Spectrally Efficient Bidirectional Decode-and-Forward Relaying for Wireless Networks

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Zusammenfassung

Relaiskonzepte werden in Zunkuft in drahtlosen Netzwerken eine zentrale Rolle spielen. In dieser Dissertation wird ein Netzwerk mit drei Stationen betrachtet, in welchem zwei Stationen mittels einer Relaisstation miteinander kommunizieren. Wir untersuchen ein Zweiphasenprotokoll ohne Rückkopplung, in dem die Relaisstation die Nachrichten für beide Richtungen gleichzeitig dekodiert und weiterleitet. Damit kann der spektrale Verlust, verursacht durch halbduplexe Stationen in unidirektionalen Protokollen, kompensiert werden.

Im ersten Abschnitt geht es um schichtübergreifende Konzepte für die bidirektionale Relaiskommunikation basierend auf dem Prinzip der Superpositionskodierung. Die Betrachtungen werden für die festgelegte und optimale Zeitaufteilung der Phasen durchgeführt. Eine ausführliche Untersuchung der Kombinatorik der erreichbaren Ratenregion für Stationen mit einer Antenne erlaubt es uns die Ratenpaare, bei denen die Summe der gewichteten Raten maximiert wird, explizit zu bestimmen. Diese Untersuchungen werden am Beispiel der paritätischen und der optimalen Zeitaufteilung ausgeführt.
Die optimalen Ratenpaare werden im Weiteren für den Entwurf einer durchsatzoptimalen Ressourcenallokationsstrategie und für ein Relaisauswahlkriterium in einem Netzwerk mit N Relaisstationen verwendet. Dabei ist es vorteilhaft, die abwechselnde Nutzung verschiedener Relaisstationen zu erlauben. Außerdem stellen wir fest, dass die Summe eines jeden Ratenpaares auf dem Rand der ergodischen Ratenregion für unabhängige und identisch rayleighverteilte Schwundkanäle asymptotisch mit $\log(\log(N))$ wächst.
Als nächstes betrachten wir den Fall, dass die Relaisstation eine Nachricht zusätzlich zur bidirektionalen Relaiskommunikation an beide Stationen übeträgt. Es zeigt sich, dass es immer optimal ist die Relaisnachricht zuerst zu dekodieren. Die gemeinsame Ressourcenallokation verbessert die Gesamteffizienz und ermöglicht einen neuen Ratenabtausch. Darüber hinaus charakterisieren und diskutieren wir das gemeinsame Summenratenmaximum beider Routingaufgaben.
Anschließend untersuchen wir die erreichbaren Ratenregionen der bidirektionalen Relaiskommunikation zwischen Stationen mit mehreren Antennen. Wir spezifizieren die optimale Sendestrategie und zeigen, dass die Summe eines jeden Ratenpaares auf dem Rand der erreichbaren Ratenregion linear gemäß der räumlichen Dimension des Vektorkanals und der Zeitaufteilung der Phasen ansteigt.

Im zweiten Abschnitt beweisen wir eine optimale Kanalkodierungsstrategie für den bidirektionalen Broadcastkanal mit einem endlichen Alphabet. Dabei werden erreichbare Raten bezüglich der maximalen Fehlerwahrscheinlichkeit betrachtet. Beim Kodierungssatz folgen wir der Philosophie der Netzwerkkodierung, die besagt, dass sich Informationsflüsse nicht wie Flüssigkeiten verhalten.

Im abschließenden Resümee geben wir einen Ausblick auf zukünftige Forschungsarbeiten und präsentieren Beispiele wie bidirektionale Relaiskommunikation in drahtlose Netzwerke integriert werden kann.

Abstract

Relaying concepts will play a central role in future wireless networks. In this thesis we consider a three-node network where two nodes communicate with each other by the support of a relay node. We study a two-phase decode-and-forward bidirectional relaying protocol without feedback. Bidirectional relaying has the ability to compensate the spectral loss due to the half-duplex constraint of nodes in wireless communications.

In the first part we study cross-layer design aspects of bidirectional relaying using superposition encoding at the relay node. For the two phases we consider the fixed and optimal time division case. For single-antenna nodes an intensive study of the combinatorial structure of the achievable rate region allows us to characterize the rate pairs which maximize the weighted rate sum for the equal and fixed time division case in closed form. These are used for the design of a throughput optimal resource allocation policy based on the backpressure strategy and to derive a relay selection criterion for routing in network with N relay nodes. It shows that it is beneficial to allow time-sharing between the usage of relay nodes. We see that the sum of any rate pair on the boundary of the ergodic rate region for independent and identical distributed Rayleigh fading channels grows asymptotically with $\log(\log(N))$.

Then we add a relay multicast to the bidirectional relay communication. The joint resource allocation of two routing tasks improves the overall efficiency and enables new rate trade-offs. It shows that it is always optimal to decode the relay message first. Furthermore, we characterize and discuss the total sum-rate maximum of both routing tasks. After that we study the achievable rate region of bidirectional relaying between nodes equipped with multiple antennas. Therefore, we specify the optimal transmit strategy and show that the achievable rate region scales linearly with respect to the spatial degrees of the vector channels and time division.

In the second part we find an optimal channel coding strategy for the bidirectional broadcast channel considering finite size alphabets. Thereby, we consider achievable rates with respect to the maximal probability of error. For the coding theorem we follow the philosophy of network coding and regard information flows not as “fluids”.

In the final conclusion we give an outlook on future research work and show how the bidirectional relaying protocol can be integrated in wireless networks.

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Abbreviations, Notation, and Symbols

Abbreviations

AWGN	additional white Gaussian noise
iid	independent and identical distributed
pdf	probability density function
cdf	cumulative distribution function
eq	equal time division
opt	optimal time division
MAC	multiple access channel
BC	broadcast channel
BR	bidirectional relaying
SISO	single-input single-output
MIMO	multiple-input multiple-output
TDMA	time division multiple access
FDMA	frequency-division multiple access
CDMA	code-division multiple access
RS	relay selection
RSTS	relay selection with time-sharing between usage of relay nodes

Notation

$\mathbb{N}$	set of all natural numbers $\{1, 2, 3, \dots\}$
$\mathbb{R}$	set of real numbers
$\mathbb{R}_+$	set of non-negative real numbers
$\mathbb{C}$	set of complex numbers
$[a, b]$	closed interval of real numbers where endpoints a to b are included
(a, b)	open interval of real numbers where endpoints a to b are not included
$[a, b), (a, b]$	half-closed intervals of real number where either a or b is included
lhs := rhs	the value of the right hand side (rhs) is assigned to the left hand side (lhs)
lhs =: rhs	the value of the left hand side (lhs) is assigned to the right hand side (rhs)
$\alpha\,\mathcal{R}$	scaled set $\alpha\,\mathcal{R} := \{\alpha\boldsymbol{R} : \boldsymbol{R} \in \mathcal{R}\}$ with $\alpha \in \mathbb{R}$ and set $\mathcal{R} \subset \mathbb{R}^n$

$f : \mathcal{D} \to \mathcal{C}$	function f from domain $\mathcal{D}$ to codomain $\mathcal{C}$
$x \mapsto f(x)$	mapping of x to $f(x)$
$\mathrm{co}(\mathcal{A})$	convex hull of set $\mathcal{A}$
$\mathrm{cl}(\mathcal{A})$	closure of set $\mathcal{A}$
$\mathrm{int}(\mathcal{A})$	interior of set $\mathcal{A}$
$\lvert\mathcal{A}\rvert$	cardinality of set $\mathcal{A}$
$\exists\, x \in \mathcal{A}$	there exists an element x in set $\mathcal{A}$
$\emptyset$	empty set
$\mathbb{1}_{\mathrm{E}}$	characteristic function which is 1 if event E is true and 0 else
$[x]_+$	abbreviation for $\max(0, x)$
$\log$	logarithm to the base two
$\ln$	natural logarithm
$\exp$	exponential function
$\wedge$, $\vee$	logical conjunction (AND), logical disjunction (OR)
$\bigwedge_{n=1}^{N}$, $\bigvee_{n=1}^{N}$	numerated logical conjunctions and disjunctions
$\lvert\lvert \boldsymbol{x} \rvert\rvert_1$	L_1-norm of vector $\boldsymbol{x}$
$\boldsymbol{H}^H$, $\boldsymbol{H}^T$	hermitian resp. transpose of the matrix or vector $\boldsymbol{H}$
$\mathrm{rank}(\boldsymbol{H})$	rank of the matrix $\boldsymbol{H}$
$\mathrm{tr}(\boldsymbol{H}), \det(\boldsymbol{H})$	trace and determinant of the matrix $\boldsymbol{H}$
$\boldsymbol{I}$	identity matrix where the dimension follows from the context
$\mathrm{diag}(a_1, a_2, \ldots, a_n)$	diagonal matrix with diagonal elements $a_1, a_2, \ldots, a_n$
$X \sim \mathcal{CN}(m, \sigma^2)$	X is complex Gaussian distributed with pdf $f_X(x) = \frac{1}{\pi\sigma^2}\,\mathrm{e}^{-\frac{\lvert z-m\rvert^2}{\sigma^2}}$
$\mathbb{E}\{X\}$	expectation of the random variable X
$\mathbb{P}\{\mathrm{E}\}$	probability of event E
$\overline{X}$	time-average of random process X
$\Theta(g(n))$	$f(n)$ is *big-theta* of $g(n)$ if $0 < \liminf\limits_{n\to\infty} \frac{f(n)}{g(n)} \leq \limsup\limits_{n\to\infty} \frac{f(n)}{g(n)} < \infty$.
$o\big(g(x)\big)$	$f(x)$ is *small-o* of $g(x)$ if $\lim\limits_{x\to\infty} \frac{g(x)}{f(x)} = 0$.

Symbols

In Chapter 2 we use boldface letters to denote vectors and matrices and calligraphic letters denote sets. In Chapter 3 we use the information theoretic typical notation where X_1^n denotes the sequence $X_{11}, X_{12}, \ldots, X_{1n}$. We often choose a generic notation for the symbols which gets at some point quite extensive. The index k enumerates the nodes 1,2, and R respectively, while the index n enumerates the relay nodes in Section 2.4. The superscripts $*$ and $\star$ are used for optimal values. The subscripts eq or opt specify the corresponding rate, rate pair, or rate region for the equal or optimal time division case. The following tables list the most important symbols.

P_k	mean power constraint at node $k \in \{1, 2, \mathrm{R}\}$
σ^2	variance of Gaussian noise
γ_k	signal-to-noise ratio P_k/σ^2
h_1	flat fading SISO channel coefficient between node 1 and the relay node
$h_{1,n}$	flat fading SISO channel coefficient between node 1 and the n^{th} relay node
h_2	flat fading SISO channel coefficient between node 2 and the relay node
$h_{2,n}$	flat fading SISO channel coefficient between node 2 and the n^{th} relay node
$\boldsymbol{H}_1$	flat fading MIMO channel matrix between node 1 and the relay node
$\boldsymbol{H}_2$	flat fading MIMO channel matrix between node 2 and the relay node
α	fraction of time in the MAC phase, i.e. fraction $1-\alpha$ of the time in BC phase
α^*	optimal α to support a certain rate pair in MAC phase
$\mathcal{B}$	set of feasible power distribution vectors $[\beta_1, \beta_2]$
β_1	proportion of P_R spend for forwarding message m_1 to node 2
β_2	proportion of P_R spend for forwarding message m_2 to node 1
β	relay power distribution with $\beta_1 = \beta$ and $\beta_2 = 1 - \beta$
β^*	optimal β to support a certain rate pair in MAC phase
$\beta^\star$	optimal relay power distribution, cf. Proposition 2.2
$\gamma_\mathrm{R}^{\min}$	minimal necessary γ_R to support a certain rate pair in $\mathcal{C}_\mathrm{MAC}$ in the BC phase
$\gamma_\mathrm{R}^{\mathrm{MAC}}$	minimal necessary γ_R to support any rate pair in $\mathcal{C}_\mathrm{MAC}$ in the BC phase
$\gamma_\mathrm{R}^{\Sigma\mathrm{MAC}}$	minimal necessary γ_R to support a rate pair $[R_1, R_2] \in \mathcal{R}_\Sigma^{\mathrm{MAC}}$ in the BC phase
$\gamma_\mathrm{R}^\dagger$	characteristic signal-to-noise ratio for the broadcast phase, cf. Corollary 2.8
$\gamma_\mathrm{R}^{\Sigma 1}$	minimal necessary γ_R to support the vertex $\boldsymbol{\nu}_{\Sigma 1} \in \mathcal{C}_\mathrm{MAC}$ in the BC phase
$\gamma_\mathrm{R}^{2\Sigma}$	minimal necessary γ_R to support the vertex $\boldsymbol{\nu}_{2\Sigma} \in \mathcal{C}_\mathrm{MAC}$ in the BC phase
$\hat{\gamma}$	characteristic signal-to-noise ratio γ_R, cf. Lemma 2.46
$\boldsymbol{\nu}_{\Sigma 1}$	vertex between $R_{\overrightarrow{1\mathrm{R}}}$ and R_Σ^{MAC} of multiple access rate region, cf. (2.5a)
$\boldsymbol{\nu}_{2\Sigma}$	vertex between $R_{\overrightarrow{2\mathrm{R}}}$ and R_Σ^{MAC} of multiple access rate region, cf. (2.5b)
R_1	achievable rate form node 1 via relay node to node 2
R_2	achievable rate from node 2 via relay node to node 1
R_Σ	bidirectional sum-rate $R_1 + R_2 = R_\Sigma$
R_R	additional relay multicast rate
$R_\mathrm{R}^{\max}$	maximal achievable additional relay rate R_R
$R^\dagger$	characteristic rate, cf. Lemma 2.46
R_tot	total sum-rate $R_\mathrm{tot} = R_1 + R_2 + R_\mathrm{R}$
R_tot^*	total sum-rate maximum of piggyback rate region $\mathcal{R}_\mathrm{Piggy}$
$R_{\overrightarrow{1\mathrm{R}}}$	individual rate constraint from node 1 to the relay node in MAC phase
$R_{\overrightarrow{1\mathrm{R}},n}$	$R_{\overrightarrow{1\mathrm{R}}}$ for the n^{th} relay node
$R_{\overrightarrow{2\mathrm{R}}}$	individual rate constraint from node 1 to the relay node in MAC phase
$R_{\overrightarrow{2\mathrm{R}},n}$	$R_{\overrightarrow{2\mathrm{R}}}$ for the n^{th} relay node
R_Σ^{MAC}	sum-rate constraint of node 1 and 2 to the relay node in the MAC phase
$R_{\Sigma,\mathrm{n}}^{\mathrm{MAC}}$	R_Σ^{MAC} for the n^{th} relay node

$R_1^{2\Sigma}$	maximal achievable rate R_1 at vertex $\boldsymbol{\nu}_{2\Sigma}$, cf. (2.5b)
$R_2^{\Sigma 1}$	maximal achievable rate R_2 at vertex $\boldsymbol{\nu}_{\Sigma 1}$, cf. (2.5a)
$R_{\overrightarrow{\mathrm{R1}}}$	individual rate constraint from the relay node to node 2 in BC phase
$R_{\overrightarrow{\mathrm{R1}},n}$	$R_{\overrightarrow{\mathrm{R1}}}$ for the n^{th} relay node
$R_{\overrightarrow{\mathrm{R2}}}$	individual rate constraint from the relay node to node 2 in BC phase
$R_{\overrightarrow{\mathrm{R2}},n}$	$R_{\overrightarrow{\mathrm{R2}}}$ for the n^{th} relay node
$R^*_{\overrightarrow{\mathrm{1R}}}$	maximal unidirectional rate R_1 achievable in $\mathcal{C}_{\mathrm{MAC}}^{\mathrm{MIMO}}$, cf. (2.6.1)
$R^*_{\overrightarrow{\mathrm{2R}}}$	maximal unidirectional rate R_2 achievable in $\mathcal{C}_{\mathrm{MAC}}^{\mathrm{MIMO}}$
$R^\star_{\overrightarrow{\mathrm{R1}}}$	sum-rate optimal rate R_1 in BC phase, cf. Proposition 2.2
$R^\star_{\overrightarrow{\mathrm{R2}}}$	sum-rate optimal rate R_2 in BC phase, cf. Proposition 2.2
R_1^{opt}	total sum-rate optimal rate R_1, cf. Theorem (2.48)
R_2^{opt}	total sum-rate optimal rate R_2, cf. Theorem (2.48)
$R_{\textcircled{1}}$	proportion of rate R_1 interchanged for relay rate R_{R}, cf. Corollary 2.49
$R_{\textcircled{2}}$	proportion of rate R_2 interchanged for relay rate R_{R}, cf. Corollary 2.49
$R_1^\star$	BC sum-rate optimal rate R_1, cf. Proposition 2.47
$R_2^\star$	BC sum-rate optimal rate R_2, cf. Proposition 2.47
R_Σ^{BC}	sum-rate maximum of broadcast phase, cf. Proposition 2.2
$\boldsymbol{R}_\Sigma$	bidirectional sum-rate optimal rate pair, cf. Corollary 2.8
$\boldsymbol{R}_{\mathrm{BC}}$	rate pair of parametrized boundary of $\mathcal{R}_{\mathrm{BC}}$
$\boldsymbol{R}_{\mathrm{opt}}$	rate pair of parametrized boundary of $\mathcal{R}_{\mathrm{BRopt}}$, cf. Corollary 2.15
$\boldsymbol{R}_{\mathrm{BC}}^\star$	BC sum-rate optimal rate pair, cf. Proposition 2.47
$\boldsymbol{R}_{\mathrm{MAC}}^{\mathrm{MIMO}}$	weighted rate sum optimal rate pair of $\mathcal{C}_{\mathrm{MAC}}^{\mathrm{MIMO}}$, cf. (2.75)
$\boldsymbol{R}_{\mathrm{BC}}^{\mathrm{MIMO}}$	weighted rate sum optimal rate pair of $\mathcal{R}_{\mathrm{BC}}^{\mathrm{MIMO}}$, cf. (2.79)
$\boldsymbol{R}_{\mathrm{eq}}^*$	weighted rate sum optimal rate pair for $\alpha = 1/2$, cf. Theorem 2.10
$\boldsymbol{R}_{\mathrm{eq},n}^*$	$\boldsymbol{R}_{\mathrm{eq}}^*$ for the n^{th} relay node
$\boldsymbol{R}_{\mathrm{opt}}^*$	weighted rate sum optimal rate pair of $\mathcal{R}_{\mathrm{BRopt}}$, cf. Theorem 2.16
$\boldsymbol{R}_{\mathrm{opt},n}^*$	$\boldsymbol{R}_{\mathrm{opt}}^*$ for the n^{th} relay node
$R_{1\mathrm{eq}}^*$	maximal unidirectional rate R_1 achievable in $\mathcal{R}_{\mathrm{BReq}}$, cf. (2.22a)
$R_{1\mathrm{eq},n}^*$	$R_{1\mathrm{eq}}^*$ for the n^{th} relay node
$R_{2\mathrm{eq}}^*$	maximal unidirectional rate R_2 achievable in $\mathcal{R}_{\mathrm{BRopt}}$, cf. (2.22b)
$R_{2\mathrm{eq},n}^*$	$R_{2\mathrm{eq}}^*$ for the n^{th} relay node
$R_{k\mathrm{eq}}^*$	maximal unidirectional rate R_k, $k=1,2$, achievable in $\mathcal{R}_{\mathrm{BReq}}$
$R_{k\mathrm{eq},n}^*$	$R_{k\mathrm{eq}}^*$ for the n^{th} relay node
$R_{k\mathrm{opt}}^*$	maximal unidirectional rate R_k, $k=1,2$, achievable in $\mathcal{R}_{\mathrm{BRopt}}$
$R_{k\mathrm{opt},n}^*$	$R_{k\mathrm{opt}}^*$ for the n^{th} relay node
$R_{k\mathrm{RSeq}}^*$	maximal unidirectional rate R_k, $k=1,2$, achievable in $\mathcal{R}_{\mathrm{RSTSeq}}$
$R_{k\mathrm{RSopt}}^*$	maximal unidirectional rate R_k, $k=1,2$, achievable in $\mathcal{R}_{\mathrm{RSTSopt}}$
$\overline{R_{k\mathrm{RSeq}}^*}$	maximal unidirectional ergodic rate $\overline{R_k}$, $k=1,2$ achievable in $\overline{\mathcal{R}_{\mathrm{RSTSeq}}}$
$\overline{R_{k\mathrm{RSopt}}^*}$	maximal unidirectional ergodic rate $\overline{R_k}$, $k=1,2$ achievable in $\overline{\mathcal{R}_{\mathrm{RSTSopt}}}$

$\mathcal{C}_{\mathrm{BC}}$	capacity region in the BC phase
$\mathcal{C}_{\mathrm{MAC}}$	capacity region in the MAC phase
$\mathcal{C}_{\mathrm{MAC},n}$	capacity region of the n^{th} relay node in the MAC phase
$\mathcal{C}_{\mathrm{MAC}}^{\mathrm{MIMO}}$	capacity region of MIMO MAC
$\mathcal{R}_{\overrightarrow{\mathrm{1R}}}$	set of rate pairs in $\mathcal{C}_{\mathrm{MAC}}$ with $R_1 = R_{\overrightarrow{\mathrm{1R}}}$
$\mathcal{R}_{\overrightarrow{\mathrm{2R}}}$	set of rate pairs in $\mathcal{C}_{\mathrm{MAC}}$ with $R_2 = R_{\overrightarrow{\mathrm{2R}}}$
$\mathcal{R}_{\Sigma}^{\mathrm{MAC}}$	set of rate pairs in $\mathcal{C}_{\mathrm{MAC}}$ with $R_1 + R_2 = R_{\Sigma}^{\mathrm{MAC}}$
$\mathcal{R}_{\mathrm{BC}}$	achievable rate region in the BC phase
$\mathcal{R}_{\mathrm{BC}}^{\mathrm{MIMO}}$	achievable rate region of MIMO BC
$\mathcal{R}_{\mathrm{BC},n}$	achievable rate region of the n^{th} relay node in the BC phase
$\mathcal{R}_1$	set of transformed rate pairs for rate constraint $R_{\overrightarrow{\mathrm{1R}}}$, Theorem 2.12
$\mathcal{R}_2$	set of transformed rate pairs for rate constraint $R_{\overrightarrow{\mathrm{2R}}}$, Theorem 2.12
$\mathcal{R}_{\Sigma}$	set of transformed rate pairs for rate constraint $R_{\Sigma}^{\mathrm{MAC}}$, Theorem 2.12
$\mathcal{R}_{\mathrm{BR}}$	bidirectional achievable rate region
$\mathcal{R}_{\mathrm{BR}}^{\mathrm{MIMO}}$	MIMO bidirectional achievable rate region
$\overline{\mathcal{R}_{\mathrm{BR}}}$	ergodic bidirectional rate region
$\mathcal{R}_{\mathrm{BReq}}$	bidirectional achievable rate region with equal time division
$\mathcal{R}_{\mathrm{BReq},n}$	$\mathcal{R}_{\mathrm{BReq}}$ achievable with the n^{th} relay node
$\mathcal{R}_{\mathrm{BRopt}}$	bidirectional achievable rate region with optimal time division
$\mathcal{R}_{\mathrm{BRopt},n}$	$\mathcal{R}_{\mathrm{BRopt}}$ achievable with the n^{th} relay node
$\mathcal{R}_{\mathrm{BRopt}}^{\mathrm{MIMO}}$	MIMO bidirectional achievable rate region with optimal time division
$\mathcal{R}_{\mathrm{RSeq}}$	achievable rate region with relay selection for equal time division
$\mathcal{R}_{\mathrm{RSopt}}$	achievable rate region with relay selection for optimal time division
$\mathcal{R}_{\mathrm{RSTSeq}}$	$\mathcal{R}_{\mathrm{RSeq}}$ with time-sharing between the usage of relay nodes
$\mathcal{R}_{\mathrm{RSTSopt}}$	$\mathcal{R}_{\mathrm{RSopt}}$ with time-sharing between the usage of relay nodes
$\overline{\mathcal{R}_{\mathrm{RSTSeq}}}$	ergodic achievable rate region of $\mathcal{R}_{\mathrm{RSTSeq}}$
$\overline{\mathcal{R}_{\mathrm{RSTSopt}}}$	ergodic achievable rate region of $\mathcal{R}_{\mathrm{RSTSopt}}$
$\mathcal{R}_{\mathrm{Piggy}}$	achievable rate region with additional relay multicast

1 Introduction

1.1 Trends and Motivations – Related Literature

Most of these days phenomenal technological advances base beside others on the continuous progress in integrated circuit design which is described by the popular Moore's law. In 1965 Moore [Moo65] observed that with continuous decreasing packages size the relative manufacturing cost per component exponentially decreases while additionally the number of components per integrated circuit exponentially increases. This trend has been observed until now which has made the development of portable computation and communication devices with an affordable price possible. With this we have an ever-growing number of wireless applications and users – the technological vision of the WWRF[1] expects "7 trillion wireless devices serving 7 billion people by 2017".

The broad consumer wireless telecommunications started with generations of cellular voice networks with a rapidly increasing number of subscribers at the turn of the millennium. With a growing demand for new high data services such as mobile Internet with video applications or interactive gaming the development of latest generations show a shift to cellular data networks which also support the voice application. However, the network bandwidth in favorable frequency bands is finite since the nowadays technically usable low frequencies are limited. Future systems will operate at higher frequencies, but there the wireless communication is more susceptible to radio propagation conditions. This will even worsen the coverage problem service providers already have. Obviously, the providers are interested to offer satisfactory services anywhere at any time, which is an engineering challenge especially where the direct link does not have the desired quality, e.g. due to shadowing or distance. In a single point-to-point connection one could simply increase the transmit power, which results in a higher interference level in the network and a higher energy consumption. Both is unattractive, because a lower interference level often allows to accommodate more users in the network and a reduced energy consumption results either in a longer battery life or a smaller battery size, which is a crucial point for miniaturization. As a consequence in traditional cellular network architectures the provider can only install more base-stations to cover the whole area, but a higher density of base-stations leads to a notable increase in

[1] WWRF-Wireless World Research Forum - http://www.wireless-world-initiative.org

infrastructure costs. Katz discusses in [Kat94] many fundamental design issues of wireless networks at that time which are still important these days.

On the other hand, wireless networks without any wired infrastructure are called ad hoc networks. Such networks utilize the broadcast nature of the wireless channel so that in principle any node can communicate to any other. But due to the attenuation of a wireless channel a reliable transmission is typically only possible to some neighboring nodes. Therefore, for a successful transmission to a node that is not in its vicinity the nodes need to cooperate to forward each others information, which is called a multi-hop communication. Ad hoc networks have some appealing properties because of this relay communication technique. First, we see that any additional node added to the network increases the overall connectivity. Further, if a node disappears due to bad channel conditions or power supply insufficiency other nodes in its vicinity have the possibility to take over its routing responsibilities. This node redundancy and its adaptability makes ad hoc networks highly robust. And since we do not require a wired backbone, it is said that an ad hoc network is easily deployed. It follows that there are many applications where an ad hoc topology is more appropriate than cellular topology, e.g. sensor networks, home environments, or networks of automobiles.

Accordingly, the integration of multi-hop capability in traditional cellular systems has the potential to improve the connectivity by closing internal gaps of a cell without installing more base-stations. It could be also used as a "last mile" technology to extend the periphery of a wireless cell. Thereby, the relay nodes may be mobile or fixed, but they do not have access to the wired backbone network of the base-stations or access points. Theoretical and measurement based radio propagation models indicate that the average received signal power typically falls off with the n-th power of the distance between transmitter and receiver with a path-loss coefficient n between 2 (free space) and 6 (obstructed in building) [Rap96, Table 4.2]. Since multi-hop transmission splits a long distance into several shorter distances, it follows that the cumulative path loss is lower than the path loss of the direct transmission. This leads to an additional reduction in power and therefore energy consumption and/or a lower power level throughout the network.

That multi-hop or relay strategies are promising technological advancements of future wireless networks indicate for instance that for all network sizes there are IEEE 802 LAN/MAN standardization working groups[2] which are focused on the developing of mesh-enabled standards:

- For wireless personal area networks (WPAN) by the Task Group 5 of IEEE 802.15 standard.
- For wireless local area networks (WLAN) in the extension IEEE 802.11s.

[2] The IEEE 802 LAN/MAN Standards Committee develops Local Area Network standards and Metropolitan Area Network standards. http://grouper.ieee.org/groups/802/

- For wireless metropolitan area networks (WMAN) in the amendment IEEE 802.16j – mobile multi-hop relay. Thereby, one allows every subscriber station to function as a backhaul, forwarding traffic until it finally arrives at the intended destination.
- Such meshed network principles will be indispensable for a future wireless network, which should realize broadband wireless access across the country (Mobile Broadband Wireless Access) and should be someday specified in a IEEE 802.20 standard.

These standardizations are guided by industrial player forums like WiMAX for WMAN, Wi-Fi for WLAN, ZigBee for low data rate WPAN which try to promote conformance and interoperability of the devices. In addition to this, many companies are developing various proprietary mesh network solutions. It is expected that relaying strategies should close the gap between WLAN-type and cellular systems. For more details we refer to [LZKS06].

For the design and development of communication networks engineers successfully used the concept of layered architecture to split the enormous complexity of modern communication networks communication. This allows the separate design of a protocol for each layer. Among them the ISO-OSI[3] reference model is still the most prominent. The IEEE 802 standards for wireless networks specify the two lower layers. However, an efficient design of a multi-hop communication represents such a significant technical challenge that a cross-layer design is imperative. This can be seen by a brief inspection of the three (media) layers at the bottom of the stack. The Physical Layer is concerned with signal transmission aspects. The Data Link Layer is concerned with the transfers of data between nodes. In IEEE 802.16, the Data Link Layer is divided in a Logical Link Control Sublayer and Media Access Control Sublayer. The Logical Link Control is concerned with the flow control, which means that it multiplexes the service provided to the higher layer and is responsible for acknowledgment and error recovery. The Media Access Control provides addressing and channel access control mechanisms which means that it is responsible for the spectral resource allocation. The Network Layer is responsible for the end-to-end transmission, which means that it is responsible for routing the data from the source to the destination. Obviously, for an efficient resource allocation for a multi-hop communication we have to route our messages via nodes which have sufficient good channels. Otherwise, if higher layers fix the route, on the Physical Layer the nodes may be coerced to use higher transmit powers to offer the desired service to the Data Link Layer. This causes a higher energy consumption and a higher interference for other nodes which will make the medium access inefficient as well. Therefore, for an efficient future wireless system a cross-layer design with an adaptive power control and resource allocation will be necessary. A more detailed discussion about these motivating thoughts about adaptive power control, energy efficient cross-layer design aspects, and

[3]Open System Interconnection (OSI) protocol stack, which was developed as a reference model by the International Standards Organizations (ISO)[Zim80]. From top to bottom the ISO-OSI reference model defines seven layers in a stack: Application Layer, Presentation Layer, Session Layer, Transport Layer, Network Layer, Data Link Layer, and Physical Layer.

efficient methods of allocating network resources can be found in [Bam98], [GW02], and [SRK03].

Wireless Communication: Cooperative Diversity

Wireless communications deals with the problem of reliable transmission of information despite of impairments introduced by the wireless transmission. Propagation impairments are usually collected in a time-varying fading channel model that collects phenomena like multi-path, Doppler-shift, path-loss, and shadowing. Additionally, noise and interference limit the communication. Diversity techniques reduce the probability that a transmission fails due to a channel which is in a deep fade. We distinguish between time, frequency, and spatial diversity according to the dimension where the robustness against fading is realized.

In a cellular network other users may receive the signal from the communication between a user and a base-station due to the broadcast nature of the wireless channel. Then we will gain spatial diversity if such a user additionally forwards the signal over an independent path to the destination. In [SEA98] and [SEA03a, SEA03b] Sendonaris, Erkip, and Aazhang introduced this concept where two users benefit by relaying their signals for each other and called it user cooperative diversity. Basically, the diversity gain is achieved by implementing an additional multi-hop communication.

The contribution [LTW01, LTW04] of Laneman, Tse, and Wornell is the next influential work in the context of cooperative diversity in wireless networks. They studied the outage behavior of several cooperative diversity protocols named according to the cooperative processing strategy: decode-and-forward, amplify-and-forward, selection, and incremental relaying. It shows that except for the decode-and-forward strategy all strategies achieve the full diversity order in the high SNR regime. Thereby, the reduced diversity order for the decode-and-forward protocol is caused by the definition of the outage event. In a subsequent work [LW03] Laneman and Wornell proposed distributed space-time codes, which also can achieve the full spatial diversity given by the number of cooperating terminals.

Those works have sparked a growing interest in the design and analysis of cooperative and therefore relay transmission protocols. Most of them focused on incorporation of various types of channel codes, so called coded cooperation, e.g. [HZF04], [HN06], [SE04], [LSS03], and [MMYZ06] to mention only a few. [NHH04] and [SHMX06] provide more elaborate introductions.

Nearly all cooperative protocols assume *half-duplex nodes*, which means that a node cannot transmit and receive at the same time using the same frequency. This is due to the fact that in wireless communications it is technically difficult and often impossible to isolate sufficiently the received signal from the transmit signal where the transmitter and receiver of the node use the same bandwidth at the same time. For that reason the protocols require a medium access strategy in time or frequency domain which allocates exclusive resources for the reception of the cooperative user signal and the transmission of its own and cooperative users signals, which can be seen as time or frequency division duplex at the relay node. Accordingly, Laneman et al. point out that for the benefits from user cooperative diversity we have a loss in spectral efficiency due to half-duplex operation, which makes a diversity-vs-multiplexing trade-off discussion necessary [LTW04, AGS05, YE06]. This concept was introduced by Zheng and Tse in [ZT03] for wireless communication of nodes with multiple antennas and can be applied if one considers the cooperative users as a distributed multi-antenna system.

The basic idea behind cooperative/multi-hop communication is that a cooperative node relays the signal or message. Accordingly, to find the optimal respectively a good cooperative strategy we should look at the results of information theory on the relay channel.

Information Theory: Relay Channel

Shannon stated the channel capacity theorem in 1948 in his seminal work "A Mathematical Theory of Communication" [Sha48], which is usually seen as the birth of information theory. In 1961 Shannon introduced in [Sha61] the two-way channel where two nodes want to communicate as effectively as possible with each other. Therein, he obtained the capacity region for the average error for the restricted two-way channel, this means that a feedback between the encoders of the two nodes is not allowed. Nowadays this work is regarded as the first work on multi-user information theory. Compared to the knowledge of the single-user case we are still in the beginning of understanding multi-user information theory.

Recently, the relay channel is experiencing a revival because multi-hop communication for meshed network architectures has the potential for a substantial coverage extension. The relay channel problem was introduced by van der Meulen in [vdM71] in the early seventies. A few years later, Cover and El Gamal established in [CG79] the capacities when the channel is degraded, reversely degraded and when feedback is added from both receivers to the sender and relay node. For the general relay channel they derived an upper bound to the capacity and an achievable rate. Later, in [GA82] the capacity of the semideterminitic relay channel, where the output of the relay is a deterministic function of the input of both senders, is found. However, the capacity of the general relay channel is still unknown.

Since then one tried to get closer by the study of channel models with a simplifying assumption. Zhang derived in [Zha88] a converse for a relay channel where he assumed a noiseless channel with a certain rate between the relay and the destination. Cover and Kim showed in [CK06] that this upper bound is actually achievable using alternatively a novel "hash-and-forward" or "compress-and-forward" strategy. In [GH05] El Gamal and Hassanpour study another simplified relay channel where the transmitted relay symbols are allowed to depend additionally on the non-causally present received symbol. It shows that with "instantaneous relaying" in a Gaussian channel an amplify-and-forward strategy achieves the cut-set bound under certain conditions.

Nonetheless, for the lower bound Cover and El Gamal established in [CG79] a coding strategy which gives us an achievable rate. An equivalent but more regular coding scheme was found by Williams in [Wil82]. Then in [SG00, GK03, XK04] the authors extend these results to the multiple relay channel. Thereby, in [XK04] an even more practical coding strategy is presented. In [KGG05] Kramer, Gastpar, and Gupta extend many cooperative strategies to relay networks with many terminals, antennas, and sources and compare the performances of the strategies under wireless communication aspects.

All works on the relay channel we considered so far assume full-duplex nodes. El Gamal noted[4] that nobody had wireless relay networks in mind when they started working on the relay channel. Accordingly, in the last years some work on the relay channel is done which considers wireless communication aspects. Firstly, in [KSA03] Khojastepour, Sabharwal, and Aazhang transfered some results in a straightforward manner to the relay channel with half-duplex nodes, which they called cheap relay nodes. They separate the transmission and reception at the relay node in the time domain. Then El Gamal and Zahedi establish in [GZ05] the capacity of the relay channels with orthogonal channels in the frequency domain from the sender to the relay and from the sender and relay to the destination. In [LLG06] Lai, Liu, and El Gamal consider cooperation strategies for a three-node wireless network. In particular they propose a cooperation strategy for the relay channel with a noisy feedback combining the decode-and-forward and compress-and-forward strategy.

In [DK04] Dawy and Kamoun determine relay regions where it is beneficial to use a relay node. Therefore, they compare the power allocation for a multi-hop transmission with and without cooperation as well as a single-hop transmission where the channel model includes the path loss between nodes. Liang and Veeravalli derive in [LV05] the optimal distribution of the orthogonal resources in frequency domain in a Gaussian relay channel. Finally, in [HMZ05] Host-Madsen and Zhang study the ergodic capacity of the relay channel in a Rayleigh distributed fading environment. All authors recognized that in the case of half-duplex nodes we need to allocate additional resources in time or frequency. This means that the relay communication is only spectrally more efficient than the direct communication if the spectral gain using the relay node compensates the "costs" of the additional resource allocation.

[4]El Gamal noted this at his talk on the MSRI Workshop: Mathematics of Relaying and Cooperation in Communication Networks, University of California, Berkeley, April 2006.

Bidirectional Relaying and Network Coding

Cooperative protocols establish a separate route to realize diversity. Therefore, the cooperating users have to split their resources which results in a loss in spectral efficiency. From the half-duplex relay channel we know that this is only spectrally more efficient if the relay route results in a sufficient large spectral gain. For instance in [SE05] we can find an example in the context of cooperative space-time codes where a user with a sufficient weak channel can even worsen the performance of the other user. Thereby, they even did not take into account the extra allocated spectral resource for the cooperative protocol. Accordingly, for a user with a good direct link it is inefficient to cooperate with a user which has a weak direct link. On the other hand the user with the weak direct channel benefits a lot from cooperation. It follows that in coverage problems only one user will benefit from a cooperative protocol, which means that for the coverage problem a simple relay or multi-hop communication protocol is more suitable. Rankov and Wittneben must have had similar thoughts and motivations when they had the ingenious idea to propose the bidirectional relaying[5] concept in [RW05a, RW05b, RW07].

In more detail, in [RW05a] two spectrally efficient amplify-and-forward protocols for unidirectional and bidirectional relaying are studied. The unidirectional protocol bases on a strategy where two relay nodes alternately forward messages from the source to the destination which we similarly proposed in [OS04]. They show that for a sufficiently bad inter-relay channel the strategy achieves roughly the performance of a full-duplex amplify-and-forward protocol. For the bidirectional amplify-and-forward protocol two nodes, namely node 1 and node 2, want to communicate through the support of a common half-duplex relay node. In the first time slot nodes 1 and 2 transmit their signal to the relay. The relay node scales the received signal according to its transmit power constraint and retransmits the signal to nodes 1 and 2 in the following time slot. Since nodes 1 and 2 know their own transmitted signals they can subtract their contribution before decoding. Thereby, each unidirectional link still suffers from the additional resource allocation due to the half-duplex node, but the sum-rate is significantly increased since the relay transmission supports two communication links at the same time. Note that for the bidirectional protocol between nodes 1 and 2 is no direct link because both nodes simultaneously transmit in the first phase and receive in the second phase. A similar strategy for a satellite communication is proposed in the patent [DMM97].

In [RW05b] Rankov and Wittneben extend the previous spectrally efficient protocols to the case where the relay nodes support the communication by a decode-and-forward strategy. The unidirectional protocol with multiple relay nodes leads to an interference channel problem [CT91] so that this protocol works well for a weak and strong inter-relay channel. The bidirectional decode-and-forward results in a multiple access channel when nodes 1 and 2

[5] The bidirectional relaying protocol is also known as the two-way relay channel.

transmit their information for each other to the relay node. After the relay node successfully decoded the messages it re-encodes them using superposition encoding and forwards the new codeword to nodes 1 and 2. Again, both nodes use the knowledge about their own message to improve their decoding capabilities. Thereby, it is important to notice that they do not allow cooperation between the encoders of nodes 1 and 2 by feedback, which may improve the performance. Nonetheless, their final sum-rate performance comparison of all proposed half-duplex spectrally efficient protocols shows that the bidirectional decode-and-forward protocol is the most efficient.

We see that if bidirectional communication between the nodes 1 and 2 is desired and the direct link is sufficiently bad, the bidirectional protocol "trades" the direct links in two half-duplex relay channels (same routing problem) for increased spectral efficiency. For that reason, we think that this is conceptually a wise approach for coverage problems where we have the problem of a weak direct link. In Chapter 2 we study cross-layer design aspects of this bidirectional decode-and-forward protocol for half-duplex nodes.

In [RW06] Rankov and Wittneben study the achievable rate region for the restricted full-duplex case. In all works they follow an analogy which Shannon phrased at his Kyoto lecture[6] in 1985 as follows *"A basic idea in information theory is that information can be treated very much like a physical quantity, such as mass or energy"*. However, in the work "Network information flow" from Ahlswede, Cai, Li, and Yeung [ACLY00] it is shown that *"it is in general not optimal to regard the information to be multicast as a "fluid" which can simply be routed or replicated. Rather, by employing coding at the nodes, which [they] refer to as network coding, bandwidth can in general be saved"*. This seminal work establishes a new direction in multiterminal source coding and is closely related to graph theory. The main result is a Max-flow Min-cut Theorem for the information flow in a multi-terminal source coding problem with one information source and error-free links. It says that the maximum rate that a sender can multicast to a set of receivers is given by the minimum cut between source and receivers. They also present a simple example from Yeung [Yeu95] which shows that a multi-source problem is not a trivial extension. Finally they conclude that *"coding by superposition is not optimal in general."* These works have caused a paradigm shift and have stimulated a new flourishing research area in information theory. The textbook [Yeu02, Chap. 11 and 15] and the tutorial [Yeu05] are good introductions into this rapidly growing research field and provide more extensive reference lists than we can give.

Ahlswede et al. do not restrict in their work the encoding functions at the terminals. Li, Yeung, and Cai prove in [LYC03] that linear coding over a certain base field suffices for a multicast problem. In [KM03], Koetter and Medard present an algebraic framework for the design of linear network codes in a finite field. Today, linear encoding functions based on the addition in finite fields, especially XOR-Operation in $\mathbb{F}_2$, have found formidable prominence in the design of network codes. This may go back to the famous "butterfly

[6] Ahlswede cited this in his Shannon Lecture 2006 [Ahl06].

network" [Yeu05, Ex. 1.1], [ACLY00, Fig.7] which shows with a striking simplicity the potential of network coding. With a specific grouping of the terminals the "butterfly network" describes a bidirectional communication network [Yeu05, Ex. 1.3] which suggests the relay node to perform an XOR-operation. Then it is interesting to see that the relay node together with the error-free channels to nodes 1 and 2 describes exactly the XOR-channel of Shannons example for the two-way channel [Sha61, Fig. 4]. Without considering channel coding aspects we find this XOR encoding example for bidirectional relaying in [Yeu05, Ex. 1.4], in a multi-hop context with a simple acknowledgment protocol in [WCK05], and with a cyclic redundancy check at the relay node in [LJS05]. Popovski and Yomo propose in [PY06] a Denoise–and–Forward protocol, where the relay tries to eliminate the noise in the received sum signal by estimation.

The proposed strategies based on network coding principles also show that bidirectional communication can increase the achievable sum-rate. The channel coding strategy of Rankov and Wittneben treats information as a physical entity, which is in a multi-terminal source coding problem in general not optimal. For that reason, in Chapter 3 we study bidirectional relaying with classical channel coding arguments following the philosophy of network coding. We see that the bidirectional relaying with decode-and-forward is a channel where superposition encoding is indeed not optimal. Unfortunately, we obtained these results after our cross-layer design studies presented in Chapter 2, where we assume separated information flows using superposition encoding.

There is lots of work ongoing to transfer the network coding idea to wireless network problems. The easiest way would be to separate channel-coding and network-coding, but this is in general not optimal. In [EMH$^+$03] it is shown that separation holds for some networks, but they also provide networks where separation fails. Likewise in [RK06] it is shown for a relay network with deterministic channels that channels separation does not hold in general. For that reason, the design of joint network-channel codes is indispensable. In [HSOB05] Hausl, Schreckenbach, Oikonomidis, and Bauch introduce a joint network-channel code based on a distributed Low-Density Parity-Check code for the two-way relay channel, where encoders and decoders are designed such that all the network coded parity bits send by the relay node are useful for the decoding at both users. Then in [HH06] Hausl and Hagenauer extend the concept to a turbo network code for the two-way relay channel. Finally in [HD06] we can see for the multiple access relay channel that the joint network-channel coding approach performs better than the separate network-channel coding or a distributed turbo code approach.

1.2 Contribution and Outline of the Thesis

In Chapter 2 we study bidirectional relaying based on superposition encoding as proposed in [RW05b], which means that we consider separated information flows. We study different cross-layer design aspects to improve the efficiency and realize synergetic benefits. As performance metric we consider the achievable rates assuming discrete-time memoryless Gaussian channels. For that reason, we briefly review some information theoretic arguments in between to introduce the achievable rates as performance metric.

- In Section 2.2 we intensively examine the achievable bidirectional rate region of a fixed and optimal time division approach where the nodes are equipped with a single-antenna element. For a complete understanding of the combinatorial properties we discuss for different desired rate-pairs the minimal necessary signal-to-noise ratios in the broadcast phase. Furthermore, we specify the sum-rate optimal rate pairs for different rate regions. This allows us to characterize in closed form the achievable rate pairs for the equal and optimal time division case where the weighted rate sum is maximized, so called Pareto optimal rate pairs[7]. In the last subsection we characterize the bidirectional achievable rate region for the optimal time division with scaled mean power constraints, which corresponds to a different interpretation of the constraint for a code word concerning its code word length. Parts of the results are published in [OB06b, OB07a], and should be published in [OB06e] and [OB07b].
- In Section 2.3 we introduce and study a queueing model and consider a throughput optimal resource allocation policy based on the maximum differential backlog algorithm developed by Tassiulas and Ephrimedes [TE92]. We prove the stability region of bidirectional relaying using standard arguments for the proof as done in [NMR03], which base on a well-developed Lyapunov drift analysis of [MT93]. Additionally, we present some numerical simulation results. Parts of the results are published in [OB06d, OB07a], and should be published in [OB06e].
- In Section 2.4 we consider the routing problem of finding the best relay node in a network where N nodes are willing to support the bidirectional communication between the nodes 1 and 2. We propose to do relay selection based on the achievable rate region using the Pareto optimal rate pairs. Finally, we prove that for independent and identical Rayleigh distributed fading channels the sum of the ergodic achievable rate pairs on the boundary of the equal and optimal time division cases asymptotically scale with $\Theta(log(log(N)))$. Parts of the results are published in [OB07a, OB07e], and should be published in [OB06a].

[7] A feasible point of a vector optimization problem is *Pareto optimal* if and only if there is no feasible point which is better in all entries of the vector-valued objective [BV04, Sect. 4.7.3.]. Accordingly, in our context it means that rate pairs on the boundary in the first quadrant are Pareto optimal since we cannot increase one rate without decreasing the other.

- In Section 2.5 we add a multicast of the relay node to the bidirectional relaying protocol. We examine the joint resource allocation for both routing tasks. Furthermore, we characterize the optimal decoding order and the total sum-rate optimum for any fixed time division. Thereby, we identify a rate trade-off property which characterizes the interchange of bidirectional rate with relay multicast rate while the total sum-rate remains constant. After that we provide an extensive combinatorial discussion regarding the trade-off, a desired relay rate, and the total sum-rate maximum with respect to the time division. Parts of the results are published in [OB07a, OB07d], and should be published in [OB06c].

- In Section 2.6 we consider bidirectional relaying between nodes equipped with multiple antennas. Therefore, we briefly review the information theoretic concepts for vector-valued processing and examine the optimal resource allocation assuming perfect channel knowledge at the transmitters as well. The look at the high power behavior shows a linear growth of the sum of any rate pair on the boundary of the achievable rate region with respect to the spatial degree of the vector channels and time division. Parts of the results are published in [OB07c].

- In Section 2.7 we sum up the main results of this chapter and discuss some connections between them.

In Chapter 3 we derive an optimal coding strategy for the two-phase bidirectional decode-and-forward relaying protocol without feedback. This means that we consider the two-phase protocol where we require that the relay node has to decode the messages in the first phase and forwards the re-encoded messages in the succeeding broadcast phase. Thereby, we do not allow cooperation between the encoders of nodes 1 and 2. We derive the capacity region with respect to the maximal error probability by proving a coding theorem and a weak converse. For the illustration of the results we present an example based on the binary erasure multiple access channel and the binary symmetric broadcast channel. After that we give a brief discussion and refer to further results regarding a strong converse and practical coding aspects. Parts of the results are published in [OSBB07, BOSB07, SOS07] and should be published in [OBSB07].

Finally, we conclude the thesis in chapter 4 where we give an outlook on future research directions and present examples how the bidirectional relaying protocol can be integrated in wireless networks. A complete publication and reference list finalizes the thesis.

Further results which are not part of the Thesis

During my work at the Technical University of Berlin we obtained further interesting results which are not part of this Thesis.

- In [OB03a] and [OB03b] we characterize the optimal power allocation for an amplify-and-forward cooperative system with and without a direct link between source and destination respectively. The single-antenna transmit nodes form a distributed antenna array, which results in a vector channel to the multi-antenna destination node. We assume a sum-power constraint between all transmitters. In [OB03a] we show that the asymptotic power distribution differs to the equal power distribution of a classical MIMO channel. Furthermore, we see that the capacity saturates in the high power regime due to the noise amplification. On the other hand, in [OB03b] we find a new behavior for a scenario where a cooperative station has a larger channel gain than of the direct link. If both channels are highly correlated it shows that it is optimal to use the cooperative node only in the low power regime. This means that with increasing sum-power it is optimal to turn off the cooperative relay node, which is again due to the additional noise of the relay node.

- In [OS04, OB05, OSB06] we propose a cooperative transmission scheme which circumvents the spectral loss due to the pre-log factor from the half-duplex restriction. Therefore, we need at least two cooperating nodes which alternatively either transmit or receive so that at any time the signal from the source is retransmitted after a linear processing by at least one relay node. It follows that we prevent the pre-log factor by allocating additional resource in space. But the retransmission scheme could also cause an unstable system. Accordingly, we find the condition for the linear processing which ensures system stability. Moreover, we present a necessary and sufficient condition so that the relay network has a finite impulse response, which limits the impact of the noise added by the relay nodes. We end up with a system equation which is equivalent to a transmission over a frequency-selective channel with additive colored noise. For the evaluation of the relay network we study the frame error performance of a proposed suboptimal Viterbi-decoder, a maximum likelihood sphere-decoder, the performance of a detector based on the semidefinite relaxation method and the diversity order[8]. Furthermore, we derived the capacity with equal power distribution between the nodes, which can be solved using Jensens formula [Rud66] and characterize channel conditions where the relay network will always have a better pair-wise error probability.

- In coauthored works with Aydin Sezgin we obtain results on the field of Space-Time Block Codes. In [SO04a] we propose a transmit strategy which bases on regular block Markov encoding. In [SO04b, SO04d, SO04c] we find and utilize spectral properties of quasi-orthogonal Space-Time Block Codes.

[8]These discussions were made in the context of the supervised diploma theses: “Analysis of Sphere Decoding in linear cooperative wireless Relay Networks" of Benjamin Schubert and “Diversity order analysis of complexity efficient decoding techniques in linear cooperative wireless relay networks" of Rafael Wyrembelski.

- In the coauthored work with Eduard Jorswieck [JOB05], we study the impact of channel correlation on diversity combiners in a flat fading single-input multiple-output system using majorization theory [MO79].

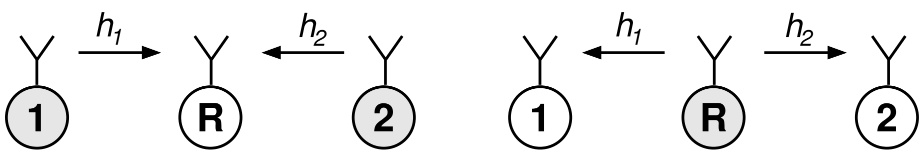
2 Bidirectional Relay Communication using Superposition Encoding

2.1 Introduction

We consider a three-node network where two nodes communicate with each other using the support of a half-duplex relay node. The inherent spectral loss of unidirectional protocols can be significantly reduced through bidirectional relaying. In this chapter we study some cross-layer design aspects of the bidirectional relaying decode-and-forward protocol which was proposed by Rankov and Wittneben in [RW05b, RW07]. Therefore, we assume discrete-time memoryless Gaussian channels between the nodes. At the relay node we apply the superposition encoding technique so that we consider separated information flows.

System Model

The bidirectional two-hop communication is established in two phases, namely the multiple access (MAC) phase and the broadcast (BC) phase, c.f. Figure 2.1. The protocol starts with the multiple access phase where node 1 transmits a message with rate R_1 and node 2 transmits a message with rate R_2 to the relay node simultaneously. The relay node decodes both messages. In the succeeding BC phase the relay broadcasts the re-encoded messages where it uses superposition encoding technique. Since for each receiving node in the BC phase one message originates from itself, each receiving node performs interference cancellation before decoding the unknown message. This results in two separated interference-free transmissions essentially.



(a) MAC phase (b) BC phase

Figure 2.1: A three-node network, where each node is equipped with one antenna element.

In this thesis we do not allow cooperation between the two phases. This means that we do not allow any feedback which could be used for cooperation between the encoders. For Shannon's two-way channel [Sha61] this is known as the *restricted* two-way channel. Furthermore, we assume that the noise and information sources at nodes 1 and 2 are independent and all nodes are perfectly synchronized.

Throughout the chapter we consider a baseband discrete-time system. We assume a multiplicative channel model which includes the physical channel, the shaping pulse at the transmitter, and the whitening matched filter at the receiver. Unless otherwise stated, the channel is considered to be a single-input single-output (SISO) linear time-invariant system, which can be seen as a snapshot of a time-varying flat fading channel.

Let $h_1 \in \mathbb{C}$ denote the channel gain between node 1 and the relay node and $h_2 \in \mathbb{C}$ denotes the channel gain between node 2 and the relay node. For notational simplicity we assume reciprocal channels. Then after symbol-rate sampling the system equation for time instants m in the MAC phase is given by

$$y_{\mathrm{R}}[m] = h_1 x_1[m] + h_2 x_2[m] + n_{\mathrm{R}}[m],$$

where $y_{\mathrm{R}}[m]$ denotes the received signal at the relay node, $x_1[m]$ and $x_2[m]$ denote the transmit signals of nodes 1 and 2, and $n_{\mathrm{R}}[m]$ denotes the additive noise. Similarly, after symbol-rate sampling the system equation for time instants m in the BC phase is given by

$$y_k[m] = h_k x_{\mathrm{R}}[m] + n_k[m], \qquad k = 1, 2,$$

where $y_k[m]$ denotes the received signal at node $k = 1, 2$, $x_{\mathrm{R}}[m]$ denotes the transmit signal of the relay node, and $n_k[m]$ denotes the additive noise at node $k = 1, 2$. We assume that the noise at each antenna is independent additive white Gaussian noise with zero mean and power σ^2. Due to the central limit theorem, the Gaussian noise assumption is reasonable for a wide class of practical channel models, e.g. thermal noise is modeled by a Gaussian distribution. For each node we have a mean transmit power constraint P_k, $k \in \{1, 2, \mathrm{R}\}$, so that we can define the signal-to-noise ratios $\gamma_k := P_k/\sigma^2$, $k \in \{1, 2, \mathrm{R}\}$. Throughout the thesis we consider normalized powers. Without further notification, we assume that $|h_1|$, $|h_2|$, and $P_k, k \in \{1, 2, R\}$ are strictly positive because otherwise bidirectional communication would not be possible.

In this thesis we consider an orthogonal resource allocation for the two phases in the time domain. Conceptually the results can be easily transfered to an orthogonal resource allocation in the frequency domain, but then we have to consider the capacity of bandlimited channels. We assume that all links are reciprocal. In practical systems this condition is fulfilled if the round-trip time of both phases is shorter than the coherence time of the channel. Similarly, if we allocate orthogonal resources in frequency domain the condition is fulfilled if the frequency separation is smaller than the coherence bandwidth. However, in this thesis we look at the rates which are achievable for a given channel realization. Therefore, channel reciprocity and equal noise powers are not necessary assumptions since with an appropriate substitution of the individual transmit power constraints we can include the general case.[1] Finally, we want to remark that in most sections we assume a time-invariant channel model. For the cases where we consider a time-variant channel model we are interested in the time averages of the achievable rates. Therefore, we assume that the channel remains constant long enough so that we can achieve the information theoretic rate. An optimization of the resource allocation over the time-variance of the channels is out of the scope of the thesis.

Outline of this Chapter

In Section 2.2 we examine the achievable bidirectional rate region for a fixed and optimized time division between the two phases. For a complete understanding of the combinatorial properties we first study minimal necessary signal-noise ratios γ_{R} to support a certain rate pair in the MAC phase and characterize the sum-rate optimal rate pairs of the rate region of each phase and of the bidirectional protocol. After that, we characterize for any weight vector the corresponding rate pair on the boundary of the rate region that maximizes the weighted rate sum. Rate pairs on the boundary are Pareto optimal since they can be seen as solutions of a multi-objective optimization problem where we want to maximize each rate individually. We use these Pareto optimal rate pairs in the following two sections. In Section 2.3 we discuss a throughput optimal resource allocation policy based on a back-pressure strategy. In Section 2.4 we consider the problem of relay selection in a network based on the achievable rate region. Furthermore, we look at the asymptotic growth of the ergodic rate region assuming independent and identical Rayleigh distributed channels.

In Section 2.5 we examine the joint resource allocation for the bidirectional relaying and an additional multicast of the relay node. We characterize the optimal decoding order and the total sum-rate maximum. Furthermore, we characterize a rate trade-off property where the total sum-rate remains constant. After that we present a combinatorial discussion of the results. In Section 2.6 we consider the achievable rate region of bidirectional relaying between nodes equipped with multiple antennas. Therefore, we briefly review the information theoretic concepts for vector-valued processing and present the optimal resource allocation assuming perfect channel knowledge at the transmitters. Finally, we identify the linear scaling law of the achievable rates in the high power regime.

For a clear presentation we illustrate many results by examples or numerical simulations. In

[1] Let h_{kl} and σ_k^2 with $k, l \in \{1, 2, \mathrm{R}\}$ denote the channel coefficient of the channel from node k to node l and the noise power at node k. Then with the substitution $\frac{|h_{\mathrm{R}k}|^2}{\sigma_k^2} = \frac{|h_k|^2}{\sigma^2}$, and $\tilde{P}_k = P_k \frac{|h_{k\mathrm{R}}|^2 \sigma_k^2}{|h_{\mathrm{R}k}|^2 \sigma_{\mathrm{R}}^2}$, for $k = 1, 2$, we can transfer our results to individual channel coefficients and noise powers.

Section 2.7 we give a comprehensive discussion about the cross-layer design aspects considered in this chapter.

2.2 Achievable Rate Region of Bidirectional Relaying

Rankov and Wittneben only looked at the sum-rate performance assuming an equal time division between the MAC and the BC phase. Since bidirectional communication is characterized by a two-dimensional rate vector, for cross-layer design aspects we need to know the whole achievable rate region in detail. Moreover, we notice that we can obviously enlarge the achievable rate region if we optimize the time division between the phases.

In this chapter we regard information as a "fluid", which means that multiple messages are encoded by using the superposition encoding technique. Then the decoding of multiple messages is done successively. This means that when we decode the codeword of a message we regard the codewords of the succeeding unknown messages as additional noise. After the successful decoding of the message the interference of the corresponding codeword is canceled, which reduces the noise level for the succeeding messages and therefore improves the decoding capability. This decoding technique is known as *successive interference cancellation*. For the encoding of multiple messages it follows that we can apply the encoding principles of the single-user point-to-point communication where the encoding has to factor in the position in decoding sequence. For that reason we first briefly look at the results for the simple single-input single-output (SISO) Gaussian channel. These can be found in most textbooks on information theory or good books on wireless communications, e.g. [Gal68], [CT91], or [TV05][2]. Then we will briefly discuss the results regarding the capacity region of the Gaussian multiple access channel for completeness. For the achievable rate region of the broadcast phase we use the results of the Gaussian channel. In the rest of the section we study the resulting achievable rate region of the bidirectional relaying protocol.

2.2.1 Gaussian Channel

We start with a simple single-input single-output discrete-time memoryless AWGN channel. We assume complex signaling, which is motivated by the baseband processing. Furthermore, we include a time-invariant multiplicative channel coefficient in this discussion. This results in a linear channel model with a multiplicative and additive distortion of the transmit signal

[2] We have argued as for an additive channel model like the AWGN channel since this is the most interesting channel for wireless communication. But the basic arguments also apply for more general settings, where we translate noise or interference with uncertainty so that interference cancellation corresponds to a reduction in uncertainty.

with a channel coefficient h, which is perfectly known at the receiver, and a complex Gaussian noise random variable $N \sim \mathcal{CN}(0, \sigma^2)$ respectively. We consider a continuous-valued input and output where the complex random variables X and Y with support sets $\mathcal{S}_Y$ and $\mathcal{S}_X$ represent single letters of our transmit and receive signals. For the transmit signal we require a transmit power constraint so that the second moment of the random variable is bounded by some constant P, i.e. $\mathbb{E}\{|X|^2\} \leq P$. We assume that the noise is independent of the transmit signal. Then the single letter input-output relation can be expressed as follows

$$Y = h\,X + N.$$

A single letter description will be sufficient since we consider a memoryless channel.

In the seminal work [Sha48] Shannon studied the problem of how best to encode the information a sender wants to transmit. For that goal he introduced the concept of mutual information which measures the amount of information one random variable reveals about another. Twenty years before, Hartley introduced in communications the concept of entropy as a measure of uncertainty about a random variable from thermodynamics. Then the mutual information characterizes the reduction in uncertainty. It therefore measures the quantity of transmitted information.

Since we consider a channel with continuous alphabets, we get the *mutual information* in terms of differential entropies[3] [CT91, Chapter 9],

$$I(X;Y) := h(Y) - h(Y|X) = h(Y) - h(N).$$

The *differential entropy* $h(Y)$ of a continuous random variables X defined on the support set $\mathcal{S}_Y$ with the density $f_Y(y)$ is defined as

$$h(X) := -\int_{\mathcal{S}_Y} f_Y(y) \log f_Y(y) dy.$$

Since the logarithm is to the base two, we measure the entropy in bits. The *conditional differential entropy* $h(Y|X)$ of the random variables (Y, X) defined on the support set $\mathcal{S}_Y \times \mathcal{S}_X$ with the joint and conditional densities $f_{Y,X}(y,x)$ and $f_{Y|X}(y|x)$ is defined as

$$h(Y|X) := -\int_{\mathcal{S}_Y \times \mathcal{S}_X} f_{Y,X}(y,x) \log f_{Y|X}(y|x) d(y,x) = h(X,Y) - h(Y). \tag{2.1}$$

[3] In the classical Gaussian channel there is no channel coefficient h. Therefore, for our channel model some authors write $I(X;Y,h)$ to indicate the channel knowledge at the receiver. Since we assume perfect channel knowledge at the receivers throughout the thesis, we skip this entry to keep the notation simple. Furthermore, often real-valued signaling is considered. Conceptually there is no difference between complex and real processing since we can consider each use of a complex AWGN channel as two uses of a real AWGN channel.

Then the differential entropy of a complex Gaussian random variable Z with density $\phi(z) = \frac{1}{\pi\sigma^2}\exp(-\frac{|z-\mu|^2}{\sigma^2})$ is given by

$$\begin{aligned} h(Z) &= -\int_{\mathcal{S}_Z} \phi(z)\log(\phi(z))d\,z = -\int_{\mathcal{S}_Z} \frac{\phi(z)}{\ln(2)}\big(\ln(\frac{1}{\pi\sigma^2}) - \frac{|z-\mu|^2}{\sigma^2}\big)d\,z \\ &= \log(\pi\sigma^2) + \frac{1}{\ln(2)} = \log(\mathrm{e}\,\pi\sigma^2). \end{aligned} \tag{2.2}$$

Therewith, we get the differential entropy of the noise as $h(N) = \log(\mathrm{e}\,\pi\sigma^2)$.

The next important quantity is the *relative entropy* $D(f||g) := \int f\log f/g$ (with $0\log(0/0) := 0$) between two densities f and g which are finite only if the support set of f is contained in the support set of g. From Jensen's Inequality[4] we know that the relative entropy between two densities f and g is always non-negative, $D(f||g) \geq 0$ [CT91, Theorem 9.6.1]. It follows that for any complex random variable Z defined on the support set $\mathcal{S}_Z$ with density $f(z)$ and $\mathbb{E}\{|z|^2\} = \sigma^2$ we have $h(Z) \leq \log(\mathrm{e}\,\pi\sigma^2)$. This can be easily seen by the following: Let $\phi(z)$ be the density of a complex Gaussian random variable according to $\mathcal{CN}(0,\sigma^2)$, then

$$\begin{aligned} 0 \leq D(f||\phi) &= \int_{\mathcal{S}_Z} f\log(f/\phi) = \int_{\mathcal{S}_Z} f\log(f) - \int_{\mathcal{S}_Z} f\log(\phi) \\ &= -h(Z) - \int_{\mathcal{S}_Z} \frac{f(z)}{\ln(2)}\Big(\ln\big(\frac{1}{\pi\sigma^2}\big) - \frac{|z|^2}{\sigma^2}\Big)d\,z = -h(Z) + \log(\mathrm{e}\,\pi\sigma^2). \end{aligned}$$

It follows that the complex normal distribution $\mathcal{CN}(0,\sigma^2)$ maximizes the entropy over all complex distributions with the same second moment.

Since the noise N is independent of the input X, for the second moment of the output we have

$$\mathbb{E}\{|Y|^2\} = \mathbb{E}\{|h\,X + N|^2\} = |h|^2\mathbb{E}\{|X|^2\} + \mathbb{E}\{|N|^2\} \leq |h|^2P + \sigma^2.$$

Then it follows from the previous that the differential entropy of the output is maximized with $h(Y) = \log(\mathrm{e}\,\pi(|h|^2P+\sigma^2))$ if the input X is distributed according to complex normal distribution $\mathcal{CN}(0,P)$. This means that Gaussian distributed codebooks are optimal.

The information capacity C of the Gaussian channel is defined as the maximum mutual information between the input and output over all input distributions that satisfy the power

[4] Jensen's Inequality can be formulated in different notations. In the notation of probability theory let X be a random variable with support $\mathcal{S}_X \subseteq \mathbb{R}$ and let f : $\mathcal{S}_X \rightarrow \mathbb{R}$ be a convex function, then we have $\mathbb{E}\{f(X)\} \geq f(\mathbb{E}\{X\})$.

constraint [CT91, Chapter 10]. With the previous considerations we can express the information capacity as follows

$$\begin{aligned} C &:= \max_{f_X(x):\, \mathbb{E}\{|X|^2\}\leq P} I(X;Y) = \max_{f_X(x):\, \mathbb{E}\{|X|^2\}\leq P} h(Y) - \log(\mathrm{e}\,\pi\sigma^2) \\ &= \log\big(\mathrm{e}\,\pi(|h|^2P+\sigma^2)\big) - \log(\mathrm{e}\,\pi\sigma^2) = \log\big(1+\frac{|h|^2P}{\sigma^2}\big). \end{aligned} \tag{2.3}$$

Then the result of Shannon's work says that one can construct a sequence of block codes that satisfies the power constraint, has a rate smaller but arbitrary close to the information capacity, and has a maximum probability of error which tends to zero with increasing block code length. Therefore, the information capacity is also the supremum of the achievable rates of the channel which means that the operational and information capacity are equal. For more details we refer to any information theory text book that covers the channel coding theorem for the Gaussian channel, e.g. [CT91].

2.2.2 Multiple Access Phase

In the multiple access phase of the bidirectional relaying protocol nodes 1 and 2 simultaneously transmit independent messages m_1 and m_2 with rates R_1 and R_2 to the relay node. Thereby, the message m_1 from node 1 is intended for node 2 and vice versa for message m_2. For the MAC phase we apply the capacity achieving coding strategy. The capacity region for the discrete memoryless multiple access channel was independently derived by Ahlswede [Ahl71a] and Liao [Lia72] using finite set alphabets and is now part of any information theory textbook that contains multiuser information theory, [Wol78], [CK81], and [CT91][5]. The result of the capacity region can be modified to apply to continuous input and output alphabets with an additional mean input power constraints, which was independently done by Wyner in [Wyn74] and Cover in [Cov75]. From wireless communication point of view the multiple access channel with Gaussian channels is most important.

Since the channel is memoryless, it is sufficient to consider single-letters only. Then the input-output relation of the discrete-time memoryless Gaussian multiple access channel is given by

$$Y_\mathrm{R} = h_1\, X_1 + h_2\, X_2 + N_\mathrm{R}.$$

The continuous random variable Y_R denotes the output at the relay node and the continuous random variables X_1 and X_2 denote the inputs from the nodes 1 and 2, which are all complex since we consider baseband signaling. The complex coefficients h_1 and h_2 model the time-invariant channel gains between the nodes. The received signal at the relay node is distorted

[5] We reproduce the theorem in Section 3.1.2 where we present a coding theorem for the BC phase of the bidirectional relay channel using finite set alphabets. Here, we only present the result for the Gaussian channel which we consider in this chapter.

by complex additive white Gaussian noise $N_{\mathrm{R}} \sim \mathcal{CN}(0,\sigma^2)$. Then the multiple access capacity region $\mathcal{C}_{\mathrm{MAC}}$ is given by the convex hull of the set of rate pairs satisfying

$$\begin{aligned} R_1 &\leq I(X_1;Y_{\mathrm{R}}|X_2), \\ R_2 &\leq I(X_2;Y_{\mathrm{R}}|X_1), \\ R_1+R_2 &\leq I(X_1,X_2;Y_{\mathrm{R}}), \end{aligned}$$

for some input distribution $f_{X_k}(x_k)$ which satisfies the input power constraints $\mathbb{E}\{|X_k|^2\} \leq P_k$, $k=1,2$. With the same arguments as in Section 2.2.1 it can easily be shown that complex Gaussian input distributions with $X_k \sim \mathcal{CN}(0,P_k)$, $k=1,2$, maximize $I(X_1;Y_{\mathrm{R}}|X_2)$, $I(X_2;Y_{\mathrm{R}}|X_1)$, and $I(X_1,X_2;Y_{\mathrm{R}})$. Since in the scalar case all mutual informations are simultaneously maximized, it follows that the rate constraints are tight so that the capacity region of the MAC phase is given by

$$\mathcal{C}_{\mathrm{MAC}} := \Big\{[R_1,R_2] \in \mathbb{R}_+^2 :\ R_1 \leq R_{\overrightarrow{1\mathrm{R}}},\ R_2 \leq R_{\overrightarrow{2\mathrm{R}}},\ R_1+R_2 \leq R_\Sigma^{\mathrm{MAC}}\Big\}$$

with the individual and sum-rate constraints

$$R_{\overrightarrow{1\mathrm{R}}} := \log\left(1+\gamma_1|h_1|^2\right), \tag{2.4a}$$

$$R_{\overrightarrow{2\mathrm{R}}} := \log\left(1+\gamma_2|h_2|^2\right), \tag{2.4b}$$

$$R_\Sigma^{\mathrm{MAC}} := \log\left(1+\gamma_1|h_1|^2+\gamma_2|h_2|^2\right), \tag{2.4c}$$

and signal-to-noise ratios $\gamma_1 = \frac{P_1}{\sigma^2}$ and $\gamma_2 = \frac{P_2}{\sigma^2}$.

Similarly to the Gaussian channel it is possible to construct a sequence of block codes that satisfies the power constraints and achieves a rate pair $[R_1,R_2] \in \mathcal{C}_{\mathrm{MAC}}$ arbitrary close while the maximum probability of error tends to zero with increasing block length [CT91]. For that reason we assume error-free decoding in the MAC phase if $[R_1,R_2] \in \mathcal{C}_{\mathrm{MAC}}$[6] for the following cross-layer design.

Since $\mathcal{C}_{\mathrm{MAC}}$ is a pentagon, it can be completely described by five vertices. Thereby, the vertices where the individual rate constraints intersect with the sum-rate constraint,

$$\boldsymbol{\nu}_{\Sigma 1} := \left[R_{\overrightarrow{1\mathrm{R}}}, R_2^{\Sigma 1}\right] \quad \text{with } R_2^{\Sigma 1} := R_\Sigma^{\mathrm{MAC}} - R_{\overrightarrow{1\mathrm{R}}} = \log\Big(1+\frac{\gamma_2|h_2|^2}{1+\gamma_1|h_1|^2}\Big), \tag{2.5a}$$

$$\boldsymbol{\nu}_{2\Sigma} := \left[R_1^{2\Sigma}, R_{\overrightarrow{2\mathrm{R}}}\right] \quad \text{with } R_1^{2\Sigma} := R_\Sigma^{\mathrm{MAC}} - R_{\overrightarrow{2\mathrm{R}}} = \log\Big(1+\frac{\gamma_1|h_1|^2}{1+\gamma_2|h_2|^2}\Big), \tag{2.5b}$$

are most interesting for the combinatoric as seen in [TH98]. To achieve the vertices, we have to apply successive interference cancellation. This means that in the first decoding step the

[6]If we can make the error of the MAC and of the BC phase arbitrary small, it follows from the union bound that we can make the error of the bidirectional relay channel arbitrary small.

codeword of the second message to decode is regarded as additional noise. After decoding the first message the interference of the corresponding codeword is canceled so that we can decode the second message without interference. Then each vertex corresponds to a specific decoding order. To achieve $\boldsymbol{\nu}_{\Sigma 1}$ we have to decode the message m_2 before m_1 and vice versa for $\boldsymbol{\nu}_{2\Sigma}$.

In the following we often consider the boundary of the rate region and therefore define the interesting sections of the boundary as follows

$$\mathcal{R}_{\overrightarrow{1\mathrm{R}}} := \Big\{[R_{\overrightarrow{1\mathrm{R}}}, R_2] \in \mathbb{R}_+^2 : \ 0 \leq R_2 \leq R_2^{\Sigma 1}\Big\}, \tag{2.6a}$$

$$\mathcal{R}_{\overrightarrow{2\mathrm{R}}} := \Big\{[R_1, R_{\overrightarrow{2\mathrm{R}}}] \in \mathbb{R}_+^2 : \ 0 \leq R_1 \leq R_1^{2\Sigma}\Big\}, \tag{2.6b}$$

$$\mathcal{R}_{\Sigma}^{\mathrm{MAC}} := \Big\{[R_1, R_{\Sigma}^{\mathrm{MAC}} - R_1] \in \mathbb{R}_+^2 : \ R_1^{2\Sigma} \leq R_1 \leq R_{\overrightarrow{1\mathrm{R}}}\Big\}. \tag{2.6c}$$

The set $\mathcal{R}_{\Sigma}^{\mathrm{MAC}}$ specifies the dominant face of the pentagon and characterizes the sum-rate optimal rate pairs in the MAC phase. We can achieve the rates on the dominant face by time-sharing between $\boldsymbol{\nu}_{\Sigma 1}$ and $\boldsymbol{\nu}_{2\Sigma}$ or by a rate splitting technique according to [RU95].

2.2.3 Broadcast Phase

In the succeeding BC phase the relay forwards the previously received message m_1 to node 2 and message m_2 to node 1. In this chapter we follow the superposition encoding strategy as proposed in [RW05b]. Therefore, the messages m_1 and m_2 are separately encoded as for the point-to-point Gaussian channel. Since we consider a memoryless channel, it is again sufficient to consider single-letters only. Then the input random variable of the relay node using superposition encoding is given by

$$X_{\mathrm{R}} = W_1 + W_2,$$

where the random variables W_1 and W_2 correspond to the codewords of the messages m_1 for node 2 and m_2 for node 1. Since the messages m_1 and m_2 are independent, the random variables W_1 and W_2 are independent as well. From this we get the output at nodes 1 and 2 as follows

$$Y_k = h_k\, X_{\mathrm{R}} + N_1 = h_k\, W_1 + h_k\, W_2 + N_k, \qquad k = 1, 2.$$

Since the messages m_1 and m_2 originate from nodes 1 and 2 respectively, at each receiving node one message and therefore one codeword is known. This a priori knowledge allows the receiving nodes to subtract the interference caused by the codeword of its own message. With this interference cancellation technique we essentially have an interference-free reception at each receiving node, which results in two equivalent single-user AWGN channels. It follows that the achievable rates in the BC phase using superposition encoding, i.e. $X_{\mathrm{R}} = W_1 + W_2$, have to fulfill the constraints

$$R_2 \leq I(X_{\mathrm{R}}; Y_1|W_1) = I(W_2; h_1 W_2 + N_1),$$
$$R_1 \leq I(X_{\mathrm{R}}; Y_2|W_2) = I(W_1; h_2 W_1 + N_2),$$

for some input distribution $f_{W_k}(w_k)$, $k = 1, 2$, satisfying the power constraint $\mathbb{E}\{|X_{\mathrm{R}}|^2\} = \mathbb{E}\{|W_1|^2\} + \mathbb{E}\{|W_2|^2\} \leq P_{\mathrm{R}}$.

Obviously, the mutual informations are only coupled by the relay input power constraint, which means that we have to distribute the power P_{R} on both codewords. Let β_1 and β_2 denote the proportion of relay transmit power P_{R} spend for the codewords W_1 and W_2 respectively. Then the simplex

$$\mathcal{B} := \{[\beta_1, \beta_2] \in [0, 1] \times [0, 1] : \ \beta_1 + \beta_2 \leq 1\}$$

characterizes the set of feasible relay power distributions that satisfy the relay transmit power constraint. For a given feasible relay power distribution $[\beta_1, \beta_2] \in \mathcal{B}$ we know from Section 2.2.1 that complex Gaussian distributed inputs $W_1 \sim \mathcal{CN}(0, \beta_1 P_{\mathrm{R}})$ and $W_2 \sim \mathcal{CN}(0, \beta_2 P_{\mathrm{R}})$ maximize the mutual informations. It follows that with the superposition encoding in the BC phase we can achieve any rate pair within the rate region

$$\mathcal{R}_{\mathrm{BC}} := \Big\{ [R_1, R_2] \in \mathbb{R}_+^2 : \ R_1 \leq R_{\overrightarrow{\mathrm{R2}}}(\beta_1), \ R_2 \leq R_{\overrightarrow{\mathrm{R1}}}(\beta_2), \text{ with } [\beta_1, \beta_2] \in \mathcal{B} \Big\}$$

with rate constraints

$$R_{\overrightarrow{\mathrm{R2}}} : [0, 1] \to \mathbb{R}_+, \quad \beta_1 \mapsto \log\left(1 + \gamma_{\mathrm{R}} \beta_1 |h_2|^2\right), \tag{2.7a}$$
$$R_{\overrightarrow{\mathrm{R1}}} : [0, 1] \to \mathbb{R}_+, \quad \beta_2 \mapsto \log\left(1 + \gamma_{\mathrm{R}} \beta_2 |h_1|^2\right), \tag{2.7b}$$

and signal-to-noise ratio $\gamma_{\mathrm{R}} = \frac{P_{\mathrm{R}}}{\sigma^2}$.

In next proposition we show that in the BC phase we do not need the convex hull operation as in the MAC phase since $\mathcal{R}_{\mathrm{BC}}$ is already convex, which means that in the BC phase time-sharing is not necessary.

Proposition 2.1. $\mathcal{R}_{\mathrm{BC}}$ *is convex.*

Proof. The proof can be found in Appendix 2.8.1. □

Similarly to the Gaussian channel it is possible to construct a sequence of block codes that satisfies the power constraint and achieves a rate pair $[R_1, R_2] \in \mathcal{R}_{\mathrm{BC}}$ arbitrary close while the maximum probability of error tends to zero with increasing block length. For that reason we assume error-free decoding in the BC phase if $[R_1, R_2] \in \mathcal{R}_{\mathrm{BC}}$ for the following cross-layer design.

For any rate pair on the boundary of the BC region we obviously have $\beta_1 + \beta_2 = 1$. Accordingly, the boundary of $\mathcal{R}_{\mathrm{BC}}$ can be parametrized by the rate pair function

$$\boldsymbol{R}_{\mathrm{BC}} : [0,1] \to \mathbb{R}_+^2, \quad \beta \mapsto \Big[R_{\overrightarrow{\mathrm{R2}}}(\beta), R_{\overrightarrow{\mathrm{R1}}}(1-\beta)\Big], \tag{2.8}$$

where we have $\beta_1 = \beta$ and $\beta_2 = 1 - \beta$. This allows us to characterize the sum-rate optimal rate pair in the BC phase.

Proposition 2.2. *The maximum sum-rate* $R_\Sigma^{\mathrm{BC}} := \max_{[R_1,R_2]\in\mathcal{R}_{\mathrm{BC}}} R_1 + R_1$ *of the broadcast rate region* $\mathcal{R}_{\mathrm{BC}}$ *is given by*

$$R_\Sigma^{\mathrm{BC}} = \begin{cases} \log(1+\gamma_{\mathrm{R}}|h_1|^2), & \text{if } \beta^\star < 0, \quad (2.9a)\\ \log\big(\frac{1}{4}(1+\gamma_{\mathrm{R}}|h_1|^2 + \frac{|h_1|^2}{|h_2|^2})(1+\gamma_{\mathrm{R}}|h_2|^2 + \frac{|h_2|^2}{|h_1|^2})\big), & \text{if } 0 \le \beta^\star \le 1, \quad (2.9b)\\ \log(1+\gamma_{\mathrm{R}}|h_2|^2), & \text{if } \beta^\star > 1, \quad (2.9c) \end{cases}$$

with $\beta^\star := \frac{1}{2} + \frac{1}{2\gamma_{\mathrm{R}}}\big(\frac{1}{|h_1|^2} - \frac{1}{|h_2|^2}\big)$ *and is attained at the rate pair*

$$[R^\star_{\overrightarrow{\mathrm{R2}}}, R^\star_{\overrightarrow{\mathrm{R1}}}] := \begin{cases} [0, R_{\overrightarrow{\mathrm{R1}}}(1)], & \text{if } \beta^\star < 0, \quad (2.10a)\\ [R_{\overrightarrow{\mathrm{R2}}}(\beta^\star), R_{\overrightarrow{\mathrm{R1}}}(1-\beta^\star)], & \text{if } 0 \le \beta^\star \le 1, \quad (2.10b)\\ [R_{\overrightarrow{\mathrm{R2}}}(1), 0], & \text{if } \beta^\star > 1, \quad (2.10c) \end{cases}$$

with $R_{\overrightarrow{\mathrm{R2}}}(\beta^\star) = \log\big(\frac{1}{2}(1+\gamma_{\mathrm{R}}|h_2|^2 + \frac{|h_2|^2}{|h_1|^2}\big)$ *and* $R_{\overrightarrow{\mathrm{R1}}}(\beta^\star) = \log\big(\frac{1}{2}(1+\gamma_{\mathrm{R}}|h_1|^2 + \frac{|h_1|^2}{|h_2|^2}\big)$.

Proof. The proof can be found in Appendix 2.8.2. □

Furthermore, for any fixed relay power distribution $\beta \in [0,1]$ with $\beta_1 = \beta$ and $\beta_2 = 1 - \beta$ in the BC phase we can achieve any rate pair in the rate region

$$\mathcal{R}_{\mathrm{BC}}(\beta) := \Big\{[R_1, R_2] : R_1 \le R_{\overrightarrow{\mathrm{R2}}}(\beta),\ R_2 \le R_{\overrightarrow{\mathrm{R1}}}(1-\beta)\Big\} \tag{2.11}$$

so that obviously $\mathcal{R}_{\mathrm{BC}} = \bigcup_{\beta\in[0,1]} \mathcal{R}_{\mathrm{BC}}(\beta)$ holds. Finally, we want to remark that $\mathcal{R}_{\mathrm{BC}}$ is larger than the rate region of the degraded broadcast channel, where for the encoding and decoding for one receiving node the own message is regarded as interference.

2.2.4 Bidirectional Achievable Rate Region

For the bidirectional relaying protocol we are in the MAC and BC phases for fraction of time only so that we have to scale the achievable rate pairs according to the time division. This means that for a time division parameter $\alpha \in [0,1]$ we can achieve rate pairs $\boldsymbol{R}$ within the rate regions $\alpha\mathcal{C}_{\mathrm{MAC}}$ in the MAC phase and $(1-\alpha)\mathcal{R}_{\mathrm{BC}}(\beta)$ in the BC phase with a relay power distribution $\beta \in [0,1]$. For a successful bidirectional relay transmission of the message m_1 with rate R_1 from node 1 to node 2 and message m_2 with rate R_2 from node 2 to node 1 the rate pair $\boldsymbol{R} = [R_1, R_2]$ has to be within the scaled MAC rate region $\alpha\mathcal{C}_{\mathrm{MAC}}$ as well as within the scaled BC rate region $(1-\alpha)\mathcal{R}_{\mathrm{BC}}(\beta)$. This means that for any given time division parameter $\alpha \in [0,1]$ and relay power distribution $\beta \in [0,1]$ the achievable rate region of the bidirectional relaying $\mathcal{R}_{\mathrm{BR}}(\alpha,\beta)$ is given by the intersection

$$\mathcal{R}_{\mathrm{BR}}(\alpha,\beta) := \alpha\mathcal{C}_{\mathrm{MAC}} \cap (1-\alpha)\mathcal{R}_{\mathrm{BC}}(\beta).$$

In the following we will call R_1 and R_2 unidirectional rates.

Since this applies to any relay power distribution $\beta \in [0,1]$, the achievable rate region of the bidirectional relaying with the optimal relay power distribution for a fixed time division parameter $\alpha \in [0,1]$ is given by the union over all possible relay power distributions

$$\begin{aligned} \mathcal{R}_{\mathrm{BR}}(\alpha) &:= \bigcup_{\beta\in[0,1]} \mathcal{R}_{\mathrm{BR}}(\alpha,\beta) = \alpha\mathcal{C}_{\mathrm{MAC}} \cap (1-\alpha) \bigcup_{\beta\in[0,1]} \mathcal{R}_{\mathrm{BC}}(\beta) \\ &= \alpha\mathcal{C}_{\mathrm{MAC}} \cap (1-\alpha)\mathcal{R}_{\mathrm{BC}}. \end{aligned} \tag{2.12}$$

The set $\mathcal{R}_{\mathrm{BR}}(\alpha)$ is convex since the intersection of convex sets, $\alpha\mathcal{C}_{\mathrm{MAC}}$ and $(1-\alpha)\mathcal{R}_{\mathrm{BC}}$, is itself convex.

We are now interested which relay power P_{R} respectively signal-to-noise ratio γ_{R} is necessary to support a rate pair $[R_1^M, R_2^M] \in \mathcal{C}_{\mathrm{MAC}}$ for a given time division parameter $\alpha \in [0,1]$. From (2.7a) and (2.7b) we can conclude

$$\begin{aligned} \alpha R_1^M &= (1-\alpha)\log(1+\beta_1\gamma_{\mathrm{R}}|h_2|^2) \quad \Rightarrow \quad \beta_1 = \frac{2^{\frac{\alpha}{1-\alpha}R_1^M}-1}{|h_2|^2\gamma_{\mathrm{R}}}, \\ \alpha R_2^M &= (1-\alpha)\log(1+\beta_2\gamma_{\mathrm{R}}|h_1|^2) \quad \Rightarrow \quad \beta_2 = \frac{2^{\frac{\alpha}{1-\alpha}R_2^M}-1}{|h_1|^2\gamma_{\mathrm{R}}}. \end{aligned}$$

It can be easily seen by contradiction that for the minimum relay power we have $1 = \beta_1 + \beta_2$. From this we get the minimum γ_{R} which is necessary to support a MAC rate pair $[R_1^M, R_2^M] \in \mathcal{C}_{\mathrm{MAC}}$ for a time division parameter α as follows

$$\gamma_{\mathrm{R}}^{\min}(R_1^M, R_2^M, \alpha) := \frac{2^{\frac{\alpha}{1-\alpha}R_1^M}-1}{|h_2|^2} + \frac{2^{\frac{\alpha}{1-\alpha}R_2^M}-1}{|h_1|^2}. \tag{2.13}$$

This allows us to characterize the minimum relay power P_{R} respectively signal-to-noise ratio γ_{R} which is necessary to support any rate pair in MAC rate region $\mathcal{C}_{\mathrm{MAC}}$ in the next proposition.

Proposition 2.3. *Given a MAC rate region $\mathcal{C}_{\mathrm{MAC}}$ and a time division parameter α where $\gamma_{\mathrm{R}}^{\Sigma 1}(\alpha) := \gamma_{\mathrm{R}}^{\min}(R_{\overrightarrow{1\mathrm{R}}}, R_2^{\Sigma 1}, \alpha)$ and $\gamma_{\mathrm{R}}^{2\Sigma}(\alpha) := \gamma_{\mathrm{R}}^{\min}(R_1^{2\Sigma}, R_{\overrightarrow{2\mathrm{R}}}, \alpha)$ denote the minimum γ_{R} necessary to support the vertices $\boldsymbol{\nu}_{\Sigma 1}$ and $\boldsymbol{\nu}_{2\Sigma}$. Then the minimum γ_{R} which is necessary to support any rate pair $[R_1^M, R_2^M] \in \mathcal{C}_{\mathrm{MAC}}$ is given by*

$$\gamma_{\mathrm{R}}^{\mathrm{MAC}}(\alpha) := \max\left\{\gamma_{\mathrm{R}}^{\Sigma 1}(\alpha), \gamma_{\mathrm{R}}^{2\Sigma}(\alpha)\right\}.$$

In the case of equal time division, $\alpha = 1/2$, we have

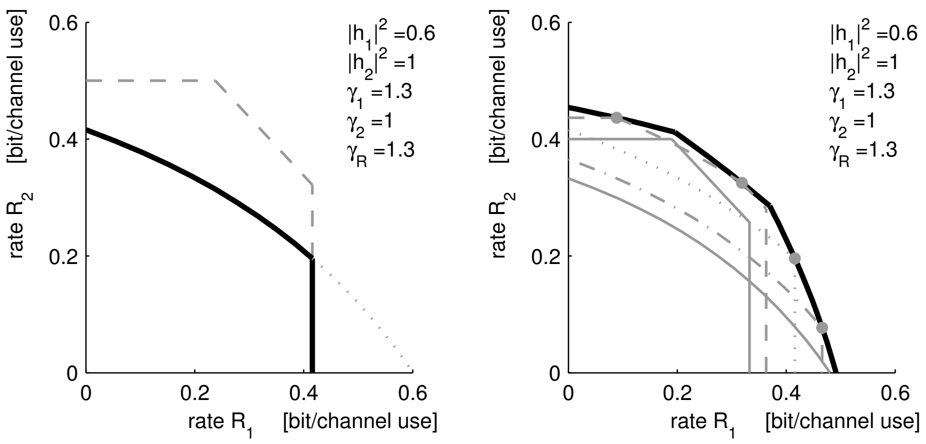
$$\begin{aligned}
\gamma_{\mathrm{R}}^{\Sigma 1}(1/2) &= \frac{\gamma_1 |h_1|^2}{|h_2|^2} + \frac{\gamma_2 |h_2|^2}{(1+\gamma_1|h_1|^2)|h_1|^2}, \\
\gamma_{\mathrm{R}}^{2\Sigma}(1/2) &= \frac{\gamma_2 |h_2|^2}{|h_1|^2} + \frac{\gamma_1 |h_1|^2}{(1+\gamma_2|h_2|^2)|h_2|^2}.
\end{aligned}$$

Proof. Since $\mathcal{R}_{\mathrm{BC}}$ is a convex set which increases with increasing γ_{R}, any rate pair $[R_1^M, R_2^M] \in \mathcal{C}_{\mathrm{MAC}}$ can be supported if γ_{R} is that large that both vertices of $\alpha\mathcal{C}_{\mathrm{MAC}}$ are in $(1-\alpha)\mathcal{R}_{\mathrm{BC}}$.

For the equal time division case we get for the vertex $\boldsymbol{\nu}_{\Sigma 1} = [R_{\overrightarrow{1\mathrm{R}}}, R_{\Sigma}^{\mathrm{MAC}} - R_{\overrightarrow{1\mathrm{R}}}] = [\log(1+\gamma_1|h_1|^2), \log(1+\frac{\gamma_2|h_2|^2}{1+\gamma_1|h_2|^2})]$ the minimum relay power $\gamma_{\mathrm{R}}^{\Sigma 1}(1/2)$ and for the vertex $\boldsymbol{\nu}_{2\Sigma} = [R_{\Sigma}^{\mathrm{MAC}} - R_{\overrightarrow{2\mathrm{R}}}, R_{\overrightarrow{2\mathrm{R}}}] = [\log(1+\frac{\gamma_1|h_1|^2}{1+\gamma_2|h_1|^2}), \log(1+\gamma_2|h_2|^2)]$ the minimum relay power $\gamma_{\mathrm{R}}^{2\Sigma}(1/2)$ using (2.13). □

Using $\gamma_{\mathrm{R}} = \gamma_{\mathrm{R}}^{\mathrm{MAC}}$ allows to support all possible rate pairs from the MAC phase, but in general it is the minimal necessary γ_{R} for one vertex only. This leads to the idea that the relay node can use the remaining power to sent an own message additionally. In Section 2.5 we study a scenario where the relay node adds a multicast communication to the bidirectional relay communication.

Remark 2.4. *For the equal time division case, $\alpha = 1/2$, we have $\frac{\alpha}{1-\alpha} = 1$ so that we can explicitly calculate $\gamma_{\mathrm{R}}^{\Sigma 1}(1/2)$ and $\gamma_{\mathrm{R}}^{2\Sigma}(1/2)$ in terms of $|h_1|^2, |h_2|^2, \gamma_1$, and γ_2. For the same reason some following closed-form results can be obtained for $\alpha = 1/2$ only. For other time division parameter such an explicit solution is not possible, but the behavior is similar. Therefore, we often consider the equal time division case exemplarily.*



(a) Bidirectional achievable rate region with equal time division $\mathcal{R}_{\mathrm{BR}}(1/2)$ (solid line), the scaled achievable rate region of MAC phase $1/2\mathcal{C}_{\mathrm{MAC}}$(dashed line), and the scaled achievable rate region of BC phase $1/2\mathcal{R}_{\mathrm{BC}}$ (dotted line)

(b) Bidirectional achievable rate region with optimal time division $\mathcal{R}_{\mathrm{BRopt}}$ (solid line) and $\mathcal{R}_{\mathrm{BR}}(0.4,\beta) = 0.4\,\mathcal{C}_{\mathrm{MAC}}$ (solid line), $\mathcal{R}_{\mathrm{BR}}(0.436,\beta)$ (dashed line), $\mathcal{R}_{\mathrm{BR}}(1/2,\beta)$ (dotted line), $\mathcal{R}_{\mathrm{BR}}(0.56,\beta)$ (dashed-dotted line), and $\mathcal{R}_{\mathrm{BR}}(0.6,\beta) = 0.4\,\mathcal{R}_{\mathrm{BC}}$ (solid line).

Figure 2.2: Bidirectional achievable rate region of an example scenario

The next natural extension is to look at the achievable rate region where we additionally allow to optimize the time division between the MAC and the BC phase. This rate region is obviously given by the union over all possible time division parameters α as follows

$$\mathcal{R}_{\mathrm{BRopt}} := \bigcup_{\alpha\in[0,1]} \mathcal{R}_{\mathrm{BR}}(\alpha) = \bigcup_{\alpha\in[0,1]} \big(\alpha\mathcal{C}_{\mathrm{MAC}} \cap (1-\alpha)\mathcal{R}_{\mathrm{BC}}\big). \tag{2.14}$$

As a union of convex sets the rate region $\mathcal{R}_{\mathrm{BRopt}}$ need not be convex. Convexity can be achieved if we allow time-sharing between bidirectional rate pairs with different time division parameters, but from the following Corollary 2.13 we will see that $\mathcal{R}_{\mathrm{BRopt}}$ is already convex. This means that for any rate pair in $\mathcal{R}_{\mathrm{BRopt}}$ there exists a time division parameters α for which this rate pair is achievable.

In Figure 2.2 we depicted the achievable rate regions of an example. In the left figure we see the bidirectional achievable rate region $\mathcal{R}_{\mathrm{BR}}(1/2)$ for the equal time division case, which is given by the intersection of the scaled achievable rate regions of the MAC and BC phases. In the right figure we show the achievable rate region $\mathcal{R}_{\mathrm{BRopt}}$ of the optimal time division case. Additionally, the rate regions $\mathcal{R}_{\mathrm{BR}}(\alpha)$ for some fixed time division parameters α are depicted. The bullets (•) mark the rate pairs where rate pairs of $\mathcal{R}_{\mathrm{BR}}(\alpha)$ achieve the boundary of $\mathcal{R}_{\mathrm{BRopt}}$. We see that we have $\mathcal{R}_{\mathrm{BR}}(\alpha) = \alpha\mathcal{C}_{\mathrm{MAC}} \subset \mathcal{R}_{\mathrm{BRopt}}$ or $\mathcal{R}_{\mathrm{BR}}(\alpha) = (1-\alpha)\mathcal{R}_{\mathrm{BC}} \subset \mathcal{R}_{\mathrm{BRopt}}$ if the time division parameter is too small or too large respectively. Furthermore, we see that for the time division parameter $\alpha = 0.436$ two rate pairs achieve the boundary. Note that the marked rate pairs on the boundary $\mathcal{R}_{\mathrm{BRopt}}$ correspond to intersection points of the boundaries of the scaled MAC and BC regions. Lemma 2.11 explains this observation and leads to an equivalent characterization of the bidirectional achievable rate region $\mathcal{R}_{\mathrm{BRopt}}$ in Theorem 2.12. The simple description of the boundary allows us to identify for any given weight vector the rate pair which maximize the weighted rate sum in Theorem 2.16.

In the following we study the properties of the bidirectional achievable rate region for the fixed and optimal time division. For the fixed time division we focus on the equal time division case[7]. For other fixed time division cases the behavior is similar, but often it cannot be calculated explicitly. After that, we identify the optimal time division and relay power distribution for an optimal rate allocation. From this we can deduce an equivalent description of $\mathcal{R}_{\mathrm{BRopt}}$. The previous studies allow us to characterize in details the boundaries of the bidirectional achievable rate regions. In particular we are able to specify in closed form the rate pairs which maximize the weighted rate sum for any weight vector.

For notationally simplicity we will use the abbreviations

$$\mathcal{R}_{\mathrm{BReq}} := \mathcal{R}_{\mathrm{BR}}(1/2) \qquad \text{and} \qquad \mathcal{R}_{\mathrm{BReq}}(\beta) := \mathcal{R}_{\mathrm{BR}}(1/2, \beta).$$

Equal time division

With the first theorem and its corollaries we explore the combinatorial structure of the achievable rate region $\mathcal{R}_{\mathrm{BReq}}$. In Theorem 2.5 we derive the maximal bidirectional sum-rate $R_\Sigma(\beta)$ for any relay power distribution $\beta \in [0, 1]$. As a direct consequence of the theorem we can characterize the achievable rate region $\mathcal{R}_{\mathrm{BR}}(\beta)$ in Corollary 2.6. But we can also use the result of Theorem 2.5 to characterize the minimal relay power which is necessary to support in the BC phase a rate pair $[R_1^M, R_2^M] \in \mathcal{C}_{\mathrm{MAC}}$ which achieves the maximal sum-rate in the MAC phase, i.e. $R_1^M + R_2^M = R_\Sigma^{\mathrm{MAC}}$. Finally in Corollary 2.8 we identify the sum-rate optimal rate pair of $\mathcal{R}_{\mathrm{BReq}}$. For the equal time division case the whole discussion leads to explicit expressions, but the combinatoric of $\mathcal{R}_{\mathrm{BR}}(\alpha)$ is the same for any $\alpha \in [0, 1]$, cf. Remark 2.4.

In Theorem 2.19 we use the explicit knowledge about the combinatoric to characterize in closed form the rate pairs on the boundary of $\mathcal{R}_{\mathrm{BReq}}$ which maximize the weighted rate sum, so called Paretto optimal rate pairs. The Pareto optimal rate pairs play a crucial role for the throughput optimal resource allocation policy in Section 2.3 and for the relay selection problem in Section 2.4.

[7] In the following the subscript eq or opt specify the corresponding rate, rate pair, or rate region for the equal time division or the optimal time division case.

Theorem 2.5. *For bidirectional relaying with equal time division the maximal sum-rate* $R_\Sigma(\beta) := \max_{\boldsymbol{R}\in\mathcal{R}_{\mathrm{BReq}}(\beta)} R_1 + R_2$ *of the bidirectional rate region* $\mathcal{R}_{\mathrm{BReq}}(\beta)$ *for a given relay power distribution parameter* $\beta \in [0,1]$ *is given by*

$$R_\Sigma(\beta) = \tfrac{1}{2}\min\left\{R_\Sigma^{\mathrm{MAC}}, R_{\overrightarrow{1\mathrm{R}}} + R_{\overrightarrow{\mathrm{R}1}}(1-\beta), R_{\overrightarrow{\mathrm{R}2}}(\beta) + R_{\overrightarrow{2\mathrm{R}}}, R_{\overrightarrow{\mathrm{R}2}}(\beta) + R_{\overrightarrow{\mathrm{R}1}}(1-\beta)\right\}$$

$$= \frac{1}{2}\begin{cases} R_{\overrightarrow{1\mathrm{R}}} + R_{\overrightarrow{\mathrm{R}1}}(1-\beta), & \text{if } \beta \in \mathcal{I}_1, & (2.15\text{a})\\ R_{\overrightarrow{\mathrm{R}2}}(\beta) + R_{\overrightarrow{2\mathrm{R}}}, & \text{if } \beta \in \mathcal{I}_2, & (2.15\text{b})\\ R_{\overrightarrow{\mathrm{R}2}}(\beta) + R_{\overrightarrow{\mathrm{R}1}}(1-\beta), & \text{if } \beta \in \mathcal{I}_{\mathrm{BC}}, & (2.15\text{c})\\ R_\Sigma^{\mathrm{MAC}}, & \text{if } \beta \in \mathcal{I}_\Sigma, & (2.15\text{d}) \end{cases}$$

with sets

$$\begin{aligned} \mathcal{I}_1 &:= [\beta_{\Sigma 1}, 1] \cap [\beta_{\mathrm{B}1}, \beta_{1\mathrm{B}}]\,,\\ \mathcal{I}_2 &:= [0, \beta_{2\Sigma}] \cap [\beta_{\mathrm{B}2}, \beta_{2\mathrm{B}}]\,,\\ \mathcal{I}_\Sigma &:= (\beta_{2\Sigma}, \beta_{\Sigma 1}) \cap \begin{cases} (\beta_{\mathrm{B}\Sigma}, \beta_{\Sigma\mathrm{B}})\,, & \text{if } \beta^{\star 2} \geq \Delta\beta,\\ \emptyset, & \text{else}, \end{cases}\\ \mathcal{I}_{\mathrm{BC}} &:= [0,1] \setminus (\mathcal{I}_1 \cup \mathcal{I}_2 \cup \mathcal{I}_\Sigma), \end{aligned}$$

and characteristic values[8]

$$\begin{aligned} \beta_{\Sigma 1} &:= 1 - \frac{\gamma_2 |h_2|^2}{\gamma_{\mathrm{R}} |h_1|^2 (1 + \gamma_1 |h_1|^2)}, & \beta_{\mathrm{B}1} &:= \frac{\gamma_1 |h_1|^2}{\gamma_{\mathrm{R}} |h_2|^2},\\ \beta_{1\mathrm{B}} &:= 1 + \frac{1}{\gamma_{\mathrm{R}} |h_1|^2}, & \beta_{2\Sigma} &:= \frac{\gamma_1 |h_1|^2}{\gamma_{\mathrm{R}} |h_2|^2 (1 + \gamma_2 |h_2|^2)},\\ \beta_{\mathrm{B}2} &:= -\frac{1}{\gamma_{\mathrm{R}} |h_2|^2}, & \beta_{2\mathrm{B}} &:= 1 - \frac{\gamma_2 |h_2|^2}{\gamma_{\mathrm{R}} |h_1|^2},\\ \beta_{\mathrm{B}\Sigma} &:= \beta^\star - \sqrt{\beta^{\star 2} - \Delta\beta}, & \beta_{\Sigma\mathrm{B}} &:= \beta^\star + \sqrt{\beta^{\star 2} - \Delta\beta},\\ \beta^\star &= \frac{1}{2} + \frac{1}{2\gamma_{\mathrm{R}}}\left(\frac{1}{|h_1|^2} - \frac{1}{|h_2|^2}\right), \quad \textit{and} & \Delta\beta &:= \frac{1}{\gamma_{\mathrm{R}}^2}\left(\frac{\gamma_2}{|h_1|^2} - \frac{\gamma_{\mathrm{R}} - \gamma_1}{|h_2|^2}\right). \end{aligned}$$

Proof. The proof can be found in Appendix 2.8.3. □

The sets $\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_\Sigma$, and $\mathcal{I}_{\mathrm{BC}}$ characterize the active restriction for a given power distribution β. If $\beta \in \mathcal{I}_1$, then the unidirectional rate R_1 is limited by $1/2 R_{\overrightarrow{1\mathrm{R}}}$ and R_2 is limited by $1/2 R_{\overrightarrow{\mathrm{R}1}}(1-\beta)$. Accordingly, if $\beta \in \mathcal{I}_2$ then the unidirectional rate R_2 is limited by $1/2 R_{\overrightarrow{2\mathrm{R}}}$

[8] The indices of the points of intersections of $\mathrm{f}_1, \mathrm{f}_2, \mathrm{f}_\Sigma$, and f_{BC}, which are used in the proof, indicate the minimal function for smaller β by the first index and for larger β by the second index.

and R_1 is limited by $1/2 R_{\overrightarrow{R2}}(\beta)$. For $\beta \in \mathcal{I}_\Sigma$ the MAC sum-rate restriction limits the sum-rate $R_1 + R_2 \leq 1/2 R_\Sigma^{\mathrm{MAC}}$. If $\beta \in \mathcal{I}_{\mathrm{BC}}$, the unidirectional rates R_1 and R_2 are limited by $1/2 R_{\overrightarrow{R2}}(\beta)$ and $1/2 R_{\overrightarrow{R1}}(1-\beta)$, i.e. the broadcast is the limiting phase. Finally, $\mathcal{R}_{\mathrm{BReq}}$ is not restricted by a certain constraint if the corresponding set is empty, e.g. if $\mathcal{I}_{\mathrm{BC}} = \emptyset$ the whole MAC region can be supported in the BC phase, i.e. $\mathcal{R}_{\mathrm{BReq}} = 1/2 \mathcal{C}_{\mathrm{MAC}}$.

Thus, as a direct consequence of the theorem, the following corollary characterizes the achievable rate region for a given relay power distribution parameter $\beta \in [0,1]$.

Corollary 2.6. *For bidirectional relaying with equal time division the achievable bidirectional rate region $\mathcal{R}_{\mathrm{BReq}}(\beta)$ for a given relay power distribution parameter $\beta \in [0,1]$ is given by*

$$\mathcal{R}_{\mathrm{BReq}}(\beta) = \frac{1}{2} \begin{cases} \left\{ \boldsymbol{R} \in \mathbb{R}_+^2 : R_1 \leq R_{\overrightarrow{1R}}, \, R_2 \leq R_{\overrightarrow{R1}}(1-\beta) \right\}, & \text{if } \beta \in \mathcal{I}_1, \\ \left\{ \boldsymbol{R} \in \mathbb{R}_+^2 : R_1 \leq R_{\overrightarrow{R2}}(\beta), \, R_2 \leq R_{\overrightarrow{2R}} \right\}, & \text{if } \beta \in \mathcal{I}_2, \\ \left\{ \boldsymbol{R} \in \mathbb{R}_+^2 : R_1 \leq R_{\overrightarrow{R2}}(\beta), \, R_2 \leq R_{\overrightarrow{R1}}(1-\beta) \right\}, & \text{if } \beta \in \mathcal{I}_{\mathrm{BC}}, \\ \left\{ \boldsymbol{R} \in \mathbb{R}_+^2 : \begin{array}{c} R_1 \leq R_{\overrightarrow{R2}}(\beta), \, R_2 \leq R_{\overrightarrow{R1}}(1-\beta), \\ R_1 + R_2 \leq R_\Sigma^{\mathrm{MAC}} \end{array} \right\}, & \text{if } \beta \in \mathcal{I}_\Sigma. \end{cases}$$

For $\beta \in \mathcal{I}_1 \cup \mathcal{I}_2 \cup \mathcal{I}_{\mathrm{BC}}$ the rate region $\mathcal{R}_{\mathrm{BReq}}(\beta)$ is a rectangular, which is characterized by two individual rate constraints. For those relay power distributions the upper right vertex of the rate region characterizes a boundary rate pair of $\mathcal{R}_{\mathrm{BReq}}$. If $\beta \in \mathcal{I}_\Sigma$, the MAC sum-rate constraint has additionally to be satisfied. For those relay power distributions the sum-rate constraint of the MAC phase restricts the achievable rate region $\mathcal{R}_{\mathrm{BReq}}$.

In the next corollary we characterize the minimal relay power respectively signal-to-noise ratio which is necessary to support a rate pair $[R_1^M, R_2^M] \in \mathcal{C}_{\mathrm{MAC}}$ that achieves the maximal MAC sum-rate, i.e. $R_1^M + R_2^M = R_\Sigma^{\mathrm{MAC}}$.

Corollary 2.7. *For bidirectional relaying with equal time division the minimal signal-to-noise ratio to support in the BC phase a rate pair $[R_1^M, R_2^M] \in \mathcal{C}_{\mathrm{MAC}}$ that achieves the maximal possible sum-rate R_Σ^{MAC} is given by*

$$\gamma_{\mathrm{R}}^{\Sigma\mathrm{MAC}} = \max\left\{ \gamma_{\mathrm{R}}^\star, \min\{\hat{\gamma}_{\mathrm{R}}^{\Sigma 1}, \gamma_{\mathrm{R}}^{\Sigma 1}(1/2)\}, \min\{\hat{\gamma}_{\mathrm{R}}^{2\Sigma}, \gamma_{\mathrm{R}}^{2\Sigma}(1/2)\} \right\} \tag{2.16}$$

with $\gamma_{\mathrm{R}}^{2\Sigma}(1/2)$ and $\gamma_{\mathrm{R}}^{\Sigma 1}(1/2)$ according to Proposition 2.3. and

$$\begin{aligned} \gamma_{\mathrm{R}}^\star &:= 2\sqrt{\frac{1 + \gamma_1|h_1|^2 + \gamma_2|h_2|^2}{|h_1|^2|h_2|^2}} - \frac{1}{|h_1|^2} - \frac{1}{|h_2|^2}, \\ \hat{\gamma}_{\mathrm{R}}^{\Sigma 1} &:= \frac{2\gamma_2|h_2|^2}{(1+\gamma_1|h_1|^2)|h_1|^2} + \frac{1}{|h_1|^2} - \frac{1}{|h_2|^2}, \\ \hat{\gamma}_{\mathrm{R}}^{2\Sigma} &:= \frac{2\gamma_1|h_1|^2}{(1+\gamma_2|h_2|^2)|h_2|^2} + \frac{1}{|h_2|^2} - \frac{1}{|h_1|^2}. \end{aligned}$$

Proof. The proof can be found in Appendix 2.8.4. □

The case where $\gamma_{\mathrm{R}}^{\Sigma_{\mathrm{MAC}}}$ is equal to $\min\{\hat{\gamma}_{\mathrm{R}}^{\Sigma 1}, \gamma_{\mathrm{R}}^{\Sigma 1}(1/2)\}$ or $\min\{\hat{\gamma}_{\mathrm{R}}^{2\Sigma}, \gamma_{\mathrm{R}}^{2\Sigma}(1/2)\}$ corresponds to the case when the vertex $\boldsymbol{\nu}_{\Sigma 1}$ or $\boldsymbol{\nu}_{2\Sigma}$ is the rate pair where the sum-rate $R_{\Sigma}^{\mathrm{MAC}}$ is achieved with the minimal γ_{R}. This is only the case when either the sum-rate optimal rate pair of the broadcast phase $[R^{\star}_{\overrightarrow{\mathrm{R2}}}, R^{\star}_{\overrightarrow{\mathrm{R1}}}]$ coincides with the vertex itself or it cannot be supported in the MAC phase, i.e. $1/2[R^{\star}_{\overrightarrow{\mathrm{R2}}}, R^{\star}_{\overrightarrow{\mathrm{R1}}}] \notin \mathcal{R}_{\mathrm{BReq}}$.

If $\gamma_{\mathrm{R}} \geq \gamma_{\mathrm{R}}^{\Sigma_{\mathrm{MAC}}}$, the maximal achievable bidirectional sum-rate is limited by the MAC sum-rate constraint. Accordingly, the maximal achievable bidirectional sum-rate is equal to $1/2 R_{\Sigma}^{\mathrm{MAC}}$. The set of sum-rate optimal bidirectional rate pairs is given by

$$\begin{aligned}\mathcal{R}^{*}_{\Sigma\mathrm{eq}} :=& \Big\{[R_1, R_2] \in \tfrac{1}{2}\mathcal{R}_{\Sigma}^{\mathrm{MAC}} : \ \exists \beta \in \mathcal{I}_{\Sigma} \text{ so that } R_1 \leq \tfrac{1}{2} R_{\overrightarrow{\mathrm{R2}}}(\beta), \ R_2 \leq \tfrac{1}{2} R_{\overrightarrow{\mathrm{R1}}}(\beta)\Big\} \\ =& \Big\{[R_1, \tfrac{1}{2} R_{\Sigma}^{\mathrm{MAC}} - R_1] \ : \ \tfrac{1}{2} R_{\overrightarrow{\mathrm{R2}}}(\beta_{\mathrm{Sum}}^{\min}) \leq R_1 \leq \tfrac{1}{2} R_{\overrightarrow{\mathrm{R2}}}(\beta_{\mathrm{Sum}}^{\max})\Big\} \qquad (2.17)\end{aligned}$$

with $\beta_{\mathrm{Sum}}^{\min} := \min_{\beta \in \mathcal{I}_{\Sigma}} \beta$ and $\beta_{\mathrm{Sum}}^{\max} := \max_{\beta \in \mathcal{I}_{\Sigma}} \beta$. In the next corollary we characterize the rate pairs where the maximal achievable sum-rate is attained if we have $\gamma_{\mathrm{R}} < \gamma_{\mathrm{R}}^{\Sigma_{\mathrm{MAC}}}$.

Corollary 2.8. *For bidirectional relaying with equal time division and $\gamma_{\mathrm{R}} < \gamma_{\mathrm{R}}^{\Sigma_{\mathrm{MAC}}}$ the rate pair which achieves the bidirectional sum-rate maximum is given by*

$$\boldsymbol{R}_{\Sigma} := \underset{[R_1,R_2] \in \mathcal{R}_{\mathrm{BReq}}}{\arg\max}\ R_1 + R_2 = \begin{cases} \boldsymbol{R}_{\mathrm{BC}}(0), & \textit{if } \gamma_{\mathrm{R}} \leq \min\{\gamma_{\mathrm{R}}^{\dagger}, \gamma_{\mathrm{R}}^{2\mathrm{B}}\}, & (2.18\mathrm{a}) \\ \boldsymbol{R}_{\mathrm{BC}}(\beta_{2\mathrm{B}}), & \textit{if } \gamma_{\mathrm{R}} > \gamma_{\mathrm{R}}^{2\mathrm{B}} \wedge \gamma_{\mathrm{R}} \geq \gamma_{\mathrm{R}}^{\star 1}, & (2.18\mathrm{b}) \\ \boldsymbol{R}_{\mathrm{BC}}(\beta^{\star}), & \textit{if } |\gamma_{\mathrm{R}}^{\dagger}| < \gamma_{\mathrm{R}} < \min\{\gamma_{\mathrm{R}}^{\star 1}, \gamma_{\mathrm{R}}^{2\star}\}, & (2.18\mathrm{c}) \\ \boldsymbol{R}_{\mathrm{BC}}(\beta_{\mathrm{B}1}), & \textit{if } \gamma_{\mathrm{R}} > \gamma_{\mathrm{R}}^{\mathrm{B}1} \wedge \gamma_{\mathrm{R}} \geq \gamma_{\mathrm{R}}^{2\star}, & (2.18\mathrm{d}) \\ \boldsymbol{R}_{\mathrm{BC}}(1), & \textit{if } \gamma_{\mathrm{R}} \leq \min\{-\gamma_{\mathrm{R}}^{\dagger}, \gamma_{\mathrm{R}}^{\mathrm{B}1}\} & (2.18\mathrm{e}) \end{cases}$$

with characteristic parameters

$$\gamma_{\mathrm{R}}^{\dagger} := \frac{1}{|h_2|^2} - \frac{1}{|h_1|^2}, \qquad \gamma_{\mathrm{R}}^{\mathrm{B}1} := \frac{\gamma_1 |h_1|^2}{|h_2|^2}, \qquad \gamma_{\mathrm{R}}^{2\mathrm{B}} := \frac{\gamma_2 |h_2|^2}{|h_1|^2},$$
$$\gamma_{\mathrm{R}}^{\star 1} := \frac{2\gamma_2 |h_2|^2}{|h_1|^2} + \frac{1}{|h_1|^2} - \frac{1}{|h_2|^2}, \quad \textit{and} \quad \gamma_{\mathrm{R}}^{2\star} := \frac{2\gamma_1 |h_1|^2}{|h_2|^2} + \frac{1}{|h_2|^2} - \frac{1}{|h_1|^2}.$$

Proof. The proof can be found in Appendix 2.8.5. □

In the following remark we just restate the interesting observation where it is sum-rate optimal not to apply the bidirectional protocol.

Remark 2.9. *If we have $\gamma_{\mathrm{R}} \leq \min\{-\gamma_{\mathrm{R}}^{\dagger}, \gamma_{\mathrm{R}}^{\mathrm{B1}}\}$, it is sum-rate optimal to communicate only from node 1 to node 2. Therefore, for $-\gamma_{\mathrm{R}}^{\dagger} \geq 0$ it is necessary that we have $|h_2|^2 \geq |h_1|^2$. Similarly, if we have $\gamma_{\mathrm{R}} \leq \min\{\gamma_{\mathrm{R}}^{\dagger}, \gamma_{\mathrm{R}}^{\mathrm{2B}}\}$, it is sum-rate optimal to communicate only from node 2 to node 1. For this case it is necessary that we have $|h_1|^2 \geq |h_2|^2$.*

The knowledge about the combinatoric allows us to characterize the boundary of the achievable rate region $\mathcal{R}_{\mathrm{BReq}}$ directly. Let $\boldsymbol{R}_{\mathrm{eq}}^{*}(\boldsymbol{q})$ denote the rate pair where the weighted rate sum with weight vector $\boldsymbol{q} = [q_1, q_2] \in \mathbb{R}_{+}^{2} \setminus \{\boldsymbol{0}\}$ is maximized,

$$\boldsymbol{R}_{\mathrm{eq}}^{*}(\boldsymbol{q}) = \underset{\boldsymbol{R}\in\mathcal{R}_{\mathrm{BReq}}}{\arg\max}\ q_1 R_1 + q_2 R_2. \tag{2.19}$$

The rate pair $\boldsymbol{R}_{\mathrm{eq}}^{*}(\boldsymbol{q})$ plays a crucial role for the throughput optimal resource allocation policy in Section 2.3 and for the relay selection problem in Section 2.4. For a convex set like $\mathcal{R}_{\mathrm{BReq}}$ the rate pair $\boldsymbol{R}_{\mathrm{eq}}^{*}(\boldsymbol{q})$ characterizes the rate pair on the boundary of $\mathcal{R}_{\mathrm{BReq}}$ where the tangential hyperplane with normal vector $\boldsymbol{q}$ intersects the boundary of $\mathcal{R}_{\mathrm{BReq}}$, i.e. the rate pairs with the maximal weighted rate sum characterize the boundary of $\mathcal{R}_{\mathrm{BReq}}$. This gives us an equivalent description of the achievable rate region using the convex hull operator

$$\mathcal{R}_{\mathrm{BReq}} = \mathrm{co}\Big(\big\{\boldsymbol{R}_{\mathrm{eq}}^{*}(\boldsymbol{q}) : \boldsymbol{q} = [q_1, q_2] \in \mathbb{R}_{+}^{2} \setminus \{\boldsymbol{0}\}\big\} \cup \big\{\boldsymbol{0}\big\}\Big).$$

In the next theorem we give a closed form solution of the optimization problem given in equation (2.19).

Theorem 2.10. *For bidirectional relaying with equal time division the weighted rate sum maximum $\boldsymbol{R}_{\mathrm{eq}}^{*}(\boldsymbol{q})$ for a weight vector $\boldsymbol{q} = [q_1, q_2] \in \mathbb{R}_{+}^{2} \setminus \{\boldsymbol{0}\}$ is given by*

$$\boldsymbol{R}_{\mathrm{eq}}^{*}(\boldsymbol{q}) = \frac{1}{2}\begin{cases}
\boldsymbol{R}_{\mathrm{BC}}(1), & \text{if } \beta_{\mathrm{B1}} > 1 \wedge q_2 \leq q_1\vartheta(1), \qquad \text{(2.20a)}\\
\boldsymbol{R}_{\mathrm{BC}}(0), & \text{if } \beta_{\mathrm{2B}} < 0 \wedge q_2 \geq q_1\vartheta(0), \qquad \text{(2.20b)}\\
\boldsymbol{\nu}_{\Sigma 1}, & \text{if } \beta_{\Sigma 1} \geq \beta_{\mathrm{B1}} \wedge q_2 \leq q_1, \qquad \text{(2.20c)}\\
\boldsymbol{\nu}_{2\Sigma}, & \text{if } \beta_{2\Sigma} \leq \beta_{\mathrm{2B}} \wedge q_2 \geq q_1, \qquad \text{(2.20d)}\\
\boldsymbol{R}_{\mathrm{BC}}(\beta_{\mathrm{B1}}), & \text{if } \beta_{\Sigma 1} < \beta_{\mathrm{B1}} \leq 1 \wedge q_2 \leq q_1\vartheta(\beta_{\mathrm{B1}}), \qquad \text{(2.20e)}\\
\boldsymbol{R}_{\mathrm{BC}}(\beta_{\mathrm{2B}}), & \text{if } 0 \leq \beta_{\mathrm{2B}} < \beta_{2\Sigma} \wedge q_2 \geq q_1\vartheta(\beta_{\mathrm{2B}}), \qquad \text{(2.20f)}\\
\boldsymbol{R}_{\mathrm{BC}}(\beta_{\mathrm{B}\Sigma}), & \begin{aligned}&\text{if } \beta^{\star 2} > \Delta\beta \ \wedge\ \beta_{2\Sigma} < \beta_{\mathrm{B}\Sigma} < \beta_{\Sigma 1}\\ &\quad \wedge\ q_1\vartheta(\beta_{\mathrm{B}\Sigma}) \leq q_2 \leq q_1,\end{aligned} \qquad \text{(2.20g)}\\
\boldsymbol{R}_{\mathrm{BC}}(\beta_{\Sigma\mathrm{B}}), & \begin{aligned}&\text{if } \beta^{\star 2} > \Delta\beta \ \wedge\ \beta_{2\Sigma} < \beta_{\Sigma\mathrm{B}} < \beta_{\Sigma 1}\\ &\quad \wedge\ q_1\vartheta(\beta_{\Sigma\mathrm{B}}) \geq q_2 \geq q_1,\end{aligned} \qquad \text{(2.20h)}\\
\boldsymbol{R}_{\mathrm{BC}}(\beta_{\mathrm{BC}}), & \text{else}, \qquad \text{(2.20i)}
\end{cases}$$

with

$$\vartheta:\ [0,1]\to\mathbb{R},\qquad \beta\mapsto\frac{|h_2|^2(1+(1-\beta)\gamma_{\mathrm{R}}|h_1|^2)}{|h_1|^2(1+\beta\gamma_{\mathrm{R}}|h_2|^2)},\ \textit{and} \tag{2.21a}$$

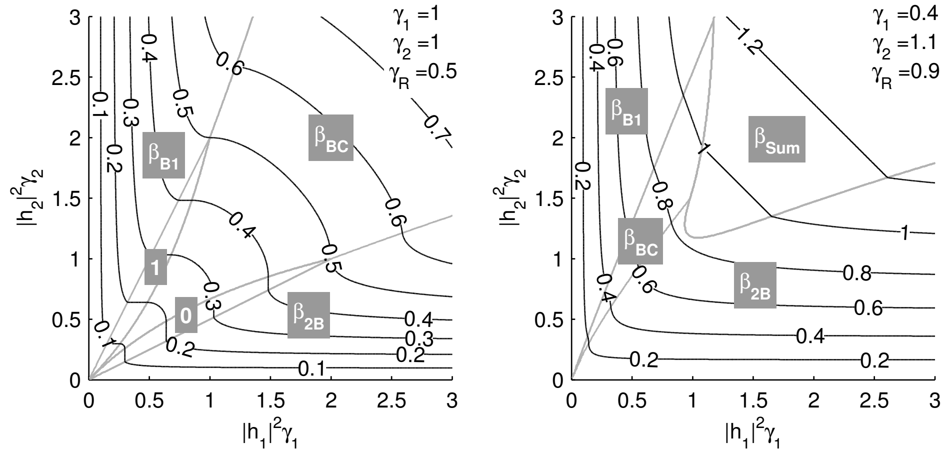
$$\beta_{\mathrm{BC}}:\ \mathbb{R}_+^2\setminus\{\mathbf{0}\}\to\mathbb{R},\qquad \boldsymbol{q}\mapsto\frac{q_1|h_2|^2(1+\gamma_{\mathrm{R}}|h_1|^2)-q_2|h_1|^2}{|h_2|^2|h_1|^2\gamma_{\mathrm{R}}(q_1+q_2)}. \tag{2.21b}$$

Proof. The proof can be found in Appendix 2.8.6. □

The function $\vartheta(\beta)$ denotes the $\arctan$ of the angle of the normal vector of the rate pair on the boundary $\boldsymbol{R}_{\mathrm{BC}}(\beta)$ for the relay power distribution $\beta\in[0,1]$. Further, $\beta_{\mathrm{BC}}(\boldsymbol{q})$ is the relay power distribution where the angle of the normal vector of the rate pair on the boundary of $\mathcal{R}_{\mathrm{BC}}$ has the same angle as the weight vector $\boldsymbol{q}$. Since the optimization problem depends on the direction of the normal vector only, the weighted rate sum maximum is attained at the same rate pair for any weight vector with the same ratio q_1/q_2. This means that only the angle of the weight vector is important. The cases (2.20a) to (2.20h) denote the rate pairs where on the boundary two restrictions apply simultaneously. Such rate pairs correspond to intersection points which have multiple tangents so that they are optimal for a range of weight vector angles. In contrast, the rate pair of the case (2.20i) is optimal for one weight vector angle only.

In more detail, the cases (2.20a) and (2.20b) characterize the intersection of the boundary of $\frac{1}{2}\mathcal{R}_{\mathrm{BC}}$ with the R_1-axis and R_2-axis respectively. The individual MAC rate constraint $R_{\overrightarrow{1\mathrm{R}}}$ limits the achievable rate region $\mathcal{R}_{\mathrm{BReq}}$ in the cases (2.20e) and (2.20c), while for the latter we additionally have $\boldsymbol{\nu}_{\Sigma 1}\in\mathcal{R}_{\mathrm{BC}}$. Similarly, $R_{\overrightarrow{2\mathrm{R}}}$ limits $\mathcal{R}_{\mathrm{BReq}}$ in the cases (2.20f) and (2.20d), while for the latter we additionally have $\boldsymbol{\nu}_{2\Sigma}\in\mathcal{R}_{\mathrm{BC}}$. We have the cases (2.20g) and (2.20h) if the MAC sum-rate constraint $R_{\Sigma}^{\mathrm{MAC}}$ limits $\mathcal{R}_{\mathrm{BReq}}$ and the boundary of $\mathcal{R}_{\mathrm{BC}}$ intersects the section between the vertices $\boldsymbol{\nu}_{\Sigma 1}$ and $\boldsymbol{\nu}_{2\Sigma}$.

The theorem also covers some special cases. If we choose $q_1=q_2=1$, we have the sum-rate maximum $\max_{\boldsymbol{R}\in\mathcal{R}_{\mathrm{BReq}}} R_1+R_2$ case, which we also studied in Corollary 2.8 for $\gamma_{\mathrm{R}}\leq\gamma_{\mathrm{R}}^{\Sigma\mathrm{MAC}}$. From this the question arises if the sum-rate maximum is Schur-convex or Schur-concave with respect to the receive signal-to-noise ratios $[\gamma_1|h_1|^2,\gamma_2|h_2|^2]$. But from the examples of Figure 2.3 we see that there is no such behavior. If the sum-rate maximum is determined by the individual rate constraints of the MAC phase, $R_{\overrightarrow{1\mathrm{R}}}$ or $R_{\overrightarrow{2\mathrm{R}}}$, i.e. at $\beta_{\mathrm{B}1}$ or $\beta_{2\mathrm{B}}$, the sum-rate increases with more balanced receive signal-to-noise ratios, which is a Schur-concave behavior. But if the maximum sum-rate is restricted by the broadcast phase or MAC sum-rate constraint, i.e. at β_{BC} or β_{Sum}, and the receive signal-to-noise ratios are relatively balanced, we see an oppositional behavior. Nevertheless, this shows that more or less balanced receive signal-to-noise ratios result in large sum-rates.



(a) Symmetric MAC power (b) Asymmetric MAC power

Figure 2.3: Contour plot of sum-rate maximum with optimal relay power distribution regions (β_{Sum} denotes the case where the MAC sum-rate constraint limits the maximal sum-rate). The cases of the sum-rate optimal relay power distribution are due to the combinatoric of the boundary of $\mathcal{R}_{\mathrm{BReq}}$.

On the other hand, if we choose either $q_1 > 0, q_2 = 0$ or $q_1 = 0, q_2 > 0$, we get the maximal unidirectional rates $R^*_{1\mathrm{eq}}$ and $R^*_{2\mathrm{eq}}$, i.e. we maximize R_1 or R_2 only. This is obviously achieved if we choose $\beta_1 = 1$ or $\beta_2 = 1$ respectively:

$$R^*_{1\mathrm{eq}} := \max_{[R_1,R_2]\in\mathcal{R}_{\mathrm{BReq}}} R_1 = \tfrac{1}{2}\log\big(1+\min\{\gamma_1|h_1|^2, \gamma_{\mathrm{R}}|h_2|^2\}\big), \tag{2.22a}$$

$$R^*_{2\mathrm{eq}} := \max_{[R_1,R_2]\in\mathcal{R}_{\mathrm{BReq}}} R_2 = \tfrac{1}{2}\log\big(1+\min\{\gamma_2|h_2|^2, \gamma_{\mathrm{R}}|h_1|^2\}\big). \tag{2.22b}$$

If we maximize R_1, the rate R_2 need not be zero. In more detail, if $R_{\overrightarrow{1\mathrm{R}}} < R_{\overrightarrow{\mathrm{R}2}}(1)$, all rate pairs in $\mathcal{R}^*_{1\mathrm{eq}} := \{[R^*_{1\mathrm{eq}}, R_2] \,:\, 0 \le R_2 \le \frac{1}{2}R_{\overrightarrow{\mathrm{R}1}}(1-\beta_{1\mathrm{B}})\}$ achieve the maximal unidirectional rate $R^*_{1\mathrm{eq}}$. Similarly, if we maximize R_2, the rate R_1 need not be zero. If $R_{\overrightarrow{2\mathrm{R}}} < R_{\overrightarrow{\mathrm{R}1}}(1)$, all rate pairs in $\mathcal{R}^*_{2\mathrm{eq}} := \{[R_1, R^*_{2\mathrm{eq}}] \,:\, 0 \le R_1 \le \frac{1}{2}R_{\overrightarrow{\mathrm{R}2}}(\beta_{\mathrm{B}2})\}$ achieve the maximal unidirectional rate $R^*_{2\mathrm{eq}}$.

Optimal time division

We now study bidirectional relaying with optimal time division between the phases. Therefore, we identify the optimal time division and relay power distribution for an optimal rate allocation, which allows us to characterize the boundary of the bidirectional achievable rate region $\mathcal{R}_{\text{BRopt}}$. For that goal, we first specify the optimal time division and relay power distribution for a fixed operating rate pair in the MAC phase. After that we use this to derive an equivalent description of the set $\mathcal{R}_{\text{BRopt}}$.

In the next lemma, we characterize for any operating rate pair $[R_1^M, R_2^M] \in \mathcal{C}_{\text{MAC}}$ in the MAC phase the corresponding optimal rate pair in the BC phase and the optimal time division α^*. It shows that the optimal rate pair is on the boundary of $(1-\alpha^*)\mathcal{R}_{\text{BC}}$. Since for any rate pair on the boundary of $\mathcal{R}_{\text{BC}}$ we have $\beta_1 + \beta_2 = 1$, we have to find the optimal relay power distribution $\beta^* \in [0,1]$ with $\beta_1 = \beta^*$ and $\beta_2 = 1 - \beta^*$.

Lemma 2.11. *For an arbitrary but fixed rate pair $[R_1^M, R_2^M] \in \mathcal{C}_{\text{MAC}}$ the feasible set of time division parameters, where operating at the rate pair $[R_1^M, R_2^M]$ in the MAC phase is possible, is given by*

$$\begin{aligned}\mathcal{A} = \big\{\alpha \in [0,1] \; : \; &\textit{there exists a relay power distribution } [\beta_1, \beta_2] \in \mathcal{B} \textit{ with} \\ &\alpha R_1^M \leq (1-\alpha) R_{\overrightarrow{\text{R2}}}(\beta_1),\; \alpha R_2^M \leq (1-\alpha) R_{\overrightarrow{\text{R1}}}(\beta_2)\big\}\end{aligned}$$

with simplex $\mathcal{B} = \{[\beta_1, \beta_2] \in [0,1] \times [0,1] : \; \beta_1 + \beta_2 \leq 1\}$. For a time division parameter $\alpha \in \mathcal{A}$ we achieve the bidirectional rate pair $[R_1, R_2] = \alpha[R_1^M, R_2^M]$. Then the optimal time division parameter $\alpha^* := \max_{\alpha \in \mathcal{A}} \alpha$ is uniquely characterized by the equations

$$\alpha^* R_1^M = (1-\alpha^*) R_{\overrightarrow{\text{R2}}}(\beta^*), \tag{2.23a}$$

$$\alpha^* R_2^M = (1-\alpha^*) R_{\overrightarrow{\text{R1}}}(1-\beta^*), \tag{2.23b}$$

which also characterizes the optimal relay power distribution $[\beta_1^*, \beta_2^*] \in \mathcal{B}$ with $\beta_1^* = \beta^*$ and $\beta_2^* = 1 - \beta^*$.

Proof. The proof can be found in Appendix 2.8.7. □

The equations (2.23a) and (2.23b) characterize the optimal time division and relay power distribution for any rate pair $[R_1^M, R_2^M] \in \mathcal{C}_{\text{MAC}}$. Obviously, $\beta^\star = 1$ is optimal if R_2^M is equal to zero. For any positive rate R_2^M the ratio

$$\frac{R_1^M}{R_2^M} = \frac{R_{\overrightarrow{\text{R2}}}(\beta^*)}{R_{\overrightarrow{\text{R1}}}(1-\beta^*)} \tag{2.24}$$

implicitly defines the optimal relay power distribution $\beta^* \in [0,1)$ with $\beta_1^* = \beta^*$ and $\beta_2^* = 1 - \beta^*$ so that $[\beta_1^*, \beta_2^*] \in \mathcal{B}$. With the optimal relay power distribution we can calculate the optimal time division coefficient

$$\alpha^* = \frac{R_{\overrightarrow{\text{R2}}}(\beta^*)}{R_1^M + R_{\overrightarrow{\text{R2}}}(\beta^*)} = \frac{R_{\overrightarrow{\text{R1}}}(1-\beta^*)}{R_2^M + R_{\overrightarrow{\text{R1}}}(1-\beta^*)}.$$

If we do not have $R_1^M = 0$ and $R_2^M = 0$ at the same time, the system of equations (2.23a) and (2.23b) has only one solution. It follows that the optimal parameters are unique.

Equation (2.24) shows that for any $[R_1^M, R_2^M] \in \mathcal{C}_{\mathrm{MAC}}$ with the same ratio we have the same optimal relay power distribution. Furthermore, the resulting bidirectional rate pair $[R_1, R_2] \in \mathcal{R}_{\mathrm{BRopt}}$ has the same ratio as the rate pair $[R_1^M, R_2^M] \in \mathcal{C}_{\mathrm{MAC}}$. This means that all lie on the same radial line. Then it is clear that for the optimal time division the boundaries of the scaled MAC and BC regions intersect, which we can observe in Figure 2.2 (b). Furthermore, it follows that the largest bidirectional rates with a certain ratio are achieved by the rate pairs on the boundaries of the MAC and of the BC region with the same ratio. This holds for any ratio and therefore for any rate pair on the boundary. With this we can find an equivalent characterization of $\mathcal{R}_{\mathrm{BRopt}}$ by transforming the rate pairs of sum and individual rate constraints of the MAC region using Lemma 2.11.

Theorem 2.12. *The bidirectional achievable rate region with optimal time division $\mathcal{R}_{\mathrm{BRopt}}$ is given by*

$$\mathcal{R}_{\mathrm{BRopt}} = \mathcal{R}_1 \cap \mathcal{R}_2 \cap \mathcal{R}_\Sigma \tag{2.25}$$

with rate regions

$$\begin{aligned}
\mathcal{R}_1 &:= \big\{\boldsymbol{R} \in \mathbb{R}_+^2 : \; R_1 \le R_{11}(\beta),\; R_2 \le R_{12}(\beta),\; \beta \in [0,1]\big\},\\
\mathcal{R}_2 &:= \big\{\boldsymbol{R} \in \mathbb{R}_+^2 : \; R_1 \le R_{21}(\beta),\; R_2 \le R_{22}(\beta),\; \beta \in [0,1]\big\},\\
\mathcal{R}_\Sigma &:= \big\{\boldsymbol{R} \in \mathbb{R}_+^2 : \; R_1 \le R_{\Sigma 1}(\beta),\; R_2 \le R_{\Sigma 2}(\beta),\; \beta \in [0,1]\big\},
\end{aligned}$$

rate constraints $R_{11}, R_{12}, R_{21}, R_{22}, R_{\Sigma 1}, R_{\Sigma 2} : \; [0,1] \to \mathbb{R}_+$ with

$$R_{11} : \; \beta \mapsto \big(1 - \alpha_1^\star(\beta)\big) R_{\overrightarrow{\mathrm{R2}}}(\beta) = \frac{R_{\overrightarrow{\mathrm{1R}}} R_{\overrightarrow{\mathrm{R2}}}(\beta)}{R_{\overrightarrow{\mathrm{1R}}} + R_{\overrightarrow{\mathrm{R2}}}(\beta)}, \tag{2.26a}$$

$$R_{12} : \; \beta \mapsto \big(1 - \alpha_1^\star(\beta)\big) R_{\overrightarrow{\mathrm{R1}}}(1-\beta) = \frac{R_{\overrightarrow{\mathrm{1R}}} R_{\overrightarrow{\mathrm{R1}}}(1-\beta)}{R_{\overrightarrow{\mathrm{1R}}} + R_{\overrightarrow{\mathrm{R2}}}(\beta)}, \tag{2.26b}$$

$$R_{21} : \; \beta \mapsto \big(1 - \alpha_2^\star(\beta)\big) R_{\overrightarrow{\mathrm{R2}}}(\beta) = \frac{R_{\overrightarrow{\mathrm{2R}}} R_{\overrightarrow{\mathrm{R2}}}(\beta)}{R_{\overrightarrow{\mathrm{2R}}} + R_{\overrightarrow{\mathrm{R1}}}(1-\beta)}, \tag{2.26c}$$

$$R_{22} : \; \beta \mapsto \big(1 - \alpha_2^\star(\beta)\big) R_{\overrightarrow{\mathrm{R1}}}(1-\beta) = \frac{R_{\overrightarrow{\mathrm{2R}}} R_{\overrightarrow{\mathrm{R1}}}(1-\beta)}{R_{\overrightarrow{\mathrm{2R}}} + R_{\overrightarrow{\mathrm{R1}}}(1-\beta)}, \tag{2.26d}$$

$$R_{\Sigma 1} : \; \beta \mapsto \big(1 - \alpha_\Sigma^\star(\beta)\big) R_{\overrightarrow{\mathrm{R2}}}(\beta) = \frac{R_\Sigma^{\mathrm{MAC}} R_{\overrightarrow{\mathrm{R2}}}(\beta)}{R_\Sigma^{\mathrm{MAC}} + R_{\overrightarrow{\mathrm{R1}}}(1-\beta) + R_{\overrightarrow{\mathrm{R2}}}(\beta)}, \tag{2.26e}$$

$$R_{\Sigma 2} : \; \beta \mapsto \big(1 - \alpha_\Sigma^\star(\beta)\big) R_{\overrightarrow{\mathrm{R1}}}(1-\beta) = \frac{R_\Sigma^{\mathrm{MAC}} R_{\overrightarrow{\mathrm{R1}}}(1-\beta)}{R_\Sigma^{\mathrm{MAC}} + R_{\overrightarrow{\mathrm{R1}}}(1-\beta) + R_{\overrightarrow{\mathrm{R2}}}(\beta)}, \tag{2.26f}$$

and optimal time division parameters

$$\alpha_1^*: \quad [0,1] \to \mathbb{R}_+, \qquad \beta \mapsto \frac{R_{\overrightarrow{\mathrm{R2}}}(\beta)}{R_{\overrightarrow{\mathrm{1R}}} + R_{\overrightarrow{\mathrm{R2}}}(\beta)}, \tag{2.27a}$$

$$\alpha_2^*: \quad [0,1] \to \mathbb{R}_+, \qquad \beta \mapsto \frac{R_{\overrightarrow{\mathrm{R1}}}(1-\beta)}{R_{\overrightarrow{\mathrm{2R}}} + R_{\overrightarrow{\mathrm{R1}}}(1-\beta)}, \tag{2.27b}$$

$$\alpha_\Sigma^*: \quad [0,1] \to \mathbb{R}_+, \qquad \beta \mapsto \frac{R_{\overrightarrow{\mathrm{R1}}}(1-\beta) + R_{\overrightarrow{\mathrm{R2}}}(\beta)}{R_\Sigma^{\mathrm{MAC}} + R_{\overrightarrow{\mathrm{R1}}}(1-\beta) + R_{\overrightarrow{\mathrm{R2}}}(\beta)}. \tag{2.27c}$$

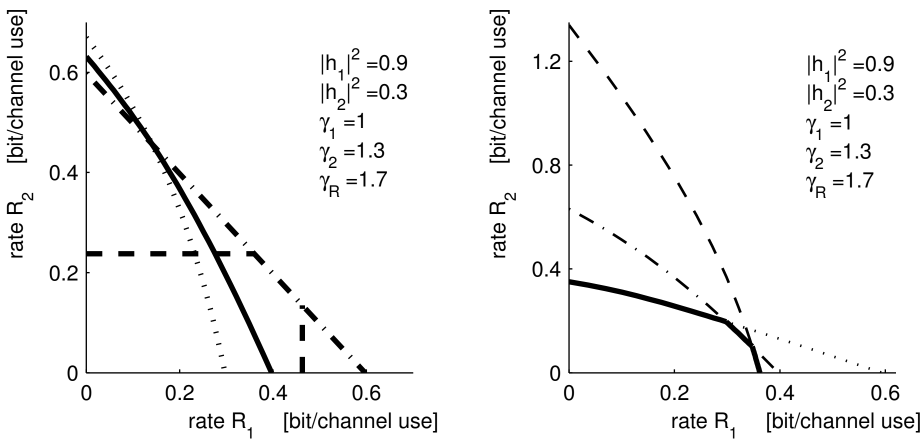
Proof. The proof can be found in Appendix 2.8.8. □

From the inspection of the equivalent characterization of the rate region we find that the information flows show a similar behavior as Kirchhoff's first law (the principle of conservation of electric charge). Here, the conservation of the information flows is due to the superposition encoding approach. In Figure 2.4 (a) we illustrate the regions $\mathcal{R}_\Sigma$, $1/2\mathcal{R}_{\mathrm{BC}}$, and $1/2\mathcal{R}_{\mathrm{MAC}}$. We see that the intersection point of $1/2R_\Sigma$ and $1/2\mathcal{R}_{\mathrm{BC}}$ represents the boundary rate pair on $\mathcal{R}_\Sigma$ for the equal time division case. It is easy to conceive that for other time division parameters α the intersection point moves on the boundary according to the scaled rate region $(1-\alpha)\mathcal{R}_{\mathrm{BC}}$ and sum-rate constraint αR_Σ of $\alpha\mathcal{R}_{\mathrm{MAC}}$. Similar arguments apply to $\mathcal{R}_1$ and $\mathcal{R}_2$. Then in Figure 2.4 (b) we illustrate the construction of $\mathcal{R}_{\mathrm{BRopt}}$ using the intersection of the sets $\mathcal{R}_1$, $\mathcal{R}_2$, and $\mathcal{R}_\Sigma$ according to Theorem 2.12.

For the throughput optimal resource allocation policy in Section 2.3 and for the relay selection problem in Section 2.4 we need to identify the angle of the normal vector of any Pareto optimal rate pair on the boundary of $\mathcal{R}_{\mathrm{BRopt}}$. In the next corollary we characterize the boundaries of $\mathcal{R}_1$, $\mathcal{R}_2$, and $\mathcal{R}_\Sigma$. Since the boundaries on the axis are obvious, we parametrized the boundaries in the first quadrant only. This also characterizes for each boundary rate pair on $\mathcal{R}_{\mathrm{BRopt}}$ the optimal relay power distribution with $\beta_1^\star = \beta$ and $\beta_2^\star = 1-\beta$. The parametrization of the boundary allows us to identify properties of the normal vectors on the boundaries from which we can conclude that $\mathcal{R}_1$, $\mathcal{R}_2$, and $\mathcal{R}_\Sigma$ are convex. Then it follows that $\mathcal{R}_{\mathrm{BRopt}}$ is convex as well. Hence, the next corollary is essential for the discussion of the weighted rate sum maximization problem considered in Theorem 2.16.

Corollary 2.13. *We can parametrize the boundaries of $\mathcal{R}_1$, $\mathcal{R}_2$, $\mathcal{R}_\Sigma$ by the rate pair functions*

$$\begin{aligned}
\boldsymbol{R}_1 &: [0,1] \to \mathbb{R}_+^2, \quad \beta \mapsto [R_{11}(\beta), R_{12}(\beta)],\\
\boldsymbol{R}_2 &: [0,1] \to \mathbb{R}_+^2, \quad \beta \mapsto [R_{21}(\beta), R_{22}(\beta)],\\
\boldsymbol{R}_\Sigma &: [0,1] \to \mathbb{R}_+^2, \quad \beta \mapsto [R_{\Sigma 1}(\beta), R_{\Sigma 2}(\beta)],
\end{aligned}$$



(a) Achievable rate regions $\mathcal{R}_\Sigma$ (solid line), $1/2\mathcal{R}_{\mathrm{BC}}$ (dotted line), $1/2\mathcal{C}_{\mathrm{MAC}}$ (dashed line), and MAC sum-rate constraint $1/2R_\Sigma^{\mathrm{MAC}}$ (dashed-dotted line).

(b) Achievable rate region $\mathcal{R}_{\mathrm{BRopt}}$ (solid line) after intersection of $\mathcal{R}_1$ (dashed line), $\mathcal{R}_\Sigma$ (dashed-dotted line), and $\mathcal{R}_2$ (dotted line).

Figure 2.4: Equivalent characterization of the bidirectional achievable rate region $\mathcal{R}_{\mathrm{BRopt}}$ for the optimal time division case using $\mathcal{R}_1$, $\mathcal{R}_2$, and $\mathcal{R}_\Sigma$.

respectively. Therewith, the angles of the normal vectors for any boundary rate pair of $\mathcal{R}_1$, $\mathcal{R}_2$, *and* $\mathcal{R}_\Sigma$ *are given by the strictly decreasing functions*

$$\varphi_1 : [0,1] \to [0,\pi/2], \quad \beta \mapsto \arctan\Big(-\frac{\mathrm{d}R_{11}(\beta)/\mathrm{d}\beta}{\mathrm{d}R_{12}(\beta)/\mathrm{d}\beta}\Big),$$

$$\varphi_2 : [0,1] \to [0,\pi/2], \quad \beta \mapsto \arctan\Big(-\frac{\mathrm{d}R_{21}(\beta)/\mathrm{d}\beta}{\mathrm{d}R_{22}(\beta)/\mathrm{d}\beta}\Big),$$

$$\varphi_\Sigma : [0,1] \to [0,\pi/2], \quad \beta \mapsto \arctan\Big(-\frac{\mathrm{d}R_{\Sigma 1}(\beta)/\mathrm{d}\beta}{\mathrm{d}R_{\Sigma 2}(\beta)/\mathrm{d}\beta}\Big),$$

respectively. It follows that $\mathcal{R}_{\mathrm{BRopt}}$ *is convex.*

Proof. The proof can be found in Appendix 2.8.9. □

Since φ_1, φ_2, and φ_Σ are continuous and strictly decreasing for each function, there exists an inverse function $\varphi_1^{-1} : [\varphi_1(1), \varphi_1(0)] \to [0,1]$, $\varphi_2^{-1} : [\varphi_2(1), \varphi_2(0)] \to [0,1]$, and $\varphi_\Sigma^{-1} : [\varphi_\Sigma(1), \varphi_\Sigma(0)] \to [0,1]$ respectively. Unfortunately, φ_1^{-1}, φ_2^{-2}, and φ_Σ^{-1} have no explicit representation.

For a complete characterization of the boundary of $\mathcal{R}_{\mathrm{BRopt}}$ we need to understand the combinatoric of the intersection (2.25). In the next proposition we see that also the combinatoric of the MAC region carries over to $\mathcal{R}_{\mathrm{BRopt}}$.

Proposition 2.14. *In the first quadrant there is exactly one intersection between the boundaries of $\mathcal{R}_1$ and $\mathcal{R}_2$, $\mathcal{R}_1$ and $\mathcal{R}_\Sigma$, and $\mathcal{R}_2$ and $\mathcal{R}_\Sigma$. In particular, for each intersection there exists exactly one relay power distributions $\beta_{12}, \beta_{1\Sigma}, \beta_{\Sigma 2} \in [0,1]$ where we have $\boldsymbol{R}_1(\beta_{12}) = \boldsymbol{R}_2(\beta_{12}) \notin \mathcal{R}_{\mathrm{BRopt}}$, $\boldsymbol{R}_1(\beta_{1\Sigma}) = \boldsymbol{R}_\Sigma(\beta_{1\Sigma}) \in \mathcal{R}_{\mathrm{BRopt}}$, and $\boldsymbol{R}_2(\beta_{\Sigma 2}) = \boldsymbol{R}_\Sigma(\beta_{\Sigma 2}) \in \mathcal{R}_{\mathrm{BRopt}}$, where the last two rate pairs are the transformed vertices $\boldsymbol{\nu}_{1\Sigma}$ and $\boldsymbol{\nu}_{\Sigma 2}$ respectively. Furthermore, we have*

$$\begin{aligned} R_{11}(1) &\leq R_{\Sigma 1}(1) \leq R_{21}(1), && (2.28a)\\ R_{12}(0) &\leq R_{\Sigma 2}(0) \leq R_{22}(0). && (2.28b)\end{aligned}$$

Proof. The proof can be found in Appendix 2.8.10. □

With the equivalent description of the rate region and the knowledge about the combinatoric it is finally easy to characterize the boundary of $\mathcal{R}_{\mathrm{BRopt}}$.

Corollary 2.15. *The section-wise defined rate pair function $\boldsymbol{R}_{\mathrm{opt}} : [0,1] \to \mathbb{R}_+^2$ with*

$$\boldsymbol{R}_{\mathrm{opt}} : \beta \mapsto \begin{cases} \boldsymbol{R}_2(\beta), & \text{for } \beta_{\Sigma 2} \geq \beta \geq 0,\\ \boldsymbol{R}_\Sigma(\beta), & \text{for } \beta_{1\Sigma} > \beta > \beta_{\Sigma 2},\\ \boldsymbol{R}_1(\beta), & \text{for } 1 \geq \beta \geq \beta_{1\Sigma}. \end{cases} \tag{2.29}$$

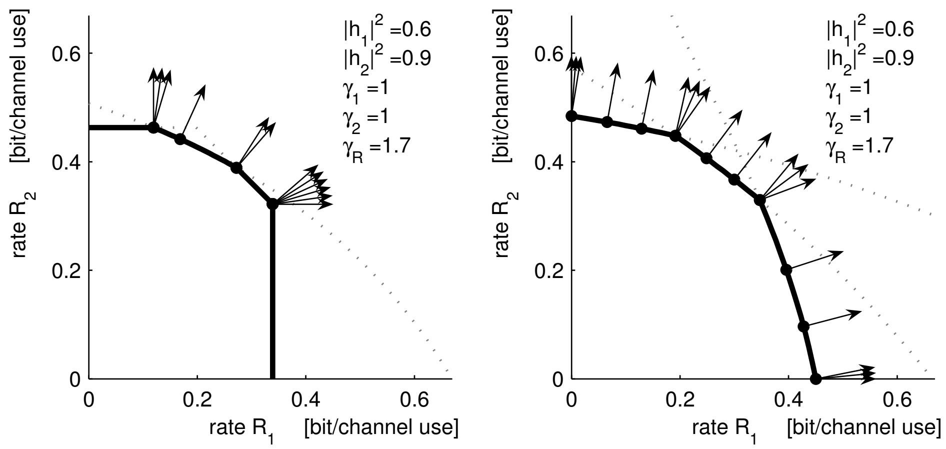
characterizes the boundary of the bidirectional achievable rate region $\mathcal{R}_{\mathrm{BRopt}}$ for the optimal time division case.

Proof. The proof can be found in Appendix 2.8.11. □

We are now ready to characterize in closed form the rate pair where the weighted rate sum is maximized. For that goal we make use of all previously introduced functions and characteristic parameters for the rate region $\mathcal{R}_{\mathrm{BRopt}}$ in the next theorem. Remember that for an optimal relay power distribution we have $\beta_1^* = 1 - \beta_2^* = \beta^*$.

Theorem 2.16. *Let $\boldsymbol{q} = [q_1, q_2] \in \mathbb{R}_+^2 \setminus \{\boldsymbol{0}\}$ denote a weight vector with nonnegative elements and angle $\theta_{\boldsymbol{q}} := \arccos\left(\frac{q_1}{\sqrt{q_1^2+q_2^2}}\right)$, then the rate pair where the weighted rate sum is maximized is given as*

$$\boldsymbol{R}^*_{\mathrm{opt}}(\boldsymbol{q}) = \operatorname*{arg\,max}_{\boldsymbol{R}\in\mathcal{R}_{\mathrm{BRopt}}} \; q_1 R_1 + q_2 R_2 = \boldsymbol{R}_{\mathrm{opt}}(\beta^*(\boldsymbol{q})) \tag{2.30}$$



(a) Equal time division: $\mathcal{R}_{\mathrm{BReq}}$ (solid line), $^1\!/_2\mathcal{C}_{\mathrm{MAC}}$ and $^1\!/_2\mathcal{R}_{\mathrm{BC}}$ (dotted lines)

(b) Optimal time division: $\mathcal{R}_{\mathrm{BRopt}}$ (solid line), $\mathcal{R}_1$, $\mathcal{R}_2$, and $\mathcal{R}_\Sigma$ (dotted lines)

Figure 2.5: Bidirectional achievable rate regions with Pareto optimal rate pairs.

with the optimal power distribution parameter $\beta^*(\boldsymbol{q})$ defined by

$$\beta^* : \mathbb{R}_+^2 \to [0,1], \quad \boldsymbol{q} \mapsto \begin{cases} 1, & \textit{if } \theta_{\boldsymbol{q}} < \varphi_1(1), \\ \varphi_1^{-1}(\theta_{\boldsymbol{q}}), & \textit{if } \varphi_1(1) \le \theta_{\boldsymbol{q}} \le \varphi_1(\beta_{1\Sigma}), \\ \beta_{1\Sigma}, & \textit{if } \varphi_1(\beta_{1\Sigma}) < \theta_{\boldsymbol{q}} < \varphi_\Sigma(\beta_{1\Sigma}), \\ \varphi_\Sigma^{-1}(\theta_{\boldsymbol{q}}), & \textit{if } \varphi_\Sigma(\beta_{1\Sigma}) \le \theta_{\boldsymbol{q}} \le \varphi_\Sigma(\beta_{\Sigma 2}), \\ \beta_{\Sigma 2}, & \textit{if } \varphi_\Sigma(\beta_{\Sigma 2}) < \theta_{\boldsymbol{q}} < \varphi_2(\beta_{\Sigma 2}), \\ \varphi_2^{-1}(\theta_{\boldsymbol{q}}), & \textit{if } \varphi_2(\beta_{\Sigma 2}) \le \theta_{\boldsymbol{q}} \le \varphi_2(0), \\ 0, & \textit{if } \varphi_2(0) < \theta_{\boldsymbol{q}}. \end{cases}$$

Then the corresponding optimal time division parameter $\alpha^*(\boldsymbol{q})$ is given by

$$\alpha^* : \mathbb{R}_+^2 \to [0,1], \quad \boldsymbol{q} \mapsto \begin{cases} \alpha_1(\beta^*(\boldsymbol{q})), & \textit{if } 1 \ge \beta^*(\boldsymbol{q}) \ge \beta_{1\Sigma}, \\ \alpha_\Sigma(\beta^*(\boldsymbol{q})), & \textit{if } \beta_{1\Sigma} > \beta^*(\boldsymbol{q}) > \beta_{\Sigma 2}, \\ \alpha_2(\beta^*(\boldsymbol{q})), & \textit{if } \beta_{\Sigma 2} \ge \beta^*(\boldsymbol{q}) \ge 0. \end{cases}$$

Proof. The proof can be found in Appendix 2.8.12. □

In Figure 2.5 we depicted the bidirectional rate regions with equal (a) and optimal (b) time division. The marked boundary rate pairs (bullets) denote the weighted rate sum optimal rate pairs for the corresponding vector (arrows). We see that the intersection points are optimal for a range of weight vectors, while the rate pairs between intersection points correspond to exactly one orthonormal vector.

As for the equal time division the theorem covers some special cases. If we choose $q_1 = q_2 = 1$, we get the sum-rate optimal rate pair. On the other hand, we maximize the unidirectional rate R_1 if we choose $q_1 > 0$ and $q_2 = 0$ so that we have $\theta_{\boldsymbol{q}} = 0$. Obviously, $\beta^* = 1$ is the optimal power distribution with the corresponding optimal rate pair $\boldsymbol{R}_{\text{opt}}(1) = \boldsymbol{R}_1(1) = [R_{11}(1), R_{12}(1)]$. This means that the maximal unidirectional rate for the optimal time division case is given by

$$R^*_{1\text{opt}} := \max_{[R_1,R_2]\in\mathcal{R}_{\text{BRopt}}} R_1 = R_{11}(1) = \frac{R_{\overrightarrow{1\text{R}}} R_{\overrightarrow{\text{R}2}}(1)}{R_{\overrightarrow{1\text{R}}} + R_{\overrightarrow{\text{R}2}}(1)}. \tag{2.31a}$$

Similarly, if we choose $q_1 = 0$ and $q_2 > 0$, we have $\theta_{\boldsymbol{q}} = \pi/2$ so that $\beta^* = 0$ is optimal. From the corresponding optimal rate pair $\boldsymbol{R}_{\text{opt}}(0) = \boldsymbol{R}_2(0) = [R_{21}(0), R_{22}(0)]$ we get the other maximal unidirectional rate as follows

$$R^*_{2\text{opt}} := \max_{[R_1,R_2]\in\mathcal{R}_{\text{BRopt}}} R_2 = R_{22}(0) = \frac{R_{\overrightarrow{2\text{R}}} R_{\overrightarrow{\text{R}1}}(1)}{R_{\overrightarrow{2\text{R}}} + R_{\overrightarrow{\text{R}1}}(1)}. \tag{2.31b}$$

Since we have $R_{12}(1) = R_{22}(0) = 0$, for the optimal time division the maximal unidirectional rate pairs are always on the R_1-axis and R_2-axis.

In the next section we will derive the achievable rate region with an optimal time-division but with a different interpretation of the block length of a code word which result in a different individual power constraint which depends on the time-division parameter.

2.2.5 Achievable Rate Regions with Power Scaling

In this section we briefly want to discuss a different interpretation of the mean power constraint assumed in the remaining sections. A more detailed discussion on the fundamentals of the following discussion can be found in [Gal68], but it is also addressed in other textbooks like [Ash65] or [CT91]. In this chapter we consider Gaussian channels, which are continuous-alphabet channels. In order to have useful results it is necessary to consider an input constraint, which is usually characterized by a real-valued function $f(x)$ on the input letters.

The most common and physically meaningful constraint is an energy constraint per transmitted symbol, i.e. $f(x) = |x|^2$. It is common to consider an equivalent power constraint, where one has normalized the energy constraint by the time duration of a symbol. This means that we are usually faced with a mean power constraint P.

But we still have to interpret the constraint per input symbol from a coding point of view. Let $\boldsymbol{x}_m = [x_{m,1}, x_{m,2}, \dots, x_{m,N}]$ denote a code word of length N, then it is common and reasonable to insist that each code word x_m satisfies the condition

$$\sum_{n=1}^{N} f(x_{m,n}) = \sum_{n=1}^{N} |x_{m,n}|^2 \leq NP. \tag{2.32}$$

Using the random coding arguments we consider the input as a random variable X. Accordingly, it is usual to bound the expected value of $f(X)$ so that (2.32) results in the commonly used mean power constraint

$$\mathbb{E}\{|X|^2\} \leq P.$$

The difference between the results obtained in this section and the remaining ones of this chapter is the interpretation of the definition of a code word. Since the relay node cannot transmit and receive at the same time using the same frequency we require orthogonal channels for the transmission and reception. As a consequence the communication is performed in two phases. In this work we consider orthogonal channels in the time domain, but the concepts and the results of this section also hold for orthogonal channels in the frequency domain where we split the total bandwidth into two channels. The time division between the two phases mean that in the first N_{MAC} time slots we are in the MAC phase. In the following N_{BC} time slots we are in the BC phase.

In the remaining of this chapter we have defined input power constraints per transmitted symbol for each phase individually, which means that the code word in the MAC phase has the length N_{MAC} and the codeword in the BC phase has the length N_{BC}. However, in this section we will assume that the code words in the MAC and BC phase have the length $N_{\mathrm{BC}} + N_{\mathrm{MAC}}$. To be conform to the time division between the phases we have to assume that in the MAC phase the last N_{BC} letters of the code words are equal to zero and in the BC phase the first N_{MAC} letters of the code words are equal to zero. Accordingly, for a code word in the MAC phase we have

$$\sum_{n=1}^{N_{\mathrm{MAC}}+N_{\mathrm{BC}}} |x_{m,n}|^2 = \sum_{n=1}^{N_{\mathrm{MAC}}} |x_{m,n}|^2 \leq (N_{\mathrm{MAC}} + N_{\mathrm{BC}})P.$$

which results in the scaled power constraint $\mathbb{E}\{|X|^2\} \leq \frac{N_{\mathrm{MAC}}+N_{\mathrm{BC}}}{N_{\mathrm{MAC}}} P$. Similarly, for a codeword in the BC phase we have

$$\sum_{n=1}^{N_{\mathrm{MAC}}+N_{\mathrm{BC}}} |x_{m,n}|^2 = \sum_{n=N_{\mathrm{MAC}}+1}^{N_{\mathrm{MAC}}+N_{\mathrm{BC}}} |x_{m,n}|^2 \leq (N_{\mathrm{MAC}} + N_{\mathrm{BC}})P.$$

which results in the scaled power constraint $\mathbb{E}\{|X|^2\} \leq \frac{N_{\mathrm{MAC}}+N_{\mathrm{BC}}}{N_{\mathrm{BC}}} P$. We see that we scale the power constraints with respect to the time division. Let α denote the ratio of N_{MAC}

to $N_{\mathrm{MAC}} + N_{\mathrm{BC}} = N$ as N gets large. Then we end up with the individual mean power constraints

$$\tilde{P}_1 := \frac{P_1}{\alpha}, \qquad \tilde{P}_2 := \frac{P_2}{\alpha}, \quad \text{and} \quad \tilde{P}_{\mathrm{R}} := \frac{P_{\mathrm{R}}}{1-\alpha} \tag{2.33}$$

of the nodes 1 and 2 in the MAC phase and the relay node in the BC phase. For that reason we consider this as bidirectional relaying with scaled powers. From (2.33) we see that this model has the property that for each phase the mean transmit power constraint gets arbitrary large if the time period of the phase is correspondingly short. However, the provided energy for each node is independent of the duration of the phases, i.e. the energy remains constant for different time division parameters.

This interpretation of the code word length is often assumed for the multiple access channel with orthogonal access (TDMA or FDMA) [CT91] and for Gaussian orthogonal relay channels in [LV05]. In the following we derive the achievable rate region of bidirectional relaying with optimal time division and scaled powers. Fortunately, we can principally follow the same ideas as before, although the proofs get more involved because of the non-linear dependence of the time division parameter on the achievable rate constraints. This also hampers closed form solutions. Accordingly, in the next sections we specify the achievable rate region of the MAC phase, the BC phase, and the achievable rate region obtained with optimal time division between the two phases.

Since all concepts of the previous section can be transfered, the symbols will have the same meaning but in a different context. For that reason we only added a tilde to the previously defined symbols to distinguish the variables and results of this section from the results of the remaining chapter. To avoid distracting case studies we exclude for notational convenience the degenerate case where one channel or power is equal to zero. However, there is no mathematical difficulty to extend the discussion for these cases.

Multiple Access Phase

In the multiple access phase node 1 and node 2 transmit their messages to the relay node. We assume that for the fraction $\alpha \in (0, 1]$ we are in the MAC phase, which specifies the effective block length of the code words. Then we assume that the relay node can decode the message of node 1 with rate R_1 and the message of node 2 with rate R_2 if the rate pair is within the capacity region

$$\tilde{\mathcal{C}}_{\mathrm{MAC}}(\alpha) := \Big\{[R_1, R_2] \in \mathbb{R}_+^2 : \; R_1 \leq \tilde{R}_{\overrightarrow{1\mathrm{R}}}(\alpha), \; R_2 \leq \tilde{R}_{\overrightarrow{2\mathrm{R}}}(\alpha), \; R_1 + R_2 \leq \tilde{R}_{\Sigma}^{\mathrm{MAC}}(\alpha)\Big\}$$

with the individual and sum-rate constraints

$$\tilde{R}_{\overrightarrow{1R}} : (0,1] \to \mathbb{R}_+, \quad \alpha \mapsto \alpha \log\left(1 + \frac{\gamma_1 |h_1|^2}{\alpha}\right), \tag{2.34a}$$

$$\tilde{R}_{\overrightarrow{2R}} : (0,1] \to \mathbb{R}_+, \quad \alpha \mapsto \alpha \log\left(1 + \frac{\gamma_2 |h_2|^2}{\alpha}\right), \tag{2.34b}$$

$$\tilde{R}_{\Sigma}^{\mathrm{MAC}} : (0,1] \to \mathbb{R}_+, \quad \alpha \mapsto \alpha \log\left(1 + \frac{\gamma_1 |h_1|^2 + \gamma_2 |h_2|^2}{\alpha}\right), \tag{2.34c}$$

and the continuously continuations $\tilde{R}_{\overrightarrow{k\mathrm{R}}}(0) := 0$, $k = 1, 2$, and $\tilde{R}_{\Sigma}^{\mathrm{MAC}}(0) := 0$. The difference to the rate constraints (2.4a), (2.4b), and (2.4c) of the previous model is that here the powers, respectively signal-to-noise ratios γ_1 and γ_2, are divided by the time division factor α. For $\alpha = 0$ the protocol degenerates and there is no multiple access phase so that we have $\tilde{\mathcal{C}}_{\mathrm{MAC}}(0) = \emptyset$. It follows that for $\alpha = 0$ no bidirectional communication will be possible. Therefore, we exclude the case $\alpha = 0$ to avoid distracting case studies caused by divisions by α.

For any fixed time division parameter $\alpha \in (0,1]$ the capacity region $\tilde{\mathcal{C}}_{\mathrm{MAC}}(\alpha)$ describes a pentagon, which is obviously convex. In the next lemma we prove monotony and concavity properties of the rate constraints, which we will extensively use in the proofs for the results on the bidirectional achievable rate region.

Lemma 2.17. *The individual and sum-rate constraint functions $\tilde{R}_{\overrightarrow{1R}}(\alpha)$, $\tilde{R}_{\overrightarrow{2R}}(\alpha)$, and $\tilde{R}_{\Sigma}^{\mathrm{MAC}}(\alpha)$ are strictly increasing and concave for $\alpha \in (0,1]$.*

Proof. The proof can be found in Appendix 2.8.13. □

Broadcast Phase

In the broadcast phase the relay node re-encodes the decoded messages from the MAC phase. Then the relay node scales the code word of the message of node 1 with rate R_1 and the code word of the message of node 2 with rate R_2 according to the relay power constraint. Again the parameter $\beta \in [0,1]$ denotes the fraction of the relay power spent for the message of node 1. Then the relay node transmits a superposition of the scaled codewords. Node 1 and node 2 subtract the interference caused by the code word of its own message before decoding the unknown message. Therefore, we can assume that node 2 and node 1 can decode the messages if the rate pair $[R_1, R_2]$ is within the achievable rate region

$$\tilde{\mathcal{R}}_{\mathrm{BC}}(\alpha) := \Big\{[R_1, R_2] \in \mathbb{R}_+^2 : \text{there exists } \beta \in [0,1] \text{ such that} \\ R_1 \leq \tilde{R}_{\overrightarrow{\mathrm{R}2}}(\alpha,\beta),\ R_2 \leq \tilde{R}_{\overrightarrow{\mathrm{R}1}}(\alpha,\beta)\Big\}$$

with rate constraints

$$\tilde{R}_{\overrightarrow{\mathrm{R2}}}: [0,1) \times [0,1] \to \mathbb{R}_+, \quad [\alpha, \beta] \mapsto (1-\alpha) \log \left(1 + \frac{\gamma_{\mathrm{R}} \beta |h_2|^2}{1-\alpha}\right), \tag{2.35a}$$

$$\tilde{R}_{\overrightarrow{\mathrm{R1}}}: [0,1) \times [0,1] \to \mathbb{R}_+, \quad [\alpha, \beta] \mapsto (1-\alpha) \log \left(1 + \frac{\gamma_{\mathrm{R}} (1-\beta) |h_1|^2}{1-\alpha}\right), \tag{2.35b}$$

and with the continuously continuation $\tilde{R}_{\overrightarrow{\mathrm{R}k}}(1, \beta) := 0$, $k = 1, 2$, for all $\beta \in [0,1]$. The difference to rate constraints (2.7a) and (2.7b) of the previous model is that here the power, respectively signal-to-noise ratio γ_{R}, is divided by the time division factor $(1-\alpha)$. For $\alpha = 1$ the protocol degenerates and there is no broadcast phase so that we have $\tilde{\mathcal{R}}_{\mathrm{BC}}(1) = \emptyset$. It follows that for $\alpha = 1$ no bidirectional communication will be possible so that we exclude the case $\alpha = 1$ to avoid distracting case studies caused by divisions by $1 - \alpha$.

We will now introduce some notation which is needed for the following studies. First, we define the following vector valued function

$$\tilde{\boldsymbol{R}}_{\mathrm{BC}}: \ [0,1) \times [0,1] \to \mathbb{R}_+^2, \quad [\alpha, \beta] \mapsto \left[\tilde{R}_{\overrightarrow{\mathrm{R2}}}(\alpha, \beta), \tilde{R}_{\overrightarrow{\mathrm{R1}}}(\alpha, \beta)\right]$$

which parametrizes the boundary of $\tilde{\mathcal{R}}_{\mathrm{BC}}(\alpha)$ for any time division parameter $\alpha \in [0,1)$ by the power distribution $\beta \in [0,1]$. Obviously, we can express any rate pair $\tilde{\boldsymbol{R}}_{\mathrm{BC}}(\alpha, \beta) \neq \mathbf{0}$ in its polar coordinates. Accordingly, the angle is given by

$$\tilde{\varphi}_{\mathrm{BC}}: [0,1) \times [0,1] \to [0, \tfrac{\pi}{2}], \quad [\alpha, \beta] \mapsto \begin{cases} \arctan \frac{\tilde{R}_{\overrightarrow{\mathrm{R1}}}(\alpha,\beta)}{\tilde{R}_{\overrightarrow{\mathrm{R2}}}(\alpha,\beta)}, & \text{if } \beta \neq 0 \\ \frac{\pi}{2}, & \text{if } \beta = 0 \end{cases}$$

and the radius can be calculated by

$$\tilde{R}_{\mathrm{BC}}: [0,1) \times [0,1] \to \mathbb{R}_+, \quad [\alpha, \beta] \mapsto \left|\left|\tilde{\boldsymbol{R}}_{\mathrm{BC}}(\alpha, \beta)\right|\right|_1$$

with $\left|\left|\tilde{\boldsymbol{R}}_{\mathrm{BC}}(\alpha, \beta)\right|\right|_1 = \sqrt{\left(\tilde{R}_{\overrightarrow{\mathrm{R1}}}(\alpha,\beta)\right)^2 + \left(\tilde{R}_{\overrightarrow{\mathrm{R2}}}(\alpha,\beta)\right)^2}$ so that we can write

$$\tilde{\boldsymbol{R}}_{\mathrm{BC}}(\alpha, \beta) = \tilde{R}_{\mathrm{BC}}(\alpha, \beta) \left[\cos\left(\tilde{\varphi}_{\mathrm{BC}}(\alpha, \beta)\right), \sin\left(\tilde{\varphi}_{\mathrm{BC}}(\alpha, \beta)\right)\right].$$

In the proofs for the results on the bidirectional achievable rate region we often use the polar representation of a rate pair. Additionally, we extensively use the monotony and concavity properties of the rate constraints, which we will prove in the next lemma.

Lemma 2.18. *The rate functions $\tilde{R}_{\overrightarrow{\mathrm{R2}}}(\alpha, \beta)$ and $\tilde{R}_{\overrightarrow{\mathrm{R1}}}(\alpha, \beta)$ are concave for $[\alpha, \beta] \in [0,1) \times [0,1]$ and strictly decreasing for $\alpha \in [0,1)$.*

Proof. The proof can be found in Appendix 2.8.14. □

The concavity of the rate constraints allows us to derive convexity of the rate region in the next proposition.

Proposition 2.19. *For any* $\alpha \in [0,1)$ *the set of rate pairs* $\tilde{\mathcal{R}}_{\mathrm{BC}}(\alpha)$ *is convex.*

Proof. The proof can be found in Appendix 2.8.15. $\square$

Proposition 2.20. *For any arbitrary but fixed time division parameter* $\alpha \in [0,1)$ *the function of the angle* $\tilde{\varphi}_{\mathrm{BC}}(\alpha,\beta)$ *is continuously and strictly decreasing for* $\beta \in [0,1]$. *It follows that there exists an inverse mapping* $\tilde{\beta}_{\mathrm{BC}}: [0,1) \times [0,\frac{\pi}{2}] \rightarrow [0,1]$ *which is implicitly defined by*

$$\tilde{\varphi}_{\mathrm{BC}}(\alpha, \tilde{\beta}_{\mathrm{BC}}(\alpha,\varphi)) = \varphi.$$

Proof. For any non-degenerate fixed time factor α it is obvious that the rates $\tilde{R}_{\overrightarrow{\mathrm{R1}}}(\alpha,\beta)$ and $\tilde{R}_{\overrightarrow{\mathrm{R2}}}(\alpha,\beta)$ are continuous and strictly decreasing and strictly increasing for $\beta \in [0,1]$. Accordingly the ratio $\frac{\tilde{R}_{\overrightarrow{\mathrm{R1}}}(\alpha,\beta)}{\tilde{R}_{\overrightarrow{\mathrm{R2}}}(\alpha,\beta)}$ is continuous and strictly decreasing. Since the trigonometric function $\arctan$ is continuous and strictly increasing the angle $\tilde{\varphi}_{\mathrm{BC}}(\alpha,\beta)$ is continuous and strictly decreasing for $\beta \in [0,1]$. Then it follows that for fixed $\alpha \in [0,1)$ there exists an inverse mapping $\tilde{\varphi}_{\mathrm{BC}}(\alpha, \tilde{\beta}_{\mathrm{BC}}(\alpha,\varphi))$. $\square$

With this we can specify the radius $\tilde{R}_{\mathrm{BC}}(\alpha,\beta)$ with respect to the angle φ by a new defined function as follows

$$\tilde{R}_{\mathrm{BC}} : [0,1) \times [0,\tfrac{\pi}{2}] \rightarrow \mathbb{R}_+, \quad [\alpha,\varphi] \mapsto \tilde{R}_{\mathrm{BC}}(\alpha, \tilde{\beta}_{\mathrm{BC}}(\alpha,\varphi)). \tag{2.36}$$

Similarly, since $\tilde{\mathcal{R}}_{\mathrm{BC}}(\alpha)$ is convex for any $\alpha \in [0,1)$ we get an unique parametrization of the boundary of $\tilde{\mathcal{R}}_{\mathrm{BC}}(\alpha)$ in the first quadrant with respect to the angle $\varphi \in [0,\frac{\pi}{2}]$ according to

$$\tilde{\boldsymbol{R}}_{\mathrm{BC}} : [0,1) \times [0,\tfrac{\pi}{2}] \rightarrow \mathbb{R}_+^2, \quad [\alpha,\varphi] \mapsto \Big[\tilde{R}_{\overrightarrow{\mathrm{R2}}}(\alpha, \tilde{\beta}_{\mathrm{BC}}(\alpha,\varphi)), \tilde{R}_{\overrightarrow{\mathrm{R1}}}(\alpha, \tilde{\beta}_{\mathrm{BC}}(\alpha,\varphi))\Big].$$

with the equality $\tilde{\boldsymbol{R}}_{\mathrm{BC}}(\alpha,\varphi) = \tilde{\boldsymbol{R}}_{\mathrm{BC}}(\alpha, \tilde{\varphi}_{\mathrm{BC}}(\alpha,\varphi))$.

Bidirectional Achievable Rate Region

In this section we study the bidirectional achievable rate region using superposition encoding and scaled powers. As in the case of non-scaled powers we first prove an equivalent characterization of the achievable rate region which then allows us to prove its convexity. Besides we can show that the combinatoric of the MAC rate region transfers for the scaled powers as well. This shows the conceptual parallelism of both interpretations of the power constraints. However, the proofs here are more involved due to the non-linear dependence of the time division parameter. This is also the reason that no closed-form solution of the weighted rate sum maximum can be obtained. But with the equivalent characterization of the boundary, including its combinatoric, and the knowledge of the convexity this can be easily obtained algorithmically.

For a successful bidirectional transmission the rate pair $[R_1, R_2]$ has to be achievable in the MAC and BC phase simultaneously. It follows that the bidirectional achievable rate region for a fixed time division parameter $\alpha \in [0, 1]$ is given by the intersection of the rate regions of the MAC and BC phase, i.e.

$$\tilde{\mathcal{R}}_{\mathrm{BR}}(\alpha) := \tilde{\mathcal{C}}_{\mathrm{MAC}}(\alpha) \cap \tilde{\mathcal{R}}_{\mathrm{BC}}(\alpha). \tag{2.37}$$

Since $\tilde{\mathcal{C}}_{\mathrm{MAC}}(\alpha)$ and $\tilde{\mathcal{R}}_{\mathrm{BC}}(\alpha)$ are convex, it follows that the intersection $\tilde{\mathcal{R}}_{\mathrm{BR}}(\alpha)$ is convex as well. Notice, that for $\alpha = 0$ or $\alpha = 1$ we have $\tilde{\mathcal{R}}_{\mathrm{BC}}(\alpha) = \emptyset$.

The next natural extension is to look at the achievable rate region where we additionally allow to optimize the time division between the MAC and the BC phase. This rate region is obviously given by the union over all possible time division parameters α as follows

$$\tilde{\mathcal{R}}_{\mathrm{BRopt}} := \bigcup_{\alpha \in [0,1]} \tilde{\mathcal{R}}_{\mathrm{BR}}(\alpha) = \bigcup_{\alpha \in (0,1)} \tilde{\mathcal{R}}_{\mathrm{BR}}(\alpha) = \bigcup_{\alpha \in (0,1)} \tilde{\mathcal{C}}_{\mathrm{MAC}}(\alpha) \cap \tilde{\mathcal{R}}_{\mathrm{BC}}(\alpha) \tag{2.38}$$

where we used the fact that $\tilde{\mathcal{R}}_{\mathrm{BC}}(\alpha) = \emptyset$ for $\alpha = 0$ or $\alpha = 1$. As a union of convex sets the rate region $\tilde{\mathcal{R}}_{\mathrm{BRopt}}$ need not be convex. Convexity can be achieved if we allow time-sharing between bidirectional rate pairs with different time division parameters, but in Theorem 2.26 we show that $\tilde{\mathcal{R}}_{\mathrm{BRopt}}$ is already convex. To this end we first prove in the next theorem an equivalent representation of the achievable rate region $\tilde{\mathcal{R}}_{\mathrm{BRopt}}$.

Theorem 2.21. *The bidirectional achievable rate region with optimal time division $\tilde{\mathcal{R}}_{\mathrm{BRopt}}$ is given by*

$$\tilde{\mathcal{R}}_{\mathrm{BRopt}} = \tilde{\mathcal{R}}_1 \cap \tilde{\mathcal{R}}_2 \cap \tilde{\mathcal{R}}_\Sigma \tag{2.39}$$

with rate regions

$$\begin{aligned}
\tilde{\mathcal{R}}_1 &:= \big\{\boldsymbol{R} \in \mathbb{R}_+^2 : \exists \varphi \in [0, \tfrac{\pi}{2}] \text{ such that } R_1 \le \tilde{R}_1(\varphi)\cos(\varphi),\ R_2 \le \tilde{R}_1(\varphi)\sin(\varphi)\big\},\\
\tilde{\mathcal{R}}_2 &:= \big\{\boldsymbol{R} \in \mathbb{R}_+^2 : \exists \varphi \in [0, \tfrac{\pi}{2}] \text{ such that } R_1 \le \tilde{R}_2(\varphi)\cos(\varphi),\ R_2 \le \tilde{R}_2(\varphi)\sin(\varphi)\big\},\\
\tilde{\mathcal{R}}_\Sigma &:= \big\{\boldsymbol{R} \in \mathbb{R}_+^2 : \exists \varphi \in [0, \tfrac{\pi}{2}] \text{ such that } R_1 \le \tilde{R}_\Sigma(\varphi)\cos(\varphi),\ R_2 \le \tilde{R}_\Sigma(\varphi)\sin(\varphi)\big\},
\end{aligned}$$

$$\tilde{R}_1 : [0, \tfrac{\pi}{2}] \to \mathbb{R}_+, \quad \varphi \mapsto \max_{\alpha \in [0,1]} \min \left\{ \tilde{R}_{\mathrm{BC}}(\alpha, \varphi), \frac{\tilde{R}_{\overrightarrow{1\mathrm{R}}}(\alpha)}{\cos(\varphi)} \right\}, \tag{2.40a}$$

$$\tilde{R}_2 : [0, \tfrac{\pi}{2}] \to \mathbb{R}_+, \quad \varphi \mapsto \max_{\alpha \in [0,1]} \min \left\{ \tilde{R}_{\mathrm{BC}}(\alpha, \varphi), \frac{\tilde{R}_{\overrightarrow{2\mathrm{R}}}(\alpha)}{\sin(\varphi)} \right\}, \tag{2.40b}$$

$$\tilde{R}_\Sigma : [0, \tfrac{\pi}{2}] \to \mathbb{R}_+, \quad \varphi \mapsto \max_{\alpha \in [0,1]} \min \left\{ \tilde{R}_{\mathrm{BC}}(\alpha, \varphi), \frac{\tilde{R}_\Sigma^{\mathrm{MAC}}(\alpha)}{\cos(\varphi)+\sin(\varphi)} \right\}. \tag{2.40c}$$

Before we refer on the proof of the theorem we will state two simple but important observations in the following remark and lemma.

Remark 2.22. *The rate regions $\tilde{\mathcal{R}}_1$, $\tilde{\mathcal{R}}_2$, and $\tilde{\mathcal{R}}_\Sigma$ are characterized by the boundaries in the first quadrant in polar coordinates. This means that $\tilde{R}_1(\varphi)$, $\tilde{R}_2(\varphi)$, $\tilde{R}_\Sigma(\varphi)$ specify the radius of the rate vectors on the boundaries of the rate regions for each $\varphi \in [0, \frac{\pi}{2}]$.*

Lemma 2.23. *For any $\varphi \in [0, \frac{\pi}{2}]$ the maximizing time division parameter α for the radii given by* (2.40a)*,* (2.40b)*, and* (2.40c) *are uniquely characterized by an α where the first and second arguments of the minima are equal. This allows us to define the optimal time division parameters for any $\varphi \in [0, \frac{\pi}{2}]$ as follows*

$$\tilde{\alpha}_1^* : [0, \tfrac{\pi}{2}] \to [0,1], \quad \varphi \mapsto \arg\max_{\alpha \in [0,1]} \min \left\{ \tilde{R}_{\mathrm{BC}}(\alpha, \varphi), \frac{\tilde{R}_{\overrightarrow{1\mathrm{R}}}(\alpha)}{\cos(\varphi)} \right\}, \tag{2.41a}$$

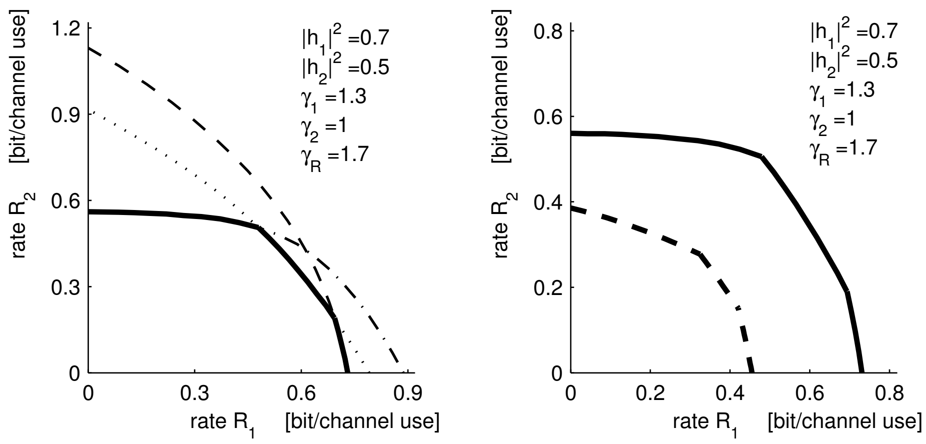
$$\tilde{\alpha}_2^* : [0, \tfrac{\pi}{2}] \to [0,1], \quad \varphi \mapsto \arg\max_{\alpha \in [0,1]} \min \left\{ \tilde{R}_{\mathrm{BC}}(\alpha, \varphi), \frac{\tilde{R}_{\overrightarrow{2\mathrm{R}}}(\alpha)}{\sin(\varphi)} \right\}, \tag{2.41b}$$

$$\tilde{\alpha}_\Sigma^* : [0, \tfrac{\pi}{2}] \to [0,1], \quad \varphi \mapsto \arg\max_{\alpha \in [0,1]} \min \left\{ \tilde{R}_{\mathrm{BC}}(\alpha, \varphi), \frac{\tilde{R}_\Sigma^{\mathrm{MAC}}(\alpha)}{\cos(\varphi)+\sin(\varphi)} \right\}. \tag{2.41c}$$

Proof. The proof of the Lemma 2.23 and the Theorem 2.21 can be found in Appendices 2.8.16 and 2.8.17 respectively. □

In Figure 2.6 (a) we depicted the intersection of the achievable rate regions $\tilde{R}_1$, $\tilde{R}_2$, and $\tilde{R}_\Sigma$ of an example. We see that for this example the rate regions are convex and that the combinatoric of the MAC rate region transfers as well. In the following we will prove that this holds in general. In addition, in Figure 2.6 (b) we depicted the achievable rate regions $\mathcal{R}_{\mathrm{BRopt}}$ and $\tilde{\mathcal{R}}_{\mathrm{BRopt}}$ with the same power constraint values. However, we want to emphasize that it is hardly possible to draw consequences from the comparison since different power constraint models are assumed.

Theorem 2.24. *In the first quadrant there is exactly one intersection between the boundaries $\tilde{\mathcal{R}}_1$, $\tilde{\mathcal{R}}_2$, and $\tilde{\mathcal{R}}_\Sigma$. The rate pairs of the intersections are transformed vertices of a corresponding MAC rate region. Let $\tilde{\varphi}_{12}$, $\tilde{\varphi}_{\Sigma 1}$, and $\tilde{\varphi}_{2\Sigma}$ denote the angles of the intersection*



(a) Equivalent characterization of $\tilde{\mathcal{R}}_{\text{BRopt}}$ as intersection of $\tilde{\mathcal{R}}_1$ (dashed line), $\tilde{\mathcal{R}}_2$ (dashed-dotted line), and $\tilde{\mathcal{R}}_\Sigma$ (dotted line).

(b) Comparison between rate region with scaled powers $\tilde{\mathcal{R}}_{\text{BRopt}}$ (solid line) and non-scaled powers $\mathcal{R}_{\text{BRopt}}$ (dashed line).

Figure 2.6: Achievable rate region $\tilde{\mathcal{R}}_{\text{BRopt}}$ (with scaled powers).

between the boundaries of $\tilde{\mathcal{R}}_1$ *and* $\tilde{\mathcal{R}}_2$*,* $\tilde{\mathcal{R}}_\Sigma$ *and* $\tilde{\mathcal{R}}_1$*, and* $\tilde{\mathcal{R}}_2$ *and* $\tilde{\mathcal{R}}_\Sigma$ *respectively. Then we have*

$$\begin{aligned}
\tilde{R}_\Sigma(\tilde{\varphi}_{12}) &\leq \tilde{R}_1(\tilde{\varphi}_{12}) = \tilde{R}_2(\tilde{\varphi}_{12}),\\
\tilde{R}_2(\tilde{\varphi}_{\Sigma 1}) &\geq \tilde{R}_\Sigma(\tilde{\varphi}_{\Sigma 1}) = \tilde{R}_1(\tilde{\varphi}_{\Sigma 1}),\\
\tilde{R}_1(\tilde{\varphi}_{2\Sigma}) &\geq \tilde{R}_2(\tilde{\varphi}_{2\Sigma}) = \tilde{R}_\Sigma(\tilde{\varphi}_{2\Sigma}).
\end{aligned}$$

Furthermore, we have

$$\tilde{R}_1(0) < \tilde{R}_\Sigma(0) < \tilde{R}_2(0) \tag{2.42a}$$

$$\tilde{R}_1(\tfrac{\pi}{2}) > \tilde{R}_\Sigma(\tfrac{\pi}{2}) > \tilde{R}_2(\tfrac{\pi}{2}) \tag{2.42b}$$

Proof. The proof can be found in Appendix 2.8.18. □

With the equivalent description of the rate region in Theorem 2.21 and the knowledge about the combinatoric we can finally characterize the boundary of $\tilde{\mathcal{R}}_{\text{BRopt}}$.

Corollary 2.25. *The section-wise defined rate pair function* $\tilde{\boldsymbol{R}}_{\text{opt}} : [0, \frac{\pi}{2}] \to \mathbb{R}_+^2$ *with*

$$\tilde{\boldsymbol{R}}_{\text{opt}} : \varphi \mapsto [\cos(\varphi), \sin(\varphi)] \begin{cases} \tilde{R}_1(\varphi), & \text{if } \varphi \in [\tilde{\varphi}_{\Sigma 1}, 0] \\ \tilde{R}_\Sigma(\varphi), & \text{if } \varphi \in (\tilde{\varphi}_{2\Sigma}, \tilde{\varphi}_{\Sigma 1}) \\ \tilde{R}_2(\varphi), & \text{if } \varphi \in [\frac{\pi}{2}, \tilde{\varphi}_{2\Sigma}] \end{cases}$$

characterizes the boundary of the bidirectional achievable rate region $\tilde{\mathcal{R}}_{\mathrm{BRopt}}$ for the optimal time division and power scaling.

Proof. The proof can be found in Appendix 2.8.19. □

The next theorem proves that $\tilde{\mathcal{R}}_{\mathrm{BRopt}}$ is convex, which is an important property to solve optimization problems.

Theorem 2.26. *The rate regions $\tilde{\mathcal{R}}_1$, $\tilde{\mathcal{R}}_2$, and $\tilde{\mathcal{R}}_\Sigma$ are convex. It follows that $\tilde{\mathcal{R}}_{\mathrm{BRopt}}$ is convex as well.*

Proof. The proof can be found in Appendix 2.8.20. □

Since the rate constraints depend non-linearly on the time division parameter α closed-form solutions are in general not possible. This includes the weighted rate sum maximum problem as discussed in the previous sections. But since we know that $\tilde{\mathcal{R}}_{\mathrm{BRopt}}$ is convex, we know that for a given weight vector $[q_1, q_2] \in \mathbb{R}_+^2$ the weighted rate sum maximum is on the boundary of $\tilde{\mathcal{R}}_{\mathrm{BRopt}}$. Then with the characterization of the boundary given in Corollary 2.25 we can easily implement an iterative algorithm which finds the optimum. For that reason for the remaining sections of this chapter we will always consider the previous model without scaled power constraints. However, there is no difficulty in transferring the following results to the case of bidirectional relaying with optimal time division and power scaling.

2.3 Throughput Optimal Resource Allocation

In this section we are interested in a cross-layer design of the Data Link Layer and the Physical Layer, which includes the Medium Access Sublayer. We are interested in an efficient power control and channel allocation strategy with respect to the higher layer traffic. Accordingly, we are concerned with an adaptive resource allocation policy which takes the exogenous stochastic arrivals at nodes 1 and 2 from the higher layer into account. To this end we look at time slotted queueing processes with ergodic arrival processes where the information theoretic bidirectional achievable rates of the previous section serve as service rates[9].

[9] In other sections of the thesis we consider achievable rates measured in bits per channel use, which describe the spectral efficiencies in [bits/s/Hz] for a given wireless system. In this section we consider service rates which are measured in [bits/s]. For notational and conceptual simplicity and without loss of generality we use the previous derived rates, which means that the system we consider has normalized band-limited channels with 1 Hz bandwidth.

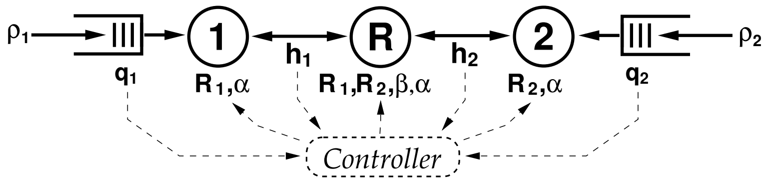


Figure 2.7: Service rate allocation by a centralized controller according to a rate allocation policy based on the current queue and channel state for bidirectional relaying with ergodic arrival processes and queues with infinite buffer size at nodes 1 and 2.

In Figure 2.7 we schematically depicted the considered queueing model. At random time instants packets arrive at nodes 1 and 2 for the other node, which are stored in queueing buffers with infinite buffer size until they are served. It is important to notice that the relay node has no internal queue, which means that we do not allow that any message will be stored at the relay node. Accordingly, any message received in the MAC phase has immediately to be forwarded in the BC phase.

We assume independent time-variant flat fading channels, where the channels stay constant for the duration of a time slot. Thereby, the duration of the time slot is long enough so that the assumption of a reliable transmission per time slot is reasonable. From time slot to time slot the channel coefficients change independently. The evolution of each channel coefficient is modeled by an ergodic and stationary random process.

At the beginning of each time slot the centralized controller adjusts the resource allocation and therefore the service rates with respect to the bidirectional rate region according to the rate allocation policy. Therefore, we assume that the centralized controller has access to the current queue and channel state. For the rate allocation the controller decides for the coding rates, which implies the power distribution factor β at the relay node and, if it is allowed, the time division parameter α. The maximum throughput policy for the bidirectional relaying protocol is an adaptation of the maximum differential backlog algorithm developed by Tassiulas and Ephrimedes in the landmark paper [TE92]. Thereby, we do not need the dynamic routing capability of the differential backlog strategy since we have fixed routes.

As we saw in Section 2.2 the bidirectional achievable rate region depends on the time division between the two phases. For the following maximum throughput policy we need to know the weighted rate sum pair of the achievable rate region. Conceptually, it makes no difference if we consider the fixed or optimal time division. Therefore, let $\mathcal{R}_{\mathrm{BR}}$ denotes the bidirectional achievable rate region which stands for the bidirectional rate region $\mathcal{R}_{\mathrm{BR}}(\alpha)$ or $\mathcal{R}_{\mathrm{BRopt}}$ respectively. From the following discussion of the maximum throughput policy we see the importance of the complete knowledge of the achievable rate region and the Pareto optimal rate pairs.

The design goal for a throughput optimal strategy in our queueing system is stability. The stability is proved by the well-developed theory of drift analysis using a quadratic Lyapunov function on the buffer levels [MT93, KM95, MLMN00, MMAW99, LMNM01]. It shows that the policy establishes stability for all arrival rate vectors within the maximal stability region, which is equal to the ergodic rate region. Our stability analysis is adapted from the cross-layer design for a satellite broadcast scenario from Neely, Modiano, and Rhors in [NMR03]. In [NMR05] they extend their policy to joint routing and power allocation for wireless networks, which means that their cross-layer design approach includes the routing problem of the Network Layer. These interesting techniques and results are also presented more a more comprehensive fashion in [Nee03] and in a more tutorial fashion in [GNT06].

The work [TG95] from Telatar and Gallager is one of first works which combines an information theoretic model with a queueing model. After that, several cross-layer design problems have been studied. In [BJH03, BW06, BW07] Boche et al. study channel aware scheduling for the MIMO multiple access channel and present an optimization-theoretic analysis of the cross-layer design as well as an iterative optimization method. Recently, in [YB05] the stability analysis for the parallel Gaussian relay channel is presented. For more references of related works we refer to reference lists of the previous mentioned works.

2.3.1 Stability Region

For this cross-layer design approach we first introduce the assumptions of the considered queueing model. Then we present the ergodic achievable rate region, which is equal to the maximal stability region; and finally, we discuss a maximum throughput policy. We consider the scenario depicted in Figure 2.7, where at nodes 1 and 2 packets for the other node arrive at random time instants. The packets are stored in queueing buffers with infinite size until they are served. For simplicity we assume that we can continuously split the data. The service of transmission to the other node is provided by bidirectional relaying protocol. Without loss of generality, we consider a normalized system with band-limited channels with 1 Hz bandwidth. Accordingly, the possible service rates are given by the bidirectional achievable rate region with rates in [bits/s] and are chosen by a centralized controller according to the rate allocation policy. Thereby, we assume that the controller does not optimize the resource allocation over time.

For the rest of this section we assume a block-fading channel model, where the flat-fading channel gains are assumed to be constant during a time period T. This allows us to consider frames in a time slotted system model where the n-th slot denotes the time period $[(n-1)T, nT]$. We assume that the frame duration T is long enough that we can transmit a codeword with a sufficient block length, which means that assuming information theoretic rates as service rates is a reasonable simplification. We model the evolution of the channel from nodes 1 and 2 to the relay node by an independent stationary and ergodic process $\boldsymbol{h}[n] = \big[h_1[n], h_2[n]\big]$ with finite state space $\boldsymbol{\mathcal{H}} = \mathcal{H}_1 \times \mathcal{H}_2$ and steady-state distribution $\pi_{\boldsymbol{h}} = \pi_{h_1}\pi_{h_2}$. Accordingly, let $\mathcal{R}_{\mathrm{BR}}\big(\boldsymbol{h}[n]\big)$ denote the bidirectional achievable rate region for the channel state $\boldsymbol{h}[n]$. The network controller adjusts the rates and therefore the resource allocation at the beginning of each frame.

Further, we assume independent ergodic arrival processes $\boldsymbol{A}[n] = \big[A_1[n], A_2[n]\big]$ at nodes 1 and 2 with bounded first and second moment. This includes homogeneous Poisson arrival processes with negative exponential distributed inter-arrival times. Let $\boldsymbol{\lambda} = [\lambda_1, \lambda_2]$ denote the average arrival rates at nodes 1 and 2. Then $\mathbb{E}\{\boldsymbol{A}\} = \boldsymbol{\lambda}T$ specifies the mean number of packet arrivals per time slot. Furthermore, we assume independent random packet lengths Z_i, $i = 1, 2$, with bounded first and second moments at each node. Thus, let $B_i[n]$, $i = 1, 2$, denote the processes of number of bits arriving in time slot n at nodes 1 and 2. Then the *bit arrival rate* at node i is given by $\rho_i = \lambda_i \mathbb{E}\{Z_i\}$, $i = 1, 2$, in [bits/s]. Note that we have $\mathbb{E}\{B_i^2\} < \infty$, $i = 1, 2$, due to the previous assumptions.

In general, the bits of the data packets cannot be sent immediately after their arrival. Therefore, at each node the arriving bits are stored in an internal queue with infinite buffer storage space until they are transmitted. The controller observes the queue lengths at the end of each time slot. Therefore, let $\boldsymbol{Q}[n] = \big[Q_1[n], Q_2[n]\big]$ represent the processes of number of remaining bits in the queues after the n-th time slot, which is obviously the same as in the beginning of the $n+1$-th time slot..

Finally, we assume a centralized controller which decides at the beginning of each time slot for the service rates $\big[R_1[n+1], R_2[n+1]\big] \in \mathcal{R}_{\mathrm{BR}}(\boldsymbol{h}[n+1])$ according to a rate allocation policy based on the current queue state $\boldsymbol{Q}[n]$ and channel state $\boldsymbol{h}[n+1]$ of the new time slot. Thereto, the controller adjusts the time division and relay power distribution parameters. Hence, the slot-to-slot dynamics of the queue lengths are given by the equation

$$Q_i[n] = \max\{Q_i[n-1] - R_i[n]T, 0\} + B_i[n], \quad i = 1, 2. \tag{2.43}$$

Since the arrival processes are memoryless and the service rates $\boldsymbol{R}[n]$ depend on the queue state $\boldsymbol{Q}(n-1)$ and the memoryless channel process $\boldsymbol{h}[n]$ only, the process $\boldsymbol{Q}[n]$ has Markov property. Therefore, $\boldsymbol{Q}[n]$ describes a discrete-time Markov chain with state space $\mathbb{R}_+$.

In the following we are interested in a maximum throughput policy. The throughput is defined as the mean number of bits transmitted from node 1 to 2 or vice versa in a unit of time. From queueing theory we know that for stability the utilization of a queueing system cannot be larger than one. This can be easily seen from the following. The utilization of a queueing system specifies the fraction of time in which a server is busy and is defined by the ratio of the mean bit arrival rate to the mean service rate. If the average number of bits that arrive is larger than the mean service rate, the queueing system is obviously unstable. Therefore, the maximum throughput is obviously upper bounded by the mean service rates. Since the information theoretic rates model the service rates, the upper bound is given by the ergodic bidirectional rate region, which we consider next.

Ergodic Bidirectional Rate Region

Let the overbar in $\overline{\boldsymbol{R}} = [\overline{R_1}, \overline{R_2}]$ and $\overline{\mathcal{R}_{\mathrm{BR}}}$ denote an ergodic rate pair and the ergodic bidirectional rate region respectively. Since $\mathcal{R}_{\mathrm{BR}}(\boldsymbol{h})$ is convex for any channel state $\boldsymbol{h} \in \boldsymbol{\mathcal{H}}$, the ergodic rate region $\overline{\mathcal{R}_{\mathrm{BR}}}$ is convex as well. It follows that the ergodic rate region can be characterized by the ergodic rate pairs on its boundary. Due to linearity the ergodic boundary rate pair for the normal vector $\boldsymbol{q} \in \mathbb{R}_+^2 \setminus \{\boldsymbol{0}\}$ is given by the mean of rate pairs which maximize the weighted rate sum with weight vector $\boldsymbol{q}$, i.e.

$$\begin{aligned}
\overline{\boldsymbol{R}^\star(\boldsymbol{q})} &= \underset{\overline{\boldsymbol{R}} \in \overline{\mathcal{R}_{\mathrm{BR}}}}{\arg\max}\; q_1 \overline{R_1} + q_2 \overline{R_2} \\
&= \mathbb{E}_{\boldsymbol{h}}\{ \underset{\boldsymbol{R} \in \mathcal{R}_{\mathrm{BR}}(\boldsymbol{h})}{\arg\max}\; q_1 R_1 + q_2 R_2 \} \\
&= \sum_{\boldsymbol{h} \in \boldsymbol{\mathcal{H}}} \pi_{\boldsymbol{h}}(\boldsymbol{h}) \underset{\boldsymbol{R} \in \mathcal{R}_{\mathrm{BR}}(\boldsymbol{h})}{\arg\max}\; q_1 R_1 + q_2 R_2,
\end{aligned}$$

which can be calculated using Theorem 2.16. Let $\overline{R_1^\star(\boldsymbol{q})}$ and $\overline{R_2^\star(\boldsymbol{q})}$ denote the components of the rate pair $\overline{\boldsymbol{R}^\star(\boldsymbol{q})}$. With this we get a characterization of the bidirectional ergodic rate region as follows

$$\overline{\mathcal{R}_{\mathrm{BR}}} = \mathrm{co}\Big(\Big\{ \boldsymbol{R} \in \mathbb{R}_+^2 : \exists \boldsymbol{q} \in \mathbb{R}_+^2 \setminus \{\boldsymbol{0}\} \text{ with } R_1 \le \overline{R_1^\star(\boldsymbol{q})}, R_2 \le \overline{R_2^\star(\boldsymbol{q})} \Big\}\Big). \tag{2.44}$$

Next, we study a resource allocation and scheduling policy for which queue stability is guaranteed for all arrival rate vectors $\boldsymbol{\rho} = [\rho_1, \rho_2]$ within the ergodic rate region $\overline{\mathcal{R}_{\mathrm{BR}}}$.

Maximum Throughput Policy

First we present a rate allocation policy derived from the maximum differential backlog algorithm presented in [TE93]. The policy achieves the maximum throughput which we will prove using the well-developed theory of drift analysis using a quadratic Lyapunov function as done in [MT93, KM95, LMNM01, NMR03], et al.

Let $f_i(M) := \limsup_{t\to\infty} \frac{1}{t} \int_0^t \mathbb{1}_{[Q_i(\tau) > M]} d\tau$, $i = 1, 2$, denote the *overflow* functions as a measure of the fraction of time that the queue length Q_i, $i = 1, 2$, is above a certain value M.

This allows us to define the meaning of *system stability* from the notion of weak stability for discrete-time Markov chains[10] as follows.

Definition 2.27 ([NMR03])**.** *For a given rate allocation policy a queueing system is stable if for all* $i = 1, 2$ *we have* $f_i(M) \to 0$ *as* $M \to \infty$.

With this, we can define the *stability region of a policy* as the set of bit arrival rate vectors $\boldsymbol{\rho}$ such that for any vector in the interior of the stability region system stability is achieved [TE93]. Accordingly, a policy dominates another policy if the stability region of the one contains the other. Further, the *stability region of a system* is the set of bit arrival rate vectors $\boldsymbol{\rho}$ such that for any vector in the interior at least one resource allocation policy exists which achieves system stability. Conversely, no stabilizing policy exists whenever $\boldsymbol{\rho}$ is outside. If $\boldsymbol{\rho}$ lies on the boundary, the system may or may not be stable. A policy that dominates any other policy is an *optimal policy*. Since the stability region of any policy is a subset of the maximum throughput region, a policy whose stability region is equal to the maximum throughput region is optimal and is called a *maximum throughput policy*. The above definitions are adopted from [TE93][11].

From the maximum differential backlog algorithm [TE93] we can deduce the following maximum throughput policy which determines the rate and resource allocation and basically tries to equalize the queue length at nodes 1 and 2. It is noteworthy that the policy does not require any knowledge about the arrival rates or channel statistics.

Maximum Throughput Policy 2.28. *At the beginning of the* $n+1$*-th time slot the centralized network controller observes the current queue states* $[q_1, q_2] := \boldsymbol{Q}[n]$ *and the new channel states* $\boldsymbol{h} := \boldsymbol{h}[n+1]$*, and adjusts the relay power distribution and, if allowed, the time division parameter on the physical layer so that we achieve the rate pair which maximizes the weighted rate sum for the weight vector* $\boldsymbol{q}$ *in* $\mathcal{R}_{\mathrm{BR}}(\boldsymbol{h})$*,*

$$\boldsymbol{R}[n+1] = \underset{[R_1,R_2]\in\mathcal{R}_{\mathrm{BR}}(\boldsymbol{h})}{\arg\max} \quad q_1 R_1 + q_2 R_2.$$

This policy is a maximum throughput policy if the stability region of the proposed policy is equal to the ergodic bidirectional rate region. To show this we consider a positive quadratic Lyapunov function on the buffer levels and show that for any arrival rate vector within the ergodic rate region, the Lyapunov function has a negative drift whenever the mean queue length

[10] A Markov chain $\boldsymbol{Q}[n]$, $n \in \mathbb{N}$, is *weakly stable* if for every $\varepsilon > 0$ there exists $B \in \mathbb{R}$ such that $\lim_{n\to\infty} \mathbb{P}\{||\boldsymbol{Q}[n]|| > B\} < \varepsilon$ holds almost surely and *strongly stable* if it is weakly stable and $\limsup_{n\to\infty} \mathbb{E}\{||\boldsymbol{Q}[n]||\} < \infty$ holds almost surely. If the Markov chain is positive recurrent, then it is weakly stable.

[11] Some authors call the maximum throughput policy a *stability optimal policy* to emphasis that this strategy stabilizes any arrival rate vector in the stability region of the system.

is large. This allows us to deduce that the discrete-time Markov chain $\boldsymbol{Q}[n]$ is aperiodic, irreducible, and positive recurrent, which means that there exists a unique steady-state distribution (equilibrium) and the Markov chain is ergodic [MT93, KM95, LMNM01, NMR03]. Note that if the Markov chain $\boldsymbol{Q}[n]$ has a steady-state distribution it is positive recurrent and therefore stable.

Theorem 2.29 ([NMR03]). *Let be given the Lyapunov function $L(\boldsymbol{q}) = \sum_{i=1}^{2} q_i^2$. If there exists a compact region $\Lambda \subseteq \mathbb{R}^2$ and $\epsilon > 0$ such that*

1. $\mathbb{E}\{L(\boldsymbol{Q}[n+1])|\boldsymbol{Q}[n] = \boldsymbol{q}\} < \infty, \forall \boldsymbol{q} \in \mathbb{R}_+^2$,
2. $\mathbb{E}\{L(\boldsymbol{Q}[n+1]) - L(\boldsymbol{Q}[n])|\boldsymbol{Q}[n] = \boldsymbol{q}\} < -\epsilon, \forall \boldsymbol{q} \notin \Lambda$,
3. *whenever $\boldsymbol{Q}[n] = \boldsymbol{q} \in \Lambda$, there exists $m \in \mathbb{N}, m < \infty$, such that the probability $\mathbb{P}(\boldsymbol{Q}[n+m] = \boldsymbol{0}) > 0$,*

then there exists a steady-state distribution for the queue state $\boldsymbol{Q}$ and hence the system is stable.

The first two conditions ensure that the mean recurrence time to the Λ region is finite. The third condition ensures that the zero state is reached infinitely often with finite mean recurrence times and therefore the Markov chain reduces to a single ergodic class. It is a necessary modification for queueing systems with uncountably infinite state space.

But an even stronger drift can be achieved which proves the stability-in-the-mean of the queues and is called the *strong stability* of the Markov chain and implies weak stability. In [NMR03], a corresponding stronger drift condition is presented. Thereto, the authors generalize results from [LMNM01, KM96] to a Markov chain with uncountably infinite state space. The fundamental idea is that the drift gets larger in magnitude as the queue lengths increase, then the mean queue length is bounded.

Corollary 2.30 ([NMR03]). *If condition three of the Theorem 2.29 holds and positive values υ and ζ exist such that*

$$\mathbb{E}\{L(\boldsymbol{Q}[n+1]) - L(\boldsymbol{Q}[n])|\boldsymbol{Q}[n] = \boldsymbol{q}\} < \upsilon - \zeta \sum_{i=1}^{2} q_i,$$

then there exists a steady-state distribution with bounded first moments $\mathbb{E}\{Q_i\} < \infty$ such that $\zeta \sum_{i=1}^{2} \mathbb{E}\{Q_i\} < \upsilon$.

Note that if the corollary holds then for any $\epsilon > 0$, the negative drift condition of Theorem 2.29 is satisfied whenever we have $\sum_{i=1}^{2} q_i > (\upsilon + \epsilon)/\zeta$, i.e. the sum of the queue lengths is sufficiently large. This means that we would have the compact region $\Lambda = \{\boldsymbol{q} \in \mathbb{R}_+^2 : \sum_{i=1}^{2} q_i \leq \frac{\upsilon+\epsilon}{\zeta}\}$.

The policy here proposed is equivalent to the dynamic power allocation policy in [NMR03]. It is therefore possible to adapt the proof in [NMR03] with the following constant

$$\upsilon := T^2 \max_{\boldsymbol{h}\in\boldsymbol{\mathcal{H}},\boldsymbol{R}\in\mathcal{R}_{\mathrm{BR}}(\boldsymbol{h})} \big(R_1^2 + R_2^2\big) + \sum_{i=1}^{2} \mathbb{E}\{B_i^2\}.$$

Since the arrival rate vector $\boldsymbol{\rho}$ is assumed to be strictly in the interior of the ergodic rate region, there exists a $\tilde{\zeta} > 0$ so that $\tilde{\zeta}[11] + \boldsymbol{\rho} \in \mathrm{int}\overline{\mathcal{R}_{\mathrm{BR}}}$ also holds. Then the proof works analog with $\zeta = 2T\tilde{\zeta}$. In Appendix 2.8.21 we reproduce the proof for completeness. □

The assumption of finite channel states and the power constraints ensure that the second moment of the service rates is bounded. The following remark extracts the necessary condition which is necessary for the previous proof.

Remark 2.31. *The finite channel state condition is used to bound in* (2.107) *the term* $\mathbb{E}\big\{\sum_{k=1}^{2} R_k^2[n] | \boldsymbol{Q}[n] = \boldsymbol{q}\big\}$. *We can weaken this condition if* $\tilde{\upsilon} < \infty$ *exists so that*

$$\mathbb{E}\Big\{\max_{\boldsymbol{R}\in\mathcal{R}_{\mathrm{BR}}(\boldsymbol{h})} \sum_{k=1}^{2} R_k^2\Big\} \leq \tilde{\upsilon}.$$

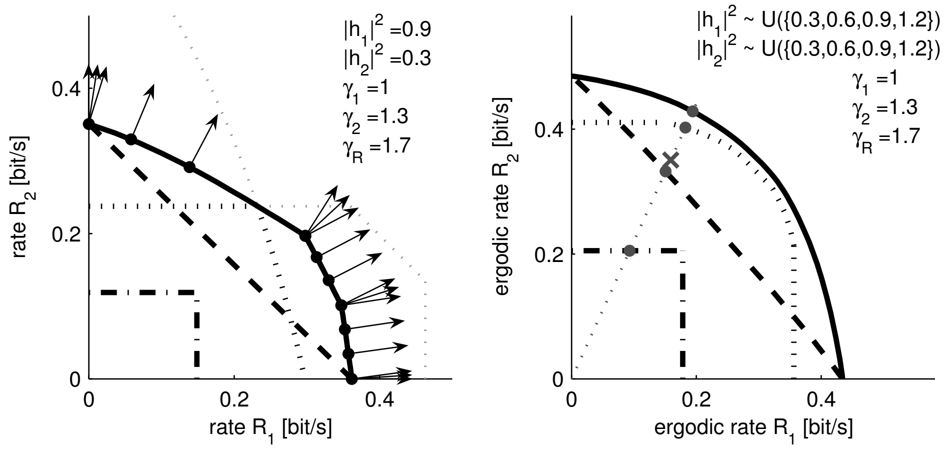
Then the proof works with $\upsilon := \tilde{\upsilon}T^2 + \sum\limits_{i=1}^{2} \mathbb{E}\{B_i^2\}$ *since* $\mathbb{E}\big\{\sum_{k=1}^{2} R_k^2[n] | \boldsymbol{Q}[n] = \boldsymbol{q}\big\} \leq \mathbb{E}\big\{\max\limits_{\boldsymbol{R}\in\mathcal{R}_{\mathrm{BR}}(\boldsymbol{h})} \sum_{k=1}^{2} R_k^2\big\}$ *for any* $\boldsymbol{q} \in \mathbb{R}_+^2$.

Finally, let $\mathbb{E}\{D_i\}$ denote the average bit delay at node i. With Little's Theorem[12] we have $\mathbb{E}\{Q_i\} = \rho_i\mathbb{E}\{D_i\}$. Therefore, we have $\upsilon/\zeta > \sum_{i=1}^{2} \mathbb{E}\{Q_i\} = \sum_{i=1}^{2} \mathbb{E}\{D_i\}\rho_i$ using the boundedness of the first moment according to Corollary 2.30. This means that the bound grows asymptotically like $1/\zeta$ as the arrival rate vector $\boldsymbol{\rho}$ is pushed towards the boundary of the ergodic rate region. A similar discussion is given in [NMR03].

2.3.2 Numerical Simulation

To clarify the previous results, we illustrate and discuss numerical simulation results of a simple example in this section. For the simulations we assumed independent homogeneous Poisson arrival processes. In order to indicate the performance gain, we present a comparison between the proposed bidirectional relaying using the optimal relay power distribution (OpR) with optimal time division (OpT) and equal time division (EqT), as well as Round-Robin

[12] Little's Theorem states that for a queueing system with almost sure finite inter-renewal interval the expected number of customers in a queueing system is equal to the expected time each customer waits in the system times the rate of arrival [Gal96, Sect. 3.6].



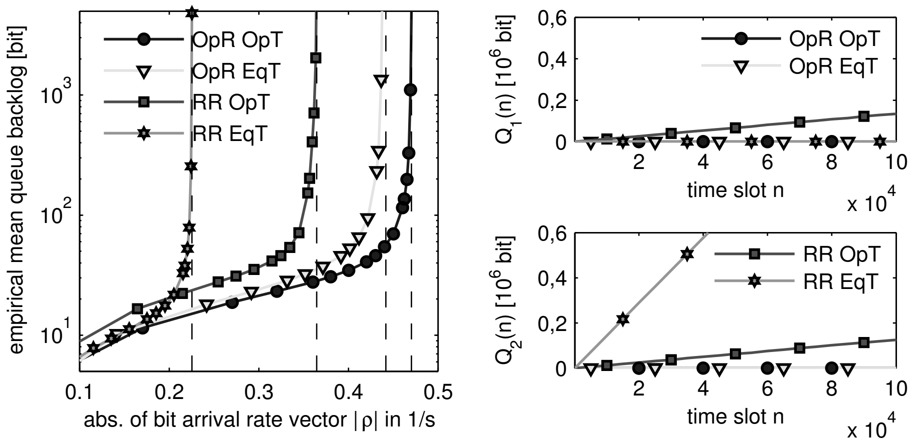
(a) Achievable Rate Regions (b) Ergodic Rate Regions

Figure 2.8: The solid, dashed, dotted, and dashed-dotted line denote OpR OpT, OpR EqT, RR OpT, and RR EqT respectively with optimal/equal time division OpT/EqT and bidirectional relaying with the optimal relay power distribution OpR and round-robin RR for a normalized system with $1Hz$ bandwidth.

(RR) scheduling with optimal and equal time division (OpT/EqT). For the Round-Robin strategy we only change the scheduling strategy so that for any communication between two nodes the time slot is subdivided into two exclusive time intervals. Thereby, we still do not allow any node to cooperate across the time intervals. For each protocol we apply the throughput optimal resource allocation policy.

In Figure 2.8 (a) the achievable rate regions for a certain channel state are depicted. The dotted line depicts $0.5\mathcal{R}_{\mathrm{MAC}}$ and $0.5\mathcal{R}_{\mathrm{BC}}$ so that its intersection denotes the corresponding achievable rate region with equal time division $\mathcal{R}_{\mathrm{BIR}}(0.5)$. Furthermore, we present the achievable rate regions for the corresponding round robin strategies. On the boundary of $\mathcal{R}_{\mathrm{BRopt}}$ some orthogonal vectors which characterize the optimal rate pair for a weight vector with the same angle are depicted, cf. Theorem 2.16. We see that several orthonormal vectors with different angles belong to a rate pair of an intersection point, while a rate pair between the intersection points corresponds to exactly one orthonormal vector. Similar arguments apply for the other rate regions. For the round robin strategies it is optimal to select the intersection points for any non-negative weight vector. In particular for the equal time division case the upper right vertex is always optimal, which means that the bidirectional relaying separates in two independent unidirectional relaying protocols.

Figure 2.8 (b) depicts the corresponding ergodic rate region assuming identically uniformly distributed channel processes. Since it is hardly possible to observe, we want to note that the



(a) Average queue length evolution (b) Temporal queue length evolution

Figure 2.9: Comparison of the queue length evolutions of different resource allocation policies where OpT and EqT specify the optimal and equal time division case and OpR and RR denote a bidirectional relaying with an optimal relay power distribution and a round-robin strategy respectively.

ergodic rate region of the RR-OpT strategy is slightly bulged. For the throughput optimal resource allocation policy the stability region is given by the ergodic rate region. For that reason we also plotted the arrival rate vectors $\boldsymbol{\rho}$ of the simulated queue results in this figure. We simulated arrival rates along the dotted radial line with an angle $\angle = 1.0081$.

Figure 2.9 (a) shows the average queue length evolutions along the radial line after 10^6 time slots with $T = 1s$. The vertical dotted lines denote the crossing of the stability region boundaries, which correspond to the bullets on the radial line in Figure 2.8 (b). It can be clearly seen that the average queue length strongly grows if $\boldsymbol{\rho}$ approaches the boundary of the stability region.

In Figure 2.9 (b) we present the temporal queue length evolutions for the different protocols with an arrival rate $\boldsymbol{\rho} = [0.16, 0.35]$ bit/s, which is marked in Figure 3 (b) by the cross ($\times$). Since the vertical axes are scaled in 10^6 bits, the random fluctuation of the queue lengths are indistinguishable. As expected, it shows that the Round-Robin strategies cannot support the bit-load. Thereby it is interesting to observe that for the equal time division case (RR-EqT) the queue at node 1 remains stable while queue at node 2 overflows. The separated queue evolutions can be explained with the separation of the bidirectional protocol in two independent unidirectional protocols. At any time slot the rate allocation policy selects for any channel and queue state the upper right vertex of the rectangular, as mentioned for Figure 2.8 (a). This results in the ergodic rate pair $[0.178, 0.205]$ bit/s, which is the upper right vertex of the corresponding ergodic rate region in Figure 2.8 (b) so that that queue evolution at node 1 will be stable and queue at node 2 will be unstable.

2.4 Relay Selection

In this section we consider the problem of relay selection in a network where N relay nodes are willing to assist the bidirectional relay communication between the nodes 1 and 2. Accordingly, we look for the "best route" which is a problem of the Network Layer. Since we propose to select a relay node based on the achievable rate region, which includes the optimal resource allocation and channel state information, this is again a cross-layer design approach.

The problem of relay selection is closely related to the question if the support of a relay node is beneficial. Already in [LTW04] Laneman, Tse, and Wornell propose a *selective* user strategy where the cooperative user decides based on the channel state if it either supports the other user or retransmits its own data. In [OS04, OSB06] we derive a sufficient condition based on the pairwise error probability when it is advantageous to cooperate using an amplify-and-forward strategy in a linear relay network. Dawy and Kamoun show in [DK04] that the power consumption in multi-hop communication depends on the route selection and derive a relay region where using the relay is better than the direct transmission. Generally, we first have to decide if the support of a certain relay node is beneficial, which obviously depends on the performance metric, the applied protocol, and the available information to the controller. Then the controller decides for the "best relay node".

In a network with N relays and independent time-variant fading channels there is the potential for a high spatial *diversity order*. To exploit this Laneman and Wornell extend in [LW03] the cooperation idea to larger networks where they introduced the distributed space-time coding concept so that all relay nodes can participate for cooperation. They show that distributed space-time coding achieves full spatial diversity in terms of the outage probability. Other works follow this concept on distributed space-time coding or beamforming, e.g. [NBK04, SO04a, LBC$^+$05, DSG$^+$03]. Thereby it is important to notice that the performance improvement of multiple relay node diversity protocols is often boosted by an increasing sum power, which means that each participating relay node adds it transmit power to the communication. Unfortunately, this often obscures the solely gain of the proposed diversity concept. In [LBC$^+$05] the authors compare different decode-and-forward schemes where one or multiple relay nodes are employed. They point out that the "effective coding gain" needs to be considered and propose a distributed power allocation between the relay nodes. The same conclusion is drawn in [ZAL06] for amplify-and-forward relay networks where the optimal distributed power allocation is obtained by an extended water-filling solution. Furthermore, they show that a selection scheme where only one relay node is chosen achieves full diversity order.

There are obvious analogies to classical receiver combining techniques as equal gain, maximum ratio, and selection combining [HM05]. Distributed beamforming and space-time coding concepts are comparable to equal and maximum ratio combining concepts with an equal and optimal power distribution among the relay nodes. Those strategies rely on coherent addition at the receiver so that the relay nodes need to be phase synchronized. Therefore, they are very sensitive to the channel state information which results for a practical system in a non-negligible technical challenge. This problem is not so serious for relay selection strategies. Relay selection is comparable to the selection combining concept. Moreover, if all nodes participate and we do not optimize the power allocation, relay selection strategies are more power efficient.

Recently, some advanced studies on relay selection for unidirectional cooperative protocols are done. They study the relay selection problem with respect to the complexity, channel state information, and energy consumption. Furthermore, distributed solutions are proposed. In more detail, Lin, Erkip, and Stefanov consider in [LES06] coded user cooperation with the pairwise error probability as performance metric. They introduce the concept of a user cooperation decision parameter based on the user cooperation gain, which is used to define user cooperative regions. Hunter and Nosratinia propose in [HN04] various distributed partner allocation protocols where a certain number of cooperating users is selected according to a random strategy, a strategy based on the receive SNR, and a strategy with a fixed priority list of nodes in its neighborhood. Relay selection with respect to the energy consumption, where the cost of acquiring the CSI is factored in, is considered in [MMMZ06]. Moreover, in [BKRL06] opportunistic relay selection is presented, which is a distributed method based on channel measurements at the relay nodes.

However, to the best of our knowledge, relay selection for bidirectional relaying has not been considered yet. In this thesis we assume a centralized decision for a relay node based on the achievable rate regions of all relay nodes. Since bidirectional communication is characterized by two rates, the decision for a "best relay node" is a vector optimization problem. Therefore, it is possible that we have several Pareto optima which may correspond to different relay nodes. We will see in the next section that we can further improve the performance if we allow time-sharing between the usage of different relay nodes. The decision for one Pareto optimum and its corresponding relay node may depend on other design aspects like the throughput optimal resource allocation policy of the previous section.

If we consider time-variant block-fading channels with sufficient block-length so that we can apply relay selection based on the achievable rate region for each channel state, the average achievable rate region grows with the number of relay nodes N. For that goal we consider independent stationary and ergodic block-fading channel processes so that the average achievable rates are given by the ergodic rate region. Thereby, we do not optimize the power allocation over the fading states. In Section 2.4.2 we show that in the iid Rayleigh fading case the growth of both maximal unidirectional achievable rates for bidirectional relaying with relay selection and equal time division is asymptotically equal to $\frac{\ln(\ln(N))}{2\ln(2)}$. We will see that the scaling law can be used to asymptotically upper and lower bound the sum of any rate pair on the ergodic achievable rate regions for the equal and optimal time division case since we can upper and lower bound the sum-rate using the maximal unidirectional ergodic rates of the equal time division case.

2.4.1 Relay Selection Criterion

Here, we improve the performance by choosing the "best relay node" in a network where N relay nodes with possibly different relay power constraints offer support for the bidirectional relay communication between the nodes 1 and 2. We will see that the "best relay node" need not be optimal for all boundary rate pairs. Accordingly, the relay selection criterion has to be subtle enough to identify the optimal relay node for each boundary rate pair. For this purpose we generically extend for the n-th relay node the notation of the already introduced variables by an additional subscript index, e.g. $h_{1,n}$ denotes the channel between node 1 and the n-th relay node, etc.

We first argue the reasoning of the relay selection criterion by means of the equal time division case. After that we briefly discuss the corresponding formulas for the optimal time division case, where exactly the same arguments apply. It shows that these concepts can be easily generalized or applied to relay selection problems in other networks.

Equal Time Division

With the first proposition we want to raise awareness of a relay selection criterion based on the two-dimensional rate pair $\boldsymbol{R} = [R_1, R_2]$.

Proposition 2.32. *We consider a network with two relay nodes, $N = 2$, and signal-to-noise ratios $\gamma_1 = \gamma_2$ and $\gamma_{\mathrm{R},1} = \gamma_{\mathrm{R},2}$. Iff we have $|h_{1,1}| \leq |h_{1,2}|$ and $|h_{2,1}| \leq |h_{2,2}|$, then $\mathcal{R}_{\mathrm{BReq},1} \subseteq \mathcal{R}_{\mathrm{BReq},2}$ with equality iff we have $|h_{1,1}| = |h_{1,2}|$ and $|h_{2,1}| = |h_{2,2}|$.*

Proof. Since $R_{\overrightarrow{1\mathrm{R}},1} \leq R_{\overrightarrow{1\mathrm{R}},2}$, $R_{\overrightarrow{2\mathrm{R}},1} \leq R_{\overrightarrow{2\mathrm{R}},2}$, $R_{\Sigma,1} \leq R_{\Sigma,2}$, $R_{\overrightarrow{\mathrm{R}2},1}(\beta_1) \leq R_{\overrightarrow{\mathrm{R}2},2}(\beta_1)$, and $R_{\overrightarrow{\mathrm{R}1},1}(\beta_2) \leq R_{\overrightarrow{\mathrm{R}2},2}(\beta_2)$ for any $\beta_1, \beta_2 \in [0, 1]$ it follows that $\mathcal{C}_{\mathrm{MAC},1} \subseteq \mathcal{C}_{\mathrm{MAC},2}$ and $\mathcal{R}_{\mathrm{BC},1} \subseteq \mathcal{R}_{\mathrm{BC},2}$ and therefore $\mathcal{R}_{\mathrm{BReq},1} \subseteq \mathcal{R}_{\mathrm{BReq},2}$. Equality follows immediately. □

From the proposition we see that we have a "good channel state" if the corresponding rate region contains the other; but if one channel gain is larger and the other one is smaller, none of both regions contains the other. Some rate pairs can be achieved with certain channel states respectively relay nodes only. This implies that there need not be one relay node which is the best for the whole two-dimensional achievable rate region. We conclude that for a reasonable relay selection criterion we need to look at each achievable rate pair individually.

First it is easy to see that in a network of N relay nodes with arbitrary channels and individual transmit power constraints we can achieve the union of all individual achievable rate regions,

$$\mathcal{R}_{\mathrm{RSeq}} := \bigcup_{n=1}^{N} \mathcal{R}_{\mathrm{BReq},n} \tag{2.45}$$

by selecting the corresponding relay node that achieves a certain rate pair. Since the union of convex regions need not be convex, the rate region using relay selection $\mathcal{R}_{\mathrm{RSeq}}$ need not be convex. This is obtained if we additionally allow *time-sharing* between the usage of the relay nodes. This means that one has to switch between two relay nodes. The time-sharing method is exactly described by the convex hull operator so that the rate region using relay selection with time-sharing is given by

$$\mathcal{R}_{\mathrm{RSTSeq}} := \mathrm{co}\big(\mathcal{R}_{\mathrm{RSeq}}\big). \tag{2.46}$$

Since a convex set can be characterized by the convex hull of the rate pairs which maximize the weighted rate sum, the set $\mathcal{R}_{\mathrm{RSTSeq}}$ can be expressed as

$$\mathcal{R}_{\mathrm{RSTSeq}} = \mathrm{co}\Big(\Big\{\operatorname*{arg\,max}_{\boldsymbol{R}\in\mathcal{R}_{\mathrm{RSeq}}} \boldsymbol{R}\boldsymbol{q}^T : \ \boldsymbol{q} \in \mathbb{R}_+^2 \setminus \{\boldsymbol{0}\}\Big\} \cup \big\{\boldsymbol{0}\big\}\Big).$$

Let $\boldsymbol{R}^*_{\mathrm{eq},n}(\boldsymbol{q})$ denote the rate pair of the n-th relay node which maximizes the weighted rate sum for the weight vector $\boldsymbol{q}$ according to Theorem 2.10. Obviously, for any weight vector $\boldsymbol{q} \in \mathbb{R}_+^2 \setminus \{\boldsymbol{0}\}$ we have

$$\max_{\boldsymbol{R}\in\mathcal{R}_{\mathrm{RSeq}}} \boldsymbol{R}\boldsymbol{q}^T = \max_{n\in\{1,2,\dots,N\}} \boldsymbol{R}^*_{\mathrm{eq},n}(\boldsymbol{q})\boldsymbol{q}^T.$$

From the previous considerations we can conclude on the following relay selection criterion for bidirectional relaying with equal time division.

Relay Selection Criterion 2.33. *For any weight vector $\boldsymbol{q} \in \mathbb{R}_+^2 \setminus \{\boldsymbol{0}\}$ we have to select the relay node whose boundary rate pair $\boldsymbol{R}^*_{\mathrm{eq},n}(\boldsymbol{q})$ maximizes the weighted rate sum, which means that we have to do relay selection for each boundary rate pair individually. If there are multiple relay nodes which all achieve the weighted rate sum maximum, we have to apply time-sharing between the usage of the relay nodes with the corresponding rate pairs to achieve all rate pairs on the boundary of $\mathcal{R}_{\mathrm{RSTSeq}}$.*

Accordingly, let

$$\mathcal{N}_{\mathrm{eq}}(\boldsymbol{q}) := \Big\{ n \in \{1, 2, \dots, N\} \;:\; \boldsymbol{R}^*_{\mathrm{eq},n}(\boldsymbol{q})\boldsymbol{q}^T = \max_{n \in \{1,2,\dots,N\}} \boldsymbol{R}^*_{\mathrm{eq},n}(\boldsymbol{q})\boldsymbol{q}^T \Big\} \tag{2.47}$$

denote the set of optimal relay nodes for a weight vector $\boldsymbol{q} \in \mathbb{R}^2_+ \setminus \{\boldsymbol{0}\}$ with respect to the achievable rate region $\mathcal{R}_{\mathrm{RSTSeq}}$. If $\mathcal{N}_{\mathrm{eq}}(\boldsymbol{q})$ is a singleton, then there is only one optimal relay node. Otherwise we have to apply time-sharing between the usage of the relay nodes with the corresponding rate pairs to achieve all rate pairs on the boundary of $\mathcal{R}_{\mathrm{RSTSeq}}$. Accordingly, the set

$$\mathcal{R}^*_{\mathrm{RSTSeq}}(\boldsymbol{q}) := \mathrm{co}\big(\{\boldsymbol{R}^*_{\mathrm{eq},n}(\boldsymbol{q}) \;:\; n \in \mathcal{N}_{\mathrm{eq}}(\boldsymbol{q})\}\big) \tag{2.48}$$

denotes the set of all optimal rate pairs for the weight vector $\boldsymbol{q}$. Note that if $\mathcal{N}_{\mathrm{eq}}(\boldsymbol{q})$ is not a singleton, than time-sharing between two relay nodes will be sufficient. This is because the elements of $\mathcal{R}_{\mathrm{RSTSeq}}$ are two-dimensional rate pairs so that time-sharing between the relay node which has the largest R_1 and the relay node which has the largest R_2 can achieve any rate pair in $\mathcal{R}_{\mathrm{RSTSeq}}$.

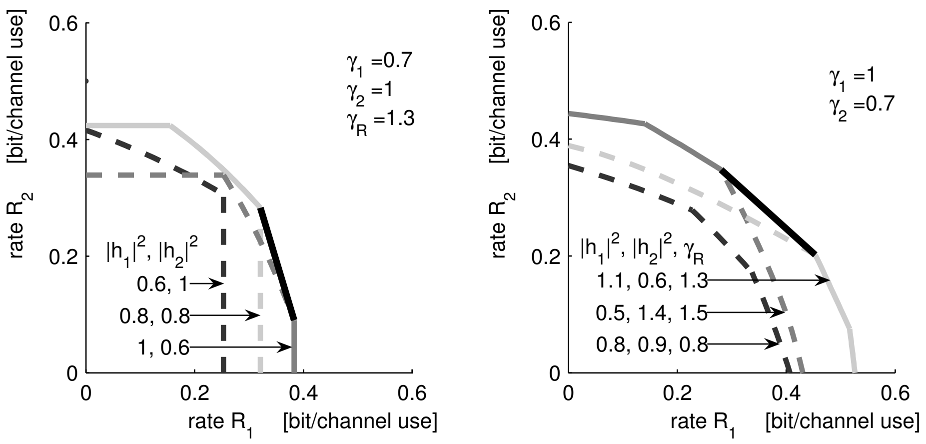
For weight vectors $\boldsymbol{q}$ which characterize the sum-rate maximum ($q_1 = q_2$), the maximal unidirectional rate R_1 ($q_1 > 0$ and $q_2 = 0$), and the maximal unidirectional rate R_2 ($q_1 = 0$ and $q_2 > 0$) it could be possible that multiple rate pairs on the boundary of $\mathcal{R}_{\mathrm{BReq},n}$ with $n \in \mathcal{N}_{\mathrm{eq}}(\boldsymbol{q})$ achieve the weighted rate sum maximum, cf. $\mathcal{R}^*_{\Sigma\mathrm{eq},n}$, $\mathcal{R}^*_{1\mathrm{eq},n}$, and $\mathcal{R}^*_{2\mathrm{eq},n}$. If this is the case we have take for $\boldsymbol{R}^*_{\mathrm{eq},n}(\boldsymbol{q})$ in (2.48) any rate pair in $\mathcal{R}^*_{\Sigma\mathrm{eq},n}$, $\mathcal{R}^*_{1\mathrm{eq},n}$, or $\mathcal{R}^*_{2\mathrm{eq},n}$ respectively.

Figure 2.10 (a) illustrates the rate region using relay selection in a network with $N = 3$ relay nodes and equal time division. Note that some rate pairs can be achieved by time-sharing between two rate pairs of different relay nodes only. This means that one has to switch between two relay nodes.

Optimal Time Division

For the optimal time division case the same arguments apply as for the equal time division case. Some rate pairs can be achieved by certain relay nodes only. Therefore, the achievable rate region using relay selection is given by the union of all individual bidirectional achievable rate regions

$$\mathcal{R}_{\mathrm{RSopt}} := \bigcup_{n=1}^{N} \mathcal{R}_{\mathrm{BRopt},n}.$$



(a) Achievable rate regions for equal time division: $\mathcal{R}_{\mathrm{RSTSeq}}$ (solid line) and $\mathcal{R}_{\mathrm{BReq},n}$ for $n = 1, 2, 3$ (dashed line).

(b) Achievable rate regions for optimal time division: $\mathcal{R}_{\mathrm{RSTSopt}}$ (solid line) and $\mathcal{R}_{\mathrm{BRopt},n}$ for $n = 1, 2, 3$ (dashed line).

Figure 2.10: Achievable rate regions of relay selection with time-sharing for $N = 3$ relay nodes. Note that some rate pairs can be only achieved by time-sharing between two rate pairs of different relay nodes.

If we additionally allow *time-sharing* between the usage of the relay nodes, then the achievable rate region is convex and can be expressed as

$$\begin{aligned}\mathcal{R}_{\mathrm{RSTSopt}} :=& \mathrm{co}\big(\mathcal{R}_{\mathrm{RSopt}}\big) \\ =& \mathrm{co}\Big(\Big\{\operatorname*{arg\,max}_{\boldsymbol{R}\in\mathcal{R}_{\mathrm{RSopt}}} \boldsymbol{R}\boldsymbol{q}^T : \boldsymbol{q} \in \mathbb{R}_+^2 \setminus \{\boldsymbol{0}\}\Big\} \cup \{\boldsymbol{0}\}\Big).\end{aligned} \tag{2.49}$$

Let $\boldsymbol{R}^*_{\mathrm{opt},n}(\boldsymbol{q})$ denote the rate pair of the n-th relay node which maximizes the weighted rate sum for the weight vector $\boldsymbol{q} = [q_1, q_2]$ according to Theorem 2.16. Then for any weight vector $\boldsymbol{q} \in \mathbb{R}_+^2 \setminus \{\boldsymbol{0}\}$ we have

$$\max_{\boldsymbol{R}\in\mathcal{R}_{\mathrm{RSopt}}} \boldsymbol{R}\boldsymbol{q}^T = \max_{n\in\{1,2,\dots,N\}} \boldsymbol{R}^*_{\mathrm{opt},n}(\boldsymbol{q})\boldsymbol{q}^T.$$

Again, we propose to do relay selection for each boundary rate pair individually.

Relay Selection Criterion 2.34. *For any weight vector $\boldsymbol{q} \in \mathbb{R}_+^2 \setminus \{\boldsymbol{0}\}$ we have to select the relay node whose boundary rate pair $\boldsymbol{R}^*_{\mathrm{opt},n}(\boldsymbol{q})$ maximizes the weighted rate sum. If there are multiple relay nodes which all achieve the weighted rate sum maximum, we have to apply time-sharing between the usage of the relay nodes with the corresponding rate pairs to achieve all rate pairs on the boundary of $\mathcal{R}_{\mathrm{RSTSopt}}$.*

Similarly, we can define for any weight vector $\boldsymbol{q} \in \mathbb{R}_+^2 \setminus \{\boldsymbol{0}\}$ the set

$$\mathcal{N}_{\mathrm{opt}}(\boldsymbol{q}) := \Big\{ n \in \{1,2,\ldots,N\} \,:\, \boldsymbol{R}^*_{\mathrm{opt},n}(\boldsymbol{q})\boldsymbol{q}^T = \max_{n\in\{1,2,\ldots,N\}} \boldsymbol{R}^*_{\mathrm{opt},n}(\boldsymbol{q})\boldsymbol{q}^T \Big\}, \quad (2.50)$$

which characterizes the optimal relay nodes with respect to the achievable rate region $\mathcal{R}_{\mathrm{RSTSopt}}$. As before, there is only one optimal relay node if $\mathcal{N}_{\mathrm{opt}}(\boldsymbol{q})$ is a singleton. Otherwise we have to apply time-sharing between the usage of the relay nodes with the corresponding rate pairs to achieve all rate pairs on the boundary of $\mathcal{R}_{\mathrm{RSTSopt}}$. Accordingly, the set

$$\mathcal{R}^*_{\mathrm{RSTSopt}}(\boldsymbol{q}) := \mathrm{co}\big(\{\boldsymbol{R}^*_{\mathrm{opt},n}(\boldsymbol{q}) \,:\, n \in \mathcal{N}_{\mathrm{opt}}(\boldsymbol{q})\}\big)$$

denotes the set of all optimal rate pairs for the weight vector $\boldsymbol{q}$. Since the elements of $\mathcal{R}_{\mathrm{RSTSopt}}$ are two-dimensional, there exist two relay nodes so that time-sharing between those two relay nodes will be sufficient to achieve any rate pair in $\mathcal{R}^*_{\mathrm{RSTSopt}}(\boldsymbol{q})$.

Figure 2.10 (b) illustrates the rate region using relay selection with time-sharing for a scenario with $N = 3$ relay nodes and optimal time division. Thereby note that some rate pairs can be achieved by time-sharing between two rate pairs of different relay nodes only.

In the next section, we study the scaling of the diversity gain in a time-variant fading context.

2.4.2 Scaling Law of the Ergodic Rate Region

We now consider a network with N relay nodes in the presence of time-variant fading. Therefore, we assume independent stationary and ergodic block-fading channel processes $\{h_{k,n}[m]\}_m$, $k = 1, 2$, $n = 1, 2, \ldots, N$. The channel remains constant over blocks consisting of M symbol periods and changes independently from block to block. Thereby, we assume a block length M so that the error-free coding assumption is reasonable. We do not optimize the power allocation over time-varying channel states, which we regard as future work. Accordingly, we are interested in the time average of the achievable rates obtained at each fading instant, which is given by the ergodic achievable rate region. We first study the ergodic rate region for the equal time division case. After that we will extend the results for the ergodic rate region with the optimal time division between the phases.

Equal Time Division

Let $\mathcal{R}_{\mathrm{RSTSeq}}(\boldsymbol{h})$ denote the achievable rate region using relay selection with time-sharing for the channel state $\boldsymbol{h}$ according to (2.46). Then let $\mathcal{R}^*_{\mathrm{RSTSeq}}(\boldsymbol{q}, \boldsymbol{h})$ denote the set of optimal achievable rates for a weight vector $\boldsymbol{q}$ and channel state $\boldsymbol{h}$. We characterize the ergodic rate region $\overline{\mathcal{R}_{\mathrm{RSTSeq}}}$ by its boundary. To this end, we choose for any weight vector $\boldsymbol{q}$ and channel state $\boldsymbol{h}$ a rate pair in $\mathcal{R}^*_{\mathrm{RSTSeq}}(\boldsymbol{q},\boldsymbol{h})$. However, if $\mathcal{R}^*_{\mathrm{RSTSeq}}(\boldsymbol{q},\boldsymbol{h})$ is not a singleton we have to consider any possible rate allocation for this weight vector and channel state. We sum this up with a definition of a rate allocation policy. For a weight vector $\boldsymbol{q}$ let $\Xi_{\mathrm{eq}}(\boldsymbol{q})$ denote a rate allocation policy $\big\{\boldsymbol{R}(\boldsymbol{h})\big\}_{\boldsymbol{h}}$ that decides for any channel realization $\boldsymbol{h}$ for a rate pair $\boldsymbol{R}(\boldsymbol{h}) \in \mathcal{R}^*_{\mathrm{RSTSeq}}(\boldsymbol{q},\boldsymbol{h})$. Then

$$\overline{\mathcal{R}^*_{\mathrm{RSTSeq}}(\boldsymbol{q})} := \Big\{\boldsymbol{R} \in \mathbb{R}^2_+ \ : \ \text{there exists a policy } \Xi_{\mathrm{eq}}(\boldsymbol{q}) \text{ with } \boldsymbol{R} = \mathbb{E}\big\{\boldsymbol{R}(\boldsymbol{h})\big\}\Big\}$$

denotes the set of ergodic rates we can achieve for a weight vector $\boldsymbol{q} \in \mathbb{R}^2_+ \setminus \{\boldsymbol{0}\}$. Therewith we can express the ergodic rate region as follows

$$\overline{\mathcal{R}_{\mathrm{RSTSeq}}} := \mathrm{co}\Big(\Big\{[R_1, R_2] \in \mathbb{R}^2_+ \ : \ \text{there exists } \boldsymbol{q} \in \mathbb{R}^2_+ \setminus \{\boldsymbol{0}\} \text{ with } R_1 \le \overline{R_1},\ R_2 \le \overline{R_2},\ [\overline{R_1}, \overline{R_2}] \in \overline{\mathcal{R}^*_{\mathrm{RSTSeq}}(\boldsymbol{q})}\Big\}\Big).$$

In the following let $\overline{\boldsymbol{R}^*_{\mathrm{RSTSeq}}(\boldsymbol{q})} \in \overline{\mathcal{R}_{\mathrm{RSTSeq}}}$ denote a rate pair on the boundary that achieves the weighted rate sum maximum with weight vector $\boldsymbol{q} \in \mathbb{R}^2_+ \setminus \{\boldsymbol{0}\}$, this means we have $\overline{\boldsymbol{R}^*_{\mathrm{RSTSeq}}(\boldsymbol{q})}\boldsymbol{q}^T = \max_{\boldsymbol{R}\in\overline{\mathcal{R}_{\mathrm{RSTSeq}}}} \boldsymbol{R}\boldsymbol{q}^T$. Due to the time-sharing operation (convex hull) the rate pair need not be unique.

We now study how the ergodic rate region using relay selection increases with an increasing number of relay nodes. To this end we look at the maximal unidirectional ergodic rates, which allows us to upper and lower bound the sum of any ergodic rate pair on the boundary of $\overline{\mathcal{R}_{\mathrm{RSTSeq}}}$. Then for iid Rayleigh fading channels we can conclude from a scaling law for the maximal unidirectional ergodic rates how the rates of rate pairs on the boundary of the ergodic rate region increase with increasing number of relay nodes.

According to (2.22a) and (2.22b) let $R^*_{1\mathrm{eq},n} := {}^1\!/\!_2 \min\{R_{\overrightarrow{1\mathrm{R}},n}, R_{\overrightarrow{\mathrm{R}2},n}(1)\}$ and $R^*_{2\mathrm{eq},n} := {}^1\!/\!_2 \min\{R_{\overrightarrow{2\mathrm{R}},n}, R_{\overrightarrow{\mathrm{R}1},n}(1)\}$ denote the maximal unidirectional rate of the n-th relay node. Therewith, we get the maximal unidirectional rates using relay selection as follows

$$R^*_{k\mathrm{RSeq}} := \max_{n\in\{1,2,\dots,N\}} R^*_{k\mathrm{eq},n}, \quad k = 1, 2.$$

We first state a technical lemma where we do some algebra for random variables. After that, we will use the lemma to characterize the statistics of the maximal unidirectional rates using relay selection. The lemma is stated for arbitrary real channel distributions but most of the succeeding results are obtained for the Rayleigh distributed fading channels.

Lemma 2.35. *Let X_n and Y_n, $n = 1, 2, \dots, N$, be pairwise independent non-negative real random variables with probability density functions (pdf) $f_{X_n}(x_n)$ and $f_{Y_n}(y_n)$. Further, let $F_{X_n}(x_n)$ and $F_{Y_n}(y_n)$ denote the corresponding cumulative distribution functions (cdf). Then the random variable $Z_n := \min\{a_n X_n, b_n Y_n\}$ with positive weights a_n and b_n has the pdf*

$$f_{Z_n}(z_n) = \tfrac{1}{a_n} f_{x_n}(\tfrac{z_n}{a_n})(1 - F_{y_n}(\tfrac{z_n}{b_n})) + \tfrac{1}{b_n} f_{y_n}(\tfrac{z_n}{b_n})(1 - F_{x_n}(\tfrac{z_n}{a_n}))$$

and the cdf

$$F_{Z_n}(z_n) = 1 - (1 - F_{x_n}(\tfrac{z_n}{a_n}))(1 - F_{y_n}(\tfrac{z_n}{b_n})).$$

The maximum $Z := \max\{Z_1, Z_2, \dots, Z_N\}$ has the pdf

$$f_Z(z) = \sum_{n=1}^{N} f_{Z_n}(z) \prod_{j=1, j\neq n}^{N} F_{Z_j}(z).$$

And finally,

$$f_R(R) = 2\ln(2) 2^{2R} f_Z(2^{2R} - 1)$$

specifies the pdf of the random variable $R = \frac{1}{2}\log[1 + Z]$ as a function of the random variable Z.

Proof. The proof can be found in Appendix 2.8.22. □

Remark 2.36. *It is interesting to observe that $Z = \min\{aX, bY\}$ in the case of exponential distributions $F_x(x) = 1 - \mathrm{e}^{-x/\sigma_x^2}$ and $F_y(y) = 1 - \mathrm{e}^{-y/\sigma_y^2}$ is again exponential distributed with $F_z(z) = 1 - \mathrm{e}^{-z/\lambda}$ and $\lambda = \frac{1}{a\sigma_x^2} + \frac{1}{b\sigma_y^2}$.*

Before deriving the asymptote of the growth of the sum of the ergodic rates on the boundary of the ergodic rate region using relay selection, we first look at the probability that the relay selection can be drawn by Proposition 2.32. In other words, we are interested in the probability that there exists a relay node η whose rate region $\mathcal{R}_{\mathrm{BReq},\eta}$ contains the regions of all others, i.e. $\mathbb{P}(\exists\, \eta : \mathcal{R}_{\mathrm{RSeq}} = \mathcal{R}_{\mathrm{BReq},\eta})$. Such a relay node exists iff we have $\mathcal{N}_{\mathrm{eq}}(\boldsymbol{q}) = \{\eta\}$ for all $\boldsymbol{q}$. The intuition says that with an increasing number of relay nodes it will be less probable.

Proposition 2.37. *For bidirectional relaying with relay selection and equal time division let $f_{|h_{k,n}|}(|h_{k,n}|)$ and $F_{|h_{k,n}|}(|h_{k,n}|)$ denote the pdf and cdf of the absolute value of the channel $h_{k,n}$ defined on the support set $\mathcal{S}_{|h_{k,n}|}$. In a network with independent channels where each relay node has the same power constraint γ_{R}, the probability that there exists a relay node*

$\eta \in \{1, 2, \ldots, N\}$ whose rate region contains the rate regions of all others is given by

$$\begin{aligned}\mathbb{P}(\exists\, \eta : \mathcal{R}_{\mathrm{RSeq}} = \mathcal{R}_{\mathrm{BR},\eta}) &= \sum_{\eta=1}^{N} \mathbb{P}\big(\mathcal{R}_{\mathrm{BReq},\eta} \supseteq \bigcup_{n=1}^{N} \mathcal{R}_{\mathrm{BReq},n}\big)\\ &= \sum_{\eta=1}^{N} \prod_{k=1}^{2} \int_{\mathcal{S}_{|h_{k,n}|}} \prod_{n=1,\, n\neq\eta}^{N} F_{|h_{k,n}|}(|h_{k,\eta}|) f_{|h_{k,\eta}|}(|h_{k,\eta}|)\, d|h_{k,\eta}|.\end{aligned}$$

In the case of iid Rayleigh fading $h_{k,n} \sim \mathcal{CN}(0, \sigma_k^2)$ we have

$$\mathbb{P}(\exists\, \eta : \mathcal{R}_{\mathrm{RSeq}} = \mathcal{R}_{\mathrm{BReq},\eta}) = \frac{1}{N}.$$

Proof. The proof can be found in Appendix 2.8.23. □

In the case of iid Rayleigh fading the intuition is proved. We see that with an increasing number of relay nodes the Proposition 2.32 is more often not applicable. In the next theorem we characterize the distributions of the maximal unidirectional rates in a network with independent Rayleigh distributed fading channels.

Theorem 2.38. *For bidirectional relaying with relay selection and equal time division, we assume independently distributed channel coefficients $h_{k,n} \sim \mathcal{CN}(0, \sigma_{k,n}^2)$ for all $k = 1, 2$ and $n = 1, 2, \ldots, N$. Then the probability density function of the maximal unidirectional rate using relay selection $R^*_{k\mathrm{RSeq}}$, $k = 1, 2$, is given by*

$$\begin{aligned}f(R^*_{k\mathrm{RSeq}}) = 2\ln(2) \sum_{n=1}^{N} \lambda_{k,n} \exp\big((1 - 2^{2R^*_{k\mathrm{RSeq}}})\lambda_{k,n}\big) 2^{2R^*_{k\mathrm{RSeq}}}\\ \prod_{m=1, m\neq n}^{N} \Big(1 - \exp\big((1 - 2^{2R^*_{k\mathrm{RSeq}}})\lambda_{k,m}\big)\Big)\end{aligned}$$

with $\lambda_{1,n} = \frac{1}{\gamma_1 \sigma_{1,n}^2} + \frac{1}{\gamma_{\mathrm{R}} \sigma_{2,n}^2}$ and $\lambda_{2,n} = \frac{1}{\gamma_2 \sigma_{2,n}^2} + \frac{1}{\gamma_{\mathrm{R}} \sigma_{1,n}^2}$.

Proof. The proof can be found in Appendix 2.8.24. □

The theorem allows us to calculate the ergodic maximal unidirectional rates.

Corollary 2.39. *The ergodic maximal unidirectional rate* $\overline{R^*_{k\mathrm{RSeq}}} = \mathbb{E}\{R^*_{k\mathrm{RSeq}}\}$*,* $k = 1, 2$*, for bidirectional relaying with relay selection and equal time division in the case of independent Rayleigh distributed fading is given by*

$$\overline{R^*_{k\mathrm{RSeq}}} = \sum_{n=1}^{N} \frac{\lambda_{k,n}}{2\ln(2)} \Big[\frac{\exp(\lambda_{k,n})\mathrm{E}_1(\lambda_{k,n})}{\lambda_{k,n}} + \sum_{m=1}^{N-1} (-1)^m \sum_{\mathcal{L}\subseteq\mathcal{J}_n, |\mathcal{L}|=m} \frac{\exp(\lambda_{k,n} + \sum_{l\in\mathcal{L}} \lambda_{k,l})\mathrm{E}_1(\lambda_{k,n} + \sum_{l\in\mathcal{L}} \lambda_{k,l})}{\lambda_{k,n} + \sum_{l\in\mathcal{L}} \lambda_{k,l}} \Big] \tag{2.51}$$

with the exponential integral $\mathrm{E}_1(x) = \int_x^\infty \frac{\mathrm{e}^{-t}}{t} dt$*. In the last sum we have to sum over all subsets* $\mathcal{L}$ *of the index set* $\mathcal{J}_n = \{1, 2, \dots, N\} \setminus \{n\}$ *with cardinality* $|\mathcal{L}| = m$*. If we have* $\lambda_{k,n} = \lambda_k$ *for all* $n = 1, 2, \dots N$*, then* (2.51) *results in*

$$\overline{R^*_{k\mathrm{RSeq}}} = N \sum_{n=0}^{N-1} \binom{N-1}{n} \frac{\exp((n+1)\lambda_k)\mathrm{E}_1((n+1)\lambda_k)}{(-1)^n(n+1)2\ln(2)}. \tag{2.52}$$

Proof. The proof can be found in Appendix 2.8.25. □

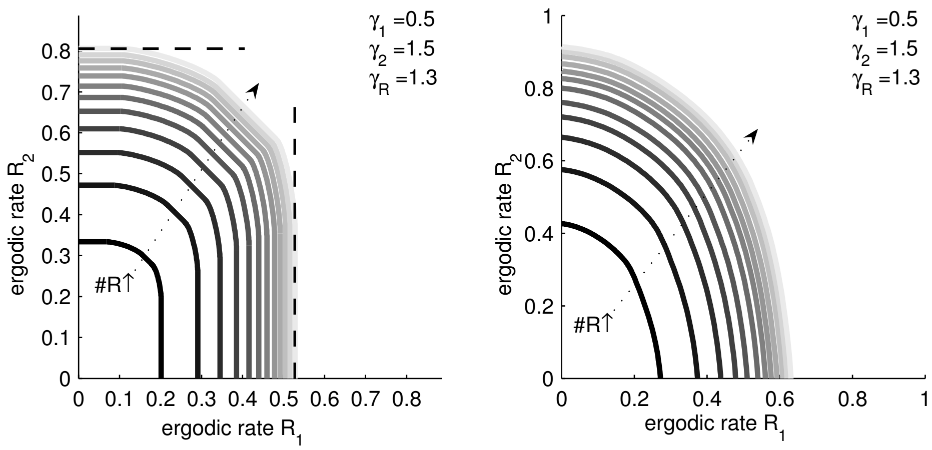
In Figure 2.11 (a) we illustrate the enlargement of the ergodic rate region due to relay selection for the equal time division case. Thereby, we assume equal relay transmit powers and iid Rayleigh fading. We see that the gain decreases with increasing number of relays. In the following we are interested in the scaling law of this growth. For its derivation, we need the analytical expression for the maximal unidirectional ergodic rate of the previous corollary.

Since $\overline{\mathcal{R}_{\mathrm{RSTSeq}}}$ is convex, we can upper and lower bound the sum-rate of any boundary rate pair $\overline{\boldsymbol{R}^*_{\mathrm{RSTSeq}}(\boldsymbol{q})}$ of the ergodic rate region as follows

$$\min\{\overline{R^*_{1\mathrm{RSeq}}}, \overline{R^*_{2\mathrm{RSeq}}}\} \;\leq\; \big|\big|\overline{\boldsymbol{R}^*_{\mathrm{RSTSeq}}(\boldsymbol{q})}\big|\big|_1 \;\leq\; \sum_{k=1}^{2} \overline{R^*_{k\mathrm{RSeq}}}. \tag{2.53}$$

The first inequality holds because $\overline{\mathcal{R}_{\mathrm{RSTSeq}}}$ is convex and the sum of any rate pair from the time-sharing between the rate pairs $[\overline{R^*_{1\mathrm{RSeq}}}, 0]$ and $[0, \overline{R^*_{2\mathrm{RSeq}}}]$ already fulfills the condition. The second inequality follows from the fact that $\overline{R^*_{1\mathrm{RSeq}}}$ and $\overline{R^*_{2\mathrm{RSeq}}}$ are both the maximal unidirectional ergodic rates so that the sum of any rate pair on the boundary of $\overline{\mathcal{R}_{\mathrm{RSTSeq}}}$ has to be smaller.

We now derive a scaling law for the growth of the sum-rate of any rate pair on the boundary of the ergodic rate region $\overline{\mathcal{R}_{\mathrm{RSTSeq}}}$ with increasing number of relay nodes in an iid Rayleigh fading scenario. To this end, we present in the next theorem an upper and lower bound on the maximal unidirectional ergodic rate $\overline{R^*_{k\mathrm{RSeq}}}$, $k = 1, 2$, which are asymptotically tight.



(a) $\overline{\mathcal{R}_{\mathrm{RSTSeq}}}$ (b) $\overline{\mathcal{R}_{\mathrm{RSTSopt}}}$

Figure 2.11: Growth of the ergodic rate region with increasing number of relay nodes $N = 1, 2, \dots, 12$ for the equal and optimal time division case. In the left plot we depicted the maximal unidirectional ergodic rates according to Corollary 2.39 for $N = 12$.

Theorem 2.40. *For bidirectional relaying with relay selection and equal time division we assume independently identical distributed channel coefficients $h_{k,n} \sim \mathcal{CN}(0, \sigma^2)$, equal signal-to-noise ratios $\gamma_{\mathrm{R},n} = \gamma_{\mathrm{R}}$, and coefficients $\lambda_k = \frac{\gamma_k+\gamma_{\mathrm{R}}}{\sigma^2\gamma_k\gamma_{\mathrm{R}}}$ for all $k = 1, 2$ and $n = 1, 2, \dots, N$. Then we can upper and lower bound the maximal unidirectional ergodic rate $\overline{R^*_{k\mathrm{RSeq}}}$, $k = 1, 2$, as follows*

$$\overline{R^*_{k\mathrm{RSeq}}} \geq \frac{1-\mathrm{e}^{-a}}{2} \log\Big(1 + \frac{1}{\lambda_k} \ln\big(\frac{N}{a}\big)\Big),$$

$$\overline{R^*_{k\mathrm{RSeq}}} \leq \frac{\mathrm{e}^{-b} + b}{2} \log\Big(1 + \frac{1}{\lambda_k} \ln(\frac{N}{b})\Big) + \frac{b}{2} \log\Big(1 + \frac{1}{\lambda_k + \ln\big(\frac{N}{b}\big)}\Big)$$

with arbitrary $a, b \in (0, N)$. The asymptotic upper and lower bound meet if we choose $a \to \infty$ and $b \to 0$ when $N \to \infty$. Therefore, the growth of the maximal unidirectional ergodic rate is asymptotically equal $\frac{1}{2}\log(\ln(N))$.

Proof. The proof can be found in Appendix 2.8.26. □

Note that the asymptote on the growth of the maximal ergodic unidirectional rate is independent of λ_k and therefore independent of the the channel gain variance σ^2 and signal-to-noise ratios γ_{R} and γ_k.

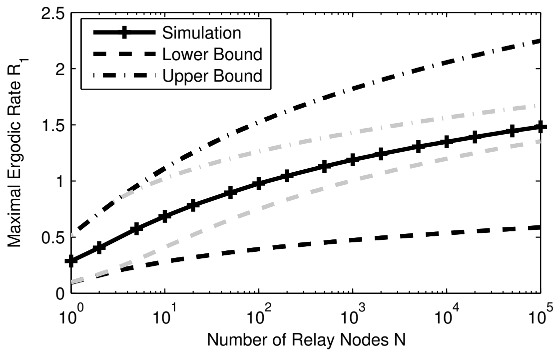


Figure 2.12: Growth of the maximal unidirectional ergodic rate $\overline{R^*_{1\mathrm{RSeq}}}$ using relay selection with an increasing number of relay nodes with upper and lower bounds with optimized (in gray) and fixed coefficients (in black) with $a = b = 1/2$, $\gamma_\mathrm{R} = 1.3$, $\gamma_1 = 1$, and $\sigma^2 = 1$.

Remark 2.41. *Since the derivation applies to any AWGN achievable rate calculation of a system where a selection combining technique between N independently exponentially distributed SNRs is utilized, the scaling law of the growth of the ergodic rate of such a system has always a $\log(\ln(N))$ asymptote.*

Since we get the coefficients a and b from the analysis in the proof, they have no physical meaning. Therefore, we can optimize the bounds for any fixed number of relay nodes N by optimizing the coefficients a and b. In Figure 2.12 we plotted the growth of the maximal unidirectional ergodic rate $\overline{R^*_{1\mathrm{RSeq}}}$ using relay selection with the number of relay nodes N (Monte Carlo Simulation) and the upper and lower bounds with optimized and fixed coefficients. The bounds are loose but they have the same growth rates. Furthermore, we see that the bounds with the optimized coefficients ("slowly") converge.

Finally, as a direct consequence of the previous theorem and the inequalities in (2.53) we can conclude on the asymptotic growth of the ergodic rate region $\overline{\mathcal{R}_\mathrm{RSTSeq}}$.

Corollary 2.42. *For bidirectional relaying using relay selection with time-sharing and equal time division in an iid Rayleigh fading scenario with $\gamma_{\mathrm{R},n} = \gamma_\mathrm{R}$ the sum of any ergodic rate pair on the boundary of the ergodic rate region $\overline{\mathcal{R}_\mathrm{RSTSeq}}$ grows with $\Theta(\log(\log(N)))$. In more detail, for any $\boldsymbol{q} \in \mathbb{R} \setminus \{\boldsymbol{0}\}$ we can asymptotically lower and upper bound the asymptotic growth of the sum-rate of any boundary rate pair $\overline{\boldsymbol{R}^*_\mathrm{RSTSeq}(\boldsymbol{q})}$ as follows*

$$\liminf_{N\to\infty} \frac{\left|\left|\overline{\boldsymbol{R}^*_\mathrm{RSTSeq}(\boldsymbol{q})}\right|\right|_1}{\log(\ln(N))} \geq \frac{1}{2}, \qquad \limsup_{N\to\infty} \frac{\left|\left|\overline{\boldsymbol{R}^*_\mathrm{RSTSeq}(\boldsymbol{q})}\right|\right|_1}{\log(\ln(N))} \leq 1.$$

The difference between the asymptotic lower and upper bound is exactly the difference between a unidirectional protocol and the bidirectional protocol with an improved spectral efficiency. As we see in Figure 2.10, the increase in spectral efficiency depends on the rate ratio which vanishes in the case of maximal unidirectional ergodic rates. Finally, we want to remark that the maximal unidirectional ergodic rates evolve according to the same asymptotic scaling law as found in [DSG+03] using distributed beamforming approach with a two-phase separation constraint.

Optimal Time Division

Let $\mathcal{R}_{\mathrm{RSTSopt}}(\boldsymbol{h})$ denote the bidirectional achievable rate region using relay selection with time-sharing and optimal time division between the phases for the channel state $\boldsymbol{h}$ according to (2.49). Then let $\mathcal{R}^*_{\mathrm{RSTSopt}}(\boldsymbol{q},\boldsymbol{h})$ denote the set of optimal achievable rates for a weight vector $\boldsymbol{q}$ and channel state $\boldsymbol{h}$. As for the equal time division case, we characterize the ergodic rate region by the rate pairs on its boundary. Therefore, we choose for any channel state $\boldsymbol{h}$ a rate pair in $\mathcal{R}^*_{\mathrm{RSTSopt}}(\boldsymbol{q},\boldsymbol{h})$. However, if $\mathcal{R}^*_{\mathrm{RSTSopt}}(\boldsymbol{q},\boldsymbol{h})$ is not a singleton, we have to take any possible rate allocation into account. Again, we sum this up with the definition of a rate allocation policy. For a weight vector $\boldsymbol{q}$ let $\Xi_{\mathrm{opt}}(\boldsymbol{q})$ denote a rate allocation policy $\{\boldsymbol{R}(\boldsymbol{h})\}_{\boldsymbol{h}}$ that decides for any channel realization $\boldsymbol{h}$ for a rate pair $\boldsymbol{R}(\boldsymbol{h}) \in \mathcal{R}^*_{\mathrm{RSTSopt}}(\boldsymbol{q},\boldsymbol{h})$. Then

$$\overline{\mathcal{R}^*_{\mathrm{RSTSopt}}(\boldsymbol{q})} := \Big\{\boldsymbol{R} \in \mathbb{R}^2_+ \ : \ \text{there exists a policy } \Xi_{\mathrm{opt}}(\boldsymbol{q}) \text{ with } \boldsymbol{R} = \mathbb{E}\big\{\boldsymbol{R}(\boldsymbol{h})\big\}\Big\}$$

denotes the set of ergodic rates we can achieve for a weight vector $\boldsymbol{q} \in \mathbb{R}^2_+ \setminus \{\boldsymbol{0}\}$. Therewith, we can express the ergodic rate region as follows

$$\overline{\mathcal{R}_{\mathrm{RSTSopt}}} := \mathrm{co}\Big(\Big\{[R_1,R_2] \in \mathbb{R}^2_+ \ : \ \text{there exists } \boldsymbol{q} \in \mathbb{R}^2_+ \setminus \{\boldsymbol{0}\} \text{ with}$$
$$R_1 \le \overline{R_1},\ R_2 \le \overline{R_2},\ [\overline{R_1},\overline{R_2}] \in \overline{\mathcal{R}^*_{\mathrm{RSTSopt}}(\boldsymbol{q})}\Big\}\Big).$$

In the following let $\overline{\boldsymbol{R}^*_{\mathrm{RSTSopt}}(\boldsymbol{q})} \in \overline{\mathcal{R}_{\mathrm{RSTSopt}}}$ denote a rate pair on the boundary that achieves the weighted rate sum maximum with weight vector $\boldsymbol{q} \in \mathbb{R}^2_+ \setminus \{\boldsymbol{0}\}$, this means we have $\overline{\boldsymbol{R}^*_{\mathrm{RSTSopt}}(\boldsymbol{q})}\boldsymbol{q}^T = \max_{\boldsymbol{R}\in\overline{\mathcal{R}_{\mathrm{RSTSopt}}}} \boldsymbol{R}\boldsymbol{q}^T$. Due to the time-sharing operation (convex hull) the rate pair need not be unique.

Clearly, the ergodic rate region $\overline{\mathcal{R}_{\mathrm{RSTSopt}}}$ of relay selection with time-sharing and optimal time division grows as well with the number of relay nodes. In Figure 2.10 (b) we illustrate the enlargement for iid Rayleigh fading channels and equal relay powers. We see a similar behavior as for the equal time division case. For the derivation of the growth we can make use of the asymptotic result of the maximal unidirectional ergodic rates for the equal time division case. To this end, we upper and lower bound in the next proposition the sum of any rate pair on the boundary of $\overline{\mathcal{R}_{\mathrm{RSTSopt}}}$ in terms of the ergodic maximal unidirectional rates for the equal time division case.

Proposition 2.43. *For bidirectional relaying using relay selection with time-sharing and optimal time division we can upper and lower bound the sum-rate of any boundary rate pair* $\overline{\boldsymbol{R}^*_{\mathrm{RSTSopt}}(\boldsymbol{q})}$ *of the ergodic rate region* $\overline{\mathcal{R}_{\mathrm{RSTSopt}}}$ *as follows*

$$\min\left\{\overline{R^*_{1\mathrm{RSeq}}}, \overline{R^*_{2\mathrm{RSeq}}}\right\} \;\leq\; \left|\left|\overline{\boldsymbol{R}^*_{\mathrm{RSTSopt}}(\boldsymbol{q})}\right|\right|_1 \;\leq\; 2\sum_{k=1}^{2}\overline{R^*_{k\mathrm{RSeq}}}. \tag{2.54}$$

Proof. The proof can be found in Appendix 2.43. □

With the upper and lower bound we can directly apply the asymptotic result for the ergodic maximal unidirectional rates $\overline{R^*_{k\mathrm{RSeq}}}$ of Theorem 2.40 as we did for Corollary 2.42.

For bidirectional relaying using relay selection with time-sharing and equal time division in an iid Rayleigh fading scenario with $\gamma_{\mathrm{R},n} = \gamma_{\mathrm{R}}$ the sum of any ergodic rate pair on the boundary of the ergodic rate region $\overline{\mathcal{R}_{\mathrm{RSTSeq}}}$ grows with $\Theta(\log(\log(N)))$. In more detail, in the next corollary we can lower and upper bound the asymptotic growth of the sum-rate of any boundary rate pair $\overline{\boldsymbol{R}^*_{\mathrm{RSTSeq}}(\boldsymbol{q})}$ for any $\boldsymbol{q} \in \mathbb{R} \setminus \{\boldsymbol{0}\}$.

Corollary 2.44. *For bidirectional relaying using relay selection with time-sharing and equal time division in an iid Rayleigh fading scenario with* $\gamma_{\mathrm{R},n} = \gamma_{\mathrm{R}}$ *the sum of any ergodic rate pair on the boundary of the ergodic rate region* $\overline{\mathcal{R}_{\mathrm{RSTSopt}}}$ *grows with* $\Theta(\log(\log(N)))$. *In more detail, for any* $\boldsymbol{q} \in \mathbb{R}^2_+ \setminus \{\boldsymbol{0}\}$ *we can asymptotically lower and upper bound the growth of the sum-rate of any boundary rate pair* $\overline{\boldsymbol{R}^*_{\mathrm{RSTSopt}}(\boldsymbol{q})}$ *as follows*

$$\liminf_{N\to\infty}\frac{\left|\left|\overline{\boldsymbol{R}^*_{\mathrm{RSTSopt}}(\boldsymbol{q})}\right|\right|_1}{\log(\ln(N))} \geq \frac{1}{2}, \qquad \limsup_{N\to\infty}\frac{\left|\left|\overline{\boldsymbol{R}^*_{\mathrm{RSTSopt}}(\boldsymbol{q})}\right|\right|_1}{\log(\ln(N))} \leq 2.$$

It is interesting to see that although the ergodic rate region of the optimal time division case is always larger than for the equal time division case, the sum of any ergodic rate pair has the same asymptotic scaling law.

2.5 Piggyback a Common Relay Message

Nowadays, there is a trend to offer multiple wireless services at a single device, which is known as the convergence of wireless services. Up to now this is realized by the coexistence of multiple wireless transceiver chains each equipped with an own antenna. This requires a careful radio segmentation with adequate transmit spectrum masks that allow a sufficient isolation of the receivers to the transmitters with respect to the receivers sensitivity levels. With a convergence on the Data Link Layer and Physical Layer we can overcome such practical problems. The joint design on the Data Link Layer and Physical Layer allows the reduction of the number of transceiver chains in the device. Moreover, the reduction of RF front-ends is interesting for further miniaturization of the devices. In addition it allows a more efficient joint resource allocation, which results in a reduction of the energy consumption.

Accordingly, in this section we add a relay multicast communication to the bidirectional relaying protocol. This means that we assume that the Network Layer assigns two routing tasks to Data Link Layer. Accordingly, we are interested in an efficient joint resource allocation design for both routing tasks across the Physical and Data Link Layer. We consider again superposition encoding, which means that the relay node encodes its own message and superimposes the codeword of the relay multicast message on the bidirectional broadcast transmit signal. In accordance to this we say that the relay messages is *piggybacked* on the bidirectional relaying.

In the seminal work [GK00] Gupta and Kumar introduce the concept of *transport capacity*, which is a measure that factors in the distance between the source and destination pairs in a wireless network. The mean feat is to use this measure to derive upper and lower bounds of scaling laws which characterize the amount of information a network can transport in the limit of large number of nodes. A weak point of the work is that they considered concurrent transmissions as interference so that in [GK03, XK04] Gupta, Xie, and Kumar consider interference as information and therewith improve the scaling laws by studying large relay networks. In this context they raised the fundamental question *"How should nodes cooperate in maximizing information transfer in a wireless network?"* In [OLT07] Özgür, Lévêque, and Tse achieve a linear scaling law with a hierarchical cooperation scheme using distributed MIMO communication. Recently, in [XK07] Xie and Kumar extend their coding scheme to multi-source, multi-destination, multi-relay networks where they group nodes along multi-relay routes. Thereby, they merge routes if they intersect at any relay node so that they can apply a joint backward decoding scheme. They point out that the coding scheme depends on the topology and note that such a coding scheme excludes two-way multi-relay networks.

The afore mentioned works study information theoretic coding strategies for large relay networks. Since it is a hard problem to find the optimal coding strategy for an arbitrary network, for most networks the optimal communication strategy is not known. However, the promising benefits of already proposed relaying concepts for wireless networks encourage researcher to study routing, media access, and power control for such problems. Accordingly, there is recently a vast growing literature on power control problems for wireless cellular and multi-hop networks which considers the joint optimization of routing, scheduling, and power control problems, which allows the development of efficient centralized and distributed resource allocation algorithms. In [Bam98] Bambos points out that general power control design goals for wireless networks are to prolong battery life, mitigate interference, and maintain link quality. For the design of resource allocation policies usually a network flow model based on flow conservation is assumed. This means that separated information flows are considered. Then the optimization is done using utility functions per user, which are often assumed to be continuous, differentiable, concave, and increasing, e.g. the capacity function of a Gaussian channel. Then the main goal is to maximize the sum of the user utility functions, which is called the network utility maximization problem. In [XJB04], Xiao, Johansson, and Boyd introduce a generic formulation of an optimization problem for the simultaneous routing and power allocation problem for wireless networks, which allows them to derive efficient solution methods based on convex optimization theory. Algorithms which compute distributed or centralized power control solution with respect to routing and/or link scheduling for multi-hop networks can be found for instance, in [TE02, CS03, Chi04]. While in [NMR05] Neely, Modiano, and Rohrs study a dynamic routing and power allocation policy for wireless networks with time-varying channels based on the maximum differential backlog strategy of Tassiulas and Ephremides [TE93]. For an introduction in the theory and algorithms of resource allocation in wireless networks with a more comprehensive literature survey on this topic we refer on the textbook [SWB06].

However, our goal in this section is not to find another algorithm which solves efficiently a power or rate allocation problem. We are interested in finding some properties of the achievable rate region, which then could be used for improving the efficiency of a resource allocation policy according to some higher layer strategies. Moreover, we can identify the optimal decoding order at nodes 1 and 2 in the BC phase. Since we consider two simple routing tasks and single-antenna nodes, some of the following results can be astonishingly obtained in closed form. But we see from the derivations that this is already a tedious combinatorial problem so that we suppose that for joint optimization of more or more complicated[13] routing tasks the resource allocation problems can only be solved algorithmically.

In the following section we extend the bidirectional broadcast by an additional relay message. We characterize the optimal decoding order and the total sum-rate maximum which can be achieved with both routing tasks. Moreover, we identify bidirectional rate pairs which result in the same total sum-rate so that we know how to interchange additional relay rate with bidirectional rate. The explicit characterization leads to combinatorial problems which we discuss for the equal time division and for $|h_1| > |h_2|$. This allows us to characterize for a desired relay rate the bidirectional rate pair which results in the largest bidirectional sum-rate. Throughout this section we illustrate the main results and the combinatoric with representative examples for a clear presentation of the arguments.

Most closed form results are obtained for the equal time division case, cf. Remark 2.4. However, the behavior will be similar for other fixed time division cases. According to this we study in the latter part of the section the maximal sum-rates with respect to the time division parameter. In the end we present for two examples the achievable rate regions for

[13] Already for the routing task where the relay node adds individual messages for nodes 1 and 2 to the bidirectional relaying protocol we have $3! = 6$ possible decoding orders at nodes 1 and 2, which results in 36 different cases to consider. For comparison, for the here considered common relay message we have $2! = 2$ decoding orders each node, which results in 4 different cases where one is always optimal, c.f. Theorem 2.45.

the equal and optimal time division case. For comparison we also depicted the achievable rate region with a simple energy-equivalent TDMA approach which realizes the same routing tasks. This illustrates the efficiency and the new established rate trade-off possibilities gained from the joint resource allocation approach.

2.5.1 Broadcast Phase with Relay Multicast

In this section, we extend the bidirectional relaying protocol by an additional relay multicast communication. This means that the relay node wants to transmit an independent common message to nodes 1 and 2. The additional relay multicast and the bidirectional broadcast are simultaneously performed in the BC phase of the bidirectional relay protocol. For the study of the problem it is enough to consider a network with one relay node only and time-invariant channels. In this subsection we assume an arbitrary but fixed time division parameter $\alpha \in [0, 1]$.

As described in Section 2.2.3, in the BC phase the relay node forwards the previously received message m_1 to node 2 and message m_2 to node 1. In addition to this, the relay node additionally encodes a relay message m_R for nodes 1 and 2 with rate R_R. For the transmission, we again follow the superposition encoding strategy. Since we have a memoryless Gaussian channel, it is sufficient to consider single-letters only. Accordingly, the random input variable of the relay node can be expressed as

$$X_\mathrm{R} = W_1 + W_2 + W_\mathrm{R}$$

where the random variables W_1, W_2, and W_R correspond to the codewords of messages m_1, m_2, and m_R respectively. Since the messages are assumed to be independent, we consider independent random variables. From this we get the output at node k, $k = 1, 2$, as follows

$$Y_k = h_k\, X_\mathrm{R} + N_1 = h_k\, W_1 + h_k\, W_2 + h_k\, W_\mathrm{R} + N_k, \qquad k = 1, 2,$$

where N_k denotes the independent additive complex Gaussian distributed noise at node k, $k = 1, 2$. Both nodes receive the signal of their own bidirectional message as interference. Therefore, before decoding the unknown messages each node subtracts the interference caused by its own message. This means that node 1 subtracts $h_1 X_1$ and node 2 subtracts $h_2 X_2$ so that for each node it remains to decode the bidirectional message from the other node and the common relay message. Accordingly, the non-negative achievable rates for the unknown messages have to fulfill the constraints

$$R_2 + R_\mathrm{R} \leq (1-\alpha) I(X_\mathrm{R}; Y_1 | W_1) = (1-\alpha) I(W_2; h_1 W_2 + h_1 W_\mathrm{R} + N_1), \tag{2.55a}$$

$$R_1 + R_\mathrm{R} \leq (1-\alpha) I(X_\mathrm{R}; Y_2 | W_2) = (1-\alpha) I(W_1; h_2 W_1 + h_2 W_\mathrm{R} + N_2), \tag{2.55b}$$

$$R_\mathrm{R} \leq \min \big\{ \underbrace{(1-\alpha) I(X_\mathrm{R}; Y_1 | W_1) - R_2}_{=:R_{\mathrm{R}@1}}, \underbrace{(1-\alpha) I(X_\mathrm{R}; Y_2 | W_2) - R_1}_{=:R_{\mathrm{R}@2}} \big\}, \tag{2.55c}$$

for some input distributions $f_{W_k}(w_k)$, $k \in \{1, 2, \mathrm{R}\}$, satisfying the power constraint $\mathbb{E}\{|X_\mathrm{R}|^2\} = \mathbb{E}\{|W_1|^2\} + \mathbb{E}\{|W_2|^2\} + \mathbb{E}\{|W_\mathrm{R}|^2\} \leq P_\mathrm{R}$. Thereby, the rates $R_{\mathrm{R}@1}$ and $R_{\mathrm{R}@2}$ denote the achievable relay rates at nodes 1 and 2.

Again, the mutual informations are coupled by the relay power distribution. This time we have to distribute the relay transmit power P_R among three messages. Therefore, as before let β_1 and β_2 denote the proportion of relay transmit power P_R spend for the codewords W_1 and W_2 and additionally let β_R denote the proportion of relay transmit power P_R spend for the codeword W_R. Now, the relay transmit power constraint requires that $\beta_1+\beta_2+\beta_\mathrm{R} \leq 1$ holds. For a feasible relay power distribution we know from Section 2.2.1 that complex Gaussian distributed inputs $W_1 \sim \mathcal{CN}(0, \beta_1 P_\mathrm{R})$, $W_2 \sim \mathcal{CN}(0, \beta_2 P_\mathrm{R})$, and $W_\mathrm{R} \sim \mathcal{CN}(0, \beta_\mathrm{R} P_\mathrm{R})$ maximize the mutual informations so that we have

$$I(X_\mathrm{R}; Y_1|W_1) = \log\big(1 + \gamma_\mathrm{R}|h_1|^2(\beta_2 + \beta_\mathrm{R})\big),$$
$$I(X_\mathrm{R}; Y_2|W_2) = \log\big(1 + \gamma_\mathrm{R}|h_2|^2(\beta_1 + \beta_\mathrm{R})\big).$$

In the following we look at the achievable rates R_1, R_2, and R_R more explicitly. Therefore, in the next theorem we identify an always optimal decoding order at nodes 1 and 2. On that score, we first look at the case where each node decodes the additional relay message first. After a successful decoding of the relay message and before decoding of the unknown bidirectional message each node subtracts the interference caused by the additional relay message. Effectively, this means that the bidirectional relay communication is essentially interference-free. For that reason, for the bidirectional communication the rate constraints (2.7a) and (2.7b) apply so that for a desired bidirectional rate pair $[R_1, R_2] \in \mathcal{R}_\mathrm{BR}(\alpha) \subseteq (1-\alpha)\mathcal{R}_\mathrm{BC}$ we need at least

$$\beta_1 = \frac{2^{\frac{R_1}{1-\alpha}} - 1}{|h_2|^2\gamma_\mathrm{R}} \quad \text{and} \quad \beta_2 = \frac{2^{\frac{R_2}{1-\alpha}} - 1}{|h_1|^2\gamma_\mathrm{R}}. \tag{2.56}$$

With this the power fraction $\beta_\mathrm{R} = 1 - \beta_1 - \beta_2$ remains at most for the multicast communication. Accordingly, for a feasible bidirectional rate pair $[R_1, R_2] \in \mathcal{R}_\mathrm{BR}(\alpha)$ the maximal achievable rate of the additional relay multicast at nodes 1 and 2 is given by

$$R_{\mathrm{R}@1}(R_1, R_2) = (1-\alpha)\log\Big(1 + \frac{\beta_\mathrm{R}|h_1|^2\gamma_\mathrm{R}}{1 + \beta_2|h_1|^2\gamma_\mathrm{R}}\Big), \tag{2.57a}$$

$$R_{\mathrm{R}@2}(R_1, R_2) = (1-\alpha)\log\Big(1 + \frac{\beta_\mathrm{R}|h_2|^2\gamma_\mathrm{R}}{1 + \beta_1|h_2|^2\gamma_\mathrm{R}}\Big). \tag{2.57b}$$

Since for a multicast both nodes should be able to decode the relay message, the maximal achievable relay rate is given by the minimum of (2.57a) and (2.57b). In the next theorem we show that this is the optimal decoding order and power distribution if we desire to achieve a certain feasible bidirectional rate pair.

Theorem 2.45. *For a given rate pair $[R_1, R_2] \in \mathcal{R}_{\mathrm{BR}}(\alpha)$ the maximal achievable additional relay rate R_{R} is achieved if nodes 1 and 2 decode the additional relay message first. Then the maximal achievable additional relay rate is given by*

$$R_{\mathrm{R}}(R_1, R_2) = \min\big\{R_{\mathrm{R}@1}(R_1, R_2), R_{\mathrm{R}@2}(R_1, R_2)\big\}. \tag{2.58}$$

Proof. The proof can be found in Appendix 2.8.28. □

Since it is optimal for any desired bidirectional rate pair that at each node the relay message is decoded first, then its interference is canceled, and finally the unknown bidirectional message is decoded without any interference of the relay message we say that the common relay message is piggybacked on the bidirectional relay communication. Moreover, this implies that potentially existing resources at the relay node are used to add a multicast communication on top of the bidirectional relaying.

The achievable rate region of the bidirectional relaying protocol with an additional relay multicast for a fixed time division parameter $\alpha \in [0, 1]$ can be expressed as

$$\mathcal{R}_{\mathrm{Piggy}}(\alpha) := \big\{[R_1, R_2, R_{\mathrm{R}}] \in \mathbb{R}_+^3 \,:\, [R_1, R_2] \in \mathcal{R}_{\mathrm{BR}}(\alpha),\ R_{\mathrm{R}} \le R_{\mathrm{R}}(R_1, R_2)\big\}.$$

Since this applies for any time division parameter, for the optimal time division case we can achieve

$$\mathcal{R}_{\mathrm{Piggy}}^{\mathrm{opt}} := \bigcup_{\alpha \in [0,1]} \mathcal{R}_{\mathrm{Piggy}}(\alpha).$$

We depicted the achievable rate region $\mathcal{R}_{\mathrm{Piggy}}(1/2)$ and $\mathcal{R}_{\mathrm{Piggy}}^{\mathrm{opt}}$ for two examples in Figure 2.18 and Figure 2.19 in the end of this section. But first we want to identify more properties of the achievable rate region $\mathcal{R}_{\mathrm{Piggy}}(\alpha)$. In particular in the next subsection we characterize the sum-rate optimal rate tuple which achieves the total sum-rate maximum.

2.5.2 Total Sum-Rate Maximum

Let $R_\Sigma = R_1 + R_2$ denote the *bidirectional sum-rate* and let $R_{\mathrm{tot}} = R_1 + R_2 + R_{\mathrm{R}}$ specify the *total sum-rate*, which includes the additional relay rate. The optimization problem for the total sum-rate maximum for a time division parameter α is given by

$$\begin{aligned} R_{\mathrm{tot}}^* &:= \max_{[R_1, R_2, R_{\mathrm{R}}] \in \mathcal{R}_{\mathrm{Piggy}}(\alpha)} R_1 + R_2 + R_{\mathrm{R}} \\ &= \max_{[R_1, R_2] \in \mathcal{R}_{\mathrm{BR}}(\alpha)} R_1 + R_2 + R_{\mathrm{R}}(R_1, R_2). \end{aligned} \tag{2.59}$$

For the characterization of the total sum-rate maximum R_{tot}^* in Theorem 2.48 we need the following lemma and proposition. In Lemma 2.46 we first characterize the bidirectional rate pair $[R_1, R_2]$ with a fixed bidirectional sum-rate $R_\Sigma = R_1 + R_2$ which maximizes the additional relay rate $R_\mathrm{R}(R_1, R_2)$. Then with Proposition 2.47 we explicitly identify the optimal rate pair. In the next subsection we present an explicit discussion and some helpful illustrations of the following results for the equal time division case.

Lemma 2.46. *Given a feasible bidirectional sum-rate R_Σ, which means that at least one feasible rate pair $[R_1, R_2] \in \mathcal{R}_\mathrm{BR}(\alpha)$ with $R_1 + R_2 = R_\Sigma$ exists, then there are $R_1^\triangleright$ and $R_1^\triangleleft$ so that for all R_1 with $R_1^\triangleright \leq R_1 \leq R_1^\triangleleft$ we have $[R_1, R_\Sigma - R_1] \in \mathcal{R}_\mathrm{BR}(\alpha)$. Furthermore, the maximal achievable additional relay rate for the desired bidirectional sum-rate R_Σ is given by*

$$R_\mathrm{R}^{\max}(R_\Sigma) := \max_{\substack{[R_1,R_2]\in\mathcal{R}_\mathrm{BR}(\alpha),\\ R_1+R_2=R_\Sigma}} R_\mathrm{R}(R_1, R_2)$$

$$= \begin{cases} (1-\alpha)\log\left(|h_1h_2|\hat{\gamma} - 2^{\frac{R_\Sigma}{2(1-\alpha)}}\right) - \frac{1}{2}R_\Sigma, & \text{if } R_1^\triangleright \leq R_1^\star \leq R_1^\triangleleft, \quad (2.60a)\\ (1-\alpha)\log\left(|h_2|^2(\hat{\gamma} - \frac{1}{|h_1|^2}2^{\frac{R_\Sigma - R_1^\triangleright}{1-\alpha}})\right) - R_1^\triangleright, & \text{if } R_1^\star < R_1^\triangleright, \quad (2.60b)\\ (1-\alpha)\log\left(|h_1|^2(\hat{\gamma} - \frac{1}{|h_2|^2}2^{\frac{R_1^\triangleleft}{1-\alpha}})\right) - R_\Sigma + R_1^\triangleleft, & \text{if } R_1^\star > R_1^\triangleleft \quad (2.60c) \end{cases}$$

with $R_1^\star := \frac{1}{2}R_\Sigma - \frac{1-\alpha}{2}R^\dagger$, $R^\dagger := \log\frac{|h_1|^2}{|h_2|^2}$, and $\hat{\gamma} := \gamma_\mathrm{R} + \frac{1}{|h_1|^2} + \frac{1}{|h_2|^2}$. The additional relay rate maximum $R_\mathrm{R}^{\max}(R_\Sigma)$ is attained at the bidirectional rate pair

$$[R_1(R_\Sigma), R_2(R_\Sigma)] = \begin{cases} [R_1^\star, R_\Sigma - R_1^\star], & \text{if } R_1^\triangleright \leq R_1^\star \leq R_1^\triangleleft,\\ [R_1^\triangleright, R_\Sigma - R_1^\triangleright], & \text{if } R_1^\star < R_1^\triangleright,\\ [R_1^\triangleleft, R_\Sigma - R_1^\triangleleft], & \text{if } R_1^\star > R_1^\triangleleft. \end{cases} \tag{2.61}$$

Proof. The proof can be found in Appendix 2.8.29. □

The lemma characterizes for any bidirectional sum-rate R_Σ the maximal achievable additional relay rate $R_\mathrm{R}^{\max}(R_\Sigma)$ and the corresponding rate pair $[R_1(R_\Sigma), R_2(R_\Sigma)] \in \mathcal{R}_\mathrm{BR}(\alpha)$ with $R_1(R_\Sigma) + R_2(R_\Sigma) = R_\Sigma$. This allows us to rewrite the total sum-rate maximum problem (2.59) as follows

$$R_\mathrm{tot}^* = \max_{\substack{R_\Sigma=R_1+R_2\\ [R_1,R_2]\in\mathcal{R}_\mathrm{BR}(\alpha)}} R_\mathrm{R}^{\max}(R_\Sigma) + R_\Sigma. \tag{2.62}$$

This means that for the total sum-rate optimum R_tot^* we have to find the optimal bidirectional sum-rate R_Σ^* which maximizes the function $R_\mathrm{tot}(R_\Sigma) := R_\mathrm{R}^{\max}(R_\Sigma) + R_\Sigma$ for feasible bidirectional sum-rates. For the discussion of $R_\mathrm{tot}(R_\Sigma)$ we need to know how $R_1^\star$ of Lemma 2.46 is related to the bidirectional rate region $\mathcal{R}_\mathrm{BR}(\alpha)$.

Let $\mathcal{R}_{\mathrm{BC}}(\gamma_{\mathrm{R}})$ denote the achievable broadcast rate region for a given signal-to-noise ratio γ_{R}. Then let $R_{\Sigma}^{\mathrm{BC}}(\gamma_{\mathrm{R}})$, $R^{\star}_{\overrightarrow{\mathrm{R2}}}(\gamma_{\mathrm{R}})$, and $R^{\star}_{\overrightarrow{\mathrm{R1}}}(\gamma_{\mathrm{R}})$ denote the maximum sum-rate and the corresponding optimal rates of $\mathcal{R}_{\mathrm{BC}}(\gamma_{\mathrm{R}})$ according to Proposition 2.2. The function $R_{\Sigma}^{\mathrm{BC}}(\gamma_{\mathrm{R}})$ is obviously bijective on the domain of non-negative γ_{R} and the range of non-negative bidirectional sum-rates. Thus, there exists a unique inverse function $\gamma_{\mathrm{R}}^{\mathrm{BC}}(R_{\Sigma})$ with $R_{\Sigma}^{\mathrm{BC}}(\gamma_{\mathrm{R}}^{\mathrm{BC}}(R_{\Sigma})) = R_{\Sigma}$ for all feasible bidirectional sum-rates. Accordingly, we can define $R^{\star}_{\overrightarrow{\mathrm{R2}}}(\gamma_{\mathrm{R}})$ and $R^{\star}_{\overrightarrow{\mathrm{R1}}}(\gamma_{\mathrm{R}})$ in terms of R_{Σ} as follows

$$R^{\star}_{\overrightarrow{\mathrm{R2}}}(R_{\Sigma}) := R^{\star}_{\overrightarrow{\mathrm{R2}}}(\gamma_{\mathrm{R}}^{\mathrm{BC}}(R_{\Sigma})) \quad \text{and} \quad R^{\star}_{\overrightarrow{\mathrm{R1}}}(R_{\Sigma}) := R^{\star}_{\overrightarrow{\mathrm{R1}}}(\gamma_{\mathrm{R}}^{\mathrm{BC}}(R_{\Sigma}))$$

with $R^{\star}_{\overrightarrow{\mathrm{R2}}}(R_{\Sigma}) + R^{\star}_{\overrightarrow{\mathrm{R1}}}(R_{\Sigma}) = R_{\Sigma}$ as well as the corresponding broadcast rate region

$$\mathcal{R}_{\mathrm{BC}}(R_{\Sigma}) := \mathcal{R}_{\mathrm{BC}}(\gamma_{\mathrm{R}}^{\mathrm{BC}}(R_{\Sigma})).$$

For a bidirectional rate pair $[R_1, R_2] \in \mathcal{R}_{\mathrm{BR}}(\alpha) = \alpha\mathcal{C}_{\mathrm{MAC}} \cap (1-\alpha)\mathcal{R}_{\mathrm{BC}}$ with sum-rate $R_1 + R_2 = R_{\Sigma}$ we have $[\frac{R_1}{1-\alpha}, \frac{R_2}{1-\alpha}] \in \mathcal{R}_{\mathrm{BC}}$ so that the relay signal-to-noise ratio γ_{R} has to be larger or equal to $\gamma_{\mathrm{R}}^{\mathrm{BC}}(\frac{R_{\Sigma}}{1-\alpha})$. The next proposition shows that $R_1^{\star}$ of Lemma 2.46 for a feasible bidirectional sum-rate R_{Σ} specifies the first sum-rate optimal rate $(1-\alpha)R^{\star}_{\overrightarrow{\mathrm{R2}}}\big(\frac{R_{\Sigma}}{1-\alpha}\big)$ of a corresponding scaled BC rate region $(1-\alpha)\mathcal{R}_{\mathrm{BC}}\big(\frac{R_{\Sigma}}{1-\alpha}\big)$ according to (2.10b) if R_{Σ} (respectively γ_{R}) is large enough, which means that we have $\beta^{\star} \in [0, 1]$.

Proposition 2.47. *The sum-rate optimal rate pair with sum-rate R_{Σ} of the broadcast rate region $(1-\alpha)\mathcal{R}_{\mathrm{BC}}(\frac{R_{\Sigma}}{1-\alpha})$ is given by the vector-valued function*

$$\boldsymbol{R}^{\star}_{\mathrm{BC}} : \mathbb{R}_+ \to \mathbb{R}^2_+, \quad R_{\Sigma} \mapsto \begin{cases} [R_1^{\star}(R_{\Sigma}), R_2^{\star}(R_{\Sigma})], & \text{if } R_{\Sigma} \geq (1-\alpha)|R^{\dagger}|, \quad (2.63a) \\ [R_{\Sigma}, 0], & \text{if } R_{\Sigma} < -(1-\alpha)R^{\dagger}, \quad (2.63b) \\ [0, R_{\Sigma}], & \text{if } R_{\Sigma} < (1-\alpha)R^{\dagger}, \quad (2.63c) \end{cases}$$

with $R_1^{\star}(R_{\Sigma}) = \frac{1}{2}R_{\Sigma} - \frac{1-\alpha}{2}R^{\dagger}$, $R_2^{\star}(R_{\Sigma}) := \frac{1}{2}R_{\Sigma} + \frac{1-\alpha}{2}R^{\dagger}$, and $R^{\dagger} = \log\frac{|h_1|^2}{|h_2|^2}$.

Proof. The proof can be found in Appendix 2.8.30. □

The function $\boldsymbol{R}_{\mathrm{BC}}(R_{\Sigma})$ of Proposition 2.47 defines for any bidirectional sum-rate R_{Σ} the unique sum-rate optimal rate pair of the rate region $(1-\alpha)\mathcal{R}_{\mathrm{BC}}(\frac{R_{\Sigma}}{1-\alpha})$ of the BC phase. This allows us to identify for a desired bidirectional sum-rate R_{Σ} the rate pair where the maximal additional relay rate $R_{\mathrm{R}}^{\max}(R_{\Sigma})$ is attained, cf. (2.61) of Lemma 2.46, explicity. With this we can characterize the total sum-rate maximum for a desired bidirectional sum-rate, $R_{\mathrm{tot}}(R_{\Sigma})$. Let us assume that the sum-rate R_{Σ} is feasible, which means that there exists at least one bidirectional rate pair with the desired sum-rate so that the intersection of $\mathcal{R}_{\mathrm{BR}}(\alpha)$ with the graph $\mathcal{G}_{\mathrm{f}_{R_{\Sigma}}}$ of the function $\mathrm{f}_{R_{\Sigma}} : [0, R_{\Sigma}] \to [0, R_{\Sigma}]$ with $R_1 \mapsto R_{\Sigma} - R_1$ is non-empty. Then, we have to distinguish between the following cases:

1. The maximal additional relay rate $R_{\mathrm{R}}^{\max}(R_\Sigma)$ is attained at $[R_1^\star(R_\Sigma), R_2^\star(R_\Sigma)]$ if the intersection point between $\mathrm{f}_{R_\Sigma}(R_1)$ and $\boldsymbol{R}_{\mathrm{BC}}^\star(R_\Sigma)$ is achievable, i.e. $[R_1^\star(R_\Sigma), R_2^\star(R_\Sigma)] \in \mathcal{R}_{\mathrm{BR}}(\alpha)$. Then with (2.60a) the total sum-rate can be expressed as

$$R_{\mathrm{tot}}(R_\Sigma) = R_{\mathrm{R}}^{\max}(R_\Sigma) + R_\Sigma = (1-\alpha)\log\left(2^{\frac{R_\Sigma}{2(1-\alpha)}}\left(|h_1h_2|\hat{\gamma} - 2^{\frac{R_\Sigma}{2(1-\alpha)}}\right)\right). \quad (2.64a)$$

2. Otherwise, if we have $[R_1^\star(R_\Sigma), R_2^\star(R_\Sigma)] \notin \mathcal{R}_{\mathrm{BR}}(\alpha)$ for a feasible bidirectional sum-rate R_Σ, we have to characterize $R_1^\triangleright$ and $R_1^\triangleleft$ from the intersection of $\mathcal{R}_{\mathrm{BR}}(\alpha)$ with the graph $\mathcal{G}_{\mathrm{f}_{R_\Sigma}}$.

   a) If $R_\Sigma < \min\{(1-\alpha)R^\dagger, \alpha R_{\overrightarrow{2\mathrm{R}}}\}$, we have $R_1^\triangleright = 0$ and $R_1^\star(R_\Sigma) < R_1^\triangleright$. Therefore, the maximal additional relay rate $R_{\mathrm{R}}^{\max}(R_\Sigma)$ is attained at $\boldsymbol{R}_{\mathrm{BC}}^\star(R_\Sigma) = [0, R_\Sigma] \in \mathcal{R}_{\mathrm{BR}}(\alpha)$. Then with (2.60b) the total sum-rate can be expressed as

$$R_{\mathrm{tot}}(R_\Sigma) = (1-\alpha)\log\left(\tfrac{|h_2|^2}{|h_1|^2} 2^{\frac{R_\Sigma}{1-\alpha}}\left(\hat{\gamma}|h_1|^2 - 2^{\frac{R_\Sigma}{1-\alpha}}\right)\right). \quad (2.64b)$$

   b) If $R_\Sigma > \min\{\alpha R_{\overrightarrow{2\mathrm{R}}}, 2\alpha R_{\overrightarrow{2\mathrm{R}}} - (1-\alpha)R^\dagger\} \Leftrightarrow R_\Sigma > \alpha R_{\overrightarrow{2\mathrm{R}}} \wedge R_1^\star(R_\Sigma) < R_\Sigma - \alpha R_{\overrightarrow{2\mathrm{R}}}$, the line $\mathrm{f}_{\mathrm{R}_\Sigma}(R_1)$ intersects the individual MAC rate constraint $\alpha R_{\overrightarrow{2\mathrm{R}}}$ so that we have $R_1^\triangleright = R_\Sigma - \alpha R_{\overrightarrow{2\mathrm{R}}}$ and $R_1^\star(R_\Sigma) < R_1^\triangleright$. Accordingly, the maximal additional relay rate $R_{\mathrm{R}}^{\max}(R_\Sigma)$ is attained at $[R_1^\triangleright, R_\Sigma - R_1^\triangleright] = [R_\Sigma - \alpha R_{\overrightarrow{2\mathrm{R}}}, \alpha R_{\overrightarrow{2\mathrm{R}}}] \in \mathcal{R}_{\mathrm{BR}}(\alpha)$. Then with (2.60b) the total sum-rate can be expressed as

$$R_{\mathrm{tot}}(R_\Sigma) = (1-\alpha)\log\left(\tfrac{|h_2|^2}{|h_1|^2} 2^{\frac{\alpha}{1-\alpha}R_{\overrightarrow{2\mathrm{R}}}}\left(\hat{\gamma}|h_1|^2 - 2^{\frac{\alpha}{1-\alpha}R_{\overrightarrow{2\mathrm{R}}}}\right)\right). \quad (2.64c)$$

   c) If $R_\Sigma < \min\{-(1-\alpha)R^\dagger, \alpha R_{\overrightarrow{1\mathrm{R}}}\}$, we have $R_1^\triangleleft = R_\Sigma$ and $R_1^\star(R_\Sigma) > R_1^\triangleleft$. Therefore, the maximum additional relay rate $R_{\mathrm{R}}^{\max}(R_\Sigma)$ is attained at $\boldsymbol{R}_{\mathrm{BC}}^\star(R_\Sigma) = [R_\Sigma, 0] \in \mathcal{R}_{\mathrm{BR}}(\alpha)$. Then with (2.60c) the total sum-rate can be expressed as

$$R_{\mathrm{tot}}(R_\Sigma) = (1-\alpha)\log\left(\tfrac{|h_1|^2}{|h_2|^2} 2^{\frac{R_\Sigma}{1-\alpha}}\left(\hat{\gamma}|h_2|^2 - 2^{\frac{R_\Sigma}{1-\alpha}}\right)\right). \quad (2.64d)$$

   d) If $R_\Sigma > \min\{\alpha R_{\overrightarrow{1\mathrm{R}}}, 2\alpha R_{\overrightarrow{1\mathrm{R}}} + (1-\alpha)R^\dagger\} \Leftrightarrow R_\Sigma > \alpha R_{\overrightarrow{1\mathrm{R}}} \wedge R_2^\star(R_\Sigma) < R_\Sigma - \alpha R_{\overrightarrow{1\mathrm{R}}}$, the line $\mathrm{f}_{\mathrm{R}_\Sigma}(R_1)$ intersects the individual MAC rate constraint $\alpha R_{\overrightarrow{1\mathrm{R}}}$ so that we have $R_1^\triangleleft = R_\Sigma - \alpha R_{\overrightarrow{1\mathrm{R}}}$ and $R_1^\star(R_\Sigma) > R_1^\triangleleft$. Accordingly, the maximal additional relay rate $R_{\mathrm{R}}^{\max}(R_\Sigma)$ is attained at $[R_1^\triangleleft, R_\Sigma - R_1^\triangleleft] = [\alpha R_{\overrightarrow{1\mathrm{R}}}, R_\Sigma - \alpha R_{\overrightarrow{1\mathrm{R}}}] \in \mathcal{R}_{\mathrm{BR}}(\alpha)$. Then with (2.60c) the total sum-rate can be expressed as

$$R_{\mathrm{tot}}(R_\Sigma) = (1-\alpha)\log\left(\tfrac{|h_1|^2}{|h_2|^2} 2^{\frac{\alpha}{1-\alpha}R_{\overrightarrow{1\mathrm{R}}}}\left(\hat{\gamma}|h_2|^2 - 2^{\frac{\alpha}{1-\alpha}R_{\overrightarrow{1\mathrm{R}}}}\right)\right). \quad (2.64e)$$

Note that if we have $R^{\dagger} < 0 \Leftrightarrow |h_1|^2 < |h_2|^2$, the case 2. a) is not possible. If the other cases occur depend on the scenario specific parameters. Similarly, we can a priori exclude the case 2.c) if we have $R^{\dagger} > 0 \Leftrightarrow |h_1|^2 < |h_2|^2$.

We have now specified the total sum-rate for any feasible bidirectional sum-rate. Therewith, we can define section-wise a function $R_{\mathrm{tot}}(R_{\Sigma})$ on the range $[0, R_{\Sigma}^{\max}]$, where $R_{\Sigma}^{\max}$ denotes the maximal feasible bidirectional sum-rate. This allows us to characterize in the next theorem the bidirectional rate pair $[R_1^{\mathrm{opt}}, R_2^{\mathrm{opt}}]$ where the total sum-rate maximum R_{tot}^{*} according to (2.62) is achieved.

Theorem 2.48. *For a given time division parameter $\alpha \in [0,1]$ the total sum-rate maximum R_{tot}^{*} of the rate region $\mathcal{R}_{\mathrm{Piggy}}(\alpha)$ is attained at the bidirectional rate pair $[R_1^{\mathrm{opt}}, R_2^{\mathrm{opt}}]$ where the function $\boldsymbol{R}_{\mathrm{BC}}^{\star}(R_{\Sigma})$ intersects the boundary of the bidirectional rate region $\mathcal{R}_{\mathrm{BR}}(\alpha)$.*

Proof. The proof can be found in Appendix 2.8.31. □

From the proof of the theorem and from the (2.64c) and (2.64e) we see that for the cases 2. b) and 2. d) the total sum-rate remains constant for a range of large bidirectional sum-rates, $R_1^{\mathrm{opt}} + R_2^{\mathrm{opt}} \leq R_{\Sigma} \leq R_{\Sigma}^{\max}$. This means that we can interchange additional relay rate for bidirectional sum-rate without losing total sum-rate optimality. The next corollary records this observation and states that such an interchange between additional relay rate and bidirectional rate exists not only for the total sum-rate maximum.

Corollary 2.49. *For any sum-rate $R_{\Sigma} \leq R_1^{\mathrm{opt}} + R_2^{\mathrm{opt}}$ with the rate pair $[R_1^{\star}, R_2^{\star}] := [R_1^{\star}(R_{\Sigma}), R_2^{\star}(R_{\Sigma})] = [\frac{1}{2}R_{\Sigma} - \frac{1-\alpha}{2}R^{\dagger}, \frac{1}{2}R_{\Sigma} + \frac{1-\alpha}{2}R^{\dagger}]$ the rate tuples*

$$[R_1, R_2, R_{\mathrm{R}}] = [R_1^{\star} + R_{①}, R_2^{\star}, R_{\mathrm{R}}(R_1^{\star} + R_{①}, R_2^{\star})]$$

with $R_{①} \geq 0$ and $[R_1^{\star} + R_{①}, R_2^{\star}] \in \mathcal{R}_{\mathrm{BR}}(\alpha)$ and

$$[R_1, R_2, R_{\mathrm{R}}] = [R_1^{\star}, R_2^{\star} + R_{②}, R_{\mathrm{R}}(R_1^{\star}, R_2^{\star} + R_{②})]$$

with $R_{②} \geq 0$ and $[R_1^{\star}, R_2^{\star} + R_{②}] \in \mathcal{R}_{\mathrm{BR}}(\alpha)$ achieve the same total sum-rate.

Proof. The proof can be found in Appendix 2.8.32. □

The corollary characterizes the bidirectional rate pairs where the total sum-rate remains constant. For a bidirectional sum-rate $R_{\Sigma} \leq R_1^{\mathrm{opt}} + R_2^{\mathrm{opt}}$ there exists the rate pair $[R_1^{\star}, R_2^{\star}]$ which determines the rate pairs where the total sum-rate remains constant. Then the corollary states that we can either increase R_1 or R_2. For a constant total sum-rate this means that if we increase one bidirectional rate, we equally decrease the relay multicast rate. Thereby, the interchange range follows from the condition that the bidirectional rate pairs $[R_1^{\star} + R_{①}, R_2^{\star}]$

or $[R_1^\star, R_2^\star + R_{②}]$ have to be within $\mathcal{R}_{\mathrm{BR}}(\alpha)$ with appropriate $R_{①}, R_{②} \geq 0$. However, if we increase both bidirectional rates we know from Theorem 2.48 that we increase the total sum-rate. At the total sum-rate optimum R_{tot}^* there may no interchange or interchange with only one bidirectional rate possible. For a better understanding we briefly discuss the low and high bidirectional sum-rate cases next.

For low sum-rates R_Σ where we have $R_1^\star(R_\Sigma)+R_2^\star(R_\Sigma) < |R^\dagger|$ we have a negative $R_1^\star(R_\Sigma)$ iff $|h_1|^2 < |h_2|^2$ and a negative $R_2^\star(R_\Sigma)$ iff $|h_2|^2 < |h_1|^2$. If $R_1^\star(R_\Sigma) < 0$ we can only interchange relay rate in the direction R_1 since for any rate $R_{②} \geq 0$ the rate pair $[R_1^\star, R_2^\star + R_{②}] \notin \mathcal{R}_{\mathrm{BR}}(\alpha)$. Furthermore, for a rate pair within the bidirectional rate region we need $R_{①} \geq R_1^\star$. Similarly, we can interchange relay rate in the direction R_2 only if $R_2^\star(R_\Sigma)$ is negative.

If we have $[R_1^\star(R_\Sigma^{\max}), R_2^\star(R_\Sigma^{\max})] \in \mathcal{R}_{\mathrm{BR}}(\alpha)$, the rate pair achieves the bidirectional sum-rate maximum $R_1^\star(R_\Sigma^{\max}) + R_2^\star(R_\Sigma^{\max}) = R_\Sigma^{\max}$, which means that the rate pair is also bidirectional sum-rate optimal. We see that this is the case iff the intersection point of the function $\boldsymbol{R}_{\mathrm{BC}}(R_\Sigma)$ with the boundary of $\mathcal{R}_{\mathrm{BR}}(\alpha)$ coincides with the intersection of $\boldsymbol{R}_{\mathrm{BC}}(R_\Sigma)$ with the boundary of $(1-\alpha)\mathcal{R}_{\mathrm{BC}}$ or the sum-rate constraint of the MAC region $\alpha\mathcal{R}_{\mathrm{MAC}}$. Since the rate pair characterizes a sum-rate optimum of a BC rate region it follows that in this case no interchange without loosing total sum-rate optimality is possible. In the next subsection we elaborate the combinatorial discussion on the previous results.

2.5.3 Combinatorial Discussion and Working Examples

In this section we present some combinatorial discussions of the achievable rate region using the previous results. We start with an explicit study of the equal time division case. After that we fix the additional relay rate and look for achievable bidirectional rate pairs. Then we discuss the total sum-rate maximum with respect to the time division parameter α. Finally, we compare the achievable rate regions with the achievable rate region of a simple TDMA approach which realizes the same routing tasks.

Equal Time Division

For the equal time division case we can explicitly express the total sum-rate optimal rate pairs and describe the possible rate trade-offs. The explicit characterization of the combinatorial results leads to simple arithmetical problems. Because of the symmetry we do the discussion for case $|h_1|^2 \geq |h_2|^2$. We first specify the combinatorial cases for which we characterize the total sum-rate optimality ranges. If we interchange the indices 1 and 2 an analog discussion gives the results for the case $|h_1|^2 \leq |h_2|^2$.

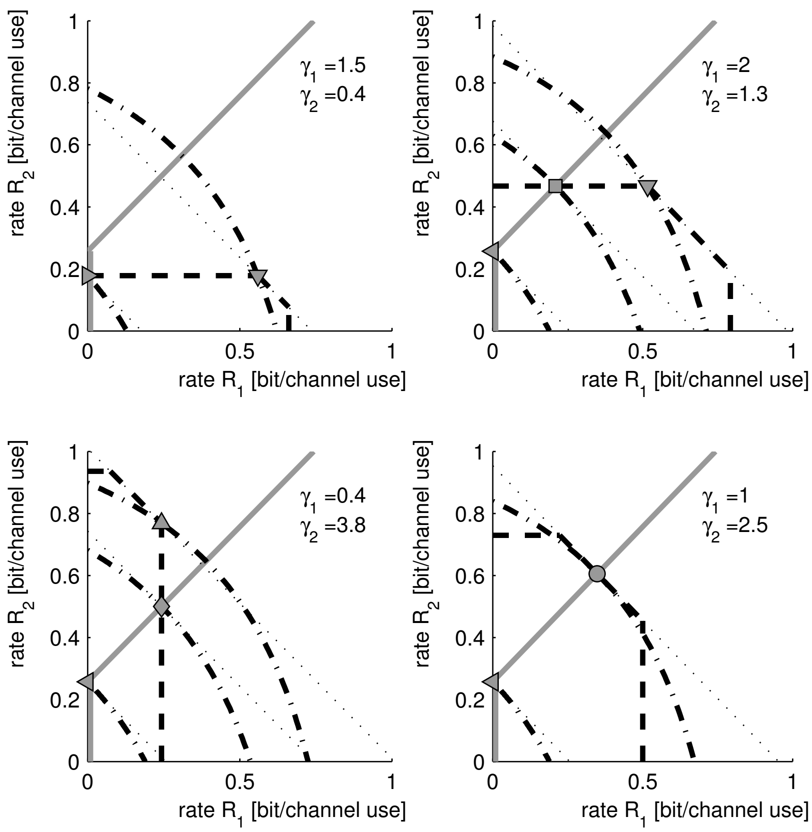


Figure 2.13: For $|h_1|^2 \geq |h_2|^2$ we can distinguish between four possible cases where $\boldsymbol{R}^{\star}_{\mathrm{BC}}(\gamma_{\mathrm{R}})$ (solid line) intersects the boundary of $1/2\mathcal{C}_{\mathrm{MAC}}$ (dashed line). Additionally, we depicted broadcast rate regions of selected relay powers (dashed-dotted line). The cases are specified by the intersection points $\triangleright$, $\square$, $\Diamond$, and $\circ$, where the function $\boldsymbol{R}^{\star}_{\mathrm{BC}}(\gamma_{\mathrm{R}})$ intersects the two individual rate constraints and the sum-rate constraint of the MAC region. For the intersection of the first individual rate constraint we additionally have to differentiate whether in the BC phase the unidirectional communication is sum-rate optimal. This is the case if the rate pair $\triangleleft$ is not achievable in the MAC phase, i.e. $\triangleleft \notin 1/2\mathcal{C}_{\mathrm{MAC}}$. Finally, the rate pairs $\triangledown$ and $\vartriangle$ denote the vertices of the MAC region $1/2\mathcal{C}_{\mathrm{MAC}}$ respectively.

For $|h_1|^2 \geq |h_2|^2$ we have $R^\dagger \geq 0$ so that there are four cases where and how the function $\boldsymbol{R}^\star_{\mathrm{BC}}(R_\Sigma)$ intersects the boundary of the MAC rate region $\frac{1}{2}\mathcal{C}_{\mathrm{MAC}}$. The four cases depend on the channel realizations and power constraints of nodes 1 and 2. In Figure 2.13 we depict the cases for four representative examples with some characteristic rate pairs. In accordance to that we can characterize the four cases as follow:

UL (upper left): The function $\boldsymbol{R}^\star_{\mathrm{BC}}(R_\Sigma)$ intersects the individual rate constraint $\frac{1}{2}R_{\overrightarrow{2\mathrm{R}}}$ for the case (2.63c) $\Leftrightarrow \frac{1}{2}R_{\overrightarrow{2\mathrm{R}}} < \frac{1}{2}R^\dagger \Leftrightarrow \gamma_2 > \frac{|h_1|^2}{|h_2|^2}\left(\frac{1}{|h_2|^2} - \frac{1}{|h_1|^2}\right)$.

UR (upper right): The function $\boldsymbol{R}^\star_{\mathrm{BC}}(R_\Sigma)$ intersects the individual rate constraint $\frac{1}{2}R_{\overrightarrow{2\mathrm{R}}}$ for the case (2.63a) $\Leftrightarrow \frac{1}{2}R_{\overrightarrow{2\mathrm{R}}} \geq \frac{1}{2}R^\dagger \wedge \frac{1}{2}R_{\overrightarrow{2\mathrm{R}}} - \frac{1}{2}R^\dagger \leq \frac{1}{2}R_1^{2\Sigma} = \frac{1}{2}(R_\Sigma^{\mathrm{MAC}} - R_{\overrightarrow{2\mathrm{R}}})$.

LL (lower left): The function $\boldsymbol{R}^\star_{\mathrm{BC}}(R_\Sigma)$ intersects the individual rate constraint $\frac{1}{2}R_{\overrightarrow{1\mathrm{R}}}$ for the case (2.63a) $\Leftrightarrow \frac{1}{2}R_{\overrightarrow{2\mathrm{R}}} \geq \frac{1}{2}R^\dagger \wedge \frac{1}{2}R_{\overrightarrow{1\mathrm{R}}} + \frac{1}{2}R^\dagger \leq \frac{1}{2}R_2^{\Sigma 1} = \frac{1}{2}(R_\Sigma^{\mathrm{MAC}} - R_{\overrightarrow{1\mathrm{R}}})$.

LR (lower right): The function $\boldsymbol{R}^\star_{\mathrm{BC}}(R_\Sigma)$ intersects the sum-rate constraint $\frac{1}{2}R_\Sigma^{\mathrm{MAC}}$ for the case (2.63a) $\Leftrightarrow \frac{1}{2}R_{\overrightarrow{2\mathrm{R}}} \geq \frac{1}{2}R^\dagger \wedge R_\Sigma^{\mathrm{MAC}} < \min\{2R_{\overrightarrow{2\mathrm{R}}} - R^\dagger, 2R_{\overrightarrow{1\mathrm{R}}} + R^\dagger\}$.

For the identification of the rate pair where the total sum-rate optimum R^*_{tot} is achieved according Theorem 2.48 we need to know the intersection point between $\boldsymbol{R}^\star_{\mathrm{BC}}(R_\Sigma)$ and the boundary of $\mathcal{R}_{\mathrm{BReq}}$. If the intersection point with the boundary of the MAC rate region $\frac{1}{2}\mathcal{C}_{\mathrm{MAC}}$ is within $\frac{1}{2}\mathcal{R}_{\mathrm{BC}}$ it coincides with the intersection point with the boundary of $\mathcal{R}_{\mathrm{BReq}}$. Otherwise, we find the intersection point on the boundary of the rate region $\frac{1}{2}\mathcal{R}_{\mathrm{BC}}$, which is given by $[\frac{1}{2}R^\star_{\overrightarrow{\mathrm{R}2}}, \frac{1}{2}R^\star_{\overrightarrow{\mathrm{R}1}}]$ according to Proposition 2.2. As a consequence, we have to characterize for each of the previous cases the condition that the intersection point is within the BC rate region. This is the case iff

UL: $\frac{1}{2}R_{\overrightarrow{\mathrm{R}1}}(1) \geq \frac{1}{2}R_{\overrightarrow{2\mathrm{R}}} \Leftrightarrow \gamma_\mathrm{R} \geq \frac{|h_2|^2}{|h_1|^2}\gamma_2$,

UR: $\frac{1}{2}R_{\overrightarrow{\mathrm{R}1}}(1-\beta^\star) \geq \frac{1}{2}R_{\overrightarrow{2\mathrm{R}}} \Leftrightarrow \gamma_\mathrm{R} \geq \gamma_\mathrm{R}^{2\mathrm{R}}$ with $\gamma_\mathrm{R}^{2\mathrm{R}} = \frac{\gamma_1|h_1|^2}{(1+\gamma_2|h_2|^2)|h_2|^2} + \gamma_2\frac{|h_2|^2}{|h_1|^2}$,

LL: $\frac{1}{2}R_{\overrightarrow{\mathrm{R}2}}(\beta^\star) \geq \frac{1}{2}R_{\overrightarrow{1\mathrm{R}}} \Leftrightarrow \gamma_\mathrm{R} \geq \gamma_\mathrm{R}^{1\mathrm{R}}$ with $\gamma_\mathrm{R}^{1\mathrm{R}} = \frac{\gamma_2|h_2|^2}{(1+\gamma_1|h_1|^2)|h_1|^2} + \gamma_1\frac{|h_1|^2}{|h_2|^2}$,

LR: $\frac{1}{2}R_\Sigma^{\mathrm{BC}} \geq \frac{1}{2}R_\Sigma^{\mathrm{MAC}} \Leftrightarrow \gamma_\mathrm{R} \geq \gamma_\mathrm{R}^\Sigma$ with $\gamma_\mathrm{R}^\Sigma = 2\sqrt{\frac{1+\gamma_1|h_1|^2+\gamma_2|h_2|^2}{|h_1|^2|h_2|^2}} - \frac{1}{|h_1|^2} - \frac{1}{|h_2|^2}$,

where we solved the conditions for γ_R. To sum up the discussion on the intersection point for the equal time division case with $|h_1|^2 \geq |h_2|^2$ we now explicitly characterize the total sum-rate optimal rate pair according to Theorem 2.48 using the previous case definitions as follows

$$[R_1^{\mathrm{opt}}, R_2^{\mathrm{opt}}] = \begin{cases} [0, \frac{1}{2}R_{\overrightarrow{\mathrm{R1}}}(1)], & \text{UL with } \gamma_{\mathrm{R}} < \frac{|h_2|^2}{|h_1|^2}\gamma_2, \quad (2.65a) \\ [0, \frac{1}{2}R_{\overrightarrow{\mathrm{2R}}}], & \text{UL with } \gamma_{\mathrm{R}} \geq \frac{|h_2|^2}{|h_1|^2}\gamma_2, \quad (2.65b) \\ [\frac{1}{2}R_{\overrightarrow{\mathrm{2R}}} - \frac{1}{2}R^{\dagger}, \frac{1}{2}R_{\overrightarrow{\mathrm{2R}}}], & \text{UR with } \gamma_{\mathrm{R}} \geq \gamma_{\mathrm{R}}^{12}, \quad (2.65c) \\ [\frac{1}{2}R_{\overrightarrow{\mathrm{1R}}}, \frac{1}{2}R_{\overrightarrow{\mathrm{2R}}} + \frac{1}{2}R^{\dagger}], & \text{LL with } \gamma_{\mathrm{R}} \geq \gamma_{\mathrm{R}}^{21}, \quad (2.65d) \\ [\frac{1}{2}R_{\Sigma}^{\mathrm{MAC}} - \frac{1}{4}R^{\dagger}, \frac{1}{2}R_{\Sigma}^{\mathrm{MAC}} + \frac{1}{4}R^{\dagger}], & \text{LR with } \gamma_{\mathrm{R}} \geq \gamma_{\mathrm{R}}^{\Sigma}, \quad (2.65e) \\ [\frac{1}{2}R_{\overrightarrow{\mathrm{R2}}}(\beta^{\star}), \frac{1}{2}R_{\overrightarrow{\mathrm{R1}}}(\beta^{\star})], & \text{else.} \quad (2.65f) \end{cases}$$

Then the total sum-rate optimal relay rate R_{R} results from (2.58) accordingly. Since UL is the case where $[R_1^{\star}(R_{\Sigma}), R_2^{\star}(R_{\Sigma})] \notin \frac{1}{2}\mathcal{C}_{\mathrm{MAC}}$ for any R_{Σ}, for the cases (2.65a) and (2.65b) we have a total sum-rate optimal bidirectional rate pair with $R_1 = 0$. This means that for the case UL there exists a unidirectional rate allocation which is total sum-rate optimal.

According to Corollary 2.49 in some cases we can increase one bidirectional rate by decreasing the relay rate. Therefore, we need that $[R_1^{\mathrm{opt}} + R_{①}, R_2^{\mathrm{opt}}]$ for any $R_{①} > 0$ or $[R_1^{\mathrm{opt}}, R_2^{\mathrm{opt}} + R_{②}]$ for any $R_{②} > 0$ is within $\mathcal{R}_{\mathrm{BReq}}$. This is impossible if the intersection point is on the boundary of the BC rate region, which corresponds to the cases (2.65a) and (2.65f), or the intersection is specified by the sum-rate constraint $1/2 R_{\Sigma}^{\mathrm{MAC}}$ of the MAC rate region $1/2\mathcal{C}_{\mathrm{MAC}}$, which corresponds to the case (2.65e). In all these cases we cannot further increase the bidirectional sum-rate, which obviously implies that we cannot increase one bidirectional rate. As a consequence it follows that in the cases (2.65a), (2.65e), and (2.65f) the total sum-rate rate pair is unique so that we can say that no interchange is possible without loosing total sum-rate optimality.

Moreover, for the cases (2.65a) and (2.65f) the intersection point is on the boundary of the BC rate region so that we have $\beta_1 + \beta_2 = 1$ and it follows that the additional relay rate R_{R} is equal to zero. This means that if γ_{R} is small enough the intersection point with the boundary of $\mathcal{R}_{\mathrm{BReq}}$ coincides with the intersection of the boundary of the BC rate region $\frac{1}{2}\mathcal{R}_{\mathrm{BC}}$. For that reason we can say that at low signal-to-noise ratios γ_{R} it is total sum-rate optimal not to have an additional relay multicast communication. Furthermore, if the signal-to-noise ratio γ_{R} is that small that the case (2.65a) applies, we can only achieve the total sum-rate with rates $R_1 = 0$ and $R_{\mathrm{R}} = 0$. This means in this case it is also necessary for total sum-rate optimality to have a relay communication in one direction only.

In the case (2.65b) the unidirectional relay communication is total sum-rate optimal as well, however we have a positive additional relay rate R_{R}. From Corollary 2.49 we know that for any interchange rate $R_{①} \geq 0$ with $[R_{①}, \frac{1}{2}R_{\overrightarrow{\mathrm{2R}}}] \in \mathcal{R}_{\mathrm{BReq}}$ we can interchange relay rate $R_{\mathrm{R}} = R_{\mathrm{R}}(R_{①}, \frac{1}{2}R_{\overrightarrow{\mathrm{2R}}})$ with bidirectional rate $R_1 = R_{①}$ without loosing total sum-rate optimality, which means that we have $R_{①} + \frac{1}{2}R_{\overrightarrow{\mathrm{2R}}} + R_{\mathrm{R}}(R_{①}, \frac{1}{2}R_{\overrightarrow{\mathrm{2R}}}) = R_{\mathrm{tot}}^{*} = \text{const}$. Obviously, we need that $R_{\mathrm{R}}(R_{①}, \frac{1}{2}R_{\overrightarrow{2\mathrm{R}}}) \geq 0$ is satisfied. The maximum relay rate for the total sum-rate optimum is attained at the bidirectional rate pair of the intersection point $[0, \frac{1}{2}R_{\overrightarrow{2\mathrm{R}}}]$. Since we have $R_{\mathrm{R}}(R_{①}, \frac{1}{2}R_{\overrightarrow{2\mathrm{R}}}) = R_{\mathrm{R}}(0, \frac{1}{2}R_{\overrightarrow{2\mathrm{R}}}) - R_{①}$ it follows that we can increase $R_{①}$ at most up to the maximum relay rate $R_{\mathrm{R}}(0, \frac{1}{2}R_{\overrightarrow{2\mathrm{R}}})$. Furthermore, since $[R_{①}, \frac{1}{2}R_{\overrightarrow{2\mathrm{R}}}]$ has to be feasible, we can increase R_1 at most up to maximum rate $\frac{1}{2}R_1^{2\Sigma}$, if $\boldsymbol{\nu}_{2\Sigma} \in \mathcal{R}_{\mathrm{BC}}$. To sum this up, for the case (2.65b) the total sum-rate optimality range with an interchange rate $R_{①}$ can be expressed as

$$\begin{aligned} [R_1, R_2, R_{\mathrm{R}}] &= [R_{①}, \tfrac{1}{2}R_{\overrightarrow{2\mathrm{R}}}, R_{\mathrm{R}}(R_{①}, \tfrac{1}{2}R_{\overrightarrow{2\mathrm{R}}})] \\ R_{①} &\in \big[0, \min\{R_{\mathrm{R}}(0, \tfrac{1}{2}R_{\overrightarrow{2\mathrm{R}}}), \frac{1}{2}R_1^{2\Sigma}\}\big]. \end{aligned} \tag{2.66}$$

In the case (2.65c) with $R_{①} \geq 0$ we can interchange relay rate $R_{\mathrm{R}} = R_{\mathrm{R}}(\frac{1}{2}R_{\overrightarrow{2\mathrm{R}}} - \frac{1}{2}R^{\dagger} + R_{①}, \frac{1}{2}R_{\overrightarrow{2\mathrm{R}}})$ with bidirectional rate $R_1 = \frac{1}{2}R_{\overrightarrow{2\mathrm{R}}} - \frac{1}{2}R^{\dagger} + R_{①}$ as well. As before we need a non-negative relay rate $R_{\mathrm{R}}(\frac{1}{2}R_{\overrightarrow{2\mathrm{R}}} - \frac{1}{2}R^{\dagger} + R_{①}, \frac{1}{2}R_{\overrightarrow{2\mathrm{R}}}) \geq 0$ and $R_1 \leq \frac{1}{2}R_1^{2\Sigma}$. Then with similar arguments it follows that for the case (2.65c) the total sum-rate optimality range with an interchange rate $R_{①}$ can be expressed as

$$\begin{aligned} [R_1, R_2, R_{\mathrm{R}}] &= \big[\tfrac{1}{2}R_{\overrightarrow{2\mathrm{R}}} - \tfrac{1}{2}R^{\dagger} + R_{①}, \tfrac{1}{2}R_{\overrightarrow{2\mathrm{R}}}, R_{\mathrm{R}}(\tfrac{1}{2}R_{\overrightarrow{2\mathrm{R}}} - \tfrac{1}{2}R^{\dagger} + R_{①}, \tfrac{1}{2}R_{\overrightarrow{2\mathrm{R}}})\big] \\ R_{①} &\in \big[0, \min\{R_{\mathrm{R}}(\tfrac{1}{2}R_{\overrightarrow{2\mathrm{R}}} - \tfrac{1}{2}R^{\dagger}, \tfrac{1}{2}R_{\overrightarrow{2\mathrm{R}}}), \frac{1}{2}R_1^{2\Sigma} - \tfrac{1}{2}R_{\overrightarrow{2\mathrm{R}}} + \tfrac{1}{2}R^{\dagger}\}\big]. \end{aligned} \tag{2.67}$$

Finally, in the case (2.65d) with $R_{②} \geq 0$ we can interchange relay rate $R_{\mathrm{R}} = R_{\mathrm{R}}(\frac{1}{2}R_{\overrightarrow{1\mathrm{R}}}, \frac{1}{2}R_{\overrightarrow{1\mathrm{R}}} + \frac{1}{2}R^{\dagger} + R_{②})$ with bidirectional rate $R_2 = \frac{1}{2}R_{\overrightarrow{1\mathrm{R}}} + \frac{1}{2}R^{\dagger} + R_{②}$. For this case we need again $R_{\mathrm{R}}(\frac{1}{2}R_{\overrightarrow{1\mathrm{R}}}, \frac{1}{2}R_{\overrightarrow{1\mathrm{R}}} + \frac{1}{2}R^{\dagger} + R_{②}) \geq 0$ and $R_2 \leq \frac{1}{2}R_2^{\Sigma 1}$. Similarly as before, it follows that for the case (2.65d) the total sum-rate optimality range with arbitrary interchange rate $R_{②}$ can be expressed as

$$\begin{aligned} [R_1, R_2, R_{\mathrm{R}}] &= \big[\tfrac{1}{2}R_{\overrightarrow{1\mathrm{R}}}, \tfrac{1}{2}R_{\overrightarrow{1\mathrm{R}}} + \tfrac{1}{2}R^{\dagger} + R_{②}, R_{\mathrm{R}}(\tfrac{1}{2}R_{\overrightarrow{1\mathrm{R}}}, \tfrac{1}{2}R_{\overrightarrow{1\mathrm{R}}} + \tfrac{1}{2}R^{\dagger} + R_{②})\big] \\ R_{②} &\in \big[0, \min\{R_{\mathrm{R}}(\tfrac{1}{2}R_{\overrightarrow{1\mathrm{R}}}, \tfrac{1}{2}R_{\overrightarrow{1\mathrm{R}}} + \tfrac{1}{2}R^{\dagger}), \tfrac{1}{2}R_2^{\Sigma 1} - \tfrac{1}{2}R_{\overrightarrow{2\mathrm{R}}} - \tfrac{1}{2}R^{\dagger}\}\big]. \end{aligned} \tag{2.68}$$

In Figure 2.14 we illustrate the total sum-rate $R_{\mathrm{tot}}(R_1, R_2)$ with respect to the bidirectional rate pairs $[R_1, R_2] \in \mathcal{R}_{\mathrm{BReq}}$ for the four different cases. The result of Corollary 2.49 is shown by some contour lines. It shows that the total sum-rate optimum is attained at the intersection of $\boldsymbol{R}_{\mathrm{BC}}^{\star}(R_{\Sigma})$ with the boundary of $\mathcal{R}_{\mathrm{BReq}}$. Furthermore, we see that if the intersection point lies on one individual rate constraint of the MAC region, there is an optimality range where we can interchange relay rate with bidirectional rate in one direction without loosing total sum-rate optimality.

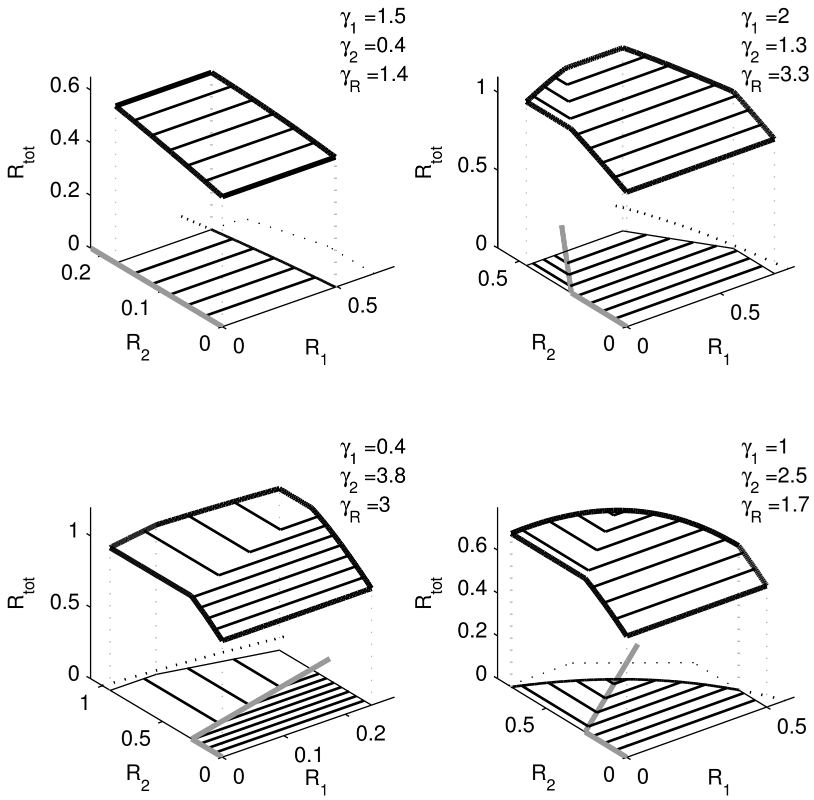


Figure 2.14: For $|h_1|^2 \geq |h_2|^2$ and equal time division, $\alpha = 1/2$, we see the total sum-rate $R_{\mathrm{tot}}(R_1, R_2)$ for $[R_1, R_2] \in \mathcal{R}_{\mathrm{BReq}}$ for the four possible cases where $\boldsymbol{R}^{\star}_{\mathrm{BC}}(\gamma_{\mathrm{R}})$ intersects the boundary of $\frac{1}{2}\mathcal{C}_{\mathrm{MAC}}$. The total sum-rate contour lines illustrate the interchange property between the additional relay rate and one bidirectional rate. Moreover, we see that the total sum-rate optimum is unique in the lower right figure only. In the upper figures we can interchange R_{R} with R_1 and in the lower left figure we can interchange R_{R} with R_2 without loosing total sum-rate optimality.

Desired Relay Rate

The explicit knowledge about the optimal rate pairs and the optimality ranges allows us to change the point of view and characterize for a desired relay rate R_{R} the bidirectional rate pair $[R_1(R_{\mathrm{R}}), R_2(R_{\mathrm{R}})] \in \mathcal{R}_{\mathrm{BR}}(\alpha)$ which maximizes the bidirectional sum-rate,

$$R_\Sigma(R_{\mathrm{R}}) := \max_{\substack{[R_1,R_2]\in\mathcal{R}_{\mathrm{BR}}(\alpha)\\ R_{\mathrm{R}}(R_1,R_2)=R_{\mathrm{R}}}} R_1 + R_2.$$

This means we have $R_\Sigma(R_{\mathrm{R}}) = R_1(R_{\mathrm{R}}) + R_2(R_{\mathrm{R}})$. Again we present the results for the case $|h_1|^2 \geq |h_2|^2$. For the case $|h_2|^2 \geq |h_1|^2$ the optimal bidirectional rate pair in a permuted order, i.e. $[R_2(R_{\mathrm{R}}), R_1(R_{\mathrm{R}})]$, follows immediately by interchanging the indices 1 and 2. However, this time we do the combinatorial discussion for an arbitrary but fixed time division parameter $\alpha \in [0, 1]$.

With increasing bidirectional rates the parameter β_{R} decreases. It follows that the relay rate R_{R} decreases with increasing bidirectional rates so that the desired relay rate is obviously bounded by the maximal possible relay rate $R_{\mathrm{R}}^{\max} := R_{\mathrm{R}}(0,0)$. At high relay rates, where we have $R_{\mathrm{R}} \geq R_{\mathrm{R}}(R_1^{\mathrm{opt}}, R_2^{\mathrm{opt}})$, the optimal bidirectional rate pairs are specified by the function $\boldsymbol{R}_{\mathrm{BC}}^\star(R_\Sigma)$. For lower desired relay rates in some cases we can additionally apply Corollary 2.49 to increase the bidirectional rate sum-rate of the intersection point $[R_1^{\mathrm{opt}}, R_2^{\mathrm{opt}}]$ while the total sum-rate remains constant. In the following we discuss the cases in more detail.

First we look at the case (2.63c) where the BC sum-rate optimal rate pair is on the R_2-axis. This means that we consider relay rates larger than $R_{\mathrm{R}}(0, \min\{(1-\alpha)R^\dagger, \alpha R_{\overrightarrow{2\mathrm{R}}}\})$. Since we have $R_1(R_{\mathrm{R}}) = 0$ for such relay rates, it follows that $R_\Sigma = R_2(R_{\mathrm{R}}) \leq \min\{(1-\alpha)R^\dagger, \alpha R_{\overrightarrow{2\mathrm{R}}}\}$. We get $R_2(R_{\mathrm{R}})$ if we solve (2.60b) with $R_1^{\triangleright} = 0$ for R_Σ. This gives us

$$[R_1(R_{\mathrm{R}}), R_2(R_{\mathrm{R}})] = \Big[0, (1-\alpha)\log\Big(|h_1|^2(\hat{\gamma} - \tfrac{2^{\frac{1}{1-\alpha}R_{\mathrm{R}}}}{|h_2|^2})\Big)\Big].$$

Next we consider relay rates for the case (2.63a) where the optimal bidirectional rate pair $[R_1^\star(R_\Sigma), R_2^\star(R_\Sigma)]$ is achievable, i.e. $[R_1^\star(R_\Sigma), R_2^\star(R_\Sigma)] \in \mathcal{R}_{\mathrm{BR}}(\alpha)$. This means we consider relay rates $R_{\mathrm{R}} \in [R_{\mathrm{R}}(R_1^{\mathrm{opt}}, R_2^{\mathrm{opt}}), R_{\mathrm{R}}(0, (1-\alpha)R^\dagger))$. This interval may be empty if the BC sum-rate optimal rate pair is on the R_2-axis, i.e. $R_1^{\mathrm{opt}} = 0$. If the interval in non-empty it follows from (2.63a) that $[R_1(R_{\mathrm{R}}), R_2(R_{\mathrm{R}})] = [\frac{1}{2}R_\Sigma - \frac{1-\alpha}{2}R^\dagger, \frac{1}{2}R_\Sigma + \frac{1-\alpha}{2}R^\dagger]$ where we get the sum-rate R_Σ if we solve (2.60a) for R_Σ. This gives us

$$[R_1(R_{\mathrm{R}}), R_2(R_{\mathrm{R}})] = \Big[(1-\alpha)\log\Big(\tfrac{|h_2|^2\hat{\gamma}}{2^{\frac{R_{\mathrm{R}}}{2(1-\alpha)}}+1}\Big), (1-\alpha)\log\Big(\tfrac{|h_1|^2\hat{\gamma}}{2^{\frac{R_{\mathrm{R}}}{2(1-\alpha)}}+1}\Big)\Big].$$

For a large signal-to-noise ratio γ_{R} we have $(1-\alpha)[R^\star_{\overrightarrow{\mathrm{R2}}}, R^\star_{\overrightarrow{\mathrm{R1}}}] \notin \mathcal{R}_{\mathrm{BR}}(\alpha)$ so that $[R_1^{\mathrm{opt}}, R_2^{\mathrm{opt}}]$ is on the boundary of $\alpha\mathcal{C}_{\mathrm{MAC}}$. In some cases we can apply Corollary 2.49 to increase the bidirectional sum-rate for a certain range of relay rate until the MAC sum-rate constraint restricts the interchange of relay rate with bidirectional rate. According to Corollary 2.49 an interchange is possible with one bidirectional rate only. For the following case study we again distinguish between the cases where $\boldsymbol{R}^\star_{\mathrm{BC}}(R_\Sigma)$ intersects the boundary of $\alpha\mathcal{C}_{\mathrm{MAC}}$.

For the case (2.63c), where $\boldsymbol{R}^\star_{\mathrm{BC}}(R_\Sigma)$ intersects the boundary of $\mathcal{R}_{\mathrm{BR}}(\alpha)$ at the rate pair $[R_1^{\mathrm{opt}}, R_2^{\mathrm{opt}}] = [0, \alpha R_{\overrightarrow{\mathrm{2R}}}]$, we have $\alpha R_{\overrightarrow{\mathrm{2R}}} < (1-\alpha)R^\dagger$. By the way if this holds, the previous considered case does not occur, i.e. $[R_1^\star(R_\Sigma), R_2^\star(R_\Sigma)] \notin \mathcal{R}_{\mathrm{BR}}(\alpha)$ for all R_Σ. In this case we are looking for relay rates which are achievable with bidirectional rate pairs $[R_1, \alpha R_{\overrightarrow{\mathrm{2R}}}] \in \mathcal{R}_{\mathrm{BR}}(\alpha)$ where the total sum-rate remains constant according to the interchange property of Corollary 2.49. The largest relay rate is achieved at $R_1 = 0$ and the smallest relay rate is either 0 or it is attained at the vertex $\alpha\boldsymbol{\nu}_{2\Sigma}$. This means that we consider relay rates $R_{\mathrm{R}} \in [\max\{0, R_{\mathrm{R}}(0, \alpha R_{\overrightarrow{\mathrm{2R}}}) - \alpha R_1^{2\Sigma}\}, R_{\mathrm{R}}(0, \alpha R_{\overrightarrow{\mathrm{2R}}}))$. Then the corresponding bidirectional rate pair with the largest bidirectional sum-rate for those relay rates R_{R} is given by

$$[R_1(R_{\mathrm{R}}), R_2(R_{\mathrm{R}})] = [R_{\mathrm{R}}(0, \alpha R_{\overrightarrow{\mathrm{2R}}}) - R_{\mathrm{R}}, \alpha R_{\overrightarrow{\mathrm{2R}}}].$$

If we have $R_{\mathrm{R}}(0, \alpha R_{\overrightarrow{\mathrm{2R}}}) > \alpha R_1^{2\Sigma}$, the interchange of relay rate and bidirectional rate is limited by the sum-rate constraint of the MAC phase. Therefore, for smaller relay rates $R_{\mathrm{R}} < R_{\mathrm{R}}(0, \alpha R_{\overrightarrow{\mathrm{2R}}}) - \alpha R_1^{2\Sigma}$ at most a maximal bidirectional sum-rate $\alpha R_\Sigma^{\mathrm{MAC}}$ is achievable, e.g. by the vertex $\alpha\boldsymbol{\nu}_{2\Sigma}$.

Next we consider the case $\alpha R_{\overrightarrow{\mathrm{2R}}} \geq (1-\alpha)R^\dagger$ so that the function $\boldsymbol{R}^\star_{\mathrm{BC}}(R_\Sigma)$ given by (2.63a) intersects the boundary of $\mathcal{R}_{\mathrm{BR}}(\alpha)$ at the rate pair $[R_1^{\mathrm{opt}}, R_2^{\mathrm{opt}}] = [\alpha R_{\overrightarrow{\mathrm{2R}}} - (1-\alpha)R^\dagger, \alpha R_{\overrightarrow{\mathrm{2R}}}]$. In this case we look for non-negative relay rates which are achievable with bidirectional rate pairs given by $[R_1^{\mathrm{opt}}, R_2^{\mathrm{opt}}]$ and the vertex $\alpha\boldsymbol{\nu}_{2\Sigma}$ where the total sum-rate remains constant according to the interchange property of Corollary 2.49. This means that we consider relay rates $R_{\mathrm{R}} \in [\max\{0, R_{\mathrm{R}}(R_1^{\mathrm{opt}}, R_2^{\mathrm{opt}}) - (\alpha R_1^{2\Sigma} - R_1^{\mathrm{opt}})\}, R_{\mathrm{R}}(R_1^{\mathrm{opt}}, R_2^{\mathrm{opt}}))$. Then the corresponding bidirectional rate pair with the largest bidirectional sum-rate for those relay rates R_{R} is given by

$$[R_1(R_{\mathrm{R}}), R_2(R_{\mathrm{R}})] = [R_1^{\mathrm{opt}} + (R_{\mathrm{R}}(R_1^{\mathrm{opt}}, R_2^{\mathrm{opt}}) - R_{\mathrm{R}}), R_2^{\mathrm{opt}}].$$

If we have $R_{\mathrm{R}}(R_1^{\mathrm{opt}}, R_2^{\mathrm{opt}}) > \alpha R_1^{2\Sigma} - R_1^{\mathrm{opt}}$, the interchange of relay rate and bidirectional rate is again limited by the sum-rate constraint of the MAC phase. It similarly follows that for smaller relay rates $R_{\mathrm{R}} < R_{\mathrm{R}}(R_1^{\mathrm{opt}}, R_2^{\mathrm{opt}}) - (\alpha R_1^{2\Sigma} - R_1^{\mathrm{opt}})$ at most a maximal bidirectional sum-rate $\alpha R_\Sigma^{\mathrm{MAC}}$ is achievable, e.g. by the vertex $\alpha\boldsymbol{\nu}_{2\Sigma}$.

Similar arguments apply for the case where the function $\boldsymbol{R}^\star_{\mathrm{BC}}(R_\Sigma)$ given by (2.63a) intersects the boundary of $\mathcal{R}_{\mathrm{BR}}(\alpha)$ at a rate pair $[R_1^{\mathrm{opt}}, R_2^{\mathrm{opt}}] = [\alpha R_{\overrightarrow{\mathrm{1R}}}, \alpha R_{\overrightarrow{\mathrm{1R}}} + (1-\alpha)R^\dagger]$. In this case we look for non-negative relay rates which are achievable with bidirectional rate pairs $[R_1^{\mathrm{opt}}, R_2^{\mathrm{opt}}]$ and the vertex $\alpha\boldsymbol{\nu}_{\Sigma 1}$ where the total sum-rate remains constant according to the interchange property of Corollary 2.49. This means that we consider relay rates $R_{\mathrm{R}} \in [\max\{0, R_{\mathrm{R}}(R_1^{\mathrm{opt}}, R_2^{\mathrm{opt}}) - (\alpha R_2^{\Sigma 1} - R_2^{\mathrm{opt}})\}, R_{\mathrm{R}}(R_1^{\mathrm{opt}}, R_2^{\mathrm{opt}}))$. Then the corresponding bidirectional rate pair with the largest bidirectional sum-rate for those relay rates R_{R} is given by

$$[R_1(R_{\mathrm{R}}), R_2(R_{\mathrm{R}})] = [R_1^{\mathrm{opt}}, R_2^{\mathrm{opt}} + (R_{\mathrm{R}}(R_1^{\mathrm{opt}}, R_2^{\mathrm{opt}}) - R_{\mathrm{R}})].$$

If we have $R_{\mathrm{R}}(R_1^{\mathrm{opt}}, R_2^{\mathrm{opt}}) > \alpha R_2^{\Sigma 1} - R_2^{\mathrm{opt}}$, the interchange of relay rate and bidirectional rate is again limited by the sum-rate constraint of the MAC phase. It similarly follows that for smaller relay rates $R_{\mathrm{R}} < R_{\mathrm{R}}(R_1^{\mathrm{opt}}, R_2^{\mathrm{opt}}) - (\alpha R_2^{\Sigma 1} - R_2^{\mathrm{opt}})$ at most a maximal bidirectional sum-rate $\alpha R_{\Sigma}^{\mathrm{MAC}}$ is achievable, e.g. by the vertex $\alpha\boldsymbol{\nu}_{\Sigma 1}$.

Finally, if $\boldsymbol{R}_{\mathrm{BC}}^{\star}(R_{\Sigma})$ intersects the boundary of $\mathcal{R}_{\mathrm{BR}}(\alpha)$ defined by the MAC sum-rate constraint, i.e. we have $[R_1^{\mathrm{opt}}, R_2^{\mathrm{opt}}] = [R_1^{\star}(R_{\Sigma}^{\mathrm{MAC}}), R_2^{\star}(R_{\Sigma}^{\mathrm{MAC}})]$, there is no interchange of relay rate and bidirectional rate possible. It follows that for relay rates $R_{\mathrm{R}} < R_{\mathrm{R}}(R_1^{\mathrm{opt}}, R_2^{\mathrm{opt}})$ bidirectional rate pairs with a sum-rate at most $\alpha R_{\Sigma}^{\mathrm{MAC}}$ are achievable, e.g. by the rate pair $[R_1^{\star}(R_{\Sigma}^{\mathrm{MAC}}), R_2^{\star}(R_{\Sigma}^{\mathrm{MAC}})]$.

Total Sum-Rate Maximum Discussion

Theorem 2.48 characterizes for any time division parameter $\alpha \in [0,1]$ the rate-pair where the total sum-rate maximum is attained. We are now interested how the total sum-rate maximum depends on the time division parameter α. For that goal, in this subsection we need to extend the notation of some symbols by an additional argument to specify the time division. Accordingly, let the function $\boldsymbol{R}_{\mathrm{BC}}^{\star}(R_{\Sigma}, \alpha)$ denote the sum-rate optimal rate pair of a broadcast region $(1-\alpha)\mathcal{R}_{\mathrm{BC}}(\frac{R_{\Sigma}}{1-\alpha})$ with sum-rate equal R_{Σ} according to Proposition 2.47. Similarly, let $R_{\mathrm{tot}}^{*}(\alpha)$ and $[R_1^{\mathrm{opt}}(\alpha), R_2^{\mathrm{opt}}(\alpha)]$ denote the total sum-rate maximum and the corresponding rate pair where the maximum is attained for a time division parameter α according to Theorem 2.48. Since the discussion of the total sum-rate with respect to the time division parameter α leads again to a case study, we restrict our discussion without loss of generality to the case where we have $|h_1|^2 \geq |h_2|^2$.

First we distinguish between the cases $\beta^{\star} < 0$ and $\beta^{\star} \geq 0$ according to Proposition 2.2. If $\beta^{\star} < 0$ we always have $[R_{\overrightarrow{\mathrm{R2}}}^{\star}, R_{\overrightarrow{\mathrm{R1}}}^{\star}] = [0, R_{\overrightarrow{\mathrm{R1}}}(1)]$ from which it follows that the function $\boldsymbol{R}_{\mathrm{BC}}^{\star}(R_{\Sigma}, \alpha)$ always intersects the boundary of $\mathcal{R}_{\mathrm{BR}}(\alpha)$ on the R_2-axis, i.e. $[R_1^{\mathrm{opt}}(\alpha), R_2^{\mathrm{opt}}(\alpha)] = [0, \min\{\alpha R_{\overrightarrow{\mathrm{2R}}}, (1-\alpha)R_{\overrightarrow{\mathrm{R1}}}(1)\}]$. This means that it is always sum-rate optimal to have a relay communication only in one direction. If we have a time division parameter $\alpha \geq \alpha_0 := \frac{R_{\overrightarrow{\mathrm{R1}}}(1)}{R_{\overrightarrow{\mathrm{2R}}} + R_{\overrightarrow{\mathrm{R1}}}(1)}$, we have $\alpha R_{\overrightarrow{\mathrm{2R}}} \geq (1-\alpha)R_{\overrightarrow{\mathrm{R1}}}(1)$ which means that the total sum-rate maximum achieving rate pair is on the boundary of the BC rate region $(1-\alpha)\mathcal{R}_{\mathrm{BC}}$. This is equivalent to the fact that the broadcast sum-rate optimal rate pair $(1-\alpha)[R^{\star}_{\overrightarrow{\mathrm{R2}}}, R^{\star}_{\overrightarrow{\mathrm{R1}}}]$ is within $\alpha\mathcal{C}_{\mathrm{MAC}}$. Accordingly, if $\alpha < \alpha_0$ we have $(1-\alpha)[R^{\star}_{\overrightarrow{\mathrm{R2}}}, R^{\star}_{\overrightarrow{\mathrm{R1}}}] \notin \alpha\mathcal{C}_{\mathrm{MAC}}$ so that the total sum-rate optimal rate pair is given by $[R_1^{\mathrm{opt}}(\alpha), R_2^{\mathrm{opt}}(\alpha)] = [0, \alpha R_{\overrightarrow{\mathrm{2R}}}]$.

On the other hand, if $\beta^\star \geq 0$ holds, we have $[R^{\star}_{\overrightarrow{\mathrm{R2}}}, R^{\star}_{\overrightarrow{\mathrm{R1}}}] = [R_{\overrightarrow{\mathrm{R2}}}(\beta^\star), R_{\overrightarrow{\mathrm{R1}}}(1-\beta^\star)]$ so that it is sum-rate optimal to communicate in both directions. With an increasing time division parameter $\alpha \in [0,1]$ we enlarge the MAC rate region $\alpha\mathcal{C}_{\mathrm{MAC}}$ and reduce the BC rate region $(1-\alpha)\mathcal{R}_{\mathrm{BC}}$. Accordingly, for some α the broadcast sum-rate optimal rate pair $(1-\alpha)[R_{\overrightarrow{\mathrm{R2}}}(\beta^\star), R_{\overrightarrow{\mathrm{R1}}}(1-\beta^\star)]$ is within $\alpha\mathcal{C}_{\mathrm{MAC}}$. As before, we denote the corresponding time division parameter by α_0. For the explicit characterization we need to distinguish between the boundary sections of $\alpha\mathcal{C}_{\mathrm{MAC}}$ where we find a rate pair which is equal to the sum-rate optimal rate pair of $(1-\alpha)\mathcal{R}_{\mathrm{BC}}$. For $\alpha \in [0,1]$ the rate pairs $(1-\alpha)[R_{\overrightarrow{\mathrm{R2}}}(\beta^\star), R_{\overrightarrow{\mathrm{R1}}}(1-\beta^\star)]$ specify a line from the origin in the first quadrant with an angle

$$\phi_{\mathrm{BC}} := \arctan \frac{R_{\overrightarrow{\mathrm{R1}}}(1-\beta^\star)}{R_{\overrightarrow{\mathrm{R2}}}(\beta^\star)} = \arctan \frac{\log(\frac{1}{2}\hat{\gamma}|h_1|^2)}{\log(\frac{1}{2}\hat{\gamma}|h_2|^2)}. \tag{2.69a}$$

Accordingly, for $\alpha \in [0,1]$ the vertices $\alpha\boldsymbol{\nu}_{\Sigma 1} = [\alpha R_{\overrightarrow{\mathrm{1R}}}, \alpha(R_\Sigma^{\mathrm{MAC}} - R_{\overrightarrow{\mathrm{1R}}})]$ and $\alpha\boldsymbol{\nu}_{2\Sigma} = [\alpha(R_\Sigma^{\mathrm{MAC}} - R_{\overrightarrow{\mathrm{2R}}}), \alpha R_{\overrightarrow{\mathrm{2R}}}]$ specify lines from the origin in the first quadrant with angles

$$\phi_{\Sigma 1} := \arctan \frac{R_\Sigma^{\mathrm{MAC}} - R_{\overrightarrow{\mathrm{1R}}}}{R_{\overrightarrow{\mathrm{1R}}}}, \tag{2.69b}$$

$$\phi_{2\Sigma} := \arctan \frac{R_{\overrightarrow{\mathrm{2R}}}}{R_\Sigma^{\mathrm{MAC}} - R_{\overrightarrow{\mathrm{2R}}}} \tag{2.69c}$$

respectively. The growing MAC rate regions $\alpha\mathcal{C}_{\mathrm{MAC}}$ and shrinking BC rate regions $(1-\alpha)\mathcal{R}_{\mathrm{BC}}$ with increasing α and the corresponding lines of the vertices and broadcast sum-rate optimal rate pairs are illustrated in Figure 2.15. Since the angles $\phi_{\Sigma 1}, \phi_{2\Sigma}$, and ϕ_{BC} do not depend on the time division parameter, we can use the angles to identify the section of the MAC boundary where we find the broadcast sum-rate optimal rate pair for α_0.

Thus, if we have $\phi_{\mathrm{BC}} \geq \phi_{2\Sigma}$ there exists a time division parameter α_0 where the sum-rate optimal rate pair $(1-\alpha_0)[R^{\star}_{\overrightarrow{\mathrm{R2}}}, R^{\star}_{\overrightarrow{\mathrm{R1}}}]$ of the BC rate region $(1-\alpha_0)\mathcal{R}_{\mathrm{BC}}$ is on the section of the boundary of $\alpha_0\mathcal{C}_{\mathrm{MAC}}$ which is characterized by the individual rate constraint $R_{\overrightarrow{\mathrm{2R}}}$, i.e. $(1-\alpha_0)[R^{\star}_{\overrightarrow{\mathrm{R2}}}, R^{\star}_{\overrightarrow{\mathrm{R1}}}] \in \{\alpha_0[R_1, R_{\overrightarrow{\mathrm{2R}}}] \in \mathbb{R}_+^2 : 0 \leq R_1 \leq R_\Sigma^{\mathrm{MAC}} - R_{\overrightarrow{\mathrm{2R}}}\}$. Accordingly, if we solve $(1-\alpha_0)R^{\star}_{\overrightarrow{\mathrm{R1}}} = \alpha_0 R_{\overrightarrow{\mathrm{2R}}}$ for α_0 we get the time division parameter $\alpha_0 = \frac{R^{\star}_{\overrightarrow{\mathrm{R1}}}}{R^{\star}_{\overrightarrow{\mathrm{R1}}} + R_{\overrightarrow{\mathrm{2R}}}}$ for the case $\phi_{\mathrm{BC}} \geq \phi_{2\Sigma}$.

Similarly, if we have $\phi_{\mathrm{BC}} \leq \phi_{\Sigma 1}$ there exists a time division parameter α_0 where the sum-rate optimal rate pair $(1-\alpha_0)[R^{\star}_{\overrightarrow{\mathrm{R2}}}, R^{\star}_{\overrightarrow{\mathrm{R1}}}]$ is on the section of the boundary of $\alpha\mathcal{C}_{\mathrm{MAC}}$ which is

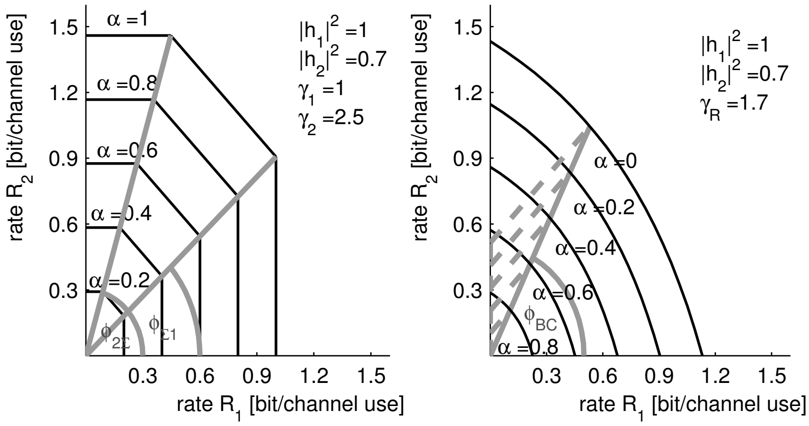


Figure 2.15: The left and right figures show the increasing MAC rate regions $\alpha\mathcal{C}_{\mathrm{MAC}}$ and decreasing BC rate regions $(1-\alpha)\mathcal{R}_{\mathrm{BC}}$ for increasing α. Furthermore, we depicted the lines characterized by the vertices and the broadcast sum-rate optimal rate pairs. In the right figure the function $\boldsymbol{R}^{\star}_{\mathrm{BC}}(R_\Sigma, \alpha)$ for $\alpha = 0, 0.2, 0.4, 0.6,$ and 0.8 (dashed lines) are additionally depicted.

characterized by the individual rate constraint $R_{\overrightarrow{\mathrm{1R}}}$. Accordingly, if we solve $(1-\alpha_0)R^{\star}_{\overrightarrow{\mathrm{R2}}} = \alpha_0 R_{\overrightarrow{\mathrm{1R}}}$ for α_0 we get $\alpha_0 = \frac{R^{\star}_{\overrightarrow{\mathrm{R2}}}}{R^{\star}_{\overrightarrow{\mathrm{R2}}}+R_{\overrightarrow{\mathrm{1R}}}}$ for the case $\phi_{\mathrm{BC}} \leq \phi_{\Sigma 1}$.

Finally, if we have $\phi_{2\Sigma} > \phi_{\mathrm{BC}} > \phi_{\Sigma 1}$ there exists a time division parameter α_0 where the sum-rate optimal rate pair of $(1-\alpha_0)\mathcal{R}_{\mathrm{BC}}$ is also sum-rate optimal with respect to the MAC rate region $\alpha_0\mathcal{C}_{\mathrm{MAC}}$, i.e. the rate pair is on the section of the boundary of $\alpha_0\mathcal{C}_{\mathrm{MAC}}$ which is characterized by the sum-rate constraint $R_{\Sigma}^{\mathrm{MAC}}$. Accordingly, if we solve $(1-\alpha_0)(R^{\star}_{\overrightarrow{\mathrm{R2}}} + R^{\star}_{\overrightarrow{\mathrm{R1}}}) = (1-\alpha_0)R_{\Sigma}^{\mathrm{BC}} = \alpha_0 R_{\Sigma}^{\mathrm{MAC}}$ for α_0 we get $\alpha_0 = \frac{R_{\Sigma}^{\mathrm{BC}}}{R_{\Sigma}^{\mathrm{BC}}+R_{\Sigma}^{\mathrm{MAC}}}$ for the case $\phi_{2\Sigma} > \phi_{\mathrm{BC}} > \phi_{\Sigma 1}$.

In the next proposition we summarize the previous considerations and characterize the total sum-rate optimal rate pair according to Theorem 2.48. Since ϕ_{BC} is greater than $\frac{\pi}{2}$ for $\beta^{\star} < 0$, we can merge the case $\beta^{\star} < 0$ with the case $\beta^{\star} \geq 0$ and $\phi_{\mathrm{BC}} > \phi_{2\Sigma}$ according to the definition of $[R^{\star}_{\overrightarrow{\mathrm{R2}}}, R^{\star}_{\overrightarrow{\mathrm{R1}}}]$ in Proposition 2.2.

Proposition 2.50. *The sum-rate optimal rate pair* $(1-\alpha)[R^{\star}_{\overrightarrow{\mathrm{R2}}}, R^{\star}_{\overrightarrow{\mathrm{R1}}}]$ *of the broadcast region* $(1-\alpha)\mathcal{R}_{\mathrm{BC}}$ *is within the scaled MAC rate region* $\alpha\mathcal{C}_{\mathrm{MAC}}$ *and therefore within* $\mathcal{R}_{\mathrm{BR}}(\alpha)$ *if* *we have a time division parameter* $\alpha \in \mathcal{A}_{\mathrm{BC}} := [\alpha_0, 1]$ *with*

$$\alpha_0 := \begin{cases} \frac{R^\star_{\overrightarrow{\mathrm{R1}}}}{R^\star_{\overrightarrow{\mathrm{R1}}} + R_{\overrightarrow{\mathrm{2R}}}}, & \text{if } \phi_{\mathrm{BC}} \geq \phi_{2\Sigma}, \\ \frac{R^{\mathrm{BC}}_\Sigma}{R^{\mathrm{BC}}_\Sigma + R^{\mathrm{MAC}}_\Sigma}, & \text{if } \phi_{2\Sigma} > \phi_{\mathrm{BC}} > \phi_{\Sigma 1}, \\ \frac{R^\star_{\overrightarrow{\mathrm{R2}}}}{R^\star_{\overrightarrow{\mathrm{R2}}} + R_{\overrightarrow{\mathrm{1R}}}}, & \text{if } \phi_{\Sigma 1} \geq \phi_{\mathrm{BC}}. \end{cases} \tag{2.70}$$

Then the BC sum-rate optimal pair is also sum-rate optimal with respect to the total sum-rate $R^*_{\mathrm{tot}}(\alpha)$. *This means for* $\alpha \geq \alpha_0$ *that we have*

$$[R_1^{\mathrm{opt}}(\alpha), R_2^{\mathrm{opt}}(\alpha)] = (1-\alpha)[R^\star_{\overrightarrow{\mathrm{R2}}}, R^\star_{\overrightarrow{\mathrm{R1}}}] \tag{2.71}$$

with total sum-rate $R^*_{\mathrm{tot}}(\alpha) = (1-\alpha)R^{\mathrm{BC}}_\Sigma = (1-\alpha)(R^\star_{\overrightarrow{\mathrm{R1}}} + R^\star_{\overrightarrow{\mathrm{R2}}})$.

The total sum-rate $R^*_{\mathrm{tot}}(\alpha)$ follows from the fact that for $\alpha \geq \alpha_0$ the total sum-rate optimal rate pair is equal to the sum-rate optimal rate pair of $(1-\alpha)\mathcal{R}_{\mathrm{BC}}$ so that β_{R} and therefore the additional relay rate R_{R} are equal to zero.

The set $\mathcal{A}_{\mathrm{BC}} = [\alpha_0, 1]$ denotes the set of time division parameters α where the sum-rate optimal rate pair of the BC phase $(1-\alpha)[R^\star_{\overrightarrow{\mathrm{R2}}}, R^\star_{\overrightarrow{\mathrm{R1}}}]$ is achievable in the MAC phase. Accordingly, for $\alpha \in \mathcal{A}_0 := [0, \alpha_0)$ the rate pair $(1-\alpha)[R^\star_{\overrightarrow{\mathrm{R1}}}, R^\star_{\overrightarrow{\mathrm{R2}}}]$ is not achievable in the MAC phase. Then it follows from Theorem 2.48 that the total sum-rate optimal rate pair is attained at the rate pair where $\boldsymbol{R}^\star_{\mathrm{BC}}(R_\Sigma, \alpha)$ intersects the boundary of $\mathcal{R}_{\mathrm{BR}}(\alpha)$. As in Section 2.5.3 we can distinguish between four possible intersections for $|h_1|^2 \geq |h_2|^2$. For the following characterization of the cases we first ignore the feasibility of the time division parameters. In the proposition after it we take the feasibility into account.

1. For small time division parameters $\alpha < \alpha_1 := \frac{R^\dagger}{R^\dagger + R_{\overrightarrow{\mathrm{2R}}}}$ the function $\boldsymbol{R}^\star_{\mathrm{BC}}(R_\Sigma, \alpha)$ intersects the boundary of $\alpha\mathcal{C}_{\mathrm{MAC}}$ on the R_2-axis, this means at the rate pair $[0, \alpha R_{\overrightarrow{\mathrm{2R}}}]$. Accordingly, α_1 characterizes the time division parameter where we have $\alpha_1 R_{\overrightarrow{\mathrm{2R}}} = (1-\alpha_1)R^\dagger$.

2. For time division parameters $\alpha_1 \leq \alpha < \alpha_2 := \frac{R^\dagger}{2R_{\overrightarrow{\mathrm{2R}}} - R^{\mathrm{MAC}}_\Sigma + R^\dagger}$ the function $\boldsymbol{R}^\star_{\mathrm{BC}}(R_\Sigma, \alpha)$ intersects the boundary of $\alpha\mathcal{C}_{\mathrm{MAC}}$ characterized by the individual rate constraint $\alpha R_{\overrightarrow{\mathrm{2R}}}$, this means at a rate pair in $\alpha\mathcal{R}_{\overrightarrow{\mathrm{2R}}} = \{\alpha[R_1, R_{\overrightarrow{\mathrm{2R}}}] : 0 \leq R_1 < R_1^{2\Sigma}\}$. Thereby, α_2 characterizes the time division parameter where $\boldsymbol{R}^\star_{\mathrm{BC}}(R_\Sigma, \alpha)$ intersects the boundary of $\alpha\mathcal{C}_{\mathrm{MAC}}$ in its vertex $\alpha\boldsymbol{\nu}_{2\Sigma}$. Accordingly, we get α_2 if we solve $R_1^\star(\alpha_2 R^{\mathrm{MAC}}_\Sigma) = \frac{\alpha_2}{2}R^{\mathrm{MAC}}_\Sigma - \frac{1-\alpha_2}{2}R^\dagger = \alpha_2(R^{\mathrm{MAC}}_\Sigma - R_{\overrightarrow{\mathrm{2R}}})$ for α_2.

3. For time division parameters $\alpha_2 \le \alpha \le \alpha_3 := \frac{R^\dagger}{R_\Sigma^{\mathrm{MAC}} - 2R_{\overrightarrow{1\mathrm{R}}} + R^\dagger}$ the function $\boldsymbol{R}^\star_{\mathrm{BC}}(R_\Sigma, \alpha)$ intersects the boundary of $\alpha\mathcal{C}_{\mathrm{MAC}}$ characterized by the sum-rate constraint $\alpha R_\Sigma^{\mathrm{MAC}}$, this means at a rate pair in $\alpha\mathcal{R}_\Sigma^{\mathrm{MAC}} = \{\alpha[R_1, R_\Sigma^{\mathrm{MAC}} - R_1] : R_1^{2\Sigma} \le R_1 \le R_{\overrightarrow{1\mathrm{R}}}\}$. Thereby, α_3 characterizes the time division parameter where $\boldsymbol{R}^\star_{\mathrm{BC}}(R_\Sigma, \alpha)$ intersects the boundary of $\alpha\mathcal{C}_{\mathrm{MAC}}$ in its vertex $\alpha\boldsymbol{\nu}_{\Sigma 1}$. Accordingly, we get α_3 if we solve $R_1^\star(\alpha_3 R_\Sigma^{\mathrm{MAC}}) = \frac{\alpha_3}{2} R_\Sigma^{\mathrm{MAC}} - \frac{1-\alpha_3}{2} R^\dagger = \alpha_3 R_{\overrightarrow{1\mathrm{R}}}$ for α_3.
4. Finally, for large time division parameters $\alpha > \alpha_3$ the function $\boldsymbol{R}^\star_{\mathrm{BC}}(R_\Sigma, \alpha)$ intersects the boundary of $\alpha\mathcal{C}_{\mathrm{MAC}}$ characterized by the individual rate constraint $\alpha R_{\overrightarrow{1\mathrm{R}}}$, this means at a rate pair in $\alpha\mathcal{R}_{\overrightarrow{1\mathrm{R}}} = \{\alpha[R_{\overrightarrow{1\mathrm{R}}}, R_2] : 0 \le R_2 < R_2^{\Sigma 1}\}$.

Since for $|h_1|^2 \ge |h_2|^2$ we have $R^\dagger \ge 0$. Then the parameters α_2 or α_3 are negative if we have $R_{\overrightarrow{2\mathrm{R}}} + R^\dagger < R_1^{2\Sigma}$ or $R_2^{\Sigma 1} < R_{\overrightarrow{1\mathrm{R}}} - R^\dagger$ respectively. Such time division parameters are obviously not feasible. The feasible time division parameters are given by the set $\mathcal{A}_0$. Then in the following proposition we show that the feasible cases are again specified by the angles ϕ_{BC}, $\phi_{\Sigma 1}$, and $\phi_{2\Sigma}$.

Proposition 2.51. *With $\mathcal{A}_0 := [0, \alpha_0)$ let $\mathcal{A}_1 := [0, \alpha_1) \cap \mathcal{A}_0$, $\mathcal{A}_2 := [\alpha_1, \alpha_2) \cap \mathcal{A}_0$, $\mathcal{A}_3 := [\alpha_2, \alpha_3) \cap \mathcal{A}_0$, and $\mathcal{A}_4 := \mathcal{A}_0 \setminus \big(\mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{A}_3\big)$ denote the sets of feasible time division parameters of the previous cases. Then we have*

$$\begin{aligned} \mathcal{A}_1 \neq \emptyset, \quad \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4 = \emptyset, &\quad \textit{if } \phi_{\mathrm{BC}} > \tfrac{\pi}{2}, \\ \mathcal{A}_1, \mathcal{A}_2 \neq \emptyset, \quad \mathcal{A}_3, \mathcal{A}_4 = \emptyset, &\quad \textit{if } \tfrac{\pi}{2} \ge \phi_{\mathrm{BC}} > \phi_{2\Sigma}, \\ \mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3 \neq \emptyset, \quad \mathcal{A}_4 = \emptyset, &\quad \textit{if } \phi_{2\Sigma} \ge \phi_{\mathrm{BC}} > \phi_{\Sigma 1}, \\ \mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4 \neq \emptyset, &\quad \textit{if } \phi_{\Sigma 1} \ge \phi_{\mathrm{BC}}. \end{aligned}$$

For a time division parameter $\alpha \in \mathcal{A}_0$ the total sum-rate optimal rate pair $[R_1^{\mathrm{opt}}(\alpha), R_2^{\mathrm{opt}}(\alpha)]$ on the boundary of the scaled MAC region $\alpha\mathcal{C}_{\mathrm{MAC}}$ is given by

$$[R_1^{\mathrm{opt}}(\alpha), R_2^{\mathrm{opt}}(\alpha)] = \begin{cases} [0, \alpha R_{\overrightarrow{2\mathrm{R}}}], & \textit{if } \alpha \in \mathcal{A}_1, \\ [\alpha R_{\overrightarrow{2\mathrm{R}}} - (1-\alpha)R^\dagger, \alpha R_{\overrightarrow{2\mathrm{R}}}], & \textit{if } \alpha \in \mathcal{A}_2, \\ [\frac{\alpha}{2} R_\Sigma^{\mathrm{MAC}} - \frac{1-\alpha}{2} R^\dagger, \frac{\alpha}{2} R_\Sigma^{\mathrm{MAC}} + \frac{1-\alpha}{2} R^\dagger], & \textit{if } \alpha \in \mathcal{A}_3, \\ [\alpha R_{\overrightarrow{1\mathrm{R}}}, \alpha R_{\overrightarrow{2\mathrm{R}}} + (1-\alpha)R^\dagger], & \textit{if } \alpha \in \mathcal{A}_4, \end{cases}$$

so that we have the total sum-rate maximum

$$R^*_{\mathrm{tot}}(\alpha) = \begin{cases} (1-\alpha)\log\left(\frac{|h_2|^2}{|h_1|^2} 2^{\frac{\alpha}{1-\alpha} R_{\overrightarrow{2\mathrm{R}}}} \left(\hat{\gamma}|h_1|^2 - 2^{\frac{\alpha}{1-\alpha} R_{\overrightarrow{2\mathrm{R}}}}\right)\right), & \textit{if } \alpha \in \mathcal{A}_1 \cup \mathcal{A}_2, \\ (1-\alpha)\log\left(2^{\frac{\alpha}{2(1-\alpha)} R_\Sigma^{\mathrm{MAC}}} \left(|h_1 h_2|\hat{\gamma} - 2^{\frac{\alpha}{2(1-\alpha)} R_\Sigma^{\mathrm{MAC}}}\right)\right), & \textit{if } \alpha \in \mathcal{A}_3, \\ (1-\alpha)\log\left(\frac{|h_1|^2}{|h_2|^2} 2^{\frac{\alpha}{1-\alpha} R_{\overrightarrow{1\mathrm{R}}}} \left(\hat{\gamma}|h_2|^2 - 2^{\frac{\alpha}{1-\alpha} R_{\overrightarrow{1\mathrm{R}}}}\right)\right), & \textit{if } \alpha \in \mathcal{A}_4. \end{cases}$$

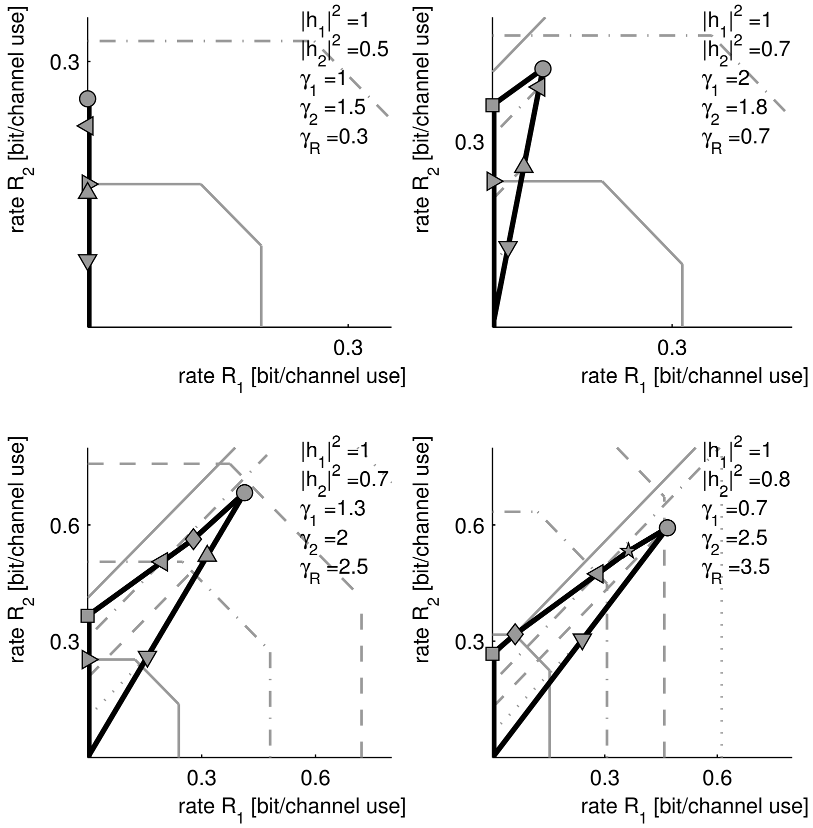


Figure 2.16: The figures show parametrized curves of the total sum-rate optimal rate pairs $[R_1^{\mathrm{opt}}(\alpha), R_2^{\mathrm{opt}}(\alpha)]$ for $\alpha \in [0,1]$ according to the four different cases for $|h_1|^2 \geq |h_2|^2$. For the upper left figure we have $\phi_{\mathrm{BC}} > \frac{\pi}{2}$, for the upper right figure we have $\frac{\pi}{2} \geq \phi_{\mathrm{BC}} > \phi_{2\Sigma}$, for the lower left figure we have $\phi_{2\Sigma} \geq \phi_{\mathrm{BC}} > \phi_{\Sigma 1}$, and for the lower right figure we have $\phi_{\Sigma 1} \geq \phi_{\mathrm{BC}}$. Furthermore, we depicted in parts $\alpha\mathcal{C}_{\mathrm{MAC}}$, $\alpha\mathcal{R}_{\mathrm{BC}}$, and $\boldsymbol{R}_{\mathrm{BC}}^{\star}(R_\Sigma, \alpha)$ for $\alpha = 0.2$ (solid), $\alpha = 0.4$ (dashed-dotted), $\alpha = 0.6$ (dashed), and $\alpha = 0.8$ (dotted) with gray lines. Additionally, we marked some characteristic optimal rate pairs specified by ($\rhd : \alpha = 0.2$), ($\lhd : \alpha = 0.4$), ($\triangle$: $\alpha = 0.6$), ($\nabla : \alpha = 0.8$), ($\circ$: $\alpha = \alpha_0$), ($\square$: $\alpha = \alpha_1$), ($\Diamond : \alpha = \alpha_2$), and (☆: $\alpha = \alpha_3$).

Proof. The proof can be found in Appendix 2.8.33. □

The parametrized curves in Figure 2.16 illustrate the optimal rate pairs $[R_1^{\text{opt}}(\alpha), R_2^{\text{opt}}(\alpha)]$ for time division parameters $\alpha \in [0,1]$ for the four possible cases for $|h_1|^2 \geq |h_2|^2$. For comparison between the cases the optimal rate pairs and in parts the broadcast and multiple access rate regions for time division parameters $\alpha = 0.2$, 0.4, 0.6, and 0.8 are depicted. Furthermore, the optimal rate pairs of the characteristic time division parameters α_0, α_1, α_2, and α_3 are marked. In more detail, in the upper left we have the case where $R^\dagger > R_{\overrightarrow{2\mathrm{R}}}$ so that we have $\mathcal{A}_{\mathrm{BC}} \cup \mathcal{A}_1 = [0,1]$. It follows that we have only two sections. For $\alpha \geq \alpha_0$ we have $[R_1^{\text{opt}}(\alpha), R_2^{\text{opt}}(\alpha)] = [0, (1-\alpha)R_{\overrightarrow{\mathrm{R}1}}(1)] \in \mathcal{R}_{\mathrm{BR}}(\alpha)$ according to Proposition 2.50 and for $\alpha < \alpha_0$ we have $[R_1^{\text{opt}}(\alpha), R_2^{\text{opt}}(\alpha)] = [0, \alpha R_{\overrightarrow{2\mathrm{R}}}] \in \mathcal{R}_{\mathrm{BR}}(\alpha)$ according to Proposition 2.51. In the upper right, we have $\mathcal{A}_2 \neq \emptyset$ so that we additionally have the case $\alpha_1 \leq \alpha < \alpha_0$ where the optimal rate pair is given by $[R_1^{\text{opt}}(\alpha), R_2^{\text{opt}}(\alpha)] = [\alpha R_{\overrightarrow{2\mathrm{R}}} - (1-\alpha)R^\dagger, \alpha R_{\overrightarrow{2\mathrm{R}}}]$ according to Proposition 2.51. Similarly in the lower left and lower right we first additionally have $\mathcal{A}_3 \neq \emptyset$ and then $\mathcal{A}_4 \neq \emptyset$ as well.

The Propositions 2.50 and 2.51 characterize for each time division parameter the corresponding total sum-rate $R_{\text{tot}}^*(\alpha)$. Since the parametrized curves $[R_1^{\text{opt}}(\alpha), R_2^{\text{opt}}(\alpha)]$ are continuous for each case, the section-wise defined function $R_{\text{tot}}^*(\alpha)$ is continuous as well. For $\alpha \in \mathcal{A}_{\mathrm{BC}}$ we know from Proposition 2.50 that with increasing α the total sum-rate $R_{\text{tot}}(\alpha)$ decreases linearly. This means that for this section the total sum-rate is always maximized at α_0. But for time division parameters $\alpha \in \mathcal{A}_0$ a closed form discussion of the total sum-rate is no longer possible, c.f. Remark 2.4. Moreover, from numerical examples there is unfortunately no clear behavior observable. In Figure 2.17 we depicted some representative numerical examples which should give some idea how the total sum-rate behaves with respect to the time division parameter α. In the upper left we see that the total sum-rate maximum is attained at α_0. In the upper right the total sum-rate is attained at some α within the interval $(0, \alpha_0)$ and in the lower left and right it is optimal with respect to the total sum-rate to have a relay multicast only.

From Theorem 2.48 we know that for a given time division parameter α the total sum-rate is maximized at the bidirectional rate pair $\boldsymbol{R}_{\mathrm{BC}}^\star(R_\Sigma, \alpha) \in \mathcal{R}_{\mathrm{BR}}(\alpha)$ of the largest possible bidirectional sum-rate R_Σ. This means that we first allocate relay power to support an efficient bidirectional rate pair then the remaining transmit power can be used for the additional relay communication. But since we a priori fixed the time division parameter we do not factor in the costs of transmitting the messages to the relay node in the previous MAC phase. In the lower left and right figure, we see that the resource costs (large fraction of time) of the MAC phase are higher than the spectral efficiency of the bidirectional broadcast phase so that it is total sum-rate optimal to have no bidirectional communication. However, since it was the aim to have two routing tasks, it is questionable if such a solution is desired. Nevertheless, it shows that for a spectrally efficient bidirectional relaying we need enough resources (transmit power at nodes 1 and 2) in the MAC phase as well.

Comparison with a TDMA protocol

To demonstrate the efficiency of the proposed protocol we compare the achievable rate regions of the piggyback on bidirectional relaying protocol with equal time division with a straightforward Round Robin TDMA approach. Thereby, we assume a TDMA protocol with five time slots of equal duration. For a fair comparison, we adapt the power constraints in the TDMA protocol so that both protocols consume the same amount of energy. Accordingly, in the first time slot node 1 transmits its message to the relay node with transmit power $\frac{5}{2}P_1$. Similarly, in the second time slot node 2 transmits its message to the relay node with transmit power $\frac{5}{2}P_2$. In the third and fourth time slot the relay node separately forwards the messages with power $\frac{5}{6}P_\mathrm{R}$. With the same power the relay message is transmitted to nodes 1 and 2 in the last time slot. This give us the achievable rates for a comparable TDMA protocol

$$\begin{aligned}
R_1^\mathrm{TDMA} &= \tfrac{1}{5}\log\bigl(1+\min\{\tfrac{5}{2}\gamma_1|h_1|^2, \tfrac{5}{6}\gamma_\mathrm{R}|h_2|^2\}\bigr),\\
R_2^\mathrm{TDMA} &= \tfrac{1}{5}\log\bigl(1+\min\{\tfrac{5}{2}\gamma_2|h_2|^2, \tfrac{5}{6}\gamma_\mathrm{R}|h_1|^2\}\bigr),\\
R_\mathrm{R}^\mathrm{TDMA} &= \tfrac{1}{5}\log\bigl(1+\tfrac{5}{6}\gamma_\mathrm{R}\min\{|h_1|^2, |h_2|^2\}\bigr).
\end{aligned}$$

Figure 2.18 illustrates for two scenarios the dominance of the bidirectional relaying protocol with an additional relay multicast for the equal time division case. For both cases we see that the achievable rate region $\mathcal{R}_\mathrm{Piggy}(1/2)$ is significantly larger in any direction than the achievable rate region of the TDMA protocol (cuboid), which means that the entire achievable rate region of the TDMA protocol (cuboid) is within the piggyback rate region. Accordingly, the piggyback approach achieves higher weighted total sum-rates, which is important for service adapted network operations. Moreover, it illustrates that the joint resource allocation allows new advantageous rate trade-offs so that we conclude that the synergy from the joint resource allocation of two routing tasks, multicast and bidirectional relaying, improves the performance significantly.

Finally, for completeness we present the achievable rate regions of bidirectional relaying with an additional relay multicast and optimal time division $\mathcal{R}_\mathrm{Piggy}^\mathrm{opt}$ of two representative examples in Figure 2.19. Since the optimal decoding order holds for any fixed time division parameter, it is also optimal to decode the relay message first for the optimal time division. Unfortunately, a closed form discussion as for the fixed time division case is not possible, cf. Remark 2.4, but we can use the equivalent characterization of the bidirectional achievable rate region $\mathcal{R}_\mathrm{BRopt}$ according to Theorem 2.12 to illustrate the rate region $\mathcal{R}_\mathrm{Piggy}^\mathrm{opt}$.

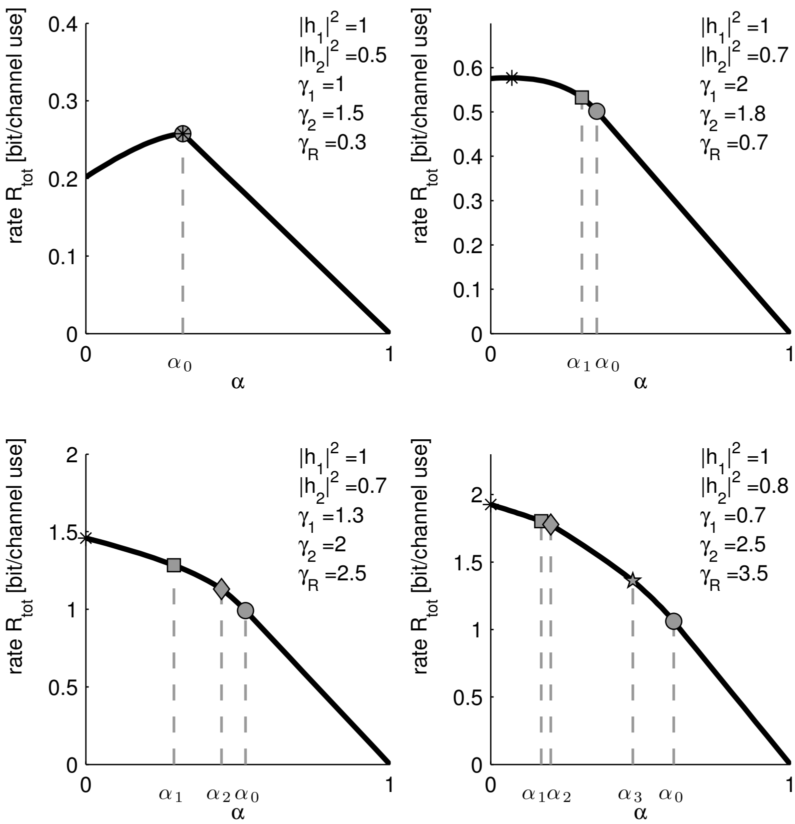


Figure 2.17: The figures show the total sum-rate maximum $R^*_{\text{tot}}(\alpha)$ with respect to the time division parameter $\alpha \in [0,1]$ for the four cases for $|h_1|^2 \geq |h_2|^2$. For the upper left figure we have $\phi_{\text{BC}} > \frac{\pi}{2}$, for the upper right figure we have $\frac{\pi}{2} \geq \phi_{\text{BC}} > \phi_{2\Sigma}$, for the lower left figure we have $\phi_{2\Sigma} \geq \phi_{\text{BC}} > \phi_{\Sigma 1}$, and for the lower right figure we have $\phi_{\Sigma 1} \geq \phi_{\text{BC}}$. ($*$) specifies the maximum for each case. Additionally, we marked the total sum-rates of some characteristic time division parameters ($\circ$: $\alpha = \alpha_0$), ($\square$: $\alpha = \alpha_1$), ($\Diamond$: $\alpha = \alpha_2$), and (☆: $\alpha = \alpha_3$).

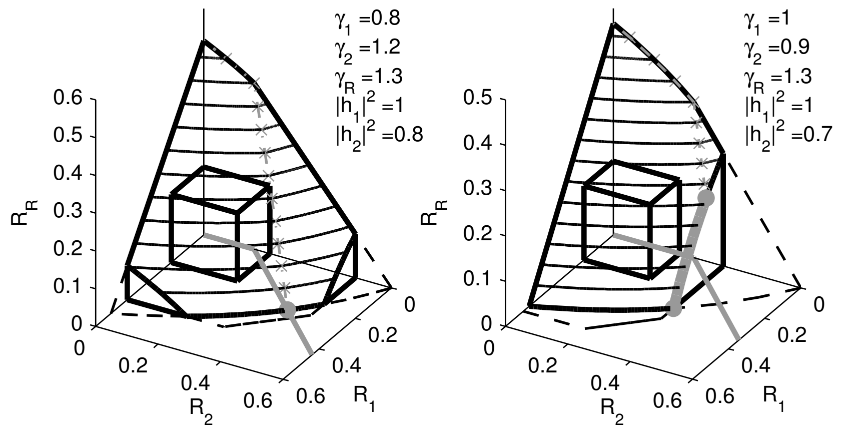


Figure 2.18: Comparison of achievable rate region $\mathcal{R}_{\text{Piggy}}(1/2)$ of piggyback a common relay message on bidirectional relaying with equal time division and a TDMA protocol realizing the same routing task (cuboid) with the same energy consumption. In the left figure we have at the bullet (•) a unique total sum-rate optimum; in the right figure we can interchange rate R_1 with R_{R} along the bold line between the the bullets without loosing total sum-rate optimality. On the contour lines for equal additional relay rates the bidirectional sum-rate optimum is marked by a cross ($\times$).

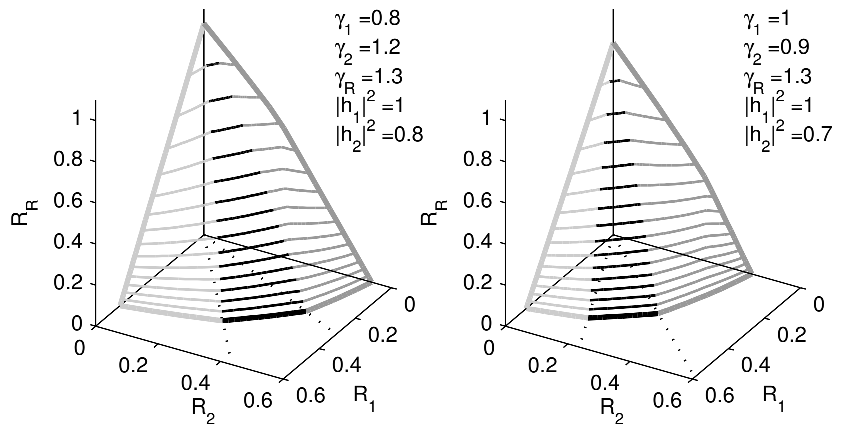
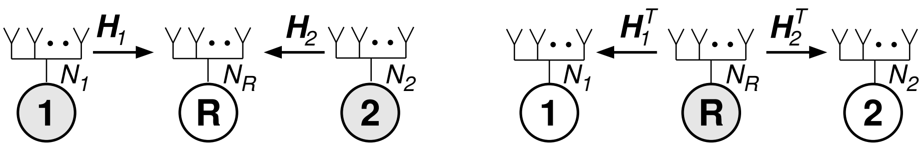


Figure 2.19: Achievable rate region $\mathcal{R}_{\text{Piggy}}^{\text{opt}}$ of piggyback a common message on bidirectional relaying with optimal time division with the same system parameters as in Figure 2.18. The contour lines denote achievable rate pairs for fixed $\beta_{\text{R}} = \frac{1}{15}, \frac{2}{15}, \dots, 1$. The contrasting colors characterize the three sections according to Therorem 2.12.

2.6 Extension to Multi-Antenna Bidirectional Relaying

In this section we extend the bidirectional relaying protocol with superposition encoding to nodes equipped with multiple antennas. The leads to a multiple-input multiple-output (MIMO) wireless system as depicted in Figure 2.20. The Gaussian vector channel was studied first by Tsybakov in [Tsy65] in 1965. The work of Teletar, Foschini and Gans have shown in [Tel99] and [FG98] that multiple-antenna systems have the ability to reach higher transmission rates than single-antenna systems. The basic idea is to exploit the spatial dimension offered by the MIMO channel. With an appropriate processing at the transmitter and receiver we obtain multiple subchannels, also called the channel *eigenmodes*, which allow data multiplexing on several substreams and lead therefore to a linear increase in capacity. On the other hand the spatial degrees of freedom can be utilized to transmit the data signal over multiple fading paths to increase the robustness of the transmission through diversity. Both works have sparked an active and flourishing research area. For a comprehensive discussion on recent results on MIMO wireless communications with large reference lists we refer to the book [BCC$^+$07] and the overview paper [GJJV03].

In [WZHM05] Wang, Zhang, and Host-Madsen can calculate upper and lower bounds for the Gaussian MIMO relay channel with a full-duplex relay node by simplifying the optimization over the joint distribution of the source and relay node input. Then they present sufficient conditions when the upper and lower bound on the ergodic capacity meet in the case of Rayleigh fading. But the spatial MIMO gains can be also obtained using multiple relay nodes. Wittneben and Rankov propose in [WR03] the relay assisted MIMO link where amplify-and-forward relays ensure the rich scattering requirement of MIMO systems. In [BNOP06] Bölcskei, Nabar, Oyman, and Paulraj derive scaling laws of the network capacity with different processing strategies at the relay nodes depending on its channel state information. The source and destination are equipped with M antennas so that the capacity scales linearly with $M/2$ in the limit of the number of half-duplex relay nodes. Moreover, there are some other interesting works which address other important MIMO relaying aspects like coverage extension in cellular networks [HKW06], linear processing for a multiuser MIMO system [TCHC06], etc.



(a) MAC phase (b) BC phase

Figure 2.20: A three-node network, where the relay node, node 1, and node 2 are equipped with N_{R}, N_1, and N_2 antennas respectively.

As before, we consider a two-phase bidirectional decode-and-forward relaying protocol without feedback. In the multiple access phase the nodes 1 and 2 transmit their messages with rate R_1 and R_2 to the relay node. We assume that the relay node decodes both messages successfully if the rates are within the capacity region of the MAC phase. In the succeeding broadcast phase the relay separately re-encodes the messages and transmits the superposition of the codewords. In [OB07c] we have studied the achievable rate region of such a multi-antenna bidirectional relaying protocol. Recently, in [HKE+07] Hammerström et. al. compare the sum-rate performance of superposition encoding and XOR precoding at the relay node based on the concept of [WCK05] and [LJS05]. Since the XOR coding approach is more power efficient, it leads to a better performance. We expect another improvement if one uses the coding idea of Chapter 3 for the Gaussian MIMO bidirectional relay channel.

The MIMO system model is similar to the SISO case considered in Section 2.1. However, in this section we require perfect channel knowledge available at each node. As before, we assume a perfectly synchronized three-node network as depicted in Figure 2.20. The relay node, nodes 1 and 2 are equipped with N_R, N_1, and N_2 antennas respectively. Let $\boldsymbol{H}_1 \in \mathbb{C}^{N_\mathrm{R}\times N_1}$ and $\boldsymbol{H}_2 \in \mathbb{C}^{N_\mathrm{R}\times N_2}$ characterize the discrete-time time-invariant multiplicative MIMO channels between the relay node and nodes 1 and 2 respectively. For simplicity we assume reciprocal channels. The rank $r_1 = \mathrm{rank}(\boldsymbol{H}_1) \leq \min\{N_1, N_\mathrm{R}\}$ and $r_2 = \mathrm{rank}(\boldsymbol{H}_2) \leq \min\{N_\mathrm{R}, N_2\}$ denote the number of spatial degree of the channels, i.e. each non-zero eigenmode of the channel can support a data stream. The mean transmit powers of each node are restricted by power constraints P_k, $k \in \{1, 2, R\}$, respectively. Furthermore, the reception at each antenna of every node is distorted by independent additive white Gaussian noise $\boldsymbol{n}_1, \boldsymbol{n}_2$, and $\boldsymbol{n}_\mathrm{R}$ with equal covariance matrices $\sigma^2\boldsymbol{I} = \rho^{-1}\boldsymbol{I}$. A generalization to individual and more general noise covariance matrices is straightforward. Then after symbol-rate sampling the system equation in the MAC phase for time instants m is given by

$$\boldsymbol{y}_\mathrm{R}[m] = \boldsymbol{H}_1\boldsymbol{x}_1[m] + \boldsymbol{H}_2\boldsymbol{x}_2[m] + \boldsymbol{n}_\mathrm{R}[m],$$

where $\boldsymbol{y}_\mathrm{R}[m] \in \mathbb{C}^{N_\mathrm{R}}$ denotes the received signal vector at the relay node, $\boldsymbol{x}_1[m] \in \mathbb{C}^{N_1}$ and $\boldsymbol{x}_2[m] \in \mathbb{C}^{N_2}$ denote the transmit signal vector of nodes 1 and 2, and $\boldsymbol{n}_\mathrm{R}[m] \in \mathbb{C}^{N_\mathrm{R}}$ denotes the additive noise vector. Similarly, after symbol-rate sampling the system equation in the BC phase for time instants m is given by

$$\boldsymbol{y}_k[m] = \boldsymbol{H}_k^T\boldsymbol{x}_\mathrm{R}[m] + \boldsymbol{n}_k[m], \quad k = 1, 2,$$

where $\boldsymbol{y}_k[m] \in \mathbb{C}^{N_k}$ denotes the received signal vector at node k, $k = 1, 2$, $\boldsymbol{x}_\mathrm{R}[m] \in \mathbb{C}^{N_\mathrm{R}}$ denotes the transmit signal vector of the relay node, and $\boldsymbol{n}_k[m] \in \mathbb{C}^{N_k}$ denotes the additive noise vector at node k, $k = 1, 2$.

In the following we first briefly review the optimal transmit strategies of the Gaussian MIMO channel with CSI at the transmitter and receiver. In the next sections we summarize the known capacity results on the Gaussian MIMO multiple access channel and present the optimal transmit strategy for the Gaussian MIMO channels in the BC phase using superposition encoding. Due to the complicated structure of the rate regions in the MIMO case the combinatoric cannot be given in closed form. Nevertheless, with the results presented here it is possible to characterize the optimal transmit strategies for any rate pair using standard methods of convex optimization. Furthermore, we will see that the achievable rate in the high power regime scales linearly with the minimum of the fraction of time weighted spatial degree of both MIMO channels.

MIMO Gaussian Channel

We first look at a $N_t \times N_r$ multiple-input multiple-output (MIMO) wireless point-to-point connection where we consider a linear time-invariant multiplicative channel with additive white Gaussian noise. We assume that the transmitter and receiver have perfect channel knowledge for encoding and decoding. In the following, we briefly reproduce the derivation of the information capacity of the MIMO Gaussian channel as done in [Tel99, TV05]. Therefore, the vector-valued linear input-output relation can be expressed as follows

$$\boldsymbol{Y} = \boldsymbol{H}\boldsymbol{X} + \boldsymbol{N}$$

where $\boldsymbol{H} \in \mathbb{C}^{N_r \times N_t}$ denotes the channel matrix and the random vectors $\boldsymbol{X} \in \mathbb{C}^{N_t \times 1}$, $\boldsymbol{Y} \in \mathbb{C}^{N_r \times 1}$, and $\boldsymbol{N} \sim \mathcal{CN}(\boldsymbol{0}, \sigma^2 \boldsymbol{I}_{N_r})$ denote the input, the output, and the complex Gaussian noise of the channel respectively. As in the scalar case we assume that the noise is independent of the input. Furthermore, we require that the input has to fulfill a mean transmit power constraint $\mathbb{E}\{\boldsymbol{X}^H \boldsymbol{X}\} \leq P$.

Similar to the scalar case we can define the mutual information for the vector-valued channel as follows

$$I(\boldsymbol{X};\boldsymbol{Y}) := h(\boldsymbol{Y}) - h(\boldsymbol{Y}|\boldsymbol{X}) = h(\boldsymbol{Y}) - h(\boldsymbol{N})$$

where $h(\boldsymbol{Y})$ and $h(\boldsymbol{Y}|\boldsymbol{X})$ denote the differential entropy and conditional differential entropy. The differential entropy for a continuous random vector $\boldsymbol{Y}$ defined on the support $\mathcal{S}_{\boldsymbol{Y}}$ with density $f_{\boldsymbol{Y}}(\boldsymbol{y})$ is defined as

$$h(\boldsymbol{Y}) := -\int_{\mathcal{S}_{\boldsymbol{Y}}} f_{\boldsymbol{Y}}(\boldsymbol{y}) \log f_{\boldsymbol{Y}}(\boldsymbol{y}) d\boldsymbol{y}.$$

The conditional differential entropy for continuous random vectors $(\boldsymbol{Y}, \boldsymbol{X})$ defined on $\mathcal{S}_{\boldsymbol{Y}} \times \mathcal{S}_{\boldsymbol{X}}$ with joint and conditional densities $f_{\boldsymbol{Y},\boldsymbol{X}}(\boldsymbol{y}, \boldsymbol{x})$ and $f_{\boldsymbol{Y}|\boldsymbol{X}}(\boldsymbol{x}|\boldsymbol{y})$ is defined as

$$h(\boldsymbol{Y}|\boldsymbol{X}) := -\int_{\mathcal{S}_{\boldsymbol{Y}} \times \mathcal{S}_{\boldsymbol{X}}} f_{\boldsymbol{Y},\boldsymbol{X}}(\boldsymbol{y}, \boldsymbol{x}) \log f_{\boldsymbol{Y}|\boldsymbol{X}}(\boldsymbol{y}|\boldsymbol{x}) d(\boldsymbol{y}, \boldsymbol{x}).$$

Then Lemma 2 in [Tel99] shows that a circularly symmetric complex Gaussian[14] distribution $\boldsymbol{X} \sim \mathcal{CN}(\boldsymbol{0}, \boldsymbol{Q})$ maximizes the differential entropy with $h(\boldsymbol{X}) = \log\det(\pi\,\mathrm{e}\,\boldsymbol{Q})$ of complex random vectors with zero mean and covariance matrix $\boldsymbol{Q}$. Since the noise vector $\boldsymbol{N}$ is independent of the the input vector $\boldsymbol{X}$ we have

$$\mathbb{E}\{\boldsymbol{Y}\boldsymbol{Y}^H\} = \boldsymbol{H}\mathbb{E}\{\boldsymbol{X}\boldsymbol{X}^H\}\boldsymbol{H}^H + \mathbb{E}\{\boldsymbol{N}\boldsymbol{N}^H\} = \boldsymbol{H}\boldsymbol{Q}\boldsymbol{H}^H + \sigma^2\boldsymbol{I}_{N_r},$$

where $\boldsymbol{Q} := \mathbb{E}\{\boldsymbol{X}\boldsymbol{X}^H\}$. It follows that a circularly symmetric complex Gaussian distributed input vector $\boldsymbol{X} \sim \mathcal{CN}(\boldsymbol{0}, \boldsymbol{Q})$ maximizes the entropy $h(\boldsymbol{Y})$ as well.

The information capacity C is again defined as the maximal mutual information over all input distributions. With a circularly symmetric complex Gaussian input vector $\boldsymbol{X} \sim \mathcal{CN}(\boldsymbol{0}, \boldsymbol{Q})$ we have

$$\begin{aligned} C &= \max_{f_{\boldsymbol{X}}(\boldsymbol{x}):\mathrm{tr}(\boldsymbol{Q})\le P} I(\boldsymbol{X};\boldsymbol{Y}) = \max_{f_{\boldsymbol{X}}(\boldsymbol{x}):\mathrm{tr}(\boldsymbol{Q})\le P} h(\boldsymbol{Y}) - \log\det(\pi\,\mathrm{e}\,\boldsymbol{I}_{N_r}\sigma^2) \\ &= \max_{\mathrm{tr}(\boldsymbol{Q})\le P} \log\det\left(\pi\,\mathrm{e}(\boldsymbol{H}\boldsymbol{Q}\boldsymbol{H}^H + \boldsymbol{I}_{N_r}\sigma^2)\right) - \log\det(\pi\,\mathrm{e}\,\boldsymbol{I}_{N_r}\sigma^2) \\ &= \max_{\mathrm{tr}(\boldsymbol{Q})\le P} \log\det(\boldsymbol{I}_{N_r} + \tfrac{1}{\sigma^2}\boldsymbol{H}\boldsymbol{Q}\boldsymbol{H}^H). \end{aligned}$$

Therefore, it remains to find the optimal covariance matrix $\boldsymbol{Q}^*$ which satisfies the power constraint $\mathrm{tr}(\boldsymbol{Q}^*) \le P$.

This convex optimization problem of finding the optimal covariance matrix can be explicitly solved using the Lagrangian method [Tel99, BV04]. This leads to the so called *water-filling solution*. In the following we briefly describe the procedure to calculate the optimal transmit covariance matrix $\boldsymbol{Q}^*$ according to the water-filling solution. Let

$$\boldsymbol{H}^H\boldsymbol{H} = \boldsymbol{V}\boldsymbol{\Sigma}\boldsymbol{V}^H, \qquad \text{with } \boldsymbol{\Sigma} := \mathrm{diag}(\lambda_1, \lambda_2, \dots, \lambda_{N_t})$$

and unitary matrix $\boldsymbol{V}$ be the eigenvalue decomposition of $\boldsymbol{H}^H\boldsymbol{H}$ with eigenvalues sorted in decreasing order, i.e. $\lambda_1 \ge \dots \ge \lambda_{N_t} \ge 0$. Then

$$\boldsymbol{Q}^* = \boldsymbol{V}\,\mathrm{diag}(\xi_1, \xi_2, \dots, \xi_{N_t})\boldsymbol{V}^H$$

with eigenvalues

$$\xi_n := \begin{cases} \max\left\{\nu - \frac{\sigma^2}{\lambda_n}, 0\right\}, & \text{if } \lambda_n \ne 0, \\ 0, & \text{if } \lambda_n = 0, \end{cases}$$

[14] The probability density function of a complex valued circularly symmetric (or spherically invariant) Gaussian random vector $\boldsymbol{X} \sim \mathcal{CN}(\boldsymbol{\mu}, \boldsymbol{Q})$ of dimension N with mean $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{Q}$ is given by

$$f_{\boldsymbol{X}}(\boldsymbol{x}) = \frac{1}{\pi^N \det(\boldsymbol{Q})}\,\mathrm{e}^{-(\boldsymbol{x}-\boldsymbol{\mu})^H\boldsymbol{Q}^{-1}(\boldsymbol{x}-\boldsymbol{\mu})}, \tag{2.72}$$

where additionally $\mathbb{E}\{(\boldsymbol{X}-\boldsymbol{\mu})(\boldsymbol{X}-\boldsymbol{\mu})^T\} = \boldsymbol{0} \in \mathbb{C}^{N\times N}$ holds. This circularity condition ensures that uncorrelated jointly Gaussian random variables are also independent[KSH00, App 3.C],[Doo53, Sec. II.3].

where the so called water-level ν follows from the transmit power constraint

$$P = \sum_{n=1}^{N_t} \xi_n = \mathrm{tr}(\boldsymbol{Q}^*),$$

which is obviously fulfilled with equality for the optimal transmit covariance matrix. For the water-filling procedure we introduce the short notation $\boldsymbol{Q}(\boldsymbol{H}^H\boldsymbol{H}, P)$ so that for the MIMO Gaussian channel the optimal transmit covariance is given by

$$\boldsymbol{Q}^* = \boldsymbol{Q}(\boldsymbol{H}^H\boldsymbol{H}, P).$$

With the optimal transmit covariance matrix $\boldsymbol{Q}^*$ the information capacity for the MIMO channel is given as follows

$$C = \log\det(\boldsymbol{I}_{N_r} + \tfrac{1}{\sigma^2}\boldsymbol{H}\boldsymbol{Q}^*\boldsymbol{H}^H) = \sum_{n=1}^{N_r} \log\left(1 + \frac{\lambda_n \xi_n}{\sigma^2}\right).$$

It depends on the rank of the channel matrix $\boldsymbol{H}$ and the transmit power constraint how many eigenmodes of the channel are used. In the so called low-power regime beamforming is optimal. This means that only the strongest eigenmode of the channel is used. Furthermore, the case where all eigenmodes of the channel are used is called the *high-power regime*.

2.6.1 MIMO Multiple Access Phase

In the first phase, nodes 1 and 2 transmit their information for each other to the relay node. The encoding and the decoding are performed as in the classical discrete-time Gaussian MIMO MAC channel. In principle the Gaussian MIMO MAC follows from the classical result from Ahlswede [Ahl71a] and Liao [Lia72], but many researchers have explicitly studied various interesting aspects [CV93, YRBC01, BJ02, VJG03, BW06] to mention only a few. Since we consider memoryless channels, it is sufficient to consider single letters only. Therefore, the vector-valued linear input-output relation can be expressed as follows

$$\boldsymbol{Y}_\mathrm{R} = \boldsymbol{H}_1\boldsymbol{X}_1 + \boldsymbol{H}_2\boldsymbol{X}_2 + \boldsymbol{N}_\mathrm{R}$$

where the continuous random vector $\boldsymbol{Y}_\mathrm{R} \in \mathbb{C}^{N_\mathrm{R}}$ denotes the output at the relay node, the continuous random vector $\boldsymbol{X}_k \in \mathbb{C}^{N_\mathrm{R}}$ denotes the input from node k, $k = 1, 2$, and $\boldsymbol{N}_\mathrm{R} \in \mathbb{C}^{N_\mathrm{R}}$ denotes the complex additive white Gaussian noise distributed according to $\mathcal{CN}(0, \boldsymbol{I}\sigma^2)$ at the relay node.

Again, we assume uncorrelated sources. Then the multiple access capacity region $\mathcal{C}_{\mathrm{MAC}}^{\mathrm{MIMO}}$ is given by the convex hull of the set of rate pairs which satisfy the following individual and sum-rate constraints

$$\begin{aligned} R_1 &\leq I(\boldsymbol{X}_1;\boldsymbol{Y}_{\mathrm{R}}|\boldsymbol{X}_2) = h(\boldsymbol{Y}_{\mathrm{R}}|\boldsymbol{X}_2) - h(\boldsymbol{Y}_{\mathrm{R}}|\boldsymbol{X}_1,\boldsymbol{X}_2) = h(\boldsymbol{H}_1\boldsymbol{X}_1+\boldsymbol{N}_{\mathrm{R}}) - h(\boldsymbol{N}_{\mathrm{R}}),\\ R_2 &\leq I(\boldsymbol{X}_2;\boldsymbol{Y}_{\mathrm{R}}|\boldsymbol{X}_1) = h(\boldsymbol{Y}_{\mathrm{R}}|\boldsymbol{X}_1) - h(\boldsymbol{Y}_{\mathrm{R}}|\boldsymbol{X}_1,\boldsymbol{X}_2) = h(\boldsymbol{H}_2\boldsymbol{X}_2+\boldsymbol{N}_{\mathrm{R}}) - h(\boldsymbol{N}_{\mathrm{R}}),\\ R_1+R_2 &\leq I(\boldsymbol{X}_1,\boldsymbol{X}_2;\boldsymbol{Y}_{\mathrm{R}}) = h(\boldsymbol{Y}_{\mathrm{R}}) - h(\boldsymbol{Y}_{\mathrm{R}}|\boldsymbol{X}_1,\boldsymbol{X}_2) = h(\boldsymbol{Y}_{\mathrm{R}}) - h(\boldsymbol{N}_{\mathrm{R}}) \end{aligned}$$

for some vector input distributions $f_{\boldsymbol{X}_1}(\boldsymbol{x}_1)$ and $f_{\boldsymbol{X}_2}(\boldsymbol{x}_2)$ which satisfy the power constraint $\mathbb{E}\{\boldsymbol{X}_k^H\boldsymbol{X}_k\} \leq P_k$, $k=1,2$.

Since we assume that the inputs $\boldsymbol{X}_1$, $\boldsymbol{X}_2$ and noise $\boldsymbol{N}_{\mathrm{R}}$ are pairwise independent, the covariance matrix of the output $\boldsymbol{Y}_{\mathrm{R}}$ can be expressed as

$$\mathbb{E}\{\boldsymbol{Y}_{\mathrm{R}}\boldsymbol{Y}_{\mathrm{R}}^H\} = \boldsymbol{H}_1\underbrace{\mathbb{E}\{\boldsymbol{X}_1\boldsymbol{X}_1^H\}}_{:=\boldsymbol{Q}_1}\boldsymbol{H}_1^H + \boldsymbol{H}_2\underbrace{\mathbb{E}\{\boldsymbol{X}_2\boldsymbol{X}_2^H\}}_{:=\boldsymbol{Q}_2}\boldsymbol{H}_2^H + \underbrace{\mathbb{E}\{\boldsymbol{N}_{\mathrm{R}}\boldsymbol{N}_{\mathrm{R}}^H\}}_{=\sigma^2\boldsymbol{I}_{N_{\mathrm{R}}}}$$

with $\mathrm{tr}(\boldsymbol{Q}_k) \leq P_k$, $k=1,2$, to fulfill the power constraints. Since a circularly symmetric complex Gaussian vector distributed according to $\mathcal{CN}(\boldsymbol{0},\boldsymbol{H}_1\boldsymbol{Q}_1\boldsymbol{H}_1^H+\boldsymbol{H}_2\boldsymbol{Q}_2\boldsymbol{H}_2^H+\sigma^2\boldsymbol{I}_{N_{\mathrm{R}}})$ maximizes the differential entropy of a random vector $\boldsymbol{Y}$ with covariance matrix $\boldsymbol{H}_1\boldsymbol{Q}_1\boldsymbol{H}_1^H+\boldsymbol{H}_2\boldsymbol{Q}_2\boldsymbol{H}_2^H+\sigma^2\boldsymbol{I}_{N_{\mathrm{R}}}$ we have

$$\begin{aligned} h(\boldsymbol{H}_1\boldsymbol{X}_1+\boldsymbol{N}_{\mathrm{R}}) &\leq \log\det\left(\pi\,\mathrm{e}(\boldsymbol{H}_1\boldsymbol{Q}_1\boldsymbol{H}_1^H+\sigma^2\boldsymbol{I}_{N_{\mathrm{R}}})\right),\\ h(\boldsymbol{H}_2\boldsymbol{X}_2+\boldsymbol{N}_{\mathrm{R}}) &\leq \log\det\left(\pi\,\mathrm{e}(\boldsymbol{H}_2\boldsymbol{Q}_2\boldsymbol{H}_2^H+\sigma^2\boldsymbol{I}_{N_{\mathrm{R}}})\right),\\ h(\boldsymbol{Y}) &\leq \log\det\left(\pi\,\mathrm{e}(\boldsymbol{H}_1\boldsymbol{Q}_1\boldsymbol{H}_1^H+\boldsymbol{H}_2\boldsymbol{Q}_2\boldsymbol{H}_2^H+\sigma^2\boldsymbol{I}_{N_{\mathrm{R}}})\right) \end{aligned}$$

with equalities if we have circularly symmetric complex Gaussian inputs $\boldsymbol{X}_k \sim \mathcal{CN}(\boldsymbol{0},\boldsymbol{Q}_k)$, $k=1,2$. It follows that the optimal input for given covariance matrix is distributed according to zero mean circularly symmetric complex Gaussian distribution. With $h(\boldsymbol{N}_{\mathrm{R}}) = \log\det\left(\pi\,\mathrm{e}\,\sigma^2\boldsymbol{I}_{N_{\mathrm{R}}}\right)$ and the optimal input distributions $\boldsymbol{X}_k \sim \mathcal{CN}(\boldsymbol{0},\boldsymbol{Q}_k)$, $k=1,2$, the individual and sum-rate constraints can be expressed as

$$R_{\overrightarrow{1\mathrm{R}}}(\boldsymbol{Q}_1) := \log\det\left(\boldsymbol{I}_{N_{\mathrm{R}}}+\tfrac{1}{\sigma^2}\boldsymbol{H}_1\boldsymbol{Q}_1\boldsymbol{H}_1^H\right), \tag{2.73a}$$

$$R_{\overrightarrow{2\mathrm{R}}}(\boldsymbol{Q}_2) := \log\det\left(\boldsymbol{I}_{N_{\mathrm{R}}}+\tfrac{1}{\sigma^2}\boldsymbol{H}_2\boldsymbol{Q}_2\boldsymbol{H}_2^H\right), \tag{2.73b}$$

$$R_{\Sigma}^{\mathrm{MAC}}(\boldsymbol{Q}_1,\boldsymbol{Q}_2) := \log\det\left(\boldsymbol{I}_{N_{\mathrm{R}}}+\tfrac{1}{\sigma^2}\boldsymbol{H}_1\boldsymbol{Q}_1\boldsymbol{H}_1^H+\tfrac{1}{\sigma^2}\boldsymbol{H}_2\boldsymbol{Q}_2\boldsymbol{H}_2^H\right). \tag{2.73c}$$

Accordingly, for covariance matrices $\boldsymbol{Q}_1$ and $\boldsymbol{Q}_2$ we can achieve rate pairs within the region

$$\begin{aligned} \mathcal{C}_{\mathrm{MAC}}^{\mathrm{MIMO}}(\boldsymbol{Q}_1,\boldsymbol{Q}_2) := \Big\{[R_1,R_2]\in\mathbb{R}_+^2 :\ & R_1+R_2 \leq R_{\Sigma}^{\mathrm{MAC}}(\boldsymbol{Q}_1,\boldsymbol{Q}_2),\\ & R_1 \leq R_{\overrightarrow{1\mathrm{R}}}(\boldsymbol{Q}_1), R_2 \leq R_{\overrightarrow{2\mathrm{R}}}(\boldsymbol{Q}_2)\Big\}. \end{aligned}$$

For each pair of covariance matrices the achievable region is described by a pentagon. As in the scalar case, the encoding methods to achieve the vertices can be deduced from the single-user MIMO Gaussian channel. For the decoding the relay node applies successive interference cancellation so that each vertex corresponds to a certain decoding order. In more detail, we can achieve the vertex $\boldsymbol{\nu}_{\Sigma 1}(\boldsymbol{Q}_1, \boldsymbol{Q}_2) := [R_{\overrightarrow{1\mathrm{R}}}(\boldsymbol{Q}_1), R_2^{\Sigma 1}(\boldsymbol{Q}_1, \boldsymbol{Q}_2)]$ with

$$\begin{aligned} R_2^{\Sigma 1}(\boldsymbol{Q}_1, \boldsymbol{Q}_2) :=& R_\Sigma^{\mathrm{MAC}}(\boldsymbol{Q}_1, \boldsymbol{Q}_2) - R_{\overrightarrow{1\mathrm{R}}}(\boldsymbol{Q}_1) \\ =& \log \frac{\det\left(\sigma^2 \boldsymbol{I}_{N_\mathrm{R}} + \boldsymbol{H}_1 \boldsymbol{Q}_1 \boldsymbol{H}_1^H + \boldsymbol{H}_2 \boldsymbol{Q}_2 \boldsymbol{H}_2^H\right)}{\det\left(\sigma^2 \boldsymbol{I}_{N_\mathrm{R}} + \boldsymbol{H}_1 \boldsymbol{Q}_1 \boldsymbol{H}_1^H\right)}, \end{aligned} \tag{2.74}$$

if node 1 encodes as in single-user Gaussian MIMO channel to achieve the rate $R_{\overrightarrow{1\mathrm{R}}}(\boldsymbol{Q}_1)$. It therefore uses a code with codewords distributed according to a circularly symmetric complex Gaussian distribution $\mathcal{CN}(\boldsymbol{0}, \boldsymbol{Q}_1)$. Node 2 encodes as in the single-user Gaussian MIMO channel to achieve the rate $R_2^{\Sigma 1}(\boldsymbol{Q}_1, \boldsymbol{Q}_2)$. The relay node first decodes the message of node 2 with high probability while it considers the interference of node 1 as additional Gaussian noise with covariance $\boldsymbol{H}_1 \boldsymbol{Q}_1 \boldsymbol{H}_1^H$. After the message of node 2 is determined, the relay can cancel the interference $\boldsymbol{H}_2 \boldsymbol{X}_2$ from the receive vector $\boldsymbol{Y}_\mathrm{R}$ so that it can decode the message of node 1 with high probability.

Similarly, we can achieve the vertex $\boldsymbol{\nu}_{2\Sigma}(\boldsymbol{Q}_1, \boldsymbol{Q}_2) := [R_1^{2\Sigma}(\boldsymbol{Q}_1, \boldsymbol{Q}_2), R_{\overrightarrow{2\mathrm{R}}}(\boldsymbol{Q}_2)]$ with

$$\begin{aligned} R_1^{2\Sigma}(\boldsymbol{Q}_1, \boldsymbol{Q}_2) :=& R_\Sigma^{\mathrm{MAC}}(\boldsymbol{Q}_1, \boldsymbol{Q}_2) - R_{\overrightarrow{2\mathrm{R}}}(\boldsymbol{Q}_2) \\ =& \log \frac{\det\left(\sigma^2 \boldsymbol{I}_{N_\mathrm{R}} + \boldsymbol{H}_1 \boldsymbol{Q}_1 \boldsymbol{H}_1^H + \boldsymbol{H}_2 \boldsymbol{Q}_2 \boldsymbol{H}_2^H\right)}{\det\left(\sigma^2 \boldsymbol{I}_{N_\mathrm{R}} + \boldsymbol{H}_2 \boldsymbol{Q}_2 \boldsymbol{H}_2^H\right)} \end{aligned}$$

by interchanging the decoding order and the corresponding encoding at nodes 1 and 2. In accordance, let π_k, $k = 1, 2$, denote the decoding order where node k is decoded last.

However, in contrast to the scalar case there are in general no input distributions so that the sum-rate and individual rate constraints are tight simultaneously. This means that the pair of covariance matrices which maximize the individual rate constraints in general do not maximize the sum-rate constraint as well. Therefore, the capacity region of the Gaussian MIMO-MAC is given by the convex hull of the union over all circularly symmetric complex Gaussian inputs that satisfy the power constraints, $\mathrm{tr}(\boldsymbol{Q}_k) \leq P_k$,

$$\mathcal{C}_{\mathrm{MAC}}^{\mathrm{MIMO}} := \mathrm{co}\left(\bigcup_{\mathrm{tr}(\boldsymbol{Q}_k) \leq P_k, k=1,2} \mathcal{C}_{\mathrm{MAC}}^{\mathrm{MIMO}}(\boldsymbol{Q}_1, \boldsymbol{Q}_2) \right).$$

This means that the capacity region $\mathcal{C}_{\mathrm{MAC}}^{\mathrm{MIMO}}$ is the convex hull of the union over all pentagons corresponding to a pair of covariance matrices. In general we will have two curved sections on the boundary, each corresponds to a certain decoding order. Thereby, each rate pair on the curved section is achieved by an individual set of covariance matrices. The sum-rate optimal section between the curved parts can only be achieved with time-sharing. Finally, we will have sections where we achieve the single-user capacities. It can be easily seen by contradiction that for any rate pair on the boundary the power constraints are satisfied with equality. In [WB04] the discussion of the boundary is presented for an arbitrary number of users, we briefly look at the sections of the boundary for our two user multiple access channel.

Since the boundary of a convex set can be characterized by rate pairs that maximize the weighted rate sum we consider the optimization problem

$$\boldsymbol{R}_{\mathrm{MAC}}^{\mathrm{MIMO}}(\boldsymbol{q}) = \operatorname*{arg\,max}_{[R_1,R_2]\in\mathcal{C}_{\mathrm{MAC}}^{\mathrm{MIMO}}} q_1 R_1 + q_2 R_2 \tag{2.75}$$

for a weight vector $\boldsymbol{q} \in \mathbb{R}_+^2 \setminus \{\boldsymbol{0}\}$. There are multiple solutions if we have $q_1 = q_2$, $q_1 = 0, q_2 > 0$, or $q_1 > 0, q_2 = 0$, which gets clearer from the following.

First, we consider weight vectors $\boldsymbol{q}$ with $q_1 > 0$ and $q_2 = 0$. This means, we want to maximize the unidirectional rate R_1 which then corresponds to the single-user capacity. Therefore, we have to maximize (2.73a) according to the water-filling procedure given in the previous section which gives us

$$R_{\overrightarrow{1\mathrm{R}}}^{*} := \max_{\boldsymbol{R}\in\mathcal{C}_{\mathrm{MAC}}} R_1 = \max_{\boldsymbol{Q}_1 \,:\, \mathrm{tr}(\boldsymbol{Q}_1)\le P_1} R_{\overrightarrow{1\mathrm{R}}}(\boldsymbol{Q}_1) = R_{\overrightarrow{1\mathrm{R}}}(\boldsymbol{Q}_1^*)$$

using the water-filling solution $\boldsymbol{Q}_1^* := \boldsymbol{Q}(\boldsymbol{H}_1^H\boldsymbol{H}_1, P_1)$. To achieve this we have to apply the decoding order π_1 which allows node 2 to transmit with rates $R_2^{\Sigma 1}(\boldsymbol{Q}_1^*, \boldsymbol{Q}_2)$. Then with the Cholesky-decomposition $\boldsymbol{C}_1\boldsymbol{C}_1^H := (\boldsymbol{I}_{N_\mathrm{R}} + \frac{1}{\sigma^2}\boldsymbol{H}_1\boldsymbol{Q}_1^*\boldsymbol{H}_1^H)^{-1}$ we can define an equivalent channel $\tilde{\boldsymbol{H}}_2 := \boldsymbol{C}_1\boldsymbol{H}_2$ so that we can rewrite (2.74) as follows

$$R_2^{\Sigma 1}(\boldsymbol{Q}_1^*, \boldsymbol{Q}_2) = \log\det\left(\boldsymbol{I}_{N_\mathrm{R}} + \tfrac{1}{\sigma^2}\tilde{\boldsymbol{H}}_2\boldsymbol{Q}_2\tilde{\boldsymbol{H}}_2^H\right).$$

Then the water-filling solution $\boldsymbol{Q}(\tilde{\boldsymbol{H}}_2^H\tilde{\boldsymbol{H}}_2, P_2)$ maximizes the rate $R_2^{\Sigma 1}(\boldsymbol{Q}_1^*, \boldsymbol{Q}_2)$. Both together gives us the first characteristic rate pair and section where node 1 achieves its single-user capacity

$$\begin{aligned}
\boldsymbol{E}_1 &:= \left[R_{\overrightarrow{1\mathrm{R}}}^{*},\, R_2^{\Sigma 1}\big(\boldsymbol{Q}_1^*, \boldsymbol{Q}(\tilde{\boldsymbol{H}}_2^H\tilde{\boldsymbol{H}}_2, P_2)\big)\right], \\
\mathcal{E}_1 &:= \left\{[R_{\overrightarrow{1\mathrm{R}}}^{*}, R_2] : 0 \le R_2 \le R_2^{\Sigma 1}\big(\boldsymbol{Q}_1^*, \boldsymbol{Q}(\tilde{\boldsymbol{H}}_2^H\tilde{\boldsymbol{H}}_2, P_2)\big)\right\}.
\end{aligned}$$

The same procedure applies to weight vectors $\boldsymbol{q}$ with $q_1 = 0$ and $q_2 > 0$ so that we get the corresponding characteristic rate pair and section where node 2 achieves its single-user capacity

$$\boldsymbol{E}_2 := \left[R_2^{\Sigma 1}\big(\boldsymbol{Q}_2^*, \boldsymbol{Q}(\tilde{\boldsymbol{H}}_1^H \tilde{\boldsymbol{H}}_1, P_1)\big)\,,\ R_{\overrightarrow{2\mathrm{R}}}^*\right],$$
$$\mathcal{E}_2 := \left\{[R_1, R_{\overrightarrow{2\mathrm{R}}}^*] : 0 \leq R_1 \leq R_1^{2\Sigma}\big(\boldsymbol{Q}(\tilde{\boldsymbol{H}}_1^H \tilde{\boldsymbol{H}}_1, P_1), \boldsymbol{Q}_2^*\big)\right\}$$

with the maximal unidirectional rate $R_{\overrightarrow{2\mathrm{R}}}^* := R_{\overrightarrow{2\mathrm{R}}}(\boldsymbol{Q}_2^*)$ achieved with the water-filling solutions $\boldsymbol{Q}_2^* := \boldsymbol{Q}(\boldsymbol{H}_2^H \boldsymbol{H}_2, P_2)$ and $\boldsymbol{Q}(\tilde{\boldsymbol{H}}_1^H \tilde{\boldsymbol{H}}_1, P_1)$ with the equivalent channel $\tilde{\boldsymbol{H}}_1 := \boldsymbol{C}_2 \boldsymbol{H}_1$ using the Cholesky-decomposition $\boldsymbol{C}_2 \boldsymbol{C}_2^H := (\boldsymbol{I}_{N_\mathrm{R}} + \frac{1}{\sigma^2} \boldsymbol{H}_2 \boldsymbol{Q}_2^* \boldsymbol{H}_2^H)^{-1}$.

For $q_1 \geq q_2$ it is shown in [BW06] that the decoding order π_1 is optimal. For that reason we can rewrite (2.75) as follows

$$\begin{aligned}\boldsymbol{R}_{\mathrm{MAC}}^{\mathrm{MIMO}}(\boldsymbol{q}) &= \underset{[R_1,R_2]\in\mathcal{C}_{\mathrm{MAC}}^{\mathrm{MIMO}}}{\arg\max}\ q_1 R_{\overrightarrow{1\mathrm{R}}}(\boldsymbol{Q}_1) + q_2 R_2^{\Sigma 1}(\boldsymbol{Q}_1, \boldsymbol{Q}_2) \\ &= \underset{[R_1,R_2]\in\mathcal{C}_{\mathrm{MAC}}^{\mathrm{MIMO}}}{\arg\max}\ (q_1 - q_2) R_{\overrightarrow{1\mathrm{R}}}(\boldsymbol{Q}_1) + q_2 R_\Sigma^{\mathrm{MAC}}(\boldsymbol{Q}_1, \boldsymbol{Q}_2).\end{aligned}$$

With $q_1 \geq q_2$ the objective is a sum of concave functions and therefore the optimization problem is convex [BW06]. Similarly, it is shown that the decoding order π_2 is optimal for weight vectors with $q_2 \geq q_1$.

In the following we look at the Lagrangian function of the optimization problem for the case $q_1 \geq q_2$,

$$\begin{aligned}L(\boldsymbol{Q}_1, \boldsymbol{Q}_2, \boldsymbol{\Psi}_1, \boldsymbol{\Psi}_2, \mu_1, \mu_2) = \quad & -(q_1 - q_2) R_{\overrightarrow{1\mathrm{R}}}(\boldsymbol{Q}_1) - q_2 R_\Sigma^{\mathrm{MAC}}(\boldsymbol{Q}_1, \boldsymbol{Q}_2) \\ & - \sum_{k=1}^{2} \mu_k\big(P_k - \mathrm{tr}(\boldsymbol{Q}_k)\big) - \sum_{k=1}^{2} \mathrm{tr}(\boldsymbol{Q}_k \boldsymbol{\Psi}_k).\end{aligned}$$

Similarly to [WB04] the optimal transmit strategies $\boldsymbol{Q}_1$ and $\boldsymbol{Q}_2$ for a rate pair $\boldsymbol{R}_{\mathrm{MAC}}^{\mathrm{MIMO}}(\boldsymbol{q})$ are uniquely characterized by the Karush-Kuhn-Tucker (KKT) conditions

$$\begin{aligned}\mu_1 \boldsymbol{I}_{N_1} + \boldsymbol{\Psi}_1 =& \boldsymbol{H}_1^H\big((q_2 - q_1)\big[\sigma^2 \boldsymbol{I}_{N_\mathrm{R}} + \boldsymbol{H}_1 \boldsymbol{Q}_1 \boldsymbol{H}_1^H\big]^{-1} \\ & - q_2\big[\sigma^2 \boldsymbol{I}_{N_\mathrm{R}} + \boldsymbol{H}_1 \boldsymbol{Q}_1 \boldsymbol{H}_1^H + \boldsymbol{H}_2 \boldsymbol{Q}_2 \boldsymbol{H}_2^H\big]^{-1}\big)\boldsymbol{H}_1, \qquad (2.76a)\\ \mu_2 \boldsymbol{I}_{N_2} + \boldsymbol{\Psi}_2 =& - q_2 \boldsymbol{H}_2^H\big(\sigma^2 \boldsymbol{I}_{N_\mathrm{R}} + \boldsymbol{H}_1 \boldsymbol{Q}_1 \boldsymbol{H}_1^H + \boldsymbol{H}_2 \boldsymbol{Q}_2 \boldsymbol{H}_2^H\big)^{-1} \boldsymbol{H}_2, \qquad (2.76b)\\ \mathrm{tr}(\boldsymbol{Q}_k \boldsymbol{\Psi}_k) = 0, & \qquad \mu_k(P_k - \mathrm{tr}(\boldsymbol{Q}_k)) = 0, \qquad k = 1, 2, \qquad (2.76c)\\ \boldsymbol{\Psi}_k \succeq \boldsymbol{0}_{N_k}, & \qquad \mu_k \geq 0, \qquad k = 1, 2, \qquad (2.76d)\\ \boldsymbol{Q}_k \succeq \boldsymbol{0}_{N_k}, & \qquad P_k \geq \mathrm{tr}(\boldsymbol{Q}_k), \qquad k = 1, 2, \qquad (2.76e)\end{aligned}$$

with complementary slackness, dual, and primal conditions (2.76c), (2.76d), and (2.76e) respectively. Since the optimization problem is convex, efficient algorithms, like the interior-point method, exist to calculate the optimal covariance matrices [BV04]. This allows us to specify the curved section on the boundary corresponding to the decoding order π_1 as follows

$$\mathcal{D}_1 := \left\{\boldsymbol{R}_{\mathrm{MAC}}^{\mathrm{MIMO}}(\boldsymbol{q}) : \boldsymbol{q} \in \mathbb{R}_+^2, q_1 \geq q_2 > 0\right\}.$$

Furthermore, we denote by $\boldsymbol{D}_1$ the sum-rate maximum $\boldsymbol{R}_{\mathrm{MAC}}^{\mathrm{MIMO}}([1,1]) \in \mathcal{D}_1$.

Similarly, in the case $q_1 \leq q_2$ the decoding order π_2 is optimal. Then the Lagrange function and KKT conditions follow by interchanging the indices 1 and 2 so that the set

$$\mathcal{D}_2 := \left\{\boldsymbol{R}_{\mathrm{MAC}}^{\mathrm{MIMO}}(\boldsymbol{q}) : \boldsymbol{q} \in \mathbb{R}_+^2, q_1 \leq q_2 > 0\right\}$$

denotes the curved section on the boundary corresponding to the decoding order π_2. Accordingly, let $\boldsymbol{D}_2 = \boldsymbol{R}_{\mathrm{MAC}}^{\mathrm{MIMO}}([1,1]) \in \mathcal{D}_2$ denote its sum-rate maximum.

The last section on the boundary is given by the connecting line between the sum-rate maxima $\boldsymbol{D}_1$ and $\boldsymbol{D}_2$,

$$\mathcal{T} := \left\{\boldsymbol{R} : \boldsymbol{R} = \tau \boldsymbol{D}_1 + (1-\tau)\boldsymbol{D}_2, \tau \in [0,1]\right\}$$

and can be reached by time-sharing between the corresponding strategies of $\boldsymbol{D}_1$ and $\boldsymbol{D}_2$ only. If one is interested in sum-rate optimal rate pairs only, one can also use an iterative water-filling algorithm presented in [YRBC01].

In Figure 2.21 we depicted the characteristic sections and rate pairs of the capacity region $\mathcal{C}_{\mathrm{MAC}}^{\mathrm{MIMO}}$ for an example where each node is equipped with two antennas.

2.6.2 MIMO Broadcast Phase

As in the scalar case in the BC phase the relay forwards the messages m_1 and m_2 which it has received in the previous MIMO MAC phase. Since we consider in this chapter superposition encoding, the messages are separately encoded and afterwards the superposition of both is transmitted. Again, since we assume a memoryless channel, it is sufficient to consider vector-valued single-letters only. To this end, let $\boldsymbol{W}_1 \in \mathbb{C}^{N_{\mathrm{R}}}$ denote a single-letters of the codeword for message m_1 for node 2 and $\boldsymbol{W}_2 \in \mathbb{C}^{N_{\mathrm{R}}}$ a single-letter of the codeword m_2 for node 1 so that the input of the relay node in the broadcast channel is given by the sum

$$\boldsymbol{X}_{\mathrm{R}} = \boldsymbol{W}_1 + \boldsymbol{W}_2.$$

We assume that the messages m_1 and m_2 are independent so that the random vectors $\boldsymbol{W}_1$ and $\boldsymbol{W}_2$ are independent as well. From this we get the vector-valued output at node k, $k = 1, 2$, as follows

$$\boldsymbol{Y}_k = \boldsymbol{H}_k^H \boldsymbol{X}_{\mathrm{R}} + \boldsymbol{N}_k = \boldsymbol{H}_k^H \boldsymbol{W}_1 + \boldsymbol{H}_k^H \boldsymbol{W}_2 + \boldsymbol{N}_k, \qquad k = 1, 2.$$

As in the scalar case, the receiving nodes perform interference cancellation so that we essentially have two interference-free MIMO Gaussian channels between the relay node and the nodes 1 and 2. It follows that the achievable rates for transmitting the unknown messages have to fulfill the constraints

$$\begin{aligned} R_2 &\leq I(\boldsymbol{X}_{\mathrm{R}}; \boldsymbol{Y}_1 | \boldsymbol{W}_1) = I(\boldsymbol{W}_2; \boldsymbol{H}_1^H \boldsymbol{W}_2 + \boldsymbol{N}_1), \\ R_1 &\leq I(\boldsymbol{X}_{\mathrm{R}}; \boldsymbol{Y}_2 | \boldsymbol{W}_2) = I(\boldsymbol{W}_1; \boldsymbol{H}_2^H \boldsymbol{W}_1 + \boldsymbol{N}_2) \end{aligned}$$

for some input distributions $f_{\boldsymbol{W}_1}(\boldsymbol{w}_1)$ and $f_{\boldsymbol{W}_2}(\boldsymbol{w}_2)$ which satisfy the power constraint $\mathbb{E}\{\boldsymbol{X}_{\mathrm{R}}^H \boldsymbol{X}_{\mathrm{R}}\} = \mathbb{E}\{\boldsymbol{W}_1^H \boldsymbol{W}_1\} + \mathbb{E}\{\boldsymbol{W}_2^H \boldsymbol{W}_2\} \leq P_{\mathrm{R}}$.

Obviously, the mutual informations are only coupled by the relay power distribution. As in the scalar case, we denote by β_1 and β_2 the proportion of relay transmit power P_{R} spend for the codewords $\boldsymbol{W}_1$ and $\boldsymbol{W}_2$ respectively. Then the simplex

$$\mathcal{B} = \{[\beta_1, \beta_2] \in [0,1] \times [0,1] : \ \beta_1 + \beta_2 \leq 1\}$$

characterizes the set of feasible relay power distributions that satisfy the relay transmit power constraint.

For any feasible relay power distribution $[\beta_1, \beta_2] \in \mathcal{B}$ we have two separated MIMO Gaussian channels so that circularly symmetric complex Gaussian distributed inputs

$$\begin{aligned} \boldsymbol{W}_1 &\sim \mathcal{CN}(\boldsymbol{0}, \boldsymbol{Q}_{\mathrm{R},1}(\beta_1)) \quad \text{with} \quad \boldsymbol{Q}_{\mathrm{R},1}(\beta_1) := \boldsymbol{Q}(\boldsymbol{H}_2 \boldsymbol{H}_2^H, \beta_1 P_{\mathrm{R}}), \\ \boldsymbol{W}_2 &\sim \mathcal{CN}(\boldsymbol{0}, \boldsymbol{Q}_{\mathrm{R},2}(\beta_2)) \quad \text{with} \quad \boldsymbol{Q}_{\mathrm{R},2}(\beta_2) := \boldsymbol{Q}(\boldsymbol{H}_1 \boldsymbol{H}_1^H, \beta_2 P_{\mathrm{R}}) \end{aligned}$$

maximize the mutual informations. Thereby, the optimal covariance matrices $\boldsymbol{Q}_{\mathrm{R},1}(\beta_1)$ and $\boldsymbol{Q}_{\mathrm{R},2}(\beta_2)$ are given by the water-filling solution as described for the Gaussian MIMO channel in Section 2.6. Therefore, in the MIMO BC phase we can achieve rate pairs within the rate region

$$\mathcal{R}_{\mathrm{BC}}^{\mathrm{MIMO}} = \Big\{[R_1, R_2] \in \mathbb{R}_+^2 : R_1 \leq R_{\overrightarrow{\mathrm{R2}}}(\beta_1), R_2 \leq R_{\overrightarrow{\mathrm{R1}}}(\beta_2), \, [\beta_1, \beta_2] \in \mathcal{B}\Big\}$$

with rate constraints

$$R_{\overrightarrow{\mathrm{R2}}}(\beta_1) := \log\det\big(\boldsymbol{I}_{N_2} + \tfrac{1}{\sigma^2} \boldsymbol{H}_2^H \boldsymbol{Q}_{\mathrm{R},1}(\beta_1) \boldsymbol{H}_2\big), \tag{2.77}$$

$$R_{\overrightarrow{\mathrm{R1}}}(\beta_2) := \log\det\big(\boldsymbol{I}_{N_1} + \tfrac{1}{\sigma^2} \boldsymbol{H}_1^H \boldsymbol{Q}_{\mathrm{R},2}(\beta_2) \boldsymbol{H}_1\big) \tag{2.78}$$

for the MIMO broadcast channel. It can be easily seen by contradiction that for any rate pair on the boundary of the achievable rate region we have $\beta_1 + \beta_2 = 1$.

From the water-filling solution we see that with increasing power it is optimal to increase the number of the used eigenmodes if the power exceeds certain thresholds. In the low-power regime we use the strongest eigenmode of the channel only, which is known as beamforming. In the high-power regime it is optimal to use all eigenmodes. The maximal number of eigenmodes is given by the rank of the channel matrix. In the next proposition we specify the threshold values which also characterize the so-called beamforming optimality range and high-power regime. To this end, let

$$\boldsymbol{H}_k\boldsymbol{H}_k^H = \boldsymbol{V}_k\boldsymbol{\Sigma}_k\boldsymbol{V}_k^H, \qquad \text{with } \boldsymbol{\Sigma}_k := \operatorname{diag}(\lambda_{k,1}, \lambda_{k,2}, \dots, \lambda_{k,N_\mathrm{R}})$$

with unitary matrix $\boldsymbol{V}_k \in \mathbb{C}^{N_\mathrm{R}\times\mathrm{R}}$ be the eigenvalue decomposition of $\boldsymbol{H}_k\boldsymbol{H}_k^H \in \mathbb{C}^{N_\mathrm{R}\times\mathrm{R}}$, $k = 1, 2$ with eigenvalues sorted in decreasing order, i.e. $\lambda_{k,1} \geq \cdots \geq \lambda_{k,N_\mathrm{R}} \geq 0$. Furthermore, we define the coefficients

$$L_{k,m} := \sum_{i=1}^{m} \frac{1}{\lambda_{k,i}}, \qquad k = 1, 2$$

for $m = 1, 2, \dots, r_2$ for $k = 1$ and $m = 1, 2, \dots, r_1$ for $k = 2$ respectively and $L_{k,0} := 0$ for $k = 1, 2$.

Proposition 2.52. *For $k = 1$ with $r_2 = \operatorname{rank}(\boldsymbol{H}_2)$ maximal possible eigenmodes it is optimal to use*

$$\begin{cases} r_2 \ \textit{eigenmodes, if} \ \beta_1 P_\mathrm{R} > \sigma^2\big(\frac{r_2-1}{\lambda_{2,r_2}} - L_{2,r_2-1}\big), \\ 1 \ \textit{eigenmode, if} \ \beta_1 P_\mathrm{R} \leq \sigma^2\big(\frac{1}{\lambda_{2,2}} - L_{2,1}\big), \textit{or} \\ m \ \textit{eigenmodes, if} \ \sigma^2\big(\frac{m}{\lambda_{2,m}} - L_{2,m-1}\big) < \beta_1 P_\mathrm{R} \leq \sigma^2\big(\frac{m}{\lambda_{2,m+1}} - L_{2,m}\big). \end{cases}$$

Similarly, for $k = 2$ with $r_1 = \operatorname{rank}(\boldsymbol{H}_1)$ maximal possible eigenmodes it is optimal to use

$$\begin{cases} r_1 \ \textit{eigenmodes, if} \ \beta_2 P_\mathrm{R} > \sigma^2\big(\frac{r_1-1}{\lambda_{1,r_1}} - L_{1,r_1-1}\big), \\ 1 \ \textit{eigenmode, if} \ \beta_2 P_\mathrm{R} \leq \sigma^2\big(\frac{1}{\lambda_{1,2}} - L_{1,1}\big), \textit{or} \\ m \ \textit{eigenmodes, if} \ \sigma^2\big(\frac{n}{\lambda_{1,m}} - L_{1,m-1}\big) < \beta_2 P_\mathrm{R} \leq \sigma^2\big(\frac{m}{\lambda_{1,m+1}} - L_{1,m}\big). \end{cases}$$

Proof. The proof can be found in Appendix 2.8.34. □

Next, we are interested in the weighted rate sum maximum

$$\boldsymbol{R}_\mathrm{BC}^\mathrm{MIMO}(\boldsymbol{q}) := \operatorname*{arg\,max}_{[R_1,R_2]\in\mathcal{R}_\mathrm{BC}^\mathrm{MIMO}} q_1R_1 + q_2R_2. \tag{2.79}$$

Since $\mathcal{R}_\mathrm{BC}^\mathrm{MIMO}$ is convex, the strategy is similar to the previous derivations where we look for the rate pair on the boundary whose normal vector has the same angle than the weight vector

$\boldsymbol{q}$. Therefore, we have to identify for each rate pair on the boundary $\mathcal{R}_{\mathrm{BC}}^{\mathrm{MIMO}}$ the number of used eigenmodes of the channels $\boldsymbol{H}_k$, $k = 1, 2$.

Proposition 2.52 characterizes the number of eigenmodes with respect to the power $\beta_1 P_{\mathrm{R}}$ and $\beta_2 P_{\mathrm{R}}$. We will look at a parametrization of the boundary in terms of $\beta = \beta_1 = 1 - \beta_2$. Therefore, we solve the power threshold values for β_1 and β_2 so that we can express the threshold values in terms of the power fraction as follows

$$\beta_{1,m} := \frac{1}{\gamma_{\mathrm{R}}}\big(\frac{m}{\lambda_{2,m+1}} - L_{2,m}\big), \qquad m = 1, 2, \dots, r_2 - 1,$$
$$\beta_{2,m} := \frac{1}{\gamma_{\mathrm{R}}}\big(\frac{m}{\lambda_{1,m+1}} - L_{1,m}\big), \qquad m = 1, 2, \dots, r_1 - 1.$$

Additionally, we define the value $\beta_{k,0} := 0$, $k = 1, 2$, as the value where no eigenmode is used. In order to identify the number of eigenmodes used for a given relay power fraction β_1 or β_2 we define the intervals

$$\mathcal{B}_{1,m} := (\beta_{1,m-1}, \beta_{1,m}], \qquad m = 1, 2, \dots, r_2 - 1,$$
$$\mathcal{B}_{2,m} := (\beta_{2,m-1}, \beta_{2,m}], \qquad m = 1, 2, \dots, r_1 - 1$$

with $\mathcal{B}_{1,r_2} := (\beta_{1,r_2-1}, \infty)$, $\mathcal{B}_{2,r_1} := (\beta_{1,r_1-1}, \infty)$, and $\mathcal{B}_{k,0} := \{0\}$, $k = 1, 2$. Therewith, we can define the following indicator functions

$$m_1 : [0, 1] \to \{0, 1, 2, \dots, r_2\}, \quad \beta_1 \mapsto m \text{ where we have } \beta_1 \in \mathcal{B}_{1,m},$$
$$m_2 : [0, 1] \to \{0, 1, 2, \dots, r_1\}, \quad \beta_2 \mapsto m \text{ where we have } \beta_2 \in \mathcal{B}_{2,m},$$

which specify the numbers of used eigenmodes for a power fraction β_1 or β_2. Obviously, for $\beta_1 = 0$ we have $R_{\overrightarrow{\mathrm{R2}}}(\beta_1) = 0$ and for $\beta_2 = 0$ we have $R_{\overrightarrow{\mathrm{R1}}}(\beta_2) = 0$. For $\beta_1, \beta_2 > 0$ the characterization of the used eigenmodes allows us to state in the next proposition a closed form solution of the rates $R_{\overrightarrow{\mathrm{R2}}}(\beta_1)$ and $R_{\overrightarrow{\mathrm{R1}}}(\beta_2)$, which we use for the derivation of the weighted rate sum optimal rate pair.

Proposition 2.53. *The rates $R_{\overrightarrow{\mathrm{R2}}}(\beta_1)$ or $R_{\overrightarrow{\mathrm{R1}}}(\beta_2)$ are zero if β_1 or β_2 are equal to zero. For $\beta_1, \beta_2 > 0$ we have*

$$R_{\overrightarrow{\mathrm{R2}}}(\beta_1) = \sum_{n=1}^{m_1(\beta_1)} \log\Big(\frac{\lambda_{2,n}}{m_1(\beta_1)}(\beta_1\gamma_{\mathrm{R}} + L_{2,m_1(\beta_1)})\Big),$$
$$R_{\overrightarrow{\mathrm{R1}}}(\beta_2) = \sum_{n=1}^{m_2(\beta_2)} \log\Big(\frac{\lambda_{1,n}}{m_2(\beta_2)}(\beta_2\gamma_{\mathrm{R}} + L_{1,m_2(\beta_2)})\Big).$$

Proof. The proof can be found in Appendix 2.8.35. □

Then

$$M_1 := \left|\left\{\beta_{1,m} \;:\; \beta_{1,m} \leq 1 \text{ for } m = 0, 1, 2, \dots, r_2 - 1\right\}\right| \quad \text{and}$$
$$M_2 := \left|\left\{\beta_{2,m} \;:\; \beta_{2,m} \leq 1 \text{ for } m = 0, 1, 2, \dots, r_1 - 1\right\}\right|$$

denote the numbers of maximal used eigenmodes. Next we define $M := M_1 + M_2$. Then let $[\tilde{\beta}_1, \tilde{\beta}_2, \dots \tilde{\beta}_M]$ denote the sorted vector of $[\beta_{1,0}, \beta_{1,1}, \dots, \beta_{1,M_1-1}, 1 - \beta_{2,0}, 1 - \beta_{2,1}, \dots, \beta_{2,M_2-1}]$ in increasing order, i.e. we have $0 = \tilde{\beta}_1 \leq \tilde{\beta}_2 \leq \cdots \leq \tilde{\beta}_M = 1$.

We are now prepared to characterize in closed form the rate pair which maximizes the weighted rate sum according to (2.79).

Theorem 2.54. *Let $\boldsymbol{q} = [q_1, q_2] \in \mathbb{R}_+^2 \setminus \{\mathbf{0}\}$ denote a weight vector with nonnegative elements and angle $\theta_{\boldsymbol{q}} := \arccos\left(\frac{q_1}{\sqrt{q_1^2+q_2^2}}\right)$, then the rate pair $\boldsymbol{R}_{\mathrm{BC}}^{\mathrm{MIMO}}(\boldsymbol{q})$ where the weighted rate sum is maximized is given by*

$$\boldsymbol{R}_{\mathrm{BC}}^{\mathrm{MIMO}}(\boldsymbol{q}) = \begin{cases} [0, R_{2,\mathrm{BC}}([0,1])], & \text{if } \theta_{\boldsymbol{q}} \geq \varphi_1, \\ [R_{1,\mathrm{BC}}(\boldsymbol{q}), R_{2,\mathrm{BC}}(\boldsymbol{q})], & \text{if } \varphi_1 > \theta_{\boldsymbol{q}} > \varphi_M, \\ [R_{1,\mathrm{BC}}([1,0]), 0], & \text{if } \varphi_M \geq \theta_{\boldsymbol{q}}, \end{cases}$$

with characteristic angles where the number of allocated eigenmodes changes

$$\varphi_n := \begin{cases} \arctan \frac{\gamma_{\mathrm{R}} + L_{1,M_1}}{M_1 L_{2,1}}, & \text{if } n = 1, \\ \arctan \frac{m_1(\tilde{\beta}_n)\left((1-\tilde{\beta}_n)\gamma_{\mathrm{R}} + L_{1,m_2(1-\tilde{\beta}_n)}\right)}{m_2(1-\tilde{\beta}_n)\left(\tilde{\beta}_n \gamma_{\mathrm{R}} + L_{2,m_1(\tilde{\beta}_n)}\right)}, & \text{if } 1 < n < M, \\ \arctan \frac{M_2 L_{1,1}}{\gamma_{\mathrm{R}} + L_{2,M_2}}, & \text{if } n = M, \end{cases}$$

which are given for a weight vector $\boldsymbol{q}$ by

$$\eta_1(\boldsymbol{q}) := \begin{cases} 0, & \text{if } \theta_{\boldsymbol{q}} \geq \varphi_1, \\ m_1(\tilde{\beta}_{n+1}), & \text{if } \varphi_n > \theta_{\boldsymbol{q}} \geq \varphi_{n+1}, \\ M_1, & \text{if } \varphi_M > \theta_{\boldsymbol{q}}, \end{cases}$$

$$\eta_2(\boldsymbol{q}) := \begin{cases} M_2, & \text{if } \theta_{\boldsymbol{q}} > \varphi_1, \\ m_2(\tilde{\beta}_n), & \text{if } \varphi_n \geq \theta_{\boldsymbol{q}} > \varphi_{n+1}, \\ 0, & \text{if } \varphi_M \geq \theta_{\boldsymbol{q}}, \end{cases}$$

so that the rates are given by

$$R_{1,\mathrm{BC}}(\boldsymbol{q}) := \sum_{i=1}^{\eta_1(\boldsymbol{q})} \log\Big(\frac{\gamma_\mathrm{R} + L_{1,\eta_2(\boldsymbol{q})} + L_{2,\eta_1(\boldsymbol{q})}}{q_1\eta_1(\boldsymbol{q}) + q_2\eta_2(\boldsymbol{q})} q_1 \lambda_{2,i}\Big),$$
$$R_{2,\mathrm{BC}}(\boldsymbol{q}) := \sum_{i=1}^{\eta_2(\boldsymbol{q})} \log\Big(\frac{\gamma_\mathrm{R} + L_{1,\eta_2(\boldsymbol{q})} + L_{2,\eta_1(\boldsymbol{q})}}{q_1\eta_1(\boldsymbol{q}) + q_2\eta_2(\boldsymbol{q})} q_2 \lambda_{1,i}\Big).$$

Proof. The proof can be found in Appendix 2.8.36. □

With this theorem we can identify for any non-negative weight vector $\boldsymbol{q} \in \mathbb{R}_+^2 \setminus \{\boldsymbol{0}\}$ the Pareto optimal rate pair on the boundary of the BC phase. In Figure 2.21 we depicted the characteristic sections and rate pairs of the achievable rate region $\mathcal{R}_\mathrm{BC}^\mathrm{MIMO}$ for an example where each node is equipped with two antennas.

2.6.3 MIMO Bidirectional Achievable Rate Region

As in the scalar case we use the MAC and BC phases for a fraction of time only. Let the time division parameter $\alpha \in [0,1]$ denote the fraction of time in the MAC phase so that $1-\alpha$ characterizes the fraction of time in the BC phase. Therefore, we have to scale the rate regions according to the time fraction. This means for a successful bidirectional relay transmission of a message m_1 with rate R_1 from node 1 to node 2 and message m_2 with rate R_2 from node 2 to node 1 the rate pair $\boldsymbol{R} = [R_1, R_2]$ has to be within $\alpha\mathcal{C}_\mathrm{MAC}^\mathrm{MIMO}$ as well as within $(1-\alpha)\mathcal{R}_\mathrm{BC}^\mathrm{MIMO}$. This means that for any given time division parameter $\alpha \in [0,1]$ the bidirectional achievable rate pairs are given by the intersection of the scaled rate regions

$$\mathcal{R}_\mathrm{BR}^\mathrm{MIMO}(\alpha) := \alpha\mathcal{C}_\mathrm{MAC}^\mathrm{MIMO} \cap (1-\alpha)\mathcal{R}_\mathrm{BC}^\mathrm{MIMO},$$

which is convex since the intersection of convex sets is itself convex. Accordingly, the boundary is characterized by the rate pairs which maximize the weighted rate sum for weight vectors $\boldsymbol{q} \in \mathbb{R}_+^2 \setminus \{\boldsymbol{0}\}$,

$$\boldsymbol{R}_\mathrm{BR}^\mathrm{MIMO}(\alpha, \boldsymbol{q}) := \operatorname*{arg\,max}_{[R_1,R_2]\in\mathcal{R}_\mathrm{BR}^\mathrm{MIMO}(\alpha)} q_1R_1 + q_2R_2.$$

Because of the difficult MIMO MAC capacity region the optimization problem cannot be solved in closed form as in the SISO case. However, for the solution we can distinguish

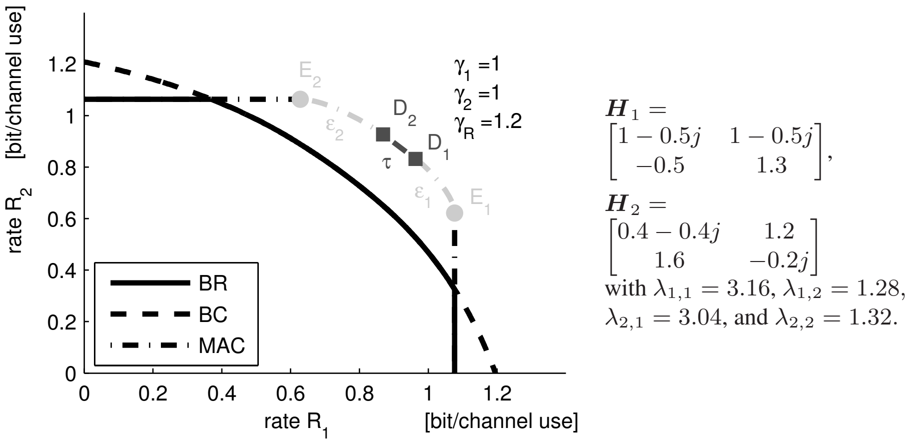


Figure 2.21: MIMO achievable rate regions $\mathcal{R}_{\mathrm{BR}}^{\mathrm{MIMO}}(1/2)$ with $N_1 = N_2 = N_{\mathrm{R}} = 2$.

between the following cases. First we consider the cases where the optimal rate pair is also the optimal rate pair of the MAC or BC phase. Accordingly, we can conclude as follows

$$\alpha \boldsymbol{R}_{\mathrm{MAC}}^{\mathrm{MIMO}}(\boldsymbol{q}) \in \mathcal{R}_{\mathrm{BR}}^{\mathrm{MIMO}}(\alpha) \quad \Rightarrow \quad \boldsymbol{R}_{\mathrm{BR}}^{\mathrm{MIMO}}(\alpha, \boldsymbol{q}) = \alpha \boldsymbol{R}_{\mathrm{MAC}}^{\mathrm{MIMO}}(\boldsymbol{q}),$$
$$(1-\alpha) \boldsymbol{R}_{\mathrm{BC}}^{\mathrm{MIMO}}(\boldsymbol{q}) \in \mathcal{R}_{\mathrm{BR}}^{\mathrm{MIMO}}(\alpha) \quad \Rightarrow \quad \boldsymbol{R}_{\mathrm{BR}}^{\mathrm{MIMO}}(\alpha, \boldsymbol{q}) = (1-\alpha) \boldsymbol{R}_{\mathrm{BC}}^{\mathrm{MIMO}}(\boldsymbol{q}).$$

If both cases do not apply the boundaries have to intersect at least once. From the previous we know that a rate pair where the boundaries intersect is optimal for a range of weight vector angles. In the case where $\alpha \boldsymbol{R}_{\mathrm{MAC}}^{\mathrm{MIMO}}(\boldsymbol{q}), (1-\alpha)\boldsymbol{R}_{\mathrm{BC}}^{\mathrm{MIMO}}(\boldsymbol{q}) \notin \mathcal{R}_{\mathrm{BR}}^{\mathrm{MIMO}}(\alpha)$ the optimal rate pair $\boldsymbol{R}_{\mathrm{BR}}^{\mathrm{MIMO}}(\alpha, \boldsymbol{q})$ is given by a rate pair where the boundaries intersect. Since multiple intersections may possible it remains to determine the intersection point which is optimal for the weight vector $\boldsymbol{q}$. Unfortunately the characterization of the intersection points is much more involved than in the SISO case and cannot be solved in closed form.

As in the scalar case we can define the rate region and the weighted rate sum optimal rate pair for the optimal time division case as follows

$$\mathcal{R}_{\mathrm{BRopt}}^{\mathrm{MIMO}} := \mathrm{co}\left(\bigcup_{\alpha \in [0,1]} \left(\alpha \mathcal{C}_{\mathrm{MAC}}^{\mathrm{MIMO}} \cap (1-\alpha) \mathcal{R}_{\mathrm{BC}}^{\mathrm{MIMO}} \right) \right),$$
$$\boldsymbol{R}_{\mathrm{BRopt}}^{\mathrm{MIMO}}(\boldsymbol{q}) := \operatorname*{arg\,max}_{[R_1, R_2] \in \mathcal{R}_{\mathrm{BRopt}}^{\mathrm{MIMO}}} q_1 R_1 + q_2 R_2,$$

which we use to study the high-power behavior in the following.

High Power Behavior

We can deduce the high power behavior from the asymptotic behavior of the maximal unidirectional rates for a fixed time division parameter α,

$$R_1^{*\mathrm{MIMO}}(\alpha) := \min\left\{\alpha R^*_{\overrightarrow{1\mathrm{R}}}, (1-\alpha)R_{\overrightarrow{\mathrm{R}2}}(1)\right\},$$
$$R_2^{*\mathrm{MIMO}}(\alpha) := \min\left\{\alpha R^*_{\overrightarrow{2\mathrm{R}}}, (1-\alpha)R_{\overrightarrow{\mathrm{R}1}}(1)\right\},$$

and for the optimal time division,

$$R_{1\mathrm{opt}}^{*\mathrm{MIMO}} := \frac{R^*_{\overrightarrow{1\mathrm{R}}} R_{\overrightarrow{\mathrm{R}2}}(1)}{R^*_{\overrightarrow{1\mathrm{R}}} + R_{\overrightarrow{\mathrm{R}2}}(1)}, \qquad \text{and} \qquad R_{2\mathrm{opt}}^{*\mathrm{MIMO}} := \frac{R^*_{\overrightarrow{2\mathrm{R}}} R_{\overrightarrow{\mathrm{R}1}}(1)}{R^*_{\overrightarrow{2\mathrm{R}}} + R_{\overrightarrow{\mathrm{R}1}}(1)},$$

where $R^*_{\overrightarrow{1\mathrm{R}}}$ and $R^*_{\overrightarrow{2\mathrm{R}}}$ denote the maximal unidirectional rates in the MAC phase as defined in Section 2.6.1 and $R_{\overrightarrow{\mathrm{R}2}}(1)$ and $R_{\overrightarrow{\mathrm{R}1}}(1)$ denote the maximal unidirectional rates in the BC phase as defined in Section 2.6.2.

In the high power regime it is asymptotically optimal to allocate equal proportions of power for each non-zero eigenmode. This means at high powers P_1, P_2, and P_R we can approximate $R^*_{\overrightarrow{k\mathrm{R}}}$ and $R_{\overrightarrow{\mathrm{R}k}}(1)$, $k = 1, 2$, by

$$R^*_{\overrightarrow{k\mathrm{R}}} \approx \sum_{n=1}^{r_k} \log\left(1 + \frac{P_k}{r_k}\frac{\lambda_{k,n}}{\sigma^2}\right) \approx \sum_{n=1}^{r_k} \log\left(\frac{P_k}{r_k}\frac{\lambda_{k,n}}{\sigma^2}\right),$$
$$R_{\overrightarrow{\mathrm{R}k}}(1) \approx \sum_{n=1}^{r_k} \log\left(1 + \frac{P_\mathrm{R}}{r_k}\frac{\lambda_{k,n}}{\sigma^2}\right) \approx \sum_{n=1}^{r_k} \log\left(\frac{P_\mathrm{R}}{r_k}\frac{\lambda_{k,n}}{\sigma^2}\right).$$

For conceptual clarity we set $P_1 = P_2 = P_\mathrm{R} = P$. Then in the high power regime the rates scale linearly with the number of eigenmodes of the corresponding MIMO channel

$$R^*_{\overrightarrow{k\mathrm{R}}} \approx r_k \log(P) + c_k, \qquad \text{and} \qquad R^*_{\overrightarrow{\mathrm{R}k}}(1) \approx r_k \log(P) + d_k,$$

with constants c_k and d_k for $k = 1, 2$.

Therewith, we get the asymptotic of the maximal unidirectional rates as follows[15]

$$R_1^{*\mathrm{MIMO}} = \min\{\alpha r_1, (1-\alpha) r_2\} \log(P) + o\big(\log(P)\big),$$
$$R_2^{*\mathrm{MIMO}} = \min\{\alpha r_2, (1-\alpha) r_1\} \log(P) + o\big(\log(P)\big),$$
$$R_{k\mathrm{opt}}^{*\mathrm{MIMO}} = \frac{r_1 r_2}{r_1 + r_2} \log(P) + o\big(\log(P)\big) \quad \text{for } k = 1, 2.$$

[15] The little-o notation collects asymptotically insignificant terms of an expression. This means, if $g(x) \in o(f(x))$ it follows that we have $\lim\limits_{x\to\infty} \frac{g(x)}{f(x)} = 0$.

This means that the maximal unidirectional rates for the fixed time division case scale with the minimum of the fraction of time weighted number of eigenmodes and for the optimal time division case scale with $\frac{r_1 r_2}{r_1+r_2}$.

Since the rate regions $\mathcal{R}_{\mathrm{BR}}^{\mathrm{MIMO}}(\alpha)$, $\alpha \in [0,1]$, and $\mathcal{R}_{\mathrm{BRopt}}^{\mathrm{MIMO}}$ are convex, for the sum of any rate pair on the boundaries we have

$$\begin{aligned}
\min\{R_1^{*\mathrm{MIMO}}, R_2^{*\mathrm{MIMO}}\} &\leq \big|\big|\boldsymbol{R}_{\mathrm{BR}}^{\mathrm{MIMO}}(\alpha, \boldsymbol{q})\big|\big|_1 \leq \big(R_1^{*\mathrm{MIMO}} + R_2^{*\mathrm{MIMO}}\big),\\
\min\{R_{1\mathrm{opt}}^{*\mathrm{MIMO}}, R_{2\mathrm{opt}}^{*\mathrm{MIMO}}\} &\leq \big|\big|\boldsymbol{R}_{\mathrm{BRopt}}^{\mathrm{MIMO}}(\boldsymbol{q})\big|\big|_1 \leq \big(R_{1\mathrm{opt}}^{*\mathrm{MIMO}} + R_{2\mathrm{opt}}^{*\mathrm{MIMO}}\big).
\end{aligned}$$

It follows that the asymptotics of the maximal unidirectional rates give us a lower and upper bound on the asymptotics of the sum-rate of the rate pairs on the boundary which we sum up in the following proposition.

Proposition 2.55. *Let $r_1 = \mathrm{rank}(\boldsymbol{H}_1)$ and $r_2 = \mathrm{rank}(\boldsymbol{H}_2)$ denote the spatial degree offered by the channel between the relay node and the nodes 1 and 2 respectively. Then the asymptotic scaling of the sum of any rate pair on the boundary of the achievable rate regions $\mathcal{R}_{\mathrm{BR}}^{\mathrm{MIMO}}(\alpha)$ and $\mathcal{R}_{\mathrm{BRopt}}^{\mathrm{MIMO}}$ with increasing power P can be upper and lower bounded as follows*

$$\begin{aligned}
\big|\big|\boldsymbol{R}_{\mathrm{BR}}^{\mathrm{MIMO}}(\alpha, \boldsymbol{q})\big|\big|_1 &\leq (s_1 + s_2)\log(P) + o\big(\log(P)\big),\\
\big|\big|\boldsymbol{R}_{\mathrm{BR}}^{\mathrm{MIMO}}(\alpha, \boldsymbol{q})\big|\big|_1 &\geq \min\{s_1, s_2\}\log(P) + o\big(\log(P)\big),\\
\big|\big|\boldsymbol{R}_{\mathrm{BRopt}}^{\mathrm{MIMO}}(\boldsymbol{q})\big|\big|_1 &\leq 2\frac{r_1 r_2}{r_1 + r_2}\log(P) + o\big(\log(P)\big),\\
\big|\big|\boldsymbol{R}_{\mathrm{BRopt}}^{\mathrm{MIMO}}(\boldsymbol{q})\big|\big|_1 &\geq \frac{r_1 r_2}{r_1 + r_2}\log(P) + o\big(\log(P)\big).
\end{aligned}$$

with $s_1 := \min\{\alpha r_1, (1-\alpha) r_2\}$ and $s_2 := \min\{\alpha r_2, (1-\alpha) r_1\}$.

We see that as for the point-to-point MIMO channel the spatial degree results in a pre-log factor in the high power regime, which depends on the spatial degree of both MIMO channels as well as on the time division between the phases.

2.7 Discussion

We considered various cross-layer design aspects for a two-phase bidirectional decode-and-forward relaying protocol using superposition encoding in the broadcast phase which was proposed first by Rankov and Wittneben in [RW05b, RW07]. In their work they propose half-duplex relaying protocols with increased spectral efficiency and show by comparing the maximal achievable ergodic sum-rates of the proposed bidirectional protocols and the achievable ergodic rates of the proposed unidirectional protocols that the bidirectional approaches can *"recover a large portion of the half-duplex loss."* After that they did not carry on research to explore further properties of their bidirectional protocols.

On that score in Section 2.2 we have extensively studied the bidirectional decode-and-forward protocol for a three-node network assuming single-antenna nodes. First, it is important to realize that a bidirectional relaying protocol is characterized by two achievable rates. This means that we have to examine a two-dimensional achievable rate region. Then it easily follows that the achievable rate region is given by the intersection of the scaled rate regions of the multiple access and broadcast phases. In this chapter, the relay node applies superposition encoding technique to forward the messages in the broadcast phase as proposed by Rankov and Wittneben. The characterization of the intersection of achievable rate regions usually results in a combinatorial problem. Theorem 2.5 and its corollaries reveal the combinatorial structure of the bidirectional achievable rate region for the equal time division case. In particular it allows us to characterize the sum-rate maximum in Corollary 2.8, which Rankov and Wittneben characterize with some simplifying assumptions only. Since the intersection of convex sets is itself convex, we know from convex theory that the boundary of the achievable rate region is characterized by the rate pairs which maximize the weighted rate sum. Theorem 2.10 characterizes in closed form the optimal rate pairs for any positive weight vector using the knowledge of the combinatorics.

Another simple extension is to relax their equal time division assumption from which it is natural to ask for the optimal time division. The achievable rate region of the optimal time division case is given by the union over all time division parameters and therefore need not be convex. However, in Corollary 2.13 we have concluded that the region is indeed convex using the equivalent description of the achievable rate region proved in Theorem 2.12. From Proposition 2.14 we see that the combinatoric of the MAC capacity region transfers to the bidirectional rate region. Again, in Theorem 2.16 we characterize the boundary of the achievable rate region with the optimal time division by the rate pairs that maximize the weighted rate sum. Since the boundary rate pairs of a convex set are Pareto optimal in the sense that we cannot increase one rate without decreasing the other, its characterization is crucial for the succeeding cross-layer designs.

For the two phase protocol there are two possibilities to interpret the mean power input constraint of a Gaussian continuous-alphabet channel with respect to a code word. We can either consider each phase separately, which means that we average over all actually transmitted symbols of each phase, or we average over both phases so that we scale the mean power constraints according to the inverse of the time division parameter. While for the equal time division case the results for both models can be obtained by an appropriate substitution, for the optimal time division case the non-linear dependence on the time division case makes a reconsideration for the second model necessary. Accordingly, in Theorem 2.39 we find an equivalent characterization of the achievable rate region. Furthermore, in Theorem 2.24 we show that the combinatoric of the MAC capacity region transfers to bidirectional achievable rate region as well. And finally, in Theorem 2.26 we establish the convexity of the rate region. The non-linear dependence of the rate constraints on the time division hinders the derivation of closed-form results. However, the obtained results makes efficient algorithmic optimization solutions feasible.

In Section 2.3 we use the results for a cross-layer design across the Data Link Layer and Physical Layer, where we are interested in an efficient resource allocation with respect to the traffic generated at higher layers. Therefore, we assume stationary and ergodic time-variant block-fading channel processes. At nodes 1 and 2 we consider queues with infinite buffer length and a centralized controller which slot-wise adapts the service rates according to the maximum weighted rate sum with weights equal to the buffer length. Since this policy is based on the maximum differential backlog algorithm of Tassiulas and Ephrimedes [TE93], we can use "standard" Lyapunov drift techniques to characterize the arrival rates under which the policy guarantees queue stability. In Corollary 2.30 we adapt the proof from [NMR03] to show that the stability region is equal to the bidirectional ergodic rate region by proving a negative drift of a quadratic Lyapunov function on the buffer levels whenever the mean number of unfinished work is large. Moreover, it follows that the mean delay behaves asymptotically to the inverse of the distance of the bit arrival rate vector to the boundary of the stability region. The following numerical simulations confirm the stability results and illustrate the efficiency of the bidirectional relay protocol for the equal and optimal time division case compared to classical Round-Robin strategies.

In a network where N relay nodes are willing to support the bidirectional communication it is a fundamental problem to identify the best relay node. But again, since bidirectional communication is characterized by two rates, this problem is a vector optimization problem so that relay selection for a bidirectional protocol is more involved than for unidirectional relaying protocols. In Section 2.4 we derive a relay selection criterion based on the achievable rate region of each relay node. From the Propositions 2.32 and 2.37 we can conclude that the probability that the achievable rate region of one relay node contains the rate region of all others decreases with an increasing number of relay nodes N, which means that there is more often no over-all best relay node. For that reason we propose to select the optimal relay node individually for any Pareto optimal rate pair on the boundary of the achievable rate region. Moreover, from Figure 2.10 we see that if two relay nodes achieve the same weighted rate sum for a certain weight vector, we can enlarge the achievable rate region if we allow time-sharing between the usage of the corresponding relay nodes.

For iid Rayleigh fading channels and for the equal time division case in Theorem 2.40 we show that the growth of the maximal unidirectional ergodic rates is asymptotically equal to $1/2 \log(\ln(N))$, which is the same scaling law achieved with the separation based relaying approach in [DSG$^+$03]. Since the sum-rate of any ergodic rate pair on the boundary of the ergodic rate region for the equal and optimal time division cases can be upper and lower bounded by the maximal unidirectional rates of the equal time division case, in the Corollaries 2.42 and 2.44 we conclude that the ergodic sum-rate of any boundary rate pair scales with $\Theta\big(log(log(N))\big)$ as well.

Relay selection is obviously a routing problem, which belongs to the Network Layer. Then for relay selection in a wireless system it remains to decide for a weight vector. We can again look at this problem from a cross-layer design perspective. This means that if we use the buffer levels as the weight vector we tie together relay selection with the maximum throughput policy of Section 2.3. It follows that we end up with a load adaptive routing protocol where the routing policy takes the resource allocation, the channel states as well as the queue states into account.

In Section 2.5 we consider the next cross-layer design approach where we study the joint resource allocation of two routing tasks. For this purpose, we add in the broadcast phase a multicast communication from the relay node to the nodes 1 and 2 to the bidirectional relaying protocol. The goal is to realize synergistic efficiency and enable rate trade-offs by converging two routing schemes. Moreover, if the relay does not fully exploit its transmit power constraint for the bidirectional communication, the relay can add the relay multicast communication without worsening the bidirectional communication. For that reason and in dependence to piggyback communication on higher layer protocols, we call this approach piggyback on bidirectional communication. We see this as the first step in integrating bidirectional relay communication in a wireless network.

In Theorem 2.45 we show that for the receiving decode-and-forward nodes 1 and 2 it is always optimal to decode the relay message first. This fixed decoding order simplifies following case studies, but the explicit characterization is still exhaustive. Then Theorem 2.48 specifies the total sum-rate maximum of the bidirectional rates and the additional relay multicast rate. We see that the total sum-rate behavior is dominated by the function which characterizes the rate pairs where the sum-rate maximum of a broadcast rate region is attained for a certain relay power. In Corollary 2.49 we identify a rate trade-off between additional relay rate and one bidirectional rate where the total sum-rate remains constant. In the following combinatorial discussion we look at some examples and work out some further aspects, which are interesting for the design and operation of service adapted network protocols. In particular we change the point of view and optimize the bidirectional communication while we desire a certain relay rate for a multicast application or we discuss the total sum-rate maximum with respect to the time division parameter. Most of the closed form results can be obtained for the equal time division case only, but the behavior for any fixed time division will be similar.

In Section 2.6 we extend the bidirectional decode-and-forward relaying protocol using superposition encoding to vector-valued processing. For this purpose, we assume that each node is equipped with multiple antennas and that each transmitter has perfect channel knowledge as well. Accordingly, we present the optimal transmit strategies at each node for both phases. In the BC phase the optimal transmit strategy is given by two point-to-point water-filling solutions which are only coupled by the relay power distribution. This allows us to characterize the rate pair which maximizes the weighted rate sum in Theorem 2.54. A combinatorial discussion of the achievable rate region as in the SISO case is not possible in the MIMO case because of the complicated structure of the MIMO-MAC capacity region. However, from the maximal unidirectional rates in the high power regime we find a scaling law for the maximal unidirectional rates, which depends on spatial degrees of the MIMO channels as well as on the time division between the phases. In Proposition 2.55 we use the scaling law for the maximal unidirectional rates to upper and lower bound the sum-rate of any rate pair on the boundary of the achievable rate regions for the fixed and optimal time division case.

Nevertheless, one can directly apply the ideas of the previous sections to the MIMO case, which means that for an optimal throughput in a system the rates should be allocated according to the queue states, the relay selection should be performed with respect to the achievable rate region, and the resource efficiency can be improved if we consider the joint resource allocation of multiple routing tasks.

2.8 Appendix: Proofs

2.8.1 Proof of Proposition 2.2.3

Let $[R_{1i}, R_{2i}] \in \mathcal{R}_{\mathrm{BC}}$, $i = 1, 2$ denote two arbitrary achievable rate pairs. It follows that there exist feasible relay power distributions $[\beta_{1i}, \beta_{2i}] \in \mathcal{B}$ with $R_{1i} = R_{\overrightarrow{\mathrm{R2}}}(\beta_{1i})$ and $R_{2i} = R_{\overrightarrow{\mathrm{R1}}}(\beta_{2i})$ for $i = 1, 2$. From the concavity of the logarithm we can conclude that for any $t \in [0, 1]$ we have

$$
\begin{aligned}
tR_{11} + (1-t)R_{12} &= tR_{\overrightarrow{\mathrm{R2}}}(\beta_{11}) + (1-t)R_{\overrightarrow{\mathrm{R2}}}(\beta_{12}) \leq R_{\overrightarrow{\mathrm{R2}}}(\beta_1(t)), \\
tR_{21} + (1-t)R_{22} &= tR_{\overrightarrow{\mathrm{R1}}}(\beta_{21}) + (1-t)R_{\overrightarrow{\mathrm{R1}}}(\beta_{22}) \leq R_{\overrightarrow{\mathrm{R1}}}(\beta_2(t))
\end{aligned}
$$

with $\beta_k(t) := t\beta_{k1} + (1-t)\beta_{k2}$, $k = 1, 2$. Since $[\beta_{1i}, \beta_{2i}] \in \mathcal{B}$, we have $\beta_{1i} + \beta_{2i} \leq 1$ for $i = 1, 2$ so that for any $t \in [0, 1]$ we have

$$
\sum_{k=1}^{2} \beta_k(t) = t\beta_{11} + (1-t)\beta_{12} + t\beta_{21} + (1-t)\beta_{22} \leq t + (1-t) = 1.
$$

This means that for any $t \in [0, 1]$ we have $[\beta_1(t), \beta_2(t)] \in \mathcal{B}$ and therefore $[R_{\overrightarrow{\mathrm{R2}}}(\beta_1(t)), R_{\overrightarrow{\mathrm{R1}}}(\beta_2(t))] \in \mathcal{R}_{\mathrm{BC}}$. It follows that any convex combination of achievable rate pairs is achievable, which proves that $\mathcal{R}_{\mathrm{BC}}$ is convex. $\square$

2.8.2 Proof of Proposition 2.2

The sum-rate maximum is obviously attained on the boundary of $\mathcal{R}_{\mathrm{BC}}$. The boundary is parametrized by (2.8). Therefore, we consider the sum-rate

$$
R_{\overrightarrow{\mathrm{R2}}}(\beta) + R_{\overrightarrow{\mathrm{R1}}}(1-\beta) = \log\big(\underbrace{(1 - \beta|h_2|^2\gamma_{\mathrm{R}})(1 - (1-\beta)|h_1|^2\gamma_{\mathrm{R}})}_{:=\mathrm{f}_{\mathrm{BC}}(\beta)}\big),
$$

which is maximized for the same β as the concave parabola $\mathrm{f}_{\mathrm{BC}}(\beta)$. The concave parabola $f(\beta)$ is maximized at its vertex

$$
[\beta^\star, \mathrm{f}_{\mathrm{BC}}(\beta^\star)] = \Big[\tfrac{1}{2} + \tfrac{1}{2\gamma_{\mathrm{R}}}\big(\tfrac{1}{|h_1|^2} - \tfrac{1}{|h_2|^2}\big), \tfrac{1}{4}(1 + \gamma_{\mathrm{R}}|h_1|^2 + \tfrac{|h_1|^2}{|h_2|^2})(1 + \gamma_{\mathrm{R}}|h_2|^2 + \tfrac{|h_2|^2}{|h_1|^2})\Big].
$$

This characterizes the maximum sum-rate if $\beta^\star$ is a feasible relay power distribution, i.e. $\beta^\star \in [0, 1]$. If we have $\beta^\star < 0$, the function $\mathrm{f}_{\mathrm{BC}}(\beta)$ is strictly decreasing on $[0, 1]$ so that the maximum sum-rate is attained at $\boldsymbol{R}_{\mathrm{BC}}(0)$. On the other hand, if we have $\beta^\star > 1$, the function $\mathrm{f}_{\mathrm{BC}}(\beta)$ is strictly increasing on $[0, 1]$ so that the maximum sum-rate is attained at $\boldsymbol{R}_{\mathrm{BC}}(1)$. $\square$

2.8.3 Proof of Theorem 2.5

For a given relay power distribution $\beta = \beta_1 = 1 - \beta_2 \in [0,1]$ the maximal sum-rate has to satisfy

$$R_\Sigma(\beta) = \max_{\boldsymbol{R} \in \mathcal{R}_{\mathrm{BReq}}(\beta)} R_1 + R_2 < {}^1\!/_2 R_\Sigma^{\mathrm{MAC}},$$

because of the MAC sum-rate constraint (2.4c). Furthermore, the unidirectional rate R_1 has to satisfy the individual rate constraint (2.4a) of the MAC phase and (2.7a) of the BC phase and similarly the unidirectional rate R_2 has to satisfy (2.4b) and (2.7b) so that we have

$$R_1 \leq {}^1\!/_2 \min\{R_{\overrightarrow{1\mathrm{R}}}, R_{\overrightarrow{\mathrm{R}2}}(\beta)\} \quad \text{and} \quad R_2 \leq {}^1\!/_2 \min\{R_{\overrightarrow{2\mathrm{R}}}, R_{\overrightarrow{\mathrm{R}1}}(1-\beta)\}.$$

The combination of all constraints gives us the maximum sum-rate

$$\begin{aligned}
R_\Sigma(\beta) &= {}^1\!/_2 \min\big\{ \min\{R_{\overrightarrow{1\mathrm{R}}}, R_{\overrightarrow{\mathrm{R}2}}(\beta)\} + \min\{R_{\overrightarrow{2\mathrm{R}}}, R_{\overrightarrow{\mathrm{R}1}}(1-\beta)\}, R_\Sigma^{\mathrm{MAC}}\big\} \\
&= {}^1\!/_2 \min\big\{R_{\overrightarrow{1\mathrm{R}}} + R_{\overrightarrow{\mathrm{R}1}}(1-\beta), R_{\overrightarrow{\mathrm{R}2}}(\beta) + R_{\overrightarrow{2\mathrm{R}}}, R_{\overrightarrow{\mathrm{R}2}}(\beta) + R_{\overrightarrow{\mathrm{R}1}}(1-\beta), \\
&\qquad\qquad R_{\overrightarrow{1\mathrm{R}}} + R_{\overrightarrow{2\mathrm{R}}}, R_\Sigma^{\mathrm{MAC}}\big\} \\
&= {}^1\!/_2 \min\big\{R_{\overrightarrow{1\mathrm{R}}} + R_{\overrightarrow{\mathrm{R}1}}(1-\beta), R_{\overrightarrow{\mathrm{R}2}}(\beta) + R_{\overrightarrow{2\mathrm{R}}}, R_{\overrightarrow{\mathrm{R}2}}(\beta) + R_{\overrightarrow{\mathrm{R}1}}(1-\beta), R_\Sigma^{\mathrm{MAC}}\big\}
\end{aligned} \tag{2.80}$$

using $R_{\overrightarrow{1\mathrm{R}}} + R_{\overrightarrow{2\mathrm{R}}} = \log\big((1+\gamma_1|h_1|^2)(1+\gamma_2|h_2|^2)\big) \geq \log(1+\gamma_1|h_1|^2+\gamma_2|h_2|^2) = R_\Sigma^{\mathrm{MAC}}$ in the last equality.

For the next equality we have to examine the combinatorics of (2.80). This becomes clearer if we discuss the geometry of the arguments of the logarithms of the rates. Accordingly, we define the following functions

$$\begin{aligned}
\mathrm{f}_\Sigma(\beta) &:= 2^{R_\Sigma^{\mathrm{MAC}}} = 1 + \gamma_1|h_1|^2 + \gamma_2|h_2|^2, \\
\mathrm{f}_1(\beta) &:= 2^{R_{\overrightarrow{1\mathrm{R}}} + R_{\overrightarrow{\mathrm{R}1}}(1-\beta)} = (1+\gamma_1|h_1|^2)(1+(1-\beta)\gamma_{\mathrm{R}}|h_1|^2), \\
\mathrm{f}_2(\beta) &:= 2^{R_{\overrightarrow{2\mathrm{R}}} + R_{\overrightarrow{2\mathrm{R}}}(\beta)} = (1+\gamma_2|h_2|^2)(1+\beta\gamma_{\mathrm{R}}|h_2|^2), \\
\mathrm{f}_{\mathrm{BC}}(\beta) &:= 2^{R_{\overrightarrow{\mathrm{R}1}}(1-\beta) + R_{\overrightarrow{\mathrm{R}2}}(\beta)} = (1+(1-\beta)\gamma_{\mathrm{R}}|h_1|^2)(1+\beta\gamma_{\mathrm{R}}|h_2|^2),
\end{aligned}$$

where we easily see that $\mathrm{f}_\Sigma(\beta)$ is a constant function, $\mathrm{f}_1(\beta)$ is a linearly decreasing function, $\mathrm{f}_2(\beta)$ is a linearly increasing function, and $\mathrm{f}_{\mathrm{BC}}(\beta)$ is a concave parabola, which we already know from the Proof of Proposition 2.2. To solve (2.80) we will study the function

$$\min\{\mathrm{f}_1(\beta), \mathrm{f}_2(\beta), \mathrm{f}_\Sigma(\beta), \mathrm{f}_{\mathrm{BC}}(\beta)\} \tag{2.81}$$

for $\beta \in [0,1]$ in the following. In Figure 2.22 we depicted two representative examples.

The intersection points characterize the combinatorics of the rate region and identify the active restriction for each feasible power distribution $\beta \in [0,1]$. In the following derivation

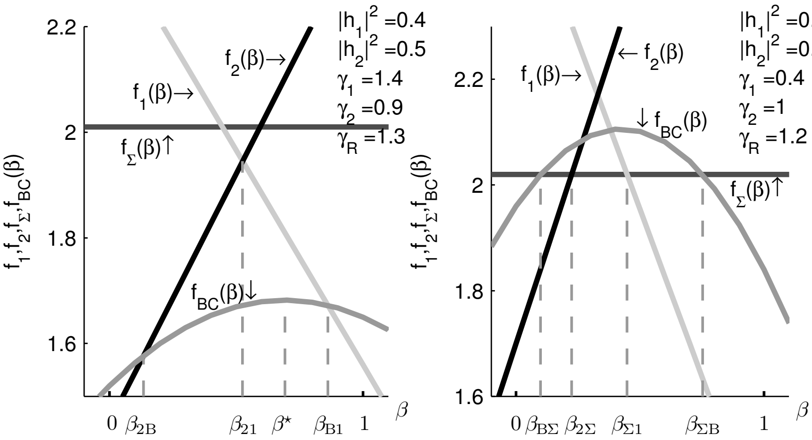


Figure 2.22: Exemplary plots for the discussion of $\mathrm{f}_1(\beta)$, $\mathrm{f}_2(\beta)$, $\mathrm{f}_\Sigma(\beta)$, and $\mathrm{f}_{\mathrm{BC}}(\beta)$ for $\beta^{\star 2} < \Delta\beta$ in the left figure and $\beta^{\star 2} \geq \Delta\beta$ in the right figure.

of the combinatorics we first neglect the feasibility, but later we will take the feasibility into account. With simple calculations of the intersection points between $\mathrm{f}_1(\beta)$, $\mathrm{f}_\Sigma(\beta)$, $\mathrm{f}_2(\beta)$, and $\mathrm{f}_{\mathrm{BC}}(\beta)$ it is easily seen that we have

$$\mathrm{f}_1(\beta) \leq \mathrm{f}_\Sigma(\beta) \text{ for } \beta \in [\beta_{\Sigma 1}, \infty) \quad \text{and} \quad \mathrm{f}_1(\beta) \geq \mathrm{f}_\Sigma(\beta) \text{ for } \beta \in (-\infty, \beta_{\Sigma 1}], \tag{2.82a}$$

$$\mathrm{f}_2(\beta) \leq \mathrm{f}_\Sigma(\beta) \text{ for } \beta \in (-\infty, \beta_{2\Sigma}] \quad \text{and} \quad \mathrm{f}_2(\beta) \geq \mathrm{f}_\Sigma(\beta) \text{ for } \beta \in [\beta_{2\Sigma}, \infty), \tag{2.82b}$$

$$\mathrm{f}_1(\beta) \leq \mathrm{f}_{\mathrm{BC}}(\beta) \text{ for } \beta \in [\beta_{\mathrm{B}1}, \beta_{1\mathrm{B}}] \quad \text{and} \quad \mathrm{f}_1(\beta) \geq \mathrm{f}_{\mathrm{BC}}(\beta) \text{ for } \beta \in (-\infty, \beta_{\mathrm{B}1}] \cup [\beta_{1\mathrm{B}}, \infty), \tag{2.82c}$$

$$\mathrm{f}_2(\beta) \leq \mathrm{f}_{\mathrm{BC}}(\beta) \text{ for } \beta \in [\beta_{\mathrm{B}2}, \beta_{2\mathrm{B}}] \quad \text{and} \quad \mathrm{f}_2(\beta) \geq \mathrm{f}_{\mathrm{BC}}(\beta) \text{ for } \beta \in (-\infty, \beta_{\mathrm{B}2}] \cup [\beta_{2\mathrm{B}}, \infty), \tag{2.82d}$$

$$\mathrm{f}_2(\beta) \leq \mathrm{f}_1(\beta) \text{ for } \beta \in (-\infty, \beta_{21}] \quad \text{and} \quad \mathrm{f}_2(\beta) \geq \mathrm{f}_1(\beta) \text{ for } \beta \in [\beta_{21}, \infty) \tag{2.82e}$$

with $\beta_{\Sigma 1}$, $\beta_{2\Sigma}$, $\beta_{\mathrm{B}1}$, $\beta_{1\mathrm{B}}$, $\beta_{\mathrm{B}2}$, and $\beta_{2\mathrm{B}}$ as given in the theorem and $\beta_{21} := \frac{(1+\gamma_1|h_1|^2)(1+\gamma_\mathrm{R}|h_1|^2)-(1+\gamma_2|h_2|^2)}{(1+\gamma_1|h_1|^2)\gamma_\mathrm{R}|h_1|^2+(1+\gamma_2|h_2|^2)\gamma_\mathrm{R}|h_2|^2}$. Furthermore, we have

$$\mathrm{f}_1(\beta_{21}) = \mathrm{f}_2(\beta_{21}) = \frac{(1+\gamma_1|h_1|^2)(1+\gamma_2|h_2|^2)(\gamma_\mathrm{R}|h_1|^2+\gamma_\mathrm{R}|h_2|^2+\gamma_\mathrm{R}^2|h_1|^2|h_2|^2)}{(1+\gamma_1|h_1|^2)\gamma_\mathrm{R}|h_1|^2+(1+\gamma_2|h_2|^2)\gamma_\mathrm{R}|h_2|^2} > 0. \tag{2.83}$$

It follows that we have $\beta_{\mathrm{B}2} < \beta_{21} < \beta_{1\mathrm{B}}$ since we have $\mathrm{f}_1(\beta_{1\mathrm{B}}) = \mathrm{f}_2(\beta_{\mathrm{B}2}) = 0$ and the linear functions $\mathrm{f}_1(\beta)$ and $\mathrm{f}_2(\beta)$ are strictly decreasing and increasing.

If we multiply the left and the right hand side of $\mathrm{f}_1(\beta_{21}) = (1+\gamma_1|h_1|^2)(1+(1-\beta_{21})\gamma_\mathrm{R}|h_1|^2)$ to the left and right hand side of $\mathrm{f}_1(\beta_{21}) = \mathrm{f}_2(\beta_{21}) = (1+\gamma_2|h_2|^2)(1+\beta_{21}\gamma_\mathrm{R}|h_2|^2)$ we get

$$\left(\mathrm{f}_1(\beta_{21})\right)^2 = \underbrace{(1+\gamma_2|h_2|^2)(1+\gamma_1|h_1|^2)}_{:=\mathrm{c}_{21}}\underbrace{(1+\beta\gamma_\mathrm{R}|h_2|^2)(1+(1-\beta_{21})\gamma_\mathrm{R}|h_1|^2)}_{=\mathrm{f}_\mathrm{BC}(\beta_{21})} \quad (2.84)$$

with $\mathrm{c}_{21} = (1+\gamma_2|h_2|^2)(1+\gamma_1|h_1|^2) > (1+\gamma_1|h_1|^2+\gamma_2|h_2|^2) = \mathrm{f}_\Sigma(\beta)$ for all β. Therewith, we will prove

$$\mathrm{f}_1(\beta_{21}) = \mathrm{f}_2(\beta_{21}) > \min\{\mathrm{f}_\Sigma(\beta_{21}), \mathrm{f}_\mathrm{BC}(\beta_{21})\} \quad (2.85)$$

next. First, we consider the case $\mathrm{f}_\Sigma(\beta_{21}) \leq \mathrm{f}_\mathrm{BC}(\beta_{21})$ so that we can conclude

$$\mathrm{f}_\Sigma(\beta_{21})^2 < \mathrm{f}_\Sigma(\beta_{21})\mathrm{c}_{21} \leq \mathrm{f}_\mathrm{BC}(\beta_{21})\mathrm{c}_{21} = \mathrm{f}_1(\beta_{21})^2$$

using (2.84). Since we know from (2.83) that $\mathrm{f}_1(\beta_{21}) > 0$ and $\mathrm{f}_\Sigma(\beta) > 0$ for all β, it follows that we have $\mathrm{f}_1(\beta_{21}) = \mathrm{f}_2(\beta_{21}) > \mathrm{f}_\Sigma(\beta_{21})$ so that (2.85) is fulfilled for this case. Otherwise, if $\mathrm{f}_\Sigma(\beta_{21}) > \mathrm{f}_\mathrm{BC}(\beta_{21})$ we have $\mathrm{f}_\mathrm{BC}(\beta_{21}) < \mathrm{c}_{21}$ since $\mathrm{f}_\Sigma(\beta_{21}) < \mathrm{c}_{21}$ holds. Then from (2.84) it follows that

$$\mathrm{f}_\mathrm{BC}(\beta_{21})^2 < \mathrm{c}_{21}\mathrm{f}_\mathrm{BC}(\beta_{21}) = \mathrm{f}_1(\beta_{21})^2$$

so that $\mathrm{f}_1(\beta_{21}) = \mathrm{f}_2(\beta_{21}) > \mathrm{f}_\mathrm{BC}(\beta_{21})$ since we have $\mathrm{f}_1(\beta_{21}), \mathrm{f}_\mathrm{BC}(\beta_{21}) > 0$. This proves the other case so that (2.85) holds.

We are now ready to derive $\mathcal{I}_1$. From (2.82a), (2.82c), and (2.82e) we have

$$\mathrm{f}_1(\beta) \leq \min\{\mathrm{f}_\Sigma(\beta), \mathrm{f}_\mathrm{BC}(\beta), \mathrm{f}_2(\beta)\} \text{ for } \beta \in [\beta_{\Sigma 1}, \infty) \cap [\beta_{\mathrm{B}1}, \beta_{1\mathrm{B}}] \cap [\beta_{21}, \infty).$$

We now claim that $[\beta_{\Sigma 1}, \infty) \cap [\beta_{\mathrm{B}1}, \beta_{1\mathrm{B}}] \cap [\beta_{21}, \infty) = [\beta_{\Sigma 1}, \infty) \cap [\beta_{\mathrm{B}1}, \beta_{1\mathrm{B}}]$ holds. To see this we first look at the case that the set on the left hand side is empty, which is equivalent to the condition that $\max\{\beta_{\Sigma 1}, \beta_{\mathrm{B}1}, \beta_{21}\} > \beta_{1\mathrm{B}}$ holds. Since we have $\beta_{\mathrm{B}1}, \beta_{21} < \beta_{1\mathrm{B}}$, it follows that $\beta_{\Sigma 1} > \beta_{1\mathrm{B}}$ has to hold so that the set on the right hand side is empty as well. On the other hand, if we have $\beta_{\Sigma 1} \leq \beta_{1\mathrm{B}}$ both sets are non-empty. Then both sets are equal if we have $\max\{\beta_{\Sigma 1}, \beta_{\mathrm{B}1}, \beta_{21}\} = \max\{\beta_{\Sigma 1}, \beta_{\mathrm{B}1}\}$, which is equivalent to the condition that $\beta_{21} \leq \max\{\beta_{\Sigma 1}, \beta_{\mathrm{B}1}\}$ holds. We will prove the last condition by contradiction. Therefore, let us assume that $\beta_{21} > \max\{\beta_{\Sigma 1}, \beta_{\mathrm{B}1}\}$ is true. This is equivalent to the condition that $\beta_{21} > \beta_{\Sigma 1}$ and $\beta_{21} > \beta_{\mathrm{B}1}$ holds simultaneously. Since we have $\beta_{21} < \beta_{1\mathrm{B}}$, it follows from (2.82a) that $\mathrm{f}_1(\beta_{21}) < \mathrm{f}_\Sigma(\beta_{21})$ and from (2.82c) that $\mathrm{f}_1(\beta_{21}) < \mathrm{f}_\mathrm{BC}(\beta_{21})$. Both together gives us $\mathrm{f}_1(\beta_{21}) < \min\{\mathrm{f}_\Sigma(\beta_{21}), \mathrm{f}_\mathrm{BC}(\beta_{21})\}$ which is a contradiction to (2.85). Accordingly, the set $\mathcal{I}_1 = [\beta_{\Sigma 1}, 1] \cap [\beta_{\mathrm{B}1}, \beta_{1\mathrm{B}}]$ denotes the set of feasible $\beta \in [0, 1]$ where the minimum of (2.81) is equal to $\mathrm{f}_1(\beta)$.

Similarly, from (2.82b), (2.82d), and (2.82e) we have

$$\mathrm{f}_2(\beta) \leq \min\{\mathrm{f}_\Sigma(\beta), \mathrm{f}_\mathrm{BC}(\beta), \mathrm{f}_1(\beta)\} \text{ for } \beta \in (-\infty, \beta_{2\Sigma}] \cap [\beta_{\mathrm{B}2}, \beta_{2\mathrm{B}}] \cap (-\infty, \beta_{21}].$$

We now claim that $(-\infty, \beta_{2\Sigma}] \cap [\beta_{B2}, \beta_{2B}] \cap (-\infty, \beta_{21}] = (-\infty, \beta_{2\Sigma}] \cap [\beta_{B2}, \beta_{2B}]$ holds. As before, we first look at the case that the set on the left hand side is empty, which is equivalent to the condition that $\min\{\beta_{2\Sigma}, \beta_{2B}, \beta_{21}\} < \beta_{B2}$ holds. Since we have $\beta_{2B}, \beta_{21} > \beta_{B2}$, it follows that $\beta_{2\Sigma} < \beta_{B2}$ has to hold so that the right hand side is empty as well. On the other hand, if we have $\beta_{2\Sigma} \geq \beta_{B2}$ both sets are non-empty. Then both sets are equal if we have $\min\{\beta_{2\Sigma}, \beta_{2B}, \beta_{21}\} = \min\{\beta_{2\Sigma}, \beta_{2B}\}$, which is equivalent to the condition that $\beta_{21} \geq \min\{\beta_{2\Sigma}, \beta_{2B}\}$ holds. Again, we will prove the last condition by contradiction. Therefore, let us assume that $\beta_{21} < \min\{\beta_{\Sigma 1}, \beta_{B1}\}$ is true. This is equivalent to the condition that $\beta_{21} < \beta_{\Sigma 1}$ and $\beta_{21} < \beta_{B1}$ holds simultaneously. Since we have $\beta_{21} > \beta_{B2}$, if follows from (2.82b) that $\mathrm{f}_2(\beta_{21}) < \mathrm{f}_\Sigma(\beta_{21})$ and from (2.82c) that $\mathrm{f}_2(\beta_{21}) < \mathrm{f}_{BC}(\beta_{21})$. Both together gives us $\mathrm{f}_2(\beta_{21}) < \min\{\mathrm{f}_\Sigma(\beta_{21}), \mathrm{f}_{BC}(\beta_{21})\}$ which is a contradiction to (2.85). Accordingly, the set $\mathcal{I}_2 = [0, \beta_{2\Sigma}] \cap [\beta_{B2}, \beta_{2B}]$ denotes the set of feasible $\beta \in [0, 1]$ where the minimum of (2.81) is equal to $\mathrm{f}_2(\beta)$.

The MAC sum-rate constraint is only active if the concave parabola $\mathrm{f}_{BC}(\beta)$ intersects $\mathrm{f}_\Sigma(\beta)$. This is the case if and only if we have $\beta^{\star 2} \geq \Delta\beta$. Then the corresponding intersection points are characterized by $\beta_{B\Sigma}$ and $\beta_{\Sigma B}$ as in the theorem so that we have $\mathrm{f}_\Sigma(\beta) \leq \mathrm{f}_{BC}(\beta)$ if $\beta \in [\beta_{B\Sigma}, \beta_{\Sigma B}]$. Furthermore, from (2.82a) and (2.82b) we know that

$$\mathrm{f}_\Sigma(\beta) < \min\{\mathrm{f}_1(\beta), \mathrm{f}_2(\beta)\} \text{ for } \beta \in (-\infty, \beta_{\Sigma 1}) \cap (\beta_{2\Sigma}, \infty)$$

so that for feasible $\beta \in \mathcal{I}_\Sigma$ as in the theorem the minimum of (2.81) is equal to $\mathrm{f}_\Sigma(\beta)$.

Finally, for all other feasible power distributions $\beta \in \mathcal{I}_{BC} = [0, 1] \setminus (\mathcal{I}_1 \cup \mathcal{I}_2 \cup \mathcal{I}_\Sigma)$ we have $\mathrm{f}_{BC}(\beta) \leq \min\{\mathrm{f}_1(\beta), \mathrm{f}_2(\beta), \mathrm{f}_\Sigma(\beta)\}$ and therefore the BC phase is more restrictive than any restriction of the MAC phase. □

2.8.4 Proof of Corollary 2.7

For the proof of the corollary we use the definitions of the proof of Theorem 2.5. Accordingly, the right plot in Figure 2.22 is a helpful illustration.

A rate pair $[R_1^M, R_2^M] \in \mathcal{C}_{MAC}$ with the sum-rate R_Σ^{MAC} can be supported in the BC phase iff the set $\mathcal{I}_\Sigma$ is non-empty. If one solves $(\beta^\star)^2 = \Delta\beta$ for γ_R we get $\gamma_R^\star$ as given in the corollary. It follows that for $\mathcal{I}_\Sigma \neq \emptyset$ we have to have $(\beta^\star)^2 \geq \Delta\beta$ so that it is necessary that $\gamma_R \geq \gamma_R^\star$.

Since $\mathcal{R}_{BC}$ increases with increasing γ_R for the minimal γ_R the set $\mathcal{I}_\Sigma$ has to be a singleton, this means that for $\gamma_R^{\Sigma MAC}$ we have $\max\{\beta_{2\Sigma}, \beta_{B\Sigma}\} = \min\{\beta_{\Sigma 1}, \beta_{\Sigma B}\}$. This is equivalent to the conditions

$$\beta_{2\Sigma} \leq \beta_{\Sigma B} \wedge \beta_{B\Sigma} \leq \beta_{\Sigma 1} \wedge \beta_{B\Sigma} \leq \beta_{\Sigma B} \wedge \beta_{2\Sigma} \leq \beta_{\Sigma 1}, \tag{2.86}$$

where we have at least one equality. Since we require $\gamma_{\mathrm{R}} \geq \gamma_{\mathrm{R}}^{\star}$ the condition $\beta_{\mathrm{B}\Sigma} \leq \beta_{\Sigma\mathrm{B}}$ is already fulfilled.

Next, we consider the case $\beta_{\mathrm{B}\Sigma} \leq \beta_{\Sigma 1}$, which is equivalent to the condition $\beta^{\star} - \beta_{\Sigma 1} \leq \sqrt{{\beta^{\star}}^2 - \Delta\beta}$. Since we already require $\gamma_{\mathrm{R}} \geq \gamma_{\mathrm{R}}^{\star}$ we have ${\beta^{\star}}^2 - \Delta\beta \geq 0$ so that the condition is fulfilled if $\beta^{\star} - \beta_{\Sigma 1} \leq 0$ holds, which is easily seen the case if we have $\gamma_{\mathrm{R}} \geq \hat{\gamma}_{\mathrm{R}}^{\Sigma 1}$. On the other hand, if $\beta^{\star} > \beta_{\Sigma 1}$ we have $\beta_{\mathrm{B}\Sigma} \leq \beta_{\Sigma 1}$ if we have $\gamma_{\mathrm{R}} \geq \gamma_{\mathrm{R}}^{\Sigma 1}(1/2)$. Both together gives us that $\beta_{\mathrm{B}\Sigma} \leq \beta_{\Sigma 1}$ is fulfilled if we have $\gamma_{\mathrm{R}} \geq \min\{\hat{\gamma}_{\mathrm{R}}^{\Sigma 1}, \gamma_{\mathrm{R}}^{\Sigma 1}(1/2)\}$.

Similarly, the case $\beta_{2\Sigma} \leq \beta_{\Sigma\mathrm{B}}$ is equivalent to the condition $\beta_{2\Sigma} - \beta^{\star} \leq \sqrt{{\beta^{\star}}^2 - \Delta\beta}$. Since we have ${\beta^{\star}}^2 - \Delta\beta \geq 0$ the condition is fulfilled if $\beta_{2\Sigma} - \beta^{\star} \leq 0$ holds, which is the case if we have $\gamma_{\mathrm{R}} \geq \hat{\gamma}_{\mathrm{R}}^{2\Sigma}$. On the hand, if $\beta^{\star} > \beta_{2\Sigma}$ we have $\beta_{2\Sigma} \leq \beta_{\Sigma\mathrm{B}}$ if we have $\gamma_{\mathrm{R}} \geq \gamma_{\mathrm{R}}^{2\Sigma}(1/2)$. Both together gives us that $\beta_{2\Sigma} \leq \beta_{\Sigma\mathrm{B}}$ is fulfilled if we have $\gamma_{\mathrm{R}} \geq \min\{\hat{\gamma}_{\mathrm{R}}^{2\Sigma}, \gamma_{\mathrm{R}}^{2\Sigma}(1/2)\}$.

Finally, we prove by contradiction that the following holds

$$\beta_{2\Sigma} \leq \beta_{\Sigma\mathrm{B}} \ \wedge \ \beta_{\mathrm{B}\Sigma} \leq \beta_{\Sigma 1} \ \wedge \ \beta_{\mathrm{B}\Sigma} \leq \beta_{\Sigma\mathrm{B}} \quad \Rightarrow \quad \beta_{2\Sigma} \leq \beta_{\Sigma 1}.$$

Therefore, let us assume that we have $\beta_{2\Sigma} > \beta_{\Sigma 1}$. Then using the first and second inequality gives us $\beta_{\Sigma\mathrm{B}} \geq \beta_{2\Sigma} > \beta_{\Sigma 1} \geq \beta_{\mathrm{B}\Sigma}$ which is a contradiction to the inequality $\beta_{\mathrm{B}\Sigma} \leq \beta_{\Sigma\mathrm{B}}$. Accordingly, the minimum necessary relay power has to satisfy the first three conditions of (2.86) and therefore is given by (2.16). □

2.8.5 Proof of Corollary 2.8

For the proof of the corollary we use the definitions of the proof of Theorem 2.5. Accordingly, the left plot in Figure 2.22 is a helpful illustration. We prove the corollary by further investigation of the combinatorial structure of the geometry of f_1, f_2, and f_{BC}. Therefore, let us define the function

$$\mathrm{r}(\beta) := \min\{\mathrm{f}_1(\beta), \mathrm{f}_2(\beta), \mathrm{f}_{\mathrm{BC}}(\beta)\}.$$

Since we assume $\gamma_{\mathrm{R}} < \gamma_{\mathrm{R}}^{\Sigma\mathrm{MAC}}$, we have $\mathrm{r}(\beta) < \mathrm{f}_{\Sigma}(\beta)$ for all $\beta \in [0,1]$. Remember that $\mathrm{f}_1(\beta)$ and $\mathrm{f}_2(\beta)$ are strictly decreasing and increasing. Furthermore, since $\mathrm{f}_{\mathrm{BC}}(\beta)$ is a concave parabola which has its vertex at $\beta^{\star}$, it is strictly increasing for $\beta < \beta^{\star}$ and strictly decreasing for $\beta > \beta^{\star}$. Therefore, the sum-rate maximum is given by the combinatoric of the vertex of the parabola at $\beta^{\star}$ and the intersection points at $\beta_{\mathrm{B}1}$ and $\beta_{2\mathrm{B}}$.

Since we have $\mathcal{I}_{\Sigma} = \emptyset$, $\beta_{1\mathrm{B}} > 1$, and $\beta_{\mathrm{B}1} < 0$, the set of feasible power distributions where $\mathrm{f}_{\mathrm{BC}}(\beta) < \mathrm{r}(\beta)$ is given by

$$\mathcal{I}_B = [0,1] \setminus \big(\mathcal{I}_1 \cup \mathcal{I}_2\big) = [0,1] \cap (\beta_{2\mathrm{B}}, \beta_{\mathrm{B}1}).$$

Since $\mathrm{f}_{\mathrm{BC}}(\beta)$ is a concave parabola, we have $\mathrm{f}_{\mathrm{BC}}(\beta^\star) = \mathrm{r}(\beta^\star) \geq \mathrm{r}(\beta)$ for all $\beta \in [0,1]$. Accordingly, if we have

$$\beta^\star \in \mathcal{I}_B \quad \Leftrightarrow \quad \beta^\star \geq 0 \ \wedge \ \beta^\star \leq 1 \ \wedge \ \beta^\star > \beta_{2\mathrm{B}} \ \wedge \ \beta^\star < \beta_{\mathrm{B}1}, \tag{2.87}$$

then the sum-rate maximum is given by $\frac{1}{2}\boldsymbol{R}_{\mathrm{BC}}(\beta^\star)$. If we solve the conditions in (2.87) for γ_{R} we get

$$\begin{aligned}
\beta^\star = \tfrac{1}{2} + \tfrac{1}{2\gamma_{\mathrm{R}}}\left(\tfrac{1}{|h_1|^2} - \tfrac{1}{|h_2|^2}\right) \geq 0 \quad &\Leftrightarrow \quad \gamma_{\mathrm{R}} \geq \gamma_{\mathrm{R}}^{\dagger} = \tfrac{1}{|h_2|^2} - \tfrac{1}{|h_1|^2},\\
\beta^\star = \tfrac{1}{2} + \tfrac{1}{2\gamma_{\mathrm{R}}}\left(\tfrac{1}{|h_1|^2} - \tfrac{1}{|h_2|^2}\right) \leq 1 \quad &\Leftrightarrow \quad \gamma_{\mathrm{R}} \geq -\gamma_{\mathrm{R}}^{\dagger} = \tfrac{1}{|h_1|^2} - \tfrac{1}{|h_2|^2},\\
\beta^\star > \beta_{2\mathrm{B}} = 1 - \tfrac{\gamma_2|h_2|^2}{\gamma_{\mathrm{R}}|h_1|^2} \quad &\Leftrightarrow \quad \gamma_{\mathrm{R}} < \gamma_{\mathrm{R}}^{\star 1} = \tfrac{2\gamma_2|h_2|^2}{|h_1|^2} + \tfrac{1}{|h_1|^2} - \tfrac{1}{|h_2|^2},\\
\beta^\star < \beta_{\mathrm{B}1} = \tfrac{\gamma_1|h_1|^2}{\gamma_{\mathrm{R}}|h_2|^2} \quad &\Leftrightarrow \quad \gamma_{\mathrm{R}} < \gamma_{\mathrm{R}}^{2\star} = \tfrac{2\gamma_1|h_1|^2}{|h_2|^2} + \tfrac{1}{|h_2|^2} - \tfrac{1}{|h_1|^2}.
\end{aligned}$$

Therefore, we have $\beta^\star \in \mathcal{I}_B$ with the sum-rate optimum $\frac{1}{2}\boldsymbol{R}_{\mathrm{BC}}(\beta^\star)$ if we have $|\gamma_{\mathrm{R}}^{\dagger}| < \gamma_{\mathrm{R}} < \min\{\gamma_{\mathrm{R}}^{\star 1}, \gamma_{\mathrm{R}}^{2\star}\}$, which proves the case (2.18c).

If we have $\beta^\star \leq \max\{\beta_{2\mathrm{B}}, 0\}$, then $\mathrm{f}_{\mathrm{BC}}(\beta)$ is decreasing for all $\beta \in \mathcal{I}_B$. Since $\mathrm{f}_2(\beta)$ and $\mathrm{f}_1(\beta)$ are strictly increasing and decreasing, the rate pair $\frac{1}{2}\boldsymbol{R}_{\mathrm{BC}}(\max\{\beta_{2\mathrm{B}}, 0\})$ is sum-rate optimal. Then the sum-rate optimal rate pair is on the R_2-axis if we additionally have

$$\beta_{2\mathrm{B}} \leq 0 \quad \Leftrightarrow \quad \gamma_{\mathrm{R}} \leq \gamma_{\mathrm{R}}^{2\mathrm{B}} = \tfrac{\gamma_2|h_2|^2}{|h_1|^2}.$$

Then it follows that for $\frac{1}{2}\boldsymbol{R}_{\mathrm{BC}}(0)$ to be sum-rate optimal we require $\beta_{2\mathrm{B}} \leq 0$ and $\beta^\star \leq 0$, which is equivalent to $\gamma_{\mathrm{R}} \leq \min\{\gamma_{\mathrm{R}}^{\dagger}, \gamma_{\mathrm{R}}^{2\mathrm{B}}\}$. This proves the case (2.18a). Accordingly, if we have $\beta_{2\mathrm{B}} > 0$ and $\beta^\star \leq \beta_{2\mathrm{B}}$ the rate pair $\frac{1}{2}\boldsymbol{R}_{\mathrm{BC}}(\beta_{2\mathrm{B}})$ is sum-rate optimal. This is case if $\gamma_{\mathrm{R}} > \gamma_{\mathrm{R}}^{2\mathrm{B}}$ and $\gamma_{\mathrm{R}} \geq \gamma_{\mathrm{R}}^{\star 1}$, which proves the case (2.18b).

Similarly, if we have $\beta^\star \geq \min\{\beta_{\mathrm{B}1}, 1\}$, then $\mathrm{f}_{\mathrm{BC}}(\beta)$ is increasing for all $\beta \in \mathcal{I}_B$. Again, since $\mathrm{f}_2(\beta)$ and $\mathrm{f}_1(\beta)$ are strictly increasing and decreasing the rate pair $\frac{1}{2}\boldsymbol{R}_{\mathrm{BC}}(\min\{\beta_{\mathrm{B}1}, 1\})$ is sum-rate optimal. Then the sum-rate optimal rate pair is on the R_1-axis if we additionally have

$$\beta_{\mathrm{B}1} \geq 1 \quad \Leftrightarrow \quad \gamma_{\mathrm{R}} \leq \gamma_{\mathrm{R}}^{\mathrm{B}1} = \tfrac{\gamma_1|h_1|^2}{|h_2|^2}.$$

Then it follows that for $\frac{1}{2}\boldsymbol{R}_{\mathrm{BC}}(1)$ to be sum-rate optimal we require $\beta_{\mathrm{B}1} \geq 1$ and $\beta^\star \geq 1$, which is equivalent to $\gamma_{\mathrm{R}} \leq \min\{-\gamma_{\mathrm{R}}^{\dagger}, \gamma_{\mathrm{R}}^{\mathrm{B}1}\}$. This proves the case (2.18e). Accordingly, if we have $\beta_{\mathrm{B}1} < 1$ and $\beta^\star \geq$ the rate pair $\frac{1}{2}\boldsymbol{R}_{\mathrm{BC}}(\beta_{\mathrm{B}1})$ is sum-rate optimal. This is case if $\gamma_{\mathrm{R}} > \gamma_{\mathrm{R}}^{\mathrm{B}1}$ and $\gamma_{\mathrm{R}} \geq \gamma_{\mathrm{R}}^{2\star}$, which proves the case (2.18d). $\square$

2.8.6 Proof of Theorem 2.10

We first separate a simple calculation in the following lemma, which also justifies $\vartheta(\beta)$.

Lemma 2.56. *The normal angle of the supporting hyperplane of the rate region $\mathcal{R}_{\mathrm{BC}}$ for a relay power distribution $\beta_1 = 1 - \beta_2 = \beta$ is given by $\varphi(\beta) = \tan\vartheta(\beta) = \tan\frac{|h_2|^2(1+(1-\beta)\gamma_{\mathrm{R}}|h_1|^2)}{|h_1|^2(1+\beta\gamma_{\mathrm{R}}|h_2|^2)}$.*

Proof of Lemma 2.56. The normal vector of the parametrized boundary $[R_{\overrightarrow{\mathrm{R2}}}(\beta), R_{\overrightarrow{\mathrm{R1}}}(1-\beta)]$ is given by $\frac{[-R'_{\overrightarrow{\mathrm{R1}}}(\beta), R'_{\overrightarrow{\mathrm{R2}}}(\beta)]}{\|[R_{\overrightarrow{\mathrm{R2}}}(\beta), R_{\overrightarrow{\mathrm{R1}}}(1-\beta)]\|}$ with $R'_{\overrightarrow{\mathrm{R1}}}(\beta) = \frac{d\,R_{\overrightarrow{\mathrm{R1}}}(1-\beta)}{d\,\beta} = \frac{-\gamma_{\mathrm{R}}|h_1|^2}{1+\gamma_{\mathrm{R}}(1-\beta)|h_1|^2}$ and $R'_{\overrightarrow{\mathrm{R2}}}(\beta) = \frac{d\,R_{\overrightarrow{\mathrm{R2}}}(\beta)}{d\,\beta} = \frac{\gamma_{\mathrm{R}}|h_2|^2}{1+\gamma_{\mathrm{R}}\beta|h_2|^2}$. Thus the angle $\varphi(\beta)$ is given by $\tan\frac{R'_{\overrightarrow{\mathrm{R2}}}(\beta)}{-R'_{\overrightarrow{\mathrm{R1}}}(\beta)}$. □

Proof of Theorem 2.10. The weighted rate sum maximum in Theorem 2.10 is given by the boundary rate pair of $\mathcal{R}_{\mathrm{BReq}}$ where the angle of the normal vector of the supporting hyperplane is equal to the angle of the weight vector, i.e. $\tan q_2/q_1$. Furthermore, two restrictions apply simultaneously at boundary intersection rate pairs. Those rate pairs are also optimal for weight vectors with an angle inbetween the angles of the normal vectors of the restrictions at this rate pair. Since for all weight vectors with non-negative elements, i.e. angles inbetween $[0, \pi/2]$, the $\tan$ is a strictly increasing function. For that reason, we prefer to characterize the different cases by the argument of the $\tan$. From Theorem 2.5 and its corollaries we know the combinatorial structure of the rate region and its boundary. Therefore, in the following we determine for any weight vector $\boldsymbol{q} \in \mathbb{R}_+^2 \setminus \{\boldsymbol{0}\}$ the corresponding boundary rate pair.

First we characterize all intersection rate pairs. If $\mathcal{I}_1 = \emptyset \Leftrightarrow \beta_{\mathrm{B1}} > 1$, the MAC single user bound $\frac{1}{2}R_{\overrightarrow{\mathrm{1R}}}$ is not achievable in the BC phase, therefore the boundary rate pair $\frac{1}{2}\boldsymbol{R}_{\mathrm{BC}}(1)$ is optimal for all weight vectors $\vartheta(1) > q_2/q_1$. Similarly, (2.20b) follows from $\mathcal{I}_2 = \emptyset$. The MAC single user bound $\frac{1}{2}R_{\overrightarrow{\mathrm{1R}}}$ is active for some weight vectors $\boldsymbol{q}$ if $\mathcal{I}_1 \neq \emptyset \Leftrightarrow \beta_{\mathrm{B1}} \leq 1$. Next we have to distinguish if γ_{R} is large enough so that the MAC vertex $\frac{1}{2}\boldsymbol{\nu}_{\Sigma 1}$ can be reached, i.e. $\beta_{\Sigma 1} \geq \beta_{\mathrm{B1}}$. Then $\frac{1}{2}\boldsymbol{\nu}_{\Sigma 1}$ is optimal for all weight vectors with $q_2/q_1 \leq 1$. If $\beta_{\Sigma 1} < \beta_{\mathrm{B1}} \leq 1$ the vertex $\boldsymbol{\nu}_{\Sigma 1}$ cannot be supported but the MAC single user bound $\frac{1}{2}R_{\overrightarrow{\mathrm{1R}}}$ is active for $\beta > \beta_{\mathrm{B1}}$. Therefore, the intersection rate pair $\frac{1}{2}\boldsymbol{R}_{\mathrm{BC}}(\beta_{\mathrm{B1}})$ is optimal for weight vectors $q_2/q_1 \leq \vartheta(\beta_{\mathrm{B1}})$. Similarly, if $\beta_{2\Sigma} \leq \beta_{2\mathrm{B}}$ the vertex $\frac{1}{2}\boldsymbol{\nu}_{2\Sigma}$ is optimal for $q_2/q_1 \geq 1$ and the case (2.20f) if not.

Finally, two possible intersection rate pairs of the MAC sum-rate $\frac{1}{2}R_{\Sigma}$ with $\mathcal{R}_{\mathrm{BC}}$ are left. For both it is necessary that $\beta^{\star 2} > \Delta\beta$. If $\beta_{2\Sigma} < \beta_{\mathrm{B}\Sigma} < \beta_{\Sigma 1}$, i.e. the MAC vertex $\frac{1}{2}\boldsymbol{\nu}_{2\Sigma}$ cannot be supported, the intersection rate pair $\frac{1}{2}\boldsymbol{R}_{\mathrm{BC}}(\beta_{\mathrm{B}\Sigma})$ is optimal for all weight vectors with $\vartheta(\beta_{\mathrm{B}\Sigma}) \leq q_2/q_1 \leq 1$. On the other hand, the MAC vertex $\frac{1}{2}\boldsymbol{\nu}_{\Sigma 1}$ cannot be supported if $\beta_{2\Sigma} < \beta_{\Sigma\mathrm{B}} < \beta_{\Sigma 1}$. Therefore $\frac{1}{2}\boldsymbol{R}_{\mathrm{BC}}(\beta_{\Sigma\mathrm{B}})$ is optimal for weight vectors with $\vartheta(\beta_{\Sigma\mathrm{B}}) \geq q_2/q_1 \geq 1$.

For all other weight vectors $\mathcal{R}_{\mathrm{BReq}}$ is bounded by the rate region of the BC phase. Therefore, the optimal rate vector $\frac{1}{2}\boldsymbol{R}_{\mathrm{BC}}(\beta_{\mathrm{BC}})$ is given by the boundary rate pair with the corresponding normal vector, where the optimal power distribution β_{BC} follows if one solves $\vartheta(\beta_{\mathrm{BC}}) = q_2/q_1$ for β_{BC}. □

2.8.7 Proof of Lemma 2.11

Since for any $\alpha \in \mathcal{A}$ a power distribution $[\beta_1, \beta_2] \in \mathcal{B}$ with $\alpha R_1^M \leq (1-\alpha) R_{\overrightarrow{\mathrm{R2}}}(\beta_1)$ and $\alpha R_2^M \leq (1-\alpha) R_{\overrightarrow{\mathrm{R1}}}(\beta_2)$ exists, it is possible for the relay node to broadcast the messages from nodes 1 and 2 with rates αR_1^M and αR_2^M. Accordingly, for any $\alpha \in \mathcal{A}$ we achieve the bidirectional rate pair $[R_1, R_2] = \alpha[R_1^M, R_2^M]$ so that the largest $\alpha^* \in \mathcal{A}$ maximizes the bidirectional rate pair in each component. Next we prove by contradiction that α^* is uniquely characterized by (2.23a) and (2.23b). Therefore, we need that $R_{\overrightarrow{\mathrm{R2}}}(\beta_1)$ and $R_{\overrightarrow{\mathrm{R1}}}(\beta_2)$ are continuous and increasing in β_1 and β_2.

Let us assume that for the largest element α^* of the set $\mathcal{A}$ at least one equality is not fulfilled. Since $\alpha^* \in \mathcal{A}$, a relay power distribution $[\beta_1, \beta_2] \in \mathcal{B}$ exists so that either $\alpha^* R_1^M < (1-\alpha^*) R_{\overrightarrow{\mathrm{R2}}}(\beta_1)$ or $\alpha^* R_2^M < (1-\alpha^*) R_{\overrightarrow{\mathrm{R1}}}(\beta_2)$.

If $\alpha^* R_1^M < (1-\alpha^*) R_{\overrightarrow{\mathrm{R2}}}(\beta_1)$, we can find a $\tilde{\beta}_1 < \beta_1$ where still $\alpha^* R_1^M < (1-\alpha^*) R_{\overrightarrow{\mathrm{R2}}}(\tilde{\beta}_1)$ holds. With $\tilde{\beta}_2 = 1 - \tilde{\beta}_1 > \beta_2$ we get $\alpha^* R_2^M < (1-\alpha^*) R_{\overrightarrow{\mathrm{R1}}}(\tilde{\beta}_2)$. Since in both cases we have a strict inequality, we can find an $\tilde{\alpha} > \alpha^*$ while both inequalities are still fulfilled, this means $\tilde{\alpha} \in \mathcal{A}$. This contradicts the assumption that α^* is the largest element in $\mathcal{A}$.

Accordingly, if $\alpha^* R_2^M < (1-\alpha^*) R_{\overrightarrow{\mathrm{R1}}}(\beta_2)$, we can find a $\tilde{\beta}_2 < \beta_2$ and a $\tilde{\beta}_1 = 1 - \tilde{\beta}_2 > \beta_1$ so that we have $\alpha^* R_1^M < (1-\alpha^*) R_{\overrightarrow{\mathrm{R2}}}(\tilde{\beta}_1)$ and $\alpha^* R_2^M < (1-\alpha^*) R_{\overrightarrow{\mathrm{R1}}}(\tilde{\beta}_2)$. This allows us to increase α while both inequalities are still fulfilled, what again contradicts the assumption.

Finally, we assume that for the largest element α^* of the set $\mathcal{A}$ we have $\beta_1 + \beta_2 < 1$. Therefore, we can find $\tilde{\beta}_1 > \beta_1$ and $\tilde{\beta}_2 > \beta_2$ with $[\tilde{\beta}_1, \tilde{\beta}_2] \in \mathcal{A}$ so that we have $\alpha^* R_1^M < (1-\alpha^*) R_{\overrightarrow{\mathrm{R2}}}(\tilde{\beta}_1)$ and $\alpha^* R_2^M < (1-\alpha^*) R_{\overrightarrow{\mathrm{R1}}}(\tilde{\beta}_2)$. This allows us to increase α while both inequalities are still fulfilled, which contradicts the assumption $\beta_1 + \beta_2 < 1$. □

2.8.8 Proof of Theorem 2.12

The equivalent description follows from a transformation of rate pairs $[R_1^M, R_2^M]$ on the boundary of the MAC rate region using Lemma 2.11. We get the sets $\mathcal{R}_1$, $\mathcal{R}_2$, and $\mathcal{R}_\Sigma$ with time division parameters $\alpha_1^\star(\beta)$, $\alpha_2^\star(\beta)$, and $\alpha_\Sigma^\star(\beta)$ if we solve (2.23a) and (2.23b) for $\alpha^\star$ with $R_1^M = R_{\overrightarrow{\mathrm{1R}}}$, $R_2^M = R_{\overrightarrow{\mathrm{2R}}}$, and $R_1^M + R_2^M = R_\Sigma^{\mathrm{MAC}}$ respectively.

For an arbitrary rate pair $\boldsymbol{R} \in \mathcal{R}_1 \cap \mathcal{R}_2 \cap \mathcal{R}_\Sigma$ we will show in the following that $\boldsymbol{R} \in \mathcal{R}_{\mathrm{BRopt}}$ holds. Since $\boldsymbol{R}$ has to be within $\mathcal{R}_1$ there exists $\tilde{\beta}_1 \in [0,1]$ so that $R_1 \leq R_{11}(\tilde{\beta}_1) = \big(1-\alpha_1^\star(\tilde{\beta}_1)\big) R_{\overrightarrow{\mathrm{R2}}}(\tilde{\beta}_1)$ and $R_2 \leq R_{12}(\tilde{\beta}_1) = \big(1-\alpha_1^\star(\tilde{\beta}_1)\big) R_{\overrightarrow{\mathrm{R1}}}(1-\tilde{\beta}_1)$ hold. This means that we have $\boldsymbol{R} \in \big(1-\alpha_1^\star(\tilde{\beta}_1)\big)\mathcal{R}_{\mathrm{BC}}$. Similarly, from $\boldsymbol{R} \in \mathcal{R}_2$ and $\boldsymbol{R} \in \mathcal{R}_\Sigma$ we know that there exist $\tilde{\beta}_2, \tilde{\beta}_\Sigma \in [0,1]$ so that we have $\boldsymbol{R} \in \big(1-\alpha_2^\star(\tilde{\beta}_2)\big)\mathcal{R}_{\mathrm{BC}}$ and $\boldsymbol{R} \in \big(1-\alpha_\Sigma^\star(\tilde{\beta}_\Sigma)\big)\mathcal{R}_{\mathrm{BC}}$ respectively. With $\tilde{\alpha} = \max\{\alpha_1^\star(\tilde{\beta}_1), \alpha_2^\star(\tilde{\beta}_2), \alpha_\Sigma^\star(\tilde{\beta}_\Sigma)\}$ we still have $\boldsymbol{R} \in (1-\tilde{\alpha})\mathcal{R}_{\mathrm{BC}}$. Furthermore, we see from

$$\begin{aligned} R_1 &\leq R_{11}(\tilde{\beta}_1) = \alpha_1^\star(\tilde{\beta}_1) R_{\overrightarrow{\mathrm{1R}}} \leq \tilde{\alpha} R_{\overrightarrow{\mathrm{1R}}}, \\ R_2 &\leq R_{22}(\tilde{\beta}_2) = \alpha_2^\star(\tilde{\beta}_2) R_{\overrightarrow{\mathrm{2R}}} \leq \tilde{\alpha} R_{\overrightarrow{\mathrm{2R}}}, \\ R_1 + R_2 &\leq R_{\Sigma 1}(\tilde{\beta}_\Sigma) + R_{\Sigma 2}(\tilde{\beta}_\Sigma) = \alpha_\Sigma^\star(\tilde{\beta}_\Sigma) R_\Sigma^{\mathrm{MAC}} \leq \tilde{\alpha} R_\Sigma^{\mathrm{MAC}} \end{aligned}$$

that we have $\boldsymbol{R} \in \tilde{\alpha}\mathcal{C}_{\mathrm{MAC}}$. It follows that $\boldsymbol{R} \in \mathcal{R}_{\mathrm{BRopt}}(\tilde{\alpha}) \subseteq \mathcal{R}_{\mathrm{BRopt}}$ so that we conclude that $\mathcal{R}_1 \cap \mathcal{R}_2 \cap \mathcal{R}_\Sigma \subseteq \mathcal{R}_{\mathrm{BRopt}}$.

For the reverse inclusion we show that for any rate pair $\boldsymbol{R} \in \mathcal{R}_{\mathrm{BRopt}}$ we also have $\boldsymbol{R} \in \mathcal{R}_1 \cap \mathcal{R}_2 \cap \mathcal{R}_\Sigma$. For any $\boldsymbol{R} \in \mathcal{R}_{\mathrm{BRopt}}$ there exists an $\tilde{\alpha} \in [0,1]$ so that $\boldsymbol{R} \in \tilde{\alpha}\mathcal{C}_{\mathrm{MAC}} \cap (1-\tilde{\alpha})\mathcal{R}_{\mathrm{BC}}$ holds. From $\boldsymbol{R} \in \tilde{\alpha}\mathcal{C}_{\mathrm{MAC}}$ it follows that $R_1 \leq \tilde{\alpha} R_{\overrightarrow{\mathrm{1R}}}$, $R_2 \leq \tilde{\alpha} R_{\overrightarrow{\mathrm{2R}}}$, and $R_1 + R_2 \leq \tilde{\alpha} R_\Sigma^{\mathrm{MAC}}$, which is equivalent to $(1-\tilde{\alpha}) \geq \frac{R_{\overrightarrow{\mathrm{1R}}} - R_1}{R_{\overrightarrow{\mathrm{1R}}}}$, $(1-\tilde{\alpha}) \geq \frac{R_{\overrightarrow{\mathrm{2R}}} - R_2}{R_{\overrightarrow{\mathrm{2R}}}}$, and $(1-\tilde{\alpha}) \geq \frac{R_\Sigma^{\mathrm{MAC}} - R_1 - R_2}{R_\Sigma^{\mathrm{MAC}}}$ respectively. Likewise, from $(1-\tilde{\alpha})\mathcal{R}_{\mathrm{BC}}$ we know that there exists a $\tilde{\beta} \in [0,1]$ where we have $R_1 \leq (1-\tilde{\alpha}) R_{\overrightarrow{\mathrm{R2}}}(\tilde{\beta})$ and $R_2 \leq (1-\tilde{\alpha}) R_{\overrightarrow{\mathrm{R1}}}(1-\tilde{\beta})$. If we solve $(1-\tilde{\alpha}) \geq \frac{R_{\overrightarrow{\mathrm{1R}}} - R_1}{R_{\overrightarrow{\mathrm{1R}}}}$ and $R_1 \leq (1-\tilde{\alpha}) R_{\overrightarrow{\mathrm{R2}}}(\tilde{\beta})$ for R_1 we get $R_1 \leq \frac{R_{\overrightarrow{\mathrm{1R}}} R_{\overrightarrow{\mathrm{R2}}}(\tilde{\beta})}{R_{\overrightarrow{\mathrm{1R}}} + R_{\overrightarrow{\mathrm{R2}}}(\tilde{\beta})} = \big(1-\alpha_1^\star(\tilde{\beta})\big) R_{\overrightarrow{\mathrm{R2}}}(\tilde{\beta}) = R_{11}(\tilde{\beta})$. We can use the previous inequalities to bound R_2 as follows

$$R_2 \leq (1-\tilde{\alpha}) R_{\overrightarrow{\mathrm{R1}}}(1-\tilde{\beta}) = \frac{R_{\overrightarrow{\mathrm{1R}}} - R_1}{R_{\overrightarrow{\mathrm{1R}}}} R_{\overrightarrow{\mathrm{R1}}}(1-\tilde{\beta}) \leq \Big(1 - \frac{R_{11}(\tilde{\beta})}{R_{\overrightarrow{\mathrm{1R}}}}\Big) R_{\overrightarrow{\mathrm{R1}}}(1-\tilde{\beta}) = R_{12}(\tilde{\beta}).$$

It follows that $\boldsymbol{R} \in \mathcal{R}_1$. If we use $(1-\tilde{\alpha}) \geq \frac{R_{\overrightarrow{\mathrm{2R}}} - R_2}{R_{\overrightarrow{\mathrm{2R}}}}$ and $(1-\tilde{\alpha}) \geq \frac{R_\Sigma^{\mathrm{MAC}} - R_1 - R_2}{R_\Sigma^{\mathrm{MAC}}}$ we can similarly conclude with $R_1 \leq (1-\tilde{\alpha}) R_{\overrightarrow{\mathrm{R2}}}(\tilde{\beta})$ and $R_2 \leq (1-\tilde{\alpha}) R_{\overrightarrow{\mathrm{R1}}}(1-\tilde{\beta})$ that $R_1 \leq R_{21}(\tilde{\beta})$ and $R_2 \leq R_{22}(\tilde{\beta})$ so that $\boldsymbol{R} \in \mathcal{R}_2$ and that $R_1 \leq R_{\Sigma 1}(\tilde{\beta})$ and $R_2 \leq R_{\Sigma 2}(\tilde{\beta})$ so that $\boldsymbol{R} \in \mathcal{R}_\Sigma$ respectively. This means that we have $\boldsymbol{R} \in \mathcal{R}_1 \cap \mathcal{R}_2 \cap \mathcal{R}_\Sigma$ from which we conclude that $\mathcal{R}_{\mathrm{BRopt}} \subseteq \mathcal{R}_1 \cap \mathcal{R}_2 \cap \mathcal{R}_\Sigma$ holds. Hence, we proved the equality $\mathcal{R}_{\mathrm{BRopt}} = \mathcal{R}_1 \cap \mathcal{R}_2 \cap \mathcal{R}_\Sigma$. $\square$

2.8.9 Proof of Corollary 2.13

For the proof of the results we basically look at the derivatives of some rate functions. For notational simplicity we omit the arguments; but we note the subtle particularities, which however do not change the main results.

For a specific but arbitrary $\beta \in [0,1]$, let $R_{\overrightarrow{\mathrm{R1}}}$, $R_{\overrightarrow{\mathrm{R2}}}$, $R'_{\overrightarrow{\mathrm{R1}}}$, $R'_{\overrightarrow{\mathrm{R2}}}$, $R''_{\overrightarrow{\mathrm{R1}}}$, and $R''_{\overrightarrow{\mathrm{R2}}}$ denote $R_{\overrightarrow{\mathrm{R1}}}(\beta)$ and $R_{\overrightarrow{\mathrm{R2}}}(1-\beta)$ and its first and second derivatives with respect to β. It can be easily seen that we have $R_{\overrightarrow{\mathrm{R1}}}$, $R_{\overrightarrow{\mathrm{R2}}}$, $R'_{\overrightarrow{\mathrm{R2}}} > 0$ and $R'_{\overrightarrow{\mathrm{R1}}}$, $R''_{\overrightarrow{\mathrm{R1}}}$, $R''_{\overrightarrow{\mathrm{R2}}} < 0$ for all $\beta \in [0,1]$ except for $\beta = 0$, where we have $R_{\overrightarrow{\mathrm{R2}}} = 0$, and for $\beta = 1$, where we have $R_{\overrightarrow{\mathrm{R1}}} = 0$. Later, we will still achieve strict monotony since we never have equality for both simultaneously.

From this we get the inequalities for the first derivatives

$$\begin{aligned}
\frac{\mathrm{d}R_{11}(\beta)}{\mathrm{d}\beta} &= \frac{R^2_{\overrightarrow{\mathrm{1R}}} R'_{\overrightarrow{\mathrm{R2}}}}{(R_{\overrightarrow{\mathrm{1R}}} + R_{\overrightarrow{\mathrm{R2}}})^2} \geq 0,\\
\frac{\mathrm{d}R_{21}(\beta)}{\mathrm{d}\beta} &= \frac{R_{\overrightarrow{\mathrm{2R}}}(R'_{\overrightarrow{\mathrm{R2}}}(R_{\overrightarrow{\mathrm{2R}}} + R_{\overrightarrow{\mathrm{R1}}}) - R_{\overrightarrow{\mathrm{R2}}} R'_{\overrightarrow{\mathrm{R1}}})}{(R_{\overrightarrow{\mathrm{2R}}} + R_{\overrightarrow{\mathrm{R1}}})^2} > 0,\\
\frac{\mathrm{d}R_{\Sigma 1}(\beta)}{\mathrm{d}\beta} &= \frac{R^{\mathrm{MAC}}_{\Sigma}(R'_{\overrightarrow{\mathrm{R2}}}(R^{\mathrm{MAC}}_{\Sigma} + R_{\overrightarrow{\mathrm{R1}}}) - R_{\overrightarrow{\mathrm{R2}}} R'_{\overrightarrow{\mathrm{R1}}}))}{(R^{\mathrm{MAC}}_{\Sigma} + R_{\overrightarrow{\mathrm{R1}}} + R_{\overrightarrow{\mathrm{R2}}})^2} > 0.
\end{aligned}$$

Therefore, $R_{11}(\beta)$ is increasing and $R_{21}(\beta)$ and $R_{\Sigma 1}(\beta)$ are strictly increasing with β. Accordingly, from

$$\begin{aligned}
\frac{\mathrm{d}R_{12}(\beta)}{\mathrm{d}\beta} &= \frac{R_{\overrightarrow{\mathrm{1R}}}(R'_{\overrightarrow{\mathrm{R1}}}(R_{\overrightarrow{\mathrm{1R}}} + R_{\overrightarrow{\mathrm{R2}}}) - R_{\overrightarrow{\mathrm{R1}}} R'_{\overrightarrow{\mathrm{R2}}})}{(R_{\overrightarrow{\mathrm{1R}}} + R_{\overrightarrow{\mathrm{R2}}})^2} < 0,\\
\frac{\mathrm{d}R_{22}(\beta)}{\mathrm{d}\beta} &= \frac{R^2_{\overrightarrow{\mathrm{2R}}} R'_{\overrightarrow{\mathrm{R1}}}}{(R_{\overrightarrow{\mathrm{2R}}} + R_{\overrightarrow{\mathrm{R1}}})^2} \leq 0,\\
\frac{\mathrm{d}R_{\Sigma 2}(\beta)}{\mathrm{d}\beta} &= \frac{R^{\mathrm{MAC}}_{\Sigma}(R'_{\overrightarrow{\mathrm{R1}}}(R^{\mathrm{MAC}}_{\Sigma} + R_{\overrightarrow{\mathrm{R2}}}) - R_{\overrightarrow{\mathrm{R1}}} R'_{\overrightarrow{\mathrm{R2}}}))}{(R^{\mathrm{MAC}}_{\Sigma} + R_{\overrightarrow{\mathrm{R1}}} + R_{\overrightarrow{\mathrm{R2}}})^2} < 0
\end{aligned}$$

we see that $R_{22}(\beta)$ is decreasing and $R_{12}(\beta)$ and $R_{\Sigma 2}(\beta)$ are strictly decreasing with β. Therefore, the functions $\boldsymbol{R}_1(\beta)$, $\boldsymbol{R}_2(\beta)$, and $\boldsymbol{R}_\Sigma(\beta)$ are parameterizations of the boundaries of $\mathcal{R}_1$, $\mathcal{R}_2$, and $\mathcal{R}_\Sigma$ in the first quadrant.

The normalized normal vector for the parametrized curve $\boldsymbol{R}_1(\beta)$ is given by $\big[-\frac{\mathrm{d}R_{12}(\beta)}{\mathrm{d}\beta}, \frac{\mathrm{d}R_{11}(\beta)}{\mathrm{d}\beta}\big]/||\boldsymbol{R}_1(\beta)||$ with the angle $\varphi_1(\beta) = \arctan -q_1(\beta)$ where $q_1(\beta) = \frac{\mathrm{d}R_{11}(\beta)}{\mathrm{d}\beta}\big/\frac{\mathrm{d}R_{12}(\beta)}{\mathrm{d}\beta}$. Since both components of the normal vector are nonnegative, the function

φ_1 has a range $[0, \pi/2]$. Since $R''_{\overrightarrow{\mathrm{R2}}} R'_{\overrightarrow{\mathrm{R1}}} - R''_{\overrightarrow{\mathrm{R1}}} R'_{\overrightarrow{\mathrm{R2}}} > 0$ for all $\beta \in [0, 1]$ the derivative

$$\frac{\mathrm{d}q_1(\beta)}{\mathrm{d}\beta} = \frac{R_{\overrightarrow{1\mathrm{R}}}(R''_{\overrightarrow{\mathrm{R2}}} R'_{\overrightarrow{\mathrm{R1}}} - R''_{\overrightarrow{\mathrm{R1}}} R'_{\overrightarrow{\mathrm{R2}}})(R_{\overrightarrow{1\mathrm{R}}} + R_{\overrightarrow{\mathrm{R2}}})}{((R'_{\overrightarrow{\mathrm{R1}}}(R_{\overrightarrow{1\mathrm{R}}} + R_{\overrightarrow{\mathrm{R2}}}) - R_{\overrightarrow{\mathrm{R1}}} R'_{\overrightarrow{\mathrm{R2}}}))^2}$$

is positive. Since $\arctan$ is strictly increasing, it follows that $\varphi_1(\beta)$ is strictly decreasing. Since $R_{11}(\beta)$ is increasing and $R_{12}(\beta)$ and $\varphi_1(\beta)$ are strictly decreasing it follows that $\mathcal{R}_1$ is convex.

Similarly, we get the angles $\varphi_2(\beta) = \arctan -q_2(\beta)$ and $\varphi_\Sigma(\beta) = \arctan -q_\Sigma(\beta)$ with $q_2(\beta) = \frac{\mathrm{d}R_{21}(\beta)}{\mathrm{d}\beta} \Big/ \frac{\mathrm{d}R_{22}(\beta)}{\mathrm{d}\beta}$ and $q_\Sigma(\beta) = \frac{\mathrm{d}R_{\Sigma 1}(\beta)}{\mathrm{d}\beta} \Big/ \frac{\mathrm{d}R_{\Sigma 2}(\beta)}{\mathrm{d}\beta}$ for the normal vectors of rate pairs on the parametrized curves $\boldsymbol{R}_2(\beta)$ and $\boldsymbol{R}_\Sigma(\beta)$ respectively. Again, $\varphi_2(\beta)$ and $\varphi_\Sigma(\beta)$ are strictly decreasing for all β since the derivatives

$$\begin{aligned}
\frac{\mathrm{d}q_2(\beta)}{\mathrm{d}\beta} &= \frac{R_{\overrightarrow{2\mathrm{R}}}(R''_{\overrightarrow{\mathrm{R2}}} R'_{\overrightarrow{\mathrm{R1}}} - R''_{\overrightarrow{\mathrm{R1}}} R'_{\overrightarrow{\mathrm{R2}}})(R_{\overrightarrow{2\mathrm{R}}} + R_{\overrightarrow{\mathrm{R1}}})}{((R'_{\overrightarrow{\mathrm{R2}}}(R_{\overrightarrow{2\mathrm{R}}} + R_{\overrightarrow{\mathrm{R1}}}) - R_{\overrightarrow{\mathrm{R2}}} R'_{\overrightarrow{\mathrm{R1}}}))^2}, \\
\frac{\mathrm{d}q_\Sigma(\beta)}{\mathrm{d}\beta} &= \frac{R_\Sigma^{\mathrm{MAC}}(R''_{\overrightarrow{\mathrm{R2}}} R'_{\overrightarrow{\mathrm{R1}}} - R''_{\overrightarrow{\mathrm{R1}}} R'_{\overrightarrow{\mathrm{R2}}})(R_\Sigma^{\mathrm{MAC}} + R_{\overrightarrow{\mathrm{R2}}} + R_{\overrightarrow{\mathrm{R1}}})}{((R'_{\overrightarrow{\mathrm{R1}}}(R_\Sigma^{\mathrm{MAC}} + R_{\overrightarrow{\mathrm{R2}}}) - R_{\overrightarrow{\mathrm{R1}}} R'_{\overrightarrow{\mathrm{R2}}}))^2}
\end{aligned}$$

are positive. Therefore, $\mathcal{R}_2$ and $\mathcal{R}_\Sigma$ are convex.

Finally, $\mathcal{R}_{\mathrm{BRopt}}$ is convex since the intersection of convex sets is itself convex. $\square$

2.8.10 Proof of Proposition 2.14

We have an intersection between $\boldsymbol{R}_1(\beta)$ and $\boldsymbol{R}_\Sigma(\beta)$ if there exist $\hat{\beta}, \check{\beta} \in [0, 1]$ with $[R_{11}(\hat{\beta}), R_{12}(\hat{\beta})] = [R_{\Sigma 1}(\check{\beta}), R_{\Sigma_2}(\check{\beta})]$. First we consider the case that the intersection point is not on the axes. Then from

$$\frac{R_{11}(\hat{\beta})}{R_{12}(\hat{\beta})} = \frac{R_{\overrightarrow{\mathrm{R2}}}(\hat{\beta})}{R_{\overrightarrow{\mathrm{R1}}}(1 - \hat{\beta})} = \frac{R_{\overrightarrow{\mathrm{R2}}}(\check{\beta})}{R_{\overrightarrow{\mathrm{R1}}}(1 - \check{\beta})} = \frac{R_{11}(\check{\beta})}{R_{12}(\check{\beta})}$$

and the fact that $R_{\overrightarrow{\mathrm{R2}}}(\beta)$ and $R_{\overrightarrow{\mathrm{R1}}}(1 - \beta)$ are strictly increasing and decreasing in β we conclude that for an intersection point we have $\hat{\beta} = \check{\beta}$. Since $\alpha_1^\star(\beta), \alpha_\Sigma^\star(\beta) < 1$ holds for any β, for an intersection point on the axes we either need $R_{\overrightarrow{\mathrm{R2}}}(\hat{\beta}) = R_{\overrightarrow{\mathrm{R2}}}(\check{\beta}) = 0$ or $R_{\overrightarrow{\mathrm{R1}}}(1 - \hat{\beta}) = R_{\overrightarrow{\mathrm{R1}}}(1 - \check{\beta}) = 0$ so that we have either $\hat{\beta} = \check{\beta} = 0$ or $\hat{\beta} = \check{\beta} = 1$. It follows that for an intersection point between $\boldsymbol{R}_1(\beta)$ and $\boldsymbol{R}_\Sigma(\beta)$ we always have $\hat{\beta} = \check{\beta}$. The same arguments apply for an intersection point between $\boldsymbol{R}_2(\beta)$ and $\boldsymbol{R}_\Sigma(\beta)$ and $\boldsymbol{R}_1(\beta)$ and $\boldsymbol{R}_2(\beta)$.

Accordingly, we have an intersection point between $\boldsymbol{R}_1(\beta)$ and $\boldsymbol{R}_\Sigma(\beta)$ iff there exists a $\beta_{1\Sigma} \in [0,1]$ with $R_{11}(\beta_{1\Sigma}) = R_{1\Sigma}(\beta_{1\Sigma})$ and $R_{21}(\beta_{1\Sigma}) = R_{2\Sigma}(\beta_{1\Sigma})$. This is fulfilled iff we have

$$\frac{R_{\overrightarrow{1\mathrm{R}}}}{R_{\overrightarrow{1\mathrm{R}}} + R_{\overrightarrow{\mathrm{R}2}}(\beta_{1\Sigma})} = \frac{R_\Sigma^{\mathrm{MAC}}}{R_\Sigma^{\mathrm{MAC}} + R_{\overrightarrow{\mathrm{R}1}}(1-\beta_{1\Sigma}) + R_{\overrightarrow{\mathrm{R}2}}(\beta_{1\Sigma})},$$

which is equivalent to the equation $\frac{R_{\overrightarrow{1\mathrm{R}}}}{R_\Sigma^{\mathrm{MAC}} - R_{\overrightarrow{1\mathrm{R}}}} = \frac{R_{\overrightarrow{\mathrm{R}2}}(\beta_{1\Sigma})}{R_{\overrightarrow{\mathrm{R}1}}(1-\beta_{1\Sigma})}$. Since this is exactly equation (2.24) we see that $\beta_{1\Sigma}$ corresponds to the unique optimal relay power distribution of the MAC rate pair $[R_1^M, R_2^M] = [R_{\overrightarrow{1\mathrm{R}}}, R_\Sigma^{\mathrm{MAC}} - R_{\overrightarrow{1\mathrm{R}}}] = \boldsymbol{\nu}_{1\Sigma} \in \mathcal{C}_{\mathrm{MAC}}$, cf. (2.5a).

With similar arguments we see that the unique intersection between $\boldsymbol{R}_1(\beta)$ and $\boldsymbol{R}_\Sigma(\beta)$ at $\beta_{\Sigma 2} \in [0,1]$ corresponds to $[R_1^M, R_2^M] = [R_\Sigma^{\mathrm{MAC}} - R_{\overrightarrow{2\mathrm{R}}}, R_{\overrightarrow{2\mathrm{R}}}] = \boldsymbol{\nu}_{\Sigma 2} \in \mathcal{C}_{\mathrm{MAC}}$.

Finally, we have an intersection of $\boldsymbol{R}_1(\beta)$ and $\boldsymbol{R}_2(\beta)$ iff there exists a $\beta_{12} \in [0,1]$ with $R_{11}(\beta_{12}) = R_{12}(\beta_{12})$ and $R_{11}(\beta_{12}) = R_{12}(\beta_{12})$. This is fulfilled iff we have

$$\frac{R_{\overrightarrow{1\mathrm{R}}}}{R_{\overrightarrow{1\mathrm{R}}} + R_{\overrightarrow{\mathrm{R}2}}(\beta_{12})} = \frac{R_{\overrightarrow{2\mathrm{R}}}}{R_{\overrightarrow{2\mathrm{R}}} + R_{\overrightarrow{\mathrm{R}1}}(1-\beta_{12}) + R_{\overrightarrow{\mathrm{R}2}}(\beta_{12})},$$

which is equivalent to the equation $\frac{R_{\overrightarrow{1\mathrm{R}}}}{R_{\overrightarrow{2\mathrm{R}}}} = \frac{R_{\overrightarrow{\mathrm{R}2}}(\beta_{12})}{R_{\overrightarrow{\mathrm{R}1}}(1-\beta_{12})}$. Let us assume that the rate pair $\boldsymbol{R}_1(\beta_{12}) \in \mathcal{R}_{\mathrm{BRopt}}$. From Lemma 2.11 we know that $[R_{\overrightarrow{1\mathrm{R}}}, R_{\overrightarrow{2\mathrm{R}}}]$ is the corresponding rate pair in the MAC phase. But since $R_{\overrightarrow{1\mathrm{R}}} + R_{\overrightarrow{2\mathrm{R}}} > R_\Sigma^{\mathrm{MAC}}$, this rate pair is not achievable so that we have $\boldsymbol{R}_1(\beta_{12}) \notin \mathcal{R}_{\mathrm{BRopt}}$.

The inequalities (2.28a) and (2.28b) follow after simple calculations using the relations $\frac{1}{R_{\overrightarrow{\mathrm{R}2}}(1)} \leq \frac{1}{R_\Sigma^{\mathrm{MAC}}} + \frac{1}{R_{\overrightarrow{\mathrm{R}2}}(1)} \leq \frac{1}{R_{\overrightarrow{1\mathrm{R}}}} + \frac{1}{R_{\overrightarrow{\mathrm{R}2}}(1)}$ and $\frac{1}{R_{\overrightarrow{\mathrm{R}1}}(1)} \leq \frac{1}{R_\Sigma^{\mathrm{MAC}}} + \frac{1}{R_{\overrightarrow{\mathrm{R}1}}(1)} \leq \frac{1}{R_{\overrightarrow{2\mathrm{R}}}} + \frac{1}{R_{\overrightarrow{\mathrm{R}1}}(1)}$. □

2.8.11 Proof of Corollary 2.15

Since $\mathcal{R}_{\mathrm{BRopt}}$ is given by the intersection (2.25), the boundary is characterized by the rate pairs on the boundaries of $\mathcal{R}_1$, $\mathcal{R}_2$, and $\mathcal{R}_\Sigma$ which are most restrictive. For $\beta = 0$ we have $R_{\overrightarrow{\mathrm{R}2}}(0) = 0 \Rightarrow R_{11}(0) = R_{21}(0) = R_{\Sigma 1}(0) = 0$. From (2.28b) we see that the rate pair $\boldsymbol{R}_2(0)$ is most restrictive for rate pairs on the ordinate $[0, R_2]$. For small $\beta > 0$ the boundary $\boldsymbol{R}_2(\beta)$ is still most restrictive until $\boldsymbol{R}_\Sigma(\beta)$ intersects the boundary at $\beta_{\Sigma 2}$. Until the next intersection, the boundary $\boldsymbol{R}_\Sigma(\beta)$ is most restrictive, i.e. for $\beta_{1\Sigma} < \beta < \beta_{\Sigma 2}$. And finally, for the last section $\boldsymbol{R}_1(\beta)$ with $\beta \geq \beta_{1\Sigma}$ is most restrictive. □

2.8.12 Proof of Theorem 2.16

For a convex set it is well-known that the weighted rate sum is attained at the boundary rate pair where the direction of the normal vector is equal to the direction of the weight vector. This can be easily seen by the following geometrical interpretation. Consider a hyperplane in $\mathbb{R}^2$ with a normal vector $\boldsymbol{q}$ which intersects the feasible set $\mathcal{R}_{\mathrm{BRopt}}$. Due to orthogonality, any rate pair of this intersection leads to the same weighted rate sum. In order to maximize the weighted rate sum we have to shift the hyperplane in the direction of the normal vector as far as possible until the hyperplane is finally tangential to the boundary of $\mathcal{R}_{\mathrm{BRopt}}$. Such a tangential hyperplane is called a supporting hyperplane. The rate pair where the supporting hyperplane with normal vector $\boldsymbol{q}$ intersects the boundary is the rate pair with the largest weighted rate sum for this weight vector.

Accordingly, the procedure to find the optimum is obvious. First we have to find the optimal relay power distribution β^* corresponding to the weight vector $\boldsymbol{q}$. Therefore, we have to find the rate pair on the boundary where the angle of the normal vector is equal to the angle of the weight vector, i.e. $\theta_{\boldsymbol{q}}$. Since the boundary is defined section-wise, we have to distinguish between the following cases: If $\theta_{\boldsymbol{q}} \in [\varphi_1(1), \varphi_1(\beta_{1\Sigma})]$, the optimal rate pair is on the boundary of $\mathcal{R}_1$. Therefore, for those angles we can calculate the optimal relay power distribution using the inverse function φ_1^{-1}. Similarly, if $\theta_{\boldsymbol{q}} \in [\varphi_\Sigma(\beta_{1\Sigma}), \varphi_\Sigma(\beta_{\Sigma 2})]$ or $\theta_{\boldsymbol{q}} \in [\varphi_2(\beta_{\Sigma 2}), \varphi_2(0)]$ the optimal rate pair is on the boundary of $\mathcal{R}_\Sigma$ or $\mathcal{R}_2$ respectively. For weight vectors with an angle $\theta_{\boldsymbol{q}} \in [0, \varphi_1(1))$, $(\varphi_1(\beta_{1\Sigma}), \varphi_\Sigma(\beta_{1\Sigma}))$, $(\varphi_\Sigma(\beta_{\Sigma 2}), \varphi_2(\beta_{\Sigma 2}))$, or $(\varphi_2(0), \pi/2)$, the rate pairs $\boldsymbol{R}_1(1)$, $\boldsymbol{R}_\Sigma(\beta_{1\Sigma})$, $\boldsymbol{R}_\Sigma(\beta_{\Sigma 2})$, or $\boldsymbol{R}_2(0)$ at the intersections of $\boldsymbol{R}_1(\beta)$, $\boldsymbol{R}_\Sigma(\beta)$, and $\boldsymbol{R}_2(\beta)$ with $\beta^* = 1$, $\beta_{1\Sigma}$, $\beta_{\Sigma 2}$, or 0 respectively are optimal. This characterizes the optimal rate pair on the boundary $\boldsymbol{R}^*_{\mathrm{opt}}(\beta^*(\boldsymbol{q}))$ and the corresponding optimal time division parameter $\alpha^*(\boldsymbol{q})$ for any weight vector $\boldsymbol{q} \in \mathbb{R}^2_+ \setminus \{\boldsymbol{0}\}$. $\square$

2.8.13 Proof of Lemma 2.17

From [AS64, 4.1.33] we have $\frac{x}{1+x} < \ln(1+x)$ for $x > -1, x \neq 0$. Therewith, we get for a constant $a > 0$ the inequality $\ln(1 + a/x) > \frac{a/x}{1+a/x}$, for $x > 0$. We can use this inequality to see that the first derivatives

$$\frac{\mathrm{d}\tilde{R}_{\overrightarrow{1\mathrm{R}}}(\alpha)}{\mathrm{d}\alpha} = \frac{1}{\ln(2)}\left(\ln\left(1+\frac{\gamma_1|h_1|^2}{\alpha}\right) - \frac{\frac{\gamma_1|h_1|^2}{\alpha}}{1+\frac{\gamma_1|h_1|^2}{\alpha}}\right) > 0$$
$$\frac{\mathrm{d}\tilde{R}_{\overrightarrow{2\mathrm{R}}}(\alpha)}{\mathrm{d}\alpha} = \frac{1}{\ln(2)}\left(\ln\left(1+\frac{\gamma_2|h_2|^2}{\alpha}\right) - \frac{\frac{\gamma_2|h_2|^2}{\alpha}}{1+\frac{\gamma_2|h_2|^2}{\alpha}}\right) > 0$$
$$\frac{\mathrm{d}\tilde{R}_{\Sigma}^{\mathrm{MAC}}(\alpha)}{\mathrm{d}\alpha} = \frac{1}{\ln(2)}\left(\ln\left(1+\frac{\gamma_1|h_1|^2+\gamma_2|h_2|^2}{\alpha}\right) - \frac{\frac{\gamma_1|h_1|^2+\gamma_2|h_2|^2}{\alpha}}{1+\frac{\gamma_1|h_1|^2+\gamma_2|h_2|^2}{\alpha}}\right) > 0$$

are positive so that $\tilde{R}_{\overrightarrow{1\mathrm{R}}}(\alpha)$, $\tilde{R}_{\overrightarrow{2\mathrm{R}}}(\alpha)$, and $\tilde{R}_{\Sigma}^{\mathrm{MAC}}(\alpha)$ are strictly increasing for $\alpha \in (0,1)$. The second derivatives of $\tilde{R}_{\overrightarrow{1\mathrm{R}}}(\alpha)$, $\tilde{R}_{\overrightarrow{2\mathrm{R}}}(\alpha)$, and $\tilde{R}_{\Sigma}^{\mathrm{MAC}}(\alpha)$ are given by

$$\frac{\mathrm{d}^2\tilde{R}_{\overrightarrow{1\mathrm{R}}}(\alpha)}{\mathrm{d}\alpha^2} = -\frac{\left(\gamma_1|h_1|^2\right)^2}{\alpha^3\left(1+\frac{\gamma_1|h_1|^2}{\alpha}\right)^2\ln(2)} \le 0$$
$$\frac{\mathrm{d}^2\tilde{R}_{\overrightarrow{2\mathrm{R}}}(\alpha)}{\mathrm{d}\alpha^2} = -\frac{\left(\gamma_2|h_2|^2\right)^2}{\alpha^3\left(1+\frac{\gamma_2|h_2|^2}{\alpha}\right)^2\ln(2)} \le 0$$
$$\frac{\mathrm{d}^2\tilde{R}_{\Sigma}^{\mathrm{MAC}}(\alpha)}{\mathrm{d}\alpha^2} = -\frac{\left(\gamma_1|h_1|^2+\gamma_2|h_2|^2\right)^2}{\alpha^3\left(1+\frac{\gamma_1|h_1|^2+\gamma_2|h_2|^2}{\alpha}\right)^2\ln(2)} \le 0.$$

Since $\tilde{R}_{\overrightarrow{1\mathrm{R}}}(\alpha)$, $\tilde{R}_{\overrightarrow{2\mathrm{R}}}(\alpha)$, and $\tilde{R}_{\Sigma}^{\mathrm{MAC}}(\alpha)$ are non-positive for all $\alpha \in (0,1)$ it follows that they are concave. □

2.8.14 Proof of Lemma 2.18

From [AS64, 4.1.33] we have $\frac{x}{1+x} < \ln(1+x)$ for $x > -1, x \neq 0$. Therewith, we get for a constant $a > 0$ the inequality $\ln(1+\frac{a}{1-x}) > \frac{\frac{a}{1-x}}{1+\frac{a}{1-x}}$, for $x \in [0,1)$. We can use this inequality to see that for an arbitrary but fixed β the first partial derivatives with respect to α

$$\frac{\partial\tilde{R}_{\overrightarrow{\mathrm{R}2}}(\alpha,\beta)}{\partial\alpha} = \frac{1}{\ln(2)}\left(\frac{\frac{\gamma_{\mathrm{R}}|h_2|^2\beta}{1-\alpha}}{1+\frac{\gamma_{\mathrm{R}}|h_2|^2\beta}{1-\alpha}} - \ln\left(1+\frac{\gamma_{\mathrm{R}}|h_2|^2\beta}{1-\alpha}\right)\right) < 0$$
$$\frac{\partial\tilde{R}_{\overrightarrow{\mathrm{R}1}}(\alpha,\beta)}{\partial\alpha} = \frac{1}{\ln(2)}\left(\frac{\frac{\gamma_{\mathrm{R}}|h_1|^2(1-\beta)}{1-\alpha}}{1+\frac{\gamma_{\mathrm{R}}|h_1|^2(1-\beta)}{1-\alpha}} - \ln\left(1+\frac{\gamma_{\mathrm{R}}|h_1|^2(1-\beta)}{1-\alpha}\right)\right) < 0$$

are negative so that $\tilde{R}_{\overrightarrow{\mathrm{R2}}}(\alpha, \beta)$ is strictly decreasing for $\alpha \in (0, 1]$. The Hessian matrix of $\tilde{R}_{\overrightarrow{\mathrm{R2}}}$ for $\alpha \in (0, 1]$ and $\beta \in [0, 1]$ can be easily calculated as follows

$$\boldsymbol{H}(\tilde{R}_{\overrightarrow{\mathrm{R2}}}) = \begin{bmatrix} \frac{\partial^2 \tilde{R}_{\overrightarrow{\mathrm{R2}}}}{\partial\alpha\,\partial\alpha} & \frac{\partial^2 \tilde{R}_{\overrightarrow{\mathrm{R2}}}}{\partial\alpha\,\partial\beta} \\ \frac{\partial^2 \tilde{R}_{\overrightarrow{\mathrm{R2}}}}{\partial\beta\,\partial\alpha} & \frac{\partial^2 \tilde{R}_{\overrightarrow{\mathrm{R2}}}}{\partial\beta\,\partial\beta} \end{bmatrix} = \frac{-\big(\gamma_{\mathrm{R}}|h_2|^2\big)^2}{\big(1+\frac{\gamma_{\mathrm{R}}\beta|h_2|^2}{1-\alpha}\big)^2\ln(2)} \begin{bmatrix} \frac{\beta^2}{(1-\alpha)^3} & \frac{\beta}{(1-\alpha)^2} \\ \frac{\beta}{(1-\alpha)^2} & \frac{1}{(1-\alpha)} \end{bmatrix}.$$

From $\prod_{i=1}^{2} \lambda_i(\tilde{R}_{\overrightarrow{\mathrm{R2}}}) = \det\Big(\boldsymbol{H}(\tilde{R}_{\overrightarrow{\mathrm{R2}}})\Big) = 0$ it follows that at least one eigenvalue $\lambda_i(\tilde{R}_{\overrightarrow{\mathrm{R2}}})$, $i = 1, 2$ is zero. Furthermore, from $\sum_{i=1}^{2} \lambda_i(\tilde{R}_{\overrightarrow{\mathrm{R2}}}) = \mathrm{tr}\Big(\boldsymbol{H}(\tilde{R}_{\overrightarrow{\mathrm{R2}}})\Big) = \frac{-\big(\gamma_{\mathrm{R}}|h_2|^2\big)^2(\beta^2+(1-\alpha)^2)}{\big(1+\frac{\gamma_{\mathrm{R}}\beta|h_2|^2}{1-\alpha}\big)^2\ln(2)(1-\alpha)^3} \le 0$ it additionally follows that the other eigenvalue is non-positive. Therefore, both eigenvalues $\lambda_i(\tilde{R}_{\overrightarrow{\mathrm{R2}}})$, $i = 1, 2$, are non-positive so that the rate function $\tilde{R}_{\overrightarrow{\mathrm{R2}}}(\alpha, \beta)$ is concave on $[0, 1) \times [0, 1]$.

Similarly, the Hessian matrix of $\tilde{R}_{\overrightarrow{\mathrm{R1}}}$ for $\alpha \in [0, 1)$ and $\beta \in [0, 1]$ is given by

$$\boldsymbol{H}(\tilde{R}_{\overrightarrow{\mathrm{R1}}}) = \begin{bmatrix} \frac{\partial^2 \tilde{R}_{\overrightarrow{\mathrm{R1}}}}{\partial\alpha\,\partial\alpha} & \frac{\partial^2 \tilde{R}_{\overrightarrow{\mathrm{R1}}}}{\partial\alpha\,\partial\beta} \\ \frac{\partial^2 \tilde{R}_{\overrightarrow{\mathrm{R1}}}}{\partial\beta\,\partial\alpha} & \frac{\partial^2 \tilde{R}_{\overrightarrow{\mathrm{R1}}}}{\partial\beta\,\partial\beta}. \end{bmatrix} = \frac{-\big(\gamma_{\mathrm{R}}|h_1|^2\big)^2}{\big(1+\frac{\gamma_{\mathrm{R}}(1-\beta)|h_1|^2}{1-\alpha}\big)^2\ln(2)} \begin{bmatrix} \frac{(1-\beta)^2}{(1-\alpha)^3} & \frac{1-\beta}{(1-\alpha)^2} \\ \frac{1-\beta}{(1-\alpha)^2} & \frac{1}{(1-\alpha)} \end{bmatrix}.$$

Again from $\prod_{i=1}^{2} \lambda_i(\tilde{R}_{\overrightarrow{\mathrm{R1}}}) = \det\Big(\boldsymbol{H}(\tilde{R}_{\overrightarrow{\mathrm{R1}}})\Big) = 0$ and $\sum_{i=1}^{2} \lambda_i(\tilde{R}_{\overrightarrow{\mathrm{R1}}}) = \mathrm{tr}\Big(\boldsymbol{H}(\tilde{R}_{\overrightarrow{\mathrm{R1}}})\Big) = \frac{-\big(\gamma_{\mathrm{R}}|h_2|^2\big)^2(\beta^2+(1-\alpha)^2)}{\big(1+\frac{\gamma_{\mathrm{R}}\beta|h_2|^2}{1-\alpha}\big)^2\ln(2)(1-\alpha)^3} \le 0$ it follows that both eigenvalues $\lambda_i(\tilde{R}_{\overrightarrow{\mathrm{R1}}})$, $i = 1, 2$, are non-positive so that the rate function $\tilde{R}_{\overrightarrow{\mathrm{R1}}}(\alpha, \beta)$ is concave on $[0, 1) \times [0, 1]$ as well. $\square$

2.8.15 Proof of Proposition 2.19

For any arbitrary but fixed time-division parameter $\alpha \in [0, 1)$ and rate pairs $\boldsymbol{R}^{(1)}, \boldsymbol{R}^{(2)} \in \tilde{\mathcal{R}}_{\mathrm{BC}}(\alpha)$ we have to show that for any $t \in [0, 1]$ we have $\boldsymbol{R}(t) := t\boldsymbol{R}^{(1)} + (1-t)\boldsymbol{R}^{(2)} \in \tilde{\mathcal{R}}_{\mathrm{BC}}(\alpha)$. Then $\tilde{\mathcal{R}}_{\mathrm{BC}}(\alpha)$ is convex.

Let $R_i(t) := tR_i^{(1)} + (1-t)R_i^{(2)}$, $i = 1, 2$, denote the components of $\boldsymbol{R}(t)$. For each rate pair $\boldsymbol{R}^{(k)}$, $k = 1, 2$, there exists a $\beta^{(k)} \in [0, 1]$ such that we have $R_1^{(k)} \le \tilde{R}_{\overrightarrow{\mathrm{R2}}}(\alpha, \beta^{(k)})$ and $R_2^{(k)} \le \tilde{R}_{\overrightarrow{\mathrm{R1}}}(\alpha, \beta^{(k)})$. For $t \in [0, 1]$ we define $\beta(t) := t\beta^{(1)} + (1-t)\beta^{(2)}$. Then we have

$$R_1(t) = tR_1^{(1)} + (1-t)R_1^{(2)} \le t\tilde{R}_{\overrightarrow{\mathrm{R2}}}(\alpha, \beta^{(1)}) + (1-t)\tilde{R}_{\overrightarrow{\mathrm{R2}}}(\alpha, \beta^{(2)}) \le \tilde{R}_{\overrightarrow{\mathrm{R2}}}(\alpha, \beta(t))$$
$$R_2(t) = tR_2^{(1)} + (1-t)R_2^{(2)} \le t\tilde{R}_{\overrightarrow{\mathrm{R1}}}(\alpha, \beta^{(1)}) + (1-t)\tilde{R}_{\overrightarrow{\mathrm{R1}}}(\alpha, \beta^{(2)}) \le \tilde{R}_{\overrightarrow{\mathrm{R1}}}(\alpha, \beta(t))$$

where the last inequalities follow from concavity of $\tilde{R}_{\overrightarrow{\mathrm{R2}}}(\alpha,\beta)$ and $\tilde{R}_{\overrightarrow{\mathrm{R1}}}(\alpha,\beta)$ for $\beta \in [0,1]$. This means that for all $t \in [0,1]$ we have $\beta(t) \in [0,1]$ and $R_1(t) \le \tilde{R}_{\overrightarrow{\mathrm{R2}}}(\alpha,\beta(t))$ and $R_1(t) \le \tilde{R}_{\overrightarrow{\mathrm{R2}}}(\alpha,\beta(t))$ so that $\boldsymbol{R}(t) \in \tilde{\mathcal{R}}_{\mathrm{BC}}(\alpha)$ follows. For $\alpha = 0$ we have $\tilde{\mathcal{R}}_{\mathrm{BC}}(0) = \emptyset$, which is convex by definition. □

2.8.16 Proof of Lemma 2.23

Since $\tilde{R}_{\overrightarrow{\mathrm{R1}}}(\alpha,\beta)$ and $\tilde{R}_{\overrightarrow{\mathrm{R2}}}(\alpha,\beta)$ are continuous, positive, and strictly decreasing for $\alpha \in [0,1)$ and a fixed β the absolute value of the rate pair $\tilde{R}_{\mathrm{BC}}(\alpha,\beta)$ is strictly decreasing as well. Likewise, $\tilde{R}_{\overrightarrow{\mathrm{1R}}}(\alpha)$, $\tilde{R}_{\overrightarrow{\mathrm{2R}}}(\alpha)$, and $\tilde{R}_{\Sigma}^{\mathrm{MAC}}(\alpha)$ are continuous, positive, and strictly increasing for $\alpha \in (0,1]$. Furthermore, for the continuous continuation we have $\tilde{R}_{\overrightarrow{\mathrm{R1}}}(1,\beta) = \tilde{R}_{\overrightarrow{\mathrm{R2}}}(1,\beta) = \tilde{R}_{\overrightarrow{\mathrm{1R}}}(0) = \tilde{R}_{\overrightarrow{\mathrm{2R}}}(0) = \tilde{R}_{\Sigma}^{\mathrm{MAC}}(0) = 0$ for all $\beta \in [0,1]$. Since both arguments of the minima are continuous and one is strictly increasing while the other is strictly decreasing it follows that for the maximizing time-division parameter both arguments are equal and the maximizing time-division parameter is unique. □

2.8.17 Proof of Theorem 2.21

First, we will prove the inclusion $\tilde{\mathcal{R}}_1 \cap \tilde{\mathcal{R}}_2 \cap \tilde{\mathcal{R}}_\Sigma \subseteq \tilde{\mathcal{R}}_{\mathrm{BRopt}}$. To this end we have to show that for any arbitrary rate pair $\boldsymbol{R} \in \tilde{\mathcal{R}}_1 \cap \tilde{\mathcal{R}}_2 \cap \tilde{\mathcal{R}}_\Sigma$ it follows that $\boldsymbol{R} \in \tilde{\mathcal{R}}_{\mathrm{BRopt}}$. Let $\varphi_{\boldsymbol{R}}$ and $R_{\boldsymbol{R}}$ denote the angle and the radius of the rate pair $\boldsymbol{R}$, i.e. the rate pair in polar coordinates is given by $\boldsymbol{R} = [R_{\boldsymbol{R}}\cos(\varphi_{\boldsymbol{R}}), R_{\boldsymbol{R}}\sin(\varphi_{\boldsymbol{R}})]$. If $\boldsymbol{R} \in \tilde{\mathcal{R}}_1 \cap \tilde{\mathcal{R}}_2 \cap \tilde{\mathcal{R}}_\Sigma$ we have $\boldsymbol{R} \in \tilde{\mathcal{R}}_1$, $\boldsymbol{R} \in \tilde{\mathcal{R}}_2$, and $\boldsymbol{R} \in \tilde{\mathcal{R}}_\Sigma$ so that we have

$$R_{\boldsymbol{R}} \le \min\left\{\tilde{R}_1(\varphi_{\boldsymbol{R}}), \tilde{R}_2(\varphi_{\boldsymbol{R}}), \tilde{R}_\Sigma(\varphi_{\boldsymbol{R}})\right\}.$$

From Lemma 2.23 we know that for each $\tilde{R}_1(\varphi_{\boldsymbol{R}})$, $\tilde{R}_2(\varphi_{\boldsymbol{R}})$, $\tilde{R}_\Sigma(\varphi_{\boldsymbol{R}})$ there exist unique maximizing time-division parameters $\tilde{\alpha}_1^*(\varphi_{\boldsymbol{R}})$, $\tilde{\alpha}_2^*(\varphi_{\boldsymbol{R}})$, and $\tilde{\alpha}_\Sigma^*(\varphi_{\boldsymbol{R}}) \in [0,1]$ where we have

$$\begin{aligned}
\tilde{R}_{\mathrm{BC}}(\tilde{\alpha}_1^*(\varphi_{\boldsymbol{R}}), \varphi_{\boldsymbol{R}}) &= \frac{\tilde{R}_{\overrightarrow{\mathrm{1R}}}(\tilde{\alpha}_1^*(\varphi_{\boldsymbol{R}}))}{\cos(\varphi_{\boldsymbol{R}})} \\
\tilde{R}_{\mathrm{BC}}(\tilde{\alpha}_2^*(\varphi_{\boldsymbol{R}}), \varphi_{\boldsymbol{R}}) &= \frac{\tilde{R}_{\overrightarrow{\mathrm{2R}}}(\tilde{\alpha}_2^*(\varphi_{\boldsymbol{R}}))}{\sin(\varphi_{\boldsymbol{R}})} \\
\tilde{R}_{\mathrm{BC}}(\tilde{\alpha}_\Sigma^*(\varphi_{\boldsymbol{R}}), \varphi_{\boldsymbol{R}}) &= \frac{\tilde{R}_\Sigma^{\mathrm{MAC}}(\tilde{\alpha}_\Sigma^*(\varphi_{\boldsymbol{R}}))}{\cos(\varphi_{\boldsymbol{R}}) + \sin(\varphi_{\boldsymbol{R}})}
\end{aligned}$$

respectively. Let $\tilde{\alpha}^* := \max\{\tilde{\alpha}_1^*(\varphi_{\boldsymbol{R}}), \tilde{\alpha}_2^*(\varphi_{\boldsymbol{R}}), \tilde{\alpha}_\Sigma^*(\varphi_{\boldsymbol{R}})\}$ denote the maximum, which corresponds to the shortest broadcast phase. With the power distribution $\tilde{\beta}^* := \tilde{\beta}_{\mathrm{BC}}(\tilde{\alpha}^*, \varphi_{\boldsymbol{R}})$, which is the power distribution of the rate pair on the boundary of $\tilde{\mathcal{R}}_{\mathrm{BC}}(\tilde{\alpha}^*)$ with angle $\varphi_{\boldsymbol{R}}$, we have $R_{\boldsymbol{R}} \leq \tilde{R}_{\mathrm{BC}}(\tilde{\alpha}^*, \tilde{\beta}^*)$ so that $\boldsymbol{R} \in \tilde{\mathcal{R}}_{\mathrm{BC}}(\tilde{\alpha}^*)$ follows. Moreover, since $\tilde{R}_{\overrightarrow{1\mathrm{R}}}(\alpha)$, $\tilde{R}_{\overrightarrow{2\mathrm{R}}}(\alpha)$, and $\tilde{R}_{\Sigma}^{\mathrm{MAC}}(\alpha)$ are increasing for $\alpha \in [0,1]$ we have for the individual rate constraints

$$\begin{aligned} R_1 &= R_{\boldsymbol{R}} \cos(\varphi_{\boldsymbol{R}}) \leq \tilde{R}_1(\varphi_{\boldsymbol{R}}) \cos(\varphi_{\boldsymbol{R}}) = \frac{\tilde{R}_{\overrightarrow{1\mathrm{R}}}(\tilde{\alpha}_1(\varphi_{\boldsymbol{R}}))}{\cos(\varphi_{\boldsymbol{R}})} \cos(\varphi_{\boldsymbol{R}}) \leq \tilde{R}_{\overrightarrow{1\mathrm{R}}}(\tilde{\alpha}^*) \\ R_2 &= R_{\boldsymbol{R}} \sin(\varphi_{\boldsymbol{R}}) \leq \tilde{R}_2(\varphi_{\boldsymbol{R}}) \sin(\varphi_{\boldsymbol{R}}) = \frac{\tilde{R}_{\overrightarrow{2\mathrm{R}}}(\tilde{\alpha}_2(\varphi_{\boldsymbol{R}}))}{\sin(\varphi_{\boldsymbol{R}})} \sin(\varphi_{\boldsymbol{R}}) \leq \tilde{R}_{\overrightarrow{2\mathrm{R}}}(\tilde{\alpha}^*) \end{aligned}$$

and for the sum-rate constraint

$$\begin{aligned} R_1 + R_2 &= R_{\boldsymbol{R}}(\cos(\varphi_{\boldsymbol{R}}) + \sin(\varphi_{\boldsymbol{R}})) \leq \tilde{R}_{\Sigma}(\varphi_{\boldsymbol{R}})(\cos(\varphi_{\boldsymbol{R}}) + \sin(\varphi_{\boldsymbol{R}})) \\ &= \frac{\tilde{R}_{\Sigma}^{\mathrm{MAC}}(\tilde{\alpha}_{\Sigma}(\varphi_{\boldsymbol{R}}))}{\cos(\varphi_{\boldsymbol{R}}) + \sin(\varphi_{\boldsymbol{R}})}(\cos(\varphi_{\boldsymbol{R}}) + \sin(\varphi_{\boldsymbol{R}})) = \tilde{R}_{\Sigma}^{\mathrm{MAC}}(\tilde{\alpha}_{\Sigma}(\varphi_{\boldsymbol{R}})) \leq \tilde{R}_{\Sigma}^{\mathrm{MAC}}(\tilde{\alpha}^*) \end{aligned}$$

so that $\boldsymbol{R} \in \tilde{\mathcal{C}}_{\mathrm{MAC}}(\tilde{\alpha}^*)$. It follows that $\boldsymbol{R} \in \tilde{\mathcal{R}}_{\mathrm{BC}}(\tilde{\alpha}^*) \cup \tilde{\mathcal{C}}_{\mathrm{MAC}}(\tilde{\alpha}^*) \subseteq \bigcup_{\alpha \in [0,1]} \tilde{\mathcal{R}}_{\mathrm{BC}}(\alpha) \cup \tilde{\mathcal{C}}_{\mathrm{MAC}}(\alpha) = \tilde{\mathcal{R}}_{\mathrm{BRopt}}$.

For the reverse inclusion we show that for any $\boldsymbol{R} \in \tilde{\mathcal{R}}_{\mathrm{BRopt}}$ it follows that $\boldsymbol{R} \in \tilde{\mathcal{R}}_1 \cap \tilde{\mathcal{R}}_2 \cap \tilde{\mathcal{R}}_{\Sigma}$. For any $\boldsymbol{R} = [R_{\boldsymbol{R}} \cos(\varphi_{\boldsymbol{R}}), R_{\boldsymbol{R}} \sin(\varphi_{\boldsymbol{R}})] \in \tilde{\mathcal{R}}_{\mathrm{BRopt}}$ there exists an $\tilde{\alpha}^* \in [0,1]$ such that $\boldsymbol{R} \in \tilde{\mathcal{C}}_{\mathrm{MAC}}(\tilde{\alpha}^*)$ and $\boldsymbol{R} \in \tilde{\mathcal{R}}_{\mathrm{BC}}(\tilde{\alpha}^*)$. This implies that we have

$$R_1 = R_{\boldsymbol{R}} \cos(\varphi_{\boldsymbol{R}}) \leq \tilde{R}_{\overrightarrow{1\mathrm{R}}}(\tilde{\alpha}^*) \;\Rightarrow\; R_{\boldsymbol{R}} \leq \frac{\tilde{R}_{\overrightarrow{1\mathrm{R}}}(\tilde{\alpha}^*)}{\cos(\varphi_{\boldsymbol{R}})}, \tag{2.88a}$$

$$R_2 = R_{\boldsymbol{R}} \sin(\varphi_{\boldsymbol{R}}) \leq \tilde{R}_{\overrightarrow{2\mathrm{R}}}(\tilde{\alpha}^*) \;\Rightarrow\; R_{\boldsymbol{R}} \leq \frac{\tilde{R}_{\overrightarrow{2\mathrm{R}}}(\tilde{\alpha}^*)}{\sin(\varphi_{\boldsymbol{R}})}, \tag{2.88b}$$

$$R_1 + R_2 = R_{\boldsymbol{R}}\big(\cos(\varphi_{\boldsymbol{R}}) + \sin(\varphi_{\boldsymbol{R}})\big) \leq \tilde{R}_{\Sigma}^{\mathrm{MAC}}(\tilde{\alpha}^*) \;\Rightarrow\; R_{\boldsymbol{R}} \leq \frac{\tilde{R}_{\Sigma}^{\mathrm{MAC}}(\tilde{\alpha}^*)}{\cos(\varphi_{\boldsymbol{R}}) + \sin(\varphi_{\boldsymbol{R}})}. \tag{2.88c}$$

For a given time-division $\tilde{\alpha}^*$ the rate pair with largest radius with the same angle as the rate pair $\boldsymbol{R}$ has the relay power distribution $\tilde{\beta}^* := \beta(\tilde{\alpha}^*, \varphi_{\boldsymbol{R}})$. This implies $\varphi_{\boldsymbol{R}} = \varphi_{\mathrm{BC}}(\tilde{\alpha}^*, \tilde{\beta}^*)$. Then it follows from $\boldsymbol{R} \in \tilde{\mathcal{R}}_{\mathrm{BC}}(\tilde{\alpha})$ that

$$\begin{aligned} R_1 &= R_{\boldsymbol{R}} \cos(\varphi_{\boldsymbol{R}}) \leq \tilde{R}_{\overrightarrow{\mathrm{R}2}}(\tilde{\alpha}^*, \tilde{\beta}^*), \\ R_2 &= R_{\boldsymbol{R}} \sin(\varphi_{\boldsymbol{R}}) \leq \tilde{R}_{\overrightarrow{\mathrm{R}1}}(\tilde{\alpha}^*, \tilde{\beta}^*). \end{aligned}$$

Furthermore, we have

$$\begin{aligned} \Big[\tilde{R}_{\overrightarrow{\mathrm{R}2}}(\tilde{\alpha}^*, \tilde{\beta}^*), \tilde{R}_{\overrightarrow{\mathrm{R}1}}(\tilde{\alpha}^*, \tilde{\beta}^*)\Big] &= \Big[\tilde{R}_{\overrightarrow{\mathrm{R}2}}\big(\tilde{\alpha}^*, \beta(\tilde{\alpha}^*, \varphi_{\boldsymbol{R}})\big), \tilde{R}_{\overrightarrow{\mathrm{R}1}}\big(\tilde{\alpha}^*, \beta(\tilde{\alpha}^*, \varphi_{\boldsymbol{R}})\big)\Big] \\ &= \tilde{\boldsymbol{R}}_{\mathrm{BC}}(\tilde{\alpha}^*, \varphi_{\boldsymbol{R}}) = \tilde{R}_{\mathrm{BC}}(\tilde{\alpha}^*, \varphi_{\boldsymbol{R}})[\cos(\varphi_{\boldsymbol{R}}), \sin(\varphi_{\boldsymbol{R}})]. \end{aligned}$$

Therefore, we have $R_{\boldsymbol{R}} \leq \tilde{R}_{\mathrm{BC}}(\tilde{\alpha}^*, \varphi_{\boldsymbol{R}})$. This together with (2.88a), (2.88b), and (2.88c) gives us

$$R_{\boldsymbol{R}} \leq \min\left\{\tilde{R}_{\mathrm{BC}}(\tilde{\alpha}^*, \varphi_{\boldsymbol{R}}), \frac{\tilde{R}_{\overrightarrow{1\mathrm{R}}}(\tilde{\alpha}^*)}{\cos(\varphi_{\boldsymbol{R}})}\right\} \leq \max_{\alpha\in[0,1]} \min\left\{\tilde{R}_{\mathrm{BC}}(\alpha, \varphi_{\boldsymbol{R}}), \frac{\tilde{R}_{\overrightarrow{1\mathrm{R}}}(\alpha)}{\cos(\varphi_{\boldsymbol{R}})}\right\} = \tilde{R}_1(\varphi_{\boldsymbol{R}})$$

$$R_{\boldsymbol{R}} \leq \min\left\{\tilde{R}_{\mathrm{BC}}(\tilde{\alpha}^*, \varphi_{\boldsymbol{R}}), \frac{\tilde{R}_{\overrightarrow{2\mathrm{R}}}(\tilde{\alpha}^*)}{\sin(\varphi_{\boldsymbol{R}})}\right\} \leq \max_{\alpha\in[0,1]} \min\left\{\tilde{R}_{\mathrm{BC}}(\alpha, \varphi_{\boldsymbol{R}}), \frac{\tilde{R}_{\overrightarrow{2\mathrm{R}}}(\alpha)}{\sin(\varphi_{\boldsymbol{R}})}\right\} = \tilde{R}_2(\varphi_{\boldsymbol{R}})$$

$$R_{\boldsymbol{R}} \leq \min\left\{\tilde{R}_{\mathrm{BC}}(\tilde{\alpha}^*, \varphi_{\boldsymbol{R}}), \frac{\tilde{R}_{\Sigma}^{\mathrm{MAC}}(\tilde{\alpha}^*)}{\cos(\varphi_{\boldsymbol{R}})+\sin(\varphi_{\boldsymbol{R}})}\right\} \leq \max_{\alpha\in[0,1]} \min\left\{\tilde{R}_{\mathrm{BC}}(\alpha, \varphi_{\boldsymbol{R}}), \frac{\tilde{R}_{\Sigma}^{\mathrm{MAC}}(\alpha)}{\cos(\varphi_{\boldsymbol{R}})+\sin(\varphi_{\boldsymbol{R}})}\right\} = \tilde{R}_{\Sigma}(\varphi_{\boldsymbol{R}})$$

This let us conclude that

$$\left.\begin{aligned} R_1 &= R_{\boldsymbol{R}}\cos(\varphi_{\boldsymbol{R}}) \leq \tilde{R}_1(\varphi_{\boldsymbol{R}})\cos(\varphi_{\boldsymbol{R}}) \\ R_2 &= R_{\boldsymbol{R}}\sin(\varphi_{\boldsymbol{R}}) \leq \tilde{R}_1(\varphi_{\boldsymbol{R}})\sin(\varphi_{\boldsymbol{R}}) \end{aligned}\right\} \Rightarrow \boldsymbol{R} = [R_1, R_2] \in \tilde{\mathcal{R}}_1,$$

$$\left.\begin{aligned} R_1 &= R_{\boldsymbol{R}}\cos(\varphi_{\boldsymbol{R}}) \leq \tilde{R}_2(\varphi_{\boldsymbol{R}})\cos(\varphi_{\boldsymbol{R}}) \\ R_2 &= R_{\boldsymbol{R}}\sin(\varphi_{\boldsymbol{R}}) \leq \tilde{R}_2(\varphi_{\boldsymbol{R}})\sin(\varphi_{\boldsymbol{R}}) \end{aligned}\right\} \Rightarrow \boldsymbol{R} = [R_1, R_2] \in \tilde{\mathcal{R}}_2,$$

$$\left.\begin{aligned} R_1 &= R_{\boldsymbol{R}}\cos(\varphi_{\boldsymbol{R}}) \leq \tilde{R}_{\Sigma}(\varphi_{\boldsymbol{R}})\cos(\varphi_{\boldsymbol{R}}) \\ R_2 &= R_{\boldsymbol{R}}\sin(\varphi_{\boldsymbol{R}}) \leq \tilde{R}_{\Sigma}(\varphi_{\boldsymbol{R}})\sin(\varphi_{\boldsymbol{R}}) \end{aligned}\right\} \Rightarrow \boldsymbol{R} = [R_1, R_2] \in \tilde{\mathcal{R}}_{\Sigma}.$$

Accordingly, we have $\boldsymbol{R} \in \tilde{\mathcal{R}}_1 \cap \tilde{\mathcal{R}}_2 \cap \tilde{\mathcal{R}}_{\Sigma}$ finally. Therewith, we have proved $\tilde{\mathcal{R}}_{\mathrm{BRopt}} = \tilde{\mathcal{R}}_1 \cap \tilde{\mathcal{R}}_2 \cap \tilde{\mathcal{R}}_{\Sigma}$. □

2.8.18 Proof of Theorem 2.24

We first prove (2.42a) and (2.42b). For $\varphi = 0$ and $\varphi = \frac{\pi}{2}$ we have $\tilde{R}_{\mathrm{BC}}(\alpha, 0) = \tilde{R}_{\overrightarrow{\mathrm{R2}}}(\alpha, 1)$ and $\tilde{R}_{\mathrm{BC}}(\alpha, \frac{\pi}{2}) = \tilde{R}_{\overrightarrow{\mathrm{R1}}}(\alpha, 0)$ respectively. Therewith, we have

$$\begin{aligned}
\tilde{R}_1(0) &= \max_{\alpha\in[0,1]} \min\left\{\tilde{R}_{\overrightarrow{\mathrm{R2}}}(\alpha, 1), \tilde{R}_{\overrightarrow{\mathrm{1R}}}(\alpha)\right\},\\
\tilde{R}_\Sigma(0) &= \max_{\alpha\in[0,1]} \min\left\{\tilde{R}_{\overrightarrow{\mathrm{R2}}}(\alpha, 1), \tilde{R}_\Sigma^{\mathrm{MAC}}(\alpha)\right\},\\
\tilde{R}_2(0) &= \lim_{\varphi\to 0} \max_{\alpha\in[0,1]} \min\left\{\tilde{R}_{\mathrm{BC}}(\alpha,\varphi), \frac{\tilde{R}_{\overrightarrow{\mathrm{2R}}}(\alpha)}{\sin(\phi)}\right\} = \max_{\alpha\in[0,1]} \tilde{R}_{\overrightarrow{\mathrm{R2}}}(\alpha, 1) = \tilde{R}_{\overrightarrow{\mathrm{R2}}}(0, 1).
\end{aligned}$$

We know that $\tilde{R}_{\overrightarrow{\mathrm{R2}}}(1, 1) = 0$ and $\tilde{R}_{\overrightarrow{\mathrm{R2}}}(\alpha, 1) > 0$ for $\alpha < 1$ and $\tilde{R}_{\overrightarrow{\mathrm{1R}}}(0) = \tilde{R}_\Sigma^{\mathrm{MAC}}(0) = 0$ and $\tilde{R}_{\overrightarrow{\mathrm{1R}}}(\alpha), \tilde{R}_\Sigma^{\mathrm{MAC}}(\alpha) > 0$ for $\alpha \in (0, 1]$. It follows that the optimal time-division parameters $\tilde{\alpha}_1(0), \tilde{\alpha}_\Sigma(0)$ are in the open interval $(0, 1)$. Since $\tilde{R}_{\overrightarrow{\mathrm{1R}}}(\alpha) < \tilde{R}_\Sigma^{\mathrm{MAC}}(\alpha)$ for all $\alpha \in (0, 1)$ it follows that $\tilde{R}_1(0) < \tilde{R}_\Sigma(0)$. Furthermore, since $\tilde{R}_{\overrightarrow{\mathrm{R2}}}(\alpha, 1) < \tilde{R}_{\overrightarrow{\mathrm{R2}}}(0, 1)$ for all $\alpha \in (0, 1)$ we have $\tilde{R}_\Sigma(0) < \tilde{R}_2(0)$ so that (2.42a) follows. Similarly, for $\varphi = \frac{\pi}{2}$ we have

$$\begin{aligned}
\tilde{R}_2(\tfrac{\pi}{2}) &= \max_{\alpha\in[0,1]} \min\left\{\tilde{R}_{\overrightarrow{\mathrm{R1}}}(\alpha, 0), \tilde{R}_{\overrightarrow{\mathrm{2R}}}(\alpha)\right\},\\
\tilde{R}_\Sigma(\tfrac{\pi}{2}) &= \max_{\alpha\in[0,1]} \min\left\{\tilde{R}_{\overrightarrow{\mathrm{R1}}}(\alpha, 0), \tilde{R}_\Sigma^{\mathrm{MAC}}(\alpha)\right\},\\
\tilde{R}_1(\tfrac{\pi}{2}) &= \lim_{\varphi\to \frac{\pi}{2}} \max_{\alpha\in[0,1]} \min\left\{\tilde{R}_{\mathrm{BC}}(\alpha,\varphi), \frac{\tilde{R}_{\overrightarrow{\mathrm{1R}}}(\alpha)}{\cos(\phi)}\right\} = \max_{\alpha\in[0,1]} \tilde{R}_{\overrightarrow{\mathrm{R1}}}(\alpha, 0) = \tilde{R}_{\overrightarrow{\mathrm{R1}}}(0, 0).
\end{aligned}$$

As before, we can conclude with $\tilde{R}_{\overrightarrow{\mathrm{2R}}}(\alpha) \leq \tilde{R}_\Sigma^{\mathrm{MAC}}(\alpha)$ and $\tilde{R}_{\overrightarrow{\mathrm{R1}}}(\alpha, 0) \leq \tilde{R}_{\overrightarrow{\mathrm{R2}}}(0, 0)$ for all $\alpha \in (0, 1)$ that (2.42b) holds.

Since the boundaries are continuous curves in the first quadrant we see from (2.42a) and (2.42b) that boundaries of $\tilde{\mathcal{R}}_1$, $\tilde{\mathcal{R}}_2$, and $\tilde{\mathcal{R}}_\Sigma$ have to intersect at least once. In the following we will see that they intersect at one boundary rate pair only.

For an intersection rate pair between the boundaries $\tilde{\mathcal{R}}_1$ and $\tilde{\mathcal{R}}_2$ there must exist an angle $\tilde{\varphi}_{12} \in (0, \frac{\pi}{2})$ where the radii are equal, i.e. we have $\tilde{R}_1(\tilde{\varphi}_{12}) = \tilde{R}_2(\tilde{\varphi}_{12})$. From Lemma 2.23 we additionally know that

$$\begin{aligned}
\tilde{R}_1(\tilde{\varphi}_{12}) &= \tilde{R}_{\mathrm{BC}}(\alpha_1^*(\tilde{\varphi}_{12}), \tilde{\varphi}_{12}) = \frac{\tilde{R}_{\overrightarrow{\mathrm{1R}}}(\tilde{\alpha}_1(\tilde{\varphi}_{12}))}{\cos(\tilde{\varphi}_{12})}\\
\tilde{R}_2(\tilde{\varphi}_{12}) &= \tilde{R}_{\mathrm{BC}}(\alpha_2^*(\tilde{\varphi}_{12}), \tilde{\varphi}_{12}) = \frac{\tilde{R}_{\overrightarrow{\mathrm{2R}}}(\tilde{\alpha}_2(\tilde{\varphi}_{12}))}{\sin(\tilde{\varphi}_{12})}
\end{aligned}$$

so that we have $\tilde{R}_{\mathrm{BC}}(\alpha_1^*(\tilde{\varphi}_{12}),\tilde{\varphi}_{12}) = \tilde{R}_{\mathrm{BC}}(\alpha_2^*(\tilde{\varphi}_{12}),\tilde{\varphi}_{12}) > 0$. Since $\tilde{R}_{\mathrm{BC}}(\alpha,\varphi)$ is continuous and strictly decreasing for $\alpha \in [0,1)$ and a fixed $\varphi \in [0,\frac{\pi}{2}]$ it follows that $\tilde{\alpha}_1^*(\tilde{\varphi}_{12}) = \tilde{\alpha}_2^*(\tilde{\varphi}_{12}) =: \tilde{\alpha}_{12}$.

The equality $\frac{\tilde{R}_{\overrightarrow{1\mathrm{R}}}(\tilde{\alpha}_{12})}{\cos(\tilde{\varphi}_{12})} = \frac{\tilde{R}_{\overrightarrow{2\mathrm{R}}}(\tilde{\alpha}_{12})}{\sin(\tilde{\varphi}_{12})}$ characterizes the radius of the intersection rate pair of the individual rate constraints of $\tilde{\mathcal{C}}_{\mathrm{MAC}}(\tilde{\alpha}_{12})$. For any α the rate region $\tilde{\mathcal{C}}_{\mathrm{MAC}}(\alpha)$ has only one intersection rate pair of the individual rate constraints, which angle we denote by $\tilde{\varphi}_{12}(\alpha)$. Since both rate constraints strictly increase with α, also the radius of the intersection strictly increases. Since $\tilde{R}_{\mathrm{BC}}(\alpha,\varphi)$ strictly decreases, we can conclude that there is only one time-division parameter where the intersection rate pair is on the boundary of $\tilde{R}_{\mathrm{BC}}(\alpha,\varphi)$ as well. It follows that $\tilde{\varphi}_{12}$ is unique.

From the equality of $\tilde{R}_1(\tilde{\varphi}_{12}) = \tilde{R}_2(\tilde{\varphi}_{12})$ we can conclude

$$\tilde{R}_{\Sigma}^{\mathrm{MAC}}(\tilde{\alpha}_{12}) < \tilde{R}_{\overrightarrow{1\mathrm{R}}}(\tilde{\alpha}_{12}) + \tilde{R}_{\overrightarrow{2\mathrm{R}}}(\tilde{\alpha}_{12}) = \tilde{R}_{\mathrm{BC}}(\tilde{\alpha}_{12},\tilde{\varphi}_{12})(\sin(\tilde{\varphi}_{12}) + \cos(\tilde{\varphi}_{12})).$$

Since $\tilde{R}_{\Sigma}^{\mathrm{MAC}}(\alpha)$ and $\tilde{R}_{\mathrm{BC}}(\alpha,\varphi)$ are strictly increasing and decreasing for α and fixed φ, it follows that for the optimal time-division parameter we have $\tilde{\alpha}_{\Sigma}^*(\tilde{\varphi}_{12}) < \tilde{\alpha}_{12}$ so that

$$\tilde{R}_1(\tilde{\varphi}_{12}) = \tilde{R}_2(\tilde{\varphi}_{12}) = \tilde{R}_{\mathrm{BC}}(\tilde{\alpha}_{12},\tilde{\varphi}_{12}) > \tilde{R}_{\mathrm{BC}}(\tilde{\alpha}_{\Sigma}^*(\tilde{\varphi}_{12}),\tilde{\varphi}_{12}) = \tilde{R}_{\Sigma}(\tilde{\varphi}).$$

Since $\tilde{R}_{\overrightarrow{2\mathrm{R}}}(\alpha,\varphi)$ and $\tilde{R}_{\mathrm{BC}}(\alpha,\varphi)$ are strictly increasing and decreasing for α and fixed φ, it follows that for the optimal time-division parameter we have $\tilde{\alpha}_2^*(\tilde{\varphi}_{\Sigma 1}) < \tilde{\alpha}_{\Sigma 1}$ so that

$$\tilde{R}_1(\tilde{\varphi}_{\Sigma 1}) = \tilde{R}_{\Sigma}(\tilde{\varphi}_{\Sigma 1}) = \tilde{R}_{\mathrm{BC}}(\tilde{\alpha}_{\Sigma 1},\tilde{\varphi}_{\Sigma 1}) < \tilde{R}_{\mathrm{BC}}(\tilde{\alpha}_2^*(\tilde{\varphi}_{\Sigma 1}),\tilde{\varphi}_{\Sigma 1}) = \tilde{R}_2(\tilde{\varphi}).$$

The discussion for the intersections between the boundaries of $\tilde{\mathcal{R}}_{\Sigma}$ and $\tilde{\mathcal{R}}_1$ or $\tilde{\mathcal{R}}_2$ is similar, which we will present here for completeness. For an intersection of the boundaries $\tilde{\mathcal{R}}_{\Sigma}$ and $\tilde{\mathcal{R}}_1$ there must exist an angle $\tilde{\varphi}_{\Sigma 1} \in (0,\frac{\pi}{2})$ where the radii are equal, i.e. we have $\tilde{R}_{\Sigma}(\tilde{\varphi}_{\Sigma 1}) = \tilde{R}_1(\tilde{\varphi}_{\Sigma 1})$. From Lemma 2.23 we additionally know that

$$\begin{aligned}
\tilde{R}_1(\tilde{\varphi}_{\Sigma 1}) = \tilde{R}_{\mathrm{BC}}(\alpha_1^*(\tilde{\varphi}_{\Sigma 1}),\tilde{\varphi}_{\Sigma 1}) &= \frac{\tilde{R}_{\overrightarrow{1\mathrm{R}}}(\tilde{\alpha}_1(\tilde{\varphi}_{\Sigma 1}))}{\cos(\tilde{\varphi}_{\Sigma 1})} \\
\tilde{R}_{\Sigma}(\tilde{\varphi}_{\Sigma 1}) = \tilde{R}_{\mathrm{BC}}(\alpha_{\Sigma}^*(\tilde{\varphi}_{\Sigma 1}),\tilde{\varphi}_{\Sigma 1}) &= \frac{\tilde{R}_{\Sigma}^{\mathrm{MAC}}(\tilde{\alpha}_{\Sigma}(\tilde{\varphi}_{\Sigma 1}))}{\sin(\tilde{\varphi}_{\Sigma 1}) + \cos(\tilde{\varphi}_{\Sigma 1})}
\end{aligned}$$

so that we have $\tilde{R}_{\mathrm{BC}}(\alpha_1^*(\tilde{\varphi}_{\Sigma 1}),\tilde{\varphi}_{\Sigma 1}) = \tilde{R}_{\mathrm{BC}}(\alpha_{\Sigma}^*(\tilde{\varphi}_{\Sigma 1}),\tilde{\varphi}_{\Sigma 1}) > 0$. Since $\tilde{R}_{\mathrm{BC}}(\alpha,\varphi)$ is continuous and strictly decreasing for $\alpha \in [0,1)$ and a fixed $\varphi \in [0,\frac{\pi}{2}]$ it follows that $\tilde{\alpha}_{\Sigma}^*(\tilde{\varphi}_{\Sigma 1}) = \tilde{\alpha}_2^*(\tilde{\varphi}_{\Sigma 1}) =: \tilde{\alpha}_{\Sigma 1}$.

The equality $\frac{\tilde{R}_{\overrightarrow{1\mathrm{R}}}(\tilde{\alpha}_{\Sigma 1})}{\cos(\tilde{\varphi}_{\Sigma 1})} = \frac{\tilde{R}_{\Sigma}^{\mathrm{MAC}}(\tilde{\alpha}_{\Sigma 1})}{\sin(\tilde{\varphi}_{\Sigma 1})+\cos(\tilde{\varphi}_{\Sigma 1})}$ characterizes the radius of the intersection rate pair of the individual rate constraint and the sum-rate constraint of $\tilde{\mathcal{C}}_{\mathrm{MAC}}(\tilde{\alpha}_{\Sigma 1})$. For any

α the rate region $\tilde{\mathcal{C}}_{\mathrm{MAC}}(\alpha)$ has only one intersection rate pair, which angle we will denote by $\tilde{\varphi}_{\Sigma 1}(\alpha)$. Since the individual and sum-rate constraints strictly increase with α, also the radius of the intersection strictly increases. Since $\tilde{R}_{\mathrm{BC}}(\alpha,\varphi)$ strictly decreases, we can conclude that there is only one time-division parameter where the intersection rate pair is on the boundary of $\tilde{R}_{\mathrm{BC}}(\alpha,\varphi)$ as well. It follows that $\tilde{\varphi}_{\Sigma 1}$ is unique.

From the equality of $\tilde{R}_1(\tilde{\varphi}_{\Sigma 1}) = \tilde{R}_\Sigma(\tilde{\varphi}_{\Sigma 1})$ we can conclude

$$\tilde{R}_{\overrightarrow{2\mathrm{R}}}(\tilde{\alpha}_{\Sigma 1}) > \tilde{R}_\Sigma^{\mathrm{MAC}}(\tilde{\alpha}_{\Sigma 1}) - \tilde{R}_{\overrightarrow{1\mathrm{R}}}(\tilde{\alpha}_{\Sigma 1}) = \tilde{R}_{\mathrm{BC}}(\tilde{\alpha}_{\Sigma 1},\tilde{\varphi}_{\Sigma 1})\sin(\tilde{\varphi}_{\Sigma 1}).$$

Since $\tilde{R}_{\overrightarrow{2\mathrm{R}}}(\alpha)$ and $\tilde{R}_{\mathrm{BC}}(\alpha,\varphi)$ are strictly increasing and decreasing for α and fixed φ, it follows that for the optimal time-division parameter we have $\tilde{\alpha}_2^*(\tilde{\varphi}_{\Sigma 1}) < \tilde{\alpha}_{\Sigma 1}$ so that

$$\tilde{R}_1(\tilde{\varphi}_{\Sigma 1}) = \tilde{R}_\Sigma(\tilde{\varphi}_{\Sigma 1}) = \tilde{R}_{\mathrm{BC}}(\tilde{\alpha}_{\Sigma 1},\tilde{\varphi}_{\Sigma 1}) < \tilde{R}_{\mathrm{BC}}(\tilde{\alpha}_2^*(\tilde{\varphi}_{\Sigma 1}),\tilde{\varphi}_{\Sigma 1}) = \tilde{R}_2(\tilde{\varphi}).$$

Finally, for an intersection of the boundaries $\tilde{\mathcal{R}}_2$ and $\tilde{\mathcal{R}}_\Sigma$ there must exist an angle $\tilde{\varphi}_{2\Sigma} \in (0,\frac{\pi}{2})$ where the radii are equal, i.e. we have $\tilde{R}_\Sigma(\tilde{\varphi}_{2\Sigma}) = \tilde{R}_2(\tilde{\varphi}_{2\Sigma})$. From Lemma 2.23 we additionally know that

$$\begin{aligned}
\tilde{R}_2(\tilde{\varphi}_{2\Sigma}) &= \tilde{R}_{\mathrm{BC}}(\alpha_1^*(\tilde{\varphi}_{2\Sigma}),\tilde{\varphi}_{2\Sigma}) = \frac{\tilde{R}_{\overrightarrow{2\mathrm{R}}}(\tilde{\alpha}_1(\tilde{\varphi}_{2\Sigma}))}{\sin(\tilde{\varphi}_{2\Sigma})}\\
\tilde{R}_\Sigma(\tilde{\varphi}_{2\Sigma}) &= \tilde{R}_{\mathrm{BC}}(\alpha_\Sigma^*(\tilde{\varphi}_{2\Sigma}),\tilde{\varphi}_{2\Sigma}) = \frac{\tilde{R}_\Sigma^{\mathrm{MAC}}(\tilde{\alpha}_\Sigma(\tilde{\varphi}_{2\Sigma}))}{\sin(\tilde{\varphi}_{2\Sigma})+\cos(\tilde{\varphi}_{2\Sigma})}
\end{aligned}$$

so that we have $\tilde{R}_{\mathrm{BC}}(\alpha_1^*(\tilde{\varphi}_{2\Sigma}),\tilde{\varphi}_{2\Sigma}) = \tilde{R}_{\mathrm{BC}}(\alpha_\Sigma^*(\tilde{\varphi}_{2\Sigma}),\tilde{\varphi}_{2\Sigma}) > 0$. Since $\tilde{R}_{\mathrm{BC}}(\alpha,\varphi)$ is continuous and strictly decreasing for $\alpha \in [0,1)$ and a fixed $\varphi \in [0,\frac{\pi}{2}]$ it follows that $\tilde{\alpha}_2^*(\tilde{\varphi}_{2\Sigma}) = \tilde{\alpha}_\Sigma^*(\tilde{\varphi}_{2\Sigma}) =: \tilde{\alpha}_{2\Sigma}$.

The equality $\frac{\tilde{R}_{\overrightarrow{2\mathrm{R}}}(\tilde{\alpha}_{2\Sigma})}{\sin(\tilde{\varphi}_{2\Sigma})} = \frac{\tilde{R}_\Sigma^{\mathrm{MAC}}(\tilde{\alpha}_{2\Sigma})}{\sin(\tilde{\varphi}_{2\Sigma})+\cos(\tilde{\varphi}_{2\Sigma})}$ characterizes the radius of the intersection rate pair of the individual rate constraint and the sum-rate constraint of $\tilde{\mathcal{C}}_{\mathrm{MAC}}(\tilde{\alpha}_{2\Sigma})$. For any α the rate region $\tilde{\mathcal{C}}_{\mathrm{MAC}}(\alpha)$ has only one intersection rate pair, which angle we will denote by $\tilde{\varphi}_{2\Sigma}(\alpha)$. Since the individual and sum-rate constraints strictly increase with α, also the radius of the intersection strictly increases. Since $\tilde{R}_{\mathrm{BC}}(\alpha,\varphi)$ strictly decreases, we can conclude that there is only one time-division parameter where the intersection rate pair is on the boundary of $\tilde{R}_{\mathrm{BC}}(\alpha,\varphi)$ as well. It follows that $\tilde{\varphi}_{2\Sigma}$ is unique.

From the equality of $\tilde{R}_2(\tilde{\varphi}_{2\Sigma}) = \tilde{R}_\Sigma(\tilde{\varphi}_{2\Sigma})$ we can conclude

$$\tilde{R}_{\overrightarrow{1\mathrm{R}}}(\tilde{\alpha}_{2\Sigma}) > \tilde{R}_\Sigma^{\mathrm{MAC}}(\tilde{\alpha}_{2\Sigma}) - \tilde{R}_{\overrightarrow{2\mathrm{R}}}(\tilde{\alpha}_{2\Sigma}) = \tilde{R}_{\mathrm{BC}}(\tilde{\alpha}_{2\Sigma},\tilde{\varphi}_{2\Sigma})\cos(\tilde{\varphi}_{2\Sigma}).$$

Since $\tilde{R}_{\overrightarrow{1\mathrm{R}}}(\alpha)$ and $\tilde{R}_{\mathrm{BC}}(\alpha,\varphi)$ are strictly increasing and decreasing for α and fixed φ, it follows that for the optimal time-division parameter we have $\tilde{\alpha}_1^*(\tilde{\varphi}_{2\Sigma}) < \tilde{\alpha}_{2\Sigma}$ so that

$$\tilde{R}_2(\tilde{\varphi}_{2\Sigma}) = \tilde{R}_\Sigma(\tilde{\varphi}_{2\Sigma}) = \tilde{R}_{\mathrm{BC}}(\tilde{\alpha}_{2\Sigma},\tilde{\varphi}_{2\Sigma}) < \tilde{R}_{\mathrm{BC}}(\tilde{\alpha}_1^*(\tilde{\varphi}_{2\Sigma}),\tilde{\varphi}_{2\Sigma}) = \tilde{R}_1(\tilde{\varphi}).$$

This finishes the proof. □

2.8.19 Proof of Corollary 2.25

Since $\tilde{\mathcal{R}}_{\text{BRopt}}$ is given by the intersection (2.39), the boundary is characterized by the rate pairs on the boundaries of $\tilde{\mathcal{R}}_1$, $\tilde{\mathcal{R}}_2$, and $\tilde{\mathcal{R}}_\Sigma$ which are most restrictive. For $\varphi = 0$ we know from (2.42a) that radius $\tilde{R}_1(0)$ is most restrictive for rate pairs on the abscissa $[R_1, 0]$. This remains the case until the boundary of $\tilde{R}_\Sigma$ intersects the boundary of $\tilde{R}_1$, i.e. for $\varphi \leq \tilde{\varphi}_{1\Sigma}$. Then the radius $\tilde{R}_\Sigma(\varphi)$ is most restrictive until the boundary of $\tilde{R}_2$ intersects the boundary of $\tilde{R}_\Sigma$, i.e. for $\varphi \in (\tilde{\varphi}_{2\Sigma}, \tilde{\varphi}_{\Sigma 1})$. Finally, for $\varphi \in [\frac{\pi}{2}, \tilde{\varphi}_{2\Sigma}]$ the radius $\tilde{R}_2(\varphi)$ is most restrictive. □

2.8.20 Proof of Theorem 2.26

We first prove that $\tilde{\mathcal{R}}_1$ is convex. Therefore, we have to show that for any two rate pairs $\boldsymbol{R}^{(k)} = [R_1^{(k)}, R_2^{(k)}] \in \tilde{\mathcal{R}}_1$, $k = 1, 2$, we have $\boldsymbol{R}(t) = [R_1(t), R_2(t)] := t\boldsymbol{R}^{(1)} + (1 - t)\boldsymbol{R}^{(2)} \in \tilde{\mathcal{R}}_1$ for all $t \in [0, 1]$. Therefore, let $\varphi_{\boldsymbol{R}^{(k)}}$ and $R_{\boldsymbol{R}^{(k)}}$ denote the angle and radius of the rate vector $\boldsymbol{R}^{(k)}$, $k = 1, 2$.

From $\boldsymbol{R}^{(k)} \in \tilde{\mathcal{R}}_1$ it follows that $R_{\boldsymbol{R}^{(k)}} \leq \tilde{R}_1(\varphi_{\boldsymbol{R}^{(k)}})$, $k = 1, 2$. According to Lemma 2.23 there exist time-division parameters $\tilde{\alpha}_1^{(k)} := \alpha_1^*(\varphi_{\boldsymbol{R}^{(k)}})$ with $R_{\boldsymbol{R}^{(k)}} \leq \frac{\tilde{R}_{\overrightarrow{1\text{R}}}(\tilde{\alpha}_1^{(k)})}{\cos(\varphi_{\boldsymbol{R}^{(k)}})}$ so that we have

$$R_1^{(k)} = R_{\boldsymbol{R}^{(k)}} \cos(\varphi_{\boldsymbol{R}^{(k)}}) \leq \tilde{R}_{\overrightarrow{1\text{R}}}(\tilde{\alpha}_1^{(k)}), \qquad k = 1, 2. \tag{2.89}$$

Then from Proposition 2.20 we additionally get the relay distribution factors $\tilde{\beta}_1^{(k)} := \tilde{\beta}_{\text{BC}}(\tilde{\alpha}_1^{(k)}, \varphi_{\boldsymbol{R}^{(k)}})$ so that we have $R_{\boldsymbol{R}^{(k)}} \leq \tilde{R}_{\text{BC}}(\tilde{\alpha}_1^{(k)}, \tilde{\beta}_1^{(k)})$, $k = 1, 2$, which is equivalent to

$$R_1^{(k)} \leq \tilde{R}_{\overrightarrow{\text{R2}}}(\tilde{\alpha}_1^{(k)}, \tilde{\beta}_1^{(k)}), \qquad R_2^{(k)} \leq \tilde{R}_{\overrightarrow{\text{R1}}}(\tilde{\alpha}_1^{(k)}, \tilde{\beta}_1^{(k)}). \tag{2.90}$$

For $t \in [0, 1]$ let us define $\tilde{\alpha}_1(t) := t\tilde{\alpha}_1^{(1)} + (1-t)\tilde{\alpha}_1^{(2)}$ and $\tilde{\beta}_1(t) := t\tilde{\beta}_1^{(1)} + (1-t)\tilde{\beta}_1^{(2)}$. From Lemma 2.17 and 2.18 we know that $\tilde{R}_{\overrightarrow{1\text{R}}}(\alpha)$, $\tilde{R}_{\overrightarrow{\text{R2}}}(\alpha, \beta)$, and $\tilde{R}_{\overrightarrow{\text{R1}}}(\alpha, \beta)$ are concave.

Then let $\tilde{\varphi}_1(t)$ be the angle of $[\min\{\tilde{R}_{\overrightarrow{\text{R2}}}(\tilde{\alpha}_1(t), \tilde{\beta}_1(t)), \tilde{R}_{\overrightarrow{1\text{R}}}(\tilde{\alpha}_1(t))\}, \tilde{R}_{\overrightarrow{\text{R1}}}(\tilde{\alpha}_1(t), \tilde{\beta}_1(t))]$, which is given by

$$\tilde{\varphi}_1(t) := \begin{cases} \arctan \frac{\tilde{R}_{\overrightarrow{\text{R1}}}(\tilde{\alpha}_1(t), \tilde{\beta}_1(t))}{\min\{\tilde{R}_{\overrightarrow{\text{R2}}}(\tilde{\alpha}_1(t), \tilde{\beta}_1(t)), \tilde{R}_{\overrightarrow{1\text{R}}}(\tilde{\alpha}_1(t))\}}, & \text{if } \tilde{\beta}_1(t) \neq 0 \\ \frac{\pi}{2}, & \text{if } \tilde{\beta}_1(t) = 0. \end{cases}$$

Note that if $\tilde{R}_{\overrightarrow{\text{R2}}}(\tilde{\alpha}_1(t), \tilde{\beta}_1(t)) \leq \tilde{R}_{\overrightarrow{1\text{R}}}(\tilde{\alpha}_1(t))$ we have $\tilde{\varphi}_1(t) = \tilde{\varphi}_{\text{BC}}(\tilde{\alpha}_1(t), \tilde{\beta}_1(t))$, else we have $\tilde{\varphi}_1(t) > \tilde{\varphi}_{\text{BC}}(\tilde{\alpha}_1(t), \tilde{\beta}_1(t))$ because the inverse function of the tangent is strictly increasing.

In the following, we show that the boundary rate pair on $\tilde{\mathcal{R}}_1$ with angle $\tilde{\varphi}_1(t)$ is component-wise larger than $\boldsymbol{R}(t)$. To this end for the radius for this boundary rate pair we have

$$\begin{aligned}\tilde{R}_1(\tilde{\varphi}_1(t)) &= \max_{\alpha\in[0,1]} \min\left\{\tilde{R}_{\mathrm{BC}}(\alpha,\tilde{\varphi}_1(t)), \frac{\tilde{R}_{\overrightarrow{1\mathrm{R}}}(\alpha)}{\cos(\tilde{\varphi}_1(t))}\right\} \\ &= \tilde{R}_{\mathrm{BC}}(\tilde{\alpha}_1^*(\tilde{\varphi}_1(t)),\tilde{\varphi}_1(t)) = \frac{\tilde{R}_{\overrightarrow{1\mathrm{R}}}(\tilde{\alpha}_1^*(\tilde{\varphi}_1(t)))}{\cos(\tilde{\varphi}_1(t))} \qquad (2.91)\\ &\geq \min\left\{\tilde{R}_{\mathrm{BC}}(\tilde{\alpha}_1(t),\tilde{\varphi}_1(t)), \frac{\tilde{R}_{\overrightarrow{1\mathrm{R}}}(\tilde{\alpha}_1(t))}{\cos(\tilde{\varphi}_1(t))}\right\}. \qquad (2.92)\end{aligned}$$

We will now show that $R_1(t) \leq \tilde{R}_1(\tilde{\alpha}_1(t))\cos(\tilde{\varphi}_1(t))$ and $R_2(t) \leq \tilde{R}_1(\tilde{\varphi}_1(t))\sin(\tilde{\varphi}_1(t))$. Therefore, we have to distinguish between two cases of the minimization in (2.92).

First, if $\tilde{R}_{\mathrm{BC}}(\tilde{\alpha}_1(t),\tilde{\varphi}_1(t)) > \frac{\tilde{R}_{\overrightarrow{1\mathrm{R}}}(\tilde{\alpha}_1(t))}{\cos(\tilde{\varphi}_1(t))}$, which implies that $\tilde{\varphi}_1(t) < \frac{\pi}{2}$, we have

$$\begin{aligned}\cos(\tilde{\varphi}_1(t))\tilde{R}_1(\tilde{\varphi}_1(t)) &= \tilde{R}_{\overrightarrow{1\mathrm{R}}}(\tilde{\alpha}_1^*(\tilde{\varphi}_1(t))) \geq \tilde{R}_{\overrightarrow{1\mathrm{R}}}(\tilde{\alpha}_1(t)) \\ &\geq t\tilde{R}_{\overrightarrow{1\mathrm{R}}}(\tilde{\alpha}_1^{(1)}) + (1-t)\tilde{R}_{\overrightarrow{1\mathrm{R}}}(\tilde{\alpha}_1^{(2)}) \geq tR_1^{(1)} + (1-t)R_1^{(2)} = R_1(t)\end{aligned}$$

where the first equality comes from (2.91) and the inequalities follow from (2.92), from the concavity of $\tilde{R}_{\overrightarrow{1\mathrm{R}}}(\alpha)$, and from the definition of $\tilde{\alpha}_1^{(k)}$, $k=1,2$, respectively. Further, we can conclude

$$\begin{aligned}\sin(\tilde{\varphi}_1(t))\tilde{R}_1(\tilde{\varphi}_1(t)) &= \tan(\tilde{\varphi}_1(t))\tilde{R}_{\overrightarrow{1\mathrm{R}}}(\tilde{\alpha}_1^*(\tilde{\varphi}_1(t))) \geq \tan(\tilde{\varphi}_1(t))\tilde{R}_{\overrightarrow{1\mathrm{R}}}(\tilde{\alpha}_1(t)) \\ &= \frac{\tilde{R}_{\overrightarrow{\mathrm{R}1}}(\tilde{\alpha}_1(t),\tilde{\beta}_1(t))\tilde{R}_{\overrightarrow{1\mathrm{R}}}(\tilde{\alpha}_1(t))}{\min\{\tilde{R}_{\overrightarrow{\mathrm{R}2}}(\tilde{\alpha}_1(t),\tilde{\beta}_1(t)),\tilde{R}_{\overrightarrow{1\mathrm{R}}}(\tilde{\alpha}_1(t))\}} \\ &\geq \frac{\tilde{R}_{\overrightarrow{\mathrm{R}1}}(\tilde{\alpha}_1(t),\tilde{\beta}_1(t))\tilde{R}_{\overrightarrow{1\mathrm{R}}}(\tilde{\alpha}_1(t))}{\tilde{R}_{\overrightarrow{1\mathrm{R}}}(\tilde{\alpha}_1(t))} = \tilde{R}_{\overrightarrow{\mathrm{R}1}}(\tilde{\alpha}_1(t),\tilde{\beta}_1(t)) \\ &\geq t\tilde{R}_{\overrightarrow{\mathrm{R}1}}(\tilde{\alpha}_1^{(1)},\tilde{\beta}_1^{(1)}) + (1-t)\tilde{R}_{\overrightarrow{\mathrm{R}1}}(\tilde{\alpha}_1^{(2)},\tilde{\beta}_1^{(2)}) \geq tR_2^{(1)} + (1-t)R_2^{(2)} = R_2(t)\end{aligned}$$

where the first equality and inequality are consequences of (2.91) and (2.92), the next follow from the definition of $\tilde{\varphi}_1(t)$, from the concavity of $\tilde{R}_{\overrightarrow{\mathrm{R}1}}(\alpha,\beta)$, and from the definition of $\tilde{\alpha}_1^{(k)}$, $k=1,2$, respectively.

For the case $\tilde{R}_{\mathrm{BC}}(\tilde{\alpha}_1(t),\tilde{\varphi}_1(t)) \leq \frac{\tilde{R}_{\overrightarrow{1\mathrm{R}}}(\tilde{\alpha}_1(t))}{\cos(\tilde{\varphi}_1(t))}$, which is equivalent to

$$\tilde{R}_{\overrightarrow{1\mathrm{R}}}(\tilde{\alpha}_1(t)) \geq \tilde{R}_{\mathrm{BC}}(\tilde{\alpha}_1(t),\tilde{\varphi}_1(t))\cos(\tilde{\varphi}_1(t)) = \tilde{R}_{\overrightarrow{\mathrm{R}2}}(\tilde{\alpha}_1(t),\tilde{\beta}_{\mathrm{BC}}(\tilde{\alpha}_1(t),\tilde{\varphi}_1(t))), \qquad (2.93)$$

we first prove the following claim.

Claim 2.57. *From* (2.93) *it follows that* $\tilde{R}_{\overrightarrow{\mathrm{R2}}}(\tilde{\alpha}_1(t), \tilde{\beta}_{\mathrm{BC}}(\tilde{\alpha}_1(t), \tilde{\varphi}_1(t))) = \tilde{R}_{\overrightarrow{\mathrm{R2}}}(\tilde{\alpha}_1(t), \tilde{\beta}_1(t))$.

Proof. If $\tilde{\varphi}_1(t) = \frac{\pi}{2}$ then we have $\tilde{\beta}_{\mathrm{BC}}(\tilde{\alpha}_1(t), \frac{\pi}{2}) = 0 = \tilde{\beta}_1(t)$ so that the claim holds. For $\tilde{\varphi}_1(t) < \frac{\pi}{2}$ the claim will be proved by contradiction. To this end we assume that from (2.93) it follows that $\tilde{R}_{\overrightarrow{\mathrm{R2}}}(\tilde{\alpha}_1(t), \tilde{\beta}_{\mathrm{BC}}(\tilde{\alpha}_1(t), \tilde{\varphi}_1(t))) \neq \tilde{R}_{\overrightarrow{\mathrm{R2}}}(\tilde{\alpha}_1(t), \tilde{\beta}_1(t))$. Then we have to distinguish between two cases. First, we assume that we have $\tilde{R}_{\overrightarrow{\mathrm{R2}}}(\tilde{\alpha}_1(t), \tilde{\beta}_{\mathrm{BC}}(\tilde{\alpha}_1(t), \tilde{\varphi}_1(t))) > \tilde{R}_{\overrightarrow{\mathrm{R2}}}(\tilde{\alpha}_1(t), \tilde{\beta}_1(t))$. Since $\tilde{R}_{\overrightarrow{\mathrm{R2}}}(\alpha, \beta)$ is strictly increasing for β we have $\tilde{\beta}_{\mathrm{BC}}(\tilde{\alpha}_1(t), \tilde{\varphi}_1(t)) > \tilde{\beta}_1(t)$. This implies

$$\tilde{R}_{\overrightarrow{\mathrm{R1}}}(\tilde{\alpha}_1(t), \tilde{\beta}_{\mathrm{BC}}(\tilde{\alpha}_1(t), \tilde{\varphi}_1(t))) < \tilde{R}_{\overrightarrow{\mathrm{R1}}}(\tilde{\alpha}_1(t), \tilde{\beta}_1(t))$$

since $\tilde{R}_{\overrightarrow{\mathrm{R1}}}(\alpha, \beta)$ is strictly decreasing for β. Further, with (2.93) we have

$$\tilde{R}_{\overrightarrow{\mathrm{1R}}}(\tilde{\alpha}_1(t)) \geq \tilde{R}_{\overrightarrow{\mathrm{R2}}}(\tilde{\alpha}_1(t), \tilde{\beta}_{\mathrm{BC}}(\tilde{\alpha}_1(t), \tilde{\varphi}_1(t))) > \tilde{R}_{\overrightarrow{\mathrm{R2}}}(\tilde{\alpha}_1(t), \tilde{\beta}_1(t))$$

so that we get the contradiction for the first case

$$\tan \tilde{\varphi}_1(t) = \frac{\tilde{R}_{\overrightarrow{\mathrm{R1}}}(\tilde{\alpha}_1(t), \tilde{\beta}_1(t))}{\tilde{R}_{\overrightarrow{\mathrm{R2}}}(\tilde{\alpha}_1(t), \tilde{\beta}_1(t))} > \frac{\tilde{R}_{\overrightarrow{\mathrm{R1}}}(\tilde{\alpha}_1(t), \tilde{\beta}_{\mathrm{BC}}(\tilde{\alpha}_1(t), \tilde{\varphi}_1(t)))}{\tilde{R}_{\overrightarrow{\mathrm{R2}}}(\tilde{\alpha}_1(t), \tilde{\beta}_{\mathrm{BC}}(\tilde{\alpha}_1(t), \tilde{\varphi}_1(t)))} = \tan \tilde{\varphi}_1(t).$$

Now we assume $\tilde{R}_{\overrightarrow{\mathrm{R2}}}(\tilde{\alpha}_1(t), \tilde{\beta}_1(t)) > \tilde{R}_{\overrightarrow{\mathrm{R2}}}(\tilde{\alpha}_1(t), \tilde{\beta}_{\mathrm{BC}}(\tilde{\alpha}_1(t), \tilde{\varphi}_1(t)))$ so that $\tilde{\beta}_1(t) > \tilde{\beta}_{\mathrm{BC}}(\tilde{\alpha}_1(t), \tilde{\varphi}_1(t))$ and $\tilde{R}_{\overrightarrow{\mathrm{R1}}}(\tilde{\alpha}_1(t), \tilde{\beta}_{\mathrm{BC}}(\tilde{\alpha}_1(t), \tilde{\varphi}_1(t))) > \tilde{R}_{\overrightarrow{\mathrm{R1}}}(\tilde{\alpha}_1(t), \tilde{\beta}_1(t))$ follow immediately. Then we get a contradiction for the second case from the following

$$\begin{aligned}
\tan \tilde{\varphi}_1(t) &= \frac{\tilde{R}_{\overrightarrow{\mathrm{R1}}}(\tilde{\alpha}_1(t), \tilde{\beta}_1(t))}{\min\{\tilde{R}_{\overrightarrow{\mathrm{R2}}}(\tilde{\alpha}_1(t), \tilde{\beta}_1(t)), \tilde{R}_{\overrightarrow{\mathrm{1R}}}(\tilde{\alpha}_1(t))\}} \\
&\leq \frac{\tilde{R}_{\overrightarrow{\mathrm{R1}}}(\tilde{\alpha}_1(t), \tilde{\beta}_1(t))}{\min\{\tilde{R}_{\overrightarrow{\mathrm{R2}}}(\tilde{\alpha}_1(t), \tilde{\beta}_{\mathrm{BC}}(\tilde{\alpha}_1(t), \tilde{\varphi}_1(t))), \tilde{R}_{\overrightarrow{\mathrm{1R}}}(\tilde{\alpha}_1(t))\}} \\
&= \frac{\tilde{R}_{\overrightarrow{\mathrm{R1}}}(\tilde{\alpha}_1(t), \tilde{\beta}_1(t))}{\tilde{R}_{\overrightarrow{\mathrm{R2}}}(\tilde{\alpha}_1(t), \tilde{\beta}_1(t))} < \frac{\tilde{R}_{\overrightarrow{\mathrm{R1}}}(\tilde{\alpha}_1(t), \tilde{\beta}_{\mathrm{BC}}(\tilde{\alpha}_1(t), \tilde{\varphi}_1(t)))}{\tilde{R}_{\overrightarrow{\mathrm{R2}}}(\tilde{\alpha}_1(t), \tilde{\beta}_{\mathrm{BC}}(\tilde{\alpha}_1(t), \tilde{\varphi}_1(t)))} = \tan \tilde{\varphi}_1(t)
\end{aligned}$$

where we used the fact that for $a < b$, $a, c \neq 0$ we have $\frac{1}{\min\{b,c\}} \leq \frac{1}{\min\{a,c\}}$, (2.93), and the monotony of $\tilde{R}_{\overrightarrow{\mathrm{R2}}}(\alpha, \beta)$ and $\tilde{R}_{\overrightarrow{\mathrm{R1}}}(\alpha, \beta)$ for β. □

Note that the claim implies $\tilde{\beta}_1(t) = \tilde{\beta}_{\mathrm{BC}}(\tilde{\alpha}_1(t), \tilde{\varphi}_1(t))$. From Claim 2.57 with (2.93) we conclude that $\tilde{R}_{\overrightarrow{\mathrm{1R}}}(\tilde{\alpha}_1(t)) \geq \tilde{R}_{\overrightarrow{\mathrm{R2}}}(\tilde{\alpha}_1(t), \tilde{\beta}_1(t))$ holds so that we have $\tilde{\varphi}_1(t) =$

$\tilde{\varphi}_{\mathrm{BC}}(\tilde{\alpha}_1(t), \tilde{\beta}_1(t))$. Accordingly, we get

$$\begin{aligned}\cos(\tilde{\varphi}_1(t))\tilde{R}_1(\tilde{\varphi}_1(t)) &= \cos(\tilde{\varphi}_1(t))\tilde{R}_{\mathrm{BC}}(\tilde{\alpha}_1(t), \tilde{\varphi}_1(t)) = \tilde{R}_{\overrightarrow{\mathrm{R2}}}(\tilde{\alpha}_1(t), \tilde{\beta}_1(t)) \\ &\geq t\tilde{R}_{\overrightarrow{\mathrm{R2}}}(\tilde{\alpha}_1^{(1)}, \tilde{\beta}_1^{(1)}) + (1-t)\tilde{R}_{\overrightarrow{\mathrm{R2}}}(\tilde{\alpha}_1^{(2)}, \tilde{\beta}_1^{(2)}) \geq tR_1^{(1)} + (1-t)R_1^{(2)} = R_1(t)\end{aligned}$$

where we used (2.93), Claim 2.57 and the concavity of $\tilde{R}_{\overrightarrow{\mathrm{R2}}}(\tilde{\alpha}_1(t), \tilde{\beta}_1(t))$. For the rate $R_2(t)$ we have

$$\begin{aligned}\sin(\tilde{\varphi}_1(t))\tilde{R}_1(\tilde{\varphi}_1(t)) &= \sin(\tilde{\varphi}_1(t))\tilde{R}_{\mathrm{BC}}(\tilde{\alpha}_1(t), \tilde{\varphi}_1(t)) = \tan(\tilde{\varphi}_1(t))\tilde{R}_{\overrightarrow{\mathrm{R2}}}(\tilde{\alpha}_1(t), \tilde{\beta}_1(t)) \\ &= \frac{\tilde{R}_{\overrightarrow{\mathrm{R1}}}(\tilde{\alpha}_1(t), \tilde{\beta}_1(t))\tilde{R}_{\overrightarrow{\mathrm{R2}}}(\tilde{\alpha}_1(t), \tilde{\beta}_1(t))}{\min\{\tilde{R}_{\overrightarrow{\mathrm{R2}}}(\tilde{\alpha}_1(t), \tilde{\beta}_1(t)), \tilde{R}_{\overrightarrow{\mathrm{1R}}}(\tilde{\alpha}_1(t))\}} \\ &= \frac{\tilde{R}_{\overrightarrow{\mathrm{R1}}}(\tilde{\alpha}_1(t), \tilde{\beta}_1(t))\tilde{R}_{\overrightarrow{\mathrm{R2}}}(\tilde{\alpha}_1(t), \tilde{\beta}_1(t))}{\min\{\tilde{R}_{\overrightarrow{\mathrm{R2}}}(\tilde{\alpha}_1(t), \tilde{\beta}_{\mathrm{BC}}(\tilde{\alpha}_1(t), \tilde{\varphi}_1(t))), \tilde{R}_{\overrightarrow{\mathrm{1R}}}(\tilde{\alpha}_1(t))\}} \\ &= \frac{\tilde{R}_{\overrightarrow{\mathrm{R1}}}(\tilde{\alpha}_1(t), \tilde{\beta}_1(t))\tilde{R}_{\overrightarrow{\mathrm{R2}}}(\tilde{\alpha}_1(t), \tilde{\beta}_1(t))}{\tilde{R}_{\overrightarrow{\mathrm{R2}}}(\tilde{\alpha}_1(t), \tilde{\beta}_{\mathrm{BC}}(\tilde{\alpha}_1(t), \tilde{\varphi}_1(t)))} = \frac{\tilde{R}_{\overrightarrow{\mathrm{R1}}}(\tilde{\alpha}_1(t), \tilde{\beta}_1(t))\tilde{R}_{\overrightarrow{\mathrm{R2}}}(\tilde{\alpha}_1(t), \tilde{\beta}_1(t))}{\tilde{R}_{\overrightarrow{\mathrm{R2}}}(\tilde{\alpha}_1(t), \tilde{\beta}_1(t))} \\ &= \tilde{R}_{\overrightarrow{\mathrm{R1}}}(\tilde{\alpha}_1(t), \tilde{\beta}_1(t)) \geq t\tilde{R}_{\overrightarrow{\mathrm{R1}}}(\tilde{\alpha}_1^{(1)}, \tilde{\beta}_1^{(1)}) + (1-t)\tilde{R}_{\overrightarrow{\mathrm{R1}}}(\tilde{\alpha}_1^{(2)}, \tilde{\beta}_1^{(2)}) \\ &\geq tR_2^{(1)} + (1-t)R_2^{(2)} = R_2(t)\end{aligned}$$

where we have used (2.93) and Claim 2.57, the definition of $\tilde{\varphi}_1(t)$, and the concavity of $\tilde{R}_{\overrightarrow{\mathrm{R1}}}(\tilde{\alpha}_1(t), \tilde{\beta}_1(t))$. It follows that $\boldsymbol{R}(t) \in \tilde{\mathcal{R}}_1$ for all $t \in [0,1]$ so that we conclude that $\tilde{\mathcal{R}}_1$ is convex!

The proof of convexity of $\tilde{\mathcal{R}}_2$ is similar, but differs at some points so that we will present it here for completeness. Again we have to show that for any two rate pairs $\boldsymbol{R}^{(k)} = [R_1^{(k)}, R_2^{(k)}] \in \tilde{\mathcal{R}}_2$, $k = 1, 2$, we have $\boldsymbol{R}(t) = [R_1(t), R_2(t)] := t\boldsymbol{R}^{(1)} + (1-t)\boldsymbol{R}^{(2)} \in \tilde{\mathcal{R}}_2$ for all $t \in [0,1]$. Therefore, let $\varphi_{\boldsymbol{R}^{(k)}}$ and $R_{\boldsymbol{R}^{(k)}}$ denote the angle and radius of the rate vector $\boldsymbol{R}^{(k)}$, $k = 1, 2$.

From $\boldsymbol{R}^{(k)} \in \tilde{\mathcal{R}}_2$ it follows that $R_{\boldsymbol{R}^{(k)}} \leq \tilde{R}_2(\varphi_{\boldsymbol{R}^{(k)}})$, $k = 1, 2$. According to Lemma 2.23 there exist time-division parameters $\tilde{\alpha}_2^{(k)} := \alpha_2^*(\varphi_{\boldsymbol{R}^{(k)}})$ with $R_{\boldsymbol{R}^{(k)}} \leq \frac{\tilde{R}_{\overrightarrow{\mathrm{2R}}}(\tilde{\alpha}_2^{(k)})}{\sin(\varphi_{\boldsymbol{R}^{(k)}})}$ so that we have

$$R_2^{(k)} = R_{\boldsymbol{R}^{(k)}} \sin(\varphi_{\boldsymbol{R}^{(k)}}) \leq \tilde{R}_{\overrightarrow{\mathrm{2R}}}(\tilde{\alpha}_2^{(k)}), \qquad k = 1, 2. \tag{2.94}$$

Then from Proposition 2.20 we additionally get the relay distribution factors $\tilde{\beta}_2^{(k)} := \tilde{\beta}_{\mathrm{BC}}(\tilde{\alpha}_2^{(k)}, \varphi_{\boldsymbol{R}^{(k)}})$ so that we have $R_{\boldsymbol{R}^{(k)}} \leq \tilde{R}_{\mathrm{BC}}(\tilde{\alpha}_2^{(k)}, \tilde{\beta}_2^{(k)})$, $k = 1, 2$, which is equivalent to

$$R_1^{(k)} \leq \tilde{R}_{\overrightarrow{\mathrm{R2}}}(\tilde{\alpha}_2^{(k)}, \tilde{\beta}_2^{(k)}), \qquad R_2^{(k)} \leq \tilde{R}_{\overrightarrow{\mathrm{R1}}}(\tilde{\alpha}_2^{(k)}, \tilde{\beta}_2^{(k)}). \tag{2.95}$$

For $t \in [0,1]$ let us define $\tilde{\alpha}_2(t) := t\tilde{\alpha}_2^{(1)} + (1-t)\tilde{\alpha}_2^{(2)}$ and $\tilde{\beta}_2(t) := t\tilde{\beta}_2^{(1)} + (1-t)\tilde{\beta}_2^{(2)}$. From Lemma 2.17 and 2.18 we know that $\tilde{R}_{\overrightarrow{2\mathrm{R}}}(\alpha)$, $\tilde{R}_{\overrightarrow{\mathrm{R}2}}(\alpha,\beta)$, and $\tilde{R}_{\overrightarrow{\mathrm{R}1}}(\alpha,\beta)$ are concave.

Then let $\tilde{\varphi}_2(t)$ be the angle of $[\tilde{R}_{\overrightarrow{\mathrm{R}2}}(\tilde{\alpha}_2(t),\tilde{\beta}_2(t)), \min\{\tilde{R}_{\overrightarrow{\mathrm{R}1}}(\tilde{\alpha}_2(t),\tilde{\beta}_2(t)), \tilde{R}_{\overrightarrow{2\mathrm{R}}}(\tilde{\alpha}_2(t))\}]$, which is given by

$$\tilde{\varphi}_2(t) := \begin{cases} \arctan \frac{\min\{\tilde{R}_{\overrightarrow{\mathrm{R}1}}(\tilde{\alpha}_2(t),\tilde{\beta}_2(t)), \tilde{R}_{\overrightarrow{2\mathrm{R}}}(\tilde{\alpha}_2(t))\}}{\tilde{R}_{\overrightarrow{\mathrm{R}2}}(\tilde{\alpha}_2(t),\tilde{\beta}_2(t))}, & \text{if } \tilde{\beta}_2(t) \neq 0 \\ \frac{\pi}{2}, & \text{if } \tilde{\beta}_2(t) = 0. \end{cases}$$

Note that if $\tilde{R}_{\overrightarrow{\mathrm{R}2}}(\tilde{\alpha}_2(t),\tilde{\beta}_2(t)) \leq \tilde{R}_{\overrightarrow{2\mathrm{R}}}(\tilde{\alpha}_2(t))$ we have $\tilde{\varphi}_2(t) = \tilde{\varphi}_{\mathrm{BC}}(\tilde{\alpha}_2(t),\tilde{\beta}_2(t))$, else we have $\tilde{\varphi}_2(t) > \tilde{\varphi}_{\mathrm{BC}}(\tilde{\alpha}_1(t),\tilde{\beta}_1(t))$ because the inverse function of the tangent is strictly increasing.

In the following, we show that the boundary rate pair on $\tilde{\mathcal{R}}_2$ with angle $\tilde{\varphi}_2(t)$ is componentwise larger than $\boldsymbol{R}(t)$. To this end for the radius for this boundary rate pair we have

$$\begin{aligned} \tilde{R}_2(\tilde{\varphi}_2(t)) &= \max_{\alpha\in[0,1]} \min\left\{\tilde{R}_{\mathrm{BC}}(\alpha,\tilde{\varphi}_2(t)), \frac{\tilde{R}_{\overrightarrow{2\mathrm{R}}}(\alpha)}{\sin(\tilde{\varphi}_2(t))}\right\} \\ &= \tilde{R}_{\mathrm{BC}}(\tilde{\alpha}_2^*(\tilde{\varphi}_2(t)),\tilde{\varphi}_2(t)) = \frac{\tilde{R}_{\overrightarrow{2\mathrm{R}}}(\tilde{\alpha}_2^*(\tilde{\varphi}_2(t)))}{\sin(\tilde{\varphi}_2(t))} && (2.96) \\ &\geq \min\left\{\tilde{R}_{\mathrm{BC}}(\tilde{\alpha}_2(t),\tilde{\varphi}_2(t)), \frac{\tilde{R}_{\overrightarrow{2\mathrm{R}}}(\tilde{\alpha}_2(t))}{\sin(\tilde{\varphi}_2(t))}\right\}. && (2.97) \end{aligned}$$

We will now show that $R_1(t) \leq \tilde{R}_2(\tilde{\varphi}_2(t))\cos(\tilde{\varphi}_2(t))$ and $R_2(t) \leq \tilde{R}_2(\tilde{\varphi}_2(t))\sin(\tilde{\varphi}_2(t))$. Therefore, we have to distinguish between two cases of the minimization in (2.97).

First, if $\tilde{R}_{\mathrm{BC}}(\tilde{\alpha}_2(t),\tilde{\varphi}_2(t)) > \frac{\tilde{R}_{\overrightarrow{2\mathrm{R}}}(\tilde{\alpha}_2(t))}{\sin(\tilde{\varphi}_2(t))}$, which implies that $\tilde{\varphi}_2(t) > 0$, we have

$$\begin{aligned} \cos(\tilde{\varphi}_2(t))\tilde{R}_2(\tilde{\varphi}_2(t)) &= \frac{\tilde{R}_{\overrightarrow{2\mathrm{R}}}(\tilde{\alpha}_2^*(\tilde{\varphi}_2(t)))}{\tan(\tilde{\varphi}_2(t))} \geq \frac{\tilde{R}_{\overrightarrow{2\mathrm{R}}}(\tilde{\alpha}_2(t))}{\tan(\tilde{\varphi}_2(t))} \\ &= \frac{\tilde{R}_{\overrightarrow{\mathrm{R}2}}(\tilde{\alpha}_2(t),\tilde{\beta}_2(t))\tilde{R}_{\overrightarrow{2\mathrm{R}}}(\tilde{\alpha}_2(t))}{\min\{\tilde{R}_{\overrightarrow{\mathrm{R}1}}(\tilde{\alpha}_2(t),\tilde{\beta}_2(t)), \tilde{R}_{\overrightarrow{2\mathrm{R}}}(\tilde{\alpha}_2(t))\}} \\ &\geq \frac{\tilde{R}_{\overrightarrow{\mathrm{R}2}}(\tilde{\alpha}_2(t),\tilde{\beta}_2(t))\tilde{R}_{\overrightarrow{2\mathrm{R}}}(\tilde{\alpha}_2(t))}{\tilde{R}_{\overrightarrow{2\mathrm{R}}}(\tilde{\alpha}_2(t))} = \tilde{R}_{\overrightarrow{\mathrm{R}2}}(\tilde{\alpha}_2(t),\tilde{\beta}_2(t)) \\ &\geq t\tilde{R}_{\overrightarrow{\mathrm{R}2}}(\tilde{\alpha}_2^{(1)},\tilde{\beta}_2^{(1)}) + (1-t)\tilde{R}_{\overrightarrow{\mathrm{R}2}}(\tilde{\alpha}_2^{(2)},\tilde{\beta}_2^{(2)}) \geq tR_1^{(1)} + (1-t)R_1^{(2)} = R_1(t) \end{aligned}$$

where the first equality and inequality are consequences of (2.96) and (2.97), the next follow from the definition of $\tilde{\varphi}_2(t)$, from the concavity of $\tilde{R}_{\overrightarrow{\mathrm{R}2}}(\alpha,\beta)$, and from the definition of

$\tilde{\alpha}_2^{(k)}$, $k = 1, 2$, respectively. Further, we can conclude

$$\begin{aligned}\sin(\tilde{\varphi}_2(t))\tilde{R}_2(\tilde{\varphi}_2(t)) = \tilde{R}_{\overrightarrow{2\mathrm{R}}}(\tilde{\alpha}_2^*(\tilde{\varphi}_2(t))) \geq \tilde{R}_{\overrightarrow{2\mathrm{R}}}(\tilde{\alpha}_2(t))\\ \geq t\tilde{R}_{\overrightarrow{2\mathrm{R}}}(\tilde{\alpha}_2^{(1)}) + (1-t)\tilde{R}_{\overrightarrow{2\mathrm{R}}}(\tilde{\alpha}_2^{(2)}) \geq tR_2^{(1)} + (1-t)R_2^{(2)} = R_2(t)\end{aligned}$$

where the first equality comes from (2.96) and the inequalities follow from (2.97), from the concavity of $\tilde{R}_{\overrightarrow{2\mathrm{R}}}(\alpha)$, and from the definition of $\tilde{\alpha}_2^{(k)}$, $k = 1, 2$, respectively.

For the case $\tilde{R}_{\mathrm{BC}}(\tilde{\alpha}_2(t), \tilde{\varphi}_2(t)) \leq \frac{\tilde{R}_{\overrightarrow{2\mathrm{R}}}(\tilde{\alpha}_2(t))}{\sin(\tilde{\varphi}_2(t))}$, which is equivalent to

$$\tilde{R}_{\overrightarrow{2\mathrm{R}}}(\tilde{\alpha}_2(t)) \geq \tilde{R}_{\mathrm{BC}}(\tilde{\alpha}_2(t), \tilde{\varphi}_2(t)) \sin(\tilde{\varphi}_2(t)) = \tilde{R}_{\overrightarrow{\mathrm{R}1}}(\tilde{\alpha}_2(t), \tilde{\beta}_{\mathrm{BC}}(\tilde{\alpha}_2(t), \tilde{\varphi}_2(t))), \quad (2.98)$$

we first prove the following claim.

Claim 2.58. *From* (2.98) *it follows that* $\tilde{R}_{\overrightarrow{\mathrm{R}1}}(\tilde{\alpha}_2(t), \tilde{\beta}_{\mathrm{BC}}(\tilde{\alpha}_2(t), \tilde{\varphi}_2(t))) = \tilde{R}_{\overrightarrow{\mathrm{R}1}}(\tilde{\alpha}_2(t), \tilde{\beta}_2(t))$.

Proof. If $\tilde{\varphi}_2(t) = 0$ then we have $\tilde{\beta}_{\mathrm{BC}}(\tilde{\alpha}_2(t), 0) = 1 = \tilde{\beta}_2(t)$ so that the claim holds. For $\tilde{\varphi}_2(t) > 0$ the claim will be proved by contradiction. To this end we assume that from (2.98) it follows that $\tilde{R}_{\overrightarrow{\mathrm{R}1}}(\tilde{\alpha}_2(t), \tilde{\beta}_{\mathrm{BC}}(\tilde{\alpha}_2(t), \tilde{\varphi}_2(t))) \neq \tilde{R}_{\overrightarrow{\mathrm{R}1}}(\tilde{\alpha}_2(t), \tilde{\beta}_2(t))$. Then we have to distinguish between two cases. First, we assume that we have $\tilde{R}_{\overrightarrow{\mathrm{R}1}}(\tilde{\alpha}_2(t), \tilde{\beta}_{\mathrm{BC}}(\tilde{\alpha}_2(t), \tilde{\varphi}_2(t))) > \tilde{R}_{\overrightarrow{\mathrm{R}1}}(\tilde{\alpha}_2(t), \tilde{\beta}_2(t))$. Since $\tilde{R}_{\overrightarrow{\mathrm{R}2}}(\alpha, \beta)$ is strictly increasing for β we have $\tilde{\beta}_{\mathrm{BC}}(\tilde{\alpha}_2(t), \tilde{\varphi}_2(t)) < \tilde{\beta}_2(t)$. This implies

$$\tilde{R}_{\overrightarrow{\mathrm{R}2}}(\tilde{\alpha}_2(t), \tilde{\beta}_{\mathrm{BC}}(\tilde{\alpha}_2(t), \tilde{\varphi}_2(t))) < \tilde{R}_{\overrightarrow{\mathrm{R}2}}(\tilde{\alpha}_2(t), \tilde{\beta}_2(t))$$

since $\tilde{R}_{\overrightarrow{\mathrm{R}2}}(\alpha, \beta)$ is strictly increasing for β. Further, with (2.98) we have

$$\tilde{R}_{\overrightarrow{2\mathrm{R}}}(\tilde{\alpha}_2(t)) \geq \tilde{R}_{\overrightarrow{\mathrm{R}1}}(\tilde{\alpha}_2(t), \tilde{\beta}_{\mathrm{BC}}(\tilde{\alpha}_2(t), \tilde{\varphi}_2(t))) > \tilde{R}_{\overrightarrow{\mathrm{R}1}}(\tilde{\alpha}_2(t), \tilde{\beta}_2(t))$$

so that we get the contradiction for the first case

$$\tan\tilde{\varphi}_2(t) = \frac{\tilde{R}_{\overrightarrow{\mathrm{R}1}}(\tilde{\alpha}_2(t), \tilde{\beta}_2(t))}{\tilde{R}_{\overrightarrow{\mathrm{R}2}}(\tilde{\alpha}_2(t), \tilde{\beta}_2(t))} < \frac{\tilde{R}_{\overrightarrow{\mathrm{R}1}}(\tilde{\alpha}_2(t), \tilde{\beta}_{\mathrm{BC}}(\tilde{\alpha}_2(t), \tilde{\varphi}_2(t)))}{\tilde{R}_{\overrightarrow{\mathrm{R}2}}(\tilde{\alpha}_2(t), \tilde{\beta}_{\mathrm{BC}}(\tilde{\alpha}_2(t), \tilde{\varphi}_2(t)))} = \tan\tilde{\varphi}_2(t).$$

Now we assume $\tilde{R}_{\overrightarrow{\mathrm{R}1}}(\tilde{\alpha}_2(t), \tilde{\beta}_2(t)) > \tilde{R}_{\overrightarrow{\mathrm{R}1}}(\tilde{\alpha}_2(t), \tilde{\beta}_{\mathrm{BC}}(\tilde{\alpha}_2(t), \tilde{\varphi}_2(t)))$ so that $\tilde{\beta}_2(t) < \tilde{\beta}_{\mathrm{BC}}(\tilde{\alpha}_2(t), \tilde{\varphi}_2(t))$ and $\tilde{R}_{\overrightarrow{\mathrm{R}2}}(\tilde{\alpha}_2(t), \tilde{\beta}_{\mathrm{BC}}(\tilde{\alpha}_2(t), \tilde{\varphi}_2(t))) > \tilde{R}_{\overrightarrow{\mathrm{R}2}}(\tilde{\alpha}_2(t), \tilde{\beta}_2(t))$ follow immediately. Then we get a contradiction for the second case from the following

$$
\begin{aligned}
\tan\tilde{\varphi}_2(t) &= \frac{\min\{\tilde{R}_{\overrightarrow{\mathrm{R1}}}(\tilde{\alpha}_2(t),\tilde{\beta}_2(t)),\tilde{R}_{\overrightarrow{\mathrm{2R}}}(\tilde{\alpha}_2(t))\}}{\tilde{R}_{\overrightarrow{\mathrm{R2}}}(\tilde{\alpha}_2(t),\tilde{\beta}_2(t))} \\
&\geq \frac{\min\{\tilde{R}_{\overrightarrow{\mathrm{R1}}}(\tilde{\alpha}_2(t),\tilde{\beta}_{\mathrm{BC}}(\tilde{\alpha}_2(t),\tilde{\varphi}_2(t))),\tilde{R}_{\overrightarrow{\mathrm{2R}}}(\tilde{\alpha}_2(t))\}}{\tilde{R}_{\overrightarrow{\mathrm{R2}}}(\tilde{\alpha}_2(t),\tilde{\beta}_2(t))} \\
&= \frac{\tilde{R}_{\overrightarrow{\mathrm{R1}}}(\tilde{\alpha}_2(t),\tilde{\beta}_2(t))}{\tilde{R}_{\overrightarrow{\mathrm{R2}}}(\tilde{\alpha}_2(t),\tilde{\beta}_2(t))} > \frac{\tilde{R}_{\overrightarrow{\mathrm{R1}}}(\tilde{\alpha}_2(t),\tilde{\beta}_{\mathrm{BC}}(\tilde{\alpha}_2(t),\tilde{\varphi}_2(t)))}{\tilde{R}_{\overrightarrow{\mathrm{R2}}}(\tilde{\alpha}_2(t),\tilde{\beta}_{\mathrm{BC}}(\tilde{\alpha}_2(t),\tilde{\varphi}_2(t)))} = \tan\tilde{\varphi}_2(t)
\end{aligned}
$$

where we used the fact that for $a < b$ we have $\min\{b,c\} \geq \min\{a,c\}$, (2.98), and the monotony of $\tilde{R}_{\overrightarrow{\mathrm{R1}}}(\alpha,\beta)$ and $\tilde{R}_{\overrightarrow{\mathrm{R2}}}(\alpha,\beta)$ for β and fixed α. □

Note that the claim implies $\tilde{\beta}_2(t) = \tilde{\beta}_{\mathrm{BC}}(\tilde{\alpha}_2(t),\tilde{\varphi}_2(t))$. From Claim 2.58 with (2.98) we conclude that $\tilde{R}_{\overrightarrow{\mathrm{2R}}}(\tilde{\alpha}_2(t)) \geq \tilde{R}_{\overrightarrow{\mathrm{R2}}}(\tilde{\alpha}_2(t),\tilde{\beta}_2(t))$ holds so that we have $\tilde{\varphi}_2(t) = \tilde{\varphi}_{\mathrm{BC}}(\tilde{\alpha}_2(t),\tilde{\beta}_2(t))$. Accordingly, we get

$$
\begin{aligned}
&\cos(\tilde{\varphi}_2(t))\tilde{R}_2(\tilde{\varphi}_2(t)) = \cos(\tilde{\varphi}_2(t))\tilde{R}_{\mathrm{BC}}(\tilde{\alpha}_2(t),\tilde{\varphi}_2(t)) \\
&\quad= \frac{\tilde{R}_{\overrightarrow{\mathrm{R1}}}(\tilde{\alpha}_2(t),\tilde{\beta}_2(t))}{\tan(\tilde{\varphi}_2(t))} = \frac{\tilde{R}_{\overrightarrow{\mathrm{R1}}}(\tilde{\alpha}_2(t),\tilde{\beta}_2(t))\tilde{R}_{\overrightarrow{\mathrm{R2}}}(\tilde{\alpha}_2(t),\tilde{\beta}_2(t))}{\min\{\tilde{R}_{\overrightarrow{\mathrm{R1}}}(\tilde{\alpha}_2(t),\tilde{\beta}_2(t)),\tilde{R}_{\overrightarrow{\mathrm{2R}}}(\tilde{\alpha}_2(t))\}} \\
&\quad= \frac{\tilde{R}_{\overrightarrow{\mathrm{R1}}}(\tilde{\alpha}_2(t),\tilde{\beta}_2(t))\tilde{R}_{\overrightarrow{\mathrm{R2}}}(\tilde{\alpha}_2(t),\tilde{\beta}_2(t))}{\min\{\tilde{R}_{\overrightarrow{\mathrm{R1}}}(\tilde{\alpha}_2(t),\tilde{\beta}_{\mathrm{BC}}(\tilde{\alpha}_2(t),\tilde{\varphi}_2(t))),\tilde{R}_{\overrightarrow{\mathrm{2R}}}(\tilde{\alpha}_2(t))\}} \\
&\quad= \frac{\tilde{R}_{\overrightarrow{\mathrm{R1}}}(\tilde{\alpha}_2(t),\tilde{\beta}_2(t))\tilde{R}_{\overrightarrow{\mathrm{R2}}}(\tilde{\alpha}_2(t),\tilde{\beta}_2(t))}{\tilde{R}_{\overrightarrow{\mathrm{R1}}}(\tilde{\alpha}_2(t),\tilde{\beta}_{\mathrm{BC}}(\tilde{\alpha}_2(t),\tilde{\varphi}_2(t)))} = \frac{\tilde{R}_{\overrightarrow{\mathrm{R1}}}(\tilde{\alpha}_2(t),\tilde{\beta}_2(t))\tilde{R}_{\overrightarrow{\mathrm{R2}}}(\tilde{\alpha}_2(t),\tilde{\beta}_2(t))}{\tilde{R}_{\overrightarrow{\mathrm{R1}}}(\tilde{\alpha}_2(t),\tilde{\beta}_2(t))} \\
&\quad= \tilde{R}_{\overrightarrow{\mathrm{R2}}}(\tilde{\alpha}_2(t),\tilde{\beta}_2(t)) \geq t\tilde{R}_{\overrightarrow{\mathrm{R2}}}(\tilde{\alpha}_2^{(1)},\tilde{\beta}_2^{(1)}) + (1-t)\tilde{R}_{\overrightarrow{\mathrm{R2}}}(\tilde{\alpha}_2^{(2)},\tilde{\beta}_2^{(2)}) \\
&\quad\geq tR_1^{(1)} + (1-t)R_1^{(2)} = R_1(t)
\end{aligned}
$$

where we have used (2.98) and Claim 2.58, the definition of $\tilde{\varphi}_2(t)$, and the concavity of $\tilde{R}_{\overrightarrow{\mathrm{R2}}}(\tilde{\alpha}_2(t),\tilde{\beta}_2(t))$. For the rate $R_2(t)$ we have

$$
\begin{aligned}
&\sin(\tilde{\varphi}_2(t))\tilde{R}_2(\tilde{\varphi}_2(t)) = \sin(\tilde{\varphi}_2(t))\tilde{R}_{\mathrm{BC}}(\tilde{\alpha}_2(t),\tilde{\varphi}_2(t)) = \tilde{R}_{\overrightarrow{\mathrm{R1}}}(\tilde{\alpha}_2(t),\tilde{\beta}_2(t)) \\
&\quad\geq t\tilde{R}_{\overrightarrow{\mathrm{R1}}}(\tilde{\alpha}_2^{(1)},\tilde{\beta}_2^{(1)}) + (1-t)\tilde{R}_{\overrightarrow{\mathrm{R1}}}(\tilde{\alpha}_2^{(2)},\tilde{\beta}_2^{(2)}) \geq tR_2^{(1)} + (1-t)R_2^{(2)} = R_2(t)
\end{aligned}
$$

where we used (2.98), Claim 2.58 and the concavity of $\tilde{R}_{\overrightarrow{\mathrm{R1}}}(\tilde{\alpha}_2(t),\tilde{\beta}_2(t))$. It follows that $\boldsymbol{R}(t) \in \tilde{\mathcal{R}}_2$ for all $t \in [0,1]$ so that we conclude that $\tilde{\mathcal{R}}_2$ is convex!

Finally, we have to prove that $\tilde{\mathcal{R}}_\Sigma$ is convex. Therefore, we have to show that for any two rate pairs $\boldsymbol{R}^{(k)} = [R_1^{(k)}, R_2^{(k)}] \in \tilde{\mathcal{R}}_\Sigma$, $k = 1,2$, we have $\boldsymbol{R}(t) = [R_1(t), R_2(t)] :=$

$t\boldsymbol{R}^{(1)} + (1-t)\boldsymbol{R}^{(2)} \in \tilde{\mathcal{R}}_\Sigma$ for all $t \in [0,1]$. Therefore, let $\varphi_{\boldsymbol{R}^{(k)}}$ and $R_{\boldsymbol{R}^{(k)}}$ denote the angle and radius of the rate vector $\boldsymbol{R}^{(k)}$, $k = 1, 2$.

From $\boldsymbol{R}^{(k)} \in \tilde{\mathcal{R}}_\Sigma$ it follows that $R_{\boldsymbol{R}^{(k)}} \leq \tilde{R}_\Sigma(\varphi_{\boldsymbol{R}^{(k)}})$, $k = 1, 2$. According to Lemma 2.23 there exist time-division parameters $\tilde{\alpha}_\Sigma^{(k)} := \alpha_\Sigma^*(\varphi_{\boldsymbol{R}^{(k)}})$ with $R_{\boldsymbol{R}^{(k)}} \leq \frac{\tilde{R}_\Sigma^{\mathrm{MAC}}(\tilde{\alpha}_\Sigma^{(k)})}{\cos(\varphi_{\boldsymbol{R}^{(k)}})+\sin(\varphi_{\boldsymbol{R}^{(k)}})}$ so that we have

$$R_1^{(k)} + R_2^{(k)} = R_{\boldsymbol{R}^{(k)}}(\cos(\varphi_{\boldsymbol{R}^{(k)}}) + \sin(\varphi_{\boldsymbol{R}^{(k)}})) \leq \tilde{R}_\Sigma^{\mathrm{MAC}}(\tilde{\alpha}_\Sigma^{(k)}), \qquad k = 1, 2. \tag{2.99}$$

Then from Proposition 2.20 we additionally get the relay distribution factors $\tilde{\beta}_\Sigma^{(k)} := \tilde{\beta}_{\mathrm{BC}}(\tilde{\alpha}_\Sigma^{(k)}, \varphi_{\boldsymbol{R}^{(k)}})$ so that we have $R_{\boldsymbol{R}^{(k)}} \leq \tilde{R}_{\mathrm{BC}}(\tilde{\alpha}_\Sigma^{(k)}, \tilde{\beta}_\Sigma^{(k)})$, $k = 1, 2$, which is equivalent to

$$R_1^{(k)} \leq \tilde{R}_{\overrightarrow{\mathrm{R2}}}(\tilde{\alpha}_\Sigma^{(k)}, \tilde{\beta}_\Sigma^{(k)}), \qquad R_2^{(k)} \leq \tilde{R}_{\overrightarrow{\mathrm{R1}}}(\tilde{\alpha}_\Sigma^{(k)}, \tilde{\beta}_\Sigma^{(k)}). \tag{2.100}$$

For $t \in [0,1]$ let us define $\tilde{\alpha}_\Sigma(t) := t\tilde{\alpha}_\Sigma^{(1)} + (1-t)\tilde{\alpha}_\Sigma^{(2)}$ and $\tilde{\beta}_\Sigma(t) := t\tilde{\beta}_\Sigma^{(1)} + (1-t)\tilde{\beta}_\Sigma^{(2)}$. From Lemma 2.17 and 2.18 we know that $\tilde{R}_\Sigma^{\mathrm{MAC}}(\alpha)$, $\tilde{R}_{\overrightarrow{\mathrm{R2}}}(\alpha, \beta)$, and $\tilde{R}_{\overrightarrow{\mathrm{R1}}}(\alpha, \beta)$ are concave.

We define the rate vector $\boldsymbol{R}_{\mathrm{BC}(\tilde{\alpha}_\Sigma(t), \tilde{\beta}_\Sigma(t))} := [\tilde{R}_{\overrightarrow{\mathrm{R2}}}(\tilde{\alpha}_\Sigma(t), \tilde{\beta}_\Sigma(t)), \tilde{R}_{\overrightarrow{\mathrm{R1}}}(\tilde{\alpha}_\Sigma(t), \tilde{\beta}_\Sigma(t))]$ with angle $\tilde{\varphi}_{\mathrm{BC}}(\tilde{\alpha}_\Sigma(t), \tilde{\beta}_\Sigma(t))$ of the polar coordinates. Furthermore, $\varphi_{\boldsymbol{R}(t)}$ specifies the angle of the rate pair $\boldsymbol{R}(t) = [R_1(t), R_2(t)]$. Then the angle $\tilde{\varphi}_\Sigma(t)$ of the component-wise dominant rate pair is given by

$$\tilde{\varphi}_\Sigma(t) := \begin{cases} \varphi_{\boldsymbol{R}(t)}, & \text{if } \tilde{R}_\Sigma^{\mathrm{BC}}(\tilde{\alpha}_\Sigma(t), \tilde{\beta}_\Sigma(t)) > \tilde{R}_\Sigma^{\mathrm{MAC}}(\tilde{\alpha}_\Sigma(t)) \\ \tilde{\varphi}_{\mathrm{BC}}(\tilde{\alpha}_\Sigma(t), \tilde{\beta}_\Sigma(t)), & \text{if } \tilde{R}_\Sigma^{\mathrm{BC}}(\tilde{\alpha}_\Sigma(t), \tilde{\beta}_\Sigma(t)) \leq \tilde{R}_\Sigma^{\mathrm{MAC}}(\tilde{\alpha}_\Sigma(t)). \end{cases} \tag{2.101}$$

with $\boldsymbol{R}_{\mathrm{BC}}(\tilde{\alpha}_\Sigma(t), \tilde{\beta}_\Sigma(t)) = [\tilde{R}_{\overrightarrow{\mathrm{R2}}}(\tilde{\alpha}_\Sigma(t), \tilde{\beta}_\Sigma(t)), \tilde{R}_{\overrightarrow{\mathrm{R1}}}(\tilde{\alpha}_\Sigma(t), \tilde{\beta}_\Sigma(t))]$.

In the following, we show that the boundary rate pair on $\tilde{\mathcal{R}}_\Sigma$ with angle $\tilde{\varphi}_\Sigma(t)$ is component-wise larger than $\boldsymbol{R}(t)$. To this end for the radius for this boundary rate pair we have

$$\begin{aligned} \tilde{R}_\Sigma(\tilde{\varphi}_\Sigma(t)) &= \max_{\alpha \in [0,1]} \min \left\{ \tilde{R}_{\mathrm{BC}}(\alpha, \tilde{\varphi}_\Sigma(t)), \frac{\tilde{R}_\Sigma^{\mathrm{MAC}}(\alpha)}{\cos(\tilde{\varphi}_\Sigma(t)) + \sin(\tilde{\varphi}_\Sigma(t))} \right\} \\ &= \tilde{R}_{\mathrm{BC}}(\tilde{\alpha}_\Sigma^*(\tilde{\varphi}_\Sigma(t)), \tilde{\varphi}_\Sigma(t)) = \frac{\tilde{R}_\Sigma^{\mathrm{MAC}}(\tilde{\alpha}_\Sigma^*(\tilde{\varphi}_\Sigma(t)))}{\cos(\tilde{\varphi}_\Sigma(t)) + \sin(\tilde{\varphi}_\Sigma(t))} \qquad (2.102) \\ &\geq \min \left\{ \tilde{R}_{\mathrm{BC}}(\tilde{\alpha}_\Sigma(t), \tilde{\varphi}_\Sigma(t)), \frac{\tilde{R}_\Sigma^{\mathrm{MAC}}(\tilde{\alpha}_\Sigma(t))}{\cos(\tilde{\varphi}_\Sigma(t)) + \sin(\tilde{\varphi}_\Sigma(t))} \right\}. \qquad (2.103) \end{aligned}$$

We will now show that $R_1(t) \leq \tilde{R}_\Sigma(\tilde{\varphi}_\Sigma(t)) \cos(\tilde{\varphi}_\Sigma(t))$ and $R_\Sigma(t) \leq \tilde{R}_\Sigma(\tilde{\varphi}_\Sigma(t)) \sin(\tilde{\varphi}_\Sigma(t))$. Therefore, we have to distinguish between two cases of the minimization in (2.103) for the two cases of possible angles $\tilde{\varphi}_\Sigma(t)$ of (2.101). To this end we will use the concavity of $\tilde{R}_\Sigma^{\mathrm{MAC}}(\tilde{\alpha}_\Sigma(t))$, i.e.

$$\begin{aligned}\tilde{R}_\Sigma^{\mathrm{MAC}}(\tilde{\alpha}_\Sigma(t)) &\geq t\tilde{R}_\Sigma^{\mathrm{MAC}(1)} + (1-t)\tilde{R}_\Sigma^{\mathrm{MAC}(2)} \geq t(R_1^{(1)} + R_2^{(1)}) + (1-t)(R_1^{(2)} + R_2^{(2)}) \\ &\geq (tR_1^{(1)} + (1-t)R_1^{(2)}) + (tR_2^{(1)} + (1-t)R_2^{(2)}) = R_1(t) + R_2(t).\end{aligned} \tag{2.104}$$

First we consider the case $\tilde{R}_{\mathrm{BC}}(\tilde{\alpha}_\Sigma(t), \tilde{\varphi}_\Sigma(t)) > \frac{\tilde{R}_\Sigma^{\mathrm{MAC}}(\tilde{\alpha}_\Sigma(t))}{\cos(\tilde{\varphi}_\Sigma(t))+\sin(\tilde{\varphi}_\Sigma(t))}$. Then we have to distinguish between the two cases of (2.103). Since for $\tilde{R}_\Sigma^{\mathrm{MAC}}(\tilde{\alpha}_\Sigma(t)) \geq \tilde{R}_\Sigma^{\mathrm{BC}}(\tilde{\alpha}_\Sigma(t), \tilde{\beta}_\Sigma(t))$ we have $\tilde{\varphi}_\Sigma(t) = \tilde{\varphi}_{\mathrm{BC}}(\tilde{\alpha}_\Sigma(t), \tilde{\beta}_\Sigma(t))$ which implies that we have $\tilde{R}_{\mathrm{BC}}(\tilde{\alpha}_\Sigma(t), \tilde{\varphi}_\Sigma(t)) = \tilde{R}_{\mathrm{BC}}(\tilde{\alpha}_\Sigma(t), \tilde{\beta}_\Sigma(t))$ it follows that this case is not possible. Furthermore, it follows that we have $\tilde{R}_\Sigma^{\mathrm{MAC}}(\tilde{\alpha}_\Sigma(t)) < \tilde{R}_\Sigma^{\mathrm{BC}}(\tilde{\alpha}_\Sigma(t), \tilde{\beta}_\Sigma(t))$ with $\tilde{\varphi}_\Sigma(t) = \varphi_{\boldsymbol{R}(t)}$. Therewith, we get

$$\begin{aligned}\cos(\tilde{\varphi}_\Sigma(t))\tilde{R}_\Sigma(\tilde{\varphi}_\Sigma(t)) &= \frac{\cos(\tilde{\varphi}_\Sigma(t))\tilde{R}_\Sigma^{\mathrm{MAC}}(\tilde{\alpha}_\Sigma^*(\tilde{\varphi}_\Sigma(t)))}{\cos(\tilde{\varphi}_\Sigma(t)) + \sin(\tilde{\varphi}_\Sigma(t))} \geq \frac{\tilde{R}_\Sigma^{\mathrm{MAC}}(\tilde{\alpha}_\Sigma(t))\cos(\tilde{\varphi}_\Sigma(t))}{\cos(\tilde{\varphi}_\Sigma(t)) + \sin(\tilde{\varphi}_\Sigma(t))} \\ &= \frac{\tilde{R}_\Sigma^{\mathrm{MAC}}(\tilde{\alpha}_\Sigma(t))}{1 + \tan(\tilde{\varphi}_\Sigma(t))} = \frac{\tilde{R}_\Sigma^{\mathrm{MAC}}(\tilde{\alpha}_\Sigma(t))}{1 + \tan(\varphi_{\boldsymbol{R}(t)})} \\ &\geq \frac{R_1(t) + R_2(t)}{1 + \tan(\varphi_{\boldsymbol{R}(t)})} = \frac{R_1(t) + R_2(t)}{1 + R_2(t)/R_1(t)} = R_1(t)\end{aligned}$$

where the first equality and inequality are consequences of (2.102) and (2.103), then we used (2.104), the fact $\tilde{\varphi}_\Sigma(t) = \varphi_{\boldsymbol{R}(t)}$, and finally the definition of $\tilde{\varphi}_\Sigma(t)$. Similarly, we can conclude that

$$\begin{aligned}\sin(\tilde{\varphi}_\Sigma(t))\tilde{R}_\Sigma(\tilde{\varphi}_\Sigma(t)) &= \frac{\sin(\tilde{\varphi}_\Sigma(t))\tilde{R}_\Sigma^{\mathrm{MAC}}(\tilde{\alpha}_\Sigma^*(\tilde{\varphi}_\Sigma(t)))}{\cos(\tilde{\varphi}_\Sigma(t)) + \sin(\tilde{\varphi}_\Sigma(t))} \geq \frac{\tilde{R}_\Sigma^{\mathrm{MAC}}(\tilde{\alpha}_\Sigma(t))\sin(\tilde{\varphi}_\Sigma(t))}{\cos(\tilde{\varphi}_\Sigma(t)) + \sin(\tilde{\varphi}_\Sigma(t))} \\ &= \frac{\tilde{R}_\Sigma^{\mathrm{MAC}}(\tilde{\alpha}_\Sigma(t))}{1 + 1/\tan(\tilde{\varphi}_\Sigma(t))} = \frac{\tilde{R}_\Sigma^{\mathrm{MAC}}(\tilde{\alpha}_\Sigma(t))}{1 + 1/\tan(\varphi_{\boldsymbol{R}(t)})} \\ &\geq \frac{R_1(t) + R_2(t)}{1 + 1/\tan(\varphi_{\boldsymbol{R}(t)})} = \frac{R_1(t) + R_2(t)}{1 + R_1(t)/R_2(t)} = R_2(t)\end{aligned}$$

so that it follows that $[R_1(t), R_2(t)] \in \tilde{\mathcal{R}}_\Sigma$ for this case.

We now consider the case $\tilde{R}_{\mathrm{BC}}(\tilde{\alpha}_\Sigma(t), \tilde{\varphi}_\Sigma(t)) \leq \frac{\tilde{R}_\Sigma^{\mathrm{MAC}}(\tilde{\alpha}_\Sigma(t))}{\cos(\tilde{\varphi}_\Sigma(t))+\sin(\tilde{\varphi}_\Sigma(t))}$. Again we have to distinguish between the two cases of (2.103). First, we assume that we have $\tilde{R}_\Sigma^{\mathrm{BC}}(\tilde{\alpha}_\Sigma(t), \tilde{\beta}_\Sigma(t)) \leq \tilde{R}_\Sigma^{\mathrm{MAC}}(\tilde{\alpha}_\Sigma(t))$ so that we have $\tilde{\varphi}_\Sigma(t) = \tilde{\varphi}_{\mathrm{BC}}(\tilde{\alpha}_\Sigma(t), \tilde{\beta}_\Sigma(t))$ which implies that we have $\tilde{\beta}_\Sigma(t) = \tilde{\beta}_{\mathrm{BC}}(\tilde{\alpha}_\Sigma(t), \tilde{\varphi}_\Sigma(t))$ or equivalently $\tilde{R}_{\mathrm{BC}}(\tilde{\alpha}_\Sigma(t), \tilde{\varphi}_\Sigma(t)) =$

$\tilde{R}_{\mathrm{BC}}(\tilde{\alpha}_\Sigma(t), \tilde{\beta}_\Sigma(t))$. Therewith, we have

$$\begin{aligned}\cos(\tilde{\varphi}_\Sigma(t))\tilde{R}_\Sigma(\tilde{\varphi}_\Sigma(t)) &= \cos(\tilde{\varphi}_\Sigma(t))\tilde{R}_{\mathrm{BC}}(\tilde{\alpha}^*_\Sigma(\tilde{\varphi}_\Sigma(t)), \tilde{\varphi}_\Sigma(t)) \\ &\geq \cos(\tilde{\varphi}_\Sigma(t))\tilde{R}_{\mathrm{BC}}(\tilde{\alpha}_\Sigma(t), \tilde{\varphi}_\Sigma(t)) = \cos(\tilde{\varphi}_\Sigma(t))\tilde{R}_{\mathrm{BC}}(\tilde{\alpha}_\Sigma(t), \tilde{\beta}_\Sigma(t)) \\ &= \tilde{R}_{\overrightarrow{\mathrm{R2}}}(\tilde{\alpha}_\Sigma(t), \tilde{\beta}_\Sigma(t)) \geq t\tilde{R}_{\overrightarrow{\mathrm{R2}}}(\tilde{\alpha}^{(1)}_\Sigma, \tilde{\beta}^{(1)}_\Sigma) + (1-t)\tilde{R}_{\overrightarrow{\mathrm{R2}}}(\tilde{\alpha}^{(2)}_\Sigma, \tilde{\beta}^{(2)}_\Sigma) \\ &\geq tR^{(1)}_1 + (1-t)R^{(2)}_1 = R_1(t)\end{aligned}$$

where the first equality and inequality are consequences of (2.102) and (2.103), then we used the fact $\tilde{R}_{\mathrm{BC}}(\tilde{\alpha}_\Sigma(t), \tilde{\varphi}_\Sigma(t)) = \tilde{R}_{\mathrm{BC}}(\tilde{\alpha}_\Sigma(t), \tilde{\beta}_\Sigma(t))$ and the concavity of $\tilde{R}_{\overrightarrow{\mathrm{R2}}}(\alpha, \beta)$. Similarly, with the concavity of $\tilde{R}_{\overrightarrow{\mathrm{R1}}}(\alpha, \beta)$ we get

$$\begin{aligned}\sin(\tilde{\varphi}_\Sigma(t))\tilde{R}_\Sigma(\tilde{\varphi}_\Sigma(t)) &= \sin(\tilde{\varphi}_\Sigma(t))\tilde{R}_{\mathrm{BC}}(\tilde{\alpha}^*_\Sigma(\tilde{\varphi}_\Sigma(t)), \tilde{\varphi}_\Sigma(t)) \\ &\geq \sin(\tilde{\varphi}_\Sigma(t))\tilde{R}_{\mathrm{BC}}(\tilde{\alpha}_\Sigma(t), \tilde{\varphi}_\Sigma(t)) = \sin(\tilde{\varphi}_\Sigma(t))\tilde{R}_{\mathrm{BC}}(\tilde{\alpha}_\Sigma(t), \tilde{\beta}_\Sigma(t)) \\ &= \tilde{R}_{\overrightarrow{\mathrm{R1}}}(\tilde{\alpha}_\Sigma(t), \tilde{\beta}_\Sigma(t)) \geq t\tilde{R}_{\overrightarrow{\mathrm{R1}}}(\tilde{\alpha}^{(1)}_\Sigma, \tilde{\beta}^{(1)}_\Sigma) + (1-t)\tilde{R}_{\overrightarrow{\mathrm{R1}}}(\tilde{\alpha}^{(2)}_\Sigma, \tilde{\beta}^{(2)}_\Sigma) \\ &\geq tR^{(1)}_2 + (1-t)R^{(2)}_2 = R_2(t)\end{aligned}$$

so that $[R_1(t), R_2(t)] \in \tilde{\mathcal{R}}_\Sigma$ for this case as well.

Finally, we assume that we have $\tilde{R}_{\mathrm{BC}}(\tilde{\alpha}_\Sigma(t), \tilde{\varphi}_\Sigma(t)) \leq \frac{\tilde{R}^{\mathrm{MAC}}_\Sigma(\tilde{\alpha}_\Sigma(t))}{\cos(\tilde{\varphi}_\Sigma(t))+\sin(\tilde{\varphi}_\Sigma(t))}$ and $\tilde{R}^{\mathrm{BC}}_\Sigma(\tilde{\alpha}_\Sigma(t), \tilde{\beta}_\Sigma(t)) > \tilde{R}^{\mathrm{MAC}}_\Sigma(\tilde{\alpha}_\Sigma(t))$ which implies that we have $\tilde{\varphi}_\Sigma(t) = \varphi_{\boldsymbol{R}(t)}$. First note that $\tilde{R}_{\mathrm{BC}}(\tilde{\alpha}_\Sigma(t), \tilde{\varphi}_\Sigma(t))$ denotes the radius of the rate pair on the boundary of $\tilde{\mathcal{R}}_{\mathrm{BC}}(\tilde{\alpha}_\Sigma(t))$ with angle $\tilde{\varphi}_\Sigma(t)$, i.e.

$$\tilde{R}_{\mathrm{BC}}(\tilde{\alpha}_\Sigma(t), \tilde{\varphi}_\Sigma(t)) = \max_{\substack{[R_1,R_2]\in\tilde{\mathcal{R}}_{\mathrm{BC}}(\tilde{\alpha}_\Sigma(t)) \\ R_1\sin\tilde{\varphi}_\Sigma(t)=R_2\cos\tilde{\varphi}_\Sigma(t)}} \sqrt{R_1^2+R_2^2}.$$

To prove the component-wise dominance of $\tilde{R}_{\mathrm{BC}}(\tilde{\alpha}_\Sigma(t), \tilde{\varphi}_\Sigma(t))[\cos(\tilde{\varphi}_\Sigma(t)), \sin(\tilde{\varphi}_\Sigma(t))]$ we define the downward comprehensive hull[16] $\hat{\mathcal{R}}(t) := \mathrm{dch}(\boldsymbol{R}_{\mathrm{BC}}(\tilde{\alpha}_\Sigma(t), \tilde{\beta}_\Sigma(t)))$ of the rate pair $\boldsymbol{R}_{\mathrm{BC}}(\tilde{\alpha}_\Sigma(t), \tilde{\beta}_\Sigma(t))$. Then, let

$$R_{\hat{\mathcal{R}}(t)}(\tilde{\varphi}_\Sigma(t)) := \max_{\substack{[R_1,R_2]\in\hat{\mathcal{R}}(t) \\ R_1\sin\tilde{\varphi}_\Sigma(t)=R_2\cos\tilde{\varphi}_\Sigma(t)}} \sqrt{R_1^2+R_2^2}$$

denote the radius of the rate pair on the boundary of the downward comprehensive hull of the vector $\boldsymbol{R}_{\mathrm{BC}}(\tilde{\alpha}_\Sigma(t), \tilde{\beta}_\Sigma(t)) = [\tilde{R}_{\overrightarrow{\mathrm{R2}}}(\tilde{\alpha}_\Sigma(t), \tilde{\beta}_\Sigma(t)), \tilde{R}_{\overrightarrow{\mathrm{R1}}}(\tilde{\alpha}_\Sigma(t), \tilde{\beta}_\Sigma(t))]$ with angle

[16] A set $\mathcal{A}$ is *downward comprehensive* in $\mathbb{R}^2_+$, if for any $\boldsymbol{x} = [x_1, x_2] \in \mathcal{A}$ it follows that for any $\boldsymbol{y} = [y_1, y_2] \in \mathbb{R}^2_+$ with $x_i \leq y_i$, $i = 1, 2$, we have $\boldsymbol{y} \in \mathcal{A}$. Then we define the *downward comprehensive hull* of the vector $\boldsymbol{x} \in \mathbb{R}^2_+$ by the set $\mathrm{dch}(\boldsymbol{x}) := \{\boldsymbol{y} \in \mathbb{R}^2_+ : y_i \leq x_i, i = 1, 2\}$.

$\tilde{\varphi}_\Sigma(t)$. Since we have $\boldsymbol{R}_{\mathrm{BC}}(\tilde{\alpha}_\Sigma(t), \tilde{\beta}_\Sigma(t)) \in \tilde{\mathcal{R}}_{\mathrm{BC}}(\tilde{\alpha}_\Sigma(t))$, it follows the inclusion $\hat{\mathcal{R}}(t) \subseteq \tilde{\mathcal{R}}_{\mathrm{BC}}(\tilde{\alpha}_\Sigma(t))$. Therefrom, we can conclude that $R_{\hat{\mathcal{R}}(t)}(\tilde{\varphi}_\Sigma(t)) \leq \tilde{R}_{\mathrm{BC}}(\tilde{\alpha}_\Sigma(t), \tilde{\varphi}_\Sigma(t))$ holds. Furthermore, from the concavity of $\tilde{R}_{\overrightarrow{\mathrm{R2}}}(\alpha, \beta)$ and $\tilde{R}_{\overrightarrow{\mathrm{R1}}}(\alpha, \beta)$ we know

$$R_1(t) \leq \tilde{R}_{\overrightarrow{\mathrm{R2}}}(\tilde{\alpha}_\Sigma(t), \tilde{\beta}_\Sigma(t)), \quad R_2(t) \leq \tilde{R}_{\overrightarrow{\mathrm{R1}}}(\tilde{\alpha}_\Sigma(t), \tilde{\beta}_\Sigma(t)).$$

It follows that $\boldsymbol{R}(t) = [R_1(t), R_2(t)] \in \hat{\mathcal{R}}(t)$. Since we have $\tilde{\varphi}_\Sigma(t) = \varphi_{\boldsymbol{R}(t)}$, we can conclude that $||\boldsymbol{R}(t)||_1 = \sqrt{(R_1(t))^2 + (R_2(t))^2} \leq R_{\hat{\mathcal{R}}(t)}(\tilde{\varphi}_\Sigma(t))$. Altogether, we have

$$\begin{aligned}\tilde{R}_\Sigma(\tilde{\varphi}_\Sigma(t)) &= \tilde{R}_{\mathrm{BC}}(\tilde{\alpha}^*_\Sigma(\tilde{\varphi}_\Sigma(t)), \tilde{\varphi}_\Sigma(t)) \geq \tilde{R}_{\mathrm{BC}}(\tilde{\alpha}_\Sigma(t), \tilde{\varphi}_\Sigma(t)) \\ &= \tilde{R}_{\mathrm{BC}}(\tilde{\alpha}_\Sigma(t), \varphi_{\boldsymbol{R}(t)}) \geq R_{\hat{\mathcal{R}}(t)}(\varphi_{\boldsymbol{R}(t)}) \geq ||\boldsymbol{R}(t)||_1\end{aligned}$$

so that we have

$$\begin{aligned}\cos(\tilde{\varphi}_\Sigma(t))\tilde{R}_\Sigma(\tilde{\varphi}_\Sigma(t)) &\geq \cos(\varphi_{\boldsymbol{R}(t)})||\boldsymbol{R}(t)||_1 = R_1 \\ \sin(\tilde{\varphi}_\Sigma(t))\tilde{R}_\Sigma(\tilde{\varphi}_\Sigma(t)) &\geq \sin(\varphi_{\boldsymbol{R}(t)})||\boldsymbol{R}(t)||_1 = R_2.\end{aligned}$$

This was the last case where we have shown that $[R_1(t), R_2(t)] \in \tilde{\mathcal{R}}_\Sigma$ for all $t \in [0, 1]$ so that we can conclude that $\tilde{\mathcal{R}}_\Sigma$ is convex as well.

Finally, since the intersection of convex sets is convex it follows that $\tilde{\mathcal{R}}_{\mathrm{BRopt}} = \tilde{\mathcal{R}}_1 \cap \tilde{\mathcal{R}}_2 \cap \tilde{\mathcal{R}}_\Sigma$ is convex as well. □

2.8.21 Proof of Corollary 2.30

From the queue evolution equation (2.43) we have

$$\begin{aligned}Q_k^2[n+1] &\leq \big(Q_k[n] - R_k[n]T\big)^2 + B_k^2[n] + 2B_k[n]\big[Q_k[n] - R_k[n]T\big]_+ \\ &\leq \big(Q_k[n] - R_k[n]T\big)^2 + B_k^2[n] + 2B_k[n]Q_k[n] \\ &\leq Q_k^2[n] - 2TQ_k[n]\big(R_k[n] - \tfrac{B_k[n]}{T}\big) + \big(R_k[n]T\big)^2 + B_k^2[n]\end{aligned}$$

for $k = 1, 2$. Using this inequality the one-step drift gives us

$$\begin{aligned}
\mathbb{E}\big\{L\big(\boldsymbol{Q}[n+1]\big) - L\big(\boldsymbol{Q}[n]\big)\big|\boldsymbol{Q}[n] = \boldsymbol{q}\big\} &= \mathbb{E}\Big\{\sum_{k=1}^{2} Q_k^2[n+1] - Q_k^2[n]\big|\boldsymbol{Q}[n] = \boldsymbol{q}\Big\} \\
&\leq \mathbb{E}\Big\{\sum_{k=1}^{2} -2TQ_k[n]\big(R_k[n] - \tfrac{B_k[n]}{T}\big) + \big(R_k[n]T\big)^2 + B_k^2[n]\big|\boldsymbol{Q}[n] = \boldsymbol{q}\Big\} \\
&= \sum_{k=1}^{2} T^2\mathbb{E}\big\{R_k^2[n]|\boldsymbol{Q}[n] = \boldsymbol{q}\big\} + \mathbb{E}\big\{B_k^2[n]\big\} \qquad (2.105) \\
&\quad - 2T\sum_{k=1}^{2} q_k\Big(\mathbb{E}\big\{R_k[n]|\boldsymbol{Q}[n] = \boldsymbol{q}\big\} - \underbrace{\mathbb{E}\big\{\tfrac{B_k[n]}{T}\big\}}_{=\lambda_k}\Big) \qquad (2.106) \\
&\leq \upsilon - 2T\sum_{k=1}^{2} q_k\Big(\mathbb{E}\big\{R_k[n]|\boldsymbol{Q}[n] = \boldsymbol{q}\big\} - \mathbb{E}\big\{\tfrac{B_k[n]}{T}\big\}\Big),
\end{aligned}$$

where we used for the first term of the sum (2.105) the inequality

$$T^2\mathbb{E}\Big\{\sum_{k=1}^{2} R_k^2[n]|\boldsymbol{Q}[n] = \boldsymbol{q}\Big\} \leq T^2 \max_{\boldsymbol{h}\in\boldsymbol{\mathcal{H}}, \boldsymbol{R}\in\mathcal{R}_{\mathrm{BR}}(\boldsymbol{h})} \big(R_1^2 + R_2^2\big) = \upsilon - \mathbb{E}\big\{B_k^2[n]\big\}. \qquad (2.107)$$

For the second term of the sum (2.106) we have

$$\sum_{k=1}^{2} q_k\mathbb{E}\big\{R_k[n]|\boldsymbol{Q}[n] = \boldsymbol{q}\big\} = \max_{\boldsymbol{R}\in\overline{\mathcal{R}_{\mathrm{BR}}}} \sum_{k=1}^{2} q_k R_k > \lambda_1 + \lambda_2,$$

since we require $[\lambda_1, \lambda_2]$ to be strictly in the interior of $\overline{\mathcal{R}_{\mathrm{BR}}}$. Accordingly, there exists $\tilde{\zeta} > 0$ so that the vector $[\lambda_1 + \tilde{\zeta}, \lambda_2 + \tilde{\zeta}]$ is still in $\overline{\mathcal{R}_{\mathrm{BR}}}$. Therefore, we get for the second term of the sum (2.106) the inequality

$$2T\sum_{k=1}^{2} q_k\Big(\mathbb{E}\big\{R_k[n]|\boldsymbol{Q}[n] = \boldsymbol{q}\big\} - \lambda_k\Big) \geq 2T\tilde{\zeta}\sum_{k=1}^{2} q_k.$$

All together we have

$$\mathbb{E}\big\{L\big(\boldsymbol{Q}[n+1]\big) - L\big(\boldsymbol{Q}[n]\big)\big|\boldsymbol{Q}[n] = \boldsymbol{q}\big\} \leq \upsilon - \zeta\sum_{k=1}^{2} q_k$$

with $\zeta = 2T\tilde{\zeta}$. For any arbitrary $\varepsilon > 0$ we can define the compact region $\Lambda := \Big\{\boldsymbol{q} \in \mathbb{R}_+^2\Big|\sum_{k=1}^{2} q_k \leq \frac{\upsilon+\varepsilon}{\zeta}\Big\}$. Whenever $\boldsymbol{q} \notin \Lambda$ we have $\sum_{k=1}^{2} q_k > \frac{\upsilon+\varepsilon}{\zeta}$ and therefore we have a negative one-step Lyapunov drift $\mathbb{E}\big\{L\big(\boldsymbol{Q}[n+1]\big) - L\big(\boldsymbol{Q}[n]\big)\big|\boldsymbol{Q}[n] = \boldsymbol{q}\big\} < -\varepsilon$. □

2.8.22 Proof of Lemma 2.35

The cdf of Z_n follows from

$$\begin{aligned}
F_{Z_n}(z_n) &= \mathbb{P}\big\{\min\{a_n x_n, b_n y_n\} \le z_n\big\} \\
&= 1 - \mathbb{P}\big\{\min\{a_n x_n, b_n y_n\} > z_n\big\} \\
&= 1 - \mathbb{P}\big\{a_n x_n > z_n\big\}\mathbb{P}\big\{b_n y_n > z_n\big\} \\
&= 1 - (1 - F_{X_n}(z_n/a_n))(1 - F_{Y_n}(z_n/b_n))
\end{aligned}$$

and the pdf is given by its derivation. Accordingly, $f_Z(z)$ is the derivation of the cdf

$$F_Z(z) = \mathbb{P}\big\{\max_n\{z_n\} \le z\big\} = \prod_n \mathbb{P}\big\{z \ge z_n\big\} = \prod_n F_{Z_n}(z_n),$$

since we assume that the random variables Z_n are pairwise independent. Finally, $R = g(Z) = \log(1+Z)/2$ is the function of the random variable Z where $g(Z)$ is strictly increasing and differentiable in Z [PP02, Chap. 5]. For that reason the pdf is given by

$$f_R(R) = \frac{f_Z(g^{-1}(R))}{|g'(g^{-1}(R))|} = 2\ln(2)2^{2R} f_Z(2^{2R} - 1),$$

where $g'(Z) = \frac{1}{2\ln(2)(1+Z)}$ and $g^{-1}(R) = 2^{2R} - 1$ denote the derivative and inverse of $g(Z)$ respectively. $\square$

2.8.23 Proof of Proposition 2.37

For independent channel realizations the achievable rate region of the η-th relay node contains the regions of all others iff we have $|h_{k,\eta}| \ge \max_{n=1,\dots,N}\{|h_{k,n}|\}$ for $k = 1, 2$ simultaneously. With $\mathbb{P}\big\{|h_{k,\eta}| \ge |h_{k,n}|\big||h_{k,\eta}|\big\} = F_{|h_{k,n}|}(|h_{k,\eta}|)$ it follows that

$$\begin{aligned}
\mathbb{P}\Big\{\mathcal{R}_{\mathrm{BReq},\eta} \supseteq \bigcup_n \mathcal{R}_{\mathrm{BReq},n}\Big\} &= \prod_{k=1}^{2} \mathbb{P}\Big\{|h_{k,\eta}| \ge \max_{n=1,\dots,N}\{|h_{k,n}|\}\Big\} \\
&= \prod_{k=1}^{2} \int \mathbb{P}\Big\{|h_{k,\eta}| \ge \max_{n=1,\dots,N}\{|h_{k,n}|\}\Big||h_{k,\eta}|\Big\} f_{|h_{k,\eta}|}(|h_{k,\eta}|)\, d|h_{k,\eta}| \\
&= \prod_{k=1}^{2} \int \prod_{\substack{n=1 \\ n\ne\eta}}^{N} \mathbb{P}\Big\{|h_{k,\eta}| \ge |h_{k,n}|\Big||h_{k,\eta}|\Big\} f_{|h_{k,\eta}|}(|h_{k,\eta}|)\, d|h_{k,\eta}|.
\end{aligned}$$

Then the probability that there exists one relay node $\eta \in \{1,\dots,N\}$ which rate region contains the others is given by the disjoint disjunction

$$\begin{aligned}\mathbb{P}\{\exists\eta : \mathcal{R}_{\mathrm{RSeq}} = \mathcal{R}_{\mathrm{BReq},\eta}\} &= \mathbb{P}\Big\{\bigvee_{\eta=1}^{N}(\mathcal{R}_{\mathrm{BReq},\eta} \supseteq \bigcup_{n}\mathcal{R}_{\mathrm{BReq},n})\Big\}\\ &= \sum_{\eta=1}^{N}\mathbb{P}\Big\{\mathcal{R}_{\mathrm{BReq},\eta} \supseteq \bigcup_{n}\mathcal{R}_{\mathrm{BReq},n}\Big\}.\end{aligned}$$

In the case of iid Rayleigh fading we have

$$\begin{aligned}&\mathbb{P}\Big\{|h_{k,\eta}| \geq \max_{n=1,\dots,N}\{|h_{k,n}|\}\Big\}\\ &\quad= \int_0^\infty \Big(1-\exp\Big(-\frac{|h_{k,\eta}|^2}{\sigma_k^2}\Big)\Big)^{N-1}\frac{2|h_{k,\eta}|}{\sigma_k^2}\exp\Big(-\frac{|h_{k,\eta}|^2}{\sigma_k^2}\Big)d|h_{k,\eta}|\\ &\quad= \frac{1}{\sigma_k^2}\sum_{n=0}^{N-1}\binom{N-1}{n}(-1)^n\int_0^\infty \exp\Big(-\frac{|h_{k,\eta}|^2(n+1)}{\sigma_k^2}\Big)d|h_{k,\eta}|^2\\ &\quad= \sum_{n=0}^{N-1}\binom{N-1}{n}\frac{(-1)^n}{n+1} = \frac{1}{N}.\end{aligned}$$

Finally, this gives us

$$\mathbb{P}\Big\{\bigvee_{\eta=1}^{N}\big(\mathcal{R}_{\mathrm{BReq},\eta} \supseteq \bigcup_{n}\mathcal{R}_{\mathrm{BReq},n}\big)\Big\} = \sum_{\eta=1}^{N}\prod_{k=1}^{2}\mathbb{P}\Big\{|h_{k,\eta}| \geq \max_{n=1,\dots,N}\{|h_{k,n}|\}\Big\} = \frac{1}{N}.$$

$\square$

2.8.24 Proof of Theorem 2.38

Since $|h_{k,n}|$ is Rayleigh distributed, $|h_{k,n}|^2$ is exponential distributed with pdf $f_{|h_{k,n}|^2}(|h_{k,n}|^2) = \frac{1}{\sigma_{k,n}^2}\exp\big(-\frac{|h_{k,n}|^2}{\sigma_{k,n}^2}\big)$ for $k=1,2$ and $n=1,\dots,N$. According to (2.22a) and (2.22b) the maximal unidirectional relay rates are given by

$$\begin{aligned}R^*_{1\mathrm{RSeq}} &= \tfrac{1}{2}\log\big(1+\max_n\{\min\{\gamma_1|h_{1,n}|^2, \gamma_{\mathrm{R}}|h_{2,n}|^2\}\}\big),\\ R^*_{2\mathrm{RSeq}} &= \tfrac{1}{2}\log\big(1+\max_n\{\min\{\gamma_2|h_{2,n}|^2, \gamma_{\mathrm{R}}|h_{1,n}|^2\}\}\big).\end{aligned}$$

Therefore, we can apply Lemma 2.35 to derive the pdf of $R^*_{k\text{RSeq}}$, $k = 1, 2$. For a given k we consider $X_n := |h_{1,n}|^2$ and $Y_n := |h_{2,n}|^2$. To be in terms of Lemma 2.35, we define $Z_n := \min\{\gamma_1 X_n, \gamma_\text{R} Y_n\}$. Then according to Remark 2.36 the random variable Z_n is again exponential distributed with pdf $f_{Z_n}(z_n) = \lambda_{k,n}\exp(-z_n\lambda_{k,n})$. with $\lambda_{1,n} = \frac{1}{\gamma_1\sigma^2_{1,n}} + \frac{1}{\gamma_\text{R}\sigma^2_{2,n}}$ and $\lambda_{2,n} = \frac{1}{\gamma_2\sigma^2_{2,n}} + \frac{1}{\gamma_\text{R}\sigma^2_{1,n}}$ respectively. We consider next the random variable $Z := \max_n\{Z_n\}$ for which we can calculate the pdf as follows

$$f_Z(z) = \sum_{n=1}^{N} f_{Z_n}(z) \prod_{j=1, j\neq n}^{N} F_{Z_j}(z) = \sum_{n=1}^{N} \lambda_{k,n}\,\mathrm{e}^{-z\lambda_{k,n}} \prod_{m=1,m\neq n}^{N} (1 - \mathrm{e}^{-z\lambda_{k,m}}) \tag{2.108}$$

according to Lemma 2.35. Finally, we can again apply Lemma 2.35 to calculate the pdf of the random variable $R^*_{k\text{RSeq}} = \log(1 + Z)$ as stated in the theorem which finishes the proof. □

2.8.25 Proof of Corollary 2.39

In a scenario with independent Rayleigh distributed fading channels and for a given k we can define a random variable Z as in the proof of Theorem 2.38 with a pdf according to (2.108)

$$\begin{aligned} f_z(z) &= \sum_{n=1}^{N} \lambda_{k,n}\,\mathrm{e}^{-z\lambda_{k,n}} \prod_{m=1,m\neq n}^{N} (1 - \mathrm{e}^{-z\lambda_{k,m}}) \\ &= \sum_{n=1}^{N} \lambda_{k,n}\,\mathrm{e}^{-z\lambda_{k,n}} \big(1 + \sum_{m=1}^{N-1} (-1)^m \sum_{\mathcal{L}\subseteq\mathcal{J}_n, |\mathcal{L}|=m} \exp(-z\sum_{l\in\mathcal{L}}\lambda_{k,l})\big) \\ &= \sum_{n=1}^{N} \lambda_{k,n}\big(\exp(-z\lambda_{k,n}) + \sum_{m=1}^{N-1} (-1)^m \sum_{\mathcal{L}\subseteq\mathcal{J}_n, |\mathcal{L}|=m} \exp\big(-z(\lambda_{k,n} + \sum_{l\in\mathcal{L}}\lambda_{k,l})\big)\big). \end{aligned}$$

Then the ergodic unidirectional rate is given by

$$\begin{aligned} \overline{R^*_{k\text{RSeq}}} &= \frac{1}{2}\int_0^\infty \log(1+z) f_z(z)\,dz = \sum_{n=1}^{N} \frac{\lambda_{k,n}}{2\ln(2)}\Big[\frac{\exp(\lambda_{k,n})\mathrm{E}_1(\lambda_{k,n})}{\lambda_{k,n}} \\ &\quad + \sum_{m=1}^{N-1}(-1)^m \sum_{\mathcal{L}\subseteq\mathcal{J}_n, |\mathcal{L}|=m} \frac{\exp(\lambda_{k,n} + \sum_{l\in\mathcal{L}}\lambda_{k,l})\mathrm{E}_1(\lambda_{k,n} + \sum_{l\in\mathcal{L}}\lambda_{k,l})}{\lambda_{k,n} + \sum_{l\in\mathcal{L}}\lambda_{k,l}}\Big] \end{aligned}$$

using the identity $\int_0^\infty \log(1+t)\exp(-ta)dt = \frac{\mathrm{e}^a\,\mathrm{E}_1(a)}{\ln(2)a}$ for $a \geq 0$.

In the case of iid Rayleigh fading, we have $\lambda_{k,n} = \lambda_k$ for all n so that we get the pdf

$$f_z(z) = N\lambda_k \, \mathrm{e}^{-\lambda_k z}(1-\mathrm{e}^{-\lambda_k z})^{N-1} = N\lambda_k \sum_{n=0}^{N-1} \binom{N-1}{n}(-1)^n \exp(-(n+1)\lambda_k z) \tag{2.109}$$

using the binomial theorem $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$. Then (2.52) follows using again the identity $\int_0^\infty \log(1+t)\exp(-ta)dt = \frac{\mathrm{e}^a \, \mathrm{E}_1(a)}{\ln(2)a}$ for $a \geq 0$. □

2.8.26 Proof of Theorem 2.40

In the case of iid Rayleigh fading, we have $f_z(z) = N\lambda_k \, \mathrm{e}^{-\lambda_k z}(1-\mathrm{e}^{-\lambda_k z})^{N-1}$ according to (2.109). By substitution of $\tau = 1 - \mathrm{e}^{-\lambda_k z}$ we get

$$\begin{aligned} \overline{R^*_{k\mathrm{RSeq}}} &= \frac{1}{2}\int_0^\infty \log(1+z) f_z(z)\, dz \\ &= \frac{N}{2\ln(2)} \int_0^1 \ln\big(1 - 1/\lambda_k \ln(1-\tau)\big)\tau^{N-1}\, d\tau. \end{aligned}$$

Therewith, we get for any $x \in (0,1)$

$$\begin{aligned} \overline{R^*_{k\mathrm{RSeq}}} &\geq \frac{N}{2\ln(2)} \int_x^1 \ln\big(1 - 1/\lambda_k \ln(1-\tau)\big)\tau^{N-1}\, d\tau \\ &\geq \frac{N}{2\ln(2)} \ln\big(1 - 1/\lambda_k \ln(1-x)\big) \int_x^1 \tau^{N-1}\, d\tau \\ &= \frac{1}{2\ln(2)} \ln\big(1 - 1/\lambda_k \ln(1-x)\big)(1-x^N), \end{aligned}$$

where we used in the first inequality that the integrand is positive and in the second that $\ln\big(1 - 1/\lambda_k \ln(1-\tau)\big)$ is increasing with τ. If we set $x = 1 - a/N$ with $a \in (0,N)$ we have

$$\overline{R^*_{k\mathrm{RSeq}}} \geq \frac{1}{2\ln(2)} \ln\big(1 - 1/\lambda_k \ln(a/N)\big)(1-(1-a/N)^N). \tag{2.110}$$

From [AS64, 4.2.36] we have the inequality $\mathrm{e}^{\frac{wy}{w+y}} < \big(1+\frac{w}{y}\big)^y$ for $w, y > 0$. Therefrom, we get with $z := \frac{wy}{w+y} > 0$ the following inequality

$$\mathrm{e}^z = \mathrm{e}^{\frac{wy}{w+y}} < \Big(1+\frac{w}{y}\Big)^y = \Big(\frac{y}{y+w}\Big)^{-y} = \Big(1-\frac{w}{y+w}\Big)^{-y} = \Big(1-\frac{z}{y}\Big)^{-y}$$

so that we have $\mathrm{e}^{-z} > \big(1-\frac{z}{y}\big)^y$ finally. With this inequality we can finally bound (2.110) so that we have

$$\overline{R^*_{k\mathrm{RSeq}}} > \frac{1}{2\ln(2)} \ln\big(1 - 1/\lambda_k \ln(a/N)\big)(1-\mathrm{e}^{-a}),$$

which is the proposed lower bound.

For the upper bound, accordingly for any $x \in (0,1)$ we have

$$\overline{R^*_{k\text{RSeq}}} \leq \frac{\ln(1-1/\lambda \ln(1-x))}{2\ln(2)} N \int_0^x \tau^{N-1}\, d\tau + \frac{N}{2\ln(2)} \int_x^1 \ln\big(1-1/\lambda_k \ln(1-\tau)\big)\, d\tau,$$

where we used the fact that τ^{N-1} takes it maximum at one. If we set $x = 1 - b/N$ with $b \in (0, N)$ we get for the first integral

$$N \int_0^x \tau^{N-1}\, d\tau = x^N = (1 - \frac{b}{N})^N < \mathrm{e}^{-b}$$

and for the second integral (integration by parts technique)

$$N \int_{1-b/N}^1 \ln\big(1-1/\lambda_k \ln(1-\tau)\big)\, d\tau = \frac{1}{b} \ln\big(1+1/\lambda_k \ln(N/b)\big) + N\,\mathrm{e}^{\lambda_k}\,\mathrm{E}_1(\lambda_k + \ln(N/b)).$$

Let $\lambda_k + \ln(N/b) =: \xi$, then

$$N\,\mathrm{e}^{\lambda_k}\,\mathrm{E}_1(\lambda_k + \ln(N/b)) = b\,\mathrm{e}^{\xi}\,\mathrm{E}_1(\xi) < b \ln(1 + 1/(\lambda_k + \ln(N/b)))$$

using the inequality $\mathrm{e}^{\xi}\,\mathrm{E}_1(\xi) < \ln(1 + 1/\xi)$ [AS64, 5.1.20]. Combining all terms leads to the upper bound.

For the asymptotic lower and upper bound we have

$$\begin{aligned}
\liminf_{N\to\infty} \frac{\overline{R^*_{k\text{RSeq}}}}{\frac{1}{2}\log(\ln(N))} &\geq \liminf_{N\to\infty} \frac{\log(1+\frac{1}{\lambda_k}\ln(\frac{N}{a}))\frac{1-\mathrm{e}^{-a}}{2}}{\frac{1}{2}\log(\ln(N))} = 1 - e^{-a} > 0,\\
\limsup_{N\to\infty} \frac{\overline{R^*_{k\text{RSeq}}}}{\frac{1}{2}\log(\ln(N))} &\leq \limsup_{N\to\infty} \frac{\log(1+\frac{1}{\lambda_k}\ln(\frac{N}{b}))\frac{\mathrm{e}^{-b}+b}{2} + \frac{b}{2}\log(1+\frac{1}{\lambda_k+\ln(\frac{N}{b})})}{\frac{1}{2}\log(\ln(N))}\\
&= \mathrm{e}^{-b} + b < \infty,
\end{aligned}$$

which meet if we choose $a \to \infty$ and $b \to 0$, i.e. $\lim_{N\to\infty} \frac{\overline{R^*_{k\text{RSeq}}}}{\frac{1}{2}\log(\ln(N))} = 1$. This means that $\overline{R^*_{k\text{RSeq}}}$ is asymptotically equal to $\frac{1}{2}\log(\ln(N))$.

Since any ergodic sum-rate on the boundary of the ergodic rate region is larger than $\min_k\{\overline{R^*_{k\text{RSeq}}}\}$ and smaller than the sum $\sum_{k=1}^2 \overline{R^*_{k\text{RSeq}}}$ the ergodic rate region grows asymptotically with $\Theta\big(\log(\log(N))\big)$ as concluded in Corollary 2.42. □

2.8.27 Proof of Proposition 2.43

The maximal unidirectional rates of the n-th relay node for the optimal time division case are given by

$$R^*_{1\mathrm{opt},n} := \frac{R_{\overrightarrow{1\mathrm{R}},n} R_{\overrightarrow{\mathrm{R}2},n}}{R_{\overrightarrow{1\mathrm{R}},n} + R_{\overrightarrow{\mathrm{R}2},n}}, \quad \text{and} \quad R^*_{2\mathrm{opt},n} := \frac{R_{\overrightarrow{2\mathrm{R}},n} R_{\overrightarrow{\mathrm{R}1},n}}{R_{\overrightarrow{2\mathrm{R}},n} + R_{\overrightarrow{\mathrm{R}1},n}}$$

according to (2.31a) and (2.31b). Then the maximal unidirectional rate using relay selection is given by

$$R^*_{k\mathrm{RSopt}} := \max_{n\in\{1,2,\dots,N\}} R^*_{k\mathrm{opt},n} \quad k = 1, 2. \tag{2.111}$$

With the maximal unidirectional rates of the equal time division case $R^*_{1\mathrm{eq},n} = 1/2 \min\{R_{\overrightarrow{1\mathrm{R}},n}, R_{\overrightarrow{\mathrm{R}2},n}\}$ and $R^*_{2\mathrm{eq},n} = 1/2 \min\{R_{\overrightarrow{2\mathrm{R}},n}, R_{\overrightarrow{\mathrm{R}1},n}\}$ for the n-th relay node according to (2.22a) and (2.22b) we get

$$R^*_{k\mathrm{eq},n} \le R^*_{k\mathrm{opt},n} \le 2R^*_{k\mathrm{eq},n} \tag{2.112}$$

using the inequalities $1/2 \min\{x, y\} \le \frac{xy}{x+y} \le \min\{x, y\}$ for $x, y > 0$, which can be easily seen from the following

$$1/2 \min\{x, y\} = \frac{1}{2\max\{\frac{1}{x}, \frac{1}{y}\}} \le \frac{1}{\frac{1}{x} + \frac{1}{y}} \le \frac{1}{\max\{\frac{1}{x}, \frac{1}{y}\}} = \min\{x, y\}.$$

Since (2.112) holds for any n, we get $R^*_{k\mathrm{RSeq}} \le R^*_{k\mathrm{RSopt}} \le 2R^*_{k\mathrm{RSeq}}$, $k = 1, 2$, for (2.111). Furthermore, since the inequalities hold for any channel realization, we get an upper and lower bound for the maximal unidirectional ergodic rates as follows

$$\overline{R^*_{k\mathrm{RSeq}}} \le \overline{R^*_{k\mathrm{RSopt}}} \le 2\overline{R^*_{k\mathrm{RSeq}}}, \quad k = 1, 2. \tag{2.113}$$

Similar to (2.53), since $\overline{\mathcal{R}_{\mathrm{RSTSopt}}}$ is convex, we can bound the sum of any ergodic rate pair $\overline{\boldsymbol{R}^*_{\mathrm{RSTSopt}}(\boldsymbol{q})}$ on the boundary of the ergodic rate region $\overline{\mathcal{R}_{\mathrm{RSTSopt}}}$ as follows

$$\min\left\{\overline{R^*_{1\mathrm{RSopt}}}, \overline{R^*_{2\mathrm{RSopt}}}\right\} \le \left|\left|\overline{\boldsymbol{R}^*_{\mathrm{RSTSopt}}(\boldsymbol{q})}\right|\right|_1 \le \sum_{k=1}^{2} \overline{R^*_{k\mathrm{RSopt}}}. \tag{2.114}$$

Then (2.54) obviously follows from (2.113) and (2.114). □

2.8.28 Proof of Theorem 2.45

Since nodes 1 and 2 cancel the interference caused by their own message before decoding the unknown messages, for the proof we have to distinguish between four possible decoding orders at nodes 1 and 2. Therefore, let the superscript $^{\mathrm{R2}}$ and $^{\mathrm{R1}}$ denote the decoding order of the unknown messages where the relay message at nodes 1 and 2 is decoded first and accordingly $^{\mathrm{2R}}$ and $^{\mathrm{1R}}$ where the relay messages is decoded last. For decoding the first unknown message the received signal of the second unknown message is regarded as interference. After successful decoding the nodes apply interference cancellation so that the nodes are interference-free in the second decoding step. Since $[R_1, R_2] \in \mathcal{R}_{\mathrm{BR}}(\alpha)$, there exists a feasible power distribution for each decoding order. Since the relay rate strictly increases with β_{R} for any decoding order, we have $\beta_{\mathrm{R}} = 1 - \beta_1 + \beta_2$.

We start with the decoding order $\mathrm{R2}, \mathrm{R1}$ where the relay message is decoded first, i.e. the bidirectional messages are decoded last. Since the second decoding step is interference-free we achieve the bidirectional rate $R_1 = (1-\alpha) R_{\overrightarrow{\mathrm{R2}}}(\beta_1)$ and $R_2 = (1-\alpha) R_{\overrightarrow{\mathrm{R1}}}(\beta_2)$. Accordingly, we need for the bidirectional relay communication the relay power distribution parameters

$$\beta_1^{\mathrm{R2,R1}} := r_1 q_2 \quad \text{and} \quad \beta_2^{\mathrm{R2,R1}} := r_2 q_1$$

with $r_1 := 2^{\frac{R_1}{1-\alpha}} - 1 \geq 0$, $r_2 := 2^{\frac{R_2}{1-\alpha}} - 1 \geq 0$, $q_1 := \frac{1}{\gamma_{\mathrm{R}} |h_1|^2} > 0$, and $q_2 := \frac{1}{\gamma_{\mathrm{R}} |h_2|^2} > 0$ to achieve a certain rate pair $[R_1, R_2] \in \mathcal{R}_{\mathrm{BR}}(\alpha)$. In the first decoding step at node 1 the message m_2 is unknown. Therefore, node 1 can successfully decode a relay message m_{R} with a rate at most

$$R_{\mathrm{R@1}}^{\mathrm{R2,R1}} := (1-\alpha) \log \Big(1 + \frac{\beta_{\mathrm{R}}^{\mathrm{R2,R1}} |h_1|^2 \gamma_{\mathrm{R}}}{1 + \beta_2^{\mathrm{R2,R1}} |h_1|^2 \gamma_{\mathrm{R}}} \Big) = (1-\alpha) \log \Big(1 + \frac{1 - r_1 q_2 - r_2 q_1}{q_1 + r_2 q_1} \Big)$$

with $\beta_{\mathrm{R}}^{\mathrm{R2,R1}} = 1 - \beta_1^{\mathrm{R2,R1}} - \beta_2^{\mathrm{R2,R1}}$. Similarly, the message m_1 is unknown at node 2 so that node 2 can successfully decode a relay message m_{R} with a rate at most

$$R_{\mathrm{R@2}}^{\mathrm{R2,R1}} := (1-\alpha) \log \Big(1 + \frac{\beta_{\mathrm{R}}^{\mathrm{R2,R1}} |h_2|^2 \gamma_{\mathrm{R}}}{1 + \beta_1^{\mathrm{R2,R1}} |h_2|^2 \gamma_{\mathrm{R}}} \Big) = (1-\alpha) \log \Big(1 + \frac{1 - r_1 q_2 - r_2 q_1}{q_2 + r_1 q_2} \Big).$$

Since both nodes have to decode the relay message only the minimum of both rates can be achieved, this means that with the decoding order $\mathrm{R2}, \mathrm{R1}$ the maximal achievable additional relay rate is given by

$$R_{\mathrm{R}}^{\mathrm{R2,R1}} := (1-\alpha) \log \Big(1 + \frac{1 - r_1 q_2 - r_2 q_1}{\max\{q_1(1+r_2), q_2(1+r_1)\}} \Big). \tag{2.115}$$

For the decoding order $\mathrm{2R}, \mathrm{1R}$ nodes 1 and 2 decode the relay message last. Since $[R_1, R_2] \in \mathcal{R}_{\mathrm{BR}}(\alpha)$ there exists a feasible relay power distribution. Since the relay message m_{R} is unknown at nodes 1 and 2 in the first decoding step, the received signal of the relay message is regarded as interference. Therefore, for the desired bidirectional rate R_1 at node 2 we have

$$\begin{aligned} R_1 &= (1-\alpha)\log\Big(1+\frac{\beta_1^{2\mathrm{R},1\mathrm{R}}|h_2|^2\gamma_\mathrm{R}}{1+\beta_\mathrm{R}^{2\mathrm{R},1\mathrm{R}}|h_2|^2\gamma_\mathrm{R}}\Big) = (1-\alpha)\log\Big(1+\frac{\beta_1^{2\mathrm{R},1\mathrm{R}}}{q_2+\beta_\mathrm{R}^{2\mathrm{R},1\mathrm{R}}}\Big) \\ \Rightarrow r_1 &= 2^{\frac{R_1}{1-\alpha}}-1 = \frac{\beta_1^{2\mathrm{R},1\mathrm{R}}}{q_2+\beta_\mathrm{R}} = \frac{1-\beta_\mathrm{R}^{2\mathrm{R},1\mathrm{R}}-\beta_2}{q_2+\beta_\mathrm{R}^{2\mathrm{R},1\mathrm{R}}} \qquad (2.116) \\ \Rightarrow \beta_2^{2\mathrm{R},1\mathrm{R}} &= 1-q_2r_1-\beta_\mathrm{R}^{2\mathrm{R},1\mathrm{R}}(1+r_1), \qquad (2.117) \end{aligned}$$

where we used $\beta_1^{2\mathrm{R},1\mathrm{R}} = 1-\beta_2^{2\mathrm{R},1\mathrm{R}}-\beta_\mathrm{R}^{2\mathrm{R},1\mathrm{R}}$ in the last equation of (2.116). Accordingly, for the bidirectional rate R_2 at node 1 we have

$$\begin{aligned} R_2 &= (1-\alpha)\log\Big(1+\frac{\beta_2^{2\mathrm{R},1\mathrm{R}}|h_1|^2\gamma_\mathrm{R}}{1+\beta_\mathrm{R}^{2\mathrm{R},1\mathrm{R}}|h_1|^2\gamma_\mathrm{R}}\Big) = (1-\alpha)\log\Big(1+\frac{\beta_2^{2\mathrm{R},1\mathrm{R}}}{q_1+\beta_\mathrm{R}^{2\mathrm{R},1\mathrm{R}}}\Big) \\ \Rightarrow r_2 &= 2^{\frac{R_2}{1-\alpha}}-1 = \frac{\beta_2^{2\mathrm{R},1\mathrm{R}}}{q_1+\beta_\mathrm{R}^{2\mathrm{R},1\mathrm{R}}} = \frac{1-q_2r_1-\beta_\mathrm{R}^{2\mathrm{R},1\mathrm{R}}(1+r_1)}{q_1+\beta_\mathrm{R}^{2\mathrm{R},1\mathrm{R}}} \qquad (2.118) \\ \Rightarrow \beta_\mathrm{R}^{2\mathrm{R},1\mathrm{R}} &= \frac{1-r_1q_2-r_2q_1}{1+r_1+r_2}, \qquad (2.119) \end{aligned}$$

where we used (2.117) in the last equation of (2.118). At both nodes the relay message m_R is decoded in the second decoding step, which is interference-free due to the interference cancellation. Therefore, node 1 can decode the relay message if the rate is at most

$$R_{\mathrm{R}@1}^{2\mathrm{R},1\mathrm{R}} := (1-\alpha)\log(1+\beta_\mathrm{R}^{2\mathrm{R},1\mathrm{R}}\gamma_\mathrm{R}|h_1|^2).$$

Similarly, node 2 can decode the relay message if the rate is at most

$$R_{\mathrm{R}@2}^{2\mathrm{R},1\mathrm{R}} := (1-\alpha)\log(1+\beta_\mathrm{R}^{2\mathrm{R},1\mathrm{R}}\gamma_\mathrm{R}|h_2|^2).$$

Since both nodes should decode the relay message the largest possible relay rate for the decoding order $2\mathrm{R},1\mathrm{R}$ is given by

$$\begin{aligned} R_\mathrm{R}^{2\mathrm{R},1\mathrm{R}} &:= \min\big\{R_{\mathrm{R}@1}^{2\mathrm{R},1\mathrm{R}}, R_{\mathrm{R}@2}^{2\mathrm{R},1\mathrm{R}}\big\} = (1-\alpha)\log(1+\beta_\mathrm{R}^{2\mathrm{R},1\mathrm{R}}\min\{\gamma_\mathrm{R}|h_1|^2,\gamma_\mathrm{R}|h_2|^2\}) \\ &=(1-\alpha)\log\big(1+\beta_\mathrm{R}^{2\mathrm{R},1\mathrm{R}}\min\{\tfrac{1}{q_1},\tfrac{1}{q_2}\}\big) = (1-\alpha)\log\Big(1+\frac{\beta_\mathrm{R}^{2\mathrm{R},1\mathrm{R}}}{\max\{q_1,q_2\}}\Big) \\ &=(1-\alpha)\log\Big(1+\frac{1-r_1q_2-r_2q_1}{\max\{q_1(1+r_1+r_2),q_2(1+r_1+r_2)\}}\Big), \qquad (2.120) \end{aligned}$$

where we used in the last equation (2.119).

Next, we study the decoding order $2\mathrm{R}, \mathrm{R}1$ where at node 1 the relay message is decoded last and at node 2 the relay message is decoded first. Therefore, for the interference-free second decoding step at node 2 we have

$$R_1 = (1-\alpha) R_{\overrightarrow{\mathrm{R2}}}(\beta_1^{2\mathrm{R},\mathrm{R}1}) \quad \Rightarrow \quad \beta_1^{2\mathrm{R},\mathrm{R}1} = r_1 q_2. \tag{2.121}$$

For the decoding of message m_2 at node 1 we regard the received signal of the relay message as the interference. Thus we have

$$\begin{aligned} R_2 &= (1-\alpha)\log\Big(1 + \frac{\beta_2^{2\mathrm{R},\mathrm{R}1}|h_1|^2\gamma_{\mathrm{R}}}{1+\beta_{\mathrm{R}}^{2\mathrm{R},\mathrm{R}1}|h_1|^2\gamma_{\mathrm{R}}}\Big) = (1-\alpha)\log\Big(1 + \frac{\beta_2^{2\mathrm{R},\mathrm{R}1}}{q_1 + \beta_{\mathrm{R}}^{2\mathrm{R},\mathrm{R}1}}\Big) \\ \Rightarrow r_2 &= \frac{\beta_2^{2\mathrm{R},\mathrm{R}1}}{q_1 + \beta_{\mathrm{R}}^{2\mathrm{R},1\mathrm{R}}} = \frac{1-\beta_1^{2\mathrm{R},\mathrm{R}1} - \beta_{\mathrm{R}}^{2\mathrm{R},\mathrm{R}1}}{q_1 + \beta_{\mathrm{R}}^{2\mathrm{R},1\mathrm{R}}} = \frac{1 - r_1 q_2 - \beta_{\mathrm{R}}^{2\mathrm{R},\mathrm{R}1}}{q_1 + \beta_{\mathrm{R}}^{2\mathrm{R},1\mathrm{R}}} \\ \Rightarrow \beta_{\mathrm{R}}^{2\mathrm{R},\mathrm{R}1} &= \frac{1 - r_1 q_2 - r_2 q_1}{1+r_2}, \end{aligned} \tag{2.122}$$

where we used $\beta_1^{2\mathrm{R},\mathrm{R}1} = 1 - \beta_2^{2\mathrm{R},\mathrm{R}1} - \beta_{\mathrm{R}}^{2\mathrm{R},\mathrm{R}1}$ and (2.121) in (2.122). At node 1 the relay message is decoded in the second interference-free decoding step so that the rate

$$R_{\mathrm{R@1}}^{2\mathrm{R},\mathrm{R}1} := (1-\alpha)\log(1+\beta_{\mathrm{R}}^{2\mathrm{R},\mathrm{R}1}\gamma_{\mathrm{R}}|h_1|^2) = (1-\alpha)\log\Big(1 + \frac{1 - r_1 q_2 - r_2 q_1}{q_1(1+r_2)}\Big)$$

is achievable. In the first decoding step at node 2 the message m_1 is unknown, therefore at node 2 we can achieve the relay rate

$$R_{\mathrm{R@2}}^{2\mathrm{R},\mathrm{R}1} := (1-\alpha)\log\Big(1 + \frac{\beta_{\mathrm{R}}^{\mathrm{R}2,\mathrm{R}1}|h_2|^2\gamma_{\mathrm{R}}}{1+\beta_1^{\mathrm{R}2,\mathrm{R}1}|h_2|^2\gamma_{\mathrm{R}}}\Big) = (1-\alpha)\log\Big(1 + \frac{1 - r_1 q_2 - r_2 q_1}{q_2(1+r1)(1+r_2)}\Big).$$

Since the message m_{R} should be decoded at both nodes, for the decoding order $2\mathrm{R}, \mathrm{R}1$ we have the maximal relay rate

$$\begin{aligned} R_{\mathrm{R}}^{2\mathrm{R},\mathrm{R}1} :=& \min\big\{R_{\mathrm{R@1}}^{2\mathrm{R},\mathrm{R}1}, R_{\mathrm{R@2}}^{2\mathrm{R},\mathrm{R}1}\big\} \\ =&(1-\alpha)\log\Big(1 + \frac{1 - r_1 q_2 - r_2 q_1}{\max\{q_1(1+r_2), q_2(1+r_1)(1+r_2)\}}\Big). \end{aligned} \tag{2.123}$$

Finally, we look at the decoding order $\mathrm{R}2, 1\mathrm{R}$, which is similar to the previous one. Form the bidirectional achievable rates R_1 and R_2 we have

$$\begin{aligned} R_2 &= (1-\alpha) R_{\overrightarrow{\mathrm{R1}}}(\beta_2^{\mathrm{R}2,1\mathrm{R}}) \quad \Rightarrow \quad \beta_2^{\mathrm{R}2,1\mathrm{R}} = r_2 q_1, \\ R_1 &= (1-\alpha)\log\Big(1 + \frac{\beta_1^{\mathrm{R}2,1\mathrm{R}}|h_2|^2\gamma_{\mathrm{R}}}{1+\beta_{\mathrm{R}}^{\mathrm{R}2,1\mathrm{R}}|h_2|^2\gamma_{\mathrm{R}}}\Big) = (1-\alpha)\log\Big(1 + \frac{\beta_1^{\mathrm{R}2,1\mathrm{R}}}{q_2 + \beta_{\mathrm{R}}^{\mathrm{R}2,1\mathrm{R}}}\Big) \\ \Rightarrow r_1 &= \frac{\beta_1^{\mathrm{R}2,1\mathrm{R}}}{q_2 + \beta_{\mathrm{R}}^{\mathrm{R}2,\mathrm{R}1}} = \frac{1-\beta_2^{\mathrm{R}2,1\mathrm{R}} - \beta_{\mathrm{R}}^{\mathrm{R}2,1\mathrm{R}}}{q_2 + \beta_{\mathrm{R}}^{\mathrm{R}2,\mathrm{R}1}} = \frac{1 - r_2 q_1 - \beta_{\mathrm{R}}^{\mathrm{R}2,1\mathrm{R}}}{q_2 + \beta_{\mathrm{R}}^{\mathrm{R}2,\mathrm{R}1}} \\ \Rightarrow \beta_{\mathrm{R}}^{\mathrm{R}2,1\mathrm{R}} &= \frac{1 - r_1 q_2 - r_2 q_1}{1+r_1} \end{aligned}$$

Accordingly, at nodes 1 and 2 we can achieve the relay rates

$$R_{\mathrm{R@1}}^{\mathrm{R2,1R}} := (1-\alpha)\log\Big(1+\frac{\beta_{\mathrm{R}}^{\mathrm{2R,1R}}|h_1|^2\gamma_{\mathrm{R}}}{1+\beta_2^{\mathrm{2R,1R}}|h_1|^2\gamma_{\mathrm{R}}}\Big) = (1-\alpha)\log\Big(1+\frac{1-r_1q_2-r_2q_1}{q_1(1+r_1)(1+r_2)}\Big),$$

$$R_{\mathrm{R@2}}^{\mathrm{R2,1R}} := (1-\alpha)\log(1+\beta_{\mathrm{R}}^{\mathrm{R2,1R}}\gamma_{\mathrm{R}}|h_2|^2) = (1-\alpha)\log\Big(1+\frac{1-r_1q_2-r_2q_1}{q_2(1+r_1)}\Big).$$

This gives us the maximal achievable relay rate for the decoding order $\mathrm{R2,1R}$

$$\begin{aligned} R_{\mathrm{R}}^{\mathrm{R2,1R}} &:= \min\big\{R_{\mathrm{R@1}}^{\mathrm{R2,1R}}, R_{\mathrm{R@2}}^{\mathrm{R2,1R}}\big\} \\ &= (1-\alpha)\log\Big(1+\frac{1-r_1q_2-r_2q_1}{\max\{q_1(1+r_1)(1+r_2), q_2(1+r_1)\}}\Big). \end{aligned} \tag{2.124}$$

Since q_1, q_2, r_1, and r_2 are non-negative we have

$$\begin{aligned} \max\{q_1(1+r_2), q_2(1+r_1)\} \le \min\big\{ &\max\{q_1(1+r_1+r_2), q_2(1+r_1+r_2)\}, \\ &\max\{q_1(1+r_2), q_2(1+r_1)(1+r_2)\}, \\ &\max\{q_1(1+r_1)(1+r_2), q_2(1+r_1)\}\big\}. \end{aligned}$$

These are the differing denominators from (2.115), (2.120), (2.123), and (2.124) so that it directly follows that

$$R_{\mathrm{R}}(R_1,R_2) := \max\big\{R_{\mathrm{R}}^{\mathrm{R2,R1}}, R_{\mathrm{R}}^{\mathrm{2R,1R}}, R_{\mathrm{R}}^{\mathrm{2R,R1}}, R_{\mathrm{R}}^{\mathrm{R2,1R}}\big\} = R_{\mathrm{R}}^{\mathrm{R2,R1}}.$$

This means that the maximal achievable additional relay rate is always achieved with the decoding order where the relay message is decoded first. □

2.8.29 Proof of Lemma 2.46

The graph $\mathcal{G}_{f_{R_\Sigma}}$ of the function $f_{R_\Sigma} : [0, R_\Sigma] \to [0, R_\Sigma]$ with $R_1 \mapsto R_\Sigma - R_1$ denotes the set of non-negative rate pairs with a sum-rate equal to R_Σ. Then the intersection of $\mathcal{G}_{f_{R_\Sigma}} \cap \mathcal{R}_{\mathrm{BR}}(\alpha)$ characterizes the set of rate pairs in $\mathcal{R}_{\mathrm{BR}}(\alpha)$ with the desired sum-rate R_Σ and is non-empty since we assume that a rate pair $[R_1, R_2] \in \mathcal{R}_{\mathrm{BR}}(\alpha)$ with the desired sum-rate $R_\Sigma = R_1 + R_2$ exists. Since $\mathcal{R}_{\mathrm{BR}}(\alpha)$ is convex there exist $R_1^{\triangleright}$ and $R_1^{\triangleleft}$ so that

$$\mathcal{G}_{f_{R_\Sigma}} \cap \mathcal{R}_{\mathrm{BR}}(\alpha) = \big\{[R_1, f_{R_\Sigma}(R_1)] : R_1^{\triangleright} \le R_1 \le R_1^{\triangleleft}\big\}.$$

This means that the interval $[R_1^{\triangleright}, R_1^{\triangleleft}]$ characterizes all bidirectional rate pair in $\mathcal{R}_{\mathrm{BR}}(\alpha)$ with a sum-rate equal to R_Σ. For any $R_1 \in [R_1^{\triangleright}, R_1^{\triangleleft}]$ we can achieve the relay rate $R_{\mathrm{R}}(R_1, R_\Sigma - R_1)$ according to Theorem 2.45. From (2.56) we have

$$\beta_1 = \frac{2^{\frac{R_1}{1-\alpha}}-1}{|h_2|^2\gamma_{\mathrm{R}}} \quad \text{and} \quad \beta_2 = \frac{2^{\frac{R_\Sigma - R_1}{1-\alpha}}-1}{|h_1|^2\gamma_{\mathrm{R}}}$$

and therefore $\beta_{\mathrm{R}} = 1 - \beta_1 - \beta_2 = 1 - \frac{2^{\frac{R_1}{1-\alpha}}-1}{|h_2|^2\gamma_{\mathrm{R}}} - \frac{2^{\frac{R_\Sigma - R_1}{1-\alpha}}-1}{|h_1|^2\gamma_{\mathrm{R}}}$. This allows us to write $R_{\mathrm{R}}(R_1, R_\Sigma - R_1)$ as follows

$$
\begin{aligned}
&R_{\mathrm{R}}(R_1, R_\Sigma - R_1)\\
&\quad= \min\Big\{(1-\alpha)\log\Big(1+\frac{\beta_{\mathrm{R}}|h_1|^2\gamma_{\mathrm{R}}}{1+\beta_2|h_1|^2\gamma_{\mathrm{R}}}\Big),(1-\alpha)\log\Big(1+\frac{\beta_{\mathrm{R}}|h_2|^2\gamma_{\mathrm{R}}}{1+\beta_1|h_2|^2\gamma_{\mathrm{R}}}\Big)\Big\}\\
&\quad= (1-\alpha)\log\Big(1+\big(1-\frac{2^{\frac{R_1}{1-\alpha}}-1}{|h_2|^2\gamma_{\mathrm{R}}}-\frac{2^{\frac{R_\Sigma-R_1}{1-\alpha}}-1}{|h_1|^2\gamma_{\mathrm{R}}}\big)\min\Big\{\frac{|h_1|^2\gamma_{\mathrm{R}}}{2^{\frac{R_\Sigma-R_1}{1-\alpha}}},\frac{|h_2|^2\gamma_{\mathrm{R}}}{2^{\frac{R_1}{1-\alpha}}}\Big\}\Big)\\
&\quad= (1-\alpha)\log\Big(\min\Big\{\underbrace{\frac{|h_1|^2}{2^{\frac{R_\Sigma}{1-\alpha}}}}_{:=\xi}\Big(\underbrace{\big(\gamma_{\mathrm{R}}+\frac{1}{|h_1|^2}+\frac{1}{|h_2|^2}\big)}_{:=\hat{\gamma}}\underbrace{2^{\frac{R_1}{1-\alpha}}}_{:=x}-\underbrace{\frac{\big(2^{\frac{R_1}{1-\alpha}}\big)^2}{|h_2|^2}}_{=x^2/\zeta}\Big),\\
&\qquad\qquad\underbrace{|h_2|^2}_{:=\zeta}\Big(\underbrace{\big(\gamma_{\mathrm{R}}+\frac{1}{|h_1|^2}+\frac{1}{|h_2|^2}\big)}_{=\hat{\gamma}}\underbrace{\frac{1}{2^{\frac{R_1}{1-\alpha}}}}_{=1/x}-\underbrace{\frac{2^{\frac{R_\Sigma}{1-\alpha}}}{|h_1|^2}}_{=1/\xi}\underbrace{\Big(\frac{1}{2^{\frac{R_1}{1-\alpha}}}\Big)^2}_{=1/x^2}\Big)\Big\}\Big)\\
&\quad= (1-\alpha)\log\Big(\min\Big\{\underbrace{\xi\big(\hat{\gamma}x-\frac{x^2}{\zeta}\big)}_{:=p(x)},\underbrace{\zeta\big(\hat{\gamma}\frac{1}{x}-\frac{1}{\xi x^2}\big)}_{:=q(x)}\Big\}\Big)\\
&\quad= (1-\alpha)\log\big(\min\{p(x),q(x)\}\big) =: R_{\mathrm{R}}(x),
\end{aligned}
$$

where the new defined function $R_{\mathrm{R}}(x)$ has the domain $[x_{\min}, x_{\max}]$ with $x_{\min} := 2^{\frac{R_1^{\triangleright}}{1-\alpha}}$ and $x_{\max} := 2^{\frac{R_1^{\triangleleft}}{1-\alpha}}$. The rate $R_{\mathrm{R}}(x)$ is non-negative for any $x \in [x_{\min}, x_{\max}]$, since we assume that a feasible bidirectional rate pair with the desired sum-rate exists. Thus $\min\{p(x), q(x)\} \geq 1$ for any $x \in [x_{\min}, x_{\max}]$.

In Figure 2.23 we illustrate the functions $p(x)$ and $q(x)$ to make the following discussion clear. Since ξ and ζ are non-negative $p(x) = \xi/\zeta(\hat{\gamma}\zeta - x)x$ is a concave parabola with roots $0, \hat{\gamma}\zeta$ and vertex $[\hat{\gamma}\zeta/2, \hat{\gamma}^2\xi\zeta/4]$. Similarly, $q(1/x) = \zeta/\xi(\hat{\gamma}\xi - x)x$ is a concave parabola for the inverse argument with roots $0, \hat{\gamma}\xi$ and vertex $[\hat{\gamma}\xi/2, \hat{\gamma}^2\xi\zeta/4]$. Therefore, $q(x)$ has a negative pole at 0, a root at $1/\hat{\gamma}\xi$, and a vertex at $[2/\hat{\gamma}\xi, \hat{\gamma}^2\xi\zeta/4]$. Furthermore, $q(x)$ is strictly increasing on the interval $(0, 2/\hat{\gamma}\xi)$ and strictly decreasing on the interval $(2/\hat{\gamma}\xi, \infty)$.

Since $p(x), q(x) \leq \hat{\gamma}^2\xi\zeta/4$ for any $x \in (0, \infty)$ and $\min\{p(x), q(1/x)\} \geq 1$ for any $x \in [x_{\min}, x_{\max}] \neq \emptyset$, we have $1 \leq \hat{\gamma}^2\xi\zeta/4$ and therefore $\hat{\gamma}\zeta/2 \geq 2/\hat{\gamma}\xi$. This means that vertex of $q(x)$ is always to the left of the vertex of $p(x)$. From a simple calculation it follows that

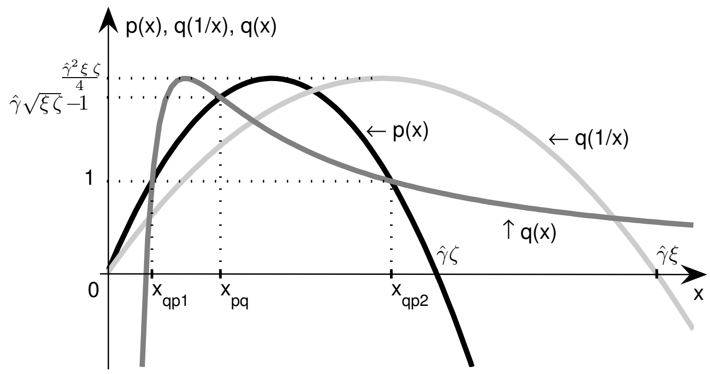


Figure 2.23: Discussion of $p(x)$, $q(1/x)$, $q(x)$ with vertexes $[\frac{\hat{\gamma}\zeta}{2}, \frac{\hat{\gamma}^2\xi\zeta}{4}]$, $[\frac{\hat{\gamma}\xi}{2}, \frac{\hat{\gamma}^2\xi\zeta}{4}]$, $[\frac{2}{\xi\hat{\gamma}}, \frac{\hat{\gamma}^2\xi\zeta}{4}]$ respectively (not explicitly depicted).

the graphs of $p(x)$ and $q(x)$ intersect at the following intersection points

$$\begin{aligned}
[x_{qp1}, p(x_{qp1})] &= [\hat{\gamma}\zeta/2 - \sqrt{(\hat{\gamma}\zeta/2)^2 - \zeta/\xi}, 1], \\
[x_{qp2}, p(x_{qp2})] &= [\hat{\gamma}\zeta/2 + \sqrt{(\hat{\gamma}\zeta/2)^2 - \zeta/\xi}, 1], \\
[x_{pq}, p(x_{pq})] &= [\sqrt{\zeta/\xi}, \hat{\gamma}\sqrt{\xi\zeta} - 1].
\end{aligned}$$

Obviously, the first two points coincide if we have $(\hat{\gamma}\zeta/2)^2 = \zeta/\xi$ which is equivalent to $\hat{\gamma}\zeta/2 = 2/\hat{\gamma}\xi$, i.e. the vertexes of $p(x)$ and $q(x)$ coincide as well. From the intersection points and the vertexes it follows that $p(x), q(x) \geq 1$ for $x \in [x_{qp1}, x_{qp2}]$ only and therefore we have $[x_{\min}, x_{\max}] \subseteq [x_{qp1}, x_{qp2}]$ because rates are always non-negative. Finally it follows that for $x \in [x_{\min}, x_{\max}]$ we have

$$\min\{p(x), q(x)\} = \begin{cases} q(x), & \text{if } x \leq x_{pq}, \\ p(x), & \text{if } x > x_{pq}. \end{cases}$$

Moreover, $\min\{p(x), q(x)\}$ is increasing for $x \in [x_{qp1}, x_{pq}]$ and decreasing for $x \in [x_{pq}, x_{qp2}]$.

Therefore, if $x_{pg} \in [x_{\min}, x_{\max}] \Leftrightarrow R_1^{\triangleright} \leq R_1^{\star} \leq R_1^{\triangleleft}$ the maximal additional relay rate $R_{\mathrm{R}}(x)$ is attained at $x_{pg} \Leftrightarrow$ at the rate pair $[R_1^{\star}, R_{\Sigma} - R_1^{\star}]$ with

$$\begin{aligned}
R_{\mathrm{R}}(R_{\Sigma}) = R_{\mathrm{R}}(x_{pg}) &= (1-\alpha)\log\big(p(x_{pg})\big) = (1-\alpha)\log\big(\hat{\gamma}\sqrt{\xi\zeta} - 1\big) \\
&= (1-\alpha)\log\Big(|h_1 h_2|\hat{\gamma} - 2^{\frac{R_{\Sigma}}{2(1-\alpha)}}\Big) - \tfrac{1}{2}R_{\Sigma},
\end{aligned}$$

otherwise if $x_{pg} < x_{\min} \Leftrightarrow R_1^\star < R_1^\triangleright$ the maximal additional relay rate $R_{\mathrm{R}}(x)$ is attained at $x_{\min} \Leftrightarrow$ at the rate pair $[R_1^\triangleleft, R_\Sigma - R_1^\triangleleft]$ with

$$R_{\mathrm{R}}(R_\Sigma) = R_{\mathrm{R}}(x_{\min}) = (1-\alpha)\log\big(q(x_{\min})\big) = (1-\alpha)\log\Big(\zeta\big(\hat{\gamma}\frac{1}{x_{\min}} - \frac{1}{\xi x_{\min}^2}\big)\Big)$$
$$= (1-\alpha)\log\Big(|h_2|^2(\hat{\gamma} - \tfrac{1}{|h_1|^2}2^{\frac{R_\Sigma - R_1^\triangleright}{1-\alpha}})\Big) - R_1^\triangleright,$$

or if $x_{pg} > x_{\max} \Leftrightarrow R_1^\star > R_1^\triangleright$ the maximal additional relay rate $R_{\mathrm{R}}(x)$ is attained at $x_{\max}$ $\Leftrightarrow$ at the rate pair $[R_1^\triangleright, R_\Sigma - R_1^\triangleright]$ with

$$R_{\mathrm{R}}(R_\Sigma) = R_{\mathrm{R}}(x_{\max}) = (1-\alpha)\log\big(p(x_{\max})\big)(1-\alpha)\log\Big(\xi\big(\hat{\gamma}x_{\max} - \frac{x_{\max}^2}{\zeta}\big)\Big)$$
$$= (1-\alpha)\log\Big(|h_1|^2(\hat{\gamma} - \tfrac{1}{|h_2|^2}2^{\frac{R_1^\triangleleft}{1-\alpha}})\Big) - R_\Sigma + R_1^\triangleleft.$$

This finishes the proof. □

2.8.30 Proof of Proposition 2.47

For a given time division parameter α we have the broadcast rate region $(1-\alpha)\mathcal{R}_{\mathrm{BC}}\big(\frac{R_\Sigma}{1-\alpha}\big) = (1-\alpha)\mathcal{R}_{\mathrm{BC}}\big(\gamma_{\mathrm{R}}^{\mathrm{BC}}(\frac{R_\Sigma}{1-\alpha})\big)$ with the sum-rate maximum $(1-\alpha)\frac{R_\Sigma}{1-\alpha} = R_\Sigma$. It follows that for the broadcast region $(1-\alpha)\mathcal{R}_{\mathrm{BC}}(\gamma_{\mathrm{R}})$ with $\gamma_{\mathrm{R}} = \gamma_{\mathrm{R}}^{\mathrm{BC}}\big(\frac{R_\Sigma}{1-\alpha}\big)$ the sum-rate maximum is attained at the rate pair $(1-\alpha)\big[R_{\overrightarrow{\mathrm{R2}}}^\star(\gamma_{\mathrm{R}}), R_{\overrightarrow{\mathrm{R1}}}^\star(\gamma_{\mathrm{R}})\big]$ according to Proposition 2.2 with a maximum sum-rate R_Σ. Therefore, we have to distinguish for $\beta^\star$ between three cases with respect to γ_{R} respectively R_Σ.

If $\beta^\star \in [0,1]$, we have

$$\begin{aligned}(1-\alpha)R_{\overrightarrow{\mathrm{R1}}}^\star(\gamma_{\mathrm{R}}) &= (1-\alpha)R_{\overrightarrow{\mathrm{R1}}}(1-\beta^\star)\\ &= (1-\alpha)\log\big(\tfrac{1}{2}(1+\gamma_{\mathrm{R}}|h_1|^2 + \tfrac{|h_1|^2}{|h_2|^2})\big)\\ &= (1-\alpha)\log\big(\tfrac{1}{2}(\tfrac{|h_2|^2}{|h_1|^2} + \gamma_{\mathrm{R}}|h_2|^2 + 1)\tfrac{|h_1|^2}{|h_2|^2}\big)\\ &= (1-\alpha)\log\big(\tfrac{1}{2}(\tfrac{|h_2|^2}{|h_1|^2} + \gamma_{\mathrm{R}}|h_2|^2 + 1)\big) + (1-\alpha)\log\big(\tfrac{|h_1|^2}{|h_2|^2}\big)\\ &= (1-\alpha)R_{\overrightarrow{\mathrm{R2}}}(\beta^\star) + (1-\alpha)R^\dagger\\ &= (1-\alpha)R_{\overrightarrow{\mathrm{R2}}}^\star(\gamma_{\mathrm{R}}) + (1-\alpha)R^\dagger.\end{aligned}$$

From the definitions of $R_1^\star(R_\Sigma)$ and $R_2^\star(R_\Sigma)$ it follows that

$$\begin{aligned}R_1^\star(R_\Sigma) + R_2^\star(R_\Sigma) &= R_\Sigma = (1-\alpha)R_{\overrightarrow{\mathrm{R2}}}(\beta^\star) + (1-\alpha)R_{\overrightarrow{\mathrm{R1}}}(\beta^\star)\\ R_1^\star(R_\Sigma) - R_2^\star(R_\Sigma) &= -(1-\alpha)R^\dagger = (1-\alpha)R_{\overrightarrow{\mathrm{R2}}}(\beta^\star) - (1-\alpha)R_{\overrightarrow{\mathrm{R1}}}(\beta^\star).\end{aligned}$$

If we solve the equation system for $R_1^\star(R_\Sigma)$ and $R_2^\star(R_\Sigma)$ we have $R_1^\star(R_\Sigma) = (1-\alpha)R_{\overrightarrow{\mathrm{R2}}}(\beta^\star)$ and $R_2^\star(R_\Sigma) = (1-\alpha)R_{\overrightarrow{\mathrm{R1}}}(1-\beta^\star)$, which means that $[R_1^\star(R_\Sigma), R_2^\star(R_\Sigma)]$ characterizes the sum-rate optimal rate pair of $(1-\alpha)\mathcal{R}_{\mathrm{BC}}\big(\frac{R_\Sigma}{1-\alpha}\big)$.

For the characterization of the case $\beta^\star \in [0,1]$ we consider the now simple equivalences

$$\begin{aligned} \beta^\star \geq 0 \quad &\Leftrightarrow \quad R_{\overrightarrow{\mathrm{R2}}}(\beta^\star) \geq 0 \quad &&\Leftrightarrow \quad R_\Sigma \geq (1-\alpha)R^\dagger, \\ \beta^\star \leq 1 \quad &\Leftrightarrow \quad R_{\overrightarrow{\mathrm{R1}}}(1-\beta^\star) \geq 0 \quad &&\Leftrightarrow \quad R_\Sigma \geq -(1-\alpha)R^\dagger. \end{aligned}$$

It follows that the case $\beta^\star \in [0,1]$ is equivalent to $R_\Sigma \geq (1-\alpha)|R^\dagger|$.

Finally, it follows that if $R_\Sigma < (1-\alpha)R^\dagger$ we have $\beta^\star < 0$ so that the sum-rate optimal rate pair is on the R_2-axis with $[0, R_\Sigma]$ and if $R_\Sigma < -(1-\alpha)R^\dagger$ we have $\beta^\star > 1$ so that the sum-rate optimal rate pair is on the R_1-axis with $[0, R_\Sigma]$. $\square$

2.8.31 Proof of Theorem 2.48

We prove the theorem for the case $|h_1|^2 \geq |h_2|^2$. The case $|h_2|^2 \geq |h_1|^2$ follows accordingly by interchanging the indices one and two. To identify the bidirectional rate pair where the total sum-rate maximum is attained we consider for any feasible bidirectional sum-rate $R_\Sigma \in [0, R_\Sigma^{\max}]$ the corresponding total sum-rate $R_{\mathrm{tot}}(R_\Sigma)$. Thereby, let $\gamma_{\mathrm{B}} := \gamma_{\mathrm{R}}^{\mathrm{BC}}(R_\Sigma) \leq \gamma_{\mathrm{R}}$ denote the necessary relay signal-to-noise ratio for a bidirectional sum-rate $R_\Sigma \leq R_\Sigma^{\max}$. Effectively, this means that we have a bidirectional relay communication with reduced relay power and a relay power distribution parameter $\beta = \frac{\gamma_{\mathrm{R}}}{\gamma_{\mathrm{B}}}\beta_1 = 1 - \frac{\gamma_{\mathrm{R}}}{\gamma_{\mathrm{B}}}\beta_2$. Accordingly, we can define $\gamma_{\mathrm{P}} := \gamma_{\mathrm{R}} - \gamma_{\mathrm{P}} \geq 0$ which denotes the relay signal-to-noise ratio of the piggyback communication. In the following we will discuss the different cases of $R_{\mathrm{tot}}(R_\Sigma)$.

At low sum-rates R_Σ, where we have $R_\Sigma < \min\{\alpha R_{\overrightarrow{\mathrm{2R}}}, (1-\alpha)R^\dagger\}$, we have $R_1^\star(R_\Sigma) < 0$ so that the bidirectional sum-rate maximum is attained at $[0, R_\Sigma] = [0, (1-\alpha)\log(1+\gamma_{\mathrm{B}}|h_1|^2)]$ with $\gamma_{\mathrm{B}} = \gamma_{\mathrm{R}}^{\mathrm{BC}}(R_\Sigma)$. For those bidirectional sum-rates the total sum-rate is given by (2.64b). Since the argument of the logarithm is a concave parabola in $2^{\frac{R_\Sigma}{1-\alpha}}$, which is maximized at its vertex, the function $R_{\mathrm{tot}}(R_\Sigma)$ is maximized at $(1-\alpha)\log\big(\frac{1}{2}|h_1|^2\hat{\gamma}\big)$. From $R_1^\star(R_\Sigma) < 0$ it follows that $\beta^\star = \frac{1}{2} - \frac{1}{2\gamma_{\mathrm{B}}}\big(\frac{1}{|h_2|^2} - \frac{1}{|h_1|^2}\big) < 0 \Leftrightarrow \gamma_{\mathrm{B}} + \frac{1}{|h_1|^2} < \frac{1}{|h_2|^2}$. Therefore, we have

$$\begin{aligned} (1-\alpha)\log\big(\tfrac{1}{2}|h_1|^2\hat{\gamma}\big) &= (1-\alpha)\log\big(\tfrac{|h_1|^2}{2}\big(\gamma_{\mathrm{R}} + \tfrac{1}{|h_1|^2} + \tfrac{1}{|h_2|^2}\big)\big) \\ &> (1-\alpha)\log\big(\tfrac{|h_1|^2}{2}\big(\gamma_{\mathrm{B}} + \tfrac{1}{|h_1|^2} + \tfrac{1}{|h_2|^2}\big)\big) \\ &> (1-\alpha)\log\big(1 + \gamma_{\mathrm{B}}|h_1|^2\big) = R_\Sigma. \end{aligned}$$

Since the vertex of the parabola is attained at a rate larger than R_Σ it follows that $R_{\text{tot}}(R_\Sigma)$ is strictly increasing on $[0, R_\Sigma]$. This implies that for this section the largest total sum-rate is attained at the largest possible bidirectional sum-rate R_Σ.

At higher sum-rates R_Σ, where we have $R_\Sigma \geq R^\dagger$, we possibly have $[R_1^\star(R_\Sigma), R_2^\star(R_\Sigma)] \in \mathcal{R}_{\text{BR}}(\alpha)$ so that (2.64a) applies for $R_{\text{tot}}(R_\Sigma)$. Now, the argument of the logarithm is a concave parabola in $2^{\frac{R_\Sigma}{2(1-\alpha)}}$ which maximizes at $R_\Sigma^\star := 2(1-\alpha)\log\left(\frac{1}{2}|h_1h_2|\hat{\gamma}\right)$. It follows that the function $R_{\text{tot}}(R_\Sigma)$ is increasing on the interval $[|R^\dagger|, R_\Sigma^\star]$. For the vertex at $R_\Sigma^\star$ we have

$$R_{\text{tot}}(R_\Sigma^\star) = R_\Sigma^\star + R_{\text{R}}^{\max}(R_\Sigma^\star) = (1-\alpha)(\log\left[|h_1h_2|\hat{\gamma} - \tfrac{1}{2}|h_1h_2|\hat{\gamma}\right] + \log[\tfrac{1}{2}|h_1h_2|\hat{\gamma}]) = R_\Sigma^\star,$$

so that we obviously have $R_{\text{R}}^{\max}(R_\Sigma^\star) = 0$. If we have $[R_1^\star(R_\Sigma^\star), R_2^\star(R_\Sigma^\star)] \in \mathcal{R}_{\text{BR}}(\alpha)$ it follows that the largest total sum-rate is attained at $R_\Sigma^\star$, which is also the largest feasible bidirectional sum-rate at all. Otherwise, if we have $[R_1^\star(R_\Sigma^\star), R_2^\star(R_\Sigma^\star)] \notin \mathcal{R}_{\text{BR}}(\alpha)$, the largest total sum-rate $R_{\text{tot}}(R_\Sigma)$ for this section is attained at the largest bidirectional sum-rate R_Σ where we have $[R_1^\star(R_\Sigma), R_2^\star(R_\Sigma)] \in \mathcal{R}_{\text{BR}}(\alpha)$. Again it follows that the largest feasible bidirectional sum-rate for this section maximizes $R_{\text{tot}}(R_\Sigma)$.

If we further increase the sum-rate R_Σ we have $\boldsymbol{R}_{\text{BC}}^\star(R_\Sigma) \notin \mathcal{R}_{\text{BR}}(\alpha)$. This means that the total sum-rate $R_{\text{tot}}(R_\Sigma)$ is given by (2.64c) if $R_1^\star(R_\Sigma) < R_1^\triangleright = R_\Sigma - \alpha R_{\overrightarrow{2\text{R}}}$ or (2.64d) if $R_1^\star(R_\Sigma) > R_1^\triangleleft = R_\Sigma - \alpha R_{\overrightarrow{1\text{R}}}$. In both cases the total sum-rate is independent of R_Σ and therefore constant.

Since the section-wise defined function $R_{\text{tot}}(R_\Sigma)$ is continuous and increasing for rate pairs $\boldsymbol{R}_{\text{BC}}(R_\Sigma) \in \mathcal{R}_{\text{BR}}(\alpha)$ and constant for rate pairs $\boldsymbol{R}_{\text{BC}}(R_\Sigma) \notin \mathcal{R}_{\text{BR}}(\alpha)$ we achieve the maximum total sum-rate R_{tot}^* at the largest sum-rate R_Σ where we have $\boldsymbol{R}_{\text{BC}}^\star(R_\Sigma) \in \mathcal{R}_{\text{BR}}(\alpha)$, which is given by the intersection point of $\boldsymbol{R}_{\text{BC}}(R_\Sigma)$ with the boundary of $\mathcal{R}_{\text{BR}}(\alpha)$. □

2.8.32 Proof of Corollary 2.49

In the following we prove the interchange of R_{R} with R_1 only. The interchange with R_2 follows accordingly.

If we can prove the equality $R_{\text{R}}(R_1^\star + R_{①}, R_2^\star) = R_{\text{R}}^{\max}(R_1^\star + R_2^\star) - R_{①}$ for non-negative $R_{①}$, it would follow from

$$\begin{aligned} R_{\text{tot}}(R_1^\star + R_2^\star) &= R_1^\star + R_2^\star + R_{\text{R}}^{\max}(R_1^\star + R_2^\star) \\ &= R_1^\star + R_{①} + R_2^\star + R_{\text{R}}^{\max}(R_1^\star + R_{①}, R_2^\star) \end{aligned} \tag{2.125}$$

that the total sum-rate remains constant. From Lemma 2.46 we know that $R_{\text{R}}^{\max}(R_1^\star + R_2^\star) = R_{\text{R}}(R_1^\star, R_2^\star)$ with $R_1^\star = \frac{1}{2}R_\Sigma - \frac{1-\alpha}{2}R^\dagger$, $R_2^\star = \frac{1}{2}R_\Sigma + \frac{1-\alpha}{2}R^\dagger$, and $R^\dagger = \log\left(\frac{|h_1|^2}{|h_2|^2}\right)$.

Therewith, it follows that

$$
\begin{aligned}
2^{\frac{1}{1-\alpha}R_{\mathrm{R@1}}(R_1^\star+R_{①},R_2^\star)}-1 &= \frac{\beta_{\mathrm{R}}|h_1|^2\gamma_{\mathrm{R}}}{1+\beta_2|h_1|^2\gamma_{\mathrm{R}}} = \frac{\Big(1-\frac{2^{\frac{R_1^\star+R_{①}}{1-\alpha}}-1}{|h_2|^2\gamma_{\mathrm{R}}}-\frac{2^{\frac{R_2^\star}{1-\alpha}}-1}{|h_1|^2\gamma_{\mathrm{R}}}\Big)|h_1|^2\gamma_{\mathrm{R}}}{1+\frac{2^{\frac{R_2^\star}{1-\alpha}}-1}{|h_1|^2\gamma_{\mathrm{R}}}|h_1|^2\gamma_{\mathrm{R}}} \\
&= \frac{|h_1|^2\gamma_{\mathrm{R}}-\frac{|h_1|^2}{|h_2|^2}\big(2^{\frac{1}{2(1-\alpha)}R_\Sigma-\frac{1}{2}R^\dagger+\frac{1}{1-\alpha}R_{①}}-1\big)-\big(2^{\frac{1}{2(1-\alpha)}R_\Sigma+\frac{1}{2}R^\dagger}-1\big)}{2^{\frac{1}{2(1-\alpha)}R_\Sigma+\frac{1}{2}R^\dagger}}\cdot\frac{2^{-R^\dagger}}{2^{-R^\dagger}} \\
&= \frac{|h_2|^2\gamma_{\mathrm{R}}-\big(2^{\frac{1}{2(1-\alpha)}R_\Sigma-\frac{1}{2}R^\dagger+\frac{1}{1-\alpha}R_{①}}-1\big)-\frac{|h_2|^2}{|h_1|^2}\big(2^{\frac{1}{2(1-\alpha)}R_\Sigma+\frac{1}{2}R^\dagger}-1\big)}{2^{\frac{1}{2(1-\alpha)}R_\Sigma-\frac{1}{2}R^\dagger}}\cdot\frac{2^{\frac{1}{1-\alpha}R_{①}}}{2^{\frac{1}{1-\alpha}R_{①}}} \\
&= 2^{\frac{1}{1-\alpha}R_{①}}\frac{\Big(1-\frac{2^{\frac{R_1^\star+R_{①}}{1-\alpha}}-1}{|h_2|^2\gamma_{\mathrm{R}}}-\frac{2^{\frac{R_2^\star}{1-\alpha}}-1}{|h_1|^2\gamma_{\mathrm{R}}}\Big)|h_2|^2\gamma_{\mathrm{R}}}{1+\frac{2^{\frac{R_1^\star+R_{①}}{1-\alpha}}-1}{|h_2|^2\gamma_{\mathrm{R}}}|h_2|^2\gamma_{\mathrm{R}}} = 2^{\frac{1}{1-\alpha}R_{①}}\frac{\beta_{\mathrm{R}}|h_2|^2\gamma_{\mathrm{R}}}{1+\beta_1|h_2|^2\gamma_{\mathrm{R}}} \\
&= 2^{\frac{1}{1-\alpha}R_{①}}\big(2^{\frac{1}{1-\alpha}R_{\mathrm{R@2}}(R_1^\star+R_{①},R_2^\star)}-1\big),
\end{aligned}
\tag{2.126}
$$

where we used $\beta_{\mathrm{R}}=1-\beta_1-\beta_2$ and (2.56). Similarly, we can follow

$$
\begin{aligned}
2^{\frac{1}{1-\alpha}R_{\mathrm{R@2}}(R_1^\star+R_{①},R_2^\star)} &= 1+\frac{\Big(1-\frac{2^{\frac{R_1^\star+R_{①}}{1-\alpha}}-1}{|h_2|^2\gamma_{\mathrm{R}}}-\frac{2^{\frac{R_2^\star}{1-\alpha}}-1}{|h_2|^2\gamma_{\mathrm{R}}}\Big)|h_2|^2\gamma_{\mathrm{R}}}{1+\frac{2^{\frac{R_1^\star+R_{①}}{1-\alpha}}-1}{|h_2|^2\gamma_{\mathrm{R}}}|h_2|^2\gamma_{\mathrm{R}}} \\
&= \frac{1+\Big(1-\frac{2^{\frac{R_2^\star}{1-\alpha}}-1}{|h_2|^2\gamma_{\mathrm{R}}}\Big)|h_2|^2\gamma_{\mathrm{R}}}{2^{\frac{R_1^\star+R_{①}}{1-\alpha}}} = 2^{-\frac{R_{①}}{1-\alpha}}\cdot\frac{1+\Big(1-\frac{2^{\frac{R_2^\star}{1-\alpha}}-1}{|h_2|^2\gamma_{\mathrm{R}}}\Big)|h_2|^2\gamma_{\mathrm{R}}}{1+\frac{2^{\frac{R_1^\star}{1-\alpha}}-1}{|h_2|^2\gamma_{\mathrm{R}}}|h_2|^2\gamma_{\mathrm{R}}} \\
&= 2^{-\frac{R_{①}}{1-\alpha}}\cdot\left(1+\frac{\Big(1-\frac{2^{\frac{R_1^\star}{1-\alpha}}-1}{|h_2|^2\gamma_{\mathrm{R}}}-\frac{2^{\frac{R_2^\star}{1-\alpha}}-1}{|h_2|^2\gamma_{\mathrm{R}}}\Big)|h_2|^2\gamma_{\mathrm{R}}}{1+\frac{2^{\frac{R_1^\star}{1-\alpha}}-1}{|h_2|^2\gamma_{\mathrm{R}}}|h_2|^2\gamma_{\mathrm{R}}}\right) \\
&= 2^{-\frac{R_{①}}{1-\alpha}}2^{\frac{1}{1-\alpha}R_{\mathrm{R@2}}(R_1^\star,R_2^\star)}.
\end{aligned}
\tag{2.127}
$$

Then for $R_{①}=0$ it follows from (2.126) that

$$
R_{\mathrm{R@1}}(R_1^\star,R_2^\star)=R_{\mathrm{R@2}}(R_1^\star,R_2^\star)=R_{\mathrm{R}}(R_1^\star,R_2^\star). \tag{2.128}
$$

Furthermore, for $R_{①}\geq 0$ we have $R_{\mathrm{R@1}}(R_1^\star+R_{①},R_2^\star)\geq R_{\mathrm{R@2}}(R_1^\star+R_{①},R_2^\star)$ so that we can conclude

$$\begin{aligned}
R_{\mathrm{R}}(R_1^\star + R_{①}, R_2^\star) &= \min\{R_{\mathrm{R}@1}(R_1^\star + R_{①}, R_2^\star), R_{\mathrm{R}@2}(R_1^\star + R_{①}, R_2^\star)\} \\
&= R_{\mathrm{R}@2}(R_1^\star + R_{①}, R_2^\star) \\
&\overset{(2.127)}{=} R_{\mathrm{R}@2}(R_1^\star, R_2^\star) - R_{①} \\
&\overset{(2.128)}{=} R_{\mathrm{R}}(R_1^\star, R_2^\star) - R_{①}.
\end{aligned}$$

This means that (2.125) holds and therefore the total sum-rate remains constant for $R_{①} \geq 0$. Similar arguments apply for $R_{②} \geq 0$. $\square$

2.8.33 Proof of Proposition 2.51

For $\phi_{\mathrm{BC}} > \frac{\pi}{2}$ we have $R_{\overrightarrow{\mathrm{R2}}}(\beta^\star) < 0$ as well as $\beta^\star < 0$. This is equivalent to $\gamma_{\mathrm{R}} < \frac{1}{|h_2|^2} - \frac{1}{|h_1|^2}$ from which it follows that $R^\star_{\overrightarrow{\mathrm{R1}}} = R_{\overrightarrow{\mathrm{R1}}}(1) = \log(1 + \gamma_{\mathrm{R}}|h_1|^2) < \log(1 + (\frac{1}{|h_2|^2} - \frac{1}{|h_1|^2})|h_1|^2) = R^\dagger$. Therefore, we have $\alpha_1 = \frac{1}{R_{\overrightarrow{\mathrm{2R}}}/R^\dagger + 1} > \frac{1}{R_{\overrightarrow{\mathrm{2R}}}/R^\star_{\overrightarrow{\mathrm{R1}}} + 1} = \alpha_0$ so that we have $\mathcal{A}_1 = [0, \alpha_0)$ and $\mathcal{A}_2 = \mathcal{A}_3 = \mathcal{A}_4 = \emptyset$.

For $\frac{\pi}{2} \geq \phi_{\mathrm{BC}}$ we have $\beta^\star \geq 0 \Rightarrow \gamma_{\mathrm{R}} \geq \frac{1}{|h_2|^2} - \frac{1}{|h_1|^2} \Rightarrow R_{\overrightarrow{\mathrm{R1}}}(1) \geq R^\dagger \Rightarrow \alpha_0 \geq \alpha_1$ so that we additionally have $\mathcal{A}_1 = [0, \alpha_1)$.

Further, for $\frac{\pi}{2} \geq \phi_{\mathrm{BC}} > \phi_{2\Sigma}$ we have $\frac{R_{\overrightarrow{\mathrm{2R}}}}{R_\Sigma^{\mathrm{MAC}} - R_{\overrightarrow{\mathrm{2R}}}} < \frac{R^\star_{\overrightarrow{\mathrm{R1}}}}{R^\star_{\overrightarrow{\mathrm{R2}}}} = \frac{R^\star_{\overrightarrow{\mathrm{R1}}}}{R^\star_{\overrightarrow{\mathrm{R1}}} - R^\dagger}$ where we used $R^\star_{\overrightarrow{\mathrm{R2}}} = R_{\overrightarrow{\mathrm{R2}}}(\beta^\star) = R_{\overrightarrow{\mathrm{R1}}}(\beta^\star) - R^\dagger = R^\star_{\overrightarrow{\mathrm{R1}}} - R^\dagger$, which is equivalent to $2R_{\overrightarrow{\mathrm{2R}}}R^\star_{\overrightarrow{\mathrm{R1}}} - R^\star_{\overrightarrow{\mathrm{R1}}}R_\Sigma^{\mathrm{MAC}} < R^\star_{\overrightarrow{\mathrm{2R}}}R^\dagger$. If we add on both sides $R_{\overrightarrow{\mathrm{R1}}}R^\dagger$ and divide both sides by $(R^\star_{\overrightarrow{\mathrm{R1}}} + R_{\overrightarrow{\mathrm{2R}}})(2R_{\overrightarrow{\mathrm{2R}}} - R_\Sigma^{\mathrm{MAC}} + R^\dagger)$ we get $\alpha_0 = \frac{R^\star_{\overrightarrow{\mathrm{R1}}}}{R^\star_{\overrightarrow{\mathrm{R1}}} + R_{\overrightarrow{\mathrm{2R}}}} < \frac{R^\dagger}{2R_{\overrightarrow{\mathrm{2R}}} - R_\Sigma^{\mathrm{MAC}} + R^\dagger} = \alpha_2$ so that we have $\mathcal{A}_2 = [\alpha_1, \alpha_0)$ and $\mathcal{A}_3 = \mathcal{A}_4 = \emptyset$.

On the other hand, if we have $\phi_{2\Sigma} \geq \phi_{\mathrm{BC}}$ we have $\alpha_0 \geq \alpha_2$ using the same arguments with the opposite relations. It follows that $\mathcal{A}_2 = [\alpha_1, \alpha_2)$.

Then, for $\phi_{2\Sigma} \geq \phi_{\mathrm{BC}} > \phi_{\Sigma 1}$ we have $\frac{R_\Sigma^{\mathrm{MAC}} - R_{\overrightarrow{\mathrm{1R}}}}{R_{\overrightarrow{\mathrm{1R}}}} < \frac{R^\star_{\overrightarrow{\mathrm{R1}}}}{R^\star_{\overrightarrow{\mathrm{R2}}}}$ from which we get $2(R^\star_{\overrightarrow{\mathrm{R1}}} + R^\star_{\overrightarrow{\mathrm{R2}}})R_{\overrightarrow{\mathrm{1R}}} > 2R^\star_{\overrightarrow{\mathrm{R2}}}R_\Sigma^{\mathrm{MAC}} = (R^\star_{\overrightarrow{\mathrm{R1}}} + R^\star_{\overrightarrow{\mathrm{R2}}} - R^\dagger)R_\Sigma^{\mathrm{MAC}} \Leftrightarrow (R^\star_{\overrightarrow{\mathrm{R1}}} + R^\star_{\overrightarrow{\mathrm{R2}}})(R_\Sigma^{\mathrm{MAC}} - 2R_{\overrightarrow{\mathrm{1R}}} + R^\dagger) < R^\dagger(R^\star_{\overrightarrow{\mathrm{R1}}} + R^\star_{\overrightarrow{\mathrm{R2}}} + R_\Sigma^{\mathrm{MAC}})$. If we divide both side with $(R^\star_{\overrightarrow{\mathrm{R1}}} + R^\star_{\overrightarrow{\mathrm{R2}}} + R_\Sigma^{\mathrm{MAC}})(R_\Sigma^{\mathrm{MAC}} - 2R_{\overrightarrow{\mathrm{1R}}} + R^\dagger)$ we get $\alpha_0 = \frac{R^\star_{\overrightarrow{\mathrm{R1}}}}{R^\star_{\overrightarrow{\mathrm{R1}}} + R_{\overrightarrow{\mathrm{2R}}}} < \frac{R^\dagger}{R_\Sigma^{\mathrm{MAC}} - 2R_{\overrightarrow{\mathrm{1R}}} + R^\dagger} = \alpha_3$ so that we have $\mathcal{A}_3 = [\alpha_2, \alpha_0)$ and $\mathcal{A}_4 = \emptyset$.

Finally, if we have $\phi_{\mathrm{BC}} \geq \phi_{2\Sigma}$ we have $\alpha_0 \geq \alpha_3$ so that $\mathcal{A}_3 = [\alpha_2, \alpha_3)$ and $\mathcal{A}_4 = [\alpha_3, \alpha_0)$.

Then the total sum-rate optimal rate pairs follow immediately. If $\alpha \in \mathcal{A}_2$ we have to solve $R_2^\star(R_\Sigma) = \frac{1}{2}R_\Sigma + \frac{1-\alpha}{2}R^\dagger = \alpha R_{\overrightarrow{2\mathrm{R}}}$ for R_Σ and plug in the solution $R_\Sigma = 2\alpha R_{\overrightarrow{2\mathrm{R}}} - (1-\alpha)R^\dagger$ in $R_1^\star(R_\Sigma)$, which gives us $R_1 = \alpha R_{\overrightarrow{2\mathrm{R}}} - (1-\alpha)R^\dagger$. Similarly, if $\alpha \in \mathcal{A}_4$ we have to solve $R_1^\star(R_\Sigma) = \alpha R_{\overrightarrow{1\mathrm{R}}}$ for R_Σ and plug in the solution in $R_2^\star(R_\Sigma)$, which gives us $R_2 = \alpha R_{\overrightarrow{1\mathrm{R}}} + (1-\alpha)R^\dagger$. Finally, for $\alpha \in \mathcal{A}_3$ we require $R_1^\star(R_\Sigma) + R_2^\star(R_\Sigma) = \alpha R_\Sigma^{\mathrm{MAC}}$ so that we get $R_\Sigma = \alpha R_\Sigma^{\mathrm{MAC}}$. Then $\boldsymbol{R}_{\mathrm{BC}}^\star(\alpha R_\Sigma^{\mathrm{MAC}}, \alpha)$ specifies the optimal rate pair.

With the optimal bidirectional rate pairs $[R_1^{\mathrm{opt}}(\alpha), R_2^{\mathrm{opt}}(\alpha)]$ we get the maximal total sum-rate $R_{\mathrm{tot}}^*(\alpha) = R_1^{\mathrm{opt}}(\alpha) + R_2^{\mathrm{opt}}(\alpha) + R_{\mathrm{R}}^{\max}(R_1^{\mathrm{opt}}(\alpha) + R_2^{\mathrm{opt}}(\alpha))$ using Lemma 2.46. In more detail, for time division parameters $\alpha\mathcal{A}_1$ we have $R_1^\star = R_{\overrightarrow{\mathrm{R}2}}(\beta^\star) < R_1^\triangleright = 0$ so that the maximum relay rate is given by (2.60b). Accordingly, we get the total sum-rate

$$\begin{aligned}
R_{\mathrm{tot}}(\alpha) &= \alpha R_{\overrightarrow{2\mathrm{R}}} + (1-\alpha)\log\Big(|h_2|^2(\hat{\gamma} - \tfrac{1}{|h_1|^2}2^{\frac{\alpha}{1-\alpha}R_{\overrightarrow{2\mathrm{R}}}})\Big)\\
&= (1-\alpha)\log\Big(\tfrac{|h_2|^2}{|h_1|^2}2^{\frac{\alpha}{1-\alpha}R_{\overrightarrow{2\mathrm{R}}}}\big(\hat{\gamma}|h_1|^2 - 2^{\frac{\alpha}{1-\alpha}R_{\overrightarrow{2\mathrm{R}}}}\big)\Big) \quad \text{for } \alpha \in \mathcal{A}_1.
\end{aligned}$$

If $\alpha \in \mathcal{A}_2 \cup \mathcal{A}_3 \cup \mathcal{A}_4$ we have $R_1^\star = R_1^{\mathrm{opt}}(\alpha)$. Since $[R_1^{\mathrm{opt}}(\alpha), R_2^{\mathrm{opt}}(\alpha)] \in \mathcal{R}_{\mathrm{BR}}(\alpha)$ we have $R_1^\triangleright \le R_1^\star \le R_1^\triangleleft$ so that the case (2.60a) applies. This gives us in the case $\alpha \in \mathcal{A}_2$ with $R_\Sigma = 2\alpha R_{\overrightarrow{2\mathrm{R}}} - (1-\alpha)R^\dagger$

$$\begin{aligned}
R_{\mathrm{tot}}(\alpha) &= \alpha R_{\overrightarrow{2\mathrm{R}}} - \tfrac{1-\alpha}{2}R^\dagger + (1-\alpha)\log\Big(|h_1h_2|\hat{\gamma} - 2^{\frac{\alpha}{1-\alpha}R_{\overrightarrow{2\mathrm{R}}} - \frac{1}{2}R^\dagger}\Big)\\
&= (1-\alpha)\log\Big(2^{\frac{\alpha}{1-\alpha}R_{\overrightarrow{2\mathrm{R}}} - \frac{1}{2}R^\dagger}\big(|h_1h_2|\hat{\gamma} - 2^{\frac{\alpha}{1-\alpha}R_{\overrightarrow{2\mathrm{R}}} - \frac{1}{2}R^\dagger}\big)\Big)\\
&= (1-\alpha)\log\Big(\tfrac{|h_2|^2}{|h_1|^2}2^{\frac{\alpha}{1-\alpha}R_{\overrightarrow{2\mathrm{R}}}}\big(\hat{\gamma}|h_1|^2 - 2^{\frac{\alpha}{1-\alpha}R_{\overrightarrow{2\mathrm{R}}}}\big)\Big),
\end{aligned}$$

where we used the equality $2^{-\frac{1}{2}R^\dagger} = \frac{|h_2|}{|h_1|}$. For $\alpha \in \mathcal{A}_3$ we have to plug in $R_\Sigma = \alpha R_\Sigma^{\mathrm{MAC}}$ in (2.60a) which gives us

$$\begin{aligned}
R_{\mathrm{tot}}(\alpha) &= \tfrac{1}{2}R_\Sigma^{\mathrm{MAC}} + (1-\alpha)\log\Big(|h_1h_2|\hat{\gamma} - 2^{\frac{R_\Sigma^{\mathrm{MAC}}}{2(1-\alpha)}}\Big)\\
&= (1-\alpha)\log\Big(2^{\frac{\alpha}{2(1-\alpha)}R_\Sigma^{\mathrm{MAC}}}\big(|h_1h_2|\hat{\gamma} - 2^{\frac{\alpha}{2(1-\alpha)}R_\Sigma^{\mathrm{MAC}}}\big)\Big).
\end{aligned}$$

Similarly, for $\alpha \in \mathcal{A}_4$ we have to plug in $R_\Sigma = 2\alpha R_{\overrightarrow{1\mathrm{R}}} + (1-\alpha)R^\dagger$ in (2.60a) which finally gives us

$$\begin{aligned}
R_{\mathrm{tot}}(\alpha) &= \alpha R_{\overrightarrow{1\mathrm{R}}} + \tfrac{1-\alpha}{2}R^\dagger + (1-\alpha)\log\Big(|h_1h_2|\hat{\gamma} - 2^{\frac{\alpha}{1-\alpha}R_{\overrightarrow{1\mathrm{R}}} + \frac{1}{2}R^\dagger}\Big)\\
&= (1-\alpha)\log\Big(\tfrac{|h_1|^2}{|h_2|^2}2^{\frac{\alpha}{1-\alpha}R_{\overrightarrow{1\mathrm{R}}}}\big(\hat{\gamma}|h_2|^2 - 2^{\frac{\alpha}{1-\alpha}R_{\overrightarrow{1\mathrm{R}}}}\big)\Big).
\end{aligned}$$

This finishes the proof. □

2.8.34 Proof of Proposition 2.52

For $k = 1$ we use all eigenmodes if for all $\lambda_{2,n} \neq 0$ we have $\xi_{1,n} \neq 0$. Therefore, we need a water-level $\nu_1 > \frac{\sigma^2}{\lambda_{2,r_2}}$. The condition follows if we look at the power constraint in the high-power regime constraint

$$\beta_1 P_{\mathrm{R}} = \sum_{n=1}^{r_2} \big(\nu_1 - \frac{\sigma^2}{\lambda_{2,n}}\big) = r_2\nu_1 - \sigma^2 L_{2,r_2} > \frac{\sigma^2 r_2}{\lambda_{2,r_2}} - \sigma^2 L_{2,r_2} = \sigma^2\big(\frac{r_2 - 1}{\lambda_{2,r_2}} - L_{2,r_2-1}\big).$$

Similarly, beamforming is optimal if we have $\xi_{2,2} = 0$. Therefore, we need a water-level $\nu_1 \leq \frac{\sigma^2}{\lambda_{2,2}}$. Again the condition follows from the power constraint

$$\beta_1 P_{\mathrm{R}} = \nu_1 - \frac{\sigma^2}{\lambda_{2,1}} \leq \frac{\sigma^2}{\lambda_{2,2}} - \sigma^2 L_{2,1}.$$

Finally, it is optimal to use m eigenmodes if we have $\xi_{2,m} \neq 0$ and $\xi_{2,m+1} = 0$. Therefore, we need a water-level within $\frac{\sigma^2}{\lambda_{2,m}} < \nu_1 \leq \frac{\sigma^2}{\lambda_{2,m+1}}$. From the power constraint we get

$$\frac{m\sigma^2}{\lambda_{2,m}} - \sigma^2 L_{2,m} < \beta_1 P_{\mathrm{R}} = \sum_{n=1}^{m} \big(\nu_1 - \frac{\sigma^2}{\lambda_{2,n}}\big) \leq \frac{m\sigma^2}{\lambda_{2,m+1}} - \sigma^2 L_{2,m}.$$

The case $k = 2$ follows accordingly. $\square$

2.8.35 Proof of Proposition 2.53

From (2.77) we have

$$\begin{aligned} R_{\overrightarrow{\mathrm{R2}}}(\beta_1) &= \log\det\big(\boldsymbol{I}_{N_2} + \rho \boldsymbol{H}_2^H \boldsymbol{Q}_{\mathrm{R},1} \boldsymbol{H}_2\big) = \log \prod_{n=1}^{r_2} \big(1 + \frac{\lambda_{2,n}\xi_{2,n}}{\sigma^2}\big) \\ &= \sum_{n=1}^{m_1(\beta_1)} \log\big(1 + \frac{\lambda_{2,n}}{\sigma^2}(\nu_1 - \frac{\sigma^2}{\lambda_{2,n}})\big) = \sum_{n=1}^{m_1(\beta_1)} \log\big(\frac{\lambda_{2,n}\nu_1}{\sigma^2}\big) \\ &= \sum_{n=1}^{m_1(\beta_1)} \log\big(\lambda_{2,n} \frac{\beta_1\gamma_{\mathrm{R}} + L_{2,m_1(\beta_1)}}{m_1(\beta_1)}\big), \end{aligned}$$

where we used in the last equality the water-level $\nu_1 = \frac{\beta_1 P_{\mathrm{R}} + \sigma^2 L_{2,m_1(\beta_1)}}{m_1(\beta_1)}$ which follows from the power constraint $\beta_1 P_{\mathrm{R}} = \sum_{n=1}^{r_2} \xi_{2,n} = \sum_{n=1}^{m_1(\beta_1)} \big(\nu_1 - \frac{\sigma^2}{\lambda_{2,n}}\big) = m_1(\beta_1)\nu_1 - \sigma^2 L_{2,m_1(\beta_1)}$. The rate $R_{\overrightarrow{\mathrm{R1}}}(\beta_2)$ follows accordingly. $\square$

2.8.36 Proof of Theorem 2.54

For a rate pair on the boundary of $\mathcal{R}_{\mathrm{BC}}^{\mathrm{MIMO}}$ we have $\beta_1 + \beta_2 = 1$. Therefore, the rate pair $[R_{\overrightarrow{\mathrm{R2}}}(\beta), R_{\overrightarrow{\mathrm{R1}}}(1-\beta)]$, $\beta \in [0,1]$ characterizes a parametrization of the boundary. Using the representation of $R_{\overrightarrow{\mathrm{R2}}}(\beta)$ for $\beta > 0$ given in Proposition 2.53 we can calculate the derivation as follows

$$\frac{\mathrm{d}R_{\overrightarrow{\mathrm{R2}}}(\beta)}{\mathrm{d}\beta} = \sum_{n=1}^{m_1(\beta)} \frac{\gamma_{\mathrm{R}}}{\beta\gamma_{\mathrm{R}} + L_{2,m_1(\beta)}} \frac{1}{\ln(2)} = \frac{m_1(\beta)\gamma_{\mathrm{R}}}{\beta\gamma_{\mathrm{R}} + L_{2,m_1(\beta)}} \frac{1}{\ln(2)}.$$

Since $R_{\overrightarrow{\mathrm{R2}}}(\beta_1)$ is continuous in $\beta_1 = 0$ we get the right hand side derivation

$$\frac{\mathrm{d}R_{\overrightarrow{\mathrm{R2}}}(\beta)}{\mathrm{d}\beta}\Big|_{\beta=0} = \lim_{\beta\downarrow 0} \frac{m_1(\beta)\gamma_{\mathrm{R}}}{\beta\gamma_{\mathrm{R}} + L_{2,m_1(\beta)}} \frac{1}{\ln(2)} = \frac{\gamma_{\mathrm{R}}}{L_{2,1}} \frac{1}{\ln(2)}.$$

Similarly, for $R_{\overrightarrow{\mathrm{R1}}}(1-\beta)$ for $\beta < 1$ we can easily calculate the derivation as follows

$$\frac{\mathrm{d}R_{\overrightarrow{\mathrm{R1}}}(1-\beta)}{\mathrm{d}\beta} = \sum_{n=1}^{m_2(1-\beta)} \frac{-\gamma_{\mathrm{R}}}{(1-\beta)\gamma_{\mathrm{R}} + L_{1,m_2(1-\beta)}} \frac{1}{\ln(2)} = \frac{-m_2(1-\beta)\gamma_{\mathrm{R}}}{(1-\beta)\gamma_{\mathrm{R}} + L_{1,m_2(1-\beta)}} \frac{1}{\ln(2)}$$

as well as the left hand side derivation at $\beta = 1$

$$\frac{\mathrm{d}R_{\overrightarrow{\mathrm{R1}}}(1-\beta)}{\mathrm{d}\beta}\Big|_{\beta=1} = \lim_{\beta\uparrow 1} \frac{m_2(1-\beta)\gamma_{\mathrm{R}}}{(1-\beta)\gamma_{\mathrm{R}} + L_{1,m_2(1-\beta)}} \frac{1}{\ln(2)} = \frac{\gamma_{\mathrm{R}}}{L_{1,1}} \frac{1}{\ln(2)}.$$

With this we get the angle of the normal vector $\varphi_{\mathrm{BC}}(\beta)$ of the parametrized boundary $[R_{\overrightarrow{\mathrm{R2}}}(\beta), R_{\overrightarrow{\mathrm{R1}}}(1-\beta)]$, $\beta \in [0,1]$, from

$$\tan\varphi_{\mathrm{BC}}(\beta) = \frac{\mathrm{d}R_{\overrightarrow{\mathrm{R2}}}(\beta)/\mathrm{d}\beta}{-\mathrm{d}R_{\overrightarrow{\mathrm{R1}}}(1-\beta)/\mathrm{d}\beta} = \begin{cases} \frac{\gamma_{\mathrm{R}}+L_{1,M_2}}{M_2 L_{2,1}}, & \text{if } \beta = 0, \\ \frac{m_1(\beta)\left((1-\beta)\gamma_{\mathrm{R}}+L_{1,m_2(1-\beta)}\right)}{m_2(1-\beta)\left(\beta\gamma_{\mathrm{R}}+L_{2,m_1(\beta)}\right)}, & \text{if } 0 < \beta < 1, \\ \frac{M_1 L_{1,1}}{\gamma_{\mathrm{R}}+L_{2,M}}, & \text{if } \beta = 1. \end{cases}$$

Obviously, $\theta_{\boldsymbol{q}}$ denotes the angle of the weight vector $\boldsymbol{q}$. Then we know from previous discussion, e.g. in the proof of Theorem 2.10 in Appendix 2.8.6, that the boundary rate pair $[R_{\overrightarrow{\mathrm{R2}}}(\beta), R_{\overrightarrow{\mathrm{R1}}}(1-\beta)]$ with a normal angle $\varphi_{\mathrm{BC}}(\beta)$ is the weighted rate sum optimal rate pair for weight vectors $\boldsymbol{q}$ where we have $\theta_{\boldsymbol{q}} = \varphi_{\mathrm{BC}}(\beta)$. Since $\mathcal{R}_{\mathrm{BC}}^{\mathrm{MIMO}}$ is convex it follows that $\varphi_{\mathrm{BC}}(\beta)$ decreases with increasing β.

For weight vectors $\boldsymbol{q}$ with an angle $\theta_{\boldsymbol{q}} \geq \varphi_{\mathrm{BC}}(0) = \varphi_{\mathrm{BC}}(\tilde{\beta}_1) = \varphi_1$ the intersection of the boundary $[R_{\overrightarrow{\mathrm{R}2}}(\beta), R_{\overrightarrow{\mathrm{R}1}}(1-\beta)]$ with the R_2-axis is optimal, this means we have $\beta = 0$. Thus, for $\theta_{\boldsymbol{q}} \geq \varphi_1$ the weighted rate sum is attained at the rate pair

$$[R_{\overrightarrow{\mathrm{R}2}}(0), R_{\overrightarrow{\mathrm{R}1}}(1)] = \Big[0, \sum_{n=1}^{M_2} \log\Big(\frac{\lambda_{1,n}}{M_2}(\gamma_{\mathrm{R}} + L_{1,M_2})\Big)\Big]$$

according to Proposition 2.53 with $m_2(1) = M_2$.

Similarly, for weight vectors $\boldsymbol{q}$ with an angle $\theta_{\boldsymbol{q}} \leq \varphi_{\mathrm{BC}}(0) = \varphi_{\mathrm{BC}}(\tilde{\beta}_M) = \varphi_M$ the intersection of the boundary $[R_{\overrightarrow{\mathrm{R}2}}(\beta), R_{\overrightarrow{\mathrm{R}1}}(1-\beta)]$ with the R_1-axis is optimal. Thus, for $\theta_{\boldsymbol{q}} \leq \varphi_M$ the weighted rate sum is attained at the rate pair

$$[R_{\overrightarrow{\mathrm{R}2}}(1), R_{\overrightarrow{\mathrm{R}1}}(0)] = \Big[\sum_{n=1}^{M_1} \log\Big(\frac{\lambda_{2,n}}{M_1}(\gamma_{\mathrm{R}} + L_{2,M_1})\Big), 0\Big]$$

according to Proposition 2.53 with $m_1(1) = M_1$.

For the characterization of the rate pair for a weight vector $\boldsymbol{q}$ with $\varphi_M < \theta_{\boldsymbol{q}} < \varphi_1$ we need to know the number of used eigenmodes for this weight vector. The functions $m_1(\beta_1)$ and $m_2(\beta_2)$ characterize the number of eigenmodes for a given relay power fraction. Furthermore, we know that the number of used eigenmodes change at relay power fractions $\tilde{\beta}_n$, $n = 1, 2, \ldots, M$ so that for all $\beta \in (\tilde{\beta}_n, \tilde{\beta}_{n+1}]$ we have $m_1(\beta) = m_1(\tilde{\beta}_{n+1})$ and similarly for all $\beta \in [\tilde{\beta}_n, \tilde{\beta}_{n+1})$ we have $m_2(\beta) = m_2(\tilde{\beta}_n)$. Since $\varphi_{\mathrm{BC}}(\beta)$ is decreasing for β we can characterize this property also in term of the angles. Therefore, let $\varphi_n = \varphi_{\mathrm{BC}}(\tilde{\beta}_n)$ denote the angle of the characteristic power fraction $\tilde{\beta}_n$, $n = 1, 2, \ldots, M$. Accordingly, for weight vectors $\boldsymbol{q}$ with angle $\theta_{\boldsymbol{q}} \in [\varphi_{n+1}, \varphi_n)$ we use $m_1(\tilde{\beta}_{n+1}) = \eta_1(\boldsymbol{q})$ eigenmodes of the channel $\boldsymbol{H}_2$ and similarly for weight vectors $\boldsymbol{q}$ with angle $\theta_{\boldsymbol{q}} \in (\varphi_{n+1}, \varphi_n]$ we use $m_2(\tilde{\beta}_n) = \eta_2(\boldsymbol{q})$ eigenmodes of the channel $\boldsymbol{H}_1$. And of course, for $\boldsymbol{q} \geq \varphi_1$ we use $\eta_1(\boldsymbol{q}) = 0$ and $\eta_2(\boldsymbol{q}) = M_2$ as well as for $\boldsymbol{q} \leq \varphi_M$ we have $\eta_1(\boldsymbol{q}) = M_1$ and $\eta_2(\boldsymbol{q}) = 0$ eigenmodes of $\boldsymbol{H}_1$ and $\boldsymbol{H}_2$ respectively.

We are now able to characterize for a weight vector $\boldsymbol{q}$ with $\varphi_M < \theta_{\boldsymbol{q}} < \varphi_1$ the rate pair on the boundary $\mathcal{R}_{\mathrm{BC}}^{\mathrm{MIMO}}$ explicitly. Therefore, we solve $\tan \varphi_{\mathrm{BC}}(\beta) = \tan \theta_{\boldsymbol{q}} = \frac{q_1}{q_2}$ for β with $0 < \beta < 1$ which gives us the corresponding optimal power fraction $\beta(\boldsymbol{q})$ as follows

$$\beta(\boldsymbol{q}) = \frac{q_1 \eta_1(\boldsymbol{q}) \left(\gamma_{\mathrm{R}} + L_{1,\eta_2(\boldsymbol{q})}\right) - q_2 \eta_2(\boldsymbol{q}) L_{2,\eta_1(\boldsymbol{q})}}{\gamma_{\mathrm{R}}(q_2 \eta_2(\boldsymbol{q}) + q_1 \eta_1(\boldsymbol{q}))}.$$

Then a simple calculation shows that $R_{\overrightarrow{\mathrm{R}2}}(\boldsymbol{q})$ and $R_{\overrightarrow{\mathrm{R}1}}(\boldsymbol{q})$ are given by $R_{\overrightarrow{\mathrm{R}2}}(\beta(\boldsymbol{q}))$ and $R_{\overrightarrow{\mathrm{R}1}}(1 - \beta(\boldsymbol{q}))$ using the representation of the rates given by Proposition 2.53. Finally, we finish the proof with the observation that the identities $R_{1,\mathrm{BC}}([1,0]) = R_{\overrightarrow{\mathrm{R}2}}(1)$ and $R_{2,\mathrm{BC}}([0,1]) = R_{\overrightarrow{\mathrm{R}1}}(1)$ hold because we defined $L_{k,0} = 0$, $k = 1, 2$. □

3 Optimal Coding Strategy for the Bidirectional Broadcast Channel

3.1 Introduction

We still consider a three-node network where two nodes want to communicate with each other using the support of the third node as a relay, which we call the *bidirectional relay channel*. Between the nodes we assume discrete memoryless channels with finite size input and output alphabets. In this chapter we present an information theoretic optimal channel coding approach for the broadcast phase based on the philosophy of network coding [ACLY00] which suggests that information should not be treated as a fluid.

In classical information theory often full-duplex nodes are assumed. However, in wireless communication this assumption is hard to fulfill. Therefore, we assume half-duplex nodes which means that a node contributes either an input or an output to the channel. A natural consequence of this assumption is that the relay communication is performed in phases. Often the relay communication should be integrated in existing infrastructures and most protocol proposals base usually on orthogonal components which require exclusive resources for each link. As a consequence they suffer from an inherent loss in spectral efficiency. As we know from the previous, this loss can be significantly reduced if bidirectional relay communication is desired, since bidirectional communication can be efficiently performed in two phases. First, we have the multiple access phase where nodes 1 and 2 transmit their information to the relay node. In the succeeding broadcast phase, the relay node simultaneously forwards the information to its destinations.

Network coding is originally a distributed source coding problem. The bidirectional relaying protocols proposed in [WCK05, LJS05, FBW06, PY06, HKE$^+$07] apply an XOR operation on the decoded data at the relay node so that they assume a source (network) and channel coding separation. The separation of network and channel coding is in general suboptimal [EMH$^+$03, RK06]. In this thesis we consider a decode-and-forward protocol and find the optimal channel coding strategy for the bidirectional broadcast channel which factors in the distributed knowledge of the sources. But the decode-and-forward approach causes an operational separation between the phases, which means that we require that the relay node successfully decodes the messages of nodes 1 and 2 in the first phase. Furthermore, we do not allow any feedback between the phases so that the encoders of nodes 1 and 2 cannot cooperate. In the first phase we have, due to the operational separation, the classical multiple access channel where the optimal coding strategy and capacity region are known. It remains to find the optimal coding strategy for the succeeding broadcast phase, which we call the *bidirectional broadcast channel*. It is important to notice that the proposed coding strategy is only optimal for the bidirectional relay channel under the restriction of the operational separation.

While for single-user channels it is of no importance whether we use vanishing average or maximal probabilities of error in the definition of the achievable rate, in a multi-terminal system the average and maximal error capacity region can be different, even in the case of asymptotically vanishing errors as shown by Dueck in [Due78]. For this reason, in this chapter we will pay attention to the consideration of the maximal and average error probabilities and the relation between them.

In the following two subsections we more precisely introduce the considered two-phase bidirectional relaying model and after that we briefly restate for completeness the multiple access channel capacity region. In Section 3.2 we prove for the bidirectional broadcast channel an optimal coding theorem and a weak converse for the maximum error probability. The proof shows that the capacity region is independent of whether we use asymptotically vanishing average or maximum probabilities of error. After we found the capacity region of the bidirectional broadcast channel we got aware of the work of Tuncel [Tun06] where a joint source and channel coding approach for a broadcast channel problem is studied. It seems that one can deduce the achievable rate region with respect to the average error from this work, however for the bidirectional broadcast channel we do not need the Slepian-Wolf source coding part since the relay node knows which side information is known at the nodes 1 and 2. For that reason we can prove the achievable rate region for the average error probability with classical channel coding arguments only. Moreover, this allows us to derive the capacity region with respect to the maximum probability of error.

After we obtained the capacity region for each phase we optimize the time division between the MAC and the BC phases. This gives us the largest achievable rate region for the discrete memoryless bidirectional relay channel using finite set alphabets with the simplifications of the operational separation of the two phases. Finally, we illustrate the bidirectional achievable rate region by means of a binary channel example and finish this chapter with a discussion as well as an outlook on further results.

3.1.1 Two-Phase Bidirectional Relay Channel

We consider a three-node network with two independent information sources W_1 and W_2 at nodes 1 and 2. The random variables W_k, $k = 1, 2$, are uniformly distributed on the message

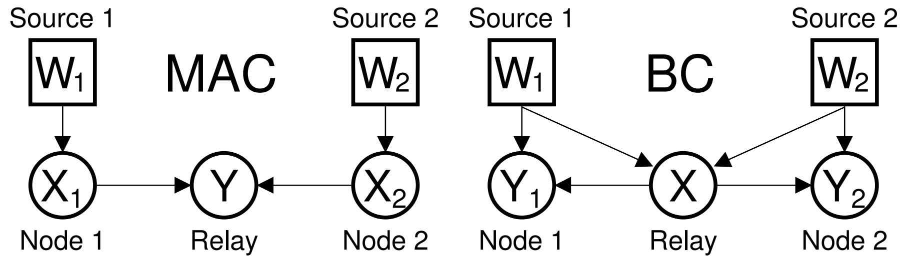


Figure 3.1: Multiple access (MAC) and broadcast (BC) phase of a two-phase decode-and-forward bidirectional relay channel without feedback.

set $\mathcal{W}_k := \{1, 2, \ldots, M_k^{(n)}\}$, $k = 1, 2$. For the bidirectional channel we want the messages $w_1 \in \mathcal{W}_1$ and $w_2 \in \mathcal{W}_2$ to be known at node 2 and node 1, respectively. For this goal, the nodes 1 and 2 use the support of the relay node. Note, that we have no direct channel between the nodes 1 and 2 due to the half-duplex assumption and the two-phase protocol.

We simplify the problem by assuming an a priori operational separation of the communication into two phases. In this thesis we consider the decode-and-forward approach, which means that we require the relay node to decode the messages from nodes 1 and 2. Furthermore, we do not allow cooperation between the encoders at nodes 1 and 2. Otherwise, a transmitted symbol could depend on previously received symbols. For a two-way channel this is known as a *restricted two-way channel*. With this simplification we end up with a multiple access phase, where the nodes 1 and 2 transmit their messages w_1 and w_2 to the relay node, and a broadcast phase, where the relay node forwards the messages to the nodes 2 and 1, respectively. Then the operational separation allows us to look at the two phases separately. After that we will briefly consider the optimal time division between the two phases.

In the multiple access phase (MAC) we have a classical multiple access channel, where the optimal coding strategy and capacity region $\mathcal{C}_{\mathrm{MAC}}$ is known [Ahl71a, Lia72]. We will restate the capacity region in the next subsection. As before, let $R_{\overrightarrow{1\mathrm{R}}}$ and $R_{\overrightarrow{2\mathrm{R}}}$ denote the achievable rates between the nodes 1 and 2 and the relay node in the MAC phase.

For the broadcast phase (BC) we assume that the relay node has successfully decoded the messages w_1 and w_2 in the multiple access phase. From the union bound[1] we know that the error probability of the two-phase protocol is at most the sum of the error probabilities of each phase. Therefore, an error-free MAC phase is reasonable if we assume rates within the MAC capacity region and a sufficient coding length. From this we end up with a broadcast channel where the message w_1 is known at node 1 and the relay node and the message w_2

[1] Let $\mathbb{P}\{E_1\}$ and $\mathbb{P}\{E_2\}$ denote the probabilities of event E_1 and E_2, then the union bound upper bounds the probability that at least one of the events happens as follows $\mathbb{P}\{E_1 \cup E_2\} \leq \mathbb{P}\{E_1\} + \mathbb{P}\{E_2\}$.

is known at node 2 and the relay node as depicted in Figure 3.1. Thereby, let x_1, x_2, and x denote the input and y_1, y_2, and y the output symbols of node 1, node 2, and the relay node, respectively. Furthermore, let $R_{\overrightarrow{\mathrm{R1}}}$ and $R_{\overrightarrow{\mathrm{R2}}}$ denote the achievable rates between the relay node and the nodes 1 and 2 in the BC phase.

The mission of the relay node is to broadcast a message to the nodes 1 and 2 which allows them to recover the unknown messages. This means that node 1 wants to recover message w_2 and node 2 wants to recover message w_1. In Section 3.2 we present an information theoretic optimal coding strategy, which gives us the capacity region of the bidirectional broadcast channel.

3.1.2 Capacity Region of the Multiple Access Phase

In this subsection, we restate the capacity region of the multiple access channel, which was independently found by Ahlswede [Ahl71a] and Liao [Lia72] and is nowadays part of any textbook on multiuser information theory [Wol78, CK81, CT91].

Definition 3.1. *A discrete memoryless multiple access channel is the family* $\{p^{(n)} : \mathcal{X}_1^n \times \mathcal{X}_2^n \to \mathcal{Y}^n\}_{n\in\mathbb{N}}$ *with finite input alphabets* $\mathcal{X}_k$, $k = 1, 2$, *and a finite output alphabet* $\mathcal{Y}$ *where the probability transition functions are given by* $p^{(n)}(y^n|x_1^n, x_2^n) := \prod_{i=1}^n p(y_i|x_{i1}, x_{i2})$ *for a given probability transition function* $\{p(y|x_1, x_2)\}_{x_1\in\mathcal{X}_1, x_2\in\mathcal{X}_2, y\in\mathcal{Y}}$.

Theorem 3.2. *The capacity region* $\mathcal{C}_{\mathrm{MAC}}$ *with respect to a vanishing average and maximum probability of error of the memoryless multiple access channel is the set of all rate pairs* $[R_{\overrightarrow{\mathrm{1R}}}, R_{\overrightarrow{\mathrm{2R}}}]$ *satisfying*

$$\begin{aligned} R_{\overrightarrow{\mathrm{1R}}} &\le I(X_1; Y|X_2, U), \\ R_{\overrightarrow{\mathrm{2R}}} &\le I(X_2; Y|X_1, U), \text{ and} \\ R_{\overrightarrow{\mathrm{1R}}} + R_{\overrightarrow{\mathrm{2R}}} &\le I(X_1, X_2; Y|U) \end{aligned}$$

for random variables $[U, X_1, X_2, Y]$ *with values in* $\mathcal{U} \times \mathcal{X}_1 \times \mathcal{X}_2 \times \mathcal{Y}$ *and the joint distribution* $\{q(u)q_1(x_1|u)q_2(x_2|u)p(y|x_1, x_2)\}_{u\in\mathcal{U}, x_1\in\mathcal{X}_1, x_2\in\mathcal{X}_2, y\in\mathcal{Y}}$. *Furthermore, the range* $\mathcal{U}$ *of the auxiliary random variable* U *has a cardinality bounded by* $|\mathcal{U}| \le 2$.

3.2 Capacity Region of the Broadcast Phase

In this section we present the capacity region of the bidirectional broadcast channel where the receiving nodes have perfect knowledge about the message which should be transmitted to the other node. This means that the independent uniformly distributed information sources

W_1 and W_2 at the relay node are also known at node 2 and node 1 respectively. We prove the capacity region by classical channel coding principles. To this end we first introduce some standard notation.

Definition 3.3. *Let $\mathcal{X}$ and $\mathcal{Y}_k$, $k = 1, 2$, be finite sets, which denote the input alphabet of the relay node and the output alphabet of nodes 1 and 2. A discrete memoryless broadcast channel is defined by a family $\{p^{(n)} : \mathcal{X}^n \to \mathcal{Y}_1^n \times \mathcal{Y}_2^n\}_{n\in\mathbb{N}}$ of probability transition functions given by $p^{(n)}(y_1^n, y_2^n|x^n) := \prod_{i=1}^n p(y_{i1}, y_{i2}|x_i)$ for a probability transition function $p : \mathcal{X} \to \mathcal{Y}_1 \times \mathcal{Y}_2$, i.e. $\{p(y_1, y_2|x)\}_{x\in\mathcal{X}, y_1\in\mathcal{Y}_1, y_2\in\mathcal{Y}_2}$ is a stochastic matrix.*

In what follows we will suppress the super-index n in the definition of the n-th extension of the channel p, i.e. we will write simply p instead of $p^{(n)}$. This should cause no confusion since it will always be clear from the context which block length is under consideration. In addition, we will use the abbreviation $\mathcal{V} := \mathcal{W}_1 \times \mathcal{W}_2$, where $\mathcal{W}_1$ and $\mathcal{W}_2$ denote the message sets.

Definition 3.4. *A $(M_1^{(n)}, M_2^{(n)}, n)$-code for the bidirectional broadcast channel consists of one encoder at the relay node*

$$x^n : \mathcal{V} \to \mathcal{X}^n$$

and decoders at nodes 1 and 2

$$\begin{aligned} g_1 :\ & \mathcal{Y}_1^n \times \mathcal{W}_1 \to \mathcal{W}_2 \cup \{0\}, \\ g_2 :\ & \mathcal{Y}_2^n \times \mathcal{W}_2 \to \mathcal{W}_1 \cup \{0\}. \end{aligned}$$

The element 0 in the definition of the decoders is included for convenience only and plays the role of an erasure symbol.

From the definition we see that we allow the decoders at nodes 1 and 2 to utilize the knowledge about its own message. When the relay node sends the message $v = [w_1, w_2]$, the receiver of node 1 is in error if $g_1(Y_1^n, w_1) \neq w_2$. The probability of this event is denoted by

$$\lambda_1(v) := \mathbb{P}\big\{g_1(Y_1^n, w_1) \neq w_2 \,|\, x^n(v) \text{ has been sent}\big\}.$$

Accordingly, we denote the probability that the receiver of node 2 is in error by

$$\lambda_2(v) := \mathbb{P}\big\{g_2(Y_2^n, w_2) \neq w_1 \,|\, x^n(v) \text{ has been sent}\big\}.$$

Hereby, Y_1^n and Y_2^n denote the random outputs at nodes 1 and 2 given that the sequence $x^n(v)$ has been sent down the channel. This allows us to introduce the notation for the maximum and average probabilities of error for the k-th node

$$\lambda_k^{(n)} := \max_{v\in\mathcal{V}} \lambda_k(v), \qquad \text{and} \qquad \mu_k^{(n)} := \tfrac{1}{|\mathcal{V}|} \sum_{v\in\mathcal{V}} \lambda_k(v).$$

Definition 3.5. *A rate pair* $[R_{\overrightarrow{\mathrm{R2}}}, R_{\overrightarrow{\mathrm{R1}}}]$ *is said to be achievable for the bidirectional broadcast channel if for any* $\delta > 0$ *there is an* $n(\delta) \in \mathbb{N}$ *and a sequence of* $(M_1^{(n)}, M_2^{(n)}, n)$*-codes such that for all* $n \geq n(\delta)$ *we have* $\frac{\log M_1^{(n)}}{n} \geq R_{\overrightarrow{\mathrm{R2}}} - \delta$ *and* $\frac{\log M_2^{(n)}}{n} \geq R_{\overrightarrow{\mathrm{R1}}} - \delta$ *while the maximum probabilities of error* $\lambda_1^{(n)}, \lambda_2^{(n)} \to 0$ *when* $n \to \infty$*. The set of all achievable rate pairs is the capacity region of the bidirectional broadcast channel and is denoted by* $\mathcal{C}_{\mathrm{BC}}$.

Remark 3.6. *Achievable rate pairs and a capacity region can be similarly defined for average probability of error* $\mu_1^{(n)}, \mu_2^{(n)} \to 0$ *when* $n \to \infty$.

We are now ready to present the main result of this chapter.

Theorem 3.7. *The capacity region* $\mathcal{C}_{\mathrm{BC}}$ *of the bidirectional memoryless broadcast channel is the set of all rate pairs* $[R_{\overrightarrow{\mathrm{R2}}}, R_{\overrightarrow{\mathrm{R1}}}]$ *satisfying*

$$\begin{aligned} R_{\overrightarrow{\mathrm{R2}}} &\leq I(X; Y_2|U), \\ R_{\overrightarrow{\mathrm{R1}}} &\leq I(X; Y_1|U) \end{aligned} \tag{3.1}$$

for random variables $[U, X, Y_1, Y_2]$ *with values in* $\mathcal{U} \times \mathcal{X} \times \mathcal{Y}_1 \times \mathcal{Y}_2$ *and the joint probability distribution* $\{q_1(u)q_2(x|u)p(y_1, y_2|x)\}_{u\in\mathcal{U}, x\in\mathcal{X}, y_1\in\mathcal{Y}_1, y_2\in\mathcal{Y}_2}$*. The cardinality of the range of* U *can be bounded by* $|\mathcal{U}| \leq 2$.

The theorem is proved in the following three subsections. In the first subsection we prove the achievability, i.e. a coding theorem. We prove a weak converse with respect to the maximum probability of error in the second subsection. Then the theorem is proved with the third subsection where we show that a cardinality of two is enough for the range of the auxiliary random variable.

3.2.1 Proof of Achievability

Here, we adapt the random coding proof for the degraded broadcast channel of [Ber73] to our context. First, we prove the achievability of all rate pairs $[R_{\overrightarrow{\mathrm{R2}}}, R_{\overrightarrow{\mathrm{R1}}}]$ satisfying

$$R_{\overrightarrow{\mathrm{R2}}} \leq I(X; Y_2), \qquad \text{and} \qquad R_{\overrightarrow{\mathrm{R1}}} \leq I(X; Y_1), \tag{3.2}$$

for some probability function $p(x)p(y_1, y_2|x)$. Then, we extend this to prove that all points in the closure of the convex hull of (3.2) are achievable, which we will see is exactly the region stated in Theorem 3.7.

Random codebook generation

For any $\delta > 0$ we define for any $R_{\overrightarrow{\mathrm{R}k}}$, $k = 1, 2$, the rate of the code $\bar{R}_{\overrightarrow{\mathrm{R}k}} := \max\{\frac{1}{n}\lfloor n(R_{\overrightarrow{\mathrm{R}k}} - \frac{\delta}{2})\rfloor, 0\}$.[2] Then we generate $M_1^{(n)} M_2^{(n)}$ independent codewords $X^n(v)$, $v = [w_1, w_2]$, of length n with $M_1^{(n)} := 2^{n\bar{R}_{\overrightarrow{\mathrm{R}2}}}$ and $M_2^{(n)} := 2^{n\bar{R}_{\overrightarrow{\mathrm{R}1}}}$ according to $\prod_{i=1}^{n} p(x_i)$.

Encoding

To send the pair $v = [w_1, w_2]$ with $w_k \in \mathcal{W}_k$, $k = 1, 2$, the relay sends the corresponding codeword $x^n(v)$.

Decoding

The receiving nodes use typical set decoding. First, we characterize the decoding sets. For the decoder at node $k = 1, 2$ let

$$I(x^n; y_k^n) := \tfrac{1}{n} \log_2 \frac{p(y_k^n | x^n)}{p(y_k^n)}$$

with average mutual information $I(X; Y_k) := \mathbb{E}_{x^n, y_k^n}\big\{I(x^n; y_k^n)\big\}$. This gives the decoding set

$$\mathcal{S}(y_k^n) := \left\{ x^n \in \mathcal{X}^n : I(x^n; y_k^n) \geq \tfrac{\bar{R}_{\overrightarrow{\mathrm{R}k}} + I(X;Y_k)}{2} \right\}$$

and indicator function

$$d(x^n, y_k^n) := \begin{cases} 1, & \text{if } x^n \notin \mathcal{S}(y_k^n), \\ 0, & \text{otherwise.} \end{cases}$$

When $x^n(v)$ with $v = [w_1, w_2]$ has been sent, and y_1^n and y_2^n have been received we say that the decoder at node k makes an error if either $x^n(v)$ is not in $\mathcal{S}(y_k^n)$ (occurring with probability $P_{e,k}^{(1)}(v)$) or if at node 1 $x^n(w_1, \hat{w}_2)$ with $\hat{w}_2 \neq w_2$ is in $\mathcal{S}(y_1^n)$ or at node 2 $x^n(\hat{w}_1, w_2)$ with $\hat{w}_1 \neq w_1$ is in $\mathcal{S}(y_2^n)$ (occurring with $P_{e,k}^{(2)}(v)$). If there is no or more than one codeword $x^n(w_1, \cdot) \in \mathcal{S}(y_1^n)$ or $x^n(\cdot, w_2) \in \mathcal{S}(y_2^n)$, the decoders map on the erasure symbol 0.

[2] We need not consider the trivial cases $\bar{R}_{\overrightarrow{\mathrm{R}k}} = 0$ for any k because then the error probability is zero by definition.

Analysis of the probability of error

From the union bound we have $\lambda_k(v) \leq P_{e,k}^{(1)}(v) + P_{e,k}^{(2)}(v)$ with

$$P_{e,k}^{(1)}(v) := \sum_{y_k^n \in \mathcal{Y}_k^n} p(y_k^n | x^n(v))\, d(x^n(v), y_k^n) \quad \text{for } k = 1, 2$$

and

$$P_{e,1}^{(2)}(v) := \sum_{y_1^n \in \mathcal{Y}_1^n} p(y_1^n | x^n(v)) \sum_{\substack{\hat{w}_2=1 \\ \hat{w}_2 \neq w_2}}^{|\mathcal{W}_2|} \big(1 - d(x^n(w_1, \hat{w}_2), y_1^n)\big),$$

$$P_{e,2}^{(2)}(v) := \sum_{y_2^n \in \mathcal{Y}_2^n} p(y_2^n | x^n(v)) \sum_{\substack{\hat{w}_1=1 \\ \hat{w}_1 \neq w_1}}^{|\mathcal{W}_1|} \big(1 - d(x^n(\hat{w}_1, w_2), y_2^n)\big).$$

For uniformly distributed messages W_1 and W_2 we define $P_{e,k}^{(m)} := \frac{1}{|\mathcal{W}_1|\,|\mathcal{W}_2|} \sum_{v \in \mathcal{W}_1 \times \mathcal{W}_2} P_{e,k}^{(m)}(v)$ for $m = 1, 2$ so that $\mu_k^{(n)} \leq P_{e,k}^{(1)} + P_{e,k}^{(2)}$.

Next, we average over all codebooks and show that $\mathbb{E}_{x^n}\{\mu_k^{(n)}\} \leq \mathbb{E}_{x^n}\{P_{e,k}^{(1)} + P_{e,k}^{(2)}\} \to 0$ as $n \to \infty$ if $R_{\overrightarrow{\mathrm{R}k}} \leq I(X, Y_k)$, $k = 1, 2$. Recall that $\bar{R}_{\overrightarrow{\mathrm{R}k}} \leq R_{\overrightarrow{\mathrm{R}k}} - \frac{\delta}{2}$ holds so that we have

$$\begin{aligned}
\mathbb{E}_{x^n}\{P_{e,k}^{(1)}\} &= \frac{1}{|\mathcal{W}_1|\,|\mathcal{W}_2|} \sum_{v \in \mathcal{W}_1 \times \mathcal{W}_2} \mathbb{E}_{x^n}\{P_{e,k}^{(1)}(v)\} \\
&\overset{\substack{\text{for any} \\ \text{fixed } v}}{=} \sum_{y_k^n \in \mathcal{Y}_k^n} \mathbb{E}_{x^n}\{p(y_k^n | x^n(v))\, d(x^n(v), y_k^n)\} \\
&= \sum_{y_k^n \in \mathcal{Y}_k^n} \sum_{x^n \in \mathcal{X}^n} p(x^n) p(y_k^n | x^n)\, d(x^n, y_k^n) \\
&= \mathbb{E}_{x^n, y_k^n}\{d(x^n, y_k^n)\} = \mathbb{P}\{d(x^n, y_k^n) = 1\} \\
&= \mathbb{P}\left\{ I(x^n; y_k^n) \leq \tfrac{\bar{R}_{\overrightarrow{\mathrm{R}k}} + I(X;Y_k)}{2} \right\} \\
&\leq \mathbb{P}\left\{ I(x^n; y_k^n) \leq I(X; Y_k) - \frac{\delta}{4} \right\} \underset{n \to \infty}{\longrightarrow} 0
\end{aligned}$$

exponentially fast by the law of large numbers. For the calculation of $\mathbb{E}_{x^n}\{P_{e,k}^{(2)}\}$ we have to distinguish between the receiving nodes. We present the analysis for $k = 1$, the case $k = 2$ follows accordingly. Thereby, we use the fact that for $v = [w_1, w_2] \neq [w_1, \hat{w}_2]$ the random variables $p(y_1^n|X^n(v))$ and $d(X^n(w_1,\hat{w}_2),y_1^n)$ are independent for each choice of $y_1^n \in \mathcal{Y}_1^n$.

$$\begin{aligned}
\mathbb{E}_{x^n}\left\{P_{e,1}^{(2)}\right\} &= \frac{1}{|\mathcal{W}_1|\,|\mathcal{W}_2|}\sum_{v\in\mathcal{W}_1\times\mathcal{W}_2}\mathbb{E}_{x^n}\left\{P_{e,1}^{(2)}(v)\right\} \\
&\overset{\substack{\text{for any}\\ \text{fixed } v}}{=} \sum_{y_1^n\in\mathcal{Y}_1^n}\mathbb{E}_{x^n}\Big\{p(y_1^n|x^n(v))\sum_{\substack{\hat{w}_2=1\\ \hat{w}_2\neq w_2}}^{|\mathcal{W}_2|}\big(1-d(x^n(w_1,\hat{w}_2),y_1^n)\big)\Big\} \\
&= \sum_{y_1^n\in\mathcal{Y}_1^n}\sum_{\substack{\hat{w}_2=1\\ \hat{w}_2\neq w_2}}^{|\mathcal{W}_2|}\mathbb{E}_{x^n}\big\{p(y_1^n|x^n(v))\big\}\mathbb{E}_{x^n}\big\{1-d(x^n(w_1,\hat{w}_2),y_1^n)\big\} \\
&= \sum_{y_1^n\in\mathcal{Y}_1^n}\sum_{\substack{\hat{w}_2=1\\ \hat{w}_2\neq w_2}}^{|\mathcal{W}_2|}p(y_1^n)\mathbb{E}_{x^n}\left\{1-d(x^n(w_1,\hat{w}_2),y_1^n)\right\} \\
&= \sum_{y_1^n\in\mathcal{Y}_1^n}\sum_{\substack{\hat{w}_2=1\\ \hat{w}_2\neq w_2}}^{|\mathcal{W}_2|}p(y_1^n)\sum_{x^n\in\mathcal{X}^n}p(x^n)\big(1-d(x^n,y_1^n)\big) \\
&= (|\mathcal{W}_2|-1)\sum_{y_1^n\in\mathcal{Y}_1^n}\sum_{x^n\in\mathcal{S}(y_1^n)}p(x^n)p(y_1^n).
\end{aligned}$$

Whenever $x^n \in \mathcal{S}(y_1^n)$, we have $I(x^n;y_1^n) = \frac{1}{n}\log_2\frac{p(y_1^n|x^n)}{p(y_1^n)} > \frac{1}{2}(\bar{R}_{\overrightarrow{\mathrm{R1}}} + I(X;Y_1))$ or $p(y_1^n) < p(y_1^n|x^n)2^{-\frac{n}{2}(\bar{R}_{\overrightarrow{\mathrm{R1}}}+I(X;Y_1))}$. Consequently,

$$\begin{aligned}
\mathbb{E}_{x^n}\left\{P_{e,1}^{(2)}\right\} &< |\mathcal{W}_2|\sum_{y_1^n\in\mathcal{Y}_1^n}\sum_{x^n\in\mathcal{S}(y_1^n)}p(x^n)p(y_1^n|x^n)2^{-\frac{n}{2}(\bar{R}_{\overrightarrow{\mathrm{R1}}}+I(X;Y_1))} \\
&\leq 2^{n\bar{R}_{\overrightarrow{\mathrm{R1}}}}2^{-\frac{n}{2}(\bar{R}_{\overrightarrow{\mathrm{R1}}}+I(X;Y_1))} \\
&= 2^{\frac{n}{2}(\bar{R}_{\overrightarrow{\mathrm{R1}}}-I(X;Y_1))} \\
&\leq 2^{\frac{n}{2}(R_{\overrightarrow{\mathrm{R1}}}-\frac{\delta}{2}-I(X;Y_1))} \\
&\leq 2^{\frac{-n\delta}{4}} \xrightarrow[n\to\infty]{} 0
\end{aligned}$$

where the last inequality holds if we have $R_{\overrightarrow{\mathrm{R1}}} \leq I(X,Y_1)$. Hence, if $R_{\overrightarrow{\mathrm{R}k}} \leq I(X,Y_k)$, $k=1,2$, the average probability of error, averaged over codebooks and codewords, gets arbitrary small for sufficiently large block length n.

Code construction with arbitrary small maximum probability of error

If $R_{\overrightarrow{\mathrm{R1}}} \leq I(X;Y_1)$ and $R_{\overrightarrow{\mathrm{R2}}} \leq I(X;Y_2)$ for any $\delta > 0$ and its code rate $\bar{R}_{\overrightarrow{\mathrm{R}k}}$, $k = 1,2$, we can choose $\varepsilon > 0$ and $n \in \mathbb{N}$ so that we have $\mathbb{E}_{x^n}\{\mu_1^{(n)} + \mu_2^{(n)}\} < \varepsilon$. Since the average probabilities of error over the codebooks is small, there exists at least one codebook $\mathcal{C}^\star$ with a small average probabilities of error $\mu_1^{(n)} + \mu_2^{(n)} < \varepsilon$. This implies that we have $\mu_1^{(n)} < \varepsilon$ and $\mu_2^{(n)} < \varepsilon$. We define sets

$$\mathcal{Q} := \{v \in \mathcal{V} : \lambda_1(v) < 8\varepsilon \text{ and } \lambda_2(v) < 8\varepsilon\},$$
$$\mathcal{R}_k := \{v \in \mathcal{V} : \lambda_k(v) \geq 8\varepsilon\}, \qquad \text{for } k = 1,2.$$

Since $\varepsilon > \frac{1}{|\mathcal{V}|}\sum_{v\in\mathcal{V}} \lambda_k(v) \geq \frac{|\mathcal{R}_k|}{|\mathcal{V}|} 8\varepsilon$, we can bound the cardinality $|\mathcal{R}_k| < \frac{|\mathcal{V}|}{8}$ for $k = 1,2$. Then from $\mathcal{V} = \mathcal{Q} \cup \mathcal{R}_1 \cup \mathcal{R}_2$ it follows

$$|\mathcal{Q}| \geq |\mathcal{V}| - |\mathcal{R}_1| - |\mathcal{R}_2| > \tfrac{3}{4}|\mathcal{V}|.$$

Now, let $\mathcal{T}$ be the set of w_1 having the property that for each w_1 there are at least $\frac{1}{2}M_2^{(n)}$ choices of w_2 so that $[w_1, w_2] \in \mathcal{Q}$. Therefore, for $w_1 \in \mathcal{T}$ there are at most $M_2^{(n)}$ choices $w_2 \in \mathcal{W}_2$ and for $w_1 \notin \mathcal{T}$ there are less than $\frac{1}{2}M_2^{(n)}$ choices $w_2 \in \mathcal{W}_2$ such that $[w_1, w_2] \in \mathcal{Q}$. Accordingly, we have

$$|\mathcal{T}|M_2^{(n)} + |\mathcal{W}_1 \setminus \mathcal{T}|\tfrac{1}{2}M_2^{(n)} > |\mathcal{Q}| > \tfrac{3}{4}M_1^{(n)}M_2^{(n)}$$

so that it follows that $|\mathcal{T}| > \frac{1}{2}M_1^{(n)}$ using $|\mathcal{W}_1 \setminus \mathcal{T}| = M_1^{(n)} - |\mathcal{T}|$. This means that there exists an index set $\mathcal{Q}_1^\star \subset \mathcal{W}_1$ with $\frac{1}{2}M_1^{(n)}$ indices w_1, to each of which we can find an index set $\mathcal{Q}_2^\star(w_1) \subset \mathcal{W}_2$ with $\frac{1}{2}M_2^{(n)}$ indices w_2 so that we have for each $w_1 \in \mathcal{Q}_1^\star$ and $w_2 \in \mathcal{Q}_2^\star(w_1)$ a maximum error $\lambda_k(w_1, w_2) < 8\varepsilon$, $k = 1,2$.

It follows that there exist one-to-one mappings $\Phi : \mathcal{V}^\star \to \mathcal{Q}^\star$, $\Phi_1 : \mathcal{W}_1^\star \to \mathcal{Q}_1^\star$, $\Phi_2^{w_1} : \mathcal{W}_2^\star \to \mathcal{Q}_2^\star(w_1)$ for each $w_1 \in \mathcal{Q}_1^\star$ with $\Phi(w_1, w_2) := [\Phi_1(w_1), \Phi_2^{w_1}(w_2)]$ with sets $\mathcal{V}^\star := \mathcal{W}_1^\star \times \mathcal{W}_2^\star$, $\mathcal{W}_k^\star := \{1, 2 \ldots, \frac{1}{2}M_k^{(n)}\}$ for $k = 1,2$, $\mathcal{Q}^\star := \{[w_1, w_2] \in \mathcal{V} : w_1 \in \mathcal{Q}_1^\star, w_2 \in \mathcal{Q}_2^\star(w_1)\} \subset \mathcal{Q}$. Accordingly, there exist mappings $\Psi_k : \mathcal{Q}^\star \to \mathcal{W}_k^\star$, $k = 1,2$, with $v = [\Psi_1(\Phi(v)), \Psi_2(\Phi(v))]$.

This allows us finally to define a $(\frac{1}{2}M_1^{(n)}, \frac{1}{2}M_2^{(n)}, n)$-code with an encoder $\tilde{x}^n : \mathcal{V}^\star \to \mathcal{X}^n$ with $\tilde{x}^n(v) := x^n(\Phi(v))$ and decoders $\tilde{g}_1 : \mathcal{Y}_1^n \times \mathcal{W}_1^\star \to \mathcal{W}_2^\star$ and $\tilde{g}_2 : \mathcal{Y}_2^n \times \mathcal{W}_2^\star \to \mathcal{W}_1^\star$ with $\tilde{g}_1(y_1^n, w_1) := \tilde{\Psi}_2(w_1, g_1(y_1^n, w_1))$ and $\tilde{g}_2(y_2^n, w_2) := \tilde{\Psi}_1(g_2(y_2^n, w_2), w_2)$ where we use the mappings $\tilde{\Psi}_k : \mathcal{V} \to W_k^\star$ given by

$$\tilde{\Psi}_k(v) := \begin{cases} \Psi_k(v), & \text{if } v \in \mathcal{Q}^\star, \\ 0, & \text{if } v \notin \mathcal{Q}^\star, \end{cases}$$

for $k = 1, 2$. The idea is that the encoder uses only codewords $x^n(v)$ of the code $\mathcal{C}^\star$ with an index $v \in \mathcal{Q}^\star$, which have a maximum error $\lambda_k(v) < 8\varepsilon$, $k = 1, 2$. Since the decoders use the typical set decoder of the code $\mathcal{C}^\star$, they could erroneously find an $x^n(v)$ with $v \in \mathcal{V} \setminus \mathcal{Q}^\star$. In this case, the mapping $\tilde{\Psi}_k$ decides on the erasure symbol 0. It was already a wrong decision by the decoder g_k, since the encoder chooses only codewords $x^n(v)$ with $v \in \mathcal{Q}^\star$. Therefore, this does not add any error to the decoding. The code has a rate pair

$$\left[\bar{R}_{\overrightarrow{\mathrm{R2}}} - \tfrac{1}{n}, \bar{R}_{\overrightarrow{\mathrm{R1}}} - \tfrac{1}{n}\right] = \left[\tfrac{1}{n}\lfloor n(R_{\overrightarrow{\mathrm{R2}}} - \tfrac{\delta}{2})\rfloor - \tfrac{1}{n}, \tfrac{1}{n}\lfloor n(R_{\overrightarrow{\mathrm{R1}}} - \tfrac{\delta}{2})\rfloor - \tfrac{1}{n}\right]$$

which is element-wise larger than $[R_{\overrightarrow{\mathrm{R2}}} - \delta, R_{\overrightarrow{\mathrm{R1}}} - \delta]$ as $n > \frac{4}{\delta}$ using $\lfloor x \rfloor \geq x - 1$. This proves the achievability of any rate pair satisfying the equation (3.2).

Closure of convex hull

Let $\mathcal{R}(p(x))$ denote the set of rates which we can achieve with the input distribution $p(x)$.

For $k = 1, 2$, we can rewrite the right hand side of (3.1) as follows

$$I(X; Y_k|U) = \sum_{u=1}^{|\mathcal{U}|} p(u) I(X; Y_k|U = u) = \sum_{u=1}^{|\mathcal{U}|} p(u) I(X; Y_k)\big|_{p(x|u)},$$

where in $I(X; Y_k)\big|_{p(x|u)}$ we choose a specific input distribution $p(x|u)$ according to the auxiliary random variable U. For the input distribution $p(x|u)$ we know from the first part of the proof that any rate pair $\boldsymbol{R}_u \in \mathcal{R}(p(x|u)) \subset \mathbb{R}^2$ is achievable. Therefore, for any convex combination $\sum_{u=1}^{m} \alpha_u \boldsymbol{R}_u$ we can regard the weights as a probability mass function with $p(u) := \alpha_u$ and $u \in \mathcal{U} := \{1, 2, \ldots, m\}$ and choose for any u an input distribution $p(x|u)$ that achieves the rate pair $\boldsymbol{R}_u$. For that reason, the conditional mutual informations given by the right hand sides of (3.1) are also achievable rates.

Finally, the set of achievable rate pairs is closed by definition of achievability since for any sequence of achievable rate pairs with $[R_1, R_2] = \lim_{m\to\infty}[R_1^{(m)}, R_2^{(m)}]$ the limit point itself is achievable. $\square$

In general in multi-terminal systems the average and maximal error capacity region can be different. Ahlswede has shown for the two-way channel in [Ahl71b] that *"one cannot reduce a code with average errors to a code with maximal errors without an essential loss in code length or error probability, whereas for one-way channels it is unessential whether one uses average or maximal errors."* The problem in the two-way channel is to find a maximal error sub-code with a Cartesian product structure. This problem is equivalent to a combinatorial problem by Zarankiewicz[3] [Ahl71b] and arises since the transmitter and receiver have disjoint partial knowledge only. Here, the relay node has full knowledge so that for the code construction with arbitrarily small maximum probability of error we need not require a sub-code with Cartesian product structure.

In the next subsection we prove the weak converse for the maximal error. Since Fano's inequalities apply for the average error as well, the weak converse for the average error follows analogously.

3.2.2 Proof of weak converse

We have to show that any given sequence of $(M_1^{(n)}, M_2^{(n)}, n)$-codes with $\lambda_1^{(n)}, \lambda_2^{(n)} \to 0$ must satisfy $\frac{1}{n}\log M_1^{(n)} \leq I(X;Y_2|U) + o(n^0)$ and $\frac{1}{n}\log M_2^{(n)} \leq I(X;Y_1|U) + o(n^0)$ for a joint probability distribution $q_1(u)q_2(x|u)\ p(y_1,y_2|x)$. For a fixed block length n we define the joint probability distribution $p(w_1,w_2,x^n,y_1^n,y_2^n) := \frac{1}{|\mathcal{W}_1|}\frac{1}{|\mathcal{W}_2|}$ $q_2(x^n|w_1,w_2)\prod_{i=1}^n p(y_{1i},y_{2i}|x_i)$ on $\mathcal{W}_1 \times \mathcal{W}_2 \times \mathcal{X}^n \times \mathcal{Y}_1^n \times \mathcal{Y}_2^n$ where the conditional distribution $q_2(x^n|w_1,w_2) = 1$ if x^n is the codeword corresponding to w_1, w_2 or is equal to 0 else. In what follows we consider for $k = 1,2$ uniformly distributed random variables W_k with values in $\mathcal{W}_k$.

Lemma 3.8. *For our context we have the Fano's inequality*

$$H(W_2|Y_1^n, W_1) \leq \lambda_1^{(n)} \log|\mathcal{W}_2| + 1 = n\varepsilon_1^{(n)} \tag{3.3}$$

with $\varepsilon_1^{(n)} = \frac{\log|\mathcal{W}_2|}{n}\lambda_1^{(n)} + \frac{1}{n} \to 0$ *for* $n \to \infty$ *as* $\lambda_1^{(n)} \to 0$.

Proof. From Y_1^n and W_1 node 1 estimates the index W_2 from the sent codeword $X^n(W_1,W_2)$. We define the event of an error at node 1 as

$$E_1 := \begin{cases} 1, & \text{if } g_1(Y_1^n, W_1) \neq W_2, \\ 0, & \text{if } g_1(Y_1^n, W_1) = W_2, \end{cases}$$

so that we have for the mean probability of error $\mu_1^{(n)} = \mathbb{P}\{E_1 = 1\} \leq \lambda_1^{(n)}$. From the chain rule for entropies we have

$$\begin{aligned} H(E_1, W_2|Y_1^n, W_1) &= H(W_2|Y_1^n, W_1) + H(E_1|Y_1^n, W_1, W_2) \\ &= H(E_1|Y_1^n, W_1) + H(W_2|E, Y_1^n, W_1). \end{aligned}$$

[3]For an $n\times n$ matrix which contains zeros and ones only Zarankiewicz [Zar51] posed the combinatorial problem to find the smallest natural number of ones, $k_j(n)$, which ensures that a $n \times n$ matrix contains a $j \times j$ minor which consists entirely of ones. In [Ahl71b] Ahlswede generalizes the problem to find the smallest number of ones, $k_{i,j}(m,n)$, which ensures that a $m \times n$ matrix contains a $i \times j$ submatrix entirely of ones.

Since E_1 is a function of W_1, W_2, and Y_1^n, we have $H(E_1|Y_1^n, W_1, W_2) = 0$. Further, since E_1 is a binary-valued random variable, we get $H(E_1|Y_1^n, W_1) \leq H(E_1) \leq 1$. So that finally with the next inequality

$$\begin{aligned} H(W_2|Y_1^n, W_1, E_1) = \mathbb{P}\big\{E_1 = 0\big\} H(W_2|Y_1^n, W_1, E_1 = 0) \\ + \mathbb{P}\big\{E_1 = 1\big\} H(W_2|Y_1^n, W_1, E_1 = 1) \\ \leq (1 - \mu_1^{(n)})0 + \mu_1^{(n)} \log(|\mathcal{W}_2| - 1) \leq \lambda_1^{(n)} \log |\mathcal{W}_2| \end{aligned}$$

we get Fano's inequality for our context. □

Therewith, we can bound the entropy $H(W_2)$ as follows

$$\begin{aligned} H(W_2) &= H(W_2|W_1) \\ &= I(W_2; Y_1^n|W_1) + H(W_2|Y_1^n, W_1) \\ &\leq I(W_2; Y_1^n|W_1) + n\varepsilon_1^{(n)} \\ &\leq I(W_1, W_2; Y_1^n) + n\varepsilon_1^{(n)} \\ &\leq I(X^n; Y_1^n) + n\varepsilon_1^{(n)} \\ &\leq H(Y_1^n) - H(Y_1^n|X^n) + n\varepsilon_1^{(n)}, \end{aligned}$$

where the equations and inequalities follow from the independence of W_1 and W_2, the definition of mutual information, Lemma 1, the chain rule for mutual information, the positivity of mutual information, and the data processing inequality. If we divide the inequality by n we get the rate

$$\begin{aligned} \tfrac{1}{n} H(W_2) &= \tfrac{1}{n}\big(H(Y_1^n) - H(Y_1^n|X^n)\big) + \varepsilon_1^{(n)} \\ &\leq \tfrac{1}{n}\sum_{i=1}^{n}\big(H(Y_{1i}|Y_1^{i-1}) - H(Y_{1i}|Y_1^{i-1}, X^n)\big) + \varepsilon_1^{(n)} \\ &\leq \tfrac{1}{n}\sum_{i=1}^{n}\big(H(Y_{1i}) - H(Y_{1i}|X_i)\big) + \varepsilon_1^{(n)} \\ &= \tfrac{1}{n}\sum_{i=1}^{n} I(Y_{1i}; X_i) + \varepsilon_1^{(n)} \end{aligned}$$

using the chain rule of mutual information, the memoryless property of the channel, and the definition of mutual information. A similar derivation for the source rate $\frac{1}{n}H(W_1)$ gives us the bound $\frac{1}{n}H(W_1) \leq \frac{1}{n}\sum_{i=1}^{n} I(Y_{2i}; X_i) + \varepsilon_2^{(n)}$ with $\varepsilon_2^{(n)} = \frac{\log|\mathcal{W}_1|}{n}\lambda_2^{(n)} + \frac{1}{n} \to 0$ for $n \to \infty$ as $\lambda_2^{(n)} \to 0$.

This means that the entropies $H(W_1)$ and $H(W_2)$ are bounded by averages of the mutual informations calculated at the empirical distribution in column i of the codebook. Therefore, we can rewrite these inequalities with an auxiliary random variable U, where $U = i \in \mathcal{U} = \{1, 2, \dots, n\}$ with probability $\frac{1}{n}$. We finish the proof of the converse with the following inequalities

$$\begin{aligned}
\tfrac{1}{n}H(W_2) &\leq \tfrac{1}{n}\sum_{i=1}^{n} I(Y_{1i}; X_i) + \varepsilon_1^{(n)} \\
&= \sum_{i=1}^{n} \mathbb{P}\big\{U = i\big\} I(Y_{1i}; X_i | U = i) + \varepsilon_1^{(n)} \\
&= I(Y_{1U}; X_U | U) + \varepsilon_1^{(n)} \\
&= I(Y_1; X | U) + \varepsilon_1^{(n)}
\end{aligned}$$

and $\frac{1}{n}H(W_1) \leq I(Y_2; X|U) + \varepsilon_2^{(n)}$ accordingly where $\varepsilon_k^{(n)} \to 0$, $k = 1, 2$, when $n \to \infty$. Thereby, $Y_k := Y_{kU}$ and $X := X_U$ are new random variables whose distributions depend on U in the same way as the distributions of Y_{ki} and X_i depend on i. □

Up to now the auxiliary random variable U is defined on a set $\mathcal{U}$ with arbitrary cardinality. Next, we will show that $|\mathcal{U}| = 2$ is enough.

3.2.3 Cardinality of set $\mathcal{U}$

With Fenchel–Bunt's extension of Carathéodory's theorem it follows that any rate pair in the convex hull $\operatorname{co}\big(\bigcup_{p(x)} \mathcal{R}(p(x))\big) = \bigcup_{u \in \mathcal{U}} \mathcal{R}(p(x|u))$ is achievable by time-sharing between two rate pairs from $\bigcup_{p(x)} \mathcal{R}(p(x))$, i.e. $|\mathcal{U}| = 2$ is enough.

Theorem 3.9 ([HUL01, Theorem 1.3.7]). *If $\mathcal{S} \subset \mathbb{R}^n$ has no more than n connected components (in particular, if $\mathcal{S}$ is connected), then any $x \in \operatorname{co}(\mathcal{S})$ can be expressed as a convex combination of n elements of $\mathcal{S}$.*

Since for any $x \in \mathcal{X}$ we have $[0, 0] \in \mathcal{R}(p(x))$, the set $\bigcup_{p(x)} \mathcal{R}(p(x))$ is connected. Therefore, any rate pair in $\mathcal{C}_{\mathrm{BC}} = \operatorname{co}\big(\bigcup_{p(x)} \mathcal{R}(p(x))\big)$ can be expressed as a convex combination of $n = 2$ rate pairs of $\bigcup_{p(x)} \mathcal{R}(p(x))$. □

This finishes the proof of the capacity region of the bidirectional broadcast channel.

Remark 3.10. *Since the coding theorem includes the achievability of rate pairs in terms of the average probability of error and the proof of the weak converse for the average error works analogously, $\mathcal{C}_{\mathrm{BC}}$ is also the capacity region in terms of the average probability of error.*

Remark 3.11. *The characterization of the bidirectional broadcast capacity region for Gaussian channels is analogous. We would have to deal with discrete channels with Gaussian channel transfer distributions and would have to add an input power constraints but the arguments are similar to the arguments considered here.*

The coding principles of the bidirectional broadcast are similar to the network coding approach with a bitwise XOR operation on the decoded messages at the relay node [WCK05, LJS05, FBW06, PY06, HKE$^+$07]. But since network coding is originally a multi-terminal source coding problem, the achievable rates with network and channel code separation in the broadcast phase using the XOR coding approach are limited by the worst receiver. This means that with a network coding approach using the XOR operation on the decoded data we can achieve in our network the rates

$$R_{\overrightarrow{\mathrm{R2}}}, R_{\overrightarrow{\mathrm{R1}}} \leq \min\{I(X;Y_1), I(X;Y_2)\}$$

for some common input distribution $p(x)$. The achievable rates depend on the common input distribution and *both* channel transfer distributions. For our coding scheme each achievable rate depends on the common input distribution and its own channel transfer distribution only. For each channel we can separately find the optimal input distribution which achieves the maximal achievable rate for this link (equal to the single link capacity), but the optimal input distribution for one channel need not be optimal for the other channel.

Accordingly, we see that the network coding approach using XOR on the decoded messages at the relay node is in general inferior, but it achieves the capacity of the bidirectional broadcast if and only if for the maximizing input distribution $p^\star(x) = \arg\max_{p(x)} \max\{I(X;Y_1), I(X;Y_2)\}$ we have $I(X;Y_1) = I(X;Y_2)$. In the following we will look at the bidirectional achievable rate region.

3.3 Achievable Bidirectional Rate Region

We will now look at the achievable bidirectional rate region where we use in each phase an optimal coding strategy. Thereby, we optimize the time division between the MAC phase with memoryless multiple access channel $p(y|x_1, x_2)$ and BC phase with memoryless broadcast channel $p(y_1, y_2|x)$. Of course, due to the a priori separation into two phases, this strategy need not be the optimal strategy for the bidirectional relay channel.

Let R_1 and R_2 denote the achievable rates for transmitting a message w_1 from node 1 to node 2 and a message w_2 from node 2 to node 1 with the support of the relay node. In more detail, node 1 wants to transmit message w_1 with rate nR_1 in n channel uses of the bidirectional relay channel to node 2. Simultaneously, node 2 wants to transmit message w_2 with rate nR_2 in n channel uses to node 1. Then let n_{MAC} and $n_{\mathrm{BC}} = n - n_{\mathrm{MAC}}$ denote the number of channel uses in the MAC phase and BC phase with the property $\frac{n_{\mathrm{MAC}}}{n} \to \alpha \in [0,1]$ and $\frac{n_{\mathrm{BC}}}{n} \to 1-\alpha$ when $n \to \infty$, respectively. As before, we call α the time division parameter the between multiple access and broadcast phase. With a sufficient block length n (respectively n_{MAC} and n_{BC}) we can achieve a bidirectional transmission of messages w_1 and w_2 with arbitrary small decoding error if rate pairs $[R_{\overrightarrow{1\mathrm{R}}}, R_{\overrightarrow{2\mathrm{R}}}] \in \mathcal{C}_{\mathrm{MAC}}$ and $[R_{\overrightarrow{\mathrm{R}2}}, R_{\overrightarrow{\mathrm{R}1}}] \in \mathcal{C}_{\mathrm{BC}}$ exist so that we have

$$nR_1 \leq \min\{n_{\mathrm{MAC}} R_{\overrightarrow{1\mathrm{R}}}, n_{\mathrm{BC}} R_{\overrightarrow{\mathrm{R}2}}\},$$
$$nR_2 \leq \min\{n_{\mathrm{MAC}} R_{\overrightarrow{2\mathrm{R}}}, n_{\mathrm{BC}} R_{\overrightarrow{\mathrm{R}1}}\}.$$

Thus, the achievable rate region of the bidirectional relay channel using the time division is given by the set of all rate pairs $[R_1, R_2]$ which are achievable with any time division parameter $\alpha \in [0,1]$ as $n \to \infty$. We collect the previous consideration in the following proposition.

Proposition 3.12. *The achievable rate region $\mathcal{R}_{\mathrm{BRC}}$ of the two-phase bidirectional relay channel is given by*

$$\begin{aligned}\mathcal{R}_{\mathrm{BRC}} = \Big\{[R_1, R_2] \in \mathbb{R}^2 : R_1 &\leq \min\{\alpha R_{\overrightarrow{1\mathrm{R}}}, (1-\alpha) R_{\overrightarrow{\mathrm{R}2}}\},\\ R_2 &\leq \min\{\alpha R_{\overrightarrow{2\mathrm{R}}}, (1-\alpha) R_{\overrightarrow{\mathrm{R}1}}\} \text{ with } \alpha \in (0,1),\\ [R_{\overrightarrow{1\mathrm{R}}}, R_{\overrightarrow{2\mathrm{R}}}] &\in \mathcal{C}_{\mathrm{MAC}}, \text{ and } [R_{\overrightarrow{\mathrm{R}2}}, R_{\overrightarrow{\mathrm{R}1}}] \in \mathcal{C}_{\mathrm{BC}}\Big\}.\end{aligned}$$

The region $\mathcal{C}_{\mathrm{BC}}$ is in general larger than the broadcast region using superposition encoding where we additionally require separated information flows and XOR on the decoded data at the relay node. It follows that the achievable bidirectional rate region $\mathcal{R}_{\mathrm{BRC}}$ is larger as well.

3.4 Example with Binary Channels

Finally, we will briefly look at an example with binary channels. Therefore, we assume to have a binary erasure multiple access channel [GW75],[CT91, Example 14.3.3] in the MAC phase and two independent binary symmetric channels [Ash65, Section 3.3] in the BC phase.

Binary Erasure Multiple Access Channel

We briefly reproduce the definitions and the resulting capacity region of the binary erasure multiple access channel from [GW75] and [CT91, Example 14.3.3]. Therefore, we have binary input alphabets $\mathcal{X}_1 = \mathcal{X}_2 = \{0, 1\}$, an output alphabet $\mathcal{Y}_\mathrm{R} = \{0, 1, E\}$, and transition probabilities

$$P_{Y_\mathrm{R}|X_1,X_2}(0|0,0) = 1, \qquad P_{Y_\mathrm{R}|X_1,X_2}(1|1,1) = 1,$$
$$P_{Y_\mathrm{R}|X_1,X_2}(E|0,1) = 1, \qquad P_{Y_\mathrm{R}|X_1,X_2}(E|1,0) = 1$$

so that we have a deterministic channel. We see that if the output of the channel is the erasure symbol E the relay node cannot uniquely determine the input. With the input distribution given by

$$P_{X_1}(0) = p_1, \qquad \text{and} \qquad P_{X_2}(0) = p_2,$$

it can be easily seen that the mutual informations $I(X_1; Y_\mathrm{R}|X_2) = H(X_1)$, $I(X_2; Y_\mathrm{R}|X_1) = H(X_2)$ and $I(X_1, X_2; Y_\mathrm{R}) = H(Y_\mathrm{R})$ are simultaneously maximized when $p_1 = p_2 = 1/2$. Then the capacity region is given by

$$\mathcal{C}_\mathrm{MAC} = \big\{[R_1, R_2] \in \mathbb{R}_+^2 \,:\, R_1 \leq 1,\ R_2 \leq 1, R_1 + R_2 \leq 1.5\big\} \tag{3.4}$$

which is shown in Figure 3.2.

Remark 3.13. *Gaarder and Wolf introduced this binary erasure multiple access channel in [GW75] to show that a noiseless feedback for the binary erasure multiple access channel can enlarge the capacity region. Accordingly, we can use this channel with a lossless broadcast channel to construct an example which shows that allowing cooperation between the encoders enlarges the bidirectional achievable rate region.*

Binary Symmetric Broadcast Channel

The binary symmetric broadcast channel consists of two independent binary symmetric channels [Ash65, Section 3.3]. For the binary symmetric broadcast channel we have binary input and output alphabets $\mathcal{X} = \mathcal{Y}_1 = \mathcal{Y}_2 = \{0, 1\}$ and transition probabilities

$$P_{Y_1|X}(0|0) = P_{Y_1|X}(1|1) = q_1, \qquad \text{and} \qquad P_{Y_2|X}(0|0) = P_{Y_2|X}(1|1) = q_2,$$

so that we have a symmetric channel, i.e. $H(Y_k|X)$ does not depend on the input distribution. Accordingly, the problem of maximizing $I(Y_k|X) = H(Y_k) - H(Y_k|X)$, $k = 1, 2$, reduces to the problem of maximizing $H(Y_k)$, $k = 1, 2$. Then it is easy to see that if all input symbols of a symmetric channel are equally likely, the output symbols are also equally likely. Therefore, the uniform input distribution maximizes both channels simultaneously so that the broadcast capacity region for the binary symmetric channel is given by

$$\mathcal{C}_\mathrm{BC} = [0, 1 - H(p_2)] \times [0, 1 - H(p_1)] \tag{3.5}$$

with $H(p_k) = -p_k \log(p_k) - (1 - p_k) \log(1 - p_k)$, $k = 1, 2$. The capacity region $\mathcal{C}_\mathrm{BC}$ is shown in Figure 3.2. Obviously, it includes the region $[0, 1 - \max\{H(p_1), H(p_2)\}] \times [0, 1 - \max\{H(p_1), H(p_2)\}]$ which is achievable using XOR on the decoded data at the relay node.

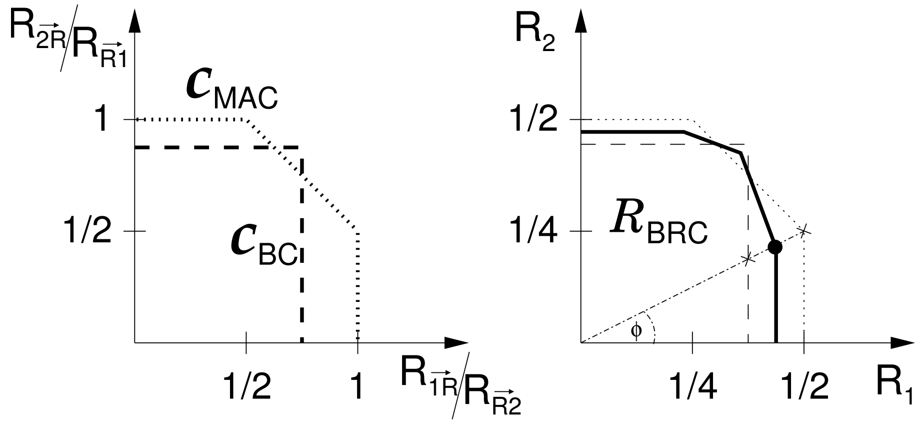


Figure 3.2: The left figure shows the capacity regions $\mathcal{C}_{\mathrm{MAC}}$ (dotted line) and $\mathcal{C}_{\mathrm{BC}}$ (dashed line), the right figure shows the corresponding achievable rate region $\mathcal{R}_{\mathrm{BRC}}$ (solid line). The dashed-dotted line exemplarily shows for one angle ϕ the achievable rate pair (•) on the boundary of $\mathcal{R}_{\mathrm{BRC}}$ with the optimal time division between the two rate pairs ($\times$) on the boundary of $\mathcal{C}_{\mathrm{MAC}}$ and $\mathcal{C}_{\mathrm{BC}}$.

Achievable Bidirectional Rate Region

In Figure 3.2 we depicted the capacity regions $\mathcal{C}_{\mathrm{MAC}}$ and $\mathcal{C}_{\mathrm{BC}}$ and the achievable rate region $\mathcal{R}_{\mathrm{BRC}}$ with a symmetric binary erasure multiple access channel, c.f. (3.4), and a binary symmetric broadcast channel, cf. (3.5). The boundary of the achievable rate region can be obtained geometrically if one takes for any angle $\phi \in [0, \pi/2]$ half of the arithmetical mean between the boundary rate pairs of the capacity regions where we have $\tan\phi = R_{\overrightarrow{2\mathrm{R}}}/R_{\overrightarrow{1\mathrm{R}}} = R_{\overrightarrow{\mathrm{R}1}}/R_{\overrightarrow{\mathrm{R}2}}$.

3.5 Discussion

The proposed coding scheme follows the network coding idea and treats information flows in a network not as physical fluids. But since network coding assumes error-free links between the nodes it does not consider channel coding aspects. Due to the half-duplex assumption we will have a natural separation of the protocol into two phases. In this thesis we assume that the encoders of nodes 1 and 2 cannot cooperate and that the relay node has to decode the messages. Then the optimal coding strategy for the MAC phase is well-known. We prove an optimal coding strategy for the remaining BC phase, which gives us the capacity region for a network with the assumed simplifications. It follows that this strategy is superior to the prior coding strategies based on superposition encoding or XOR operation on the decoded data at the relay node.

For a given broadcast input distribution we achieve on each link the corresponding discrete memoryless channel rate. But it is important to notice that the achievable rates are coupled by the common input distribution, which means that we cannot optimize the rates on each link separately. Then it is curious to see that if we transfer the result to scalar Gaussian channels with a mean power constraint, obviously the Gaussian input distribution will maximize both links simultaneously. However, for the vector valued Gaussian channel this is no longer the case. For the optimal input distribution we can apply the same arguments as for the MIMO MAC in Section 2.6.1. For each MIMO channel there is an optimal input distribution. This leads to two boundary rate pairs where at each rate pair one rate is maximal. Then the boundary inbetween the two rate pairs will be curved and can be calculated by convex optimization methods.

Further Results

Finally, we want briefly mention two further results which we derived in this context. The the main contribution to the the strong converse result is due to Igor Bjelakovic and to the practical coding aspects is due to Clemens Schnurr.

Strong Converse

In [BOSB07, OBSB07] we prove the strong converse for the maximum error probability and show that this implies that $[\varepsilon_1, \varepsilon_2]$-capacity region in terms of average probability of error is constant for small values of error parameters ε_1 and ε_2.

Let $\mathcal{C}_{\mathrm{BC,max}}(\varepsilon_1, \varepsilon_2)$ denote the set of all achievable rate pairs with maximum errors less than ε_1 and ε_2 as $n \to \infty$. Then we can prove the strong converse for the memoryless bidirectional broadcast channel which means that we have $\mathcal{C}_{\mathrm{BC}} = \mathcal{C}_{\mathrm{BC,max}}(\varepsilon_1, \varepsilon_2)$ for all $\varepsilon_1, \varepsilon_2 \in (0, 1)$. The proof is based on the *blowing-up technique* of Ahlswede, Gács, Körner [AGK76]. Therefore, we "blow-up" the decoding sets and use the Blowing-up Lemma to bound the error event in a variant of Fano's inequality. Then we follow the line of the weak converse.

Similarly, we define $\mathcal{C}_{\mathrm{BC,av}}(\varepsilon_1, \varepsilon_2)$ as the capacity region for the average errors. For sufficiently small $\varepsilon_1, \varepsilon_2$ we can construct a sub-code with bounded maximum error probability from which it follows from the strong converse for the maximum error that the corresponding rate pair has to be within $\mathcal{C}_{\mathrm{BC}}$. As a corollary we have that for the memoryless bidirectional broadcast channel it holds that $\mathcal{C}_{\mathrm{BC}} = \mathcal{C}_{\mathrm{BC,av}}(\varepsilon_1, \varepsilon_2)$ for all $[\varepsilon_1, \varepsilon_2] \in (0, \frac{1}{2}) \times (0, \frac{1}{4})$ or $[\varepsilon_1, \varepsilon_2] \in (0, \frac{1}{4}) \times (0, \frac{1}{2})$.

Practical Coding Aspects

The coding theorem offers a hint how to design a good practical channel code. Accordingly, in [SOS07] an interesting coset coding strategy for symmetric channels is discussed.

The coding theorem suggests to design for each receiving node and for each side information one triplet a code, an encoder, and a decoder. Let $\{E_{1,w_1}(w_2)\}_{w_2 \in \mathcal{W}_2}$ denote the code for node 1 with side information $w_1 \in \mathcal{W}_1$ and similarly let $\{E_{2,w_2}(w_1)\}_{w_1 \in \mathcal{W}_1}$ denote the code for node 2 with side information $w_2 \in \mathcal{W}_2$. Then the coding theorem requires that the two code families are interwoven so that we have $E_{1,w_1}(w_2) = E_{2,w_2}(w_1)$ for each message pair $[w_1, w_2] \in \mathcal{W}_1 \times \mathcal{W}_2$ the relay wants to transmit.

Then we make the observation that for symmetric channels ("channel distortion" is input independent) the cosets of the code perform as good as the original code. This suggests that we can construct the code families for each user as coset codes by simply shifting the base codes. If the two single codes base on an Ablelian additive group this leads to a simple superposition of the codewords in the group, i.e. $\psi(w_1, w_2) = E_1(w_2) + E_2(w_1)$. Then each code will perform as good as the base code. We also present a simple counter example which shows that this construction rule does not extend to non-symmetric channels.

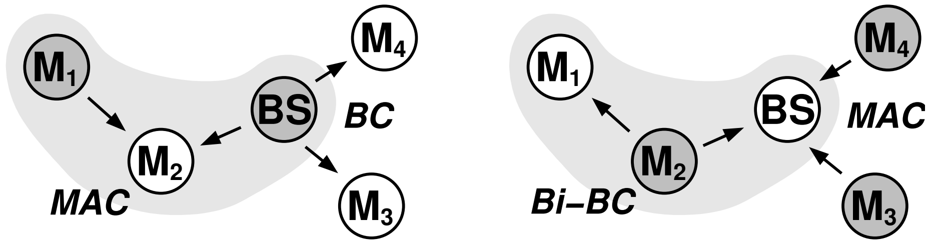
4 Conclusion

Historically, information theory has played a crucial role in the development of one-way point-to-point communication. With the development of the wireless networks and the Internet the research focus has recently shifted to multi-terminal problems. A one-hop transmission over a long distance or of a shadowed node needs high transmit power which causes a large interference for a wide range and results in a high energy consumption. Relaying protocols have the potential to enhance the coverage and the throughput in wireless networks by utilizing the broadcast nature of the wireless medium. For that reason, it is common sense that relaying concepts will play a central role in future wireless communication systems.

In this thesis we study bidirectional decode-and-forward relaying which has the ability to reduce the spectral loss due to the half-duplex constraint of wireless nodes. From the half-duplex constraint the relay communication is naturally separated into two phases where the relay node either transmits or receives. The two-phase separation of the protocol has the appealing property that the bidirectional relaying protocol can be easily integrated in a conventional cellular network or extended to multi-hop communication as depicted in Figure 4.1 and Figure 4.2. Accordingly, we think that the study of the two-phase bidirectional decode-and-forward protocol is of great interest. We consider the optimal and fixed time division case. The optimal case is interesting on its own, while the fixed may be interesting if the bidirectional protocol should be integrated in an existing wireless network.

The scarce of radio resources and the limited energy supply of mobile nodes in wireless networks make efficient resource and power allocation strategies absolutely necessary. It is widely known that a spectrally efficient wireless network cannot be designed with a layered architecture where we optimize each layer separately. Furthermore, the performance can be substantially enhanced if we allow interaction between layers, which leads to the cross-layer design concept. In Chapter 2 we studied different cross-layer design aspects for bidirectional relaying based on the achievable rates using the superposition encoding strategy at the relay node.

From Section 2.3 we see that for the maximal throughput over a bidirectional relaying link, the optimal resource allocation not just aims for the maximal sum-rate with respect to the channel states. An optimal resource allocation policy depends on the achievable rate region and factors in the traffic of the higher layers. This means that a throughput optimal



(a) MAC phase/cellular downlink (b) BC phase/cellular uplink

Figure 4.1: The integration of the bidirectional broadcast channel (Bi-BC) can be used for coverage extension in a cellular network, where BS denotes the basestation and M_i, $i = 1, 2, 3, 4$ denote the mobile terminals. Mobile M_2 works as the relay node. In the cellular downlink the bidirectional relay channel works in its MAC phase, while M_1 may cause interference at other nodes. In the cellular uplink the bidirectional relay channel works is in its broadcast phase (Bi-BC).

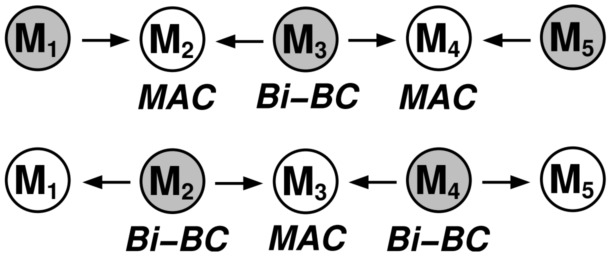


Figure 4.2: Bidirectional relaying can be easily extended to multi-hop communication. In the upper figure, node M_1, M_3, and M_5 transmit and M_2 and M_4 receive. The signal of node M_1 and M_5 cause interference at nodes M_4 and M_2 respectively. In the lower figure, node M_2 and M_4 transmit and M_1, M_3, and M_5 receive. The signal of node M_2 and M_4 cause interference at nodes M_5 and M_1 respectively. (Chain topology as proposed in [KRH$^+$06])

resource allocation policy depends, of course, on the channel states but also on the load at each node.

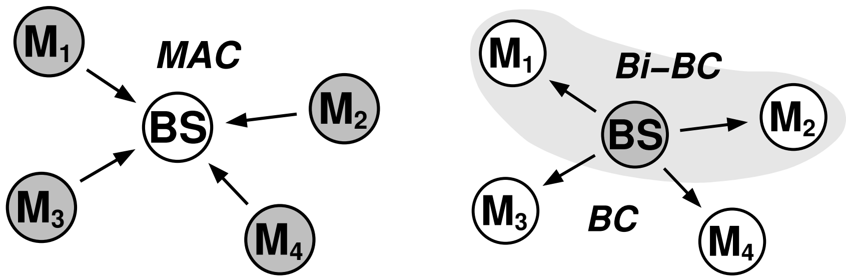
This insight has a direct consequence for the routing problem in a wireless network considered in Section 2.4. While for unidirectional relay protocols we select the relay node which results in the largest achievable rate, for bidirectional protocols we have to select the relay node with respect to the achievable rate pair at which we want to operate. It is therefore a vector optimization problem. Moreover, this means that if we apply the throughput optimal policy from Section 2.3 the route we choose depends again on the channel state as well as on the load, which therefore leads to a load adaptive routing strategy. Furthermore, we see that the performance can be improved if we allow time-sharing between the usage of relay nodes.

However, the scaling law of the ergodic rate region for relay nodes with iid Rayleigh fading channels shows that the spatial diversity gain decreases with increasing number of relay nodes. Since in real wireless networks the channels of the relay nodes are usually not identically distributed we conclude that for a practical implementation it will be sufficient to decide between a small number of preselected relay nodes.

In Section 2.5 we see from the joint resource allocation for the bidirectional relaying and an additional relay multicast communication that it is always optimal to decode the relay message first. Although we specified the total sum-rate maximum for this simple problem, we see that the closed form discussion is tedious and it is probably impossible for more or more difficult routing problems. This lets us conclude that joint optimization of multiple routing tasks always improves the overall performance and makes new trade-offs possible, but it also increases the complexity of the problem.

The previous insights were obtained from the discussion of single-antenna nodes, but all principles transfer to the multiple antenna case which we studied in Section 2.6. The difficult optimal transmit strategy for the vector valued processing in the MIMO-MAC prevents a closed form discussion as in the scalar case. Nevertheless, multiple antennas improve the achievable rate for each link in the usual manner so that it follows that the achievable rates linearly scale in the high power regime according to the spatial degrees of the MIMO channels and the time division between the phases.

In Chapter 3 we find the optimal channel coding strategy with respect to asymptotically vanishing maximal probability of error for the bidirectional broadcast channel of the bidirectional decode-and-forward protocol without feedback. The important improvement to the superposition coding approach in Chapter 2 is that we do not treat information as a fluid, which is in accordance with the philosophy of network coding [ACLY00]. Although we obtained the results for discrete memoryless channels with finite-size alphabets, the coding strategy can be easily transfered using standard arguments to continuous Gaussian channels with an input mean power constraint.



(a) MAC phase/cellular uplink (b) BC phase/cellular downlink

Figure 4.3: If two nodes within one cell (here M_1 and M_2) want to communicate with each other, the integration of the coding idea of the bidirectional broadcast channel (Bi-BC) can improve the downlink performance. Thereby, the base-station works as the relay node.

For multi-terminal channel coding problems it generally makes a difference if one considers achievable rates with respect to the average or maximal probability of error. We follow the classical strategy and extract a sub-code of essentially the same size with vanishing maximal error probability from the code which suffices the average error probability condition. Thereby, we surprisingly do not rely on a Cartesian mapping of the message indexes at the relay node. This is because of the special distribution of the knowledge about the messages for the bidirectional broadcast channel. But this will cause a difference for the achievable rates if one considers channel coding for other network problems, e.g. [Ahl71b].

A more explicit discussion of the obtained results is given in Sections 2.7 and 3.5.

Future Work

Unfortunately, we obtained the optimal coding result in Chapter 3 after we have studied the cross-layer design aspects presented in Chapter 2. However, all insights, techniques, and conclusions can be transfered to the optimal coding approach. In particular, all results based on the maximal unidirectional rates will be the same. It will be future work to characterize the results explicitly.

In Figure 4.3 we present an idea how bidirectional relaying can be used to improve the performance in a cellular wireless network. If two nodes in one cell want to communicate with each other we can use the broadcast coding techniques to improve the broadcast in the downlink. In the future we will work out this example and similar networks as depicted in Figure 4.1 and Figure 4.2.

It follows that bidirectional relaying can be applied to cellular, ad-hoc, or hybrid networks in order to enhance the coverage or increase the overall throughput and we are sure that there are more situations where the bidirectional relaying protocol can be integrated in a meshed wireless network architecture. Moreover, we see that this will again open up new possibilities for further improvement. For example in Figure 4.1 the transmission of M_1 interferes with the broadcast of the basestation so that node M_1 causes additional interference at other nodes in the cell. If those regard the interference as additional information, it should be possible to improve the overall performance again. This example illustrates the potential improvement of the performance in a meshed architecture, which makes a careful study necessary. But we also see that these concepts will increase the system‘s complexity. For that reason it will be necessary to specify conditions where additional cooperation is still beneficial in a wireless system.

A more closely related extension to our results is to find the optimal coding strategy if we drop the assumptions we made in Chapter 3. In particular this means that we do not require that the relay node has to decode the messages and we allow the encoders at nodes 1 and 2 to cooperate via a feedback.

In some situations in this thesis we considered the averages of the achievable rates with a short-term (over a frame) average power constraint. Therefore, it would be interesting to consider adaptive resource allocation policies which take advantage of a time-varying channel as done in [KH95, GV97, TH98, BPS98]. Next, it would be interesting if we allow queueing at the relay node. Additional queueing and adaptive resource allocation policies are closely related to problems concerning the delay.

For an implementation we have to re-evaluate the performance where we take into account the costs for providing the system state information at a centralized controller and/or other nodes. In accordance, for adaptive resource allocation policies it would be interesting to find decentralized decision strategies based on the local information or to develop new strategies which base on partial knowledge only.

Of course, one has to answer many further questions before considering a practical implementation. However, we are absolutely convinced that further studies on bidirectional relaying will pay off, because we think that bidirectional relaying is conceptually a wise approach. It avoids the spectral loss of two separated unidirectional relaying protocols due to the half-duplex constraint, and in addition to this it can fully exploit the network coding idea. All this lets us finally conclude that bidirectional relaying is spectrally efficient and has the potential to enhance the throughput and coverage in wireless networks.

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