

Strain localization in two-dimensional lattices

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Abstract. Two-dimensional localized strain wave solutions of the nonlinear equation for shear waves in two-dimensional lattices are studied. The corresponding equation does not possess an invariance in one of the spatial direction while its exact plane traveling wave solution does not reflect that. However, the numerical simulation of a two-dimensional localized wave reveals a non-symmetric evolution.

Introduction

Recently it was found that the Fourier based parabolic law of heat conduction is insufficient in some cases [1]. Other heat conduction equations were developed, in particular, a hyperbolic one called “ballistic heat equation” was obtained as a mathematical consequence of the equations of lattice dynamics [2, 3]. The equation was obtained for a one-dimensional lattice chain. A development of this approach is needed in two-dimensional lattice models. Recently various two-dimensional nonlinear lattices were studied in [4, 5]. As a rule, an analytical study of the discrete nonlinear problem is impossible, and a continuum limit was applied in [4, 5] in order to obtain the governing nonlinear partial differential equations for the strains.

It was found in [4, 5] that longitudinal strains obey the famous Kadomtsev-Petviashvili equation [6] whose solutions including localized ones are well known. An equation with other features has been derived for shear strains. First studies of it were performed in [4, 5] for a transverse instability of plane shear waves. In this paper, we consider the dynamics of two-dimensional localized shear strain waves. Of special interest is the non-symmetric propagation of localized waves.

1. Equation for shear strains and its solution

The governing equation for shear waves reads [4, 5]

$$v_{xt} + c_1 v_{xxxx} + c_2 v_x^2 v_{xx} + \lambda v_{yy} + \gamma v_y v_{xx} = 0, \quad (1)$$

where $v(x, t)$ is a shear displacement (for details see Ref. [5]), c_i , λ , γ are the constants.

One can see that Eq. (1) is not invariant under the transformation $y \rightarrow -y$. Consider its plane traveling wave solution depending only on the phase variable $\xi = x + my - Vt$ and assume



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that $u = v_x$. The solution, obtained by using direct integration, reads

$$u = \frac{A}{Q + \cosh(\beta\xi)}, \quad (2)$$

where

$$A_{\pm} = \pm \frac{6c_1\beta^2}{\sqrt{\gamma^2 m^2 + 6c_1 c_2 \beta^2}}, \quad Q_{\pm} = \pm \frac{\gamma m}{\sqrt{\gamma^2 m^2 + 6c_1 c_2 \beta^2}}, \quad V = \beta^2 c_1 + m^2 \lambda, \quad (3)$$

where β is a free parameter. We consider only bounded solutions when $Q < 1$. This is guaranteed if $c_1 c_2 > 0$, and the solutions with both Q_+ and Q_- being bounded. Otherwise only positive Q_{\pm} give rise to a bounded solution. The sign of Q depends on the sign of the parameter m responsible for the angle of the wave front propagation relative to the x -axis and the sign of the denominator. However, the presence of \pm in the expression for Q_{\pm} discards the dependence on the sign of m . The sign of the amplitude $A/(Q+1)$ of the bounded solution is not sensitive to the sign of m . For the wave propagating along x axis we have $m = 0$ and $Q = 0$. Then the

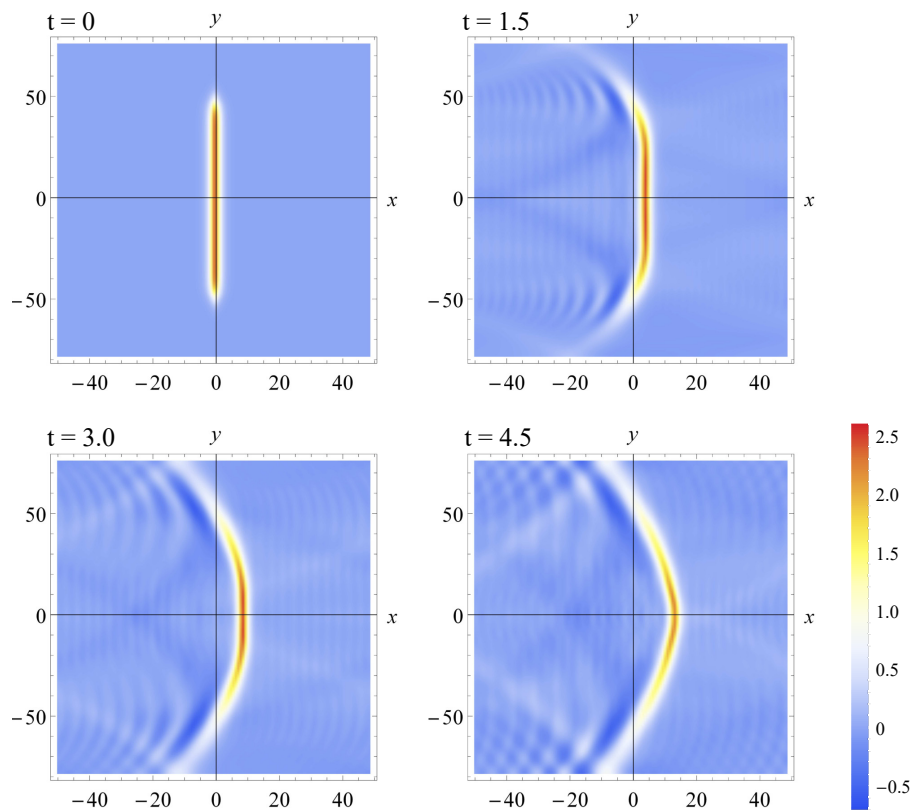


Figure 1. Evolution of the shear strain wave(6) in Case 1.

familiar solitary wave solution to the modified Korteweg-de Vries equation appears from (2),

$$u = \pm \frac{6c_1\beta}{\sqrt{6c_1 c_2}} \cosh^{-1}(\beta\xi) \quad (4)$$

that exists only when $c_1 c_2 > 0$.

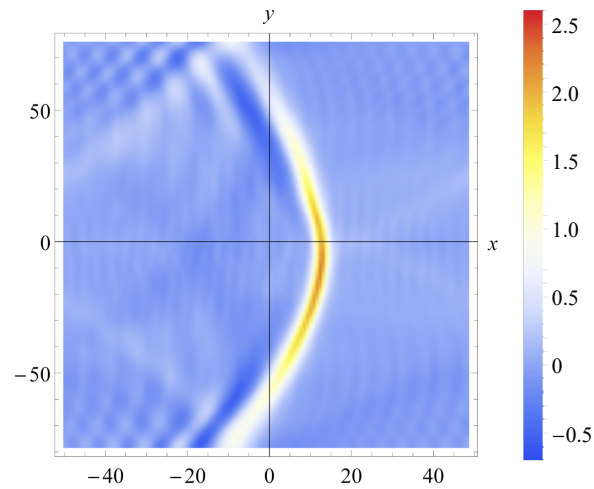


Figure 2. Transformation to an asymmetric profile in Case 2 at $t = 4.5$.

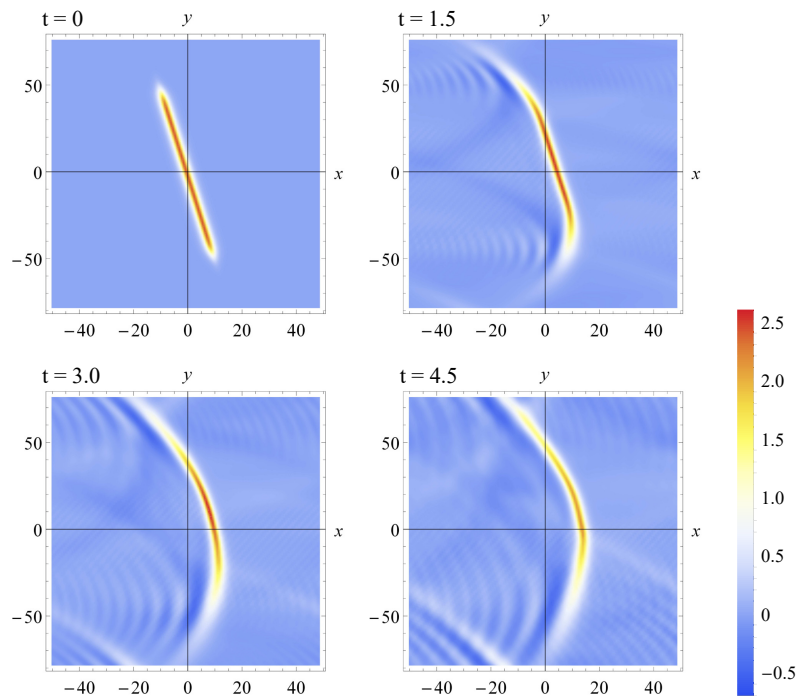


Figure 3. Asymmetry for the initial condition corresponding to Case 3.

2. Numerical study of nonlinear shear strains localization

Let us re-write Eq. (1) through the shear strain,

$$\partial_x \left(\partial_t u + c_1 \partial_{xxx} u + \frac{c_2}{3} \partial_x u^3 \right) + \lambda \partial_{yy} u + \gamma \partial_x \left(\partial_x u \int_{-\infty}^x \partial_y u dx' \right) = 0. \quad (5)$$

The Fourier spectral method is used here to solve Eq. (5) numerically. More detailed information about the method can be found in [5]. In all the simulations we put $c_1 = 1$, $c_2 = 1$ and $\lambda = 3$.

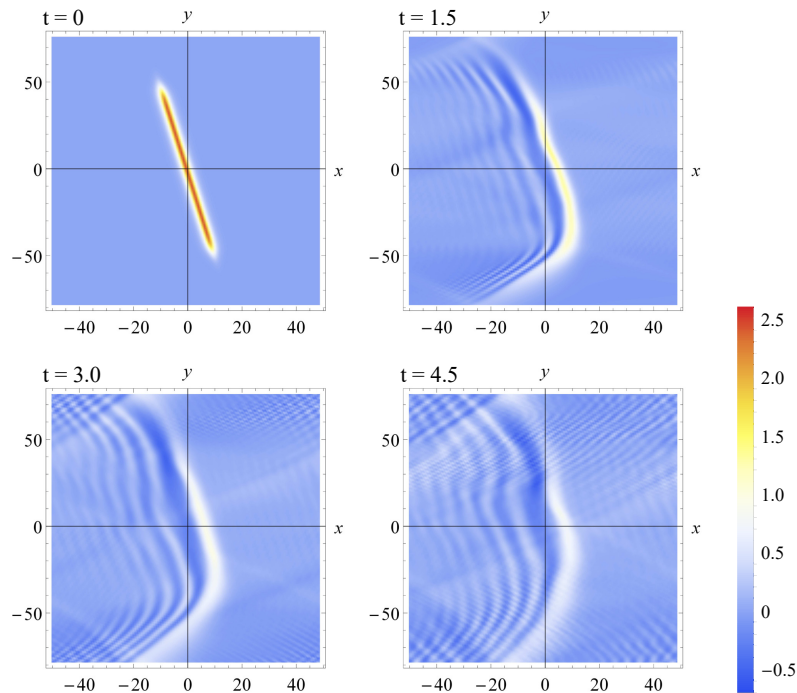


Figure 4. Scattering of the initial condition corresponding to Case 4.

The initial condition is assumed as

$$u_0 = \frac{\sqrt{6} \operatorname{sech}(X)}{10^{-8} \cosh(0.6Y) + 1}. \quad (6)$$

The values of X, Y and γ are varied as shown in Table 1.

	X	Y	γ
Case 1	x	y	0
Case 2	x	y	1
Case 3	$x \cos(0.2) + y \sin(0.2)$	$x \cos(0.2) + y \sin(0.2)$	1
Case 4	$x \cos(0.2) + y \sin(0.2)$	$x \cos(0.2) + y \sin(0.2)$	5

Table 1. Expressions of X, Y for the initial conditions.

Case 1. In Fig. 1 the symmetric relative to the x -axis propagation of the strain wave (6) is shown. A curvature of the wave profile develops in time and a scattering of the wave happens behind the curved wave front. The amplitude and the velocity of the wave decrease in time.

Case 2. The initial profile suffers from asymmetric variations relatively to the x -axis, see Fig. 2. The highest amplitude is not reached on the Ox axis. However, changing the sign of γ does not provide additional non-symmetry.

Case 3. The role of the quadratic nonlinearity on the initially inclined wave is explored in Fig. 3. It is quite similar to what is shown in Fig. 2. The presence of the nonlinear term changes the direction of wave propagation in the similar way as in Case 2. The amplitude and velocity also decrease in time.

Case 4. One can see in Fig. 4 that an increase in the value of γ results in a scattering behind the wave front.

3. Conclusions

Numerical simulations reveal an asymmetry of the two-dimensional localized initial condition due to the presence of the nonlinear term with the coefficient γ . It differs from the analytical solution for localized plane waves. However, variations of the sign of γ do not reveal an additional non-symmetry of the wave form.

4. Acknowledgment

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