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Abstract

We study relations between Weyl geometries and Codazzi structures (see [1]) and investigate examples of Weyl geometries on affine hypersurfaces.

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0 Introduction

In section 1 we summarize basic facts on Weyl structures. In the literature there are differences in the definition of a Weyl structure depending on the choice of a constant; for this reason we define a Weyl structure using an arbitrary constant $\mathfrak{w}$: A Weyl structure on a C^∞-manifold M is given by a quadruple $\mathbf{W} = \{\widetilde{\nabla}, \mathcal{C}, \mathfrak{w}, \mathcal{T}\}$ (where $\widetilde{\nabla}$ is a torsion-free connection, $\mathcal{C}$ a conformal class of semi-Riemannian metrics, $\mathfrak{w} \in \mathbb{R} \setminus \{0\}$ and $\mathcal{T}$ a class of one-forms) satisfying the compatibility condition (1.4) below.

In [1] the authors investigate the relation between Codazzi structures $\mathbf{C} = \{\mathcal{P}, \mathcal{C}\}$ (where $\mathcal{P}$ is a projective class of connections and $\mathcal{C}$ a conformal class of semi-Riemannian metrics) and Weyl structures $\mathbf{W}$ on a manifold M. They show how a Weyl structure can be constructed from a Codazzi structure and vice versa. We give a short introduction to these constructions (with an arbitrary non-zero real number $\mathfrak{w}$) in section 2 and prove a necessary and sufficient condition for a bijective relation between Weyl and Codazzi structures.

In section 3 we investigate two naturally arising one-forms in the theory of affine hypersurfaces, i.e. the connection form $\hat{\tau}$ and the Tchebychev form $\widehat{T}$. We show that the vanishing of the exterior derivative of one of them is equivalent to the vanishing of the the exterior derivative of the other, so the construction of a non trivial Weyl structure (see definition 1.3.1 below) is either possible with both of them or none of them. For this Weyl geometry, we prove the following two results:

(i) Only on hyperquadrics, the induced connection can be realised as either a Weyl connection or the Levi-Civita connection of the affine metric h.

(ii) The induced connection of the centroaffine hypersurface geometry is invariant under gauge transformations of the Weyl geometry.

We refer the reader to [11] or [6] for definitions and facts on affine differential geometry, for more detailed proofs see [9]. In this work we always assume that the C^∞- manifolds are simply connected and real.

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1 Weyl structures

Let M be a connected C^∞–manifold of dimension $n \geq 2$, let $\mathfrak{X}(M)$ the set of tangent vector fields and let $u, v, w, \ldots$ denote elements from $\mathfrak{X}(M)$.

1.1 Projective structure and conjugate connections

Definition 1.1.1 *Let ∇, ∇^* be torsion-free connections on a manifold M.*

(i) ∇ *and* ∇^* *are called* projectively equivalent *iff there exists a one-form $\hat{\alpha}$ such that, for all $u, v \in \mathfrak{X}(M)$,*

$$\nabla_u v = \nabla^*_u v + \hat{\alpha}(u)v + \hat{\alpha}(v)u, \tag{1.1}$$

we will write $\nabla \sim \nabla^*$.

(ii) *The set $\mathcal{P}(\nabla) := \{\nabla^* | \nabla^* \sim \nabla\}$ is called the* projective class *of ∇.*

(iii) *A connection ∇ is called* flat *iff its curvature tensor is identical zero.*

(iv) ∇ *is called* projectively flat *if it is projectively equivalent to a flat connection.*

(v) *The* projective curvature tensor W *with respect to ∇ is defined by*

$$W(u,v)w = R(u,v)w - P(u,w)v + P(v,w)u + P(u,v)w - P(v,u)w$$

where $P(u,v) := \frac{1}{n^2-1}\{n\mathrm{Ric}(u,v) - \mathrm{Ric}(v,u)\}$.

The projective flatness of a connection is related to the projective curvature tensor W and to P in the following way (for the proof see [2]):

Theorem 1.1.2 (Weyl) *A connection ∇ on a manifold M is projectively flat if and only if*

(i) *W is equal to zero on M and*

(ii) *$(\nabla_u P)(v,w) = (\nabla_v P)(u,w)$ for all $u, v, w \in \mathfrak{X}(M)$.*

Definition 1.1.3 *Let h be a semi-Riemannian metric on a manifold M and $u, v, w \in \mathfrak{X}(M)$.*

(i) *Two connections ∇ and ∇^* are called* conjugate w.r.t. h *iff they satisfy*

$$u\, h(v,w) = h(\nabla_u v, w) + h(v, \nabla^*_u w). \tag{1.2}$$

The triple $\{\nabla, h, \nabla^*\}$ is called a conjugate triple.

(ii) *Let ∇ be a connection and $\hat{\tau}$ a one-form such that $\nabla h + \hat{\tau} \otimes h$ is totally symmetric. Define a torsion-free connection $\nabla^\#$ by*

$$uh(v,w) = h(\nabla^\#_u v, w) + h(v, \nabla_u w)) + \hat{\tau}(v)h(u,w). \tag{1.3}$$

Then $\nabla^\#$ is said to be semi-conjugate to ∇ relative to h by $\hat{\tau}$.

Remark 1.1.4 Let $\nabla^\#$ be semi-conjugate to ∇ relative to h by $\hat{\tau}$. Define $\check{\nabla}$ via $\check{\nabla}_u v := \nabla_u v + h(u,v)\tau$, then it is easy to see that $\{\nabla, h, \check{\nabla}\}$ is a conjugate triple.

Proposition 1.1.5 ([7]) *Let $\{\nabla, h, \nabla^*\}$ be a conjugate triple and R, R^* the curvature tensors of ∇ and ∇^* respectively. For all $u, v, w, z \in \mathfrak{X}(M)$, we have the relation $h(R(u,v)w, z) + h(w, R^*(u,v)z) = 0$.*

1.2 Weyl geometry

Definition 1.2.1 *Let $n \geq 2$ and M be a n-dimensional manifold endowed with a torsion-free connection $\widetilde{\nabla}$. Consider a conformal class of semi-Riemannian metrics $\mathcal{C} := \{\beta\, h |\, 0 < \beta \in C^\infty(M)\}$, a constant $\mathfrak{w} \in \mathbb{R}\backslash\{0\}$ and a set of one-forms $\mathcal{T} := \{\hat{\theta}_h \,|\, h \in \mathcal{C}\}$.*

(i) *The quadruple $\{\widetilde{\nabla}, \mathcal{C}, \mathfrak{w}, \mathcal{T}\}$ is called a* Weyl structure **W** on M *iff for the connection $\widetilde{\nabla}$ and all semi-Riemannian metrics $h \in \mathcal{C}$ the following condition is satisfied:*

$$\widetilde{\nabla} h = \mathfrak{w}\, \hat{\theta}_h \otimes h =: \mathfrak{w}\, \hat{\theta} \otimes h. \tag{1.4}$$

The condition (1.4) *is called* compatibility condition *and* (M, W) *is called a* Weyl manifold.

(ii) *If for the connection $\widetilde{\nabla}$, an arbitrary semi-Riemannian metric h and $\mathfrak{w} \in \mathbb{R} \setminus \{0\}$ given there exists a one-form $\hat{\theta}$ such that the compatibility condition is satisfied, then $\widetilde{\nabla}$ is called* Weyl connection *and we write $\widetilde{\nabla} =: \widetilde{\nabla}(h, \mathfrak{w}, \hat{\theta})$.*

If we make a conformal change of the metric there exists a transformation of the one-form that preserves the compatibility condition for all metrics in $\mathcal{C}$.

Definition 1.2.2 *Let $0 < \beta \in C^\infty(M)$ and $\widetilde{\nabla} = \widetilde{\nabla}(h, \mathfrak{w}, \hat{\theta})$. The mapping*

$$h \to \beta h, \quad \hat{\theta} \to \hat{\theta} + \tfrac{1}{\mathfrak{w}} d \ln \beta. \tag{1.5}$$

is called gauge transformation.

Remark 1.2.3 From definition 1.2.2 one easily verifies that the compatibility condition is invariant under gauge transformations.

From a given metric $h \in \mathcal{C}$, a real non-zero number $\mathfrak{w}$ and a one-form $\hat{\theta}$ one can construct a torsion-free connection $\widetilde{\nabla}$ satisfying (1.4); thus $(h, \mathfrak{w}, \hat{\theta})$ induce a Weyl geometry.

Lemma 1.2.4 *Let $h \in \mathcal{C}$, $\mathfrak{w} \in \mathbb{R} \setminus \{0\}$, $\hat{\theta}$ be a one-form and $\widehat{\nabla}$ the Levi-Civita connection of h. Then*

(i) *the Weyl connection $\widetilde{\nabla} = \widetilde{\nabla}(h, \mathfrak{w}, \hat{\theta})$ can be expressed in terms of $h, \mathfrak{w}$ and $\hat{\theta}$ by*

$$\widetilde{\nabla}_u v = \widehat{\nabla}_u v - \tfrac{\mathfrak{w}}{2} \Big\{\hat{\theta}(u) v + \hat{\theta}(v) u + h(u, v) \theta\Big\} \text{ for all } u, v \in \mathfrak{X}(M), \tag{1.6}$$

where θ is defined by $\hat{\theta}(u) =: h(u, \theta)$; this construction is invariant under gauge transformations;

(ii) *the connection $\widetilde{\nabla}$ defined in* (i) *satisfies the compatibility condition.*

Proof. For the proof see [12]. □

Remark 1.2.5 For a Weyl connection $\widetilde{\nabla}$ it is obvious that its curvature tensor $\widetilde{R}$ defined by $\widetilde{R}(u, v) w := \{\widetilde{\nabla}_u \widetilde{\nabla}_v - \widetilde{\nabla}_v \widetilde{\nabla}_u - \widetilde{\nabla}_{[u,v]}\} w$ and its Ricci tensor $\widetilde{\mathrm{Ric}}$ defined by $\widetilde{\mathrm{Ric}}(v, w) := \mathrm{trace}\{u \mapsto \widetilde{R}(u, v) w\}$ are invariant under gauge transformations.

1.3 Weyl curvature

Definition 1.3.1 *Let $\tilde{\nabla} = \tilde{\nabla}(h, \mathfrak{w}, \hat{\theta})$ be a Weyl connection and $u, v, w \in \mathfrak{X}(M)$.*

(i) *The* length curvature *$F\colon \mathfrak{X}(M) \times \mathfrak{X}(M) \to C^\infty(M)$ is defined by*

$$F(u,v) = -\mathfrak{w}\, d\hat{\theta}(u,v) \tag{1.7}$$

where $d\hat{\theta}$ denotes the exterior derivative of $\hat{\theta}$. We call a Weyl structure trivial *iff $F \equiv 0$.*

(ii) *The* directional curvature *is the mapping $K\colon \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ defined by*

$$K(u,v)w = \tilde{R}(u,v)w - F(u,v)w. \tag{1.8}$$

Remark 1.3.2 F is obviously gauge invariant and so is K, too.

Lemma 1.3.3 *The following relations for $\tilde{R}$, F and K*

(i)		$h(\tilde{R}(u,v)w,w)$	$=$	$F(u,v)h(w,w)$,
(ii)		$h(K(u,v)w,z)$	$=$	$-h(w,K(u,v)z)$,
(iii)	(a)	$K(u,v)w \perp_h w$	*and*	(b) $F(u,v)w \parallel w$,
(iv)		$2F(u,v)h(w,z)$	$=$	$h(\tilde{R}(u,v)w,z) + h(w,\tilde{R}(u,v)z)$

hold for all $h \in C$ and $u, v, w, z \in \mathfrak{X}(M)$.

Proof. (i) – (iii) are well known, see [12]. (iv) follows directly from (i) and (iii). □

The relation 1.3.3 (iv) is similar to a relation of curvature tensors of conjugate connections. As a consequence of proposition 1.1.5 and lemma 1.3.3 (iv) we get for an arbitrary choice of a semi-Riemannian metric $h \in C$:

Corollary 1.3.4 *Let $\tilde{\nabla} = \tilde{\nabla}(h, \mathfrak{w}, \hat{\theta})$ be a Weyl connection, $\{\tilde{\nabla}, h, \tilde{\nabla}^*\}$ the conjugate triple and $\tilde{R}$, $\tilde{R}^*$ the curvature tensors of $\tilde{\nabla}$ respectively $\tilde{\nabla}^*$. The equation $\tilde{R}^*(u,v)w = \tilde{R}(u,v)w - 2F(u,v)w$ is valid for all $u, v, w \in \mathfrak{X}(M)$. This relation is gauge invariant.*

Remark 1.3.5 The conjugate connection $\tilde{\nabla}^*$ of $\{\tilde{\nabla}, h\}$ is torsion-free if $\hat{\theta} \equiv 0$; then we have $\tilde{\nabla}^* = \tilde{\nabla} = \hat{\nabla}$.

Theorem 1.3.6 ([3]) *If and only if $F \equiv 0$ then the Weyl connection $\tilde{\nabla} = \tilde{\nabla}(h, \mathfrak{w}, \hat{\theta})$ is the Levi-Civita connection of an appropriate metric $h^\# \in C$.*

Furthermore the symmetry of the Ricci tensor $\widetilde{\mathrm{Ric}}$ is closely related to the vanishing of the length curvature, a straightforward computation shows $\widetilde{\mathrm{Ric}}(u,v) - \widetilde{\mathrm{Ric}}(v,u) = n\,F(u,v)$. This proves:

Proposition 1.3.7 *Let $\tilde{\nabla} = \tilde{\nabla}(h, \mathfrak{w}, \hat{\theta})$ be a Weyl connection, $\widetilde{\mathrm{Ric}}$ the Ricci tensor and F the length curvature* w.r.t. *$\tilde{\nabla}$. Then we have that $F \equiv 0$ if and only if $\widetilde{\mathrm{Ric}}$ is symmetric.*

The projective flatness of a Weyl connection is also related to the vanishing of the length curvature.

Theorem 1.3.8 *Let $\tilde{\nabla} = \tilde{\nabla}(h, \mathfrak{w}, \hat{\theta})$ be a Weyl connection and $\widetilde{W}$ the projective curvature tensor* w.r.t. *$\tilde{\nabla}$. If $n \geq 3$ then we get that $\widetilde{W} = 0$ implies $F = 0$.*

Proof. Let $n \geq 3$ and $u, v, w \in \mathfrak{X}(M)$ be h-orthogonal by pairs. Then

$$0 = \widetilde{W}(u,v)w = K(u,v)w - \widetilde{P}(u,w)v + \widetilde{P}(v,w)u - F(u,v)w + \widetilde{P}(u,v)w - \widetilde{P}(v,u)w.$$

It is sufficient to show $F(u,v) \neq \widetilde{P}(u,v) - \widetilde{P}(v,u)$ because the other terms are h-orthogonal to w. Using proposition 1.3.7 we have $\widetilde{P}(u,v) - \widetilde{P}(v,u) = \frac{n}{n+1}F(u,v)$. □

Remark 1.3.9 If the length curvature does not vanish the Weyl connection cannot be projectively flat in $\dim(M) \geq 3$. In dimension two an analogon to this is not known.

2 Codazzi structures

2.1 Introduction

Definition 2.1.1 *Let $\mathcal{P}^* := \mathcal{P}(\nabla^*)$ be a projective class of torsion-free connections*[1]*, $\mathcal{C}$ a conformal class of semi-Riemannian metrics and $u, v, w \in \mathfrak{X}(M)$.*

(i) *If for an arbitrary $h \in \mathcal{C}$ there exists a $\nabla^* \in \mathcal{P}^*$ such that the equation*

$$(\nabla^*_u h)(v,w) = (\nabla^*_v h)(u,w) \tag{2.1}$$

holds, then the pair $\{\nabla^*, h\}$ will be called a Codazzi pair.

(ii) *Equation* (2.1) *is called* Codazzi equation *for ∇^* and h.*

(iii) *If there exists a Codazzi pair $\{\nabla^*, h\}$ in $\{\mathcal{P}^*, \mathcal{C}\}$, then $\{\mathcal{P}^*, \mathcal{C}\}$ is called a* Codazzi structure *on M.*

(iv) *A* Codazzi transformation *is the mapping*

$$h \mapsto \beta h =: h^{\#}, \quad \nabla^*_u v \mapsto \nabla^*_u v + d\ln\beta(u)v + d\ln\beta(v)u =: \nabla^{*\#}. \tag{2.2}$$

Remark 2.1.2 If $\{\mathcal{P}^*, \mathcal{C}\}$ is a Codazzi structure then for every $h \in \mathcal{C}$ there exists a $\nabla \in \mathcal{P}^*$ such that $\{h, \nabla\}$ is a Codazzi pair; the mapping $h \mapsto \nabla$ is injective, not surjective.

2.2 The constructions

In [1], the authors considered two constructions: (i) Given a Weyl structure[2] one constructs a Codazzi structure and (ii) vice versa, construct a Weyl structure from a given Codazzi structure.

(i) Construct a Codazzi structure from a given Weyl structure $\{\widetilde{\nabla} = \widetilde{\nabla}(h, \mathfrak{w}, \hat{\theta}), \mathcal{C}, \mathfrak{w}, \mathcal{T}\}$ on M: Using lemma 1.2.4 (i) we have $\widetilde{\nabla}_u v = \widehat{\nabla}_u v - \frac{\mathfrak{a}}{2}\{\hat{\theta}(u)v + \hat{\theta}(v)u - h(u,v)\theta\}$. Define

$$\nabla^*_u v = \widehat{\nabla}_u v + \tfrac{\mathfrak{w}}{2}\left\{\hat{\theta}(u) + \hat{\theta}(v)u + h(u,v)\theta\right\} \tag{2.3}$$

and a connection ∇ that is conjugate to ∇^* with respect to h:

$$\nabla_u v := \widehat{\nabla}_u v - \tfrac{\mathfrak{w}}{2}\left\{\hat{\theta}(u) + \hat{\theta}(v)u + h(u,v)\theta\right\}.$$

The result is that $\{\nabla^*, h\}$ and $\{\nabla, h\}$ are Codazzi pairs and ∇^* is projectively equivalent to $\widetilde{\nabla}$.

[1] that do not necessarily have a symmetric Ricci tensor

[2] with $\mathfrak{w} = 2$, which is not a necessary condition. One can choose an arbitrary $\mathfrak{w} \in \mathbb{R} \setminus \{0\}$.

Remark 2.2.1 For the curvature tensors R^* of ∇^* and R of ∇ of conjugate connections we have proposition 1.1.5 (i.e. $h(R(u,v)w,z)+h(w,R^*(u,v)z)=0$); given a Weyl structure one can construct a Codazzi structure, in this situation proposition 1.1.5 is equivalent to lemma 1.3.3 (iv) (i.e. $h(\widetilde{R}(u,v)w,z)+h(w,\widetilde{R}(u,v)z)=2F(u,v)h(w,z)$). Lemma 1.1.5 is not conformally invariant but lemma (1.3.3) (iv) obviously is invariant under gauge transformations that include a conformal change of the metric.

(ii) Again, following [1], we can construct a Weyl structure from a Codazzi structure: let $\{\mathcal{P},\mathcal{C}\}$ be a Codazzi structure. For a fixed Codazzi pair $\{\nabla,h\}\in\{\mathcal{P},\mathcal{C}\}$, we define the (1.2)-tensor C:

$$C := \widehat{\nabla} - \nabla. \tag{2.4}$$

∇ and $\widehat{\nabla}$ are torsion-free, therefore C is a symmetric (1.2)-tensor. For C define the associated one-form

$$n\,\widehat{T}(v) := \operatorname{trace}\{u \mapsto C(u,v)\}. \tag{2.5}$$

A Codazzi transformation induces the following transformation formulas for C and $\widehat{T}$ (see [11], proposition 5.1.3.)

$$C^{\#}(u,v) = C(u,v) - \tfrac{1}{2}\{d\ln\beta(u)v + d\ln\beta(v)u + h(u,v)\operatorname{grad}_h \ln\beta\}, \tag{2.6}$$

$$\tfrac{n}{n+2}\widehat{T}^{\#} = \tfrac{n}{n+2}\widehat{T} - \tfrac{1}{2}d\ln\beta. \tag{2.7}$$

Here we see that the one-form $\frac{n}{n+2}\widehat{T}$ transforms like the one-form $\hat{\theta}$ that appears in the gauge transformation (1.5) with $\mathfrak{w}=-2$. Therefore this one-form is eligible to construct a Weyl connection:

$$\widetilde{\nabla}_u v := \widehat{\nabla}_u v + \tfrac{n}{n+2}\left\{\widehat{T}(u) + \widehat{T}(v)u - h(u,v)T\right\} \tag{2.8}$$

where T is defined by $\widehat{T}(u) =: h(u,T)$.

Remark 2.2.2 The connection $\widetilde{\nabla}$ is not necessarily projectively equivalent to the given ∇; the invariant formulation of lemma 1.1.5 for the curvature tensors is only possible if ∇ and $\widetilde{\nabla}$ are in the same projective class.

Lemma 2.2.3 ([1]) *Consider two Codazzi structures $\{\mathcal{P},\mathcal{C}\}$ and $\{\mathcal{P}^*,\mathcal{C}\}$ and define the symmetric* (1.2)-*tensor field γ for any two Codazzi pairs $\{\nabla,h\}$ and $\{\nabla^*,h\}$ by $\gamma(v,w):=\nabla_v w-\nabla^*_v w$. The Codazzi structures define the same Weyl structure if and only if γ is apolar, which means* $\operatorname{trace}\{u\mapsto\gamma(u,v)\}=0$.

Following [1] prescribe a Weyl structure and construct a Codazzi structure. From this Codazzi structure again a Weyl structure can be constructed. This latter Weyl structure coincides with the given one.

On the other hand, if we start with a given Codazzi structure and first construct a Weyl structure and then from this a Codazzi structure again, the latter Codazzi structure need not to coincide with the given one.

Theorem 2.2.4 *Let $\{\mathcal{P},\mathcal{C}\}$ be a Codazzi structure and, for a fixed $h\in\mathcal{C}$, construct a Weyl connection $\widetilde{\nabla}=\widetilde{\nabla}(h,-\frac{2n}{\mathfrak{w}(n+2)},\widehat{T})$ following* (2.4) - (2.8). *From that Weyl connection construct a projective class $\mathcal{P}^*$ as in* (2.3)*; this implies that $\{\mathcal{P}^*,\mathcal{C}\}$ is a Codazzi structure. Then*

$$\mathcal{P}=\mathcal{P}^* \text{ if and only if, in } \mathcal{P}, \text{ there exists a Weyl connection compatible with } h.$$

Proof. Let $\tilde{\tilde{\nabla}}\in\mathcal{P}$ be a Weyl connection and show that $\tilde{\tilde{\nabla}}$ is equal to $\widetilde{\nabla}(h,-\frac{2n}{a(n+2)},\widehat{T})\in\mathcal{P}^*$, using the projective equivalence of $\tilde{\tilde{\nabla}}$ and a $\nabla\in\mathcal{P}$ and lemma 1.2.4 (i). □

Remark 2.2.5 Let $\mathcal{C}$ be a given conformal class; then there are two types of projective classes: projective classes that contain a Weyl connection compatible with $\mathcal{C}$ and others that do not contain such a connection.

3 Affine differential geometry of hypersurfaces

3.1 Introduction

Let M be orientable and A a real affine space of dimension $n+1$ equipped with the canonical flat connection $\overline{\nabla}$; let V be the real vector space associated to A and V^* its dual space. Let $x: M \to A$ be an immersion with an arbitrary transversal field y. Then we have the structure equations

$$\overline{\nabla}_u dx(v) = dx(\nabla_u v) + h(u,v)y, \tag{3.1}$$

$$dy(v) = dx(-Sv) + \hat{\tau}(v)y. \tag{3.2}$$

Here, h is a symmetric (0.2)-tensor field, ∇ a torsion-free connection, called the *induced connection*, S a (1.1)-tensor field, called the *shape operator* and $\hat{\tau}$ a one-form called the *connection form*.
Choose a *conormal field* $Y: M \to V^*$ as the unique solution of

$$Y(y) = 1 \quad \text{and} \quad Y(dx(v)) = 0 \quad (v \in \mathfrak{X}(M)). \tag{3.3}$$

If x is a *regular* hypersurface — i.e. h is nondegenerate — then h is called the *affine metric*.

Remark 3.1.1 The regularity of x is independent of the choice of y and equivalent to $\operatorname{rank}(dY, Y) = n+1$.

Let x be regular, then we can consider Y as a hypersurface $Y: M \to V^*$ with transversal field $(-Y)$ and structure equation

$$\overline{\nabla}_u dY(v) = dY(\nabla^*_u v) + \hat{S}(u,v)(-Y). \tag{3.4}$$

∇^* is a torsion-free connection, called the *conormal connection*, and $\hat{S}$ is a symmetric (0.2)-tensor field.

In this section we assume all hypersurfaces to be regular; then the pair $\{Y, y\}$ satisfying (3.3) is called a *normalisation*.

Lemma 3.1.2 *Some basic relations of the coefficients of the structure equations are:*

(i) $\{\nabla^*, h\}$ *is a Codazzi pair,*

(ii) ∇ *and* ∇^* *are semi-conjugate relative to* h *by* $\hat{\tau}$ *and*

(iii) $\hat{S}(u,v) = h(Su,v) + (\nabla^*_u\hat{\tau})(v) - \hat{\tau}(u)\hat{\tau}(v)$.

Proof. The proof of (i) follows [7]; use that $\check{\nabla}_u v$, as defined in remark 1.1.4, and ∇^* are torsion-free and conjugate with respect to h. For (ii) show that $Y(dy(v)) =: < Y, dy(v) > = \hat{\tau}(v)$, use (i) and the fact that for a conjugate triple $\{\nabla^*, h, \check{\nabla}\}$ we have: $\nabla^* h$ is totally symmetric iff $\check{\nabla} h$ is totally symmetric, see [11], 4.4.1; (iii) follows from (i), (ii) and the structure equations (3.1), (3.2) and (3.4). $\square$

Corollary 3.1.3 *The Levi-Civita connection* $\widehat{\nabla}$ *of* h *in terms of* ∇, ∇^* *and* $\hat{\tau}$ *is given, using the notation of remark* 1.1.4, *by* $\widehat{\nabla}_v u = \frac{1}{2}(\nabla^*_v u + \nabla_v u + h(u,v)\tau) = \frac{1}{2}(\nabla^*_v u + \check{\nabla}_v u)$.

The integrability conditions for the hypersurfaces x and Y in terms of ∇ and ∇^* read

$$\begin{aligned} h(v,Su)-h(u,Sv) &= 2d\hat{\tau}(v,u), & (3.5)\\ (\nabla_v S)u-(\nabla_u S)v &= \hat{\tau}(v)Su-\hat{\tau}(u)Sv, & (3.6)\\ R(u,v)w &= h(v,w)Su-h(u,w)Sv & (3.7)\\ (\nabla_v h)(u,w)+\hat{\tau}(v)h(u,w) &= (\nabla_u h)(v,w)+\hat{\tau}(u)h(v,w), & (3.8)\\ R^*(u,v)w &= \widehat{S}(v,w)u-\widehat{S}(u,w)v, & (3.9)\\ (\nabla^*_v\widehat{S})(u,w) &= (\nabla^*_u\widehat{S})(v,w); & (3.10) \end{aligned}$$

the proof is analogous to [11], 4.8.1 and 4.8.2, compare also [8] and [4].

Remark 3.1.4 By the equations (3.9) and (3.10) it can be seen that, like in the case of relative[3] normalisations, ∇^* is projectively flat because the integrability conditions of ∇^* are the same as in the case of relative normalisations (for the proof see [11], 4.10.3.2.). Moreover the Ricci tensor Ric* of ∇^* is symmetric (see [11], 4.8.1.).

3.2 The vanishing of the derivative of the connection form

A natural question that arises is: under which conditions for the connection form is it possible to construct non-trivial Weyl structures. The existence of a connection form with non-vanishing exterior derivative is proved by Opozda:

Theorem 3.2.1 ([8]) *Let M be a simply connected n-dimensional manifold endowed with a connection ∇, a symmetric bilinear form h, a $(1,1)$-tensor field S and a one-form $\hat{\tau}$ such that equations* (3.5) – (3.8) *are satisfied. Then there are an nondegenerate immersion $x\colon M\to A$ and a vector field y transversal to x such that ∇, h, S and $\hat{\tau}$ are the objects induced by $\{x,y\}$ via* (3.1) *and* (3.2).

Here we can see that there are no further restrictions to $\hat{\tau}$, so we can assume that the exterior derivative of the connection form does not vanish. In this case we can construct a non-trivial Weyl connection using $\hat{\tau}$. The following lemma gives conditions to ∇, h and S, resp., which imply $d\hat{\tau}\equiv 0$.

Lemma 3.2.2 *Let x be a hypersurface with transversal field y and conormal field Y. The following properties are equivalent:*

(i) $d\hat{\tau}\equiv 0$;

(ii) *S is selfadjoint* w.r.t. *h;*

(iii) *∇ has a symmetric Ricci tensor;*

(iv) *$d\widehat{T}\equiv 0$ where $\widehat{T}(u) := \frac{1}{n}\mathrm{trace}\,\{v\mapsto(\nabla_v u-\nabla^*_v u)\}$; $\widehat{T}$ is called the* Tchebychev form.

Proof. For (i) $\Leftrightarrow$ (ii) use (3.5); (ii) $\Leftrightarrow$ (iii) follows from (3.7) and (iii) $\Leftrightarrow$ (iv) is shown in [7], proposition 4.1 and 4.4. $\square$

Remark 3.2.3 Because of the equivalence of (i) and (iv) there are either two one-forms to construct a non-trivial Weyl connection, or none.

[3] where y is chosen such that $\hat{\tau}$ is equal to zero

Lemma 3.2.4 *Let $x\colon M \to A$ be a hypersurface with transversal field y and conormal field Y. Let C be the conformal class of metrics such that $h \in C$, $\widehat{\nabla}$ the Levi-Civita connection of h and $u, v \in \mathfrak{X}(M)$. If any one of the conditions (i) – (iv) is satisfied, then $d\hat{\tau} \equiv 0$:*

(i) *the induced connection is a Weyl connection,*

(ii) *there exists $h^{\#} \in C$ such that $\{\nabla, h\}$ is a Codazzi pair,*

(iii) $\nabla_u v = \widehat{\nabla}_u v - h(u,v)\tau$ *with $\hat{\tau}(u) =: h(u,\tau)$ for all ($u \in \mathfrak{X}(M)$),*

(iv) ∇ *is projectively equivalent to* $\widehat{\nabla}$.

Proof. (i) Let ∇ be a Weyl connection. From lemma 1.2.4 (i) we know that there exists a one-form $\hat{\alpha}$ such that $\nabla_u v = \widehat{\nabla}_u v - \frac{\mathfrak{w}}{2}\{\hat{\alpha}(u)v + \hat{\alpha}(v)u - h(u,v)\alpha\}$, where α is defined by $\hat{\alpha}(u) =: h(u,\alpha)$. Using corollary 3.1.3 we get $\nabla^*_u v = \widehat{\nabla}_u v + \frac{\mathfrak{w}}{2}\{\hat{\alpha}(u)v + \hat{\alpha}(v)u - h(u,v)\alpha\} - h(u,v)\hat{\tau}$. Additionally we have $(\nabla^*_u h)(v,w) = -\mathfrak{w}\,\hat{\alpha}(u)h(v,w) + \hat{\tau}(v)h(u,w) + \hat{\tau}(w)h(u,v)$. Lemma 3.1.2 (i) implies $\alpha = -\frac{1}{\mathfrak{w}}\hat{\tau}$; we get

$$\nabla^*_u v = \widehat{\nabla}_u v - \tfrac{1}{2}\left\{\hat{\tau}(u)v + \hat{\tau}(v)u + h(u,v)\tau\right\}. \tag{3.11}$$

This and remark 3.1.4 imply $d\hat{\tau} \equiv 0$.
(ii) A straightforward computation shows $d\hat{\tau} \equiv 0$.
(iii) Using corollary 3.1.3 we have $\nabla^*_u v = \widehat{\nabla}_u v - 2\,h(u,v)\tau$. Lemma 1.2.4 (i) shows $\hat{\tau} = 0$, this implies $d\hat{\tau} \equiv 0$.
(iv) Let $\hat{\alpha}$ be a one-form such that $\nabla_u v = \widehat{\nabla}_u v - \hat{\alpha}(u)v - \hat{\alpha}(v)u$. A similar calculation as in the proof of (i) shows $\nabla^*_u v = \widehat{\nabla}_u v + \hat{\alpha}(u)v + \hat{\alpha}(v)u - h(u,v)\tau$. The Codazzi property of $\{\nabla^*, h\}$ leads us to

$$\nabla^*_u v = \widehat{\nabla}_u v - \hat{\tau}(u)v - \hat{\tau}(v)u - h(u,v)\tau. \tag{3.12}$$

With the same argumentation as in (i) we get $d\hat{\tau} \equiv 0$. □

Remark 3.2.5 In parts (ii) and (iv) of the above proof, it can be seen from the equations (3.11) and (3.12) that the cubic form $\widehat{C}(u,v,w) := h(\widehat{\nabla}_u v - \nabla^*_u v, w)$ has the form $\widehat{C}(u,v,w) = \hat{\alpha}(u)h(v,w) + \hat{\alpha}(v)h(w,u) + \hat{\alpha}(w)h(u,v)$. This is a necessary and sufficient condition for $x(M)$ to be a quadric — as shown in [5], theorem 8. Therefore, only on quadrics, the induced connection can be realised as a connection that is projectively equivalent to the Levi-Civita connection, or as a Weyl connection.

3.3 Transformations of the transversal field

Lemma 3.3.1 (Transformation Lemma) *Consider a hypersurface $x\colon M \to A$ with two normalisations $\{Y, y\}$ and $\{Y^{\#}, y^{\#}\}$ with the same orientation. There are a function $0 < \phi \in C^\infty(M)$ and a vector field $\eta \in \mathfrak{X}(M)$ such that $y^{\#} = \phi^{-1}\{y + dx(\eta)\}$. Then we have*

(i) $Y^{\#} = \phi Y$,

(ii) $\nabla^{*\#}_u v = \nabla^*_u v + d\ln\phi(u)v + d\ln\phi(v)u$,

(iii) $h^{\#} = \phi\, h$,

(iv) $\nabla^{\#}_u v = \nabla_u v - h(u,v)\eta$ *and*

(v) $\hat{\tau}^{\#}(v) = \hat{\tau}(v) + \hat{\eta}(v) - d\ln\phi(v)$, *where η is given by $\hat{\eta}(u) = h(u,\eta)$ for all $u \in \mathfrak{X}(M)$.*

Proof. Straightforward calculations. $\square$

If we choose $\phi = \beta$ and $\eta = (1+\frac{1}{a})\mathrm{grad}_h \ln\beta$ then we have a gauge transformation $(h, \hat{\tau}) \mapsto (\beta\, h, \hat{\tau} + \frac{1}{a} d\ln\beta)$, which is induced by a transformation of the transversal field. Moreover, if we set $a = -1$ it is easy to see that ∇ is invariant under transformations of the transversal field with ϕ and η chosen as above. If we assume that ∇ is a Weyl connection then lemma 3.2.4 shows that it is a trivial Weyl connection.

3.4 The centroaffine connection as a gauge invariant connection

Definition 3.4.1 *Let $x\colon M \to V$ be a hypersurface with $0 \notin x(M)$, and x be transversal to $x(M)$. Let $\{Y, y\}$ be a normalisation and define the associated* support function

$$\rho := < Y, -x >.$$

As x and y are transversal we can express y in terms of x and dx; by straightforward computations we get $y = -\rho^{-1}x + dx(\mathrm{grad}_h \ln\rho + \tau)$ and the decomposition

$$\overline{\nabla}_u dx(v) = dx(\nabla_u v + h(u,v)\,\{\mathrm{grad}_h \ln\rho + \tau\}) - \rho^{-1}h(u,v)x.$$

Definition 3.4.2 *Let $x\colon M \to V$ be a hypersurface with normalisation $\{Y, y\}$ and associated support function $\rho \neq 0$. Let ∇ be the induced connection and $\hat{\tau}$ the connection form. Define the connection*

$$\overset{\circ}{\nabla}_u v = \nabla_u v + h(u,v)\,\{\mathrm{grad}_h \ln\rho + \tau\})$$

where τ is given by $\hat{\tau}(u) = h(u,\tau)$ for all $u \in \mathfrak{X}(M)$.

Remark 3.4.3 $\overset{\circ}{\nabla}$ again is a torsion-free connection (∇ torsion-free, h symmetric). The geometric interpretation of $\overset{\circ}{\nabla}$ is that the pregeodesics of this connection are intersections of the hypersurface with planes that contain the point x_0. This connection is studied in [10] for the case of relative normalisations.

For a transformation $y \mapsto \beta^{-1}y + (1+\frac{1}{a})dx(\mathrm{grad}_h \ln\beta) =: y^{\#}$ we have $\rho \mapsto \beta\rho := \rho^{\#}$; the other quantities change as in the transformation lemma. We get

$$\begin{aligned} \overset{\circ}{\nabla}{}^{\#}_u v &= \nabla^{\#}_u v + h^{\#}(u,v)\,\{\mathrm{grad}_{h^{\#}} \ln\rho^{\#} + \tau^{\#}\} \\ &= \nabla_u v - (1+\tfrac{1}{a})h(u,v)\mathrm{grad}_h \ln\beta + \beta h(u,v)\beta^{-1}(\mathrm{grad}_h \ln\beta\rho + \tau + \tfrac{1}{a}\mathrm{grad}_h \ln\beta) \\ &= \overset{\circ}{\nabla}_u v. \end{aligned}$$

We have proved:

Proposition 3.4.4 *Let $x : M \to V$ be a hypersurface as in definition 3.4.1. Then $\overset{\circ}{\nabla}$ is gauge invariant.*

The connection induced by a transversal field that is a multiple of its position vector field x is well-known in affine differential geometry of hypersurfaces: it is known as *centroaffine* connection. Therefore we have

Corollary 3.4.5 *The centroaffine connection is gauge invariant.*

Lemma 3.4.6 *If $\overset{\circ}{\nabla}$ is a Weyl connection, then it is trivial.*

Proof. From the definition of $\overset{\circ}{\nabla}$ we can see that $\overset{\circ}{\nabla}$ is induced by the transversal field $-\rho^{-1}x$. Lemma 3.2.4 gives that the induced connection is a trivial Weyl connection. $\square$

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References

[1] N. Bokan, P. Gilkey, and U. Simon, *Asymptotic spectra for Weyl geometries*, Analele Stiintifice ale Universitatii "Al.I.Cuza" DiN IASI, **42**, Supliments Ia Mathematica (1996), 59 – 71.

[2] L. P. Eisenhart, *Non-Riemannian Geometry*, AMS Colloquium Publications, Volume VIII, 1927.

[3] T. Higa, *Weyl manifolds and Einstein-Weyl manifolds*, Comm. Math. Univ. St. Pauli **42** (1993), 143 – 160.

[4] S. Ivanov, *On dual-projectively flat affine connections*, Journ. of Geom. **53** (1995), 89–99.

[5] K. Nomizu and U. Pinkall, *Cubic form theorem for affine immersions*, Res. in Math. **13** (1988), 338 – 362.

[6] K. Nomizu and T. Sasaki, *Affine Differential Geometry*, Cambridge University Press, 1994.

[7] K. Nomizu and U. Simon, *Notes on conjugate connections*, Geometry and Topology of Submanifolds IV (F. Dillen and L. Verstraelen, eds.), Leuven and Brussel, Belgium 1991, World Scientific, Singapore, 1992, pp. 152 – 172.

[8] B. Opozda, *Some extensions of Radon's theorem*, Lecture Notes in Math., vol. 1481, Springer, Berlin, 1991, pp. 185 – 191.

[9] M. Peikert, *Beispiele von Weyl-Geometrien in der affinen Differentialgeometrie*, Diploma thesis, Technische Universität Berlin, 1997.

[10] U. Simon, *Connections and conformal structure in affine differential geometry*, Differential Geometry and its Applications (D. Krupka and A. Svec, eds.), J.E. Pyrkyne University, CSSR, 1987, pp. 315 – 327.

[11] U. Simon, A. Schwenk-Schellschmidt, and H. Viesel, *Introduction to the Affine Differential Geometry of Hypersurfaces*, Lecture Notes, ISBN 3 7983 1529 9, Science University Tokyo, 1991.

[12] H. Weyl, *Raum – Zeit – Materie*, 8. edition, edited and added by J. Ehlers, 1^{st} edition 1918 (in German), Springer Verlag, Berlin, Heidelberg, New York, 1993.