Spectral properties of the random conductance model

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Abstract

Charge and exciton transport in disordered media plays an essential role in modern technologies. Classical examples are amorphous and organic semiconductors where the disorder can give rise to localized electron states. These localized electrons effectively behave like discrete particles hopping between discrete sites in an inhomogeneous environment.

A popular model for such a hopping process is the random walk among random conductances where the jump rate between any two sites is the same in both directions. The long-time behavior of such a random walk, when it is killed at the boundary of a large box, is intimately linked to the first eigenvectors and eigenvalues of its generator with zero Dirichlet boundary condition. This follows directly from the spectral decomposition of the associated heat equation.

In this thesis, we study these first eigenvectors and eigenvalues when the underlying lattice is $\mathbb{Z}^d$. In addition to the spectrum, we study the homogenization properties of the corresponding Poisson equation.

Regarding the spectrum, we find that in dimensions $d \geq 2$ and for independent and identically distributed positive conductances, there is a sharp transition between a completely localized and a completely homogenized regime. This transition hinges on the exponent $q = \sup\{r \geq 0 \colon \mathbb{E}[\omega^{-r}] < \infty\}$ where $\mathbb{E}[\omega^{-r}]$ is the inverse rth moment of the conductance ω.

If $q < 1/4$, then we show that for almost every environment the first Dirichlet eigenvectors asymptotically concentrate in a single site and the corresponding eigenvalues scale subdiffusively. We further prove weak convergence of the rescaled eigenvalues to non-degenerate random variables. Our proofs are based on a spatial extreme value analysis of the local speed measure, Borel-Cantelli arguments, the Rayleigh-Ritz formula, results from percolation theory, path arguments and the Bauer-Fike theorem.

On the other hand, if $q > 1/4$, then we show that the properly rescaled first eigenvectors and eigenvalues converge almost-surely to the first eigenvectors and eigenvalues of a homogenized operator. For this result it is sufficient to assume stationary and ergodic conductances, which are positive between nearest neighbors. Apart from that, we further allow unbounded-range connections. In this general case we need a stronger integrability condition on the lower tail of the conductances, which coincides with a well-known necessary condition for the validity of a local central limit theorem for the random walk among random conductances. The main result on the way to spectral homogenization is the homogenization of the corresponding Poisson equation. More precisely, we prove two-scale convergence of the solutions and their gradients. As an application of spectral homogenization, we prove a quenched large deviation principle for the normalized and rescaled local times of the random walk in a growing box. Our proofs are based on a compactness result for the Laplacian's Dirichlet energy, Poincaré inequalities, Moser iteration and two-scale convergence.

Zusammenfassung

Die elektronischen Transporteigenschaften ungeordneter Medien spielen eine wichtige Rolle in vielen modernen Technologien. Klassische Beispiele sind amorphe und organische Halbleiter, deren Eigenschaften stark davon geprägt sind, dass viele Elektroneneigenzustände auf Grund der Unordnung lokalisiert sind. Diese lokalisierten Elektronen verhalten sich wie diskrete Teilchen, die in der inhomogenen Umgebung zwischen diskreten Orten hin und her springen.

Ein verbreitetes Modell für einen solchen Sprungprozess ist die Irrfahrt (*random walk*) mit zufälligen Leitfähigkeiten, für die die Sprungrate zwischen zwei Orten in beide Richtungen die gleiche ist. Das Langzeitverhalten einer solchen Irrfahrt, die zusätzlich darauf bedingt wird eine große Box nicht zu verlassen, wird stark durch die ersten Eigenvektoren und Eigenwerte ihres Generators mit Null-Dirichlet-Randbedingungen bestimmt. Dies folgt direkt aus der spektralen Zerlegung der zugehörigen Diffusionsgleichung.

In dieser Dissertation untersuchen wir diese ersten Eigenvektoren und Eigenwerte, wenn das zu Grunde liegende Gitter $\mathbb{Z}^d$ ist. Zusätzlich zum Spektrum untersuchen wir die Homogenisierungseigenschaften der verwandten Poissongleichung.

Für das Spektrum beobachten wir in Dimensionen $d \geq 2$ für unabhängige gleichverteilte Leitfähigkeiten einen scharfen Übergang zwischen einem komplett lokalisierten und einem komplett homogenisierten Regime. Dieser Übergang wird durch den Exponenten $q = \sup\{r \geq 0 \colon \mathbb{E}[\omega^{-r}] < \infty\}$ bestimmt, wobei $\mathbb{E}[\omega^{-r}]$ das inverse r-te Moment der Leitfähigkeiten ω ist.

Für den Fall $q < 1/4$ zeigen wir, dass sich die ersten Dirichlet-Eigenvektoren für fast alle Realisierungen der Umgebung in einem einzigen Ort konzentrieren und, dass der zugehörige Eigenwert subdiffusiv skaliert. Weiterhin zeigen wir, dass die reskalierten Eigenwerte in Verteilung zu nichttrivialen Zufallsvariablen konvergieren. Unsere Beweise benutzen eine räumliche Extremwertanalyse des lokalen Geschwindigkeitsmaßes, Borel-Cantelli-Argumente, das Rayleigh-Ritz-Prinzip, Ergebnisse aus der Perkolationstheorie, Pfadargumente und das Bauer-Fike-Theorem.

Auf der anderen Seite, wenn $q > 1/4$, dann zeigen wir, dass die richtig reskalierten ersten Eigenvektoren und Eigenwerte fast sicher zu den ersten Eigenvektoren und Eigenwerten eines homogenisierten Operators konvergieren. Für dieses Resultat können wir sogar annehmen, dass die Leitfähigkeiten stationär und ergodisch sind und langreichweitige Verbindungen existieren, solange wir weiterhin annehmen, dass die Leitfähigkeiten zwischen nächsten Nachbarn positiv bleiben. In diesem allgemeinen Fall brauchen wir allerdings eine stärkere Integrabilitätsbedingung an die unteren Schwänze der Leitfähigkeiten. Diese Bedingung fällt mit einer bekannten notwendigen Bedingung für den lokalen zentralen Grenzwertsatz der zugehörigen Irrfahrt zusammen. Unser Hauptresultat auf dem Weg zu spektraler Homogenisierung ist die Homogenisierung der verwandten Poissongleichung. Genauer gesagt beweisen wir die Zweiskalenkonvergenz der Lösungen und ihrer Gradienten. Als eine Anwendung der spektralen Homogenisierung zeigen wir ein fast-sicheres Prinzip der großen Abweichungen für die normierten und reskalierten Lokalzeiten der Irrfahrt in einer wachsenden Box. Unsere Beweise basieren auf einem Kompaktheitsresultat für die Dirichletenergie des Generators, Poincaré-Ungleichungen, Moseriteration und Zweiskalenkonvergenz.

Contents

Chapter 1
Introduction

For our modern-life technologies, it is of paramount importance to understand and control charge transport in a wide range of materials. In recent years, industry and research are especially interested in the transport properties of amorphous and organic materials since they combine several electronic, chemical and mechanical advantages.

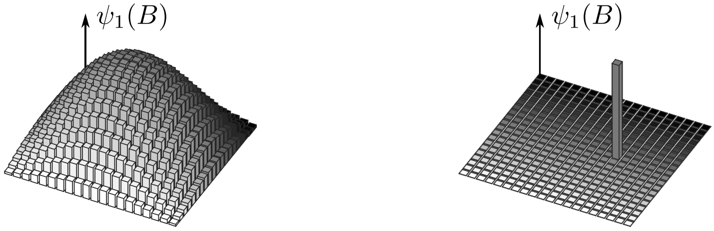
In contrast to inorganic crystals, these materials are highly disordered and it is therefore not a trivial question whether a certain material admits effective properties as, for example, an effective conductivity. Many disordered media do indeed display an effective behavior on large scales but this is definitely not the case for some chalcogenide glasses where the conductivity depends on the thickness of the material [SM75].

To explain why in some materials the disorder averages out and in others it does not is an intriguing quest for both physicists and mathematicians. Several models for disordered media have therefore enjoyed considerable interest in the recent decades. Prominent among these is the class of hopping models where discrete particles jump between discrete sites according to specific jump rates.

In this thesis, we consider such a model, the *random conductance model*, where the jump rate between any two sites is the same in both directions. A particle that moves according to these jump rates, performs a *random walk among random conductances*.

In the 1970's physicists were very interested in this model because it describes the high-temperature limit of the diffusion of optical excitations in fluorescence line-narrowing experiments, see [HHB77]. In recent years, it has received further mathematical attention and it was found that, under a wide range of conditions, the disorder does indeed average out, see [ABDH13, ADS16]. This means that, in many respects, we observe a homogeneous behavior on large scales. On the other hand, there are also regimes where particular properties display an anomalous behavior, see [BBHK08, BB12].

In this thesis, we are interested in the generator $\mathcal{L}^\omega$ of the random walk among random conductances, which we also call *random conductance Laplacian*. In particular, we study the spectrum of this Laplacian on a bounded



(a) Light tails near zero. **(b)** Heavy tails near zero.

Fig. 1.1: Principal eigenvector $\psi_1(B)$ of the Laplacian with i.i.d. conductances and zero Dirichlet conditions at the boundary of the domain $B \subset \mathbb{Z}^2$. Depending on the tails of the conductances near zero, the principal Dirichlet eigenvector either (a) asymptotically homogenizes for large domains B, or (b) it localizes in a single site.

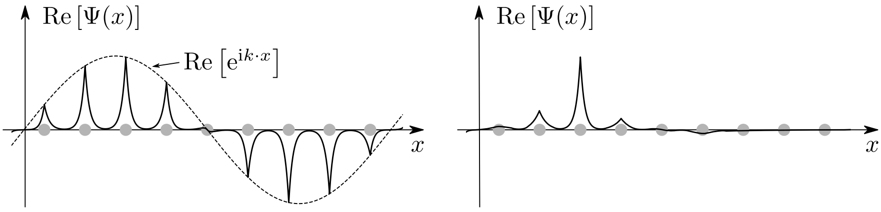
domain B with zero Dirichlet boundary conditions as well as the corresponding Poisson equation. The Poisson equation arises in many physical situations as for example in electrostatics when we wish to determine the profile of the stationary electric potential due to charges inserted into the system. We are going to see that, under a wide range of conditions, the solution to the Poisson equation homogenizes on large scales and we indeed recover an effective conductivity.

The top of the spectrum is linked to the long-time behavior of the random walk among random conductances when it is conditioned to stay in the domain B. This is easily seen from an eigenfunction expansion of the heat equation. What is remarkable, is that the spectrum of the Laplacian with independent and identically distributed (i.i.d.) conductances displays a sharp transition between a completely homogenized and a completely localized regime depending on the tails of the conductances near zero, see Figure 1.1.

Before we give a more extensive overview over our results and properly introduce all our main objects in Sections 1.2 and 1.3, we wish to start with some details on the physical and historical background of the class of hopping models.

1.1 Hopping transport

In this section we wish to motivate why physicists are interested in hopping models. Not all of our story relates to the random conductance model itself, which requires three main ingredients: First, that the randomness is on the connections between two sites, second, that the jump rates are symmetric, and third, that particles do not interact. Therefore, it is a strong simplification for many real situations. However, these real situations are often so complicated that we have to first understand the essence of some simpler models before we can move on to the more complicated ones. This is why



(a) Extended electron wave in a periodic potential (courtesy of Jannick Weißhaupt).

(b) Localized electron wave in a disordered potential.

Fig. 1.2: Real part of the solution Ψ of the Schrödinger equation (1.1). In (a) the potential around each of the atoms (gray circles) is the same and therefore the overall potential is periodic. Consequently, the wave function Ψ extends over the whole crystal. In (b) an inhomogeneous potential causes the solution Ψ to localize on a few atoms (Anderson localization).

we find it important to place the random conductance model in some wider context.

Nevertheless, we will give an application for the random conductance model at the end of Section 1.1.2, i.e., the transport of optical excitations among impurity ions in fluorescence line-narrowing experiments.

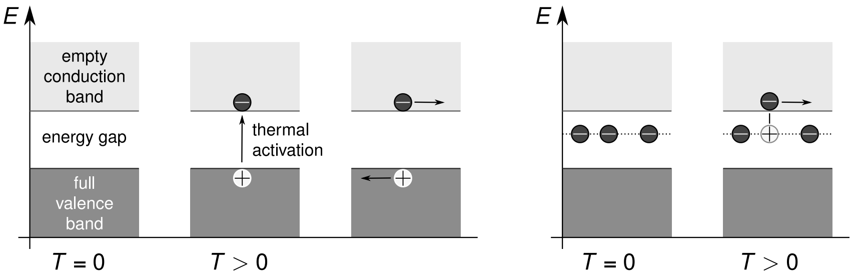
1.1.1 Electrons in inorganic crystals

Hopping transport is dominant in highly disordered systems, as for example in amorphous materials. This is in contrast to the wave-like electron transport usually prevailing in periodic structures such as inorganic crystals. The reason is that the spatial and energetic disorder localizes the quantum mechanical electron states between which the electrons move by *tunnel effect.*[1] This is even the case in the low-temperature regime of many inorganic semiconductors, which are nearly perfect crystals with some randomly distributed impurities.

Before we trace back the experiments and ideas that finally confirmed the hypothesis of hopping transport in inorganic semiconductor crystals, let us first understand the dominant transport mechanism at room temperature. In inorganic semiconductor crystals this is the so-called band or wave-like transport.

Wave-like transport. In inorganic crystals the atoms are ordered in a periodic structure and thus give rise to a periodic potential. Quantum theory

[1] The tunnel effect is a quantum mechanical phenomenon where the electron hops over a high potential barrier, which it could not overcome in the framework of classical physics.



(a) Pure semiconductor crystal. **(b)** n-type semiconductor.

Fig. 1.3: Simplified band model. In the pure semiconductor crystal (a) all valence electrons are bound in the full valence band at zero temperature and there are no free charge carriers. An energy gap divides both bands. At higher temperatures, electrons are excited into the conduction band and can move as free (negative) charge carriers in the conduction band. The resulting holes in the valence band also move as free (positive) charge carriers. At room temperature, however, there are still very few of these *intrinsic* free carriers. Electronically active impurities, as for example electron donors as in (b), generate new allowed energy states in the energy gap between valence and conductance band and therefore lower the activation energy. These *impurity states* are localized in space. A semiconductor where most of the impurities are donors (acceptors) is called n-type (p-type) semiconductor.

predicts that electrons behave like waves with a probability density $|\Psi|^2$, where Ψ is the solution to the time-independent Schrödinger equation

$$\hat{H}\Psi = E\Psi \tag{1.1}$$

with $\hat{H}$ the corresponding Schrödinger operator and E the energy of the solution Ψ. In a periodic structure, these solutions have the form

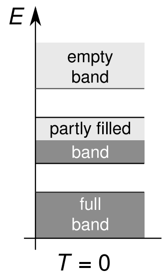


Fig. 1.4: Allowed energy bands.

$$\Psi(x) = \mathrm{e}^{\mathrm{i}k\cdot x}\, u_k(x) \qquad (x \in \mathbb{R}^3)\,, \tag{1.2}$$

where $k \in \mathbb{R}^3$ is the so-called wave vector and $u_k(x)$ is a complex-valued periodic function with the same periodicity as the crystal itself. In Figure 1.2a we have depicted a schematic view of the real part of Ψ along one coordinate axis. It follows that the probability density of an electron extends over the whole crystal. What we have to keep in mind is that in a perfect crystal, all the electron states are of this form even if, for illustrative reasons, we sketch the electrons as particles as in Figure 1.3a, which we are going to explain in a later paragraph.

If we would solve (1.1) in the case of a periodic potential, we would further see that the possible values of the energy E lie in certain allowed energy bands, which are separated by energy gaps, see Figure 1.4.

At zero temperature, these bands are populated by the *valence electrons*[2] from the lowest values of E to the higher values until all electrons are used up. Due to Pauli's exclusion principle, the number of electrons in each band is bounded from above. When the energy bands are filled up, it might happen that the highest one is only partly filled. Then a small electric field can induce an electric current through the material and we say that the electrons are effectively free. If this is the case, we call the material a *metal*.

In contrast, if there exists only completely full and completely empty bands, a small electric field cannot induce an electric current since, in this case, the electrons cannot move inside the band – again due to Pauli's exclusion principle. We call such a material either a *semiconductor* or an *insulator* – depending on the energy gap that separates the highest occupied band (*valence band*) and the lowest empty one (*conduction band*). At zero temperature both types of material act as insulators but at higher temperatures some electrons are thermally activated into the conduction band where they contribute to an electric current, see Figure 1.3a. Since in insulators the energy gap is larger than in semiconductors, it takes higher temperatures to excite the electrons from the valence band into the conduction band. Nevertheless, even in semiconductors this gap is usually large in comparison to room temperature, and therefore we will observe only a small number of electrons in the conduction band of a pure semiconductor crystal. On the other hand, in the presence of electronically active impurities (*donors* and *acceptors*), the energy gap is substantially reduced because these impurities produce new energy levels between the valence and the conduction band, see Figure 1.3b. The intentional introduction of such impurities is called *doping*.

Doped semiconductors at room temperature. We can picture the situation in a doped semiconductor as follows: In a pure semiconductor crystal, say silicon, all valence electrons are bound in the interatomic bonds. In the case of silicon, these are four electrons. If such a crystal is doped with phosphorus, which has five valence electrons, four of these electrons will contribute to the bonds to the neighboring silicon atoms but one electron remains free. At zero temperature, the Coulomb attraction between the phosphorus nucleus and the fifth electron still binds the electron to the phosphorus atom. But since this electron does not contribute to an interatomic bond, its activation energy to the conduction band is much smaller than the activation energy of the other electrons. We call such impurities that have an excess valence electron *donors* and semiconductors that are mostly doped with donors *n-type semiconductors*.[3]

We can also dope the semiconductor with impurities that have one electron less in the outermost shell, as for example boron with three valence electrons. Then the current would not result from thermally activated electrons in the conduction band but rather from thermally activated (positively

[2] Electrons in the outermost shell of the atom.

[3] "n" for "negative" in contrast to "p" for "positive".

charged) holes in the valence band. We call these impurities *acceptors* and semiconductors that are mostly doped with acceptors *p-type semiconductors*. The mechanisms behind hole transport are analogous to the mechanisms behind electron transport but less intuitive on first sight. Therefore we will concentrate on n-type semiconductors in what follows.

Tunnelhaftleitung. The concept of electrons moving as free carriers in allowed energy bands dates back to Wilson in 1931 [Wil31]. Although widely accepted, this concept was nevertheless disputed in the subsequent years because some experimental puzzle pieces did not fit the picture. One of these puzzle pieces was that the band model implies that in a doped semiconductor the conductivity σ depends on the temperature T as

$$\sigma = A\mathrm{e}^{-\frac{B}{k_{\mathrm{B}}T}}\,, \tag{1.3}$$

where k_{B} is the Boltzmann constant and A and B are other constants. As Gudden and Schottky argued in their talk at the 11th Deutschen Physiker-Tagung in 1935 [GS35], the number B should be a material constant independent of the temperature whereas A should mostly vary with impurity concentration.[4] This was, however, in contrast to experimental data, for example by Fritsch [Fri35]. Therefore Schottky suggested that there might be a second transport mechanism. Plausible to him was a model where the charges hop between impurity centers by tunnel effect, something he called *Tunnelhaftleitung* ("tunnel adherence conduction").

This is something qualitatively different from the wave-like transport of the previous paragraphs. There the electrons were always viewed as waves that extend over the entire crystal. A current was produced by a subtle change in the distribution of the electrons among the allowed wave states. Now, on the other hand, Schottky treats the electrons as if they were particles and the current is produced simply by many particles moving into the same direction.

Experimental evidence of hopping transport. After the talk by Gudden and Schottky in 1935, the next important progress in the direction of understanding and verifying hopping transport in semiconductors is attributed[5] to Busch and Labhart in 1946, who carefully measured the conductivity and Hall constants of silicon carbide over a wide range of temperature and impurity concentrations [BL46].

[4] Actually A is also temperature dependent because it represents the mobility of the charge carriers, which depends on temperature. But compared to the exponential term, it is usually a good approximation to consider A as constant in T.

[5] See for example Mott & Twose [MT61] and Shklovskii & Efros [SE84, Chapter 4].

In fact, it was still difficult to obtain genuine and noise-free data at that time and in his article [Bus46], Busch lists some examples of how easily the silicon carbide measurements can be distorted. For simplicity, let us concentrate on the conductivity measurements by Busch, which exhibited three temperature regimes,[6] see Figure 1.5. Most important for us is the change of slope between regimes II and III, which could not be explained by pure band theory. Indeed, band theory would predict that the conductivity decreases to zero with the same slope as in regime II. But instead, the slope in regime III is significantly smaller the one in regime II.

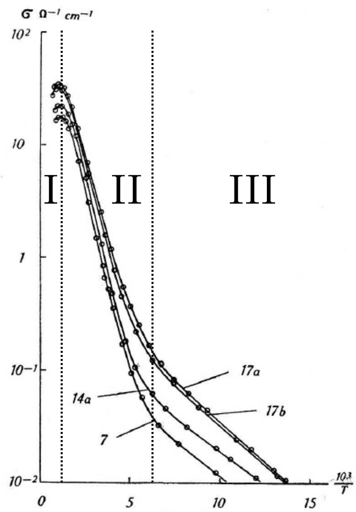


Fig. 1.5: Figure 6 from [Bus46] shows the conductivity σ in a logarithmic scaling in dependence of the inverse temperature $1/T$. Important for us is the change of slope between regimes II and III.

In 1950 Hung and Gliessmann [HG50] performed similar measurements in germanium and confirmed the observations by Busch. Hung conjectured that the change of slope was due to the presence of another *impurity band* [Hun50].[7] Furthermore, five years later, Fritzsche [Fri55] excluded a large number of possible other reasons and therefore concluded that the explanation by Hung seemed quite plausible.

But it was Conwell who, in 1956, combined a few arguments that gave the hopping hypothesis more evidence [Con56]. First, she stresses that conduction at low temperatures and small impurity concentrations depends greatly on the amount of compensation, i.e., the amount of impurities that are of the opposite kind to the majority. This means that in a lightly-doped n-type semiconductor where most of the impurities are electron donors, the presence of electron acceptors is crucial for low-temperature conduction. This is because of the Coulomb interaction, each impurity state, which is localized around a donor, can be occupied by at most one electron,[8] similar to the situation in a mathematical *exclusion process*. If all donor states are occupied, conduction is not possible. On the other hand, if electron acceptors are present, these will attract electrons from the donors so that other electrons can hop into the resulting gaps, see Figure 1.6. The importance of compensation for low-concentration-low-temperature conduction is there-

[6] The Hall measurements are important as well but their discussion would exceed the scope of this introduction.

[7] What Hung called "band" is actually the hopping regime. We assume that he used the term "impurity band" for want of a better word.

[8] Shklovskii and Efros point out that this is indeed not due to Pauli's exclusion principle, see [SE84, Section 2.1].

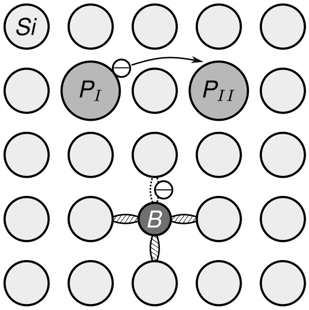


Fig. 1.6: A silicon crystal doped with phosphorus (P) and compensated with boron (B). The phosphorus atoms act like donors in silicon and the boron atoms as acceptors. Since the boron atom has only three valence electrons, there is one electron missing in order to fill the bond to the fourth neighboring silicon atom. At non-zero temperature, the boron atom therefore attracts one electron from one of the two phosphorus atoms, for example from the one we labeled P_{II}. This produces an empty donor state at P_{II} and the electron at P_I can hop to P_{II}.

fore in agreement with the hopping hypothesis and the exclusion principle by Coulomb interaction.

Second, Conwell cites data from spin-resonance experiments of Fletcher *et. al.*, which show hyperfine lines for lightly-doped but not for heavily-doped silicon crystals [FYPM54]. These hyperfine lines correspond to electrons bound to donors. The fact that they vanish for higher impurity concentrations is a consequence of several localization-delocalization transitions, which for example Baltensperger predicted already in 1953 [Bal53]. Therefore this is also in good agreement with the theory.

Localization transitions. Above a critical impurity concentration both the data from Fletcher and the model by Baltensperger predict that there is something like an "impurity band formation".[9] This happens when the overlap between the highly localized wave functions is large enough, such that tunneling is no longer necessary and we arrive at a degenerate electron gas [MT61]. Above this concentration, the electrons should move by yet another transport mechanism and, indeed, Conwell observes that the low-temperature conduction properties of highly impure samples differ qualitatively from the ones for low concentration samples. Similar observations were found by Mott in the same year.

Although the model by Baltensperger and also a much-celebrated model by Mott, see [SE84, Chapter 2.1], deliver theoretical explanations for such a transition, they both consider impurities that form an ordered pattern inside the semiconductor crystal. In contrast, in reality there is a huge energetic and spatial disorder and we now know that the full theoretical explanation must therefore include Anderson localization (energetic disorder but spatial order) as well as Lifshitz localization (energetic order but spatial disorder), see [SE84, Chapter 2]. What these two models neglect, however, is any electron-electron interaction. This, on the other hand, is a feature of the Mott transition, which must therefore also contribute to the real situation.

Now we understand some of the background of hopping transport. Let us conclude with the implications for crystalline semiconductors and then learn about a model for electrons in amorphous semiconductors.

[9] Note that this is something different than the "impurity band" meant by Hung.

Conclusion for crystalline semiconductors. In order to conserve energy, the charge carriers have to absorb or emit phonons when they hop from one site to another. This requires an activation energy ε_3. We now know that this activation energy is much smaller than the activation energy ε_1 from the impurity states into the conduction band.[10] At the same time, the hopping mobility μ_3 is much smaller than the mobility μ_1 in band transport, which is why, at higher temperatures, the conductivity is dominated by band transport.

Since the mobility is proportional to the conductivity, we obtain in good approximation for the temperature regimes II and III:

$$\sigma(T) = \sigma_1 \exp\left(-\frac{\varepsilon_1}{k_\mathrm{B}T}\right) + \sigma_3 \exp\left(-\frac{\varepsilon_3}{k_\mathrm{B}T}\right). \tag{1.4}$$

This fits the experimental data in Figure 1.5.

1.1.2 Electrons in amorphous semiconductors

Although (1.4) is in good agreement with measurements in doped semiconductor crystals, Clark noticed in 1967 that it fails for amorphous semiconductors as for example amorphous germanium [Cla67]. In fact, in this material the conductivity follows the law

$$\sigma(T) \propto \exp\left[-(T_0/T)^{1/4}\right], \tag{1.5}$$

with a characteristic temperature T_0. In 1968 Mott introduced the so-called *variable-range hopping* [Mot68] and gave a heuristic explanation for the $T^{1/4}$-law, nowadays referred to as *Mott's law*. A more systematic approach was later given by Ambegaokar, Halperin and Langer [AHL71] who used the resistor network approach by Miller and Abrahams [MA60].[11] Actually they start with a model where the jump rates between two sites depend heavily on the energy difference of these sites and are not at all symmetric. Furthermore they have to start with an exclusion process. But due to a detailed balance relation and several other assumptions, they finally arrive at a resistor network, where the conductance σ_{ij} (the inverse resistance) between the sites i and j is given by

$$\sigma_{ij} = \exp\left[-2\alpha r_{ij} - \frac{|E_i| + |E_j| + |E_i - E_j|}{k_\mathrm{B}T}\right], \tag{1.6}$$

[10] There is an intermediate temperature regime to which belongs an activation energy ε_2 but this is beyond the scope of this introduction, see also [SE84, p. 79].

[11] For a good overview, see also [SE84, Chapter 9].

where $\alpha > 0$ is a constant, r_{ij} the distance between i and j, E_i the energy of the site i and k_B the Boltzmann constant. Note that this conductance is symmetric, i.e., $\sigma_{ij} = \sigma_{ji}$.

With the help of this model and a percolation ansatz, Ambegaokar and his co-authors deduce Mott's law if the energies $|E_i|$ are distributed uniformly between 0 and $E_{\max} > 0$. This has been mathematically confirmed very recently by Faggionato and Mimun [FM17].

The Miller-Abrahams model is already very near to the random conductance model. However, the conductances between different sites have a very specific dependence structure.

1.1.3 Optical excitations among impurity ions

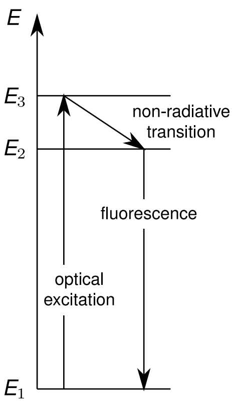


Fig. 1.7: An electron is first optically excited from the energy level E_1 to the energy level E_3. Then it relaxes to the level E_2 without emitting any light. Afterwards it falls to E_1 and emits a photon with the energy $E_2 - E_1$.

In order to understand the diverse properties of various types of materials and molecules, science has invented several experimenting strategies. One of these strategies is to ray a sample with a laser pulse, which *optically excites* electrons in the shell of the atoms to higher energy levels. Afterwards, these electrons typically first relax to slightly lower energy levels by *non-radiative transitions*[12] and then fall back to the original level by emitting a photon in the visible spectrum (*fluorescence*), see Figure 1.7. When we measure the spectrum of these photons, i.e. the *fluorescence spectrum*, then we can obtain information about the inner structure of the material.

For inhomogeneous materials, however, it can be a challenge to distinguish between effects arising from the inhomogeneity and effects arising from the inner structure of the individual atoms. That is, the peaks in the fluorescence spectrum that belong to certain energetic transitions are broadened by I) *homogeneous line-broadening*, or II) *inhomogeneous line-broadening.* Homogeneous line-broadening already occurs for a single atom, for example because there might exist multiple intermediate states like E_2 in Figure 1.7. This causes the emitted photons to have multiple energies. On the other hand, inhomogeneous line-broadening is an ensemble effect that is caused by the disorder of a system, i.e., different atoms experience a different environment and have therefore different optical properties. Since the measured fluorescence spec-

[12] This means that they do not emit light.

trum is a superposition of all the spectra of the single atoms, this leads to a broadening.

In order to distinguish these two origins, Szabo reported in 1970 that he had developed a technique where a ruby sample was rayed with a laser pulse that had a very narrow frequency distribution and therefore only excited very specific electrons that belonged to atoms of the same energetic population [Sza70]. With this technique he was able to isolate the homogeneous linewidth of the fluorescence lines.

We point out that, as in Section 1.1.1, ruby is an impurity-doped crystal where the disorder is mainly due to the randomly distributed impurities.

What is interesting for us in this experiment is that after some time the fluorescence spectrum begins to broaden again because some excitations did not relax back in their original energy levels but rather the impurities transferred the excitation to their neighboring impurities, which have themselves a different homogeneous fluorescence spectrum, see Figure 1.8. We say that the excitations hops from one impurity to another. This leads again to an inhomogeneous line-broadening.

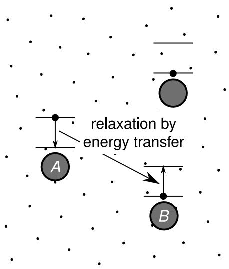


Fig. 1.8: An electron in the impurity A has been excited due to the laser pulse but instead of relaxing directly, it transfers the excitation to a neighboring impurity B, whose energy levels are different from those of impurity A because of the disorder in the material.

The question is now how we best describe the dynamics of the excitation hopping. Huber, Hamilton and Barnett proposed the following model [HHB77]. Let us assume that, initially, only a small number of impurities is excited and therefore we can neglect any exclusion effects in the temporal evolution. When we take the average over disorder afterwards, we can further assume that exactly one impurity is excited, for example the one at the origin. Then we model the probability $p_t(x)$ that the site x is excited with the evolution equation

$$\frac{\mathrm{d}p_t(x)}{\mathrm{d}t} = \sum_{\substack{\text{sites } y, \\ \text{excluding } x}} \left[W_{yx} p_t(y) - W_{xy} p_t(x) \right], \quad (1.7)$$

where W_{xy} is the hopping rate from site x to site y. Although this hopping rate W_{xy} is a complicated object that involves the energy difference and different forms of coupling between the impurities, Huber, Hamilton and Barnett argue that in the high-temperature limit, the rate is symmetric, i.e., $W_{xy} = W_{yx}$. In addition, instead of ruby, they consider the compound $Pr_{0.2}La_{0.8}F_3$ where experimental data suggest that the rate W_{xy} is even independent of the energy mismatch but depends rather on the distance r_{xy} between the ions. It is certainly reasonable to assume that the disorder does not make W_{xy} a deterministic function of the distance r_{xy}. It follows that in this case we arrive indeed at a random conductance model on a lattice with randomly distributed sites.

So, after an overview about hopping transport in anorganic semiconductor crystals, we learned about a model that describes electron transport in amorphous semiconductors by a random resistor network. This model is near to the random conductance model but also different in some respects. Now we have also learned about a different kind of transport in disordered media, that is, the transport of optical excitations. Here, the diffusion is indeed described by a random conductance model.

1.2 Random conductance model

Let us now focus on the mathematical definition of the random conductance model and some aspects of its large-scale behavior.

1.2.1 Notation and key concepts

In principle, we can define the random conductance model on a general graph (V, E) with a vertex set V and an *undirected* edge set E. Here, we wish to restrict ourselves to the vertex set $V = \mathbb{Z}^d$ and an edge set E that is either the full set

$$\mathfrak{E} \;=\; \{\{x, y\} : x, y \in \mathbb{Z}^d \text{ and } x \neq y\} \tag{1.8}$$

or the set of nearest-neighbor edges

$$\mathfrak{E}_d \;=\; \{\{x, y\} : x, y \in \mathbb{Z}^d, |x - y|_1 = 1\}\,. \tag{1.9}$$

If two sites $x, y \in \mathbb{Z}^d$ are neighbors according to $\mathfrak{E}_d$, we also write $x \sim y$.

To each edge $e \in E$ we assign a non-negative random variable $\omega(e)$. In analogy to a resistor network, we call the random weight $\omega(e)$ *conductance* of the edge e. If $e = \{x, y\}$ for $x, y \in \mathbb{Z}^d$, we will also write $\omega_{x,y}$ or ω_{xy} instead of $\omega(e)$. Since the edges in E are undirected, the conductances are symmetric, i.e., $\omega_{xy} = \omega_{xy}$.

The family $\boldsymbol{\omega} = (\omega(e))_{e \in E}$ is called *environment* or *landscape* and we assume that it is governed by the probability space

$$(\Omega, \mathcal{F}, \mathbb{P}) \;=\; \big([0, \infty]^E, \mathcal{B}([0, \infty])^{\otimes E}, \mathbb{P}\big)\,. \tag{1.10}$$

Moreover, we let $\mathbb{E}$ denote the expectation with respect to the law $\mathbb{P}$.

Random walk among random conductances. Given a realization $\boldsymbol{\omega}$ of the *environment*, we consider the Markov chain on $\mathbb{Z}^d$ with transition rates related to the conductances $\omega(e)$. In the literature, one usually finds two different rules on how to let the conductances govern the Markov chain.

The first one is called the *variable-speed* random walk. When this random walk is at a site $x \in \mathbb{Z}^d$, it waits for an exponential time with parameter

$$\pi_x = \sum_{y\colon \{x,y\}\in E} \omega_{xy} \tag{1.11}$$

and then it jumps to one of the adjacent sites y with probability $p_{xy} = \omega_{xy}/\pi_x$. Its mean waiting time at x is therefore $1/\pi_x$. Since the random variable π_x is related to the speed of the random walk at the site x, we call the field $\{\pi_x \colon x \in \mathbb{Z}^d\}$ the *local speed measure.* The generator of this random walk is the Laplacian $\mathcal{L}^\omega$ that acts on real-valued functions $u \in \ell^2(\mathbb{Z}^d)$ as

$$(\mathcal{L}^\omega u)(x) = \sum_{y\colon \{x,y\}\in E} \omega_{xy}(u(y) - u(x)) \qquad (x \in \mathbb{Z}^d)\,. \tag{1.12}$$

The other notion of the random walk among random conductances is the so-called *constant-speed* random walk. Its spatial trajectory looks the same as the one of the variable-speed random walk but regardless where it is, it always waits for an exponential time with mean 1. It follows that its generator is the Laplacian $\mathcal{L}^\omega_{\mathrm{cs}}$ that acts on real-valued functions $u \in \ell^2(\mathbb{Z}^d)$ as

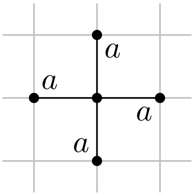
$$(\mathcal{L}^\omega_{\mathrm{cs}} u)(x) = (\pi_x)^{-1} \sum_{y\colon \{x,y\}\in E} \omega_{xy}(u(y) - u(x)) \qquad (x \in \mathbb{Z}^d)\,. \tag{1.13}$$

Although it is the behavior of the two different random walks that justifies the attributes “variable-speed” and “constant-speed”, we will use the term “variable-speed random conductance *model*”, when we merely refer to the Laplace operator defined in (1.12). In this thesis we only prove results for the variable-speed random conductance model although the constant-speed model is very popular among probabilists. We assert that most of the results can be rewritten to suit the constant-speed case (see Remark 3.10) but their proofs become even more technically involved.

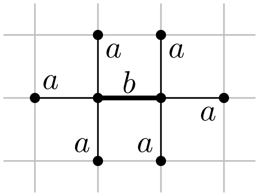
It makes sense to understand one huge difference in the behavior of the two random walks and this is in the shape of suitable trapping structures.

How to trap the random walks. Consider a site $x \in \mathbb{Z}^d$ that is surrounded by conductances of some order $a > 0$ as in Figure 1.9a. A variable-speed random walk that is located at x waits for an exponential time with mean $(2da)^{-1}$. If a is very small, then we call x a variable-speed trap because, on average, the variable speed random has to wait for a long time at x. On the other hand, the structure in Figure 1.9a is not a trap for the constant-speed random walk since the constant-speed random walk always jumps with rate 1.

Let us now consider a structure as in Figure 1.9b where we have an edge with conductance of order b and all the adjacent conductances to this *edge* are of order a. If we assume that the quotient b/a is very large, then both the



(a) Variable-speed trap.



(b) Constant-speed trap.

Fig. 1.9: Different shapes of traps.

variable-speed and constant-speed random walk hop very often between the sites incident to the conductance of order b before leaving the structure. The difference between the two random walks is in *how much time* they need to hop back and forth. For the constant-speed random walk the time is of order b/a regardless of whether a is small or b large. The variable-speed random walk, on the other hand, needs a long time if b is of average size and a is very small. In contrast, if b is very large and a is of average size, then it performs the many hops very fast and leaves the structure after an average amount of time. It follows that the constant-speed random walk can be trapped by a single very large conductance but the variable-speed random walk cannot.

The heat kernel. With the random walk we now have two kinds of randomness in our model: One from the random environment and one from the random walk. In what follows, we will only consider the variable-speed random walk, which we denote by $(X_t\colon t \geq 0)$. Given an environment $\boldsymbol{\omega}$ and an initial site x, we will denote the law associated with the random walk by P_x^ω, i.e., $\mathrm{P}_x^\omega[X_0 = x] = 1$. By E_x^ω we denote the corresponding expectation.

The *heat kernel* $(p_t, t \geq 0)$ of the random walk encodes the probability that a random walker that started at site x at time zero is at site y at time t, i.e.,

$$p_t(x,y) = \mathrm{P}_x^\omega[X_t = y]\,, \tag{1.14}$$

where the superscript $\boldsymbol{\omega}$ is usually suppressed in the expression $p_t(x,y)$.

The fact that the Laplacian $\mathcal{L}^\omega$ is the generator of the random walk X, expresses itself in the heat equation

$$\partial_t p_t(x,\cdot) = \mathcal{L}^\omega p_t(x,\cdot)\,, \tag{1.15}$$

with initial condition $p_0(x,\cdot) = \delta_x$ where

$$\delta_x\colon \mathbb{Z}^d \to \mathbb{R}_+,\ y \mapsto \begin{cases} 1 & \text{for } y = x \\ 0 & \text{else.} \end{cases} \tag{1.16}$$

Dirichlet spectrum. The main object of this thesis is to study the behavior of the first k Dirichlet eigenvalues $\lambda_1^{(n)} \leq \ldots \leq \lambda_k^{(n)}$ and eigenvectors $\psi_1^{(n)}, \ldots, \psi_k^{(n)}$ of the sign-inverted generator $-\mathcal{L}^\omega$ in the ball

$$B_n := \big\{x \in \mathbb{Z}^d \colon |x|_\infty < n\big\} = (-n, n)^d \cap \mathbb{Z}^d \tag{1.17}$$

with zero Dirichlet conditions at the boundary. This means we study the solution to the problem

$$\begin{aligned} -\mathcal{L}^\omega \psi &= \lambda \psi \quad \text{in } B_n\,, \\ \psi &= 0 \qquad \text{else.} \end{aligned} \tag{1.18}$$

For a symmetric operator as $\mathcal{L}^\omega$, the Courant-Fischer theorem states that the solutions to (1.18) are given by the minima and minimizers of the variational problem (1.21). For this purpose let us introduce some more notation.

For a subset $A \subset \mathbb{Z}^d$ we define the function space

$$\ell^2(A) := \left\{ f \colon \mathbb{Z}^d \to \mathbb{R} \text{ such that } \operatorname{supp} f \subseteq A \text{ and } \sum_{x \in A} f(x)^2 < \infty \right\} \subset \ell^2(\mathbb{Z}^d)\,, \tag{1.19}$$

where we let "supp f" denote the support of the function f. Accordingly, for functions $f_1, f_2 \in \ell^2(\mathbb{Z}^d)$ we define the scalar product

$$\langle f_1, f_2 \rangle_{\ell^2(A)} = \sum_{x \in A} f_1(x) f_2(x)\,.$$

For a real-valued function $f \in \ell^2(\mathbb{Z}^d)$ let us define the Dirichlet energy $\mathcal{E}^\omega(f)$ with respect to the operator $-\mathcal{L}^\omega$ by

$$\mathcal{E}^\omega(f) = \langle f, -\mathcal{L}^\omega f \rangle_{\ell^2(\mathbb{Z}^d)}\,. \tag{1.20}$$

Then, according to the Courant-Fischer theorem, the kth Dirichlet eigenvalue is given by the variational formula

$$\lambda_k^{(n)} = \inf_{\substack{\mathcal{M} \leq \ell^2(B_n), \\ \dim \mathcal{M} = k}} \; \sup_{\substack{f \in \mathcal{M}, \\ \|f\|_2 = 1}} \mathcal{E}^\omega(f) \tag{1.21}$$

where $\mathcal{M} \leq \ell^2(B_n)$ means that $\mathcal{M}$ is a linear subspace of $\ell^2(B_n)$. Note that $\lambda_k^{(n)} = \mathcal{E}^\omega\big(\psi_k^{(n)}\big)$.

Divergence form. In this context, we point out that we can also write the Laplace operator of (1.12) as a divergence-form operator. In order to explain this, let us assume for simplicity that the edge set E is equal to $\mathfrak{E}_d$. For the case $E = \mathfrak{E}$, the definitions are analogous, see also Section 2.5.3.

For a function $u\colon \mathbb{Z}^d \to \mathbb{R}$, we define the discrete derivative in $x \in \mathbb{Z}^d$ for the directions $\boldsymbol{e}_1, \dots, \boldsymbol{e}_{2d}$ by

$$\partial_i u(x) = u(x + \boldsymbol{e}_i) - u(x)\,, \qquad (i = 1, \dots 2d)$$

and similarly the gradient $\nabla u(x) = (\partial_1 u(x), \dots, \partial_{2d} u(x))$. Although it seems peculiar that we define the derivative and the gradient with $2d$ directions instead of d, this procedure becomes clear when we consider the case $E = \mathfrak{E}$ in Section 2.5.3. For a function $v\colon \mathbb{Z}^d \to \mathbb{R}^{2d}$, we define the divergence

$$\operatorname{div} v(x) = \sum_{i=1}^{2d} (v_i(x) - v_i(x - \boldsymbol{e}_i))\,,$$

where v_i is the ith component of the vector $v(x)$. Last, we define the space-dependent tensor $A_\omega(x) = \operatorname{diag}(\omega_{x,x+\boldsymbol{e}_1}, \dots, \omega_{x,x+\boldsymbol{e}_{2d}})$.

Then we see that

$$\mathcal{L}^\omega u = \frac{1}{2} \operatorname{div}\left(A_\omega \nabla u\right), \tag{1.22}$$

where the factor $1/2$ is due to the $2d$ instead of d directions.

With these definitions, we can further write the Dirichlet energy as

$$\mathcal{E}^\omega(f) = \frac{1}{2} \left\langle \nabla f, -A_\omega \nabla f \right\rangle_{\ell^2(\mathbb{Z}^d)}\,.$$

Poisson equation. Besides the Dirichlet spectrum, our other object of interest is the Poisson equation on the box B_n with zero Dirichlet boundary conditions, i.e.,

$$\begin{aligned} -n^2 \mathcal{L}^\omega u_n &= f_n \qquad \text{on } B_n, \\ u_n &= 0 \qquad \text{else.} \end{aligned} \tag{1.23}$$

We are especially interested in the question under what conditions on the environment $\boldsymbol{\omega}$ and the right-hand side f_n, the solution u_n converges to the solution of a homogenized Poisson equation.

1.2.2 Different aspects of large-scale behavior

In this section we wish to give an overview over different kinds of large-scale behavior and existing results for the random conductance model. Since the different classes of hopping models, i.e., random walks in random environments were invented to account for the microscopic inhomogeneities of complex materials, it is a natural question whether the microscopic inhomogeneities average out when zooming to larger scales.

Mean squared displacement. Among physicists, the random walk among independent and identically distributed conductances enjoys the reputation that it *behaves mostly homogeneous* in dimensions $d \geq 2$, given that the edge set E equals $\mathfrak{E}_d$ and the conductances are $\mathbb{P}$-a.s. positive. This is due to the fact that there is a simple percolation argument why the effective conductivity σ_{eff} has to be positive, see [BG90, p. 177] and [Ale81]. Since the effective conductivity σ_{eff} and the effective diffusion constant D_{eff} are related by the Einstein relation, it follows that the effective diffusion constant is positive and diffusion therefore *normal*, i.e., it *is to be expected* that

$$\mathrm{E}_0\left[|X_t|^2\right] \sim D_{\text{eff}} \cdot t\,. \tag{1.24}$$

This implies that the random walk cannot behave subdiffusively and it follows that the higher-dimensional random conductance model cannot explain the anomalous conductivity behavior that we observe in some amorphous materials, see e.g. [SM75].

Invariance principles. It is remarkable that in 2012 Andres, Barlow, Deuschel and Hambly showed that for the graph $(\mathbb{Z}^d, \mathfrak{E}_d)$ with $d \geq 2$ and independent and identically distributed conductances ω, one can even allow the conductances to be zero as long as

$$\mathbb{P}[\omega > 0] > p_{\mathrm{c}}(d)\,, \tag{1.25}$$

where $p_{\mathrm{c}}(d)$ is the critical probability for bond percolation on the graph $(\mathbb{Z}^d, \mathfrak{E}_d)$. These conditions are sufficient to prove a *quenched functional central limit theorem* (QFCLT). This means that for $\mathbb{P}$-a.e. environment $\boldsymbol{\omega}$ where the origin is in the infinite connected cluster, the rescaled walk

$$\left(X_t^{(n)} : t \geq 0\right) := \left(\frac{1}{n} X_{n^2 t} : t \geq 0\right)$$

converges (under P_0^{ω}) in law to a Brownian motion on $\mathbb{R}^d$ with a deterministic covariance matrix Σ_X^2. This is, of course, a much stronger result than (1.24).

Local limit theorem. An even stronger result is the local central limit theorem (LCLT) that states that the heat kernel converges point-wise. As Andres, Deuschel and Slowik prove in [ADS16], this theorem holds on the graph $(\mathbb{Z}^d, \mathfrak{E}_d)$ with dimension $d \geq 2$ if there exist $p, q \in (1, \infty]$ satisfying $1/p + 1/q < 2/d$ such that $\mathbb{E}[\omega(e)^p] < \infty$ and $\mathbb{E}[\omega(e)^{-q}] < \infty$ for any edge $e \in \mathfrak{E}_d$. That is, for $\mathbb{P}$-a.e. environment $\boldsymbol{\omega}$ and for every $T_2 > T_1 > 0$ and $K > 0$ we have

$$\lim_{n\to\infty} \sup_{|x|\leq K} \sup_{t\in[T_1,T_2]} \left|n^d p_{n^2 t}(0, \lfloor nx \rfloor) - k_t\right| = 0\,, \tag{1.26}$$

where k_t is the heat kernel of the Brownian motion from the QFCLT.

A similar result was shown by Boukhadra, Kumagai and Mathieu in [BKM15] for independent and identically distributed conductances under the condition that $\mathbb{P}[\omega \leq a] = a^\gamma$ ($a \in [0,1]$) for a $\gamma > 1/4$, i.e., when there exists $q > 1/4$ such that $\mathbb{E}[\omega^{-q}] < \infty$. Their specific choice of $\mathbb{P}$ was for simplicity reasons since their results can easily be generalized to a wider class of distributions.

When the LCLT fails. In contrast to the last paragraph, let us assume that there exists $q' < d/2$ such that $\mathbb{E}[\omega^{-q'}] = \infty$. Then in analogy to the arguments in the proof of [ADS16, Theorem 5.4], one can show that there might exist a sequence $(x_n)_{n\in\mathbb{N}}$ of sites such that $|x_n| < n$ and $\pi_{x_n} \in o(n^{-2})$. More precisely, one can show that for any $q < d/2$ we can construct a stationary and ergodic environment such that $\mathbb{E}[\omega^{-q}] < \infty$ and such that a sequence $(x_n)_{n\in\mathbb{N}}$ as described above exists. We call the sites x_n *traps.* As Andres, Deuschel and Slowik argue in Step 5 of the proof of [ADS16, Theorem 5.4], the presence of such traps contradicts the validity of a local central limit theorem. Note, however, that this is not a contradiction to the quenched functional limit theorem since the QFCLT associates with macroscopic properties of the random walk, in contrast to the local limit theorem, which is sensitive to microscopic trapping structures.

For positive i.i.d. conductances ω on $(\mathbb{Z}^d, \mathfrak{E}_d)$, the traps occur even for $\mathbb{P}$-a.e. environment $\boldsymbol{\omega}$ if there exists $q' < 1/4$ such that $\mathbb{E}[\omega^{-q'}] = \infty$, see [BKM15, Remark 1.10(1)] and also our discussion on page 23.

Anomalous heat-kernel decay. In addition to the failure of the local limit theorem, there is another phenomenon for heavy-tailed conductances: This is the anomalous heat-kernel decay for dimensions four and higher. For example, in [BBHK08] the authors Berger, Biskup, Hoffmann and Kozma consider the discrete-time version of the constant-speed random walk for dimensions $d \geq 5$. They prove that for every increasing sequence $(a_n)_n$ with $a_n \to \infty$, there exists an i.i.d. law $\mathbb{P}$ on bounded, nearest-neighbor conductances with $\mathbb{P}[\omega > 0] > p_c(d)$ and a $\mathbb{P}$-a.s. positive random variable $C(\omega)$, such that we have

$$\mathrm{P}_0^\omega(X_n = 0) \geq \frac{C(\omega)}{a_n n^2} \tag{1.27}$$

along a non-random subsequence for $\mathbb{P}$-a.e. environment $\boldsymbol{\omega}$ where the origin is in the infinite connected cluster. The *diffusive* or normal polynomial order would be $n^{-d/2}$.

A similar result holds for $d = 4$ as Biskup and Boukhadra show in [BB12] with (1.27) replaced by

$$\mathrm{P}_0^\omega(X_n = 0) \geq \frac{C(\omega)\log n}{a_n n^2}\,.$$

Similar as for the failure of the LCLT, these results are not a contradiction to the QFCLT.

Local times. Another natural quantity to study is how much time the random walker X has spend in any site $z \in \mathbb{Z}^d$ up to a time t, i.e., the family of *occupation time measures*

$$l_t(X,z) \; := \; \int_0^t \delta_z(X_s)\,\mathrm{d}s \qquad (z \in \mathbb{Z}^d, t > 0)\,. \tag{1.28}$$

For the random walk X, which takes values in the discrete space $\mathbb{Z}^d$, these measures are also called *local times*. For a set $A \subset \mathbb{Z}^d$, we accordingly write

$$l_t(X,A) \; = \; \int_0^t \mathbb{1}_A(X_s)\,\mathrm{d}s \qquad (t > 0)\,, \tag{1.29}$$

where $\mathbb{1}_A$ is the indicator function on the set A. When we embed $\mathbb{Z}^d$ into $\mathbb{R}^d$ in the canonical way, then the above formula also makes sense for $A \in \mathbb{R}^d$. Therefore we write for the Brownian motion B from the quenched functional limit theorem

$$l_t(B,A) \; = \; \int_0^t \mathbb{1}_A(B_s)\,\mathrm{d}s \qquad (t > 0)\,. \tag{1.30}$$

Since for any open set $A \in \mathbb{R}^d$ and any $T > 0$ the measure $l_T(\cdot, A)$ is a bounded and continuous function on the Skorohod space $D([0,T], \mathbb{R}^d)$, it follows that the QFCLT implies that

$$\mathrm{E}_0^\omega\left[l_T(X^{(n)}, A)\right] \to \mathrm{E}_0^{\mathrm{BM}}[l_T(B,A)]\,, \tag{1.31}$$

where $\mathrm{E}_0^{\mathrm{BM}}$ is the expectation with respect to the law of the Brownian motion.

Apart from this immediate result about the limit law of the local times, there is a beautiful connection between the large-deviation behavior of the local times with the Dirichlet energy in (1.20). That is, for a bounded and connected domain $B \subset \mathbb{Z}^d$, the famous result by Donsker, Varadhan and Gärtner (cf. [DV75-83] and [Gär77]) states that for a fixed environment $\boldsymbol{\omega}$ and under the measures $\mathrm{P}_0[\,\cdot \mid \operatorname{supp} l_t \subseteq B]$, the normalized local times $\frac{1}{t} l_t$ satisfy a large deviation principle (LDP) with the rate function $I^\omega - \inf_{\mathcal{M}} I^\omega$, where $I^\omega(\mu) = \mathcal{E}^\omega(\sqrt{\mu})$ and

$$\mathcal{M} = \left\{\mu \colon \mu \in \ell^1(\mathbb{Z}^d), \operatorname{supp} f \subseteq B, \|f\|_1 = 1\right\}. \tag{1.32}$$

Roughly speaking, this means that for large times t we have

$$\tfrac{1}{t}\log \mathrm{P}_0^\omega\left[\tfrac{1}{t}l_t \approx \mu \,|\, \operatorname{supp} l_t \subseteq B\right] \;\sim\; -\Big(I^\omega(\sqrt{\mu}) - \inf_{\mathcal{M}} I^\omega\Big)\,. \tag{1.33}$$

In Section 2.2.4 we explain what an LDP is in greater detail.

For the annealed setting, where the underlying measure is averaged over the environment, König, Salvi and Wolff [KSW12] investigate the case where the law of the conductances has the form

$$\log \mathbb{P}[\omega_{xy} \le \varepsilon] \sim -D\varepsilon^{-\eta}\,, \quad \varepsilon \to 0\,,$$

with the parameters $D, \eta \in (0,\infty)$. They show that under the annealed sub-probability law $\mathbb{E}\mathrm{P}_0[\,\cdot\, \cap \{\text{supp } l_t \subseteq B\}]$, the process of normalized local times satisfies a large deviation principle with speed $t^{\frac{\eta}{1+\eta}}$ and an explicit rate function. In a subsequent paper, König and Wolff [KW15] extend their result to growing boxes and find an interesting sharp transition in the parameter η: For $\eta > d/2$, i.e. for light-enough tails near zero, an LDP holds and the rate function is of a continuous form. On the other hand, if $\eta \le d/2$, then this continuous rate function does no longer have compact level sets and therefore one cannot expect an LDP to be valid. Moreover, in this case, the leading-order logarithmic asymptotics of the non-exit probability of the time-dependent set do not depend on the growing set at all but are the same as for the static setting. König and Wolff interpret this as a sign for clumping behavior, i.e., that the random walk gets trapped in a finite region. However, it is still an open problem to actually determine the asymptotic shape of the local times conditioned on the event that the random walker has not left the box. This is definitely an interesting open task.

Let us now go back to (1.33) and the quenched setting, i.e., where we fix a realization of the environment. If ψ_1^B is the minimizer of $\mathcal{E}^\omega$ over all ℓ^2-normalized functions with support enclosed in B, then the right-hand side of (1.33) is maximal for $\sqrt{\mu} = \psi_1^B$, i.e., then it is zero. This means that for large times t and under the condition that the random walker has not exited the domain B, the asymptotic shape of the normalized local times is $\frac{1}{t}l_t \approx \big(\psi_1^B\big)^2$. Since ψ_1^B is the principal Dirichlet eigenvalue on the domain B with zero Dirichlet conditions, the large deviations of the local times are closely connected to the spectral properties of the random conductance Laplacian. In addition, the Donsker-Varadhan-Gärtner LDP implies that the non-exit probability $\mathrm{P}_0^\omega[\operatorname{supp} l_t \subseteq B]$ behaves like $\exp\!\big(-\lambda_1^B t\big)$, where λ_1^B is the principal Dirichlet eigenvalue of the Laplacian $-\mathcal{L}^\omega$ on the domain B.

Spectral asymptotics and Poisson equation. Since these are the main objects in this thesis, we review the earlier results regarding these topics in Section 1.4. Nevertheless, we wish to comment on a variant of the Poisson equation because of its important relation to heat-kernel convergence. Let

$$-\mathcal{L}^\omega u - \lambda u = f \qquad \text{on } \mathbb{Z}^d \tag{1.34}$$

where we have to subtract the massive term λu in order to eliminate the kernel of the operator $-\mathcal{L}^\omega$. The Laplacian $\mathcal{L}^\omega$ itself is not invertible on $\mathbb{Z}^d$. We call the massive term also *spectral shift* since it shifts the whole spectrum by the constant λ. The convergence of solutions to (1.34) when the lattice constant approaches zero, is connected with the convergence of the heat kernel: By virtue of, for example, [ZP06, Theorem 9.2] (a generalization of the Trotter-Kato Theorem, as Faggionato [Fag08, Section 7] calls it), the convergence of solutions to (1.34) implies the convergence of the semigroups.

Γ-convergence. Very recently Neukamm, Schäffner and Schlömerkemper [NSS17] proved stochastic homogenization of a very large class of energy functionals in the sense of Γ-*convergence*. Roughly speaking, we say that a sequence F_n of functionals Γ-converges to a limit F if the following two properties are fulfilled:

I) $F(x) \leq \liminf_{n\to\infty} F_n(x_n)$ for every sequence x_n that converges to x.
II) For every x there exists a *recovery sequence* x_n that converges to x such that $F(x) \geq \limsup_{n\to\infty} F_n(x_n)$.

For an introduction to Γ-convergence, see for example [Mas93].

For a *nice* domain $A \subset \mathbb{R}^d$, the authors of [NSS17] consider a sequence of energy functionals $(\mathcal{E}_\varepsilon(\omega;\,\cdot\,, A))_\varepsilon$ on the sequence of rescaled graphs $\varepsilon\mathcal{G}$ where ε plays the role of $1/n$ in the notation of above. Under certain moment and convexity conditions they show that this sequence Γ-converges to a continuous deterministic functional

$$\mathcal{E}_{\mathrm{hom}}(u, A) = \int_A W_{\mathrm{hom}}(\nabla u(x))\,\mathrm{d}x$$

with the energy density W_{hom}.

Their setting is more general than ours in the sense that they also consider the *so-called* system case where, for example, $u\colon \mathbb{Z}^d \to \mathbb{R}^n$ with $n \in \mathbb{N}$ not necessarily equal to one. On the other hand, they consider only bounded-range connections. As a special case, their results imply that in the random conductance model the Dirichlet energy Γ-converges and the minimizers have a strongly convergent subsequence if $\mathbb{E}[\omega(e)^{-d/2}] < \infty$ for nearest-neighbor edges e. Similar as for the local limit theorem, this condition can be improved to $\mathbb{E}[\omega(e)^{-1/4}] < \infty$ in the i.i.d. case, see also Remark A.1. We will comment on how this relates to spectral homogenization and the Poisson equation in Section 1.4.

1.3 Results in a nutshell and heuristics

Let us briefly summarize our main results and some elements of the proofs in a simplifying way. Rigorous statements are in Section 2.2 for the homogenization results and Section 3.1 for the localization results.

We group our results into three sections: First, the dichotomy between a completely homogenized and a completely localized regime in the case of independent and identically distributed (i.i.d.) conductances on nearest-neighbor edges. Second, the localization of the first k Dirichlet eigenvectors when the tail of the conductances near zero is heavy enough and the conductances are i.i.d., and third, the homogenization of the first k Dirichlet eigenvectors and eigenvalues when the tail of the conductances is light enough.

1.3.1 Dichotomy in the i.i.d. case

Let us assume that exactly the nearest-neighbor conductances are positive and that these are independent and identically distributed. Let us further recall that $\lambda_1^{(n)} \leq \ldots \leq \lambda_k^{(n)}$ are the first k Dirichlet eigenvalues of $-\mathcal{L}^\omega$ in the box $B_n = (-n,n)^d \cap \mathbb{Z}^d$ and that $\psi_1^{(n)}, \ldots, \psi_k^{(n)}$ are the corresponding eigenvectors. We call $n^2\lambda_1^{(n)}, \ldots, n^2\lambda_k^{(n)}$ the *diffusively rescaled* eigenvalues and say that the eigenvalues *scale subdiffusively* if $\lambda_1^{(n)}, \ldots, \lambda_k^{(n)} \in o(n^{-2})$.

Moreover, let us define

$$q = \sup\{r \geq 0 \colon \mathbb{E}[\omega^{-r}] < \infty\}\,. \tag{1.35}$$

Note that by Jensen's inequality, it follows that $\mathbb{E}[\omega^{-r}] < \infty$ for all $r < q$.

With these definitions, the results in Chapter 2 imply as a special case that

$$q > 1/4 \Rightarrow \begin{cases} \text{a.s. complete hom. of first } k \text{ Dirichlet eigenvectors and} \\ \text{a.s. convergence of diffus. rescaled first } k \text{ Dirichlet eigenvalues.} \end{cases} \tag{1.36}$$

In contrast, under some further regularity assumptions and in dimensions $d \geq 2$,[13] we show in Chapter 3 that

$$q < 1/4 \Rightarrow \begin{cases} \text{a.s. complete localization of first } k \text{ Dirichlet eigenvectors and} \\ \text{a.s. \textit{subdiffusive scaling} of first } k \text{ Dirichlet eigenvalues.} \end{cases} \tag{1.37}$$

[13] For dimension one, see Remark 3.14 and [Fag12].

Together, (1.36) and (1.37) imply that the spectrum of the Laplacian displays a sharp transition between a completely homogenized and a completely localized regime. With the results of Neukamm, Schäffner and Schlömerkemper in [NSS17] and a small improvement in their calculations, we can even infer that $\mathbb{E}[\omega^{-1/4}] < \infty$ is sufficient for spectral homogenization, see Remark A.1. However, there still remains a critical regime around $q = q_{\mathrm{c}}$ and it might be that there localization and homogenization coexist along different random subsequences as n tends to infinity. But this is still an open problem.

It is not very surprising that the critical exponent $q_{\mathrm{c}} = 1/4$ is the same as for the validity of the local limit theorem of the corresponding random walk. Indeed, the reason why the local limit theorem fails for $q < 1/4$ (see e.g. [BKM15, Remark 1.10(1)]) is the same as for the subdiffusive scaling of the principal Dirichlet eigenvalue. We are going to discuss this in the next paragraphs.

Under what circumstances should the principal Dirichlet eigenvalue scale subdiffusively? Let us recall the variational formula (1.21), which implies that

$$\lambda_1^{(n)} = \inf_{\substack{\operatorname{supp} f \subseteq B_n, \\ \|f\|_2 = 1}} \langle f, -\mathcal{L}^\omega f \rangle = \inf_{\substack{\operatorname{supp} f \subseteq B_n, \\ \|f\|_2 = 1}} \sum_{\{x,y\} \in \mathfrak{E}_d} \omega_{xy} \big(f(x) - f(y)\big)^2 . \quad (1.38)$$

Furthermore, let us now suppose that the box B_n contains a site z_n where all the surrounding conductances are much smaller than n^{-2}, see Figure 1.10. We call such a site a trap. Then we choose a test function $f = \delta_{z_n}$, insert it into the variational formula (1.38) and obtain that

$$\lambda_1^{(n)} \leq \pi_{z_n} \in o(n^{-2}) .$$

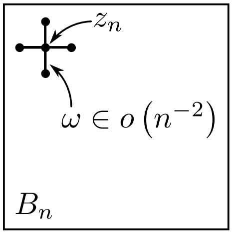


Fig. 1.10: Trap z_n.

This means that the presence of such a trap z_n is a contradiction to spectral homogenization where we would expect that $n^2 \lambda_1^{(n)}$ converges almost-surely to a *non-trivial* limit. At the same time, it is a contradiction to $p_t(z_n, z_n) \lesssim t^{-d/2}$ when t is of order n^2, see [BKM15, Remark 1.10(1)], and therefore it is a contradiction to the validity of a local limit theorem.

Under what conditions does such a trap z_n exist? Let us suppose that all the $2d$ conductances connecting a given site with its neighbors are less than or equal to some value $g(n)$. We call such a site a $g(n)$-trap. For a given site the probability to be a $g(n)$-trap is $\mathbb{P}[\omega \leq g(n)]^{2d}$. Since the number of sites in the box B_n is of order n^d, the expected number of $g(n)$-traps in B_n is of order

$$\Lambda_g(n) := n^d \mathbb{P}[\omega \leq g(n)]^{2d} . \quad (1.39)$$

In the critical case we set $g(n) = n^{-2}$. We furthermore assume, for simplicity, that $\mathbb{P}[\omega \le a] = a^\gamma$ for $a \in [0,1]$. Note that in this case, the exponent q equals γ, although $\mathbb{E}[\omega^{-\gamma}] = \infty$. Now we observe that

$$n^d \mathbb{P}\big[\omega \le n^{-2}\big]^{2d} = n^{d-4d\gamma} \to \begin{cases} 0 & \text{if } \gamma > 1/4\,, \\ \infty & \text{if } \gamma < 1/4\,. \end{cases} \tag{1.40}$$

Thus, for $\gamma < 1/4$ we expect to have many traps in the box B_n and for $\gamma > 1/4$ we expect to have none.

This is, of course, neither a proof of that a localized test function as $f = \delta_{z_n}$ is indeed the best choice for the variational formula (1.38), nor is it a proof that for $\gamma > 1/4$ the principal eigenvalue $\lambda_1^{(n)}$ scales diffusively. This is, amongst other things, the work done in this thesis.

1.3.2 Localization

We proceed with summarizing the results of Chapter 3 where we still assume that exactly the nearest-neighbor conductances are positive and that these are independent and identically distributed.

Principal Dirichlet eigenvalue. Let $g\colon (0,\infty) \to (0,\infty)$ be a function that decreases monotonically to zero and is asymptotically smaller than n^{-2}. As we have discussed in the previous section, the function $\Lambda_g\colon (0,\infty) \to (0,\infty)$ defined in (1.39) carries the information about how many $g(n)$-traps we can expect in the box B_n. We make this rigorous with the help of the Borel-Cantelli arguments in Lemmas 3.19 and 3.24. Generally, if Λ_g diverges fast enough, then $\mathbb{P}$-a.s. for n large enough, the box B_n contains at least one $g(n)$-trap. In this case, $2dg(n)$ is a trivial upper bound for the principal Dirichlet eigenvalue by the same reasoning as in the previous section. On the other hand, if Λ_g decreases fast enough to zero, then $\mathbb{P}$-a.s. for n large enough, the box B_n does not contain a $g(n)$-trap and it is possible to show that then $g(n)$ is an asymptotic lower bound for the principal Dirichlet eigenvalue $\lambda_1^{(n)}$.

In between, however, there is a regime where Λ_g neither increases nor decreases fast enough and the principal Dirichlet eigenvalue will sometimes be smaller than and sometimes be greater than $g(n)$. This is especially the case when Λ_g is constant.

In Section 3.1.1 our aim is to find optimal conditions to determine whether $g(n)$ is an asymptotic upper or asymptotic lower bound for the principal Dirichlet eigenvalue. We summarize these conditions in Figure 3.1. In this figure we see that for the lower bound the sharp condition is whether or not $\int_0^\infty u^{-1}\Lambda_g(u)\,\mathrm{d}u < \infty$. For the upper bound, on the other hand, the sufficient and the necessary conditions differ by a double-logarithmic order.

Heuristics on the lower bound. For the lower bound of the principal Dirichlet eigenvalue we have to put in significantly more work than for the upper bound. The key idea, however, is linked to the considerations for the shape theorem of first-passage percolation, see e.g. Cox and Durrett [CD81] for the sample case $d = 2$. The philosophy is that we have to show that each site in the box B_n is sufficiently well reachable by conductances that are significantly greater than the lower bound candidate $g(n)$.

Note that this is similar to the idea of Lemma 4.6 in [BKM15] where the authors prove this for a polynomial tail of the conductances with parameter γ and the candidate $g(n) = n^{-\alpha}$ with $\alpha > 1/(2\gamma)$. Indeed, for the lower bound on the principal Dirichlet eigenvalue $\lambda_1^{(n)}$, we adapt a path argument from [BKM15, Lemma 5.1], see Section 3.6.

It turns out that a crucial element of the proof is to give a condition that implies that $\mathbb{P}$-a.s. for n large enough all sites in the box B_n have at least one incident edge with conductance greater than $g(n)$, similar to [CD81, p. 585] and [Kes86, Theorem (1.7)]. In general, we let $g\colon \mathbb{R}_+ \to \mathbb{R}_+$ be a continuous function that decreases to zero and define $g^{-1}(v) := \inf\{u\colon g(u) = v\}$ as its inverse. Further, we let $\omega_1 \dots, \omega_{2d}$ be $2d$ independent copies of the conductance ω. Then

$$\mathbb{E}\Big[g^{-1}(\max\{\omega_1, \dots, \omega_{2d}\})^d\Big] = d\int_0^\infty u^{-1}\Lambda_g(u)\,\mathrm{d}u < \infty \tag{1.41}$$

implies that $\mathbb{P}$-a.s. for n large enough, all sites in the box B_n have at least one incident edge with conductance greater than $g(n)$. This together with a path argument, which we adapt from [BKM15], gives the $\mathbb{P}$-a.s. lower bound for the principal Dirichlet eigenvalue $\lambda_1^{(n)}$ (given that $g(n)$ is not asymptotically larger than n^{-2}). On the other hand, if Condition (1.41) is violated, then the same arguments as in Cox and Durrett [CD81, p. 585] yield that, $\mathbb{P}$-a.s. as n tends to infinity, the box B_n contains a $g(n)$-trap infinitely often, see Lemma 3.19.

Eigenvectors. Under certain regularity assumptions on the distribution function F of the conductances[14] we show that the principal Dirichlet eigenvector $\psi_1^{(n)}$ asymptotically localizes in the site $z_{(1,n)}$ that minimizes the local speed measure π over the box B_n. For this purpose, we first show that all the mass of the eigenvector $\psi_1^{(n)}$ has to be concentrated on a sparse set $\mathscr{I}^{(n)}$ (see Definition 3.34 and Lemma 3.44) and then we use an extreme value analysis of the local speed measure π in B_n to infer complete localization. This requires some technical effort since the random variables $\{\pi_x\colon x \in B_n\}$ are not independent.

In order to infer the complete localization, we need the Perron-Frobenius property of Remark 3.3, i.e., that we can assume that $\psi_1^{(n)}$ is non-negative.

[14] See Assumption 3.11.

This is the reason why it is not trivial to generalize the localization result to the higher order eigenvectors. We achieve this with the help of auxiliary eigenvectors and the Bauer-Fike theorem. That is, we prove that the kth Dirichlet eigenvector $\psi_k^{(n)}$ asymptotically concentrates in the site $z_{(k,n)}$ where the local speed measure π attains its kth minimum π_{k,B_n} over the box B_n.

As a direct consequence of the localization, the kth Dirichlet eigenvalue $\lambda_k^{(n)}$ behaves asymptotically like π_{k,B_n}.

Weak convergence of the eigenvalues. Roughly speaking, if the distribution function F varies regularly at zero with index $\gamma \in (0, 1/4)$, then we show that there exists a slowly varying function $L^*(n)$ such that $L^*(n)n^{\frac{1}{2\gamma}}\lambda_k^{(n)}$ converges in distribution to a non-degenerate random variable. This is in analogy to [Fag12, Theorem 2.5(i)] for the one-dimensional case.[15]

1.3.3 Homogenization

Let us now assume a more general setting: First, instead of i.i.d. conductances, we only assume that the conductances are stationary and ergodic with respect to spatial translations. Next, we still assume that all nearest-neighbor conductances are positive but now also conductances on long-range connections are allowed to be non-zero as long as

$$\mathbb{E}\left[\sum_{z\in\mathbb{Z}^d} \omega_{0z}|z|^2\right] < \infty\,.$$

This is the minimal assumption that we need in order to use the usual L^2-theory of Section 2.5.2. Heuristically, this condition ensures that we do not enter the superdiffusive regime whose investigation lies outside the scope of this thesis.

Regarding the lower tail of the conductances, we are going to see that we have a critical moment condition as well. Let us recall the definition of q in (1.35), where we take only nearest-neighbor conductances into account, and let us assume that

$$q \;>\; q_{\mathrm{c}} \;=\; \begin{cases} d/2, & \text{for general stationary, ergodic conductances and } d \geq 2, \\ 1/4, & \text{for i.i.d. nearest-neighbor conductances and } d \geq 2, \\ 1, & \text{for } d = 1. \end{cases} \tag{1.42}$$

[15] Note the different scale: For $d = 1$ it would be $n^{1+1/\gamma}$ instead of $n^{1/(2\gamma)}$.

Thus, the critical exponent q_c coincides with the critical exponent for the local central limit theorem. Under the condition that $q > q_c$ we show that the diffusively rescaled solution to the Poisson equation (1.23) converges almost surely to the solution of a homogenized Poisson equation. By virtue of [JKO94, Chapter 11] it follows that the diffusively rescaled eigenvectors and eigenvalues converge almost surely as well. As an application of the spectral homogenization, we prove a quenched large deviation principle for the occupation time measures, given that the random walk stays in a slowly growing box, see Proposition 2.12. Thereby, we extend the results of [KW15, Theorem 1.8], where the authors use the close connection between the Dirichlet energy of the Laplace operator and the Donsker-Varadhan rate function of the occupation time measures of the associated random walk – a fact that we already mentioned on page 19.

Poincaré inequality. For self-adjoint Laplace operators, a crucial condition for many kinds of asymptotic homogenization is – apart from ergodicity – the validity of the Poincaré inequalities. This means that we need the inequalities (2.27) and (2.35).

For spectral homogenization this is immediately evident since the optimal constant C in (2.35) is exactly the inverse of the principal Dirichlet eigenvalue of the Laplacian (see Remark 2.10). In the situation of this thesis, we will see that the Poincaré inequalities (2.27) and (2.35) are not only necessary but carry us quite far, although they are not completely sufficient for our results. In fact, $\mathbb{E}[\omega(e)^{-q_c}] < \infty$ is sufficient for the Poincaré inequalities but not for the Moser iteration, which we use in Section 2.3.2.

Moreover, if we would take into account the long-range connections, we could even improve the sufficient condition for the Poincaré inequalities.

Optimality. For the moment, let us assume that only nearest neighbors are connected, or equivalently, that only nearest-neighbor conductances carry a positive conductance. As we explain in Remark 2.10, excepting the critical case $q = q_c$, the condition in (1.42) is optimal.

If $q < q_c$, then it is possible (and in the i.i.d. case even almost sure, see Chapter 3) that trapping structures as in Figure 1.10 appear, which immediately contradict the Poincaré inequality.

If $q = q_c$, then we have to look closer. As we have already mentioned in Section 1.3.1, the results of Neukamm, Schäffner and Schlömerkemper in [NSS17] imply that even $\mathbb{E}[\omega(e)^{-d/2}] < \infty$ is sufficient for homogenization, see also Remark 2.3. In dimension one, Faggionato [Fag12, Theorem 2.6] also showed that $\mathbb{E}[\omega(e)^{-1}] < \infty$ is sufficient. The reason why we do not reach this condition is that we need $q > q_c$ for the Moser iteration, as we have already mentioned in the previous paragraph.

1.4 Comparison with earlier results

Our investigation on the spectral behavior of the random conductance Laplacian fits well with the results of earlier research.

Dimension one. In the special case of dimension one, Faggionato [Fag12] showed that a finite inverse moment of ω is sufficient for spectral homogenization [Fag12, Proposition 2.6]. We reproduce this result in Chapter 2. Further, if the inverse conductances ω^{-1} are i.i.d. and in the domain of attraction of an α-stable law with $0 < \alpha < 1$, then Faggionato showed that the vector of the first k Dirichlet eigenvalues rescaled by $n^{1+1/\alpha}$ times a slowly varying function converges in distribution to the vector of the first k Dirichlet eigenvalues of a random generalized differential operator [Fag12, Theorem 2.5]. This compares to our result Corollary 3.17.

Spectral homogenization for $d \geq 2$. Boivin and Depauw [BD03] proved spectral homogenization for stationary and ergodic conductances that fulfill the uniform ellipticity condition, i.e., where there exist positive and finite constants that uniformly bound the conductances from above and below.

As we have already mentioned in Section 1.2.2, when the conductances have a bounded range, Neukamm, Schäffner, and Schlömerkemper [NSS17, Corollary 3.4, Remark 3.6, Proposition 3.24] have recently proved that for $q > q_{\mathrm{c}}$ the Dirichlet energy of $-\mathcal{L}^\omega$ Γ-converges to a deterministic, homogeneous integral and the minimizers admit a strongly convergent subsequence. This holds also if $\mathbb{E}[\omega^{-q_{\mathrm{c}}}] < \infty$.[16] This together with [Mas93, Theorem 13.5] implies that Conditions I–IV of [JKO94, Chapter 11] are fulfilled and spectral convergence follows. On the other hand, in Chapter 2 we use the method of stochastic two-scale convergence by Zhikov and Pyatniskii [ZP06] to show that the Poisson equation homogenizes. There our approach is similar to the one of Faggionato [Fag08] who already employed two-scale convergence in order to show homogenization for a Laplacian with bounded conductances and shifted spectrum as in (1.34). From the homogenization of the Poisson equation, the spectral homogenization follows again by [JKO94, Chapter 11].

The basis for both [NSS17] and Chapter 2 are Poincaré and Sobolev inequalities that were already used by Andres, Deuschel, and Slowik [ADS16] to prove a quenched local CLT under suitable moment conditions. This is one reason why the critical exponents match each other.

Relation to heat-kernel upper bounds. In this paragraph we explain why the subdiffusive scaling of the principal Dirichlet eigenvalue contradicts the validity of a local central limit theorem. As we have already mentioned in Section 1.2.2, the local CLT was established in 2015 by the two teams of authors Andres, Deuschel, Slowik [ADS16] and Boukhadra, Kumagai, Mathieu

[16] On how to obtain the result for $q = q_{\mathrm{c}}$ in the i.i.d. case, see our Remark A.1.

[BKM15]. For independent and identically conductances both require that there exists $\varepsilon > 0$ such that both $\mathbb{E}[\omega]$ and $\mathbb{E}[\omega^{-1/4-\varepsilon}]$ are finite.

Let τ_A be the escape time from a set $A \subset \mathbb{Z}^d$, i.e., $\tau_A = \inf\{t \geq 0 \colon X_t \notin A\}$. There exists a natural relation between the principal Dirichlet eigenvalue of the operator $-\mathcal{L}^\omega$ and the expected escape time $\mathrm{E}_x^\omega[\tau_{B_n}]$ from the box B_n, see [BdH15, Section 8.4.1]:

$$\lambda_1^{(n)} \geq \left(\max_{z \in B_n} \mathrm{E}_z^\omega \left[\tau_{B_n} \right] \right)^{-1} .$$

Thus, an upper bound for the principal Dirichlet eigenvalue $\lambda_1^{(n)}$ implies a lower bound on the maximal expected escape time from the box B_n.

Now Lemma 2.1(i) of [BKM15] implies that if the heat kernel $p_t(x, y)$ has a diffusive on-diagonal upper bound, i.e. there exists $c \in (0, \infty)$ and a random $n_0 \in \mathbb{N}$ such that

$$p_{n^2}(x, y) \leq c n^{-d} \qquad \forall x, y \in B_n \, , \; n \geq n_0 \, ,$$

then $\max_{z \in B_n} \mathrm{E}_z^\omega \left[\tau_{B_n} \right] \sim n^{-2}$. Diffusive heat-kernel upper bounds are a necessary condition for the validity of a local CLT.

But if we assume that the principal Dirichlet eigenvalue scales subdiffusively, i.e., $\lambda_1^{(n)} \in o(n^{-2})$, then $\max_{z \in B_n} \mathrm{E}_z^\omega \left[\tau_{B_n} \right]$ grows faster than n^{-2} and therefore a subdiffusively scaling principal Dirichlet eigenvalue contradicts the validity of a local CLT.

We can explain the exploding escape times by showing that a large box contains some sites where the expected time to even leave the initial position is anomalously long (see Section 3.2). Although this effect is related to the one responsible for the anomalous heat-kernel decay observed in [BBHK08], it is still a different one. In [BBHK08], the dominating effect is that a random walk finds a trap elsewhere and then returns to its initial position. This behavior has a more complex dependence on the Laplacian's eigenvalues. In [FM06], on the other hand, the situation is different due to the averaging over the environment. There the dominating strategy is indeed to not leave the initial position at all.

Localization for $d \geq 2$. The results of Chapter 3 for the random conductance Laplacian compare well to similar results of the random Schrödinger operator $\Delta + \xi$ with random potential $\xi \colon \mathbb{Z}^d \to \mathbb{R}$, see [BK16] and [Ast16, Ch. 6].

Homogenization of non-local operators. A related problem of a non-local operator was recently studied by Piatnitski and Zhizhina [PZ17] in the periodic case, where, as in the present article, the limit operator is a deterministic second order elliptic operator.

Chapter 2
Homogenization[1]

In this chapter we assume a very general situation where the conductances are stationary and ergodic with respect to spatial translations and, in addition, we allow unbounded-range connections. This means that the underlying graph is the complete graph $(\mathbb{Z}^d, \mathfrak{E})$.

Under the condition that very long connections are sufficiently weak and that the conductances have a sufficiently light tail near zero, we prove homogenization of the Poisson equation and spectral homogenization on a bounded domain with zero Dirichlet boundary condition. As an application of spectral homogenization, we prove a quenched large deviation principle for the occupation time measures, given that the random walk stays in a slowly growing box.

Besides ergodic theory, the main ingredient for homogenization are Poincaré and Sobolev inequalities, which are only valid if the lower tail of the conductances fulfills a well-known moment condition, that is also crucial for the validity of a local limit theorem of the random walk among random conductances. In addition to Poincaré and Sobolev inequalities, our proofs rely on stochastic two-scale convergence, an analytic method that is based on the ergodic theorem and was introduced in [ZP06].

2.1 Notation and assumptions

For any $\boldsymbol{\omega} \in \Omega$, we denote the set of *open edges* by

$$\mathcal{O} \;\equiv\; \mathcal{O}(\boldsymbol{\omega}) \;:=\; \{e \in \mathfrak{E} \,:\, \omega(e) > 0\} \;\subset\; \mathfrak{E}.$$

[1] The content of this chapter is joint work with Martin Heida (Weierstraß-Institut Berlin) and Martin Slowik (Technische Universität Berlin). It is accepted at the journal *Annales de l'Institute Henri Poincaré* with the title "Homogenization theory for the random conductance model with degenerate ergodic weights and unbounded-range jumps" and it will soon be available at the DOI 10.1214/18-AIHP917.

Moreover, we let τ_x denote the translation by a vector $x \in \mathbb{Z}^d$, i.e., we write $\omega_{x,y} = (\tau_x \boldsymbol{\omega})_{0,y-x}$.

2.1.1 Assumptions

In this chapter we will usually assume that the law $\mathbb{P}$ fulfills the following conditions.

Assumption 2.1.

(a) The law $\mathbb{P}$ is stationary and ergodic with respect to spatial translations $(\tau_x)_{x\in\mathbb{Z}^d}$.
(b) $\mathbb{E}\big[\sum_{z\in\mathbb{Z}^d} \omega_{0,z}|z|^2\big] < \infty$.
(c) For $\mathbb{P}$-a.e. environment $\boldsymbol{\omega}$, the set $\mathcal{O}(\boldsymbol{\omega})$ of open edges contains the set $\mathfrak{E}_d$ of nearest-neighbor edges of $\mathbb{Z}^d$.

Note that $\mathcal{L}^\omega$ is $\mathbb{P}$-a.s. well-defined under Assumption 2.1(b).

In addition to Assumption 2.1, our main results rely on an integrability condition for the lower tails of the conductances, for which we need to define the notion of paths in $(\mathbb{Z}^d, \mathfrak{E}_d)$. A path of length l between x and y in $(\mathbb{Z}^d, \mathfrak{E}_d)$ is a sequence $(x_i : i = 0, \ldots, l)$ with the property that $x_0 = x$, $x_l = y$ and $\{x_i, x_{i+1}\} \in \mathfrak{E}_d$ for any $i = 0, \ldots, l-1$. If $\zeta = (x_i : i = 0, \ldots, l)$ is a path and there exists $i \in \{1, \ldots, l-1\}$ such that $\{x_i, x_{i+1}\} = e$, then we use the shorthand notation $e \in \zeta$.

For any $e \in \mathfrak{E}_d$ and $\mathbb{N} \ni l < \infty$, let $\Gamma_l(e)$ be a collection of paths in $(\mathbb{Z}^d, \mathfrak{E}_d)$ between the vertices of the edge e with length at most l such that no two paths in $\Gamma_l(e)$ share an edge. We define the measures ν^ω and ν_l^ω on $\mathbb{Z}^d$ by

$$\nu^\omega(x) := \sum_{e\in\mathfrak{E}_d:\ x\in e} \omega(e)^{-1} \qquad \text{and} \qquad \nu_l^\omega(x) := \sum_{e\in\mathfrak{E}_d:\ x\in e} \omega_l(e)^{-1}, \tag{2.1}$$

where

$$\omega_l(e)^{-1} := \min_{\zeta\in\Gamma_l(e)} \sum_{e'\in\zeta} \omega(e')^{-1}. \tag{2.2}$$

We let $\zeta_l^{\mathrm{opt}}(e)$ denote the minimizer of the RHS of (2.2). For an example of how to choose Γ_9 reasonably for the nearest-neighbor lattice $(\mathbb{Z}^d, \mathfrak{E}_d)$ if the conductances are independent and identically distributed, see Figure 2.1.

Assumption 2.2 (Lower moment condition).
If $d = 1$, then $\mathbb{E}[1/\omega(e)] < \infty$ for any $e \in \mathfrak{E}_d$. In addition, if $d \geq 2$, then

(a) there exists $l \in \mathbb{N}$ such that $\mathbb{E}\big[(\nu_l^\omega(0))^{d/2}\big] < \infty$.
(a') there exists $l \in \mathbb{N}$ and $q > d/2$ such that $\mathbb{E}[(\nu_l^\omega(0))^q] < \infty$.

Remark 2.3. *Note that Assumption 2.2(a) is sufficient for the Poincaré inequalities (Section 2.3.1) and the compact embedding (Section 2.4). The only reason why we need Assumption 2.2(a') is the Moser iteration in the proof of Proposition 2.17, which we need for the Auxiliary Lemma 2.36. In fact, if we would assume that the length of the connections was bounded, or in other words, there exists $R < \infty$ such that $\mathbb{P}$-a.s. $\mathcal{O}(\boldsymbol{\omega}) \subseteq \{e\colon |e| < R\}$, then the authors of [NSS17] proved Γ-convergence under Assumption 2.2(a). Therefore the compact embedding of Section 2.4 implies the homogenization result Theorem 2.5 and thus Assumption 2.2(a) is sufficient.*

Remark 2.4 *Generally, $\mathbb{E}[\omega(e)^{-d/2}] < \infty$ for edges $e \in \mathfrak{E}_d$ is sufficient for Assumption 2.2(a). This can even be improved if the conductances $\omega(e)$ $(e \in E)$ are independent and identically distributed (i.i.d.) and $d \geq 2$. For example, on the nearest-neighbor lattice $(\mathbb{Z}^d, \mathfrak{E}_d)$ with independent and identically distributed conductances, Assumption 2.2(a) holds if $\mathbb{E}\big[\omega(e)^{-1/4}\big] < \infty$ for any edge $e \in \mathfrak{E}_d$. Similarly, Assumption 2.2(a') holds if there exists $q > q_{\mathrm{c}} = 1/4$ such that $\mathbb{E}\big[\omega(e)^{-q}\big] < \infty$ for any edge $e \in \mathfrak{E}_d$. This follows because any two sites in $\mathbb{Z}^d$ are connected through $2d$ independent nearest-neighbor paths (see Figure 2.1, cf. [ADS16, Fig. 2], [Kes86, Fig. 2.1]).*

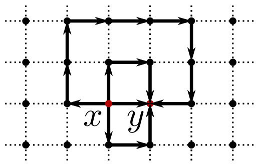


Fig. 2.1:
Independent paths

If we added further links to the edge set $E = \mathfrak{E}_d$, the number of independent paths between any two sites would increase whence the critical exponent q_{c} would decrease. If we assumed that the edge set E would contain all the links of $\mathfrak{E}$, then it would even be sufficient to assume that there exists $q > 0$ such that $\mathbb{E}\big[\omega(e)^{-q}\big] < \infty$. Note that in order not to violate Assumption 2.1(b), we would assume in this case that $\omega(e) = \tilde{\omega}(e)/|e|^{\alpha}$ where the $(\tilde{\omega}(e))_e$ are i.i.d., $\alpha > d + 2$ and $|e|$ is the euclidean length of the edge e.

2.1.2 The rescaled lattice

We aim to consider the behavior of the operator $\mathcal{L}^{\omega}$ in boxes of the form $B_n := (-n, n)^d \cap \mathbb{Z}^d$ with zero Dirichlet boundary conditions. More precisely, we fix an environment $\boldsymbol{\omega}$ on the entire $\mathbb{Z}^d$, let the box size n grow to infinity and want to characterize the behavior of solutions to the Poisson equation and the spectral problem. For this purpose we use analytic techniques as introduced in Section 2.5. Regarding these techniques, it is more natural to replace the lattice $\mathbb{Z}^d$ by the rescaled lattice $\mathbb{Z}^d_\varepsilon := \varepsilon\mathbb{Z}^d$ and the growing box B_n by the box $Q_\varepsilon := Q \cap \mathbb{Z}^d_\varepsilon$ with $Q = (-1, 1)^d$ and $\varepsilon = n^{-1}$.

In this context, the Laplacian defined in (1.12) corresponds to the accelerated operator $\mathcal{L}_\varepsilon^\omega$ which acts on real-valued functions $f \in \ell^2(\mathbb{Z}_\varepsilon^d)$ as

$$(\mathcal{L}_\varepsilon^\omega f)(x) \;=\; \varepsilon^{-2} \sum_{z \in \mathbb{Z}_\varepsilon^d} \omega_{\frac{x}{\varepsilon},\frac{z}{\varepsilon}} [f(z) - f(x)]\,, \qquad \big(x \in \mathbb{Z}_\varepsilon^d\big)\,, \tag{2.3}$$

where the conductances $\omega_{\frac{x}{\varepsilon},\frac{z}{\varepsilon}}$ remain random variables associated with the links in the edge set $\mathfrak{E}$, i.e., the links between sites in $\mathbb{Z}^d$. Note that if $\mathcal{L}^\omega$ is the generator of a Markov process $(X_t)_{t\geq 0}$, then $\mathcal{L}_\varepsilon^\omega$ is the generator of the diffusively rescaled Markov process $(X_t^\varepsilon)_{t\geq 0}$, which fulfills $X_t^\varepsilon = \varepsilon X_{\varepsilon^{-2}t}$.

For $\varepsilon, p > 0$ and $A_\varepsilon \subseteq \mathbb{Z}_\varepsilon^d$, we define the function spaces

$$\ell_\varepsilon^p(A_\varepsilon) := \Big\{ v \,:\, \mathbb{Z}_\varepsilon^d \to \mathbb{R} \colon\; \varepsilon^d \sum_{x \in A_\varepsilon} v(x)^p < \infty \Big\}$$

$$\text{with } \|v\|_{\ell_\varepsilon^p(A_\varepsilon)} := \Big(\varepsilon^d \sum_{x \in A_\varepsilon} v(x)^p \Big)^{1/p}\,. \tag{2.4}$$

We abbreviate $\ell_\varepsilon^p := \ell_\varepsilon^p(\mathbb{Z}_\varepsilon^d)$.

Analogously to ℓ_ε^p, we introduce the Hilbert spaces $\mathcal{H}_0, \mathcal{H}_\varepsilon$ through

$$\mathcal{H}_0 \;=\; \big\{ v \in L^2\big(\mathbb{R}^d\big) \,:\, \operatorname{supp} v \subseteq Q \big\}, \quad \mathcal{H}_\varepsilon \;=\; \big\{ v \in \ell_\varepsilon^2(\mathbb{Z}_\varepsilon^d) \,:\, \operatorname{supp} v \subseteq Q_\varepsilon \big\}$$

and let $\mathcal{H}_0$ and $\mathcal{H}_\varepsilon$ be equipped with the scalar products

$$\langle u, v \rangle_{\mathcal{H}_0} \;=\; \int_{\mathbb{R}^d} u(x) v(x)\, \mathrm{d}x\,, \quad \langle u^\varepsilon, v^\varepsilon \rangle_{\mathcal{H}_\varepsilon} \;=\; \varepsilon^d \sum_{z \in \mathbb{Z}_\varepsilon^d} u^\varepsilon(z)\, v^\varepsilon(z)\,.$$

In analogy to (1.20), we define

$$\mathcal{E}_\varepsilon^\omega(u^\varepsilon) \;:=\; \langle u^\varepsilon, -\mathcal{L}_\varepsilon^\omega u^\varepsilon \rangle_{\mathcal{H}_\varepsilon}\,. \tag{2.5}$$

For $z \in \mathbb{Z}_\varepsilon^d$, we let $b(z, \varepsilon/2)$ denote the half-open ball $z + (-\varepsilon/2, \varepsilon/2]^d$. We define the local averaging operator $\mathcal{R}_\varepsilon \colon \mathcal{H}_0 \to \mathcal{H}_\varepsilon$ acting on functions $f \in \mathcal{H}_0$ by

$$(\mathcal{R}_\varepsilon f)(z) \;=\; \varepsilon^{-d} \int_{b\left(z, \frac{\varepsilon}{2}\right)} f(x)\, \mathrm{d}x \quad z \in \mathbb{Z}_\varepsilon^d\,. \tag{2.6}$$

A direct calculation shows that its adjoint operator $\mathcal{R}_\varepsilon^* \colon \mathcal{H}_\varepsilon \to \mathcal{H}_0$ is given by

$$\mathcal{R}_\varepsilon^* v^\varepsilon \;=\; \sum_{z \,\in\, \mathbb{Z}_\varepsilon^d} v^\varepsilon(z) \mathbb{1}_{b\left(z, \frac{\varepsilon}{2}\right)} \qquad (v^\varepsilon \in \mathcal{H}_\varepsilon)\,, \tag{2.7}$$

where we write $\mathbb{1}_{b\left(z, \frac{\varepsilon}{2}\right)}$ for the characteristic function of $b\big(z, \frac{\varepsilon}{2}\big)$.

2.2 Main results

2.2.1 Poisson equation

Given a function $f^\varepsilon\colon \mathbb{Z}^d_\varepsilon \to \mathbb{R}$, we are interested in the solution $u^\varepsilon \in \mathcal{H}_\varepsilon$ of the Poisson problem

$$-\mathcal{L}^\omega_\varepsilon u^\varepsilon \;=\; f^\varepsilon \qquad \text{on } Q_\varepsilon \tag{2.8}$$

with zero Dirichlet conditions. The above problem has a unique solution because $-\mathcal{L}^\omega_\varepsilon$ is invertible on $\mathcal{H}_\varepsilon$.

Theorem 2.5. *Let $f^\varepsilon\colon Q_\varepsilon \to \mathbb{R}$ be a sequence of functions such that $\mathcal{R}^*_\varepsilon f^\varepsilon \rightharpoonup f$ weakly in $L^2(Q)$ for some $f \in L^2(Q)$. If Assumptions 2.1 and 2.2(a') hold, then for almost all $\boldsymbol{\omega} \in \Omega$ the sequence of solutions $u^\varepsilon \in \mathcal{H}_\varepsilon$ to the problem* (2.8) *satisfies $\mathcal{R}^*_\varepsilon u^\varepsilon \to u$ strongly in $L^2(Q)$, where $u \in H^1_0(Q)\cap H^2(Q)$ solves the limit problem*

$$-\nabla\cdot(A_{\text{hom}}\nabla u) = 2f\,, \tag{2.9}$$

almost everywhere in Q with A_{hom} defined through (2.54).

We prove this theorem at the end of Section 2.6. In Lemma 2.25 we prove that A_{hom} is strictly positive definite and by standard arguments A_{hom} is symmetric.

Based on Theorem 2.5, we introduce the operator

$$\forall u \in L^2(Q) \qquad \mathcal{L}^\omega_0 u := \nabla\cdot(A_{\text{hom}}\nabla u)\,,$$

such that $-\mathcal{L}^\omega_0$ is a symmetric positive definite operator on $L^2(Q)$ with domain $H^2(Q)$.

Remark 2.6 *With our methods, Theorem 2.5 can be easily generalized for other lattices than $\mathbb{Z}^d$. In order to apply our methods directly, we just have to require that the lattice is translationally invariant (for the two-scale convergence, see Section 2.5.4) and fulfills a Sobolev inequality (as in* (2.24) *or* (2.25)*) with isoperimetric dimension d_{ISO} (to obtain the necessary Poincaré inequalities and make the Moser iteration work). For example, the triangular lattice in Figure 2.2 is translationally invariant and has isoperimetric dimension $d_{\text{ISO}} = 2$. If we therefore replace $\mathbb{Z}^d$ by the triangular lattice and the dimension d in Assumption 2.2 by the isoperimetric dimension d_{ISO}, Theorem 2.5 still holds.*

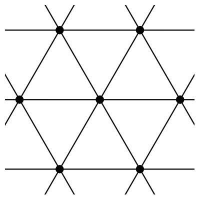


Fig. 2.2: Triangular lattice.

Note that in view of Remark 2.4, we observe that in the case of independent and identically distributed conductances on the triangular lattice, Assumption 2.2(a) holds if $\mathbb{E}[\omega(e)^{-1/6}] < \infty$.

Remark 2.7. *Although we focus here on the random conductance model with long-range jumps and positive nearest-neighbor conductances, our arguments do not require the full strength of this assumption. For instance, we can also extend the homogenization result to the* nearest-neighbor *percolation case. More precisely, we can relax Assumption 2.1(c) such that the set of open edges $\mathcal{O}(\omega) \subset \mathfrak{E}_d$ forms a unique infinite cluster that satisfies both a volume regularity condition and a (weak) relative isoperimetric inequality on large scales, cf. [DNS18]. Notice that in the nearest-neighbor percolation setting, similar homogenization results have also been obtained by Faggionato in [Fag08] under the additional assumption that the conductances are bounded from above.*

2.2.2 Spectral homogenization

In order to infer the large deviation principle Proposition 2.12, let us now consider the spectrum of the operators $-\mathcal{L}_\varepsilon^\omega + \mathcal{R}_\varepsilon V$ with an arbitrary bounded, continuous potential $V \colon \mathbb{R}^d \to \mathbb{R}$. On the domain Q_ε with zero Dirichlet conditions we can represent $-\mathcal{L}_\varepsilon^\omega + \mathcal{R}_\varepsilon V$ as a real symmetric matrix and therefore we can choose the set $\{\psi_j^\varepsilon\}_{j=1,\dots,k}$ of Dirichlet eigenvectors such that they form an orthonormal system. By virtue of the Perron-Frobenius theorem (see e.g. [Sen81, Chapter 1]) the principal Dirichlet eigenvalue λ_1^ε is unique. Thus, we now consider the problem

$$\begin{aligned}
&\psi_k^\varepsilon \in \mathcal{H}_\varepsilon, \quad (-\mathcal{L}_\varepsilon^\omega + \mathcal{R}_\varepsilon V)\psi_k^\varepsilon = \lambda_k^\varepsilon \psi_k^\varepsilon, \quad k = 1, 2, \dots, \\
&\lambda_1^\varepsilon < \lambda_2^\varepsilon \le \dots \le \lambda_k^\varepsilon \dots, \\
&\langle \psi_k^\varepsilon, \psi_l^\varepsilon \rangle_{\mathcal{H}_\varepsilon} = \delta_{kl}\,.
\end{aligned} \tag{2.10}$$

Similarly, we consider the spectrum of the operator $-\mathcal{L}_0^\omega + V$, i.e.,

$$\begin{aligned}
&\psi_k^0 \in \mathcal{H}_0, \quad (-\mathcal{L}_0^\omega + V)\psi_k^0 = \lambda_k^0 \psi_k^0, \quad k = 1, 2, \dots, \\
&\lambda_1^0 < \lambda_2^0 \le \dots \le \lambda_k^0 \dots, \\
&\langle \psi_k^0, \psi_l^0 \rangle_{\mathcal{H}_\varepsilon} = \delta_{kl}\,.
\end{aligned} \tag{2.11}$$

In order to study the homogenization of (2.10) with a non-trivial potential V, we need the following result.

Proposition 2.8. *Let $f^\varepsilon \colon Q_\varepsilon \to \mathbb{R}$ be a sequence of functions such that $\mathcal{R}_\varepsilon^* f^\varepsilon \rightharpoonup f$ weakly in $L^2(Q)$ for some $f \in L^2(Q)$. Let $V \colon \mathbb{R}^d \to \mathbb{R}$ be a bounded, continuous potential such that $\liminf_{\varepsilon\to 0} \lambda_1^\varepsilon > 0$. If Assumptions*

2.1 and 2.2(a') hold, then for almost all $\boldsymbol{\omega} \in \Omega$ the sequence of solutions $u^\varepsilon \in \mathcal{H}_\varepsilon$ to the problem

$$(-\mathcal{L}_\varepsilon^\omega + \mathcal{R}_\varepsilon V)u^\varepsilon \;=\; f^\varepsilon \tag{2.12}$$

satisfies $\mathcal{R}_\varepsilon^* u^\varepsilon \to u$ strongly in $L^2(Q)$, where $u \in H_0^1(Q) \cap H^2(Q)$ solves the limit problem

$$-\nabla \cdot (A_{\mathrm{hom}} \nabla u) + 2Vu \;=\; 2f\,, \tag{2.13}$$

almost everywhere in Q with A_{hom} defined through (2.54).

We prove this proposition in Section 2.7. Note that under Assumption 2.2(a'), the condition $V \geq 0$ is sufficient for $\liminf_{\varepsilon\to 0} \lambda_1^\varepsilon > 0$.

By virtue of [JKO94, Lemma 11.3, Theorem 11.5], Proposition 2.8 implies the following result, see Section 2.7. Note that for the spectral result we can drop the assumption $\liminf_{\varepsilon\to 0} \lambda_1^\varepsilon > 0$ as we explain in Section 2.7.

Theorem 2.9. *Let $V\colon \mathbb{R}^d \to \mathbb{R}$ be a bounded, continuous potential and let $k \in \mathbb{N}$. If Assumptions 2.1 and 2.2(a') hold, then*

$$\lambda_k^\varepsilon \to \lambda_k^0 \qquad \mathbb{P}\text{-a.s. as } \varepsilon \to 0\,. \tag{2.14}$$

Further, the following statements are true:

(i) Let $k \in \mathbb{N}$ and let ε_m be a null sequence. Then there $\mathbb{P}$-a.s. exists a family $\{\psi_j^0\}_{1\leq j\leq k}$ of eigenvectors of the operator $-\mathcal{L}_0^\omega + V$ and a subsequence, still indexed by ε_m, along which the vector

$$\left(\mathcal{R}_{\varepsilon_m}^* \psi_1^{\varepsilon_m}, \dots, \mathcal{R}_{\varepsilon_m}^* \psi_k^{\varepsilon_m}\right) \to \left(\psi_1^0, \dots, \psi_k^0\right) \qquad \text{strongly in } L^2(Q)\,.$$

(ii) On the other hand, if the multiplicity of λ_k^0 is equal to s, i.e.,

$$\lambda_{k-1}^0 < \lambda_k^0 = \ldots = \lambda_{k+s-1}^0 < \lambda_{k+1}^0 \qquad (\text{with } \lambda_0^0 < \lambda_1^0 \text{ arbitrary})\,,$$

then there $\mathbb{P}$-a.s. exists a sequence $\psi^\varepsilon \in \mathcal{H}_\varepsilon$ such that

$$\lim_{\varepsilon\to 0} \|\psi^\varepsilon - \mathcal{R}_\varepsilon \psi_k^0\|_{\mathcal{H}_\varepsilon} = 0\,, \tag{2.15}$$

where ψ^ε is a linear combination of the eigenfunctions of the operator $-\mathcal{L}_\varepsilon^\omega + \mathcal{R}_\varepsilon V$ corresponding to the eigenvalues $\lambda_k^\varepsilon, \dots, \lambda_{k+s-1}^\varepsilon$.

Note that Biskup, Fukushima and König [BFK16] proved a spectral homogenization theorem for a random bounded potential and the standard lattice Laplacian. They later extended their result to unbounded potentials in [BFK17].

2.2.3 A note on optimality

Remark 2.10 *Let us discuss in what sense Assumption 2.2 is optimal for the result of Theorem 2.9 with $V = 0$. Since the principal Dirichlet eigenvalue has the variational representation*

$$\lambda_1^\varepsilon \;=\; \inf\{\langle u^\varepsilon, -\mathcal{L}_\varepsilon^\omega u^\varepsilon\rangle_{\mathcal{H}_\varepsilon} \,:\, u^\varepsilon \in \mathcal{H}_\varepsilon \text{ and } \|u^\varepsilon\|_{\mathcal{H}_\varepsilon} = 1\}$$

(also known as the Rayleigh-Ritz formula, or the Courant-Fischer theorem), it is necessary for spectral homogenization that $\mathbb{P}$-a.s. there exists $C < \infty$ such that

$$\|u^\varepsilon\|_{\mathcal{H}_\varepsilon}^2 \;\le\; C\langle u^\varepsilon, -\mathcal{L}_\varepsilon^\omega u^\varepsilon\rangle_{\mathcal{H}_\varepsilon} \qquad \text{for all } u^\varepsilon \in \mathcal{H}_\varepsilon \tag{2.16}$$

and for all $\varepsilon > 0$ (uniform Poincaré inequality).

If we assume that $\mathbb{P}$-a.s. only nearest-neighbor connections carry a positive conductance, i.e., $\mathcal{O}(\boldsymbol{\omega}) = \mathfrak{E}_d$, then Assumption 2.2 is optimal for the uniform Poincaré inequality up to the critical case $\sup\{r\colon \mathbb{E}[\omega(e)^{-r}] < \infty\} = q_{\mathrm{c}}$ *(cf.* (1.42)*). This means that if* $\sup\{r\colon \mathbb{E}[\omega(e)^{-r}] < \infty\} < q_{\mathrm{c}}$*, then it is possible to construct an environment where the uniform Poincaré inequality does not hold as ε tends to zero.*

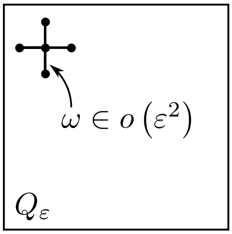


Fig. 2.3: Variable-speed trap in $d \ge 2$, cf. Fig. 1.10.

For $d \ge 2$, this is due to trapping structures as in Figure 2.3 where u^ε can concentrate its entire mass, see e.g. the survey on eigenvalue upper bounds on page 79. The construction of stationary, ergodic environments with such trapping structures is analogous to the one of a trap for the constant-speed random walk in [ADS16, Theorem 5.4]. In the i.i.d. case and if $\sup\{r\colon \mathbb{E}[\omega(e)^{-r}] < \infty\} < 1/4$*, the traps occur even $\mathbb{P}$-a.s. for ε small enough and the principal Dirichlet eigenvector localizes $\mathbb{P}$-a.s. in a single site Theorem 3.13.*

In $d = 1$ and if $\sup\{r\colon \mathbb{E}[\omega(e)^{-r}] < \infty\} < 1$*, even an i.i.d. environment contradicts the uniform Poincaré inequality: By a Borel Cantelli argument we can show that $\mathbb{P}$-a.s. for ε small enough there exist edges $e_1 = \{x_1, y_1\}$ and $e_2 = \{x_2, y_2\}$ such that $x_1 \in (-\varepsilon^{-1}, -\varepsilon^{-1}/2) \cap \mathbb{Z}$ and $x_2 \in (\varepsilon^{-1}/2, \varepsilon^{-1}) \cap \mathbb{Z}$, respectively, and such that both $\omega(e_1)$ and $\omega(e_2)$ decay much faster than ε. When we insert a function $u^\varepsilon \in \mathcal{H}_\varepsilon$ into* (2.16) *that is 1 on the interval* $[\max(\varepsilon x_1, \varepsilon y_1), \min(\varepsilon x_2, \varepsilon y_2)]$ *and zero otherwise, then we see that C diverges as ε tends to zero, which is a contradiction to a uniform Poincaré inequality.*

If we assume that $\mathcal{O}(\boldsymbol{\omega})$ is $\mathbb{P}$-a.s. strictly larger that $\mathfrak{E}_d$ but contains only connections of bounded length, an analogous construction as in [ADS16, Theorem 5.4] shows that $q_{\mathrm{c}} = d/2$ is still optimal in the general stationary ergodic case with $d \ge 2$. For independent conductances however, q_{c} decreases when the upper bound for the length of the connections increases, see also Re-

mark 2.4. On the other hand, if we assume that $\mathcal{O}(\boldsymbol{\omega})$ contains connections of unbounded length, all the suggested counterexamples fail and the question about the optimal conditions requires further research.

2.2.4 Local times

Our main motivation for this chapter is to prove a quenched large deviation principle (LDP) for the family of local times defined in (1.28) given that the random walk stays in a certain growing region of the lattice. More precisely, we define a spatial scaling α_t with $1 \ll \alpha_t \ll \sqrt{t}$ and consider the rescaled local times

$$L_t(z) \;:=\; \frac{\alpha_t^d}{t}\, l_t(\lfloor \alpha_t z \rfloor) \qquad (z \in \mathbb{R}^d, t > 0)\,. \tag{2.17}$$

Further, let $Q = (-1,1)^d$ and define $Q_t = \alpha_t Q \cap \mathbb{Z}^d$. In [KW15, Theorem 1.8], the authors prove a quenched large deviation principle for the function L_t given that $\operatorname{supp}(l_t) \subset Q_t$ and under the assumption that the conductances are i.i.d. and uniformly elliptic. Our aim is to generalize this result to stationary and ergodic conductances and replace the uniform ellipticity condition by a suitable moment condition.

Let us recall some facts about the local times of the simple random walk. We define the set

$$\mathcal{F} \;=\; \left\{ f^2 \colon f \in L^2(Q), \|f\|_2 = 1 \right\} \tag{2.18}$$

and equip $\mathcal{F}$ with the weak topology of integrals against bounded continuous functions $V \colon Q \to \mathbb{R}$. Notice that on the event $\{\operatorname{supp}(l_t) \subset Q_t\}$ the function L_t is an element of the set $\mathcal{F}$ and an L^1-normalized random step function on $\mathbb{R}^d$.

In the case of a simple random walk, i.e., when $\omega_{x,z} \equiv 1$, it is known that on the event $\{\operatorname{supp}(l_t) \subset Q_t\}$ the function L_t satisfies a large deviation principle on $\mathcal{F}$ with scale $t\alpha_t^{-2}$ and rate function $I_0 = I^{\mathrm{SRW}} - \inf_{\mathcal{F}} I^{\mathrm{SRW}}$, where

$$I^{\mathrm{SRW}}(f) \;=\; \begin{cases} \sum_{i=1}^d \int_Q (\partial_i f(y))^2 \,\mathrm{d}y = \|\nabla f\|_2^2, & f \in H_0^1(Q)\,, \\ \infty\,, & \text{else,} \end{cases} \tag{2.19}$$

see [KW15] for further explanation and [GKS07]. We prove that under quite general conditions, this is also true for the random conductance model, see

Proposition 2.12 and Corollary 2.13. For general stationary and ergodic conductances the resulting rate function reads

$$I_0 = I - \inf_{\mathcal{F}} I \qquad \text{where } I(f) \;=\; \begin{cases} \int_Q (\nabla f) \cdot A_{\mathrm{hom}} \nabla f\,, & f \in H_0^1(Q), \\ \infty\,, & \text{else,} \end{cases} \tag{2.20}$$

and the matrix $A_{\mathrm{hom}} \in \mathbb{R}^d \times \mathbb{R}^d$ is defined as in (2.54).

Assumption 2.11 (Heat kernel lower bounds). *There exists $c > 0$ such that $\mathbb{P}$-a.s. for t large enough*

$$\mathrm{P}_0^\omega[X_t = x] \;\geq\; ct^{-d/2} \tag{2.21}$$

for all $x \in \mathbb{Z}^d$ with $|x| \leq \sqrt{t}$.

Proposition 2.12. *Let Assumptions 2.1, 2.2(a') and 2.11 be fulfilled. Then $\mathbb{P}$-a.s. the rescaled local times L_t satisfy a large deviation principle with respect to the weak topology of integrals against bounded continuous functions $V\colon Q \to \mathbb{R}$ under $\mathrm{P}_0^\omega[\,\cdot \mid \mathrm{supp}(l_t) \subset \alpha_t Q]$ on $\mathcal{F}$. The scale is $t\alpha_t^{-2}$ and the rate function I_0 is defined in (2.20).*

We prove this proposition in Section 2.8 as a consequence of Theorem 2.9.

In the special case where only nearest-neighbor conductances are positive, Proposition 2.12 together with the heat kernel bounds of [ADS16, Proposition 4.7] respectively, implies the following corollary.

Corollary 2.13. *Let the conductances be stationary and ergodic with law $\mathbb{P}$ and let $\mathbb{P}$-a.s. $\mathcal{O}(\boldsymbol{\omega}) = \mathfrak{E}_d$. For $p, q \in [1, \infty]$ satisfying $1/p + 1/q < 2/d$ assume that $\mathbb{E}[\omega(e)^p] < \infty$ and $\mathbb{E}[\omega(e)^{-q}] < \infty$ for any $e \in \mathfrak{E}_d$. Then the large deviation principle from Proposition 2.12 holds.*

2.3 Inequalities

In analogy to the definition of ℓ^p_ε in (2.4), we define the following space-averaged norms for functions $f\colon \mathbb{Z}^d \to \mathbb{R}$. Let $A \subseteq \mathbb{Z}^d$ be a non-empty set and $p \in [1, \infty)$. Then

$$\|f\|_{p,A} \;:=\; \Bigg(\frac{1}{|A|} \sum_{x \in A} |f(x)|^p\Bigg)^{1/p} \qquad \text{and} \qquad \|f\|_{\infty,A} \;:=\; \max_{x \in A} |f(x)|\,, \tag{2.22}$$

where $|A|$ is the counting measure on A. Moreover, we let

$$(f)_A \;:=\; |A|^{-1} \sum_{x \in A} f(x) \tag{2.23}$$

abbreviate the average of f over the set A.

2.3.1 Poincaré and Sobolev inequalities

The main objective in this subsection is to prove weighted Poincaré and Sobolev inequalities. The Poincaré inequalities of Proposition 2.14 and (2.35) are the main tools in the proof of Lemma 2.19, whereas the Sobolev inequality of Proposition 2.15 with $\rho > 1$ ensures uniform ℓ^∞-bounds of the solution to the Poisson equation (see Section 2.3.2).

Starting point for our further considerations is the fact that the underlying unweighted Euclidean lattice $(\mathbb{Z}^d, \mathfrak{E}_d)$ satisfies the classical Sobolev inequality for any $d \geq 1$. Let $B \subset \mathbb{Z}^d$ be finite and connected and $u : \mathbb{Z}^d \to \mathbb{R}$. Then,

$$\inf_{a \in \mathbb{R}} \left\| u - a \right\|_{\infty, B} \;\leq\; C_1 \, |B|^{1/d} \left(\frac{1}{|B|} \sum_{\substack{x,y \in B \\ \{x,y\} \in \mathfrak{E}_d}} \left| u(x) - u(y) \right| \right) \tag{2.24}$$

for $d = 1$, whereas for any $d \geq 2$ and $\alpha \in [1, d)$ we have

$$\inf_{a \in \mathbb{R}} \left\| u - a \right\|_{\frac{d\alpha}{d-\alpha}, B} \;\leq\; C_1 \, |B|^{1/d} \left(\frac{1}{|B|} \sum_{\substack{x,y \in B \\ \{x,y\} \in \mathfrak{E}_d}} \left| u(x) - u(y) \right|^\alpha \right)^{1/\alpha}. \tag{2.25}$$

For $d \geq 2$ this Sobolev inequality follows from the isoperimetric inequality of the underlying Euclidean lattice, see e.g. [Kum14, Theorem 3.2.7].

Proposition 2.14 (local Poincaré inequality). *For any $x_0 \in \mathbb{Z}^d$ and $n \geq 1$, let $B(n) \equiv B(x_0, n) \subset \mathbb{Z}^d$. Suppose that $d = 1$ and that $\nu^\omega(x) < \infty$ for all $x \in \mathbb{Z}^d$. Then, there exists $C_{\mathrm{PI}} < \infty$ such that*

$$\left\| u - (u)_{B(n)} \right\|_{2, B(n)}^2 \;\leq\; C_{\mathrm{PI}} \left\| \nu^\omega \right\|_{1, B(n)} \frac{n^2}{|B(n)|} \sum_{x,y \in B(n)} \omega_{xy} \left| u(x) - u(y) \right|^2 \tag{2.26}$$

for any $u \colon \mathbb{Z} \to \mathbb{R}$.

Furthermore, for every $d \geq 2$ and $l \in [1, \infty)$ with $\nu_l^\omega(x) < \infty$ for all $x \in \mathbb{Z}^d$, there exist constants $C_{\mathrm{PI}} \equiv C_{\mathrm{PI}}(d, l) < \infty$ and $C_{\mathrm{W}} \equiv C_{\mathrm{W}}(l) < \infty$ with $C_{\mathrm{W}}(1) = 1$ such that

$$\begin{aligned} &\left\| u - (u)_{B(n)} \right\|_{2, B(n)}^2 \\ &\qquad \leq\; C_{\mathrm{PI}} \left\| \nu_l^\omega \right\|_{\frac{d}{2}, B(n)} \frac{n^2}{|B(n)|} \sum_{x,y \in B(C_{\mathrm{W}} n)} \omega_{xy} \left| u(x) - u(y) \right|^2, \end{aligned} \tag{2.27}$$

for any $u \colon \mathbb{Z}^d \to \mathbb{R}$, where the measure ν_l is given by (2.1) with suitable path sets Γ_l.

Proof of Proposition 2.14. As in [ADS16, Proposition 2.1 or 6.1], the assertion is an immediate consequence of (2.25) and Hölder's inequality (see also [GM18, Lemma 2.3]). Nevertheless, we will repeat the argument here for the reader's convenience.

Since $\|u-(u)_{B(n)}\|_{2,B(n)} = \inf_{a\in\mathbb{R}}\|u-a\|_{2,B(n)} \leq \inf_{a\in\mathbb{R}}\|u-a\|_{\infty,B(n)}$, the assertion (2.26) follows from (2.24) by an application of the Cauchy-Schwarz inequality.

Let us now consider (2.27), i.e., the case $d \geq 2$. For $e = \{x,y\} \in \mathfrak{E}_d$ we let $|\nabla u(e)|$ denote the difference $|u(x) - u(y)|$. For any $e \in \mathfrak{E}_d$ we observe that by the Cauchy-Schwarz inequality

$$|\nabla u(e)| \;\leq\; \left(\frac{1}{\omega_l(e)}\right)^{1/2} \left(\sum_{e'\in\mathfrak{E}_d} \omega(e')\,|\nabla u(e')|^2\,\mathbb{1}_{e'\in\zeta_l^{\mathrm{opt}}(e)}\right)^{1/2},$$

where we recall the definitions of ω_l and ζ_l^{opt} in (2.2) and below. Thus, for any $\alpha \in [1,2)$, Hölder's inequality yields

$$\begin{aligned}\left(\frac{1}{|B(n)|}\sum_{\substack{x,y\in B(n)\\ \{x,y\}\in\mathfrak{E}_d}} |\nabla u(\{x,y\})|^{\alpha}\right)^{1/\alpha}&\\ \leq \big\|\nu_l^{\omega}\big\|_{\alpha/(2-\alpha),B(n)}^{1/2} &\left(\frac{1}{|B(n)|}\sum_{e'\in\mathfrak{E}_d}\omega(e')\,|\nabla u(e')|^2\,N_l(e')\right)^{1/2},\end{aligned} \tag{2.28}$$

where

$$N_l(e') \;:=\; \sum_{\substack{x,y\in B(n)\\ \{x,y\}\in\mathfrak{E}_d}} \mathbb{1}_{e'\in\zeta_l^{\mathrm{opt}}(\{x,y\})} \quad \text{for any } e'\in\mathfrak{E}_d\,.$$

Note that there exists $c < \infty$ such that $N_l(e') \leq cl^d$ for any $e' \in \mathfrak{E}_d$. In addition, there exists $C_{\mathrm{W}} < \infty$ such that $N_l(x,y) = 0$ if $x, y \notin B(C_{\mathrm{W}}n)$. Thus, when we choose $\alpha = 2d/(d+2)$, then (2.27) follows from (2.25). □

Our next task is to establish the corresponding versions of (2.24) and (2.25) on the weighted graph $(\mathbb{Z}^d, \mathfrak{E}_d, \boldsymbol{\omega})$. For this purpose, for $d \geq 2$ and $q \geq 1$ we define

$$\rho \;\equiv\; \rho(d,q) \;:=\; \frac{d}{d-2+d/q}. \tag{2.29}$$

Notice that $\rho(d,q)$ is monotonically increasing in q and converges to $d/(d-2)$ as q tends to infinity. Moreover, $\rho(d,d/2) = 1$. For the following propositions, recall the definitions of the Dirichlet energy in (1.20) and (2.5).

Proposition 2.15 (Sobolev inequality). *Let $x_0 \in \mathbb{Z}^d$ and $n \in \mathbb{N}$. Suppose that $d = 1$ and that $\nu^\omega(x) < \infty$ for all $x \in \mathbb{Z}^d$. Then there exists $C_\mathrm{S} < \infty$ such that*

$$\left\|u^2\right\|_{\infty,B(x_0,n)} \;\leq\; C_\mathrm{S}\, n^2 \left\|\nu^\omega\right\|_{1,B(x_0,n)} \frac{\mathcal{E}^\omega(u)}{|B(x_0,n)|} \tag{2.30}$$

for any $u\colon \mathbb{Z} \to \mathbb{R}$ with $\operatorname{supp} u \subset B(x_0,n)$.

Furthermore, for every $d \geq 2$, $q \in [1,\infty)$ and $l \in [1,\infty)$ with $\nu_l^\omega(x) < \infty$ for all $x \in \mathbb{Z}^d$, there exists $C_\mathrm{S} \equiv C_\mathrm{S}(d,q,l) < \infty$ such that

$$\left\|u^2\right\|_{\rho,B(x_0,n)} \;\leq\; C_\mathrm{S}\, n^2 \left\|\nu_l^\omega\right\|_{q,B(x_0,n)} \frac{\mathcal{E}^\omega(u)}{|B(x_0,n)|} \tag{2.31}$$

for any $u\colon \mathbb{Z}^d \to \mathbb{R}$ with $\operatorname{supp} u \subset B(x_0,n)$, where the measure ν_l is given by (2.1) with suitable path sets Γ_l.

We prove Proposition 2.15 after the following remark.

Remark 2.16. *For $d = 1$, Proposition 2.15 implies that*

$$\max_{x \in Q_\varepsilon} (u^\varepsilon(x))^2 \;\leq\; C_\mathrm{S} \left\|\nu^\omega\right\|_{1,B_{1/\varepsilon}} \mathcal{E}_\varepsilon^\omega(u^\varepsilon)\,. \tag{2.32}$$

For $d \geq 2$, Proposition 2.15 implies that

$$\|(u^\varepsilon)^2\|_{\ell^\rho_\varepsilon(Q_\varepsilon)} \;\leq\; C_\mathrm{S} \left\|\nu_l^\omega\right\|_{q,B_{1/\varepsilon}} \mathcal{E}_\varepsilon^\omega(u^\varepsilon)\,. \tag{2.33}$$

When we insert $q = d/2$ into (2.33), we especially obtain that

$$\|u^\varepsilon\|^2_{\ell^2_\varepsilon(Q_\varepsilon)} \;\leq\; C_\mathrm{S} \left\|\nu_l^\omega\right\|_{\frac{d}{2},B_{1/\varepsilon}} \mathcal{E}_\varepsilon^\omega(u^\varepsilon)\,. \tag{2.34}$$

Under Assumption 2.2(a) and by virtue of the ergodic theorem, (2.34) and (2.30) imply that for $d \geq 1$ there exists a $\mathbb{P}$-a.s. finite $C(\boldsymbol{\omega})$ such that for all $\varepsilon > 0$ and all $u^\varepsilon \in \mathcal{H}_\varepsilon$ we have

$$\|u^\varepsilon\|^2_{\mathcal{H}_\varepsilon} \leq C(\boldsymbol{\omega})\, \mathcal{E}_\varepsilon^\omega(u^\varepsilon) \quad \textit{(uniform Poincaré inequality)}\,. \tag{2.35}$$

Proof of Proposition 2.15. In the sequel we will give a proof only for (2.31). The assertion (2.30) follows by similar arguments. To lighten notation, set $B(n) \equiv B(x_0,n)$ and define $A(n) := B(2n)\backslash B(n)$. The constant $c \in (0,\infty)$ appearing in the computations below is independent of α but may change from line to line. Let $a \in \mathbb{R}$ and $\alpha \in [1,d)$. Since $u(x) = 0$ for $x \in A(n)$, we have

$$|a| = \frac{1}{|A(n)|} \sum_{x \in A(n)} |u(x) - a| \leq \frac{|B(2n)|}{|A(n)|} \left\|u - a\right\|_{1,B(2n)} \leq c \left\|u - a\right\|_{\frac{d\alpha}{d-\alpha},B(2n)}.$$

Hence, an application of Minkowski's inequality yields

$$\big\|u\big\|_{\frac{d\alpha}{d-\alpha},B(n)} \;\leq\; \big\|u-a\big\|_{\frac{d\alpha}{d-\alpha},B(n)} \,+\, |a| \;\leq\; c\,\big\|u-a\big\|_{\frac{d\alpha}{d-\alpha},B(2n)}.$$

Thus, for any $q \geq 1$ the assertion (2.31) follows as in the previous proof from (2.25) combined with (2.28) by choosing $\alpha = 2q/(q+1) \in [1,2)$. □

2.3.2 Maximal inequality

Proposition 2.17 (ℓ^∞-bound for solution of Poisson equation in $d \geq 2$). *Let $d \geq 2$ and suppose that $u^\varepsilon : \mathbb{Z}^d_\varepsilon \to \mathbb{R}$ is a solution of* (2.8)*. For some fixed $l \in [1,\infty)$ consider the measure ν^ω_l on $\mathbb{Z}^d$ as defined in* (2.1) *and assume that $\nu^\omega_l(x) < \infty$ for all $x \in \mathbb{Z}^d$. Then, for any $q > d/2$ there exist $\zeta \in (0,1]$, $\kappa \equiv \kappa(d,q)$, and $C_1 \equiv C_1(d,q)$ such that*

$$\max_{x \in Q_\varepsilon} |u(x)| \;\leq\; C_1 \Big(1 \vee \big\|\nu^\omega_l\big\|_{q,B_{1/\varepsilon}} \big\|f^\varepsilon\big\|_{\ell^\infty(Q_\varepsilon)}\Big)^\kappa \, \big\|u\big\|^\zeta_{\ell^2_\varepsilon}. \tag{2.36}$$

We prove this proposition after the following remark.

Remark 2.18. *Note that if $u^\varepsilon : \mathbb{Z}^d_\varepsilon \to \mathbb{R}$ is a solution of* (2.8)*, then due to* (2.32)*,* (2.34) *and the Cauchy-Schwarz inequality it follows for any dimension $d \geq 1$ that*

$$\|u^\varepsilon\|^2_{\ell^2_\varepsilon} \;\leq\; C_{\mathrm{S}} \big\|\nu^\omega_l\big\|_{\frac{d}{2},B_{1/\varepsilon}} \,\mathcal{E}^\omega_\varepsilon(u^\varepsilon) \;\leq\; C_{\mathrm{S}} \big\|\nu^\omega_l\big\|_{\frac{d}{2},B_{1/\varepsilon}} \,\|u^\varepsilon\|_{\ell^2_\varepsilon} \,\|f^\varepsilon\|_{\ell^2_\varepsilon(Q_\varepsilon)}\,. \tag{2.37}$$

Let Assumption 2.2(a) be fulfilled. Then $\sup_{\varepsilon>0} \|f^\varepsilon\|_{\ell^2_\varepsilon(Q_\varepsilon)} < \infty$ implies by the ergodic theorem that both $\sup_{\varepsilon>0} \|u^\varepsilon\|_{\ell^2_\varepsilon}$ and $\sup_{\varepsilon>0} \mathcal{E}^\omega_\varepsilon(u^\varepsilon)$ are bounded as well. Thus, (2.32) *implies that in dimension one $\sup_{\varepsilon>0} \|u^\varepsilon\|_\infty$ is bounded. Furthermore, if even Assumption 2.2(a') is fulfilled and $\sup_{\varepsilon>0} \|f^\varepsilon\|_{\ell^\infty(Q_\varepsilon)} < \infty$, then* (2.36) *implies that $\sup_{\varepsilon>0} \|u^\varepsilon\|_\infty$ is bounded for $d \geq 2$ as well.*

Proof of Proposition 2.17. We use the Moser iteration scheme. Let us fix $\varepsilon > 0$ and consider $u_\varepsilon : \mathbb{Z}^d_\varepsilon \to \mathbb{R}$ with supp $u_\varepsilon \in Q_\varepsilon$. We define $\tilde{u}^\alpha := |u|^\alpha \operatorname{sign} u$ for any $\alpha \geq 1$. By virtue of Eq. (A.2) in [ADS15] we obtain the following energy estimate

$$\mathcal{E}^\varepsilon_\omega(\tilde{u}^\alpha_\varepsilon) \leq \frac{\alpha^2}{2\alpha-1}\, \varepsilon^d \sum_{x \in \mathbb{Z}^d} \tilde{u}^{2\alpha-1}_\varepsilon(\varepsilon x)(-\mathcal{L}^\omega_\varepsilon u_\varepsilon)(\varepsilon x)\,. \tag{2.38}$$

Since u^ε is a solution to the Poisson equation (2.8), the energy estimate (2.38) implies that

$$\begin{aligned}\mathcal{E}_\omega^\varepsilon((\tilde{u}^\varepsilon)^\alpha) &\leq \frac{\alpha^2}{2\alpha-1}\|f^\varepsilon\|_{\ell^\infty(Q_\varepsilon)}\,\varepsilon^d \sum_{x\in Q_\varepsilon}(\tilde{u}^\varepsilon(x))^{2\alpha-1} \\ &= \frac{\alpha^2}{2\alpha-1}\|f^\varepsilon\|_{\ell^\infty(Q_\varepsilon)}\|u^\varepsilon\|_{\ell_\varepsilon^{2\alpha-1}}^{2\alpha-1}\end{aligned}$$

By the Sobolev inequality (2.33) and Jensen's inequality it follows that

$$\|u^\varepsilon\|_{\ell_\varepsilon^{2\alpha\rho}}^{2\alpha} \leq C_{\mathrm{S}}\frac{\alpha^2}{2\alpha-1}\|f^\varepsilon\|_{\ell^\infty(Q_\varepsilon)}\big\|\nu_l^\omega\big\|_{q,B_{1/\varepsilon}}\|u^\varepsilon\|_{\ell_\varepsilon^{2\alpha}}^{2\alpha-1}, \tag{2.39}$$

We define $\alpha_j = \rho^j$ for $j \in \mathbb{N}_0$. Further, we set $\zeta_j := 1-1/(2\alpha_j)$ for $\|u^\varepsilon\|_{\ell_\varepsilon^{2\alpha_j}} < 1$ and $\zeta_j := 1$ for $\|u^\varepsilon\|_{\ell_\varepsilon^{2\alpha_j}} \geq 1$. Recall that $\rho \equiv \rho(d,q) > 1$ for any $q > d/2$. Furthermore, we observe that for any $\beta > 0$ we have $\max_{x\in Q_\varepsilon}|u(x)| \leq (2/\varepsilon)^{d/\beta}\|u\|_{\ell_\varepsilon^\beta}$. Thus, by iterating the inequality (2.39) and using the fact that $\sum_{j=1}^\infty j/\alpha_j < \infty$, we obtain that there exists $C_1 \equiv C_1(d,q) < \infty$ such that

$$\|u^\varepsilon\|_\infty \leq (2/\varepsilon)^{d\varepsilon}\|u^\varepsilon\|_{\ell_\varepsilon^{1/\varepsilon}} \leq C_1\|u^\varepsilon\|_{\ell_\varepsilon^2}^\zeta \prod_{j=0}^m \big(1 \vee \|f^\varepsilon\|_{\ell^\infty(Q_\varepsilon)}\,\|\nu_l^\omega\|_{q,B_{1/\varepsilon}}\big)^{\frac{1}{2\rho^{j-1}}}$$

where $\zeta = \prod_{j=0}^m \zeta_j \leq 1$ and m such that $2\alpha_m > 1/\varepsilon$. Choosing $\kappa = \sum_{j=0}^\infty 1/(2\alpha_j) < \infty$, we complete the proof. □

2.4 Compact embedding

The very first step to prove homogenization of the operator $\mathcal{L}_\varepsilon^\omega$ is to show that a sequence $\mathcal{R}_\varepsilon^* u^\varepsilon$ $(u^\varepsilon \in \mathcal{H}_\varepsilon)$ has a strongly convergent subsequence if $\sup_\varepsilon \mathcal{E}_\varepsilon^\omega(u^\varepsilon) < \infty$. The Dirichlet energy $\mathcal{E}_\varepsilon^\omega$ is defined in (2.5).

For any $m \in \mathbb{N}$ consider a partition of Q into m^d congruent open subcubes $(Q_j^m)_{j=1,\dots,m^d}$ with side length $2/m$. For a fixed m we further define $Q_j^\varepsilon := \operatorname{supp} \mathcal{R}_\varepsilon^*\Big(\mathcal{R}_\varepsilon \mathbb{1}_{Q_j^m}\Big)$, where we suppress the superscript "m" although Q_j^ε depends on m. Then $Q_j^m \subset Q_j^\varepsilon$ and $|Q_j^\varepsilon \backslash Q_j^m| \to 0$ as $\varepsilon \to 0$.

Lemma 2.19. *Let $\boldsymbol{\omega} \in \Omega$ and assume that the uniform Poincaré inequality* (2.35) *holds with a finite $C(\boldsymbol{\omega})$ and that for any $m \in \mathbb{N}$ there exists $\varepsilon_m^* > 0$ such that for all $\varepsilon < \varepsilon_m^*$ we have*

$$\max_{1\leq j\leq m^d}\big\|\nu_l^\omega\big\|_{q,\varepsilon^{-1}Q_j^\varepsilon} \leq 2\mathbb{E}[(\nu_l^\omega(0))^q]^{1/q}. \tag{2.40}$$

Then the Poincaré inequality (2.27) *implies that for any sequence* $u^\varepsilon \in \mathcal{H}_\varepsilon$ ($\varepsilon^{-1} \in \mathbb{N}$) *with* $\sup_{\varepsilon>0} \mathcal{E}^\varepsilon_\omega(u^\varepsilon) < \infty$*, the sequence* $(\mathcal{R}^*_\varepsilon u^\varepsilon)_{\varepsilon>0}$ *has a strongly convergent subsequence in* $L^2(\mathbb{R}^d)$*.*

This result also follows from [NSS17, Lemma 3.3, Lemma 3.14].

Remark 2.20. *If Assumptions 2.1 and 2.2(a) are fulfilled, then for* $\mathbb{P}$*-a.e. realization* $\boldsymbol{\omega} \in \Omega$ *the hypotheses of Lemma 2.19 are fulfilled. That is, by virtue of Assumptions 2.1(a), (c) and 2.2(a) as well as Remark 2.16, there exists a* $\mathbb{P}$*-a.s. finite* $C(\boldsymbol{\omega})$ *such that* (2.35) *is fulfilled. Furthermore, the same assumptions together with the ergodic theorem imply that* $\mathbb{P}$*-a.s. there exists* $\varepsilon^*_m > 0$ *such that for all* $\varepsilon < \varepsilon^*_m$ (2.40) *holds.*

Proof of Lemma 2.19. First of all we observe that by virtue of (2.35) we have

$$\|\mathcal{R}^*_\varepsilon u^\varepsilon\|_2 \; = \; \|u^\varepsilon\|_{\ell^2_\varepsilon} \leq C(\boldsymbol{\omega})\mathcal{E}^\varepsilon_\omega(u^\varepsilon)\,,$$

which implies that $\sup_{\varepsilon>0} \|\mathcal{R}^*_\varepsilon u^\varepsilon\|_2$ is finite by assumption. By the Banach-Alaoglu theorem it follows that there exists a subsequence, which we still index by ε, and $u \in \mathcal{H}_0$ such that

$$\mathcal{R}^*_\varepsilon u^\varepsilon \rightharpoonup u \quad \text{weakly in } L^2(Q)\,.$$

We now show that u is also a strong limit. We estimate

$$\|\mathcal{R}^*_\varepsilon u^\varepsilon - u\|_2^2 \leq 3\sum_{j=1}^{m^d} \Bigg(\|\mathcal{R}^*_\varepsilon u^\varepsilon - (\mathcal{R}^*_\varepsilon u^\varepsilon)_{Q^\varepsilon_j}\|^2_{L^2(Q^\varepsilon_j)} + \\ + \|(\mathcal{R}^*_\varepsilon u^\varepsilon - u)_{Q^\varepsilon_j}\|^2_{L^2(Q^\varepsilon_j)} + \|(u)_{Q^\varepsilon_j} - u\|^2_{L^2(Q^\varepsilon_j)} \Bigg)\,, \tag{2.41}$$

where, in analogy to (2.23), we abbreviate

$$(v)_{Q^\varepsilon_j} \; := \; |Q^\varepsilon_j|^{-1} \int\limits_{Q^\varepsilon_j} v(x)\,\mathrm{d}x \quad \text{for } v\colon\, \mathbb{R}^d \to \mathbb{R}.$$

Since $\mathcal{R}^*_\varepsilon u^\varepsilon$ converges weakly in $L^2(Q)$ to u, the sum over the second term on the right-hand side of (2.41) vanishes as ε tends to zero. It remains to show that, as $\varepsilon \to 0$, the limit superior of the sum of the first and third term is zero as well.

We use arguments similar to the ones given in [ADS15, Proposition 2.9], see also [NSS17, Lemma 3.3, Lemma 3.14]. Let $\boldsymbol{e}_i$ ($i = 1, \dots, d$) be the unit base vectors of $\mathbb{R}^d$. By virtue of Proposition 2.14 there exists $C_{\mathrm{PI}} < \infty$ such that $\mathbb{P}$-a.s. for ε small enough the first term in the brackets of the right-hand side in (2.41) can be estimated by

$$
\begin{aligned}
\|\mathcal{R}_\varepsilon^* u^\varepsilon - (\mathcal{R}_\varepsilon^* u^\varepsilon)_{Q_j^\varepsilon}\|^2_{L^2(Q_j^\varepsilon)} &= \|u^\varepsilon - (u^\varepsilon)_{Q_j^\varepsilon}\|^2_{\ell^2_\varepsilon(Q_j^\varepsilon)} \\
&\le C_{\mathrm{PI}} \big\|\nu_l^\omega\big\|_{q,\varepsilon^{-1}Q_j^\varepsilon} \frac{4\varepsilon^d}{m^2} \sum_{i=1}^d \sum_{x,x+\boldsymbol{e}_i \in C_{\mathrm{W}}\varepsilon^{-1}Q_j^\varepsilon} \omega_{x,x+\boldsymbol{e}_i} \big(\partial^\varepsilon_{\boldsymbol{e}_i} u^\varepsilon(\varepsilon x)\big)^2
\end{aligned}
\tag{2.42}
$$

where for $d = 1$ we set $l = C_{\mathrm{W}} = q = 1$. For $d \ge 2$ we set $q = d/2$. Since any edge $e \in \mathfrak{E}_d$ is contained in at most $C_{\mathrm{o}} := 2dC_{\mathrm{W}}$ cubes $C_{\mathrm{W}}\varepsilon^{-1}Q_j^\varepsilon$, summing over $j = 1, \dots, m^d$ yields

$$
\sum_{j=1}^{m^d} \|\mathcal{R}_\varepsilon^* u^\varepsilon - (\mathcal{R}_\varepsilon^* u^\varepsilon)_{Q_j^\varepsilon}\|^2_{L^2(Q_j^\varepsilon)} \le 4m^{-2} C_{\mathrm{PI}} C_{\mathrm{o}}\, \mathcal{E}^\varepsilon_\omega(u^\varepsilon) \max_{1\le j\le m^d} \big\|\nu_l^\omega\big\|_{q,\varepsilon^{-1}Q_j^\varepsilon} .
\tag{2.43}
$$

Note that C_{o} is independent of m and $\mathcal{E}^\varepsilon_\omega(u^\varepsilon)$ is bounded in ε by assumption.

By virtue of (2.40), (2.41) and (2.43) it follows that there exists $C < \infty$ independent of m such that $\mathbb{P}$-almost surely

$$
\begin{aligned}
\limsup_{\varepsilon\to 0} \|\mathcal{R}_\varepsilon^* u^\varepsilon - u\|_2^2 &\le Cm^{-2} + 3\sum_{j=1}^{m^d} \limsup_{\varepsilon\to 0} \|(u)_{Q_j^\varepsilon} - u\|^2_{L^2(Q_j^\varepsilon)} \\
&= Cm^{-2} + 3\|u - \mathcal{R}^*_{2/m}\mathcal{R}_{2/m} u\|_2^2 .
\end{aligned}
$$

Since m might be arbitrarily large and $u \in L^2(Q)$ has bounded support, the claim follows. □

2.5 Analytic tools

In this section we always assume that the law $\mathbb{P}$ is stationary and ergodic with respect to spatial translations.

2.5.1 An ergodic theorem

In what follows, we will generalize a result by Boivin and Depauw.

Theorem 2.21 (Ergodic Theorem by [BD03, Theorem 3]). *For every $f \in L^1(\Omega, \mathbb{P})$, for $\mathbb{P}$-almost every $\boldsymbol{\omega} \in \Omega$ it holds*

$$
\lim_{\varepsilon\to 0} \varepsilon^d \sum_{x\in\varepsilon^{-1}Q_\varepsilon} v(\varepsilon x) f(\tau_x \boldsymbol{\omega}) = \mathbb{E}[f] \int_Q v(x)\,\mathrm{d}x \qquad \forall v \in C(\overline{Q}) ,
\tag{2.44}
$$

and the Null-set depends on f but not on v.

Remark 2.22. *Evidently, we can also choose* v *as the characteristic function of any relatively open or compact set* $A \subset Q$ *and we obtain the Tempel'man ergodic theorem.*

We will use both Theorem 2.21 and Remark 2.22 in order to prove the following theorem.

Theorem 2.23. *For every* $f \in L^1(\Omega, \mathbb{P})$*, for* $\mathbb{P}$*-almost every* $\boldsymbol{\omega} \in \Omega$ *the following holds: Let* $(u^\varepsilon)_{\varepsilon>0}$ *be a sequence of functions from* $\varepsilon\mathbb{Z}^d \to \mathbb{R}$ *with support in* Q_ε *such that* $\mathcal{R}_\varepsilon^* u^\varepsilon \to u$ *pointwise a.e. in* Q*. Furthermore, let* $\sup_{\varepsilon>0}\|u^\varepsilon\|_\infty < \infty$*. Then* $u \in L^\infty(Q)$ *and*

$$\lim_{\varepsilon\to 0} \varepsilon^d \sum_{x\in\varepsilon^{-1}Q_\varepsilon} u^\varepsilon(\varepsilon x) f(\tau_x\boldsymbol{\omega}) \;=\; \mathbb{E}[f] \int_Q u(x)\,\mathrm{d}x \tag{2.45}$$

and the Null-set depends on f *but not on the sequence* u^ε*.*

Proof. First we note that $u \in L^\infty(Q)$ since $\sup_{\varepsilon>0}\|u^\varepsilon\|_\infty < \infty$. Now we let $\eta > 0$ and let ρ_δ be a sequence of mollifiers approximating the identity. By Egorov's theorem, there exists a compact set K_η with $\mathscr{L}(Q\backslash K_\eta) < \eta$ such that both $\mathcal{R}_\varepsilon^* u^\varepsilon \to u$ and $u_\delta := u * \rho_\delta \to u$ uniformly on K_η. We now make the following decomposition:

$$\begin{aligned} &\left| \varepsilon^d \sum_{x\in\varepsilon^{-1}Q_\varepsilon} u^\varepsilon(\varepsilon x) f(\tau_x\boldsymbol{\omega}) - \mathbb{E}[f] \int_Q u(x)\,\mathrm{d}x \right| \\ &\qquad \leq \left| \varepsilon^d \sum_{x\in\varepsilon^{-1}Q_\varepsilon} (u^\varepsilon(\varepsilon x) - u_\delta(\varepsilon x)) f(\tau_x\boldsymbol{\omega}) \right| \\ &\qquad\quad + \left| \varepsilon^d \sum_{x\in\varepsilon^{-1}Q_\varepsilon} u_\delta(\varepsilon x) f(\tau_x\boldsymbol{\omega}) - \mathbb{E}[f] \int_Q u_\delta(x)\,\mathrm{d}x \right| \\ &\qquad\quad + \left| \mathbb{E}[f] \int_Q (u_\delta(x) - u(x))\,\mathrm{d}x \right| \end{aligned} \tag{2.46}$$

Since $u_\delta \in C(\overline{Q})$, the second summand on the above right-hand side converges to zero by virtue of Theorem 2.21. For the first summand on the right-hand side of (2.46) we estimate that

$$\begin{aligned} &\lim_{\varepsilon\to 0} \left| \varepsilon^d \sum_{x\in\varepsilon^{-1}Q_\varepsilon} (u^\varepsilon(\varepsilon x) - u_\delta(\varepsilon x)) f(\tau_x\boldsymbol{\omega}) \right| \\ &\qquad \leq \lim_{\varepsilon\to 0} \sup_{x\in K_\eta} |u^\varepsilon(x) - u_\delta(x)|\, \varepsilon^d \sum_{x\in\varepsilon^{-1}(K_\eta\cap Q_\varepsilon)} |f(\tau_x\boldsymbol{\omega})| \\ &\qquad\quad + \lim_{\varepsilon\to 0} (\|u_\delta\|_\infty + \|u^\varepsilon\|_\infty)\, \varepsilon^d \sum_{x\in\varepsilon^{-1}Q_\varepsilon\backslash K_\eta} |f(\tau_x\boldsymbol{\omega})|\,. \end{aligned} \tag{2.47}$$

Since the function $\mathcal{R}_\varepsilon^* u^\varepsilon$ converges uniformly in ε to u on K_η, we can estimate by virtue of Remark 2.22 that

$$\lim_{\varepsilon\to 0} \sup_{x\in K_\eta} |u^\varepsilon(x) - u_\delta(x)|\, \varepsilon^d \sum_{x\in\varepsilon^{-1}(K_\eta\cap Q_\varepsilon)} |f(\tau_x\boldsymbol{\omega})| \\ \leq \sup_{x\in K_\eta} |u_\delta(x) - u(x)|\, |Q|\, \mathbb{E}[f]\,.$$

We further estimate the second summand on the right-hand side of (2.47) by

$$\lim_{\varepsilon\to 0} (\|u_\delta\|_\infty + \|u^\varepsilon\|_\infty)\, \varepsilon^d \sum_{x\in\varepsilon^{-1}Q_\varepsilon\setminus K_\eta} |f(\tau_x\boldsymbol{\omega})| \;\leq\; 2\eta \sup_{\varepsilon>0} \|u^\varepsilon\|_\infty \mathbb{E}[f]\,,$$

where we have used Remark 2.22. Thus, as $\varepsilon\to 0$, we obtain that

$$\lim_{\varepsilon\to 0} \left| \varepsilon^d \sum_{x\in\varepsilon^{-1}Q_\varepsilon} u^\varepsilon(\varepsilon x) f(\tau_x\boldsymbol{\omega}) - \mathbb{E}[f] \int_Q u(x)\,\mathrm{d}x \right| \\ \leq \sup_{x\in K_\eta} |u_\delta(x) - u(x)|\, |Q|\, \mathbb{E}[f] \;+\; 2\eta \sup_{\varepsilon>0} \|u^\varepsilon\|_\infty \mathbb{E}[f] \\ + \left| \mathbb{E}[f] \int_Q (u_\delta(x) - u(x))\,\mathrm{d}x \right|$$

As $\delta\to 0$, the uniform convergence $u_\delta\to u$ on K_η yields

$$\lim_{\varepsilon\to 0} \left| \varepsilon^d \sum_{x\in\varepsilon^{-1}Q_\varepsilon} u^\varepsilon(\varepsilon x) f(\tau_x\boldsymbol{\omega}) \;-\; \mathbb{E}[f] \int_Q u(x)\,\mathrm{d}x \right| \;\leq\; 2\eta \sup_{\varepsilon>0} \|u^\varepsilon\|_\infty \mathbb{E}[f]\,.$$

Since the last inequality holds for every $\eta > 0$, the claim follows. $\square$

2.5.2 Function spaces

In what follows, we always assume that Assumption 2.1(b) holds. We first note that the probability space given in (1.10) is generated from the compact metric space $[0,\infty]^E$, and therefore the notion of continuity on Ω makes sense. We say that a function $\varphi : \Omega\times\mathbb{Z}^d \to \mathbb{R}$ is *shift covariant* if it fulfills

$$\varphi(\boldsymbol{\omega}, x+z) - \varphi(\boldsymbol{\omega}, x) \;=\; \varphi(\tau_x\boldsymbol{\omega}, z) \tag{2.48}$$

for all $x, z\in\mathbb{Z}^d$ (cf. [Bis11] Eq. (3.14)). Note that shift covariant functions φ fulfill $\varphi(\boldsymbol{\omega}, 0) = 0$. Then (2.48) directly implies that

$$\varphi(\boldsymbol{\omega}, x) = -\varphi(\tau_x\boldsymbol{\omega}, -x)\,. \tag{2.49}$$

We define on $\Omega \times \mathbb{Z}^d$ the space

$$L^2_{\mathrm{cov}} := \left\{ \varphi : \Omega \times \mathbb{Z}^d \to \mathbb{R} \; : \; \varphi \text{ satisfies (2.48) and } \|\varphi\|_{L^2_{\mathrm{cov}}} < \infty \right\},$$

$$\text{where } \|\varphi\|^2_{L^2_{\mathrm{cov}}} := \mathbb{E}\left[\sum_{z \in \mathbb{Z}^d} \omega_{0,z} \varphi(\boldsymbol{\omega}, z)^2 \right].$$

Accordingly, we define the scalar product between $\varphi_1, \varphi_2 \in L^2_{\mathrm{cov}}$ by

$$\langle \varphi_1, \varphi_2 \rangle_{L^2_{\mathrm{cov}}} := \mathbb{E}\left[\sum_{z \in \mathbb{Z}^d} \omega_{0,z} \varphi_1(\boldsymbol{\omega}, z) \varphi_2(\boldsymbol{\omega}, z) \right]. \tag{2.50}$$

Note that L^2_{cov} is a closed subspace of $\bigotimes_{z \in \mathbb{Z}^d} L^2(\Omega, \mu_z)$, where μ_z is the measure on Ω defined by $\mathrm{d}\mu_z(\boldsymbol{\omega}) = \omega_{0,z}\, \mathrm{d}\mathbb{P}(\boldsymbol{\omega})$. Since Ω is a compact metric space, $L^2(\Omega, \mu_z)$ is separable for all $z \in \mathbb{Z}^d$ and thus also the countable product space $\bigotimes_{z \in \mathbb{Z}^d} L^2(\Omega, \mu_z)$ and its subspace L^2_{cov} are separable.

Further, we note that for all $\phi : \Omega \to \mathbb{R}$ it holds that $\mathrm{D}\phi(\boldsymbol{\omega}, z) := \mathrm{D}_z\phi(\boldsymbol{\omega}) := \phi(\tau_z\boldsymbol{\omega}) - \phi(\boldsymbol{\omega})$ satisfies $\mathrm{D}\phi(\boldsymbol{\omega}, x+z) - \mathrm{D}\phi(\boldsymbol{\omega}, x) = \mathrm{D}\phi(\tau_x\boldsymbol{\omega}, z)$. Therefore $\mathrm{D}\phi$ is in L^2_{cov}. A *local function* on Ω is a bounded, continuous function that only depends on finitely many coordinates of $[0,\infty]^E$. Following the outline of Chapter 3 in [Bis11], we define the closed subspace

$$L^2_{\mathrm{pot}} \; := \; \overline{\{\mathrm{D}\phi \; : \; \phi \text{ local}\}}^{L^2_{\mathrm{cov}}}.$$

Let L^2_{sol} be the orthogonal complement of L^2_{pot} in L^2_{cov} and let us define

$$\mathrm{div}(\boldsymbol{\omega} b) \; := \; \sum_z \omega_{0,z} \big(b(\boldsymbol{\omega}, z) - b(\tau_z\boldsymbol{\omega}, -z) \big).$$

Note that since b satisfies (2.49), the last equation also reads

$$\mathrm{div}(\boldsymbol{\omega} b) \; = \; 2 \sum_z \omega_{0,z} b(\boldsymbol{\omega}, z). \tag{2.51}$$

Then we have the following lemma.

Lemma 2.24 ([Bis11, Lemma 3.6]).

$$\mathrm{div}(\boldsymbol{\omega} b) \; = \; 0 \qquad \textit{for all } b \in L^2_{\mathrm{sol}} \textit{ and } \mathbb{P}\textit{-a.a. } \omega. \tag{2.52}$$

Using the above notation, we define $\chi \in \left(L^2_{\mathrm{pot}}\right)^d$ through

$$\chi = \operatorname{argmin}\left\{ \mathbb{E}\left[\sum_{z \in \mathbb{Z}^d} \omega_{0,z} |z + \tilde{\chi}(\boldsymbol{\omega}, z)|^2 \right] \,:\, \tilde{\chi} \in \left(L^2_{\mathrm{pot}}\right)^d \right\}, \tag{2.53}$$

i.e., $-\chi_j$ is the orthogonal projection of $z_j \in L^2_{\mathrm{cov}}$ on the space L^2_{pot} with respect to the scalar product defined in (2.50). We will see below that we can write the homogenized matrix as

$$(A_{\mathrm{hom}})_{i,j} = \mathbb{E}\left[\sum_{z \in \mathbb{Z}^d} \omega_{0,z} (\boldsymbol{e}_i \cdot [z + \chi(\boldsymbol{\omega}, z)])(\boldsymbol{e}_j \cdot [z + \chi(\boldsymbol{\omega}, z)]) \right], \tag{2.54}$$

where the $\boldsymbol{e}_i$, $i = 1, \dots, d$, denote the unit base vectors of $\mathbb{R}^d$. In analogy to [Fag08, Lemma 4.5] we know the following result.

Lemma 2.25. *Suppose that* $\mathbb{E}\left[\nu^\omega_l(0)\right] < \infty$ *with* ν^ω_l *as defined in* (2.1)*. Then the matrix* A_{hom} *is positive definite. In particular, the vectorial space spanned by the following vectors*

$$\mathbb{E}\left[\textstyle\sum_{z \in \mathbb{Z}^d} \omega_{0,z}\, z b(\boldsymbol{\omega}, z)\right] \in \mathbb{R}^d, \qquad b \in L^2_{\mathrm{sol}} \tag{2.55}$$

coincides with $\mathbb{R}^d$*.*

Proof. First we notice that $\psi(\cdot, \boldsymbol{e}_i) \in L^1(\Omega, \mathbb{P})$ for any $\psi \in L^2_{\mathrm{cov}}$ and $i = 1, \dots, d$, provided that $\mathbb{E}[\nu^\omega_l(0)] < \infty$. Indeed, by the Cauchy-Schwarz inequality and the shift covariance (2.48), we observe that

$$\begin{aligned} \mathbb{E}\left[|\psi(\boldsymbol{\omega}, \boldsymbol{e}_i)|\right] \;&\leq\; \mathbb{E}\left[1/\omega_l(0, \boldsymbol{e}_i)\right]^{1/2} \left(\mathbb{E}\Big[\sum_{x,y \in \zeta^{\mathrm{opt}}_l} \omega_{xy} |\psi(\tau_x \boldsymbol{\omega}, y - x)|^2 \Big] \right)^{1/2} \\ &\leq\; \sqrt{l\,|\Gamma_l|}\, \mathbb{E}\left[\nu^\omega_l(0)\right]^{1/2} \|\psi\|_{L^2_{\mathrm{cov}}}, \end{aligned} \tag{2.56}$$

where we abbreviate $\zeta^{\mathrm{opt}}_l = \zeta^{\mathrm{opt}}_l(\{0, \boldsymbol{e}_i\})$, recall (2.2). Moreover, by adapting the argument given in [Bis11, Proof of Lemma 4.8], it follows that $\mathbb{E}[\psi(\boldsymbol{\omega}, \boldsymbol{e}_i)] = 0$ for any $\psi \in L^2_{\mathrm{pot}}$ and $i = 1, \dots, d$. In particular,

$$\mathbb{E}[\chi_j(\boldsymbol{\omega}, \boldsymbol{e}_i)] = 0$$

for any $i, j = 1, \dots, d$.

Now let $v \in \mathbb{R}^d \backslash \{0\}$. Since $\mathbb{E}[v \cdot \chi(\boldsymbol{\omega}, \boldsymbol{e}_i)] = 0$, it follows that

$$\begin{aligned}(v \cdot \boldsymbol{e}_i)^2 &= (v \cdot \boldsymbol{e}_i)\, \mathbb{E}\Big[\big(v \cdot [\boldsymbol{e}_i + \chi(\omega, \boldsymbol{e}_i)]\big)\Big] \\ &\overset{(2.56)}{\leq} |v \cdot \boldsymbol{e}_i| \sqrt{l\, |\Gamma_l|}\, \mathbb{E}\big[\nu_l^\omega(0)\big]^{1/2}\, \mathbb{E}\left[\sum_{z \in \mathbb{Z}^d} \omega_{0,z} \big(v \cdot [z + \chi(\boldsymbol{\omega}, z)]\big)^2\right].\end{aligned}$$

Thus, by summing both sides over $i = 1, \ldots, d$, we obtain

$$\begin{aligned}\sqrt{(v, A_{\mathrm{hom}} v)} &= \mathbb{E}\left[\sum_{z \in \mathbb{Z}^d} \omega_{0,z} \big(v \cdot [z + \chi(\boldsymbol{\omega}, z)]\big)^2\right] \\ &\geq \frac{|v|_2^2}{|v|_1} \big(l\, |\Gamma_l|\, \mathbb{E}\big[\nu_l^\omega(0)\big]\big)^{-1/2} > 0\,.\end{aligned}$$

Thus, the matrix A_{hom} is positive definite. By following literally the proof of [Fag08, Lemma 4.5] we obtain the claim. □

Bochner spaces. We will use the concept of Bochner spaces, which are a special case of the theory outlined in [Ma02]. Let X be a normed space with norm $\|\cdot\|_X$ with the corresponding topology and Borel-σ-algebra and let $U \subset \mathbb{R}^d$ be a Lebesgue-measurable set. Then, for $1 \leq p < \infty$, we define

$$\|f\|_{L^p(U;X)} := \left(\int_U \|f(x)\|_X^p dx\right)^{\frac{1}{p}},$$

$$L^p(U;X) := \left\{ f : U \to X \;:\; f \text{ is measurable and } \int_U \|f(x)\|_X^p \,\mathrm{d}x < \infty \right\}.$$

Given a measure space $(\Omega, \mathcal{F}, \mathbb{P})$, it turns out that $L^p(U; L^p(\Omega, \mathbb{P}))$ and $L^p(U \times \Omega; \mathscr{L} \otimes \mathbb{P})$ are isometrically isomorph via the trivial identification $f(x)(\boldsymbol{\omega}) = f(x, \boldsymbol{\omega})$. Here, $\mathscr{L}$ denotes the Lebesgue measure and $\mathscr{L} \otimes \mathbb{P}$ denotes the product measure. While not being necessary, this notation has proved useful in homogenization theory since the introduction of two-scale convergence in [All92]. In particular, it gives a clear and intuitive meaning to spaces such as

$$\begin{aligned}L^2(Q; L^2_{\mathrm{cov}}) := \Big\{ \varphi : Q \times \Omega \times \mathbb{Z}^d \to \mathbb{R} \;:\; &\int_Q \|\varphi(x, \cdot, \cdot)\|_{L^2_{\mathrm{cov}}} \,\mathrm{d}x < \infty\,, \\ &\varphi(x, \cdot, \cdot) \in L^2_{\mathrm{cov}} \text{ for a.e. } x \in Q \Big\}\end{aligned}$$

or, equivalently, $L^2(Q; L^2_{\mathrm{pot}})$.

If $\tilde{X} \subset X$ is a family of vectors in X, we denote

$$C(\overline{Q}) \otimes \tilde{X} := \operatorname{span}\Big\{ xf \ : \ f \in C(Q)\,, \ x \in \tilde{X} \Big\}\,.$$

If $\tilde{X}$ is a countable dense subset of X, i.e. X is separable, every element of $L^2(Q;X)$ can be approximated by finite sums of elements of $C(\overline{Q}) \otimes \tilde{X}$ [Ma02].

2.5.3 Discrete derivatives

With the following definitions of discrete derivatives, we can write the operator $\mathcal{L}_\varepsilon^\omega$ in divergence form.

Definition 2.26 (Discrete derivatives). *For $u\colon \mathbb{Z}_\varepsilon^d \to \mathbb{R}$ we define the ε-forward derivative in the direction $z \in \mathbb{Z}^d$ by*

$$\partial_z^\varepsilon u(x) = \varepsilon^{-1}(u(x+\varepsilon z) - u(x))\,, \tag{2.57}$$

and the analogous backward derivative,

$$\partial_z^{\varepsilon-} u(x) = \varepsilon^{-1}(u(x) - u(x-\varepsilon z))\,. \tag{2.58}$$

Further, we define $\nabla^\varepsilon u(x,z) := \partial_z^\varepsilon u(x)$ and write $\nabla^\varepsilon u(x)$ for the function that maps $z \in \mathbb{Z}^d$ to $\nabla^\varepsilon u(x,z)$. Accordingly, we define $\nabla^{\varepsilon-} u(x,z) := \partial_z^{\varepsilon-} u(x)$ and $\nabla^{\varepsilon-} u(x)$. Moreover, for a function $v : \mathbb{Z}_\varepsilon^d \times \mathbb{Z}^d \to \mathbb{R}$ we define

$$\operatorname{div}^\varepsilon v(x) = \sum_{z\in\mathbb{Z}^d} \partial_z^{\varepsilon-} v(x,z)\,. \tag{2.59}$$

We use this notation to clearly distinguish between ∇^ε, an operator on discrete functions, and ∇, an operator on the Sobolev space $H^1\big(\mathbb{R}^d\big)$. A direct calculation shows that when A_ω^ε maps $v(x,z) \mapsto \omega_{\frac{x}{\varepsilon},\frac{x}{\varepsilon}+z} v(x,z)$, then

$$-\mathcal{L}_\varepsilon^\omega u^\varepsilon = -\frac{1}{2}\operatorname{div}^\varepsilon(A_\omega^\varepsilon \nabla^\varepsilon u^\varepsilon)\,. \tag{2.60}$$

Moreover, for $v^\varepsilon\colon\ \mathbb{Z}_\varepsilon^d \to \mathbb{R}$ we observe that

$$\langle -\mathcal{L}_\varepsilon^\omega u^\varepsilon, v^\varepsilon\rangle_{\mathcal{H}_\varepsilon} = \frac{\varepsilon^d}{2} \sum_{x\in\mathbb{Z}^d}\sum_{z\in\mathbb{Z}^d} \omega_{x,x+z}\big(\partial_z^\varepsilon u^\varepsilon(\varepsilon x)\big)\big(\partial_z^\varepsilon v^\varepsilon(\varepsilon x)\big)\,. \tag{2.61}$$

When we compare the divergence form of the operator $\mathcal{L}_\varepsilon^\omega$ in (2.60) with the limit operator in (2.9), we better understand the result of Theorem 2.5. Furthermore (2.61) implies that $\mathcal{L}_\varepsilon^\omega$ is strictly positive definite on any bounded domain with zero Dirichlet conditions at the boundary.

2.5.4 Two-scale convergence

We adapt the concept of stochastic two-scale convergence by Zhikov and Piatnitsky [ZP06] to our setting.

We denote by z_i the function that maps $z \in \mathbb{Z}^d$ onto its i'th coordinate and observe that, since $\mathbb{E}\big[\sum_{z\in\mathbb{Z}^d} \omega_{0,z}|z|^2\big]$ is finite, $z_i \in L^2_{\mathrm{cov}}$ for $i = 1, \dots, d$.

Since L^2_{cov} is separable, there exist countable sets $\Phi_{\mathrm{sol}} \subset L^2_{\mathrm{sol}}$ and $\Phi_{\mathrm{pot}} \subset L^2_{\mathrm{pot}}$ such that $\Phi := \Phi_{\mathrm{sol}} \oplus \Phi_{\mathrm{pot}} \oplus \{z_1, \dots, z_d\} \oplus \{1\}$ is dense in L^2_{cov}. We can assume that every $\varphi \in \Phi_{\mathrm{pot}}$ is the gradient of a local function. Furthermore, there exists a countable subspace $\Psi \subset C^\infty_{\mathrm{c}}(\mathbb{R}^d)$ such that Ψ is dense both in $L^2(\mathbb{R}^d)$ and in $C^\infty_{\mathrm{c}}(\mathbb{R}^d)$. We then find that $\Psi \otimes \Phi$ is dense in $L^2(\mathbb{R}^d; L^2_{\mathrm{cov}})$.

Definition 2.27 (Typical realizations). *We denote by $\Omega_\Phi \subset \Omega$ the set of all $\boldsymbol{\omega} \in \Omega$ such that Theorem 2.21 holds*

a) for all $f(\boldsymbol{\omega}) := \sum_{z\in\mathbb{Z}^d} \omega_{0,z}\varphi(\boldsymbol{\omega}, z)$, where $\varphi \in \Phi$,
b) for all $f(\boldsymbol{\omega}) := \sum_{z\in\mathbb{Z}^d} \omega_{0,z}(\varphi_i\varphi_j)(\boldsymbol{\omega}, z)$, where $\varphi_i, \varphi_j \in \Phi$, and
c) and for all $f(\boldsymbol{\omega}) := \sum_{z\in\mathbb{Z}^d\setminus Z} \omega_{0,z}|z|^2$, where Z is a finite subset of $\mathbb{Z}^d$,
d) $\mathrm{div}(\boldsymbol{\omega} b) \circ \tau_x = 2\sum_z \omega_{x,x+z} b(\tau_x\boldsymbol{\omega}, z) = 0$ for all $b \in \Phi_{sol}$ and all $x \in \mathbb{Z}^d$.

We call Ω_Φ the set of typical realizations*.*

Remark 2.28. *Note that $\mathbb{P}(\Omega_\Phi) = 1$ (compare to [Fag08, Lemma 4.4]).*

Definition 2.29 (Two-scale convergence). *Let $\mathrm{w}_\varepsilon : \varepsilon\mathbb{Z}^d \times \mathbb{Z}^d \to \mathbb{R}$. We say that w_ε* converges weakly in two scales *to $\mathrm{w} \in L^2(\mathbb{R}^d; L^2_{\mathrm{cov}})$ if*

$$\lim_{\varepsilon\to 0} \varepsilon^d \sum_{x\in\mathbb{Z}^d} v(\varepsilon x) \sum_{z\in\mathbb{Z}^d} \omega_{x,x+z} \mathrm{w}_\varepsilon(\varepsilon x, z)\varphi(\tau_x\boldsymbol{\omega}, z) = \int_{\mathbb{R}^d} v(x)\,\mathbb{E}\left[\sum_{z\in\mathbb{Z}^d} \omega_{0,z}\mathrm{w}(x, \boldsymbol{\omega}, z)\varphi(\boldsymbol{\omega}, z)\right] \mathrm{d}x \tag{2.62}$$

for all $v \in C^\infty_{\mathrm{c}}(\mathbb{R}^d)$ and all $\varphi \in \Phi$. In this case we write $\mathrm{w}_\varepsilon \xrightharpoonup{2s} \mathrm{w}$.

Proposition 2.30. *For all typical realizations $\boldsymbol{\omega} \in \Omega_\Phi$ it holds: If $\mathrm{w}_\varepsilon : \varepsilon\mathbb{Z}^d \times \mathbb{Z}^d \to \mathbb{R}$ and $C < \infty$ are such that*

$$\varepsilon^d \sum_{x\in\mathbb{Z}^d} \sum_{z\in\mathbb{Z}^d} \omega_{x,x+z} \mathrm{w}^2_\varepsilon(\varepsilon x, z) \;\le\; C \qquad \forall \varepsilon > 0\,, \tag{2.63}$$

then there exists a subsequence $\mathrm{w}_{\varepsilon_k}$ and $\mathrm{w} \in L^2(\mathbb{R}^d; L^2_{\mathrm{cov}})$ such that

$$\mathrm{w}_{\varepsilon_k} \xrightharpoonup{2s} \mathrm{w}\,. \tag{2.64}$$

Proof. The proof goes along the lines of classical proofs of two-scale convergence like for example in [ZP06], Section 5.

We observe that for every $v \in \Psi$ and $\varphi \in \Phi$ we find

$$\begin{aligned}\limsup_{\varepsilon\to 0} \varepsilon^d & \left| \sum_{x\in\mathbb{Z}^d} v(\varepsilon x) \sum_{z\in\mathbb{Z}^d} \omega_{x,x+z} \mathrm{w}_\varepsilon(\varepsilon x, z)\varphi(\tau_x\boldsymbol{\omega}, z)\right| \\ & \overset{(2.63)}{\leq} \limsup_{\varepsilon\to 0} \sqrt{C}\left(\varepsilon^d \sum_{x\in\mathbb{Z}^d}\sum_{z\in\mathbb{Z}^d} \omega_{x,x+z} v^2(\varepsilon x)\varphi^2(\tau_x\boldsymbol{\omega}, z)\right)^{1/2} \\ & \overset{(2.44)}{\leq} \sqrt{C}\|v\|_{L^2}\|\varphi\|_{L^2_{\mathrm{cov}}}\,, \end{aligned} \tag{2.65}$$

where, in the last step, we have also used that v has bounded support. It follows that since Ψ and Φ are countable, we can choose a subsequence $\varepsilon_k \to 0$ as $k \to \infty$ such that the limit $I(v\varphi)$ of

$$I_{\varepsilon_k}(v\varphi) := \varepsilon_k^d \sum_{x\in\mathbb{Z}^d} v(\varepsilon_k x) \sum_{z\in\mathbb{Z}^d} \omega_{x,x+z}\mathrm{w}_{\varepsilon_k}(\varepsilon_k x, z)\varphi(\tau_x\boldsymbol{\omega}, z)$$

exists for every $v \in \Psi$ and $\varphi \in \Phi$. We notice that the functional $I(\cdot)$ is linear in $v\varphi \in \Psi\otimes\Phi$. Furthermore, due to (2.65), $I(\cdot)$ is continuous on $\mathrm{span}\{\Psi\otimes\Phi\}$. It follows by Riesz representation theorem that we can find $\mathrm{w} \in L^2(\mathbb{R}^d; L^2_{\mathrm{cov}})$ such that

$$I(v\varphi) = \int_{\mathbb{R}^d} v(x)\,\mathbb{E}\left[\sum_{z\in\mathbb{Z}^d} \omega_{0,z}\mathrm{w}(x,\boldsymbol{\omega},z)\varphi(\boldsymbol{\omega},z)\right]\,\mathrm{d}x\,.$$

Since $\Psi\otimes\Phi$ is dense in $L^2(\mathbb{R}^d; L^2_{\mathrm{cov}})$, we obtain that w is uniquely defined. Since, in addition, Ψ is dense in C_{c}^∞, we find for every $v \in C_c^\infty(\mathbb{R}^d)$ and $\varphi\in\Phi$ that $I_{\varepsilon_k}(v\varphi) \to I(v\varphi)$ as $\varepsilon \to 0$ and hence (2.64) holds. □

Lemma 2.31. *For all typical realizations $\boldsymbol{\omega} \in \Omega_\Phi$ and all Lipschitz functions $v : \mathbb{R}^d \to \mathbb{R}$ there exists $C(\boldsymbol{\omega}) \in (0,\infty]$, which depends only on* $\mathrm{supp}\, v$ *and $\boldsymbol{\omega}$, such that*

$$\sup_{\varepsilon>0} \varepsilon^d \sum_{x\in\mathbb{Z}^d}\sum_{z\in\mathbb{Z}^d} \omega_{x,x+z}\left(\partial_z^\varepsilon v(\varepsilon x)\right)^2 < C(\boldsymbol{\omega})\|\nabla v\|_\infty^2\,. \tag{2.66}$$

If $\mathrm{supp}\, v$ *is bounded, then $C(\boldsymbol{\omega})$ is $\mathbb{P}$-a.s. finite.*

Proof. We observe that we can interchange the order of the sums and estimate

$$
\begin{aligned}
&\varepsilon^d \sum_{z\in\mathbb{Z}^d} \sum_{x\in\mathbb{Z}^d} \omega_{x,x+z} (\partial_z^\varepsilon v(\varepsilon x))^2 \\
&\qquad\le \varepsilon^d \, \|\nabla v\|_\infty^2 \sum_{z\in\mathbb{Z}^d} \sum_{x\in\varepsilon^{-1}(\operatorname{supp} v \cup (\operatorname{supp} v - \varepsilon z))} \omega_{x,x+z} |z|^2 \\
&\qquad\le \varepsilon^d \, \|\nabla v\|_\infty^2 \sum_{x\in\varepsilon^{-1}\operatorname{supp} v} \sum_{z\in\mathbb{Z}^d} \omega_{x,x+z} |z|^2 \\
&\qquad\qquad + \varepsilon^d \, \|\nabla v\|_\infty^2 \sum_{z\in\mathbb{Z}^d} \sum_{x\in(\varepsilon^{-1}\operatorname{supp} v)-z} \omega_{x,x+z} |z|^2
\end{aligned}
$$

The first term on the above right-hand side is finite by virtue of the ergodic theorem. This also holds for the second term after an index shift in x and a rearrangement of the two sums. $\square$

Lemma 2.32. *Let $Q_\varepsilon = Q \cap \varepsilon\mathbb{Z}^d$. For all typical realizations $\boldsymbol{\omega} \in \Omega_\Phi$ it holds:*

$$
\limsup_{\varepsilon\to 0} \varepsilon^d \sum_{x\in\mathbb{Z}^d} \sum_{z\in\mathbb{Z}^d} \omega_{x,x+z} (\partial_z^\varepsilon v(\varepsilon x) - \nabla v(\varepsilon x)\cdot z)^2 = 0 \text{ for all } v \in C_c^\infty(\mathbb{R}^d)\,. \tag{2.67}
$$

Proof. Let $\delta > 0$. Since $\mathbb{E}\Big[\sum_{z\in\mathbb{Z}^d} \omega_{0,z}|z|^2\Big] < \infty$, we can choose a finite point-symmetric subset $Z_\delta \subset \mathbb{Z}^d$ such that

$$
\mathbb{E}\left[\sum_{z\in\mathbb{Z}^d\setminus Z_\delta} \omega_{0,z}|z|^2\right] < \delta\,.
$$

Then we split the sum in (2.67) into a sum over $z \in Z_\delta$ and a sum over $z \notin Z_\delta$.

For $z \in Z_\delta$ we observe that, since $v \in C_c^\infty(\mathbb{R}^d)$, we have

$$
\frac{v(x+\varepsilon z) - v(x)}{\varepsilon} - \nabla v(x)\cdot z \to 0 \tag{2.68}
$$

uniformly in $x \in \mathbb{R}^d$. Further, we observe that there exists $\varepsilon^* > 0$ such that for $z \in Z_\delta$ and for all $\varepsilon < \varepsilon^*$, the statement $\varepsilon x \notin 2\operatorname{supp} v$ implies that $\varepsilon x + \varepsilon z \notin \operatorname{supp} v$. It follows that for ε small enough, we have

$$\varepsilon^d \sum_{x\in\mathbb{Z}^d} \sum_{z\in Z_\delta} \omega_{x,x+z} \left(\frac{v(\varepsilon x + \varepsilon z) - v(\varepsilon x)}{\varepsilon} - \nabla v(\varepsilon x) \cdot z \right)^2$$
$$\leq \varepsilon^d \sum_{x\in 2\varepsilon^{-1} \operatorname{supp} v} \sum_{z\in Z_\delta} \omega_{x,x+z} \left(\frac{v(\varepsilon x + \varepsilon z) - v(\varepsilon x)}{\varepsilon} - \nabla v(\varepsilon x) \cdot z \right)^2 .$$

This together with (2.68) and the ergodic theorem implies that

$$\varepsilon^d \sum_{x\in\mathbb{Z}^d} \sum_{z\in Z_\delta} \omega_{x,x+z} \left(\frac{v(\varepsilon x + \varepsilon z) - v(\varepsilon x)}{\varepsilon} - \nabla v(\varepsilon x) \cdot z \right)^2 \to 0 \qquad \text{as } \varepsilon \to 0 .$$

Let us now consider the case $z \notin Z_\delta$. As in the proof of Lemma 2.31, we interchange the sums and observe that

$$\varepsilon^d \sum_{x\in\mathbb{Z}^d} \sum_{z\notin Z_\delta} \omega_{x,x+z} \left(\frac{v(\varepsilon x + \varepsilon z) - v(\varepsilon x)}{\varepsilon} - \nabla v(\varepsilon x) \cdot z \right)^2$$
$$\leq 4\|\nabla v\|_\infty^2 \varepsilon^d \sum_{z\notin Z_\delta} \sum_{x\in\varepsilon^{-1}(\operatorname{supp} v \cup \operatorname{supp} v - \varepsilon z)} \omega_{x,x+z} |z|^2$$
$$\leq 4\|\nabla v\|_\infty^2 \varepsilon^d \sum_{x\in\varepsilon^{-1} \operatorname{supp} v} \sum_{z\notin Z_\delta} \left(\omega_{x,x+z}|z|^2 + \omega_{x,x-z}|z|^2 \right)$$

By the ergodic theorem and the choice of Z_δ it follows that the limit superior of the above right-hand side is bounded from above by a constant times $\delta|\operatorname{supp} v|$, which we can choose arbitrarily small. □

Corollary 2.33 (of Lemma 2.32). *For all typical realizations $\boldsymbol{\omega} \in \Omega_\Phi$ it holds: If $\mathrm{w}_\varepsilon \overset{2\mathrm{s}}{\rightharpoonup} \mathrm{w}$, then for all $v \in C_c^\infty(\mathbb{R}^d)$ the limit*

$$\lim_{\varepsilon\to 0} \varepsilon^d \sum_{x\in\mathbb{Z}^d} \sum_{z\in\mathbb{Z}^d} \omega_{x,x+z} \mathrm{w}_\varepsilon(\varepsilon x, z) \partial_z^\varepsilon v(\varepsilon x)$$
$$= \int_{\mathbb{R}^d} \mathbb{E}\left[\sum_{z\in\mathbb{Z}^d} \omega_{0,z} \mathrm{w}(x, \boldsymbol{\omega}, z) (\nabla v(x) \cdot z) \right] \mathrm{d}x . \tag{2.69}$$

Proof. First we observe that $z_i \in \Phi$ for $i = 1, \dots, d$ and $\partial_{\boldsymbol{e}_i} v \in C_c^\infty(\mathbb{R}^d)$ where the $\boldsymbol{e}_i$, $i = 1, \dots, d$, denote the unit base vectors of $\mathbb{R}^d$. Therefore the assumption that $\mathrm{w}_\varepsilon \overset{2\mathrm{s}}{\rightharpoonup} \mathrm{w}$ implies that

$$\lim_{\varepsilon\to 0} \varepsilon^d \sum_{x\in\mathbb{Z}^d} \partial_{\boldsymbol{e}_i} v(\varepsilon x) \sum_{z\in\mathbb{Z}^d} \omega_{x,x+z} z_i \mathrm{w}_\varepsilon(\varepsilon x, z)$$
$$= \int_{\mathbb{R}^d} \partial_{\boldsymbol{e}_i} v(x) \mathbb{E}\left[\sum_{z\in\mathbb{Z}^d} \omega_{0,z} z_i \mathrm{w}(x, \boldsymbol{\omega}, z) \right] \mathrm{d}x ,$$

where the $\boldsymbol{e}_i$, $i = 1, \dots, d$, denote the unit base vectors of $\mathbb{R}^d$. It follows that

$$\lim_{\varepsilon\to 0} \varepsilon^d \sum_{x\in\mathbb{Z}^d} \nabla v(\varepsilon x) \cdot \sum_{z\in\mathbb{Z}^d} \omega_{x,x+z} z \mathrm{w}_\varepsilon(\varepsilon x, z) = \int_{\mathbb{R}^d} \nabla v(x) \cdot \mathbb{E}\left[\sum_{z\in\mathbb{Z}^d} \omega_{0,z} z \mathrm{w}(x, \boldsymbol{\omega}, z)\right] \mathrm{d}x$$

for all $v \in C_{\mathrm{c}}^\infty(\mathbb{R}^d)$. In order to prove (2.69), it thus remains to show that

$$\lim_{\varepsilon\to 0} \left| \varepsilon^d \sum_{x\in\mathbb{Z}^d} \sum_{z\in\mathbb{Z}^d} \omega_{x,x+z} (\partial_z^\varepsilon v(\varepsilon x) - \nabla v(\varepsilon x) \cdot z) \mathrm{w}_\varepsilon(\varepsilon x, z) \right| \to 0\,.$$

This follows from Cauchy-Schwarz, i.e.,

$$\begin{aligned} &\left| \varepsilon^d \sum_{x\in\mathbb{Z}^d} \sum_{z\in\mathbb{Z}^d} \omega_{x,x+z} (\partial_z^\varepsilon v(\varepsilon x) - \nabla v(\varepsilon x) \cdot z) \mathrm{w}_\varepsilon(\varepsilon x, z) \right| \\ &\leq \left(\varepsilon^d \sum_{x,z\in\mathbb{Z}^d} \omega_{x,x+z} \mathrm{w}_\varepsilon^2(\varepsilon x, z) \right)^{1/2} \left(\varepsilon^d \sum_{x,z\in\mathbb{Z}^d} \omega_{x,x+z} (\partial_z^\varepsilon v(\varepsilon x) - \nabla v(\varepsilon x) \cdot z)^2 \right)^{1/2}. \end{aligned}$$

The first factor on the right-hand side is bounded by assumption and the second factor converges to zero by virtue of Lemma 2.32. □

2.5.5 Convergence of gradients

Let us start with the following auxiliary lemma.

Lemma 2.34. *For all $\boldsymbol{\omega} \in \Omega_\Phi$ and all $b \in \Phi_{\mathrm{sol}}$ the following is true:*

$$\sum_{x\in\mathbb{Z}^d} \sum_{z\in\mathbb{Z}^d} \partial_z^\varepsilon v(\varepsilon x)\, \omega_{x,x+z} b(\tau_x \boldsymbol{\omega}, z) = 0 \tag{2.70}$$

for all $v \in \ell^\infty(\mathbb{Z}_\varepsilon^d)$ with bounded support.

Proof. We write the left-hand side of (2.70) as

$$\varepsilon^{-1} \sum_{x\in\mathbb{Z}^d} \sum_{z\in\mathbb{Z}^d} v(\varepsilon x + \varepsilon z) \omega_{x,x+z} b(\tau_x \omega, z) - \varepsilon^{-1} \sum_{x\in\mathbb{Z}^d} v(\varepsilon x) \sum_{z\in\mathbb{Z}^d} \omega_{x,x+z} b(\tau_x \omega, z)$$

The second term is immediately zero since $\boldsymbol{\omega} \in \Omega_\Phi$ and $\mathrm{div}(\boldsymbol{\omega} b) \circ \tau_x = 2 \sum_{z\in\mathbb{Z}^d} \omega_{x,x+z} b(\tau_x \omega, z)$ by (2.51). The first term is absolutely convergent and therefore we can interchange the sums. After an additional index shift

$x \rightsquigarrow x - z$, we obtain that the above first term is equal to

$$\varepsilon^{-1} \sum_{x\in\mathbb{Z}^d} v(\varepsilon x) \sum_{z\in\mathbb{Z}^d} \omega_{x-z,x} b(\tau_{x-z}\omega, z) \\ = -\varepsilon^{-1} \sum_{x\in\mathbb{Z}^d} v(\varepsilon x) \sum_{z\in\mathbb{Z}^d} \omega_{x,x-z} b(\tau_x \omega, -z)\,,$$

where we have used (2.49) as well as the symmetry of the conductances. The claim follows from (2.51) and $b \in \Phi_{\mathrm{sol}}$ since $\boldsymbol{\omega} \in \Omega_\Phi$. □

We can now prove the convergence of gradients. Our result is the natural transfer to the corresponding original result by Nguetseng [Ngu89] to the present setting.

Lemma 2.35 (Two-scale convergence for gradients). *For all $\boldsymbol{\omega} \in \Omega_\Phi$ such that the Poincaré-inequality (2.35) holds uniformly in ε, also the following holds true. If $u^\varepsilon : \mathbb{Z}^d_\varepsilon \to \mathbb{R}$ is a family of functions with $\operatorname{supp}(u^\varepsilon) \subseteq Q \cap \mathbb{Z}^d_\varepsilon$ for all ε and*

$$\sup_{\varepsilon>0} \left(\varepsilon^d \sum_{x\in\mathbb{Z}^d} \sum_{z\in\mathbb{Z}^d} \omega_{x,x+z} \big(\partial^\varepsilon_z u^\varepsilon(\varepsilon x)\big)^2 + \|u^\varepsilon\|_\infty \right) \;<\; \infty\,, \tag{2.71}$$

then there exists a subsequence $u^{\varepsilon'}$, $u \in H^1_0(Q)$ and $\nu \in L^2(\mathbb{R}^d; L^2_{\mathrm{pot}})$ such that

$$\mathcal{R}^*_{\varepsilon'} u^{\varepsilon'} \rightharpoonup u \text{ in } L^2(\mathbb{R}^d)\,, \qquad \partial^{\varepsilon'}_z u^{\varepsilon'}(x) \stackrel{2s}{\rightharpoonup} \nabla u(x)\cdot z + \nu(x,\boldsymbol{\omega},z) \quad \text{as } \varepsilon' \to 0\,. \tag{2.72}$$

Further, if the compact embedding of Lemma 2.19 holds, then $\mathcal{R}^*_{\varepsilon'} u^{\varepsilon'} \to u$ strongly in $L^2(\mathbb{R}^d)$.

Proof. Condition (2.71) together with Lemma 2.19 implies that there exists a subsequence, which we still index by ε, and $u \in L^2(Q)$ such that $\mathcal{R}^*_\varepsilon u^\varepsilon \to u$ in $L^2(\mathbb{R}^d)$. It remains to show that $u \in H^1_0(Q)$ and to proof the second statement in (2.72).

By virtue of Proposition 2.30, Condition (2.71) further implies that there exists a subsequence, which we still index by $\varepsilon \to 0$, and $\mathrm{w} \in L^2(\mathbb{R}^d; L^2_{\mathrm{cov}})$ such that $\nabla^\varepsilon u^\varepsilon \stackrel{2s}{\rightharpoonup} \mathrm{w}$ in the two-scale sense. We choose $b \in \Phi_{\mathrm{sol}}$ and $v \in C^\infty_{\mathrm{c}}(\mathbb{R}^d)$ and apply (2.70) to the discrete product rule

$$\partial^\varepsilon_z (v u^\varepsilon)(\varepsilon x) = v(\varepsilon x)\partial^\varepsilon_z u^\varepsilon(\varepsilon x) + u^\varepsilon(\varepsilon x + \varepsilon z)\partial^\varepsilon_z v(\varepsilon x)$$

to obtain that

$$0 = \varepsilon^d \sum_{x\in\mathbb{Z}^d} \sum_{z\in\mathbb{Z}^d} \omega_{x,x+z} \Big(v(\varepsilon x)\partial^\varepsilon_z u^\varepsilon(\varepsilon x) + u^\varepsilon(\varepsilon x + \varepsilon z)\partial^\varepsilon_z v(\varepsilon x) \Big) b(\tau_x \omega, z)\,. \tag{2.73}$$

For the first term on the right-hand side of (2.73), we obtain from the two-scale convergence of $\nabla^\varepsilon u^\varepsilon$ that

$$\varepsilon^d \sum_{x\in\mathbb{Z}^d} v(\varepsilon x) \sum_{z\in\mathbb{Z}^d} \omega_{x,x+z}\, \partial_z^\varepsilon u^\varepsilon(\varepsilon x) b(\tau_x\omega, z) \\ \to \int_{\mathbb{R}^d} v(x)\mathbb{E}\left[\sum_{z\in\mathbb{Z}^d} \omega_{0,z} \mathrm{w}(x,\boldsymbol{\omega},z) b(\boldsymbol{\omega},z)\right] \mathrm{d}x\,. \tag{2.74}$$

For the second term on the right-hand side of (2.73), we first notice that the sum is absolutely convergent since

$$\begin{aligned} &\varepsilon^d \sum_{x\in\mathbb{Z}^d}\sum_{z\in\mathbb{Z}^d} \left| \omega_{x,x+z}\, b(\tau_x\boldsymbol{\omega},z) u^\varepsilon(\varepsilon x+\varepsilon z)\left(\frac{v(\varepsilon x+\varepsilon z)-v(\varepsilon x)}{\varepsilon}\right)\right| \\ &= \varepsilon^d \sum_{z\in\mathbb{Z}^d}\sum_{x\in\varepsilon^{-1}Q_\varepsilon - z} \left| \omega_{x,x+z}\, b(\tau_x\boldsymbol{\omega},z) u^\varepsilon(\varepsilon x+\varepsilon z)\left(\frac{v(\varepsilon x+\varepsilon z)-v(\varepsilon x)}{\varepsilon}\right)\right| \\ &= \varepsilon^d \sum_{z\in\mathbb{Z}^d}\sum_{x\in\varepsilon^{-1}Q_\varepsilon} \left| \omega_{x-z,x}\, b(\tau_{x-z}\boldsymbol{\omega},z) u^\varepsilon(\varepsilon x)\left(\frac{v(\varepsilon x)-v(\varepsilon x-\varepsilon z)}{\varepsilon}\right)\right| \\ &= \varepsilon^d \sum_{x\in\varepsilon^{-1}Q_\varepsilon}\sum_{z\in\mathbb{Z}^d} \left| \omega_{x,x+z}\, b(\tau_x\boldsymbol{\omega},z) u^\varepsilon(\varepsilon x)\left(\frac{v(\varepsilon x+\varepsilon z)-v(\varepsilon x)}{\varepsilon}\right)\right|, \end{aligned} \tag{2.75}$$

where for the last equality we have used the symmetry of the conductances, the shift covariance (2.49) and the substitution $z \rightsquigarrow -z$. We now use the fact that u^ε and ∇v are bounded, apply the Cauchy-Schwarz inequality and the ergodic theorem to obtain that the above sum is indeed finite. It follows that for the second term on the right-hand side of (2.73), we can exchange the order of the sums. By the same arguments as those that led to (2.75), we obtain that

$$\varepsilon^d \sum_{x\in\mathbb{Z}^d}\sum_{z\in\mathbb{Z}^d} \omega_{x,x+z}\, u^\varepsilon(\varepsilon x+\varepsilon z)\partial_z^\varepsilon v(\varepsilon x) b(\tau_x\omega,z) \\ = \varepsilon^d \sum_{x\in\mathbb{Z}^d}\sum_{z\in\mathbb{Z}^d} \omega_{x,x+z}\, u^\varepsilon(\varepsilon x)\partial_z^\varepsilon v(\varepsilon x) b(\tau_x\omega,z)\,.$$

Further we notice that since u^ε has support only in $\varepsilon^{-1}Q_\varepsilon$, we can estimate

$$
\begin{aligned}
&\varepsilon^d \sum_{x\in\mathbb{Z}^d} u^\varepsilon(\varepsilon x) \sum_{z\in\mathbb{Z}^d} \omega_{x,x+z}\,(\partial_z^\varepsilon v(\varepsilon x) - \nabla v(\varepsilon x)\cdot z) b(\tau_x\omega, z) \\
&\quad \le \|u^\varepsilon\|_\infty^2 \left(\varepsilon^d \sum_{x\in\mathbb{Z}^d} \sum_{z\in\mathbb{Z}^d} \omega_{x,x+z}\,(\partial_z^\varepsilon v(\varepsilon x) - \nabla v(\varepsilon x)\cdot z)^2 \right)^{1/2} \\
&\qquad\qquad \times \left(\varepsilon^d \sum_{x\in\varepsilon^{-1}Q_\varepsilon} \sum_{z\in\mathbb{Z}^d} \omega_{x,x+z}\, b^2(\tau_x\boldsymbol{\omega}, z) \right)^{1/2} .
\end{aligned}
$$

The limit superior of the second factor vanishes due to Lemma 2.32 and the third factor is finite due to the ergodic theorem. Thus, for $\boldsymbol{\omega}\in\Omega_\Phi$ we have

$$
\limsup_{\varepsilon\to 0} \varepsilon^d \sum_{x\in\mathbb{Z}^d} u^\varepsilon(\varepsilon x) \sum_{z\in\mathbb{Z}^d} \omega_{x,x+z}\,(\partial_z^\varepsilon v(\varepsilon x) - \nabla v(\varepsilon x)\cdot z) b(\tau_x\boldsymbol{\omega}, z) = 0 . \tag{2.76}
$$

To summarize, for the second term on the right-hand side of (2.73), it follows that

$$
\begin{aligned}
&\lim_{\varepsilon\to 0} \varepsilon^d \sum_{x\in\mathbb{Z}^d} \sum_{z\in\mathbb{Z}^d} \omega_{x,x+z}\, u^\varepsilon(\varepsilon x + \varepsilon z)\, \partial_z^\varepsilon v(\varepsilon x)\, b(\tau_x\boldsymbol{\omega}, z) \\
&\qquad = \lim_{\varepsilon\to 0} \varepsilon^d \sum_{x\in\mathbb{Z}^d} u^\varepsilon(\varepsilon x) \nabla v(\varepsilon x) \cdot \left(\sum_{z\in\mathbb{Z}^d} z\, \omega_{x,x+z}\, b(\tau_x\boldsymbol{\omega}, z) \right) .
\end{aligned}
$$

By the assumptions on $\boldsymbol{\omega}$ and b, the last bracket on the above right-hand side is in $L^1(\Omega,\mathbb{P})$. Since we already know that the subsequence $\mathcal{R}_\varepsilon^* u^\varepsilon \to u$ in $L^2(\mathbb{R}^d)$, there exists a further subsequence, which we still index by $\varepsilon\to 0$, where u^ε converges pointwise a.e. in Q [Bre11, Theorem 4.9]. Moreover, u^ε has support in Q_ε and $\sup_{\varepsilon>0}\|u^\varepsilon\|_\infty < \infty$ by assumption. It follows that we can apply Theorem 2.23 along the above subsequence and obtain that

$$
\begin{aligned}
&\varepsilon^d \sum_{x\in\mathbb{Z}^d} u^\varepsilon(\varepsilon x) \nabla v(\varepsilon x) \cdot \left(\sum_{z\in\mathbb{Z}^d} z\, \omega_{x,x+z}\, b(\tau_x\boldsymbol{\omega}, z) \right) \\
&\qquad \to \int_{\mathbb{R}^d} u(x) \nabla v(x) \cdot \mathbb{E}\left[\sum_{z\in\mathbb{Z}^d} z\omega_{0,z} b(\boldsymbol{\omega}, z) \right] \mathrm{d}x .
\end{aligned} \tag{2.77}
$$

Thus we obtain by (2.73) that

$$\int_{\mathbb{R}^d} v(x)\mathbb{E}\left[\sum_{z\in\mathbb{Z}^d} \omega_{0,z}\mathrm{w}(x,\boldsymbol{\omega},z)b(\boldsymbol{\omega},z)\right]\mathrm{d}x = -\int_{\mathbb{R}^d} u(x)\nabla v(x)\cdot\mathbb{E}\left[\sum_{z\in\mathbb{Z}^d} z\omega_{0,z}b(\boldsymbol{\omega},z)\right]\mathrm{d}x\,. \tag{2.78}$$

Let us now argue that (2.78) implies that $\nabla u \in L^2(\mathbb{R}^d)$. By virtue of (2.55), for any $i = 1,\ldots,d$ we choose b^i such that $\mathbb{E}\big[\sum_{z\in\mathbb{Z}^d}\omega_{0,z} z b^i(\boldsymbol{\omega},z)\big] = \boldsymbol{e}_i$. Then (2.78) implies that

$$\begin{aligned}\left|\int_{\mathbb{R}^d} u(x)\partial_i v(x)\,\mathrm{d}x\right| &= \left|\int_{\mathbb{R}^d} v(x)\mathbb{E}\left[\sum_{z\in\mathbb{Z}^d}\omega_{0,z}\mathrm{w}(x,\boldsymbol{\omega},z)b^i(\boldsymbol{\omega},z)\right]\mathrm{d}x\right| \\ &\le \|b^i\|_{L^2_{\mathrm{cov}}}\int_{\mathbb{R}^d}|v(x)|\left(\mathbb{E}\left[\sum_{z\in\mathbb{Z}^d}\omega_{0,z}\mathrm{w}^2(x,\boldsymbol{\omega},z)\right]\right)^{1/2}\mathrm{d}x \\ &\le \|v\|_2\|b^i\|_{L^2_{\mathrm{cov}}}\left(\int_{\mathbb{R}^d}\mathbb{E}\left[\sum_{z\in\mathbb{Z}^d}\omega_{0,z}\mathrm{w}^2(x,\boldsymbol{\omega},z)\right]\mathrm{d}x\right)^{1/2}.\end{aligned}$$

Since $\mathrm{w}\in L^2(\mathbb{R}^d, L^2_{\mathrm{cov}})$, there exists $C<\infty$ such that for any $i = 1,\ldots,d$ we observe that

$$\left|\int_{\mathbb{R}^d} u(x)\partial_i v(x)\,\mathrm{d}x\right| \le C\|v\|_2\,.$$

By virtue of [Bre11, Proposition 9.3] it follows that $\nabla u\in L^2(\mathbb{R}^d)$. Since $u|_{\mathbb{R}^d\setminus Q} = 0$, we conclude $u\in H^1_0(Q)$.

We now use integration by parts on the right-hand side of (2.78) and obtain that

$$\int_{\mathbb{R}^d} v(x)\mathbb{E}\left[\sum_{z\in\mathbb{Z}^d}\omega_{0,z}(\mathrm{w}(x,\boldsymbol{\omega},z) - \nabla u(x)\cdot z)b(\boldsymbol{\omega},z)\right]\mathrm{d}x = 0\,. \tag{2.79}$$

Since the last equation holds for all $v\in\Psi$ and all $b\in\Phi_{\mathrm{sol}}$, we find that

$$\mathrm{w}(x,\boldsymbol{\omega},z) = \nabla u(x)\cdot z + \nu(x,\boldsymbol{\omega},z) \qquad \text{with } \nu\in L^2(\mathbb{R}^d; L^2_{\mathrm{pot}})\,.$$

□

2.6 Proof of Theorem 2.5

We start with an auxiliary lemma.

Lemma 2.36. *Let $f^\varepsilon \colon Q \cap \mathbb{Z}^d_\varepsilon \to \mathbb{R}$ be a sequence of functions such that $\mathcal{R}^*_\varepsilon f^\varepsilon \rightharpoonup f$ weakly in $L^2(Q)$ for some $f \in L^2(Q)$ and such that $\sup_\varepsilon \|f^\varepsilon\|_\infty < \infty$. Then for almost all $\boldsymbol{\omega} \in \Omega$ it holds: The sequence of solutions $u^\varepsilon \in \mathcal{H}_\varepsilon$ to the problem (2.8) satisfies $\mathcal{R}^*_\varepsilon u^\varepsilon \to u$ strongly in $L^2(Q)$, where $u \in H^1_0(Q) \cap H^2(Q)$ solves the limit problem (2.9).*

Proof. We test Equation (2.8) with an arbitrary test function $g^\varepsilon : \mathbb{Z}^d_\varepsilon \to \mathbb{R}$ with $\operatorname{supp} g^\varepsilon \subseteq Q \cap \mathbb{Z}^d_\varepsilon$ and obtain by Notation (2.60) and Equation (2.61) that

$$\langle -\mathcal{L}^\omega_\varepsilon u^\varepsilon, g^\varepsilon \rangle_{\mathcal{H}_\varepsilon} = \tfrac{1}{2} \langle A^\varepsilon_\omega \nabla^\varepsilon u^\varepsilon, \nabla^\varepsilon g^\varepsilon \rangle_{\mathcal{H}_\varepsilon} = \langle f^\varepsilon, g^\varepsilon \rangle_{\mathcal{H}_\varepsilon} . \tag{2.80}$$

We now choose $g^\varepsilon = u^\varepsilon$ and apply (2.35) and Cauchy-Schwarz to obtain that

$$\|u^\varepsilon\|^2_{\mathcal{H}_\varepsilon} \le C\varepsilon^d \sum_{x\in\mathbb{Z}^d} \sum_{z\in\mathbb{Z}^d} \omega_{x,x+z} \left(\partial^\varepsilon_z u^\varepsilon(\varepsilon x)\right)^2 \le 2C \|u^\varepsilon\|_{\mathcal{H}_\varepsilon} \|f^\varepsilon\|_{\mathcal{H}_\varepsilon} . \tag{2.81}$$

Hence, in combination with Remark 2.18 we conclude that

$$\|u^\varepsilon\|^2_{\ell^2(Q\cap\mathbb{Z}^d_\varepsilon)} + \varepsilon^d \sum_{x\in\mathbb{Z}^d} \sum_{z\in\mathbb{Z}^d} \omega_{x,x+z} \left(\partial^\varepsilon_z u^\varepsilon(\varepsilon x)\right)^2 \le 4C^2 \sup_{\varepsilon>0} \|f^\varepsilon\|^2_{\ell^2(Q\cap\mathbb{Z}^d_\varepsilon)} \tag{2.82a}$$

$$\text{and } \sup_{\varepsilon>0} \|u^\varepsilon\|_\infty < \infty . \tag{2.82b}$$

It follows that by virtue of Lemma 2.19 and Lemma 2.35, there exists $u \in H^1_0(Q)$, $\nu \in L^2(Q; L^2_{\mathrm{pot}})$ and a subsequence, which we still index by ε, such that

$$\mathcal{R}^*_\varepsilon u^\varepsilon \to u \,, \text{ strongly in } L^2(Q) \text{ and } \partial^\varepsilon_z u^\varepsilon(x) \overset{2s}{\rightharpoonup} \nabla u(x) \cdot z + \nu(x, \boldsymbol{\omega}, z) \text{ as } \varepsilon \to 0 \tag{2.83}$$

for all $x, z \in \mathbb{Z}^d$ and $\boldsymbol{\omega} \in \Omega_\Phi$.

Let us choose $v \in C^\infty_c(\mathbb{R}^d)$ with $\operatorname{supp} v \in Q$ and $\varphi \in \Phi_{\mathrm{pot}}$ with $\varphi = D\tilde\varphi$ for some bounded local function $\tilde\varphi$. When we insert $g^\varepsilon = \varepsilon v \tilde\varphi$ into (2.80), then we observe for all $\varepsilon > 0$ that

$$\begin{aligned}
&\varepsilon^d \sum_{x\in\mathbb{Z}^d} 2 f^\varepsilon(\varepsilon x)(\varepsilon v(\varepsilon x) \tilde\varphi(\tau_x \boldsymbol{\omega})) \\
&\qquad = \varepsilon^d \sum_{x\in\mathbb{Z}^d} \sum_{z\in\mathbb{Z}^d} \omega_{x,x+z}\, \partial^\varepsilon_z u^\varepsilon(\varepsilon x) \left(v(\varepsilon x + \varepsilon z) \tilde\varphi(\tau_{x+z}\boldsymbol{\omega}) - v(\varepsilon x) \tilde\varphi(\tau_x \boldsymbol{\omega}) \right) \\
&\qquad = \varepsilon^d \sum_{x\in\mathbb{Z}^d} \sum_{z\in\mathbb{Z}^d} \omega_{x,x+z}\, \partial^\varepsilon_z u^\varepsilon(\varepsilon x) \\
&\qquad\qquad\qquad \times \left[v(\varepsilon x)(\tilde\varphi(\tau_{x+z}\boldsymbol{\omega}) - \tilde\varphi(\tau_x\boldsymbol{\omega})) + \varepsilon \tilde\varphi(\tau_{x+z}\boldsymbol{\omega}) \partial^\varepsilon_z v(\varepsilon x) \right] ,
\end{aligned}$$

which, by definition of $\tilde{\varphi}$, means that

$$\begin{aligned}
&\varepsilon^d \sum_{x\in\mathbb{Z}^d} 2f^\varepsilon(\varepsilon x)(\varepsilon v(\varepsilon x)\tilde{\varphi}(\tau_x\boldsymbol{\omega})) \\
&\qquad = \varepsilon^d \sum_{x\in\mathbb{Z}^d}\sum_{z\in\mathbb{Z}^d} \omega_{x,x+z}\,\partial_z^\varepsilon u^\varepsilon(\varepsilon x)\,v(\varepsilon x)\varphi(\tau_x\boldsymbol{\omega},z) \\
&\qquad\qquad + \varepsilon^d \sum_{x\in\mathbb{Z}^d}\sum_{z\in\mathbb{Z}^d} \omega_{x,x+z}\,\partial_z^\varepsilon u^\varepsilon(\varepsilon x)\,\varepsilon\tilde{\varphi}(\tau_{x+z}\boldsymbol{\omega})\partial_z^\varepsilon v(\varepsilon x)\,. \qquad (2.84)
\end{aligned}$$

The second summand on the above right-hand side vanishes as $\varepsilon\to 0$ since

$$\begin{aligned}
&\varepsilon^d \Big| \sum_{x\in\mathbb{Z}^d}\sum_{z\in\mathbb{Z}^d} \omega_{x,x+z}\,\partial_z^\varepsilon u^\varepsilon(\varepsilon x)\,\varepsilon\tilde{\varphi}(\tau_{x+z}\boldsymbol{\omega})\partial_z^\varepsilon v(\varepsilon x)\Big| \\
&\qquad \le \varepsilon^{d+1}\|\tilde{\varphi}\|_\infty \left| \sum_{x\in\mathbb{Z}^d}\sum_{z\in\mathbb{Z}^d} \omega_{x,x+z}\,\partial_z^\varepsilon u^\varepsilon(\varepsilon x)\,\partial_z^\varepsilon v(\varepsilon x)\right| \\
&\qquad \le \varepsilon\|\tilde{\varphi}\|_\infty \left(\varepsilon^d \sum_{x\in\mathbb{Z}^d}\sum_{z\in\mathbb{Z}^d} \omega_{x,x+z}\,(\partial_z^\varepsilon u^\varepsilon(\varepsilon x))^2\right)^{1/2} \\
&\qquad\qquad \times \left(\varepsilon^d \sum_{x\in\mathbb{Z}^d}\sum_{z\in\mathbb{Z}^d} \omega_{x,x+z}\,(\partial_z^\varepsilon v(\varepsilon x))^2\right)^{1/2}\,. \qquad (2.85)
\end{aligned}$$

By assumption $\|\tilde{\varphi}\|_\infty$ is bounded. The second factor is bounded due to (2.82) and the third factor is bounded by virtue of Lemma 2.31. Since the left-hand side of (2.84) vanishes as well, (2.83) and (2.84) imply that in the limit $\varepsilon\to 0$ and along the chosen subsequence we obtain

$$\int_Q v(x)\mathbb{E}\left[\sum_{z\in\mathbb{Z}^d} \omega_{0,z}(\nabla u(x)\cdot z + \nu(x,\boldsymbol{\omega},z))\varphi(\boldsymbol{\omega},z)\right]\mathrm{d}x = 0\,. \qquad (2.86)$$

Since Φ_{pot} is dense in L^2_{pot} and Ψ is dense in $H^1_0(Q)$, Equation (2.86) holds for all $\varphi\in L^2_{\mathrm{pot}}$ and all $v\in H^1_0(Q)$.

Let $\chi\in\left(L^2_{\mathrm{pot}}\right)^d$ be given through (2.53). Since $u\in H^1_0(Q)$ is given, the function $\nu(x,\tilde{\boldsymbol{\omega}},z) := \nabla u(x)\cdot\chi(\tilde{\boldsymbol{\omega}},z)$ is the unique solution to (2.86). We have thus identified ν.

Now we observe that if we test (2.80) by an arbitrary $g\in C_c^\infty(\mathbb{R}^d)$ with support in Q, we obtain that

$$\varepsilon^d \sum_{x\in\mathbb{Z}^d}\sum_{z\in\mathbb{Z}^d} \omega_{x,x+z}\,\partial_z^\varepsilon u^\varepsilon(\varepsilon x)\,\partial_z^\varepsilon g(\varepsilon x) = \varepsilon^d \sum_{x\in\mathbb{Z}^d} 2f^\varepsilon(\varepsilon x) g(\varepsilon x)\,.$$

Passing to the limit, we obtain by virtue of Corollary 2.33 and $\nu(x,\boldsymbol{\omega},z) = \nabla u(x)\cdot\chi(\boldsymbol{\omega},z)$ that

$$\int_{\mathbb{R}^d} \mathbb{E}\left[\sum_{z\in\mathbb{Z}^d} \omega_{0,z}(\nabla u(x)\cdot(z+\chi))(\nabla g(x)\cdot z)\right] \mathrm{d}x = \int_{\mathbb{R}^d} 2f(x)g(x)\,\mathrm{d}x\,. \quad (2.87)$$

When we now insert $v = \partial_i g$ and $\varphi = \chi_i$ for $i = 1,\dots,d$ into (2.86) and add the resulting equations to (2.87), then we obtain that

$$\int_{\mathbb{R}^d} \mathbb{E}\left[\sum_{z\in\mathbb{Z}^d} \omega_{0,z}(\nabla u(x)\cdot(z+\chi))(\nabla g(x)\cdot(z+\chi))\right] \mathrm{d}x = \int_{\mathbb{R}^d} 2f(x)g(x)\,\mathrm{d}x\,. \quad (2.88)$$

A comparison with the definition of A_{hom} in (2.54) finally yields that u solves

$$\int_Q \nabla u\cdot(A_{\mathrm{hom}}\nabla g) = \int_Q 2f\,g \qquad \text{for all } g\in C_{\mathrm{c}}^\infty(\mathbb{R}^d) \text{ with } \operatorname{supp} g\subseteq Q\,. \quad (2.89)$$

Since A_{hom} is non-degenerate, we find that (2.89) is the weak formulation of (2.9). Hence, from elliptic regularity theory [Eva10, Chapter 6], we obtain that $u\in H^2(Q)\cap H_0^1(Q)$.

Since the solution u of (2.9) is unique, it follows that (2.83) holds for the entire sequence. □

As for the last ingredient for the proof of Theorem 2.5, we observe the following: On the cube Q the operator $-\mathcal{L}_\varepsilon^\omega$ with zero Dirichlet conditions is strictly positive definite (see e.g. (2.61)) and thus it follows that on Q its inverse $\mathcal{B}_\varepsilon\colon\,\mathcal{H}_\varepsilon\to\mathcal{H}_\varepsilon$ is well-defined. Similarly, the inverse $\mathcal{B}_0\colon\,\mathcal{H}_0\to\mathcal{H}_0$ of $-\mathcal{L}_0^\omega$ on Q is well-defined. We have the following lemma.

Lemma 2.37. *The operators $\mathcal{B}_\varepsilon,\mathcal{B}_0$ are $\mathbb{P}$-a.s. positive, compact and self-adjoint. The norms $\|\mathcal{B}_\varepsilon\|$ are $\mathbb{P}$-a.s. bounded by a constant independent of ε.*

Proof. Since A_{hom} is positive definite (see e.g. proof of Lemma 2.25) and symmetric (by definition), the properties of $\mathcal{B}_0$ follow from the theory of elliptic partial differential equations, see e.g. [Eva10, Chapter 6].

The operator $\mathcal{B}_\varepsilon$ is uniformly bounded in ε by virtue of (2.82a). Moreover, $\mathcal{B}_\varepsilon$ is real and symmetric by construction and therefore self-adjoint. Finally, its range $\mathcal{H}_\varepsilon$ is finite-dimensional and thus $\mathcal{B}_\varepsilon$ is compact. □

Proof of Theorem 2.5. Let us first show that

$$\lim_{\varepsilon\to 0}\int\limits_{\mathbb{R}^d}\left(\mathcal{R}_\varepsilon^* u^\varepsilon\right) v = \int\limits_{\mathbb{R}^d} uv \quad \text{for all } v \in C(\overline{Q})\,, \tag{2.90}$$

where $u \in H^2(Q) \cap H_0^1(Q)$ is the solution to (2.9). Indeed, since $\mathcal{B}_\varepsilon$ is self-adjoint, we observe that

$$\int\limits_{\mathbb{R}^d}\left(\mathcal{R}_\varepsilon^* u^\varepsilon\right) v = \int\limits_{\mathbb{R}^d}\left(\mathcal{R}_\varepsilon^* \mathcal{B}_\varepsilon f^\varepsilon\right) v = \left\langle f^\varepsilon,\, \left(\mathcal{B}_\varepsilon \mathcal{R}_\varepsilon v\right)\right\rangle_{\mathcal{H}_\varepsilon}.$$

Since $\mathcal{R}_\varepsilon^*\mathcal{R}_\varepsilon v \rightharpoonup v$ in L^2 and $\sup_{\varepsilon>0}\|\mathcal{R}_\varepsilon v\|_\infty < \infty$, Lemma 2.36 implies that $\mathcal{B}_\varepsilon\mathcal{R}_\varepsilon v$ converges strongly in L^2 to $\mathcal{B}_0 v$. It follows that

$$\lim_{\varepsilon\to 0}\left\langle f^\varepsilon,\, \left(\mathcal{B}_\varepsilon \mathcal{R}_\varepsilon v\right)\right\rangle_{\mathcal{H}_\varepsilon} = \int\limits_{\mathbb{R}^d} f\left(\mathcal{B}_0 v\right) = \int\limits_{\mathbb{R}^d}\left(\mathcal{B}_0 f\right) v = \int\limits_{\mathbb{R}^d} uv\,,$$

where we have used that the operator $\mathcal{B}_0$ is self-adjoint, see Lemma 2.37.

We further note that $\sup_{\varepsilon>0}\|\mathcal{R}_\varepsilon^* u^\varepsilon\|_2 < \infty$ by the same arguments as for (2.82a). Since $C(\overline{Q})$ is dense in $L^2(Q)$, it thus follows that $\mathcal{R}_\varepsilon^* u^\varepsilon \rightharpoonup u$. By virtue of Lemma 2.19 and (2.82a) we conclude that $\mathcal{R}_\varepsilon^* u^\varepsilon \to u$ strongly in L^2. □

2.7 Proofs of Proposition 2.8 and Theorem 2.9

Proof of Proposition 2.8. The existence of solutions to (2.12) follows from positivity of the first eigenvalue for small ε. Hence we can calculate the apriori estimates similar to (2.81) by testing (2.12) with u^ε and using $\liminf_{\varepsilon\to 0}\lambda_1^\varepsilon > 0$ to obtain

$$\|u^\varepsilon\|_{\mathcal{H}_\varepsilon}^2 \le (\lambda_1^\varepsilon)^{-1}\langle -\mathcal{L}_\varepsilon^\omega u^\varepsilon + \mathcal{R}_\varepsilon V u^\varepsilon, u^\varepsilon\rangle_{\mathcal{H}_\varepsilon} \le 2(\lambda_1^\varepsilon)^{-1}\|u^\varepsilon\|_{\mathcal{H}_\varepsilon}\|f^\varepsilon\|_{\mathcal{H}_\varepsilon}.$$

Since V is bounded, this implies that $\langle -\mathcal{L}_\varepsilon^\omega u^\varepsilon, u^\varepsilon\rangle_{\mathcal{H}_\varepsilon}$ is bounded in ε. From Lemma 2.19 it follows that $\mathcal{R}_\varepsilon^* u^\varepsilon \to u$ strongly in $L^2(Q)$ and hence $\mathcal{R}_\varepsilon^*(\mathcal{R}_\varepsilon V\, u^\varepsilon) \rightharpoonup Vu$. Hence from Theorem 2.5 we obtain that u solves (2.13). □

Proof of Theorem 2.9. First, we notice that without loss of generality, we can assume that the function V is nonnegative. Otherwise, we simply substitute V for $V - \min_{x\in Q} V(x)$ and prove the result for the new V. Then we notice that the substitution has simply shifted the spectrum by the constant $\min_{x\in Q} V(x)$ and the new eigenvectors are the same as the old ones. Thus, it suffices to prove the claim for $V \ge 0$. Note that (2.61) directly implies that if $V \ge 0$, then λ_1^ε is positive.

Then Lemmas 2.39 and 2.37 ensure that Conditions I-IV of [JKO94, Section 11.1] are satisfied and Theorem 2.9 follows by virtue of [JKO94, Theorems 11.4, 11.5]. □

As in the paragraph before Lemma 2.37, we now define the operators $\mathcal{B}_\varepsilon(V)$ and $\mathcal{B}_0(V)$ as the inverses of $-\mathcal{L}_\varepsilon^\omega + \mathcal{R}_\varepsilon V$ and $-\mathcal{L}_0^\omega + V$, respectively. For $V \geq 0$, we further consider the spectrum of the operators $\mathcal{B}_\varepsilon(V)$, where we drop the argument "(V)" for readability:

$$\begin{aligned} &\psi_k^\varepsilon \in \mathcal{H}_\varepsilon, \quad \mathcal{B}_\varepsilon \psi_k^\varepsilon = \mu_k^\varepsilon \psi_k^\varepsilon, \quad k = 1, 2, \dots, \\ &\mu_1^\varepsilon \geq \mu_2^\varepsilon \geq \dots \geq \mu_k^\varepsilon \dots, \quad \mu_k^\varepsilon > 0\,, \\ &\langle \psi_k^\varepsilon, \psi_l^\varepsilon \rangle_{\mathcal{H}_\varepsilon} = \delta_{kl}\,, \end{aligned} \tag{2.91}$$

as well as the spectrum of the operator $\mathcal{B}_0(V)$, where we also drop the argument "(V)" for readability:

$$\begin{aligned} &\psi_k^0 \in \mathcal{H}_0, \quad \mathcal{B}_0 \psi_k^0 = \mu_k^0 \psi_k^0, \quad k = 1, 2, \dots, \\ &\mu_1^0 \geq \mu_2^0 \geq \dots \geq \mu_k^0 \dots, \quad \mu_k^0 > 0\,, \\ &\langle \psi_k^0, \psi_l^0 \rangle_{\mathcal{H}_\varepsilon} = \delta_{kl}\,. \end{aligned} \tag{2.92}$$

Remark 2.38. *The eigenfunctions $\{\psi_k^\varepsilon\}_k$ of the operator $\mathcal{B}_\varepsilon$ and the eigenfunctions $\left\{\psi_k^0\right\}_k$ of the operator $\mathcal{B}_0$ coincide with the eigenfunctions of the operators $-\mathcal{L}_\varepsilon^\omega + \mathcal{R}_\varepsilon V$ and $-\mathcal{L}_0^\omega + V$, respectively. Their eigenvalues relate to those of $-\mathcal{L}_\varepsilon^\omega + \mathcal{R}_\varepsilon V$ and $-\mathcal{L}_0^\omega + V$ by*

$$\mu_k^\varepsilon = (\lambda_k^\varepsilon)^{-1}\,, \quad \mu_k^0 = \left(\lambda_k^0\right)^{-1} \quad k = 1, 2, \dots\,.$$

Lemma 2.39.

(i) For any $u \in \mathcal{H}_0$, the following is true:

$$\|\mathcal{R}_\varepsilon u\|_{\mathcal{H}_\varepsilon} \leq \|u\|_{\mathcal{H}_0}\,. \tag{2.93}$$

Further,

$$\lim_{\varepsilon \to 0} \langle u^\varepsilon, v^\varepsilon \rangle_{\mathcal{H}_\varepsilon} = \langle u, v \rangle_{\mathcal{H}_0}\,. \tag{2.94}$$

provided that $u, v \in \mathcal{H}_0$ and $u^\varepsilon, v^\varepsilon \in \mathcal{H}_\varepsilon$ and

$$\lim_{\varepsilon \to 0} \|u^\varepsilon - \mathcal{R}_\varepsilon u\|_{\mathcal{H}_\varepsilon} = 0, \quad \textit{and} \quad \lim_{\varepsilon \to 0} \|v^\varepsilon - \mathcal{R}_\varepsilon v\|_{\mathcal{H}_\varepsilon} = 0\,. \tag{2.95}$$

Let $V\colon \mathbb{R}^d \to \mathbb{R}$ be a non-negative, continuous potential. If Assumptions 2.1 and 2.2(a') are fulfilled, then furthermore the following statements hold.

(ii) Let $f \in \mathcal{H}_0$ and let $f^\varepsilon \in \mathcal{H}_\varepsilon$. Then the following is true:
If

$$\lim_{\varepsilon\to 0} \|f^\varepsilon - \mathcal{R}_\varepsilon f\|_{\mathcal{H}_\varepsilon} = 0\,, \tag{2.96}$$

then

$$\lim_{\varepsilon\to 0} \|\mathcal{B}_\varepsilon f^\varepsilon - \mathcal{R}_\varepsilon \mathcal{B}_0 f\|_{\mathcal{H}_\varepsilon} = 0 \quad \mathbb{P}\text{-}a.s. \tag{2.97}$$

(iii) For any sequence $f^\varepsilon \in \mathcal{H}_\varepsilon$ such that $\sup_\varepsilon \|f^\varepsilon\|_{\mathcal{H}_\varepsilon} < \infty$, there exists a subsequence $f^{\varepsilon'}$ and a vector $w^0 \in \mathcal{H}_0$ such that

$$\lim_{\varepsilon'\to 0}\left\|\mathcal{R}^*_{\varepsilon'}\mathcal{B}_{\varepsilon'} f^{\varepsilon'} - w^0\right\|_{\mathcal{H}_0} = \lim_{\varepsilon'\to 0}\left\|\mathcal{B}_{\varepsilon'} f^{\varepsilon'} - \mathcal{R}_{\varepsilon'} w^0\right\|_{\mathcal{H}_{\varepsilon'}} = 0\,.$$

Proof. For (i): Let $u \in \mathcal{H}_0$. By Jensen's inequality it follows that

$$\|\mathcal{R}_\varepsilon u\|^2_{\mathcal{H}_\varepsilon} = \varepsilon^d \sum_{z\in\mathbb{Z}^d_\varepsilon} \varepsilon^{-2d}\Big(\int\limits_{b\left(z,\frac{\varepsilon}{2}\right)} u\,\mathrm{d}x\Big)^2 \le \varepsilon^d \sum_{z\in\mathbb{Z}^d_\varepsilon} \varepsilon^{-d}\Big(\int\limits_{b\left(z,\frac{\varepsilon}{2}\right)} u^2\,\mathrm{d}x\Big) = \|u\|^2_{\mathcal{H}_0}\,.$$

For (2.94) we first observe that

$$\begin{aligned}|\langle u^\varepsilon, v^\varepsilon\rangle_{\mathcal{H}_\varepsilon} - \langle u, v\rangle_{\mathcal{H}_0}| &\le \left|\langle v^\varepsilon, u^\varepsilon - \mathcal{R}_\varepsilon u\rangle_{\mathcal{H}_\varepsilon}\right| + \left|\sum_{z\in\mathbb{Z}^d_\varepsilon}\int\limits_{b\left(z,\frac{\varepsilon}{2}\right)} u(\mathcal{R}^*_\varepsilon v^\varepsilon(z) - v)\,\mathrm{d}x\right| \\ &\le \|v^\varepsilon\|_{\mathcal{H}_\varepsilon}\|u^\varepsilon - \mathcal{R}_\varepsilon u\|_{\mathcal{H}_\varepsilon} + \|u\|_{\mathcal{H}_0}\,\|v^\varepsilon - \mathcal{R}_\varepsilon v\|_{\mathcal{H}_\varepsilon}\,.\end{aligned} \tag{2.98}$$

The second term on the above right-hand side converges to zero by assumption. For the first term we note that the triangle inequality together with (2.93) yields

$$\|v^\varepsilon\|_{\mathcal{H}_\varepsilon} \le \|\mathcal{R}_\varepsilon v\|_{\mathcal{H}_\varepsilon} + \|v^\varepsilon - \mathcal{R}_\varepsilon v\|_{\mathcal{H}_\varepsilon} \le \|v\|_{\mathcal{H}_0} + \|v^\varepsilon - \mathcal{R}_\varepsilon v\|_{\mathcal{H}_\varepsilon}\,,$$

which is bounded from above. It follows that the first term on the right-hand side of (2.98) converges to zero as well.

Part (ii) follows directly from Proposition 2.8 and (2.94). Similarly, Part (iii) follows from Proposition 2.8 and (2.94) since $\sup_\varepsilon \|f^\varepsilon\|_2 < \infty$ implies that there exists a subsequence ε' along which $\mathcal{R}^*_{\varepsilon'} f^{\varepsilon'} \rightharpoonup f$ in L^2. □

2.8 Proof of Proposition 2.12

Proof of Proposition 2.12. This proof is an application of the Gärtner-Ellis theorem and goes along the lines of [KW15, Theorem 1.8]. For the convenience of the reader, we outline the main steps here.

Let $V\colon \mathbb{R}^d \to \mathbb{R}$ be a bounded, continuous function. We define the generating cumulant function

$$\Lambda_t(V) := \frac{\alpha_t^2}{t} \log \mathrm{E}_0^\omega \left[\exp\left\{ -\frac{t}{\alpha_t^2} \int\limits_Q V(y) L_t(y) \, \mathrm{d}y \right\} \middle| X_{[0,t]} \subset \alpha_t Q \right]. \quad (2.99)$$

As in [KW15], it suffices to show that

$$\Lambda(V) := \lim_{t\to\infty} \Lambda_t(V) = -\lambda_1(V) + \lambda_1(0)\,, \quad (2.100)$$

where $\lambda_1(V)$ denotes the principal Dirichlet eigenvalue of $-\mathcal{L}_0^\omega + V$ on Q with zero Dirichlet boundary conditions. Then the claim follows by the Gärtner-Ellis theorem.

In order to show (2.100), we define the operator $\mathcal{P}_t^{\omega,V}$ acting on real-valued functions $f \in \ell^2(\alpha_t Q \cap \mathbb{Z}^d)$ by

$$\left(\mathcal{P}_t^{\omega,V} f\right)(z) := \mathrm{E}_z^\omega \left[\exp\left\{ -\frac{t}{\alpha_t^2} \int\limits_Q V(y) L_t(y) \, \mathrm{d}y \right\} \mathbb{1}_{\{X_{[0,t]} \subset \alpha_t Q\}} f(X_t) \right] \quad (2.101)$$

for all $z \in \alpha_t Q \cap \mathbb{Z}^d$. Since L_t is a step function, $\mathcal{P}_t^{\omega,V}$ admits the semigroup representation

$$\mathcal{P}_t^{\omega,V} = \exp\left\{ -t\alpha_t^{-2} \left[-\alpha_t^2 \mathcal{L}^\omega + V_t \right] \right\}, \quad (2.102)$$

where the operator in the exponent is considered with zero Dirichlet conditions at the boundary of $\alpha_t Q \cap \mathbb{Z}^d$ and

$$V_t(z) := \int\limits_{[-\frac{1}{2},\frac{1}{2}]} V\left(\frac{z+y}{\alpha_t} \right) \mathrm{d}y \qquad (z \in \alpha_t Q \cap \mathbb{Z}^d)\,.$$

Let $\lambda_1^{(t)}(V)$ denote the principal Dirichlet eigenvalue of $-\alpha_t^2 \mathcal{L}^\omega + V_t$ on $\alpha_t Q \cap \mathbb{Z}^d$ with zero Dirichlet boundary conditions. Let $\psi_1^{(t)}(V)$ be the corresponding principal Dirichlet eigenfunction. Then, in order to show (2.100), for any $V \in \mathcal{C}_\mathrm{b}(\mathbb{R}^d)$ we have to show that

$$\lim_{t\to\infty} \frac{\alpha_t^2}{t} \log\left(\mathcal{P}_t^{\omega,V} \mathbb{1} \right)(0) = \lim_{t\to\infty} \lambda_1^{(t)}(V) = \lambda_1(V)\,. \quad (2.103)$$

The second equality follows by virtue of Theorem 2.9. It remains to prove the first equality. For this purpose we notice that an eigenvalue expansion together with Cauchy-Schwarz and Parseval's identity yields that

$$\Big(\mathcal{P}_t^{\omega,V}\mathbb{1}\Big)(0) \leq \sqrt{|\alpha_t Q|}\,\exp\Big\{-\frac{t}{\alpha_t^2}\lambda_1^{(t)}(V)\Big\}\,.$$

On the other hand, since $\mathcal{P}_t^{\omega,V} \geq 0$, we can estimate from below

$$\Big(\mathcal{P}_t^{\omega,V}\mathbb{1}\Big)(0) \;\geq\; \frac{1}{\sup_{\alpha_t Q}\psi_1^{(t)}}\Big(\mathcal{P}_t^{\omega,V}\psi_1^{(t)}\Big)(0) \;\geq\; \psi_1^{(t)}(0)\exp\Big\{-\frac{t}{\alpha_t^2}\,\lambda_1^{(t)}(V)\Big\}$$

since $\psi_1^{(t)}$ is a normalized eigenfunction. Thus, if $\psi_1^{(t)}(0)$ decays at most polynomially, we have proved the claim.

Similar to the proof in [KW15], we obtain that

$$\psi_1^{(t)}(0) \geq \mathrm{e}^{-\lambda(V)-V^*}\Big(\max_{x\in\alpha_t Q\cap\mathbb{Z}^d}\psi_1^{(t)}\Big)\Big(\min_{x\in\alpha_t Q\cap\mathbb{Z}^d}\mathrm{P}_0^{\alpha_t^2\omega}[X_1 = x]\Big)\,, \qquad (2.104)$$

where V^* is an upper bound for V. Since $\psi_1^{(t)}$ is normalized and

$$\min_{x\in\alpha_t Q\cap\mathbb{Z}^d}\mathrm{P}_0^{\alpha_t^2\omega}[X_1 = x] = \min_{x\in\alpha_t^2 Q\cap\mathbb{Z}^d}\mathrm{P}_0^{\omega}[X_{\alpha_t} = x]$$

decays at most polynomially by Assumption 2.11, the claim follows. □

Chapter 3
Localization[1]

In this chapter we assume that the conductances ω are independent and identically distributed positive random variables between nearest neighbors and zero elsewhere. This means that we assume that our underlying graph is $(\mathbb{Z}^d, \mathfrak{E}_d)$ with the edge set $\mathfrak{E}_d$ defined in (1.9). We restrict ourselves to dimensions $d \geq 2$. For the one-dimensional case, see [Fag12].

In what follows, let ω be a copy of the conductance variables. Moreover, we call $F\colon [0,\infty) \to [0,1]$, $u \mapsto \mathbb{P}[\omega \leq u]$ the distribution function of the conductances.

Previously, we have seen that, if $\mathbb{E}[\omega^{-1/4}] < \infty$, then the top of the Dirichlet spectrum of the Laplacian $\mathcal{L}^\omega$ in the box B_n homogenizes. Now we would like to study the case $\gamma < 1/4$ where $\gamma = \sup\{q \geq 0\colon \mathbb{E}[w^{-q}] < \infty\}$, i.e., when the conductances have a very heavy tail near zero. Note that a critical regime remains, to which belongs the case $\mathbb{P}[\omega \leq a] = a^{1/4}$ for $a \in [0,1]$. Here, we are going to see that for $\gamma < 1/4$, the principal Dirichlet eigenvalue $\lambda_1^{(n)}$ scales subdiffusively almost-surely, i.e., it approaches zero much faster than n^{-2}. If, in addition, certain regularity assumptions apply, then the first k Dirichlet eigenvectors $\psi_1^{(n)}, \dots, \psi_k^{(n)}$ in the box B_n concentrate in a single site as n tends to infinity.

In this chapter, the box B_n is defined as

$$B_n := [-n,n]^d \cap \mathbb{Z}^d$$

instead of $(-n,n)^d \cap \mathbb{Z}^d$ as in (1.17).

Let us now recall the definition of the local speed measure π and introduce its order statistics.

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[1] Part of this chapter is published as F. Flegel: “Localization of the principal Dirichlet eigenvector in the heavy-tailed random conductance model”, *Electron. J. Probab.*, 23:no. 68, 1–43, 2018. The other part is currently available as the preprint “Eigenvector localization in the heavy-tailed random conductance model” (arXiv:1801.05684).

Definition 3.1 (Local speed measure and its order statistics). *We define the local speed measure* π *by*

$$\pi_z = \sum_{x\colon x\sim z} \omega_{xz} \qquad (z \in \mathbb{Z}^d) \tag{3.1}$$

and we label the order statistics of the set $\{\pi_z \colon z \in B_n\}$ *by*

$$\pi_{1,B_n} \leq \pi_{2,B_n} \leq \ldots \leq \pi_{|B_n|,B_n} \,. \tag{3.2}$$

Furthermore, for $k, n \in \mathbb{N}$ *let* $z_{(k,n)}$ *be the site where* π *attains its kth minimum over* B_n*, i.e.,* $\pi_{z_{(k,n)}} = \pi_{k,B_n}$.

Let F_π be the distribution function of the random variable π, i.e., the distribution function of the sum of $2d$ independent copies of the conductance ω.

Remark 3.2. *If the distribution function* F *is continuous, then* F_π *is continuous as well and therefore* $\mathbb{P}$*-almost surely* $\pi_{1,B_n} < \pi_{2,B_n} < \ldots < \pi_{|B_n|,B_n}$ *for all* $n \in \mathbb{N}$*. It follows that the minimizers* $z_{(k,n)}$ *are* $\mathbb{P}$*-a.s. unique.*

We show that the kth Dirichlet eigenvector $\psi_k^{(n)}$ approaches the δ-function in the site $z_{(k,n)}$ where the local speed measure π attains its kth minimum in the box B_n. This further implies that the kth Dirichlet eigenvalue $\lambda_k^{(n)}$ is asymptotically equivalent to the kth minimum π_{k,B_n} of π over the box B_n. If the conductances vary regularly at zero with positive index, then the kth minimum of $\{\pi_x \colon x \in B_n\}$ converges weakly as if it was the kth minimum of an independent field, see the proof of Corollary 3.17. It follows that, in this case, the properly rescaled kth eigenvalue $\lambda_k^{(n)}$ converges in distribution to a non-degenerate random variable. This relates to a similar result in dimension $d = 1$, see [Fag12, Theorem 2.5(i)].

Generally, the results of this chapter for the random conductance Laplacian compare well to similar results of the random Schrödinger operator $\Delta+\xi$ with random potential $\xi \colon \mathbb{Z}^d \to \mathbb{R}$, see [BK16] and [Ast16, Ch. 6]. For more references, we refer the reader to Section 1.4.

Let us briefly outline the strategy of this chapter. First we give asymptotic lower and upper bounds for the principal Dirichlet eigenvalue $\lambda_1^{(n)}$ where we aim to find as optimal conditions as possible. Indeed, for the lower bound the condition that we find is sharp, see Theorem 3.7. In contrast, for the upper bound the sufficient and the necessary conditions differ by a double-logarithmic order, see Theorem 3.5. For this part we use path arguments that are adapted from [BKM15] as well as Borel-Cantelli arguments and percolation results.

Second, we state the localization of the principal Dirichlet eigenvector $\psi_1^{(n)}$, see Theorem 3.13, which relies heavily on the extreme value analysis of Section 3.7.3. Afterwards we use the localization of the principal eigenvector as an inductive base case for the localization of the higher order eigenvectors, see Theorem 3.16 and its proof in Section 3.9.

The reason why we first concentrate on the principal Dirichlet eigenvector is that we can assume that all its entries are non-negative due to the Perron-Frobenius theorem, see the remark below.

Remark 3.3 (Perron-Frobenius). *For a given box B_n the operator $\mathcal{L}^\omega$ together with the zero Dirichlet boundary conditions can be written as a $|B_n| \times |B_n|$-matrix with non-negative entries everywhere except on the diagonal. Since the matrix is finite-dimensional, we can add a multiple of the identity to obtain a non-negative primitive matrix without changing the matrix' spectrum. By the Perron-Frobenius theorem (see e.g. [Sen81, Chapter 1]) it follows that its principal eigenvalue is simple and we can assume without loss of generality that its principal eigenvector is positive.*

This remark is an essential ingredient for Lemma 3.45 and it does not hold for any higher order eigenvectors. Therefore, in order to infer localization of the second eigenvector $\psi_2^{(n)}$, we first prove localization of an auxiliary principal eigenvector $\phi_{2,1}^{(n)}$ where we simply impose another zero Dirichlet condition at the localization center $z_{(1,n)}$ of the principal eigenvector $\psi_1^{(n)}$, see Definition 3.52. For this auxiliary eigenvector we can again apply the Perron-Frobenius theorem. Then we show that the eigenvector $\phi_{2,1}^{(n)}$ is indeed close to the eigenvector $\psi_2^{(n)}$ by invoking the Bauer-Fike lemma, see Section 3.8.3 as well as the proof of Theorem 3.16. The localization of the other eigenvectors follows by induction.

3.1 Main Results

3.1.1 The principal Dirichlet eigenvalue

Let $g\colon (0,\infty) \to (0,\infty)$ be a function that decreases monotonically to zero and let us recall the definition of the function $\Lambda_g\colon (0,\infty) \to (0,\infty)$ in (1.39). As we have explained above (1.39), the function Λ_g relates to the expected number of $g(n)$-traps in the box B_n.

In our results, we often require recurring conditions on the function $g\colon (0,\infty) \to (0,\infty)$. Let us recall that a function g varies regularly at infinity (zero) with index $\rho \in \mathbb{R}$ if $g(u) = u^\rho L(u)$ where L is a slowly varying function at infinity (zero), i.e., for all $C > 0$ we have

$$\frac{L(Cu)}{L(u)} \to 1 \qquad \text{as } u \to \infty \text{ (as } u \to 0)$$

see e.g. [BGT89, Chapter 1].

Assumption 3.4. *Let* $g\colon (0,\infty) \to (0,\infty)$.

(a) The function g *varies regularly at infinity with an index strictly less than* -2.
(b) The function $u \mapsto u^2 g(u)$ *is monotone and has a finite limit as* u *tends to infinity.*
(b') The function $u \mapsto u^2 g(u)$ *converges monotonically to zero as* u *tends to infinity.*

Our first theorem gives a sufficient and a necessary condition for the function g being is an asymptotic upper bound for the principal Dirichlet eigenvalue $\lambda_1^{(n)}$. Note that, given one of the Assumptions 3.4 (a) or (b') is true, then the sufficient and necessary conditions coincide up to the case where Λ_g scales exactly like $\log\log n$. We summarize all the conditions of the following two theorems in a graphical overview (see Figure 3.1).

Theorem 3.5 (Upper bound). *Let* $g : (0,\infty) \to (0,\infty)$ *be a function that converges monotonically to zero and let* Λ_g *be as in* (1.39)*. Then the following statements are true:*

(i) If there exists $\varepsilon > 0$ *such that for all* n *large enough*

$$\frac{\Lambda_g(n)}{\log\log n} \geq 2 + \varepsilon\,, \tag{3.3}$$

then $\mathbb{P}$*-a.s. for* n *large enough* $\lambda_1^{(n)} \leq 2dg(n)$.
(ii) On the other hand, if

$$\lim_{u\to\infty} \frac{\Lambda_g(u)}{\log\log u} = 0\,, \tag{3.4}$$

and one of the Assumptions 3.4 (a) or (b') is true, then $\mathbb{P}$*-almost surely* $\limsup_{n\to\infty} \frac{\lambda_1^{(n)}}{g(n)} = \infty$.

We prove part (i) of this theorem in Section 3.2 and part (ii) in Section 3.3. Note that in (ii) the Assumptions 3.4 (a) and (b') correspond to the fact that we can only deduce that the limit superior diverges if we assume that g is in $o(n^{-2})$. This is because in the diffusive regime $\lambda_1^{(n)}$ scales like n^{-2}.

In the case where the distribution function $F(a)$ varies regularly at zero with index $\gamma > 0$, Theorem 3.5 (i) implies the following corollary. Since its proof is immediate, we omit it.

Corollary 3.6. *Let* $\delta > 0$*. If* F *varies regularly at zero with index* $\gamma > 0$*, then* $\mathbb{P}$*-a.s. for* n *large enough the function* $g(n) = n^{-\frac{1}{2\gamma}+\delta}$ *is an asymptotic upper bound for* $\lambda_1^{(n)}$*. If even* $F(a) = a^\gamma$ *for* $a \in [0,1]$*, then the upper bound can be improved to* $g(n) = n^{-\frac{1}{2\gamma}}((2+\varepsilon)\log\log n)^{\frac{1}{2d\gamma}}$.

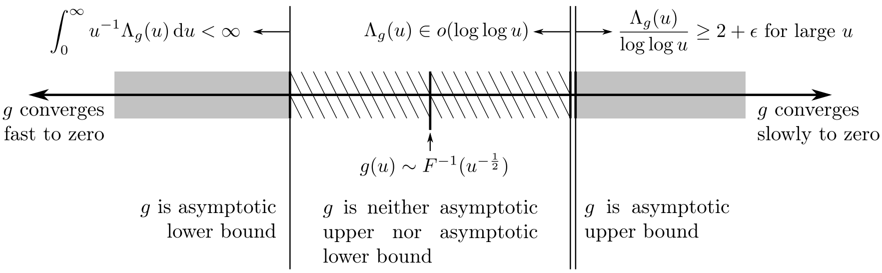


Fig. 3.1: Visualization of our results from Theorems 3.5 and 3.7 for a fixed distribution function F that is continuous and strictly monotone near zero. The figure shows the space of functions $g : (0, \infty) \to (0, \infty)$ that decrease to zero. The space is depicted such that if $f \in o(g)$, then f appears left of g. For simplicity we assume that g fulfills one of the Assumptions 3.4 (a) or (b'). If $F^{-1}(u^{-1/2}) \in o(g(u))$, then $\Lambda_g(u)$ diverges. If g even decays slowly enough such that condition (3.3) is fulfilled, then $\mathbb{P}$-a.s. for n large enough $\lambda_1^{(n)} \leq 2dg(n)$. On the other hand, if $g(u) \in o(F^{-1}(u^{-1/2}))$, then $\Lambda_g(u)$ converges to zero. If g even decays fast enough such that (3.5) is fulfilled, then there exists $c > 0$ such that $\mathbb{P}$-a.s. for n large enough $\lambda_1^{(n)} \geq cg(n)$. The figure also shows that around $g(u) \sim F^{-1}(u^{-1/2})$ there is an interval where g is definitely neither an a.s. asymptotic upper nor an a.s. asymptotic lower bound, see e.g. Corollary 3.9.

Note that if F varies regularly at zero with index $\gamma > 0$, then $\gamma = \sup\{q \geq 0\colon \mathbb{E}[\omega^{-q}] < \infty\}$, as defined in the introduction. Further note that if $\gamma \in [0, 1/4)$, then there exists $\eta > 0$ such that the expectation $\mathbb{E}\big[\omega^{-1/4+\eta}\big]$ diverges, cf. the conditions of [ADS16, Theorem 1.13].

The second theorem gives conditions for when the function g is an asymptotic lower bound of the principal Dirichlet eigenvalue $\lambda_1^{(n)}$. Note that this theorem implies that, given one of the Assumptions 3.4 (a) or (b) is true, then the condition in (3.5) is sharp. We further comment on these conditions in Section 3.3. As with the conditions of Theorem 3.5, we summarize them in the graphical overview Figure 3.1.

Theorem 3.7 (Lower Bound). *Let $g : (0, \infty) \to (0, \infty)$ be a continuous decreasing function that fulfills one of the Assumptions 3.4 (a) or (b). Let Λ_g be as in* (1.39)*. Then the following statements are true: If*

$$\int_0^\infty u^{-1}\Lambda_g(u)\ \mathrm{d}u < \infty\,, \tag{3.5}$$

then there exists a constant $c > 0$ such that $\mathbb{P}$-a.s. for n large enough $\lambda_1^{(n)} \geq cg(n)$. If, on the other hand, Condition (3.5) *does not hold, then $\mathbb{P}$-a.s. $\liminf_{n\to\infty} \frac{\lambda_1^{(n)}}{g(n)} = 0$.*

We prove the first part of this theorem in Section 3.3. The second part, i.e., where Condition (3.5) does not hold, is covered in Section 3.2.

Similarly as for Theorem 3.5, we obtain the following corollary. As before, its proof is immediate and therefore we omit it.

Corollary 3.8. *Let $\delta > 0$. If F varies regularly at zero with index $\gamma \in (0, 1/4]$, then $\mathbb{P}$-a.s. for n large enough the function $g(n) = n^{-\frac{1}{2\gamma}-\delta}$ is an asymptotic lower bound for $\lambda_1^{(n)}$. If even $F(a) = a^\gamma$ for $a \in [0, 1]$, then the lower bound can be further improved. For example $g(n) = n^{-\frac{1}{2\gamma}}(\log n)^{-\frac{1}{2d\gamma}-\delta}$ is an asymptotic lower bound in this case.*

Furthermore, if F varies regularly at zero with index $\gamma > 1/4$, then there exists $c > 0$ such that cn^{-2} is an asymptotic lower bound for $\lambda_1^{(n)}$. Then $\mathcal{E}^\omega(f) \geq cn^2\|f\|_2$ for all $f \in \ell^2(B_n)$, which is a Poincaré inequality for functions with bounded support.

Note that for i.i.d. conductances with finite expectation of $w^{-1/4}$ the Poincaré inequality for functions with bounded support is also a consequence of [ADS16, Proposition 2.4] (with $q = d/2$, η a step function and ν_ω replaced a $\tilde{\nu}_\omega$ which for each neighbor sums over the optimal detour from the $2d$ independent paths in Figure 2 of [ADS16]).

When we assume that F is bijective near zero and set $g(u) = F^{-1}(u^{-1/2})$, then Theorems 3.5 (ii) and 3.7 directly imply the following corollary. Its proof is immediate once we have observed that $\Lambda_g(u)$ is constant in u.

Corollary 3.9. *Assume that there exists $v > 0$ such that $F\colon [0, v) \to F([0, v))$ is bijective and that the function $u \mapsto u^2F^{-1}\big(u^{-\frac{1}{2}}\big)$ converges monotonically to zero. Then*

$$\liminf_{n\to\infty} \frac{\lambda_1^{(n)}}{F^{-1}\Big(n^{-\frac{1}{2}}\Big)} = 0 \quad \textit{and} \quad \limsup_{n\to\infty} \frac{\lambda_1^{(n)}}{F^{-1}\Big(n^{-\frac{1}{2}}\Big)} = \infty \qquad \mathbb{P}\textit{-a.s.} \tag{3.6}$$

We comment on this behavior in Remark 3.18 in Section 3.2.

Note that in the special case where there exists $\gamma \in (0, 1/4)$ such that the law $\mathbb{P}$ of the conductances fulfills $\mathbb{P}[\omega \leq a] = a^\gamma$ for $a \in [0, 1]$, Corollary 3.9 implies that

$$\liminf_{n\to\infty} n^{\frac{1}{2\gamma}}\lambda_1^{(n)} = 0 \quad \text{and} \quad \limsup_{n\to\infty} n^{\frac{1}{2\gamma}}\lambda_1^{(n)} = \infty \qquad \mathbb{P}\text{-a.s.} \tag{3.7}$$

Remark 3.10 (Constant speed). *If the conductances are bounded from above, we conjecture that, qualitatively, the results below should also hold for the constant-speed random conductance model, i.e., where the Laplacian is given by*

$$(\mathcal{L}^\omega_{\mathrm{cs}}f)(x) = \pi_x^{-1} \sum_{y\colon |x-y|_1=1} \omega_{xy}(f(y) - f(x)) \qquad (x \in \mathbb{Z}^d, f \in \ell^2(\mathbb{Z}^d))\,.$$

In this case, the critical exponent γ_{c}^π should be $\frac{1}{8}\frac{d}{d-1/2}$ (cf. [BKM15, Theorem 1.8 (1)]). Further, the typical trapping structures are not single sites but *pairs of sites (cf. [ADS16, Figure 1]). In a similar way as we adapt the proof techniques of [BKM15] for the variable-speed case, this should be possible for the constant-speed model. However, the proofs become much more technical.*

3.1.2 Localization of the principal Dirichlet eigenvector

Now we show that $\mathbb{P}$-a.s. as n tends to infinity, the principal Dirichlet eigenvector $\psi_1^{(n)}$ localizes in the sequence of sites $(z_n)_{n\in\mathbb{N}}$ that minimize π over the box B_n. Since we assume that the distribution function F is continuous, this sequence is $\mathbb{P}$-a.s. uniquely defined. Note that by virtue of Remark 3.3 we can assume without loss of generality that $\psi_1^{(n)}$ is non-negative.

Further, since the distribution function F is continuous, for each $a \in [0,1)$ there exists $s \geq 0$ such that $F(s) = a$. For what follows, we thus define the function g as

$$g : [0,\infty) \to [0,\infty),\ u \mapsto \sup\left\{s \geq 0\colon F(s) = u^{-1/2}\right\}. \tag{3.8}$$

Assumption 3.11. *Let F be continuous and vary regularly at zero with index $\gamma \in [0, 1/4)$. Assume that there exists $a^* > 0$ such that $F(ab) \geq bF(a)$ for all $a \leq a^*$ and all $0 \leq b \leq 1$. In the case where $\gamma = 0$, we assume additionally that there exists $\varepsilon_1 \in (0,1)$ such that the product $n^{2+\varepsilon_1} g(n)$ converges monotonically to zero as n grows to infinity.*

Remark 3.12. *In the case where $\gamma > 0$, it follows that $(1/F(1/s))^2$ varies regularly at infinity with index 2γ. Further, $(1/F(1/s))^2$ diverges as $s \to \infty$. It follows that $1/g(u) = \inf\left\{s \geq 0\colon (1/F(1/s))^2 = u\right\}$ varies regularly at infinity with index $1/(2\gamma)$ by virtue of [Res87, Prop. 0.8(v)] and thus g varies regularly at infinity with index $-1/(2\gamma)$. Since in addition $\gamma < 1/4$, there exists $\varepsilon_1 \in (0,1)$ such that $-1/(2\gamma) < -(2+\varepsilon_1)$.*

Theorem 3.13 (Localization of the principal Dirichlet eigenvector). *Let the distribution function F be such that Assumption 3.11 holds. For $n \in \mathbb{N}$ let z_n be the site that minimizes π over B_n. Then $\mathbb{P}$-a.s. the mass of the principal Dirichlet eigenvector $\psi_1^{(n)}$ with zero Dirichlet conditions outside the box B_n increasingly concentrates in the site z_n. More precisely, $\mathbb{P}$-a.s. for n large enough*

$$\psi_1^{(n)}(z_n)^2 \geq 1 - n^{-\varepsilon_1/4}, \tag{3.9}$$

where for $\gamma > 0$ the value of $\varepsilon_1 \in (0,1)$ is chosen such that $1/(2\gamma) > 2 + \varepsilon_1$.

We prove this theorem in Section 3.7.4.

Remark 3.14 (Dimension one). *Note that we cannot expect that a result like Theorem 3.13 holds in dimension one. This is because in dimension one, the probabilistic cost to generate a hardly reachable area is independent of the area's diameter.*

As a consequence of Theorem 3.13 we have the following corollary, which we prove in Section 3.7.5, as well as the weak convergence of the principal eigenvalue, see Section 3.1.4. Similar results hold for the higher order eigenvalues and -vectors, see the next section.

Corollary 3.15. *Assume that the distribution function F fulfills the conditions of Assumption 3.11. Then the principal Dirichlet eigenvalue $\lambda_1^{(n)}$ $\mathbb{P}$-a.s. behaves like $\min_{x\in B_n}\pi_x$ for large n, i.e.,*

$$\mathbb{P}\left[\lim_{n\to\infty}\frac{\lambda_1^{(n)}}{\min_{x\in B_n}\pi_x}=1\right]=1\,. \tag{3.10}$$

3.1.3 Higher order eigenvalues and -vectors

After we have understood the base case $k=1$, we can move to the higher order eigenvalues and eigenvectors. In the main theorem of this section, we see that the kth Dirichlet eigenvector $\psi_k^{(n)}$ increasingly concentrates in the location $z_{(k,n)}$ of the kth minimum π_{k,B_n} of the field $\{\pi_x\colon x\in B_n\}$, where the local speed measure π and its order statistics are defined in Definition 3.1. Likewise, the kth Dirichlet eigenvalue asymptotically behaves like the kth minimum of π_{k,B_n}.

Theorem 3.16. *Let $k\in\mathbb{N}$. If Assumption 3.11 holds, then the kth Dirichlet eigenvalue $\lambda_k^{(n)}$ with zero Dirichlet conditions outside the box B_n fulfills*

$$\mathbb{P}\left[\lim_{n\to\infty}\frac{\lambda_k^{(n)}}{\pi_{k,B_n}}=1\right]=1 \tag{3.11}$$

and the mass of the kth Dirichlet eigenvector $\psi_k^{(n)}$ asymptotically concentrates in the site $z_{(k,n)}$. More precisely, if $\varepsilon_1>0$ is as in Assumption 3.11 or Remark 3.12, then $\mathbb{P}$-a.s. for n large enough

$$1-n^{-\varepsilon/8}\le\frac{\lambda_k^{(n)}}{\pi_{k,B_n}}\le 1\qquad\textit{for all } \varepsilon<\varepsilon_1 \tag{3.12}$$

and

$$\psi_k^{(n)}\big(z_{(k,n)}\big)\ge\sqrt{1-n^{-\varepsilon/4}}\qquad\textit{for all } \varepsilon<\varepsilon_1\,. \tag{3.13}$$

We prove this theorem in Section 3.9.

3.1.4 Weak convergence of eigenvalues

Similar to [Fag12, p. 7], we define

$$h\colon (0,\infty) \to (0,\infty),\ u \mapsto \inf\left\{s\colon \frac{1}{F_\pi(1/s)} = u\right\}. \tag{3.14}$$

If F varies regularly at zero with index $\gamma > 0$, then by virtue of Lemma 3.47, it follows that F_π varies regularly at zero with index $2d\gamma$. It thus follows by virtue of [Res87, Proposition 0.8(v)] that h varies regularly at infinity with index $1/(2d\gamma)$. Therefore there exists a function L^* that varies slowly at infinity such that

$$h(|B_n|) = n^{\frac{1}{2\gamma}} L^*(n)\,. \tag{3.15}$$

Corollary 3.17. *Assume that F fulfills Assumption 3.11 with $\gamma > 0$ and let L^* be as in* (3.15)*. Let $k \in \mathbb{N}$. Then as n tends to infinity, the product $L^*(n) n^{\frac{1}{2\gamma}} \lambda_k^{(n)}$ converges in distribution to a non-degenerate random variable. More precisely,*

$$\lim_{n\to\infty} \mathbb{P}\Big[L^*(n) n^{\frac{1}{2\gamma}} \lambda_k^{(n)} > \zeta\Big] = \exp\big(-\zeta^{2d\gamma}\big) \sum_{j=0}^{k-1} \frac{\zeta^{2d\gamma j}}{j!} \qquad \textit{for all } \zeta \in [0,\infty)\,. \tag{3.16}$$

We prove this corollary in Section 3.10.

3.2 Survey on proofs for upper bounds.

Let us consider the variational formula (1.21). The equation implies that for any real-valued test function $f \in \ell^2(\mathbb{Z}^d)$ with $\operatorname{supp} f \subseteq B_n$ and $\|f\|_2 = 1$ we can estimate

$$\lambda_1^{(n)} \le \langle f, -\mathcal{L}^\omega f\rangle = \frac{1}{2} \sum_{x\in\mathbb{Z}^d} \sum_{y\colon y\sim x} \omega_{xy} (f(x) - f(y))^2\,.$$

Suppose that z_n is a random site that minimizes π (see (1.13)) in B_n. Now we choose the function f such that its whole mass is concentrated in the site $z_n \in B_n$, i.e., $f = \delta_{z_n}$. When we insert this into the variational formula (1.21), then we obtain that

$$\lambda_1^{(n)} \le \min_{x\in B_n} \pi_x \le 2d \min_{x\in B_n} \max_{y\colon x\sim y} \omega_{xy}\,. \tag{3.17}$$

It remains to find conditions under which the above right-hand side can be bounded from above by a decreasing function $g(n)$. As we have already mentioned before, a quantity which carries this information, is the function Λ_g defined in (1.39), as we see in the two following proofs.

Proof of Theorem 3.5 (i). Condition (3.3) together with Lemma 3.24 implies that $\mathbb{P}$-a.s. for n large enough there exists a site $z_n \in B_n$ such that $\max_{y\colon y\sim z_n} \omega_{z_n y} \le g(n)$. Choose the test function $f_n = \delta_{z_n}$ and insert it into the variational formula (1.21). The claim follows. $\square$

Proof of Theorem 3.7 if Condition (3.5) **fails.** If Condition (3.5) fails, then we have $c\int_0^\infty u^{-1}\Lambda_g(u)\,\mathrm{d}u = \infty$ for any $c > 0$. For a site $z \in \mathbb{Z}^d$ let $z + \mathfrak{N} = \{e \in \mathfrak{E}_d\colon z \in e\}$ be the set of edges incident to the site z and note that $|z+\mathfrak{N}| = 2d$ for all $z \in \mathbb{Z}^d$. A substitution of variables and Lemma 3.19 imply that for any $c > 0$ the following event occurs $\mathbb{P}$-a.s. infinitely often as $n \to \infty$: There exists a site $z_n \in B_{n+1}$ such that all edges in $z_n + \mathfrak{N}$ have conductance smaller than or equal to $g(cn)$.

Every time this event occurs, we choose the test function $f_n = \delta_{z_n}$ (as in the proof of Theorem 3.5 (i)), insert it into the variational formula (1.21) and immediately obtain that

$$\liminf_{n\to\infty} \frac{\lambda_1^{(n)}}{g(cn)} \le 2d \quad \mathbb{P}\text{-a.s. for any } c > 0\,.$$

We now show that this implies the claim. Let $c > 1$ and recall that we have assumed that one of the Assumptions 3.4 (a) or (b) is true. In any case it follows that eventually $2g(n) \ge c^2 g(cn)$. It follows that $\mathbb{P}$-a.s. $\liminf_{n\to\infty} \frac{\lambda_1^{(n)}}{g(n)} \le 4dc^{-2}$. This holds for any $c > 1$, implying that $\mathbb{P}$-a.s. $\liminf_{n\to\infty} \lambda_1^{(n)}/g(n) = 0$. $\square$

Remark 3.18. *Now we can intuitively understand the result of Corollary 3.9: For the choice $g(u) = F^{-1}(u^{-1/2})$, the function Λ_g is constant one. Therefore for every $c > 0$ $\mathbb{P}$-a.s. there exists an infinite subsequence n_k where the box B_{n_k} contains a $(cg(n_k))$-trap. However, as we will see in Section 3.3, $\mathbb{P}$-a.s. there also exists an infinite subsequence n'_k where the box $B_{n'_k}$ does not contain a sufficiently good trap. It follows that the asymptotics of $\lambda_1^{(n)}$ fluctuate around the asymptotics of $F^{-1}(u^{-1/2})$.*

3.3 Survey on proofs for lower bounds

In what follows we give a survey on the proofs of Theorem 3.5 (ii) as well as Theorem 3.7 if Condition (3.5) holds. We recall the relation in (1.41), i.e.,

$$\mathbb{E}\Big[g^{-1}(\max\{\omega_1,\ldots,\omega_{2d}\})^d\Big] = d\int_0^\infty u^{-1}\Lambda_g(u)\,\mathrm{d}u\,, \tag{3.18}$$

and use the line of thought of that paragraph. The arguments of that section are made rigorous in several auxiliary lemmas, which we present in the subsequent sections.

Proof of Theorem 3.7 if Condition (3.5) **holds..** By virtue of Lemma 3.19 with $\mathfrak{A} = \{e \in \mathfrak{E}_d \colon 0 \in e\}$ it follows that $\mathbb{P}$-a.s. there exists $n_1^* \in \mathbb{N}$ such that for all $n \geq n_1^*$ all sites $z \in B_n$ have an incident edge with conductance greater than $g(n)$.

Further, if we understand the expression $g^{-1}(v)$ as $\inf\{u \colon g(u) = v\}$, then

$$\Lambda_g(u) \leq u^d \, \mathbb{P}\big[g^{-1}(\max\{\omega_1, \ldots, \omega_{2d}\}) \geq u\big]$$

and Markov's inequality implies that $\Lambda_g(u) \leq \mathbb{E}\Big[g^{-1}(\max\{\omega_1, \ldots, \omega_{2d}\})^d\Big]$ for all $u \in [0, \infty)$. By virtue of (3.18) and Condition (3.5), it follows that Λ_g is bounded from above. Therefore Corollary 3.22 (with $m = 2d$ and $\kappa = d$) implies that there exists $\varepsilon > 0$ such that $\mathbb{P}$-a.s. there exists $n_2^* \in \mathbb{N}$ such that for all $n \geq n_2^*$ and for all $z \in B_{n+3d}$ the box $B_{3d}(z)$ contains at most $3d - 1$ edges with conductance less than or equal to $g(n^{1-\epsilon})$. Now we choose ξ small enough such that $\mathbb{P}$-a.s. there exists n_3^* such that for all $n \geq n_3^*$ Assumptions (ii), (iii) and (iv) of Proposition 3.38 are fulfilled. This is possible by virtue of (3.34) and Lemmas 3.27 and 3.28. Then the claim follows by virtue of Proposition 3.39 with $n_k = k + \max(n_1^*, n_2^*, n_3^*)$. □

Proof of Theorem 3.5 (ii). Let $\overline{c} > 1$. In any case of 3.4 (a) or (b'), we observe that for n large enough $\overline{c}g(n) \leq g\big(\overline{c}^{-1/2}n\big)$. It follows that the quotient $\Lambda_{\overline{c}g(u)}/\log\log u$ is bounded as u tends to infinity. Thus we know the following by Corollary 3.23: There exists $\varepsilon > 0$ such that $\mathbb{P}$-a.s. for n large enough, there are at most $2d$ edges in any subbox $B_{3d}(z) \subset B_{n+3d}$ with conductance smaller than or equal to $\overline{c}g(n^{1-\epsilon})$. This implies Assumption (i) of Proposition 3.38 with $\overline{c}g$ instead of g. Now we choose ξ small enough such that $\mathbb{P}$-a.s. for n large enough Assumptions (ii), (iii) and (iv) of Proposition 3.38 are fulfilled. This is possible by virtue of (3.34) and Lemmas 3.27 and 3.28.

Further, Condition (3.4) together with Lemma 3.25 implies that $\mathbb{P}$-a.s. as the box size n grows to infinity, there exists a random subsequence $n' = n'(\boldsymbol{\omega})$ along which each site $z \in B_{n'}$ has at least one incident edge e such that $\omega(e) > \overline{c}g(n')$. It follows that we can apply Proposition 3.39 with $\overline{c}g$ instead of g and obtain that there exists $C > 0$ (independent of $\overline{c}$ since we have assumed 3.4 (a) or (b')) such that along the random subsequence n'_k and for k large enough

$$\mathcal{E}^\omega(f) \geq C\overline{c}g(n'_k)\|f\|_2^2 \qquad \text{for any } f \colon \mathbb{Z}^d \to \mathbb{R} \text{ with supp } f \subseteq B_{n'_k}.$$

Since this holds for any $\overline{c} > 1$, this implies the claim. □

3.4 Borel-Cantelli arguments

In this section we always assume that the dimension $d \geq 2$ and that the conductances are i.i.d. with law $\mathbb{P}$. We further let $g\colon (0,\infty) \to (0,\infty)$ be a function that decreases monotonically to zero. Moreover, we use the following abbreviations: For $\alpha > 0$ and an edge set $\mathfrak{A} \subseteq \mathfrak{E}_d$ we define the event

$$J_\alpha(\mathfrak{A}) = \{\exists e \in \mathfrak{A}\colon \omega(e) > \alpha\}\,. \tag{3.19}$$

For a set $A \subset \mathbb{Z}^d$ we define $\mathfrak{E}(A)$ to be the set of edges that connect a site in A with a neighbor in positive axes direction (i.e., right, above, in front, etc.), i.e.,

$$\mathfrak{E}(A) = \{\{x,y\} \in \mathfrak{E}_d\colon x \in A \text{ and } \exists j \in \{1,\dots,d\} \text{ such that } y = x + \boldsymbol{e}_j\}\,,$$

where $\{\boldsymbol{e}_j\}$ is the canonical basis of $\mathbb{Z}^d$. For $\mathfrak{A} \subseteq \mathfrak{E}_d$ we write $z + \mathfrak{A}$ for the translation of $\mathfrak{A}$ by $z \in \mathbb{Z}^d$, i.e., for $x, y, z \in \mathbb{Z}^d$ with $\{x,y\} \in \mathfrak{E}_d$ we define $z + \{x,y\} = \{x+z, y+z\}$.

Further, for a sequence of events $(E_n)_{n\in\mathbb{N}}$ we recall the definitions

$$\liminf_{n\to\infty} E_n = \bigcup_{n=1}^\infty \left(\bigcap_{k=n}^\infty E_n\right) \text{ and } \limsup_{n\to\infty} E_n = \bigcap_{n=1}^\infty \left(\bigcup_{k=n}^\infty E_n\right).$$

Lemma 3.19. *If $b \in \mathbb{N}$ and $\mathfrak{A} \subseteq \mathfrak{E}(B_b)$ is an edge set with $|\mathfrak{A}| = m$, then*

$$\mathbb{P}\left[\liminf_{n\to\infty} \bigcap_{z\in B_{n+b}} J_{g(n)}(z+\mathfrak{A})\right] = \begin{cases} 1, \text{ if } \int_0^\infty u^{d-1}\mathbb{P}[\omega \leq g(u)]^m \,\mathrm{d}u < \infty, & (3.20\text{a}) \\ 0, \text{ otherwise.} & (3.20\text{b}) \end{cases}$$

This means that if and only if the integral $\int_0^\infty u^{d-1}\mathbb{P}[\omega \leq g(u)]^m \,\mathrm{d}u$ is finite, then $\mathbb{P}$-a.s. for n large enough for all sites $z \in B_{n+b}$ the edge set $z + \mathfrak{A}$ contains a conductance greater than $g(n)$. Otherwise the complement of this event occurs for infinitely many n.

Remark 3.20. *The result of Lemma 3.19 as well as the proof are generalizations of the considerations of Cox and Durrett [CD81] and Kesten [Kes03, p. 108] (there, $m = 2d$ and $g(n) = n^{-1}$). For the sake of completeness, we included the proofs here.*

Proof of Lemma 3.19. For (3.20a): We first show that

$$0 = 1 - \mathbb{P}\left[\liminf_{|z|_\infty\to\infty} J_{g(|z|_\infty - b)}(z+\mathfrak{A})\right] = \mathbb{P}\left[\limsup_{|z|_\infty\to\infty} \left(J_{g(|z|_\infty - b)}(z+\mathfrak{A})\right)^{\mathrm{c}}\right]. \tag{3.21}$$

We achieve this by applying the first Borel-Cantelli lemma, i.e., we have to estimate

$$\sum_{z\in\mathbb{Z}^d\setminus B_b} \mathbb{P}\big[\big(J_{g(|z|_\infty-b)}(z+\mathfrak{A})\big)^{\mathrm{c}}\big] = \sum_{k=b+1}^{\infty}\sum_{|z|_\infty=k} \mathbb{P}[\omega \le g(|z|_\infty - b)]^m$$
$$\le 2d\sum_{k=b+1}^{\infty}(2k+1)^{d-1}\mathbb{P}[\omega \le g(k-b)]^m\,.$$

Since $g(\cdot)$ is monotonically decreasing, $\mathbb{P}[\omega \le g(\cdot)]$ is monotonically decreasing as well. Further, there exists an index k_b such that $k-b \ge 2^{-1}(k+1)$ for all $k \ge k_b$. It follows that there exists $C<\infty$ such that

$$\sum_{z\in\mathbb{Z}^d\setminus B_b} \mathbb{P}\big[\big(J_{g(|z|_\infty-b)}(z+\mathfrak{A})\big)^{\mathrm{c}}\big]$$
$$\le C + 4^d d\sum_{k=b+1}^{\infty}(2^{-1}(k+1))^{d-1}\mathbb{P}\big[\omega \le g\big(2^{-1}(k+1)\big)\big]^m\,.$$

This implies that there exists $c<\infty$ such that the left-hand side is bounded from above by $C + c\int_0^\infty u^{d-1}\mathbb{P}[\omega \le g(u)]^m\,\mathrm{d}u$, which is finite by assumption. The claim (3.21) follows from the first Borel-Cantelli lemma.

To arrive at the claim of the lemma, we observe that (3.21) implies that $\mathbb{P}$-a.s. there exists $n^*\in\mathbb{N}$ such that for all $|z|_\infty \ge n^*$ the set $z+\mathfrak{A}$ contains at least one conductance greater than $g(|z|_\infty - b)$. If $n>n^*$ and $z\in B_{n+b}\setminus B_{n^*}$, i.e., $|z|_\infty \in (n^*, n+b]$, this means that $z+\mathfrak{A}$ contains at least one conductance greater than $g(n)$ (recall that g is monotonically decreasing). Since n^* is finite and g decreases monotonically to zero, it also follows that there exists a finite $n' \ge n^*$ such that for all edges $e\in\mathfrak{E}(B_{n^*+1})$ we have $g(n')<\omega(e)$. Thus, $\bigcap_{z\in B_{n+b}} J_{g(n)}(z+\mathfrak{A})$ is true $\mathbb{P}$-a.s. for n large enough.

For (3.20b): Let $\int_0^\infty u^{d-1}\mathbb{P}[\omega\le g(u)]^m\,\mathrm{d}u = \infty$. We want to show that this implies

$$\mathbb{P}\left[\liminf_{n\to\infty}\bigcap_{z\in B_{n+b}} J_{g(n)}(z+\mathfrak{A})\right] = 0\,. \tag{3.22}$$

Let us define the set $A_b = (2b+1)\mathbb{Z}^d$. It suffices to prove the claim (3.22) for the intersection over $z\in B_{n+b}\cap A_b$, which in turn follows by the second Borel-Cantelli lemma if

$$\sum_{z\in A_b}\mathbb{P}\big[\big(J_{g(|z|+b)}(z+\mathfrak{A})\big)^{\mathrm{c}}\big] = \infty\,,$$

since the events $\big\{J_{g(|z|+b)}(z+\mathfrak{A})\big\}_{z\in A_b}$ are independent. To prove that the above sum diverges, we observe that there exists a constant $C>0$ such that

$$\sum_{z\in A_b} \mathbb{P}\big[\big(J_{g(|z|+b)}(z+\mathfrak{A})\big)^{\mathrm{c}}\big]$$
$$\geq 2d(2b+1)^d \sum_{k=1}^{\infty} (2k-1)^{d-1}\mathbb{P}[\omega \leq g((2b+1)k+b)]^m$$
$$\geq C \int_0^{\infty} u^{d-1}\mathbb{P}[\omega \leq g(u)]^m \,\mathrm{d}u\,.$$

By the assumption $\int_0^\infty u^{d-1}\mathbb{P}[\omega \leq g(u)]^m \,\mathrm{d}u = \infty$, the sum diverges. □

Corollary 3.21 (of Lemma 3.19). *Let $b \in \mathbb{N}$ and $m \leq |\mathfrak{E}(B_b)|$. Then the following equivalence holds:*

$$\int_0^{\infty} u^{d-1}\mathbb{P}[\omega \leq g(u)]^m \,\mathrm{d}u < \infty$$
$$\Leftrightarrow\; \mathbb{P}\left[\liminf_{n\to\infty} \bigcap_{\substack{\mathfrak{A}\subseteq\mathfrak{E}(B_b),\\ |\mathfrak{A}|\geq m}} \;\bigcap_{z\in B_{n+b}} J_{g(n)}(z+\mathfrak{A})\right] = 1\,. \tag{3.23}$$

Proof of Corollary 3.21. For "$\Leftarrow$", we apply Lemma 3.19 for an arbitrary $\mathfrak{A} \subseteq \mathfrak{E}(B_b)$ with $|\mathfrak{A}| = m$. For "$\Rightarrow$", note that since B_b is finite, the intersection over the edge sets $\mathfrak{A}$ on the right-hand side of (3.23) runs over finitely many events. By virtue of Lemma 3.19 the claim holds for each of these events and therefore also for the finite intersection. □

Corollary 3.22 (of Corollary 3.21). *Let $b, m, \kappa \in \mathbb{N}$ with $m < |\mathfrak{E}(B_b)|$. If $u^d\mathbb{P}[\omega \leq g(u)]^m$ is bounded from above, then*

$$\mathbb{P}\left[\liminf_{n\to\infty} \bigcap_{\substack{\mathfrak{A}\subseteq\mathfrak{E}(B_b),\\ |\mathfrak{A}|\geq m+\kappa}} \;\bigcap_{z\in B_{n+b}} J_{g(n^{1-\epsilon})}(z+\mathfrak{A})\right] = 1 \quad \textit{for all } \varepsilon \in [0, \kappa(m+\kappa)^{-1})\,. \tag{3.24}$$

Proof. We show that the integral $\int_0^\infty v^{d-1}\mathbb{P}\big[\omega \leq g\big(v^{1-\varepsilon}\big)\big]^{m+\kappa} \,\mathrm{d}v$ is finite and then we apply Corollary 3.21.

The change of variable $v^{1-\varepsilon} = u$ yields

$$\int_0^{\infty} v^{d-1}\mathbb{P}\big[\omega \leq g\big(v^{1-\varepsilon}\big)\big]^{m+\kappa} \,\mathrm{d}v$$
$$= (1-\varepsilon)^{-1} \int_0^{\infty} u^{d(1-\varepsilon)^{-1}-1}\mathbb{P}[\omega \leq g(u)]^{m+\kappa} \,\mathrm{d}u\,.$$

Now we consider that

$$u^{d(1-\varepsilon)^{-1}-1}\mathbb{P}[\omega \le g(u)]^{m+\kappa} = u^{d\left((1-\varepsilon)^{-1}-1-\frac{\kappa}{m}\right)-1}\left(u^d\mathbb{P}[\omega \le g(u)]^m\right)^{1+\frac{\kappa}{m}}.$$

Since both $u^d\mathbb{P}[\omega \le g(u)]^m$ and $\mathbb{P}[\omega \le g(u)]^m$ are bounded from above, we obtain that

$$\int_0^\infty u^{d(1-\varepsilon)^{-1}-1}\mathbb{P}[\omega \le g(u)]^{m+\kappa}\,\mathrm{d}u \le \int_0^{u_1} u^{d(1-\varepsilon)^{-1}-1}\,\mathrm{d}u + C\int_{u_1}^\infty u^{d\left((1-\varepsilon)^{-1}-1-\frac{\kappa}{m}\right)-1}\,\mathrm{d}u < \infty$$

for any $u_1 \in (0,\infty)$ and a suitable $C < \infty$. Since $\varepsilon \in [0,1)$ and $d \ge 2$, the first integral on the right-hand side is finite. Further, since $\varepsilon < \kappa(m+\kappa)^{-1}$, the second integral on the right-hand side is finite as well. □

For the next three results, we define Λ_g as in (1.39).

Corollary 3.23 (of Corollary 3.21). *Let $\Lambda_g(u)/\log\log u$ be bounded from above for u large enough and let $b \ge 2$. Then*

$$\mathbb{P}\left[\liminf_{n\to\infty} \bigcap_{\substack{\mathfrak{A}\subseteq\mathfrak{E}(B_b),\\ |\mathfrak{A}|\ge 2d+1}} \bigcap_{z\in B_{n+b}} J_{g(n^{1-\epsilon})}(z+\mathfrak{A})\right] = 1 \qquad \textit{for all } \varepsilon \in \left(0,(2d+1)^{-1}\right). \tag{3.25}$$

Proof. We show that the integral in (3.23) is finite for $m = 2d+1$ and $g(u^{1-\epsilon})$ instead of $g(u)$. The assumption on the function Λ_g implies that there exists $C < \infty$ such that for u large enough

$$\frac{u^{\left(d+\frac{1}{2}\right)(1-\epsilon)}\mathbb{P}\left[\omega \le g(u^{1-\epsilon})\right]^{2d+1}}{\left(\log\log u^{1-\epsilon}\right)^{1+\frac{1}{2d}}} < C \quad \text{for all } \epsilon \in (0,1).$$

It follows that for u large enough

$$u^d\mathbb{P}\left[\omega \le g(u^{1-\epsilon})\right]^{2d+1} \le Cu^{-\frac{1}{2}+\epsilon\left(d+\frac{1}{2}\right)}\left(\log\log u^{1-\epsilon}\right)^{1+\frac{1}{2d}}.$$

This implies that the integral $\int_0^\infty u^{d-1}\mathbb{P}\left[\omega \le g(u^{1-\epsilon})\right]^{2d+1}\,\mathrm{d}u$ is finite for all $\epsilon < (2d+1)^{-1}$. The claim follows by virtue of Corollary 3.21. □

For the next lemma we need the following definition: For $i,k \in \mathbb{N}$ with $i \le k$ we define $A_{i,k}$ as the set which "has residue class i modulo k", i.e.,

$$A_{i,k} = \left\{z = (z_1,\ldots,z_d) \in \mathbb{Z}^d \colon z_1+\ldots+z_d \equiv i \bmod k\right\}. \tag{3.26}$$

Note that for fixed k, the sets $A_{i,k}$ are disjoint and eventually the sets $B_n \cap A_{i,k}$ have cardinality greater than $(2n)^d/k$. For $k = 2$ we especially define the even lattice as the set

$$A_{\mathrm{e}} = \left\{ z \in \mathbb{Z}^d \colon |z|_1 \equiv 0 \bmod 2 \right\}. \tag{3.27}$$

Accordingly, the odd lattice is $A_{\mathrm{o}} = \mathbb{Z}^d \setminus A_{\mathrm{e}}$.

Lemma 3.24. *Let $k \in \mathbb{N}$ with $k \geq 2$. Further, let $\mathfrak{N} = \{e \in \mathfrak{E}_d \colon 0 \in e\}$. Then the following implication is true: If there exists $\varepsilon > 0$ and $n^* \in \mathbb{N}$ such that*

$$\frac{\Lambda_g(n)}{\log\log n} \geq k + \varepsilon \textit{ for all } n \geq n^*,$$

$$\textit{then } \mathbb{P}\left[\limsup_{n\to\infty}\left(\bigcap_{z \in B_n \cap A} J_{g(n)}(z + \mathfrak{N})\right)\right] = 0$$

for any $A \in \{A_{1,k}, \dots, A_{k,k}\}$, i.e., $\mathbb{P}$-a.s. for n large enough there exists a site $z_n \in B_n \cap A$ that is completely surrounded by edges with conductance less than or equal to $g(n)$. It follows that $\mathbb{P}$-a.s. for n large enough there exist k distinct sites $z_{(1,n)}, \dots, z_{(k,n)} \in B_n$ that all fulfill $\pi_{z_{(i,n)}} \leq 2dg(n)$ $(i \leq k)$.

Proof. We first prove the claim for the subsequence $n_j = 2^j$ with $j \in \mathbb{N}$ and with $g(2n)$ instead of $g(n)$. Then we show how to infer the claim along the whole sequence $n \in \mathbb{N}$.

For the first part, let $\omega_1, \dots, \omega_{2d}$ be $2d$ independent copies of w. Since $k \geq 2$, it follows that for any $\alpha > 0$ and any fixed i the events $\{J_\alpha(z + \mathfrak{N})\}_{z \in A_{i,k}}$ are independent and thus we can estimate

$$\begin{aligned}
\mathbb{P}\left[\bigcap_{z \in B_{n_j} \cap A_{i,k}} J_{g(2n_j)}(z + \mathfrak{N})\right] &= \mathbb{P}[\max\{\omega_1, \dots, \omega_{2d}\} > g(2n_j)]^{\left|B_{n_j} \cap A_{i,k}\right|} \\
&\leq \left(1 - \mathbb{P}[\omega \leq g(2n_j)]^{2d}\right)^{(2n_j)^d/k} \leq \exp\left(-\frac{1}{k}(2n_j)^d \mathbb{P}[\omega \leq g(2n_j)]^{2d}\right).
\end{aligned}$$

The assumption on Λ_g implies that the right-hand side is summable along the sequence $n_j = 2^j$. Thus, it follows directly by the Borel-Cantelli lemma that the statement of this lemma holds along the subsequence n_j and with $g(2n_j)$ instead of $g(n_j)$.

To infer the claim of the lemma along the entire sequence, we define

$$M_n := \inf_{x \in B_n \cap A} \sup_{e \in x + \mathfrak{N}} \omega(e),$$

where $A \in \{A_{1,k}, \dots, A_{k,k}\}$.

Note that M_n is monotonically decreasing in n. By the first part of the proof we know that

$$\mathbb{P}\left[\liminf_{j\to\infty} \frac{M_{n_j}}{g(2n_j)} \le 1\right] = 1\,. \tag{3.28}$$

For $n \in \mathbb{N}$ we now choose j_n such that

$$2^{j_n} \le n \le 2^{j_n+1}\,.$$

Since g and $M_{(\,\cdot\,)}$ are both monotonically decreasing, this implies that $M_{2^{j_n}} \ge M_n$ and $g(n) \ge g\big(2^{j_n+1}\big)$. Thus, the claim follows by (3.28). $\square$

Lemma 3.25. *Let $\mathfrak{N}$ be as in Lemma 3.24. If the function $u \mapsto ug(u)$ decreases monotonically to zero or g varies regularly at infinity with index less than -1 and in any case*

$$\lim_{u\to\infty} \frac{\Lambda_g(u)}{\log\log u} = 0\,, \quad \textit{then} \quad \mathbb{P}\left[\limsup_{n\to\infty}\left(\bigcap_{z\in B_n} J_{cg(n)}(z+\mathfrak{N})\right)\right] = 1 \quad \forall c > 0\,. \tag{3.29}$$

Proof. For $A \subset \mathbb{Z}^d$, a fixed $c > 0$, and a fixed function g let us abbreviate

$$H_A^n = \bigcap_{z\in A} J_{cg(n)}(z+\mathfrak{N})\,.$$

Let us briefly outline the idea of the proof: It is sufficient to show that the claim is true along the subsequence $n_j = j^j$. First we show that

$$\sum_{j=1}^{\infty} \mathbb{P}\left[H_{B_{n_j}}^{n_j}\right] = \infty \tag{3.30}$$

which, since $H_{B_{n_j}}^{n_j} \subset H_{B_{n_j}\setminus B_{n_{j-1}+1}}^{n_j}$, implies that $\sum_{j=1}^{\infty} \mathbb{P}\left[H_{B_{n_j}\setminus B_{n_{j-1}+1}}^{n_j}\right] = \infty$. Note that since for $i, j \in \mathbb{N}$ with $i \ne j$ the intersection

$$\left(\bigcup_{z\in B_{n_j}\setminus B_{n_{j-1}+1}} z+\mathfrak{N}\right) \cap \left(\bigcup_{z\in B_{n_i}\setminus B_{n_{i-1}+1}} z+\mathfrak{N}\right) = \emptyset\,,$$

the events $\left\{H_{B_{n_j}\setminus B_{n_{j-1}+1}}^{n_j}\right\}_{j\ge 2}$ are independent. Thus, given (3.30), we can infer by the second Borel-Cantelli lemma that

$$\mathbb{P}\left[\limsup_{j\to\infty} H_{B_{n_j}\setminus B_{n_{j-1}+1}}^{n_j}\right] = 1\,. \tag{3.31}$$

Then we show that

$$\mathbb{P}\Big[\liminf_{j\to\infty} H^{n_j}_{B_{n_{j-1}+1}}\Big] = 1\,. \tag{3.32}$$

Since by definition

$$H^{n_j}_{B_{n_j}} = H^{n_j}_{B_{n_j}\setminus B_{n_{j-1}+1}} \cap H^{n_j}_{B_{n_{j-1}+1}}\,,$$

(3.32) together with (3.31) implies the claim of the lemma.

Let us start with the proof of (3.30). We note that for A_{e} and A_{o} as defined in (3.27) the FKG-inequality implies that

$$\mathbb{P}\Big[H^{n_j}_{B_{n_j}}\Big] = \mathbb{P}\Big[H^{n_j}_{A_{\mathrm{e}}\cap B_{n_j}} \cap H^{n_j}_{A_{\mathrm{o}}\cap B_{n_j}}\Big] \geq \mathbb{P}\Big[H^{n_j}_{A_{\mathrm{e}}\cap B_{n_j}}\Big]^2\,.$$

Then we recall that A_{e} was constructed such that $H^{n_j}_{A_{\mathrm{e}}\cap B_{n_j}}$ is the intersection of less than $(2n+1)^d$ i.i.d. subevents $\big\{J_{cg(n_j)}(z+\mathfrak{N})\big\}_{z\in A_{\mathrm{e}}\cap B_{n_j}}$, each with probability

$$\mathbb{P}\big[J_{cg(n_j)}(\mathfrak{N})\big] = 1 - \mathbb{P}[\omega \leq cg(n_j)]^{2d}\,.$$

Thus for j large enough, there exists $C < \infty$ such that

$$\begin{aligned}
\mathbb{P}\Big[H^{n_j}_{B_{n_j}}\Big] &\geq \Big(1 - \mathbb{P}[\omega \leq cg(n_j)]^{2d}\Big)^{2(2n_j+1)^d} \\
&= \bigg(\Big(1 - \mathbb{P}[\omega \leq cg(n_j)]^{2d}\Big)^{\mathbb{P}[\omega\leq cg(n_j)]^{-2d}}\bigg)^{2(2n_j+1)^d\mathbb{P}[\omega\leq cg(n_j)]^{2d}} \\
&\geq \exp\Big(-Cn_j^d\mathbb{P}[\omega \leq cg(n_j)]^{2d}\Big) = \exp(-C\Lambda_{cg}(n_j))\,. \qquad (3.33)
\end{aligned}$$

Now we explain why the assumptions on g and Λ_g imply that the right-hand side of (3.33) is not summable for any $c > 0$. If $c \leq 1$, then $\Lambda_{cg}(n) \leq \Lambda_g(n)$ for all $n \in \mathbb{N}$. It follows that for any $\varepsilon > 0$ there exists $j^* \in \mathbb{N}$ such that for all $j > j^*$ we have

$$\Lambda_{cg}(n_j) < \varepsilon(\log j + \log\log j) < 2\varepsilon \log j\,.$$

When we choose $\varepsilon < (2C)^{-1}$, then we see that the right-hand side of (3.33) is not summable. Let us now assume that $c > 1$. If $u \mapsto ug(u)$ decreases monotonically to zero, then $cg(n) \leq g(n/c)$ for all n. If g varies regularly at infinity with index less than -1, then for any $\tilde{c} > c$ and for n large enough $cg(n) \leq g(n/\tilde{c})$. This implies that for n large enough $\Lambda_{cg}(n) \leq \tilde{c}^d\Lambda_g(n/\tilde{c})$. Thus, by similar arguments as for the case $c \leq 1$, we obtain that the right-hand side of (3.33) is not summable. This concludes the argument for (3.30).

Let us proceed with the proof of (3.32). Note that for any $\epsilon > 0$ we have $n_{j+1} \geq h(n_j)$ with $h(u) = u(\log u)^{1-\epsilon}$. This is because for j large enough and any $\epsilon > 0$ we have

$$n_{j+1} = (j+1)\left(1+\frac{1}{j}\right)^j j^j \geq (j+1)n_j \geq n_j(\log n_j)^{1-\epsilon}.$$

Thus, (3.32) is a consequence of

$$\mathbb{P}\left[\liminf_{n\to\infty} \bigcap_{z\in B_{n+1}} J_{cg(h(n))}(z+\mathfrak{A})\right] = 1 \quad \forall c > 0\,.$$

By virtue of Lemma 3.19 we can thus verify (3.32) by showing that for all $c > 0$ the integral

$$\int_0^\infty u^{d-1}\mathbb{P}[\omega \leq cg(h(u))]^{2d}\,\mathrm{d}u < \infty\,.$$

To see that the integral is indeed finite, we consider the following: There exists a constant $C < \infty$ such that

$$\begin{aligned}
&\int_0^\infty u^{d-1}\mathbb{P}[\omega \leq cg(h(u))]^{2d}\,\mathrm{d}u \\
&\qquad\leq C + \int_2^\infty u^{-1}\left(\frac{u}{h(u)}\right)^d \left(h(u)^d\,\mathbb{P}[\omega \leq cg(h(u))]^{2d}\right)\,\mathrm{d}u \\
&\qquad= C + \int_2^\infty u^{-1}(\log u)^{-d(1-\epsilon)}\Lambda_{cg}(h(u))\,\mathrm{d}u\,.
\end{aligned}$$

Again, we distinguish the cases $c \leq 1$ and $c > 1$. If $c \leq 1$, then $cg \leq g$. If $c > 1$, then we observe, as before, that $cg(u) \leq g(u/c)$ for u large enough. Therefore the condition on Λ_g implies that also $\Lambda_{cg}(u)/\log\log u \to 0$ as u tends to infinity. Since h diverges, it follows that there exists $u^* < \infty$ such that $\Lambda_{cg}(h(u)) \leq \log\log h(u)$ for all $u \geq u^*$. Further, since on the interval $[2, u^*]$ the function $\Lambda_{cg}(h(\cdot))$ is bounded, the claim follows since $\int_{u^*}^\infty u^{-1}(\log u)^{-d(1-\epsilon)}\log\log h(u)\,\mathrm{d}u$ is finite. $\square$

3.5 Percolation results

In this section we adapt three standard percolation results that we need for the path arguments of the next section in order to establish the lower bound for the principal Dirichlet eigenvalue.

Let us consider the standard Bernoulli bond percolation on the graph $(\mathbb{Z}^d, \mathfrak{E}_d)$, i.e., we assume that the conductances are independent random variables with common law $\mathbb{P}$ such that an individual conductance is 1 with probability p and 0 otherwise. For an introduction to percolation we refer the reader to [Gri99]. As in the previous section, we call $\boldsymbol{\omega} = (\omega(e))_{e\in\mathfrak{E}_d} \in \{0,1\}^{\mathfrak{E}_d}$ an environment and we denote the law of the environment by $\mathbb{P}$. If the conductance $\omega(e)$ of an edge e is equal to 1, then we call e an open edge. Otherwise we call the edge e closed. Given a realization $\boldsymbol{\omega}$ of the environment, we denote the set of open edges by $\mathfrak{E}_{\mathrm{O}} \subset \mathfrak{E}_d$.

Consider the random graph $(\mathbb{Z}^d, \mathfrak{E}_{\mathrm{O}})$. Following the terminology of Grimmet [Gri99], we call the connected components of this graph *open clusters* and, for $x \in \mathbb{Z}^d$, we write $\mathscr{C}(x)$ for the open cluster that contains the site x. Note that $\mathscr{C}(x) \subset (\mathbb{Z}^d, \mathfrak{E}_{\mathrm{O}})$ is a graph. We define the clusters in this way in order to make sense of Dirichlet forms defined as in (3.35) below. However, when we write $|\mathscr{C}(x)|$, we refer to the number of sites in $\mathscr{C}(x)$. Furthermore, when $\mathscr{C}$ is a cluster and y is a site in the vertex set of $\mathscr{C}$, then we use the shorthand notation $y \in \mathscr{C}$. Similarly, if e is in the edge set of $\mathscr{C}$, then we write $e \in \mathscr{C}$.

We say that a path $l = (x_0, \dots, x_m)$ is open if and only if $\{x_{i-1}, x_i\} \in \mathfrak{E}_{\mathrm{O}}$ for all $i \in \{1, \dots, m\}$.

Let $p_c(d)$ be the critical probability such that $\mathbb{P}$-a.s. there exists an infinite open cluster $\mathscr{C}_\infty$. This cluster is $\mathbb{P}$-a.s. unique. We assume that $p_c(d) < p < 1$. Note that $\mathscr{C}_\infty$ contains all sites x that are connected to infinity through an open path *as well as* all open edges that are incident to a site in $\mathscr{C}_\infty$. We further define $\mathscr{H}$ as the complement of $\mathscr{C}_\infty$ in $\mathbb{Z}^d$, i.e., we regard $\mathscr{H}$ as a set of sites.

The main object of this section is to collect results from the literature and adapt the details such that they exactly fit our needs.

Lemma 3.26 ([BKM15, Lemma 4.2]). *Let $\eta \in (0,1)$. Then for p sufficiently close to one, there exist constants $C < \infty$ and $c > 0$ such that*

$$\mathbb{P}[|B_n \cap \mathscr{C}_\infty| \leq \eta |B_n|] \leq C\mathrm{e}^{-cn} \quad \textit{for all } n \geq 1. \tag{3.34}$$

The second lemma is an implication of Lemma 3.26 above.

Lemma 3.27. *Let $d \geq 2$ and choose p such that Lemma 3.26 holds with $\eta = \frac{1}{2}$. Then $\mathbb{P}$-a.s. for n large enough there exists an injective map $\varphi_1 \colon \mathscr{H} \cap B_n \to \mathscr{C}_\infty$ such that for any site $x \in \mathscr{H} \cap B_n$ the distance $|x - \varphi_1(x)|_1 \leq 2d(\log n)^{(d+1)}$.*

Proof of Lemma 3.27. The proof of this lemma follows the lines of the first paragraph of the proof of [BKM15, Lemma 4.7] but we included the proof here for completeness. For $z \in \mathbb{Z}^d$ and $m \geq 0$, we denote $B_m(z) = \{x \in \mathbb{Z}^d \colon |x - z|_\infty \leq m\}$. Choose the percolation parameter p such that (3.34) is fulfilled with $\eta = \frac{1}{2}$. Let $m = \lfloor (\log n)^{d+1} \rfloor$ and consider the disjoint partition $\mathcal{P}_m := \{B_m((2m+1)z)\}_{z\in\mathbb{Z}^d}$ of $\mathbb{Z}^d$. Then Lemma 3.26 implies that there exist $c, C \in (0,\infty)$ such that

$$\begin{aligned}\mathbb{P}\Bigg[\bigcup_{\substack{B\in\mathcal{P}_m,\\ B\cap B_n\neq\emptyset}} \left\{|B\cap\mathscr{C}_\infty| \le \tfrac{1}{2}|B|\right\}\Bigg] &\le \sum_{\substack{B\in\mathcal{P}_m,\\ B\cap B_n\neq\emptyset}} \mathbb{P}\left[|B\cap\mathscr{C}_\infty| \le \tfrac{1}{2}|B|\right] \\ &\le C(2n+1)^d \exp\Big(-c(\log n)^{d+1}\Big)\,,\end{aligned}$$

which is summable. By the Borel-Cantelli lemma it follows that $\mathbb{P}$-a.s. for n large enough we have $|B\cap\mathscr{C}_\infty| > |B|/2$ in any $B\in\mathcal{P}_m$ with $B\cap B_n\neq\emptyset$.

Now we construct φ_1 as follows: For $x\in\mathscr{H}\cap B_n$ choose $B\in\mathcal{P}_m$ (unique) such that $x\in B$. Choose $\varphi_1(x)\in B\cap\mathscr{C}_\infty$ in an injective way - this is possible since $|\mathscr{H}\cap B| < |\mathscr{C}_\infty\cap B|$. The ℓ_1-distance between x and $\varphi_1(x)$ is thus smaller than or equal to $2d(\log n)^{(d+1)}$. □

For $f:\mathbb{Z}^d\to\mathbb{R}$ with $\|f\|_2^2<\infty$ we define the Dirichlet-form $\mathcal{E}_{\mathscr{C}_\infty}(f)$:

$$\mathcal{E}_{\mathscr{C}_\infty}(f) = \sum_{\{x,y\}\in\mathscr{C}_\infty} (f(x)-f(y))^2\,, \tag{3.35}$$

as well as the norm $\|f\|_{\ell^2(\mathscr{C}_\infty)} = \sum_{x\in\mathscr{C}_\infty} f^2(x)$.

In the following lemma we give a lower bound for the principal Dirichlet eigenvalue on $B_n\cap\mathscr{C}_\infty$. The lemma is similar to Theorem 1.3 from [MR04] with the difference that $B_n\cap\mathscr{C}_\infty$ is in general not connected and we do not include the condition that $0\in\mathscr{C}_\infty$.

Lemma 3.28. *Let $d\ge 2$ and choose p such that Lemma 3.26 holds with $\eta>\frac{1}{2}$. Then there exists a (deterministic) constant $c>0$ such that $\mathbb{P}$-a.s. for n large enough and all real-valued functions $f\in\ell^2(\mathbb{Z}^d)$ with* supp $f\subseteq B_n$ *we have*

$$\|f\|^2_{\ell^2(\mathscr{C}_\infty)} \le cn^2\mathcal{E}_{\mathscr{C}_\infty}(f)\,. \tag{3.36}$$

The proof of this lemma is rather standard given the relative isoperimetric inequality from Theorem 3.29 below (see e.g. [Sal97, p. 83]) but since the details are slightly different, we include the proof for the convenience of the reader. Let $A\subseteq\mathscr{C}_\infty$ be a set of sites. We define the relative edge boundary of A with respect to $\mathscr{C}_\infty$ as the edge set

$$\partial_{\mathrm{E}}(A|\mathscr{C}_\infty) = \{\{x,y\}\in\mathscr{C}_\infty : x\in A \text{ and } y\in\mathscr{C}_\infty\backslash A\}\,.$$

Further, as in [BBHK08], given a percolation environment $\boldsymbol{\omega}$, we call the set $A\subseteq\mathbb{Z}^d$ $\boldsymbol{\omega}$-connected if every two sites in A can be connected by a finite path that uses only open edges and runs only through sites in A. Then we have the following theorem.

Theorem 3.29 ([BBHK08], Theorem A.1). *For all $d \geq 2$ and $p > p_c(d)$, there are positive and finite constants $c_1 = c_1(d,p)$ and $c_2 = c_2(d,p)$ and a $\mathbb{P}$-a.s. finite random variable $n_0 = n_0(\boldsymbol{\omega})$ such that for each $n \geq n_0$ and each $\boldsymbol{\omega}$-connected A satisfying $A \subset \mathscr{C}_\infty \cap B_n$ and $|A| \geq (c_1 \log n)^{d/(d-1)}$ we have*

$$\partial_{\mathrm{E}}(A|\mathscr{C}_\infty) \geq c_2 |A|^{(d-1)/d}. \tag{3.37}$$

Remark 3.30. *Let $A \subset \mathscr{C}_\infty \cap B_n$ and n_0, c_1, c_2 be as in Theorem 3.29. If A is $\boldsymbol{\omega}$-connected and $|A| \geq (c_1 \log n)^{d/(d-1)}$, then the relative isoperimetric inequality* (3.37) *yields*

$$\frac{|\partial_{\mathrm{E}}(A|\mathscr{C}_\infty)|}{|A|} \geq \frac{c_2}{|A|^{1/d}} \geq \frac{c_2}{3n},$$

where we have used that $A \subseteq B_n$ and thus $|A|^{1/d} \leq (2n+1)^d$. If, on the other hand, $|A| < (c_1 \log n)^{d/(d-1)}$, then eventually

$$\frac{|\partial_{\mathrm{E}}(A|\mathscr{C}_\infty)|}{|A|} \geq \frac{1}{|A|} \geq \frac{1}{(c_1 \log n)^{d/(d-1)}} \geq \frac{1}{n}.$$

It follows that there exists $c > 0$ such that for n large enough and all $\boldsymbol{\omega}$-connected $A \subset \mathscr{C}_\infty \cap B_n$ we have

$$\frac{|\partial_{\mathrm{E}}(A|\mathscr{C}_\infty)|}{|A|} \geq \frac{c}{n}. \tag{3.38}$$

If A is not $\boldsymbol{\omega}$-connected, then similar to the arguments in [MR04, Section 3.1], we write $A = \bigcup_i A_i$ where the A_i are the $\boldsymbol{\omega}$-connected components of the set A. Thus,

$$\frac{|\partial_{\mathrm{E}}(A|\mathscr{C}_\infty)|}{|A|} = \frac{1}{|A|} \sum_i \frac{|\partial_{\mathrm{E}}(A_i|\mathscr{C}_\infty)|}{|A_i|} \cdot |A_i| \geq \frac{c}{n|A|} \sum_i |A_i| = \frac{c}{n}.$$

It follows that (3.38) *holds for all sets $A \subset \mathscr{C}_\infty \cap B_n$.*

Proof of Lemma 3.28. Let n_0 be as in Theorem 3.29 and let $n \geq n_0$. Further let $f \colon \mathbb{Z}^d \to \mathbb{R}$ such that $\operatorname{supp} f \subseteq B_n$. We apply the mean value inequality and Hölder's inequality to obtain

$$\begin{aligned} \sqrt{4d}\, \|f\|_{\ell^2(\mathscr{C}_\infty)} \sqrt{\mathcal{E}_{\mathscr{C}_\infty}(f)} &\geq \sqrt{\sum_{\{x,y\} \in \mathscr{C}_\infty} (f(x)+f(y))^2} \sqrt{\sum_{\{x,y\} \in \mathscr{C}_\infty} (f(x)-f(y))^2} \\ &\geq \sum_{\{x,y\} \in \mathscr{C}_\infty} \left| f^2(x) - f^2(y) \right|. \end{aligned} \tag{3.39}$$

Now we use a standard approach which is known as the co-area formula (see e.g. [Sal97, p. 83]):

$$\sum_{\{x,y\}\in\mathscr{C}_\infty} \left|f^2(x) - f^2(y)\right| = \sum_{x\in\mathscr{C}_\infty} \sum_{\substack{y\colon \{x,y\}\in\mathscr{C}_\infty,\\ f(x)\geq f(y)}} \int_0^\infty \mathbb{1}_{\{f^2(x)>t\geq f^2(y)\}}\,\mathrm{d}t\,.$$

If for $t \geq 0$ we define the set of sites $A_t = \left\{x \in \mathscr{C}_\infty \colon f^2(x) > t\right\}$, then we see that

$$\sum_{x\in\mathscr{C}_\infty} \sum_{\substack{y\colon \{x,y\}\in\mathscr{C}_\infty,\\ f(x)\geq f(y)}} \mathbb{1}_{\{f(x)^2>t\geq f^2(y)\}} = \left|\partial_{\mathrm{E}}(A_t|\mathscr{C}_\infty)\right|.$$

By virtue of Theorem 3.29 and Remark 3.30 it follows that there exists $c > 0$ such that eventually

$$\sum_{\{x,y\}\in\mathscr{C}_\infty} \left|f^2(x) - f^2(y)\right| \geq \frac{c}{n}\int_0^\infty |A_t|\,\mathrm{d}t = \frac{c}{n}\sum_{x\in\mathscr{C}_\infty} f^2(x)\,.$$

Together with (3.39) this implies that

$$\sqrt{\mathcal{E}_{\mathscr{C}_\infty}(f)} \geq \frac{c}{\sqrt{4d}\cdot n}\|f\|_{\ell^2(\mathscr{C}_\infty)}\,.$$

□

3.6 Path argument

In this section we give the two Propositions 3.38 and 3.39, which transfer the knowledge we obtained by the Borel-Cantelli arguments in Section 3.4 to lower bounds on Dirichlet energies. In order to achieve this, Lemma 3.33 generalizes and modifies the path argument in [BKM15, Lemma 4.7]. Before we start, we give a definition which is crucial for the remaining part of the paper.

Definition 3.31. *Let $\mathscr{G} = (V, \mathfrak{E})$ be an undirected graph and $\boldsymbol{\omega} = (\omega(e))_{e\in\mathfrak{E}}$. For $f\colon V \to \mathbb{R}$, we define the Dirichlet energy on $\mathscr{G}$ as*

$$\mathcal{E}^{\omega}_{\mathscr{G}}(f) = \frac{1}{2}\sum_{x\in V}\sum_{\substack{y\in V,\\ \{x,y\}\in\mathfrak{E}}} \omega_{xy}(f(x) - f(y))^2\,. \tag{3.40}$$

Remark 3.32. *For $\xi > 0$ let us define $a(e) = \mathbb{1}_{\{\omega(e)\geq\xi\}}$ ($e \in \mathfrak{E}_d$). Let us call an edge e open if and only if $a(e) = 1$ and let $\mathscr{C}$ be an open cluster in the environment $\boldsymbol{a} = (a(e))_{e\in\mathfrak{E}_d}$. Then, with reference to (3.35), we obtain that $\xi\mathcal{E}_{\mathscr{C}}(f) \leq \mathcal{E}^{\omega}_{\mathscr{C}}(f)$ for all real-valued functions $f \in \ell^2(\mathbb{Z}^d)$.*

Since we apply a similar argument for two slightly different situations (i.e., once for the proofs of Theorems 3.5 and 3.7, see Proposition 3.39, and once for the proof of Theorem 3.13, see Proposition 3.38), we kept the conditions of the following lemma as general as necessary.

Lemma 3.33. *Let $\mathscr{G} = (V, \mathfrak{E})$ be a subgraph of $(\mathbb{Z}^d, \mathfrak{E}_d)$ and let $\mathscr{C} = (V_{\mathscr{C}}, \mathfrak{E}_{\mathscr{C}})$ be a subgraph of $\mathscr{G}$. Assume that $\nu, L \in (0,\infty)$ and $B \subseteq V$ are such that the following conditions are fulfilled:*

(i) There exists a constant $\mu > 0$ such that for all $f\colon V \to \mathbb{R}$ with $\operatorname{supp} f \subseteq B$ the following inequality holds:

$$\mathcal{E}^{\omega}_{\mathscr{C}}(f) \geq \mu \|f\|^2_{\ell^2(\mathscr{C})}. \tag{3.41}$$

(ii) There exists an injective map $\varphi : B\backslash V_{\mathscr{C}} \to V_{\mathscr{C}}$ such that the following holds: From any $x \in B\backslash V_{\mathscr{C}}$ there exists a (self-avoiding) directed path $l(x, \varphi(x))$ to $\varphi(x)$ in $\mathscr{G}$ such that

a. all $e \in l(x, \varphi(x))$ fulfill $\omega(e) > \nu$,
b. $|l(x, \varphi(x))| \leq L$.

Then for all $f\colon V \to \mathbb{R}$ with $\operatorname{supp} f \subseteq B$ the following holds:

$$\mathcal{E}^{\omega}_{\mathscr{G}}(f) \geq \left((2L)^{d+1}\nu^{-1} + 3\mu^{-1}\right)^{-1} \|f\|^2_{\ell^2(\mathscr{G})}\,. \tag{3.42}$$

Proof of Lemma 3.33. We generalize the proof of [BKM15, Lemma 5.1], which uses arguments from [Bou10, Lemma 3.4]. Let $f\colon V \to \mathbb{R}$ with $\operatorname{supp} f \subseteq B$. For the following calculation we abbreviate $f(y) - f(z) = \mathrm{d}f((y,z))$ where (y,z) is the (directed) edge from site y to its neighbor z. For $x \in B\backslash V_{\mathscr{C}}$ we write $f(x)$ as a telescopic sum

$$f(x) = \sum_{b \in l(x,\varphi(x))} \mathrm{d}f(b) + f(\varphi(x))\,.$$

We apply the Cauchy-Schwarz inequality and expand the terms on the right-hand side by the conductances:

$$f^2(x) \leq \frac{2|l(x,\varphi(x))|}{\nu} \sum_{b \in l(x,\varphi(x))} \omega(b)\,(\mathrm{d}f(b))^2 + 2f^2(\varphi(x))\,.$$

Now we sum over all $x \in B\backslash V_{\mathscr{C}}$ and use the upper bound for $|l(x,\varphi(x))|$ according to Condition (ii)b:

$$\sum_{x \in B\backslash V_{\mathscr{C}}} f^2(x) \leq \frac{2L}{\nu} \sum_{x \in B\backslash V_{\mathscr{C}}} \sum_{b \in l(x,\varphi(x))} \omega(b)\,(\mathrm{d}f(b))^2 + 2 \sum_{x \in B\backslash V_{\mathscr{C}}} f^2(\varphi(x))\,. \tag{3.43}$$

Let us look at the last term on the right-hand side: By definition φ is injective and its image is in $V_{\mathscr{C}}$. This means that

$$\sum_{x\in B\setminus V_{\mathscr{C}}} f^2(\varphi(x)) \leq \sum_{x\in V_{\mathscr{C}}} f^2(x)\,.$$

Since the path $l(x,\varphi(x))$ has a length of at most L, any path that uses a given edge b must have started in an ℓ_1-ball of radius L around $b =: \{b_1, b_2\}$ with $b_1, b_2 \in \mathbb{Z}^d$. Thus, if the path $l(x,\varphi(x))$ runs through the edge b, then

$$x \in \big\{z\in\mathbb{Z}^d\colon \|z-b_1\|_1 \leq L-1\big\} \cup \big\{z\in\mathbb{Z}^d\colon \|z-b_2\|_1 \leq L-1\big\}\,.$$

Since in dimension $d\geq 2$ and for $L\geq 2$, the cardinality of either one of the above ℓ_1-balls is bounded from above[2] by $2^{d-1}L^d$, it follows that the cardinality of the whole set on the above right-hand side is bounded from above by $(2L)^d$. Thus, the sum over $b\in l(x,\varphi(x))$ on the right-hand side in (3.43) uses each edge not more than $(2L)^d$ times, whence

$$\sum_{x\in B\setminus V_{\mathscr{C}}} \sum_{b\in l(x,\varphi(x))} \omega(b)\,(\mathrm{d}f(b))^2 \leq (2L)^d \mathcal{E}^{\omega}_{\mathscr{G}}(f)\,.$$

Completing the sum to all sites $x\in\mathscr{G}$ and using the comparability between $\mathcal{E}^{\omega}_{\mathscr{G}}(f)$ and $\mathcal{E}^{\omega}_{\mathscr{C}}(f)$, we obtain by virtue of Condition (i):

$$\sum_{x\in V} f^2(x) \leq \frac{(2L)^{d+1}}{\nu}\mathcal{E}^{\omega}_{\mathscr{G}}(f) + 3\sum_{x\in V_{\mathscr{C}}} f^2(x) \leq \left(\frac{(2L)^{d+1}}{\nu} + \frac{3}{\mu}\right)\mathcal{E}^{\omega}_{\mathscr{G}}(f)\,.$$

□

3.6.1 Asymptotics of the principal Dirichlet eigenvalue

From the path argument in Lemma 3.33 we can use our observations from Section 3.4 to obtain lower bounds of the Dirichlet forms. We use similar arguments as in [BKM15, Lemma 5.1]. Let us fix $\xi > 0$ such that

$$\mathbb{P}[\omega > \xi] > p_c(d)\,. \tag{3.44}$$

Moreover, we fix an environment $\boldsymbol{\omega}$ and define a new environment $\boldsymbol{a}$ by setting

$$a(e) = \mathbb{1}_{\{\omega(e)>\xi\}} \qquad (e\in\mathfrak{E}_d)\,, \tag{3.45}$$

[2] In dimension $d=2$, the cardinality of an ℓ^1-ball with radius R is $1+2R(R+1) < 2(R+1)^2$. If $V_d^{(1)}(R)$ is the cardinality of an ℓ^1-ball with radius R in dimension d, then one convinces oneself that $V_d^{(1)}(R) < 2(R+1)V_{d-1}^{(1)}(R)$.

as in Remark 3.32. We denote the unique infinite cluster of the environment $\boldsymbol{a}$ by $\mathscr{C}^\xi$ and we use the same shorthand notations as explained at the beginning of Section 3.5. Further we define $\mathscr{C}_n^\xi$ as the restriction of $\mathscr{C}^\xi$ to the box B_n and similarly the holes $\mathscr{H}_n^\xi$.

Additionally, we define a second percolation environment $\tilde{\boldsymbol{\omega}}_{g(n)}$ for $g : (0,\infty) \to (0,\infty)$ by setting

$$\tilde{w}_{g(n)}(e) = \omega(e)\mathbb{1}_{\{\omega(e)>g(n)\}} \qquad (e \in \mathfrak{E}_d)\,. \tag{3.46}$$

Thus, edges with conductance less than or equal to $g(n)$ are considered to be closed and all others keep their original conductance. With this terminology we can now define the following clusters.

Definition 3.34. *For a fixed function g and a fixed $\varepsilon > 0$, let $\mathscr{D}_n$ be the unique infinite open cluster of $\tilde{\boldsymbol{\omega}}_{g(n^{1-\varepsilon})}$. Regarding this cluster, we use the same shorthand notations as introduced at the beginning of Section 3.5. Furthermore, let $\mathscr{I}_n = B_n \backslash \mathscr{D}_n$ be the set of holes in B_n.*

Definition 3.35. *We call a set $\mathscr{I} \subset \mathbb{Z}^d$* **sparse** *if the set $\mathscr{I}$ does not contain any neighboring sites. Further, a set $\mathscr{I} \subset \mathbb{Z}^d$ is* **b-sparse** *if any box $B_b(z) \subset \mathbb{Z}^d$ with $z \in \mathbb{Z}^d$ contains at most one site of the set $\mathscr{I}$.*

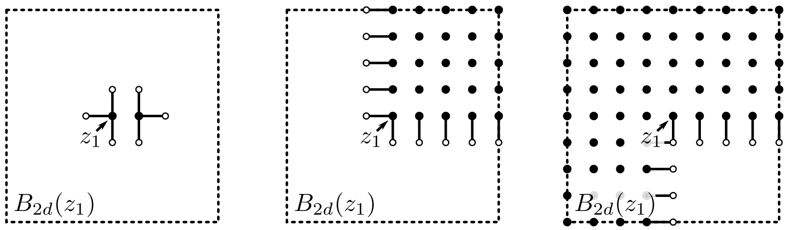
Remark 3.36. *Let $b_1 < b_2$ be natural numbers. If a set $\mathscr{I} \subset \mathbb{Z}^d$ is b_2-sparse, it is also b_1-sparse and sparse.*

Lemma 3.37. *Let $b \in \mathbb{N}$ with $b \geq 2d$ and $g\colon (0,\infty) \to (0,\infty)$ be a decreasing function. For a fixed environment $\boldsymbol{\omega}$ assume that for n large enough for all $z \in B_{n+b}$ the edge set $\mathfrak{E}(B_b(z))$ contains at most $3d-1$ edges with conductance less than or equal to $g(n^{1-\varepsilon})$. Further, let $\mathscr{D}_n$ be as in Definition 3.34 the unique infinite cluster. Then, for n large enough the set $\mathscr{I}_n = B_n \backslash \mathscr{D}_n$ is b-sparse.*

Proof. To show that for n large enough the set $\mathscr{I}_n$ is b-sparse, we first show that for n large enough the set $\mathscr{I}_n$ is sparse. We define $\tilde{\mathscr{I}_n} = \mathbb{Z}^d \backslash \mathscr{D}_n$. Let us assume that for infinitely many n there exists a pair of neighbors z_1, z_2 in the set $\mathscr{I}_n = \tilde{\mathscr{I}_n} \cap B_n$. Since by assumption $\mathscr{D}_n$ is the unique infinite cluster, it follows that for n large enough $\mathscr{D}_n \cap B_n \neq \emptyset$. Thus, we can assume without loss of generality that z_1 has a neighbor $x \in \mathscr{D}_n$. If z_1 does not have a neighbor in $\mathscr{D}_n$, then we consider a self-avoiding path l inside B_n from z_1 to a site $x \in \mathscr{D}_n \cap B_n$. Let x' be the first site on the path l that is in $\mathscr{D}_n$ and let z_1' be the preceding site to x' on the path l. Since z_1 does not have a neighbor in $\mathscr{D}_n$, the site z_1' is different from z_1 and thus z_1' has a further predecessor $z_2' \in \mathscr{I}_n$ on the path l. It follows that the neighbors z_1', z_2' are in $\mathscr{I}_n \cap B_n$ and further z_1' has a neighbor $x' \in \mathscr{D}_n \cap B_n$.

In the context of this proof, for $z \in \tilde{\mathscr{I}_n}$ we define $\tilde{\mathscr{I}_n}(z) \subset \tilde{\mathscr{I}_n}$ as the connected component that contains z, i.e., $y \in \tilde{\mathscr{I}_n}(z)$ if there exists a path $l \subset \mathfrak{E}_d$ between the sites z and y that runs only through sites in $\tilde{\mathscr{I}_n}$.

Let z_1 be as above. We distinguish two cases now:



(a) Case 1 **(b)** Case 2 **(c)** Case 2

Fig. 3.2: Boundary edges needed to separate the set $\tilde{\mathscr{I}}_n(z_1)$ from the infinite cluster $\mathscr{D}_n$. The full circles represent sites of $\tilde{\mathscr{I}}_n$ while open circles represent sites of $\mathscr{D}_n$. In Figure 3.2a the component $\tilde{\mathscr{I}}_n(z_1)$ is a subset of $B_{2d}(z_1)$, in Figures 3.2b and 3.2c it is not.

1. $\tilde{\mathscr{I}}_n(z_1)$ is a subset of $B_{2d}(z_1)$.
2. $\tilde{\mathscr{I}}_n(z_1)$ is *not* a subset of $B_{2d}(z_1)$.

In the first case, the edge boundary that separates the cluster $\mathscr{D}_n$ and the component $\tilde{\mathscr{I}}_n(z_1)$, consists of at least $4d-2$ edges that are in the edge set $\mathfrak{E}(B_{2d}(z_1))$. For a sketch see Figure 3.2a. This is a contradiction to the first claim of this lemma. Since we have assumed that $\mathscr{D}_n$ is the infinite cluster, in the second case the edge boundary that separates the cluster $\mathscr{D}_n$ and the component $\tilde{\mathscr{I}}_n(z_1)$, has to link two (not necessarily different) faces of the cube $B_{2d}(z_1)$ and at the same time enclose the site z_1 (e.g. as in Figure 3.2b) in the very middle of the cube $B_{2d}(z_1)$ or enclose one of its neighbors (e.g. as in Figure 3.2c). It follows that the set $\mathfrak{E}(B_{2d}(z_1))$ consists of more than $4d$ edges with conductance less than or equal to $g(n^{1-\varepsilon_2})$. This is again a contradiction and it follows that $\mathscr{I}_n$ is sparse.

Now we further show that $\mathbb{P}$-a.s. for n large enough the set $\mathscr{I}_n$ is b-sparse. Let us assume that for infinitely many n there exists $z \in B_{n+b}$ such that $B_b(z)$ contains two sites of $\mathscr{I}_n$. Since we already know that $\mathbb{P}$-a.s. for n large enough the set $\mathscr{I}_n$ is sparse and each site has $2d$ incident edges, it follows that for infinitely many n there exists $z \in B_{n+b}$ such that the edge set $\mathfrak{E}(B_{b+1}(z))$ contains $4d$ edges with conductance less than or equal to $g(n^{1-\varepsilon_2})$. This is a contradiction to the first claim of this lemma. $\square$

Proposition 3.38. *Fix an environment $\boldsymbol{\omega} \in \Omega$. Let g be a positive function decreasing to zero and let $\epsilon, \xi, c_1 \in (0, \infty)$. Let $(n_k)_{k\in\mathbb{N}}$ be any (possibly empty) subsequence, along which the following assumptions are true:*

(i) For all $z \in B_{n_k+3d}$ the edge set $\mathfrak{E}(B_{3d}(z))$ contains at most $3d-1$ edges with conductance less than or equal to $g(n^{1-\epsilon})$.
(ii) $\mathscr{C}^{\xi}$ is the unique infinite open cluster of the environment $\boldsymbol{a}$ defined through (3.45).
(iii) All real-valued functions $f \in \ell^2(\mathbb{Z}^d)$ with $\operatorname{supp} f \subseteq B_n$ *fulfill*

$$\|f\|^2_{\ell^2(\mathscr{C}^\xi)} \le c_1 n_k^2 \mathcal{E}_{\mathscr{C}^\xi}(f) \le \xi^{-1} c_1 n_k^2 \mathcal{E}^\omega_{\mathscr{C}^\xi}(f)\,. \tag{3.47}$$

(iv) There exists an injective map $\varphi_1\colon \mathscr{H}^\xi \cap B_n \to \mathscr{C}^\xi$ such that for any $x \in \mathscr{H}^\xi \cap B_n$ there exists a directed path $l_1(x,\varphi_1(x))$ in $(\mathbb{Z}^d, \mathfrak{E}_d)$ from x to $\varphi_1(x)$ of length $|l_1(x,\varphi_1(x))| \le 2d(\log n)^{(d+1)}$.

Then for k large enough, $\mathscr{D}_{n_k}$ is the unique infinite open cluster of the environment $\tilde{\boldsymbol{\omega}}_{g(n^{1-\varepsilon})}$ (see Definition 3.34) and

$$\mathcal{E}^\omega_{\mathscr{D}_{n_k}}(f) \ge \left(2^{d+1}(\log n_k)^{4d^2} g\big(n_k^{1-\varepsilon}\big)^{-1} + 3c_1\xi^{-1}n_k^2\right)^{-1} \|f\|^2_{\ell^2(\mathscr{D}_{n_k})}\,,$$

for all real-valued functions $f \in \ell^2(\mathbb{Z}^d)$ with $\operatorname{supp} f \subseteq B_{n_k}$ *and with $\mathcal{E}^\omega_{\mathscr{D}_{n_k}}$ as in Definition 3.31.*

Proposition 3.39. *Let the assumptions of Proposition 3.38 be true for a subsequence $(n_k)_{k\in\mathbb{N}}$ as well as one of Assumptions 3.4 (a) or (b). Further, assume that along the same subsequence for all $z \in B_{n_k}$ there exists an incident edge with conductance greater than $g(n_k)$. Then there exists $c > 0$ such that for k large enough*

$$\mathcal{E}^\omega\big(f\big) \ge c g(n_k)\|f\|_2^2$$

for all real-valued functions $f \in \ell^2(\mathbb{Z}^d)$ with $\operatorname{supp} f \subseteq B_{n_k}$. *If one of the Assumptions 3.4 (a) or (b') is fulfilled, then the constant c can be chosen independently of g.*

We prove these propositions in the next section.

3.6.2 Proofs of Propositions 3.38 and 3.39

Proof of Proposition 3.38. In this proof we shortly write n for a member of the subsequence $(n_k)_{k\in\mathbb{N}}$. The fact that for n large enough there exists a unique infinite open cluster of the environment $\tilde{\boldsymbol{\omega}}_{g(n^{1-\varepsilon})}$, follows from Assumption (ii) when we choose n such that $g\big(n^{1-\varepsilon}\big) \le \xi$, i.e., when $\mathscr{C}^\xi \subset \mathscr{D}_n$.

For the actual claim we apply Lemma 3.33 with $\mathscr{G}$ given by the cluster $\mathscr{D}_n$ and $\mathscr{C}$ given by $\mathscr{C}^\xi$. Further, let $\nu_n = g\big(n^{1-\varepsilon}\big)$. Further, if we choose $\mu_n = \frac{\xi}{c_1 n^2}$ in the place of μ in (3.41), then Condition (i) of Lemma 3.33 is fulfilled.

We are now going to construct the map $\varphi\colon B_n \cap \mathscr{D}_n \cap \mathscr{H}^\xi \to \mathscr{C}^\xi$ and the path $l(x,\varphi(x))$. For the next paragraph we say that a conductance is “bad” if it is smaller than or equal to $g(n^{1-\epsilon})$. Let $\varphi = \varphi_1|_{\mathscr{H}^\xi\cap B_n \cap \mathscr{D}_n}$ (see Assumption (iv)). By Assumption (i), each subbox $B_{3d}(z)$ with $z \in B_{n+3d}$ contains at most $3d-1$ bad conductances. We thus construct the path $l(x,\varphi(x))$ by the following algorithm: The path $l(x,\varphi(x))$ follows $l_1(x,\varphi(x))$ until it hits an

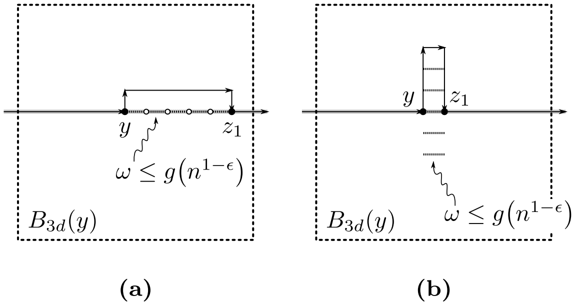


Fig. 3.3: Construction of the path $l(x,\varphi(x))$ (solid black line with arrows) from $l_1(x,\varphi(x))$ (thick gray line). Inside the box $B_{3d}(y)$ there are at most $3d-1$ bad conductances (dotted lines). The path $l(x,\varphi(x))$ follows $l_1(x,\varphi(x))$ until it hits an edge with a bad conductance at site y. Let $e = \{z_1, z_2\}$ be the first good conductance on $l_1(x,\varphi(x))$ after y, where the site z_1 comes before z_2 on the path $l_1(x,\varphi(x))$. Then between the sites y and z_1 the path $l(x,\varphi(x))$ takes the shortest detour from y to z_1 without using any edge with a bad conductance. If, for this purpose, the path has to take a loop, as depicted in Example 3.3c, then we delete the loop.

edge with a bad conductance. Let y be the last site that the path $l(x,\varphi(x))$ reached before hitting this bad conductance. Let $e = \{z_1, z_2\}$ be the first good conductance on $l_1(x,\varphi(x))$ after y, where the site z_1 comes before z_2 on the path $l_1(x,\varphi(x))$. Then between the sites y and z_1 the path $l(x,\varphi(x))$ takes the shortest detour from y to z_1 without using any edge with a bad conductance, see Figure 3.3 for a sketch. This is always possible, even if $y = x$ or $z_1 = \varphi(x)$, since $x, \varphi(x) \in \mathscr{D}_n$. If, for the purpose of the detour, the path has to take a loop as depicted in Example 3.3c, then we simply delete the entire loop. Since the box $B_{3d}(y)$ contains only $3d-1$ bad conductances, the detour is always contained in the edge set $\mathfrak{E}(B_{3d}(y))$. Thus the length of each detour is bounded by a constant $C \leq |\mathfrak{E}(B_{3d}(0))|$. After the detour, $l(x,\varphi(x))$ continues again on $l_1(x,\varphi(x))$ until it hits the next bad conductance and so on. For all $x \in B_n \cap \mathscr{D}_n \cap \mathscr{H}^\xi$ and for n large enough it follows that $|l(x,\varphi(x))| \leq 2dC(\log n)^{(d+1)} < (\log n)^{2d}$.

We can now apply Lemma 3.33 to obtain the claim. □

Proof of Proposition 3.39. Again, we shortly write n for a member of the subsequence $(n_k)_{k\in\mathbb{N}}$ and, again, we apply Lemma 3.33. Let $\mathscr{G} = (\mathbb{Z}^d, \mathfrak{E}_d)$ and $\nu_n = g(n)$. Further, let $\mathscr{D}_n$ be as in Proposition 3.38 and let $\mathscr{C}$ be given by $\mathscr{D}_n$. Then Condition (i) of Lemma 3.33 is fulfilled with

$$\mu_n = \left(2^{d+1}(\log n)^{4d^2} g\left(n^{1-\varepsilon}\right)^{-1} + 3c_1 n^2 \xi^{-1}\right)^{-1}.$$

By virtue of Assumption (i) of 3.38 we can apply Lemma 3.37 and thus each site $x \in \mathscr{I}_n = B_n \backslash \mathscr{D}_n$ has only neighbors in $\mathscr{D}_n$. By assumption there exists a neighbor $\varphi(x)$ of x such that the conductance $\omega_{x,\varphi(x)} > g(n)$. Further, since $\mathscr{I}_n$ is $3d$-sparse, any neighbor y of $x \in \mathscr{I}_n$ has no second neighbor in $\mathscr{I}_n$.

It follows that the map $\varphi\colon \mathscr{I}_n \to \mathscr{D}_n$ is injective and the path $l(x,\varphi(x)) = (x,\varphi(x))$ fulfills the requirements of Lemma 3.33.

It follows that for n large enough

$$\mathcal{E}^\omega(f) \geq \left(2^{d+1}g(n)^{-1} + 2^{d+3}(\log n)^{4d^2} g\big(n^{1-\varepsilon}\big)^{-1} + 9c_1 n^2 \xi^{-1}\right)^{-1} \|f\|_2^2$$

for all $f\colon \mathbb{Z}^d \to \mathbb{R}$ with supp $f \subseteq B_n$:

We have assumed that one of Assumptions 3.4 (a) or (b) is fulfilled. Let us first assume that Assumption 3.4 (b) is true and that the limit of $u^2 g(u)$ is smaller than $c_2 \in (0,\infty)$. It follows that eventually $9c_1\xi^{-1}n^2 g(n) < 9c_1c_2\xi^{-1}$ and

$$2^{d+3}(\log n)^{4d^2} \frac{g(n)}{g\big(n^{1-\varepsilon}\big)} < 2^{d+5}(\log n)^{4d^2} n^{-2\varepsilon} < 1\,,$$

and therefore for n large enough

$$\mathcal{E}^\omega(f) \geq \frac{g(n)}{1 + 2^{d+1} + 9c_1c_2\xi^{-1}} \|f\|_2^2 \qquad (\operatorname{supp} f \subseteq B_n)\,.$$

If we assume that Assumption 3.4 (b') is fulfilled, then eventually even $9c_1\xi^{-1}n^2 g(n) < 1$ and thus the lower bound becomes independent of c_1, c_2, and ξ.

Let us now assume that Assumption (a) is true. Then there exists $\rho < -2$ such that we can write $g(n) = n^\rho L(n)$ where L varies slowly at infinity. It follows that eventually

$$9c_1\xi^{-1}n^2 g(n) < 1 \text{ and } 2^{d+3}(\log n)^{4d^2} \frac{g(n)}{g\big(n^{1-\varepsilon}\big)} = n^{\rho\varepsilon}\, \frac{2^{d+3}(\log n)^{4d^2} L(n)}{L(n^{1-\varepsilon})} < 1\,.$$

It follows that in this case for n large enough

$$\mathcal{E}^\omega(f) \geq \frac{g(n)}{2^{d+1}+2} \|f\|_2^2\,.$$

$\square$

3.7 Localization of the principal eigenvector

For the proof of Theorem 3.13 we need to analyze the extreme value statistics of the dependent field of random variables $(\pi_z)_{z\in B_n}$. Heuristically speaking, since the smallest values of $(\pi_z)_{z\in B_n}$ are far apart (see e.g. Lemma 3.43), they are asymptotically independent. In order to make this argument rigorous (see e.g. Lemma 3.49), we first introduce a decomposition of the lattice $\mathbb{Z}^d$. Then we continue with a number of auxiliary lemmas in Section 3.7.2 and the extreme value analysis in Section 3.7.3. Finally, we give the proof of Theorem 3.13 in Section 3.7.4. In Section 3.7.5, we prove Corollaries 3.15 and 3.17.

3.7.1 Decomposition of the lattice

To reduce the number of indices, we fix $k \in \mathbb{N}$ throughout this section, i.e., although most of the quantities discussed in this section depend on some $k \in \mathbb{N}$, it will not show as an index.

We define the cube $\mathscr{N} \subset \mathbb{Z}^d$ as

$$\mathscr{N} := \{1, \dots, 2(k+1)\}^d ,$$

and the vertex set $\mathscr{V}$ as

$$\mathscr{V} := \dot{\bigcup_{y \in (2k+3)\mathbb{Z}^d}} (y + \mathscr{N}) ,$$

where $y + \mathscr{N} = \left\{z \in \mathbb{Z}^d \colon z - y \in \mathscr{N}\right\}$.

The important two features of the set $\mathscr{V}$ are that first, for all $a, b \geq 0$ and all $x, y \in (2k+3)\mathbb{Z}^d$ with $x \neq y$, we see that

$$\mathbb{P}\left[\min_{z \in x + \mathscr{N}} \pi_z \leq a, \min_{z \in y + \mathscr{N}} \pi_z \leq b\right] = \mathbb{P}\left[\min_{z \in x + \mathscr{N}} \pi_z \leq a\right] \mathbb{P}\left[\min_{z \in y + \mathscr{N}} \pi_z \leq b\right] , \tag{3.48}$$

and second, the following lemma.

Lemma 3.40. *For any vertex set $\mathscr{A} \subset \mathbb{Z}^d$ with cardinality $|\mathscr{A}| \leq 2(k+1)$ there exists $x \in B_{k+1} = \{-k-1, -k, \dots, k, k+1\}^d$ such that $\mathscr{A} \subset x + \mathscr{V} = \left\{z \in \mathbb{Z}^d \colon z - x \in \mathscr{V}\right\}$.*

Proof. First we note that the set $\mathscr{V}$ is equal to the set

$$\left\{y = (y_1, \dots y_d) \in \mathbb{Z}^d \colon (y_1 \not\equiv 0 \bmod (2k+3)), \dots, (y_d \not\equiv 0 \bmod (2k+3))\right\} .$$

Let $\mathscr{A} = \{v_1, \dots, v_{2k+2}\}$ with $v_1, \dots, v_{2k+2} \in \mathbb{Z}^d$ and let $v_{1,1}, \dots, v_{2k+2,1}$ be the first components of the vectors $v_1, \dots, v_{2k+2}$. Then we choose the first component x_1 of the translation vector $x = (x_1, \dots, x_d) \in B_{k+1}$ such that its residue class modulo $(2k+3)$ is not among the residue classes of $-v_{1,1}, \dots, -v_{2k+2,1}$ modulo $(2k+3)$. This is possible since $-v_{1,1}, \dots, -v_{2k+2,1}$ assume at most $2k+2$ different residue classes modulo $2k+3$. The other components of the translation vector x are chosen likewise. $\square$

Let us now define the random variable χ as

$$\chi := \min_{z \in \mathscr{N}} \pi_z \tag{3.49}$$

and, for $x \in \mathbb{Z}^d$, analogously χ_x as

$$\chi_x := \min_{z \in x + \mathscr{N}} \pi_z \,. \tag{3.50}$$

Lemma 3.41. *For any $a \geq 0$, the value of $F_\chi(a) := \mathbb{P}[\chi \leq a]$ is bounded by*

$$(2k+2)^d F_\pi(a) - \binom{(2k+2)^d}{2} F(a)^{4d-1} \;\leq\; F_\chi(a) \;\leq\; (2k+2)^d F_\pi(a)\,.$$

Proof. First we note that

$$\mathbb{P}[\chi \leq a] \;=\; \mathbb{P}\left[\min_{z \in \mathscr{N}} \pi_z \leq a\right] \;=\; \mathbb{P}\left[\bigvee_{z \in \mathscr{N}} (\pi_z \leq a)\right]. \tag{3.51}$$

Since the set $\mathscr{N}$ contains $(2k+2)^d$ vertices, the above right-hand side is bounded from above by $(2k+2)^d \mathbb{P}[\pi \leq a]$. For the lower bound, we simply expand the right-hand side of (3.51) by one term more, i.e.,

$$\mathbb{P}\left[\bigvee_{z \in \mathscr{N}} (\pi_z \leq a)\right] \geq (2k+2)^d \mathbb{P}[\pi \leq a] - \sum_{\substack{z_1, z_2 \in \mathscr{N},\\ z_1 \neq z_2}} \mathbb{P}[\pi_{z_1} \leq a, \pi_{z_2} \leq a]\,. \tag{3.52}$$

The claim follows since there are $\binom{(2k+2)^d}{2}$ pairs $z_1, z_2 \in \mathscr{N}$ with $z_1 \neq z_2$ and in order to achieve that simultaneously $\pi_{z_1} \leq a$ and $\pi_{z_2} \leq a$, at least $4d-1$ independent conductances have to be less than or equal to a. $\square$

Lemma 3.42. *Let F be continuous and let F_χ be as in Lemma 3.41. Then the random variable $F_\chi(\chi)$ is uniformly distributed on* $[0,1]$.

Proof. Since π is the sum of $2d$ independent random variables with continuous distribution function, it has a continuous distribution function as well. It follows that $F_\chi \colon [0,\infty) \to [0,1]$ is also continuous and thus surjective.

Let $a \in [0,1)$. Since F_χ is surjective, there exists b such that $F_\chi(b) = a$. Since F_χ is also monotonically increasing, it follows that

$$\mathbb{P}[F_\chi(\chi) \leq a] \;\leq\; \mathbb{P}[\chi \leq \sup\{b\colon F_\chi(b) = a\}] \;=\; F_\chi(\sup\{b\colon F_\chi(b) = a\}) \;=\; a$$

and

$$\mathbb{P}[F_\chi(\chi) \leq a] \;\geq\; \mathbb{P}[\chi \leq \inf\{b\colon F_\chi(b) = a\}] \;=\; F_\chi(\inf\{b\colon F_\chi(b) = a\}) \;=\; a\,.$$

$\square$

3.7.2 Auxiliary lemmas

Throughout this section we assume that the distribution function F is continuous. We recall that $\psi_1^{(n)}$ is the principal Dirichlet eigenvector and that it is associated with the principal Dirichlet eigenvalue $\lambda_1^{(n)}$. We normalize it such that $\|\psi_1^{(n)}\|_2 = 1$. By virtue of the Perron-Frobenius Theorem we can assume without loss of generality that $\psi_1^{(n)}$ is nonnegative everywhere, see Remark 3.3.

In Lemma 3.44 we are going to see that $\psi_1^{(n)}$ concentrates on the cluster $\mathscr{D}_n$, which we defined in Definition 3.34. Further, when the sites $z_{(1,n)}, z_{(2,n)}, \ldots, z_{(k,n)}$ are the locations of the smallest, the second-smallest up to the kth smallest value of π_z for $z \in B_n$, then Lemma 3.45 implies that the smaller the quotient $\pi_{z_{(1,n)}}/\pi_{z_{(2,n)}}$, the more $\psi_1^{(n)}$ tends to concentrate in the site $z_{(1,n)}$. Since F is continuous, these minimizers $z_{(1,n)}, \ldots, z_{(k,n)}$ are $\mathbb{P}$-a.s. unique.

In order to bound the quotient $\pi_{z_{(1,n)}}/\pi_{z_{(2,n)}}$ from above in Section 3.7.3, we collect some further structural properties of the environment in this section.

For what follows it is important to note that with g as defined in (3.8), we have

$$\Lambda_g(n) = n^d \mathbb{P}\Big[\omega \leq \sup\Big\{s\colon F(s) = n^{-1/2}\Big\}\Big]^{2d} = 1\,.$$

We thus have the following lemma.

Lemma 3.43. *Let g be as in* (3.8) *and $\varepsilon_2 \in (0, 1/3)$. Let $b, k \in \mathbb{N}$. Then $\mathbb{P}$-a.s. for n large enough and for all $z \in B_{n+b}$ the edge set $\mathfrak{E}(B_b(z))$ contains at most $3d-1$ edges with conductance less than or equal to $g(n^{1-\varepsilon_2})$. Furthermore, if $\mathscr{D}_n$ is as in Definition 3.34 with $\varepsilon = \varepsilon_2$, then $\mathbb{P}$-a.s. for n large enough the set $\mathscr{I}_n = B_n \backslash \mathscr{D}_n$ is b-sparse and $z_{(1,n)}, \ldots, z_{(k,n)} \in \mathscr{I}_n$.*

Proof. Since Λ_g is constant and therefore bounded, the first claim follows by virtue of Corollary 3.22 (with $m = 2d$ and $\kappa = d$).

Since the function $n \mapsto g(n^{1-\varepsilon_2})$ decreases to zero, $\mathbb{P}$-a.s. for n large enough the cluster $\mathscr{D}_n$ is the unique infinite cluster of the environment $\tilde{\boldsymbol{\omega}}_{g(n^{1-\varepsilon_2})}$. We can thus apply Lemma 3.37 and obtain that $\mathbb{P}$-a.s. for n large enough the set $\mathscr{I}_n$ is b-sparse.

For the last statement: Since the quotient $\Lambda_{g\left((\cdot)^{1-\varepsilon_2/2}\right)}(n)/\log\log n$ diverges for n growing to infinity, Lemma 3.24 implies that $\mathbb{P}$-a.s. for n large enough $\pi_{z_{(1,n)}} \leq \ldots \leq \pi_{z_{(k,n)}} < 2dg(n^{1-\varepsilon_2/2})$. This implies that eventually $z_{(1,n)}, \ldots, z_{(k,n)} \in \mathscr{I}_n$. □

Lemma 3.44. *Let the function g be as in* (3.8)*. Assume that there exists $\varepsilon_1 \in (0, 1)$ such that one of the two cases occurs: g varies regularly at infinity with index $\rho < -(2 + \varepsilon_1)$ or the product $n^{2+\varepsilon_1} g(n)$ converges monotonically* *to zero as n grows to infinity. Further, let $\varepsilon = \varepsilon_2 = \frac{7\varepsilon_1}{8(2+\varepsilon_1)}$ and $\mathscr{D}_n$ be as in Definition 3.34. Then $\mathbb{P}$-a.s. for n large enough*

$$\|\psi_1^{(n)}\|^2_{\ell^2(\mathscr{D}_n)} \leq n^{-\varepsilon_1/2} . \tag{3.53}$$

Proof. We aim to apply Proposition 3.38 to the set $\mathscr{D}_n$. By virtue of Lemma 3.43 with $b = 3d$, it follows that Assumption (i) of Proposition 3.38 is fulfilled $\mathbb{P}$-a.s. for n large enough. Further we choose $\xi > 0$ small enough such that $\mathbb{P}$-a.s. for n large enough Assumptions (ii), (iii) and (iv) are fulfilled. This is possible by virtue of the Lemmas 3.26, 3.27 and 3.28. It follows that there exists $c > 0$ such that $\mathbb{P}$-a.s. for n large enough

$$\mathcal{E}^{\boldsymbol{\omega}}_{\mathscr{D}_n}(f) \geq \Big(2^{d+1}(\log n)^{4d^2} g\big(n^{1-\varepsilon_2}\big)^{-1} + cn^2\Big)^{-1} \|f\|^2_{\ell^2(\mathscr{D}_n)} , \tag{3.54}$$

for any function $f\colon \mathbb{Z}^d \to \mathbb{R}$ with $\operatorname{supp} f \subseteq B_n$. In any case, the assumptions imply that the product $n^{2+\varepsilon_1} g(n)$ converges to zero as n grows to infinity. Therefore $n^2 g\big(n^{1-\varepsilon_2}\big)/(\log n)^{4d^2}$ converges to zero as well. It follows that if $C = 2^{d+1} + 1$, then (3.54) implies that $\mathbb{P}$-a.s. for n large enough

$$\mathcal{E}^{\boldsymbol{\omega}}_{\mathscr{D}_n}(f) \geq \frac{1}{C} \frac{g\big(n^{1-\varepsilon_2}\big)}{(\log n)^{4d^2}} \|f\|^2_{\ell^2(\mathscr{D}_n)} .$$

On the other hand, we know that the term $\Lambda_{g((\cdot)^{1-\varepsilon_3})}(n)/\log\log n$ diverges for any $\varepsilon_3 > 0$. Let us specifically choose $\varepsilon_3 = \varepsilon_1(8(2+\varepsilon_1))^{-1}$. Now we use Theorem 3.5 (i) and the fact that the Dirichlet energy $\mathcal{E}^{\omega}$ majorizes $\mathcal{E}^{\boldsymbol{\omega}}_{\mathscr{D}_n}$ to infer that $\mathbb{P}$-a.s. for n large enough

$$2dg(n^{1-\varepsilon_3}) \geq \lambda_1^{(n)} = \mathcal{E}^{\omega}\Big(\psi_1^{(n)}\Big) \geq \mathcal{E}^{\boldsymbol{\omega}}_{\mathscr{D}_n}\big(\psi_1^{(n)}\big) \geq \frac{1}{C} \frac{g\big(n^{1-\varepsilon_2}\big)}{(\log n)^{4d^2}} \|\psi_1^{(n)}\|^2_{\ell^2(\mathscr{D}_n)} . \tag{3.55}$$

When we solve this inequality for $\|\psi_1^{(n)}\|^2_{\ell^2(\mathscr{D}_n)}$, we obtain that

$$\|\psi_1^{(n)}\|^2_{\ell^2(\mathscr{D}_n)} \leq c_1 C \frac{g(n^{1-\varepsilon_3})(\log n)^{4d^2}}{g(n^{1-\varepsilon_2})} .$$

To finish the proof, we use one of the additional assumptions about g: If g varies regularly at infinity with index $\rho < -(2+\varepsilon_1)$, then we can write $g(n) = n^{\rho} L(n)$ where L varies slowly at infinity. In this case we observe that eventually

$$c_1 C \frac{g(n^{1-\varepsilon_3})(\log n)^{4d^2}}{g(n^{1-\varepsilon_2})} = c_1 C n^{\frac{3\rho\varepsilon_1}{4(2+\varepsilon_1)}} \frac{(\log n)^{4d^2} L(n^{1-\varepsilon_3})}{L(n^{1-\varepsilon_2})} \leq n^{-\varepsilon_1/2} ,$$

which implies the claim. In the other case, i.e., if the product $n^{2+\varepsilon_1}g(n)$ converges monotonically to zero as n tends to infinity, we observe that eventually

$$c_1 C\,\frac{g(n^{1-\varepsilon_3})(\log n)^{4d^2}}{g(n^{1-\varepsilon_2})} \leq c_1 C n^{-(2+\varepsilon_1)(\varepsilon_2-\varepsilon_3)}(\log n)^{4d^2} \leq n^{-\varepsilon_1/2}\,,$$

which implies the claim as well. □

Lemma 3.45. *Let $y, z \in B_n$ with $\pi_z < \pi_y$ and $y \nsim z$. Assume that $\psi_1^{(n)}$ is nonnegative. Further, define $m_y = 2\max_{x\colon x\sim y}\psi_1^{(n)}(x)$. Then the mass $\psi_1^{(n)}(y)$ is bounded from above by*

$$\psi_1^{(n)}(y) \leq \frac{m_y}{1-\frac{\pi_z}{\pi_y}}\,. \tag{3.56}$$

Proof. We assume the contrary, i.e., we assume that

$$m_y\pi_y + \psi_1^{(n)}(y)(\pi_z - \pi_y) < 0\,. \tag{3.57}$$

Then we define a new function $\phi : \mathbb{Z}^d \to \mathbb{R}_+$ by setting

$$\phi(x) = \begin{cases} \psi_1^{(n)}(x)\,, & \text{for } x \notin \{y, z\}\,,\\ m_y\,, & \text{for } x = y\,,\\ \sqrt{\psi_1^{(n)}(y)^2 + \psi_1^{(n)}(z)^2 - m_y^2}\,, & \text{for } x = z\,. \end{cases} \tag{3.58}$$

Note that since (3.57) implies that $\psi_1^{(n)}(y) > m_y$, it must be $\phi(z) > \psi_1^{(n)}(z)$. Obviously, supp $\phi \subseteq B_n$ and $\|\phi\|_2 = 1$. Therefore, by the variational formula (1.21) and Remark 3.3, the Dirichlet energy $\langle\phi, -\mathcal{L}^\omega\phi\rangle$ is larger than the principal Dirichlet eigenvalue $\lambda_1^{(n)}$.

However, the Dirichlet energy $\langle\phi, -\mathcal{L}^\omega\phi\rangle$ of ϕ is given by

$$\begin{aligned} \lambda_1^{(n)} &+ \left[\sum_{x:x\sim y}\omega_{xy}\Big(\psi_1^{(n)}(x) - m_y\Big)^2 - \sum_{x:x\sim y}\omega_{xy}\Big(\psi_1^{(n)}(x) - \psi_1^{(n)}(y)\Big)^2\right]\\ &+ \left[\sum_{x:x\sim z}\omega_{xz}\Big(\psi_1^{(n)}(x) - \phi(z)\Big)^2 - \sum_{x:x\sim z}\omega_{xz}\Big(\psi_1^{(n)}(x) - \psi_1^{(n)}(z)\Big)^2\right]. \end{aligned} \tag{3.59}$$

Evaluation of the first bracketed summand on the right-hand side gives:

$$\sum_{x:x\sim y}\omega_{xy}\Big(\psi_1^{(n)}(x)-m_y\Big)^2-\sum_{x:x\sim y}\omega_{xy}\Big(\psi_1^{(n)}(x)-\psi_1^{(n)}(y)\Big)^2$$
$$=\sum_{x:x\sim y}\omega_{xy}\Big(\psi_1^{(n)}(y)-m_y\Big)\Big(2\psi_1^{(n)}(x)-m_y-\psi_1^{(n)}(y)\Big)$$
$$\le -\psi_1^{(n)}(y)\sum_{x:x\sim y}\omega_{xy}\Big(\psi_1^{(n)}(y)-m_y\Big)\,, \tag{3.60}$$

where the last inequality follows by the definition of m_y and since Assumption (3.57) implies that $\psi_1^{(n)}(y)>m_y$. Further, we evaluate the second bracketed summand in (3.59) as

$$\sum_{x:x\sim z}\omega_{xz}\Big(\psi_1^{(n)}(x)-\phi(z)\Big)^2-\sum_{x:x\sim z}\omega_{xz}\Big(\psi_1^{(n)}(x)-\psi_1^{(n)}(z)\Big)^2$$
$$=\sum_{x:x\sim z}\omega_{xz}\Big(\phi(z)-\psi_1^{(n)}(z)\Big)\Big(\phi(z)+\psi_1^{(n)}(z)-2\psi_1^{(n)}(x)\Big)\,.$$

Since $\psi_1^{(n)}$ is nonnegative and since Assumption (3.57) implies that $\phi(z)>\psi_1^{(n)}(z)$, we conclude that

$$\sum_{x:x\sim z}\omega_{xz}\Big(\psi_1^{(n)}(x)-\phi(z)\Big)^2-\sum_{x:x\sim z}\omega_{xz}\Big(\psi_1^{(n)}(x)-\psi_1^{(n)}(z)\Big)^2$$
$$\le\sum_{x:x\sim z}\omega_{xz}\Big(\phi(z)^2-\psi_1^{(n)}(z)^2\Big)=\sum_{x:x\sim z}\omega_{xz}\Big(\psi_1^{(n)}(y)^2-m_y^2\Big)\,, \tag{3.61}$$

where the last equality follows by the definition of $\phi(z)$. When we insert (3.60) and (3.61) into (3.59), then we obtain that the Dirichlet energy $\langle\phi,-\mathcal{L}^\omega\phi\rangle$ is bounded from above by

$$\lambda_1^{(n)}-\psi_1^{(n)}(y)\sum_{x:x\sim y}\omega_{xy}\Big(\psi_1^{(n)}(y)-m_y\Big)+\sum_{x:x\sim z}\omega_{xz}\Big(\psi_1^{(n)}(y)^2-m_y^2\Big)$$
$$=\lambda_1^{(n)}+\psi_1^{(n)}(y)^2(\pi_z-\pi_y)+m_y\Big(\psi_1^{(n)}(y)\pi_y-m_y\pi_z\Big)$$
$$\le\lambda_1^{(n)}+\psi_1^{(n)}(y)\Big[m_y\pi_y+\psi_1^{(n)}(y)(\pi_z-\pi_y)\Big]\,. \tag{3.62}$$

Under Assumption (3.57) and because $\psi_1^{(n)}(y)$ is nonnegative, it follows that the Dirichlet energy of ϕ is not larger than $\lambda_1^{(n)}$. This is a contradiction to the considerations above. $\square$

Let F_π be as defined before (3.14). Then we have the following two lemmas.

Lemma 3.46. *If there exists* $a^*>0$ *such that* $F(ab)\ge bF(a)$ *for all* $a\le a^*$ *and all* $0\le b\le 1$*, then* $F_\pi(ab)\ge b^{2d}F_\pi(a)$ *for all* $a\le a^*$ *and all* $0\le b\le 1$*.*

Proof. Let $a \le a^*$, $0 \le b \le 1$ and let ω_1, ω_2 be two independent copies of ω. Then

$$\begin{aligned}\mathbb{P}[\omega_1 + \omega_2 \le ab] &= \int_0^\infty \mathbb{P}[\omega_2 \le ab - t]\,\mathrm{d}\mathbb{P}[\omega_1 \le t] \\ &\ge b \int_0^\infty \mathbb{P}\left[\omega_2 \le a - \frac{t}{b}\right] \mathrm{d}\mathbb{P}[\omega_1 \le t]\,.\end{aligned}$$

where we have used that $\mathbb{P}[\omega_2 \le ab - t] \ge b\mathbb{P}[\omega_2 \le a - t/b]$ and $\mathrm{d}\mathbb{P}[\omega_1 \le t] \ge 0$ for all $t \in [0, \infty)$. It follows that

$$\mathbb{P}[\omega_1 + \omega_2 \le ab] \ \ge\ b\mathbb{P}[\omega_2 \le a - \omega_1/b] \ =\ b\mathbb{P}[\omega_1 \le ab - bw_2]\,.$$

Similarly, we infer that $\mathbb{P}[\omega_1 + \omega_2 \le ab] \ge b^2 \mathbb{P}[\omega_1 + \omega_2 \le a]$. The claim follows by induction. □

Lemma 3.47. *If there exists $\gamma \in [0, 1/4)$ such that F varies regularly at zero with index γ, then F_π varies regularly at zero with index $2d\gamma$.*

Proof. Let $\mathscr{L}[F]$ be the Laplace transform of F. Then the Laplace transform of F_π fulfills

$$\mathscr{L}[F_\pi] = (\mathscr{L}[F])^{2d}\,.$$

By virtue of the Tauberian theorems, more precisely by virtue of Theorem 3 in [Fel71, XIII.5] (or, equivalently Theorem 1.7.1' of [BGT89]), $\mathscr{L}[F]$ varies regularly at infinity with index $-\gamma$. It follows that $\mathscr{L}[F_\pi]$ varies regularly at infinity with index $-2d\gamma$. Hence, by another application of Theorem 3 in [Fel71, XIII.5] we obtain that F_π varies regularly at zero with index $2d\gamma$. □

Lemma 3.48. *Let $\sigma_1, \sigma_2, \ldots$ be a sequence of i.i.d. random variables with continuous distribution. For $N \in \mathbb{N}$, let $\sigma_{1,N} \ge \sigma_{2,N} \ge \ldots \ge \sigma_{N,N}$ be the Nth order statistics. Let $a > 0$ and $i, j, k, l, m \in \mathbb{N}$ with $N > j > i$ as well as $N > l > m > k$. Then the events $\{\sigma_{i,N} - \sigma_{j,N} \le a\}$ and $\{\sigma_{k,l} > \sigma_{k,m}\}$ are independent.*

Proof. As in the proof of [Res87, Proposition 4.3], we observe that since the distribution of the σ's is continuous, we can assume without loss of generality that there are no ties between the σ's and we observe that each of the $N!$ orderings $\sigma_{q_1} < \ldots < \sigma_{q_N}$ is equally likely where $q_1, \ldots, q_N$ is a permutation of $1, \ldots, N$. Now we note that whether or not $\{\sigma_{k,l} > \sigma_{k,m}\}$ is univocally given by the specific ordering $(q_1, \ldots, q_N)$. On the other hand, the difference between any two order statistics $\sigma_{i,N}$ and $\sigma_{j,N}$ is completely independent of the ordering $(q_1, \ldots, q_N)$. □

3.7.3 Extreme value analysis

In what follows, we let $\pi_{1,B_n} \leq \pi_{2,B_n} \leq \dots \leq \pi_{|B_n|,B_n}$ denote the order statistics of the set $\{\pi_z : z \in B_n\}$. For $l \in \mathbb{N}$ we let $z_{(l,n)}$ denote the site in B_n that fulfills $\pi_{z_{(l,n)}} = \pi_{l,B_n}$. Note that since F is continuous, the sites $z_{(l,n)}$ are $\mathbb{P}$-a.s. unique. The main lemma of this section is the following one.

Lemma 3.49 (Quotient of order statistics). *Let F be continuous and assume that there exists $a^* > 0$ such that $F(ab) \geq bF(a)$ for all $a \leq a^*$ and all $0 \leq b \leq 1$. Let $\varepsilon > 0$ and $k \in \mathbb{N}$. Then $\mathbb{P}$-a.s. for n large enough*

$$1 - \frac{\pi_{k,B_n}}{\pi_{k+1,B_n}} > n^{-\varepsilon} . \tag{3.63}$$

We prove the lemma after the proof of Lemma 3.50. The main difficulty here is that the random variables in $\{\pi_z : z \in B_n\}$ are not independent. However, as Lemma 3.43 shows, the smallest values are located very far apart and therefore they are *heuristically* independent. In order to make this idea rigorous, we use the notation of Section 3.7.1. For an $x \in B_{k+1}$, let $\chi^{(x)}_{(1,n)} \leq \chi^{(x)}_{(2,n)} \leq \dots$ denote the order statistics of $\big\{\chi_y : y \in \mathbb{Z}^d, y + \mathscr{N} \subset B_{n+2k+1} \cap (x + \mathscr{V})\big\}$. Here, we have to define the order statistics with respect to B_{n+2k+1} since

$$\text{it is } \textbf{not} \text{ necessarily} \quad \bigcup_{\substack{y\in\mathbb{Z}^d,\\ (y+\mathscr{N})\subset x+\mathscr{V},\\ (y+\mathscr{N})\cap B_n\neq\emptyset}} (y + \mathscr{N}) \subseteq B_n \tag{3.64}$$

$$\text{but instead} \quad \bigcup_{\substack{y\in\mathbb{Z}^d,\\ (y+\mathscr{N})\subset x+\mathscr{V},\\ (y+\mathscr{N})\cap B_n\neq\emptyset}} (y + \mathscr{N}) \subseteq B_{n+2k+1} , \tag{3.65}$$

see also Figure 3.4. Our aim is to compare the $\pi_{1,B_n}, \dots, \pi_{k+1,B_n}$ with the $\chi^{(x)}_{(1,n)}, \chi^{(x)}_{(2,n)}, \dots$ for a suitable $x \in B_{k+1}$ and we will explain how to do this in the proof of Lemma 3.50.

Since for any j and any $x \in B_{k+1}$, the random variable $\chi^{(x)}_{(j,n)}$ $\mathbb{P}$-a.s. decreases monotonically to zero as n tends to infinity, it follows that $\mathbb{P}$-a.s. for n large enough $\chi^{(x)}_{(k+2,n-2k-3)} \leq a^*$ with a^* as in Lemma 3.49.

Now we let the function g be as defined in (3.8) and let $\varepsilon_2 \in (1, 1/3)$. Since the quotient $\Lambda_{g\left((\cdot)^{1-\varepsilon_2/2}\right)}(n)/\log\log n$ diverges for n growing to infinity, Lemma 3.24 implies that $\mathbb{P}$-a.s. for n large enough

$$\pi_{1,B_{n-2k-3}} \leq \dots \leq \pi_{k+1,B_{n-2k-3}} < 2dg((n-2k-3)^{1-\varepsilon_2/2}) ,$$

which is eventually smaller than $g(n^{1-\varepsilon_2})$. Since the distribution function F is continuous, it follows that $\mathbb{P}$-a.s. the above inequalities are even strict.

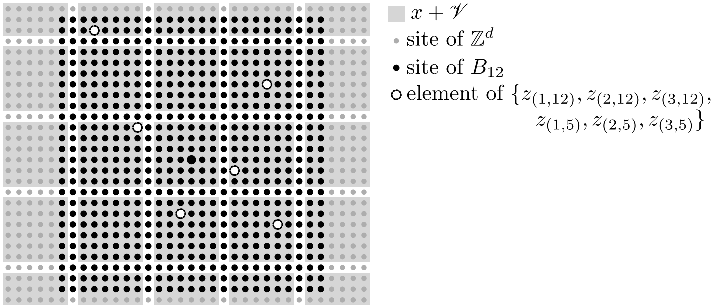


Fig. 3.4: Decomposition of the box B_{12} into subcubes $(y + \mathscr{N})_{y \in (7\mathbb{Z}^d + x)}$ of $x + \mathscr{V}$ with the translation vector $x = (-4, -3) \in B_3$. We have chosen $k = 2$ and thus $\mathscr{N} = \{1, \ldots, 6\}^2$. The gray dots depict the sites of $\mathbb{Z}^2$ and the black dots especially the sites of B_{12}. The slightly larger black circle in the middle is the origin. The gray squares depict the subcubes $y + \mathscr{N}$ and the white dots indicate the sites $\{z_{(1,12)}, z_{(2,12)}, z_{(3,12)}, z_{(1,5)}, z_{(2,5)}, z_{(3,5)}\}$. These six sites will become important in (3.81). In the depicted situation, the choice of $x \in B_3$ is unique given these six sites. Furthermore, this figure illustrates that when we want to account for all the important sites $\{z_{(1,12)}, z_{(2,12)}, z_{(3,12)}, z_{(1,5)}, z_{(2,5)}, z_{(3,5)}\}$, then we have to consider the union of all $(y + \mathscr{N}) \subset (x + \mathscr{V})$ with y such that $y + \mathscr{N} \cap B_{12} \neq \emptyset$. Since we do not want to cut the subcubes at the boundary of B_{12}, some subcubes protrude into the space B_{12}^{c}, see also (3.65).

The above considerations together with Lemma 3.43 imply that the event

$$\begin{aligned} G_n^* := &\left\{\pi_{1,B_{n-2k-3}} < \ldots < \pi_{k+1,B_{n-2k-3}} < g(n^{1-\varepsilon_2})\right\} \\ &\cap \left\{\max_{x \in B_{k+1}} \chi^{(x)}_{(k+2,n-2k-3)} \leq a^*\right\} \\ &\cap \left\{\forall z \in B_{n+2k+1} \colon \left|\left\{e \in \mathfrak{E}(B_{k+1}(z)) \colon \omega(e) \leq g(n^{1-\varepsilon_2})\right\}\right| \leq 3d - 1\right\} \end{aligned}$$

occurs $\mathbb{P}$-a.s. for n large enough, i.e., $\mathbb{P}[\liminf_{n\to\infty} G_n^*] = 1$.

Here comes the lemma that connects the $\pi_{1,B_n}, \ldots, \pi_{k+1,B_n}$ with the $\chi^{(x)}_{(1,n)}, \chi^{(x)}_{(2,n)}, \ldots$. Let us define the event G_n as

$$\begin{aligned} G_n^* \cap \bigcup_{x \in B_{k+1}} \Bigg(&\left\{\left\{z_{(1,n)}, \ldots, z_{(k+1,n)}, z_{(1,n-2k-3)}, \ldots, z_{(k+1,n-2k-3)}\right\} \subset x + \mathscr{V}\right\} \\ &\cap \left\{\left\{\pi_{k,B_n}, \pi_{k+1,B_n}\right\} \subset \left\{\chi^{(x)}_{(k,n)}, \chi^{(x)}_{(k+1,n)}, \chi^{(x)}_{(k+2,n)}\right\}\right\} \\ &\cap \left\{\pi_{k+1,n-2k-3} \in \left\{\chi^{(x)}_{(k+1,n-2k-3)}, \chi^{(x)}_{(k+2,n-2k-3)}\right\}\right\}\Bigg). \end{aligned} \tag{3.66}$$

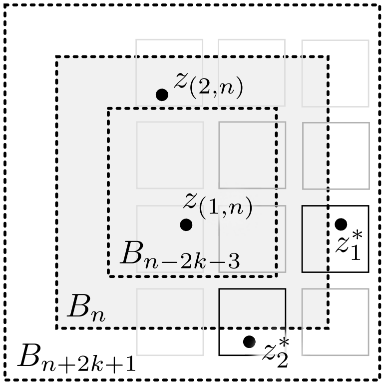


Fig. 3.5: Let $k=1$ and $z_{(1,n)}, z_{(2,n)} \in x + \mathscr{V}$. If $\pi_{2,B_n} > \chi^{(x)}_{(3,n)}$, then there exist two sites $z_1^*, z_2^* \in (B_{n+2k+1}\backslash B_n) \cap x + \mathscr{V}$ with $\pi_{z_1^*} < \pi_{2,B_n}$ and $\pi_{z_2^*} < \pi_{2,B_n}$: The first one exists because $\pi_{2,B_n} > \chi^{(x)}_{(2,n)}$ and the other one exists because even $\pi_{2,B_n} > \chi^{(x)}_{(3,n)}$. On the event G_n^* these two sites z_1^*, z_2^* have to be located in different cubes $y_1^* + \mathscr{N}$ and $y_2^* + \mathscr{N}$, indicated here as the small squares. Since these subcubes contain an element in $B_{n+2k+1}\backslash B_n$ and have side length $2(k+1)$, it follows that $(y^*_{1,2} + \mathscr{N}) \cap B_{n-2k-3} = \emptyset$.

Lemma 3.50.

$$\mathbb{P}\Big[\liminf_{n\to\infty} G_n\Big] = 1\,. \tag{3.67}$$

Proof. In what follows, we always assume the event G_n^*, which occurs $\mathbb{P}$-a.s. for n large enough. Since

$$G_n^* \subset \Big\{\forall z \in B_{n+2k+1}\colon \Big|\Big\{e \in \mathfrak{E}(B_{k+1}(z))\colon \omega(e) \le g(n^{1-\varepsilon_2})\Big\}\Big| \le 3d-1\Big\}\,,$$

there is no $y \in \mathbb{Z}^d$ such that $y + \mathscr{N}$ contains more than one site z with $\pi_z < g(n^{1-\varepsilon_2})$. Since for any $l \in \mathbb{N}$ the value of π_{l,B_n} is monotonically decreasing in n, we also have

$$G_n^* \subset \Big\{\pi_{1,B_n} < \ldots < \pi_{k+1,B_n} < g(n^{1-\varepsilon_2})\Big\}\,.$$

This especially means that there is no $y \in \mathbb{Z}^d$ such that $y + \mathscr{N}$ contains more than one site of $\big\{z_{(1,n)}, \ldots, z_{(k+1,n)}\big\}$. It moreover implies that if there is a site $z \in B_{n+2k+1}\backslash B_n$ such that $\pi_z \le \pi_{k+1,B_n}$, then this site cannot be in the same subcube $y + \mathscr{N}$ with any of the $\big\{z_{(1,n)}, \ldots, z_{(k+1,n)}\big\}$. Since $\chi_y = \min_{z\in(y+\mathscr{N})} \pi_z$, it follows that if $z_{(l,n)} \in (y+\mathscr{N})$, then $\pi_{l,B_n} = \chi_y$. Therefore, on the event $\big\{\big\{z_{(1,n)}, \ldots, z_{(k+1,n)}\big\} \subset x + \mathscr{V}\big\}$, this has two implications: First,

$$\pi_{k,B_n} \ge \chi^{(x)}_{(k,n)}\,, \tag{3.68}$$

where equality does not necessarily hold since there might exist a site $z \in B_{n+2k+1}\backslash B_n$ such that $\pi_z \le \pi_{k,B_n}$. Second, with (3.65) we infer that

$$\big\{\pi_{1,B_n}, \ldots, \pi_{k+1,B_n}\big\} \subset \Big\{\chi_y\colon y \in \mathbb{Z}^d, (y+\mathscr{N}) \subset B_{n+2k+1} \cap (x+\mathscr{V})\Big\}\,. \tag{3.69}$$

Let us now show that on the event $\{\{z_{(1,n)},\dots,z_{(k+1,n)}\}\subset x+\mathscr{V}\}$ $\mathbb{P}$-a.s. for n large enough

$$\pi_{k+1,B_n}\le \chi^{(x)}_{(k+2,n)}\,. \tag{3.70}$$

We assume the counter event, i.e., that $\pi_{k+1,B_n}>\chi^{(x)}_{(k+2,n)}$. It follows that there exist two sites $z_1^*,z_2^*\in(B_{n+2k+1}\backslash B_n)\cap x+\mathscr{V}$ with $\pi_{z_1^*}<\pi_{k+1,B_n}$ and $\pi_{z_2^*}<\pi_{k+1,B_n}$. Since on G_n^* each edge set $\mathfrak{E}(y+\mathscr{N})$ with $y\in B_{n+2k+1}$ contains at most $3d-1$ edges with conductance less than or equal to $g(n^{1-\varepsilon_2})$ and further $\pi_{k+1,B_n}<g(n^{1-\varepsilon_2})$, it follows that for n large enough, these two sites z_1^*,z_2^* have to be located in different cubes $y_1^*+\mathscr{N}$ and $y_2^*+\mathscr{N}$, with

$$\begin{aligned} y_1^*,y_2^*\in\{y\in((2k+3)\mathbb{Z}^d+x)\colon (y+\mathscr{N})\cap B_n\neq\emptyset,&\\ (y+\mathscr{N})\cap B_{n-2k-3}=\emptyset\}\,,&\end{aligned} \tag{3.71}$$

see also Figure 3.5. Thus $\chi_{y_1^*}$ and $\chi_{y_2^*}$ are new records in the sense that both $\chi_{y_1^*}<\chi^{(x)}_{(k+1,n-2k-3)}$ and $\chi_{y_2^*}<\chi^{(x)}_{(k+1,n-2k-3)}$. Now we observe that the cardinality of the set on the right-hand side of (3.71) is of order n^{d-1}. Further, by virtue of [Res87, Proposition 4.3], the probability that one specific value χ_{y^*} with y^* in the set on the right-hand side of (3.71) fulfills $\chi_{y^*}<\chi^{(x)}_{(k+1,n-2k-3)}$, is of order n^{-d}. It follows that the probability of the event $\pi_{k+1,B_n}>\chi^{(x)}_{(n,k+2)}$ is of order $\left(n^{d-1}/n^d\right)^2=n^{-2}$ and thus the claim (3.70) follows by the Borel-Cantelli lemma.

Because of (3.69) and since by definition $\pi_{k,B_n}\le\pi_{k+1,B_n}$, it follows that $\mathbb{P}$-a.s. for n large enough both values π_{k,B_n} and π_{k+1,B_n} are in $\left\{\chi^{(x)}_{(k,n)},\chi^{(x)}_{(k+1,n)},\chi^{(x)}_{(k+2,n)}\right\}$.

Similar to the considerations leading to (3.68), (3.69) and (3.70), we infer that on the event $\{\{z_{(1,n-2k-3)},\dots,z_{(k+1,n-2k-3)}\}\subset x+\mathscr{V}\}$ we have $\mathbb{P}$-a.s. for n large enough

$$\pi_{k+1,B_{n-2k-3}}\in\left\{\chi^{(x)}_{(k+1,n-2k-3)},\chi^{(x)}_{(k+2,n-2k-3)}\right\}.$$

□

Proof of Lemma 3.49. Without loss of generality we may and will assume that $\varepsilon<1$.

We abbreviate $E_n=\left\{n^{-\varepsilon}\ge 1-\frac{\pi_{k,B_n}}{\pi_{k+1,B_n}}\right\}$. Since $\mathbb{P}[\limsup_{n\to\infty}G_n^{\mathrm{c}}]=0$ by Lemma 3.50, we already know that

$$\mathbb{P}\Big[\limsup_{n\to\infty} E_n\Big] = \mathbb{P}\Big[\limsup_{n\to\infty} (E_n \cap G_n)\Big]$$

$$= \lim_{n\to\infty} \mathbb{P}\Bigg[(E_n \cap G_n) \cup \ldots \cup (E_{n+2k+2} \cap G_{n+2k+2}) \cup \bigcup_{m=n+2k+3}^{\infty} (E_m \cap G_m \cap (E_{m-2k-3} \cap G_{m-2k-3})^{\mathrm{c}})\Bigg].$$

Since again $\mathbb{P}[\limsup_{m\to\infty} G_m^{\mathrm{c}}] = 0$, it follows that

$$\mathbb{P}\Big[\limsup_{n\to\infty} E_n\Big] \leq (2k+3) \limsup_{n\to\infty} \mathbb{P}[E_n \cap G_n] + + \lim_{n\to\infty} \mathbb{P}\Bigg[\bigcup_{m=n+2k+3}^{\infty} \big(E_m \cap G_m \cap E_{m-2k-3}^{\mathrm{c}}\big)\Bigg]. \quad (3.72)$$

We now treat the first and second term on the above right-hand side separately.

Let us first show that

$$\limsup_{n\to\infty} \mathbb{P}[E_n \cap G_n] = 0\,. \quad (3.73)$$

We decompose $E_n \cap G_n$ as

$$E_n \cap G_n = \bigcup_{x \in B_{k+1}} \big(E_n \cap G_n \cap \big\{\{z_{(1,n)}, \ldots, z_{(k+1,n)}\} \subset x + \mathscr{V}\big\}\big) \quad (3.74)$$

and note that on the event $G_n \cap \big\{\{z_{(1,n)}, \ldots, z_{(k+1,n)}\} \subset x + \mathscr{V}\big\}$ we have

$$\frac{\pi_{k,B_n}}{\pi_{k+1,B_n}} \leq \max_{k \leq i < j \leq k+2} \frac{\chi^{(x)}_{(i,n)}}{\chi^{(x)}_{(j,n)}} \quad (3.75)$$

by definition of the event G_n in (3.66).

Now we define F_χ as in Lemma 3.41. By definition, F_χ is an increasing function and thus on the event $G_n \cap \big\{z_{(1,n)}, \ldots, z_{(k+1,n)} \in x + \mathscr{V}\big\}$ it follows that $n^{-\varepsilon} > 1 - \pi_{k,B_n}/\pi_{k+1,B_n}$ implies that

$$\Bigg(\exists\, k \leq i < j \leq k+2 : F_\chi\Big(\chi^{(x)}_{(i,n)}\Big) \geq F_\chi\Big((1-n^{-\varepsilon})\chi^{(x)}_{(j,n)}\Big)\Bigg).$$

Our next aim is to extract the factor $(1 - n^{-\varepsilon})$ from inside the function argument of F_χ. Since on the event G_n we have $\chi^{(x)}_{(j,n)} \leq a^*$ for all $j \leq k+2$, we estimate by virtue of Lemmas 3.41 and 3.46

$$
\begin{aligned}
&F_\chi\Big((1-n^{-\varepsilon})\chi^{(x)}_{(j,n)}\Big)\\
&\overset{3.41}{\geq} (2k+2)^d F_\pi\Big((1-n^{-\varepsilon})\chi^{(x)}_{(j,n)}\Big) - \tbinom{(2k+2)^d}{2} F\Big((1-n^{-\varepsilon})\chi^{(x)}_{(j,n)}\Big)^{4d-1}\\
&\overset{3.46}{\geq} (1-n^{-\varepsilon})^{2d}(2k+2)^d F_\pi\Big(\chi^{(x)}_{(j,n)}\Big) - \tbinom{(2k+2)^d}{2} F\Big((1-n^{-\varepsilon})\chi^{(x)}_{(j,n)}\Big)^{4d-1}\\
&\overset{3.41}{\geq} (1-n^{-\varepsilon})^{2d} F_\chi\Big(\chi^{(x)}_{(j,n)}\Big) - \tbinom{(2k+2)^d}{2} F\Big((1-n^{-\varepsilon})\chi^{(x)}_{(j,n)}\Big)^{4d-1}.
\end{aligned}
$$

Thus, on the event $G_n \cap \big\{\{z_{(1,n)},\dots,z_{(k+1,n)}\} \subset x+\mathscr{V}\big\}$ the event E_n implies that there exist $k \leq i < j \leq k+2$ such that

$$
\frac{F_\chi\Big(\chi^{(x)}_{(i,n)}\Big)}{F_\chi\Big(\chi^{(x)}_{(j,n)}\Big)} \geq (1-n^{-\varepsilon})^{2d} - \binom{(2k+2)^d}{2}\frac{F\Big((1-n^{-\varepsilon})\chi^{(x)}_{(j,n)}\Big)^{4d-1}}{F_\chi\Big(\chi^{(x)}_{(j,n)}\Big)}.
$$

Now we observe that by virtue of Lemma 3.42, the random variable $F_\chi(\chi)$ is uniform on $[0,1]$. It follows that the random variable $\sigma := -\log F_\chi(\chi)$ is exponentially distributed with parameter 1. In analogy to the definitions above, we define $\sigma_z := -\log F_\chi(\chi_z)$, $\sigma^{(x)}_{(j,n)} := -\log \chi^{(x)}_{(j,n)}$ for $j=1,\dots,k+2$.

Thus, using the decomposition (3.74), we can bound $\mathbb{P}[E_n \cap G_n]$ by

$$
\begin{aligned}
&\mathbb{P}[E_n \cap G_n]\\
&\leq \sum_{x\in B_{k+1}} \sum_{\substack{i,j=k,\\ i<j}}^{k+2} \mathbb{P}\Bigg[G_n \cap \big\{\{z_{(1,n)},\dots,z_{(k+1,n)}\} \subset x+\mathscr{V}\big\}\\
&\quad \cap \Bigg\{\sigma^{(x)}_{(i,n)} - \sigma^{(x)}_{(j,n)} \leq -\log\Bigg((1-n^{-\varepsilon})^{2d} - \tbinom{(2k+2)^d}{2}\frac{F\big((1-n^{-\varepsilon})\chi^{(x)}_{(j,n)}\big)^{4d-1}}{F_\chi\big(\chi^{(x)}_{(j,n)}\big)}\Bigg)\Bigg\}\Bigg]\\
&\leq \sum_{x\in B_{k+1}} \sum_{k\leq i<j\leq k+2} \mathbb{P}\Big[\sigma^{(x)}_{(i,n)} - \sigma^{(x)}_{(j,n)} \leq -\log\big((1-n^{-\varepsilon})^{2d} - n^{-\varepsilon}\big)\Big] +\\
&\qquad + \sum_{x\in B_{k+1}} \sum_{k\leq i<j\leq k+2} \mathbb{P}\Bigg[\binom{(2k+2)^d}{2}\frac{F\big((1-n^{-\varepsilon})\chi^{(x)}_{(j,n)}\big)^{4d-1}}{F_\chi\big(\chi^{(x)}_{(j,n)}\big)} > n^{-\varepsilon}\Bigg].
\end{aligned}
\tag{3.76}
$$

In the first summand on the above right-hand side we have the difference between any pair of the kth to $(k+2)$th largest values of a sequence of independent exponential variables with parameter 1. By virtue of [Dev86, Chapter 5, Theorem 2.3], we know that the normalized spacings $\Big\{i\cdot\Big(\sigma^{(x)}_{(i,n)} - \sigma^{(x)}_{(i+1,n)}\Big)\Big\}_{i=k,k+1}$ are i.i.d. exponential variables with parameter 1. It follows that

$$\sum_{k\le i<j\le k+2} \mathbb{P}\left[\sigma^{(x)}_{(i,n)} - \sigma^{(x)}_{(j,n)} \le -\log\bigl((1-n^{-\varepsilon})^{2d} - n^{-\varepsilon}\bigr)\right]$$
$$\le 3\,\mathbb{P}\left[\sigma^{(x)}_{(k+1,n)} - \sigma^{(x)}_{(k+2,n)} \le -\log\bigl((1-n^{-\varepsilon})^{2d} - n^{-\varepsilon}\bigr)\right]$$
$$= 3\Bigl(1-\mathrm{e}^{(k+1)\log((1-n^{-\varepsilon})^{2d}-n^{-\varepsilon})}\Bigr) \le 3(k+1)(2d+1)n^{-\varepsilon}\,, \qquad (3.77)$$

which converges to zero.

For the second summand on the right-hand side of (3.76), we infer that since F is increasing, and by virtue of Lemma 3.41, that

$$\mathbb{P}\left[\tbinom{(2k+2)^d}{2}\frac{F\bigl((1-n^{-\varepsilon})\chi^{(x)}_{(j,n)}\bigr)^{4d-1}}{F_\chi\bigl(\chi^{(x)}_{(j,n)}\bigr)} > n^{-\varepsilon}\right] \le \mathbb{P}\left[\tbinom{(2k+2)^d}{2}n^{\varepsilon} > \frac{F_\chi\bigl(\chi^{(x)}_{(j,n)}\bigr)}{F\bigl(\chi^{(x)}_{(j,n)}\bigr)^{4d-1}}\right]$$
$$\le \mathbb{P}\left[\tbinom{(2k+2)^d}{2}(n^{\varepsilon}+1) > \frac{(2k+2)^d F_\pi\bigl(\chi^{(x)}_{(j,n)}\bigr)}{F\bigl(\chi^{(x)}_{(j,n)}\bigr)^{4d-1}}\right].$$

Since π is the sum of $2d$ independent copies of the conductance ω, we can bound $F_\pi(a) \ge F(a/(2d))^{2d}$ for all $a \ge 0$. Together with the assumption in the present lemma this implies that

$$F_\pi(a) \ge (2d)^{-2d}F(a)^{2d} \qquad \text{for all } a \le a^* \qquad (3.78)$$

and therefore

$$\mathbb{P}\left[\binom{(2k+2)^d}{2}(n^{\varepsilon}+1) > \frac{(2k+2)^d F_\pi\bigl(\chi^{(x)}_{(j,n)}\bigr)}{F\bigl(\chi^{(x)}_{(j,n)}\bigr)^{4d-1}}\right]$$
$$\le \mathbb{P}\left[\bigl((2k+2)^d-1\bigr)n^{\varepsilon} > (2d)^{-2d}F\Bigl(\chi^{(x)}_{(j,n)}\Bigr)^{1-2d}\right]$$

where we have furthermore used that $n^{\varepsilon} \ge 1$ for all $n \in \mathbb{N}$. Since F is continuous and increasing, it follows that there exists a constant $A < \infty$ such that

$$\mathbb{P}\left[\bigl((2k+2)^d-1\bigr)n^{\varepsilon} > (2d)^{-2d}F\Bigl(\chi^{(x)}_{(j,n)}\Bigr)^{1-2d}\right]$$
$$\le \mathbb{P}\left[\chi^{(x)}_{(j,n)} > \inf\bigl\{b\colon F(b) = An^{-\frac{\varepsilon}{2d-1}}\bigr\}\right].$$

Let β_n be the cardinality of the set $\bigl\{y \in \mathbb{Z}^d\colon (y+\mathscr{N}) \subset B_{n+2k+1} \cap (x+\mathscr{V})\bigr\}$. Then for any $a \ge 0$ and $j \in \{k, k+1, k+2\}$ we know that

$$\begin{aligned}\mathbb{P}\Big[\chi^{(x)}_{(j,n)} > a\Big] &\le \mathbb{P}\Big[\chi^{(x)}_{(k+2,n)} > a\Big] \\ &= \mathbb{P}[\chi > a]^{\beta_n} + \beta_n \mathbb{P}[\chi > a]^{\beta_n - 1}(1 - \mathbb{P}[\chi > a]) \\ &\quad + \ldots + \binom{\beta_n}{k+1}\mathbb{P}[\chi > a]^{\beta_n - k - 1}(1 - \mathbb{P}[\chi > a])^{k+1} \\ &\le (k+2)\beta_n^{k+1}\mathbb{P}[\chi > a]^{\beta_n - k - 1}\,.\end{aligned}$$

By virtue of Lemma 3.41 and (3.78) we thus obtain for all $0 \le a \le a^*$ that

$$\mathbb{P}\left[\chi^{(x)}_{(j,n)} > a\right] \le (k+2)\beta_n^{k+1}\Big(1 - \big(\tfrac{k+1}{2d^2}\big)^d F(a)^{2d} + \big(\tfrac{(2k+2)^d}{2}\big) F(a)^{4d-1}\Big)^{\beta_n - k - 1}.$$

We now insert $a = \inf\big\{s\colon F(s) = An^{-\frac{\varepsilon}{2d-1}}\big\}$ and observe that, with this choice, $F(a)^{4d-1}$ is $o\big(F(a)^{2d}\big)$ viewed as functions of n. Furthermore, there exist $C_1, C_2 \in (0,\infty)$ such that $C_1 n^d + k + 1 \le \beta_n \le C_2 n^d$. It follows that there exists $B > 0$ depending on the dimension d and the index k such that for n large enough

$$\mathbb{P}\left[\chi^{(x)}_{(j,n)} > F^{-1}\big(An^{-\frac{\varepsilon}{2d-1}}\big)\right] \le (k+2)C_2^{k+1} n^{2d}\Big(1 - Bn^{-\frac{2d\varepsilon}{2d-1}}\Big)^{C_1 n^d}. \tag{3.79}$$

Since we have assumed that $\varepsilon < 1$ at the beginning of this proof, this converges to zero and is even summable. This concludes the proof of (3.73).

Let us now treat the second term on the right-hand side of (3.72). We split the event E_m into

$$\big(E_m \cap \big\{\pi_{k+1,B_m} < \pi_{k+1,B_{m-2k-3}}\big\}\big) \cup \big(E_m \cap \big\{\pi_{k+1,B_m} = \pi_{k+1,B_{m-2k-3}}\big\}\big)$$

and we observe that

$$\mathbb{P}\big[E_m \cap \big\{\pi_{k+1,B_m} = \pi_{k+1,B_{m-2k-3}}\big\} \cap E^{\mathrm{c}}_{m-2k-3}\big] = 0\,. \tag{3.80}$$

This is because since F is continuous, we have $\mathbb{P}$-a.s. $\pi_{1,B_m} < \pi_{2,B_m} < \ldots < \pi_{|B_m|,B_m}$. Therefore $\pi_{k+1,B_m} = \pi_{k+1,B_{m-2k-3}}$ $\mathbb{P}$-a.s. implies that when the box B_{m-2k-3} was enlarged to B_m, there was no new record value smaller than $\pi_{k+1,B_{m-2k-3}}$. On this event E^{c}_{m-2k-3} implies E^{c}_m.

Equation (3.80) implies that

$$\begin{aligned}&\lim_{n\to\infty}\mathbb{P}\left[\bigcup_{m=n+2k+3}^{\infty}\big(E_m \cap G_m \cap E^{\mathrm{c}}_{m-2k-3}\big)\right] \\ &\qquad\le \lim_{n\to\infty}\sum_{m=n+2k+3}^{\infty}\mathbb{P}\big[E_m \cap G_m \cap \big\{\pi_{k+1,B_m} < \pi_{k+1,B_{m-2k-3}}\big\}\big]\,.\end{aligned}$$

By Lemma 3.40 we decompose and estimate

$$
\begin{aligned}
&\mathbb{P}\Big[E_m \cap G_m \cap \big\{\pi_{k+1,B_m} < \pi_{k+1,B_{m-2k-3}}\big\}\Big] \\
&\quad \leq \sum_{x \in B_{k+1}} \mathbb{P}\Big[E_m \cap G_m \cap \big\{\pi_{k+1,B_m} < \pi_{k+1,B_{m-2k-3}}\big\} \\
&\qquad\qquad \cap \ \big\{\{z_{(1,m)}, \dots, z_{(k+1,m)}\} \subset x + \mathscr{V}\big\} \\
&\qquad\qquad \cap \big\{\{z_{(1,m-2k-3)}, \dots, z_{(k+1,m-2k-3)}\} \subset x + \mathscr{V}\big\}\Big]. \qquad (3.81)
\end{aligned}
$$

On the event $G_n \cap \big\{\{z_{(1,m-2k-3)}, \dots, z_{(k+1,m-2k-3)}\} \subset x + \mathscr{V}\big\}$, we know that

$$
\pi_{k+1,B_{m-2k-3}} \in \Big\{\chi^{(x)}_{(k+1,m-2k-3)}, \chi^{(x)}_{(k+2,m-2k-3)}\Big\}.
$$

Therefore we further decompose the sample space Ω by

$$
\begin{aligned}
\Omega = \bigcup_{i=k}^{k+1} \bigcup_{j=i+1}^{k+2} \bigcup_{l=k+1}^{k+2} \Big\{\pi_{k,B_m} = \chi^{(x)}_{(i,m)}\Big\} \cap \Big\{\pi_{k+1,B_m} = \chi^{(x)}_{(j,m)}\Big\} \\
\cap \Big\{\pi_{k+1,B_{m-2k-3}} = \chi^{(x)}_{(l,m-2k-3)}\Big\}.
\end{aligned}
$$

And this we split into the three cases $l < j$, $l = j$ and $l > j$, i.e.,

$$
\begin{aligned}
&\sum_{x \in B_{k+1}} \sum_{i=k}^{k+1} \sum_{j=i+1}^{k+2} \sum_{l=k+1}^{k+2} \mathbb{P}[\dots] \\
&\quad = \sum_{x \in B_{k+1}} \sum_{i=k}^{k+1} \left(\sum_{j=i+1}^{k+2} \sum_{l=k+1}^{j-1} \mathbb{P}[\dots] + \sum_{\substack{j=i+1, \\ l=j}}^{k+2} \mathbb{P}[\dots] + \sum_{j=i+1}^{k+2} \sum_{l=j+1}^{k+2} \mathbb{P}[\dots] \right). \qquad (3.82)
\end{aligned}
$$

Let us consider the first term on the above right-hand side. It has to be $j = k+2$ and $l = k+1$. Thus it contains

$$
\begin{aligned}
\mathbb{P}\Big[E_m \cap G_m \cap \big\{\{z_{(1,m)}, \dots, z_{(k+1,m)}\} \subset x + \mathscr{V}\big\} \\
\cap \big\{\{z_{(1,m-2k-3)}, \dots, z_{(k+1,m-2k-3)}\} \subset x + \mathscr{V}\big\} \\
\cap \big\{\pi_{k+1,B_m} < \pi_{k+1,B_{m-2k-3}}\big\} \cap \Big\{\pi_{k,B_m} = \chi^{(x)}_{(i,m)}\Big\} \\
\cap \Big\{\pi_{k+1,B_m} = \chi^{(x)}_{(k+2,m)}\Big\} \cap \Big\{\pi_{k+1,B_{m-2k-3}} = \chi^{(x)}_{(k+1,m-2k-3)}\Big\}\Big],
\end{aligned}
$$

which is less than or equal to $\mathbb{P}\Big[\chi^{(x)}_{(k+2,m)} < \chi^{(x)}_{(k+1,m-2k-3)}\Big]$. But if $\chi^{(x)}_{(k+2,m)} < \chi^{(x)}_{(k+1,m-2k-3)}$, then there are at least two sites

$$y_1^*, y_2^* \in \big\{y \in ((2k+3)\mathbb{Z}^d + x)\colon (y+\mathscr{N}) \cap B_k \neq \emptyset, (y+\mathscr{N}) \cap B_{m-2k-3} = \emptyset\big\}$$

such that $\chi_{y_1^*}, \chi_{y_2^*} < \chi^{(x)}_{(k+1,m-2k-3)}$. Since the cubes $(y+\mathscr{N})$, $y \in ((2k+3)\mathbb{Z}^d + x)$, are disjoint, the probability that this happens is of order m^{-2}, see e.g. the considerations in the proof of Lemma 3.50 or [Res87, Proposition 4.3]. This implies that the sum over m is finite.

Now we consider the second term on the right-hand side of (3.82). It contains

$$\begin{aligned}
\sum_{j=i+1}^{k+2} \mathbb{P}\big[E_m \cap G_m \cap \big\{z_{(1,m)}, \dots, z_{(k+1,m)}\big\} \subset x + \mathscr{V}\big\} \qquad\qquad\qquad\qquad& \\
\cap \big\{z_{(1,m-2k-3)}, \dots, z_{(k+1,m-2k-3)}\big\} \subset x + \mathscr{V}\big\}& \\
\cap\ \big\{\pi_{k+1,B_m} < \pi_{k+1,B_{m-2k-3}}\big\} \cap \Big\{\pi_{k,B_m} = \chi^{(x)}_{(i,m)}\Big\}& \\
\cap \Big\{\pi_{k+1,B_m} = \chi^{(x)}_{(j,m)}\Big\} \cap \Big\{\pi_{k+1,B_{m-2k-3}} = \chi^{(x)}_{(j,m-2k-3)}\Big\}\Big]& \\
\le \sum_{j=i+1}^{k+2} \mathbb{P}\left[\left\{m^{-\varepsilon} > 1 - \frac{\chi^{(x)}_{(i,m)}}{\chi^{(x)}_{(j,m)}}\right\} \cap \Big\{\chi^{(x)}_{(j,m)} < \chi^{(x)}_{(j,m-2k-3)}\Big\}\right]& \\
\le \sum_{j=i+1}^{k+2} \mathbb{P}\Big[\Big\{\sigma^{(x)}_{(i,m)} - \sigma^{(x)}_{(j,m)} \le -\log\big((1-m^{-\varepsilon})^{2d} - m^{-\varepsilon}\big)\Big\}& \\
\cap \Big\{\sigma^{(x)}_{(j,m)} > \sigma^{(x)}_{(j,m-2k-3)}\Big\}\Big]& \\
+ \sum_{j=i+1}^{k+2} \mathbb{P}\left[\binom{(2k+2)^d}{2} \frac{F\Big((1-m^{-\varepsilon})\chi^{(x)}_{(j,m)}\Big)^{4d-1}}{F_\chi\Big(\chi^{(x)}_{(j,m)}\Big)} > m^{-\varepsilon}\right]&,
\end{aligned}$$

where we have applied the same considerations as for (3.76). The second summand on the above right-hand side is already summable over m as we have shown in (3.79). For the first term we recall that the $\sigma^{(x)}_z$ are independent exponential random variables and by virtue of Lemma 3.48 the two events $\Big\{\sigma^{(x)}_{(i,m)} - \sigma^{(x)}_{(j,m)} \le -\log\big((1-m^{-\varepsilon})^{2d} - m^{-\varepsilon}\big)\Big\}$ and $\Big\{\sigma^{(x)}_{(j,m)} > \sigma^{(x)}_{(j,m-2k-3)}\Big\}$ are independent. The probability of the event

$$\Big\{\sigma^{(x)}_{(i,m)} - \sigma^{(x)}_{(j,m)} \le -\log\big((1-m^{-\varepsilon})^{2d} - m^{-\varepsilon}\big)\Big\}$$

is of order $m^{-\varepsilon}$, see (3.77), whereas the probability of the event

$$\left\{\sigma^{(x)}_{(j,m)} > \sigma^{(x)}_{(j,m-2k-3)}\right\}$$

is of order m^{-1}, see e.g. [Res87, Proposition 4.3]. It follows that the second term on the right-hand side of (3.82) is summable over m as well.

Now we consider the third term on the right-hand side of (3.82). Here, $i = k$, $j = k+1$ and $l = k+2$. Since thus $\pi_{k+1,B_{m-2k-3}} = \chi^{(x)}_{(k+2,m-2k-3)}$, it follows that there exists a site $z^* \in (B_{m-2} \backslash B_{m-2k-3}) \cap x + \mathscr{V}$ such that $\pi_{z^*} < \pi_{k+1,B_{m-2k-3}}$. Thus, the cube $y + \mathscr{N}$ with $y \in (2k+3)\mathbb{Z}^d + x$ that contains this site z^*, is associated with a χ_y that is a new record in the sense that $\chi_y < \chi^{(x)}_{(k+1,m-4k-6)}$. It follows that $\chi^{(x)}_{(k+1,m-2k-3)} < \chi^{(x)}_{(k+1,m-4k-6)}$. Therefore we arrive at

$$\begin{aligned}
&\mathbb{P}\Big[E_m \cap G_m \cap \big\{z_{(1,m)}, \ldots, z_{(k+1,m)}\big\} \subset x + \mathscr{V}\big\} \\
&\qquad \cap \big\{z_{(1,m-2k-3)}, \ldots, z_{(k+1,m-2k-3)}\big\} \subset x + \mathscr{V}\big\} \\
&\qquad \cap \big\{\pi_{k+1,B_m} < \pi_{k+1,B_{m-2k-3}}\big\} \cap \Big\{\pi_{k,B_m} = \chi^{(x)}_{(k,m)}\Big\} \\
&\qquad \cap \Big\{\pi_{k+1,B_m} = \chi^{(x)}_{(k+1,m)}\Big\} \cap \Big\{\pi_{k+1,B_{m-2k-3}} = \chi^{(x)}_{(k+2,m-2k-3)}\Big\}\Big] \\
&\quad \le \mathbb{P}\left[\left\{m^{-\varepsilon} > 1 - \frac{\chi^{(x)}_{(k,m)}}{\chi^{(x)}_{(k+1,m)}}\right\} \cap \Big\{\chi^{(x)}_{(k+1,m-2k-3)} < \chi^{(x)}_{(k+1,m-4k-6)}\Big\}\right] \\
&\quad \le \mathbb{P}\Big[\Big\{\sigma^{(x)}_{(k,m)} - \sigma^{(x)}_{(k+1,m)} \le -\log\big((1-m^{-\varepsilon})^{2d} - m^{-\varepsilon}\big)\Big\} \\
&\qquad\qquad \cap \Big\{\sigma^{(x)}_{(k+1,m-2k-3)} > \sigma^{(x)}_{(k+1,m-4k-6)}\Big\}\Big] \\
&\qquad + \mathbb{P}\left[\binom{(2k+2)^d}{2} \frac{F\Big((1-m^{-\varepsilon})\chi^{(x)}_{(k+1,m)}\Big)^{4d-1}}{F_\chi\Big(\chi^{(x)}_{(k+1,m)}\Big)} > m^{-\varepsilon}\right].
\end{aligned}$$

Both summands are summable over m by the same considerations as above. We thus conclude the proof. □

3.7.4 Proof of Theorem 3.13

Let us recall that we assumed that one of the two following cases occurs: $\gamma \in (0, 1/4)$ or $\gamma = 0$ and there exists $\varepsilon_1 \in (0,1)$ such that the product $n^{2+\varepsilon_1} g(n)$ converges monotonically to zero as n grows to infinity.

In the case where $\gamma > 0$, it follows that $(1/F(1/s))^2$ varies regularly at infinity with index 2γ. Further, $(1/F(1/s))^2$ diverges as $s \to \infty$. It follows by virtue of [Res87, Prop. 0.8(v)] that $1/g(u) = \inf\{s \ge 0\colon (1/F(1/s))^2 = u\}$ varies regularly at infinity with index $1/(2\gamma)$ and thus g varies regularly at infinity with index $-1/(2\gamma)$. Since in addition $\gamma < 1/4$, there exists $\varepsilon_1 \in (0,1)$ such that $-1/(2\gamma) < -(2+\varepsilon_1)$.

In any case, we define $\mathscr{D}_n$ as in Definition 3.34 with $\varepsilon = \varepsilon_2 = \frac{7\varepsilon_1}{8(2+\varepsilon_1)} \in (0,1/3)$. Let $\mathscr{I}_n = B_n \backslash \mathscr{D}_n$. By virtue of Lemma 3.43 and Remark 3.36 we know that $\mathbb{P}$-a.s. for n large enough the set $\mathscr{I}_n$ is sparse in the sense of Definition 3.35 and further $z_{(1,n)}, z_{(2,n)} \in \mathscr{I}_n$. It follows that $\mathbb{P}$-a.s. for n large enough the sites $z_{(1,n)}$ and $z_{(2,n)}$ are no neighbors. We abbreviate $z_n = z_{(1,n)}$.

Now we let $\alpha_n = n^{-\varepsilon_1/8}$ and note that

$$\Big\{\big\|\psi_1^{(n)}\big\|^2_{\ell^2(B_n\backslash\{z_n\})} > \alpha_n^2\Big\} \subseteq \Big\{\big\|\psi_1^{(n)}\big\|^2_{\ell^2(\mathscr{D}_n)} > \frac{\alpha_n^2}{2}\Big\} \cup \Big\{\big\|\psi_1^{(n)}\big\|^2_{\ell^2(\mathscr{I}_n\backslash\{z_n\})} > \frac{\alpha_n^2}{2}\Big\}. \tag{3.83}$$

However, by virtue of Lemma 3.44 we know that $\mathbb{P}$-a.s. for n large enough

$$\big\|\psi_1^{(n)}\big\|^2_{\ell^2(\mathscr{D}_n)} \le \alpha_n^4\,, \tag{3.84}$$

and thus $\mathbb{P}$-a.s. the limit superior of the first event on the right-hand side of (3.83) vanishes.

In order to estimate the probability of the second event on the right-hand side of (3.83), we now estimate $\big\|\psi_1^{(n)}\big\|_{\ell^2(\mathscr{I}_n\backslash\{z_n\})}$ in terms of $\big\|\psi_1^{(n)}\big\|_{\ell^2(\mathscr{D}_n)}$. By virtue of Remark 3.3, we can assume without loss of generality that $\psi_1^{(n)}$ nonnegative. Let $y \in \mathscr{I}_n\backslash\{z_n\}$ and define $m_y = 2\max_{x:x\sim y}\psi_1^{(n)}(x)$. On the event where $\mathscr{I}_n$ is sparse, $y \nsim z_n$. Therefore we know by virtue of Lemma 3.45 that $\psi_1^{(n)}(y) \le m_y\Big(1-\frac{\pi_{1,B_n}}{\pi_y}\Big)^{-1}$. By definition $\pi_y \ge \pi_{2,B_n}$ and thus it follows that $\mathbb{P}$-a.s. for n large enough

$$\big\|\psi_1^{(n)}\big\|^2_{\ell^2(\mathscr{I}_n\backslash\{z_n\})} \le \Big(1-\frac{\pi_{1,B_n}}{\pi_{2,B_n}}\Big)^{-2} \sum_{y\in\mathscr{I}_n\backslash\{z_n\}} m_y^2\,.$$

Moreover, on the event where $\mathscr{I}_n$ is sparse, any neighbor of $y \in \mathscr{I}_n$ is in $\mathscr{D}_n$ and therefore

$$\big\|\psi_1^{(n)}\big\|^2_{\ell^2(\mathscr{I}_n\backslash\{z_n\})} \le 8d\Big(1-\frac{\pi_{1,B_n}}{\pi_{2,B_n}}\Big)^{-2} \big\|\psi_1^{(n)}\big\|^2_{\ell^2(\mathscr{D}_n)}\,. \tag{3.85}$$

On the event where (3.84) is true and $\mathscr{I}_n$ is sparse, we hence infer that

$$\left\{\|\psi_1^{(n)}\|^2_{\ell^2(\mathscr{I}_n\setminus\{z_n\})} > \frac{\alpha_n^2}{2}\right\} \subseteq \left\{4\sqrt{d}\,\alpha_n > 1 - \frac{\pi_{1,B_n}}{\pi_{2,B_n}}\right\}.$$

However, by virtue of Lemma 3.49 we know that $\mathbb{P}$-a.s. for n large enough $4\sqrt{d}\,\alpha_n < 1 - \frac{\pi_{1,B_n}}{\pi_{2,B_n}}$. The claim follows.

3.7.5 Asymptotics of principal Dirichlet eigenvalue

Proof of Corollary 3.15. By virtue of (3.17), we already know that $\lambda_1^{(n)} \le \min_{z\in B_n} \pi_z$. By (3.9) we further know that there exists $\varepsilon_1 > 0$ such that $\mathbb{P}$-a.s. for n large enough

$$\psi_1^{(n)}(z_n)^2 \ge 1 - n^{-\varepsilon_1/4}\,.$$

It follows that $\mathbb{P}$-a.s. for n large enough

$$\begin{aligned}\lambda_1^{(n)} = \langle \psi_1^{(n)}, \mathcal{L}^\omega \psi_1^{(n)}\rangle &\ge \sum_{x\colon\, x\sim z_n} \omega_{xz_n}\Big(\psi_1^{(n)}(z_n) - \psi_1^{(n)}(x)\Big)^2\\ &\ge \Big(n^{-\varepsilon_1/8} - \sqrt{1-n^{-\varepsilon_1/4}}\Big)^2 \min_{z\in B_n} \pi_z\,. \qquad (3.86)\end{aligned}$$

The claim (3.10) follows. □

3.8 Auxiliary spectral problems

Definition 3.51 (Auxiliary lattice and Laplacian). *We define the set*

$$\mathscr{B}_l^{(n)} = B_n\setminus\big\{z_{(1,n)},\dots,z_{(l-1,n)}\big\} \qquad (3.87)$$

and abbreviate the operator $\mathcal{L}^\omega$ with zero Dirichlet conditions outside $\mathscr{B}_l^{(n)}$ as $\mathcal{L}^\omega_{(l,n)}$, i.e., we define

$$\mathcal{L}^\omega_{(l,n)} := \mathbb{1}_{\mathscr{B}_l^{(n)}}\,\mathcal{L}^\omega\,\mathbb{1}_{\mathscr{B}_l^{(n)}}\,, \qquad (3.88)$$

where the operator $\mathbb{1}_{\mathscr{B}_l^{(n)}}$ is the identity on $\mathscr{B}_l^{(n)}$ and zero otherwise.

Since the operator $-\mathcal{L}^\omega$ is self-adjoint, the operator $-\mathcal{L}^\omega_{(l,n)}$ is self-adjoint as well. This justifies the next definition.

Definition 3.52 (Auxiliary eigenvectors and values). *We define the eigenvalues of the operator* $-\mathcal{L}^{\omega}_{(l,n)}$ *restricted to* $\ell^2\left(\mathscr{B}_l^{(n)}\right)$ *by*

$$\mu_{l,1}^{(n)} \leq \mu_{l,2}^{(n)} \leq \ldots \leq \mu_{l,|\mathscr{B}_l^{(n)}|}^{(n)} \tag{3.89}$$

and its eigenvectors by

$$\phi_{l,1}^{(n)}, \phi_{l,2}^{(n)}, \ldots, \phi_{l,|\mathscr{B}_l^{(n)}|}^{(n)} \in \ell^2\left(\mathscr{B}_l^{(n)}\right) \qquad \text{with} \quad \left\langle \phi_{l,i}^{(n)}, \phi_{l,j}^{(n)} \right\rangle = \delta_{ij}\,. \tag{3.90}$$

Note that $\mathscr{B}_1^{(n)} = B_n$ and thus $\mu_{1,k}^{(n)} = \lambda_k^{(n)}$ and $\phi_{1,k}^{(n)} = \psi_k^{(n)}$. Moreover the variational formula for the auxiliary eigenvalues reads

$$\mu_{l,m}^{(n)} = \inf_{\substack{\mathcal{M} \leq \ell^2(\mathscr{B}_l^{(n)}),\\ \dim \mathcal{M} = m}} \sup_{\substack{f \in \mathcal{M},\\ \|f\|_2 = 1}} \mathcal{E}^{\omega}(f)\,. \tag{3.91}$$

Similar to Remark 3.3, we have the following remark about the principal auxiliary eigenvector and eigenvalue.

Remark 3.53 (Perron-Frobenius). *For a given box* B_n *the operator* $\mathcal{L}^{\omega}_{(l,n)}$ *can be written as a* $(|B_n| - l + 1) \times (|B_n| - l + 1)$*-matrix with non-negative entries everywhere except on the diagonal. Since the matrix is finite-dimensional, we can add a multiple of the identity to obtain a non-negative primitive matrix without changing the matrix' spectrum. By the Perron-Frobenius theorem (see e.g. [Sen81, Chapter 1]) it follows that its principal eigenvalue* $-\mu_{l,1}^{(n)}$ *is simple and we can assume without loss of generality that its principal eigenvector is positive, which implies that* $\phi_{l,1}^{(n)}$ *is nonnegative.*

Lemma 3.54. *For any* $l \in \mathbb{N}$ *and* $m \in \{1, \ldots, |B_n| - l + 1\}$ *the eigenvalue* $\mu_{l,m}^{(n)}$ *is bounded from above by*

$$\mu_{l,m}^{(n)} \leq \pi_{l+m-1,B_n}\,. \tag{3.92}$$

Proof. We choose

$$\mathcal{M} = \operatorname{span}\left\{\delta_{z_{(l,n)}}, \delta_{z_{(l+1,n)}}, \ldots, \delta_{z_{(l+m-1,n)}}\right\}$$

and insert it as a test space into the variational formula (3.91). □

3.8.1 Principal eigenvectors

The following lemma is the analogue of Lemma 3.45, where we need the Perron-Frobenius property.

Lemma 3.55. *Let $k \in \mathbb{N}$ and let $y, z \in B_n \cap \mathscr{B}_k^{(n)}$ with $\pi_z < \pi_y$ and $y \nsim z$. Assume that $\phi_{k,1}^{(n)}$ is nonnegative. Further, define $m_y = 2\max_{x\colon x\sim y} \phi_{k,1}^{(n)}(x)$. Then the mass $\phi_{k,1}^{(n)}(y)$ is bounded from above by*

$$\phi_{k,1}^{(n)}(y) \le \frac{m_y}{1-\frac{\pi_z}{\pi_y}}\,. \tag{3.93}$$

The proof of this lemma is analogous to the proof of Lemma 3.45 and therefore we omit it here.

Remark 3.56. *Let us recall that in Assumption 3.11 we assume that one of the two following cases occurs: $\gamma \in (0,1/4)$ or $\gamma = 0$ and there exists $\varepsilon_1 \in (0,1)$ such that the product $n^{2+\varepsilon_1} g(n)$ converges monotonically to zero as n grows to infinity. In the case where $\gamma \in (0,1/4)$, we define ε_1 as in Remark 3.12.*

In both cases we define $\mathscr{D}^{(n)}$ and $\mathscr{I}^{(n)}$ as in Definition 3.34 with $\varepsilon = \varepsilon_2 := \frac{7\varepsilon_1}{8(2+\varepsilon_1)}$. By virtue of Lemma 3.43 and Remark 3.36 we know that for any fixed $b \in \mathbb{N}$ the set $\mathscr{I}^{(n)}$ is b-sparse and therefore sparse $\mathbb{P}$-a.s. for n large enough in the sense of Definition 3.35. Moreover, Lemma 3.43 implies that for any $k \in \mathbb{N}$ we have $\mathbb{P}$-a.s. for n large enough $z_{(1,n)}, \dots, z_{(k+1,n)} \in \mathscr{I}^{(n)}$ and thus $\mathbb{P}$-a.s. for n large enough there is no pair of neighbors among the the sites $z_{(1,n)}, \dots, z_{(k+1,n)}$. Since F is continuous, the sites $z_{(1,n)}, \dots, z_{(k+1,n)}$ are $\mathbb{P}$-a.s. unique.

The next lemma about the principal Dirichlet eigenvector $\phi_{k,1}^{(n)}$ of the auxiliary operator $-\mathcal{L}^{\omega}_{(k,n)}$ is very similar to Lemma 3.44. Indeed, we can nearly copy the proof since the deleted sites $z_{(1,n)}, \dots, z_{(k-1,n)}$ are in $\mathscr{I}^{(n)}$, see Remark 3.56.

Lemma 3.57. *Let the function g be as in (3.8). Assume that there exists $\varepsilon_1 \in (0,1)$ such that one of the two cases occurs: g varies regularly at infinity with index $\rho < -(2+\varepsilon_1)$ or the product $n^{2+\varepsilon_1} g(n)$ converges monotonically to zero as n grows to infinity. Further, let $\varepsilon = \varepsilon_2 := \frac{7\varepsilon_1}{8(2+\varepsilon_1)}$ and $\mathscr{D}^{(n)}$ be as in Definition 3.34. Then $\mathbb{P}$-a.s. for n large enough*

$$\left\|\phi_{k,1}^{(n)}\right\|^2_{\ell^2(\mathscr{D}^{(n)})} \le n^{-\varepsilon_1/2}\,. \tag{3.94}$$

Proof. The proof follows the lines of the proof of Lemma 3.44 until *right before* (5.8). Here, we then apply Lemma 3.54 to infer that

$$\pi_{k,B_n} \ge \mu_{k,1}^{(n)} = \mathcal{E}^{\omega}\left(\phi_{k,1}^{(n)}\right).$$

Moreover, by virtue of Lemma 3.24 there exists $c_1 < \infty$ such that $\mathbb{P}$-a.s. for n large enough

$$c_1 g(n^{1-\varepsilon_3}) \ge \pi_{k,B_n}$$

with $\varepsilon_3 = \varepsilon_1(8(2+\varepsilon_1))^{-1}$. The rest of the proof follows again the lines of the proof of Lemma 3.44. □

From Lemma 3.57 to localization in a single site, the main two ingredients are Lemma 3.55 and Lemma 3.49 about the order statistics of $\{\pi_x\}_{x\in B_n}$.

The next lemma therefore follows.

Lemma 3.58. *Let $k \in \mathbb{N}$. Under Assumption 3.11, it follows that $\mathbb{P}$-a.s. for n large enough*

$$\phi_{k,1}^{(n)}\big(z_{(k,n)}\big) \geq \sqrt{1-n^{-\varepsilon_1/4}}\,. \tag{3.95}$$

This implies that $\mathbb{P}$-a.s. for n large enough

$$\mu_{k,1}^{(n)} \geq \Big(1-2n^{-\varepsilon_1/8}\Big)\pi_{k,B_n}\,. \tag{3.96}$$

Proof. In view of Remark 3.56, Lemma 3.55 and the extreme value result Lemma 3.49, the proof of (3.95) is completely analogous to the proof of Theorem 3.13 and thus we omit it here. For (3.96) we observe that since $\mu_{k,1}^{(n)} = \langle \phi_{k,1}^{(n)}, \mathcal{L}^\omega \phi_{k,1}^{(n)}\rangle$ it follows that $\mathbb{P}$-a.s. for n large enough

$$\begin{aligned}
\mu_{k,1}^{(n)} \;&\geq \sum_{x:\, x\sim z_{(k,n)}} \omega_{xz_{(k,n)}}\Big(\phi_{k,1}^{(n)}(z_{(k,n)}) - \phi_{k,1}^{(n)}(x)\Big)^2 \\
&\geq \Big(n^{-\varepsilon_1/8} - \sqrt{1-n^{-\varepsilon_1/4}}\Big)^2 \pi_{k,B_n}\,.
\end{aligned}$$

□

3.8.2 Orthogonality of eigenvectors

The next very simple ingredient of our proof is due to the orthogonality of the eigenvectors.

Lemma 3.59. *Let $\varepsilon > 0$, let $j,l,m,n \in \mathbb{N}$ with $j < m$ and let $\phi_{l,j}^{(n)}(z) \geq \sqrt{1-n^{-\varepsilon/4}}$.*

$$\Big|\phi_{l,m}^{(n)}(z)\Big| \leq n^{-\varepsilon/8}\,. \tag{3.97}$$

Proof. For $n=1$ the claim is immediate. For $n \geq 2$ we observe that since the eigenvectors $\phi_{l,j}^{(n)}$ and $\phi_{l,m}^{(n)}$ are orthogonal to each other, it follows that

$$\phi_{l,m}^{(n)}(z) = -\frac{\sum_{x\neq z}\phi_{l,j}^{(n)}(x)\phi_{l,m}^{(n)}(x)}{\phi_{l,j}^{(n)}(z)}\,.$$

By the Cauchy-Schwarz inequality it follows that for n greater than one

$$\left(\phi_{l,m}^{(n)}(z)\right)^2 \leq \frac{\left(\sum_{x\neq z}\left(\phi_{l,j}^{(n)}(x)\right)^2\right)\left(1-\left(\phi_{l,m}^{(n)}(z)\right)^2\right)}{\left(\phi_{l,j}^{(n)}(z)\right)^2}$$

$$\leq \frac{n^{-\varepsilon/4}}{1-n^{-\varepsilon/4}}\left(1-\left(\phi_{l,m}^{(n)}(z)\right)^2\right)$$

where we have used that the assumption implies $\sum_{x\neq z}\left(\phi_{l,j}^{(n)}(x)\right)^2 \leq n^{-\varepsilon/4}$. The claim follows. □

3.8.3 Higher eigenvalues and -vectors

We establish the connection to the original eigenvalues and -vectors via the Bauer-Fike theorem [BF60], which we cite below from [JKO94, Lemma 11.2].

Lemma 3.60 ([JKO94, Lemma 11.2]). *Let $A\colon H \to H$ be a linear self-adjoint compact operator in a Hilbert space H. Let $\mu \in \mathbb{R}$, and let $u \in H$ be such that $\|u\|_H = 1$ and*

$$\|Au - \mu u\|_H \leq \alpha\,, \qquad \alpha > 0\,. \tag{3.98}$$

Then there exists an eigenvalue μ_i of the operator A such that

$$|\mu_i - \mu| \leq \alpha\,. \tag{3.99}$$

Moreover, for any $\beta > \alpha$, there exists a vector $\overline{u}$ such that

$$\|u - \overline{u}\|_H \leq 2\alpha\beta^{-1}\,, \qquad \|\overline{u}\|_H = 1 \tag{3.100}$$

and $\overline{u}$ is a linear combination of the eigenvectors of operator A corresponding to the eigenvalues from the interval $[\mu-\beta, \mu+\beta]$.

Here comes the first application of Lemma 3.60.

Lemma 3.61. *Let $l \in \mathbb{N}$ and $m \in \{1, \ldots, |B_n| - l + 1\}$. Under Assumption 3.11 there exists $i \in \{1, \ldots, |B_n| - l + 1\}$ such that*

$$\left|\mu_{l,i}^{(n)} - \mu_{l+m,1}^{(n)}\right| \leq n^{-\varepsilon_1/4} \cdot \pi_{l+m-1,B_n}\,. \tag{3.101}$$

Proof. We aim to apply Lemma 3.60 with the operator $A = -\mathcal{L}^{\omega}_{(l,n)}$, the Hilbert space $H = \ell^2(\mathscr{B}^{(n)}_l)$, the value $\mu = \mu_{l+m,1}$ and the vector $u = \phi^{(n)}_{l+m,1}$. First, we note that $\|\phi^{(n)}_{l+m,1}\|_{\ell^2(\mathscr{B}^{(n)}_l)} = 1$. Next, we recall that $\phi^{(n)}_{l+m,1}$ is an eigenvector of the operator $-\mathcal{L}^{\omega}_{(l+m,n)}$ to the eigenvalue $\mu^{(n)}_{l+m,1}$ and therefore

$$\left\|\mathcal{L}^{\omega}_{(l,n)}\phi^{(n)}_{l+m,1} + \mu^{(n)}_{l+m,1}\phi^{(n)}_{l+m,1}\right\|^2_{\ell^2(\mathscr{B}^{(n)}_l)} = \sum_{z\in\mathscr{B}^{(n)}_l\setminus\mathscr{B}^{(n)}_{l+m}} \left(\mathcal{L}^{\omega}_{(l,n)}\phi^{(n)}_{l+m,1}(z) + \mu^{(n)}_{l+m,1}\phi^{(n)}_{l+m,1}(z)\right)^2,$$

where all other summands vanish. Recall that

$$\mathscr{B}^{(n)}_l\setminus\mathscr{B}^{(n)}_{l+m} = \left\{z_{(l,n)}, \dots z_{(l+m-1,n)}\right\}$$

and by definition we have $\phi^{(n)}_{l+m,1}(z) = 0$ for all $z \in \left\{z_{(l,n)}, \dots z_{(l+m-1,n)}\right\}$. It follows that for all $z \in \left\{z_{(l,n)}, \dots, z_{(l+m-1,n)}\right\}$ we have

$$\begin{aligned}\mathcal{L}^{\omega}_{(l,n)}\phi^{(n)}_{l+m,1}(z) &= \sum_{x:\, x\sim z} \omega_{xz}\left(\phi^{(n)}_{l+m,1}(x) - \phi^{(n)}_{l+m,1}(z)\right)\\ &= \sum_{x:\, x\sim z} \omega_{xz}\,\phi^{(n)}_{l+m,1}(x)\,.\end{aligned}$$

Since $\pi_{l+m-1,B_n} \geq \pi_{l+m-2,B_n} \geq \dots \geq \pi_{l,B_n}$, it follows that

$$\left\|\mathcal{L}^{\omega}_{(l,n)}\phi^{(n)}_{l+m,1} + \mu^{(n)}_{l+m,1}\phi^{(n)}_{l+m,1}\right\|^2_{\ell^2(\mathscr{B}^{(n)}_l)} \leq \pi^2_{l+m-1,B_n} \sum_{z\in\mathscr{B}^{(n)}_l\setminus\mathscr{B}^{(n)}_{l+m}} \max_{x:\, x\sim z}\left(\phi^{(n)}_{l+m,1}(x)\right)^2.$$

Since by virtue of Remark 3.56 the sites $z_{(1,n)}, \dots, z_{(l+m-1,n)}$ are in $\mathscr{I}^{(n)}$ and are neither neighbors nor do they share a common neighbor $\mathbb{P}$-a.s. for n large enough, it follows that $\mathbb{P}$-a.s. for n large enough

$$\sum_{z\in\mathscr{B}^{(n)}_l\setminus\mathscr{B}^{(n)}_{l+m}} \max_{x:\, x\sim z}\left(\phi^{(n)}_{l+m,1}(x)\right)^2 \leq \sum_{x\in\mathscr{D}^{(n)}}\left(\phi^{(n)}_{l+m,1}(x)\right)^2 \leq n^{-\varepsilon_1/2},$$

where the last bound is due to Lemma 3.57. The claim follows by virtue of Lemma 3.60. $\square$

Here comes the second application of Lemma 3.60.

Lemma 3.62. *Let $\varepsilon > 0$, $l, m \in \mathbb{N}$. If Assumption 3.11 holds and $\mathbb{P}$-a.s. for n large enough*

$$\phi_{l,j}^{(n)}\big(z_{(l+j-1,n)}\big) \geq \sqrt{1-n^{-\varepsilon/4}} \qquad \text{for all } 1 \leq j \leq m\,, \tag{3.102}$$

then $\mathbb{P}$-a.s. for n large enough there exists $j \in \{1, \ldots, |B_n| - l - m + 1\}$ such that

$$\left|\mu_{l,m+1}^{(n)} - \mu_{l+m,j}^{(n)}\right| \leq \pi_{l+m-1,B_n} \sqrt{\frac{mn^{-\varepsilon/4}}{1-mn^{-\varepsilon/4}}}\,. \tag{3.103}$$

Proof. We aim to apply Lemma 3.60 with the operator $A = -\mathcal{L}_{(l+m,n)}^{\omega}$, the Hilbert space $H = \ell^2(\mathscr{B}_{l+m}^{(n)})$, the value $\mu = \mu_{l,m+1}^{(n)}$ and the vector $u = \phi_{l,m+1}^{(n)}/\|\phi_{l,m+1}^{(n)}\|_{\ell^2(\mathscr{B}_{l+m}^{(n)})}$. First, we note that by definition $\|u\|_{\ell^2(\mathscr{B}_{l+m}^{(n)})} = 1$ and $\mathbb{P}$-a.s. for n large enough

$$\|\phi_{l,m+1}^{(n)}\|_{\ell^2(\mathscr{B}_{l+m}^{(n)})}^2 = 1 - \sum_{z \in \mathscr{B}_l^{(n)} \setminus \mathscr{B}_{l+m}^{(n)}} \Big(\phi_{l,m+1}^{(n)}(z)\Big)^2 \geq 1 - mn^{-\varepsilon/4} \tag{3.104}$$

by virtue of Condition (3.102) and Lemma 3.59.

Next, as we show in detail in (3.107), we can estimate

$$\begin{aligned}\Big\|\mathcal{L}_{(l+m,n)}^{\omega}\phi_{l,m+1}^{(n)} + \mu_{l,m+1}^{(n)}\phi_{l,m+1}^{(n)}\Big\|_{\ell^2(\mathscr{B}_{l+m}^{(n)})}^2 & \\ \leq \max_{z \in \mathscr{B}_l^{(n)} \setminus \mathscr{B}_{l+m}^{(n)}} \Big(\phi_{l,m+1}^{(n)}(z)\Big)^2 \sum_{x \in B_n} \Bigg(\sum_{\substack{z:\, z \sim x \\ z \in \mathscr{B}_l^{(n)} \setminus \mathscr{B}_{l+m}^{(n)}}} \omega_{xz}\Bigg)^2 .\end{aligned} \tag{3.105}$$

Since by virtue of Remark 3.56 we have $\mathbb{P}$-a.s. for n large enough

$$\mathscr{B}_l^{(n)} \setminus \mathscr{B}_{l+m}^{(n)} = \{z_{l,n}, \ldots, z_{l+m-1,n}\} \subset \mathscr{I}^{(n)}$$

and $\mathscr{I}^{(n)}$ is 1-sparse, it follows that on the right-hand side of (3.105) for each $x \in B_n$ the sum over $\Big\{z \in \mathscr{B}_l^{(n)} \setminus \mathscr{B}_{l+m}^{(n)} : z \sim x\Big\}$ contains at most one summand. Therefore $\mathbb{P}$-a.s. for n large enough we can pull the square into the inner sum. Then we rearrange both sums and use that for all z we have $\sum_{x:\, x \sim z} \omega_{xz}^2 \leq \pi_z^2$ to infer that $\mathbb{P}$-a.s. for n large enough

$$\left\| \mathcal{L}^{\omega}_{(l+m,n)} \phi^{(n)}_{l,m+1} + \mu^{(n)}_{l,m+1} \phi^{(n)}_{l,m+1} \right\|^2_{\ell^2(\mathscr{B}^{(n)}_{l+m})}$$

$$\leq \max_{z \in \mathscr{B}^{(n)}_l \setminus \mathscr{B}^{(n)}_{l+m}} \left(\phi^{(n)}_{l,m+1}(z) \right)^2 \sum_{z \in \mathscr{B}^{(n)}_l \setminus \mathscr{B}^{(n)}_{l+m}} \pi_z^2 \,.$$

By virtue of Lemma 3.59 and Assumption (3.102), we know for all $z \in \{z_{(l,n)}, \dots, z_{(l+m-1,n)}\}$ that $\mathbb{P}$-a.s. for n large enough

$$\left| \phi^{(n)}_{l,m+1}(z) \right| \leq n^{-\varepsilon/8} \,.$$

Furthermore, $\sum_{z \in \mathscr{B}^{(n)}_l \setminus \mathscr{B}^{(n)}_{l+m}} \pi_z^2 \leq m \pi^2_{l+m-1,B_n}$. It follows that $\mathbb{P}$-a.s. for n large enough

$$\left\| \mathcal{L}^{\omega}_{(l+m,n)} \phi^{(n)}_{l,m+1} - \mu^{(n)}_{l,m+1} \phi^{(n)}_{l,m+1} \right\|^2_{\ell^2(\mathscr{B}^{(n)}_{l+m})} \leq m n^{-\varepsilon/4} \pi^2_{l+m-1,B_n} \,.$$

Together with (3.104) it follows that $\mathbb{P}$-a.s. for n large enough

$$\left\| \mathcal{L}^{\omega}_{(l+m,n)} u - \mu^{(n)}_{l,m+1} u \right\|^2_{\ell^2(\mathscr{B}^{(n)}_{l+m})} \leq \frac{m n^{-\varepsilon/4}}{1 - m n^{-\varepsilon/4}} \, \pi^2_{l+m-1,B_n} \,. \tag{3.106}$$

and therefore the claim follows by virtue of Lemma 3.60.

It remains to justify (3.105). We start by inserting the definition of the Laplacian, i.e.,

$$\sum_{x \in \mathscr{B}^{(n)}_{l+m}} \left(\mathcal{L}^{\omega}_{(l+m,n)} \phi^{(n)}_{l,m+1}(x) + \mu^{(n)}_{l,m+1} \phi^{(n)}_{l,m+1}(x) \right)^2$$

$$= \sum_{x \in \mathscr{B}^{(n)}_{l+m}} \left(\sum_{z:\, z \sim x} \omega_{xz} \left(\left(\phi^{(n)}_{l,m+1} \mathbb{1}_{\mathscr{B}^{(n)}_{l+m}} \right)(z) - \phi^{(n)}_{l,m+1}(x) \right) + \mu^{(n)}_{l,m+1} \phi^{(n)}_{l,m+1}(x) \right)^2$$

Now we rearrange the terms in order to cancel $\mu^{(n)}_{l,m+1} \phi^{(n)}_{l,m+1}(x)$, i.e.,

$$\text{above left-hand side} =$$
$$= \sum_{x \in \mathscr{B}^{(n)}_{l+m}} \Bigg(\sum_{z:\, z \sim x} \omega_{xz} \left(\left(\phi^{(n)}_{l,m+1} \mathbb{1}_{\mathscr{B}^{(n)}_l} \right)(z) - \phi^{(n)}_{l,m+1}(x) \right) + \mu^{(n)}_{l,m+1} \phi^{(n)}_{l,m+1}(x)$$
$$- \sum_{z:\, z \sim x} \omega_{xz} \left(\phi^{(n)}_{l,m+1} \mathbb{1}_{\mathscr{B}^{(n)}_l \setminus \mathscr{B}^{(n)}_{l+m}} \right)(z) \Bigg)^2 ,$$

where the first two terms cancel. The last term simplifies to

$$\begin{aligned}\text{LHS} &= \sum_{x\in\mathscr{B}_{l+m}^{(n)}}\left(\sum_{z\in\mathscr{B}_l^{(n)}\setminus\mathscr{B}_{l+m}^{(n)}\,:\,z\sim x}\omega_{xz}\,\phi_{l,m+1}^{(n)}(z)\right)^2\\ &\leq \max_{z\in\mathscr{B}_l^{(n)}\setminus\mathscr{B}_{l+m}^{(n)}}\left(\phi_{l,m+1}^{(n)}(z)\right)^2\sum_{x\in B_n}\left(\sum_{z\in\mathscr{B}_l^{(n)}\setminus\mathscr{B}_{l+m}^{(n)}\,:\,z\sim x}\omega_{xz}\right)^2 . \end{aligned} \tag{3.107}$$

□

Both Lemmas 3.61 and 3.62 imply the following lemma.

Lemma 3.63. *Let $\varepsilon\in(0,\varepsilon_1)$ and $l,m\in\mathbb{N}$. If Assumption 3.11 holds and $\mathbb{P}$-a.s. for n large enough*

$$\phi_{l,j}^{(n)}\left(z_{(l+j-1,n)}\right)\geq\sqrt{1-n^{-\varepsilon/4}}\qquad\textit{for all } 1\leq j\leq m\,, \tag{3.108}$$

then

$$\mu_{l,m+1}^{(n)}\geq\left(1-(2+\sqrt{m})n^{-\varepsilon/8}\right)\pi_{l+m,B_n}\,. \tag{3.109}$$

Proof. Let us first assume that $\mu_{l,m+1}^{(n)}\leq\mu_{l+m,1}^{(n)}$. Due to Assumption (3.108) we can apply Lemma 3.62. Because of the ordering $\mu_{l+m,1}^{(n)}\leq\mu_{l+m,2}^{(n)}\leq\cdots$, it follows that Relation (3.103) holds with $j=1$ and $\varepsilon=\varepsilon$. On the other hand, if $\mu_{l,m+1}^{(n)}>\mu_{l+m,1}^{(n)}$, then (3.101) holds with an index $i\leq m+1$. Let us now argue why (3.101) holds with exactly $i=m+1$ $\mathbb{P}$-a.s. for n large enough. We assume the contrary, i.e., that $i\leq m$ infinitely often as n tends to infinity. Then (3.101) together with (3.96) implies that

$$\mu_{l,i}^{(n)}\geq\mu_{l+m,1}^{(n)}-n^{-\varepsilon_1/4}\pi_{l+m-1,B_n}\geq\left(1-2n^{-\varepsilon_1/8}-n^{-\varepsilon_1/4}\right)\pi_{l+m,B_n}$$

Note that (3.92) implies that $\mu_{l,i}^{(n)}\leq\pi_{l+i-1,B_n}$, which we assumed to be less than or equal to π_{l+m-1,B_n} infinitely often as n tends to infinity. Thus

$$\frac{\pi_{l+m-1,B_n}}{\pi_{l+m,B_n}}\geq 1-3n^{-\varepsilon_1/8}$$

infinitely often as n tends to infinity. This is a contradiction to Lemma 3.49.

Thus, since $\varepsilon<\varepsilon_1$, it follows regardless of whether $\mu_{l,m+1}^{(n)}\leq\mu_{l+m,1}^{(n)}$ or $\mu_{l,m+1}^{(n)}>\mu_{l+m,1}^{(n)}$ that $\mathbb{P}$-a.s. for n large enough

$$\left|\mu_{l,m+1}^{(n)}-\mu_{l+m,1}^{(n)}\right|\leq\sqrt{\frac{mn^{-\varepsilon/4}}{1-mn^{-\varepsilon/4}}}\,\pi_{l+m-1,B_n}\leq\sqrt{m}n^{-\varepsilon/8}\cdot\pi_{l+m,B_n}\,. \tag{3.110}$$

Therefore $\mathbb{P}$-a.s. for n large enough $\mu^{(n)}_{l,m+1}$ is bounded from below by

$$\begin{aligned}\mu^{(n)}_{l,m+1} &\geq \mu^{(n)}_{l+m,1} - \sqrt{m} n^{-\varepsilon/8} \cdot \pi_{l+m,B_n} \\ &\overset{(3.96)}{\geq} \left(1 - (2+\sqrt{m}) n^{-\varepsilon/8}\right) \pi_{l+m,B_n} \,. \end{aligned} \tag{3.111}$$

□

Now we have the ingredients to prove the main theorem by induction.

3.9 Proof of Theorem 3.16

By virtue of Lemma 3.54, we already know that

$$\lambda^{(n)}_k \leq \pi_{k,B_n} \qquad \text{for all } k \in \mathbb{N}\,.$$

In what follows, we further prove (3.13) and that $\mathbb{P}$-a.s. for n large enough

$$\lambda^{(n)}_k \geq \left(1 - n^{-\varepsilon/8}\right) \pi_{k,B_n} \qquad \text{for all } \varepsilon < \varepsilon_1\,. \tag{3.112}$$

We prove the claim by induction over k.

Base case: $k = 1$. In this case, Claim (3.13) is a consequence of (3.9) of Theorem 3.13. Furthermore, Claim (3.112) follows because

$$\lambda^{(n)}_1 \geq \left(1 - 2n^{-\varepsilon_1/8}\right) \pi_{1,B_n} > \left(1 - n^{-\varepsilon/8}\right) \pi_{1,B_n} \qquad \text{for all } \varepsilon < \varepsilon_1 \tag{3.113}$$

by virtue of (3.86).

Inductive step: $(k-1) \rightsquigarrow k$. Suppose that the claims (3.12) and (3.13) hold for some $k-1 \in \mathbb{N}$. We now show that this implies that the claims also hold for k instead of $k-1$.

For (3.12) this already follows by Lemma 3.63 with $l = 1$ and $m = k-1$. Note that here Condition (3.108) holds for all $\varepsilon < \varepsilon_1$ and therefore (3.109) holds even without the multiplicative constants. For (3.13) we apply the second part of Lemma 3.60: Let $0 < \delta < \varepsilon_1/16$ and

$$\beta^{(n)}_k = 2\sqrt{k-1}\, n^{-\delta} \pi_{k,B_n}\,. \tag{3.114}$$

Since $\pi_{k-1,B_n} \leq \pi_{k,B_n}$, it follows that $\beta^{(n)}_k > \alpha^{(n)}_k$ with

$$\alpha^{(n)}_k := \sqrt{k-1}\, n^{-\varepsilon_1/8} \pi_{k-1,B_n}\,.$$

Therefore Lemma 3.60 and (3.106) with $l = 1$ and $m = k-1$ imply that there exists a function $\overline{u}\colon \mathbb{Z}^d \to \mathbb{R}$ such that

$$\left\| \psi_k^{(n)} - \overline{u} \right\|_{\ell^2(B_n)} \le \frac{2\sqrt{k-1}\, n^{-\varepsilon_1/8} \pi_{k-1,B_n}}{\beta_k^{(n)}} \tag{3.115}$$

where $\overline{u}$ is a linear combination of the eigenvectors $\{\phi_{k,j}\}_{j\ge 1}$ corresponding to the eigenvalues from the interval $\left[\lambda_k^{(n)} - \beta_k^{(n)}, \lambda_k^{(n)} + \beta_k^{(n)}\right]$ of the operator $-\mathcal{L}^{\omega}_{(k,n)}$. We now show that $\mathbb{P}$-a.s. for n large enough $\overline{u} = \phi_{k,1}^{(n)}$, i.e., that $\mathbb{P}$-a.s. for n large enough

$$\operatorname{spec} \mathcal{L}^{\omega}_{(k,n)} \cap \left[\lambda_k^{(n)} - \beta_k^{(n)}, \lambda_k^{(n)} + \beta_k^{(n)}\right] = \left\{\mu_{k,1}^{(n)}\right\}. \tag{3.116}$$

It suffices to show that $\mathbb{P}$-a.s. for n large enough $\mu_{k,2}^{(n)} > \lambda_k^{(n)} + \beta_k^{(n)}$. We note that Lemma 3.54 implies that

$$\lambda_k^{(n)} + \beta_k^{(n)} \le \left(1 + 2\sqrt{k-1}\, n^{-\delta}\right) \pi_{k,B_n} . \tag{3.117}$$

By virtue of Lemma 3.49 we have $\mathbb{P}$-a.s. for n large enough $\frac{\pi_{k,B_n}}{\pi_{k+1,B_n}} < 1 - 2\sqrt{k-1}\, n^{-\delta}$, whence it follows that $\mathbb{P}$-a.s. for n large enough

$$\lambda_k^{(n)} + \beta_k^{(n)} < \left(1 - 4(k-1) n^{-2\delta}\right) \pi_{k+1,B_n} \le \mu_{k,2}^{(n)} ,$$

where the last inequality follows since by the inductive assumption the relation (3.108) holds for all $\varepsilon < \varepsilon_1$ and therefore (3.109) holds for all $\varepsilon < \varepsilon_1$ with $l = k$ and $m = 1$. Therefore (3.116) is true.

It follows that for any $0 < \delta < \varepsilon_1/16$ we have $\mathbb{P}$-a.s. for n large enough

$$\left| \psi_k^{(n)}(z_{(k,n)}) - \phi_{k,1}^{(n)}(z_{(k,n)}) \right| \;\le\; \frac{n^{\delta - \varepsilon_1/8} \pi_{k-1,B_n}}{\pi_{k,B_n}} \;<\; n^{\delta - \varepsilon_1/8} .$$

By virtue of Lemma 3.58, we already know that $\mathbb{P}$-a.s. for n large enough $\left| \phi_{k,1}^{(n)}(z_{(k,n)}) \right| \ge \sqrt{1 - n^{-\varepsilon_1/4}}$. It follows that

$$\left(\psi_k^{(n)}(z_{(k,n)}) \right)^2 \;\ge\; 1 - n^{-\varepsilon_1/4} + n^{2\delta - \varepsilon_1/4} - 2n^{\delta - \varepsilon_1/8} \;\ge\; 1 - 2n^{\delta - \varepsilon_1/8} .$$

The claim follows since we can choose δ arbitrarily small.

3.10 Weak convergence of the eigenvalues

Because of Corollary 3.15 and Theorem 3.16, in order to prove Corollary 3.17, it remains to prove that

$$\lim_{n\to\infty} \mathbb{P}\left[\pi_{k,B_n} > \frac{\zeta}{n^{\frac{1}{2\gamma}} L^*(n)}\right] = \exp\bigl(-\zeta^{2d\gamma}\bigr) \sum_{j=0}^{k-1} \frac{\zeta^{2d\gamma j}}{j!} \quad \text{for all } \zeta \geq 0\,. \tag{3.118}$$

where the difficulty lies in the dependence of the random variables $(\pi_x)_{x\in B_n}$. However, the dependence is very short-ranged since π_{x_1} and π_{x_2} are dependent if and only if the sites x_1, x_2 are neighbors. We strongly rely on the ideas of [Wat54], which we easily adapt to our needs.

In what follows, we always mean that a statement holds for all $\zeta \geq 0$ even if we do not explicitly write so. We define

$$a_n := \Bigl(n^{\frac{1}{2\gamma}} L^*(n)\Bigr)^{-1} = \frac{1}{h(|B_n|)} = \sup\bigl\{t\colon F_\pi(t) = |B_n|^{-1}\bigr\}\,.$$

with h as in (3.14) and $L^*(n)$ as in (3.15). Then $|B_n| = (\mathbb{P}[\pi_0 \leq a_n])^{-1}$ and therefore

$$\lim_{n\to\infty} |B_n|\mathbb{P}[\pi_0 \leq a_n\zeta] = \lim_{n\to\infty} \frac{F_\pi(a_n\zeta)}{F_\pi(a_n)} = \zeta^{2d\gamma} \tag{3.119}$$

since $a_n \to 0$ as $n \to \infty$ and F_π varies regularly at zero with index $2d\gamma$ by virtue of Lemma 3.47. We further note that if $\boldsymbol{e}_1 \in \mathbb{Z}^d$ is a neighbor of the origin, then $\mathbb{P}[\{\pi_0 \leq a_n\zeta\} \cap \{\pi_{\boldsymbol{e}_1} \leq a_n\zeta\}] \leq F(a_n\zeta)^{4d-1}$ since for the event $\{\pi_0 \leq a_n\zeta\} \cap \{\pi_{\boldsymbol{e}_1} \leq a_n\zeta\}$ at least $4d-1$ independent conductances ω have to be smaller than or equal to $a_n\zeta$. By virtue of (3.78) and since F varies regularly at zero with index γ, it follows that

$$|B_n|\,\mathbb{P}[\{\pi_0 \leq a_n\zeta\} \cap \{\pi_{\boldsymbol{e}_1} \leq a_n\zeta\}] \to 0 \qquad \text{as } n \to \infty\,. \tag{3.120}$$

We start with the auxiliary Lemma 3.65, for which we need some further definitions. For a set $A \subset \mathbb{Z}^d$ we define $CC(A)$ as the set of connected components of A. Furthermore, we define the outer site boundary of the set A as

$$\partial A := \bigl\{z \in \mathbb{Z}^d \backslash A\colon \exists x \in A \text{ with } x \sim z\bigr\}\,. \tag{3.121}$$

For the natural numbers $q \leq m$ we further define the number

$$C^{(n)}_{m,q}(A) := \bigl|\{M \subset B_n \backslash (A \cap \partial A)\colon |M| = m, |CC(M)| = q\}\bigr|\,. \tag{3.122}$$

If A is the empty set, we simply write $C^{(n)}_{m,q}$.

Remark 3.64. *Note that if we fix $k \in \mathbb{N}$, then as n tends to infinity we have $C^{(n)}_{m,m}(A_n) = |B_n|^m/m! + O\bigl(|B_n|^{m-1}\bigr)$ for all sequences of subsets $A_n \in B_n$ with the constraint $|A_n| = k-1$. Moreover, for $q \leq m-1$ there exists a constant $c_q < \infty$ such that for all $n \in \mathbb{N}$ and all sequences of subsets*

$A_n \subset B_n$ with $|A_n| = k-1$, we have $C^{(n)}_{m,q}(A_n) < c_q|B_n|^q$. Note that this c_q is independent of the specific choice of A_n.

Lemma 3.65. *For any fixed $k, l \in \mathbb{N}$ the relations* (3.119) *and* (3.120) *imply that*

$$\lim_{n\to\infty} \sup_{\substack{A_n\subset B_n,\\ |A_n|=k-1}} \sum_{m=1}^{l}\sum_{q=1}^{m-1} \sum_{\substack{M\subset B_n\setminus(A_n\cap\partial A_n),\\ |M|=m,\\ |CC(M)|=q}} \mathbb{P}\left[\bigcap_{x\in M}\{\pi_x \le a_n\zeta\}\right] = 0 \text{ for all } \zeta \ge 0\,. \tag{3.123}$$

Proof. We are summing over sets M with the constraint $|CC(M)| = q < m = |M|$. This means that here all the sets M contain at least one connected component $\mathscr{C}$ with a neighboring pair of sites, i.e., $\mathbb{P}[\bigcap_{x\in\mathscr{C}}\{\pi_x \le a_n\zeta\}] \le \mathbb{P}[\{\pi_0 \le a_n\zeta\}\cap\{\pi_{\boldsymbol{e}_1} \le a_n\zeta\}]$. Since π_x and π_y are independent if the sites x and y are in two different connected components of M, it follows that

$$\begin{aligned}
&\sum_{m=1}^{l}\sum_{q=1}^{m-1} \sum_{\substack{M\subset B_n\setminus(A_n\cap\partial A_n),\\ |M|=m,\\ |CC(M)|=q}} \mathbb{P}\left[\bigcap_{x\in M}\{\pi_x \le a_n\zeta\}\right]\\
&\qquad\le \sum_{m=1}^{l}\sum_{q=1}^{m-1} C^{(n)}_{m,q}(A_n)\mathbb{P}[\pi_0 \le a_n\zeta]^{q-1}\mathbb{P}[\{\pi_0 \le a_n\zeta\}\cap\{\pi_{\boldsymbol{e}_1} \le a_n\zeta\}]
\end{aligned}$$

By Remark 3.64 there exists a constant $c_q < \infty$ such that $C^{(n)}_{m,q}(A_n) \le c_q|B_n|^q$ for all sequences of subsets $A_n \subset B_n$ with the constraint that $|A_n| = k-1$. Therefore the claim follows by (3.119) and (3.120). □

Proof of Corollary 3.17. As already stated at the beginning of this section, it remains to prove (3.118). We prove the claim by induction over k.

Base case: $\boldsymbol{k = 1}$. For any even integer $l \le |B_n|$ we estimate

$$\begin{aligned}
&1-\sum_{m=1}^{l-1}(-1)^{m-1}\sum_{q=1}^{m} \sum_{\substack{M\subset B_n,\\ |M|=m,\\ CC(M)=q}} \mathbb{P}\left[\bigcap_{x\in M}\{\pi_x \le a_n\zeta\}\right]\\
&\qquad\le\ \mathbb{P}\left[\min_{x\in B_n}\pi_x > a_n\zeta\right]\\
&\qquad\le\ 1-\sum_{m=1}^{l}(-1)^{m-1}\sum_{q=1}^{m} \sum_{\substack{M\subset B_n,\\ |M|=m,\\ CC(M)=q}} \mathbb{P}\left[\bigcap_{x\in M}\{\pi_x \le a_n\zeta\}\right].
\end{aligned} \tag{3.124}$$

The first term in the above sums over m, i.e. $\sum_{x\in B_n}\mathbb{P}[\pi_x \le a_n\zeta]$, is equal to $|B_n|\mathbb{P}[\pi_0 \le a_n\zeta]$ and converges to $\zeta^{2d\gamma}$. For the rest of the terms, for example on the left-most side of (3.124), we observe that

$$\begin{aligned}\sum_{m=1}^{l-1}(-1)^{m-1}\sum_{q=1}^{m}\sum_{\substack{M\subset B_n,\\ |M|=m,\\ CC(M)=q}}\mathbb{P}\Bigg[\bigcap_{x\in M}\{\pi_x\le a_n\zeta\}\Bigg] &= \sum_{m=1}^{l-1}(-1)^{m-1}\sum_{q=1}^{m-1}\sum_{\substack{M\subset B_n,\\ |M|=m,\\ CC(M)=q}}\mathbb{P}\Bigg[\bigcap_{x\in M}\{\pi_x\le a_n\zeta\}\Bigg]\\ &\quad+\sum_{m=1}^{l-1}(-1)^{m-1}\sum_{\substack{M\subset B_n,\\ |M|=m,\\ CC(M)=m}}\mathbb{P}\Bigg[\bigcap_{x\in M}\{\pi_x\le a_n\zeta\}\Bigg].\end{aligned}$$

The first sum on the above right-hand side converges to zero as n tends to infinity by virtue of Lemma 3.65 (with $k=1$). The second sum contains only sets M with $CC(M)=|M|$, i.e. sets which are completely disconnected. For such sets all the events $\{\pi_x\le a_n\zeta\}_{x\in M}$ are independent. It follows that the second sum on the above right-hand side is

$$\sum_{m=1}^{l-1}(-1)^{m-1}\sum_{\substack{M\subset B_n,\\ |M|=m,\\ CC(M)=m}}\mathbb{P}\Bigg[\bigcap_{x\in M}\{\pi_x\le a_n\zeta\}\Bigg]=\sum_{m=1}^{l-1}(-1)^{m-1}C_{m,m}^{(n)}\mathbb{P}[\pi_0\le a_n\zeta]^m\,,$$

which, by virtue of (3.119) and Remark 3.64, converges to

$$\sum_{m=1}^{l-1}(-1)^{m-1}\zeta^{2md\gamma}/m!$$

as n tends to infinity.

The same considerations hold also for the right-most side in (3.124) and it follows that for any even integer l we have

$$\sum_{m=0}^{l-1}\frac{(-\zeta)^{2md\gamma}}{m!}\le\lim_{n\to\infty}\mathbb{P}\Big[\min_{x\in B_n}\pi_x>a_n\zeta\Big]\le\sum_{m=0}^{l}\frac{(-\zeta)^{2md\gamma}}{m!}\,.$$

Therefore the claim follows for $k=1$.

Inductive step: $(k-1)\rightsquigarrow k$. For the inductive step we consider

$$\mathbb{P}[\pi_{k,B_n}>a_n\zeta]=\mathbb{P}[\pi_{k-1,B_n}>a_n\zeta]+\mathbb{P}[\{\pi_{k,B_n}>a_n\zeta\}\cap\{\pi_{k-1,B_n}\le a_n\zeta\}]\,.$$

Let us now assume that the claim (3.118) holds for some $k-1$. It follows that it remains to show that

$$\lim_{n\to\infty} \mathbb{P}[\pi_{k,B_n} > a_n\zeta,\, \pi_{k-1,B_n} \le a_n\zeta] = \frac{\zeta^{2(k-1)d\gamma}}{(k-1)!} \exp\bigl(-\zeta^{2d\gamma}\bigr)\,.$$

Let us start with the decomposition

$$\begin{aligned}
&\mathbb{P}[\pi_{k,B_n} > a_n\zeta,\, \pi_{k-1,B_n} \le a_n\zeta] \qquad (3.125)\\
&\quad= \sum_{\substack{A\subset B_n,\\ |A|=k-1}} \mathbb{P}\left[\bigcap_{x\in A}\{\pi_x \le a_n\zeta\} \cap \bigcap_{y\in B_n\backslash A}\{\pi_y > a_n\zeta\}\right]\\
&\quad= \sum_{\substack{A\subset B_n,\\ |A|=k-1}} \mathbb{P}\left[\bigcap_{x\in A}\{\pi_x \le a_n\zeta\} \cap \bigcap_{y\in B_n\backslash(A\cap\partial A)}\{\pi_y > a_n\zeta\}\right]\\
&\qquad\quad - \sum_{\substack{A\subset B_n,\\ |A|=k-1}} \mathbb{P}\left[\bigcap_{x\in A}\{\pi_x \le a_n\zeta\} \cap \left(\bigcup_{y\in\partial A}\{\pi_y \le a_n\zeta\}\right)\right].
\end{aligned} \tag{3.126}$$

Let us argue that the second term on the above right-hand side converges to zero as n tends to infinity. We observe that

$$\begin{aligned}
&\sum_{\substack{A\subset B_n,\\ |A|=k-1}} \mathbb{P}\left[\bigcap_{x\in A}\{\pi_x \le a_n\zeta\} \cap \left(\bigcup_{y\in\partial A}\{y \le a_n\zeta\}\right)\right]\\
&\qquad\le \sum_{\substack{A\subset B_n,\\ |A|=k-1}} \sum_{y\in\partial A} \mathbb{P}\left[\{\pi_y \le a_n\zeta\} \cap \bigcap_{x\in A}\{\pi_x \le a_n\zeta\}\right]\\
&\qquad\le \sum_{\substack{A\subset B_n,\\ |A|=k,\\ |CC(A)|\le k-1}} \mathbb{P}\left[\bigcap_{x\in A}\{\pi_x \le a_n\zeta\}\right]
\end{aligned}$$

which converges to zero by virtue of Lemma 3.65.

Let us now consider the first term on the right-hand side of (3.126). Since for any $y \in B_n\backslash(A\cap\partial A)$ the random variable π_y is independent of $\{\pi_x\}_{x\in A}$, the first sum on the right-hand side of (3.126) is

$$
\begin{aligned}
&\sum_{\substack{A\subset B_n,\\|A|=k-1}} \mathbb{P}\left[\bigcap_{x\in A}\{\pi_x \le a_n\zeta\}\right]\mathbb{P}\left[\min_{y\in B_n\setminus(A\cap\partial A)}\pi_y > a_n\zeta\right] \\
&\qquad\ge \mathbb{P}\left[\min_{y\in B_n}\pi_y > a_n\zeta\right]\sum_{\substack{A\subset B_n,\\|A|=k-1}} \mathbb{P}\left[\bigcap_{x\in A}\{\pi_x \le a_n\zeta\}\right] \\
&\qquad= \mathbb{P}\left[\min_{y\in B_n}\pi_y > a_n\zeta\right]\left(\sum_{q=1}^{k-1}\sum_{\substack{M\subset B_n,\\|M|=k-1,\\CC(M)=q}} \mathbb{P}\left[\bigcap_{x\in M}\{\pi_x \le a_n\zeta\}\right]\right).
\end{aligned}
\tag{3.127}
$$

Due to the inductive base case $k = 1$, the first factor in the above right-hand side converges to $\exp\left(-\zeta^{2d\gamma}\right)$. Similar to our arguments for the inductive base case, we also argue that the second factor converges to $\zeta^{2(k-1)d\gamma}/(k-1)!$. Therefore, we already have the desired lower bound for the left-hand side of (3.127). It remains to prove the desired upper bound. Similar to the proof for $k = 1$, we let l be an even integer and estimate for all sequences of subsets $A_n \subset B_n$ with the constraint $|A_n| = k - 1$ that

$$
\begin{aligned}
&\mathbb{P}\left[\min_{y\in B_n\setminus(A_n\cap\partial A_n)}\pi_y > a_n\zeta\right] \\
&\qquad\le 1 + \sum_{m=1}^{l}(-1)^m \sum_{\substack{M\subset B_n\setminus(A_n\cap\partial A_n),\\|M|=m}} \mathbb{P}\left[\bigcap_{x\in M}\{\pi_x \le a_n\zeta\}\right] \\
&\qquad= 1 + \sum_{m=1}^{l}(-1)^m \sum_{\substack{M\subset B_n\setminus(A_n\cap\partial A_n),\\|M|=m,\\CC(M)=m}} \mathbb{P}\left[\bigcap_{x\in M}\{\pi_x \le a_n\zeta\}\right] \\
&\qquad\quad + \sum_{m=1}^{l}(-1)^m \sum_{q=1}^{m-1} \sum_{\substack{M\subset B_n\setminus(A_n\cap\partial A_n),\\|M|=m,\\CC(M)=q}} \mathbb{P}\left[\bigcap_{x\in M}\{\pi_x \le a_n\zeta\}\right]
\end{aligned}
$$

According to Lemma 3.65, the supremum of the last sum on the above right-hand side taken over all sequences $A_n \subset B_n$ with $|A_n| = k - 1$ converges to zero. For the first sum we observe that since $|CC(M)| = |M|$, the set M is sparse and therefore $\{\pi_x\}_{x\in M}$ is a set of independent random variables. It follows that

$$\sum_{m=1}^{l}(-1)^m \sum_{\substack{M\subset B_n\setminus(A_n\cap\partial A_n),\\ |M|=m,\\ CC(M)=m}} \mathbb{P}\Bigg[\bigcap_{x\in M}\{\pi_x \le a_n\zeta\}\Bigg]$$

$$= \sum_{m=1}^{l}(-1)^m C^{(n)}_{m,m}(A_n)\mathbb{P}[\pi_0 \le a_n\zeta]^m$$

$$= \sum_{m=1}^{l}(-1)^m \big(|B_n|^m/m! + O\big(|B_n|^{m-1}\big)\big)\mathbb{P}[\pi_0 \le a_n\zeta]^m$$

by Remark 3.64. Taking the supremum over all sequences of subsets $A_n \subset B_n$ with the constraint $|A_n| = k-1$, this still converges to $\sum_{m=0}^{l}\zeta^{2d\gamma m}/m!$. Since this holds for every $l \in 2\mathbb{N}$ and we already have the lower bound (3.127), the claim follows. □

Appendix A
Improved moment condition for i.i.d. conductances

Remark A.1. *The authors of [NSS17] prove for nearest-neighbor connections* e*, that the moment condition* $\mathbb{E}\big[\omega(\mathrm{e})^{-d/2}\big] < \infty$ *is sufficient for homogenization, see [NSS17, Corollary 3.4, Remark 3.6, Lemma 3.14]. When they assume that the conductances are independent and identically distributed, they can improve this moment condition to* $\mathbb{E}[\omega(\mathrm{e})^{-q}] < \infty$ *for* $q > 1/4$*, see Proposition 3.24 in [NSS17]. However, with a small alteration in the proof of Proposition 3.24, we can show that even* $\mathbb{E}\big[\omega(\mathrm{e})^{-1/4}\big] < \infty$ *is sufficient.*

For brevity, we assume that the reader is familiar with the article [NSS17] and we copy the notation used therein. We have reached our goal, when we can allow $\beta = 2d\gamma$ *in Step 2 of the proof of [NSS17, Proposition 3.24], i.e., if* $\mathbb{E}[\omega^{-\gamma}] < \infty$ *implies* $\mathbb{E}\big[\mu(\,\cdot\,;\mathrm{e})^{\beta p}\big] < \infty$ *for* $\beta = 2d\gamma$*.*

We will not use (94) but instead

$$\infty > \mathbb{E}\big[\omega^{-\gamma}\big] = \gamma \int_0^\infty u^{\gamma-1}\mathbb{P}\big[\omega^{-1} > u\big]\,\mathrm{d}u\,. \tag{A.1}$$

Now we follow the lines of the proof in [NSS17], i.e.,

$$\begin{aligned}\mathbb{E}\big[\mu(\mathrm{e})^{\beta p}\big] &= \beta p \int_0^\infty t^{\beta p-1}\mathbb{P}\big[\mu(\mathrm{e})^{-1} > t\big]\,\mathrm{d}t \\ &= \beta p \int_0^\infty t^{\beta p-1}\prod_{i=1}^{2d}\mathbb{P}\left[\sum_{\mathrm{b}\in\ell_i(\mathrm{e})}\omega(\mathrm{b})^{-\frac{1}{p-1}} > t^{\frac{p}{p-1}}\right]\mathrm{d}t\end{aligned}$$

With the same arguments as in [NSS17], we infer that

$$\mathbb{E}\big[\mu(\mathrm{e})^{\beta p}\big] \le 9\beta p \int_0^\infty t^{\beta p-1}\mathbb{P}\left[\omega(\mathrm{b})^{-1} > \frac{t^p}{9^{p-1}}\right]^{2d}\mathrm{d}t$$

By a change of variables we obtain that

$$
\begin{aligned}
9\beta p \int_0^\infty t^{\beta p-1}\mathbb{P}\Big[\omega(\mathrm{b})^{-1} > \frac{t^p}{9^{p-1}}\Big]^{2d}\,\mathrm{d}t \\
&= \frac{9\beta}{9^{\beta(p-1)}}\int_0^\infty u^{\beta-1}\mathbb{P}\big[\omega^{-1} > u\big]^{2d}\,\mathrm{d}u \\
&= \frac{9\beta}{9^{\beta(p-1)}}\int_0^\infty u^{\gamma-1}\mathbb{P}\big[\omega^{-1} > u\big]\cdot\Big(u^{\beta-\gamma}\mathbb{P}\big[\omega^{-1} > u\big]^{2d-1}\Big)\,\mathrm{d}u\,.
\end{aligned}
\tag{A.2}
$$

By Markov's inequality we obtain that

$$
u^{\beta-\gamma}\mathbb{P}\big[\omega^{-1} > u\big]^{2d-1} \le u^{\beta-\gamma}\frac{\mathbb{E}[\omega^{-\gamma}]^{2d-1}}{u^{\gamma(2d-1)}}\,.
$$

Since $\mathbb{E}[\omega^{-\gamma}] < \infty$ *and* $\beta \le 2d\gamma$, *this is bounded from above. Together with* (A.2) *it follows that*

$$
\mathbb{E}\big[\mu(\mathrm{e})^{\beta p}\big] \lesssim \gamma\int_0^\infty u^{\gamma-1}\mathbb{P}\big[\omega^{-1} > u\big]\,\mathrm{d}u = \mathbb{E}\big[\omega^{-\gamma}\big] < \infty\,. \tag{A.3}
$$

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