Asymptotic properties of the parabolic Anderson model

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Zusammenfassung

Das parabolische Anderson-Modell ist die Wärmeleitungsgleichung auf dem $\mathbb{Z}^d$ mit zufälligem Potential. Charakteristisch für die Langzeitasymptotik der Lösung ist das Auftreten von sich immer weiter voneinander entfernenden Inseln, in denen fast alle Masse konzentriert ist. Dieser Effekt wird als Intermittenz bezeichnet. Es gibt im Wesentlichen zwei Möglichkeiten, das Verhalten der Lösung zu betrachten. Einerseits kann man die fast sichere (quenched) Asymptotik betrachten, nachdem eine Realisierung des Potentials fixiert wurde. Andererseits kann man die (annealed) Asymptotik betrachten, nachdem der Erwartungswert über das Potential genommen wurde.

Eine mögliche Charakterisierung der Intermittenz erfolgt über den Vergleich des asymptotischen Wachstums verschiedener Momente der Lösung. Wir leiten asymptotische Formeln her für die zeitliche Korrelation von regulär variierenden Funktionen der Lösung für den Fall einer homogenen Anfangsbedingung und eines geeigneten zeitunabhängigen Potentials. Mit Hilfe dieser Formeln lassen sich unter anderem alle Momente der Lösung bis auf asymptotische Äquivalenz genau bestimmen. Weiterhin beschreiben wir die Geometrie der Intermittenzpeaks, die das gemittelte Verhalten der Lösung bestimmen, insbesondere die Höhe, die Größe und die relative Häufigkeit der Peaks. Darüber hinaus untersuchen wir die Alterungseigenschaften des Modells anhand verschiedener Definitionen. Im Besonderen bestimmen wir, wie lange einzelne Intermittenzpeaks relevant bleiben.

Eine weitere Charakterisierung der Intermittenz erfolgt über den Vergleich der Quenched-Asymptotik mit der Annealed-Asymptotik. Betrachtet man bei homogener Anfangsbedingung eine mit der Zeit wachsende Teilbox des $\mathbb{Z}^d$ und mittelt die Lösung in dieser, so erhält man bei langsamem Wachstum der Box dieselbe Asymptotik wie im Quenched-Fall, wohingegen bei schnellem Wachstum dasselbe Verhalten eintritt wie im Annealed-Fall. Wir geben für geeignete Potentialklassen stabile Grenzwertsätze für die in der Box gemittelte Lösung in Abhängigkeit vom Potential und der Wachstumsrate der Box an, um den Übergang zwischen dem Quenched- und dem Annealed-Regime zu beschreiben. Desweitern leiten wir hinreichende Bedingungen an das Wachstum der Box her für ein starkes Gesetz der großen Zahlen.

Abschließend leiten wir asymptotische Formeln her für die räumliche und die zeitliche Korrelation im parabolischen Anderson-Modell mit (zeitabhängigem) weißem Rauschen als Potential.

Abstract

The parabolic Anderson model is the heat equation on the lattice with a random potential. A characteristic feature of the large time asymptotics of the solution is the occurrence of small islands where almost all mass is concentrated. This effect is called intermittency. There are basically two ways of looking at the long time behaviour of the solution. On the one hand one can consider the almost sure asymptotics after one realisation of the potential is fixed (the quenched setting). On the other hand one can consider he asymptotics after taking expectation with respect to the potential (the annealed setting).

One possible characterisation of intermittency is to compare the asymptotics of different moments of the solution. We derive asymptotic formulas for time correlations of regularly varying functions of the solution in the case of a homogeneous initial condition and an appropriate time-independent potential. These formulas can be used, for instance, to calculate moments of the solution of all orders up to asymptotic equivalence. Furthermore, we show what the geometry of the intermittency peaks that determine the annealed behaviour looks like. More precisely, we show what the height, the size and the frequency of the relevant peaks are. We also investigate ageing properties of the model under different definitions. In particular, we examine for how long intermittency peaks remain relevant.

Another characterisation of intermittency is to compare the quenched asymptotics of the solution with the annealed asymptotics. If one considers the averaged solution to the parabolic Anderson model with homogeneous initial condition in a growing box that is time-dependent, then one observes different regimes. If the growth rate of the box is small, then one observes quenched behaviour, whereas if the box grows fast one observes annealed behaviour. We derive stable limit theorems for the averaged solution depending on the growth rate of the box and the potential tails for suitable potentials to describe the transition from quenched to annealed behaviour. Furthermore, we give sufficient conditions on the growth of the box for a strong law of large numbers to hold.

Finally, we derive asymptotic formulas for time and spatial correlations in the parabolic Anderson model with a (time-dependent) white-noise potential.

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Contents

1 Introduction

1.1 The parabolic Anderson model

The *parabolic Anderson model* (PAM) is the heat equation on the lattice with a random potential, given by

$$\begin{cases} \frac{\partial}{\partial t} u(t,x) = \kappa \Delta u(t,x) + \xi(t,x) u(t,x), & (t,x) \in (0,\infty) \times \mathbb{Z}^d, \\ u(0,x) = u_0(x), & x \in \mathbb{Z}^d, \end{cases} \tag{1.1}$$

where $\kappa > 0$ denotes a diffusion constant, u_0 a nonnegative function, and Δ the *discrete Laplacian*, defined by

$$\Delta f(x) := \sum_{\substack{y \in \mathbb{Z}^d: \\ |x-y|_1 = 1}} \left[f(y) - f(x) \right], \qquad x \in \mathbb{Z}^d,\ f\colon \mathbb{Z}^d \to \mathbb{R}.$$

Furthermore, $\xi := \left\{ \xi(t,x), x \in \mathbb{Z}^d, t \in (0,\infty) \right\}$ is a (possibly time-dependent) *random potential* or *random medium*. If not stated otherwise, and in particular in Chapters 2 and 3, we will assume that ξ is an i.i.d. field of time-independent random variables, i.e. $\xi = \left\{ \xi(x), x \in \mathbb{Z}^d \right\}$. Chapter 4 deals with a time-dependent potential field. Typical choices for u_0 are the homogeneous initial condition $u_0 \equiv 1$, and the localised initial condition $u_0 = \delta_0$. The former makes the solution $u(t,\cdot)$ spatially homogeneous and ergodic for every t.

The PAM is an example of a reaction-diffusion equation with random coefficients where on the one hand interesting effects occur that are not observable in the deterministic case but which, on the other hand, is still mathematically tractable. Therefore, it has been an active field of research during the last decades.

One possible interpretation is that u describes a heat flow through a field of random sources (sites x with $\xi(x) > 0$) and sinks (sites x with $\xi(x) < 0$). Another interpretation is that $u(t,x)$ describes the averaged concentration of a branching random walk in a random potential field at time t in the lattice point x. If $\xi(x) > 0$, then the random walk produces offspring, whereas if $\xi(x) < 0$ then it is killed with a certain rate. The case $\xi(x) = -\infty$ corresponds to a hard trap where the random walk is killed immediately.

The solution u depends on two effects. On the one hand, the Laplacian tends to make it flat, whereas the potential causes the occurrence of small regions where almost all mass of the system is located. The latter effect is called *intermittency*. For time-independent potentials it is always present unless the potential is almost surely constant, see [GM90, Theorem 3.2]. It turns out that, the more heavy tailed the potential is, the more dominant this effect becomes. These regions are often referred to as *intermittency islands*, and the solution $u(t, \cdot)$ develops high peaks on these islands. As time increases the islands become more scattered in space. Commonly the almost sure behaviour of u is referred to as *quenched*, whereas the behaviour after averaging over the potential ξ is called *annealed*. By $\langle . \rangle$ we denote expectation with respect to ξ. The corresponding probability measure is denoted by $\mathbf{P}$.

The PAM with i.i.d. time-independent potential was introduced in the seminal paper [GM90] where existence and uniqueness of the solution were investigated as well as first order asymptotics of the statistical moments and of the almost sure behaviour of the solution. An overview of the rich literature and recent results on the PAM can be found in [GK05]. Applications of the PAM are summarised, for instance, in [M94].

Further versions of the PAM, which are not considered in this thesis, are those with correlated potential fields as considered for instance in [GM00], those with drift, see [D08], or those with continuous space as in [GK00] and [GKM00]. Relatives of the PAM are models concerning Brownian motion and random walks among random obstacles that were surveyed in [S98].

The name parabolic Anderson model originates from the fact that the heat equation is a parabolic equation and that (1.1) contains the *Anderson Hamiltonian* $\mathscr{H} := \Delta + \xi$, named after P.W. Anderson who originally, in the late 1950s, investigated the discrete Schrödinger equation with a random potential as a quantum mechanic model for electron transport in an alloy, see [A58]. For that model, called the Anderson model, he showed complete localisation of the electrons. He was awarded the Nobel price in physics in 1977 for his resarch on that model. Anderson Hamiltonians are a special case of random Schrödinger operators. They are still a wide and active field of research. For an overview of this topic see for instance [K07]. The PAM was introduced into the physics literature in the early 1980s as a tractable model for intermittency with possible applications to magneto-hydrodynamics and other fields. For more details see [M94] or [ZMRS88].

Representations of the solution

There are basically two ways of representing the solution u. One is a more probabilistic approach via path-integral formulae, the *Feynman-Kac representation*. The other one is a more analytic approach via a Fourier expansion of the eigenpairs of the Anderson Hamiltonian in a large box with suitable boundary conditions, the *spectral representation*. Throughout this thesis we will heavily benefit from both of them.

Feynman-Kac representation

The solution to (1.1) admits the following Feynman-Kac representation (see [GM90, Theorem 2.1]),

$$u(t,x) = \mathbb{E}_x \exp\left\{ \int_0^t \xi\left(X_s\right) \,\mathrm{d}s \right\} u_0\left(X_t\right), \qquad (t,x) \in [0,\infty) \times \mathbb{Z}^d, \tag{1.2}$$

where X is a simple, symmetric, continuous time random walk with generator $\kappa\Delta$ and $\mathbb{P}_x$ ($\mathbb{E}_x$) denotes the corresponding probability measure (expectation) if $X_0 = x$ a.s. Here a new kind of randomness, the random walk X, is introduced. From this representation one can justify the interpretation of an averaged random walk in a random potential field.

It is in particular appropriate to characterise the asymptotic behaviour of u as t tends to infinity by the optimal path of X through the potential field that contributes most to the expectation. On the one hand the random walk prefers to stay at high potential peaks but on the other hand it might have a long way through regions with low potential values to get there and it has to pay for staying in a high peak without moving for a long time.

Spectral representation

With high probability a random walk will not leave a large box up to time t. Therefore, it follows from the Feyman-Kac formula that for large t we can asymptotically approximate the solution to (1.1) by the solution in a large time-dependent box with suitable boundary conditions. Usual choices are Dirichlet (or zero), periodic and free boundary conditions. Let $Q_R := [-\lceil R\rceil, \lceil R\rceil]^d \cap \mathbb{Z}^d$ be the d-dimensional centered lattice cube of radius $\lceil R\rceil \geq 1$, u_R be the solution to the PAM in this box and $\mathscr{H}_R$ be the corresponding Anderson Hamiltonian. Then u_R admits the spectral representation

$$u_R(t,x) = \sum_{k=1}^{|Q_R|} \mathrm{e}^{\lambda_k^R t} \left(e_k^R, \mathbb{1}\right) e_k^R(x), \qquad (t,x) \in [0,\infty) \times Q_R, \tag{1.3}$$

where $\lambda_1^R > \lambda_2^R \geq \cdots \geq \lambda_{|Q_R|}^R$ is an order statistics of the eigenvalues of $\mathscr{H}_R$, and $e_1^R, e_2^R, \cdots, e_{|Q_R|}^R$ is a corresponding orthonormal basis of ℓ^2-eigenfunctions such that

$e_1^R > 0$. Furthermore, $(\cdot, \cdot)$ denotes the standard scalar product. More details on this construction will be given in Chapter 2.

1.2 Intermittency

Classical approaches to intermittency

The asymptotic property of the PAM that has attracted the most attention is the phenomenon of *intermittency* as mentioned above. While heuristically it is quite clear what intermittency looks like it is challenging to give it a precise mathematical meaning. The following features are usually associated with intermittency:

i) The technique of homogenisation does not work, i.e. it is not possible to approximate the PAM by a heat equation with a deterministic potential.

ii) The eigenfunctions in (1.3) are exponentially localised and the principal eigenvalue is dominant.

iii) The annealed solution grows asymptotically much faster than the quenched one, i.e. there are very unlikely potential configurations that determine the moments due to high exceedances of its extreme values.

iv) Higher moments grow asymptotically much faster than lower moments. This was the first rigorous definition of intermittency given in [GM90]. They call an ergodic field u intermittent if for all $p, q \in \mathbb{N}$ with $p > q \geq 1$,

$$\lim_{t\to\infty}\left(\frac{\log\langle u(t,0)^p\rangle}{p} - \frac{\log\langle u(t,0)^q\rangle}{q}\right) = \infty. \tag{1.4}$$

An argument why the definition in iv) is consistent with the heuristic picture from above is given in [GM90, page 616].

The disadvantage of all these characterisations is that they do not provide much information on what the geometric picture of intermittency peaks looks like. More precisely, one wants to know more about the height, location, frequency, size and shape of ξ and u within the intermittency islands. Therefore, more sophisticated methods have been established recently. The drawback of these approaches is that they require stronger assumptions on ξ and that different techniques are required depending on the distribution of $\xi(0)$. We will come back to this later.

Item ii) is closely related to the topic of *Anderson localisation* where the exponential localisation of eigenfunctions is proven for Anderson Hamiltonians. One expects that the intermittency peaks in the PAM look like these eigenfunctions.

Transition from quenched to annealed behaviour

As already mentioned, those realisations of ξ that govern the quenched behaviour of u differ heavily from those that govern the annealed behaviour. Therefore, there has recently been some effort to understand the transition mechanism from the quenched to the annealed regime.
To this end, one looks at expressions such as $\frac{1}{|Q|}\sum_{x\in Q} u(t,x)$ with $u_0 \equiv 1$, where Q is a large centred box. If Q has a fixed size, then $\frac{1}{|Q|}\sum_{x\in Q} u(t,x)$ admits quenched behaviour as t tends to infinity. On the other hand if we fix t and let the size of Q tend to infinity then (due to the homogeneous initial condition) by Birkhoff's ergodic theorem $\frac{1}{|Q|}\sum_{x\in Q} u(t,x)$ admits annealed behaviour almost surely. In [BAMR07] the authors examine the transition mechanism between these two regimes by investigating $\frac{1}{|Q|}\sum_{x\in Q} u(t,x)$ for boxes with time-dependent growth rate. More precisely, they give necessary and sufficient conditions on the growth rate for a central limit theorem and a weak law of large numbers. More details on their work will be given in Chapter 3 where we extend their results to stable limit theorems and give sufficient conditions for a strong law of large numbers.

Universality classes

In [HKM06] the authors show that under some mild regularity assumptions on the tail behaviour of $\xi(0)$ there are only four universality classes of intermittency.

i) The first universality class contains potentials with tails that decay faster than those of a double-exponential distribution. In this class the quenched as well as the annealed intermittency peaks consist asymptotically of single lattice points. This class corresponds to the case $\rho = \infty$ in [GM98].

ii) The second universality class contains potentials with tails that decay asymptotically as those of a double-exponential distribution, i.e. $-\log \mathbb{P}(\xi(0) > h) \sim \exp\{h/\rho\}$, $h \to \infty$, for some parameter $\rho > 0$. In this class the intermittency peaks consist of islands with asymptotically fixed size. It turns out that the shape of the potential field of the annealed intermittency islands is the same as for the quenched ones but the islands differ in height. This class was investigated in [GM98] where the first universality class is included as a special case.

iii) The third universality class contains the so called almost bounded potentials whose tails decay more slowly than those of a double-exponential distribution. This class includes both bounded and unbounded potentials and was studied in [HKM06]. In this class the growth rate of the size of the intermittency islands is of slowly varying order.

iv) The fourth universality class contains bounded from above potentials whose tails decay more slowly than those of the third universality class. It was studied in [BK01]. In this class the growth rate of the size of intermittency islands is of regularly varying order.

In all these classes, one can derive the leading terms of the quenched and the annealed asymptotics of u by considering large deviations for the random walk in the Feynman-Kac formula that give its optimal way through an optimally shaped potential field. Alternatively, one can derive these results by applying a large deviation principle for the principal eigenvalue in the spectral representation. Heuristics for both approaches can be found in [GK05]. In Section 2.4 we derive exact moment asymptotics for the first universality class. We show that only the principal eigenvalue in the spectral representation contributes and give its precise asymptotic behaviour, i.e. we identify all other terms in the asymptotic expansion.

Geometric description of intermittency

If the potential tails are Pareto distributed, i.e. $\mathbb{P}(\xi(0) > x) = x^{-\alpha}$, $\alpha > d$, $x \geq 1$, then an even stronger notion of intermittency can be proven. In [KLMS09] the authors show that for $u_0 = \delta_0$ asymptotically there is only one intermittency peak in probability and that almost surely there are only two for any large t. This result is known as the two cities theorem. For an overview of the PAM with heavy-tailed potential see [M09].

Another more geometrical approach aims to define time-dependent sets $\Gamma_t \in Q_t$, with Q_t a large box, such that $\sum_{x\in\Gamma_t} u(t,x) \sim \sum_{x\in Q_t} u(t,x), \quad t \to \infty$. In [GKM07] the authors show that in the first and second universality class with $u_0 = \delta_0$ in a box of sidelength $t \log^2 t$ the number of quenched intermittency islands increases more slowly than any power of t and their distance increases almost like t. Furthermore, they find that $|\Gamma_t| = t^{o(1)}$. Their rather technical approach is based on the Feynman-Kac formula as well as on the spectral representation of u. Shape and size of the intermittency islands are determined by the shape and the support of the eigenfunctions. In Section 2.5 we give a precise picture of the geometry of the annealed intermittency islands in the first universality class.

Ageing

While so far most of the literature has dealt with a static picture of intermittency there have recently been new approaches to get a more dynamic picture of the whole process in terms of ageing. More precisely, one wants to determine for how long intermittency peaks remain relevant and whether the time for which they are relevant increases in t.

There are different approaches to ageing in different models. For instance [MOS10] deals with ageing in the heavy-tailed case with $u_0 = \delta_0$ where the two cities theorem holds. We will come back to this issue in more detail in Section 2.6 where we compare two approaches to ageing for the first universality class with $u_0 \equiv 1$ and potential tails that are lighter than those of an exponential distribution where a two cities theorem does not hold.

This might be regarded as a first step towards a description of the whole heat flow, i.e. of the process $\big(u(t,0)\big)_{t\in[0,\infty)}$.

1.3 Time dependent potentials

While at least the rough asymptotic behaviour of the PAM with time-independent potential is now understood quite well, there has recently been some effort to understand various models with time-dependent potential.

In [CM94] a PAM with time-dependent potential was introduced for the first time. One part of that monography deals with a potential that satisfies $\xi(t,x) = \mathrm{d}W_t(x)$, where $\mathrm{d}W$ denotes white noise. We will call this *white-noise potential.* There has been extensive research on this model following the seminal work [CM94]. For instance ageing properties and transition from quenched to annealed were investigated in [DD07] and [CM07], respectively. We will come back to this model in Chapter 4 where we derive formulas for time and spatial correlations therein.

More recently, potentials have been investigated that consist of finitely or infinitely many independent random walks, exclusion- or voter dynamics. Here one often interprets the PAM as a model for chemical reactions, where $u(t,x)$ describes the concentration of reactants and $\xi(t,x)$ describes the concentration of catalysts at site x and time t. See the survey article [GdHM09] and the references therein for details.

One common feature of all time-dependent potentials is that, in contrast to the time-independent case, there is not always intermittent behaviour. Indeed, one finds that in small dimensions the systems is always intermittent, whereas in higher dimensions the system is only intermittent if κ is small enough, i.e. if the heat flow is not too fast compared to the speed of the potential. The exact dimension where this phase transition occurs, as well as the appropriate definition of intermittency, are model dependent.

The main tool for these models is the Feynman-Kac formula for time-dependent potentials given by

$$u(t,x) = \mathbb{E}_x \exp\left\{ \int_0^t \xi(t-s, X_s)\, ds \right\} u_0(X_t), \qquad (t,x) \in [0,\infty) \times \mathbb{Z}^d. \tag{1.5}$$

An analog to the spectral representation does not exist but it is often possible to treat the problem with the help of spectral analysis of certain deterministic Schrödinger operators.

1.4 Results of this thesis

The scientific work done in this thesis contributes to some important aspects of the PAM. Each chapter contains a short introduction with the main results of that chapter and the relevant notation such that they can be read independently of each other.

Exact asymptotics for time correlations and moments, geometric picture of annealed intermittency peaks, and ageing

In Chapter 2 we derive exact asymptotics of time correlation functions for the PAM with homogeneous initial condition and time-independent potential tails that are in the first universality class and have a finite cumulant generating function. We use these results to give precise asymptotics for statistical moments of positive order. Furthermore, we show what the potential peaks that contribute to the annealed intermittency picture look like and how they are distributed in space. We also investigate for how long intermittency peaks remain relevant in terms of ageing properties of the model.

Transition from quenched to annealed behaviour

Chapter 3 is devoted to the transition from quenched to annealed asymptotics. We consider the solution to the PAM with homogeneous initial condition in large time-dependent boxes. We derive stable limit theorems (ranging over all possible scaling parameters) for the rescaled sum over the solution depending on the growth rate of the boxes. Furthermore, we give sufficient conditions for a strong law of large numbers.

Correlations for the PAM with white noise-potential

In Chapter 4 we derive time and spatial correlations for the PAM with a time-dependent white-noise potential.

2 Time correlations for the parabolic Anderson model

2.1 Introduction

Main results

In this chapter we will restrict to the case that we have the homogeneous initial condition $u_0 \equiv 1$, and that the potential is i.i.d. and time-independent. We deal with potential tails that decay more slowly than those of a double exponentially (Gumbel) distributed variable X, e.g. $\mathbb{P}(X > r) = \exp\{-\mathrm{e}^r\}$ but still have a finite cumulant generating function. Examples that satisfy all conditions that we impose later include the Weibull distribution, i.e., $\mathbb{P}(X > h) = \exp\{-h^\gamma\}$ for $\gamma \in (1, \infty)$. Hence, we are in the first universality class in the classification of [HKM06]. This class was studied in [GM98], where some little evidence was gained that the main contribution to the moments of the solution comes from delta-like peaks in the ξ-landscape which are far away from each other. Among other results, they derived the first two terms of the logarithmic asymptotics for the moments of the total mass of the solution. One main result of the present chapter are the exact asymptotics of these moments. Furthermore, we give a generalisation to more complex functions of the solution evaluated at different times.

Another main result describes the height of the intermittency peaks that determine the annealed behaviour. Furthermore, we prove that the complement of the intermittency islands is indeed negligible with respect to the peaks. Since we consider the homogeneous initial condition $u_0 \equiv 1$, we will investigate the solution in extremely large boxes in which many of these peaks contribute.

Another aspect that we study in this chapter are ageing properties of the model. To this end, we compare two notions of ageing, one in terms of time correlations and one in terms of stability of intermittency peaks. In particular, we analyse mixed moments of the solution on two time scales.

Let us formulate more precisely our main assumptions and introduce some notation. Recall that $\langle . \rangle$ denotes expectation with respect to ξ and that the corresponding probability measure is denoted by $\mathbf{P}$. Let $\bar{F}(h) := \mathbf{P}(\xi(0) > h)$ denote the tail of $\xi(0)$ and $\varphi := -\log \bar{F}$. Furthermore, let $H(t) := \log \left\langle \mathrm{e}^{t\xi(0)} \right\rangle$ be the cumulant generating function of $\xi(0)$. We will make the following assumption on the tails of ξ:

Assumption (F):

i) If $x \neq y$, then for all $c > 0$,

$$\mathbf{P}\left(\frac{\xi(x)+\xi(y)}{2} > h - c\right) = o\left(\bar{F}(h)\right), \qquad h \to \infty.$$

ii) $H(t) < \infty$ for all $t \geq 0$.

Item ii) is equivalent to the existence of moments of the solution of all orders, see [GM90]. Item i) means that it is much more likely to have one very high peak than to have two quite high peaks. Under Assumption (F) we know that $\lim_{t\to\infty} H(t)/t = \infty$, i.e., the potential is unbounded to infinity.

To keep the proofs as simple as possible we assume that ξ is bounded from below although analogous results hold true if the potential is unbounded from below. This allows us to assume without loss of generality that $\operatorname{essinf} \xi = 0$. If $\operatorname{essinf} \xi = c$, we can use the transformation $u \mapsto \mathrm{e}^{ct} u$ which shifts $\operatorname{essinf} \xi$ to the origin.

Time correlations

Theorem 1 provides us with a formula how to compute asymptotically the time correlations for regularly varying functions of the solution u. It is also the main proof tool for all further applications. Spatial correlations for potentials with double exponential or heavier tails can be found in [GdH99], whereas time correlations have not been investigated so far. Let $Q_R := [-\lceil R \rceil, \lceil R \rceil]^d \cap \mathbb{Z}^d$ be the d-dimensional centered lattice cube of radius $\lceil R \rceil \geq 1$ and let

$$\widehat{\xi}_R := \max_{x \in Q_R \setminus \{0\}} \xi(x).$$

We impose free and zero boundary conditions on the boundary of Q_R, denoted by $* = \mathrm{f}$ and $* = 0$, respectively. The corresponding Laplacians are denoted by Δ_R^*, that is, for $f\colon Q_R \to \mathbb{R}$,

$$\Delta_R^{\mathrm{f}} f(x) = \sum_{y \in Q_R \colon y \sim x} (f(y) - f(x)), \qquad \Delta_R^0 f(x) = \sum_{y \in \mathbb{Z}^d \colon y \sim x} (f(y) - f(x)),$$

where for $\Delta_R^0 f$ we extend f trivially to $\mathbb{Z}^d$ with the value zero. Zero boundary conditions correspond to $\xi(x) = -\infty$ for $x \notin Q_R$. Its law and expectation will be denoted by $\mathbb{P}_x^{R,0}$ and $\mathbb{E}_x^{R,0}$, respectively. The random walk generated by Δ_R^{f} just remains at its current site at the boundary when the random walk generated by Δ would jump out of Q_R. Its law and expectation will be denoted by $\mathbb{P}_x^{R,\mathrm{f}}$ and $\mathbb{E}_x^{R,\mathrm{f}}$, respectively. The corresponding Dirichlet form is given by

$$\left(-\Delta_R^{\mathrm{f}} u, u\right)_{Q_R} = \sum_{\substack{\{x,y\}\in Q_R:\\ |x-y|_1=1}} \big(u(x)-u(y)\big)^2.$$

Let $\lambda_1^{R,*} = \lambda_1^{R,*}(\xi)$ be the principal (i.e., largest) eigenvalue of the Anderson Hamiltonian $\mathscr{H}_R^* := \kappa\Delta_R^* + \xi$ on $\ell^2(Q_R)$ with free and zero boundary condition, respectively.

Recall that regularly varying functions are those positive functions f that can be written as $x^\gamma L(x)$, where $\gamma \in \mathbb{R}$ is called the index of variation and L is a slowly varying function called slowly varying part of f.

Let $\mathcal{R}, \mathcal{R}_\gamma$ and $\mathcal{R}_+$ be the set of regularly varying functions, regularly varying functions with index of variation γ, and regularly varying functions with positive index of variation, respectively, with non-decreasing or bounded away from zero and infinity, regularly varying part. Let

$$\mathcal{F} := \left\{ f \in C^1 \colon f \in \mathcal{R}_+, f'(x) > 0\,\forall x > 0, f(0) = 0, \lim_{t\to\infty} f(t) = \infty \right\}$$

and

$$\mathcal{T} := \left\{ f \in C^1 \colon f'(x) > 0\,\forall x > 0, f(0) = 0, \lim_{t\to\infty} f(t) = \infty \right\}.$$

Remark. The fact $f \in \mathcal{R}_+$ already implies that $\lim_{t\to\infty} f(t) = \infty$, see [BGT87, Proposition 1.5.1].

Theorem 1 (Time correlations)**.** *Let Assumption (F) be satisfied. Furthermore, let $f_1, \dots, f_p \in \mathcal{F}$ and $t_1, \dots, t_p \in \mathcal{T}$ be given such that for all $a \geq 0$,*

$$\max_{1\le j\le p} \mathrm{e}^{t_j(t)a} = o\left(\min_{1\le i\le p} \left\langle f_i\left(\mathrm{e}^{t_i(t)\xi(0)}\right)\right\rangle \right), \qquad \textit{as } t\to\infty. \tag{2.1}$$

Then for every $R \geq 1$ *and* $0 < \underline{C} < 1 < \overline{C} < \infty$ *we find that for all* c, t *large enough,*

$$\begin{aligned} &\underline{C} \int_c^\infty \left[\frac{\mathrm{d}}{\mathrm{d}h} \prod_{i=1}^p f_i \left(\mathrm{e}^{t_i(t)h} \right) \right] \mathbf{P} \left(\lambda_1^{R,0}(\xi) > h \middle| \widehat{\xi}_R \leq h - c \right) \mathrm{d}h \\ \leq\ & \left\langle \prod_{i=1}^p f_i \big(u\left(t_i(t), 0\right) \big) \right\rangle \\ \leq\ & \overline{C} \int_c^\infty \left[\frac{\mathrm{d}}{\mathrm{d}h} \prod_{i=1}^p f_i \left(\mathrm{e}^{t_i(t)h} \right) \right] \mathbf{P} \left(\lambda_1^{R,\mathrm{f}}(\xi) > h \middle| \widehat{\xi}_R \leq h - c \right) \mathrm{d}h. \end{aligned}$$

Condition (2.1) determines of what order the functions t_i can be chosen. It is always possible to choose $\max t_i = a \cdot \min t_i$, $a > 0$.

Note that Assumption (F) is given in terms of the distribution of the potential, while the asymptotics themselves are expressed in terms of the conditional distribution of the eigenvalues. The asymptotics may be understood as follows. A Fourier expansion in terms of the eigenvalues of $\mathscr{H}_R^*$ yields that

$$u(t, \cdot) \approx \mathrm{e}^{t\lambda_1^{R,*}(\xi)} \big(e_1^{R,*}, \mathbb{1} \big) e_1^{R,*}(\cdot), \tag{2.2}$$

where $e_1^{R,*}$ is the positive ℓ^2-normalised principal eigenfunction. Under Assumption (F), it turns out that the eigenfunction $e_1^{R,*}$ is extremely delta-like peaked. Due to the the requirement $\widehat{\xi}_R \leq h - c$, the peak centre lies in the origin since $\xi(0)$ and $\lambda_1^{R,*}(\xi)$ differ by at most $2d\kappa$.

Exact moment asymptotics

Our first application of Theorem 1 are exact asymptotics for all moments of positive order. The second order asymptotics for integer moments for a large class of potentials, including the ones that satisfy Assumption (F), can be found in [GM98]: For any $p \in \mathbb{N}$,

$$\langle u(t, 0)^p \rangle = \mathrm{e}^{H(pt) - 2d\kappa pt}\, \mathrm{e}^{o(t)}, \qquad t \to \infty.$$

We now present much finer asymptotics which are even up to asymptotic equivalence. To the best of our knowledge, this precision has not yet been achieved for the PAM. We need the tails of the principal eigenvalue, conditional on having an extremely high peak at the origin:

$$\varphi_R^*(h) := -\log \mathbf{P} \left(\lambda_1^{R,*}(\xi) > h \,\middle|\, \widehat{\xi}_R \leq h^\alpha \right).$$

Here α is picked according to the following condition which is slightly stronger than Assumption (F).

Assumption (F*):

i) $\exists\, \alpha < 1\colon \quad \bar{F}(h) \cdot \bar{F}(h^\alpha) = o\big(\bar{F}(h + 2d\kappa)\big), \qquad h \to \infty.$

ii) $H(t) < \infty$ for all $t \geq 0$.

Let h_t be a solution to

$$\sup_{h \in (0,\infty)} (th - \varphi(h)) = th_t - \varphi(h_t) =: \psi(t).$$

If φ is ultimately convex, then h_t is unique for any large t.
Now we introduce a condition on the function $\varphi(h) = -\log \mathbb{P}(\xi(0) > h)$. A function $f(t) = o(t)$ is called *self-neglecting* if

$$f\big(t + af(t)\big) \sim f(t), \qquad t \to \infty, \tag{2.3}$$

locally uniformly in $a \in (0, \infty)$. The convergence in (2.3) is already locally uniform in a if f is continuous (see for instance [BGT87, Theorem 2.11.1]).
Let $h_t^{R,*}$ be a solution to

$$\sup_{h \in (0,\infty)} (th - \varphi_R^*(h)) = th_t^{R,*} - \varphi_R^*\left(h_t^{R,*}\right) =: \psi_R^*(t).$$

If φ is ultimately convex, then $h_t^{R,*}$ is unique for any large t.

Condition (B): The map $t \mapsto \sqrt{\varphi''(h_t)}$ is self-neglecting.

Again Condition (B) and Assumption (F*) concern ξ and not λ_1^R.

Theorem 2 (Moment asymptotics)**.** *Let $\varphi \in C^2$ be ultimately convex, Assumption (F*) and Condition (B) be satisfied and $p \in (0, \infty)$. Then, for any sufficiently large R,*

$$\langle u(t,0)^p \rangle \sim \exp\left\{ pth_{pt}^{R,*} - \varphi_R^*(h_{pt}^{R,*}) + \log pt + \frac{1}{2} \log \frac{\pi}{(\varphi_R^*)''(h_{pt}^{R,*})} \right\}, \qquad t \to \infty.$$

We see from (2.2) and Theorem 1 that Theorem 2 basically follows from an application of the Laplace method.

Note that Weibull tails with parameter $\gamma > 1$ satisfy both Condition (F*) and Condition (B). For $\gamma \in (1, 3)$, we give an explicit identification of all terms of the asymptotics, see Corollary 22.

Relevant potential peaks and intermittency

While originally intermittency was studied by comparing the asymptotics of successive moments of u, there have recently been efforts to describe intermittency in a more geometric way by determining time dependent random sets in $\mathbb{Z}^d$ in which the solution is asymptotically concentrated. These sets are closely related to the support of the leading eigenfunctions of the Anderson Hamiltonian. Clearly, the quenched intermittency picture differs from the annealed one. The height of the quenched intermittency peaks is basically determined by the almost sure growth of the maximal potential peak in a time-dependent box. Its radius depends on the distance that the random walk in the Feynman-Kac representation can make by time t. In [GKM07] the authors describe the geometry of the quenched intermittency peaks for the localised initial condition $u_0 = \delta_0$. They find that size and shape of the islands are deterministic, whereas number and location are random. They also give rough bounds on the number and location. They show that under Assumption (F) the quenched intermittency peaks consist of single lattice points.

In contrast, the annealed peaks are significantly higher and occur less frequently. Their geometry has not been investigated so far. Theorem 3 below determines the height of those potential peaks that contribute to the annealed intermittency peaks, and it proves that the complement contributes a negligible amount. It turns out that the peaks consist of single lattice points as well.

We will assume from now on that the box Q_{L_t} is chosen so large that the following weak law of large numbers holds true, see [BAMR07, Theorem 1]:

$$\frac{1}{|Q_{L_t}|} \sum_{x \in Q_{L_t}} u(t,x) \sim \langle u(t,0) \rangle, \qquad \text{as } t \to \infty, \text{ in probability.} \tag{2.4}$$

To this end, it is sufficient to pick $L(t)$ much larger than $\exp\{H(t)\}$. Let

$$\Upsilon_t^a = \left[h_t - \frac{a}{\sqrt{\varphi''(h_t)}}, h_t + \frac{a}{\sqrt{\varphi''(h_t)}} \right], \qquad a > 0.$$

In our result it turns out that the set of intermittency peaks may be taken as the set of those sites in which the potential height lies in Υ_t^a:

Theorem 3 (Intermittency)**.** *Let Assumption (F*) and Condition (B) be satisfied. Then for every $\varepsilon > 0$ there exists a_ε such that*

$$\lim_{t \to \infty} \mathbf{P}\left(1 - \frac{\sum_{x \in Q_{L_t}} u(t,x) \mathbb{1}_{\xi(x) \in \Upsilon_t^a}}{\sum_{x \in Q_{L_t}} u(t,x)} > \varepsilon \right) = \begin{cases} 1 & \text{if } a < a_\varepsilon, \\ 0 & \text{if } a > a_\varepsilon. \end{cases}$$

The locations of the peaks form a Bernoulli process, see Corollary 24 for details.

Ageing

In this section, we present our results on the dynamic picture of intermittency in the PAM. We will investigate two types of ageing behaviours, *correlation ageing* and *intermittency ageing*. While the first type gives only rather indirect information about the intermittency peaks, intermittency ageing explicitly describes for how long the intermittency peaks remain relevant. Nevertheless, both approaches give very similar results.

Roughly speaking, a system is ageing if the time it spends in a certain state increases as a function of its current age. An overview of the topic of ageing can be found, for instance, in [BA02]. For the PAM, there have been two approaches. In the case of a (time-dependent) white noise potential ξ as defined in [CM94], a variant of correlation ageing was investigated in [DD07] and [AD11]. The authors found that there is no ageing.

In [MOS10] the authors consider a localised initial condition and a time-independent i.i.d. potential with Pareto-distributed tails. They find that intermittency ageing holds. Their proofs rely on the two cities theorem proved in [KLMS09, Theorem 1.1] which states that, at any sufficiently late time, all the mass is concentrated in no more than two lattice points, almost surely.

Let us describe our result on intermittency ageing. As we know from Theorem 3, there are infinitely many intermittency peaks in our setting, possibly due to the homogeneous initial condition and to the lighter tails, so we have to use a modified definition and different techniques. To define the notion, introduce, for a scale function $s\colon (0,\infty) \to (0,\infty)$,

$$\mathscr{A}_s(t) := \mathbf{P}\left(\left| \frac{\sum_{x \in Q_{L_{t+s(t)}}} u(t,x) \mathbb{1}_{\xi(x) \in \Upsilon_t^a}}{\sum_{x \in Q_{L_{t+s(t)}}} u(t,x)} - \frac{\sum_{x \in Q_{L_{t+s(t)}}} u(t+s(t),x) \mathbb{1}_{\xi(x) \in \Upsilon_t^a}}{\sum_{x \in Q_{L_{t+s(t)}}} u(t+s(t),x)} \right| < \varepsilon \right), \quad t > 0.$$

Recall that $Q_{L_{t+s(t)}}$ is chosen such that the weak law of large numbers from (2.4) holds. We will consider only $a > a_\varepsilon$ as in Theorem 3. Roughly speaking, $\mathscr{A}_s$ measures whether those potential points that are intermittency peaks at time t are still relevant after time $t + s$.

We define *intermittency ageing* by requiring that for any small $\varepsilon > 0$ there is $a > 0$ and two scale functions s_1, s_2 satisfying $\lim_{t\to\infty} s_1(t) = \lim_{t\to\infty} s_2(t) = \infty$ such that

$$\lim_{t\to\infty} |\mathscr{A}_{s_1}(t) - \mathscr{A}_{s_2}(t)| > 0, \tag{2.5}$$

i.e., the two limits of $\mathscr{A}_{s_1}$ and $\mathscr{A}_{s_2}$ both exist and are different.

By the *length of intermittency ageing* we understand the class of functions

$$\mathscr{A} := \left\{ s\colon \mathbb{R} \to \mathbb{R}\colon \lim_{t\to\infty} s(t) = \infty, \exists \theta \in (0,\infty)\colon \lim_{t\to\infty} |\mathscr{A}_s(t) - \mathscr{A}_{\theta s}(t)| > 0 \right\}.$$

Theorem 4 (Intermittency ageing). *Let Assumption (F*) and Condition (B) be satisfied. Then the PAM ages in the sense of intermittency ageing if and only if* $\lim_{t\to\infty} H''(t) = 0$. *In this case* $\mathscr{A} \ni 1/\sqrt{H''(t)} = o(t)$.

For the study of correlation ageing we investigate the following time correlation coefficient

$$A_f(s,t) = \operatorname{corr}\Big(f\big(u(t,0)\big), f\big(u(t+s(t),0)\big)\Big) = \frac{\operatorname{cov}\Big(f\big(u(t,0)\big), f\big(u(t+s(t),0)\big)\Big)}{\sqrt{\operatorname{var}\Big(f\big(u(t,0)\big)\Big)\operatorname{var}\Big(f\big(u(t+s(t),0)\big)\Big)}}.$$

Here $f \in C$ is a strictly increasing function with $\lim_{t\to\infty} f(t) = \infty$.We define *correlation ageing* by requiring that there exist two scale functions s_1, s_2 satisfying $\lim_{t\to\infty} s_1(t) = \lim_{t\to\infty} s_2(t) = \infty$ such that

$$\lim_{t\to\infty} |A_f(s_1,t) - A_f(s_2,t)| > 0. \tag{2.6}$$

By the *length of correlation ageing* we understand the class of functions

$$\mathcal{A} := \left\{ s\colon \mathbb{R} \to \mathbb{R}\colon \lim_{t\to\infty} s(t) = \infty, \exists \theta \in (0,\infty)\colon \lim_{t\to\infty} |A_f(s,t) - A_f(\theta s,t)| > 0 \right\}.$$

Theorem 5 (Correlation ageing). *Let Assumption (F*) and Condition (B) be satisfied and* $\varphi \in C^2$ *be ultimately convex. Then the PAM ages for* $f(x) = x^p, p \in \mathbb{R}_+$ *in the sense of correlation ageing if and only if* $\lim\limits_{t\to\infty} H''(t) = 0$. *In this case* $\mathcal{A} \ni 1/\sqrt{H''(t)} = o(t)$.

Notice that for both definitions ageing happens for lighter tails. In Theorem 27 we show that Theorem 5 can be extended to more general potentials if we weaken the requirement (2.6).

Overview

In Section 2.2 we prove Theorem 1 which forms the basis of this chapter. In Section 2.3 we show how the conditional probability in Theorem 1 can be evaluated. After that we will give several applications. In Section 2.4 we apply Theorem 1 to prove Theorem 2 and to derive exact asymptotics for statistical moments and more general functionals of the PAM. In Section 2.5 we prove Theorem 3. We conclude how the intermittency peaks are distributed in space and give precise estimates on their frequency. In Section 2.6 we investigate the ageing behaviour of the PAM and prove Theorems 4, 5 and 27.

2.2 Time correlations

In this section we prove Theorem 1. The strategy of the proof is to show that asymptotically only those realisations of the potential ξ contribute to the expectation $\langle\prod_{i=1}^{p} f_i\left(u\left(t_i(t),0\right)\right)\rangle$, where the highest potential peak $\xi_R^{(1)}$ in the large centered box Q_R is significantly higher than the second one, and where $\xi_R^{(1)}$ is located in the origin. It turns out that for those realisations we can neglect all eigenpairs but the principal one in the spectral representation, and the first eigenfunction becomes delta like. We will see that it is sufficient to consider a large box with time independent size. The following universal bounds are always true (see for instance [GM90, Theorem 3.1] and [GM98, Proof of Theorem 2.16]).

Lemma 6. *Let $t \geq 0$ then for every $R > 1$ and $p \in \mathbb{N}$,*

i) $\lambda_1^{R,*}(\xi) \leq \xi_R^{(1)} \leq \lambda_1^{R,*}(\xi) + 2d\kappa$,

ii) $\mathrm{e}^{H(pt)-2d\kappa pt} \leq \langle u(t,0)^p\rangle \leq \mathrm{e}^{H(pt)}$.

Remark. The lower bound in Lemma 6 ii) can be proven by forcing the random walk X from the Feynman-Kac representation to stay in the origin up to time t. Hence, it remains true if we replace the power function by an arbitrary nonnegative function f. Then it reads $\left\langle f(\mathrm{e}^{t\xi(0)-2d\kappa t})\right\rangle \leq \left\langle f\big(u(t,0)\big)\right\rangle$.

Now we show that we can restrict our calculations to an increasing box $Q_{R_{\widehat{t}}}$ with zero boundary conditions where $R_t := t\log^2 t$ and $\widehat{t} := \max_{i=1,\dots,p} t_i$.
By $\tau_U := \inf\{t > 0\colon X_t \in U\}$ we denote the first hitting time of a set U by the random walk X. For $x \in \mathbb{Z}^d$ we write τ_x instead of $\tau_{\{x\}}$. Let u_R be the solution to the PAM in Q_R with Dirichlet boundary conditions. Its Feynman-Kac representation is given by

$$u_R\left(t,x\right) = \mathbb{E}_x \exp\left\{\int_0^t \xi(X_s)\,\mathrm{d}s\right\}\mathbb{1}_{\tau_{Q_R^{\mathrm{c}}}\geq t}, \qquad (t,x) \in [0,\infty)\times\mathbb{Z}^d.$$

Proposition 7. *Let ξ be i.i.d., nonnegative and unbounded from above. If $f_1,\dots,f_p \in \mathcal{F}$ and $t_1,\dots,t_p \in \mathcal{T}$, then*

$$\left\langle\prod_{i=1}^{p} f_i\big(u\left(t_i(t),0\right)\big)\right\rangle \sim \left\langle\prod_{i=1}^{p} f_i\left(u_{R_{\widehat{t}(t)}}\left(t_i(t),0\right)\right)\right\rangle, \qquad t\to\infty.$$

Proof. Let

$$\widetilde{u}_R\left(t,x\right) := u\left(t,x\right) - u_R\left(t,x\right) = \mathbb{E}_x \exp\left\{\int_0^t \xi(X_s)\,\mathrm{d}s\right\}\mathbb{1}_{\tau_{Q_R^{\mathrm{c}}}< t}, \quad (t,x) \in [0,\infty)\times\mathbb{Z}^d.$$

Then for every $\delta > 0$ we find that

$$\begin{aligned}
&\left\langle \prod_{i=1}^{p} f_i\left(u\left(t_i,0\right)\right)\right\rangle \\
= &\left\langle \prod_{i=1}^{p} f_i\left(u_{R_{\hat{t}}}\left(t_i,0\right)+\widetilde{u}_{R_{\hat{t}}}\left(t_i,0\right)\right)\left[\mathbb{1}_{\forall i:\, \widetilde{u}_{R_{\hat{t}}}(t_i,0)\le \delta u_{R_{\hat{t}}}(t_i,0)}+\mathbb{1}_{\exists i:\, \widetilde{u}_{R_{\hat{t}}}(t_i,0)> \delta u_{R_{\hat{t}}}(t_i,0)}\right]\right\rangle \\
\le &\sum_{T\in\mathcal{P}(p)}\left\langle \prod_{i\in T} f_i\left(u_{R_{\hat{t}}}\left(t_i,0\right)(1+\delta)\right)\prod_{j\in T^{\mathrm{c}}} f_j\left(\widetilde{u}_{R_{\hat{t}}}\left(t_j,0\right)\left(1+\frac{1}{\delta}\right)\right)\right\rangle . \qquad (2.7)
\end{aligned}$$

Here $\mathcal{P}(p)$ denotes the power set of $\{1,\ldots,p\}$ and T^{c} denotes the complement of T within $\{1,\ldots,p\}$. Since all f_i are regularly varying, it follows that for every $\theta > 1$ there exists $\delta = \delta(\theta)$ with $\lim\limits_{\theta\to 1}\delta(\theta) = 0$, and C_θ such that

$$\max_{i=1,\ldots,p}\frac{f_i\big((1+\delta)u\big)}{f_i\left(u\right)}\le\theta, \qquad u > C_\theta. \qquad (2.8)$$

Now choose $\theta > 1$ arbitrary and fix $\delta > 0$ such that (2.8) is satisfied.
Because all f_i are also increasing to infinity we get for large t,

$$\left\langle \prod_{i=1}^{p} f_i\left(u_{R_{\hat{t}}}\left(t_i,0\right)(1+\delta)\right)\right\rangle \le \theta\left\langle \prod_{i=1}^{p} f_i\left(u_{R_{\hat{t}}}\left(t_i,0\right)\right)\right\rangle + \prod_{i=1}^{p} f_i\big((1+\delta)C_\theta\big). \qquad (2.9)$$

Since almost surely $u_{R_{\hat{t}}}(t_i(t),0)\overset{t\to\infty}{\longrightarrow}\infty$ for all i, we can apply Fatou's lemma and see that the asymptotic behaviour of the right hand side of (2.9) is determined by

$$\theta\left\langle \prod_{i=1}^{p} f_i\left(u_{R_{\hat{t}}}\left(t_i,0\right)\right)\right\rangle .$$

By similar arguments we find that there exists C_u such that for sufficiently large t,

$$\begin{aligned}
&\left\langle \prod_{i\in T} f_i\left(u_{R_{\hat{t}}}\left(t_i,0\right)(1+\delta)\right)\prod_{j\in T^{\mathrm{c}}} f_j\left(\widetilde{u}_{R_{\hat{t}}}\left(t_i,0\right)\left(1+\frac{1}{\delta}\right)\right)\right\rangle \\
\le\ & C_u\left\langle \prod_{i\in T} f_i\left(u_{R_{\hat{t}}}\left(t_i,0\right)\right)\prod_{j\in T^{\mathrm{c}}} f_j\left(\widetilde{u}_{R_{\hat{t}}}\left(t_i,0\right)\right)\right\rangle .
\end{aligned}$$

In a next step we show that

$$\lim_{t\to\infty}\frac{\left\langle \prod\limits_{i\in T} f_i\left(u_{R_{\hat{t}}}\left(t_i,0\right)\right)\prod\limits_{j\in T^{\mathrm{c}}} f_j\left(\widetilde{u}_{R_{\hat{t}}}\left(t_i,0\right)\right)\right\rangle}{\left\langle \prod\limits_{i\in T} f_i\left(u(t_i,0)\right)\prod\limits_{j\in T^{\mathrm{c}}} f_j\left(u(t_j,0)\right)\right\rangle}=0. \qquad (2.10)$$

For simplicity we only look at $\lim_{t\to\infty} \langle f_1(\widetilde{u}(t,0))\rangle / \langle f_1(u(t,0))\rangle$, which may easily be generalised. Recall that f_1 is regularly varying, so it can be written as $f_1(x) = x^\beta L(x)$, for some $\beta > 0$ and some slowly varying function L. By forcing the random walk in the Feynman-Kac formula to stay in the origin up to time t we find that

$$\langle f_1(u(t,0))\rangle = \langle u(t,0)^\beta L(u(t,0))\rangle \geq \langle \exp\{\beta t\xi(0) - 2d\kappa\beta t\} L(u(t,0))\rangle .$$

Furthermore, together with [GM98, Lemma 2.5] we find that

$$\begin{aligned}
&\langle f_1(\widetilde{u}(t,0))\rangle \\
&\leq \left\langle \left(\exp\{t\xi^{(1)}_{Q_{R_t}}\}\mathbb{P}_0\left(\tau_{Q^c_{R_t}} \leq t\right)\right)^\beta L(\widetilde{u}(t,0)) \right\rangle \\
&= \left\langle \left(\sum_{x\in Q_{R_t}} \exp\{t\xi(x)\}\mathbb{1}_{\xi(x)=\xi^{(1)}_{Q_{R_t}}} \mathbb{P}_0\left(\tau_{Q^c_{R_t}} \leq t\right)\right)^\beta L(\widetilde{u}(t,0)) \right\rangle \\
&\leq \max(1, |Q_{R_t}|^{\beta-1}) \sum_{x\in Q_{R_t}} \left\langle \left(\exp\{t\xi(x)\} 2^{d+1} \exp\left\{ -R_t \log\frac{R_t}{\kappa dt} + nR_t \right\} \right)^\beta L(\widetilde{u}(t,0)) \right\rangle \\
&= 2^{(d+1)\beta} \exp\left\{ -\beta R_t \log\frac{R_t}{\kappa dt} + o\left(R_t \log\frac{R_t}{\kappa dt}\right)\right\} \langle \exp\{\beta t\xi(0)\} L(\widetilde{u}(t,0))\rangle .
\end{aligned}$$

In the third line we use that (due to Jensen's inequality for $\beta > 1$) for any real numbers $a_1, \ldots, a_n$, and $\beta > 0$,

$$\left(\sum_{k=1}^n a_k\right)^\beta \leq \begin{cases} n^{\beta-1} \sum_{k=1}^n |a_k|^\beta, & \text{if } \beta \geq 1, \\ \sum_{k=1}^n |a_k|^\beta, & \text{if } \beta \leq 1. \end{cases}$$

Altogether, this proves (2.10) since $\widetilde{u} \leq u$ for all t by definition and L is bounded away from zero and infinity or non-decreasing. Therefore, we can conclude that

$$\text{rhs of (2.7)} - \left\langle \prod_{i=1}^p f_i\left(u_{R_{\hat{t}}}(t_i,0)\right)\right\rangle = o\left(\left\langle \prod_{i=1}^p f_i\big(u(t_i,0)\big)\right\rangle\right), \qquad t\to\infty.$$

Now the claim follows because θ can be chosen arbitrarily close to 1 and because

$$\left\langle \prod_{i=1}^p f_i\left(u_{R_{\hat{t}}}(t_i,0)\right)\right\rangle \leq \left\langle \prod_{i=1}^p f_i\left(u(t_i,0)\right)\right\rangle, \qquad t \geq 0,$$

is true by the monotonicity and the nonnegativity of $f_1, \dots, f_p$, and because $u_{R_t} \le u$ for all t.

□

The next lemma allows us to consider only those realisations of the potential where the highest potential peak is significantly higher than the second one. Let $\xi^{(1)}_{R_{\hat t}} \ge \xi^{(2)}_{R_{\hat t}} \ge \dots$ be an order statistics of the potential.

Lemma 8. *For all $c > 0$ and $f_1, \dots, f_p \in \mathcal{F}$, $t_1, \dots, t_p \in \mathcal{T}$ satisfying* (2.1),

$$\left\langle \prod_{i=1}^p f_i\left(u^{\mathrm{f}}_{R_{\hat t}}(t_i,0)\right) \mathbb{1}_{\xi^{(1)}_{R_{\hat t}} - \xi^{(2)}_{R_{\hat t}} \le c} \right\rangle = o\Bigg(\left\langle \prod_{i=1}^p f_i\left(u^{\mathrm{f}}_{R_{\hat t}}(t_i,0)\right)\right\rangle\Bigg), \qquad t \to \infty.$$

Proof. Let $x_0, x_1 \in Q_{R_{\hat t}}$ be two arbitrarily chosen points. Then

$$\begin{aligned}
\left\langle \prod_{i=1}^p f_i\left(u^{\mathrm{f}}_{R_{\hat t}}(t_i,0)\right) \mathbb{1}_{\xi^{(1)}_{R_{\hat t}} - \xi^{(2)}_{R_{\hat t}} \le c} \right\rangle &\le \left\langle \prod_{i=1}^p f_i\left(\mathrm{e}^{t_i \xi^{(1)}_{R_{\hat t}}}\right) \mathbb{1}_{\xi^{(1)}_{R_{\hat t}} - \xi^{(2)}_{R_{\hat t}} \le c} \right\rangle \\
&\le \sum_{x \in Q_{R_{\hat t}}} \sum_{y \in Q_{R_{\hat t}} \setminus \{x\}} \left\langle \prod_{i=1}^p f_i\left(\mathrm{e}^{t_i \xi(x)}\right) \mathbb{1}_{\xi(x) > \xi(y) \ge \xi(x) - c} \right\rangle \\
&\le |Q_{R_{\hat t}}|^2 \left\langle \prod_{i=1}^p f_i\left(\mathrm{e}^{\frac{t_i}{2}(\xi(x_0)+\xi(x_1)+c)}\right)\right\rangle
\end{aligned}$$

Since all f_i are regularly varying, they can be written as $f_i(x) = x^{\beta_i} L_i(x)$, where $L_1, \dots, L_p$ are slowly varying functions and $\beta_1, \dots, \beta_i > 0$. Therefore, we find that

$$\begin{aligned}
&|Q_{R_{\hat t}}|^2 \left\langle \prod_{i=1}^p f_i\left(\mathrm{e}^{\frac{t_i}{2}(\xi(x_0)+\xi(x_1)+c)}\right)\right\rangle \\
= &\left\langle \prod_{i=1}^p \mathrm{e}^{\beta_i \frac{t_i}{2}(\xi(x_0)+\xi(x_1)+2c) + 2d\log(\hat t \log^2 \hat t)} \mathrm{e}^{-\beta_i \frac{t_i}{2} c} L\left(\mathrm{e}^{-\frac{t_i}{2}c} \mathrm{e}^{\frac{t_i}{2}(\xi(x_0)+\xi(x_1)+2c)}\right)\right\rangle \\
= &\left\langle \prod_{i=1}^p \underbrace{\mathrm{e}^{-\beta_i \frac{t_i}{2} c + 2d\log(\hat t\log^2 \hat t)} \frac{L_i\left(\mathrm{e}^{-\frac{t_i}{2}c} \mathrm{e}^{\frac{t_i}{2}(\xi(x_0)+\xi(x_1)+2c)}\right)}{L_i\left(\mathrm{e}^{\frac{t_i}{2}(\xi(x_0)+\xi(x_1)+2c)}\right)}}_{\overset{t\to\infty}{\longrightarrow} 0} f_i\left(\mathrm{e}^{\frac{t_i}{2}(\xi(x_0)+\xi(x_1)+2c)}\right)\right\rangle.
\end{aligned}$$

Assumption (F) states for $c > 2d\kappa$:

For all $\delta > 0$ it exists $h_0 = h_0(\delta) > 0$ such that for all $h > h_0$:

$$\mathbf{P}\left(\frac{\xi(x_0)+\xi(x_1)}{2} > h-c\right) \le \delta \mathbf{P}\left(\xi(0) > h+2d\kappa\right).$$

Furthermore, it follows with Lemma 6 ii) that

$$\left\langle \prod_{i=1}^{p} f_i\left(u^{\mathrm{f}}_{R_{\hat{t}}}(t_i,0)\right)\right\rangle \ge \int_0^\infty \prod_{i=1}^{p} f_i\left(\mathrm{e}^{t_i h}\right) \mathbf{P}\left(\xi(0) > h+2d\kappa\right) \mathrm{d}h,$$

and therefore, since all f_i are nonnegative and increasing, and because of (2.1),

$$\begin{aligned}
\frac{\left\langle \prod_{i=1}^{p} f_i\left(\mathrm{e}^{\frac{t_i}{2}(\xi(x_0)+\xi(x_1)+2c)}\right)\right\rangle}{\left\langle \prod_{i=1}^{p} f_i\left(u^{\mathrm{f}}_{R_{\hat{t}}}(t_i,0)\right)\right\rangle}
&\le \frac{\int_0^{h_0} \prod_{i=1}^{p} f_i\left(\mathrm{e}^{t_i h}\right) \mathbf{P}\left(\frac{\xi(x_0)+\xi(x_1)}{2} > h-c\right) \mathrm{d}h}{\left\langle \prod_{i=1}^{p} f_i\left(u^{\mathrm{f}}_{R_{\hat{t}}}(t_i,0)\right)\right\rangle} + \delta \\
&\le \frac{h_0 \prod_{i=1}^{p} f_i\left(\mathrm{e}^{t_i h_0}\right)}{\left\langle \prod_{i=1}^{p} f_i\left(u^{\mathrm{f}}_{R_{\hat{t}}}(t_i,0)\right)\right\rangle} + \delta \overset{t\to\infty,\delta\to 0}{\longrightarrow} 0.
\end{aligned}$$

□

Now we prove that the first eigenfunction $e_1^{R,*}$ decays at least exponentially fast. To this end we give probabilistic representations for $e_1^{R,*}$.

Lemma 9. *Let $\xi_R^{(1)} - \xi_R^{(2)} = c > 2d\kappa$ and $\xi_R^{(1)} = \xi(0)$. Then*

$$e_1^{R,*}(x) = e_0^* \mathbb{E}_x^{R,*} \exp\left\{\int_0^{\tau_0} \left(\xi\left(X_s\right) - \lambda_1^{R,*}(\xi)\right) \mathrm{d}s\right\}, \qquad x \in \mathbb{Z}^d,$$

where $e_0^* = e_0^*(R)$ is a normalising constant.

Proof. *i)* $* = \mathrm{f}$. The eigenvalue equation for $\lambda_1^{R,\mathrm{f}}(\xi)$ may be rewritten as

$$\begin{cases} \kappa\Delta e_1^{R,\mathrm{f}}(x) + (\xi(x) - \lambda_1^{R,\mathrm{f}}) e_1^{R,\mathrm{f}}(x) = 0, & x \in Q_R \setminus \{0\}, \\ e_1^{R,\mathrm{f}}(0) = e_0^{\mathrm{f}}. \end{cases}$$

Since we are working on a finite state space, we know that $\mathbb{E}_x^{R,\mathrm{f}}\tau_0 < \infty$ and because $c > 2d\kappa$ it also follows that $\xi(x) - \lambda_1^{R,\mathrm{f}}(\xi) < 0$ for all $x \in Q_R \setminus \{0\}$, and hence the Feynman-Kac representation of this boundary problem is given by

$$e_1^{R,\mathrm{f}}(x) = e_0^{\mathrm{f}} \mathbb{E}_x^{R,\mathrm{f}} \exp\left\{\int_0^{\tau_0} \left(\xi\left(X_s\right) - \lambda_1^{R,\mathrm{f}}(\xi)\right) \mathrm{d}s\right\}.$$

ii) $* = 0$. Analogously, the eigenvalue equation for $\lambda_1^{R,0}(\xi)$ may be rewritten as

$$\begin{cases} \kappa\Delta e_1^{R,0}(x) + (\xi(x) - \lambda_1^{R,0})e_1^{R,0}(x) = 0, & x \in Q_R \setminus \{0\}, \\ e_1^{R,0}(0) = e_0^0, \quad e_1^0(x) = 0, \quad x \notin Q_R. \end{cases}$$

Notice that

$$\mathbb{E}_x^{R,0} \exp\left\{ \int_0^{\tau_0} \left(\xi\left(X_s\right) - \lambda_1^{R,0}(\xi)\right) \mathrm{d}s \right\} = \mathbb{E}_x \exp\left\{ \int_0^{\tau_0} \left(\xi\left(X_s\right) - \lambda_1^{R,0}(\xi)\right) \mathrm{d}s \right\} \mathbb{1}_{\tau_0 < \tau_{(Q_R)^c}}.$$

Thus, it admits the desired probabilistic representation. □

Let

$$\sigma_1 := \inf\{t > 0 \colon X_t \neq X_0\}$$

be the time of the first jump of X. The stopping time σ_1 is exponentially distributed with parameter $2d\kappa$. Now we define recursively

$$\sigma_n := \inf\left\{t > 0 \colon X_{t+\sigma_{n-1}} \neq X_{\sigma_{n-1}}\right\}.$$

The sequence $\{\sigma_n, n \in \mathbb{N}\}$ is i.i.d. by the definition of X.

Lemma 10. *It exists $K = K(c) > 0$ such that if $\xi(0) - \widehat{\xi}_R > c > 2d\kappa$ then $e_1^{R,*}(x) \leq K\mathrm{e}^{-|x|\log\frac{c}{2d\kappa}}$ for all $x \in Q_R$.*

Proof. Using the probabilistic representation of $e_1^{R,*}$ from Lemma 9 and because of Lemma 6 we find that

$$\begin{aligned} \frac{e_1^{R,*}(x)}{e_0^*} &= \mathbb{E}_x^{R,*} \exp\left\{ \int_0^{\tau_0} \left(\xi(X_s) - \lambda_1^{R,*}(\xi)\right) \mathrm{d}s \right\} \\ &\leq \mathbb{E}_x^{R,*} \exp\left\{\tau_0(2d\kappa - c)\right\} \\ &= \sum_{n=1}^{\infty} \mathbb{E}_x^{R,*} \exp\left\{ (2d\kappa - c) \sum_{k=1}^{n} \sigma_k \right\} \mathbb{1}_{\tau_0 = \sum_{k=1}^{n} \sigma_k} \\ &\leq \sum_{n=|x|}^{\infty} \left(\frac{2d\kappa}{c}\right)^n = \frac{1}{1 - \frac{2d\kappa}{c}} \mathrm{e}^{-|x|\log\frac{c}{2d\kappa}}. \end{aligned}$$

□

Now we give two facts about regularly varying functions.

Lemma 11. *Let $f \in \mathcal{R}_\beta$ be an increasing function and $g \in C$ such that $\lim_{t\to\infty} g(t) = \infty$.*

i) If $\beta > 0$, then for any $c > 0$ there exists $C(c)$ with $\lim_{c\to\infty} C(c) = 1$ such that

$$\sum_{x\in\mathbb{Z}^d} f\left(\mathrm{e}^{g(t)-c|x|}\right) \sim C(c) f(\mathrm{e}^{g(t)}), \qquad t \to \infty.$$

ii) For every $h\colon \mathbb{R} \to \mathbb{R}$ with $\lim_{t\to\infty} h(t) = 1$: $\lim_{t\to\infty} \frac{f(\mathrm{e}^{g(t)})}{f\left(\mathrm{e}^{g(t)}h(t)\right)} = 1$.

Proof. *i*) It follows from the uniform convergence theorem for regularly varying functions (see [BGT87, Theorem 1.5.2]) that

$$\begin{aligned}\sum_{x\in\mathbb{Z}^d} f\left(\mathrm{e}^{g(t)-c|x|}\right) &= f\left(\mathrm{e}^{g(t)}\right) \sum_{x\in\mathbb{Z}^d} \frac{f\left(\mathrm{e}^{g(t)}\mathrm{e}^{-c|x|}\right)}{f\left(\mathrm{e}^{g(t)}\right)} \\ &\sim f\left(\mathrm{e}^{g(t)}\right) \sum_{x\in\mathbb{Z}^d} \mathrm{e}^{-\beta c|x|} = C(c) f(\mathrm{e}^{g(t)}), \qquad t \to \infty.\end{aligned}$$

ii) follows similarly. □

Next we show that we can neglect all eigenvalues but the principal one. To this end we cut off a time-independent centred inner box Q_R from the time-dependent box Q_{R_t} and put free boundary conditions on the inner as well as on the outer side of the boundary of Q_R. The principal eigenvalue of the outer box will be denoted by $\lambda_1^{R_t\setminus R,\mathrm{f}}$. Under these circumstances it holds almost surely:

$$\max\left(\lambda_1^{R,\mathrm{f}}, \lambda_1^{R_t\setminus R,\mathrm{f}}\right) \geq \lambda_1^{R_t,*} \geq \lambda_1^{R,0}. \tag{2.11}$$

The lower bound for $\lambda_1^{R_t,*}$ follows immediately from the Rayleigh-Ritz formula. For the upper bound see for instance [K07, Chapter 5.2]. Recall that $\lambda_1^{R,*} > \lambda_2^{R,*} \geq \cdots \geq \lambda_{|Q_R|}^{R,*}$ are the eigenvalues of $\mathscr{H}_R^*$, and that the solution u_R^* admits the following spectral representation

$$u_R^*(t,x) = \sum_{k=1}^{|Q_R|} \mathrm{e}^{\lambda_k^{R,*} t} \left(e_k^{R,*}, \mathbb{1}\right) e_k^{R,*}(x), \tag{2.12}$$

where $e_1^{R,*}, e_2^{R,*}, \cdots, e_{|Q_R|}^{R,*}$ is a corresponding orthonormal basis of ℓ^2 eigenfunctions.

Proposition 12. *Let* $f_1, \cdots, f_p \in \mathcal{F}$, $t_1, \cdots, t_p \in \mathcal{T}$, (2.1) *be satisfied, and fix* $R > 0$. *Then as* $t \to \infty$,

$$\begin{aligned} & \frac{1+o(1)}{|Q_R|} \Big\langle \prod_{i=1}^{p} f_i \left(\mathrm{e}^{\lambda_1^{R,0} t_i(t)} \right) \Big| \xi_R^{(1)} = \xi(0) \Big\rangle \\ \leq \; & \Big\langle \prod_{i=1}^{p} f_i \left(u_{R_{\hat{t}(t)}}(t_i(t), 0) \right) \Big\rangle \\ \leq \; & \frac{1+o(1)}{|Q_{R_{\hat{t}(t)}}|} \Big\langle \prod_{i=1}^{p} f_i \left(\mathrm{e}^{\lambda_1^{R,\mathrm{f}} t_i(t)} \right) \Big| \xi_{R_t}^{(1)} = \xi(0) \Big\rangle. \end{aligned}$$

Proof. We will restrict to the case $p = 1$ and write $\lambda_k^{R_t}$ instead of $\lambda_k^{R_t,0}$ to keep the notation simple.
Upper bound.
It follows by Lemma 8 that for every $c > 0$,

$$\big\langle f_1\big(u_{R_t}(t,0)\big)\big\rangle = \Big\langle f_1\big(u_{R_t}(t,0)\big)\mathbb{1}_{\xi_{R_t}^{(1)} - \xi_{R_t}^{(2)} > c}\Big\rangle + o\big(\big\langle f_1\big(u_{R_t}(t,0)\big)\big\rangle\big), \qquad t \to \infty.$$

Using the spectral representation (2.12) of u_{R_t}, we find that there exists $C(c) \in [1, \infty)$ such that as t tends to infinity

$$\begin{aligned} & \Big\langle f_1\big(u_{R_t}(t,0)\big)\mathbb{1}_{\xi_{R_t}^{(1)} - \xi_{R_t}^{(2)} > c}\Big\rangle \\ = \; & \sum_{x \in Q_{R_t}} \Big\langle f_1\Big(\sum_{k=1}^{|Q_{R_t}|} \mathrm{e}^{\lambda_k^{R_t} t} (e_k, \mathbb{1}) e_1(0) \Big) \mathbb{1}_{\xi_{R_t}^{(1)} - \xi_{R_t}^{(2)} > c} \mathbb{1}_{\xi_{R_t}^{(1)} = \xi(x)} \Big\rangle \\ = \; & \frac{1}{|Q_{R_t}|} \cdot \\ & \sum_{x \in Q_{R_t}} \Big\langle f_1\Big(\mathrm{e}^{\lambda_1^{R_t} t} \Big((e_1, \mathbb{1}) e_1(0) + \sum_{k=2}^{|Q_{R_t}|} \mathrm{e}^{(\lambda_k^{R_t} - \lambda_1^{R_t}) t} (e_k, \mathbb{1}) e_k(0) \Big) \Big) \mathbb{1}_{\xi_{R_t}^{(1)} - \xi_{R_t}^{(2)} > c} \Big| \xi_{R_t}^{(1)} = \xi(x) \Big\rangle \\ = \; & \frac{1+o(1)}{|Q_{R_t}|} C(c) \Big\langle f_1 \left(\mathrm{e}^{\lambda_1^{R_t} t} \right) \mathbb{1}_{\xi_{R_t}^{(1)} - \xi_{R_t}^{(2)} > c} \Big| \xi_{R_t}^{(1)} = \xi(0) \Big\rangle \\ \leq \; & \frac{1+o(1)}{|Q_{R_t}|} C(c) \Big\langle f_1 \left(\mathrm{e}^{\lambda_1^{R,\mathrm{f}} t} \right) \Big| \xi_{R_t}^{(1)} = \xi(0) \Big\rangle. \end{aligned}$$

Here we use that f_1 is increasing, regularly varying (see Lemma 11), and $f_1(0) = 0$. The last inequality follows from the upper bound in (2.11). The expression $(e_k, \mathbb{1}) e_1(0)$

becomes delta-like as c tends to infinity due to Lemma 10. Notice that by the Cauchy-Schwarz inequality and Parseval's identity,

$$\sum_{k=2}^{|Q_{R_t}|} \mathrm{e}^{(\lambda_k^{R_t}-\lambda_1^{R_t})t}(e_k,\mathbb{1})e_k(0) \le \mathrm{e}^{-ct}|Q_{R_t}|^2 = \mathrm{e}^{-ct+4d\log(t\log^2 t)}.$$

It follows by Lemma 11 i) that $C(c) \overset{c\to\infty}{\longrightarrow} 1$.
Lower bound.
Similarly as for the upper bound we find asymptotically

$$\begin{aligned}
\langle f_1\left(u_{R_t}(t,0)\right)\rangle &\ge \left\langle f_1\left(u_{R_t}(t)\mathbb{1}_{\xi_R^{(1)}=\xi(0)}\right)\right\rangle\\
&= \frac{1}{|Q_R|}\left\langle f_1\left(\sum_{k=1}^{|Q_{R_t}|}\mathrm{e}^{\lambda_k^{R_t}t}(e_k,\mathbb{1})e_k(0)\right)\middle|\xi_R^{(1)}=\xi(0)\right\rangle\\
&\ge \frac{1+o(1)}{|Q_R|}\left\langle f_1\left(\mathrm{e}^{\lambda_1^{R,0}t}\right)\middle|\xi_R^{(1)}=\xi(0)\right\rangle.
\end{aligned}$$

The last inequality follows by the lower bound in (2.11) and arguments similar to those of the upper bound. Notice that $(e_1,\mathbb{1})e_1(0)\ge(e_1,e_1)=1$ if $\xi_{R_t}^{(1)}=\xi(0)$. □

Recall that $\widehat{\xi}_R = \max_{x\in Q_R\setminus\{0\}}\xi(x)$.

Lemma 13. *Let $f_1,\dots,f_p\in\mathcal{F}$, $t_1,\dots,t_p\in\mathcal{T}$ and (2.1) be satisfied. Then for $R>0$ and $c\in(2d\kappa,\infty)$, as $t\to\infty$,*

$$\begin{aligned}
&\left\langle\prod_{i=1}^p f_i\left(\mathrm{e}^{\lambda_1^{R,\mathrm{f}}t_i}\right)\middle|\xi_{R_{\hat t}}^{(1)}=\xi(0)\right\rangle\\
\sim&|Q_{R_{\hat t}}|\int_c^\infty\left[\frac{\mathrm{d}}{\mathrm{d}h}\prod_{i=1}^p f_i\left(\mathrm{e}^{t_i(t)h}\right)\right]\mathbf{P}\left(\lambda_1^{R,\mathrm{f}}(\xi)>h\middle|\widehat{\xi}_R\le h-c\right)\mathrm{d}h.
\end{aligned}$$

Proof. Let $F_{\lambda_1^{R,\mathrm{f}}}(h)=\mathbf{P}(\lambda_1^{R,\mathrm{f}}\le h|\xi_{R_t}^{(1)}=\xi(0))$. Integration by parts yields

$$\begin{aligned}
&\left\langle\prod_{i=1}^p f_i\left(\mathrm{e}^{\lambda_1^{R,\mathrm{f}}t_i}\right)\middle|\xi_{R_{\hat t}}^{(1)}=\xi(0)\right\rangle = \int_0^\infty\prod_{i=1}^p f_i\left(\mathrm{e}^{ht_i}\right)\mathrm{d}F_{\lambda_1^{R,\mathrm{f}}}(h)\\
=&\prod_{i=1}^p f_i(1)+\int_0^\infty\left[\frac{\mathrm{d}}{\mathrm{d}h}\prod_{i=1}^p f_i\left(\mathrm{e}^{t_ih}\right)\right]\mathbf{P}\left(\lambda_1^{R,\mathrm{f}}>h\middle|\xi_{R_{\hat t}}^{(1)}=\xi(0)\right)\mathrm{d}h.
\end{aligned}$$

For $c > 2d\kappa$, it follows by Lemma 6 and the law of total probability

$$\begin{aligned}\mathbf{P}\left(\lambda_1^{R,\mathrm{f}} > h \middle| \xi^{(1)}_{R_{\widehat{t}}} = \xi(0)\right) =& |Q_{R_{\widehat{t}}}| \Big[\mathbf{P}\left(\lambda_1^{R,\mathrm{f}} > h \middle| \widehat{\xi}_R \le h - c\right)\mathbf{P}\left(\widehat{\xi}_R \le h-c\right) \\ &+ \mathbf{P}\left(\lambda_1^{R,\mathrm{f}} > h, \xi^{(1)}_{R_{\widehat{t}}} = \xi(0) \middle| \widehat{\xi}_R > h - c\right)\mathbf{P}\left(\widehat{\xi}_R > h-c\right)\Big].\end{aligned}$$

Furthermore, elementary calculus yields

$$\frac{\mathrm{d}}{\mathrm{d}h}\prod_{i=1}^p f_i\left(\mathrm{e}^{t_i(t)h}\right) = \sum_{i=1}^p t_i(t)t_i'(t)e^{t_i(t)h} f_i'\left(e^{t_i(t)h}\right)\prod_{\substack{j=1\\ j\neq i}}^p f_j\left(e^{t_j(t)h}\right).$$

It remains to show

$$\begin{aligned}&\int_c^\infty \mathrm{e}^{t_1 h} f_1'\left(\mathrm{e}^{t_1 h}\right)\prod_{j=2}^p f_j\left(\mathrm{e}^{t_j h}\right)\mathbf{P}\left(\lambda_1^{R,\mathrm{f}} > h \middle| \widehat{\xi}_R \le h-c\right)\mathbf{P}\left(\widehat{\xi}_R \le h - c\right)\,\mathrm{d}h \\ \sim\ & \int_c^\infty \mathrm{e}^{t_1 h} f_1'\left(\mathrm{e}^{t_1 h}\right)\prod_{j=2}^p f_j\left(\mathrm{e}^{t_j h}\right)\mathbf{P}\left(\lambda_1^{R,\mathrm{f}} > h \middle| \widehat{\xi}_R \le h-c\right)\,\mathrm{d}h, \qquad t\to\infty, \qquad (2.13)\end{aligned}$$

and

$$\begin{aligned}&\int_0^\infty \mathrm{e}^{t_1 h} f_1'\left(\mathrm{e}^{t_1 h}\right)\prod_{j=2}^p f_j\left(\mathrm{e}^{t_j h}\right)\mathbf{P}\left(\lambda_1^{R,\mathrm{f}} > h, \xi^{(1)}_{R_{\widehat{t}}} = \xi(0) \middle| \widehat{\xi}_R > h-c\right)\mathbf{P}\left(\widehat{\xi}_R > h - c\right)\,\mathrm{d}h \\ =\ & o\left(\int_0^\infty \mathrm{e}^{t_1 h} f_1'\left(\mathrm{e}^{t_1 h}\right)\prod_{j=2}^p f_j\left(\mathrm{e}^{t_j h}\right)\mathbf{P}\left(\lambda_1^{R,\mathrm{f}} > h\right)\,\mathrm{d}h\right), \qquad t\to\infty. \qquad (2.14)\end{aligned}$$

First we show (2.14).
Assumption (F) implies that for $c > 2d\kappa$:

For all $\delta > 0$ it exists $h_0 = h_0(\delta) > 0$ such that for all $h > h_0$:

$$\mathbf{P}\left(\xi(0) > h - c\right)^2 \le \delta \mathbf{P}\left(\xi(0) > h + 2d\kappa\right).$$

It follows with (2.1) and Lemma 6,

$$\begin{aligned}
&\frac{\text{lhs of (2.14)}}{\int\limits_0^\infty \mathrm{e}^{t_1 h} f_1'\left(\mathrm{e}^{t_1 h}\right) \prod\limits_{j=2}^p f_j\left(\mathrm{e}^{t_j h}\right) \mathbf{P}\left(\lambda_1^{R,\mathrm{f}} > h\right) \,\mathrm{d}h} \\
\leq\ & |Q_R| \left(\frac{\int\limits_0^{h_0} \mathrm{e}^{t_1 h} f_1'\left(\mathrm{e}^{t_1 h}\right) \prod\limits_{j=2}^p f_j\left(\mathrm{e}^{t_j h}\right) \mathbf{P}\left(\xi(0) > h - c\right)^2 \,\mathrm{d}h}{\int\limits_0^\infty \mathrm{e}^{t_1 h} f_1'\left(\mathrm{e}^{t_1 h}\right) \prod\limits_{j=2}^p f_j\left(\mathrm{e}^{t_j h}\right) \mathbf{P}\left(\xi(0) > h + 2d\kappa\right) \,\mathrm{d}h} + \delta \right) \\
\leq\ & |Q_R| \left(\frac{\prod\limits_{j=1}^p f_j\left(\mathrm{e}^{t_j h_0}\right) \mathrm{e}^{t_1 h_0}}{\prod\limits_{j=2}^p f_j(1) \left\langle f_1\left(\mathrm{e}^{t_1 \xi}\right)\right\rangle} + \delta \right) \overset{t\to\infty,\delta\to 0}{\longrightarrow} 0.
\end{aligned}$$

Now we show (2.13).
Since $\mathbf{P}(\widehat{\xi}_R \leq h - c) = (F(h-c))^{|Q_R|-1} \overset{h\to\infty}{\longrightarrow} 1$ for fixed R we find for every $\vartheta \in (0,1)$ an $a = a(R, c, \vartheta)$ such that

$$\begin{aligned}
&\int\limits_c^\infty \mathrm{e}^{t_1 h} f_1'\left(\mathrm{e}^{t_1 h}\right) \prod_{j=2}^p f_j\left(\mathrm{e}^{t_j h}\right) \mathbf{P}\left(\lambda_1^{R,\mathrm{f}} > h \middle| \widehat{\xi}_R \leq h - c\right) \mathbf{P}\left(\widehat{\xi}_R \leq h - c\right) \,\mathrm{d}h \\
\geq\ & \vartheta \int\limits_a^\infty \mathrm{e}^{t_1 h} f_1'\left(\mathrm{e}^{t_1 h}\right) \prod_{j=2}^p f_j\left(\mathrm{e}^{t_j h}\right) \mathbf{P}\left(\lambda_1^{R,\mathrm{f}} > h \middle| \widehat{\xi}_R \leq h - c\right) \,\mathrm{d}h.
\end{aligned}$$

Therefore, it is sufficient to show for every $a \geq 0$,

$$\frac{\int\limits_c^a \mathrm{e}^{t_1 h} f_1'\left(\mathrm{e}^{t_1 h}\right) \prod\limits_{j=2}^p f_j\left(\mathrm{e}^{t_j h}\right) \mathbf{P}\left(\lambda_1^{R,\mathrm{f}} > h \middle| \widehat{\xi}_R \leq h - c\right) \,\mathrm{d}h}{\int\limits_c^\infty \mathrm{e}^{t_1 h} f_1'\left(\mathrm{e}^{t_1 h}\right) \prod\limits_{j=2}^p f_j\left(\mathrm{e}^{t_j h}\right) \mathbf{P}\left(\lambda_1^{R,\mathrm{f}} > h\right) \,\mathrm{d}h} \overset{t\to\infty}{\longrightarrow} 0. \tag{2.15}$$

Using the bounds from Lemma 6 for $\lambda_1^{*,R}$ and (2.1), we find

$$\text{lhs of (2.15)} \ \leq \exp\left\{t(a + 4d\kappa) + \log t - H(t)\right\} \overset{t\to\infty}{\longrightarrow} 0. \tag{2.16}$$

In the last line we use that $\lim_{t\to\infty} H(t)/t = \infty$. □

Lemma 14. *Let $f_1, \dots, f_p \in \mathcal{F}$, $t_1, \dots, t_p \in \mathcal{T}$ and (2.1) be satisfied. Then for $R > 0$ and $c \in (0, \infty)$, as $t \to \infty$,*

$$\begin{aligned}
&\left\langle \prod_{i=1}^{p} f_i \left(\mathrm{e}^{\lambda_1^{0,R} t_i}\right) \middle| \xi_R^{(1)} = \xi(0) \right\rangle \\
\geq &\frac{|Q_R|}{\big(1+o(1)\big)} \int_c^\infty \left[\frac{\mathrm{d}}{\mathrm{d}h} \prod_{i=1}^{p} f_i \left(\mathrm{e}^{t_i h}\right)\right] \mathbf{P}\left(\lambda_1^{0,R}(\xi) > h \middle| \widehat{\xi}_R \leq h - c\right) \mathrm{d}h.
\end{aligned}$$

Proof. Analog to the proof of Lemma 13 we find

$$\begin{aligned}
&\left\langle \prod_{i=1}^{p} f_i \left(\mathrm{e}^{\lambda_1^{0,R} t_i}\right) \middle| \xi_R^{(1)} = \xi(0) \right\rangle \\
= &\prod_{i=1}^{p} f_i(1) + \int_c^\infty \left[\frac{\mathrm{d}}{\mathrm{d}h} \prod_{i=1}^{p} f_i \left(\mathrm{e}^{t_i h}\right)\right] \mathbf{P}\left(\lambda_1^{0,R} > h \middle| \xi_R^{(1)} = \xi(0)\right) \mathrm{d}h.
\end{aligned}$$

Now the claim follows because

$$\begin{aligned}
\mathbf{P}\left(\lambda_1^{0,R} > h \middle| \xi_R^{(1)} = \xi(0)\right) &\geq \mathbf{P}\left(\lambda_1^{0,R} > h \middle| \widehat{\xi}_R \leq h - c\right) \frac{\mathbf{P}\left(\widehat{\xi}_R \leq h - c\right)}{\mathbf{P}\left(\xi_R^{(1)} = \xi(0)\right)} \\
&\sim |Q_R| \mathbf{P}\left(\lambda_1^{0,R} > h \middle| \widehat{\xi}_R \leq h - c\right), \qquad h \to \infty.
\end{aligned}$$

The last asymptotics are proven in the same way as in (2.13). □

Proof of Theorem 1. The upper bound is an immediate consequence of Propositions 7 and 12 and Lemma 13, and the lower bound follows from Propositions 7 and 12 and Lemma 14. □

Remark. If we only consider integer moments (i.e. $f_i(x) = x^p, p \in \mathbb{N}$), then the proofs can be simplified and Assumption (F+) below suffices in Theorem 1 because we can use periodic boundary conditions for the upper bound due to [GM98, Lemma 1.4].

<u>Assumption (F+)</u>:

i) $\left(\bar{F}(h-c)\right)^2 = o\left(\bar{F}(h)\right), \qquad h \to \infty, \qquad$ for all $c > 0$.

ii) $H(t) < \infty$ for all $t \geq 0$.

Remark. Theorem 1 also holds true if we consider the initial condition $u_0 = \delta_0$ which implies that in this case for all $f \in \mathcal{R}_+$,

$$\sum_{x\in\mathbb{Z}^d\setminus\{0\}} \langle f(u(t,x))\rangle = o\big(\big\langle f\big(u(t,0)\big)\big\rangle\big), \qquad t\to\infty.$$

2.3 The conditional probability

In this section we investigate how to calculate $\mathbf{P}\big(\lambda_1^{*,R}(\xi) > h \big| \widehat{\xi}_R \le h-c\big)$. Let

$$E_R^*(h) := \kappa \sum_{|y|_1=1} \mathbb{E}_y^{R,*} \exp\left\{\int_0^{\tau_0} (\xi\,(X_s)-h)\ \mathrm{d}s\right\}.$$

Lemma 15. *Let $h \ge c > 2d\kappa$. Then*

$$\mathbf{P}\left(\lambda_1^{R,\mathrm{f}}(\xi) > h \middle| \widehat{\xi}_R \le h-c\right) = \left\langle \bar{F}\left(h+2d\kappa - E_R^*(h)\right)\middle|\widehat{\xi}_R \le h-c\right\rangle.$$

Proof. Using the probabilistic representation of $e_1^{R,*}$ from Lemma 9, we find that

$$\xi(0) = \lambda_1^{R,*}(\xi) + 2d\kappa - E_R^*\left(\lambda_1^{R,*}(\xi)\right),$$

whose right hand side is strictly increasing in $\lambda_1^{R,\mathrm{f}}$. Hence,

$$\lambda_1^{R,*}(\xi) > h \quad\Longleftrightarrow\quad \xi(0) > h + 2d\kappa - E_R^*(h).$$

Now the statement follows immediately. □

Lemma 16. *If $R>0$, $|y|_1 = 1$, $n\in\mathbb{N}_0$ and $h > \widehat{\xi}_R - 2d\kappa$, then*

$$\begin{aligned}
&\sum_{k=0}^{R} n!\binom{n+2k}{n}\mathbb{P}_y\left(\tau_0 = \sum_{l=1}^{2k+1}\sigma_l\right)\frac{(2d\kappa)^{2k+1}}{(h+2d\kappa)^{n+2k+1}}\\
\le\ & \frac{\mathrm{d}^n}{\mathrm{d}h^n}\mathbb{E}_y^{R,*}\exp\left\{\int_0^{\tau_0}(\xi(X_s)-h)\ \mathrm{d}s\right\}\\
\le\ & \sum_{k=0}^{\infty} n!\binom{n+2k}{n}\mathbb{P}_y\left(\tau_0 = \sum_{l=1}^{k}\sigma_l\right)\frac{(2d\kappa)^{2k+1}}{(h+2d\kappa-\widehat{\xi}_R)^{n+2k+1}}.
\end{aligned}$$

Proof. Let $Y_m := X_{\sum_{i=1}^{m} \sigma_i}$ be the embedded discrete time random walk. We get

$$\begin{aligned}
&\frac{\mathrm{d}^n}{\mathrm{d}h^n}\mathbb{E}_y \exp\left\{\int_0^{\tau_0} (\xi(X_s)-h)\ \mathrm{d}s\right\} = \mathbb{E}_y \exp\left\{(-\tau_0)^n \int_0^{\tau_0} (\xi(X_s)-h)\ \mathrm{d}s\right\} \\
= &\sum_{k=0}^{\infty} \mathbb{E}_y \left(-\sum_{l=1}^{2k+1} \sigma_l\right)^n \exp\left\{\sum_{l=1}^{2k+1} \sigma_l \left(\xi(Y_{l-1})-h\right)\right\} \mathbb{1}_{\tau_0 = \sum_{l=1}^{2k+1} \sigma_l}.
\end{aligned} \tag{2.17}$$

Since $\sum_{l=1}^{2k+1} \sigma_l \sim$ Erlang $(2k+1, 2d\kappa)$, we find for $z \in \{0, \widehat{\xi}_R\}$,

$$\begin{aligned}
\mathbb{E}_y \mathrm{e}^{(z-h)\sum_{l=1}^{k}\sigma_l} \left(-\sum_{l=1}^{k} \sigma_l\right)^n &= \frac{(2d\kappa)^k}{(k-1)!} \int_0^{\infty} \mathrm{e}^{(z-h-2d\kappa)x} x^{n+k-1}\,\mathrm{d}x \\
&= \frac{(n+k-1)!}{(k-1)!} \frac{(2d\kappa)^k}{(h+2d\kappa-z)^{n+k}}.
\end{aligned}$$

Now the claim follows because $0 \le \xi(Y_{l-1}) \le \widehat{\xi}_R$ for all $Y_{l-1} \ne 0$ and because jumptimes and jumps are independent. □

Lemma 17. *Let $\alpha < 1$ and fix $R > 0$. Then there exists $0 < c_1, c_2 < \infty$ such that for any large h,*

i) $$\begin{aligned}
&\bar{F}\left(h + 2d\kappa - \frac{c_2}{h}\right) \\
\le &\mathbf{P}\left(\lambda_1^{R,*}(\xi) > h, \widehat{\xi}_R < h^\alpha \middle| \widehat{\xi}_R \le h - c\right) \\
\le &\bar{F}\left(h + 2d\kappa - \frac{c_1}{h}\right),
\end{aligned}$$

ii) $$\begin{aligned}
&\bar{F}'\left(h + 2d\kappa - \frac{c_1}{h}\right)\left(1 - \frac{c_1}{(h+2d\kappa)^2}\right) \\
\le &\frac{\mathrm{d}}{\mathrm{d}h}\mathbf{P}\left(\lambda_1^{R,*}(\xi) > h, \widehat{\xi}_R < h^\alpha \middle| \widehat{\xi}_R \le h - c\right) \\
\le &\bar{F}'\left(h + 2d\kappa - \frac{c_2}{h}\right)\left(1 - \frac{c_2}{(h+2d\kappa)^2}\right),
\end{aligned}$$

iii) $$\begin{aligned}
&\bar{F}''\left(h + 2d\kappa - \frac{c_2}{h}\right)\left(1 - \frac{c_1}{(h+2d\kappa)^2}\right)^2 + \bar{F}'\left(h + 2d\kappa - \frac{c_1}{h}\right)\frac{c_1}{(h+2d\kappa)^3} \\
\le &\frac{\mathrm{d}^2}{\mathrm{d}h^2}\mathbf{P}\left(\lambda_1^{R,*}(\xi) > h, \widehat{\xi}_R < h^\alpha \middle| \widehat{\xi}_R \le h - c\right) \\
\le &\bar{F}''\left(h + 2d\kappa - \frac{c_1}{h}\right)\left(1 - \frac{c_2}{(h+2d\kappa)^2}\right)^2 + \bar{F}'\left(h + 2d\kappa - \frac{c_2}{h}\right)\frac{c_2}{(h+2d\kappa)^3}.
\end{aligned}$$

Proof. By Lemma 16 we find $0 < c_1 \leq c_2 < \infty$ such that for $|y|_1 = 1$, any large h, and $n \in \mathbb{N}$,

$$\begin{aligned} & \frac{c_1}{(h+2d\kappa)^{n+1}} \\ \leq \quad & \frac{\mathrm{d}^n}{\mathrm{d}h^n} \mathbb{E}_y^{R,*} \exp\left\{ \int_0^{\tau_0} (\xi(X_s) - h) \, \mathrm{d}s \right\} \\ \leq \quad & \frac{c_1}{(h+2d\kappa-\widehat{\xi}_R)^{n+1}} \leq \frac{c_2}{(h+2d\kappa-h^\alpha)^{n+1}} \leq \frac{c_2}{(h+2d\kappa)^{n+1}}. \end{aligned} \tag{2.18}$$

For large h, Lemma 15 yields

$$\mathbf{P}\left(\lambda_1^{R,*}(\xi) > h, \widehat{\xi}_R < h^\alpha \middle| \widehat{\xi}_R \leq h - c\right) = \left\langle \bar{F}\left(h + 2d - E_R^*(h)\right) \middle| \widehat{\xi}_R \leq h^\alpha \right\rangle.$$

The differentiation lemma guarantees that we can differentiate under the integral.

i) follows by substituting the deterministic estimates (2.18) in

$$\left\langle \bar{F}\left(h + 2d\kappa - E_R^*(h)\right) \middle| \widehat{\xi}_R \leq h^\alpha \right\rangle.$$

ii) holds because

$$\begin{aligned} & \frac{\mathrm{d}}{\mathrm{d}h} \left\langle \bar{F}\big(h + 2d\kappa - E_R^*(h)\big) \middle| \widehat{\xi}_R \leq h^\alpha \right\rangle \\ = \quad & \left\langle \bar{F}'\left(h + 2d\kappa - E_R^*(h)\right)\left(1 - (E_R^*)'(h)\right) \middle| \widehat{\xi}_R \leq h^\alpha \right\rangle. \end{aligned}$$

Now the claim follows by substituting the deterministic estimates (2.18).

iii) holds because

$$\begin{aligned} & \frac{\mathrm{d}^2}{\mathrm{d}h^2} \left\langle \bar{F}\big(h + 2d\kappa - E_R^*(h)\big) \middle| \widehat{\xi}_R \leq h^\alpha \right\rangle \\ = \quad & \Big\langle \bar{F}'\big(h + 2d\kappa - E_R^*(h)\big) \big(1 - (E_R^*)'(h)^2\big)^2 \\ & + (E_R^*)''(h) \bar{F}'\big(h + 2d\kappa - E_R^*(h)\big) \Big| \widehat{\xi}_R \leq h^\alpha \Big\rangle. \end{aligned}$$

Now the claim follows by substituting the deterministic estimates (2.18).

$\square$

2.4 Exact moment asymptotics

In this section we apply Theorem 1 to compute exact moment asymptotics via Theorem 2. To do so we need to impose Assumption (F*) which is a slightly stronger condition than Assumption (F).

Lemma 18. *Let Assumption (F*) be satisfied. Then there exists $\alpha \in (0,1)$ such that for any $R > 0$ and $c > 2d\kappa$,*

$$\frac{\left\langle \bar{F}\Big(h + 2d\kappa - E_R^*(h)\Big)\mathbb{1}_{\widehat{\xi}_R > h^\alpha}\Big|\widehat{\xi}_R \leq h - c\right\rangle}{\left\langle \bar{F}\Big(h + 2d\kappa - E_R^*(h)\Big)\mathbb{1}_{\widehat{\xi}_R \leq h^\alpha}\Big|\widehat{\xi}_R \leq h - c\right\rangle} \xrightarrow{h\to\infty} 0. \tag{2.19}$$

Proof. Asymptotically, we find

$$\text{lhs of (2.19)} \;\leq\; \frac{\bar{F}(h)\mathbf{P}\left(\widehat{\xi}_R > h^\alpha\right)}{\bar{F}(h+2d\kappa)\mathbf{P}\left(\widehat{\xi}_R \leq h^\alpha\right)} = \frac{\bar{F}(h)\bar{F}(h^\alpha)}{\bar{F}(h+2d\kappa)} \underbrace{\frac{(|Q_R|-1)}{(1-\mathrm{e}^{-h^{\alpha\gamma}})^{|Q_R|-1}}}_{\xrightarrow{h\to\infty} 1} \xrightarrow{h\to\infty} 0.$$

□

In the remainder of this section we will give explicit results for the case that the tails of $\xi(0)$ have a Weibull distribution (i.e. $\bar{F}(h) = \exp\{-h^\gamma\}$) with parameter $\gamma > 1$. One easily checks that Weibull tails satisfy Assumption (F*) for $\alpha > (\gamma-1)/\gamma$.

Lemma 19. *Let $\xi \sim$ Weibull (γ). Then for any $R > 0$ and $c > 2d\kappa$,*

$$\begin{aligned}
&\mathbf{P}\left(\lambda_1^{R,\mathrm{f}}(\xi) > h\Big|\widehat{\xi}_R \leq h - c\right)\\
=\;& \exp\left\{-(h+2d\kappa)^\gamma + 2d\kappa^2\gamma(h+2d\kappa)^{\gamma-2} + \mathcal{O}\left((h+2d\kappa)^{\gamma-3}\right)\right\}, \qquad h\to\infty.
\end{aligned}$$

Proof. It follows from Lemmas 15, 16 and 18 and because the lower and the upper bound in Lemma 16 are asymptotically equivalent on a exponential scale that for $\alpha > (\gamma-1)/\gamma$,

$$\begin{aligned}
&\mathbf{P}\left(\lambda_1^{R,\mathrm{f}}(\xi) > h\Big|\widehat{\xi}_R \leq h - c\right)\\
\sim\;& \left\langle \bar{F}\Big(h + 2d\kappa - E_R^*(h)\Big)\mathbb{1}_{\widehat{\xi}_R \leq h^\alpha}\Big|\widehat{\xi}_R \leq h - c\right\rangle\\
=\;& \exp\left\{-(h+2d\kappa)^\gamma + 2d\kappa^2\gamma(h+2d\kappa)^{\gamma-2} + \mathcal{O}\left((h+2d\kappa)^{\gamma-3}\right)\right\}, \qquad h\to\infty.
\end{aligned}$$

In the last line we use that

$$\begin{aligned}
&\left(h + 2d\kappa - \kappa \sum_{|y|_1=1} \mathbb{E}_y^{R,\mathrm{f}} \exp\left\{ \int_0^{\tau_0} \left(\xi\left(X_s\right) - h\right) \, \mathrm{d}s \right\}\right)^\gamma \\
= \quad & (h+2d\kappa)^\gamma - \gamma\,(h+2d\kappa)^{\gamma-2} \sum_{|y|_1=1} \frac{h+2d\kappa}{h+2d\kappa-\xi(y)} + \mathcal{O}\big((h+2d\kappa)^{\gamma-4}\big)
\end{aligned}$$

and

$$\frac{h+2d\kappa}{h+2d\kappa-\xi(y)} = \sum_{n=0}^{\infty} \left(\frac{\xi(y)}{h+2d\kappa}\right)^n = 1 + \xi(y)\left((h+2d\kappa)^{-1} + \mathcal{O}\left(h^\alpha (h+2d\kappa)^{\gamma-4}\right)\right).$$

$\square$

Lemma 20. *Let Assumption (F*) be satisfied. Then there exists $R_0 = R_0(\bar{F})$ such that for every $R > R_0$ and $c > 2d\kappa$,*

$$\mathbf{P}\left(\lambda_1^{R,\mathrm{f}}(\xi) > h \middle| \widehat{\xi}_R \le h - c\right) \sim \mathbf{P}\left(\lambda_1^{R,0}(\xi) > h \middle| \widehat{\xi}_R \le h - c\right), \qquad h \to \infty.$$

Proof. Using the representation from Lemma 15, we see that we only have to look at those paths with $\tau_0 \geq \tau_{(Q_R)^c}$ in the left hand side of (2.17) in the proof of Lemma 16. In this case the random walk X must jump at least $2R+1$ times until it reaches the origin for the first time. Therefore, we have to consider only those summands in the right hand side of (2.17) where $k \geq R$. Together with Lemma 18 this yields

$$\mathbb{E}_y \exp\left\{ \int_0^{\tau_0} \left(\xi\left(X_s\right) - h\right) \, \mathrm{d}s \right\} \mathbb{1}_{\tau_0 \geq \tau_{(Q_R)^c}} \mathbb{1}_{\widehat{\xi}_R \leq h-c} = \mathcal{O}\left(h^{-2-2R}\right).$$

$\square$

Remark. If $\xi \sim$ Weibull(γ), $\gamma > 1$, then we can choose $R_0 > \frac{\gamma-1}{2}$.

Now we would like to use a variant of the Laplace method to calculate exact moment asymptotics. This is possible due to the strong Tauberian theorem, Theorem 21, in the spirit of [FY83]. Recall that $\varphi = -\log \bar{F}$.

Theorem 21. *Let $\varphi \in C^2$ be ultimately convex and Condition (B) be satisfied. Then*

$$\exp\left\{H(t)\right\} \sim \exp\left\{t h_t - \varphi(h_t)\right\} \sqrt{\frac{2\pi}{\varphi''(h_t)}}, \qquad t \to \infty.$$

Remark. It follows that

$$H'(t) \sim h_t \text{ and } H''(t) \sim (h_t)' = 1/\varphi''(h_t), \qquad t \to \infty.$$

Proof of Theorem 2. We find that all conditions of Theorem 21 are satisfied. Therefore, the claim follows together with Theorem 1, where we can take $p = 1$, $f_1(x) = x^p$ and $t_1(t) = t$. □

In particular we find together with Lemma 19 and an asymptotic expansion of $h_t^{R,*}$ for Weibull tails:

Corollary 22. *Let $\xi(0) \sim$ Weibull (γ), $\gamma > 1$ and $p \in (0, \infty)$. Then for $t \to \infty$,*

$$\begin{aligned} \langle u(t,0)^p \rangle \;&=\; \exp\Bigg\{ (\gamma-1)\left(\frac{p}{\gamma}t\right)^{\frac{\gamma}{\gamma-1}} - 2d\kappa pt + 2d\kappa^2\gamma\left(\frac{p}{\gamma}t\right)^{\frac{\gamma-2}{\gamma-1}} \\ &\qquad + \log pt + \frac{1}{2}\log\frac{2\pi}{\gamma(\gamma-1)}\left(\frac{p}{\gamma}t\right)^{-\frac{\gamma-2}{\gamma-1}} + \mathcal{O}\left(t^{\frac{\gamma-3}{\gamma-1}}\right)\Bigg\}. \end{aligned}$$

2.5 Relevant potential peaks and intermittency

In this section we prove Theorem 3 which tells us what the potential peaks that contribute to the intermittency picture look like and how frequently they occur. Fix $R > 0$ and let

$$\widetilde{\Upsilon}_t^a := \left[h_t^{R,*} - \frac{a}{\sqrt{(\varphi_R^*)''(h_t^{R,*})}},\, h_t^{R,*} + \frac{a}{\sqrt{(\varphi_R^*)''(h_t^{R,*})}}\right], \qquad a > 0,$$

and recall that

$$\Upsilon_t^a = \left[h_t - \frac{a}{\sqrt{\varphi''(h_t)}},\, h_t + \frac{a}{\sqrt{\varphi''(h_t)}}\right], \qquad a > 0.$$

Notice that $1/\varphi''(h_t) = (h_t)'$ and $1/(\varphi_R^*)''(h_t^{R,*}) = (h_t^{R,*})'$.

Lemma 23. *Let Assumption (F) and Condition (B) be satisfied, and R be sufficiently large. Then for every $\varepsilon > 0$ there exists a_ε such that*

$$\lim_{t\to\infty} \frac{\left\langle u(t,0)\mathbb{1}_{\lambda_1^{R,*}\in\widetilde{\Upsilon}_t^a}\right\rangle}{\langle u(t,0)\rangle} \begin{cases} > 1-\varepsilon & \text{if } a > a_\varepsilon, \\ < 1-\varepsilon & \text{if } a < a_\varepsilon. \end{cases}$$

Proof. Recall that $\psi_R^*(t) = th_t^{R,*} - \varphi(h_t^{R,*})$. By Theorems 1 and 21 and with the help of a first order Taylor expansion we see that there exists $\eta_t \in \widetilde{\Upsilon}_t^a$ such that

$$\begin{aligned}
\left\langle u(t,0)\mathbb{1}_{\lambda_1^{R,*}\in\widetilde{\Upsilon}_t^a}\right\rangle &\sim t\int\limits_{\widetilde{\Upsilon}_t^a} \exp\left\{th - \varphi_R^*(h)\right\}\,\mathrm{d}h \\
&= t\frac{\exp\left\{\psi_R^*(t)\right\}}{\sqrt{(\varphi_R^*)''(h_t^{R,*})}}\int\limits_{-a}^{a}\exp\left\{-\frac{(\varphi_R^*)''(\eta_t)}{2(\varphi_R^*)''(h_t)}u^2\right\}\mathrm{d}u \\
&\sim t\exp\left\{\psi_R^*(t)\right\}\sqrt{\frac{2\pi}{(\varphi_R^*)''(h_t^{R,*})}}\big(\Phi(a)-\Phi(-a)\big) \\
&\sim \langle u(t,0)\rangle\big(\Phi(a)-\Phi(-a)\big), \qquad t\to\infty.
\end{aligned}$$

Here, Φ denotes the distribution function of the standard normal distribution. The asymptotic equivalence in the third line is due to Condition (B). □

Since $H''(t) \sim 1/(\varphi_R^*)''(h_t^{R,*}) \sim 1/\varphi''(h_t)$ as t tends to infinity, Lemma 23 implies that for all $a > 0$,

$$|\widetilde{\Upsilon}_t^a| \sim |\Upsilon_t^a| \asymp \sqrt{H''(t)}, \qquad t\to\infty.$$

Proof of Theorem 3. It follows from [GM98] that only those realisations of ξ contribute to the annealed behaviour where $\lambda_1^{R,*} = \xi_R^{(1)} - 2d\kappa + o(1)$. Therefore, we find that $h_t^{R,*} = h_t + o(1)$ and hence it follows from Lemma 23,

$$\lim_{t\to\infty}\frac{\left\langle u(t,0)\mathbb{1}_{\xi(0)\in\Upsilon_t^a}\right\rangle}{\langle u(t,0)\rangle}\begin{cases} > 1-\varepsilon & \text{if } a > a_\varepsilon, \\ < 1-\varepsilon & \text{if } a < a_\varepsilon. \end{cases}.$$

Since we have chosen $Q_{L(t)}$ sufficiently large, we can apply the weak LLN and find

$$\frac{\sum\limits_{x\in Q_{L(t)}} u(t,x)\mathbb{1}_{\xi(x)\in\Upsilon_t^a}}{\sum\limits_{x\in Q_{L(t)}} u(t,x)} \xrightarrow{\mathbb{P}} \frac{\left\langle u(t,0)\mathbb{1}_{\xi(0)\in\Upsilon_t^a}\right\rangle}{\langle u(t,0)\rangle},$$

which completes the proof. □

Now we consider the random set

$$\Gamma_t^a := \left\{x\in Q_{L(t)} \colon \xi(x)\in\Upsilon_t^a\right\}.$$

Furthermore, let Ber_p be a Bernoulli process on the lattice with parameter p and let

$$i_t^a = \exp\left\{ -\varphi\left(h_t + \frac{a}{\sqrt{\varphi''(h_t)}} \right) \right\}.$$

We find that the spatial picture of the intermittency peaks looks as follows:

Corollary 24. *Asymptotically,*

$$\Gamma_t^a \sim \mathrm{Ber}_{i_t^a}, \qquad t \to \infty.$$

Proof. The fact that Γ_t^a is a Bernoulli process follows since ξ is i.i.d.
The value of the parameter follows because

$$\mathbf{P}\left(\xi(0) \in \Upsilon_t^a\right) = i_t^a - i_t^{-a} \sim i_t^a, \qquad t \to \infty.$$

$\square$

If $\xi(0) \sim \mathrm{Weibull}(\gamma)$, $\gamma > 1$, we find with a first order Taylor expansion around h_t that

$$i_t^a = \exp\left\{ -\left(\frac{t}{\gamma}\right)^{\gamma/\gamma-1} - \frac{a t^{-\gamma/2(\gamma-1)}}{\gamma^{1/(\gamma-1)}(\gamma-1)^{1/2}} - \frac{1}{2}a^2 \right\}.$$

If we are interested in those potential peaks that are relevant to the p-th intermittency peak ($p \in (0,\infty)$), we only have to replace i_t^a by i_{pt}^a. It becomes obvious that the p-th intermittency peaks at time t correspond to the q-th intermittency peaks at time p/q.

2.6 Ageing

In this section we prove Theorems 4 and 5 to gain a better understanding on how stable the intermittency peaks are.

Intermittency Ageing

We start with the first approach.

Proof of Theorem 4. By Theorem 3 we find

$$\lim_{t\to\infty} \mathscr{A}_s(t) = \lim_{t\to\infty} \mathbf{P}\left(\left|1 - \frac{\sum\limits_{x\in Q_{L_{t+s(t)}}} u(t+s(t),x)\mathbb{1}_{\xi(x)\in\Upsilon_t^a}}{\sum\limits_{x\in Q_{L_{t+s(t)}}} u(t+s(t),x)}\right| < \varepsilon\right).$$

Recall that $1/\sqrt{\varphi''(h_t)} = \sqrt{(h_t)'} \sim \sqrt{H''(t)}$. Hence, $\Upsilon_t^a = \left[h_t - a\sqrt{(h_t)'}, h_t + a\sqrt{(h_t)'}\right]$ and therefore $\Upsilon_t^a \cap \Upsilon_{t+s(t)}^a = \emptyset$ if and only if $h_t + a\sqrt{(h_t)'} - h_{t+s} + a\sqrt{(h_{t+s})'} < 0$. For $s = o(t)$ a Taylor expansion yields

$$h_t + a\sqrt{(h_t)'} - h_{t+s} + a\sqrt{(h_{t+s})'} = -s(t)(h_t)'\big(1+o(1)\big) + 2a\sqrt{(h_t)'}\big(1+o(1)\big). \quad (2.20)$$

If $\lim_{t\to\infty} H''(t) > 0$ then $\lim_{t\to\infty}(h_t)' > 0$ and hence, it follows that the right hand side of (2.20) becomes eventually negative and therefore, $\Upsilon_t^a \cap \Upsilon_{t+s(t)}^a = \emptyset$ for t large and all $s(t)$ tending to infinity.
Furthermore, with the help of Fatou's lemma we see that asymptotically

$$\frac{1}{|Q_{L_{t+s(t)}}|} \sum_{x\in Q_{L_{t+s(t)}}} u(t+s(t),x)\mathbb{1}_{\xi(x)\in\Upsilon_t^a} \le \left\langle u(t+s(t),0)\mathbb{1}_{\xi(0)\in\Upsilon_t^a}\right\rangle.$$

Now we can conclude that

$$\lim_{t\to\infty} \mathscr{A}_s(t) = \lim_{t\to\infty} \mathbf{P}\left(\left|1 - \frac{\left\langle u(t+s(t),0)\mathbb{1}_{\xi(0)\in\Upsilon_t^a}\right\rangle}{\langle u(t+s(t),0)\rangle}\right| < \varepsilon\right) = 0.$$

On the other hand, if $\lim_{t\to\infty} H''(t) = 0$ then $\lim_{t\to\infty}(h_t)' = 0$ and hence, if $s(t) = o\left(1/\sqrt{(h_t)'}\right)$ then eventually $\Upsilon_t^a \cap \Upsilon_{t+s(t)}^a = \emptyset$ as above, whereas if $1/\sqrt{(h_t)'} = o(s(t))$ then $\lim_{t\to\infty} |\Upsilon_t^a \triangle \Upsilon_{t+s(t)}^a| = 0$.
In the first case we can proceed as above, while in the second case we find that asymptotically

$$\begin{aligned}
&\frac{1}{|Q_{L_{t+s(t)}}|} \sum_{x\in Q_{L_{t+s(t)}}} u(t+s(t),x)\mathbb{1}_{\xi(x)\in\Upsilon_t^a} \\
\sim &\frac{1}{|Q_{L_{t+s(t)}}|} \sum_{x\in Q_{L_{t+s(t)}}} u(t+s(t),x)\mathbb{1}_{\xi(x)\in\Upsilon_{t+s(t)}^a} = \left\langle u(t+s(t),0)\mathbb{1}_{\xi(0)\in\Upsilon_{t+s(t)}^a}\right\rangle, \quad t\to\infty.
\end{aligned}$$

and hence,

$$\lim_{t\to\infty} \mathscr{A}_s(t) = \lim_{t\to\infty} \mathbf{P}\left(\left|1 - \frac{\left\langle u(t+s(t),0)\mathbb{1}_{\xi(0)\in\Upsilon_{t+s(t)}^a}\right\rangle}{\langle u(t+s(t),0)\rangle}\right| < \varepsilon\right) = 1.$$

□

Remark. The result remains true if we replace u by u^p, $p \in (0, \infty)$.

Corollary 25. *Let $\xi(0) \sim$ Weibull (γ), $\gamma > 1$. Then the PAM ages in the sense of intermittency ageing if and only if $\gamma > 2$.*

Proof. It follows from Theorem 2 that all conditions of Theorem 4 are satisfied. Now the assumption follows by Theorem 4 using the asymptotics in Theorem 2. □

If we want to know for how long a large peak of height t_1 remains relevant, we have to find $t_2 = t_2(t_1, a)$ such that

$$h_{t_1} + \frac{a}{\sqrt{\varphi''(h_{t_1})}} = h_{t_2} - \frac{a}{\sqrt{\varphi''(h_{t_2})}}.$$

Correlation Ageing

Now we come to the second approach.

Proof of Theorem 5. For simplicity we will only consider $p = 1$. Higher powers can be treated analogously. The PAM is always intermittent for stationary potentials, i.e. the second moments are growing much faster than the squares of the first moments. Therefore, it holds

$$\lim_{t\to\infty} A_{id}(s,t) = \lim_{t\to\infty} \frac{\langle u(t,0)u(t+s(t),0)\rangle}{\sqrt{\langle u(t,0)^2\rangle \langle u(t+s(t),0)^2\rangle}}.$$

It has been shown in [GM90, (3.13)] that $\langle u(t,0)^p u(s,0)^p\rangle = \langle u(t+s,0)^p\rangle$ for all $t, s \geq 0$ if $u_0 \equiv 1$ for $p = 1$. The claim for $p \in \mathbb{N}$ follows by similar techniques. Therefore, together with Theorem 21 we find

$$\Lambda_p(t) := \log \langle u(t,0)^p\rangle = \psi_R^*(pt) + \frac{1}{2}\log \frac{2\pi}{(\varphi_R^*)''(\widetilde{h}_{pt}^*)} + \epsilon(pt) \text{ with } \epsilon(t) = o(1).$$

Under the given assumptions we find as t tends to infinity,

$$\begin{aligned}
&\frac{\langle u(t,0)u(t+s(t),0)\rangle}{\sqrt{\langle u(t,0)^2\rangle\,\langle u(t+s(t),0)^2\rangle}} = \frac{\langle u(2t+s(t),0)\rangle}{\sqrt{\langle u(2t,0)\rangle\,\langle u(2(t+s(t)),0)\rangle}}\\
= \quad &\exp\left\{\Lambda_1\big(2t+s(t)\big) - \frac{1}{2}\left[\Lambda_1\left(2t\right) + \Lambda_1\big(2t+2s(t)\big)\right]\right\}\\
\sim \quad &\underbrace{\exp\left\{\psi_R^*\big(2t+s(t)\big) - \frac{1}{2}\left[\psi_R^*\left(2t\right) + \psi_R^*\big(2t+2s(t)\big)\right]\right\}}_{=:B\left(t,s(t)\right)}\\
&\times \exp\underbrace{\left\{\epsilon\big(2t+s(t)\big) - \frac{1}{2}\left[\epsilon\left(2t\right) + \epsilon\big(2t+2s(t)\big)\right]\right\}}_{\overset{t\to\infty}{\longrightarrow}0}\\
&\times \exp\underbrace{\left\{\frac{1}{2}\log\frac{2\pi}{(\varphi_R^*)''\left(\widetilde{h}^*_{2t+s(t)}\right)} - \frac{1}{4}\left[\log\frac{2\pi}{(\varphi_R^*)''\left(\widetilde{h}^*_{2t}\right)} + \log\frac{2\pi}{(\varphi_R^*)''\left(\widetilde{h}^*_{2t+2s(t)}\right)}\right]\right\}}_{=:D\left(t,s(t)\right)}.
\end{aligned}$$

It is well known that $\Lambda_p \in C^\infty$ and $(\psi_R^*)'' > 0$.
Expanding $B\big(t,s(t)\big)$ into first order Taylor polynomials around $2t+s(t)$ we see that there exist $\eta_1(t) \in [2t, 2t+s(t)]$ and $\eta_2(t) \in [2t+s(t), 2t+2s(t)]$ such that

$$B\big(t,s(t)\big) = -\frac{1}{2}s(t)^2\Big((\psi_R^*)''\big(\eta_1(t)\big) + (\psi_R^*)''\big(\eta_2(t)\big)\Big).$$

Using the estimates from Lemma 17, we find that

$$(\psi_R^*)''(t) \sim \psi''(t) \sim H''(t), \qquad t\to\infty.$$

Case 1: $\lim\limits_{t\to\infty} H''(t) > 0$.
In this case it follows that $\lim_{t\to\infty} B\big(t,s(t)\big) = -\infty$ which implies $\lim_{t\to\infty} A_{\mathrm{id}}(s(t),t) = 0$, for all s.

Case 2: $\lim\limits_{t\to\infty} H''(t) = 0$.
Remember that under Assumption F we have $t = o(H(t))$ and hence $t^{-1} = o(H''(t))$. In this case we find two constants $0 < C_1 < C_2 < \infty$ such that

$$-C_1 H''(2t)s(t)^2 \le B\big(t,s(t)\big) \le -C_2 H''\big(2t+s(t)\big)s(t)^2.$$

Consequently, if we choose s such that $\lim_{t\to\infty} H''\big(2t+2s(t)\big)s(t)^2 = \infty$ it follows that $A_{id}(s) = 0$, whereas if we choose s such that $\lim_{t\to\infty} H''(2t)s(t)^2 = 0$ it follows that

$\lim_{t\to\infty} A_{\mathrm{id}}(s,t) = 1$.
We observe that $1/\sqrt{H''(t)} \in \mathcal{A}$ which implies that both regimes can occur for functions s of order $o(t)$. Because of Condition (B), we find that $1/(\varphi_R^*)''\left(\widetilde{h}_t^*\right) = o(t)$ and hence $D\big(t, s(t)\big)$ tends to zero as t tends to infinity if $s = o(t)$. Theorem 1 is applicable for $s = o(t)$ and implies that

$$\left\langle u(t,0)^p u\big(t+s(t)\big)^p\right\rangle \sim \left\langle u\Big(p\big(2t+s(t)\big),0\Big),0\big)\right\rangle, \qquad t\to\infty,$$

for $p \in (0,\infty)$. Hence, we can generalise the result to positive real exponents which completes the proof. □

Notice that the PAM ages if and only if the length of the intervals Υ_t^a tends to zero as t tends to infinity. In this case we find that $1/|\Upsilon_t^a| \in \mathcal{A}$, for all $a > 0$.

Corollary 26. *Let $\xi(0) \sim$ Weibull (γ), $\gamma > 1$. Then the PAM ages for $f = x^p, p \in \mathbb{R}_+$ in the sense of correlation ageing if and only if $\gamma > 2$.*

Proof. Analog as in Corollary 25. □

We see that for Weibull tails the order of the length of ageing is increasing in γ.

The main obstacle in proving Theorem 5 is that we have to show that $\frac{\mathrm{d}^2}{\mathrm{d}t^2}\log\langle u(t,0)\rangle$ is not fluctuating too much. We have proven this under Assumptoin (F*). For more general potentials we are still able to prove correlation ageing if we replace (2.6) by only requiring that

$$\liminf_{t\to\infty} |A_f(s_1,t) - A_f(s_2,t)| > 0,$$

and by modifying the definition of $\mathcal{A}$ accordingly. We call this *weak correlation ageing.* It has been proven in [HKM06] that under some mild regularity assumptions on ξ there exists a non-decreasing and regularly varying scale function $\alpha\colon (0,\infty) \to (0,\infty)$ with $\alpha(t) = o(t)$ and a constant $\chi \in \mathbb{R}$ such that

$$\log\langle u(t,0)\rangle = H\left(\frac{t}{\alpha(t)^d}\right)\alpha(t)^d + \frac{t}{\alpha(t)^2}\left(\chi + o(1)\right). \tag{2.21}$$

Let $H_\alpha(t) := H\left(\frac{t}{\alpha(t)^d}\right)\alpha(t)^d + \frac{t}{\alpha(t)^2}\chi$ and $h(t) := \log\langle u(t,0)\rangle - H_\alpha(t)$. Then we find:

Theorem 27. *Let Assumptions (H) and (K) from [HKM06] be satisfied. If $\lim\limits_{t\to\infty}\alpha''(t)$ exists and $\lim\limits_{t\to\infty} H''(t) = 0$, then the PAM is weakly correlation ageing for $f = id$.*

Proof. Since $\alpha(t) = o(t)$ and $\lim_{t\to\infty} \alpha''(t)$ exists this limit must be zero. Therefore, and because $\lim_{t\to\infty} H''(t) = 0$ it follows that $\lim_{t\to\infty} H''_\alpha(t) = 0$, as well. It follows from (2.21) that $h(t) = o\left(H_\alpha(t)\right)$. Altogether, this and the convexity of $\log\langle u(t,0)\rangle$ imply that for every sequence of intervals $I(t) := [l(t), 2l(t)]$ with $\lim_{t\to\infty} l(t) = \infty$,

$$\frac{\lambda\Big(s \in I(t)\colon h''(s) \notin [-H''_\alpha(t), H''_\alpha(t)]\Big)}{\lambda\Big(s \in I(t)\colon h''(s) \in [-H''_\alpha(t), H''_\alpha(t)]\Big)} \xrightarrow{t\to\infty} 0,$$

where λ denotes the Lebesgue measure. Now the claim follows with a Taylor expansion as in the proof of Theorem 5. $\square$

For the first and the second universality class in the classification of [HKM06] we find that α is constant, and hence, the requirement that $\lim\limits_{t\to\infty} \alpha''(t)$ exists is fulfilled for all distributions in these classes. Here it holds that $1/\sqrt{H''(t)} \in \mathcal{A}$.

3 The parabolic Anderson model between quenched and annealed behaviour

3.1 Introduction

The problem

We will stick in this chapter to the homogeneous initial condition $u_0 \equiv 1$ and potentials that are i.i.d. and time-independent.

Basically, there are two ways of looking at the solution. On the one hand one can pick one realisation of the potential field and consider the almost sure behaviour of u, i.e. one looks at the *quenched* setting. On the other hand one can take expectation with respect to the potential and consider the averaged behaviour of u, i.e. one looks at the *annealed* setting. Recall that expectation with respect to ξ is denoted by $\langle\cdot\rangle$, and the corresponding probability measure is denoted by $\mathbf{P}$. Those realisations of ξ that govern the quenched behaviour of u differ heavily from those that govern the annealed behaviour, see [GM90], or Chapter 2. Therefore, it is interesting to understand the transition mechanism from quenched to annealed behaviour.

To this end we are interested in expressions such as $\frac{1}{|Q|}\sum_{x\in Q} u(t,x)$ where Q is a large centred box. If Q has a fixed size, then $\frac{1}{|Q|}\sum_{x\in Q} u(t,x)$ admits quenched behaviour as t tends to infinity. This can be deduced from the Feynman-Kac representation (1.2) of u, which, in this case, reads

$$u(t,x) = \mathbb{E}_x \exp\left\{\int_0^t \xi\left(X_s\right)\,\mathrm{d}s\right\}, \qquad (t,x) \in [0,\infty)\times\mathbb{Z}^d.$$

On the other hand if we fix t and let the size of Q tend to infinity then (due to the homogeneous initial condition) by Birkhoff's ergodic theorem $\frac{1}{|Q|}\sum_{x\in Q} u(t,x)$ admits annealed behaviour almost surely. Therefore, a natural question is what happens if the box Q is time-dependent.

More precisely, we want to find for all $\alpha \in (0,2)$ a large box $Q_{L_\alpha(t)}$, with $Q_{r(t)} = [-r(t), r(t)]^d \cap \mathbb{Z}^d$, for any $r(t) > 0$, and numbers $A(t), B_\alpha(t)$ such that

$$\sum_{x \in Q_{L_\alpha(t)}} \frac{u(t,x) - A(t)}{B_\alpha(t)} \overset{t\to\infty}{\Longrightarrow} \mathcal{F}_\alpha,$$

with $\mathcal{F}_\alpha$ a suitable stable distribution.

In the case $\kappa = 0$, i.e. if the solutions at different sites are independent, the problem has been addressed in [BABM05] under the assumption that the cumulant generating function is normalised regularly varying. As a byproduct we generalise their results to a wider class of cumulant generating functions. In [BABM05] the authors also give sufficient and necessary conditions on the growth rate of Q for a weak law of large numbers (WLLN) and for a central limit theorem (CLT) to hold. Corresponding results for a WLLN and a CLT for the PAM were derived in [BAMR05] and in [BAMR07]. They state that, under appropriate regularity assumptions, there exist $J(t)$ and $\gamma_1 < \gamma_2$, all depending on the tails of ξ, such that:

i) $\frac{1}{|Q_{\gamma J(t)}|} \sum_{x \in Q_{\gamma J(t)}} u(t,x) \sim \langle u(t,0) \rangle$, as $t \to \infty$ if $\gamma > \gamma_1$, in probability,
$\frac{1}{|Q_{\gamma J(t)}|} \sum_{x \in Q_{\gamma J(t)}} u(t,x) = o\left(\langle u(t,0) \rangle\right)$, as $t \to \infty$ if $\gamma < \gamma_1$, in probability.

ii) $\frac{1}{|Q_{\gamma J(t)}|} \sum_{x \in Q_{\gamma J(t)}} \frac{u(t,x) - \langle u(t,0) \rangle}{\sqrt{\langle u(t,0)^2 \rangle}} = \mathcal{N}(0,1)$, as $t \to \infty$ if $\gamma > \gamma_2$, in distribution,
$\frac{1}{|Q_{\gamma J(t)}|} \sum_{x \in Q_{\gamma J(t)}} \frac{u(t,x) - \langle u(t,0) \rangle}{\sqrt{\langle u(t,0)^2 \rangle}} = o(1)$, as $t \to \infty$ if $\gamma < \gamma_2$, in probability.

Here, $\mathcal{N}(0,1)$ denotes the law of a standard normal distribution with variance 1. However, α-stable limits for the PAM have not been investigated so far.

Furthermore, we give sufficient conditions on the growth rate of Q for a strong law of large numbers to hold. So far this has neither be done for the PAM nor for the $\kappa = 0$ case.

A WLLN and a CLT for the PAM with time-dependent white-noise potential using rather different techniques can be found in [CM07]. Similar questions concerning a version of the random energy model were investigated in [BKL02].

Main results

To state the main results we need to introduce some notation. Let, as in Chapter 2,

$$\varphi(h) := -\log \mathbf{P}(\xi(0) > h)$$

and h_t being a solution to

$$\sup_{h\in(0,\infty)} (th - \varphi(h)) = th_t - \varphi(h_t).$$

If φ is ultimately convex, then h_t is unique for any large t. Throughout this chapter we will assume that $\xi(0)$ is unbounded from above and has finite exponential moments of all orders. Under these circumstances the left-continuous inverse of φ,

$$\psi(s) := \min\{r : \varphi(r) \geq s\}, \qquad s > 0,$$

is well-defined. Furthermore, this implies that the cumulant generating function

$$H(t) := \log \langle \exp\{t\xi(0)\}\rangle, \qquad t \geq 0,$$

is well-defined and that $H(t) < \infty$ for all t with $\lim_{t\to\infty} H(t)/t = \infty$. If $\varphi \in C^2$ is ultimately convex and satisfies some mild regularity assumptions, then the Laplace method yields that $H(t) = th_t - \varphi(h_t) + o(t)$, see Theorem 21. In the sequel we will frequently need the following regularity assumptions.
Assumption (G):
There exists $\rho \in [0, \infty]$ such that for all $c \in (0, 1)$,

$$\lim_{t\to\infty} [\psi(ct) - \psi(t)] = \rho c \log c.$$

Assumption (H):
There exists $\rho \in [0, \infty]$ such that for all $c \in (0, 1)$,

$$\lim_{t\to\infty} \frac{1}{t} [H(ct) - cH(t)] = \rho c \log c.$$

In [GM98, Theorems 1.2 and 2.2] the authors prove that there exists $\chi = \chi(\rho) \in [0, 2d\kappa]$ such that

$$\frac{\log u(t,0)}{t} = \xi^{(1)}_{Q_t} - \chi + o(1), \qquad \text{a.s.,} \tag{3.1}$$

with $\xi^{(1)}_A = \sup\{\xi(x) : x \in A\}$, if Assumption (G) is satisfied, and

$$\frac{\log \langle u(t,0)^p\rangle}{t} = \frac{H(pt)}{t} - p\chi + o(1), \qquad p \in \mathbb{N}, \tag{3.2}$$

if Assumption (H) is satisfied. Notice that Assumption (G) implies Assumption (H). Furthermore, it turns out that $\chi = \chi(\rho)$ is strictly increasing in ρ with $\chi(0) = 0$ and $\chi(\infty) = 2d\kappa$. For details see [GM98].

Prominent examples satisfying Assumption (G) are the double exponential distribution, i.e. $\mathbf{P}(X > x) = \exp\{-\exp\{x/\rho\}\}$, $x > 0$, for $\rho \in (0,\infty)$ and the Weibull distribution, i.e. $\mathbf{P}(X > x) = \exp\{-x^\gamma\}$, $x > 0$ with $\gamma > 1$ for $\rho = \infty$.

For $\alpha \in (0,2)$ let $\mathcal{F}_\alpha$ be the α-stable distribution with characteristic function

$$\phi_\alpha(u) = \begin{cases} \exp\left\{-\Gamma(1-\alpha)|u|^\alpha \exp\left\{\frac{-i\pi\alpha}{2}\operatorname{sign} u\right\}\right\}, & \alpha \neq 1, \\ \exp\left\{iu(1-\gamma) - \frac{\pi}{2}|u|\left(1 + 2\pi i \log|u| \operatorname{sign} u\cdot\right)\right\}, & \alpha = 1. \end{cases}$$

Moreover, let

$$L_\alpha(t) := \exp\{\varphi\big(h_{\alpha t}\big)\} \quad \text{and} \quad B_\alpha(t) := \exp\{t \cdot \big(h_{\alpha t} - \chi + o(1)\big)\},$$

where the error term of $B_\alpha(t)$ is chosen in a suitable way. Then we find:

Theorem 28 (Stable limit laws)**.** *Let $\varphi \in C^2$ be ultimately convex and Assumption (G) be satisfied. Then for $\alpha \in (0,2)$,*

$$\sum_{x \in Q_{L_\alpha(t)}} \frac{u(t,x) - A(t)}{B_\alpha(t)} \overset{t\to\infty}{\Longrightarrow} \mathcal{F}_\alpha,$$

with

$$A(t) = \begin{cases} 0, & \text{if } \alpha \in (0,1), \\ \langle u(t,0)\rangle, & \text{if } \alpha \in (1,2), \\ \big\langle u(t,0)\mathbb{1}_{u(t,0)\leq B_\alpha(t)}\big\rangle, & \text{if } \alpha = 1. \end{cases}$$

and

Theorem 29 (Strong law of large numbers)**.** *Let Assumption (H) be satisfied, and $r(t)$ be so large that $\lim_{t\to\infty} \frac{1}{t}\big(\log|Q_{r(t)}| - H(2t) + 2H(t)\big) > 0$ then for every sequence $(t_n)_{n\in\mathbb{N}}$ satisfying $\sum_{t_n} \exp\{-t_n\} < \infty$,*

$$\frac{1}{|Q_{r(t_n)}|} \sum_{x \in Q_{r(t_n)}} \left(\frac{u(t_n,x)}{\langle u(t_n,0)\rangle} - 1\right) \overset{t_n\to\infty}{\longrightarrow} 0 \quad a.s.$$

Notice that the necessary growth rate of Q for a WLLN to hold is the same as in Theorem 28 for $\alpha = 1$ and that the necessary growth rate of Q for a CLT to hold corresponds to $\alpha = 2$, see [BAMR07]. The growth rate in Theorem 29 is of the same

order as in the CLT case.
To get a feeling for the numbers involved we give them for the two examples mentioned above in the table below.

Distribution	$\boldsymbol{\varphi(x)}$	$\boldsymbol{\log L_\alpha(t)}$	$\boldsymbol{\log B_\alpha(t)}$
Weibull	x^γ, $\gamma > 1$	$\left(\frac{\alpha t}{\gamma}\right)^{\gamma/(\gamma-1)}$	$t\left(\frac{\alpha t}{\gamma}\right)^{1/(\gamma-1)} - 2d\kappa\alpha t + o(t)$
Double-exponential	$\exp\{x/\rho\}$	$\rho\alpha t$	$t\rho\log\rho\alpha t - \chi(\rho)\alpha t + o(t)$

Notice that in the Weibull case we have

$$\log B_\alpha(t) = \frac{1}{\alpha}\left(\log\langle u(t,0)^\alpha\rangle + \log|Q_{L_\alpha(t)}|\right) + o(t),$$

see Chapter 2. Because of (3.2) and our considerations in Section 3.2 this relationship seems to be true in the double exponential case, as well.

3.2 Stable limit laws

Let us explain our strategy of the proof of Theorem 28. We decompose the large box $Q_{L_\alpha(t)}$ into boxes $Q^{(i)}_{l(t)}$, $i = 1, \cdots, \lfloor|Q_{L_\alpha(t)}|/|Q_{l(t)}|\rfloor$ of much smaller size. In each subbox we approximate u by $u^{(i)}$, the solution with Dirichlet boundary conditions in $Q^{(i)}_{l(t)}$. In this way, we reduce the problem to the case of i.i.d. random variables. A spectral representation shows that $\sum_{x\in Q^{(i)}_{l(t)}} u^{(i)}(t,x)$ can be approached by $\mathrm{e}^{t\lambda_1^{(i)}}$, where $\lambda_1^{(i)}$ is the principal Dirichlet eigenvalue of $\Delta + \xi$ in $Q^{(i)}_{l(t)}$. Then a classical result on stable limits for sums of t-dependent i.i.d. random variables yields the result.

Note that we cannot apply the results of [BABM05] since they require the function φ to be normalised regularly varying, which is, for instance, not true in the important case of double-exponential tails.

Let us turn to the details. We first work on the $u^{(i)}$ and show in the end how to approach u by $u^{(i)}$. We assume that $Q^{(i)}_{l(t)}$ are translated copies of $Q_{l(t)}$. We consider the solution $u^{(i)}$ to the PAM in $Q^{(i)}_{l(t)}$ with Dirichlet boundary conditions, i.e. $\xi(x) = -\infty$ for all $x \notin Q^{(i)}_{l(t)}$, where $l(t) = \max\{t^2\log^2 t, H(4t)\}$. The corresponding Laplacian will be denoted $\Delta^0_{Q^{(i)}_{l(t)}}$. The Feynman-Kac representation of $u^{(i)}$ reads

$$u^{(i)}(t,x) = \mathbb{E}_x \exp\left\{\int\limits_0^\infty \xi(X_s)\,\mathrm{d}s\right\}\mathbb{1}_{\tau_{(Q^{(i)}_{l(t)})^c}>t}, \qquad (t,x)\in[0,\infty)\times Q^{(i)}_{l(t)}.$$

By $\tau_U := \inf\{t > 0\colon X_t \in U\}$ we denote the first hitting time of a set U by a random walk X. Let $\lambda_1^{(i)}, \cdots, \lambda_{|Q_{l(t)}|}^{(i)}$ be an order statistics of the eigenvalues of the Anderson Hamiltonian $\Delta^0_{Q^{(i)}_{l(t)}} + \xi$ and $e_1^{(i)}, \cdots, e_{|Q_{l(t)}|}^{(i)}$ be a corresponding orthonormal basis. Then we have the following spectral representation

$$\sum_{x \in Q^{(i)}_{l(t)}} u^{(i)}(t,x) = \sum_{x,y \in Q^{(i)}_{l(t)}} \sum_{k=1}^{|Q_{l(t)}|} \mathrm{e}^{\lambda_k^{(i)} t} e_k^{(i)}(x) e_k^{(i)}(y), \qquad t \in [0,\infty). \tag{3.3}$$

From Parseval's inequality, the fact that $l(t)$ is of subexponential order, and the proof of Theorem 2.2 in [GM98] it follows that there exists $\widetilde{\varepsilon}^{(i)}(t) = \widetilde{\varepsilon}^{(i)}(\xi,t) = o(1)$ such that

$$\sum_{x \in Q^{(i)}_{l(t)}} u^{(i)}(t,x) = \mathrm{e}^{\mu_t^{(i)}}, \qquad \text{where} \qquad \mu_t^{(i)} = \mu_t^{(i)}(\xi) = \lambda_1^{(i)} + \widetilde{\varepsilon}^{(i)}(t).$$

Sometimes we will write μ_t instead of μ_t^1, λ_1 for $\lambda_1^{(i)}$ and $\widetilde{\varepsilon}(t)$ for $\widetilde{\varepsilon}^{(i)}(t)$.
Remark. The above already implies that for $\log r(t) = o\big(H(t)\big)$ the quenched setting is prominent in the following sense,

$$\lim_{t\to\infty} \frac{\log u(t,0)}{\log \sum\limits_{x \in Q_{r(t)}} u(t,x)} = 1, \qquad \text{a.s.}$$

In the next lemma we show how the distributions of μ_t and $\xi(0)$ are linked.

Lemma 30. *Let Assumption (G) be satisfied. Then asymptotically for all $h(t)$ with $\lim_{t\to\infty} h(t) = \infty$ there exists $\varepsilon(t) = \varepsilon(\xi,t) = o(1)$ such that for all $s > 0$,*

$$\mathbf{P}\Big(\mu_t > sh(t)\Big) \sim |Q_{l(t)}| \mathbf{P}\big(\xi(0) > sh(t) + \chi - \varepsilon(t)\big), \qquad t \to \infty.$$

Proof. In [GM98, Proof of Theorem 2.16] the authors show that the first eigenvalue of $\Delta^0_{Q_{l(t)}} + \xi$ satisfies

$$\lambda_1 = \xi^{(1)}_{Q_{l(t)}} - \chi + \bar{\varepsilon}(t),$$

with $\bar{\varepsilon}(t) = \bar{\varepsilon}(\xi,t) = o(1)$. Let

$$\varepsilon(t) := \widetilde{\varepsilon}(t) + \bar{\varepsilon}(t).$$

Then

$$\begin{aligned}\mathbf{P}\Big(\mu_t > sh(t)\Big) &= \mathbf{P}\Big(\xi^{(1)}_{Q_{l(t)}} > sh(t) + \chi - \varepsilon(t)\Big) \\ &= 1 - \bigg(1 - \mathbf{P}\Big(\xi(0) > sh(t) + \chi - \varepsilon(t)\Big)\bigg)^{|Q_{l(t)}|} \\ &\sim 1 - \exp\Big\{ - |Q_{l(t)}|\mathbf{P}\Big(\xi(0) > sh(t) + \chi - \varepsilon(t)\Big)\Big\} \\ &\sim |Q_{l(t)}|\mathbf{P}\Big(\xi(0) > sh(t) + \chi - \varepsilon(t)\Big), \qquad t \to \infty.\end{aligned}$$

In the third line we use L'Hopital's rule. □

Let

$$\widetilde{\varphi}_t(x) = -\log \mathbf{P}\left(\mu_t > x\right),$$

and $\widetilde{h}_t = h_t + \chi + o(1)$, where the error term is chosen in a suitable way. In particular, it is chosen such that if $\widetilde{\varphi}_t$ is ultimately convex then $\widetilde{h}_t$ is the unique solution to

$$\sup_{h\in(0,\infty)} \left(th - \widetilde{\varphi}_t(h)\right) = t\widetilde{h}_t - \widetilde{\varphi}_t\left(\widetilde{h}_t\right).$$

Then, an application of the Laplace method yields

$$\langle u(t,0)\rangle \sim \langle u^{(1)}(t,0)\rangle \sim t\int_0^\infty \exp\{th - \widetilde{\varphi}(h)\}\,\mathrm{d}h = \exp\{\big[t\widetilde{h}_t - \widetilde{\varphi}\big(\widetilde{h}_t\big)\big]\big(1 + o(1)\big)\}.$$

The first asymptotics follow from Proposition 7. Hence, we obtain together with Lemma 30 that

$$\log B_\alpha(t) = t\widetilde{h}_{\alpha t} = \frac{1}{\alpha}\left(\log\langle u(t,0)^\alpha\rangle + \log|Q_{L_\alpha(t)}|\right)\left(1 + o(1)\right).$$

To prove convergence of $\sum_{i:\, Q^{(i)}_{l(t)}\subset Q_{L_\alpha(t)}}\big(\mathrm{e}^{t\mu^{(i)}_t} - \widetilde{A}(t)\big)/B_\alpha(t)$, as $t \to \infty$, to an infinitely divisible distribution with characteristic function equal to

$$\phi(u) = \exp\left\{iau - \frac{\sigma^2u^2}{2} + \int\limits_{|x|>0}\left(\mathrm{e}^{iux} - 1 - \frac{iux}{1+x^2}\right)\mathrm{d}\widetilde{L}(x)\right\}, \qquad (3.4)$$

we have to verify the following condition (see [P75, Chapter IV]).

Condition (P):

i) Condition of infinity smallness:

$$\lim_{t\to\infty} \max_{i\colon Q^{(i)}_{l(t)}\subset Q_{L_\alpha(t)}} \mathbf{P}\bigg(\frac{\mathrm{e}^{t\mu^{(i)}_t}}{B_\alpha(t)} \geq \varepsilon\bigg) = 0, \qquad \varepsilon > 0.$$

ii) In all points x of continuity, the function $\widetilde{L}$ satisfies:

$$\widetilde{L}(x) = -\lim_{t\to\infty} \frac{|Q_{l(t)}|}{|Q_{L_\alpha(t)}|}\mathbf{P}\bigg(\frac{\mathrm{e}^{t\mu_t}}{B_\alpha(t)} > x\bigg).$$

iii) The constant σ^2 satisfies:

$$\begin{aligned}\sigma^2 &= \lim_{\tau\to 0}\limsup_{t\to\infty} \frac{|Q_{l(t)}|}{|Q_{L_\alpha(t)}|}\mathrm{Var}\bigg(\frac{\mathrm{e}^{t\mu_t}}{B_\alpha(t)}\mathbb{1}_{\frac{\mathrm{e}^{t\mu_t}}{B_\alpha(t)}\leq\tau}\bigg)\\ &= \lim_{\tau\to 0}\liminf_{t\to\infty} \frac{|Q_{l(t)}|}{|Q_{L_\alpha(t)}|}\mathrm{Var}\bigg(\frac{\mathrm{e}^{t\mu_t}}{B_\alpha(t)}\mathbb{1}_{\frac{\mathrm{e}^{t\mu_t}}{B_\alpha(t)}\leq\tau}\bigg).\end{aligned}$$

iv) For every $\tau > 0$ the constant a satisfies:

$$\begin{aligned}&\lim_{t\to\infty}\bigg\{\frac{|Q_{l(t)}|}{|Q_{L_\alpha(t)}|}\bigg\langle\frac{\mathrm{e}^{t\mu_t}}{B_\alpha(t)}\mathbb{1}_{\frac{\mathrm{e}^{t\mu_t}}{B_\alpha(t)}\leq\tau}\bigg\rangle - \frac{\widetilde{A}(t)}{B_\alpha(t)}\bigg\}\\ =&a + \int_0^\tau \frac{x^3}{1+x^2}\,\mathrm{d}L(x) - \int_\tau^\infty \frac{x}{1+x^2}\,\mathrm{d}L(x).\end{aligned}$$

Items i) and ii) will follow from the next lemma, and iii) and iv) from the next proposition.

Lemma 31. *Let Assumption (G) be satisfied and $\varphi \in C^2$ be ultimately convex. Then, for any $x > 0$,*

$$\lim_{t\to\infty}\mathbf{P}\bigg(\mu_t > \frac{\log B_\alpha(t)}{t} + \frac{\log x}{t}\bigg) = x^{-\alpha}.$$

Proof. Lemma 30 and a first order Taylor expansion yield

$$\begin{aligned}&\mathbf{P}\bigg(\mu_t > \frac{\log B_\alpha(t)}{t} + \frac{\log x}{t}\bigg) \sim |Q_{l(t)}|\mathbf{P}\bigg(\xi(0) > \frac{\log B_\alpha(t)}{t} + \frac{\log x}{t} + \chi + \varepsilon(t)\bigg)\\ &= \exp\bigg\{\log|Q_{l(t)}| - \varphi\bigg(\frac{\log B_\alpha(t)}{t} + \chi + o(1)\bigg)\\ &\qquad - \varphi'\bigg(\frac{\log B_\alpha(t)}{t} + \chi + o(1)\bigg)\frac{\log x}{t} + o(1)\bigg\}.\end{aligned}$$

Since φ is ultimately convex and ξ is unbounded from above we find that $\varphi''(h_{\alpha t}) = 1/h'_{\alpha t} = o(t^2)$. From this we can conclude that the error term in the Taylor expansion above vanishes asymptotically. Moreover, by our choice of $l(t)$ and $B_\alpha(t)$ it follows that

$$\log|Q_{l(t)}| = \varphi\left(\frac{\log B_\alpha(t)}{t} + \chi + o(1)\right) \qquad \text{and} \qquad \lim_{t\to\infty} \frac{\varphi'\left(\frac{\log B_\alpha(t)}{t} + \chi + o(1)\right)}{t} = \alpha.$$

□

Proposition 32. *Let Assumption (G) be satisfied and $\varphi \in C^2$ be ultimately convex. Then, for any $\tau > 0$,*

i) if $p > \alpha$ then

$$\lim_{t\to\infty} \frac{|Q_{l(t)}|}{|Q_{L_\alpha(t)}|} \left\langle \frac{e^{pt\mu_t}}{B_\alpha(t)^p} \mathbb{1}_{\frac{e^{t\mu_t}}{B_\alpha(t)} \le \tau} \right\rangle = \frac{\alpha}{p-\alpha}\tau^{p-\alpha}.$$

ii) if $p < \alpha$ then

$$\lim_{t\to\infty} \frac{|Q_{l(t)}|}{|Q_{L_\alpha(t)}|} \left\langle \frac{e^{pt\mu_t}}{B_\alpha(t)^p} \mathbb{1}_{\frac{e^{t\mu_t}}{B_\alpha(t)} > \tau} \right\rangle = \frac{\alpha}{\alpha-p}\tau^{p-\alpha}.$$

iii) if $p = \alpha$ then

$$\lim_{t\to\infty} \frac{|Q_{l(t)}|}{|Q_{L_\alpha(t)}|} \left\langle \frac{e^{pt\mu_t}}{B_\alpha(t)^p} \left(\mathbb{1}_{\frac{e^{t\mu_t}}{B_\alpha(t)} \le \tau} - \mathbb{1}_{\frac{e^{t\mu_t}}{B_\alpha(t)} \le 1}\right) \right\rangle = \alpha \log \tau.$$

Proof. i) Integration by parts yields

$$\left\langle e^{pt\mu_t} \mathbb{1}_{\frac{e^{t\mu_t}}{B_\alpha(t)} \le \tau} \right\rangle = \int_0^{\widetilde{h}_{\alpha t} + \frac{\log\tau}{t}} e^{tpx}\, \mathrm{d}\left(1 - \bar{F}_{\mu_t}(x)\right)$$
$$= -\left[e^{tpx - \widetilde{\varphi}(x)}\right]_{x=0}^{x=\widetilde{h}_{\alpha t} + \frac{\log\tau}{t}} + pt \int_0^{\widetilde{h}_{\alpha t} + \frac{\log\tau}{t}} e^{tpx - \widetilde{\varphi}(x)}\, \mathrm{d}x.$$

Here $\bar{F}_{\mu_t}$ denotes the tail distribution function of μ_t. Similarly as in the proof of Lemma 31 we find with the help of a first order Taylor expansion of φ that uniformly in τ,

$$\widetilde{\varphi}\big(\widetilde{h}_{\alpha t} + \frac{\log\tau}{t}\big) \sim \widetilde{\varphi}(\widetilde{h}_{\alpha t}) - \alpha\log\tau, \qquad t\to\infty.$$

Substituting $x = \widetilde{h}_{\alpha t} + \frac{\log \tau}{t} u$ for $\tau \neq 1$ we find that

$$pt \int_0^{\widetilde{h}_{\alpha t} + \frac{\log \tau}{t}} \mathrm{e}^{tpx - \widetilde{\varphi}(x)} \, \mathrm{d}x \sim \mathrm{e}^{pt\widetilde{h}_{\alpha t} - \widetilde{\varphi}(\widetilde{h}_{\alpha t})} p \log \tau \int_{-\infty \cdot \operatorname{sign} \log \tau}^{1} \mathrm{e}^{u(p-\alpha) \log \tau} \, \mathrm{d}u$$
$$\sim \frac{p}{p-\alpha} \mathrm{e}^{pt\widetilde{h}_{\alpha t} - \widetilde{\varphi}(\widetilde{h}_{\alpha t}) + (p-\alpha)\log \tau}, \qquad t \to \infty.$$

Altogether this proves the claim.

ii) and iii) follow similarly.

□

Overall we find:

Theorem 33. *Let Assumption (G) be satisfied and $\varphi \in C^2$ be ultimately convex. Then for $\alpha \in (0, 2)$,*

$$\sum_{i \colon Q_{l(t)}^{(i)} \subset Q_{L_\alpha(t)}} \frac{\mathrm{e}^{t\mu_t^{(i)}} - \widetilde{A}(t)}{B_\alpha(t)} \overset{t \to \infty}{\Longrightarrow} \mathcal{F}_\alpha,$$

with

$$\widetilde{A}(t) = \begin{cases} 0, & \text{if } \alpha \in (0,1), \\ \langle \mathrm{e}^{t\mu_t} \rangle, & \text{if } \alpha \in (1,2) \\ \langle \mathrm{e}^{t\mu_t} \mathbb{1}_{\mu_t \leq 1} \rangle, & \text{if } \alpha = 1. \end{cases}$$

Proof. Since the $u^{(i)}$ are i.i.d., the $\mu_t^{(i)}$ are as well. Hence, we have to check the four points of Condition (P). Items i) and ii) follow from Lemma 31. We find that $\widetilde{L}(x) = x^{-\alpha}$. It follows from Proposition 32 that $\sigma^2 = 0$. Furthermore, Proposition 32 together with [BABM05, Proposition 6.4] yields the constant a from which we can deduce ϕ. The stability of the limit law follows from [P75, Theorem IV.12] since $\sigma^2 = 0$ and $\widetilde{L}(x) = x^{-\alpha}$. □

Remark. An infinitely divisible law with characteristic function as in (3.4) is stable if and only if either $\widetilde{L} \equiv 0$ or $\sigma^2 = 0$ and $\widetilde{L}(x) = cx^{-\alpha}$, $c > 0$, $\alpha \in (0, 2)$, see [P75, Theorem IV.12].

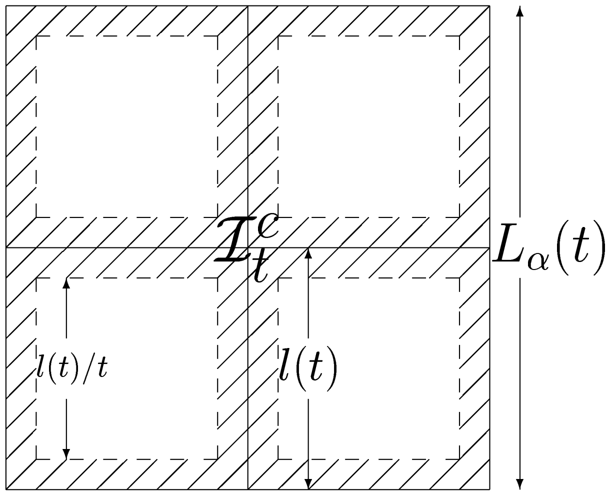
We extend the functions $u^{(i)}$ to a function $\widetilde{u} \colon Q_{L_\alpha(t)} \to [0, \infty)$ by putting $\widetilde{u}(t, x) = u^{(i)}(t, x)$ for $x \in Q_{l(t)}^{(i)}$. Now it remains to show that

$$\sum_{x\in Q_{L_\alpha(t)}} \frac{u(t,x)-A(t)}{B_\alpha(t)} \qquad \text{and}$$

$$\sum_{x\in Q_{L_\alpha(t)}} \frac{\widetilde{u}(t,x)-\widetilde{A}(t)/|Q_{l(t)}|}{B_\alpha(t)}$$

$$= \sum_{i\colon Q^{(i)}_{l(t)}\subset Q_{L_\alpha(t)}} \frac{\exp\{t\mu^{(i)}_t\}-\widetilde{A}(t)}{B_\alpha(t)}$$

have the same α-stable limit distribution.



To this end let $\mathcal{I}^c_t = \bigcup_{i\colon Q^{(i)}_{l(t)}\subset Q_{L_\alpha(t)}} Q^{(i)}_{l(t)} \setminus Q^{(i)}_{l(t)/t}$ and $\mathcal{I}_t = Q_{L_\alpha(t)} \setminus \mathcal{I}^c_t$. Notice that

$$u(t,x)-\widetilde{u}(t,x) = \mathbb{E}_x \exp\left\{\int_0^\infty \xi(X_s)\,\mathrm{d}s\right\}\mathbb{1}_{\tau_{(Q^{(i)}_{l(t)})^c}\le t}, \qquad (t,x)\in[0,\infty)\times Q^{(i)}_{l(t)}.$$

Lemma 34. *For all $\varepsilon>0$,*

i) if $\alpha\in(0,1]$ then

$$\lim_{t\to\infty}\mathbf{P}\left(\frac{1}{B_\alpha(t)}\sum_{x\in Q_{L_\alpha(t)}}\big[u(t,x)-\widetilde{u}(t,x)\big]>\varepsilon\right)=0.$$

ii) if $\alpha\in[1,2)$ then

$$\lim_{t\to\infty}\mathbf{P}\left(\frac{1}{B_\alpha(t)}\sum_{x\in Q_{L_\alpha(t)}}\big[u(t,x)-\langle u(t,x)\rangle-\widetilde{u}(t,x)+\langle\widetilde{u}(t,x)\rangle\,\big]>\varepsilon\right)=0.$$

iii) if $\alpha=1$ then

$$\lim_{t\to\infty}\mathbf{P}\left(\sum_{x\in Q_{L_\alpha(t)}}\frac{u(t,x)-\big\langle u(t,x)\mathbb{1}_{u(t,x)\le B_\alpha(t)}\big\rangle-\widetilde{u}(t,x)+\big\langle\widetilde{u}(t,x)\mathbb{1}_{\widetilde{u}(t,x)\le B_\alpha(t)}\big\rangle}{B_\alpha(t)}>\varepsilon\right)=0.$$

Proof. i) If $x \in \mathcal{I}_t$, then $\lim_{t\to\infty} u(t,x) - \widetilde{u}(t,x) = 0$ a.s. (see [GM98]). Furthermore, $|\mathcal{I}_t|/B_\alpha(t) \overset{t\to\infty}{\longrightarrow} 0$. Hence, we find for large t,

$$\mathbf{P}\left(\frac{1}{B_\alpha(t)}\sum_{x\in Q_{L_\alpha(t)}} u(t,x) - \widetilde{u}(t,x) > \varepsilon\right) \sim \mathbf{P}\left(\frac{1}{B_\alpha(t)}\sum_{x\in \mathcal{I}_t^c} u(t,x) - \widetilde{u}(t,x) > \varepsilon\right).$$

By the definition of $B_\alpha(t)$ and by Markov's inequality it follows that

$$\begin{aligned}\mathbf{P}\left(\frac{1}{B_\alpha(t)}\sum_{x\in \mathcal{I}_t^c} u(t,x) - \widetilde{u}(t,x) > \varepsilon\right) &\leq \mathbf{P}\left(\sum_{x\in\mathcal{I}_t^c}\frac{u(t,x)}{|Q_{L_\alpha(t)}|^{1/\alpha}\left\langle u(t,0)^\alpha\right\rangle^{1/\alpha}} > \varepsilon\right)\\ &\leq \frac{1}{\varepsilon^\alpha}\frac{\left\langle\left(\sum_{x\in\mathcal{I}_t^c} u(t,x)\right)^\alpha\right\rangle}{|Q_{L_\alpha(t)}|\left\langle u(t,0)^\alpha\right\rangle}\\ &\leq \frac{1}{\varepsilon^\alpha}\frac{|Q_{L_\alpha(t)}|}{|\mathcal{I}_t|}\frac{\left\langle u(t,0)^\alpha\right\rangle}{|Q_{L_\alpha(t)}|\left\langle u(t,0)^\alpha\right\rangle} \overset{t\to\infty}{\longrightarrow} 0.\end{aligned}$$

ii) Similarly as in case i) we find that asymptotically

$$\begin{aligned}&\mathbf{P}\left(\frac{1}{B_\alpha(t)}\sum_{x\in Q_{L_\alpha(t)}}\left[u(t,x) - \langle u(t,x)\rangle - \widetilde{u}(t,x) + \langle\widetilde{u}(t,x)\rangle\right] > \varepsilon\right)\\ \leq &\frac{1}{\varepsilon^\alpha}\frac{\left\langle\left(\sum_{x\in\mathcal{I}_t^c} u(t,x)\right)^\alpha\right\rangle}{|Q_{L_\alpha(t)}|\left\langle u(t,0)^\alpha\right\rangle} + o(1).\end{aligned}$$

Furthermore, we have

$$\begin{aligned}&\left\langle\left(\sum_{x\in\mathcal{I}_t^c} u(t,x)\right)^\alpha\right\rangle\\ \leq &\left\langle\left(\sum_{x\in\mathcal{I}_t^c}\left(u(t,x)^2 + \sum_{\substack{y\in\mathcal{I}_t^c:\\ |x-y|\leq l(t)/t}} u(t,x)u(t,y) + \sum_{\substack{y\in\mathcal{I}_t^c:\\ |x-y|> l(t)/t}} u(t,x)u(t,y)\right)\right)^{\alpha/2}\right\rangle\\ \leq &\sum_{x\in\mathcal{I}_t^c}|l(t)/t|\left\langle u(t,x)^\alpha\right\rangle + \sum_{\substack{x,y\in\mathcal{I}_t^c:\\ |x-y|> l(t)/t}}\left\langle\left(u(t,x)u(t,y)\right)^{\alpha/2}\right\rangle.\end{aligned}$$

The first summand can be treated as in case i), whereas the second summand can be treated similarly as in the proof of Lemma 35.

iii) follows analogously.

□

Now we are able to prove Theorem 28.

Proof of Theorem 28. We only consider the case $\alpha \in (0,1)$. The other cases follow similarly. It follows from Lemma 34 that for every $\varepsilon > 0$,

$$\lim_{t\to\infty} \mathbf{P}\left(\sum_{i:\, Q_{l(t)}^{(i)} \subset Q_{L_\alpha(t)}} \left| \frac{\sum_{x\in Q_{l(t)}^{(i)}} u(t,x) - \exp\{t\mu_t^{(i)}\}}{B_\alpha(t)} \right| > \varepsilon \right) = 0,$$

while Theorem 33 states that under the same conditions as in Theorem 28,

$$\sum_{i:\, Q_{l(t)}^{(i)} \subset Q_{L_\alpha(t)}} \frac{\mathrm{e}^{t\mu_t^{(i)}}}{B_\alpha(t)} \overset{t\to\infty}{\Longrightarrow} \mathcal{F}_\alpha .$$

Therefore, the claim follows from Slutzky's theorem. □

Remark. We expect that a similar result as Theorem 28 with the same stable limit distribution also holds for potential tails that are bounded from above as considered in [BK01] and [HKM06]. However, since in that case we do not have such a close link between between μ_t and $\xi^{(1)}_{Q_{l(t)}}$ we cannot determine the distribution of μ_t and therefore $L_\alpha(t) = -\log \mathbf{P}(\mu_t > \widetilde{h}_t)$ cannot be made as explicit as under Assumption (G).

3.3 Strong law of large numbers

Recall that $l(t) = \max\{t^2 \log^2 t, H(4t)\}$ and that $x + Q_{l(t)}$ is the lattice box with centre x and sidelength $l(t)$.

Lemma 35. *Let Assumption (H) be satisfied and $r(t)$ be chosen as in Theorem 29, then*

$$\lim_{t\to\infty} \frac{1}{|Q_{r(t)}|^2} \sum_{\substack{x,y\in Q_{r(t)}:\\ |x-y|>2l(t)}} \left(\frac{\langle u(t,x)u(t,y)\rangle}{\langle u(t,0)\rangle^2} - 1 \right) = 0.$$

Proof. For $t > 0$ and $x \in Q_{r(t)}$ let

$$u^{(1)}(t,x) = \mathbb{E}_x \exp\left\{\int_0^\infty \xi(X_s)\,\mathrm{d}s\right\} \mathbb{1}_{\tau_{(x+Q_{l(t)})^c} \geq t},$$

and

$$u(t,x,y) = \mathbb{E}_{x,y} \exp\left\{\int_0^\infty \xi(X_s)\,\mathrm{d}s\right\} \exp\left\{\int_0^\infty \xi(Y_s)\,\mathrm{d}s\right\} \mathbb{1}_{\tau^X_{(x+Q_{l(t)})^c} < t \text{ or } \tau^Y_{(y+Q_{l(t)})^c} < t},$$

where X and Y are two independent random walks starting in x, y, respectively, $\mathbb{E}_{x,y}$ is their joint expectation, and τ_A^X, τ_A^Y are their exit times from a set $A \subset \mathbb{Z}^d$, respectively. If $|x - y| > 2l(t)$, then $u^{(1)}(t,x)$ and $u^{(1)}(t,y)$ are independent, and hence

$$\sum_{\substack{x,y \in Q_{r(t)}: \\ |x-y| > 2l(t)}} \left(\frac{\langle u(t,x)u(t,y)\rangle}{\langle u(t,0)\rangle^2} - 1 \right) = \sum_{\substack{x,y \in Q_{r(t)}: \\ |x-y| > 2l(t)}} \frac{\langle u(t,x,y)\rangle}{\langle u(t,0)\rangle^2}.$$

Hölder's inequality and [GM90, Lemma 2.4 and Theorem 3.1] yield for all $x, y \in Q_{r(t)} \setminus Q_{l(t)}$,

$$\begin{aligned}\langle u(t,x,y)\rangle &\leq \sqrt{\langle u(t,0)^4\rangle\, 2\mathbb{P}_x\left(\tau_{(x+Q_{l(t)})^c} < t\right)} \\ &\leq \exp\left\{\frac{1}{2}(l(t) - l(t)\log l(t) + o(t)\right\} \stackrel{t\to\infty}{\longrightarrow} 0.\end{aligned}$$

□

Proof of Theorem 29. By Chebyshev's inequality we find that for every $s > 0$,

$$\begin{aligned}&\mathbf{P}\left(\sup_{t_n > s} \frac{1}{|Q_{r(t_n)}|} \sum_{x \in Q_{r(t_n)}} \left(\frac{u(t_n,x)}{\langle u(t_n,0)\rangle} - 1\right) > \varepsilon\right) \\ &\leq \sum_{t_n > s} \frac{1}{\varepsilon^2} \mathrm{Var}\left(\frac{1}{|Q_{r(t_n)}|} \sum_{x \in Q_{r(t_n)}} \left(\frac{u(t_n,x)}{\langle u(t_n,0)\rangle} - 1\right)\right).\end{aligned}$$

As t tends to infinity it follows with Lemma 35 that

$$\begin{aligned}
\operatorname{Var}\bigg(\frac{1}{|Q_{r(t)}|}\sum_{x\in Q_{r(t)}}\Big(\frac{u(t,x)}{\langle u(t,0)\rangle}-1\Big)\bigg) &\sim \frac{1}{|Q_{r(t)}|^2}\sum_{\substack{x,y\in Q_{r(t)}:\\ |x-y|<2l(t)}}\bigg(\frac{\langle u(t,x)u(t,y)\rangle}{\langle u(t,0)\rangle^2}-1\bigg)\\
&\sim \frac{1}{|Q_{r(t)}|}\sum_{x\in Q_{l(t)}}\bigg(\frac{\langle u(t,0)u(t,x)\rangle}{\langle u(t,0)\rangle^2}-1\bigg)\\
&\le \frac{|Q_{l(t)}|}{|Q_{r(t)}|}\frac{\langle u(t,0)^2\rangle}{\langle u(t,0)\rangle^2}\\
&= \exp\{-\log|Q_{r(t)}|+H(2t)-2H(t)+o(t)\}.
\end{aligned}$$

The last asymptotics are due to (3.2). Now the claim follows because for our choice of $r(t)$,

$$\lim_{s\to\infty}\sum_{t_n>s}\exp\{-\log|Q_{r(t_n)}|+H(2t_n)-2H(t_n)+o(t_n)\}=0.$$

□

4 Correlations for the parabolic Anderson model with white-noise potential

4.1 Introduction

In this chapter we investigate time and spatial correlations for the parabolic Anderson model (PAM) with a white-noise potential, i.e. the following initial value problem for the heat equation with discrete space variable and homogeneous initial value

$$\begin{cases} \frac{\partial}{\partial t} u(t,x) = \kappa \Delta u(t,x) + \gamma \mathrm{d}W_t(x) u(t,x), & (t,x) \in (0,\infty) \times \mathbb{Z}^d, \\ u(0,x) = 1, & x \in \mathbb{Z}^d. \end{cases}$$

Here, $\{W(x), x \in \mathbb{Z}^d\}$ denotes a family of independent one-dimensional Brownian motions, and κ, γ are positive constants. This problem can be reformulated as an infinite-dimensional system of stochastic differential equations which reads

$$u(t,x) = 1 + \kappa \int_0^t \Delta u(s,x)\,\mathrm{d}s + \gamma \int_0^t u(t,x)\,\mathrm{d}W_s(x), \qquad (t,x) \in [0,\infty) \times \mathbb{Z}^d.$$

Existence and uniqueness in the space of nonnegative functions were shown in [CM94]. We want to understand the above problem in the sense of Itô. Thus, it follows by the martingale property and the Feynman-Kac formula (1.5) that for all $t \geq 0$, $x \in \mathbb{Z}^d$:

$$\langle u(t,x) \rangle = \left\langle \mathbb{E}_x^X \exp\left\{ \gamma \int_0^t \mathrm{d}W_s(X_{t-s}) - \frac{\gamma^2}{2} t \right\} \right\rangle = 1,$$

where X denotes a simple symmetric continuous time random walk with generator $\kappa\Delta$ that starts in x under $\mathbb{E}_x^X$ and $\langle \cdot \rangle$ denotes expectation with respect to the medium. We are going to investigate the following correlation coefficient

$$\operatorname{corr}(u(t,0), u(s,x)) := \frac{\langle u(t,0) u(s,x) \rangle - 1}{\sqrt{(\langle u(t,0)^2 \rangle - 1)(\langle u(s,x)^2 \rangle - 1)}}.$$

Let $X^{(1)}$ and $X^{(2)}$ be two independent random walks with generator $\kappa\Delta$ and $Z := X^{(2)} - X^{(1)}$. Then Z is a random walk with generator $2\kappa\Delta$ that starts in x under $\mathbb{E}_x^Z$. Let $\mathcal{H} := 2\kappa\Delta + \gamma^2\delta_0$. Then the semigroup representation of $\mathcal{H}$ acting on $\ell^2(\mathbb{Z}^d)$ is given by

$$(\mathrm{e}^{\mathcal{H}t} f)(z) = \mathbb{E}_z^Z \exp\left\{\gamma^2 \int_0^t \mathrm{d}u\, \delta_0\left(Z_u\right)\right\} f(Z_t).$$

It is well known (see for instance [GdH06, Lemma 1.3]) that the spectrum of $\mathcal{H}$ has the following shape

$$\sigma(\mathcal{H}) = \begin{cases} [-8d\kappa, 0] \cup \{\mu\}, & \text{if } 2\kappa < \gamma^2 G(0), \\ [-8d\kappa, 0], & \text{if } 2\kappa \geq \gamma^2 G(0). \end{cases}$$

Here,

$$G(x) := \mathbb{E}_x^Y \int_0^\infty \delta_0(Y_s)\, \mathrm{d}s = \int_0^\infty p_s(x)\, \mathrm{d}s, \qquad x \in \mathbb{Z}^d,$$

denotes the Green's function of a random walk Y with generator Δ, $p_s(x) = \mathbb{P}_0^Y\left(Y_s = x\right)$, and $\mu > 0$ is the principal eigenvalue of $\mathcal{H}$ corresponding to a positive eigenfunction v. Notice that the Green's function is finite if and only if the random walk is transient, i.e. if and only if $d \geq 3$. Now we can formulate our main theorem.

Theorem 36. *i) There exists $c(\kappa) > 0$ such that as $s \to \infty$,*

$$\lim_{t\to\infty} \mathit{corr}(u(t,0), u(t+s,0)) = \begin{cases} \mathrm{e}^{-\mu\gamma^2 s/2 - \mathcal{O}(\log s)}, & \text{if } 2\kappa < \gamma^2 G(0), \\ \frac{1}{G(0)} \int_{\kappa s}^\infty p_{2\kappa u}(0)\, \mathrm{d}u \sim c(\kappa) s^{1-d/2}, & \text{if } 2\kappa \geq \gamma^2 G(0). \end{cases}$$

ii) For any $z \in \mathbb{Z}^d$,

$$\lim_{t\to\infty} \mathit{corr}(u(t,0), u(t,z)) = \begin{cases} \frac{v(z)}{v(0)}, \ t \to \infty & \text{if } 2\kappa < \gamma^2 G(0), \\ \frac{G(z)}{G(0)}, \ t \to \infty & \text{if } 2\kappa \geq \gamma^2 G(0). \end{cases}$$

This is to the best of our knowledge the first time that spatial correlations for this model have been derived. For time correlations only upper bounds have been computed, so far, see [DD07] and [AD11].

Remark. If $2\kappa \geq \gamma^2 G(0)$, then we find that for $x \in \mathbb{Z}^d$, $s \geq 0$,

$$\lim_{t\to\infty} \mathrm{corr}(u(t,0), u(t+s,x)) = \frac{G(x) + \int_{\kappa s}^\infty p_{2\kappa u}(x)\, \mathrm{d}u - G(x)\delta_0(x)}{G(0)}.$$

Remark. Transforming the corresponding eigenvalue-equation yields for $z \in \mathbb{Z}^d$,

$$\frac{v(z)}{v(0)} = \gamma(R_\mu \delta_0)(z),$$

where $R_\mu := (\mu - \Delta)^{-1}$ denotes the resolvent of the discrete Laplacian, see [GdH06].

Recall the definition of correlation ageing from Section 2.1.

Corollary 37. *There is no correlation ageing for $f = id$ in the PAM with white-noise potential.*

Remark. The ageing result from Corollary 37 has already been found in [DD07] where the authors use different techniques.

In [CM94, Chapter III] the authors show that the PAM with white-noise potential is intermittent (in the sense of exponential growth of moments as defined in (1.4)) if and and only if $2\kappa < \gamma^2 G(0)$. Theorem 36 makes plausible why for $2\kappa < \gamma^2 G(0)$ we have intermittency but no ageing. It seems as if the islands are moving around but do not increase their size.

4.2 Time and spatial correlations

We need the following representation of *corr*.

Proposition 38. *For all $t, s \geq 0$ and $x \in \mathbb{Z}^d$,*

$$\begin{aligned} &corr(u(t,0), u(t+s,x)) \\ = &\frac{\mathbb{E}_x^{X^{(2)}} \mathbb{E}_{X_s^{(2)}}^Z \exp\left\{\gamma^2 \int_0^t \mathrm{d}u\, \delta_0\left(Z_u\right)\right\} - 1}{\sqrt{\left(\mathbb{E}_0^Z \exp\left\{\gamma^2 \int_0^t \mathrm{d}u\, \delta_0\left(Z_u\right)\right\} - 1\right)\left(\mathbb{E}_0^Z \exp\left\{\gamma^2 \int_0^{s+t} \mathrm{d}u\, \delta_0\left(Z_u\right)\right\} - 1\right)}}. \end{aligned}$$

Proof. By the Feynman-Kac formula (1.5) and Fubini's theorem it follows that

$$\begin{aligned} &\langle u(t,0)u(t+s,x)\rangle \\ = \; &\mathrm{e}^{\frac{-\gamma^2}{2}(2t+s)} \mathbb{E}_0^{X^{(1)}} \mathbb{E}_x^{X^{(2)}} \left\langle \exp\left\{ \underbrace{\gamma \int_0^t \mathrm{d}W_u(X_{t-u}^{(1)}) + \gamma \int_0^{t+s} \mathrm{d}W_v(X_{t+s-v}^{(2)})}_{\sim \mathcal{N}(0,\sigma^2)} \right\} \right\rangle. \end{aligned}$$

Here, $\mathcal{N}(0,\sigma^2)$ denotes the law of the standard normal distribution with variance σ^2. The exponent is normally distributed because $X^{(1)}$ and $X^{(2)}$ are piecewise constant and W has independent increments.

Now we compute its variance σ^2. This can be done in the following way

$$\begin{aligned}\sigma^2 &= \left\langle \Big(\gamma \int_0^t \mathrm{d}W_u(X^{(1)}_{t-u}) + \gamma \int_0^{t+s} \mathrm{d}W_v(X^{(2)}_{t+s-v})\Big)^2 \right\rangle \\ &= \gamma^2 \left[(2t+s) + 2 \left\langle \int_0^t \mathrm{d}W_u(X^{(1)}_{t-u}) \int_0^t \mathrm{d}W_v(X^{(2)}_{t+s-v}) \right\rangle \right].\end{aligned}$$

Here, we use the fact that $\int_0^t \mathrm{d}W_u(X^{(1)}_{t-u})$ and $\int_t^{t+s} \mathrm{d}W_v(X^{(2)}_{t+s-v})$ are independent. Furthermore, we have

$$\begin{aligned}\left\langle \int_0^t \mathrm{d}W_u(X^{(1)}_{t-u}) \int_0^t \mathrm{d}W_v(X^{(2)}_{t+s-v}) \right\rangle &= \int_0^t \int_0^t \left\langle \mathrm{d}W_u(X^{(1)}_{t-u}) \mathrm{d}W_v(X^{(2)}_{t+s-v}) \right\rangle \\ = \int_0^t \int_0^t \delta_0(u-v)\delta_0(X^{(1)}_{t-u} - X^{(2)}_{t+s-v})\,\mathrm{d}u\,\mathrm{d}v &= \int_0^t \mathrm{d}u\,\delta_0(X^{(1)}_u - X^{(2)}_{s+u}).\end{aligned}$$

Hence, using the strong Markov property of $X^{(2)}$ and Fubini's theorem it follows that

$$\begin{aligned}\langle u(t,0)u(t+s,x)\rangle &= \mathbb{E}_0^{X^{(1)}} \mathbb{E}_x^{X^{(2)}} \exp\left\{ \gamma^2 \int_0^t \mathrm{d}u\,\delta_0(X^{(1)}_u - X^{(2)}_{s+u}) \right\} \\ &= \mathbb{E}_0^{X^{(1)}} \mathbb{E}_x^{X^{(2)}} \mathbb{E}_{X^{(2)}_s}^{X^{(2)}} \exp\left\{ \gamma^2 \int_0^t \mathrm{d}u\,\delta_0(X^{(1)}_u - X^{(2)}_u) \right\} \\ &= \mathbb{E}_x^{X^{(2)}} \mathbb{E}_{X^{(2)}_s}^{Z} \exp\left\{ \gamma^2 \int_0^t \mathrm{d}u\,\delta_0(Z_u) \right\}.\end{aligned}$$

By similar arguments we find that

$$\left\langle u(t,0)^2 \right\rangle = \mathbb{E}_0^Z \exp\left\{ \gamma^2 \int_0^t \mathrm{d}u\,\delta_0\left(Z_u\right) \right\} \text{ and } \left\langle u(t+s,x)^2 \right\rangle = \mathbb{E}_0^Z \exp\left\{ \gamma^2 \int_0^{s+t} \mathrm{d}u\,\delta_0\left(Z_u\right) \right\},$$

which proves the claim. □

Lemma 39. *a) The function* $u(z) := \mathbb{E}_z^Z \exp\left\{\gamma^2 \int\limits_0^\infty \delta_0(Z_s)\,\mathrm{d}s\right\}$ *solves the following boundary value problem if* $4d\kappa > \gamma^2$*:*

$$\begin{cases} 2\kappa\Delta u(z) + \gamma^2\delta_0(z)u(z) = 0, & z \in \mathbb{Z}^d, \\ \lim_{|z|\to\infty} u(z) = 1. \end{cases} \tag{4.1}$$

b) $u(z) = 1 + \frac{\gamma^2}{2\kappa - \gamma^2 G(0)} G(z)$ *is the unique solution to the boundary value problem* (4.1)*.*

Proof. a) Let

$$\sigma := \inf\{t \geq 0 \colon Z_t \neq Z_0\}$$

be the stopping time of the first jump of Z.

First we look at the case $z \neq 0$. It follows by the strong Markov property at time σ that

$$\begin{aligned} u(z) &= \mathbb{E}_z^Z \exp\left\{\gamma^2 \int\limits_\sigma^\infty \delta_0(Z_s)\,\mathrm{d}s\right\} \\ &= \mathbb{E}_z^Z \mathbb{E}_{Z_\sigma}^Z \exp\left\{\gamma^2 \int\limits_0^\infty \delta_0(Z_s)\,\mathrm{d}s\right\} \\ &= \mathbb{E}_z^Z u(Z_\sigma) = \frac{1}{2d} \sum_{\substack{y\in\mathbb{Z}^d: \\ |z-y|=1}} u(y). \end{aligned}$$

The last equality holds since Z arrives at one of the $2d$ neighbours of z with equal probability after the first jump. Therefore, it follows that $2du(z) = \Delta u(z) + 2du(z)$ and consequently $\Delta u(z) = 0$.

Now we look at the case $z = 0$. There, we have

$$\begin{aligned} u(0) &= \mathbb{E}_0^Z \mathrm{e}^{\gamma^2\sigma} \exp\left\{\gamma^2 \int\limits_\sigma^\infty \delta_0(Z_s)\,\mathrm{d}s\right\} \\ &= \mathbb{E}_0^Z \mathrm{e}^{\gamma^2\sigma} \mathbb{E}_{Z_\sigma}^Z \exp\left\{\gamma^2 \int\limits_0^\infty \delta_0(Z_s)\,\mathrm{d}s\right\} \\ &= \mathbb{E}_0^Z \mathrm{e}^{\gamma^2\sigma} u(Z_\sigma). \end{aligned}$$

Note that σ is exponentially distributed with parameter $4d\kappa$, which implies that $\mathbb{E}_0^Z \mathrm{e}^{\gamma^2\sigma} = 4d\kappa/(-\gamma^2 + 4d\kappa)$. Since σ and Z_σ are independent we therefore obtain

$$u(0) = \mathbb{E}_0^Z \mathrm{e}^{\gamma^2\sigma} \mathbb{E}_0^Z u(Z_\sigma) = \frac{4d\kappa}{4d\kappa - \gamma^2} \frac{1}{2d} \sum_{\substack{y \in \mathbb{Z}^d \\ |y|=1}} u(y) = \frac{2\kappa}{4d\kappa - \gamma^2} \big(\Delta u(0) + 2du(0)\big),$$

which is equivalent to

$$2\kappa \Delta u(0) + \gamma^2 u(0) = 0.$$

Sumarising we get

$$2\kappa \Delta u(z) + \gamma^2 \delta_0(z) u(z) = 0.$$

The boundary condition follows because $\lim\limits_{|z|\to\infty} G(z) = 0$.

b) Substituting the above expression shows existence. Uniqueness follows by the maximum principle. □

Now we are able to prove Theorem 36. Recall that v denotes the ℓ^2-normalized positive eigenfunction of $\mathcal{H}$ corresponding to μ.

Proof of Theorem 36. i) Assume that $2\kappa < \gamma^2 G(0)$. In [SW10, Theorem 5] the authors show that in this case for every $z \in \mathbb{Z}^d$,

$$\mathbb{E}_z^Z \exp\left\{ \int_0^t \mathrm{d}u\, \delta_0\left(Z_u\right) \right\} \sim v(z) \|v\|_1 \mathrm{e}^{t\mu}, \qquad t \to \infty,$$

where $\|\cdot\|_1$ denotes the ℓ^1-norm in $\mathbb{Z}^d$. Hence, we find on the one hand as t tends to infinity

$$\mathbb{E}_0^{X^{(2)}} \mathbb{E}_{X_s^{(2)}}^Z \exp\left\{ \gamma^2 \int_0^t \mathrm{d}u\, \delta_0\left(Z_u\right) \right\} \le \mathbb{E}_0^Z \exp\left\{ \gamma^2 \int_0^t \mathrm{d}u\, \delta_0\left(Z_u\right) \right\} \sim v(0)\|v\|_1 \mathrm{e}^{\gamma^2\mu t}.$$

On the other hand we find asymptotically as $t \to \infty$,

$$\begin{aligned} &\mathbb{E}_0^{X^{(2)}} \mathbb{E}_{X_s^{(2)}}^Z \exp\left\{ \gamma^2 \int_0^t \mathrm{d}u\, \delta_0\left(Z_u\right) \right\} \\ \ge &\mathbb{E}_0^Z \exp\left\{ \gamma^2 \int_0^t \mathrm{d}u\, \delta_0\left(Z_u\right) \right\} \mathbb{P}_0^{X^{(2)}}\left(X_s^{(2)} = 0\right) \sim v(0)\|v\|_1 \mathrm{e}^{\gamma^2\mu t - \mathcal{O}(\log s)}, \end{aligned}$$

where $\mathcal{O}(\log s)$ does not depend on t. Now we see that asymptotically as $s \to \infty$,

$$\lim_{t\to\infty} \operatorname{corr}(u(t,0), u(t+s,0)) = \lim_{t\to\infty} \frac{\mathrm{e}^{\gamma^2 \mu t}}{\sqrt{\mathrm{e}^{\gamma^2 \mu t}\mathrm{e}^{\mu\gamma^2(t+s)}}} \mathrm{e}^{-\mathcal{O}(\log s)} = \mathrm{e}^{-\mu\gamma^2 s/2 - \mathcal{O}(\log s)}.$$

Assume now that $2\kappa \geq \gamma^2 G(0)$. This case only occurs in dimensions $d \geq 3$ where the random walk Z is transient and we know that almost surely $\int_0^\infty \mathrm{d}u\, \delta_0(Z_u) < \infty$. Therefore, it follows that

$$\begin{aligned}
&\lim_{t\to\infty} \operatorname{corr}\big(u(t,0), u(t+s,0)\big) \\
&= \frac{\mathbb{E}_0^{X^{(2)}} \mathbb{E}_{X_s^{(2)}}^{Z} \exp\left\{\gamma^2 \int\limits_0^\infty \mathrm{d}u\, \delta_0(Z_u)\right\} - 1}{\sqrt{\left(\mathbb{E}_0^Z \exp\left\{\gamma^2 \int\limits_0^\infty \mathrm{d}u\, \delta_0(Z_u)\right\} - 1\right)\left(\mathbb{E}_0^Z \exp\left\{\gamma^2 \int\limits_0^\infty \mathrm{d}u\, \delta_0(Z_u)\right\} - 1\right)}}.
\end{aligned}$$

Since $G(0) \geq 1/2d$ we can apply Lemma 39, which yields that for every $z \in \mathbb{Z}^d$,

$$\mathbb{E}_z^Z \exp\left\{\gamma^2 \int\limits_0^\infty \mathrm{d}u\, \delta_0(Z_u)\right\} = 1 + \frac{\gamma^2}{2\kappa - \gamma^2 G(0)} G(z).$$

Consequently, it follows from Fubini's theorem and the Chapman-Kolmogorov equation that

$$\begin{aligned}
\lim_{t\to\infty} \operatorname{corr}\big(u(t,0), u(t+s,0)\big) &= \frac{\mathbb{E}_0^{X^{(2)}} G(X_s^{(2)})}{G(0)} = \frac{1}{G(0)} \int\limits_0^\infty \mathbb{E}_0^{X^{(2)}} p_u(X_s^{(2)})\, \mathrm{d}u \\
&= \frac{1}{G(0)} \int\limits_{\kappa s}^\infty p_{2\kappa u}(0)\, \mathrm{d}u \sim c(\kappa) s^{1-d/2}, \qquad s \to \infty.
\end{aligned}$$

ii) follows similarly. □

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