Logarithmic inapproximability results for the minimum shortest path routing conflict problem*

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Abstract

Nowadays most data networks use shortest path protocols such as OSPF or IS-IS to route traffic. Given administrative routing lengths for the links of a network, all data packets are sent along shortest paths with respect to these lengths from their source to their destination. One of the most fundamental problems in planning shortest path networks is to decide whether a given set $\mathcal{S}$ of routing paths forms a valid routing and, if this is not the case, to find a small subset $\mathcal{R} \subseteq \mathcal{S}$ of paths that cannot occur together in any valid routing. In this paper we show that it is $\mathcal{NP}$-hard to approximate the minimal size or the minimal weight of a shortest path conflict $\mathcal{R} \subseteq \mathcal{S}$ by a factor less than $c \log |\mathcal{S}|$ for some $c > 0$.

Keywords: shortest path routing; computational complexity

1 Introduction

One of the most fundamental problems in planning networks that employ a shortest path routing protocol such as OSPF or IS-IS is to decide whether a given set of routing paths forms a valid routing and, if this is the case, to find a routing metric for which these paths are shortest paths between their respective terminals. If the given path set does not form a valid shortest path routing, one often wishes to find a small subset of the given paths that form a shortest path conflict, i.e., that cannot be shortest paths simultaneously for any routing metric.

In this paper, we consider the problem of finding a shortest path routing conflict of minimum size in a given path set. This problem naturally arises in integer linear programming approaches for shortest path routing optimization, where invalid routing patterns are cut off the feasible solution space using inequalities based on such shortest path routing conflicts; see [3, 4, 6, 7, 10, 12, 15, 14]. The separation problem for these conflict inequalities is exactly the problem of finding a shortest path routing conflict of minimum weight. Depending on the shortest path routing version and the integer linear programming formulation

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used to optimize the routing paths, slightly different notions of 'conflict' and of the 'weight' of a shortest path routing conflict have to be used. For the unsplittable shortest path routing version, Bley [3, 4] proposed greedy algorithms to compute conflicts that are inclusion-wise minimal, but not necessarily minimal in terms of size or weight. For the problem version associated with an arc-flow formulation for the shortest multi-path routing version, Tomaszewski et al. [14] proposed an integer programming approach, while Broström and Holmberg [8, 9] derived efficient polynomial time algorithms to optimize (and separate) over a special subclass of the corresponding conflicts.

In this paper, we prove a logarithmic inapproximability threshold for the problem of finding a shortest path routing conflict of minimum size or minimum weight. This improves the previously best known constant factor threshold derived in [3, 5]. We only discuss the case corresponding to the path flow formulation of the unsplittable shortest path routing variant explicitly. Our inapproximability results, however, generalize in a straightforward way to the problem of finding minimum size or minimum weight conflicts in path sets corresponding to the shortest multi-path routing variant or in alternative representations of shortest path routings based on arc flows or shortest path graphs.

2 Notation and Preliminaries

Throughout this paper, let $D = (V, A)$ be a directed graph, $\mathcal{P}(s, t)$ be the set of all simple (s, t)-paths, and $\mathcal{P}$ be the set of all simple paths in D. We denote the source and the target of a path P by s_P and t_P, respectively. We say that the arc lengths $\lambda \in \mathbb{Z}^A$ are *compatible* with a given path set $\mathcal{S} \subset \mathcal{P}$, if each path $P \in \mathcal{S}$ is the uniquely determined shortest path between its terminals with respect to λ. A path set $\mathcal{S}$ is a *Unique Shortest Path System (USPS)* if there exists a vector of compatible arc lengths $\lambda \in \mathbb{Z}^A$ for $\mathcal{S}$. Otherwise we say that $\mathcal{S}$ is a *non-USPS* . If $\mathcal{S}$ is a non-USPS, but any proper subset of $\mathcal{S}$ is a USPS, then $\mathcal{S}$ is called an *irreducible non-USPS* or a *(unique) shortest path conflict*.

Clearly, any subset of an USPS is an USPS as well. Also the empty path set $\mathcal{S} = \emptyset$ is an USPS. The family of USPSs in a digraph D forms a so-called independence system (or hereditary family) $\mathcal{I} \subset 2^{\mathcal{P}}$. The circuits of this independence system are exactly the irreducible non-USPSs. A given path set $\mathcal{S}$ forms a valid unique shortest path routing if and only if it does not fully contain any of these irreducible non-USPSs. Using the linear programming techniques discussed in [1, 2], for example, one can decide in polynomial time whether a given path set $\mathcal{S} \subset \mathcal{P}$ is a USPS or not. One easily finds that $\mathcal{S}$ is an USPS if and only if the linear system

$$\sum_{a \in Q} \lambda_a - \sum_{a \in P} \lambda_a \geq 1 \qquad \text{for all } P \in \mathcal{S},\, Q \in \mathcal{P}(s_P, t_P) \setminus \{P\} \tag{1a}$$
$$\lambda_a \geq 1 \qquad \text{for all } a \in A \tag{1b}$$

has a solution. As the separation problem for the inequalities (1a) boils down to a two shortest path problem, which can be solved efficiently with the algorithm of Katoh, Ibaraki, and Mine [11] for example, the overall system (1) can be solved in polynomial time. Using scaling and rounding techniques, any fractional solution of (1) can be turned into an integer-valued compatible metric for the given path set $\mathcal{S}$ [1]. If (1) has no solution, then $\mathcal{S}$ is a non-USPS. In this case, we are interested in finding a small non-USPS contained in $\mathcal{S}$, which can be formally described as follows:

Problem:	MIN-NON-USPS
Instance:	A digraph $D = (V, A)$ and a non-USPS $\mathcal{S} \subseteq \mathcal{P}$.
Solution:	An (irreducible) non-USPS $\mathcal{R} \subseteq \mathcal{S}$.
Objective:	$\min \lvert\mathcal{R}\rvert$.

The more general problem of finding a non-USPS of minimum weight can be formalized as follows:

Problem:	MIN-WEIGHT-NON-USPS
Instance:	A digraph $D = (V, A)$, a non-USPS $\mathcal{S} \subseteq \mathcal{P}$, and strictly positive weights $w_P \in \mathbb{Z}_+$ for all $P \in \mathcal{S}$.
Solution:	An (irreducible) non-USPS $\mathcal{R} \subseteq \mathcal{S}$.
Objective:	$\min \sum_{P \in \mathcal{R}} w_P$.

This problem arises when we seek for an inequality that separates a given fractional routing from the unique shortest path routing polytope, which is defined by the path sets of all valid unique shortest path routings. Its computational complexity therefore is of great practical importance. If MIN-WEIGHT-NON-USPS cannot be solved in polynomial time, one cannot hope to optimize over the unique shortest path routing polytope in polynomial time.

3 Hardness Results

For any fixed $k \in \mathbb{Z}$, we can find a minimum weight non-USPS $\mathcal{R} \subseteq \mathcal{S}$ with $\lvert\mathcal{R}\rvert \leq k$ (or prove that no irreducible non-USPS with $\lvert\mathcal{R}\rvert \leq k$ exists) in polynomial time by solving the linear system (1) for all subsets $\mathcal{R} \subseteq \mathcal{S}$ with $\lvert\mathcal{R}\rvert \leq k$. In special digraphs where the size of irreducible non-USPSs is bounded by some constant, MIN-NON-USPS and MIN-WEIGHT-NON-USPS are therefore solvable in polynomial time. It was shown in [5] that it is $\mathcal{NP}$-hard to approximate these problems within a factor strictly smaller than 7/6 in general. In the following, we prove that it is even hard to approximate these problems within a logarithmic factor.

Theorem 3.1 *There exists some $c > 0$ such that it is $\mathcal{NP}$-hard to approximate* MIN-NON-USPS *within a factor of $c \log \lvert\mathcal{S}\rvert$. This holds even if each path $P \in \mathcal{S}$ is a shortest (s_P, t_P)-path w.r.t. the number of arcs and $\lvert P\rvert = 2$ for all $P \in \mathcal{S}$.*

Proof. We construct an approximation preserving reduction from the optimization problem MINIMUM DOMINATING SET. The latter problem is defined a follows: Given an undirected graph $H = (W, F)$, find a minimum cardinality set $X \subseteq W$ such that, for all $w \in W \setminus X$ there is a node $v \in X$ for which $(v, w) \in F$. Raz and Safra [13] proved that, unless $\mathcal{P} = \mathcal{NP}$, this problem is not approximable within a factor $c \log \lvert W\rvert$ for some $c > 0$.

Suppose we are given an instance $H = (W, F)$ of MINIMUM DOMINATING SET consisting of the nodes w_i with $i \in I := \{1, \ldots, n\}$ and the edges f_k with $k \in K = \{1, \ldots, m\}$. For the sake of notational simplicity (and without loss of generality) we assume that $\{w_i w_i : w_i \in W\} \subseteq F$. Let $\alpha \in \mathbb{Z}$ be a large integer number. At the end of the proof, we discuss how to choose α appropriately.

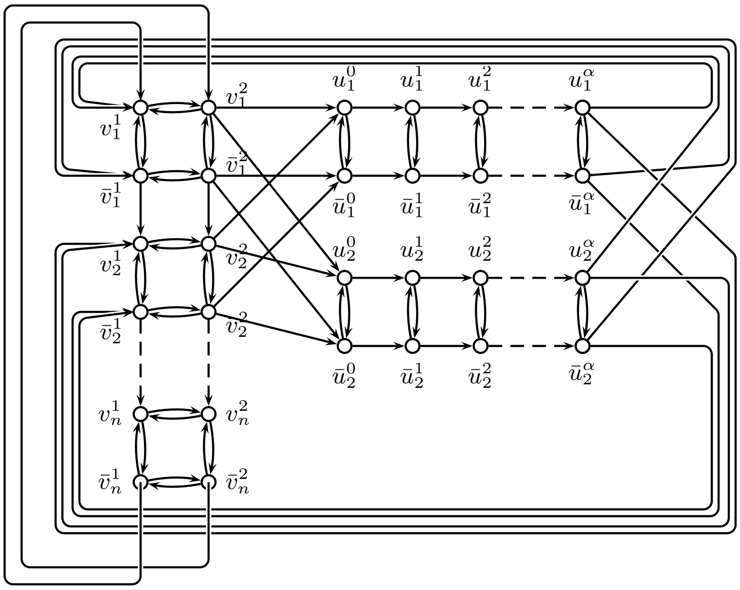

Figure 1: Subgraph of D containing nodes and arcs corresponding to the edges w_1w_2 in F.

We construct a MIN-NON-USPS instance consisting of a digraph $D = (V, A)$ and a path set $\mathcal{S}$ as follows.

For each $i \in I$, we introduce 4 nodes v_i^1, v_i^2, $\bar{v}_i^1$, $\bar{v}_i^2$ and $2\alpha+6$ nodes $u_i^0, \ldots, u_i^\alpha$ and $\bar{u}_i^0, \ldots, \bar{u}_i^\alpha$. The v-nodes are connected by the arcs (v_k^1, v_k^2), (v_k^2, v_k^1), $(\bar{v}_k^1, \bar{v}_k^2)$, $(\bar{v}_k^2, \bar{v}_k^1)$, $(v_k^1, \bar{v}_k^1)$, $(\bar{v}_k^1, v_k^1)$, $(v_k^2, \bar{v}_k^2)$, and $(\bar{v}_k^2, v_k^2)$ for all $k \in K$. Furthermore, we add two arcs $(\bar{v}_k^1, v_{k+1}^1)$ and $(\bar{v}_k^2, v_{k+1}^2)$ for each $k = 1, \ldots, m-1$, and two arcs $(\bar{v}_m^1, v_1^2)$ and $(\bar{v}_m^2, v_1^1)$ for $k = m$. The u-nodes are connected by the arcs $(u_i^l, \bar{u}_i^l)$ and $(\bar{u}_i^l, u_i^l)$, for $l = 0, \ldots, \alpha$, and by the arcs (u_i^l, u_i^{l+1}) and $(\bar{u}_i^l, \bar{u}_i^{l+1})$, for $l = 0, \ldots, \alpha - 1$.

For each edge $w_iw_j \in F$, we we add the 8 arcs (v_i^2, u_j^0), $(\bar{v}_i^2, \bar{u}_j^0)$, (u_j^α, v_i^1), $(\bar{u}_j^\alpha, \bar{v}_i^1)$ and (v_j^2, u_i^0), $(\bar{v}_j^2, \bar{u}_i^0)$, (u_i^α, v_j^1), $(\bar{u}_i^\alpha, \bar{v}_j^1)$.

The resulting digraph D is illustrated in Figure 1.

The given path set consists of four different types of paths. For each node $w_i \in W$, we have 2α many paths in the set

$$\mathcal{S}_i^1 := \left\{ (\bar{u}_i^l, u_i^l, u_i^{l+1}),\ (u_i^l, \bar{u}_i^l, \bar{u}_i^{l+1}) \mid l = 0, \ldots, \alpha - 1 \right\}$$

and 4 paths in the sets

$$\begin{aligned} \mathcal{S}_i^2 &:= \left\{ (\bar{v}_i^1, v_i^1, v_i^2),\ (v_i^2, v_i^1, \bar{v}_i^1),\ (\bar{v}_i^1, \bar{v}_i^2, v_{i+1}^2),\ (\bar{v}_i^2, \bar{v}_i^1, v_{i+1}^1) \right\} \quad &\text{if } i \neq n, \text{ and} \\ \mathcal{S}_m^2 &:= \left\{ (\bar{v}_i^1, v_i^1, v_i^2),\ (v_i^2, v_i^1, \bar{v}_i^1),\ (\bar{v}_i^1, \bar{v}_i^2, v_{i+1}^1),\ (\bar{v}_i^2, \bar{v}_i^1, v_{i+1}^2) \right\} \quad &\text{for } i = m. \end{aligned}$$

For each edge $w_iw_j \in F$, we consider the 4 paths in the sets

$$\mathcal{S}_{i,j}^3 := \{ (v_i^2, \bar{v}_i^2, \bar{u}_j^0),\ (\bar{v}_i^2, v_i^2, u_j^0) \}, \qquad \mathcal{S}_{i,j}^4 := \{ (u_j^\alpha, \bar{u}_j^\alpha, \bar{v}_i^1),\ (\bar{u}_j^\alpha, u_j^\alpha, v_i^1) \},$$

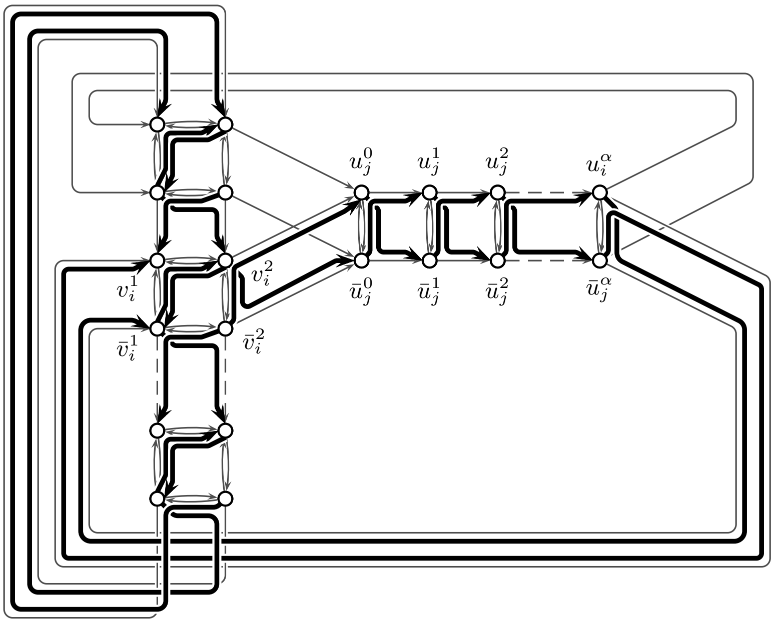

Figure 2: Union of the path sets $\mathcal{S}_i^2$ for all $i \in I$ and the path sets $\mathcal{S}_i^1$, $\mathcal{S}_{i,j}^3$, and $\mathcal{S}_{i,j}^4$ for some edge $w_i w_j \in F$.

and the 4 paths in the sets $\mathcal{S}_{j,i}^3$ and $\mathcal{S}_{j,i}^4$ obtained by exchanging the indices i and j. The path set $\mathcal{S}$ is the union of all those sets, i.e.,

$$\mathcal{S} := \bigcup_{w_i \in W} (\mathcal{S}_i^1 \cup \mathcal{S}_i^2) \ \cup \bigcup_{w_i w_j \in F} (\mathcal{S}_{i,j}^3 \cup \mathcal{S}_{j,i}^3 \cup \mathcal{S}_{i,j}^4 \cup \mathcal{S}_{j,i}^4) \ .$$

Figure 2 illustrates these paths. Note that each path in $\mathcal{S}$ is a shortest path between its terminal nodes and contains exactly two arcs.

In the first part of the proof, we show that any dominating vertex set $X \subseteq W$ for H can be transformed into a non-USPS $\mathcal{R} := \mathcal{R}(X) \subseteq \mathcal{S}$ in D with $|\mathcal{R}| = 2\alpha\,|X| + 8m$. As a byproduct, this also proves that the constructed path system $\mathcal{S}$ is indeed a non-USPS.

Let $X \subseteq W$ be a dominating vertex set for H. For each node $w_i \in W$, we denote by $x(w_i)$ the lexicographically first node $w_j \in X$ such that $w_i w_j \in F$. Since X is a vertex cover, such a node exists for each $w_i \in W$. We define the path set corresponding to X as

$$\mathcal{R} = \mathcal{R}(X) := \bigcup_{i\colon w_i \in X} \mathcal{S}_i^1 \ \cup \bigcup_{i \in I} \mathcal{S}_i^2 \ \cup \bigcup_{i,j \in I\colon w_j = x(w_i)} \mathcal{S}_{i,j}^3 \cup \mathcal{S}_{i,j}^4 \ .$$

The size of this path set is

$$|\mathcal{R}| = 2\alpha \cdot |X| + 8n \ . \tag{2}$$

The path set $\mathcal{R}$ is a USPS if and only if the linear system (1) has a feasible solution. In order to show that $\mathcal{R}$ is no USPS, it is therefore sufficient to show that the following sub-system of (1) has no solution:

$$\lambda_{(\bar{v}_i^1,v_i^1)} + \lambda_{(v_i^1,v_i^2)} + 1 \le \lambda_{(\bar{v}_i^1,\bar{v}_i^2)} + \lambda_{(\bar{v}_i^2,v_i^2)} \qquad \forall\, i \tag{3a}$$
$$\lambda_{(v_i^1,\bar{v}_i^1)} + \lambda_{(v_i^2,v_i^1)} + 1 \le \lambda_{(\bar{v}_i^2,\bar{v}_i^1)} + \lambda_{(v_i^2,\bar{v}_i^2)} \qquad \forall\, i \tag{3b}$$
$$\lambda_{(\bar{v}_i^1,\bar{v}_i^2)} + \lambda_{(\bar{v}_i^2,v_{i+1}^2)} + 1 \le \lambda_{(\bar{v}_i^1,v_{i+1}^1)} + \lambda_{(v_{i+1}^1,v_{i+1}^2)} \qquad \forall\, i \neq n \tag{3c}$$
$$\lambda_{(\bar{v}_i^2,\bar{v}_i^1)} + \lambda_{(\bar{v}_i^1,v_{i+1}^1)} + 1 \le \lambda_{(\bar{v}_i^2,v_{i+1}^2)} + \lambda_{(v_{i+1}^2,v_{i+1}^1)} \qquad \forall\, i \neq n \tag{3d}$$
$$\lambda_{(\bar{v}_n^1,\bar{v}_n^2)} + \lambda_{(\bar{v}_n^2,v_1^1)} + 1 \le \lambda_{(\bar{v}_n^1,v_1^2)} + \lambda_{(v_1^2,v_1^1)} \tag{3e}$$
$$\lambda_{(\bar{v}_n^2,\bar{v}_n^1)} + \lambda_{(\bar{v}_n^1,v_1^2)} + 1 \le \lambda_{(\bar{v}_n^2,v_1^1)} + \lambda_{(v_1^1,v_1^2)} \tag{3f}$$
$$\lambda_{(\bar{u}_i^l,u_i^l)} + \lambda_{(u_i^l,u_i^{l+1})} + 1 \le \lambda_{(\bar{u}_i^l,\bar{u}_i^{l+1})} + \lambda_{(\bar{u}_i^{l+1},u_i^{l+1})} \qquad \forall\, i,l:\, w_i \in X,\, l \neq \alpha \tag{3g}$$
$$\lambda_{(u_i^l,\bar{u}_i^l)} + \lambda_{(\bar{u}_i^l,\bar{u}_i^{l+1})} + 1 \le \lambda_{(u_i^l,u_i^{l+1})} + \lambda_{(u_i^{l+1},\bar{u}_i^{l+1})} \qquad \forall\, i,l:\, w_i \in X,\, l \neq \alpha \tag{3h}$$
$$\lambda_{(\bar{v}_i^2,v_i^2)} + \lambda_{(v_i^2,u_j^0)} + 1 \le \lambda_{(\bar{v}_i^2,\bar{u}_j^0)} + \lambda_{(\bar{u}_j^0,u_j^0)} \qquad \forall\, i,j:\, w_j = x(w_i) \tag{3i}$$
$$\lambda_{(v_i^2,\bar{v}_i^2)} + \lambda_{(\bar{v}_i^2,\bar{u}_j^0)} + 1 \le \lambda_{(v_i^2,u_j^0)} + \lambda_{(u_j^0,\bar{u}_j^0)} \qquad \forall\, i,j:\, w_j = x(w_i) \tag{3j}$$
$$\lambda_{(u_j^\alpha,\bar{u}_j^\alpha)} + \lambda_{(\bar{u}_j^\alpha,\bar{v}_i^1)} + 1 \le \lambda_{(u_j^\alpha,v_i^1)} + \lambda_{(v_i^1,\bar{v}_i^1)} \qquad \forall\, i,j:\, w_j = x(w_i) \tag{3k}$$
$$\lambda_{(\bar{u}_j^\alpha,u_j^\alpha)} + \lambda_{(u_j^\alpha,v_i^1)} + 1 \le \lambda_{(\bar{u}_j^\alpha,\bar{v}_i^1)} + \lambda_{(\bar{v}_i^1,v_i^1)} \qquad \forall\, i,j:\, w_j = x(w_i) \tag{3l}$$

Inequalities (3a) ensure that, for each i, the path $(\bar{v}_i^1, v_i^1, v_i^2)$ is strictly shorter than the other two-arc path $(\bar{v}_i^1, \bar{v}_i^2, v_k^2)$ from $\bar{v}_i^1$ to v_i^2. Together, (3a)–(3f) express that each path in $\bigcup_i \mathcal{S}_i^2$ is strictly shorter than its alternative other two-arc path. Analogously, inequalities (3g) and (3h) enforce that each path in $\bigcup_{i:w_i \in X} \mathcal{S}_i^1$ is shorter than its respective alternative two-arc path, while inequalities (3i)–(3l) ensure that each path in $\bigcup_{i,j:\, w_j = x(w_i)} \mathcal{S}_{i,j}^3 \cup \mathcal{S}_{i,j}^4$ is shorter than the corresponding other two-arc path.

To verify that this linear system has no solution, we apply Farkas' Lemma. For each i, l with $w_i \in X$ and $l \neq \alpha$, we multiply both inequalities (3g) and (3h) with a factor of $\mu(i) := |\{w_j \in W : w_i = x(w_j)\}|$, which yields the equivalent inequalities (3g') and (3h'). Adding all inequalities (3a)–(3f), (3g'), (3h'), and (3i)–(3l) yields an inequality that contains each variable λ_a, $a \in A$, with the same coefficient on the left and on the right hand side plus a positive constant on the left hand side. As this inequality cannot be satisfied, $\mathcal{R}$ is no USPS.

In the second part of the proof, we now show that any irreducible non-USPS $\mathcal{R} \subseteq \mathcal{S}$ in D can be transformed back into a dominating set $X := X(\mathcal{R}) \subseteq W$ for H with $2\alpha\,|X| \ge |\mathcal{R}| - 8n$. It is sufficient to define such a backward transformation only for irreducible (i.e., inclusion-wise minimal) non-USPSs, because any non-USPS $\mathcal{R}' \subseteq \mathcal{S}$ in D can be reduced to an irreducible non-USPS $\mathcal{R} \subseteq \mathcal{R}'$ in polynomial time (using the greedy algorithm proposed in [3], for example).

In order to define the backward transformation properly, we first need to show that all irreducible non-USPSs in D have a structure that is similar to that of the non-USPSs $\mathcal{R}(X)$ constructed in the first part of the proof. So, let $\mathcal{R} \subseteq \mathcal{S}$ be an irreducible non-USPS.

First, observe that all paths in $\bigcup_i \mathcal{S}_i^2$ must be contained in $\mathcal{R}$. Suppose there is some $k \in I$ such that the path $(v_k^2, v_k^1, \bar{v}_k^1)$ does not belong to $\mathcal{R}$. Without loss of generality, we may assume that $k = 1$. Let $M \ge 2|A|$ and consider the metric

$$\lambda_a := \begin{cases} M+l+1, & \text{if } a \in \{(u_i^l, \bar{u}_i^l),\ (\bar{u}_i^l, u_i^l)\}, \\ M+\alpha+2, & \text{if } a \in \{(v_i^1, \bar{v}_i^1),\ (\bar{v}_i^1, v_i^1)\}, \\ M+1, & \text{if } a \in \{(\bar{v}_i^1, v_{i+1}^1),\ (\bar{v}_i^2, v_{i+1}^2),\ (\bar{v}_n^1, v_1^2),\ (\bar{v}_n^2, \bar{v}_1^1)\} \\ M+(2n-1)(\alpha+4)+1, & \text{if } a = (v_1^2, v_1^1), \\ M+(i-2)(\alpha+4)+1, & \text{if } a \in \{(v_i^2, v_i^1) : i \neq 1\}, \\ M+(i-1)(\alpha+4), & \text{if } a = (\bar{v}_i^2, \bar{v}_i^1), \\ M+(n+i-2)(\alpha+4)+1, & \text{if } a = (v_i^1, v_i^2), \\ M+(n+i-1)(\alpha+4), & \text{if } a = (\bar{v}_i^1, \bar{v}_i^2), \text{ and} \\ M, & \text{otherwise.} \end{cases}$$

Note that $M \leq \lambda_a < 3/2M$ for all $a \in A$. Since all paths in $\mathcal{S}$ contain exactly two arcs, no path with three or more arcs in D can be shorter than any path in $\mathcal{S}$. For each path $P \in \mathcal{S} \setminus \{(v_1^2, v_1^1, \bar{v}_1^1)\}$, however, there is only one alternative (s_P, t_P)-path with only two arcs. Knowing this, one can easily verify that each path in $\mathcal{S} \setminus \{(v_1^2, v_1^1, \bar{v}_1^1)\}$ is indeed shorter than the corresponding alternative two-arc path. Hence, $\mathcal{S} \setminus \{(v_1^2, v_1^1, \bar{v}_1^1)\}$ is a USPS, which implies that the path $(v_1^2, v_1^1, \bar{v}_1^1)$ must be contained in any non-USPS $\mathcal{R} \subseteq \mathcal{S}$. Analogously, it follows that any other path $P \in \bigcup_i \mathcal{S}_i^2$ is contained in $\mathcal{R}$.

Using the same technique, we can show that, for any $i \in I$, there exists some w_j with $w_i w_j \in F$ such that $\mathcal{S}_{i,j}^3 \subset \mathcal{R}$. Without loss of generality, let $i = 1$ and suppose that for each j with $w_i w_j$ at least one of the two paths $P_{ij} = (\bar{v}_i^2, v_i^2, u_j^0)$ or $\bar{P}_{ij} = (v_i^2, \bar{v}_i^2, \bar{u}_j^0)$ is not contained in $\mathcal{R}$. Then, with λ defined as above, the metric

$$\lambda'_a := \begin{cases} \lambda_a + 2n(\alpha+4) & \text{if } a = (v_i^2, u_j^0) \text{ and } P_{ij} \notin \mathcal{R}, \\ \lambda_a + 2n(\alpha+4) & \text{if } a = (\bar{v}_i^2, \bar{u}_j^0) \text{ and } \bar{P}_{ij} \notin \mathcal{R}, \\ \lambda_a & \text{otherwise,} \end{cases}$$

is compatible with the path set $\mathcal{S}' := \mathcal{S} \setminus (\{P_{ij} : w_i w_j \in F,\ P_{ij} \notin \mathcal{R}\} \cup \{\bar{P}_{ij} : w_i w_j \in F, /, \bar{P}_{ij} \notin \mathcal{R}\})$ obtained by removing the missing paths P_{ij} and $\bar{P}_{ij}$ from the entire path set $\mathcal{S}$. To verify that the metric λ' is compatible with this path set, it is again sufficient to check that each path is shorter than the corresponding alternative two-arc path between its terminals. As the path set $\mathcal{R}$ is fully contained in $\mathcal{S}'$ and $\mathcal{R}$ is assumed to be a non-USPS, there must be some j with $w_i w_j \in F$, such that both paths P_{ij} and $\bar{P}_{ij}$ of $\mathcal{S}_{i,j}^3$ are contained in $\mathcal{S}$. For each $i \in I$, we let $x(i)$ be the lexicographically first index j with $\mathcal{S}_{i,j}^3 \subset \mathcal{R}$.

Analogously, one can show that $\mathcal{S}_j^1 \subset \mathcal{R}$ for each $j = x(i)$ and $i \in I$. Furthermore, one finds that, for any $i \in I$, there exist some $j = x'(i)$ with $w_i w_j \in F$ such that $\mathcal{S}_{i,j}^4 \subset \mathcal{R}$.[1]

Now we can define the dominating set corresponding to the irreducible non-USPS $\mathcal{R}$ as

$$X = X(\mathcal{R}) := \{w_{x(i)} \ : \ i \in I\}.$$

[1] Note that $x(i)$ and $x'(i)$ may be different. There may exist an irreducible non-USPS $\mathcal{R} \subset \mathcal{S}$ in the constructed digraph D that contains only one of the two path sets $\mathcal{S}_{i,j}^3$ and $\mathcal{S}_{i,j}^4$ for each $w_i w_j \in F$.

Because $w_i w_j \in F$ for any $j = x(i)$, the set X is indeed a dominating set for H. (Recall that we assumed $\{w_i w_i : i \in I\} \subseteq F$ for notational simplicity.) The above observations imply that

$$|\mathcal{R}| \geq \sum_{i \in I} \left(|\mathcal{S}_i^2| + |\mathcal{S}_{i,x(i)}^3| + |\mathcal{S}_{i,x'(i)}^4| \right) + \sum_{i \in I:\, w_i \in X} |\mathcal{S}_i^1| \geq 8n + 2\alpha\,|C| \ . \qquad (4)$$

Now it follows in a straightforward way that computing an approximate solution for Min-Non-USPS is at least as hard as computing an approximate solution for Minimum Dominating Set. Let $\alpha := n^p$ for some $p \geq 2$. With this choice of α, both the construction of the Min-Non-USPS instance and the backward transformation of an irreducible non-USPS to a dominating set are polynomial in the encoding size of H.

Due to (2) and (4), we have $|\mathcal{R}^*| = 8n + 2\alpha|X^*|$ for any *minimum* dominating set X^* for H and any *minimum* non-USPS $\mathcal{R}^* \subseteq \mathcal{S}$ in D. Suppose we are given a $c \log|\mathcal{S}|$-approximate solution $\mathcal{R}$ for the constructed Min-Non-USPS instance and apply the backward transformation described above to construct a dominating set $X(\mathcal{R})$ for H. Then (4) implies

$$\frac{2\alpha|X(\mathcal{R})| + 8m}{2\alpha|X^*| + 8m} \leq \frac{|\mathcal{R}|}{|\mathcal{R}^*|} \leq c \log|\mathcal{S}| \ .$$

With $\alpha = n^p$ and $|\mathcal{S}| = (2\alpha + 4)n + 8m \in \Theta(n^{p+1})$, it follows furthermore that

$$\begin{aligned} \frac{|X(\mathcal{R})|}{|X^*|} &\in \Theta(c \log n^{p+1}) + \mathcal{O}\Big(\frac{8n(c \log n^{p+1} - 1)}{2n^p|X^*|}\Big) \\ &\in \Theta(c \log n) \ . \end{aligned}$$

Consequently, any $c \log|\mathcal{S}|$-approximation algorithm for Min-Non-USPS leads to a $\beta c \log|W|$-approximation algorithm for the Minimum Dominating Set problem, for some $\beta > 0$. Since the exists some $c' > 0$ such that Minimum Dominating Set is inapproximable within a factor of $c' \log|W|$, there also is a $c > 0$ such that Min-Non-USPS is inapproximable within a factor of $c \log|\mathcal{S}|$. □

The logarithmic inapproximability threshold carries over directly to the weighted problem version Min-Weight-Non-USPS.

Corollary 3.2 *There exists some $c > 0$ such that it is $\mathcal{NP}$-hard to approximate* Min-Weight-Non-USPS *within a factor of* $c \log|\mathcal{S}|$.

4 Concluding remarks

In this paper, we showed that it is $\mathcal{NP}$-hard to approximate the problems Min-Non-USPS and Min-Weight-Non-USPS of finding a minimum size or minimum weight non-USPS contained in a given path set $\mathcal{S}$ within a factor of $c \log|\mathcal{S}|$ for some $c > 0$. This implies that the problem of finding an inequality that separates a given fractional path-routing from the unique shortest path routing polytope is hard to approximate within a logarithmic factor as well.

These results and the presented proof carry over in a straightforward way also to the case where we seek for a minimum weight conflict in an invalid shortest multi-path routing, where a conflict is given as a set of shortest paths and a set of non-shortest paths, that cannot be shortest and non-shortest paths simultaneously.

With a slight modification, the presented proofs also carry over to the problem where we seek for a minimum size or minimum weight conflict in a given collection of shortest path graphs (see [8, 3]) for both the unsplittable shortest and the shortest multi-path routing variant. The (extended) shortest path graph for a destination node defines which arcs must be contained in any shortest path towards this destination and which arcs must not be contained in any shortest path towards this destination. A conflict in this representation is a pair of two sets. One set contains the prescribed destination-arc pairs, where the arc must be contained in a shortest path towards the destination, while the other set contains the forbidden destination-arc pairs, where the arc must not be contained in any shortest path towards the destination. These variants of the problem of finding a minimum weight shortest path routing conflict arise in the separation problem over the polytopes associated with integer linear programming formulations of shortest path routing problems that are based on arc routings or shortest path graphs.

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