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Existence, Uniqueness and Asymptotic Behavior of Parametric Anisotropic (p, q)-Equations with Convection

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Abstract

In this paper we study anisotropic weighted (p, q)-equations with a parametric right-hand side depending on the gradient of the solution. Under very general assumptions on the data and by using a topological approach, we prove existence and uniqueness results and study the asymptotic behavior of the solutions when both the $q(\cdot)$-Laplacian on the left-hand side and the reaction term are modulated by a parameter. Moreover, we present some properties of the solution sets with respect to the parameters.

Keywords Anisotropic eigenvalue problem · Anisotropic (p, q)-Laplace differential operator · Asymptotic behavior · Convection term · Gradient dependence · Pseudomonotone operators · Uniqueness

Mathematics Subject Classification 35A02 · 35B40 · 35J15 · 35J25 · 35J62

1 Introduction

Let $\Omega \subseteq \mathbb{R}^N$ $(N \geq 2)$ be a bounded domain with smooth boundary $\partial\Omega$. We consider the following nonlinear Dirichlet problem with parameter dependence in the leading term and with gradient and parameter dependence in the reaction term

$$\begin{aligned} -\Delta_{p(\cdot)}u - \mu\Delta_{q(\cdot)}u &= \lambda f(x, u, \nabla u) \quad && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

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where $\mu \geq 0$ and $\lambda > 0$ are the parameters to be specified, the exponents $p, q \in C(\overline{\Omega})$ are such that $1 < q(x) < p(x)$ for all $x \in \overline{\Omega}$ and $\Delta_{r(\cdot)}$ denotes the $r(\cdot)$-Laplace differential operator defined by

$$\Delta_{r(\cdot)} u = \operatorname{div} \left(|\nabla u|^{r(\cdot)-2} \nabla u \right) \quad \text{for all } u \in W_0^{1,r(\cdot)}(\Omega).$$

In the right-hand side of problem (1.1) we have a parametric reaction term in form of a Carathéodory function $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ which satisfies very general structure conditions, see hypotheses (H2) and (H3) in Sects. 2 and 3. Since the reaction term $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ also depends on the gradient ∇u of the solution u (that phenomenon is called convection), problem (1.1) does not have a variational structure and so we cannot apply tools from critical point theory. Instead we will use a topological approach based on the surjectivity of pseudomonotone operators.

We will not only present existence results under very general structure conditions but also sufficient conditions for the uniqueness of the solution of (1.1). Further, we study the asymptotic behavior of the solutions of (1.1) and prove some properties of the solution sets depending on the two parameters $\mu \geq 0$ and $\lambda > 0$ which are controlling the $q(\cdot)$-Laplacian on the left-hand side and the reaction on the right-hand side. This leads to interesting results on certain ranges of μ and λ.

The novelty in our paper is the fact that we have an anisotropic nonhomogeneous differential operator and a parametric convection term on the right-hand side. If $\mu = 0$ in (1.1), the operator becomes the anisotropic p-Laplacian and such equations have been studied for $\lambda = 1$ in the recent paper of Wang–Hou–Ge [25]. For constant exponents there exist several works but without parameter on the right-hand side. Precisely, constant exponent p-Laplace problems with convection can be found in the papers of de Figueiredo–Girardi–Matzeu [4] for the Laplacian, Fragnelli–Papageorgiou–Mugnai [11] and Ruiz [24] both for the p-Laplacian. For (p, q)-equation with constant exponents, convection term and $\lambda = 1$, we refer to the works of Averna–Motreanu–Tornatore [1] for weighted (p, q)-equations, El Manouni–Marino–Winkert [6] for double phase problems depending on Robin and Steklov eigenvalues for the p-Laplacian, Faria–Miyagaki–Motreanu [10] using a comparison principle and an approximation process, Gasiński–Winkert [13] for double phase problems, Liu–Papageorgiou [17] for resonant reaction terms using the frozen variable method together with the Leray–Schauder alternative principle, Marano–Winkert [18] with nonlinear boundary condition, Motreanu–Winkert [19] via sub-supersolution approach, Papageorgiou–Vetro–Vetro [20] for right-hand sides with a parametric singular term and a locally defined perturbation and [21] for semilinear Neumann problems, see also the references therein.

To the best of our knowledge, this is the first work dealing with an anisotropic differential operator and a parametric convection term. Such equations provide mathematical models of anisotropic materials. The parameter $\mu \geq 0$ modulates the effect of the $q(\cdot)$-Laplace operator, and hence controls the geometry of the composite made of two different materials. In general, equations driven by the sum of two differential operators of different nature arise often in mathematical models of physical processes. We refer to the works of Bahrouni–Rădulescu–Repovš [2] for transonic flow problems, Cherfils–Il'yasov [3] for reaction diffusion systems and Zhikov [26] for elasticity problems.

Finally, we mention that there are several relevant differences when dealing with anisotropic equations in contrast to constant exponent problems. We refer to the books of Diening–Harjulehto–Hästö–Růžicka [5], Harjulehto–Hästö [14] and Rădulescu–Repovš [23] for more information on the differences.

The paper is organized as follows. In Sect. 2 we collect some properties on variable exponent Sobolev spaces as well as on the $p(\cdot)$-Laplacian and we present the hypotheses on the data of problem (1.1). Section 3 is devoted to the existence and uniqueness results as well as the asymptotic behavior when the parameter μ moves to 0 and $+\infty$, respectively. We also show the boundedness of the set of solutions to problem (1.1). In Sect. 4 we complete the characterization of the set of solutions with respect to compactness and closedness.

2 Preliminaries and Hypotheses

In this section we give a brief overview about variable exponent Lebesgue and Sobolev spaces, see the books of Diening–Harjulehto–Hästö–Růžička [5], Harjulehto–Hästö [14] and the papers of Fan–Zhao [7], Kováčik–Rákosník [16]. Moreover, we recall some facts about pseudomonotone operators and we state the hypotheses on the data of problem (1.1).

To this end, let Ω be a bounded domain in $\mathbb{R}^N$ $(N \geq 2)$ with smooth boundary $\partial\Omega$. For $r \in C_+(\overline{\Omega})$, where $C_+(\overline{\Omega})$ is given by

$$C_+(\overline{\Omega}) = \left\{h \in C(\overline{\Omega}) \, : \, 1 < h(x) \text{ for all } x \in \overline{\Omega}\right\},$$

we denote

$$r^- := \inf_{x\in\overline{\Omega}} r(x) \quad \text{and} \quad r^+ := \sup_{x\in\overline{\Omega}} r(x).$$

Moreover, denoting by $M(\Omega)$ the space of all measurable functions $u\colon \Omega \to \mathbb{R}$, the variable exponent Lebesgue space $L^{r(\cdot)}(\Omega)$ for a given $r \in C_+(\overline{\Omega})$ is defined as

$$L^{r(\cdot)}(\Omega) = \left\{u \in M(\Omega) \, : \, \int_\Omega |u|^{r(x)} \, \mathrm{d}x < \infty\right\}$$

equipped with the Luxemburg norm given by

$$\|u\|_{r(\cdot)} = \inf\left\{\lambda > 0 \, : \, \int_\Omega \left|\frac{u}{\lambda}\right|^{r(x)} \mathrm{d}x \leq 1\right\}.$$

Here the corresponding modular $\rho_r\colon L^{r(\cdot)}(\Omega) \to \mathbb{R}$ is given by

$$\rho_r(u) = \int_\Omega |u|^{r(x)} \, \mathrm{d}x \quad \text{for all } u \in L^{r(\cdot)}(\Omega).$$

We know that $(L^{r(\cdot)}(\Omega), \|\cdot\|_{r(\cdot)})$ is a separable, reflexive and uniformly convex Banach space.

The following proposition gives the relation between the norm $\|\cdot\|_{r(\cdot)}$ and the modular $\rho_r(\cdot)$.

Proposition 2.1 *For all $u \in L^{r(\cdot)}(\Omega)$ we have the following assertions:*

(i) $\|u\|_{r(\cdot)} < 1$ *(resp.* $= 1, > 1$*) if and only if* $\rho_r(u) < 1$ *(resp.* $= 1, > 1$*);*
(ii) *if* $\|u\|_{r(\cdot)} > 1$, *then* $\|u\|_{r(\cdot)}^{r^-} \le \rho_r(u) \le \|u\|_{r(\cdot)}^{r^+}$*;*
(iii) *if* $\|u\|_{r(\cdot)} < 1$, *then* $\|u\|_{r(\cdot)}^{r^+} \le \rho_r(u) \le \|u\|_{r(\cdot)}^{r^-}$.

Remark 2.2 A direct consequence of Proposition 2.1 is the following relation

$$\|u\|_{r(\cdot)}^{r^-} - 1 \le \rho_r(u) \le \|u\|_{r(\cdot)}^{r^+} + 1 \quad \text{for all } u \in L^{r(\cdot)}(\Omega). \tag{2.1}$$

Let $r' \in C_+(\overline{\Omega})$ be the conjugate variable exponent to r, that is,

$$\frac{1}{r(x)} + \frac{1}{r'(x)} = 1 \quad \text{for all } x \in \overline{\Omega}.$$

We know that $L^{r(\cdot)}(\Omega)^* = L^{r'(\cdot)}(\Omega)$ and Hölder's inequality holds, that is,

$$\int_\Omega |uv| \mathrm{d}x \le \left[\frac{1}{r^-} + \frac{1}{r'^-}\right] \|u\|_{r(\cdot)} \|v\|_{r'(\cdot)} \le 2\|u\|_{r(\cdot)} \|v\|_{r'(\cdot)}$$

for all $u \in L^{r(\cdot)}(\Omega)$ and for all $v \in L^{r'(\cdot)}(\Omega)$.

If $r_1, r_2 \in C_+(\overline{\Omega})$ and $r_1(x) \le r_2(x)$ for all $x \in \overline{\Omega}$, then we have the continuous embedding

$$L^{r_2(\cdot)}(\Omega) \hookrightarrow L^{r_1(\cdot)}(\Omega).$$

For $r \in C_+(\overline{\Omega})$ we define the variable exponent Sobolev space $W^{1,r(\cdot)}(\Omega)$ by

$$W^{1,r(\cdot)}(\Omega) = \left\{u \in L^{r(\cdot)}(\Omega) \, : \, |\nabla u| \in L^{r(\cdot)}(\Omega)\right\}$$

endowed with the norm

$$\|u\|_{1,r(\cdot)} = \|u\|_{r(\cdot)} + \|\nabla u\|_{r(\cdot)},$$

where $\|\nabla u\|_{r(\cdot)} = \| \, |\nabla u| \, \|_{r(\cdot)}$. Furthermore, we define

$$W_0^{1,r(\cdot)}(\Omega) = \overline{C_0^\infty(\Omega)}^{\|\cdot\|_{1,r(\cdot)}}.$$

The spaces $W^{1,r(\cdot)}(\Omega)$ and $W_0^{1,r(\cdot)}(\Omega)$ are both separable and reflexive Banach spaces, in fact uniformly convex Banach spaces. In the space $W_0^{1,r(\cdot)}(\Omega)$, we have Poincaré's inequality, that is,

$$\|u\|_{r(\cdot)} \leq c\|\nabla u\|_{r(\cdot)} \quad \text{for all } u \in W_0^{1,r(\cdot)}(\Omega)$$

with some $c > 0$. As a consequence, we consider on $W_0^{1,r(\cdot)}(\Omega)$ the equivalent norm

$$\|u\| = \|\nabla u\|_{r(\cdot)} \quad \text{for all } u \in W_0^{1,r(\cdot)}(\Omega).$$

For $r \in C_+(\overline{\Omega})$ we introduce the critical variable Sobolev exponent r^* defined by

$$r^*(x) = \begin{cases} \frac{Nr(x)}{N-r(x)} & \text{if } r(x) < N, \\ \infty & \text{if } N \leq r(x), \end{cases} \quad \text{for all } x \in \overline{\Omega}. \tag{2.2}$$

The following proposition states the Sobolev embedding theorem for variable exponent Sobolev spaces.

Proposition 2.3 *If $r \in C_+(\overline{\Omega})$, $s \in C(\overline{\Omega})$ and $1 \leq s(x) < r^*(x)$ for all $x \in \overline{\Omega}$, then there exists a compact embedding $W^{1,r(\cdot)}(\Omega) \hookrightarrow L^{s(\cdot)}(\Omega)$.*

Let us now recall some definitions which are used in the sequel.

Definition 2.4 Let X be a reflexive Banach space, X^* its dual space and denote by $\langle \cdot\,, \cdot \rangle$ its duality pairing. Let $A\colon X \to X^*$, then A is called

(i) to satisfy the (S_+)-property if $u_n \rightharpoonup u$ in X and $\limsup_{n\to\infty} \langle A(u_n), u_n - u\rangle \leq 0$ imply $u_n \to u$ in X;
(ii) pseudomonotone if $u_n \rightharpoonup u$ in X and $\limsup_{n\to+\infty} \langle A(u_n), u_n - u\rangle \leq 0$ imply

$$\liminf_{n\to+\infty} \langle A(u_n), u_n - v\rangle \geq \langle A(u), u - v\rangle \quad \text{for all } v \in X;$$

(iii) coercive if

$$\lim_{\|u\|_X \to +\infty} \frac{\langle A(u), u\rangle}{\|u\|_X} = +\infty.$$

Remark 2.5 We point out that if the operator $A\colon X \to X^*$ is bounded, then the definition of pseudomonotonicity in Definition 2.4 (ii) is equivalent to $u_n \rightharpoonup u$ in X and $\limsup_{n\to+\infty} \langle A(u_n), u_n - u\rangle \leq 0$ imply $A(u_n) \rightharpoonup A(u)$ and $\langle A(u_n), u_n\rangle \to \langle A(u), u\rangle$. In the following we are going to use this definition since our operators involved are bounded.

Pseudomonotone operators exhibit remarkable surjectivity properties. In particular, we have the following result, see, for example, Papageorgiou–Winkert [22, Theorem 6.1.57].

Theorem 2.6 *Let X be a real, reflexive Banach space, let $A\colon X \to X^*$ be a pseudomonotone, bounded, and coercive operator, and $b \in X^*$. Then, a solution of the equation $Au = b$ exists.*

Next, we introduce the nonlinear operator $A_{r(\cdot)}: W_0^{1,r(\cdot)}(\Omega) \to W^{-1,r'(\cdot)}(\Omega) = W_0^{1,r(\cdot)}(\Omega)^*$ defined by

$$\langle A_{r(\cdot)}(u), h\rangle = \int_\Omega |\nabla u|^{r(x)-2}\nabla u \cdot \nabla h \,\mathrm{d}x \quad \text{for all } u, h \in W_0^{1,r(\cdot)}(\Omega).$$

This operator has the following properties, see Fan–Zhang [9, Theorem 3.1].

Proposition 2.7 *The operator $A_{r(\cdot)}(\cdot)$ is bounded (that is, it maps bounded sets to bounded sets), continuous, monotone (thus maximal monotone) and of type* (S_+).

Now we can formulate the hypotheses on the data of problem (1.1).

(H1) $p, q \in C_+(\overline{\Omega})$ with $q(x) < p(x)$ for all $x \in \overline{\Omega}$ and there exists $\xi_0 \in \mathbb{R}^N \setminus \{0\}$ such that for all $x \in \Omega$ the function $p_x: \Omega_x \to \mathbb{R}$ defined by $p_x(z) = p(x+z\xi_0)$ is monotone, where $\Omega_x := \{z \in \mathbb{R} : x + z\xi_0 \in \Omega\}$.

Remark 2.8 Hypothesis (H1) implies that

$$\widehat{\lambda} := \inf_{u \in W_0^{1,p(\cdot)}(\Omega)\setminus\{0\}} \frac{\displaystyle\int_\Omega |\nabla u|^{p(x)}\,\mathrm{d}x}{\displaystyle\int_\Omega |u|^{p(x)}\,\mathrm{d}x} > 0. \tag{2.3}$$

This follows from the paper of Fan–Zhang–Zhao [8, Theorem 3.3].

(H2) $f: \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a Carathéodory function such that

(i) there exist $\sigma \in L^{\alpha'(\cdot)}(\Omega)$ with $1 < \alpha(x) < p^*(x)$ for all $x \in \overline{\Omega}$ and $c > 0$ such that

$$|f(x,s,\xi)| \leq c\left(\sigma(x) + |s|^{\alpha(x)-1} + |\xi|^{\frac{p(x)}{\alpha'(x)}}\right)$$

for a. a. $x \in \Omega$, for all $s \in \mathbb{R}$ and for all $\xi \in \mathbb{R}^N$, where p^* is the critical exponent to p given in (2.2) for $r = p$;
(ii) there exist $a_0 \in L^1(\Omega)$ and $b_1, b_2 > 0$ such that

$$f(x,s,\xi)s \leq a_0(x) + b_1|s|^{p(x)} + b_2|\xi|^{p(x)}$$

for a. a. $x \in \Omega$, for all $s \in \mathbb{R}$ and for all $\xi \in \mathbb{R}^N$.

Example 2.9 Let $d_1, d_2 > 0$ and consider the function defined by

$$f(x,s,\xi) = \sigma(x) - d_1|s|^{p(x)-2}s + d_2|\xi|^{p(x)-1}$$

for a. a. $x \in \Omega$, for all $s \in \mathbb{R}$ and for all $\xi \in \mathbb{R}^N$ with $0 \neq \sigma \in L^{p'(\cdot)}(\Omega)$. It is easy to see that f fulfills hypotheses (H2).

Recall that $u \in W_0^{1,p(\cdot)}(\Omega)$ is a weak solution to (1.1) if

$$\left\langle A_{p(\cdot)}(u), h\right\rangle + \mu \left\langle A_{q(\cdot)}(u), h\right\rangle = \lambda \int_\Omega f(x, u, \nabla u) h \, \mathrm{d}x \tag{2.4}$$

is satisfied for all $h \in W_0^{1,p(\cdot)}(\Omega)$.

We also recall the following result, see Gasiński–Papageorgiou [12, Lemma 2.2.27, p. 141].

Lemma 2.10 *If X,Y are two Banach spaces such that $X \subseteq Y$, the embedding is continuous and X is dense in Y, then the embedding $Y^* \subseteq X^*$ is continuous. Moreover, if X is reflexive, then Y^* is dense in X^*.*

3 Existence and Uniqueness Results and Asymptotic Behavior

Now we state and prove the following existence result for problem (1.1). In the sequel we use the abbreviation

$$\lambda^* := \left(b_1 \widehat{\lambda}^{-1} + b_2\right)^{-1} > 0.$$

Theorem 3.1 *Let hypotheses (H1) and (H2) be satisfied. Then problem (1.1) admits at least one weak solution $u \in C^{0,\beta}(\overline{\Omega})$ for some $\beta \in]0, 1]$ for all $\mu \geq 0$ and for all $\lambda \in]0, \lambda^*[$.*

Proof Let $N_f^*\colon W_0^{1,p(\cdot)}(\Omega) \subset L^{\alpha(\cdot)}(\Omega) \to L^{\alpha'(\cdot)}(\Omega)$ be the Nemytskij operator corresponding to the Carathéodory function f, that is,

$$N_f^*(u)(\cdot) = f(\cdot, u(\cdot), \nabla u(\cdot)) \quad \text{for all } u \in W_0^{1,p(\cdot)}(\Omega).$$

Hypothesis (H2)(i) implies that $N_f^*(\cdot)$ is well-defined, bounded and continuous, see Fan–Zhao [7] and Kováčik–Rákosník [16]. By Lemma 2.10, the embedding $i^*\colon L^{\alpha'(\cdot)}(\Omega) \to W^{-1,p'(\cdot)}(\Omega)$ is continuous and hence the operator $N_f\colon W_0^{1,p(\cdot)}(\Omega) \to W^{-1,p'(\cdot)}(\Omega)$ defined by $N_f = i^* \circ N_f^*$ is bounded and continuous. We fix $\mu \geq 0$ as well as $\lambda \in]0, \lambda^*[$ and consider the operator $V\colon W_0^{1,p(\cdot)}(\Omega) \to W^{-1,p'(\cdot)}(\Omega)$ defined by

$$V(u) = A_{p(\cdot)}(u) + \mu A_{q(\cdot)}(u) - \lambda N_f(u) \quad \text{for all } u \in W_0^{1,p(\cdot)}(\Omega).$$

Evidently $V(\cdot)$ is bounded and continuous. Next we show that $V(\cdot)$ is pseudomonotone in the sense of Remark 2.5. To this end, let $\{u_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,p(\cdot)}(\Omega)$ be a sequence such that

$$u_n \rightharpoonup u \quad \text{in } W_0^{1,p(\cdot)}(\Omega) \quad \text{and} \quad \limsup_{n\to+\infty} \langle V(u_n), u_n - u\rangle \leq 0. \tag{3.1}$$

Since $\{u_n\}_{n\in\mathbb{N}}$ converges weakly in $W_0^{1,p(\cdot)}(\Omega)$, it is bounded in its norm and so $\{N_f^*(u_n)\}_{n\in\mathbb{N}}$ is bounded. Using this fact along with Hölder's inequality and the compact embedding $W_0^{1,p(\cdot)}(\Omega) \hookrightarrow L^{\alpha(\cdot)}(\Omega)$ (see Proposition 2.3), we get

$$\begin{aligned} &\left|\int_\Omega f(x,u_n,\nabla u_n)(u_n-u)\,\mathrm{d}x\right| \\ &\leq 2\left\|N_f^*(u_n)\right\|_{\frac{\alpha(\cdot)-1}{\alpha(\cdot)}}\|u-u_n\|_{\alpha(\cdot)} \\ &\leq 2\sup_{n\in\mathbb{N}}\left\|N_f^*(u_n)\right\|_{\frac{\alpha(\cdot)-1}{\alpha(\cdot)}}\|u-u_n\|_{\alpha(\cdot)} \to 0 \quad \text{as } n\to\infty. \end{aligned} \tag{3.2}$$

Therefore, if we pass to the limit in the weak formulation in (2.4) replacing u by u_n and h by $u_n - u$ and using (3.2), it follows that

$$\limsup_{n\to+\infty}\left[\left\langle A_{p(\cdot)}(u_n), u_n-u\right\rangle + \mu\left\langle A_{q(\cdot)}(u_n), u_n-u\right\rangle\right] \leq 0.$$

Since $A_{q(\cdot)}(\cdot)$ is monotone, this implies

$$\limsup_{n\to+\infty}\left[\left\langle A_{p(\cdot)}(u_n), u_n-u\right\rangle + \mu\left\langle A_{q(\cdot)}(u), u_n-u\right\rangle\right] \leq 0.$$

Therefore, by the weak convergence of $\{u_n\}_{n\in\mathbb{N}}$,

$$\limsup_{n\to+\infty}\left\langle A_{p(\cdot)}(u_n), u_n-u\right\rangle \leq 0.$$

Taking the (S_+)-property of $A_{p(\cdot)}(\cdot)$ into account (see Proposition 2.7) along with (3.1) gives $u_n \to u$ in $W_0^{1,p(\cdot)}(\Omega)$. From the strong convergence and the continuity of V, we conclude that $V(u_n) \to V(u)$ in $W_0^{1,p(\cdot)}(\Omega)^*$. Therefore, V is pseudomonotone.

Let us now prove that $V(\cdot)$ is coercive. From (2.3) we have

$$\int_\Omega |u|^{p(x)}\,\mathrm{d}x \leq \widehat{\lambda}^{-1}\int_\Omega |\nabla u|^{p(x)}\,\mathrm{d}x \quad \text{for all } u \in W_0^{1,p(\cdot)}(\Omega). \tag{3.3}$$

Applying (H2)(ii) and (3.3) along with Proposition 2.1(ii), we obtain for $u \in W_0^{1,p(\cdot)}(\Omega)$ with $\|u\| > 1$

$$
\begin{aligned}
&\langle V(u), u\rangle \\
&= \int_\Omega |\nabla u|^{p(x)}\,\mathrm{d}x + \mu \int_\Omega |\nabla u|^{q(x)}\,\mathrm{d}x - \lambda \int_\Omega f(x,u,\nabla u)u\,\mathrm{d}x \\
&\geq \int_\Omega |\nabla u|^{p(x)}\,\mathrm{d}x - \lambda \int_\Omega |a_0(x)|\,\mathrm{d}x - b_1\lambda \int_\Omega |u|^{p(x)}\,\mathrm{d}x - b_2\lambda \int_\Omega |\nabla u|^{p(x)}\,\mathrm{d}x \\
&\geq (1-\lambda b_2)\int_\Omega |\nabla u|^{p(x)}\,\mathrm{d}x - \lambda\|a_0\|_1 - b_1\lambda\widehat{\lambda}^{-1}\int_\Omega |\nabla u|^{p(x)}\,\mathrm{d}x \\
&\geq \left(1-\lambda(\lambda^*)^{-1}\right)\|\nabla u\|_{p(\cdot)}^{p^-} - \lambda\|a_0\|_1.
\end{aligned}
$$

Since $\lambda \in]0, \lambda^*[$, we see that $V(\cdot)$ is coercive. Hence, the operator $V\colon W_0^{1,p(\cdot)}(\Omega) \to W^{-1,p'(\cdot}(\Omega)$ is bounded, pseudomonotone and coercive. Then, Theorem 2.6 implies the existence of a function $u \in W_0^{1,p(\cdot)}(\Omega)$ which turns out to be a weak solution of problem (1.1). From Ho–Kim–Winkert–Zhang [15, Theorem 5.1] we know that $u \in C^{0,\beta}(\overline{\Omega})$ for some $\beta \in]0, 1]$. □

Let us now consider equation (1.1) under stronger assumptions in order to prove a uniqueness result. We suppose the additional assumptions.

(H3) (i) There exists a constant $a_1 > 0$ such that

$$
(f(x,s,\xi) - f(x,t,\xi))(s-t) \leq a_1|s-t|^2
$$

for a. a. $x \in \Omega$, for all $s, t \in \mathbb{R}$ and for all $\xi \in \mathbb{R}^N$.

(ii) There exist a function $\psi \in L^{r'(\cdot)}(\Omega)$ with $r \in C_+(\overline{\Omega})$ such that $r(x) < p^*(x)$ for all $x \in \overline{\Omega}$ and a constant $a_2 > 0$ such that the function $\xi \mapsto f(x,s,\xi) - \psi(x)$ is linear for a. a. $x \in \Omega$, for all $s \in \mathbb{R}$ and

$$
|f(x,s,\xi) - \psi(x)| \leq a_2|\xi|
$$

for a. a. $x \in \Omega$, for all $s \in \mathbb{R}$ and for all $\xi \in \mathbb{R}^N$.

Example 3.2 The following function satisfies hypotheses (H1)–(H3), where we drop the s-dependence:

$$
f(x,\xi) = \sum_{i=1}^N \beta_i\xi_i + \psi(x) \quad \text{for a. a. } x \in \Omega \text{ and for all } \xi \in \mathbb{R}^N,
$$

with $p^- = 2$, $0 \neq \psi \in L^2(\Omega)$ and $\beta = (\beta_1, \ldots, \beta_N) \in \mathbb{R}^N$.

Let

$$
\overline{\lambda} = \left(a_1\lambda_1^{-1} + a_2\lambda_1^{-\frac{1}{2}}\right)^{-1} > 0,
$$

with $\lambda_1 > 0$ being the first eigenvalue of the Laplacian with Dirichlet boundary condition given by

$$\lambda_1 := \inf_{u \in W_0^{1,2}(\Omega)\setminus\{0\}} \frac{\|\nabla u\|_2^2}{\|u\|_2^2}. \tag{3.4}$$

Our uniqueness result reads as follows.

Theorem 3.3 *Let hypotheses (H1)–(H3) be satisfied and let $q(x) \equiv 2$ for all $x \in \overline{\Omega}$. Then problem (1.1) admits a unique weak solution $u \in C^{0,\beta}(\overline{\Omega})$ for some $\beta \in]0, 1]$ for all $\mu > 0$ and for all $\lambda \in]0, \min\{\lambda^*, \mu\overline{\lambda}\}[$.*

Proof The existence of a weak solution follows from Theorem 3.1. Let us assume there are two weak solutions $u, v \in W_0^{1,p(\cdot)}(\Omega)$ of (1.1). We test the corresponding weak formulations given in (2.4) with $h = u - v$ and subtract these equations. This leads to

$$\begin{aligned}
&\int_\Omega \left(|\nabla u|^{p(x)-2}\nabla u - |\nabla v|^{p(x)-2}\nabla u\right) \cdot \nabla(u-v)\,\mathrm{d}x + \mu \int_\Omega |\nabla(u-v)|^2 \mathrm{d}x \\
&\quad = \lambda \int_\Omega (f(x,u,\nabla u) - f(x,v,\nabla u))(u-v)\,\mathrm{d}x \\
&\qquad + \lambda \int_\Omega (f(x,v,\nabla u) - f(x,v,\nabla v))(u-v)\,\mathrm{d}x.
\end{aligned} \tag{3.5}$$

First, it is easy to see that the left-hand side of (3.5) can be estimated via

$$\begin{aligned}
&\int_\Omega \left(|\nabla u|^{p(x)-2}\nabla u - |\nabla v|^{p(x)-2}\nabla u\right) \cdot \nabla(u-v)\,\mathrm{d}x + \mu \int_\Omega |\nabla(u-v)|^2 \mathrm{d}x \\
&\quad \geq \mu \int_\Omega |\nabla(u-v)|^2\,\mathrm{d}x.
\end{aligned} \tag{3.6}$$

Now we apply the conditions in (H3) along with Hölder's inequality and (3.4) to the right-hand side of (3.5) in order to obtain

$$\begin{aligned}
&\lambda \int_\Omega (f(x,u,\nabla u) - f(x,v,\nabla u))(u-v)\,\mathrm{d}x \\
&\qquad + \lambda \int_\Omega (f(x,v,\nabla u) - f(x,v,\nabla v))(u-v)\,\mathrm{d}x \\
&\quad \leq \lambda a_1 \|u-v\|_2^2 + \lambda \int_\Omega \left(f\left(x, v, \nabla\left(\frac{1}{2}(u-v)^2\right)\right) - \psi(x)\right)\mathrm{d}x \\
&\quad \leq \lambda a_1 \|u-v\|_2^2 + \lambda a_2 \int_\Omega |u-v||\nabla(u-v)|\,\mathrm{d}x \\
&\quad \leq \lambda(\overline{\lambda})^{-1} \|\nabla(u-v)\|_2^2.
\end{aligned} \tag{3.7}$$

From (3.5), (3.6) and (3.7) we conclude that

$$\left(\mu - \lambda(\overline{\lambda})^{-1}\right) \|\nabla(u - v)\|_2^2 \leq 0. \tag{3.8}$$

Since $\lambda < \mu\overline{\lambda}$, from (3.8) it follows $u = v$. $\square$

Now, we study the asymptotic behavior of problem (1.1) as the parameters μ and λ vary in an appropriate range. We introduce the following two sets

$$\mathcal{S}_\mu(\lambda) = \Big\{ u \ : \ u \text{is a solution of problem (1.1) for fixed } \mu \geq 0 \text{ and } \lambda \in \left]0, \lambda^*\right[\Big\},$$
$$\mathcal{S}(\lambda) = \bigcup_{\mu \geq 0} \mathcal{S}_\mu(\lambda) = \Big\{ \text{set of solutions of problem (1.1) for fixed } \lambda \in \left]0, \lambda^*\right[\Big\}.$$

First, we show the boundedness of $\mathcal{S}_\mu(\lambda)$ and $\mathcal{S}(\lambda)$ in $W_0^{1,p(\cdot)}(\Omega)$.

Proposition 3.4 *Let hypotheses (H1) and (H2) be satisfied. Then $\mathcal{S}_\mu(\lambda)$ is bounded in $W_0^{1,p(\cdot)}(\Omega)$ for all $\mu \geq 0$ and for all $\lambda \in]0, \lambda^*[$.*

Proof Let $\mu \geq 0$, $\lambda \in]0, \lambda^*[$ be fixed and let $u \in W_0^{1,p(\cdot)}(\Omega)$ be a solution of problem (1.1). Taking $h = u$ in the weak formulation in (2.4) and applying (H2)(ii) as well as (3.3), we have

$$\begin{aligned}
\int_\Omega |\nabla u|^{p(x)}\, \mathrm{d}x &\leq \left\langle A_{p(\cdot)}(u), u \right\rangle + \mu \left\langle A_{q(\cdot)}(u), u \right\rangle \\
&= \lambda \int_\Omega f(x, u, \nabla u) u \, \mathrm{d}x \\
&\leq \lambda \int_\Omega \left(a_0(x) + b_1 |u|^{p(x)} + b_2 |\nabla u|^{p(x)} \right) \mathrm{d}x \\
&\leq \lambda \|a_0\|_{L^1(\Omega)} + \lambda \left(b_1 \widehat{\lambda}^{-1} + b_2 \right) \int_\Omega |\nabla u|^{p(x)}\, \mathrm{d}x.
\end{aligned}$$

This implies by (2.1) that

$$\|\nabla u\|_{p(\cdot)}^{p^-} \leq \frac{\|a_0\|_1}{1 - \lambda(\lambda^*)^{-1}} \lambda + 1. \tag{3.9}$$

It follows that $\mathcal{S}_\mu(\lambda)$ is bounded in $W_0^{1,p(\cdot)}(\Omega)$. $\square$

Remark 3.5 Since the right hand side in (3.9) does not dependent on μ, we derive that $\mathcal{S}(\lambda) = \cup_{\mu \geq 0} \mathcal{S}_\mu(\lambda)$ is bounded in $W_0^{1,p(\cdot)}(\Omega)$ for all $\lambda \in]0, \lambda^*[$.

For a subset $\Lambda \subset]0, \lambda^*[$ we associate the following two sets

$$\mathcal{S}_\mu(\Lambda) = \bigcup_{\lambda \in \Lambda} \mathcal{S}_\mu(\lambda) \quad \text{for fixed } \mu \geq 0,$$
$$\mathcal{S}(\Lambda) = \bigcup_{\mu \geq 0} \mathcal{S}_\mu(\Lambda).$$

Remark 3.6 From (3.9) we deduce that $\mathcal{S}_\mu(\Lambda)$ is bounded in $W_0^{1,p(\cdot)}(\Omega)$ for all $\mu \geq 0$ whenever $\sup \Lambda < \lambda^*$. We also obtain that $\mathcal{S}(\Lambda)$ is bounded in $W_0^{1,p(\cdot)}(\Omega)$ whenever $\sup \Lambda < \lambda^*$. In particular, if $\Lambda \subset]0, \lambda^*[$ is a closed subset of $\mathbb{R}$, then $\mathcal{S}_\mu(\Lambda)$ and $\mathcal{S}(\Lambda)$ are bounded in $W_0^{1,p(\cdot)}(\Omega)$.

Now, we consider the limit case of (1.1) as $\mu \to 0^+$.

Proposition 3.7 *Let hypotheses (H1) and (H2) be satisfied. Further, let $\{\lambda_n\}_{n\in\mathbb{N}} \subset]0, \lambda^*[$ be a given sequence converging to $\lambda \in]0, \lambda^*[$, $\{\mu_n\}_{n\in\mathbb{N}}$ be a sequence of parameters converging to 0^+ and $\{u_n\}_{n\in\mathbb{N}}$ be a sequence of solutions to equation (1.1) such that $u_n \in \mathcal{S}_{\mu_n}(\lambda_n)$ for all $n \in \mathbb{N}$. Then there is a subsequence of $\{u_n\}_{n\in\mathbb{N}}$ (not relabeled) such that $u_n \to u$ in $W_0^{1,p(\cdot)}(\Omega)$ with $u \in W_0^{1,p(\cdot)}(\Omega)$ being a solution of (1.1).*

Proof Since $u_n \in \mathcal{S}_{\mu_n}(\lambda_n)$ for all $n \in \mathbb{N}$ and $\Lambda = \{\lambda_n : n \in \mathbb{N}\} \cup \{\lambda\}$ is such that $\sup \Lambda < \lambda^*$, we deduce by Remark 3.6 that $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $W_0^{1,p(\cdot)}(\Omega)$. So, we may assume (for a subsequence if necessary) that

$$u_n \rightharpoonup u \quad \text{in } W_0^{1,p(\cdot)}(\Omega) \quad \text{and} \quad u_n \to u \quad \text{in } L^{\alpha(\cdot)}(\Omega)$$

for some $u \in W_0^{1,p(\cdot)}(\Omega)$, see Proposition 2.3.

Returning to the proof of Theorem 3.1, from (3.2) we know that

$$\int_\Omega f(x, u_n, \nabla u_n)(u_n - u)\,\mathrm{d}x \to 0 \quad \text{as } n \to +\infty,$$

since $u_n \to u$ in $L^{\alpha(\cdot)}(\Omega)$ and by hypothesis (H2)(i).

Now, $u_n \in \mathcal{S}_{\mu_n}(\lambda_n)$ for all $n \in \mathbb{N}$ ensures that

$$\left\langle A_{p(\cdot)}(u_n), h\right\rangle + \mu_n \left\langle A_{q(\cdot)}(u_n), h\right\rangle = \lambda_n \int_\Omega f(x, u_n, \nabla u_n) h\,\mathrm{d}x \tag{3.10}$$

for all $h \in W_0^{1,p(\cdot)}(\Omega)$. Choosing $h = u_n - u \in W_0^{1,p(\cdot)}(\Omega)$ in (3.10), we deduce that

$$\begin{aligned} &\left\langle A_{p(\cdot)}(u_n), u_n - u\right\rangle + \mu_n \left\langle A_{q(\cdot)}(u_n), u_n - u\right\rangle \\ &= \lambda_n \int_\Omega f(x, u_n, \nabla u_n)(u_n - u)\,\mathrm{d}x \end{aligned} \tag{3.11}$$

for all $n \in \mathbb{N}$. Consequently, passing to the limit as $n \to +\infty$ in (3.11) and using $\mu_n \to 0^+$, we obtain

$$\lim_{n\to+\infty} \left\langle A_{p(\cdot)}(u_n), u_n - u \right\rangle = 0,$$

which by the (S_+)-property of $A_{p(\cdot)}(\cdot)$ (see Proposition 2.7) results in $u_n \to u$ in $W_0^{1,p(\cdot)}(\Omega)$.

Recall that the Nemytskij operator $N_f : W_0^{1,p(\cdot)}(\Omega) \to W^{-1,p'(\cdot)}(\Omega)$ is bounded and continuous due to hypothesis (H2)(i). Hence, we have

$$N_f(u_n) \to N_f(u) \quad \text{in } W^{-1,p'(\cdot)}(\Omega).$$

On the other hand,

$$\left\langle A_{p(\cdot)}(u_n), h \right\rangle \to \left\langle A_{p(\cdot)}(u), h \right\rangle \quad \text{and} \quad \left\langle A_{q(\cdot)}(u_n), h \right\rangle \to \left\langle A_{q(\cdot)}(u), h \right\rangle.$$

Therefore, taking the limit in (3.10) as $n \to +\infty$, we conclude that $u \in W_0^{1,p(\cdot)}(\Omega)$ is a weak solution of (1.1) with $\mu = 0$, that is, a weak solution of the following problem

$$\begin{aligned} -\Delta_{p(\cdot)} u &= \lambda f(x, u, \nabla u) && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega. \end{aligned}$$

□

Let us now study the case when $\mu \to +\infty$.

Proposition 3.8 *Let hypotheses (H1) and (H2) be satisfied. Further, let $\{\lambda_n\}_{n\in\mathbb{N}} \subset]0, \lambda^*[$ be a given sequence with $\sup_{n\in\mathbb{N}} \lambda_n < \lambda^*$ and $\{\mu_n\}_{n\in\mathbb{N}}$ be a sequence such that $\mu_n \to +\infty$. Then every $\{u_n\}_{n\in\mathbb{N}}$ such that $u_n \in \mathcal{S}_{\mu_n}(\lambda_n)$ for all $n \in \mathbb{N}$ converges to zero in $W_0^{1,q(\cdot)}(\Omega)$.*

Proof Repeating the arguments from the proof of Proposition 3.7 and using again Remark 3.6, we know that $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $W_0^{1,p(\cdot)}(\Omega)$. Hence,

$$u_n \rightharpoonup u \quad \text{in } W_0^{1,p(\cdot)}(\Omega) \quad \text{and} \quad u_n \to u \quad \text{in } L^{\alpha(\cdot)}(\Omega)$$

for some $u \in W_0^{1,p(\cdot)}(\Omega)$

We can rewrite (3.10) as

$$\frac{1}{\mu_n} \left\langle A_{p(\cdot)}(u_n), h \right\rangle + \left\langle A_{q(\cdot)}(u_n), h \right\rangle = \frac{\lambda_n}{\mu_n} \int_\Omega f(x, u_n, \nabla u_n) h \, \mathrm{d}x \tag{3.12}$$

for all $h \in W_0^{1,p(\cdot)}(\Omega)$.

For (3.12) we can follow the proof of Proposition 3.7 by changing the roles of $A_p(\cdot)$ with $A_q(\cdot)$. We have that $u_n \to u$ in $W_0^{1,q(\cdot)}(\Omega)$. Therefore, taking the limit in (3.12) as $n \to +\infty$, we obtain that u is a solution of the equation

$$\begin{aligned} -\Delta_{q(\cdot)} u &= 0 \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

Hence, $u = 0$ in $\overline{\Omega}$. Since our arguments apply to every convergent subsequence of $\{u_n\}_{n\in\mathbb{N}}$, we conclude that it holds for the whole sequence. So, we have $u_n \to 0$ in $W_0^{1,q(\cdot)}(\Omega)$. □

4 Properties of the Solution Sets

In this section we are going to prove some properties of the solution sets introduced in Sect. 3 concerning compactness and closedness. Recall that from Proposition 3.4 and Remarks 3.5, 3.6, we already know the boundedness of $\mathcal{S}_\mu(\lambda)$, $\mathcal{S}(\lambda)$, $\mathcal{S}_\mu(\Lambda)$ and $\mathcal{S}(\Lambda)$ in $W_0^{1,p(\cdot)}(\Omega)$ for all $\lambda \in]0, \lambda^*[$ and $\Lambda \subset]0, \lambda^*[$ with $\sup \Lambda < \lambda^*$.

Proposition 4.1 *Let hypotheses (H1) and (H2) be satisfied. Then $\mathcal{S}_\mu(\Lambda)$ is compact in $W_0^{1,p(\cdot)}(\Omega)$ for all $\mu \geq 0$ and $\Lambda \subset]0, \lambda^*[$ being closed in $\mathbb{R}$.*

Proof Let $u \in \overline{\mathcal{S}_\mu(\Lambda)} \setminus \mathcal{S}_\mu(\Lambda)$. Then there exists a sequence $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{S}_\mu(\Lambda)$ such that $u_n \to u$.

Claim 1: $\mathcal{S}_\mu(\Lambda)$ is closed for all $\mu \in [0, +\infty[$ and $\Lambda \subset]0, \lambda^*[$ being closed in $\mathbb{R}$.

First we note that for each $n \in \mathbb{N}$ there is $\lambda_n \in \Lambda$ such that $u_n \in \mathcal{S}_\mu(\lambda_n)$. Since the sequence $\{\lambda_n\}_{n\in\mathbb{N}}$ is bounded, we can assume, for a subsequence if necessary, that $\lambda_n \to \lambda \in \Lambda$. Since $u_n \in \mathcal{S}_\mu(\lambda_n)$ for all $n \in \mathbb{N}$, we obtain

$$\left\langle A_{p(\cdot)}(u_n), h \right\rangle + \mu \left\langle A_{q(\cdot)}(u_n), h \right\rangle = \lambda_n \int_\Omega f(x, u_n, \nabla u_n) h \, \mathrm{d}x \tag{4.1}$$

for all $h \in W_0^{1,p(\cdot)}(\Omega)$. Thus, passing to the limit as $n \to +\infty$ in (4.1), it follows that

$$\left\langle A_{p(\cdot)}(u), h \right\rangle + \mu \left\langle A_{q(\cdot)}(u), h \right\rangle = \lambda \int_\Omega f(x, u, \nabla u) h \, \mathrm{d}x$$

for all $h \in W_0^{1,p(\cdot)}(\Omega)$. This implies that $u \in \mathcal{S}_\mu(\lambda) \subset \mathcal{S}_\mu(\Lambda)$ and so $\mathcal{S}_\mu(\Lambda)$ is closed in $W_0^{1,p(\cdot)}(\Omega)$. This proves Claim 1.

Claim 2: Each $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{S}_\mu(\Lambda)$ admits a subsequence converging to some $u \in \mathcal{S}_\mu(\Lambda)$.

Remark 3.6 ensures that every sequence $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{S}_\mu(\Lambda)$ is bounded. So, we may assume, for a subsequence if necessary, that

$$u_n \rightharpoonup u \quad \text{in } W_0^{1,p(\cdot)}(\Omega) \quad \text{and} \quad u_n \to u \quad \text{in } L^{\alpha(\cdot)}(\Omega)$$

for some $u \in W_0^{1,p(\cdot)}(\Omega)$.

Next, let $\lambda_n \in \Lambda$ be such that $u_n \in \mathcal{S}_\mu(\lambda_n)$ for all $n \in \mathbb{N}$. Returning to the proof of Theorem 3.1, from (3.9) we can deduce that

$$\int_\Omega f(x, u_n, \nabla u_n)(u_n - u)\,\mathrm{d}x \to 0 \quad \text{as } n \to +\infty,$$

as $u_n \to u$ in $L^{\alpha(\cdot)}(\Omega)$ along with hypotheses (H2)(i). If we take $h = u_n - u \in W_0^{1,p(\cdot)}(\Omega)$ in (4.1), we have that

$$\left\langle A_{p(\cdot)}(u_n), u_n - u\right\rangle + \mu \left\langle A_{q(\cdot)}(u_n), u_n - u\right\rangle = \lambda_n \int_\Omega f(x, u_n, \nabla u_n)(u_n - u)\,\mathrm{d}x \tag{4.2}$$

for all $n \in \mathbb{N}$. Passing to the limit as $n \to +\infty$ in (4.2) and considering that $A_q(\cdot)$ is monotone, we obtain

$$\limsup_{n\to+\infty} \left\langle A_{p(\cdot)}(u_n), u_n - u\right\rangle \le 0.$$

Therefore, $u_n \to u$ in $W^{1,p(\cdot)}(\Omega)$ by Proposition 2.7 and so, $u \in \mathcal{S}_\mu(\Lambda)$ by Claim 1. This shows Claim 2.

From Claims 1 and 2 we conclude that $\mathcal{S}_\mu(\Lambda)$ is compact in $W_0^{1,p(\cdot)}(\Omega)$. $\square$

From the previous proposition, we deduce the following corollary.

Corollary 4.2 *Let hypotheses (H1) and (H2) be satisfied. Then $\mathcal{S}_\mu(\lambda)$ is compact in $W_0^{1,p(\cdot)}(\Omega)$ for all $\mu \ge 0$ and for all $\lambda \in]0, \lambda^*[$.*

Next, we give a sufficient condition when $\mathcal{S}(\Lambda)$ is closed.

Proposition 4.3 *Let hypotheses (H1) and (H2) be satisfied. Then $\mathcal{S}(\Lambda)$ is closed for all $\Lambda \subset]0, \lambda^*[$ whenever $0 \in \mathcal{S}(\Lambda)$ and Λ is closed in $\mathbb{R}$. In particular, $\mathcal{S}(\Lambda) \cup \{0\}$ is a closed subset of $W_0^{1,p(\cdot)}(\Omega)$ for all $\Lambda \subset]0, \lambda^*[$ being closed in $\mathbb{R}$.*

Proof From Proposition 3.8 we know that $0 \in \overline{\mathcal{S}(\Lambda)}$. So, let $u \in \overline{\mathcal{S}(\Lambda)} \setminus (\mathcal{S}(\Lambda) \cup \{0\})$. We are going to show that $u \in \mathcal{S}(\Lambda)$. Since $u \in \overline{\mathcal{S}(\Lambda)} \setminus (\mathcal{S}(\Lambda) \cup \{0\})$ we can find a sequence $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{S}(\Lambda)$ such that $u_n \to u$ in $W_0^{1,p(\cdot)}(\Omega)$. First, observe that for every $n \in \mathbb{N}$ there exist $\mu_n \ge 0$ and $\lambda_n \in \Lambda$ such that $u_n \in \mathcal{S}_{\mu_n}(\lambda_n)$. This means that

$$\left\langle A_{p(\cdot)}(u_n), h\right\rangle + \mu_n \left\langle A_{q(\cdot)}(u_n), h\right\rangle = \lambda_n \int_\Omega f(x, u_n, \nabla u_n) h\,\mathrm{d}x \tag{4.3}$$

for all $h \in W_0^{1,p(\cdot)}(\Omega)$.

Applying again Proposition 3.8 leads to the fact that $\{\mu_n\}_{n\in\mathbb{N}}$ is a bounded sequence and so we can assume that $\mu_n \to \mu$ for some $\mu \in [0, +\infty[$. Since the sequence

$\{\lambda_n\}_{n\in\mathbb{N}}$ is bounded we can assume that $\lambda_n \to \lambda \in \Lambda$. From $u_n \to u$, we get that

$$\begin{aligned}
&\langle N_f(u_n), h\rangle \to \langle N_f(u), h\rangle,\\
&\langle A_p(u_n), h\rangle \to \langle A_p(u), h\rangle,\\
&\langle A_q(u_n), h\rangle \to \langle A_q(u), h\rangle \quad \text{for all } h \in W_0^{1,p(\cdot)}(\Omega).
\end{aligned}$$

Therefore, taking the limit in (4.3) as $n \to +\infty$, we see that

$$\left\langle A_{p(\cdot)}(u), h\right\rangle + \mu \left\langle A_{q(\cdot)}(u), h\right\rangle = \lambda \int_\Omega f(x, u, \nabla u) h \, \mathrm{d}x$$

for all $h \in W_0^{1,p(\cdot)}(\Omega)$. Thus, $u \in \mathcal{S}_\mu(\lambda) \subset \mathcal{S}(\Lambda)$. Consequently, we have that $\mathcal{S}(\Lambda)$ is closed whenever $0 \in \mathcal{S}(\Lambda)$, that is, $\mathcal{S}(\Lambda) \cup \{0\}$ is closed in $W_0^{1,p(\cdot)}(\Omega)$. □

We have the following corollary.

Corollary 4.4 *Let hypotheses (H1) and (H2) be satisfied. Then $\mathcal{S}(\lambda)$ is closed for all $\lambda \in]0, \lambda^*[$ whenever $0 \in \mathcal{S}(\lambda)$. Therefore, $\mathcal{S}(\lambda) \cup \{0\}$ is a closed subset of $W_0^{1,p(\cdot)}(\Omega)$ for all $\lambda \in]0, \lambda^*[$.*

In the last part of this paper, we introduce the set-valued map $\mathfrak{S}_\Lambda : [0, +\infty[\to 2^{W_0^{1,p(\cdot)}(\Omega)}$ defined by $\mathfrak{S}_\Lambda(\mu) = \mathcal{S}_\mu(\Lambda)$ for all $\mu \in [0, +\infty[$ with $\Lambda \subset]0, \lambda^*[$ being closed in $\mathbb{R}$. $\mathfrak{S}_\Lambda$ is the Λ-solution map of (1.1).

We have the following properties of $\mathfrak{S}_\Lambda : [0, +\infty[\to 2^{W_0^{1,p(\cdot)}(\Omega)}$.

Proposition 4.5 *Let hypotheses (H1) and (H2) be satisfied. Then the set-valued map $\mathfrak{S}_\Lambda$ is upper semicontinuous for all $\Lambda \subset]0, \lambda^*[$ being closed in $\mathbb{R}$.*

Proof The set-valued map $\mathfrak{S}_\Lambda$ is upper semicontinuous if for each closed subset C of $W_0^{1,p(\cdot)}(\Omega)$ the set

$$\mathfrak{S}_\Lambda^-(C) = \{\mu \in [0, +\infty[: \ \mathfrak{S}_\Lambda(\mu) \cap C \neq \emptyset\}$$

is closed in $[0, +\infty[$. To this end, let $\{\mu_n\}_{n\in\mathbb{N}} \subset \mathfrak{S}_\Lambda^-(C)$ be such that $\mu_n \to \mu$ in $[0, +\infty[$. Obviously, for every $n \in \mathbb{N}$ there exists $u_n \in \mathfrak{S}_\Lambda(\mu_n) \cap C$. From Remark 3.6 it follows that the sequence $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $W_0^{1,p(\cdot)}(\Omega)$. Similar to the proof of Proposition 3.7 we can show that $u_n \to u$ in $W_0^{1,p(\cdot)}(\Omega)$.

Arguing as in the proof of Proposition 4.3 (since $u_n \in \mathcal{S}_{\mu_n}(\Lambda)$), we deduce that $u \in \mathcal{S}_\mu(\Lambda) = \mathfrak{S}_\Lambda(\mu)$. On the other hand, $u \in C$ since C is closed. Hence, $\mu \in \mathfrak{S}_\Lambda^-(C)$. This completes the proof. □

Proposition 4.6 *Let hypotheses (H1) and (H2) be satisfied. Then the set-valued map $\mathfrak{S}_\Lambda$ is compact, that is, $\mathfrak{S}_\Lambda$ maps bounded sets in $[0, +\infty[$ into relatively compact subsets of $W_0^{1,p(\cdot)}(\Omega)$.*

Proof Let $\Theta \subset [0, +\infty[$ be a bounded set, $\{u_n\}_{n\in\mathbb{N}} \subset \mathfrak{S}_\Lambda(\Theta)$ and $\mu_n \in \Theta$ be such that $u_n \in \mathcal{S}_{\mu_n}(\Lambda)$ for all $n \in \mathbb{N}$.

We distinguish the following two situations:

Case 1: If the set $\{\mu_n : n \in \mathbb{N}\}$ is finite, then there exists some $\mu \in \Theta$ such that $\mu = \mu_n$ for infinite values of n. We deduce that $\{u_n\}_{n\in\mathbb{N}}$ admits a subsequence $\{u_{n_k}\}_{k\in\mathbb{N}} \subset \mathcal{S}_\mu(\Lambda)$. Since $\mathcal{S}_\mu(\Lambda)$ is compact, we have that $\{u_{n_k}\}_{k\in\mathbb{N}}$ admits a subsequence converging to some $u \in \mathcal{S}_\mu(\Lambda) \subset \mathfrak{S}_\Lambda(\Theta)$.

Case 2: If the set $\{\mu_n : n \in \mathbb{N}\}$ has infinite elements, then $\{\mu_n\}_{n\in\mathbb{N}}$ has a convergent subsequence (not relabeled). If we assume that $\mu_n \to \mu$ for some $\mu \in \overline{\Theta}$, then we have

$$u_n \rightharpoonup u \quad \text{in } W_0^{1,p(\cdot)}(\Omega) \quad \text{for some } u \in W_0^{1,p(\cdot)}(\Omega),$$

since $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $W_0^{1,p(\cdot)}(\Omega)$. Then we can show that $u_n \to u$ in $W_0^{1,p(\cdot)}(\Omega)$. It is easy to verify that $u \in \mathcal{S}_\mu(\Lambda)$ and $u \in \overline{\mathfrak{S}_\Lambda(\Theta)}$.

Next, let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence in $\overline{\mathfrak{S}_\Lambda(\Theta)} \setminus \mathfrak{S}_\Lambda(\Theta)$. From $\mathfrak{S}_\Lambda(\Theta) \subset \mathcal{S}(\Lambda)$, we deduce that $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{S}(\Lambda)$ and hence it is bounded. This implies that for a subsequence of $\{u_n\}_{n\in\mathbb{N}}$ (not relabeled), we have

$$u_n \to u \quad \text{in } W_0^{1,p(\cdot)}(\Omega) \quad \text{for some } u \in W_0^{1,p(\cdot)}(\Omega),$$

Therefore, $u \in \overline{\mathfrak{S}_\Lambda(\Theta)}$ and so, $\mathfrak{S}_\Lambda(\Theta)$ is a relatively compact subset of $W_0^{1,p(\cdot)}(\Omega)$. This proves that the set-valued map $\mathfrak{S}_\Lambda$ is compact. □

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