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ISOPARAMETRIC SURFACES IN 3-DIMENSIONAL DE SITTER SPACE AND ANTI-DE SITTER SPACE

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ISOPARAMETRIC SURFACES IN 3-DIMENSIONAL DE SITTER SPACE AND ANTI-DE SITTER SPACE

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ABSTRACT. A spacelike surface M in 3-dimensional de Sitter space $\mathbb{S}_1^3$ or 3-dimensional anti-de Sitter space $\mathbb{H}_1^3$ is called isoparametric, if M has constant principal curvatures. A timelike surface is called isoparametric, if its minimal polynomial of the shape operator is constant. In this paper, We determine the spacelike isoparametric surfaces and the timelike isoparametric surfaces in $\mathbb{S}_1^3$ and $\mathbb{H}_1^3$.

§1. Introduction.

A hypersurface M of a complete simply-connected Riemannian manifold $\mathbb{R}^{n+1}(c)$ of constant curvature c is isoparametric if M has constant principal curvatures. There are many results about the isoparametric hypersurfaces in Riemannian space forms (cf. [CE], [CHEN], [N-R]). For the isoparametric hypersurfaces in the indefinite space forms, Nomizu [NO] derived the Cartan formula for spacelike isoparametric hypersurfaces in Lorentzian space forms. Hahn [HA] considered the general case of indefinite space forms of curvature c and obtained the Cartan-type formula. Magid [MA] studied Lorentzian hypersurfaces in the Minkowski space $\mathbb{E}_1^n$. He obtained a complete classification of isoparametric hypersurfaces in $\mathbb{E}_1^n$. In this paper, we consider the problem in 3-dimensional de Sitter space $\mathbb{S}_1^3$ and 3-dimensional anti-de Sitter space $\mathbb{H}_1^3$. A spacelike surface M in 3-dimensional de Sitter space $\mathbb{S}_1^3$ or 3-dimensional anti-de Sitter space $\mathbb{H}_1^3$ is called isoparametric, if M has constant principal curvatures. A timelike surface M in 3-dimensional de Sitter space $\mathbb{S}_1^3$ or 3-dimensional anti-de Sitter space $\mathbb{H}_1^3$ is called isoparametric, if its minimal polynomial of the shape operator is constant. we will prove the following theorems.

Theorem 1.1. *Let $x : \mathrm{M} \to \mathbb{S}_1^3$ be a spacelike isoparametric surface, then, by a transformation in $\mathbb{E}_1^4$, it can be written as the one of the following surfaces:*

(i) *the totally umbilical surface;*

(ii) $x(u,v) = (a\sin(u), a\cos(u), b\sinh(v), b\cosh(v)), \quad a^2 - b^2 = 1.$

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Theorem 1.2. *Let $x : \mathbf{M} \to \mathbb{H}^3_1$ be a spacelike isoparametric surface, then, by a transformation in $\mathbb{E}^4_2$, it can be written as the one of the following surfaces:*
(i) *the totally umbilical surface;*
(ii) $x(u, v) = (a\sinh(u), b\sinh(v), a\cosh(u), b\cosh(v)), \quad a^2 + b^2 = 1.$

Theorem 1.3. *Let $x : \mathbf{M} \to \mathbb{S}^3_1$ be a timelike isoparametric surface such that the mean curvature H and the Gauss curvature κ satisfy $H^2 - \kappa + 1 \neq 0$, then, by a transformation in $\mathbb{E}^4_1$, it can be written as the following surface:*

$$x(u, v) = (a\sin\frac{1}{\sqrt{2}a}(u + v), a\cos\frac{1}{\sqrt{2}a}(u + v), b\cosh\frac{1}{\sqrt{2}b}(u - v), b\sinh\frac{1}{\sqrt{2}b}(u - v)),$$

where $a^2 + b^2 = 1$.

Theorem 1.4. *Let $x : \mathbf{M} \to \mathbb{H}^3_1$ be a timelike isoparametric surface such that the mean curvature H and the Gauss curvature κ satisfy $H^2 - \kappa - 1 > 0$, then, by a transformation in $\mathbb{E}^4_2$, it can be written as the one of the following surfaces:*

(i) $x(u, v) = (a\sinh\frac{1}{\sqrt{2}a}(u + v), b\cosh\frac{1}{\sqrt{2}b}(u - v), a\cosh\frac{1}{\sqrt{2}a}(u + v), b\sinh\frac{1}{\sqrt{2}b}(u - v)),$

where $a^2 - b^2 = 1$.

(ii) $x(u, v) = (a\sin\frac{1}{\sqrt{2}a}(u + v), a\cos\frac{1}{\sqrt{2}a}(u + v), b\sin\frac{1}{\sqrt{2}b}(u - v), b\cos\frac{1}{\sqrt{2}b}(u - v)),$

where $a^2 - b^2 = -1$.

§2. Preliminaries.

Let $\mathbb{E}^m_q$ be the m-dimensional pseudo-Euclidean space with the natural basis $e_1, ..., e_m$, its metric $< \, , \, >$ is given by

$$< x, y >= \sum_{i=1}^{m-q} x_i y_i - \sum_{j=m-q+1}^{m} x_j y_j, \quad x, y \in \mathbb{E}^m_q, \tag{2.1}$$

where $x = (x_1, x_2, ..., x_m)$, $y = (y_1, y_2, ..., y_m)$. The n-dimensional de Sitter space $\mathbb{S}^n_1$ and n-dimensional anti-de Sitter space $\mathbb{H}^n_1$ are defined by

$$\mathbb{S}^n_1 = (x \in \mathbb{E}^{n+1}_1 :< x, x >= 1), \tag{2.2}$$

$$\mathbb{H}^n_1 = (x \in \mathbb{E}^{n+1}_2 :< x, x >= -1). \tag{2.3}$$

It is well known that $\mathbb{S}^n_1$ and $\mathbb{H}^n_1$ are the complete connected pseudo-Riemannian hypersurfaces with constant sectional curvature 1 and -1 in $\mathbb{E}^{n+1}_1$ and $\mathbb{E}^{n+1}_2$, respectively ([O]).

Let $\mathbf{N}$ be a pseudo-Riemannian manifold with the pseudo-Riemannian metric $\bar{g}$ and $\mathbf{M}$ be a submanifold of $\mathbf{N}$. If the pseudo-Riemannian metric $\bar{g}$ of N induces a Riemannian metric

g (respectively, a pseudo-Riemannian metric, a degenerate quadric form) on $\mathbf{M}$, then $\mathbf{M}$ is called a spacelike (respectively, timelike, degenerate) submanifold.

We denote by $\tilde{\nabla}$ the covariant differentiation with respect to the indefinite Riemannian metric of $\mathbb{E}^4_1$ (or $\mathbb{E}^4_2$) and by $\overline{\nabla}$ and ∇ the covariant differentiations with respect to the induced metric of $\mathbb{S}^3_1$ (or $\mathbb{H}^3_1$) and $\mathbf{M}$, respectively. We denote by $\eta(x) = -\varepsilon x$, ($x \in \mathbb{S}^3_1$, $\varepsilon = 1$, $x \in \mathbb{H}^3_1$, $\varepsilon = -1$), the normal vector field of $\mathbb{S}^3_1$ (or $\mathbb{H}^3_1$) in $\mathbb{E}^4_1$ (or $\mathbb{E}^4_2$); ξ, the normal vector field of $\mathbf{M}$ in $\mathbb{S}^3_1$ (or $\mathbb{H}^3_1$). Then, considering that $\mathbf{M}$ is locally embedded in $\mathbb{S}^3_1$ (or $\mathbb{H}^3_1$), we have the following Gauss's and Weingarten's formulas.

$$\begin{cases} \tilde{\nabla}_X Y = \overline{\nabla}_X Y + < X, Y > \eta \\ \overline{\nabla}_X Y = \nabla_X Y + h(X,Y)\xi \\ \overline{\nabla}_X \xi = -A(X), \end{cases} \tag{2.4}$$

where X and Y are tangent vector fields on $\mathbf{M}$, and A is a field of type (1,1) tensor (Weingarten operator) on $\mathbf{M}$ corresponding to ξ, i.e.,

$$< A(X), Y >= h(X,Y) < \xi, \xi > . \tag{2.5}$$

Proposition 2.1. *Let $x : \mathbf{M} \to \mathbb{S}^3_1$ (or $\mathbb{H}^3_1$) be a timelike surface in $\mathbb{S}^3_1$ (or $\mathbb{H}^3_1$). Then the Weingarten operator A of x has real eigenvalues if and only if the mean curvature H and the Gauss curvature κ of x satisfying $H^2 - \kappa \geq 0$.*

Proof. Let $x : \mathbf{M} \to \mathbb{S}^3_1$ (or $\mathbb{H}^3_1$) be a timelike surface and $\{e_1, e_2\}$ be a local pseudo-orthonormal basis of $\mathbf{TM}$ such that the metric of x is given by

$$\mathrm{d}s^2 = e^w(\mathrm{d}u^2 - \mathrm{d}v^2).$$

From (2.4) we have

$$A(e_1) = h_{11}e_1 - h_{12}e_2$$
$$A(e_2) = h_{21}e_1 - h_{22}e_2,$$

where $h_{ij} = h(e_i, e_j)$. Thus A has real eigenvalues if and only if

$$(h_{11} - h_{22})^2 - 4(h_{12}^2 - h_{11}h_{22}) = 4(H^2 - \kappa) \geq 0.$$

□

It is easy to see by Theorem 1.3 and Proposition 2.1:

Corollary 2.1. *Let $x : \mathbf{M} \to \mathbb{S}^3_1$ be a timelike isoparametric surface such that its Weingarten operator has real eigenvalues, then, by a transformation in $\mathbb{E}^4_1$, it can be written as the following surface:*

$$x(u,v) = (a \sin \frac{1}{\sqrt{2}a}(u+v), a \cos \frac{1}{\sqrt{2}a}(u+v), b \cosh \frac{1}{\sqrt{2}b}(u-v), b \sinh \frac{1}{\sqrt{2}b}(u-v)),$$

where $a^2 + b^2 = 1$.

§3. Spacelike isoparematric surfaces in $\mathbb{S}_1^3$ and $\mathbb{H}_1^3$.

In this section, we prove the Theorem 1.1 and Theorem 1.2 given in section 1. Let $x : \mathrm{M} \to \mathbb{S}_1^3$ (or $\mathbb{H}_1^3$) $\subset \mathbb{E}_1^4$ (or $\mathbb{E}_2^4$) be a spacelike surface of 3-dimensional de Sitter space $\mathbb{S}_1^3$ (or anti-de Sitte space $\mathbb{H}_1^3$) with the metric given by

$$g = 2e^w(\mathrm{d}u^2 + \mathrm{d}v^2) = 2e^w|\mathrm{d}z|^2 = e^w(\mathrm{d}z \otimes \mathrm{d}\bar{z} + \mathrm{d}\bar{z} \otimes \mathrm{d}z), \tag{3.1}$$

where $z = u + iv$, $\mathrm{d}z = \mathrm{d}u + i\mathrm{d}v$. Then from $< x, x >= \varepsilon$ (for $\mathbb{S}_1^3$, $\varepsilon = 1$; for $\mathbb{H}_1^3$, $\varepsilon = -1$) and

$$g =< \mathrm{d}x, \mathrm{d}x >= - < x, \mathrm{d}^2 x >= e^w(\mathrm{d}z \otimes \mathrm{d}\bar{z} + \mathrm{d}\bar{z} \otimes \mathrm{d}z)$$

we have

$$\begin{cases} < x_z, x > =< x_{\bar{z}}, x >=< x_z, x_z >=< x_{\bar{z}}, x_{\bar{z}} >=< x, x_{zz} >= 0 \\ < x, x_{\bar{z}\bar{z}} > =< x_z, x_{zz} >=< x_{\bar{z}}, x_{\bar{z}\bar{z}} >=< x_z, x_{z\bar{z}} >=< x_{\bar{z}}, x_{z\bar{z}} >= 0 \\ < x_z, x_{\bar{z}} > = - < x, x_{z\bar{z}} >= e^w. \end{cases} \tag{3.2}$$

We use

$$\partial_z = \frac{\partial}{\partial z} = \frac{1}{2}(\frac{\partial}{\partial u} - i\frac{\partial}{\partial v}), \quad \partial_{\bar{z}} = \frac{\partial}{\partial_{\bar{z}}} = \frac{1}{2}(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v}).$$

Let Δ be the Laplacian of g, then

$$\Delta = 2e^{-w}\partial_z\partial_{\bar{z}}, \qquad \kappa = -e^{-w}w_{z\bar{z}},$$

where κ is the Gauss curvature of g.

We choose $\xi \in \mathbb{S}_1^3$ (or $\mathbb{H}_1^3$) such that

$$< \xi, x_z >=< \xi, x_{\bar{z}} >=< \xi, x >= 0, \quad < \xi, \xi >= -1.$$

Then we have

$$x_{zz} = w_z x_z + \varphi\xi, \quad \varphi = - < x_{zz}, \xi > . \tag{3.3}$$

The mean curvture H of x is given by

$$H\xi = e^{-w}x_{z\bar{z}} + \varepsilon x. \tag{3.4}$$

If $H \neq 0$, we have

$$\varphi = -H^{-1}e^{-w} < x_{zz}, x_{z\bar{z}} > . \tag{3.5}$$

Let $\Phi = \varphi \mathrm{d}z^2$. Then Φ is global defined and Φ^2 is independent of the choise of ξ.

For the surface x we have the following structure equations:

$$\begin{cases} x_{zz} = w_z x_z + \varphi\xi \\ x_{z\bar{z}} = -\varepsilon e^w x + He^w\xi \\ x_{\bar{z}\bar{z}} = w_{\bar{z}} x_{\bar{z}} + \bar{\varphi}\xi \\ \xi_z = Hx_z + \varphi e^{-w}x_{\bar{z}} \\ \xi_{\bar{z}} = \bar{\varphi}e^{-w}x_z + Hx_{\bar{z}}. \end{cases} \tag{3.6}$$

From $x_{z\bar{z}z} = x_{zz\bar{z}}$ we obtain the integrability conditions for the structure equations of x:

$$\begin{cases} w_{z\bar{z}} + e^{-w}|\varphi|^2 = -\varepsilon e^w + H^2 e^w \\ \varphi_{\bar{z}} = H_z e^w \end{cases} \tag{3.7}$$

that is

$$\begin{cases} \varphi_{\bar{z}} = H_z e^w \\ -\kappa + e^{-2w}|\varphi|^2 = H^2 - \varepsilon. \end{cases} \tag{3.8}$$

By (2.4) and (2.5) we have

$$\varphi = -\frac{1}{8}e^{-w}(h_{11} - h_{22} + 2ih_{12}), \tag{3.9}$$

where h_{ij} is the second fundamental form of x.

If $x : \mathrm{M} \to \mathbb{S}^3_1$ (or $\mathbb{H}^3_1$) is a isoparametric surface, then by the definition we know that the Gauss curvature κ and the mean curvature H of x are constant. From (3.8) we know that $\varphi_{\bar{z}} = 0$ and $e^{-2w}|\varphi|^2$ is constant. If $\varphi \equiv 0$, the surface is totally umbilical. If $\varphi \neq 0$, from (3.8) we have

$$0 = \Delta(\log e^{-2w}|\varphi|^2) = \Delta(-2w + \log\varphi + \log\bar{\varphi}) = -2\Delta w.$$

Therefore we get

$$\kappa = -e^{-w}w_{z\bar{z}} = -\frac{1}{2}\Delta w = 0.$$

The surface is flat and we can choose the coordinate z such that $w \equiv 0$. In this case, (3.8) becomes

$$\begin{cases} \varphi_{\bar{z}} = H_z = 0 \\ |\varphi|^2 = H^2 - \varepsilon. \end{cases} \tag{3.10}$$

But $\varphi_{\bar{z}} = 0$ and $|\varphi|^2 =$ *constant* yield that φ is constant. Then we have the following structure equations for the surface x:

$$\begin{cases} x_{zz} = \varphi\xi \\ x_{z\bar{z}} = -\varepsilon x + H\xi \\ x_{\bar{z}\bar{z}} = \bar{\varphi}\xi \\ \xi_z = Hx_z + \varphi x_{\bar{z}} \\ \xi_{\bar{z}} = \bar{\varphi}x_z + Hx_{\bar{z}}, \end{cases} \tag{3.11}$$

where $|\varphi|^2 = H^2 - \varepsilon$. By a transformation of z we can assume that $\varphi = \bar{\varphi} > 0$. We solve the equations (3.11) under the conditions $\varphi^2 = H^2 - \varepsilon$ and $\varphi > 0$. From $\varphi\xi = x_{zz} = x_{\bar{z}\bar{z}}$ we get $x_{uv} = 0$. Then the surface x can be written as

$$x = f(u) + g(v), \quad f(u), g(v) \in \mathbb{E}^4_1 \ (\text{or } \mathbb{E}^4_2). \tag{3.12}$$

From (3.11) and (3.12) we have

$$(H-\varphi)x_{uu}-(H+\varphi)x_{vv}=4\varepsilon\varphi x;$$

then

$$\begin{cases}(H-\varphi)f''(u)-4\varepsilon\varphi f(u)=a\\ (H+\varphi)g''(v)+4\varepsilon\varphi g(v)=a,\end{cases}$$

where a is constant vector in $\mathbb{E}^4_1$ (or $\mathbb{E}^4_2$).

By a translation in $\mathbb{E}^4_1$ (or $\mathbb{E}^4_2$) we may assume that $a=0$. Then we obtain

$$\begin{cases}f''(u)=\dfrac{4\varepsilon\varphi}{H-\varphi}f(u)\\ g''(v)=\dfrac{-4\varepsilon\varphi}{H+\varphi}g(v)\\ \varphi^2=H^2-\varepsilon.\end{cases}\tag{3.13}$$

(a) When $\varepsilon=1$, (3.13) is

$$\begin{cases}f''(u)=\dfrac{4\varphi}{H-\varphi}f(u)\\ g''(v)=\dfrac{-4\varphi}{H+\varphi}g(v)\\ (H+\varphi)(H-\varphi)=1.\end{cases}\tag{3.14}$$

Therefore

$$\begin{cases}f(u)=c_3\sinh\left(\sqrt{\dfrac{4\varphi}{H-\varphi}}u\right)+c_4\cosh\left(\sqrt{\dfrac{4\varphi}{H-\varphi}}u\right)\\ g(v)=c_1\sin\left(\sqrt{\dfrac{4\varphi}{H+\varphi}}v\right)+c_2\cos\left(\sqrt{\dfrac{4\varphi}{H+\varphi}}v\right),\end{cases}\tag{3.15}$$

for $H-\varphi>0$ or

$$\begin{cases}f(u)=c_1\sin\left(\sqrt{\dfrac{-4\varphi}{H-\varphi}}u\right)+c_2\cos\left(\sqrt{\dfrac{-4\varphi}{H-\varphi}}u\right)\\ g(v)=c_3\sinh\left(\sqrt{\dfrac{-4\varphi}{H+\varphi}}v\right)+c_4\cosh\left(\sqrt{\dfrac{-4\varphi}{H+\varphi}}v\right),\end{cases}\tag{3.16}$$

for $H-\varphi<0$. The surface is congruent to the surface (ii) of Theorem 1.1.

(b) When $\varepsilon=-1$, (3.13) is

$$\begin{cases}f''(u)=\dfrac{4\varphi}{\varphi-H}f(u)\\ g''(v)=\dfrac{4\varphi}{\varphi+H}g(v)\\ \varphi^2=H^2+1.\end{cases}\tag{3.17}$$

Therefore

$$\left\{\begin{array}{l} f(u)=c_1\sinh\left(\sqrt{\dfrac{4\varphi}{\varphi-H}}u\right)+c_3\cosh\left(\sqrt{\dfrac{4\varphi}{\varphi-H}}u\right) \\ g(v)=c_2\sinh\left(\sqrt{\dfrac{4\varphi}{\varphi+H}}v\right)+c_4\cosh\left(\sqrt{\dfrac{4\varphi}{\varphi+H}}v\right), \end{array}\right. \tag{3.18}$$

$\varphi-H>0$. The surface is congruent to the surface (ii) of Theorem 1.2.

This completes the proof of the Theorem 1.1 and Theorem 1.2.

§4. Timelike isoparematric surfaces in $\mathbb{S}^3_1$ and $\mathbb{H}^3_1$.

In this section, we prove the Theorem 1.3 and Theorem 1.4 given in section 1. Let $x:\mathrm{M}\to\mathbb{S}^3_1$ (or $\mathbb{H}^3_1$) $\subset\mathbb{E}^4_1$ (or $\mathbb{E}^4_2$) be a timelike surface of 3-dimensional de Sitter space $\mathbb{S}^3_1$ (or anti-de Sitte space $\mathbb{H}^3_1$) with the metric given by

$$g=e^w(\mathrm{d}u\otimes\mathrm{d}v+\mathrm{d}v\otimes\mathrm{d}u). \tag{4.1}$$

Then from $<x,x>=\varepsilon$ (for $\mathbb{S}^3_1$, $\varepsilon=1$; for $\mathbb{H}^3_1$, $\varepsilon=-1$) and

$$g=<\mathrm{d}x,\mathrm{d}x>=-<x,\mathrm{d}^2x>=e^w(\mathrm{d}u\otimes\mathrm{d}v+\mathrm{d}v\otimes\mathrm{d}u)$$

we have

$$\left\{\begin{array}{l} <x_u,x>=<x_v,x>=<x_u,x_u>=<x_v,x_v>=<x,x_{uu}>=0 \\ <x,x_{vv}>=<x_u,x_{uu}>=<x_v,x_{vv}>=<x_u,x_{uv}>=<x_v,x_{uv}>=0 \\ <x_u,x_v>=-<x,x_{uv}>=e^w. \end{array}\right. \tag{4.2}$$

We use

$$\partial_u=\frac{\partial}{\partial u},\quad \partial_v=\frac{\partial}{\partial_v}.$$

Let Δ be the Laplacian of g, then

$$\Delta=2e^{-w}\partial_u\partial_v,\qquad \kappa=-e^{-w}w_{uv},$$

where κ is the Gauss curvature of g.

We choose $\xi\in\mathbb{S}^3_1$ (or $\mathbb{H}^3_1$) such that

$$<\xi,x_u>=<\xi,x_v>=<\xi,x>=0,\quad <\xi,\xi>=1.$$

Then we have

$$\left\{\begin{array}{ll} x_{uu}=w_ux_u+\varphi\xi, & \varphi=<x_{uu},\xi> \\ x_{vv}=w_vx_v+\psi\xi, & \psi=<x_{vv},\xi>. \end{array}\right. \tag{4.3}$$

The mean curvture H of x is given by

$$H\xi=e^{-w}x_{uv}+\varepsilon x. \tag{4.4}$$

If $H \neq 0$, we have

$$\begin{cases} \varphi = H^{-1}e^{-w} < x_{uu}, x_{uv} > \\ \psi = H^{-1}e^{-w} < x_{vv}, x_{uv} > . \end{cases} \tag{4.5}$$

Let $\Phi = \varphi du^2$, $\Psi = \psi dv^2$. Then Φ and Ψ are global defined and Φ^2 and Ψ^2 are independent of the choise of ξ.

For the surface x we have the following structure equations:

$$\begin{cases} x_{uu} = w_u x_u + \varphi\xi \\ x_{uv} = -\varepsilon e^w x + He^w \xi \\ x_{vv} = w_v x_v + \psi\xi \\ \xi_u = -Hx_u - \varphi e^{-w} x_v \\ \xi_v = -\psi e^{-w} x_u - Hx_v. \end{cases} \tag{4.6}$$

From $x_{uvu} = x_{uuv}$ and $x_{uvv} = x_{vvu}$ we obtain the integrability conditions for the structure equations of x:

$$\begin{cases} w_{uv} - e^{-w}\varphi\psi = -\varepsilon e^w - H^2 e^w \\ \varphi_v = H_u e^w \\ \psi_u = H_v e^w \end{cases} \tag{4.7}$$

that is

$$\begin{cases} \varphi_v = H_u e^w \\ \psi_u = H_v e^w \\ \kappa + e^{-2w}\varphi\psi = H^2 + \varepsilon. \end{cases} \tag{4.8}$$

If $x : \mathbf{M} \to \mathbb{S}^3_1$ (or $\mathbb{H}^3_1$) is a isoparametric surface, then by the definition we know that the Gauss curvature κ and the mean curvature H of x are constant. From (4.8) we know that $\varphi_v = 0$, $\psi_u = 0$ and $e^{-2w}\varphi\psi$ is constant. If $H^2 - \kappa + \varepsilon \neq 0$, by (4.8) we know that $\varphi\psi \neq 0$. Then from (4.8) we get

$$0 = \Delta(\log e^{-2w}\varphi\psi) = \Delta(-2w + \log\varphi + \log\psi) = -2\Delta w.$$

Therefore we get

$$\kappa = -e^{-w}w_{uv} = -\frac{1}{2}\Delta w = 0. \tag{4.9}$$

The surface is flat and we can choose the coordinate (u, v) such that $w \equiv 0$. In this case, (4.8) becomes

$$\begin{cases} \varphi_v = H_u = 0 \\ \psi_u = H_v = 0 \\ \varphi\psi = H^2 + \varepsilon. \end{cases} \tag{4.10}$$

But $\varphi_v = 0$, $\psi_u = 0$ and $\varphi\psi = \textit{constant}$ yield that φ and ψ are constant. Then we have the following structure equations for the surface x:

$$\begin{cases} x_{uu} = \varphi\xi \\ x_{uv} = -\varepsilon x + H\xi \\ x_{vv} = \psi\xi \\ \xi_u = -Hx_u - \varphi x_v \\ \xi_v = -\psi x_u - Hx_v, \end{cases} \tag{4.11}$$

where $\varphi\psi = H^2 + \varepsilon$. We solve the equations (4.11) under the conditions $\varphi\psi > 0$. From (4.11) we have

$$\begin{cases} x_{uuuu} + 2\varphi H x_{uu} - \varepsilon\varphi^2 x = 0 \\ x_{vvvv} + 2\psi H x_{vv} - \varepsilon\psi^2 x = 0. \end{cases} \tag{4.12}$$

By a parametric transformation $(u, v) \to (\sqrt{|\varphi|}u, \sqrt{|\psi|}v)$ we obtain

$$\begin{cases} x_{uuuu} + 2H(\mathrm{sign}\varphi)x_{uu} - \varepsilon x = 0 \\ x_{vvvv} + 2H(\mathrm{sign}\psi)x_{vv} - \varepsilon x = 0, \end{cases} \tag{4.13}$$

where $\mathrm{sign}\varphi = 1$, when $\varphi > 0$; $\mathrm{sign}\varphi = -1$, when $\varphi < 0$. If $\varphi\psi > 0$, then (4.13) becomes:

$$\begin{cases} x_{uuuu} + 2H(\mathrm{sign}\varphi)x_{uu} - \varepsilon x = 0 \\ x_{vvvv} + 2H(\mathrm{sign}\varphi)x_{vv} - \varepsilon x = 0. \end{cases} \tag{4.14}$$

(a) When $\varepsilon = 1$,

$$x(u, v) = c_1 \sin\lambda(u + v) + c_2 \cos\lambda(u + v) + c_3 \cosh\mu(u - v) + c_4 \sinh\mu(u - v), \tag{4.15}$$

where $c_1, c_2, c_3, c_4 \in \mathbb{E}^4_1$ are constant vectors and

$$\lambda^2 = H(\mathrm{sign}\varphi) + \sqrt{H^2 + 1}, \quad \mu^2 = -H(\mathrm{sign}\varphi) + \sqrt{H^2 + 1}.$$

The surface is congruent to the surface given by Theorem 1.3.
(b) When $\varepsilon = -1$,

$$x(u, v) = c_1 \sinh\lambda(u + v) + c_2 \cosh\lambda(u - v) + c_3 \cosh\mu(u + v) + c_4 \sinh\mu(u - v), \tag{4.16}$$

where $c_1, c_2, c_3, c_4 \in \mathbb{E}^4_2$ are constant vectors and

$$\lambda^2 = -H(\mathrm{sign}\varphi) + \sqrt{H^2 - 1}, \quad \mu^2 = -H(\mathrm{sign}\varphi) - \sqrt{H^2 - 1}, \quad -H(\mathrm{sign}\varphi) > 0.$$

The surface is congruent to the surface (i) given by Theorem 1.4.

$$x(u, v) = c_1 \sin\lambda(u + v) + c_2 \cos\lambda(u + v) + c_3 \sin\mu(u - v) + c_4 \cos\mu(u - v), \tag{4.17}$$

where $c_1, c_2, c_3, c_4 \in \mathbb{E}^4_2$ are constant vectors and

$$\lambda^2 = H(\mathrm{sign}\varphi) - \sqrt{H^2 - 1}, \quad \mu^2 = H(\mathrm{sign}\varphi) + \sqrt{H^2 - 1}, \quad H(\mathrm{sign}\varphi) > 0.$$

The surface is congruent to the surface (ii) given by Theorem 1.4.

This completes the proof of the Theorem 1.3 and Theorem 1.4.

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