2R)}F(Z).$$

But the space of filled Landau level wave functions is one-dimensional, hence there exists a $c \in \mathbb{R}$ such that

$$\sqrt{N!}\tilde{\phi}_0 \wedge .. \wedge \tilde{\phi}_{N-1} = cG. \tag{2.67}$$

This gives

$$\begin{aligned}\Phi_N^T(z_1,..,z_N) &= \frac{1}{\sqrt{N!}} c^p e^{i(\sum_{s=1}^p X_s)(\sum_{j=1}^N y_j)/(pNl^2)} \prod_{s=1}^p G(z_1 - \frac{Z_s}{N},..,z_N - \frac{Z_s}{N}) \\ &= \frac{1}{\sqrt{N!}} c^p F^{cm}(\sum_{j=1}^N z_j) \prod_{1\le j<k\le N} \theta_1(\frac{i(z_j-z_k)}{2R} \mid \frac{iL}{2\pi R})^p e^{-\frac{1}{2l^2}\sum_{j=1}^N x_j^2} \qquad (2.68) \\ F^{cm}(Z) &= F(Z-Z_1)F(Z-Z_2)..F(Z-Z_p). \qquad (2.69)\end{aligned}$$

Conversely, any torus wave function is of the form (2.68) for some holomorphic function $F^{cm}$ satisfying the conditions (1.7). We know from the theory of theta functions ([FK01], Chapter 7) that any such function can be written, up to a multiplicative constant, as a product of the form (2.69), where $Z_1,..,Z_p$ are related to the zeroes of $F^{cm}$ and fulfill $\sum X_s \in L\mathbb{Z}$, $\sum Y_s \in 2\pi R\mathbb{Z}$, and $F$ is an odd or even theta function depending on the parity of $N$. As a consequence, every torus Laughlin wave function can be written in the form of Definition 2.19. □

*Remark.* A more direct way of getting (2.67) uses a *generalized Vandermonde identity* involving theta functions, see [For06], Proposition 3.1.

Now that we have defined the generalized functions, we can turn to their thermodynamic limits. We start with the cylinder functions.

### 2.2.2 Cylinder wave functions

In this subsection we give some properties of the modified Laughlin function and investigate the solvable monomer-dimer model. In Lemma 2.22, we prove that the modified cylinder function is associated with a polymer system; this holds true for general $f$. In Proposition 2.23, we choose $f$ as a function of compact support and show how the size of the support compared to the "unit cell" $[-p\gamma l/2, p\gamma l/2]$ affects the corresponding polymer system. When the support of $f$ is contained in $[-p\gamma l, p\gamma l]$, the associated polymer system is a monomer-dimer system. This system is solvable. In Proposition 2.24, we take $p = 3$ and determine the normalization constant and the one-particle density for the solvable model (finite volumes), from which we deduce the corresponding expressions in the infinite volume limit (Corollary 2.25).

Let $N \in \mathbb{N}$ and $\pi \in \mathcal{S}_{\{0,..,N-1\}}$. We define the set of renewal points of the permutation $\pi$ as:

$$R(\pi) := \{k \in \{0,..,N-2\} \mid \pi(\{0,..,k\}) \subset \{0,..,k\}\}.$$

For $n \in \mathbb{N}$, $j \in \mathbb{Z}$ and $p$ odd, define

$$\begin{aligned}u_{\{0,..,n-1\}} := \sqrt{n!}A\Big( &\sum_{\substack{\sigma_2,..,\sigma_p \in \mathcal{S}_n: \\ R(\sigma_2)\cap..\cap R(\sigma_p)=\emptyset}} \operatorname{sgn}(\sigma_2..\sigma_p) \\ &\phi_0\phi_{\sigma_2(1)}..\phi_{\sigma_p(1)} \otimes .. \otimes \phi_{N-1}\phi_{\sigma_2(N-1)}..\phi_{\sigma_p(N-1)}\Big) \qquad (2.70) \\ u_{\{j,..,j+n-1\}} := t(jp\gamma l\mathbf{e}_x)^{\otimes n} u_{\{0,..,n-1\}}. &\qquad (2.71)\end{aligned}$$

For even $p$, the antisymmetrization $A$ is replaced with the symmetrization $S$. In the following $||\cdot||$ refers to the $L^2((\mathbb{R}\times[0,2\pi R])^N)$ for suitable $N$. The one-particle density of $\Phi_N^C$ is denoted $\rho_N(z)$.

**Lemma 2.22.** *Let $p\in\mathbb{N}$ odd, $N\in\mathbb{N}$, then*

$$\Phi_N^C=\sum_{X_1,..,X_D} u_{X_1}\wedge..\wedge u_{X_D} \qquad \rho_N(z)=\sum_X \rho_N^P(X)v_X(z)$$
$$C_N:=||\Phi_N^C||^2=\sum_{n_1+..+n_D=N}\alpha_{n_1}..\alpha_{n_D}$$

*where the first sum ranges over ordered partitions $X_1<..<X_D$ of $\{0,..,N-1\}$ and*

$$\alpha_{N(X)}=||u_X||^2,\quad v_X(z):=\frac{N(X)}{||u_X||^2}\int |u_X|^2 dx_2..dy_{N(X)}.$$

Similar formulas hold for even $p$, provided we replace the wedge product with the symmetric product and, in the definition of $u_X$, the antisymmetrization $A$ with the symmetrization operator $S$.

*Proof.* To fix ideas, we take $p=3$. $\Psi_N^C$ can be expanded as

$$\Psi_N=\sqrt{N!}\,\mathrm{A}\Big(\sum_{\sigma,\tau}\mathrm{sgn}(\sigma\tau)\phi_0\phi_{\tau(0)}\phi_{\sigma(0)}\otimes..\otimes\phi_{N-1}\phi_{\tau(N-1)}\phi_{\sigma(N-1)}\Big), \tag{2.72}$$

where the sum ranges over permutations of $\{0,1,..,N-1\}$. The sum can be written as a sum over (possibly empty) sets $K\subset\{0,..,N-2\}$ and permutations $\sigma,\tau$ with $R(\sigma)\cap R(\tau)=K$. The empty set $K=\emptyset$ gives $(N!)^{-1/2}u_{\{0,..,N-1\}}$. If $K=\{k_1,..,k_D\}$ with $k_1<..<k_D$, the summation over corresponding $\sigma,\tau$ gives rise to $\prod(N(X_j))^{-1/2}u_{X_1}\otimes..\otimes u_{X_D}$ with $X_j=\{k_j,..,k_{j+1}-1\}$, $k_0=0, k_{D+1}=N$. Using $A(Af\otimes Ag)=A(f\otimes g)$, we obtain

$$\Psi_N^C=\sum_{X_1,..,X_D}\Big(\frac{N!}{N(X_1)!...N(X_D)!}\Big)^{1/2}A(u_{X_1}\otimes..\otimes u_{X_D})=\sum_{X_1,..,X_D}u_{X_1}\wedge..\wedge u_{X_D}.$$

Let $X_j=\{k_j,..,k_{j+1}-1\}$, $0=k_0<k_1<..<k_{D+1}=N$. Then

$$u_{X_1}\wedge..\wedge u_{X_D}=A(\sum_{m_1<..<m_N}{}' e^{i(m_1y_1+..+m_Ny_N)/R}f_m(x_1,...,x_N))$$

for suitable functions $f_m$, and the sum goes only over $N$-admissible partitions with renewal points $R(m)=\{k_1,..,k_{D+1}\}$. This can be used to show

$$\{X_1,..,X_D\}\neq\{Y_1,..,Y_{D'}\}\Rightarrow\int\overline{u_{X_1}\wedge..\wedge u_{X_D}}u_{Y_1}\wedge..\wedge u_{Y_{D'}}dy_2..dy_N=0 \tag{2.73}$$

from which the representations of the norm $C_N$ and the one-particle density can be deduced. □

The previous lemma is of particular interest when the Gaussian is replaced with a function $f$ of compact support $\mathrm{supp}f\subset[-s,s]$. The parameter $s$ controls the size of the support and the overlap between the function and its translates $f(\cdot-kp\gamma l)$, $k\in\mathbb{Z}$. It takes the role of $\gamma$ as a parameter controlling the activities.

**Proposition 2.23.** *For large $\gamma$, the associated polymer system is a monomer system. As $\gamma$ is decreased, it becomes first a monomer-dimer system and then a system with polymers of arbitrary length:*

1. *If $p\gamma \geq 2s$, $\phi_k\phi_n = 0$ unless $k = l$, and $u_X$ vanishes unless $N(X) = 1$.*
2. *If $p\gamma \geq s$, $\phi_k\phi_n = 0$ when $|k - l| \geq 2$, and $u_X$ vanishes unless $N(X) \in \{1, 2\}$.*
3. *If $\phi_k\phi_{k+2} \neq 0$, $u_X \neq 0$ for all rods $X$.*

*Proof. 1.* Suppose $p\gamma \geq 2s$ so that $\mathrm{supp} f \subset [-p\gamma/2, p\gamma/2]$ and $\phi_k\phi_l = 0$ unless $k = l$. To fix ideas, consider the case $p = 3$. The only permutations contributing to the sum in (2.72) are $\sigma = \tau = \mathrm{id}$, so that

$$\Phi_N^C = \phi_0^3 \wedge .. \wedge \phi_{N-1}^3 = u_{\{0\}} \wedge .. \wedge u_{\{N-1\}}, \quad u_{\{n\}} = \phi_n^3.$$

*2.* Suppose $p\gamma \geq s$. Again, chose $p = 3$. If $\sigma, \tau$ contribute to the sum in (2.72), they must fulfill

$$\forall k \in \{0, .., N-1\} : \ |\sigma(k) - k| \leq 1, \ \ |\tau(k) - k| \leq 1, \ |\sigma(k) - \tau(k)| \leq 1. \tag{2.74}$$

This condition can also be expressed in terms of the permutation matrices $P_{\mathrm{id}} = \mathbf{1}$, $P_\sigma$, $P_\tau$: the matrices $P_\sigma$, $P_\tau$ and $P_\sigma P_\tau^{-1}$ must be band-diagonal (only matrix elements $p_{j,j}$ and $p_{j,j\pm1}$ are allowed to be non-zero). The first inequality in (2.74) implies that the cycle decomposition of $\sigma$ consists of nearest neighbor transpositions $(r\ r+1)$. If $(r\ r+1)$ is in the cycle decomposition of $\sigma$, the condition $\forall k : \ |\sigma(k) - \tau(k)| \leq 1$ prevents $(r-1\ r)$ and $(r+1\ r+2)$ from appearing in the cycle decomposition of $\tau$. Thus there is a set of disjoint nearest neighbor transpositions $(i_1\ i_1+1), .., (i_k\ i_k+1)$ consisting of the transpositions appearing in the decomposition of $\sigma$ or $\tau$. The supports of the transpositions $\{i_1, i_1+1\}, .., \{i_k, i_k+1\}$ can be completed by singletons $\{n\}$ to give a partition of $\{0, .., N-1\}$ into monomers and dimers. Following the arguments in the proof of Lemma 2.22, one sees that $\Phi_N^C$ can be written as a sum over monomer-dimer partitions.

*3.* Suppose $\phi_k\phi_{k+2} \neq 0$. Let $\sigma_1\sigma_2, .., \sigma_p \in \mathcal{S}_{\{0,..,n-1\}}$ and $(m_0, .., m_{N-1}) := (\sigma_1(0) + \sigma_2(0) + .. + \sigma_p(0), .., \sigma_1(N-1) + .. + \sigma_p(N-1))$. The $N$-tuple $(m_1, .., m_N)$ can be considered as a set of angular momenta, since

$$(\phi_{\sigma_1(0)}..\phi_{\sigma_p(0)}) \otimes .. \otimes (\phi_{\sigma_1(N-1)}..\phi_{\sigma_p(N-1)}) = e^{i\sum_{j=1}^N m_j y_j/R} F_{\sigma,\tau}(x_1, .., x_N)$$

for some function $F$. Suppose that $(m_0, m_1, .., m_{N-1}) = (1, p, 2p, .., (N-2)p, (N-1)p-1)$. We wish to determine the corresponding permutations $\sigma_1, .., \sigma_p$. To do this, we start by looking at the partition of $m_0 = 1$ obtained by arranging $\sigma_1(0), .., \sigma_p(0)$ in descending order, then at the partition of $m_1$, etc. The only way of writing 1 as a sum of $p$ non-negative integers arranged in decreasing order is $1 = 1 + 0 + .. + 0$, hence the partition of $m_0$ may be written as $(1\ 0^{p-1})$. Thus the sum $m_1 = p = \sigma_1(1) + .. + \sigma_p(1)$ has $p$ non-negative numbers, among which at least one is different from 1 and at most one is equal to 0. This leaves $(2\ 1^{p-2}\ 0)$ as the only possible partition. Hence the partition associated to $2p$ consists of $p$ integers $\geq 1$ among which at least one is different from 2 and at most one is equal to 1. The only possibility is $(3\ 2^{p-1}\ 1)$. Proceeding in this way, we get the sequence of partitions

$$(1\ 0^{p-1}),\ (2\ 1^{p-2}\ 0),\ (3\ 2^{p-2}\ 1), ..., (N-1\ (N-2)^{p-2}\ N-3),\ ((N-1)^{p-1}\ N-2).$$

The corresponding product of functions

$$\phi_1\phi_0^{p-1} \otimes \phi_2\phi_1^{p-2} \otimes .. \otimes \phi_{N-1}\phi_{N-2}^{p-2}\phi_{N-3} \otimes \phi_{N-1}^{p-1}\phi_{N-2}$$

does not vanish since we assume that nearest neighbor functions overlap. Now we have to figure out the corresponding permutations. Up to a numbering of the permutations, the only way to get the first partition is by choosing $\sigma_1(0) = 1$, $\sigma_2(0) = .. = \sigma_p(0) = 0$. Therefore $\sigma_2(1), ..\sigma_p(1)$ must be different from 0. Thus in order to obtain the partition $(2, 1^{p-2}, 0)$ we must have, up to a renumbering of $\sigma_2, .., \sigma_p$, $\sigma_1(1) = 0$, $\sigma_2(1) = 2$, $\sigma_3(1) = .. = \sigma_p(1) = 1$. It follows that $\sigma_2(2) = 3$ and one of the permutations $\sigma_0, \sigma_3, ., \sigma_p$ takes the value 1. Continuing in this way and restoring the degree of freedom in the numbering of the permutations, we see that the only cycles in the decompositions of $\sigma_1, .., \sigma_p$ are transpositions $(r\ r+1)$, and each such transposition occurs in the decomposition of exactly one permutation. There are $p$ choices for the permutation that takes (0 1), $p-1$ possibilities for the one taking (1 3), etc., in total $p(p-1)^{N-2}$ choices. For each of these, the product of signs is $\mathrm{sgn}(\sigma_1..\sigma_p) = (-1)^{N-1}$. In total we get

$$\begin{aligned} & u_{\{0,..,n-1\}} \\ &= p(p-1)^{N-2}(-1)^{N-1}\phi_1\phi_0^{p-1} \otimes \phi_2\phi_1^{p-2} \otimes .. \otimes \phi_{N-1}\phi_{N-2}^{p-2}\phi_{N-3} \otimes \phi_{N-1}^{p-1}\phi_{N-2} + r_n \end{aligned} \quad (2.75)$$

where the remainder $r_n$ has no contribution of the momentum type $(1, p, .., p(N-2), pN - p - 1)$ and in particular is orthogonal to the first part of the sum. It follows that

$$u_{\{0,..,n-1\}} \neq 0$$

and the associated polymer system has polymers of arbitrary length and non-vanishing activity $\Phi(\{0, .., n-1\}) = ||u(\{0, .., n-1\})||^2$. □

*Remark:* Formula (2.75) should be compared to the exact expression (2.5) of $b_N(1, p, .., pN - 2p, pN - p - 1)$.

When the associated polymer system is a monomer-dimer system, the normalization constants and correlation functions can be determined explicitly. The following proposition gives the one-particle density at filling factor $1/p = 1/3$.

**Proposition 2.24.** *Let $p = 3$ and suppose that $\phi_k\phi_m = 0$ if $|k-m| \geq 2$. Let $\alpha := ||\phi_0^2\phi_1||^2 = ||\phi_0\phi_1^2||^2$ be a measure of the overlap between nearest neighbor functions [4]. The one-particle density of $\Phi_N^C$ is*

$$\rho_N(x) = \sum_{j=0}^{N-1} \frac{C_j C_{N-j-1}}{C_N} |\phi_j^3|^2(x) + \sum_{j=0}^{N-2} \frac{C_j C_{N-j-2}}{C_N} 9\alpha\big(|\phi_j^2\phi_{j+1}|^2(x) + |\phi_j\phi_{j+1}^2|^2(x)\big)$$

*where*

$$C_N = \frac{\lambda_+^{N+1} - \lambda_-^{N+1}}{\lambda_+ - \lambda_-}, \quad \lambda_\pm = \frac{1}{2} \pm \sqrt{\frac{1}{4} + 9\alpha^2}.$$

*Proof.* We start by computing the monomer and dimer functions $u_{\{n\}}$, $u_{\{n,n+1\}}$. This can be done directly from their definition (2.70), but it is more instructive to determine them from Lemma 2.22. Remark that $\Phi_1^C = u_{\{0\}}$ by that lemma, and $\Phi_1^C = \phi_0^3$ by definition, thus $u_{\{0\}} = \phi_0^3$ and we get from (2.71) $u_{\{n\}} = \phi_n^3$. Next, looking at the 2-particle wave function

[4] The equality $||\phi_0^2\phi_1||^2 = ||\phi_0\phi_1^2||^2$ is checked using that $f$ is even.

$\Phi_2^C$, we get

$$\begin{aligned}\Phi_2^C &= \frac{1}{\sqrt{2}}(\phi_0 \otimes \phi_1 - \phi_1 \otimes \phi_0)^3 \\ &= \frac{1}{\sqrt{2}}(\phi_0^3 \otimes \phi_1^3 - 3\phi_0^2\phi_1 \otimes \phi_0\phi_1^2 + 3\phi_0\phi_1^2 \otimes \phi_0^2\phi_1 - \phi_1^3 \otimes \phi_0^3) \\ &= \phi_0^3 \wedge \phi_1^3 - 3\phi_0^2\phi_1 \wedge \phi_0\phi_1^2 \\ &\stackrel{!}{=} u_{\{0\}} \wedge u_{\{1\}} + u_{\{0,1\}}\end{aligned}$$

which gives

$$u_{\{0,1\}} = -3\phi_0^2\phi_1 \wedge \phi_0\phi_1^2, \quad u_{\{n,n+1\}} = -3\phi_n^2\phi_{n+1} \wedge \phi_n\phi_{n+1}^2.$$

We know from the previous propositions that $C_N$ is the polymer partition function of a monomer-dimer system. The recurrence relation (2.36) becomes

$$C_0 = 1, \quad C_1 = 1, \quad C_{N+2} = C_{N+1} + 9\alpha^2 C_N$$

where we used that $\alpha_1 = ||\phi_0^3||^2 = 1$, $||u_{\{0,1\}}||^2 = 9\alpha^2$. The normalization constant is then expressed in terms of the roots $\lambda_\pm$ of the characteristic equation $\lambda^2 = \lambda + 9\alpha^2$ as

$$C_N = \frac{\lambda_+^{N+1} - \lambda_-^{N+1}}{\lambda_+ - \lambda_-}.$$

By Lemma 2.22, the one-particle density is a sum of monomer and dimer contributions

$$\begin{aligned}\rho_N(x) &= \sum_{j=0}^{N-1} \frac{C_j C_{N-j-1}}{C_N} v_{\{j\}}(x) + \sum_{j=0}^{N-1} \frac{C_j \alpha_2 C_{N-j-2}}{C_N} v_{\{j,j+1\}}(x), \\ v_{\{j\}}(x) &= |\phi_j^3|^2(x), \\ v_{\{j,j+1\}}(x) &= \frac{2}{||v_{\{j,j+1\}}||^2} \int_{\mathbb{R}\times[0,2\pi R]} |v_{\{j,j+1\}}|^2(x, x_2 + iy_2) dx_2 dy_2. \\ &= \frac{9\alpha}{\alpha_2}\big(|\phi_j^2\phi_{j+1}|^2(x) + |\phi_j\phi_{j+1}^2|^2(x)\big).\end{aligned}$$

□

The previous proposition leads immediately to the existence of the thermodynamic limit of the one-particle density, away from the boundaries of the finite cylinder $[0, pN\gamma l] \times [0, 2\pi R]$.

**Corollary 2.25.** *Under the assumptions of the previous proposition, the one-particle density has a limit away form the boundaries of the support of $\Phi_N^C$: for $x \in [0, pN\gamma l]$, $|\rho_N(x) - \rho(x)| \leq f(|x| + |pN\gamma l - x|)$ with $f(m) \to_{m\to\infty} 0$ and*

$$\rho(x) = \frac{1}{\lambda_+ - \lambda_-} \sum_{j\in\mathbb{Z}} \Big(|\phi_0^3|^2 + \frac{1}{\lambda_+}\big(|\phi_0^2\phi_1|^2 + |\phi_0\phi_1^2|^2\big)\Big)(x - 3j\gamma l).$$

In particular, we see that the one-particle density is periodic with period $3\gamma l$. We have observed in Proposition 2.23 that the system becomes a pure monomer system as $\gamma$ gets large. Our results allow us to check that, loosely speaking, the transition is smooth: indeed, recall the roots $\lambda_\pm$ are defined are simple functions of the overlap parameter $\alpha := ||\phi_0^2\phi_1||^2$ (see Proposition 2.24). The transition from the monomer-dimer to the monomer system corresponds to the transition from $\alpha > 0$ to $\alpha = 0$, and we see that $\lambda_\pm$ is a smooth function of the overlap parameter $\alpha$.

### 2.2.3 Torus wave functions

The Laughlin-type torus wave functions proposed in [HR85b] form a $p$-dimensional space; they are eigenfunctions of suitable center-of-mass translations. This invariance entails the periodicity of the one-particle density, which is thus much simpler as in the cylinder case. In Lemma 2.26, we check that this feature is reproduced by the generalized torus functions. This holds for general functions $f$.

Next, we take $f$ as a function of compact support such that the associated cylinder polymer model is a monomer-dimer system and show that for a suitable choice of $Z_1, .., Z_p$ (the degrees of freedom in the definition of torus functions), the torus function is associated with a monomer-dimer system on a ring, with one additional long polymer covering the whole ring (Proposition 2.27). As a corollary, we obtain that the one-particle density of the corresponding torus function equals the one-particle density of the cylinder function in the limit $N \to \infty$ (Corollary 2.28).

Let us start by looking at the translational invariance of the torus functions. We have seen that the torus functions are quasiperiodic in each complex variable $z_1, .., z_N$ of periods $L = pN\gamma l$ and $i2\pi R$. The space of such functions is invariant with respect to center-of-mass translations by multiples of $\gamma l \mathbf{e}_x$ and $2\pi R/(pN)\mathbf{e}_y$ (the lattice $\gamma l\mathbb{Z} \times (2\pi R/pN)\mathbb{Z}$ already appeared on p.8; see also p.94). The modified torus functions are invariant with respect to overall shifts by $p$ times the lattice vectors $\gamma l \mathbf{e}_x$ and $(2\pi R/pN)\mathbf{e}_y$:

**Lemma 2.26.** *For $a \in \mathbb{R}^2$, let $T(a) := t(a)^{\otimes N}$ be the magnetic translation of all the particles. Then the torus wave functions are invariant with respect to overall shifts by $p\gamma l\mathbf{e}_x$ and $2\pi R/N\mathbf{e}_y$:*

$$T(p\gamma l\mathbf{e}_x)\Phi_N^T = (-1)^{p(N-1)}\Phi_N^T, \quad T(\frac{2\pi R}{N}\mathbf{e}_y)\Phi_N^T = (-1)^{p(N-1)}\Phi_N^T.$$

*The periodicity is enhanced for particular choices of $Z_1, .., Z_p$. For example, if $Z_j = ij2\pi R/N$, $j = 1, .., p$:*

$$T(\frac{2\pi R}{pN}\mathbf{e}_y)\Phi_N^T = (-1)^{N-1}\Phi_N^T.$$

*Proof.* Using $e^{iy/R}\tilde{\phi}_k(z - p\gamma) = \tilde{\phi}_{k+1}(z)$ and $\tilde{\phi}_0 = \tilde{\phi}_N$, we see that

$$\tilde{\phi}_1 \wedge .. \wedge \tilde{\phi}_N = (-1)^{N-1}\tilde{\phi}_0 \wedge .. \wedge \tilde{\phi}_{N-1}$$

and

$$\begin{aligned}(t(p\gamma l))^{\otimes N}\Phi_N^T(z_1, .., z_N) &= \frac{1}{\sqrt{N!}}e^{ip\sum_j y_j/R}\prod_{s=1}^{p}\Big(\det(e^{iX_s y_j/(pNl^2)}\tilde{\phi}_{k-1}(z_j - \frac{X_s}{N} - p\gamma))_{j,k}\Big)\\ &= \frac{1}{\sqrt{N!}}\prod_{s=1}^{p}\Big(\det(e^{iX_s y_j/(pNl^2)}\tilde{\phi}_k(z_j - \frac{X_s}{N}))_{j,k}\Big)\\ &= \frac{1}{\sqrt{N!}}(-1)^{p(N-1)}\Phi_N^T(z_1, .., z_N).\end{aligned}$$

Similarly, $\tilde{\phi}_n(z - i2\pi R/N) = e^{-i2\pi n/N}\tilde{\phi}_n$, hence

$$\begin{aligned} T(i2\pi R/N)\Phi_N^T &= \frac{1}{\sqrt{N!}}\prod_{s=1}^{p}\Big(\det(e^{iX_s(y_j - i2\pi R/N)/(pNl^2)}e^{-i2\pi(k-1)/N}\tilde{\phi}_{k-1}(z_j - \frac{X_s}{N}))_{j,k}) \\ &= e^{-i(\sum_{s=1}^{N}X_s)2\pi R/(pNl^2)}e^{-i2\pi p\sum_{k=1}^{N}(k-1)/N}\Phi_N^T \\ &= (-1)^{p(N-1)}\Phi_N^T. \end{aligned}$$

Suppose in addition that $Z_j = ij2\pi R/(pN)$, then

$$\begin{aligned} T(i2\pi R/pN)\Phi_N^T &= \frac{1}{\sqrt{N!}}\prod_{s=1}^{p}\Big(\det(\tilde{\phi}_{k-1}(z_j - ij\frac{2\pi R}{pN} - i\frac{2\pi R}{pN}))_{1\le j,k\le N}) \\ &= \frac{1}{\sqrt{N!}}(-1)^{N-1}\prod_{s=1}^{p}\Big(\det(\tilde{\phi}_{k-1}(z_j - ij\frac{2\pi R}{pN}))_{1\le j,k\le N}) \\ &= (-1)^{N-1}\Phi_N^T. \end{aligned}$$

□

The previous lemma implies that the one-particle density of the torus wave functions is doubly periodic with the period $p\gamma l$ in the $x$-direction and $2\pi R/N$ in the $y$-direction. The periodicity of the one-particle density of the cylinder function would follow if we knew that both functions are equivalent in the thermodynamic limit. We prove the equivalence only for the special case of nearest neighbor overlapping.

Note that there is only one Laughlin function, but a $p$-dimensional space of torus functions. One must thus be careful as to which torus function one compares the cylinder function. The most natural candidates are the torus functions $\Phi_{N,1}^T$ and $\Phi_{N,2}^T$ obtained by choosing $Z_1 = .. = Z_p = 0$ for $\Phi_{N,1}^T$ and $Z_j = ij2\pi R/(pN)$ for $\Phi_{N,2}^T$. The first function has in common with $\Phi_N^C$ the representation as the power of a determinant, while $\Phi_{N,2}^T$ and $\Phi_N^C$ are both eigenvectors of $T(i2\pi R/(pN))$ for the eigenvalue $(-1)^{N-1}$.

For $X = \{j, .., j+n-1\}$, $N \in \mathbb{N}$, let $u_X$ be defined as in the previous section and

$$\hat{u}_X := \sum_{k\in\mathbb{Z}} t(kpN\gamma l\mathbf{e}_x)^{\otimes N(X)}u_X$$

its periodification. The torus wave functions for $Z_1 = .. = Z_s = 0$ have a representation of the form

$$\Phi_N^T = \sum_{X_1,..,X_D} \pm\hat{u}_{X_1}\wedge..\wedge\hat{u}_{X_D} + \tilde{u}_N$$

where $X_1, .., X_D$ are partitions of the ring $\mathbb{Z}/N\mathbb{Z}$. However, in general $\tilde{u}_N$ is not orthogonal anymore to $\sum_{X_1,..,X_D}$. For this reason we restrict to nearest neighbor overlapping. For simplicity, we take $p = 3$.

**Proposition 2.27.** *Let $N \ge 3$. Suppose $\phi_k\phi_l = 0$ if $|k-l| \ge 2$, take $p = 3$ and let $\hat{u}_X$ be defined as above. Let $\hat{u}_{\{0,N-1\}} := \hat{u}_{\{-1,0\}}$ and*

$$\tilde{u}_N = 3(-1)^{N-1}\tilde{\phi}_0^2\tilde{\phi}_1\wedge..\wedge\tilde{\phi}_{N-1}^2\tilde{\phi}_N + 3\tilde{\phi}_0\tilde{\phi}_1^2\wedge..\wedge\tilde{\phi}_{N-1}\tilde{\phi}_N^2.$$

*Then the torus function $\Phi_N^T$ for $Z_1 = .. = Z_p = 0$ is represented as a polymer system on a ring, made up of monomers, dimers, and one additional long chain:*

$$\Phi_N^T = \sum_{X_1,...,X_D}{}' \epsilon_{\{X_j\}} \hat{u}_{X_1} \wedge .. \wedge \hat{u}_{X_D} + \hat{u}_N \tag{2.76}$$

*The sum ranges over two types of partitions:*

1. *partitions $X_1 < .. < X_D$ of $\{0, .., N-1\}$ into monomers $\{j\}$ and dimers $\{j, j+1\}$. For these partitions, $\epsilon_{\{X_j\}} = +1$;*
2. *$X_1 = \{0, N-1\}$, $X_2 < .. < X_D$ partition of $\{1, .., N-2\}$ into monomers and dimers. For this type of partition, $\epsilon_{\{X_j\}} = (-1)^{N-1}$.*

*The different contributions to (2.76) are orthogonal and the normalization constants in $L^2(([0, 3N\gamma l] \times [0, 2\pi R])^N)$ as well as the one-particle density have representations similar to Lemma 2.22, for the correct associated polymer system.*

*Sketch of proof.* We start with

$$\Phi_N^T = \frac{1}{\sqrt{N!}} A( \sum_{\sigma,\tau \in \mathcal{S}_{\{0,..,N-1\}}} \mathrm{sgn}(\sigma\tau) \tilde{\phi}_0 \tilde{\phi}_{\sigma(0)} \tilde{\phi}_{\tau(0)} \otimes .. \otimes \tilde{\phi}_{N-1} \tilde{\phi}_{\sigma(N-1)} \tilde{\phi}_{\tau(N-1)} ).$$

To derive (2.76), we use a reasoning similar to the proof of *2.* in Proposition 2.23. If $\sigma, \tau$ give a non-vanishing contribution to $\Phi_N^T$, $(k, \sigma(k))$ must be nearest neighbors on the ring $\mathbb{Z}/N\mathbb{Z}$, and similarly for $(k, \tau(k))$, $(\tau(k), \sigma(k))$, for all $k \in \mathbb{Z}$. This leaves two possibilities. First, $\sigma$ and $\tau$ can be products of nearest neighbor transpositions (including $(0\ N-1)$) with disjoint support. Second, we can have $\sigma, \tau \in \{\mathbf{1}, c\}$ or $\sigma, \tau \in \{\mathbf{1}, c^{-1}\}$ where $c$ is the cycle $c = (0\ 1\ 2\ ..N-1)$. The transpositions give the sum over $\sum_{X_1,..,X_D}$ in (2.76). The sign $\epsilon_{\{X_j\}}$ is $(-1)^{N-1}$ if the transposition $(0\ N-1)$ occurs: in this case we have to permute in the wedge product. The cycles give $\tilde{u}_N$ (note that $\tilde{\phi}_N = \tilde{\phi}_0$). For $N = 2$, the cycle $c$ is a nearest neighbor transposition; this is why we assume $N = 3$. In order to get representations of the normalization constant and the one-particle density, one has to check that an orthogonality relation similar to (2.73) holds. Again, this is done by looking at the $y$-momenta. The only difference is that $y$-momenta are only defined modulo $3N$. □

*Remark.* Similar representations can be derived for torus functions with $X_1 = .. = X_p = 0$ but $Y_1, ., Y_p$ possibly non-zero. The shift in $y$ by $Y_s$ then leads to phase factors in front of the polymer functions $\hat{u}_X$. If the zeroes are chosen in a way that $T(i2\pi R/(pN)\mathbf{e}_y)\Phi_N^T = (-1)^{N-1}\Phi_N^T$, we obtain a formula with $\tilde{u}_N = 0$, there is no "ring contribution".

The previous proposition allows to check the equivalence of modified cylinder and torus functions in the nearest neighbor overlapping case.

**Corollary 2.28.** *Under the assumptions of the previous proposition, and with the overlap parameter $\alpha = ||\phi_0^2 \phi_1||^2_{L^2(\mathbb{R}\times[0,2\pi R])}$, and $\lambda_\pm$ as in Proposition 2.24, the torus normalization constant satisfies*

$$C_N^T \underset{N\to\infty}{\sim} \lambda_+^N$$

*and the one-particle density of the torus function has as thermodynamic limit the function $\rho$ given in Corollary 2.25.*

*Sketch of proof.* Let $\hat{\Omega} = [0, pN\gamma l]\times[0, 2\pi R]$ and $\Omega = \mathbb{R}\times[0, 2\pi R]$. Note that $||u_X||_{L^2(\Omega^{N(X)})} = ||\hat{u}_X||_{L^2(\hat{\Omega}^{N(X)})}$ and $||\tilde{u}_n||^2_{L^2(\hat{\Omega}^n)} = 18\alpha^n$. For $N \geq 3$, write $||\Phi^T_N||^2_{L^2(\hat{\Omega}^N)} = D_N + 18\alpha^n$. $D_N$ is a sum over monomers and dimers and can be written in terms of the cylinder normalization constant as

$$D_N = C_{N-1} + 2\alpha_2 C_{N-2}, \quad \alpha_2 = ||u_{\{0,1\}}||^2 = 9\alpha^2$$

(see also [HL72], Section III, equation (3.8)). The first summand corresponds to partitions where 0 is covered by a monomer, while $2C_{N-2}$ comes from partitions where 0 is in the dimer $\{0, 1\}$ or $\{N-1, 0\}$. Thus we obtain, using $\lambda_+\lambda_- = -9\alpha^2$ and $\lambda_+ + \lambda_- = 1$:

$$C^T_N \underset{N\to\infty}{\sim} \frac{1}{\lambda_+ - \lambda_-}(1 + 2\cdot 9\alpha^2 \frac{1}{\lambda_+})\lambda_+^N = \lambda_+^N$$

We get for the finite volume probabilities of finding a monomer or a dimer

$$\rho^P_{N,T}(\{0\}) = \frac{C_{N-1}}{C^T_N} \sim \frac{\lambda_+^N}{\lambda_+ - \lambda_-}\frac{1}{\lambda_+^N} = \frac{1}{\lambda_+ - \lambda_-}$$

$$\rho^P_{N,T}(\{0,1\}) = 9\alpha^2\frac{C_{N-2}}{C^T_N} \sim \frac{\lambda_+^{N-1}}{\lambda_+ - \lambda_-}\frac{1}{\lambda_+^N} = \frac{9\alpha^2}{\lambda_+}\frac{1}{\lambda_+ - \lambda_-}.$$

The limits are the same as for the polymer system with non-periodic boundary conditions; the long chain $\tilde{u}_N$ does not contribute in the limit $N \to \infty$. We conclude with a representation of the type

$$\rho_{N,T}(z) = \sum_X \rho^P_{N,T}(X)\hat{v}_X(z). \qquad \square$$

Thus for the solvable monomer-dimer case, the modified cylinder and suitable torus functions give the same one-particle density.

## 2.3 Jellium tubes

In the beginning of the thesis, we have presented the plasma analogy: the modulus squared of Laughlin's wave function is essentially the Boltzmann weight of a classical one-component plasma, with a logarithmic Coulomb interaction. The results on the normalization constants and correlation functions of Laughlin's wave function lead to results on the free energy and correlation functions of the plasma system.

The plasma system related to Laughlin's cylinder wave function lives on a cylinder, or semiperiodic strip, and following [AGL01] we will refer to it as a jellium tube. When the cylinder is very thin, the system is close to a one-dimensional jellium system, while for large cylinder radius, the jellium tube approaches a fully two-dimensional one-component plasma.

For one-dimensional jellium systems, the symmetry breaking has been proved a while ago [Kun74, BL75]. When $N$ electrons are placed on a line of length $L$ and interact between themselves and with a neutralizing background through a one-dimensional Coulomb interaction, they tend to minimize their Coulomb interaction by taking positions on a lattice. The spacing between two consecutive positions is $L/N$. If we view the jellium tube as a quasi one-dimensional system rather than a two-dimensional system, we may expect the same periodicity. The tubes we consider have length $pNl^2/R$. In the jellium context, the period $pl^2/R$ is thus quite natural.

Forrester [For91] has examined the asymptotics of jellium strip free energies as the strip width gets large; he has also looked at torus Coulomb free energy finite size corrections, motivated by the question of universal finite size corrections and a relation to Sine-Gordon fields (see [For06] and the references therein).

In the first subsection, we look at the jellium tube in both limits: large and thin strips. We give an expression for the jellium tube free energy. We compare the free energy and the correlation functions of thin cylinders to the corresponding quantities for one-dimensional systems and find that they are in good agreement with existing results. We also compare the free energy of broad jellium strips to the bulk free energy; this leads to a conjecture consistent with results by [For91] and with the bounds on the normalization constant of Laughlin's wave function derived in Lemma 2.10.

The plasma analogy also sheds some light on the recurrence relation satisfied by the normalization constants. In the second subsection, we recall Lenard's combinatorial treatment of the one-dimensional two-component plasma. Lenard derived a renewal equality for the isothermal-isobaric partition functions and interpreted it in terms of minimal electrically neutral components. Then we derive a recurrence inequality for jellium tubes with the help of Newton's electrostatic theorem; the inequality looks similar to a renewal equation. Again, there is a relation to minimal neutral blocks.

### 2.3.1 Interpolation between one- and two-dimensional jellium

The plasma analogy relates Laughlin's cylinder wave function to a one-component plasma on a semiperiodic strip of width $2\pi R$, with a "Coulomb" interaction $V_C(z) = -\log|\frac{2R}{R_0}\sinh\frac{z}{2R}|$, where $R_0 > 0$ is a reference length scale. When the width of the strip goes to infinity, we recover the full two-dimensional logarithmic interaction: if $z \neq 0$ and $R_0 > 0$ are kept fixed,

$$\lim_{R\to\infty} V_C(z) = -\log|z/R_0| = V_D(z).$$

In the opposite limit of thin strips, for all $z \in \mathbb{C}$ with $x = \Re z \neq 0$,

$$V_C(z) = -\log\frac{R}{R_0} - \frac{|x|}{2R} + o(1) \quad \text{as } R \to 0.$$

The right-hand side represents a one-dimensional Coulomb interaction. Thus the semiperiodic strip interpolates in a natural way between the two-dimensional and the one-dimensional systems, and it is instructive to compare our results to existing results on these systems. We start by giving an expression for the cylinder free energy.

Let $C := [-L/2, L/2] \times [0, 2\pi R]$. The free energy of the jellium tube is

$$\beta F := -\log\Big(\frac{1}{N!h^{2N}}\int_{(\mathbb{R}^2)^N\times C^N} e^{-\beta(\frac{1}{2m}\sum_{j=1}^N |p_j|^2 + U_C)}\Big) \quad = -\log\Big(\frac{1}{N!\lambda_\beta^{2N}}\int_{C^N} e^{-\beta U_C}\Big) \tag{2.77}$$

where $\lambda_\beta := \sqrt{\beta h^2/(2\pi m)}$ is the *thermal wavelength* and $h$ Planck's constant. We will be interested in the free energy per particle in the thermodynamic limit $N \to \infty$ with a fixed density $n = N/|\Omega|$. For the cylinder and torus systems, we keep the radius $R$ fixed when taking the limit. The corresponding free energies (per particle) are

$$\beta f = \beta f_0 - \lim_{N\to\infty}\frac{1}{N}\log\Big(\frac{1}{(2\pi RL)^N}\int_{C^N} e^{-\beta U_C}\Big), \qquad \beta f_0 = \log n\lambda_\beta^2 - 1. \tag{2.78}$$

$f_0$ is the free energy of a system of noninteracting particles: $U_C = 0$ would give $f = f_0$. In view of the plasma analogy (1.36), if $\Gamma = \beta q^2 = 2p$ and $n = (p \cdot 2\pi l^2)^{-1}$ for $p \in \mathbb{N}$, the free energy can be written as

$$\beta f = \beta f_0 + \frac{\lambda^2}{12} - \lim_{N\to\infty} \frac{1}{N} \log \frac{N! l^N \sqrt{\pi}^N}{L^N} \int_{(I\times[0,2\pi R])^N} |\Psi_N^C|^2$$

where $I := [-\frac{L}{2} + \frac{p(N-1)\gamma l}{2}, \frac{L}{2} + \frac{p(N-1)\gamma l}{2}]$. In the following we assume that the result is unchanged if integrations are carried out on the infinite cylinder instead of $I \times [0, 2\pi R]$ (remember that $\Psi_N^C$ decays exponentially outside $(I \times [0, 2\pi R])^N$. Under this assumption, using Stirling's formula, $L = Np\gamma l$ and

$$\lambda^2 = \frac{\Gamma}{4\pi n R^2} = \frac{2p}{4\pi (p \cdot 2\pi l^2)^{-1} R^2} = p^2\gamma^2, \tag{2.79}$$

we get

$$\beta f = \beta f_0 + \frac{\lambda^2}{12} + \log \frac{e\lambda}{\sqrt{\pi}} + \log r_p(\gamma) \tag{2.80}$$

where $r_p(\gamma)$ comes from Lemma 2.10.

**Comparison with a one-dimensional Coulomb gas.** Let $V(x) = -|x|/(2R)$ and for $L > 0$, $N \in \mathbb{N}$, $n_0 := N/L$

$$U(x_1, .., x_N) := q^2 \sum_{1\le j<k\le N} V(x_j - x_k) - n_0 q^2 \sum_{j=1}^{N} \int_0^L V(x_j - x)dx + \frac{1}{2} n_0^2 q^2 \int_0^L \int_0^L V(x - x')dxdx'$$

be the energy of a one-dimensional Coulomb gas of $N$ particles moving in a neutralizing background. A useful quantity is $\Gamma_1 := \frac{\beta q^2}{2\pi R n_0}$, the *plasma parameter* or *coupling constant*; it is the ratio of a typical potential and kinetic energy. Large coupling constants correspond to kinetic energies that are small compared to typical potential energies. In the thermodynamic limit $N, L \to \infty$ at fixed density $n$, the free energy per particle including kinetic energy can be expressed as

$$\beta f^{(1)} = \beta f_0^{(1)} + \frac{\Gamma_1}{12} - \log \frac{z_0(\sqrt{\Gamma_1})}{e}, \quad \beta f_0^{(1)} = \log n_0 \lambda_\beta - 1, \tag{2.81}$$

where $z_0(\lambda)$ is an analytical function such that $\lim_{\lambda\to\infty} z_0(\lambda)\lambda/\sqrt{\pi} = 1$ see [Kun74], and the one-particle density $\rho(x)$ is

$$\rho(x) = n_0 \sum_{k=-\infty}^{\infty} e^{-\Gamma_1 (n_0 x - k)^2} F_{\Gamma_1}(n_0 x - k), \qquad \lim_{\Gamma_1\to\infty} F_{\Gamma_1}(x) = \chi_{[-1,1]}(x), \tag{2.82}$$

see [Kun74, BL75]. If we look at a one-dimensional system with density $n_0 = 2\pi R n$, the one-dimensional plasma parameter is nothing else but the quantity $\lambda^2$ of (2.79):

$$\Gamma_1 = \frac{\beta q^2}{2R \cdot 2\pi R n} = \lambda^2.$$

Comparing (2.80) and (2.81), we see that the one- and two-dimensional free energies $f - f_0$ and $f^{(1)} - f_0^{(1)}$ get close as the strip gets thin, i.e., $R \to 0$ or $\lambda = p\gamma = pl/R \to \infty$ at fixed two-dimensional density $n$. (Recall that $\lim_{\gamma\to\infty} r_p(\gamma) = 1$.) The one-particle density of Laughlin's wave function on thin strips is

$$\rho(x) = \frac{1}{p \cdot 2\pi l^2} \sum_{k\in\mathbb{Z}} n_k e^{-(x-k\gamma l)^2/l^2} \text{ with } n_k = \begin{cases} 1 + O(e^{-\gamma^2}), & \text{if } k \in p\mathbb{Z}, \\ O(e^{-\gamma^2}), & \text{else.} \end{cases}$$

This compares well to the one-dimensional density, since

$$\Gamma_1(n_0 x - k)^2 = (p\gamma \frac{2\pi R}{p \cdot 2\pi l^2} x - pk\gamma)^2 = \frac{1}{l^2}(x - p\gamma l)^2.$$

**Comparison with a two-dimensional Coulomb gas.** The free energy $\beta F$ of a two-dimensional jellium system (particles of charge $q$ + neutralizing background of charge density $-nq$) with logarithmic Coulomb interaction $V_D(z) = -\log|z/R_0|$ is defined by a formula similar to (2.77). The thermodynamic limit of the free energy per particle exists and is given by an expression of the type

$$\beta f^{(2)} = \beta f_0 + g(\Gamma) - \frac{\Gamma}{4} \log nR_0^2, \quad \beta f_0 = \log n\lambda_\beta^2 - 1$$

for a suitable function $g$ and $\Gamma = \beta q^2$, see [SM76]. The value of the function at coupling constant $\Gamma = 2$ is known and equals

$$g(2) = -\frac{1}{2}\log(2\pi) + \log\frac{e}{\sqrt{\pi}}.$$

We expect that the strip free energy (2.79) approaches the two-dimensional energy as $R \to \infty$ if the length scale $R_0$ is chosen consistently; we will choose $R = R_0$. Forrester [For91] derives more refined asymptotics for the strip free energies. He shows that if the thermodynamic limit of correlation functions in the bulk (i.e., in the fully two-dimensional geometry) and in the strip geometry exist, if the strip correlation functions converge to the bulk correlation functions as the strip width goes to infinity, and if in addition the bulk correlation functions satisfy sum rules typical for conducting Coulomb systems, then the following asymptotics hold:

$$\beta(f - f^{(2)}) = \frac{\pi}{6n(2\pi R)^2} + O(\frac{1}{R^4}) \quad \text{as } R \to \infty. \tag{2.83}$$

Using (2.80), the expressions of $\beta f^{(2)}$ and $g(2)$ given above, and the identifications $\Gamma = \beta q^2 = 2p$, $n = (p2\pi l^2)^{-1}$, one finds that (2.83) is equivalent to

$$\log A_p(\gamma) = \frac{(p - p^2)\gamma^2}{12} + g(2p) - pg(2) + O(\gamma^4) \text{ as } \gamma \to 0 \tag{2.84}$$

where the auxiliary quantity

$$A_p(\gamma) = r_p(\gamma)(\frac{e\gamma}{\sqrt{\pi}})^{1-p} p^{1-\frac{p}{2}}$$

is exactly the quantity on which we have given bounds in Lemma 2.10. As observed by [For91], for $\Gamma = 2$, the relation (2.83) can be checked explicitly: the normalization constant for the filled Landau level ($p = 1$) is $C_N = 1$ whence $r_1(\gamma) = 1$, $A_1(\gamma) = 1$ and (2.84) holds. For $p \geq 2$, we do not have a proof of (2.83) (note that Forrester's assumptions are quite strong); however, (2.84) is consistent with Lemma 2.10 if

$$p(g(2) - 1) \leq g(2p) \leq pg(2).$$

**Is there a phase transition?** The previous paragraphs picture the jellium tube as an interpolation between one- and two-dimensional jellium systems. It can be characterized in terms of the two parameters $\Gamma = \beta q^2$, $\lambda^2 = \beta q^2/(4\pi n R^2)$ that are the coupling constants of the related two- and one-dimensional plasmas. An interesting feature is that the one-dimensional system exhibits no phase transition (Kunz [Kun74] proved that the free energy is an analytic function of the coupling constant). On the other hand, there is numerical evidence of a phase transition for jellium systems of dimensions 2 or 3 (see e.g. [BST66] for three dimensions, [CLWH82] for two dimensions). It is believed that above $\Gamma \approx 140$, the one-component plasma is crystalline.
We proved that for *all* even plasma parameters $\Gamma = 2p$, there is a value $\gamma_p$ such that for $\gamma = \ell/R \geq \gamma_p$ the free energy is an analytic function of $\gamma$, or equivalently of $\lambda^2 = (p\gamma)^2$. One might speculate that for small $\gamma$, the nature of the limiting two-dimensional system and hence the magnitude of $\Gamma = 2p$ should play a role for large strips.

### 2.3.2 Minimal electrically neutral components

The normalization constants of Laughlin's wave function satisfy a recurrence relation, the renewal equation described in Section 2.1. This relation was used extensively to prove the existence of thermodynamic limits. However, we did not give any physical meaning to it. The aim of this subsection is to show that the plasma analogy gives some intuition that makes the relation less surprising.

**One-dimensional two-component plasma: Lenard's combinatorial treatment.** It is interesting to observe that the renewal equality already appears in Lenard's combinatorial treatment of the two-component, one-dimensional plasma [Len61]. Let us briefly summarize the model. Consider $2N$ particles of charges $\sigma_1, .., \sigma_N \in \{\pm\sigma\}$ moving on a semi-infinite line $[0, \infty[$. The potential energy of the system is

$$U(q_1, \sigma_1; ..; q_N, \sigma_N) = -2\pi \sum_{1 \leq i < j \leq N} \sigma_i \sigma_j |q_i - q_j|.$$

The system is assumed to be neutral, $\sum_i \sigma_i = 0$. Let $\beta > 0$ be an inverse temperature and $P > 0$ an external pressure. It is convenient to work with a different set of parameters: write $\sigma_j = \epsilon_j \sigma$, $\epsilon_j = \pm 1$, $x_j = 2\pi\beta\sigma^2 q_j$, $\bar{\gamma} = P/2\pi\sigma^2$. Lenard considers the isothermal-isobaric partition function

$$e^{-\beta G} := \frac{1}{N!^2} \sum_{\sigma_1, .., \sigma_N = \pm\sigma} \int_{\mathbb{R}^{2N} \times \mathbb{R}_+^{2N}} e^{-\beta \left( \sum_{j=1}^{2N} p_j^2/2 + U(q_1, \sigma_1; ..; q_N, \sigma_N) + P \max_j q_j \right)} dpdq \qquad (2.85)$$

$$= \sqrt{\frac{2\pi}{\beta}}^{2N} \frac{1}{(2\pi\beta\sigma^2)^{2N}} Q_N(\bar{\gamma}),$$

$$Q_N(\bar{\gamma}) := \sum_{\epsilon_1, .., \epsilon_N = \pm 1} \int_0^\infty dx_{2N} \int_0^{x_{2N}} dx_{2N-1} .. \int_0^{x_2} dx_1 e^{\sum_{1 \leq i < j \leq 2N} \epsilon_i \epsilon_j |x_i - x_j| - \bar{\gamma} x_{2N})}. \qquad (2.86)$$

The sequence $(\epsilon_1, .., \epsilon_N)$ can be characterized in terms of the charges $\nu_i = \sum_{k=1}^{i} \epsilon_k$ accumulated to the left of charge number $i$. Lenard calls a configuration $(\epsilon_1, .., \epsilon_N)$ *irreducible* if $\nu_i \neq 0$ unless $i = 0$ or $i = 2N$. Let $\bar{Q}_N(\bar{\gamma})$ be defined through (2.86) except that the summation is over irreducible $\epsilon$-sequences only. Then

$$Q_N = \sum_{k=1}^{N} \sum_{n_1 + .. + n_k = N} \bar{Q}_{n_1} .. \bar{Q}_{n_k}.$$

This can be expressed in terms of a renewal equation and as a formal power series identity.

$$Q(z) = \frac{1}{1-\bar{Q}(z)}, \quad Q(z) = 1 + \sum_{n=1}^{\infty} Q_n z^n, \quad \bar{Q}(z) = \sum_{n=1}^{\infty} \bar{Q}_n z^n.$$

Lenard then proves (p. 688) that $Q(z)$ is meromorphic in a disk of radius $R$ with a unique pole $r$ inside that disk; $r$ is real and

$$Q_N r^N = \frac{1}{r\bar{Q}'(r)} + O((r/R)^N), \quad r\bar{Q}'(r) < \infty.$$

Later (p. 691), Lenard comes back to the significance of the irreducible configurations and characterizes them as "the *minimal electrically neutral parts* of a general configuration". Let $P_n := r^n \bar{Q}_n$. The numbers $(P_n)$ define a probability distribution on $\mathbb{N}$, they give the probability that an irreducible block of a given length appears. The first moment

$$\sum_{n=1}^{\infty} nP_n = r\bar{Q}'(r)$$

is finite for all $\bar{\gamma} > 0$ and represents "a measure of the efficacy of electrostatic shielding". However, in the limit $\bar{\gamma} \to \infty$, the moment gets infinite: "the number of particles within a shielding distance tends to infinity".

**A renewal inequality for jellium tubes.** Newton's electrostatic theorem plays a crucial role in the investigation of thermodynamic limits of Coulomb systems [LL72]. In one dimension, the Coulomb potential is essentially $-|x|$ and disjoint neutral subsystems on a line do not interact: this may be viewed as a one-dimensional version of Newton's electrostatic theorem. The $y$-average of the semi-periodic version of the logarithmic interaction gives the one-dimensional Coulomb interaction. These observations and a clever partitioning of the configuration state into neutral blocks allow the derivation of a renewal inequality.

Consider $N$ particles of charge $q$ moving on a cylinder $[0, NT] \times [0, 2\pi R]$ in a neutralizing background of charge density $-nq = -q/(2\pi RT)$. The Coulomb interaction is taken as $V_C(z) = -\log|2\sinh\frac{z}{2R}|$, $z = x + iy \equiv (x, y) \in \mathbb{R}^2$. Let

$$Z_N := \frac{1}{N!} \int_{([0,NT]\times[0,2\pi R])^N} e^{-\beta U_N}$$

be the associated partition function. We will be interested in a splitting of the system into neutral subsystem and define the total charge accumulated to the left of $x \in [0, NT]$ as

$$Q(x; x_1, .., x_N) = q|\{j \in \{1, .., N\} \mid 0 \le x_j \le x\}| - nq \cdot 2\pi Rx.$$

If the subsystem to the left of $x$ is neutral, the background charge $qn2\pi Rx = x/T$ compensates an integer number of charged particles, so that $x$ must be a multiple of $T$. It is convenient to label particles from left to right and partition the configuration space according to the set of zeroes of $Q(\cdot; x_1, .., x_N)$: for $k \in \{1, .., N\}$, let $\mathcal{R}_{N,k}$ be the set of $((x_1, y_1), .., (x_N, y_N)) \in ([0, NT] \times [0, 2\pi R])^N$ such that

$$x_1 \le .. \le x_N \text{ and } \min\{x > 0 \mid Q(x; x_1, .., x_N) = 0\} = kT.$$

The set $\mathcal{R}_{N,k}$ may be seen as a minimal electrically neutral component, in the spirit of Lenard's treatment.

Now we are ready to state the renewal inequality:

**Proposition 2.29.** *For $N \in \mathbb{N}$, let $\alpha_N := \int_{\mathcal{R}_{N,N}} e^{-\beta U_N}$. Then $Z_1 = \alpha_1$ and for $N \geq 2$,*

$$Z_N \geq \alpha_1 Z_{N-1} + \alpha_2 Z_{N-2} + ... + \alpha_{N-1} Z_1 + \alpha_N.$$

*Proof.* $Z_1 = \alpha_1$ is obvious; for $N \geq 2$, note that $Z_N = \sum_{k=1}^{N} \int_{\mathcal{R}_{N,k}} e^{-\beta U_N}$ so that the proposition follows from $\int_{\mathcal{R}_{N,k}} e^{-\beta U_N} \geq \alpha_k Z_{N-k}$. We prove the inequality for $k = 1$, the case $k \geq 2$ is analogous. If $Q(T; x_1, .., x_N) = 0$, the system splits into two neutral subsystems, consisting of the charges and the background located to the left /right of $x = T$; there is only one charge to the left of $x = T$. The potential energy can be written as

$$U_N(z_1, .., z_N) = U_{11}(z_1) + U_{12}(z_1, .., z_N) + U_{22}(z_2, .., z_N)$$

where $U_{11}$, $U_{22}$ represent the energies of the separate subsystems and $U_{N,12}$ their interaction. Using Jensen's inequality, we get

$$\int_{\mathcal{R}_{N,1}} e^{-\beta U_N} \geq e^{-\beta \int_{\mathcal{R}_{N,1}} U_{12} e^{-\beta(U_{11}+U_{22})}} \int_{\mathcal{R}_{N,1}} e^{-\beta(U_{11}+U_{22})}. \tag{2.87}$$

The neutrality of the subsystems located to the left and right of $x = T$ implies

$$\int_{[0,2\pi R]^N} U_{12}(z_1, .., z_N) dy_1 dy_2 .. dy_N = 0.$$

Indeed,

$$-\frac{1}{2\pi R} \int_0^{2\pi R} \log |2 \sinh \frac{x + iy}{2R}| dy = -\frac{|x|}{2R},$$

hence the $y$ average of $U_{12}$ reduces to a one-dimensional interaction which we may write

$$\langle U_{12} \rangle_y (x_1, ., x_N) = -\int_{[0,T]\times[T,NT]} \frac{|x_1' - x_2'|}{2R} d\rho_1(x_1') d\rho_2(x_2') = 0$$

where $\rho_1 := \delta_{x_1} - n dx_1'$ and $\rho_2 = \sum_{j=2}^{N} \delta_{x_j} - n dx_2'$ represent the charge densities of the two subsystems, and $\rho_1([0, T]) = 0 = \rho_2([T, NT])$ due to neutrality. Since $e^{-\beta(U_{11}+U_{22})}$ and $\mathcal{R}_{N,1}$ are invariant with respect to translations $(y_1, y_2, .., y_N) \mapsto (y_1 - a, .., y_N - a)$, it follows that

$$\int_{\mathcal{R}_{N,1}} U_{12} e^{-\beta(U_{11}+U_{22})} = 0. \tag{2.88}$$

Due to translational invariance,

$$\int_{\mathcal{R}_{N,1}} e^{-\beta U_{11}} e^{-\beta U_{22}} = \alpha_1 Z_{N-1}.$$

Combining this with (2.87) and (2.88), we obtain $\int_{\mathcal{R}_{N,1}} e^{-\beta U_N} \geq \alpha_1 Z_{N-2}$. □

The previous proposition is an analogue of Lenard's formula for the two-component plasma partition function. The result is not completely satisfying and we would prefer an equality instead of an inequality. However, we believe that it connects the renewal inequality to physical intuition, at least in the jellium picture. It would be interesting to have an interpretation of the renewal equality in the quantum-mechanical setting of Laughlin's wave function.

## 2.4 A Lieb-Schultz-Mattis type argument

Up to now, we have not used the characterization of Laughlin's wave function as the ground state of a truncated interaction. Laughlin described his state as incompressible; in order to apply the characterizations of incompressibility of Section 1.5, we need a Hamiltonian that has Laughlin's state as its ground state. The truncated Hamiltonians of Section 1.4 serve this purpose. In this section, we show that *if* the truncated Hamiltonian reproduces the incompressibility of the ground state, then there must symmetry breaking. This can be shown with a Lieb-Schultz-Mattis method as given by Koma [Kom04]. A drawback is that it has not been proved that the truncated Hamiltonian give an incompressible ground state, see the discussion by Haldane in [PG87], p.319 and the end of this section. Moreover, the argument in this form applies to short-range, translationally invariant interactions, and cannot be applied to the lowest Landau level projected Coulomb Hamiltonian. Note that not only is the Coulomb interaction long-ranged, but the neutralizing background on the cylinder gives a potential attracting the electrons to the center of the cylinder: the background creates a potential that is not translationally invariant.

Koma [Kom04] considers infinite volume ground states of electrons on a cylinder, confined to finitely many Landau levels and interacting via a short range potential. Using a Lieb-Schultz-Mattis type argument, he establishes a relation between the existence of a gap, spatial periods and average density of a ground state. This section gives a simplified version of his arguments, adapted to the truncated Hamiltonian that has Laughlin's $p = 3$ wave function as its ground state. Before we turn to the proof, we give a glimpse of the Lieb-Schultz-Mattis argument for spin chains, its extension to electrons on a lattice by [Tas04, YOA97] (see also [Osh00]), and point out a connection to the usual argument for degeneracy of ground states on a torus.

In [LSM61], Appendix B, Lieb, Schultz and Mattis prove the nondegeneracy of the ground state and absence of an energy gap for a Heisenberg spin chain with periodic boundary conditions. The non-degeneracy is proved first; the existence of low-energy excitations is then inferred from a nice interplay of two transformations ([LSM61] p. 457): an overall shift $T_x$ and a "twist" $\mathcal{O}^k = \exp(ik\sum_{n=1}^{N} nS_n^z)$, where $N$ is the length of the chain. Using the commutation rule

$$T_x \mathcal{O}^k T_x^{-1} = \mathcal{O}^k e^{ikNS_1^z} e^{-ik\sum_{n=1}^{N} S_n^z}, \tag{2.89}$$

it is shown that for suitable $k$ the candidate excited state $\mathcal{O}^k\Psi_0$ is orthogonal to the ground state $\Psi_0$. A related argument is given in a setting with infinite spin chains in [AL86], where the global twist was replaced with a local twist $\exp(i\frac{\pi}{\ell}\sum_{j=-\ell}^{\ell} S_j^z(j+\ell))$.
Later, the argument has been adapted to fermions on a one-dimensional lattice, for the finite-volume [YOA97] and the infinite volume [Tas04] case. The twist operator that is used is of the type $\exp(ik\sum_j j\hat{n}_j)$, where $\hat{n}_j$ is the occupation number operator of lattice site $j$. This is applied in [Kom04] to electrons in a magnetic field on a lattice.

For electrons in a magnetic field on a torus, there is a simple argument establishing a relation between the density and the degeneracy. It rests on the non-trivial commutation rules of magnetic translations, reminiscent of (2.89). Let us briefly recall the argument: consider a Hamiltonian of the type

$$H_N = \frac{1}{2m}\sum_{j=1}^{N}(p_j + eA(x_j, y_j))^2 + \sum_{1\le j<k\le N} V(z_j - z_k)$$

in $L^2([0,L]\times[0,2\pi R])$ with quasiperiodic boundary conditions as discussed on p.8. Define the center of mass translations

$$T_x := t(\frac{L}{N_f}\mathbf{e}_x)^{\otimes N}, \quad T_y := t(\frac{2\pi R}{N_f}\mathbf{e}_y)^{\otimes N}.$$

Then $t(L/N_f\mathbf{e}_x)t(2\pi R/N_f\mathbf{e}_y) = e^{i2\pi/N_f}t(2\pi R/N_f\mathbf{e}_y)t(L/N_f\mathbf{e}_x)$ and

$$[H_N, T_x] = 0, \quad [H_N, T_y] = 0, \quad T_xT_y = e^{i2\pi N/N_f}T_yT_x.$$

Suppose that $H$ has an eigenspace of finite dimension $d$. Consider $T_x$, $T_y$ as unitary operators in this space, then

$$\det T_x \det T_y = \det T_xT_y = \det e^{i2\pi N/N_f}T_yT_x = e^{i2\pi dN/N_f}\det T_y \det T_x$$

whence $e^{i2\pi dN/N_f} = 1$ and $dN/N_f \in \mathbb{N}$. Write $N/N_f = p/q$ with $p, q$ coprime and suppose $\nu = p/q \notin \mathbb{N}$. Then $q$ divides $d$, and every eigenspace of $H_N$ is at least $q$-fold degenerate. Moreover, if $\Psi_0$ is a translationally invariant state, i.e. $T_x\Psi_0 = e^{i\alpha}\Psi_0$ for some $\alpha \in \mathbb{R}$, it breaks the $y$-invariance:

$$T_xT_y\Psi_0 = e^{i2\pi p/q}T_yT_x\Psi_0 = e^{i2\pi p/q}e^{i\alpha}T_y\Psi_0,$$

$\Psi_0$ and $T_y\Psi_0$ belong to different eigenspaces of $T_x$ and thus $\langle\Psi_0, T_y\Psi_0\rangle = 0$.

This can be nicely related to the Lieb-Schultz-Mattis type argument for electrons on a lattice, as given in [YOA97], if we restrict to the lowest Landau level. The lowest Landau level has a basis $\tilde{\psi}_k$, $k \in \mathbb{Z}/N_f\mathbb{Z}$. Each basis function can be thought of as a lattice site. The effect of the magnetic translations is $t(L/N_f\mathbf{e}_x)\tilde{\psi}_k = \tilde{\psi}_{k+1}$, $t(2\pi R/N_f\mathbf{e}_y)\tilde{\psi}_k = e^{-i2\pi k/N_f}\tilde{\psi}_k$. Thus the shift in the $y$-direction may be rewritten as

$$T_y = \exp(-i\sum_{j=1}^{N_f} j\hat{n}_j)$$

and we recognize the twist operator of [YOA97]. From this point of view, Koma's approach looks like an infinite volume version of the torus degeneracy argument (let us mention however, that Koma allows a one-particle potential $V(y)$). The overall $y$-shift is replaced by a local twist, and instead of orthogonality of $\Psi_0$ and $T_y\Psi_0$ we obtain orthogonality only in the limit where the local twist becomes global (see Lemma 2.34 below).

In the following we restrict to the lowest Landau level. Let $\mathcal{A}_{LL}$ be the canonical anti-commutation algebra over the lowest Landau level, i.e., the sub-algebra of $\mathcal{A}$ generated by $\mathbf{1}$ and $c_k, c_m^*, k, m \in \mathbb{Z}$. For $\Lambda \subset \mathbb{Z}$, let $\mathcal{A}_\Lambda$ be the sub-algebra of $\mathcal{A}_{LL}$ generated by $\mathbf{1}$ and $c_k, c_m^*, k, m \in \Lambda$, and let $\mathcal{A}_{loc} := \cup_{\Lambda\subset\mathbb{Z},|\Lambda|<\infty}\mathcal{A}_\Lambda$ be the set of local observables. Let $\hat{n}_k := c_k^*c_k$ and consider the formal expressions

$$H = \sum_{k,k',n,n'\in\mathbb{Z}} \delta_{k+k',n+n'} f(k-k')f(n-n')c_{k'}^*c_k^*c_nc_{n'}, \qquad \hat{N} = \sum_{k\in\mathbb{Z}}\hat{n}_k$$

where $f(n) := n\exp(-\frac{\gamma^2}{4}n^2)$ and $\hat{n}_k = c_k^*c_k$. The Hamiltonian $H$ is the second quantized form of the truncated interaction of Definition 1.3, see Corollary 1.7. For $A \in \mathcal{A}_{loc}$, the sums

$$[H, A] := \sum_{k,k',n,n'\in\mathbb{Z}} f(n-n')f(k-k')[c_{k'}^*c_k^*c_nc_{n'}, A] \qquad [\hat{N}, A] := \sum_{k\in\mathbb{Z}}[\hat{n}_k, A].$$

are absolutely convergent: suppose $A \in \mathcal{A}_\Lambda$, $\Lambda \subset \mathbb{Z}$, $|\Lambda| < \infty$. The commutator $[\hat{n}_k, A]$ vanishes unless $k \in \Lambda$, thus $[\hat{N}, A]$ is a finite sum. Similarly, $[c^*_{k'}c^*_k c_n c_{n'}, A]$ vanishes unless one of the indices $k, k', n, n'$ is in $\Lambda$. Let $C, c > 0$ such that $|f(n)| \leq C\exp(-cn^2)$. Then

$$\begin{aligned}
&\sum_{k\in\Lambda}\sum_{k',n,n'\in\mathbb{Z}} \delta_{k+k',n+n'}||f(n-n')f(k-k')[c^*_{k'}c^*_k c_n c_{n'}, A]|| \\
&\quad\leq 2C||A||\sum_{k\in\Lambda}\sum_{k',n,n'\in\mathbb{Z}} \delta_{k+k',n+n'}|f(n-n')f(k-k')| \\
&\quad\leq 2C||A||\sum_{k\in\Lambda,n\in\mathbb{Z}}\sum_{\substack{k',n'\in\mathbb{Z}:\\ n'-k'=k-n}} e^{-\frac{c}{2}((n'+k'-n-k)^2+4(k-n)^2)} \\
&\quad\leq 2C||A||\sum_{k\in\Lambda,n\in\mathbb{Z}} e^{-2c(k-n)^2}\sum_{t\in\mathbb{Z}} e^{-\frac{c}{2}t^2} < \infty.
\end{aligned}$$

For $\mu \in \mathbb{R}$, $A \in \mathcal{A}_{loc}$, let $\delta_\mu(A) := i[H - \mu\hat{N}, A] = i[H, A] - i\mu[\hat{N}, A]$. Then $\delta_\mu$ is a symmetric derivation on $\mathcal{A}_{loc}$. Its closure is the generator of a strongly continuous group $(\alpha^\mu_t)_{t\in\mathbb{R}}$of $C^*$-automorphisms of $\mathcal{A}_{LL}$:

$$\forall t \in \mathbb{R},\ \forall A \in \mathcal{A}_{loc}: \quad \frac{d}{dt}\alpha^\mu_t(A) = i\alpha^\mu_t([H - \mu\hat{N}, A])$$

see [Mat96], Theorems 1.1 and 2.9. The algebra $\mathcal{A}^{U(1)}$ is the subalgebra of operators of $\mathcal{A}_{LL}$ that preserve the total particle number. More precisely, let $\gamma_t$ be the $^*$-automorphism of $\mathcal{A}$ such that for all $k \in \mathbb{Z}$, $\gamma_t(c_k) = e^{-it}c_k$ and $\mathcal{A}^{U(1)}$ the set of $A \in \mathcal{A}_{LL}$ such that $\gamma_t(A) = A$ for all $t \in \mathbb{R}$. Note that $\partial_t\gamma_t(c_k) = +i\gamma_t([\hat{N}, c_k])$, $\gamma_t$ is associated to the derivation $a \mapsto i[\hat{N}, a]$. A local observable is in $\mathcal{A}^{U(1)}$ if and only if $[\hat{N}, A] = 0$.

**Definition 2.30.** *Let $\mu \in \mathbb{R}$ and $\omega$ be a state on $\mathcal{A}_{LL}$.*

- *$\omega$ is a ground state of $H - \mu\hat{N}$ if*

$$\forall a \in \mathcal{A}_{loc}: \quad \omega(a^*[H - \mu\hat{N}, a]) \geq 0. \tag{2.90}$$

- *$\omega$ is a ground state at fixed density, or ground state for $\mathcal{A}^{U(1)}$, if it is gauge-invariant ($\omega \circ \gamma_t = \omega$ for all $t \in \mathbb{R}$) and (2.90) holds for all $a \in \mathcal{A}^{U(1)} \cap \mathcal{A}_{loc}$.*

For *2.*, the precise value of the chemical potential is irrelevant. This is due to $[H - \mu\hat{N}, a] = [H, a]$ for all $a \in \mathcal{A}^{U(1)} \cap \mathcal{A}_{loc}$. It follows from *1.* that $\omega \circ \alpha^\mu_t = \omega$ (see Lemma 5.3.16 in [BR79b]), and there exists a positive Hamiltonian $H_\omega$ such that

$$H_\omega\Omega_\omega = 0, \quad \pi_\omega(\alpha^\mu_t(a)) = e^{itH_\omega}\pi_\omega(a)e^{-itH_\omega},$$

where $(\pi_\omega, H_\omega, \Omega_\omega)$ is the cyclic representation associated to $\omega$. We will say that the state $\omega$ is *gapped* if $H_\omega$ has a gap above its (possibly degenerate) ground state. An equivalent, more intrinsic definition is the following (see also [AL86], p.65):

**Definition 2.31.**

*1. Let $\mu \in \mathbb{R}$, and $\omega$ be a ground state of $H - \mu\hat{N}$. Then $\omega$ has a gap $w_0 > 0$ if for all $a, b \in \mathcal{A}_{LL}$, the function $\rho_{ab}(t) = \omega(a\alpha^\mu_t(b))$ has a distributional Fourier transform $\hat{\rho}_{ab}$ with* $\mathrm{supp}\hat{\rho}_{ab} \cap ]0, w_0[ = \emptyset$.

*2. Let $\omega$ be a ground state at fixed density. Then $\omega$ is gapped with respect to $\mathcal{A}^{U(1)}$, i.e., excitations preserving the number of particles, with gap $\geq w_0$, if the previous condition on $\rho_{ab}$ holds for all $a, b \in \mathcal{A}^{U(1)}$.*

The thermodynamic limits of cylinder Laughlin wave functions are ground states at fixed density, but also of $H - \mu\hat{N}$ at chemical potential $\mu = 0$. This follows from the positivity of the truncated Hamiltonians. To see this, let $(a_n)_{n\in\mathbb{N}}$ be such that $a_n \to -\infty$, $n + a_n \to \infty$ and let $\omega_n$ be the state on $\mathcal{A}_{LL}$ associated to $t(a_n 3\gamma l \mathbf{e}_x)^{\otimes N}\Psi_N^C$. The weak-* limit points of $\omega$ are ground states of $H$. Recall that the states form a weak*-compact set.

**Proposition 2.32.** *Suppose that for a subsequence $(\omega)_{n_k}$ of $(\omega_n)_n$, $\omega_{n_k} \stackrel{*}{\rightharpoonup} \omega$. Then $\omega$ is a ground state of $H$.*

*Proof.* For simplicity suppose $\omega_n \stackrel{*}{\rightharpoonup} \omega$. Let $\Lambda_n := \{3a_n, 3a_n + 1, .., 3(a_n + n - 1)\}$ and let $H_{\Lambda_n} := \sum_{3a_n \le k,k',n,n' \le 3a_n+3n-3} \delta_{k+k',n+n'} f(k-k')f(n-n')c^*_{k'}c^*_k c_n c_{n'}$ be the finite volume truncated Hamiltonian. The shifted Laughlin state is a ground state of $H_{\Lambda_n}$ considered as an operator in $\wedge^n\text{span}\{\psi_k \mid k \in \Lambda_n\}$, of energy 0. But $H_{\Lambda_n}$ is a positive operator in $\wedge\text{span}\{\psi_k \mid k \in \Lambda_n\}$, and the shifted Laughlin state can also be considered as a ground state in this larger space (i.e., changing the number of particles does not lower the energy, and we can take $\mu = 0$). Thus

$$\forall a \in \mathcal{A}_{\Lambda_n} : \ \omega_n(a^*[H_n, a]) \ge 0.$$

Let $a \in \mathcal{A}_{loc}$. By the absolute convergence of the sum $[H, a]$, $[H_n, a]$ converges to $[H, a]$. Thus

$$|\omega(a^*[H,a]) - \omega_n(a^*[H_n,a])| \le |\omega(a^*[H,a]) - \omega_n(a^*[H,a])| + ||a^*[H - H_n, a]|| \to 0$$

thus $\omega(a^*[H, a]) = \lim_n \omega_n(a^*[H_n, a]) \ge 0$. □

*Remarks: 1. Equivalence of torus and cylinder functions.* In Section 2.2.3, we compared modified cylinder and torus wave functions and proved their equivalence in special cases. The description in terms of truncated Hamiltonians allows an indirect approach to a similar question. The torus wave functions can be used to define lowest Landau level states (we map $\tilde{\psi}_k$ to $\psi_k$); these are ground states of $H$ on finite chains, with periodic boundary conditions, and their limit points should be ground states of $H$ as well. Thus in the thermodynamic limit, torus and cylinder states are ground states of the *same* Hamiltonian. If they are different, the infinite volume Hamiltonian has a degenerate ground state.

*2. Equivalence of ensembles.* It is often easily checked that the limits of $N$-particle finite volume ground states are ground states at fixed density. To prove that they are also ground states of $H - \mu\hat{N}$ for suitable $\mu$ (equivalence of canonical and grand-canonical ensembles) may require more effort and can be done by using general theorems (see [Mat96]).

*3. Value of the chemical potential.* The limit of Laughlin's wave function at $p = 1/3$ is a ground state of $H - 0 \cdot \hat{N}$, but not the only one: the limits of Laughlin's wave functions at lower filling fractions $1/p'$ give ground states as well, with a lower density. However, in view of the incompressibility, we expect that there exists a positive chemical potential $\mu > 0$ such that the limits of the $1/3$ functions are ground states of $H - \mu\hat{N}$, and the $1/p$ wave functions, $p > 3$, are not ($\mu > 0$ favors higher density). In Theorem 2.33 below, we have in mind such a positive chemical potential $\mu > 0$.

The remainder of the section aims at the proof of the following theorem on ground states of $H - \mu\hat{N}$ (among which we find Laughlin's states):

**Theorem 2.33.** *Let $\mu \in \mathbb{R}$ be a fixed chemical potential.*

1. *Suppose $H - \mu\hat{N}$ has a unique ground state $\omega$ and $\omega$ is gapped. Then $\omega$ must be translationally invariant and have an integer filling factor $\nu$.*
2. *Suppose all ground states of $H - \mu\hat{N}$ have a spatial period. Let $\omega$ be a pure, gapped ground state with filling factor $\nu$. Write $\nu = p/q$, $p, q \in \mathbb{N}$ relatively prime. Then any period of $\omega$ is a multiple of $q$, and the ground state of $\alpha_t^\mu$ is at least $q$-fold degenerate.*

The theorem says essentially that at non-integer filling factor, the existence of a gap implies symmetry breaking. If there is a gap, there are either ground states with no translational period at all, or all ground states have a period related to the average density. If one tries to write this down as a mathematical statement, one has to assign an average density to a state that may have no translational period. The theorem is formulated slightly differently to avoid that trouble.

In view of the discussion on gaps in the first chapter, let us remark that the low-lying energy excitations are constructed by means of local twists $U_n \in \mathcal{A}^{U(1)} \cap \mathcal{A}_{loc}$. Thus we actually only use the gappedness of $\omega$ with respect to excitations that do not change the number of particles. Nevertheless, we chose to formulate the theorem in terms of $H - \mu\hat{N}$ because ground states for $\mathcal{A}^{U(1)}$ are less likely to fulfill the assumptions on the degeneracy of the ground states (e.g. *1.* will never hold, since there are different ground states for different densities).

The strategy of the proof is to take a gapped ground state $\omega$ with period $q$ and construct candidate low energy states using a local twist. One proves that under suitable assumptions among which an incommensurability of the period $q$ and the filling factor, these candidate states are orthogonal in the limit of global twists (Lemma 2.34) and have energies converging to the ground state energy (Lemma 2.35). Thus if $\omega$ is gapped, the constructed low energy states are close to a ground state, but orthogonal to $\omega$. This gives a ground state degeneracy and *1.* follows. Lemma 2.36 below then allows to show that the candidate states are close to ground states with no translational period at all, which is used to prove *2.*

**Lemma 2.34.** *(Orthogonality.) Let $\omega$ be a state on $\mathcal{A}_{LL}$ with a period $q \in \mathbb{N}$, i.e., $\omega \circ \tau_x^q = \omega$. Let $\nu := \frac{1}{q}\sum_{j=1}^{q} \omega(\hat{n}_j)$ be the filling factor. For $n \in \mathbb{N}$, let*

$$U_n := \exp(i\frac{2\pi}{n}\sum_{j=0}^{n-1} j\hat{n}_j).$$

*Suppose $q\nu \notin \mathbb{N}$ and $\lim_{n\to\infty}(\frac{1}{n^2}\sum_{i,j=0}^{n-1} \hat{n}_i\hat{n}_j) = \nu^2$ (this is the case if $\omega$ is ergodic with respect to $\tau_x^q$). Then $\lim_{n\to\infty} \omega(U_n) = 0$.*

*Proof.* Let $\ell \in \mathbb{N}$. By a direct computation,

$$\tau_x^q(U_\ell) = U_\ell e^{-i\frac{2\pi q}{\ell}\sum_{j=q}^{\ell+q-1}\hat{n}_j} R_\ell, \quad R_\ell := e^{i\frac{2\pi}{\ell}\sum_{j=0}^{q-1} j(\hat{n}_{j+l}-\hat{n}_j)}.$$

Thus

$$\omega(U_\ell) = \omega(\tau_x^q(U_\ell)) = \omega(U_\ell e^{-i\frac{2\pi q}{\ell}\sum_{j=q}^{\ell+q-1}\hat{n}_j} R_\ell)$$
$$(1 - e^{-i2\pi q\nu})\omega(U_\ell) = \omega(R_\ell U_\ell(e^{-i\frac{2\pi}{\ell}\sum_{j=q}^{\ell+q-1}\hat{n}_j} - e^{-i2\pi q\nu}\mathbf{1})) + e^{-i2\pi q\nu}\omega(U_\ell(R_\ell - \mathbf{1})).$$

The second member of the sum goes to zero since $||R_\ell - \mathbf{1}|| = O(1/\ell)$. Note that for $A \in \mathcal{A}$, $\alpha \in \mathbb{C}$,

$$(e^{iA} - e^{i\alpha})^*(e^{iA} - e^{i\alpha}) = 4|\sin\frac{A-\alpha}{2}|^2 \leq (A-\alpha)^*(A-\alpha).$$

$R_\ell U_\ell$ is unitary. Using Cauchy-Schwarz, we see

$$|\omega(R_\ell U_\ell(e^{-i\frac{2\pi}{\ell}\sum_{j=q}^{\ell+q-1}\hat{n}_j} - e^{-i2\pi q\nu}\mathbf{1}))|^2 \leq 4\pi^2 q^2 \omega((\frac{1}{\ell}\sum_{j=q}^{\ell+q-1}\hat{n}_j - \nu)^2).$$

The upper bound goes to zero by the assumptions of the Lemma. Thus

$$(1 - e^{-i2\pi q\nu})\omega(U_\ell) \to 0 \quad (\ell \to \infty).$$

If $q\nu \notin \mathbb{N}$, this implies $\omega(U_\ell) \to 0$. □

**Lemma 2.35.** *(Low energy states.) Let $\omega$ be a state on $\mathcal{A}$, $\mu \in \mathbb{R}$. Then*

$$\lim_{\ell\to\infty}(\omega(U_\ell^*[H_\mu, U_\ell]) + \omega(U_\ell[H_\mu, U_\ell^*])) = 0.$$

*If $\omega$ is a ground state at fixed density, the two parts of the sum go to zero individually.*

*Proof.* Let $\ell \in \mathbb{N}$. First note $U_\ell \in \mathcal{A}_{loc}$, and $U_\ell$ commutes with every number operator $\hat{n}_j$. Thus $[H - \mu\hat{N}, U_\ell]$ is well-defined and equals $[H, U_\ell]$. Similarly, $[H - \mu\hat{N}, U_\ell^*] = [H, U_\ell^*]$. Note also $U_\ell, U_\ell^* \in \mathcal{A}_{loc}^{U(1)}$. Thus if $\omega$ is a ground state for $\mathcal{A}^{U(1)}$, $\omega(U_\ell[H, U_\ell^*]) \geq 0$ and $\omega(U_\ell^*[H, U_\ell]) \geq 0$. In this case the sum converges to zero if and only if each summand goes to zero. Let $g(k) := k\chi_{[1,\ell-1]}(k)$. so that $U_\ell c_k U^* = \exp(i\frac{2\pi}{\ell}g(k)\hat{n}_k)c_k$. Then, using $\cos x - 1 = -2\sin^2(x/2)$, we get

$$\omega(U_\ell^*[H_\mu, U_\ell]) + \omega(U_\ell[H_\mu, U_\ell^*]) = -4\sum_{k,k',n,n'\in\mathbb{Z}} \delta_{k+k',n+n'} f(n-n')f(k-k') \sin^2(\frac{2\pi}{\ell}(g(n)+g(n')-g(k)-g(k')))c_{k'}^* c_k^* c_n c_{n'}$$

hence

$$|\omega(U_\ell^*[H_\mu, U_\ell]) + \omega(U_\ell[H_\mu, U_\ell^*])| \leq 4C' \sum_{\substack{k,k',n,n'\in\mathbb{Z}\\ k+k'=n+n'}} e^{-c((n-n')^2+(k-k')^2)} \sin^2(\frac{2\pi}{\ell}(g(n)+g(n')-g(k)-g(k')))$$

where $c, C'$ are chosen so that $|f(n)| \leq C'e^{-cn^2}$. The sum on the right-hand side can be split into two parts. Let $\delta \in ]0, 1/3[$. Let $\Sigma_1$ denote the sum over indices such that $k - n = n' - k'$ has modulus smaller or equal to $\ell^\delta/2$ for some $\delta \in ]0, 1/3[$, and $\Sigma_2$ the sum over indices where $k - n = n' - k'$ has modules greater than $\ell^\delta/2$. We claim that there is a constant $C > 0$ so that for sufficiently large $\ell$,

$$|\Sigma_1| \leq C\frac{\ell^{3\delta}}{\ell}, \quad |\Sigma_2| \leq C\ell \sum_{k\geq\ell^\delta/2} e^{-ck^2}.$$

Thus $\Sigma_1$ and $\Sigma_2$ go to zero as $\ell \to \infty$.

*Estimate of* $\Sigma_2$. The only contributions to the sum come from indices $k, k', n, n'$ where at least one of them is in $\{1, .., \ell - 1\}$. By a reasoning similar to the one used to prove the convergence of $[H, A]$ for $A \in \mathcal{A}_{loc}$, we get

$$\sum_{k\in\{1,..,\ell-1\}} \sum_{\substack{k,k',n',n\in\mathbb{Z} \\ k-n=n'-k', |k-n|>\ell^\delta/2}} e^{-c((n-n')^2+(k-k')^2)}$$
$$\leq \sum_{k\in\{1,..,\ell-1\}} \sum_{n\in\mathbb{Z}:|k-n|>\ell^\delta/2} e^{-2c(k-n)^2} \sum_{t\in\mathbb{Z}} e^{-ct^2/2}$$

The parts of the sum where $k', n'$ or $n$ lie in $\{1, .., \ell - 1\}$ are treated in a similar way. This gives the claimed estimate on $\Sigma_2$.

*Estimate of* $\Sigma_1$. Let $k, n \in \mathbb{Z}$ such that $|k - n| \leq \ell^\delta/2$. We claim that $\mathrm{dist}\,(g(k) - g(n), \{0, -\ell, \ell\}) \leq \ell^\delta$. If $k, n \in \{1, .., \ell - 1\}$. $|g(k) - g(n)| = |k - n| \leq \ell^\delta/2$. If $n \geq \ell > k \geq 1$, $g(k) - g(n) = k \geq \ell - \ell^\delta$, thus $|g(k) - g(n) - \ell| \leq \ell^\delta/2$. The other cases are treated in a similar fashion. Using $\sin(x + \pi) = -\sin x$ and $|\sin x| \leq |x|$, we see

$$\sum_{k\in\{1,..,\ell-1\}} \sum_{\substack{k',n,n'\in\mathbb{Z} \\ k+k'=n+n', |k-n|\leq\ell^\delta/2}} e^{-c((n-n')^2+(k-k')^2)} \sin^2(\frac{2\pi}{\ell}(g(n) + g(n') - g(k) - g(k')))$$
$$\leq 4\pi^2 \frac{\ell^{2\delta}}{\ell} \sum_{k\in\{1,..,\ell-1\}} \sum_{\substack{n\in\mathbb{Z}: \\ |n-k|\leq\ell^\delta/2}} (\sum_{n'\in\mathbb{Z}} e^{-c(n-n')^2})(\sum_{k'\in\mathbb{Z}} e^{-c(k-k')^2})$$
$$\leq 4\pi^2 (\sum_{t\in\mathbb{Z}} e^{-ct^2})^2 \frac{\ell^{3\delta}}{\ell}.$$

Again, the cases $k', n'$ or $n \in \{1, .., \ell - 1\}$ can be treated in a similar way, and we get the bound on $|\Sigma_2|$. □

**Lemma 2.36.** *Let $\omega$ be a fixed state on $\mathcal{A}$ with a period $q$, $l \in \mathbb{N}$. Let $\omega_\ell(A) := \omega(U_\ell^* A U_\ell)$. $\omega_\ell$ defines a state on $\mathcal{A}$. Suppose that $\omega_\ell$ is close to a periodic state, i.e., there exist an $\epsilon > 0$ and a periodic state $\omega'$ with $||\omega' - \omega_\ell|| < \epsilon$. Then $\omega_\ell$ is close to $\omega$:*

$$||\omega_\ell - \omega|| \leq 2\epsilon.$$

*Proof.* Let $\omega'$ be a periodic state on $\mathcal{A}$, i.e. $\omega' \circ \tau_x^T = \omega'$ for some $T \in \mathbb{N}$. Without loss of generality we may assume that $T$ is a multiple of $q$. Let $A \in \mathcal{A}_{loc}$. Then for $n \in \mathbb{Z}$

$$||\omega' - \omega||\,||A|| \geq |\omega'(\tau_x^{nT}(A)) - \omega_\ell(\tau_x^{nT}(A))|| = |\omega'(A) - \omega(U_\ell^* \tau_x^{nT}(A) U_\ell)|$$

But for sufficiently large $n$, $U_\ell^* \tau_x^{nT}(A) U_\ell = \tau_x^{nT}(A)$ and $\omega(U_\ell^* \tau_x^{nT}(A) U_\ell) = \omega(\tau_x^{nT}(A)) = \omega(A)$, and we get

$$|\omega'(A) - \omega(A)| \leq ||\omega' - \omega||\,||A||$$

for all local elements. Since $\mathcal{A}_{loc}$ is norm-dense in $\mathcal{A}$, this shows $||\omega' - \omega|| \leq ||\omega' - \omega_\ell||$, whence

$$||\omega - \omega_\ell|| \leq ||\omega' - \omega_\ell|| + ||\omega' - \omega|| \leq 2||\omega' - \omega_\ell|| \leq 2\epsilon.$$ □

*Proof of Theorem 2.33. 1.* Let $\omega$ be the ground state of $H - \mu\hat{N}$. Since $\alpha_t^\mu$ is translationally invariant, $\omega \circ \tau_x$ is a ground state too and by the unicity of the ground state $\omega \circ \tau_x = \omega$. Let $\nu := \omega(\hat{n}_1)$ be the filling factor. Suppose $\nu \notin \mathbb{N}$. Since $\omega$ is the unique ground state and any extremal ground state is pure, $\omega$ is a pure state. In particular, it is ergodic with respect to $\tau_x$. By the previous lemmata,

$$\lim_{\ell\to\infty} \omega(U_\ell) = 0, \quad \lim_{\ell\to\infty} \omega(U_\ell^*[H, U_\ell]) = 0. \tag{2.91}$$

Let $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$ be the cyclic representation associated to $\omega$ and $H_\omega$ the unique self-adjoint operator in $\mathcal{H}_\omega$ such that $H_\omega\Omega_\omega = 0$ and $\pi_\omega(\alpha_t^\mu(A)) = e^{itH_\omega}\pi_\omega(A)e^{-itH_\omega}$. Let $\psi_\ell := \pi_\omega(U_\ell)\Omega_\omega$. Note $||\psi_\ell|| = 1$. (2.91) can be rewritten as

$$\lim_{\ell\to\infty} \langle\Omega_\omega, \psi_\ell\rangle = 0, \quad \lim_{\ell\to\infty} \langle\psi_\ell, H_\omega\psi_\ell\rangle = 0. \tag{2.92}$$

Write $\psi_\ell = \tilde{\psi}_\ell + \langle\Omega_\omega, \psi_\ell\rangle\Omega_\omega$. Then

$$||\tilde{\psi}_\ell|| \to 1, \ \langle\tilde{\psi}_\ell, H_\omega\tilde{\psi}_\ell\rangle \to 0, \ \langle\tilde{\psi}_\ell, \Omega_\omega\rangle = 0,$$

contradicting the uniqueness of the ground state and the existence of a gap above the ground state.

*2.* Let $\omega$ be a gapped pure ground state of $\alpha_t^\mu$ with spatial period $T$ and filling factor $\nu = p/q$, $p, q$ coprime. Suppose $T\nu \notin \mathbb{N}$. Let $\mathcal{H}_\omega, \pi_\omega, \Omega_\omega, H_\omega, \psi_\ell$ be as above. Let $G$ be the projection on $\mathcal{N}(H_\omega)$ in $\mathcal{H}_\omega$. Then (2.92) stays valid, thus $\langle\psi_\ell, H_\omega\psi_\ell\rangle \to 0$. The existence of a gap above the ground state implies $\lim_{\ell\to\infty}(\mathbf{1} - G)\psi_\ell = 0$. Thus for $\epsilon > 0$ there exists $l \in \mathbb{N}$ and a normalized $\psi' \in \mathcal{N}(H_\omega)$ such that $||\psi' - \psi_\ell|| < \epsilon$ and $|(\Omega_\omega, \psi_\ell)| \leq \epsilon$. Let $\omega'$ be the state defined through $\omega'(A) = \langle\psi', \pi_\omega(A)\psi'\rangle$. $\omega'$ is a ground state of $H - \mu\hat{N}$, therefore by our assumptions it has a period. Moreover, $||\omega' - \omega_\ell|| \leq 2\epsilon$. Thus by Lemma 2.36, $||\omega_\ell - \omega|| \leq 4\epsilon$, i.e.,

$$\forall A \in \pi_\omega(\mathcal{A}_{LL}) : \ |\langle\psi_\ell, A\psi_\ell\rangle - \langle\Omega_\omega, A\Omega_\omega\rangle| \leq 4\epsilon||A||. \tag{2.93}$$

Since the bicommutant $\pi_\omega(\mathcal{A}_{LL})'' \subset \mathcal{B}(\mathcal{H}_\omega)$ is the closure of $\pi_\omega(\mathcal{A}_{LL})$ with respect to the seminorms $|\langle\xi, \cdot\eta\rangle|$, (2.93) extends to $\pi_\omega(\mathcal{A}_{LL})''$. But since $\omega$ is pure, $\mathcal{H}_\omega$ is irreducible with respect to $\pi_\omega(\mathcal{A}_{LL})$ and $(\pi_\omega(A))'' = (\mathbb{C}\mathbf{1})' = \mathcal{B}(\mathcal{H}_\omega)$. Thus we may take $A = |\Omega_\omega\rangle\langle\Omega_\omega|$, the projection on $\mathbb{C}\Omega_\omega$ in (2.93) and obtain

$$\Big||\langle\psi_\ell, \Omega_\omega\rangle|^2 - 1\Big| \leq 4\epsilon, \quad |\langle\psi_\ell, \Omega_\omega\rangle| \leq \epsilon.$$

This is contradictory for sufficiently small $\epsilon$. Thus $qT \in \mathbb{N}$, i.e., $T$ is a multiple of $q$.

Finally, recall that a state $\omega$ is called pure if for any positive functional $\omega'$, $\omega' \leq \omega$ implies $\omega' = \lambda\omega$ for some $\lambda \in [0, 1]$. From this we see that $\omega \circ \tau_x^j$ ($j \in \mathbb{Z}$) inherits the purity of $\omega$. Let $T$ be the minimal period of $\omega$. Then $\omega, \omega \circ \tau_x, .., \omega \circ \tau_x^{T-1}$ must be distinct. The translational invariance of $\alpha_t^\mu$ implies that they are all ground states. Thus we have $T$ distinct pure ground states, and the affine dimension of the ground state is greater or equal to $T$. Since $T$ is a multiple of $q$, this concludes the proof. □

**Is the truncated Hamiltonian gapped?** Theorem 2.33 shows that the existence of a gap above the ground state, at filling factor 1/3, implies some kind of symmetry breaking. However, the question whether the the truncated interaction has a gap above its ground state is open. Numerical computations [SFL$^+$05] suggest that the truncated Hamiltonian has a

non-vanishing gap above its ground state, for all values of the radius, but to our knowledge this is yet unproven.

In the previous section on jellium tubes, we mentioned that the jellium tube interpolates between a system with a phase transition (the one-dimensional jellium) and a system without (the two-dimensional system). A similar aspect appears in relation to the question of gaps of truncated interactions on cylinders. We start by observing that in the limit of thin cylinders, the truncated interaction at filling factor 1/3 may formally be written as

$$H = H_{TT} + O(e^{-5\gamma^2}), \quad H_{TT} := \sum_{k\in\mathbb{Z}} (e^{-\gamma^2}\hat{n}_k\hat{n}_{k+1} + 4e^{-2\gamma^2}\hat{n}_k\hat{n}_{k+2}) \tag{2.94}$$

see also [SFL$^+$05], equation (9). The limiting Hamiltonian $H_{TT}$ has obviously the ground state energy 0. Ground states consist of configurations where electrons sit in lattice sites spaced apart at least by 3 sites. At density 1/3, the unique ground state having a particle in 0 is the state which we loosely write

$$... \wedge \psi_{-3} \wedge \psi_0 \wedge \psi_3 \wedge \psi_6 ...$$

(the Tao-Thouless state). This state is exactly the limit of Laughlin's state as $\gamma \to \infty$, on finite samples (see the remark on p.54) as well as on infinite samples (this is implicit in the proof of the second part of Theorem 2.17). Furthermore, the Hamiltonian $H_{TT}$ has obviously a gap of size $4e^{-2\gamma^2}$ above the ground state at density 1/3.

By the results of Section 1.4, the cylinder Hamiltonian may be considered as a periodified version of the plane interaction (see also the expressions of the Hamiltonians in [RH94] and the corresponding expressions of the disk interaction in [PG87], p.307). Thus we expect that in the limit $\gamma \to 0$, the truncated interaction $H$ on the cylinder has a gap above its ground state if the disk truncated interaction has.

More generally, we can consider truncated interactions for different filling fractions $1/p$, $p \in \mathbb{N}$ and ask whether these have a gap. In the limit $\gamma \to \infty$, we will again obtain a gapped Hamiltonian without hopping terms as in (2.94) and thus expect that the Hamiltonian on thin cylinders is gapped. In the limit $\gamma \to 0$, the Hamiltonian approaches the disk truncated interactions. In [PG87], p.319, Haldane argues that the disk Hamiltonian should be gapped for $p < 70$ but gapless for $p > 70$; this is related to the expected phase transition of the two-dimensional one-component plasma at coupling constant $\Gamma \sim 140$.

# Chapter 3

# Charge transport

Laughlin's work [Lau81, Lau83] is important for the understanding of both the integer and the fractional quantum Hall effect. *Laughlin's argument* refers to the subtle use of a clever geometry - the cylinder - and gauge periodicity in an explanation of the integer effect. Laughlin's wave function is considered as a good approximation of fractional Hall effect ground states at simple filling fractions. It has been suggested [TW84] that a degeneracy in the ground states of electrons on cylinders is required in order to reconcile Laughlin's cylinder argument for the integer quantum Hall effect with fractional Hall conductances. If this is the case, and if Laughlin's function is indeed a good approximation to the ground state, it should reflect this degeneracy. This is a heuristic argument; nevertheless, it is interesting to observe that it is consistent with the symmetry breaking (on thin cylinders) proved in the previous chapter.

For quantum Hall systems on tori, the Hall conductance can be characterized in terms of a Chern number, and fractional quantization of the Hall conductance requires indeed a ground state degeneracy. This is a rigorous result. It is less clear what the significance, beyond heuristics, of our symmetry breaking results is for the FQHE. This chapter is devoted to that question. Let us anticipate and say that we do not give a full answer. The Hall conductance for systems without impurities is related to the electronic density, and our results do imply that the charge transport in Laughlin's state has the correct quantization. However, as we will see below, this uses only the fact that the density is periodic and has average value $(p \cdot 2\pi l^2)^{-1}$, but *not* that the minimal period is $p\gamma l$. Another drawback is that the results only pertain to samples without impurities. However, we think that this chapter is useful in replacing our results in their FQHE context.

We start with a brief summary of the gedanken experiments involving slow addition of a flux quantum, as described in [Lau81, Lau83]. Then we turn to a simple model of bulk charge transport on a cylinder. First we consider cylinders with quasiperiodic boundary conditions in the direction along the cylinder axis, i.e., we look at a system on a torus. We show that in the absence of a background potential, the adiabatic curvature appearing in the Chern number approach can be expressed in terms of the one-particle density. We pursue with spectral boundary conditions, more suitable for the cylinder Laughlin wave functions, and give an account on the number of approximations involved to recover the widely accepted fact that Laughlin's function describes a fractionally quantized charge transport.

## 3.1 Laughlin's argument(s)

### Integer quantum Hall effect: cylinder geometry

In his paper on the integer quantum Hall effect [Lau81], Laughlin considers a gas of electrons (charge $-e$) on a finite cylinder of radius $R$. A magnetic field is perpendicular to the surface, an electric field of strength $E_x$ is present on the cylinder in the direction along the cylinder axis, and a flux $\phi$ threads the cylinder. This system is represented by the Hamiltonian

$$H(\phi) = \frac{1}{2m}\sum_{j=1}^{N}\big((-i\hbar\partial_{x_j})^2 + (-i\hbar\partial_{y_j} + e(B_z - \frac{\phi}{2\pi R})^2\big) + \sum_{j=1}^{N} eE_x x_j \\ + \sum_{j=1}^{N} W(x_j, y_j) + \sum_{1\le j<k\le N} V(x_j - x_k, y_j - y_k)$$

$V$ is an interaction potential and $W$ a background potential. For the moment, we leave open the modelization of the cylinder edges. At fixed flux $\phi$, the electric and magnetic fields lead to a Hall current around the cylinder axis. The corresponding current operator is

$$I_y = \sum_{j=1}^{N} \frac{1}{2\pi R}\frac{-e}{m}(-i\partial_{y_j} + e(B_z - \frac{\phi}{2\pi R})) = \frac{\partial H}{\partial \phi}.$$

Now imagine that we are given a projection $P(0)$ onto a many particle wave function at flux $\phi = 0$. If we consider the electrons as non-interacting, we may fill the one-particle states up to a given Fermi energy and take $P(0)$ to be the projection onto the corresponding many-particle state. We wish to evaluate the current $\langle I_y(0)\rangle := \mathrm{tr}P(0)I_y(0)$. Suppose that $P(0)$ can be extended to a smooth family of projections $P(\phi)$ commuting with the Hamiltonian $H(\phi)$: $[H(\phi), P(\phi)] = 0$. Then

$$\langle I_y(0)\rangle = \mathrm{tr}P(0)\frac{\partial H}{\partial \phi}(0) = \frac{\partial \mathrm{tr}PH}{\partial \phi}(0). \tag{3.1}$$

If slow variation of the flux $\phi$ results in an adiabatic time evolution $U_A(\phi)$, we may take $P(\phi) = U_A(\phi)P(0)U_A(\phi)^*$. Therefore the derivatives in (3.1) are referred to as *adiabatic derivatives*. Now, the Hamiltonian is gauge periodic:

$$H(\phi + \phi_0) = e^{i\sum_{j=1}^{N} y_j/R} H(\phi) e^{-i\sum_{j=1}^{N} y_j/R}$$

for all $\phi$. The period is the flux quantum $\phi_0 = h/e$. This suggests the following approximation:

$$\langle I_y(0)\rangle \simeq \frac{\mathrm{tr}P(\phi_0)H(\phi_0) - \mathrm{tr}P(0)H(0)}{\phi_0}, \tag{3.2}$$

i.e., the derivative is approximated by a difference quotient with a flux difference $\Delta\phi := \phi_0$. Now comes the key sentence of [Lau81]:

> "Since, by gauge invariance, adding $\Delta\phi$ maps the system back into itself , the energy increase due to it results form the net transfer of $n$ electrons (..) from one edge to the other. The current is thus
>
> $$I = c\frac{neV}{\Delta\phi} = \frac{ne^2V}{h}."$$

(Here, $V$ is the difference in electrochemical potential between the left and right edge.) Several difficulties are encountered when one tries to turn Laughlin's argument into a mathematical statement, giving rise to different versions of the argument, compare e.g. [FGW00, ASY87]. We do not give a detailed discussion of the various difficulties encountered.

Suppose first $H(\phi)$ has a non-degenerate ground state, for each $\phi$, and let $P_0(\phi)$ be the projection onto the ground state. If the ground state energy stays separated from the rest of the spectrum by a gap $g(\phi) > g_{min} > 0$ (see hypothesis H2 below), adiabatic time evolution follows the ground state: $P_0(\phi) = P(\phi)$. In this case, $P(\phi)$ is gauge periodic and the difference quotient in (3.2) is 0. This is obviously not a good approximation to $\langle I_y(0)\rangle$.

To gain a better understanding, we have to take into account edges. We will consider only independent particles ($V = 0$). The one-particle Hilbert space can be written as direct sum of three orthogonal subspaces representing left edge, bulk and right edge states. Let $P_0(\phi)$ represent a state where bulk states are filled up to a given Fermi energy and edge states up to local chemical potentials. The one-particle Hamiltonian restricted to the bulk state has discrete spectrum with large gaps, whereas the restriction to the edge states has a spectrum that becomes continuous in the limit of large radii. Thus intuitively, adiabatic time evolution follows the ground state in the bulk, but not in the edges. In the words of [Lau81]:

> "*isothermal* differentiation with respect to $\phi$ (..) is equivalent to adiabatic differentiation in the sample interior."

Therefore the adiabatically evolved bulk state should be gauge periodic, but changes may occur in the edge states, the "net transfer of $n$ electrons".

However, it is not clear how to make sense of the separate application to edge and bulk of adiabatic time evolution. The decomposition into edge and bulk Hilbert spaces is flux dependent. During the flux increase left edge states typically flow into the bulk and bulk states into the right edge. Another problem is related to the integrality of the charge transfer.

## Fractional quantum Hall effect: infinite plane with flux tube

It is instructive to compare the previously described argument with the effect of slow addition of a flux quantum as performed in [Lau83]. Consider a gas of electrons moving in a plane with a perpendicular magnetic field. A solenoid carrying a flux $\phi$ pierces the plane at the origin. A one-particle Hamiltonian $H_1(\phi)$ can be defined whose spectrum consists again of the eigenvalues $\hbar\omega(n + 1/2)$, $n \in \mathbb{N}_0$, and the lowest Landau level $\mathcal{N}(H_1(\phi) - \hbar\omega/2)$ has a complete orthonormal set

$$\psi_{n,\phi}(z) = c_{n,\phi}|z|^{\phi/\phi_0} z^n e^{-|z|^2/4l^2}, \quad n \in \mathbb{Z} : n \geq -\phi/\phi_0,$$

see [EŠV02], Section III. Now consider the $N$-particle Hamiltonian

$$H(\phi) = \sum_{j=1}^{N} H_{1,j}(\phi) + W(z_j) + \sum_{1\leq j<k\leq N} V(z_j - z_k)$$

describing $N$ particles. Imagine that we start in a many-body ground state $\Psi_N$ at flux $\phi = 0$ and slowly increase the flux from 0 to the flux quantum $\phi_0$. If a suitable adiabatic theorem can be applied, the final state in the adiabatic limit will be an eigenstate of $H(\phi_0)$. But $H(\phi_0)$ is gauge periodic, thus applying a unitary $U$ to the adiabatically evolved state we

obtain an eigenvector of $H(0)$. We wish to apply this procedure to Laughlin's state as a starting vector. (We suppose that it is an eigenvector for some interaction.) In [Lau83], the adiabatically evolved state is given as

$$U^{-1}U_A\Psi_N^L(z_1,..,z_N) \simeq C_N \prod_{j=1}^{N} z_j \Psi_N^L(z_1,..,z_N), \tag{3.3}$$

$C_N \in \mathbb{C}$. In [PG87], p. 260, Laughlin explains that (3.3) should be considered as a good approximation to the adiabatically evolved state. This is motivated through the help of several approximations. The background potential is chosen to be a lowest Landau level projected Gaussian: $W(z_j) = \Pi\alpha|z_j|^2\Pi$. With this choice, $W\psi_{n,0} = 2\alpha(n+1)\psi_{n,0}$. It is stated that adiabatic time evolution away from the solenoid is not affected by interactions, so that we can set $V = 0$. The one-particle states $\psi_{n,0}$ evolve, up to a phase, to $\psi_{n,\phi_0}$, which is unitarily related to $\psi_{n+1,0}$. The $n$-dependence of the normalization constants is neglected: $\psi_{n+1,0}(z) \simeq z\psi_{n,0}(z)$. Thus if $\Psi_N^L = \sum a(m_1,..,m_N)\psi_{m_1,0} \wedge .. \wedge \psi_{m_N,0}$, the adiabatically evolved state will be, up to the unitary $U$

$$\begin{aligned} U^{-1}U_A\Psi_N^L(z_1,..,z_N) &\simeq \sum_{m_1,..,m_N} a(m_1,..,m_N)\psi_{m_1+1,0} \wedge .. \wedge \psi_{m_1+1,0} \\ &\simeq \sum_{m_1,..,m_N} a(m_1,..,m_N) z_1..z_N \psi_{m_1,0} \wedge .. \wedge \psi_{m_1,0} \\ &= z_1..z_N \Psi_N^L(z_1,..,z_N). \end{aligned}$$

A plasma analogy gives access to the one-particle density $\rho_h(z)$ of the obtained excited state: $\rho_h(z)$ should converge very fast to $(3\cdot 2\pi l^2)^{-1}$ away from the origin ($|z| \to \infty$), and

$$\int_{\mathbb{R}^2} (\rho_h(x,y) - \frac{1}{3\cdot 2\pi l^2}) dx\, dy = -\frac{1}{3}. \tag{3.4}$$

The excited state is therefore called a *quasihole state*: in the limit of infinitely many particles, it represents a uniform gas of charge density $-e(3\cdot 2\pi l^2)^{-1}$ except that there is a hole at the origin which leads to an overall missing charge $-e/3$ (fractional charge of the quasihole). Hence during the adiabatic flux increase, a charge $-e/3$ has been transferred from the origin to infinity. This leads to a fractional Hall conductance.

It is interesting to observe that there is a double use of the adiabatic flux increase: it is invoked to show that the Hall conductance is fractionally quantized, but also to show the existence of exited states describing a fractionally charged hole.

**Hall conductance as a relative index.** The (integer) Hall conductance for non-interacting electrons can be characterized as a Chern number when there is a doubly periodic background potential on the plane, or when the electrons are placed on a torus. When the background potential is not periodic, but macroscopically homogeneous, it can be expressed as a non-commutative Chern number (see [BvESB94]), which in turn can be expressed as the relative index of a pair of projections. In [ASS94] the Hall conductance is related to an index without non-commutative geometry. Let $H_1(\phi)$ be a one-particle Hamiltonian representing a particle in an infinite plane with a flux tube in the origin and a magnetic field perpendicular to the surface. Let $P_1(\phi)$ be the projection onto the ground state of $H_1(\phi)$. Due to gauge periodicity, $P_1(\phi_0) = UP(0)U^{-1}$. The relative index gives a precise meaning to $\mathrm{tr}P_1(\phi_0) - \mathrm{tr}P_1(0)$ in case the traces are infinite-dimensional. The difference may

be interpreted as a charge deficiency, i.e., the difference in the number of electrons in the ground states at the two values of the flux. If we regard $P_1(\phi)$ as the one-particle reduced matrix of a many-particle state, $\mathrm{tr}P_1(\phi_0) - \mathrm{tr}P_1(0)$ is an analogue of (3.4). Note however that subtleties show up: due to gauge invariance, the one-particle densities of $P_1(\phi_0)$ and $P_1(0)$ are identical, so that a strict analogue of (3.4) would give zero.

## 3.2 Charge transport on a cylinder

Based on Laughlin's cylinder argument, Tao and Wu [TW84] have suggested that a fractional transport on the cylinder requires ground state degeneracy. They present the picture that, at filling factor 1/3, there are three ground states; adding one flux quantum shifts the system from one ground state to the other. Only after addition of three units of flux does the system return to itself, so that Laughlin's argument would guarantee an integer transferred charge only after addition of three flux units, leading to a fractional charge.

For Hall systems on tori, the Chern number approach requires indeed a ground state degeneracy in order to explain fractional quantization of the Hall conductance. For cylinder systems, the significance of ground state degeneracy is not quite clear. Since the connection to charge transport has motivated the investigations of this thesis, let us go into some detail.

In the following, units are chosen so that $\hbar = 1, eB_z = -1, l = 1, m = 1$. We adopt a simple model for the bulk charge transport, induced by a variable flux threading a cylinder $[0, L] \times \mathbb{R}/(2\pi R\mathbb{Z})$. Consider the $N$-particle Hamiltonian

$$H(\phi) = \sum_j \frac{1}{2}(v_{x_j}^2 + v_{y_j}(\phi)^2) + \sum_j W(z_j) + \sum_{j<k} V(z_j - z_k), \tag{3.5}$$

where the velocity operators are

$$v_{x_j} = -i\partial_{x_j}, \quad v_{y_j}(\phi) = -i\partial_{y_j} - (x - \frac{\phi}{2\pi R}).$$

We do not yet specify boundary conditions. Let $\chi$ be a switch function, i.e., a function with derivative $\chi' \in C_0^\infty(]0, L[)$ such that $\chi' \geq 0$ and $\chi(0) = 0$, $\chi(L) = 1$. We suppose that $\chi'$ has its support in the middle of the cylinder, away from boundaries. According to Faraday's law, a time-dependent flux $\phi(t)$ will generate an electric field of strength $\dot{\phi}/(2\pi R)$ around the cylinder axis which together with the magnetic field gives rise to a Hall current along the cylinder axis. The $\chi'$-averaged current in the $x$-direction is

$$I_\chi = -\frac{1}{2}\sum_{j=1}^N (\chi'(x_j)v_{x_j} + v_{x_j}\chi'(x_j)).$$

The flux $\phi$ is slowly increased from 0 to the flux quantum $\phi_0 = 2\pi$: let $f \in C^1([0,1];\mathbb{R})$ be a monotone increasing function with $f(0) = 0$, $f(1) = 2\pi$ and let $\phi(t) := f(t/\tau)$ for some time scale $\tau > 0$. The associated propagator in the rescaled time $s = t/\tau$ satisfies

$$i\partial_s U_\tau(s) = \tau H(f(s))U_\tau(s), \quad U_\tau(0) = \mathbf{1}. \tag{3.6}$$

Let the system be initially in a state described by a projection $P$ of trace $q$, and let $P_\tau(s) := U_\tau(s)P(0)U_\tau(s)^*$ be the time evolved projection. The current and total charge transport are

given by

$$I_\tau(s) := \frac{1}{q}\mathrm{tr}P_\tau(s)I_\chi, \quad Q_\tau(s) := \tau\int_0^1 I_\tau(s)ds. \tag{3.7}$$

We are interested in expectation values of this current when the flux $\phi$ is adiabatically changed, i.e. in limits of the quantities above in the limit $\tau \to \infty$. In view of $E_y = \dot{\phi}(t)/(2\pi R) = -f'(t/\tau)/(\tau 2\pi R)$ this corresponds to the limit of small electric fields $E_y$. The time average over the period of the flux increase of the field is $\phi_0/2\pi R$. A fractionally quantized Hall conductance $\sigma_H = \frac{1}{3}e^2/h$ arises when the charge transported during the flux increase by one flux quantum $\phi_0 = h/e$ is fractional: $Q = e/p$, or when the charge transported during an increase by three flux units is $Q = e$.

We will look at two different choices of boundary conditions: quasiperiodic boundary conditions lead to the characterization of Hall conductance as a Chern number, whereas spectral boundary conditions are the natural setting for the cylinder Laughlin wave function.

It is commonly stated that the quantization of charge transport in Laughlin's state follows from the correct value of the average density. This is based on the picture that, in the absence of impurities, the (bulk) conductance equals the electronic charge density. This can be checked for the torus geometry (see the next subsection), where the adiabatic increase of the flux results in an overall translation of the system. As a result, one gets, for $W = 0$, a charge transport in terms of the one-particle density $\rho(x, y)$

$$Q_A(\phi) \propto \frac{1}{2\pi R}\int_0^\phi \Big(\int_{[0,L]\times[0,2\pi R]} \chi'(x)\rho(x - \frac{\phi'}{2\pi R}, y)dxdy\Big)d\phi'. \tag{3.8}$$

For a cylinder with boundaries, such a formula holds at best in the limit of infinitely long cylinders. If the one-particle density has period $3/R$ in the direction of the cylinder axis, and we extend the $x$-integration in (3.8) to $\mathbb{R}$, the average charge transport after addition of three flux units is proportional to

$$\begin{aligned}&\frac{1}{2\pi R}\int_0^{3\cdot 2\pi} \Big(\int_{\mathbb{R}\times[0,2\pi R]} \chi'(x)\rho(x - \frac{\phi}{2\pi R}, y)dxdy\Big)d\phi \\ &\quad = \Big(\int_{\mathbb{R}} \chi'(x)dx\Big)\Big(\int_{[0,3/R]\times[0,2\pi R]} \rho(u, y)dudy\Big) = 3\cdot 2\pi\bar{\rho}\end{aligned}$$

where $\bar{\rho}$ is the average density. Thus if the average density is $\bar{\rho} = (3\cdot 2\pi)^{-1}$, we obtain an integer transport after addition of three flux units and a correct quantization of the Hall conductance. The question whether $3/R$ is the smallest period of the one-particle density does not play a role. However, the symmetry breaking is consistent with the heuristic that "it takes three flux units before the system returns to itself".

The following two subsections are devoted to the proof of formulas of the type (3.8) on tori (i.e., cylinders with periodic boundary conditions) and cylinders with spectral boundary conditions. The main interest on tori is to compare the formula to the interpretation of the Hall conductance in terms of Chern numbers.

### 3.2.1 Periodic boundary conditions

This subsection presents the classical Chern number approach (see [KS90, ASS94]) in a way slightly different from the usual way. Our presentation simplifies the comparison with charge

transport on cylinders as defined in the next section. This section also aims at clarifying the relationship between the characterization of the Hall conductance as a Chern number and the common saying "the correct quantization of Hall conductance in Laughlin's state is obvious provided it has the right one-particle density".
Quasiperiodic boundary conditions in the direction along the cylinder axis are defined in terms of the magnetic translations $t(L)\psi(z) = e^{iLy}\psi(z-L)$. It proves useful to consider the family of boundary conditions $t(L)\psi = e^{i\beta}\psi$. Thus we should consider the Hamiltonian of (3.5) with these boundary conditions; call it $\hat{H}(\beta,\phi)$. In order to make the $\beta$-dependence more explicit and get a nice representation of the current $I_\chi$, we will work with the unitarily related Hamiltonian

$$H(\beta,\phi) = \sum_{j=1}^{N} \frac{1}{2}(v_{x_j}(\beta)^2 + v_{y_j}(\phi)^2) + \sum_j W(z_j) + \sum_{j<k} V(z_j - z_k), \tag{3.9}$$

$$v_{x_j}(\beta) = -i\partial_{x_j} - \beta\chi'(x_j), \quad v_{y_j}(\phi) = -i\partial_{y_j} - (x - \frac{\phi}{2\pi R}) \tag{3.10}$$

with the $\beta$-independent boundary conditions $t(L)\psi = \psi$. The current can be represented as a derivative:

$$I_\chi(\beta,\phi) = \frac{\partial H}{\partial\beta}(\beta,\phi).$$

We will need the following hypotheses:

**H1:** We say a function $\psi : [0,L]\times[0,2\pi R]$ fulfills the condition (QP) if it can be extended to a function $\tilde{\psi} \in C^\infty(\mathbb{R}^2)$ satisfying the quasiperiodicity condition

$$\tilde{\psi}(x+L,y) = e^{-iLy}\tilde{\psi}(x,y), \quad \tilde{\psi}(x,y+2\pi R) = \tilde{\psi}(x,y). \tag{3.11}$$

Let $D_0$ be the set of $N$-particle functions $\psi \in L^2(([0,L]\times[0,2\pi R])^N)$ that satisfy (QP) in each variable. There is a family of self-adjoint operators $H(\beta,\phi)$ with common domain $D$ and common core $D_0$, given on $D_0$ by the expression (3.9).

**H2:** The family of Hamiltonians is smooth and has a band bordered by gaps:

(i) Consider $(\beta,\phi) \mapsto H(\beta,\phi)$ as a map from $\mathbb{R}^2$ to $\mathcal{B}(D,\mathcal{H}_N)$, the space of bounded operators from $D$ equipped with the graph norm of $H(0,0)$ to $\mathcal{H}_N$. Then $H(\cdot,\cdot)$ is twice continuously differentiable.

(ii) There are two real-valued, continous functions $g_- \le g_+$ and $\epsilon > 0$ such that for all $\beta,\phi$:

$$\operatorname{dist}\big\{g_+(\beta,\phi), g_-(\beta,\phi)\}, \sigma(H(\beta,\phi))\big) > \epsilon$$

and the spectral projection $P(\beta,\phi)$ of of $H(\beta,\phi)$ associated with $[g_-,g_+](\beta,\phi)$ is finite-dimensional, of (constant) dimension $q > 0$.

*Remarks: 1.* The family of Hamiltonians is gauge-periodic:

$$H(\beta+2\pi,\phi) = e^{i2\pi\sum_{j=1}^N \chi(x_j)} H(\beta,\phi) e^{-i2\pi\sum_{j=1}^N \chi(x_j)}$$
$$H(\beta,\phi+2\pi) = e^{i\sum_{j=1}^N y_j/R} H(\beta,\phi) e^{-i\sum_{j=1}^N y_j/R}.$$

*2.* The Hamiltonian $H(\beta,\phi)$ is unitarily related to a more standard operator, used in [ASS94]: let $\tilde{H}(\beta,\phi)$ be defined exactly as $H(\beta,\phi)$ except that the velocity operator in the $y$-direction is $v_y(\beta) = -i\partial_y - \beta/L$. Then

$$H(\beta,\phi) = e^{i\beta(\chi(x_j)-x_j/L)}\tilde{H}(\beta,\phi)e^{-i\beta\sum_{j=1}^{N}(\chi(x_j)-x_j/L)}.$$

H1 and H2(i) ensure the existence and uniqueness of the propagator $U_\tau(\beta,s)$ ($i\partial_s U_\tau(\beta,s) = \tau H(\beta,f(s))U_\tau(\beta,s)$). Let $P_\tau(\beta,s) := U_\tau(\beta,s)P(\beta,0)U_\tau(\beta,s)^*$ be the time evolved projection. The adiabatic limits ($\tau\to\infty$) of the current and charge transport are characterized by the following two theorems:

**Theorem 3.1.** *([ASY87, KS90]) Suppose H1 and H2 hold and define the current $I_\tau$ and charge transport $Q_\tau$ as in (3.7). Then the current is a $2\pi$-periodic function of $\beta$ whose average in the large $\tau$ limit is*

$$\frac{1}{2\pi}\int_0^{2\pi} I_\tau(\beta,s)d\beta = \frac{f'(s)}{\tau}\frac{1}{2\pi q}\int_0^{2\pi}\mathrm{tr}P[(\partial_\beta P),(\partial_\phi P)](\beta,f(s))d\beta + O(1/\tau^2) \tag{3.12}$$

*uniformly in $s\in[0,1]$. As a consequence, the average charge transport is*

$$\frac{1}{2\pi}\int_0^{2\pi} Q_\tau(\beta)d\beta = \frac{1}{q}c(P) + O(1/\tau)$$

*where $c(P) = \frac{i}{2\pi}\int_{[0,2\pi]\times[0,2\pi]}\mathrm{tr}P[(\partial_\phi P),(\partial_\beta P)](\beta,\phi)d\beta d\phi$ is an integer, the* Chern number *associated with the gauge periodic family of projections $P(\beta,\phi)$.*

**Theorem 3.2.** *Suppose in addition to H1 and H2 that there is no background potential: $W=0$ in (3.6). Let $\rho(x,y;\beta,\phi)$ be the one-particle density associated to the density operator $q^{-1}P(\beta,\phi)$. Then*

$$I_\tau(\beta,s) = -\frac{f'(s)}{\tau}\frac{1}{2\pi R}\int_{[0,L]\times[0,2\pi R]}\rho(x,y;\beta,f(s))\chi'(x)dxdy + O(1/\tau^2) \tag{3.13}$$

$$\rho(x,y;\beta,\phi) = \rho(x-\frac{\phi}{2\pi R},y+\frac{\beta}{L};0,0). \tag{3.14}$$

*(3.13) holds uniformly in $(\beta,s)\in\mathbb{R}\times[0,1]$. The current and the charge transport are independent of $\beta$.*

*Remark.* The one-particle density is the function $\rho:[0,L]\times[0,2\pi R]\to\mathbb{R}$ such that for all continuous $f:[0,L]\times[0,2\pi R]\to\mathbb{C}$,

$$\frac{1}{q}\mathrm{tr}P\sum_{j=1}^{N}f(x_j,y_j) = \int_{[0,L]\times[0,2\pi R]}\rho(x,y)f(x,y)dxdy.$$

In (3.14) we consider $\rho$ as a doubly periodic function from $\mathbb{R}^2$ to $\mathbb{R}$.

*Proof of Theorem 3.1.* Details of the proof can be found in [KS90]. We recall the key idea behind (3.12): the current can be rewritten in terms of the *persistent current formula*

$$I_\tau(\beta,s) = \frac{\partial E_\tau}{\partial\beta}(\beta,s) + \frac{i}{q\tau}\mathrm{tr}P_\tau[(\partial_s P_\tau),(\partial_\beta P_\tau)](\beta,s). \tag{3.15}$$

where $E_\tau(\beta,s) := q^{-1}\mathrm{tr}P(\beta,s)H(\beta,f(s))$ is a $2\pi$ - periodic function of $\beta$. The adiabatic theorem allows to prove

$$\mathrm{tr}P_\tau[(\partial_s P_\tau),(\partial_\beta P_\tau)](\beta,s) = f'(s)\mathrm{tr}P[(\partial_\phi P),(\partial_\beta P)](\beta,f(s)) + O(1/\tau)$$

uniformly in $\beta, s$. Inserting this into (3.15) and taking the average over $\beta$, we get (3.12). □

*Proof of Theorem 3.2.* The key observation is that the Hamiltonian is not only gauge-periodic, but gauge-constant. For $\psi \in L^2([0,L]\times[0,2\pi R])$, let

$$u(\beta,\phi)\psi(x,y) := e^{i\beta\chi(x)}\psi(x-\frac{\phi}{2\pi R}, y+\frac{\beta}{L})$$

where $\psi$ is identified with its quasiperiodic extension as in (3.11). The unitary $u$ preserves the set of smooth quasiperiodic functions. To see this, consider $\chi$ as a function in $C^\infty(\mathbb{R})$ with the quasiperiodicity $\chi(x+L) = \chi(x)+1$ (this is compatible with $\chi' \in C_0^\infty(]0,L[)$, $\chi(0)=0$ and $\chi(L)=1$). For $\psi \in D_0$, let $\tilde\psi := u(\beta,\phi)\tilde\psi$. Then $\tilde\psi \in C^\infty(\mathbb{R}^2)$ and

$$\begin{aligned}\tilde\psi(x+L,y) &= e^{i\beta\chi(x+L)}\psi(x+L,y+\frac{\beta}{L})\\ &= e^{i\beta(\chi(x)+1)}e^{-iL(y+\beta/L)}\psi(x,y+\frac{\beta}{L})\\ &= e^{i\beta\chi(x)}e^{-iLy}\psi(x,y+\frac{\beta}{L}) = e^{-iLy}\tilde\psi(x,y).\end{aligned}$$

The $y$-periodicity of $\tilde\psi$ is obtained in a similar way. Thus $u(\beta,\phi)$ satisfies (QP). Let $V(\beta,\phi) := u(\beta,\phi)^{\otimes N}$. Then $V(\beta,\phi)$ leaves the domain of definition of $H(\beta,\phi)$ invariant, and

$$H(\beta,\phi)V(\beta,\phi) = V(\beta,\phi)H(0,0)V(\beta,\phi)^{-1}, \quad H(\beta,\phi) = V(\beta,0)H(0,\phi)V(\beta,0)^{-1} \tag{3.16}$$

for all $\beta,\phi\in\mathbb{R}$, and (3.14) follows from $P(\beta,\phi) = V(\beta,\phi)P(0,0)V(\beta,\phi)^{-1}$. Moreover, the propagators are related through $U_\tau(\beta,s) = V(\beta,0)U_\tau(0,s)$ so that the time evolved projections $P_\tau(\beta,s)$ and $P_\tau(0,s)$ are unitarily related. As a consequence, the energy $E_\tau(\beta,s) = E_\tau(0,s)$ does not depend on $\beta$, the persistent current $\partial_\beta E_\tau$ in (3.15) vanishes and

$$I_\tau(\beta,s) = \frac{if'(s)}{q\tau}\mathrm{tr}P[(\partial_\phi P),(\partial_\beta P)](\beta,f(s)) + O(1/\tau^2) \tag{3.17}$$

uniformly in $\beta, s$. In order to evaluate $\mathrm{tr}P[(\partial_\phi P),(\partial_\beta P)]$, we use the Chern-Simons formula (see [KS90], Lemma 3.2) which gives

$$\begin{aligned}\mathrm{tr}P[\partial_\beta P,\partial_\phi P](\beta,\phi) &= \partial_\beta\mathrm{tr}P(0,\phi)V(\beta,0)^{-1}(\partial_\phi(V(\beta,0))) - \partial_\phi\mathrm{tr}P(0,\phi)V(\beta,0)^{-1}(\partial_\beta(V(\beta,0))\\ &= -\partial_\phi\mathrm{tr}P(0,\phi)V(\beta,0)^{-1}(\partial_\beta V)(\beta,0).\end{aligned}$$

By an explicit computation, $i\partial_\beta u(\beta,0)\psi = (\frac{i}{L}\partial_y - \chi(x))u(\beta,0)\psi$, whence

$$\mathrm{tr}P[\partial_\beta P,\partial_\phi P] = -\partial_\phi\mathrm{tr}P(\beta,\phi)\sum_{j=1}^N(\frac{i}{L}\partial_{y_j} - \chi(x_j)). \tag{3.18}$$

Now let $\psi \in L^2([0,L]\times[0,2\pi R])$ fulfill (QP). For $\beta,\phi\in\mathbb{R}$, let $\tilde{\psi} := u(\beta,\phi)\psi$. Again, consider $\chi$ as a quasiperiodic function ($\chi(x+L)=\chi(x)+1$).

$$\begin{aligned}
&\langle u(\beta,\phi)\psi, (\frac{i}{L}\partial_y - \chi(x))u(\beta,\phi)\psi\rangle \\
&= \int_{\phi/2\pi R}^{L+\phi/2\pi R}\int_0^{2\pi R} (\overline{\psi(x,y)}\frac{i}{L}(\partial_y\psi)(x,y) - \chi(x+\frac{\phi}{2\pi R})|\psi(x,y)|^2)dxdy \\
&= \int_{\phi/2\pi R}^{L}\int_0^{2\pi R} (\overline{\psi(x,y)}\frac{i}{L}(\partial_y\psi)(x,y) - \chi(x+\frac{\phi}{2\pi R})|\psi(x,y)|^2)dxdy \\
&\quad + \int_0^{\phi/2\pi R}\int_0^{2\pi R} (\overline{\psi(x+L,y)}\frac{i}{L}(\partial_y\psi)(x+L,y) - \chi(x+\frac{\phi}{2\pi R}+L)|\psi(x+L,y)|^2)dxdy \\
&= \int_0^{L}\int_0^{2\pi R} (\overline{\psi(x,y)}\frac{i}{L}(\partial_y\psi)(x,y) - \chi(x+\frac{\phi}{2\pi R})|\psi(x,y)|^2)dxdy.
\end{aligned}$$

In passing to the last line, we used the relation

$$\overline{\psi(x+L,y)}\partial_y\psi(x+L,y) = \overline{\psi(x,y)}\partial_y\psi(x,y) - iL|\psi(x,y)|^2$$

deduced from (3.11). Thus we get

$$\langle u(\beta,\phi')\psi, (\frac{i}{L}\partial_y - \chi(x))u(\beta,\phi')\psi\rangle|_{\phi'=0}^{\phi'=\phi} = \int_0^L\int_0^{2\pi R}(\chi(x)-\chi(x+\frac{\phi}{2\pi R}))|\psi(x,y)|^2dxdy.$$

A similar computation gives

$$\mathrm{tr}P(\beta,\phi')\sum_{j=1}^N(\frac{i}{L}\partial_{y_j}-\chi(x_j))|_{\phi'=0}^{\phi'=\phi} = \int_0^L\int_0^{2\pi R}(\chi(x)-\chi(x+\frac{\phi}{2\pi R})\rho(x,y)dxdy.$$

If we differentiate this equation, change variables (note that $\chi'$ and $\rho$ are periodic) and use (3.18), we get

$$\mathrm{tr}P[\partial_\beta P,\partial_\phi P](\beta,\phi) = \frac{1}{2\pi R}\int_0^L\int_0^{2\pi R}\chi'(x)\rho(x-\frac{\phi}{2\pi R},y)dxdy$$

which gives, together with (3.17) and (3.14), the desired identity (3.13). □

Thus we see how the representation of the Hall conductance through an adiabatic curvature and the saying "Hall conductance = one-particle density" (in the absence of background potential) fit together.

Theorem 3.2 is just one among several arguments relating the Hall conductance to the one-particle density *in the absence of a background potential*. This way of proving it is interesting insofar as it allows a direct comparison with the Chern number approach and facilitates the comparison with the spectral boundary conditions setting of the next subsection.

Furthermore, this theorem generalizes a result by Tao and Haldane. In [TH86], they compute the Chern number associated to the family of projections onto flux-dependent Laughlin-type wave functions on a torus.

### 3.2.2 Spectral boundary conditions

Instead of (quasi)periodic boundary conditions, we can use the *spectral* or *chiral* boundary conditions introduced in [AANS98]. The treatment of charge transport in this subsection follows closely the PhD thesis by Richter [Ric00]. Before we turn to the charge transport, let us recall some facts about spectral boundary conditions.

**Spectral boundary conditions: one particle.** Let $[a,b] \subset \mathbb{R}$, $R > 0$ and $\phi \in \mathbb{R}$. Consider the formal operator in $L^2([a,b] \times [0, 2\pi R])$ given by

$$H(\phi) = \frac{1}{2}(-i\partial_x)^2 + \frac{1}{2}(-i\partial_y - (x - \frac{\phi}{2\pi R}))^2.$$

We will consider $H(\phi)$ with periodic boundary conditions in the $y$-direction, thus it is useful to work with Fourier series: let $U$ be the unitary

$$U : \bigoplus_{k\in\mathbb{Z}} L^2([a,b]) \to L^2([a,b] \times [0,2\pi R]),\ (f_k)_{k\in\mathbb{Z}} \mapsto \sum_{k\in\mathbb{Z}} e^{iky/R} f_k(x).$$

Then formally,

$$U^{-1}H(\phi)U = \bigoplus_{k\in\mathbb{Z}} h((k + \frac{\phi}{2\pi})\frac{1}{R}), \quad h(\rho) = \frac{1}{2}(-\partial_x^2 + (x-\rho)^2). \tag{3.19}$$

For $\rho \in \mathbb{R}$, let $B_\rho$ be the set of functions $f : [a,b] \to \mathbb{C}$ that are $C^\infty$ and satisfy the $\rho$-dependent boundary conditions

$$\begin{aligned} \rho < a: &\quad f'(a) = 0, && f'(b) = (\rho - b)f(b) \\ a \le \rho \le b: &\quad f'(a) = (\rho - a)f(a), && f'(b) = (\rho - b)f(b) \\ \rho > b: &\quad f'(a) = (\rho - a)f(a), && f'(b) = 0. \end{aligned}$$

Then $B_\rho \to L^2([0,L])$, $\quad f \mapsto \frac{1}{2}(-\partial_x^2 + (x-\rho)^2)f$ is essentially self-adjoint. We take $h(\rho)$ as its self-adjoint closure and define $H(\phi)$ through (3.19). The spectrum of $h(\rho)$ consists of simple eigenvalues $e_0(\rho) < e_1(\rho) < ...$ where the smallest eigenvalue $e_0(\rho)$ is always $\ge 1/2$ and $e_0(\rho) = 1/2$ if and only if $\rho \in [a,b]$. The resolvents $\rho \mapsto (h(\rho) + i)^{-1}$ are norm-continuous, but not norm-differentiable: the smallest eigenvalue has a non-zero left derivative in $a$. From this we get the following properties of $H(\phi)$:

1. $H(\phi)$ is norm-resolvent continuous in $\phi$.

2. For $n \in \mathbb{Z}$, $\phi \in \mathbb{R}$, let $\psi_n(x,y;\phi) = \exp(iny/R)\exp(-(x - \frac{n+\phi/2\pi}{R})^2/2)$. The Hamiltonian has the finite dimensional ground state

$$\mathcal{N}(H(\phi) - \frac{1}{2}) = \mathrm{span}\{\psi_n \mid n \in \mathbb{Z} : a \le \frac{n + \phi/2\pi}{R} \le b\}. \tag{3.20}$$

   Loosely speaking, one may say that the spectral boundary condition leaves the form of the lowest Landau level basis functions unchanged; the only effect of the boundary conditions is to pick out those Gaussians that are centered inside $[a,b]$.

3. $H(\phi)$ is gauge periodic: $H(\phi + 2\pi) = e^{iy/R}H(\phi)e^{-iy/R}$.

In view of the definition of charge transport, it is important to notice that $H(\phi)$ does not satisfy H1, since the domain of definition is $\phi$-dependent. Moreover, the ground state does not satisfy H2. Indeed, the gap above the ground state

$$g(\phi) := \min_{E\in\sigma(H(\phi)\setminus\{1/2\}} (E - \frac{1}{2}) \leq \min\{e_0(\frac{n+\phi/2\pi}{R}) - \frac{1}{2} \mid \frac{n+\phi/2\pi}{R} \notin [a,b]\}.$$

is strictly positive for all values of the flux $\phi$, but discontinuous for flux values where $n + \phi/2\pi \in \{Ra, Rb\}$ for some integer $n \in \mathbb{Z}$ and approaches zero in those points. These values also correspond to discontinuities of the projection onto the ground state (3.20) of $H(\phi)$.

The fact that $H(\phi)$ does not satisfy H1 and H2 leads to technical complications. For example, the existence of time evolution becomes more complicated.

**Proposition 3.3.** *Let $\tau > 0$ and $\phi(\cdot) \in C^\infty([0,1])$ such that $\dot\phi > 0$. Then there exists a unique map of unitaries $U_\tau(s), s \in [0,1]$ that is strongly differentiable, and such that $\mathcal{D}H(\phi(0)) \subset \mathcal{D}(H(\phi(s))$ for all $s \in [0,1]$ and*

$$i\partial_s U_\tau(s) = \tau H(\phi(s))U_\tau(s), \quad U_\tau(0) = \mathbf{1}.$$

*Proof.* We give only a sketch of proof, following [Ric00], Lemma 1.4.1 but correcting a small mistake done there. The main idea is to note that the Hamiltonian is equivalent to a (non self-adjoint) operator with *constant* boundary conditions. First remark that it is enough to prove the existence of the solution to the time-dependent Schrödinger equation

$$i\partial_s u(s) = \tau h(\rho(s))u(s) \tag{3.21}$$

in $L^2([a,b])$. Define the $\rho$-dependent function

$$f_\rho(s) = \begin{cases} \frac{1}{2}\frac{b-\rho}{b-a}(x-a)^2, & \rho < a, \\ \frac{1}{2}(x-\rho)^2, & a \leq \rho \leq b. \\ \frac{1}{2}\frac{a-\rho}{a-b}(x-b)^2, & \rho > b. \end{cases}$$

A smooth function $g$ satisfies the boundary conditions $(B_\rho)$ if and only if $\tilde g = e^{f_\rho} g$ satisfies the von Neumann boundary conditions $\tilde g'(a) = \tilde g'(b) = 0$. Moreover, if $i\partial_s g(s) = \tau h(\rho(s))g(s)$, then

$$i\partial_s \tilde g(s) = (\tau e^{f_{\rho(s)}} h(\rho(s)) e^{-f_{\rho(s)}} + i\frac{\partial f_\rho}{\partial \rho}\frac{d\rho}{ds})\tilde g(s) =: A(s)\tilde g(s)$$

The right-hand side is well-defined for $\rho(s) \notin \{a,b\}$. On those intervals the operator $A(s)$, although not self-adjoint, is better behaved than $h(\rho(s))$ and a classical result (cf. [RS75]) ensures the solvability of the equation. We can patch together a continous, piecewise differentiable solution and then apply $e^{-f_\rho}$ to obtain a continuous, piecewise differentiable solution $g(s)$ of the time-dependent Schrödinger equation (in particular, $g(s) \in \mathcal{D}(h(\rho(s))$ for all $s \in [0,1]$). Then (3.21) and the continuity of $h(\rho)$ imply that $g(\cdot)$ is actually strongly differentiable. □

**Multiparticle Hamiltonian.** The previous considerations lead us to a different set of assumptions on $H(\phi)$. Let $W$ be a background potential and $V$ an interaction potential between the particles.

**H'1:** For each $\phi \in \mathbb{R}$, the operator

$$H(\phi) = \sum_{j=1}^{N} H_j(\phi) + \sum_{j=1}^{N} W(z_j) + \sum_{1 \le j < k \le N} V(z_j - z_k)$$

is selfadjoint, and its domain is the same as for $W = V = 0$; $H_j(\phi)$ is the spectral boundary Hamiltonian for the $j$-th particle.

**H'2:** The time evolution exists and has an adiabatic limit:

(i) Let $\phi(\cdot) \in C^\infty([0,1])$ with $\dot{\phi} > 0$, and $\tau > 0$. There is a strongly differentable unitary $U_\tau(\cdot)$ such that $U_\tau(0) = \mathbf{1}$, $U_\tau(s)\mathcal{D}\big(H(\phi(0))\big) \subset \mathcal{D}\big(H(\phi(s))\big)$ for all $s \in [0,1]$, and $i\partial_s U_\tau(s) = \tau H(\phi(s))U_\tau(s)$.

(ii) There is a continuous, piecewise differentiable family of finite-dimensional projections $P(\phi)$ leaving the domain of $H(\phi)$ invariant and commuting with $H(\phi)$, such that

$$\lim_{\tau \to \infty} U_\tau(s)P(\phi(0))U_\tau(s)^* = P(\phi(s))$$

for all $s \in [0,1]$.

**Lemma 3.4.** *Assume H'1 and H'2. Let the charge transport be defined as in (3.7). Let* $q := \mathrm{tr}P(\phi)$ *and* $\rho(x,y;\phi)$ *be the one-particle density associated to* $q^{-1}P(\phi)$. *Then the charge transport in the adiabatic limit is*

$$\begin{aligned}\lim_{\tau \to \infty} Q_\tau &= -q^{-1}\mathrm{tr}P(\phi(s)) \sum_{j=1}^{N} \chi(x_j)|_{s=0}^{s=1} \\ &= -\int_a^b \int_0^{2\pi R} \chi(x)(\rho(x,y;\phi(1)) - \rho(x,y;\phi(0)))dxdy.\end{aligned}$$

*Sketch of proof.* We observe that $i[H(\phi), -\sum_{j=1}^{N} \chi(x_j)] = I_\chi$ in the sense of quadratic forms, i.e. for all functions $\psi$ in the domain of $H(\phi)$, we have

$$i\langle H(\phi)\psi, -\sum_{j=1}^{N} \chi(x_j)\psi\rangle - i\langle -\sum_{j=1}^{N} \chi(x_j)\psi, H(\phi)\psi\rangle = \langle \psi, I_\chi \psi\rangle.$$

This is shown by an integration by parts. The spectral boundary conditions on $\psi$ are needed to ensure that the boundary terms arising in the integration by parts vanish. (With quasiperiodic boundary conditions, the previous commutator relation does *not* hold in the sense of quadratic forms.) It follows that for $\psi \in \mathcal{D}(H(\phi))$,

$$\partial_s \langle U_\tau \psi, -\sum_{j=1}^{N} U_\tau \psi\rangle = \langle U_\tau \psi, I_\chi U_\tau \psi\rangle.$$

This implies that

$$Q_\tau = -\frac{1}{q}\mathrm{tr}U_\tau(s)P(0)U_\tau(s)^* \sum_{j=1}^{N} \chi(x_j)|_{s=0}^{s=1}.$$

Passing to the limit $\tau \to \infty$, we obtain the desired result. □

Lemma 3.4 motivates the following definition:

**Definition 3.5.** *Let $P(\phi), \phi \in \mathbb{R}$ be a norm-continuous family of projections in $L^2(([a,b] \times [0,2\pi R])^N)$ and $\chi \in C^\infty([a,b])$ a switch function of finite dimension $q > 0$. Then*

$$Q_A(\phi) := -q^{-1}\mathrm{tr}(P(\phi) - P(0)) \sum_{j=1}^{N} \chi(x_j)$$

*is called the adiabatic charge transport through* $\mathrm{supp}\chi'$ *when the flux is varied from* $0$ *to* $\phi$.

**Charge transport as adiabatic curvature.** The map $\psi \in L^2([a,b] \times [0,2\pi R]) \mapsto e^{i\beta\chi(x)}\psi$ leaves the spectral boundary conditions unchanged. Define $H(\beta,\phi)$ through (3.5) with spectral boundary conditions. Then $H(\beta,\phi)$ is unitarily related to $H(\phi)$ through

$$H(\beta,\phi) = e^{i\beta\sum_{j=1}^N \chi(x_j)} H(\phi) e^{-i\sum_{j=1}^N \beta\chi(x_j)}.$$

(This is to be compared with (3.16).) If $P(\phi)$ is a projection as in the previous lemma, let $P(\beta,\phi) := e^{i\beta\sum_{j=1}^N \chi(x_j)} P(\phi) e^{-i\beta\sum_{j=1}^N \chi(x_j)}$. Then the adiabatic charge transport $Q_A(\phi)$ can be expressed as an integral of the adiabatic curvature $P[\partial_\beta P, \partial_\phi P]$ and is independent of $\beta$:

$$Q_A(\phi) = -q^{-1} \int_0^\phi \mathrm{tr} P[\partial_\beta P, \partial_\phi P](\beta,\phi')d\phi'.$$

This is to be compared with Theorem 3.12. Thus the adiabatic charge transport on a finite cylinder (spectral boundary conditions) is an integral of an adiabatic curvature, just as it is for the torus. The projections $P(\beta,\phi)$ that arise typically are gauge constant in $\beta$, but *not* gauge periodic in $\phi$. Hence contrary to the torus case, there is no Chern number associated to the projection.

**Flux dependent Laughlin functions.** The advantage of Lemma 3.4 is that it allows to associate a "charge transport" to a family of projections without making reference to a Hamiltonian or a time evolution. If we wish to apply this to Laughlin's state, we need to define a flux-dependent wave function that coincides with Laughlin's wave function at flux $\phi = 0$. There are two natural candidates:

$$\Psi^{(1)}_{N,\phi}(z_1,..,z_N) := \prod_{1\le j<k\le N} (e^{z_j/R} - e^{z_k/R})^3 e^{-\sum_{j=1}^N (x_j - \frac{\phi}{2\pi R})^2/2} \tag{3.22}$$

$$\Psi^{(2)}_{N,\phi}(z_1,...,z_N) := \sum_{0\le m_1<..<m_N\le 3N-3} a_N(m_1,..,m_N)\psi_{m_1,\phi} \wedge .. \wedge \psi_{m_N,\phi} \tag{3.23}$$

In (3.23), $a_N(m_1,..,m_N)$ are the expansion coefficients of Chapter 2, and $\psi_{m,\phi}$ are normalized ground state functions of $H(\phi)$ (spectral boundary conditions) restricted to the $y$-momentum sector $i\partial_y\psi = \frac{m}{R}\psi$: let $f_\rho \in L^2([a,b])$ be a normalized ground state function of $h(\rho)$, then

$$\psi_{m,\phi}(z) = \frac{1}{\sqrt{2\pi R}} e^{imy/R} f_{(m+\phi/2\pi)/R}(x).$$

Note that

$$a \le \frac{m+\phi/2\pi}{R} \le b \;\Rightarrow\; \psi_{m,\phi}(z) \propto e^{imy/R} e^{-(x-(m+\phi/2\pi)/R)^2}.$$

Both choices have their advantages and inconvenients. The function (3.22) shares the multiplicity of zeroes of Laughlin's wave function: as two particles get close $z_j - z_k \to 0$, it vanishes

as $O((z_j - z_k)^3)$. It is the cylinder version of the function given in [TG91] for electrons on an annulus. However, it is made of wedge products of eigenfunctions of the one-particle Hamiltonian with spectral boundary conditions in $L^2([a,b] \times [0, 2\pi R])$ only as long as $a \leq \phi/2\pi R$, $(3N - 3 + \phi/2\pi)/R \leq b$.
The function (3.23) is built using adiabatic time evolution in the absence of interactions and background potential. It is always a sum of Slater determinants of eigenfunctions of $H_1(\phi)$. But it does not have zeroes of order 3 as two particles approach.
Of course, we expect that both functions have the same bulk properties as the cylinder gets infinite. Intuitively, in the bulk, for both functions, increasing the flux from 0 to $\phi$ just shifts the function by an amount of $\phi/2\pi R$ along the cylinder axis, and we expect that the charge transport is essentially given by a formula of the type (3.8). We check this for $\Psi_N^{(2)}$ and time evolutions that leave the wave function in the bulk.

**Proposition 3.6.** *Let $\phi \in \mathbb{R}$, and $P(\phi)$ be the projection in $L^2(([a,b] \times [0, 2\pi R])^N)$ on $\Psi_{N,\phi}^{(2)}$, $d > 0$, and $\chi$ a switch function with $\mathrm{supp}\chi' \subset ]a + d, b - d[$. Let $Q_A(\phi)$ be the charge transport associated with $P(\phi)$ through Definition 3.5. Suppose $\Psi_{N,0}^{(2)}$ and $\Psi_{N,\phi}^{(2)}$ are bulk states, i.e., $a \leq 0 \leq (3N-3)/R \leq b$ and $a - \phi/2\pi \leq 0 \leq (3N-3)/R \leq b - \phi/2\pi$. Then*

$$Q_A(\phi) = \int_{[a,b]\times[0,2\pi R]} (\chi(x + \phi/2\pi R) - \chi(x))\rho(x,y)dxdy + o(1) \tag{3.24}$$

*as* $\mathrm{dist}\,(\mathrm{supp}\chi', \{a + |\phi|/2\pi R, b - |\phi|/2\pi R\}) \to \infty$, *uniformly in the number of particles $N$. Here, $\rho(x,y)$ is the one-particle density of $\Psi_{N,\phi}^{(2)}$.*

*Proof.* Let $\rho(x;\phi) := \int_0^{2\pi R} \rho(x,y;\phi)dy$ be the $y$-integrated density of $\Psi_{N,\phi}^{(2)}$. For $\phi \in \mathbb{R}$, let $\phi' := \phi/2\pi$. We consider only $\phi > 0$ and suppose $d \geq \phi'$. Then

$$\rho_N(x;\phi) = \sum_{k=0}^{3N-3} n_k d_{k+\phi'} e^{-(x-(k+\phi')/R)^2}$$

where $0 \leq n_k \leq 1$ are $\phi$-independent, and $d_\rho = (\int_a^b e^{-(x-\rho/R)^2}dx)^{-1}$. Let $0 \leq m < M \leq 3N-3$ and set

$$\rho^L(x;\phi) = \sum_{k=0}^{m-1} n_k d_{k+\phi'} e^{-(x-(k+\phi')/R)^2} \qquad \rho^B(x;\phi) = \sum_{k=m}^{M} n_k d_{k+\phi'} e^{-(x-(k+\phi')/R)^2}$$
$$\rho^R(x;\phi) = \sum_{k=M+1}^{3N-3} n_k d_{k+\phi'} e^{-(x-(k+\phi')/R)^2}$$

(roughly, left edge, bulk and right edge contributions). We have to estimate the difference

$$\int_a^b \chi(x)(\rho(x;\phi) - \rho(x;0))dx - \int_a^b (\chi(x + \phi'/R) - \chi(x))\rho(x;0)dx = R + B + L$$

where $R, B, L$ are obtained from the left-hand side by replacing $\rho$ with $\rho^R, \rho^B, \rho^L$. The three contributions are estimated separately.

*Bulk.* The bulk remainder can estimated as

$$\begin{aligned}|B| &= |\int_{a+\phi'}^{b} \chi(x)(\rho^B(x;\phi) - \rho^B(x-\phi'/R;0))dx + \int_{b-\phi'}^{b} \rho^B(x;0)dx| \\ &\leq \sum_{k=m}^{M} n_k \int_{\mathbb{R}} |d_{k+\phi'} - d_k| e^{-(x-(k+\phi')/R)^2} dx + \sum_{k=m}^{M} n_k d_k \int_{b-\phi'}^{\infty} e^{-(x-k/R)^2} dx \\ &\leq \sqrt{\pi} \sum_{k=m}^{M} |d_{k+\phi'} - d_k| + \frac{\sqrt{\pi}}{2} (\max_{m\leq k\leq M} d_k)(\sum_{k=m}^{M} \operatorname{erfc}(b - (k+\phi')/R)).\end{aligned}$$

where $\operatorname{erfc}(u) := \frac{2}{\sqrt{\pi}} \int_u^\infty e^{-x^2} dx$ is the complementary error function. Let $c \geq \max_{\rho/R\in[a,b]} d_\rho$ ($c$ is essentially $2/\sqrt{\pi}$ when $b-a$ is large). Then one can show that

$$|d_{k+\phi'} - d_k| \leq c^2 \frac{\sqrt{\pi}}{2} (\operatorname{erfc}(\frac{k}{R} - a) + \operatorname{erfc}(b - \frac{k+\phi'}{R})).$$

By the good decay properties of erfc, we get a bound of the type

$$|B| \leq F(\frac{m}{R} - a) + F(b - \frac{M+\phi'}{R}),$$

where $F$ is independent of $N, m, M, \phi$ and goes to zero at infinity.

*Left edge.* Suppose $(m+\phi')/R \leq a+d$. Then

$$\begin{aligned}|L| &\leq \int_{a+d-\phi'/R}^{b} \rho(x;\phi)dx + \int_{a+d}^{b} \rho(x;0)dx \\ &\leq \sqrt{\pi} \sum_{k=0}^{m} \operatorname{erfc}(a+d - \frac{k+\phi'}{R}) \leq \sqrt{\pi} \sum_{k=0}^{m} \operatorname{erfc}(a+d - \frac{k+\phi'}{R}) \\ &\leq F(a+d - \frac{m+\phi'}{R})\end{aligned}$$

for some function $F$ going to zero at infinity.

*Right edge.* Using

$$\int_a^b \rho^R(x;\phi)dx = \sum_{k=M+1}^{3N-3} n_k = \int_a^b \rho^R(x;0)dx,$$

one rewrite $R$ as an integral involving $\chi_c := 1 - \chi$. As for the left edge, we obtain a bound

$$|R| \leq F(\frac{M+\phi'}{R} - (b-d))$$

if $(M+\phi')/R \geq b-d$.

Choosing $m/R$ well between $a$ and $a+d-\phi'/R$, $(M+\phi')/R$ well in between $b-d$ and $b$ and putting together the estimates of $B, L, R$ one sees that the sum $B+L+R$ is small when $d$ is large. □

Differentiating (3.24) with respect to $\phi$ and passing to the limit of infinite cylinders, we obtain a formula for the charge transport of the type of (3.8). The discussion of the beginning of

the section applies, and we obtain the "correct" charge transport if the one-particle density has the "correct" value.

To summarize, we went into some lengths to show that if Laughlin's wave function has the right one-particle density, it is associated to a fractionally quantized charge transport. The picture behind this is that in the absence of impurities, slow addition of a flux acts like translation. This can be proved rigorously on cylinders with periodic boundary conditions, i.e., tori. In contrast, cylinders with spectral boundary conditions model samples with boundaries. We expect that in the bulk, adiabatic addition of a flux quantum still acts like a translation. A proof of this would require a simultaneous on an adiabatic and a thermodynamic limit. Especially for systems of interacting particles, this involves some work.

Additional difficulties come from the flux-dependence of the boundary conditions and the fact that eigenvalues cross. The first problem can be handled, on finite cylinders, by relating the Hamiltonian to an operator with flux independent boundary conditions, at the price of obtaining an operator that is not self-adjoint. On an infinite plane pierced by a flux tube, the situation is more delicate, see [AHŠ05]. The second problem can be addressed by restricting to an invariant subspace (e.g., a given $y$-momentum) where the crossing does not occur, or by the use of an adiabatic theorem without a gap condition [AE99].

The violation of the gap condition reflects the fact that adiabatic time evolution does *not* follow the ground state. This is clearly behind Laughlin's use of the adiabatic flux increase to obtain the quasihole excitation. On the cylinder, the situation is as described in [TG91, GT93]: as the flux increases, a bulk ground state moves into the right edge. This is accompanied by an energy increase. Meanwhile, a many-body state partially living on the edge moves into the bulk, thereby decreasing its energy. In the process, energy levels may cross.

# Conclusion

We have shown that on sufficiently thin cylinders, the thermodynamic limit at filling factor $1/p$ of Laughlin's state exists. The limiting state is translationally invariant and mixing with respect to shifts by multiples of $pl^2/R$ along the cylinder axis. The state is not $l^2/R$-periodic. Since Laughlin's state is the ground state of a $l^2/R$-periodic Hamiltonian, this means that there is symmetry breaking.

The key ingredient to our proof is the use of a product rule of expansion coefficients in the expansion of Laughlin's wave function into lowest Landau level basis functions. The product rule allows the representation of Laughlin's cylinder function as a quantum polymer. The normalization constant is the partition function of a polymer system with translationally invariant activity. Using bounds on the normalization constant, we have shown that the activity can be rescaled to a stable activity. The polymer system is associated with a one-dimensional renewal process. Our proof of symmetry breaking relies on a condition on the interarrival distribution of the renewal process. Using information on the radius dependence of the activities, we have shown that for thin cylinders, the associated renewal process has finite mean. What happens for broad cylinders remains open.

A torus can be considered as a cylinder with periodic boundary conditions. In this sense, cylinder and torus wave functions differ merely by boundary conditions. It is commonly accepted that Laughlin's wave function, for small values of $p$, describes an incompressible liquid. Thus intuitively, boundary conditions should not affect the bulk behavior and we expect that the torus and cylinder functions are equivalent in the limit of long cylinders. We have defined a class of solvable models, using functions of compact support. Under certain conditions on the supports of the functions (next nearest neghbors overlap), the torus and cylinder functions are related to monomer-dimer systems and we have shown their equivalence in the limit of long cylinders. We believe that the solvable model gives a hint on the mechanism behind the equivalence of the cylinder and Laughlin-type wave functions and that the method of proof may be extended to the general case.

In view of the plasma analogy, our results can be interpreted in terms of jellium tubes. Semiperiodic Coulomb interactions are invariant with respect to any shift. But the limiting state of jellium tubes cannot be completely translationally invariant, since any period must be a multiple of $l^2/R$. Thus there is symmetry breaking on jellium tubes, for *all* radii. If we view jellium tubes as quasi one-dimensional systems and invoke known results on the one-dimensional one-component plasma, we expect a periodicity $L/N = pl^2/R$. From the plasma point of view, the smaller period $l^2/R$ is surprising, and it would be interesting to have a physical interpretation.

In the limit of thin strips, we recover the expressions of the excess free energies and one-particle densities for one-dimensional jellium systems. The free energy at large radii should approach the fully two-dimensional free energy. This leads to a conjecture consistent with

bounds on the normalization constants that we have derived.

Thus the tube interpolates nicely between one- and fully two-dimensional systems. By what is known on those systems, it follows that it interpolates between a system with no phase transition and a system with a phase transition. This observation might be relevant for large strips; in particular, the magnitude of $p$ should play a role.

It is widely accepted that the ground state of FQHE system is a gapped, incompressible liquid at FQHE fractions. It is believed that the truncated interactions reproduce this feature, at least for small $p$. We have shown that *if* the truncated interaction have a gapped ground state, then at fractional filling there must be some kind of symmetry breaking. This leaves open the question whether they do indeed have a gapped ground state, and of course says nothing about "real" FQHE systems.

Finally, our results are consistent with heuristic arguments adapting Laughlin's IQHE argument on the cylinder to the fractional effect. In systems without background interaction, the Hall conductance is directly related to the one-particle density; we have given one derivation of this common wisdom using the Chern number approach. On cylinders with spectral boundary conditions, we have obtained a similar result. The periodicity and the correct average value of the density imply fractional quantization of the Hall conductance. However, we have used a number of assumptions that are rather unsatisfying, although they are commonly used. It would be worthwhile to obtain better results on the significance of cylinder ground state degeneracy for the FQHE. We hope that our detailed presentation will be useful for future investigations in this direction.

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