Four Essays on Equilibrium Selection, Fair Allocation and Voting

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Abstract

This dissertation contains three chapters on equilibrium selection, fair allocation and voting.

The first paper in chapter 2 contains results that make it easier to use global games for deriving equilibrium predictions in games of strategic complementarities. Moreover in a second paper, we analyse the relationship between noise independence of this global game selection and the property of equilibria to be robust to incomplete information.

Chapter 3 considers the problem of achieving a fair and efficient allocation of indivisible goods. Here we find that a number of fairness criteria are incompatible with one another. Besides such impossibility results, we identify a new solution that satisfies a maximal number of our fairness criteria as well as Pareto efficiency.

Chapter 4 analyses voting procedures with respect to their ability to aggregate voters preferences despite the fact that voters may vote strategically. In particular we characterize the Borda Rule and Approval Voting according to a small number of intuitive axioms.

Zusammenfassung

Die vorliegende Dissertation umfasst drei Kapitel zur Gleichgewichtsauswahl, fairer Allokation und Wahlverfahren.

Die erste Arbeit in Kapitel 2 beinhaltet Resultate, die eine Anwendung Globaler Spiele zum Zweck der Herleitung von Vorhersagen zur Gleichgewichtsauswahl in Spielen mit strategischen Komplementen erleichtern. Darüber hinaus analysieren wir in einem zweiten Aufsatz die Beziehung zwischen der sogenannten Noise Independence dieser Vorhersagen und der Eigenschaft von Gleichgewichten robust unter unvollständiger Information zu sein.

Kapitel 3 widmet sich dem Problem unteilbare Güter fair und effizient zu allozieren. Hier stoßen wir auf Unvereinbarkeiten zwischen verschiedenen Fairnesskriterien. Neben diesen Unmöglichkeitsresultaten beschreiben wir eine neue Lösung, die eine maximale Anzahl der von uns identifizierten Fairnesskriterien erfüllt und gleichzeitig Pareto-Effizienz garantiert.

Kapitel 4 untersucht Wahlverfahren mit Blick auf ihre Fähigkeit selbst dann Wählerpräferenzen zu aggregieren, wenn Wähler sich strategisch verhalten. Insbesondere charakterisieren wir die Borda-Wahl und die Wahl durch Zustimmung anhand einer kleinen Zahl intuitiver Axiome.

CONTENTS

1. INTRODUCTION

This dissertation considers three diverse economic problems that all study the relationship – and potential conflict – between individual strategic decision-making and the implementation of solutions that satisfy certain normative criteria such as efficiency and equity. This conflict arises in particular where agents lack an effective coordination device or where preferences are private information that agents reveal only strategically.

Global Games and Equilibrium Selection

In Chapter 2 we focus on strategic uncertainty that arises in games of strategic complementarities. In such games, agents' strategies can be ordered in such a way that a move by some agents to a 'higher' strategy increases the incentives of others to also move to 'higher' strategies. Applications include speculative attacks, bank runs or investment problems and typically feature multiple - Pareto ranked - equilibria.

Here, global games are widely used to predict behaviour by selecting a particular equilibrium. For that, the original complete information game is embedded in a global game with a continuum of payoff relevant states on which players receive a noisy signal. As the noise in signals goes to zero, the global game can be seen as (locally) approximating the original complete information game and the unique rationalizable strategy profile in the global game may serve as an equilibrium refinement for the original game.

By changing payoff parameters of a complete information game and tracking the changes in the global game selection that result, we arrive at recommendations on how to achieve coordination on a Pareto efficient equilibrium. For example in a model of banking crises, global games can be used to show how a *partial* deposit insurance may be able to prevent the emergence of a bank run – the continued existence of an inefficient bank run equilibrium notwithstanding.

Another approach to apply global games in achieving a desired equilibrium outcome is given in Section 2.1.6. Here we consider the possibility of introducing new strategies that are able to shift the global game selection from one pre-existing equilibrium to another. For example in the refinancing game, we present a situation where 3 lenders are unable to coordinate on an efficient investment project when the only way to fund the project are unsecured loans – fearing that the unwillingness of others to extend such loans jeopardizes the project. Then, as we introduce costly secured loans as a third option, lenders will be willing to extend *un*secured loans. Intuitively, this shift between equilibria is due to reduced strategic uncertainty in that now lenders expect others to at least extend secured loans and fund the project.

Fair Solutions to the Random Assignment Problem

Chapter 3 studies the problem of assigning indivisible goods to agents where each is to receive one good. To guarantee fairness in the absence of monetary compensation, we consider random assignments and formulate a number of equity criteria in this context. Adherence to such formal equity criteria may be seen as particularly important where the goods in question are publicly provided (or subsidised) private goods – such as for example school seats. Here, neither individuals nor groups should be discriminated against and receive less that their 'fair share'.

Perhaps surprisingly, we find that while each of the identified equity criteria is compatible with Pareto efficiency, some equity requirements are in conflict with one another. Hence, no solution can be implemented that jointly satisfied these criteria even if agents truthfully reveal their preferences.

To bridge this gap, we introduce a new solution based on Walrasian equilibria from equal incomes that satisfies a maximal number of our identified equity criteria and guarantees Pareto efficiency with respect to reported preferences. Since it is based on Walrasian equilibria, the only way in which agents may gainfully misreport their preferences, is if they can thereby influence prices. Moreover, any price changes to ones' advantage need to be sufficiently large to overcome the disadvantage that arises from the fact that the market agent will now maximize another than the true preference relation, subject to the budget constraint. Thus in applications – in particular in large markets where the effect of ones' own report on equilibrium prices is small and hard to foresee – the new solution should be expected to elicit agents' true preferences.

Scoring Rules and Implementation in Iteratively Undominated Strategies

Chapter 4 considers a classical mechanism design problem where the conflict between individual strategic behaviour and the implementation of a desired social choice correspondence that maps agents preferences to aggregate outcomes takes center stage. Here, we characterize and compare voting procedures according to the social choice correspondences that they implement in iteratively undominated strategies. While this solution concept plays a prominent role in the literature on voting – where it is also known as sophisticated voting – a complete characterization of social choice correspondences that can be implement in this way, still outstanding. Restricting attention to elections with three alternatives and a finite number of voters who have strict preferences over alternatives we are able to derive 3 main characterization as well as 2 impossibility results:

First, in the class of positional scoring rules (including among others Plurality-, Antiplurality- and the Borda-Rule), the Borda Rule is the unique voting procedure implementing a social choice correspondence that satisfies *unanimity* **(U)** (i.e., elects an alternative whenever it is unanimously preferred) and is *majoritarian after elimi-*

nating a worst alternative **(MEW)** (i.e., if there is a unanimously disliked alternative, the majority-preferred alternative among the other two is elected).

Second, in the larger class of direct mechanism scoring rules (including e.g. all positional scoring rules as well as Approval Voting), Approval Voting is characterized by a single axiom – it is the unique voting procedure that is *majoritarian after eliminating a Pareto-dominated alternative* **(MEPD)** (i.e., if there is a Pareto-dominated alternative, the majority-preferred alternative among the other two is elected).

Third, in the class of direct mechanism scoring rules, the Borda Rule is the unique voting procedure implementing a social choice correspondence that satisfies **U**, **MEW** and *monotonicity* **(MON)**(i.e., an alternative that is elected for some preference profile should still be elected for a preference profile where every voter ranks this alternative weakly higher).

Finally, there exists no direct mechanism scoring rule implementing a social choice correspondence that satisfies both **MON** and **MEPD** or *Condorcet consistency* **(CON)** (i.e. an alternative that is majority preferred over the other two is elected).

2. GLOBAL GAMES AND EQUILIBRIUM SELECTION

2.1 Characterising Equilibrium Selection in Global Games with Strategic Complementarities

Section 2.1 has been published as

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2.1.1 Abstract

Global games are widely used to predict behaviour in games with strategic complementarities and multiple equilibria. We establish two results on the global game selection. First, we show that, for any supermodular complete information game, the global game selection is independent of the payoff functions chosen for the gameÊijs global game embedding. Second, we give a simple sufficient criterion to derive the selection and establish noise independence in many-action games by decomposing them into games with smaller action sets, to which we may often apply simple criteria. We also report in which small games noise independence may be established by counting the number of players or actions.

2.1.2 Introduction

Games of strategic complementarities, for instance models of financial crises or network externalities, often have multiple equilibria. An important issue, from a theoretical as well as a policy perspective, is how to predict the equilibrium that will be played. One widely used approach to predict behaviour in such games is to turn them into "global games". A global game extends a complete information game to an incomplete information game with a one-dimensional state space, such that the original game can be viewed as one particular realisation of the random state. This state is usually interpreted as an "economic fundamental". Each player receives a noisy private signal about the true state. Then, under certain supermodularity and monotonicity conditions,[1] Frankel, Morris and Pauzner [2003] ("FMP") prove *limit uniqueness*: as the noise in private signals vanishes, for almost all realisations of the

[1] FMP assume that there exists an ordering on actions such that players' incentive to switch to a higher action is increasing in both the state and the actions of others; and that at sufficiently low (high) states, each player's lowest (highest) action is strictly dominant. See Section 2.1.3 for precise definitions.

state parameter, players coordinate on a Nash equilibrium of the complete information game given by the payoffs at the true state. This *global game selection* ("GGS") may be used as a prediction and to derive comparative statics results. Applications include models of speculative attacks (Morris and Shin [1998], Cukierman et al. [2004], Guimaraes and Morris [2007], Corsetti et al. [2004], Corsetti et al. [2006]), banking crises (Goldstein [2005], Rochet and Vives [2004]), and investment problems (Sákovics and Steiner [2012]). Experiments by Heinemann et al. [2004, 2009] show that the GGS is useful for predicting subjects' behaviour.

Unfortunately, there are many ways to extend a complete information game to a global game, and which equilibrium is selected may depend on the details of the chosen extension. If multiple equilibria are replaced by multiple global game selections, this reduces the value of the GGS as a selection criterion. For instance, it is well-known that the GGS may depend on the distribution of the noisy private signals. To circumvent this problem, FMP provide some conditions under which the GGS is *noise independent*, that is, independent of this distribution.

Furthermore, most applications are limited to binary-action games. In symmetric binary-action games, the GGS can be easily determined as a player's best reply to the belief that the fraction of other players choosing either action is uniformly distributed (see FMP or Morris and Shin [2003]; Sákovics and Steiner [2012] give a generalisation for a class of asymmetric binary-action games).

In this paper, we extend results on global games in two directions. First, we show that for any supermodular complete information game, the GGS is independent of the extended payoff function, as long as it satisfies the usual supermodularity assumptions. Hence, the GGS does not depend, for example, on the choice of economic fundamental used as a state parameter in the global game. Our result implies that the distribution of private signals is the only source of multiplicity for the GGS.

Second, we provide a new and simple method to determine the GGS and check its noise independence in games with many actions. The GGS is noise independent if the game can be suitably decomposed into smaller noise independent games. For example, we may split up an n-action game into many binary-action games and apply simple known criteria for deriving the GGS. If the smaller games are noise independent and their selections point in direction of the same action profile, then this action profile is the noise independent selection of the larger game.

Our second result gives a new heuristic for the global game equilibrium prediction that is useful in economic applications. For instance, introducing a third action in a binary-action game may change the selection between the two original equilibria. We give an example of a refinancing model where introducing collateralised loans as a third action besides withdrawing and extending unsecured loans changes the GGS from the inefficient withdrawal equilibrium to that of *un*secured refinancing. Other models of interest may have many actions and potentially a large number of equilibria. As an example, we analyse a generalised version of Bryant's [1983] minimum effort game, to which none of the known easy heuristics may be applied. By decomposing it into binary-action games, we may derive the GGS and establish its noise independence straightforwardly.

The decomposition of large games is particularly useful, because noise independence is typically easier to establish for smaller games, often simply by counting the number of players and actions. In their seminal paper introducing global games, Carlsson and van Damme [1993] proved that all two-player two-action games with multiple equilibria are noise independent. In another application of our methods, we prove the same of all supermodular two-player games in which one player's action set is binary. Together with known results, this completes the characterisation of supermodular games for which noise independence can be established by counting.

Our paper proceeds as follows. Section 2.1.3 contains preliminaries. The rest of the paper is organised around a characterisation of the GGS given in Section 2.1.4. Instead of analysing a sequence of global games with vanishing noise, we show that we may determine the GGS from a single incomplete information game with simple payoffs. This game is independent of the state-dependent payoff function that is chosen as an extension of the original game. We use the characterisation to prove generic uniqueness of the GGS and extend it to discontinuous global games. In Section 2.1.5 we discuss the decomposition method and establish our results on noise independence. Section 2.1.6 contains applications. Section 2.1.7 concludes. Proofs are in Appendix A.

2.1.3 Setting and Definitions

Throughout this paper we consider games played by a finite set of players I, who have finite action sets $A_i = \{0, 1, \ldots, m_i\}$, $i \in I$, which we endow with the natural ordering. We define the joint action set A as $\prod_{i \in I} A_i$ and write A_{-i} for $\prod_{j \neq i} A_j$. For action profiles $a = (a_i)_{i \in I}$ and $a' = (a'_i)_{i \in I}$ in A, we write $a \leq a'$ if and only if $a_i \leq a'_i$ for all $i \in I$. The lowest and highest action profiles in A are denoted by 0 and m. A complete information game $\mathbf{g}$ is specified by its real-valued payoff functions $g_i(a_i, a_{-i})$, $i \in I$, where a_i denotes i's action and $a_{-i} \in A_{-i}$ denotes the opposing action profile. A game $\mathbf{g}$ is *supermodular* if for all i, $a_i \leq a'_i$, and $a_{-i} \leq a'_{-i}$,

$$g_i(a_i, a_{-i}) - g_i(a'_i, a_{-i}) \leq g_i(a_i, a'_{-i}) - g_i(a'_i, a'_{-i}). \tag{2.1}$$

This implies a player's best reply is non-decreasing in the actions of her opponents (see Topkis [1998]; note that property (2.1) is what FMP refer to as *strategic complementarities*).

We define a global game $G(v)$ as in FMP: consider payoff functions $u_i(a_i, a_{-i}, \theta)$ that depend on an additional state parameter $\theta \in \mathbb{R}$. For each θ, let the game given by the $u_i(\cdot, \theta)$ be a supermodular game (Assumption **A1**). In addition, assume there are *dominance regions*: there exist thresholds $\underline{\theta} < \overline{\theta}$ such that the lowest and highest actions in A_i are strictly dominant when payoffs are given by, respectively, $u_i(\cdot, \underline{\theta})$ and $u_i(\cdot, \overline{\theta})$ (Assumption **A2**). Furthermore, each u_i satisfies *state monotonicity* in the sense that higher states make higher actions more appealing (Assumption **A3**):

there exists $K > 0$ such that for all $a_i \leq a_i'$ and $\underline{\theta} \leq \theta \leq \theta' \leq \overline{\theta}$ we have

$$0 \leq K(a_i' - a_i)(\theta' - \theta) \leq \big(u_i(a_i', a_{-i}, \theta') - u_i(a_i, a_{-i}, \theta')\big) - \big(u_i(a_i', a_{-i}, \theta) - u_i(a_i, a_{-i}, \theta)\big).$$

Finally, each u_i is continuous in θ (Assumption **A4**). In Section 2.1.4.3, we will relax **(A3)** and **(A4)**.

In the global game $G(v)$, the state θ is realised according to a continuous density with connected support that includes the thresholds $\underline{\theta}$ and $\overline{\theta}$ in its interior. Players observe θ with some noise, and then act simultaneously. Formally, let $f = (f_i)_{i \in I}$ denote a tuple of probability densities, whose supports are subsets of $[-\frac{1}{2}, \frac{1}{2}]$. Each player i receives a private signal $x_i = \theta + v\eta_i$, where each η_i is drawn independently according to the density f_i, and $v \in (0, 1]$ is a scale factor. Thus, a global game $G(v)$ is specified by the payoff functions u_i, a prior distribution of states, a noise distribution f, and a scale factor v, each of which is common knowledge among players.

A strategy s_i is a function that maps a player's signal onto an action.[2] Joint strategy profiles are denoted by $s = (s_i)_{i \in I}$ and $s_{-i} = (s_j)_{j \in I - \{i\}}$. Slightly abusing notation, denote by $s(x)$ the joint action profile obtained when each player receives the same signal $x_i = x$. We write $s \leq s'$ if and only if $s(x) \leq s'(x)$ for all x. A strategy profile s is *increasing*, if each s_i is weakly increasing in x_i; it is a (Bayes-Nash) equilibrium if each $s_i(x_i)$ is a best reply against s_{-i}, given u_i and using Bayes' rule to derive the conditional densities of θ and x_{-i}, given x_i.

Following FMP, we also define the *simplified* global game $G^*(v)$, differing from the global game $G(v)$ in that its prior is uniform and payoffs are given by $u_i(a_i, a_{-i}, x_i)$ and thus depend directly on the signals. FMP prove that the equilibrium strategy profile in $G^*(v)$ is unique up to its points of discontinuity (Lemma A1 in FMP) while $G(v)$ may have multiple equilibrium strategy profiles. However, the key result on global games says that as the scale factor v goes to zero, the equilibrium strategy profiles of the games $G(v)$ and $G^*(v)$ all converge to the same limiting strategy profile, which is increasing and unique up to its finitely many discontinuities.

Fact 1. Theorem 1 in FMP. *The global games $G(v)$ and $G^*(v)$ have an essentially unique, common limit equilibrium strategy profile as the scale factor v goes to zero. More precisely, there exists an increasing pure limit strategy profile s such that, for each $v > 0$, if s_v is an equilibrium strategy profile of $G(v)$ and s_v^* is the unique equilibrium strategy profile of $G^*(v)$, then $\lim_{v \to 0} s_v(x) = \lim_{v \to 0} s_v^*(x) = s(x)$ for all x except possibly at the finitely many discontinuities of s.*

Since a global game's limit strategy profile s is well-defined up to its points of discontinuity, we use $\overline{s}$ and $\underline{s}$ to denote its right and left continuous versions.

[2] Focusing on pure strategies is without loss of generality and simplifies notation. Supermodularity implies that highest and lowest equilibrium strategy profiles exist in pure strategies and are bounds on any mixed strategy profile Since these two equilibria converge almost everywhere to a common limit as the scale factor v goes to zero (Theorem 0), surviving strategies may only mix on a null set of signals, which has no effect on other players' incentives.

Finally, we will use the following fact about supermodular incomplete information games. For a given game and strategy profile s, let $\beta(s)$ denote the joint upperbest reply to s, i.e. the profile in which each player uses her highest best reply to s_{-i}. Then, β is monotonically increasing in s. Moreover, if $s \leq \beta(s)$, the upperbest reply iteration $s, \beta(s), \beta(\beta(s)), \ldots$ converges monotonically to an equilibrium strategy profile. This applies in particular to the global games $G(v)$ and $G^*(v)$.

2.1.4 Equilibrium Selection via Global Games

Let $\mathbf{g}$ be a given supermodular complete information game. Often, such a game has multiple Nash equilibria. In that case, a global game gives a method to resolve equilibrium indeterminacy. Consider a global game, in which payoffs depend on a state parameter, and suppose that for some fixed state θ^*, these payoffs coincide with those of $\mathbf{g}$. We say that the global game *embeds* $\mathbf{g}$ at state θ^*. If – as the noise in private signals vanishes – the global game's limit equilibrium strategy profile is continuous at θ^*, its value at that state determines a particular Nash equilibrium of the complete information game.[3] More generally, consider the left continuous version $\underline{s}$ and right continuous version $\overline{s}$ of the limit equilibrium strategy profile. These determine two (perhaps distinct) Nash equilibria: $\underline{s}(\theta^*)$ and $\overline{s}(\theta^*)$.[4] We refer to them as the lowest and, respectively, highest *global game selection* (GGS, following Heinemann et al. [2009]). If they coincide, the GGS is *unique*.

In this section, we give a simple characterisation of the GGS, which shows that it depends only on the payoff functions of the complete information game $\mathbf{g}$ and the noise distribution of private signals f in the global game. The additional modelling choices of the global game – its prior and its payoff functions at states other than θ^* – do not affect the GGS. We also show that (for a given noise distribution) the lowest GGS is identical to the highest GGS for almost all supermodular games. Thus, the GGS is generically unique.

Let us first note that this approach applies to any supermodular complete information game.

Lemma 1. *For any supermodular game* $\mathbf{g}$*, there is a global game that embeds it.*

To see this, extend the payoff functions g_i of the complete information game with a state parameter θ by setting $u_i(a_i, a_{-i}, \theta) = g_i(a_i, a_{-i}) + \theta a_i$. The payoff functions u_i satisfy the global game assumptions **(A1)**–**(A4)**. Choose an appropriate prior with sufficiently wide support and some noise distribution f for private signals, and our claim is satisfied at $\theta = 0$.

[3] Theorem 2 in FMP implies that the limit strategy profile selects a Nash equilibrium of $\mathbf{g}$ if it is continuous at θ.

[4] Continuity of the payoff functions u_i ensures that $\underline{s}(\theta^*)$ and $\overline{s}(\theta^*)$ are also Nash equilibria of $\mathbf{g}$.

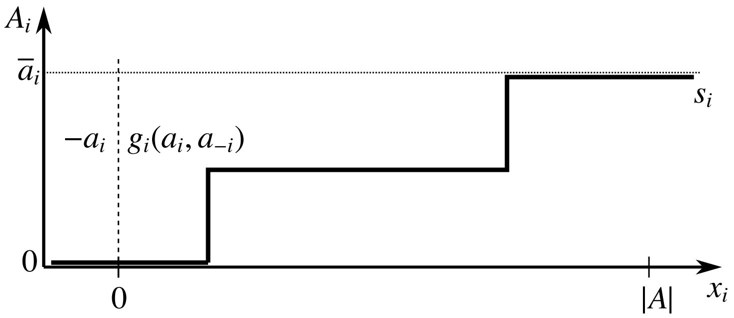


Fig. 2.1: The lower-f-elaboration E and the attained action $\overline{a}_i$

2.1.4.1 A Characterisation of the Global Game Selection

We now show that the GGS induced by the global game may be determined without analysing the full global game under vanishing noise. Instead, we introduce a much simpler game that allows us to establish the GGS more directly. Following FMP, consider a new incomplete information game E, in which each agent i has a payoff function $\tilde{u}_i(a_i, a_{-i}, x_i)$ that depends directly on her signal x_i. The payoff functions $\tilde{u}_i$ are equal to the payoff functions g_i of the complete information game $\mathbf{g}$ for high signals, but for low signals they make the lowest action dominant (Figure 2.1):

$$\tilde{u}_i(a_i, a_{-i}, x_i) := \begin{cases} -a_i & \text{if } x_i < 0, \\ g_i(a_i, a_{-i}) & \text{if } x_i \geq 0. \end{cases} \tag{2.2}$$

Players' signals are given by $x_i = \theta + \eta_i$, where the common part θ is drawn uniformly from a large interval[5] and, for each player i, the idiosyncratic part η_i is drawn according to a noise distribution f_i, just like in a global game. We refer to E as the (*lower-f-*)*elaboration* of $\mathbf{g}$.

Standard results on supermodular games ensure that the elaboration E has a highest equilibrium strategy profile s and ensure it is increasing. Therefore, s must consist of monotonic step functions, where at each step at least one player switches to a higher action. Since the joint action set is finite, s must reach a highest action profile (see again Figure 2.1). We denote this profile by $\overline{a}$ and refer to it as the action profile *attained* in E. Furthermore, s must be constant for signals above $|A|$. This is because the number of steps must be less than the size of the action space; and a player with signal x_i knows that all her opponents receive signals within $[x_i - 1, x_i + 1]$. Therefore, the distance between the steps must be weakly less than 1 – otherwise the distance can be shortened without affecting expected payoffs at the steps, contradicting the maximality of s. Thus, the attained action profile $\overline{a}$ is equal to $s(|A|)$.

Dually, define the *upper-f-elaboration* E^∂ of the complete information game $\mathbf{g}$, similar to the lower-f-elaboration E, except that payoffs are given by g_i if the signal

[5] We assume this interval is a superset of $[-|A| - 1, |A| + 1]$, where $|A|$ denotes the cardinality of the joint action set.

$x_i \le 0$ and by a_i if $x_i > 0$. We consider the lowest equilibrium strategy profile in E^∂, and denote by $\underline{a}$ the lowest action profile that is attained in it. It is found by evaluating the lowest equilibrium strategy profile at $x = -|A|$.

The following result gives (for the noise distribution f) the relation between the GGS, the action profile $\overline{a}$ attained in the elaboration E, and the action profile $\underline{a}$ attained in the elaboration E^∂.

Theorem 1. *Let $G(v)$ be any global game with noise distribution f that embeds the complete information game $\mathbf{g}$ at some state θ^*. Let $\overline{s}$ and $\underline{s}$ be the left and right continuous versions of the essentially unique[6] limit equilibrium strategy profile in $G(v)$. Then $\overline{s}(\theta^*) = \overline{a}$ and $\underline{s}(\theta^*) = \underline{a}$.*

Note the following implication. In any global game, at any state, as the noise in private signals vanishes, the value taken by its essentially unique limit strategy profile depends only on the complete information game given by the payoffs at this state and the noise distribution f. This holds because our choice of the embedded complete information game $\mathbf{g}$ at the start of the section was entirely arbitrary, and $\mathbf{g}$ and f are the only ingredients of the global game appearing in E and E^∂.

The irrelevance of the prior distribution in the global game for establishing its limit equilibrium strategy profile was already shown by FMP. It may be surprising that the extended payoff function is also irrelevant. If one thinks of the GGS as determined by an infection process starting from the dominance regions, one might imagine that if the complete information game $\mathbf{g}$ is embedded at some state θ^* close to the lower dominance region, this may influence the GGS in such a way that it selects a lower equilibrium compared to a global game in which $\mathbf{g}$ is embedded near the upper dominance region. However, Theorem 1 tells us this is not the case.

A practical way to think about Theorem 1 is the following. In economic applications, the state θ is typically interpreted as an economic fundamental affecting the decision problem of players. But several economic variables may be candidates for the state θ. Theorem 1 says that the choice of the fundamental used to perturb the decision problem is irrelevant: the GGS will be the same.

We sketch the proof of Theorem 1, concentrating on the highest GGS. To this end, we connect the elaboration E and its attained action profile $\overline{a}$ with the simplified global game $G^*(v)$ that determines the GGS. We do this by introducing two additional parameters in E: the scale factor v and an explicit threshold signal y at which its payoffs change (normalised to 0 in Equation (2.2)). Formally, let $E_y(v)$ denote a version of E in which agents receive scaled signals $x_i = \theta + v\eta_i$, and in which payoffs are given by g_i for $x_i \ge y$ and by $-a_i$ for $x_i < y$. Note that due to the simple structure of E, any equilibrium strategy profile of E has a scaled and shifted counterpart in $E_y(v)$. In particular, the highest equilibrium strategy profile s of E has a scaled counterpart in $E_y(v)$ where the distance between steps is scaled down by a factor v and all steps are shifted to the right by y. Thus, it prescribes that player i plays action $\overline{a}_i$ for all signals higher than $x_i = y + v|A|$.

[6] Recall that the limit strategy profile is unique up to its (finitely many) points of discontinuity.

Now, we compare the simplified global game $G^*(v)$ and the elaboration $E_{\theta^*}(v)$ (i.e., the threshold y is set to the state θ^*). For any signal x_i and any opposing strategy profile, the upper best reply in $G^*(v)$ is weakly higher than in $E_{\theta^*}(v)$: for signals below θ^* the lowest action is strictly dominant in $E_{\theta^*}(v)$; for signals above θ^* a player's incentive to increase her action increases with rising x_i in $G^*(v)$ but is constant in $E_{\theta^*}(v)$. In particular, this means that any equilibrium in $E_{\theta^*}(v)$ must be a lower bound on the unique equilibrium of $G^*(v)$. Letting the scale factor v go to zero, the fact that action $\overline{a}_i$ is played at $x_i = \theta^* + v|A|$ in an equilibrium of $E_{\theta^*}(v)$, together with the right-continuity of the limit profile $\overline{s}$ of $G^*(v)$, establishes that the attained action profile $\overline{a}$ is a *lower bound* on the highest GGS at θ^*. This argument is also used in the proof of Theorem 4 in FMP.

The key observation behind Theorem 1 is that a converse result also holds: the attained action profile $\overline{a}$ also determines an *upper bound* for the highest GGS. To see this, compare the simplified global game $G^*(v)$ with the elaboration $E_{\underline{\theta}}(v)$, so that the scale factor v and dominance regions of both games coincide (the threshold y is set to the lower dominance threshold $\underline{\theta}$ of $G^*(v)$). For signals $x_i \in [\underline{\theta}, \theta^*]$, payoffs in $E_{\underline{\theta}}(v)$ are given by $g_i(\cdot) = u_i(\cdot, \theta^*)$, while payoffs in $G^*(v)$ are given by $u_i(\cdot, x_i)$. State monotonicity of payoff functions u_i (Assumption **A3**) implies that for all signals up to θ^*, best replies in $E_{\underline{\theta}}(v)$ will be weakly higher than in $G^*(v)$. This suggest that, at least for small v, $\overline{a}$ can serve as an upper bound on the action profile played at θ^* in the unique equilibrium of $G^*(v)$. In Appendix A, we prove this intuition to be correct.

2.1.4.2 The Global Game Selection is Generically Unique

FMP define a global game as a family of supermodular games ordered along a one-dimensional state space, and find that the GGS is unique at almost all states. Theorem 1 says that for any given supermodular game, the lowest and highest GGS depend only on the choice of the noise distribution f and can be determined without reference to any particular global game embedding. We will complement FMP's observation by showing that (given f), if one picks an individual supermodular game at random, the lowest GGS and highest GGS typically coincide.

More precisely, consider the set of games with player set I and joint action set A, which may be identified with the Euclidean space $\mathbb{R}^{|I\times A|}$. E.g., for two-player two-action games, $|I \times A| = 8$, corresponding to the number of entries that characterise the payoff matrix. Let $S \subseteq \mathbb{R}^{|I\times A|}$ be the subset of supermodular games. For any fixed noise distribution f, denote by S^f the subset of supermodular games in which the GGS is unique; S^{-f} denotes its complement in S, the set of games in which the GGS is not unique. Then the set S^{-f} is small relative to S^f, both in measure and in a topological sense.

Theorem 2. *For any noise distribution f, the set S^f of supermodular games with a unique GGS is open and dense in S, while its complement S^{-f} is closed and nowhere dense in S. Moreover, S^f is of infinite Lebesgue measure, while S^{-f} is of zero Lebesgue measure.*

			$\alpha > \xi$	$\alpha \le \xi$
player i	("don't attack")	0	0	0
	("attack")	1	-1	1

Recall that S^f is dense in S if each supermodular game $\mathbf{g} \in S^{-f}$ can be approximated by games in S^f. As jumps in the limit strategy profile of a global game are isolated, we can approximate any $\mathbf{g} \in S^{-f}$ if we embed it in a global game $G(v)$ with noise distribution f and choose a sequence of games in S^f along the one-dimensional state space of $G(v)$. In the proof, we also establish that S^f is open in S, and thus S^{-f} is closed and nowhere dense. We then show that S^{-f} has Lebesgue measure zero, by applying a result that connects its topological properties to its measure.

2.1.4.3 Global Game Selection in Discontinuous Global Games

Since Theorem 1 says that the GGS may be determined without reference to any particular global game embedding, this suggests that the monotonicity **(A3)** and continuity **(A4)** assumptions imposed on the embedding can be weakened. To see why this may be important in an applied context, consider the following n-player speculative attack game:
where α is the fraction of players who choose 0. It is an exemplary regime change game, where players' payoffs depend on whether they reach a critical mass ξ. Intuitively, there are at least two ways to embed the game in a global game: one can perturb the payoffs as in Lemma 1, or one can perturb the critical mass by setting $\xi = \theta$. In the latter case, which is often considered in the applied literature (e.g. Corsetti et al. [2004], Guimaraes and Morris [2007], Morris and Shin [1998], Sákovics and Steiner [2012]), payoffs remain unchanged for most changes in the state, and jump at the point where a change in the state leads to a regime change. Thus the global game payoff functions violate state monotonicity **(A3)** and continuity **(A4)**. But even in this case, they satisfy the following assumption:

(**A3***) *Weak state monotonicity.* Higher states make higher actions weakly more appealing:

$$\text{(2.3)} \quad \text{for all } i, a_{-i} \text{ and } a_i < a_i', \;\; u_i(a_i', a_{-i}, \theta) - u_i(a_i, a_{-i}, \theta) \text{ is weakly increasing in } \theta.$$

We can show that, even if a global game embedding satisfies only these weakened conditions, the GGS may be determined analogous to Theorem 1. As before, let $\mathbf{g}$ be a given supermodular complete information game. Suppose it is embedded at state θ^* in a *generalised global game*, with payoffs differing from an ordinary global game in that they satisfy (**A3***) but not necessarily (**A3**) or (**A4**). By standard results on supermodular games, for each scale factor $v > 0$, the highest and lowest equilibrium strategy profiles $\hat{s}_v$ and $\check{s}_v$ exist nonetheless. We define their pointwise limits $\hat{s} = \limsup_{v \to 0} \hat{s}_v$ and $\check{s} = \liminf_{v \to 0} \check{s}_v$. Also, for each state θ, consider the complete information game with payoffs $u_i(\cdot, \theta)$. We may determine its attained action profiles $\overline{a}_\theta$ and $\underline{a}_\theta$ from the associated elaborations E_θ and E_θ^∂; these attained action profiles may be regarded as functions of θ.[7] Using these definitions, we obtain the following result, parallel to Theorem 1:

Theorem 3. *Let $\tilde{G}(v)$ be a generalised global game with noise distribution f that embeds the complete information game $\mathbf{g}$ at θ^*. Let $\hat{s}$ and $\check{s}$ be its highest and lowest limit equilibrium strategy profiles. If (i) the attained equilibria $\overline{a}$ and $\underline{a}$ of the game $\mathbf{g}$ coincide and (ii) the functions $\overline{a}_\theta$ and $\underline{a}_\theta$ are continuous at θ^*, then $\hat{s}(\theta^*) = \overline{a} = \underline{a} = \check{s}(\theta^*)$.*

To prove this result, we "sandwich" the payoff function of the global game $\tilde{G}(v)$ between that of two "ordinary" global games that approximate it. Then, we use Theorem 1 to show that their limit strategy profiles coincide at θ^*. This pins down the limit strategy profile of $\tilde{G}(v)$ at θ^* as well.

The regulatory conditions (i) and (ii) are needed because of the weakening of (**A3**) and (**A4**). Note that Theorem 2 guarantees that (i) is satisfied for almost all supermodular games. As for (ii), (**A3***) implies that the attained profiles $\overline{a}_\theta$ and $\underline{a}_\theta$ are increasing in θ and thus have only finitely many discontinuities, as the joint action set is finite. But unlike in an ordinary global game, (i) does not imply (ii), as $\overline{a}_\theta$ and $\underline{a}_\theta$ may jump because of a discontinuity in the payoff difference (2.3).

2.1.5 A Decomposition Approach to Noise Independence

A supermodular complete information game $\mathbf{g}$, embedded in a global game at a state θ^*, is called *noise independent* if, as the noise in private signals vanishes, the global game's limit strategy profile takes on the same value at θ^* regardless of the choice of the noise distribution f. Theorem 1 implies that noise independence is a well-defined property of the game $\mathbf{g}$: it is noise independent in one global game embedding if and only if it is noise independent in every other.

Many small games, with few players or few actions, are noise independent. Typically, for such games there are also easy heuristics to find the GGS. For instance, a well-known elementary condition to judge whether a game is noise independent is the "p-dominance" criterion.

Definition. Given a tuple $p = (p_i)_{i\in I}$, an action profile a^* is said to be p-dominant if each player i who expects her opponents to play a^*_{-i} with probability p_i would choose a^*_i as a best reply.

If a supermodular game $\mathbf{g}$ has a p-dominant action profile a^* with $\sum_{i\in I} p_i < 1$, then a^* is an equilibrium robust to incomplete information in the sense of Kajii and Morris [1997]. Moreover, a^* is robust in all games with payoffs close to those of $\mathbf{g}$. This implies that a^* is the unique GGS in $\mathbf{g}$, regardless of the noise distribution.[8]

[7] We formally define these concepts by substituting $u_i(\cdot, \theta)$ for $g_i(\cdot)$ in the definition of E, E^∂, $\overline{a}$, and $\underline{a}$.

[8] See Oury and Tercieux Oury and Tercieux [2007] or our working paper version Basteck et al. [2010]; Morris and Shin [2003] provide a heuristic argument.

Symmetric Games			
actions:	2 each	3 each	4 each
2 players	✓[a]	✓[c]	×[b]
3 players	✓[b]	×[d]	
n players	✓[b]		

Asymmetric Games			
actions:	2 each	2 by n	3 each
2 players	✓[a]	✔[g]	×[c]
3 players	×[e]	n/a	
n players	×[f]	n/a	

✓ Always noise independent. × Counterexample to noise independence exists. For empty cells noise dependence follows from an example in smaller games. [a] Carlsson and van Damme [1993]. [b] Frankel et al. [2003]. [c] Basteck and Daniëls [2011]. [d] Basteck et al. [2010]. [e] Carlsson [1989]. [f] Corsetti et al. [2004]. [g] This paper, see Section 2.1.6: Two-player games with 2 by n actions.

Tab. 2.1: Noise (In)dependence in Supermodular Games

For some games, an even simpler way to determine noise independence is to count the number of players and actions. Table 2.1 summarises when this is possible; for supermodular games indicated with a "✓", noise independence always holds.

Table 2.1 also shows that noise independence may fail quickly as we enlarge the action sets of players. In this section, we will show that a game may nevertheless be noise independent if we can suitably decompose it into smaller games. We start by making a more basic observation: in certain games with large action sets, the GGS may be determined by solving smaller games.

Definition. Consider a supermodular complete information game $\mathbf{g}$ with joint action set A. For action profiles $a \leq a'$, we define $[a, a'] := \{\tilde{a} \in A \mid a \leq \tilde{a} \leq a'\}$. The *restricted* game $\mathbf{g}|[a, a']$ and elaboration $E|[a, a']$ are given by restricting the joint action set of $\mathbf{g}$ and its elaboration E to $[a, a']$.

Figure 2.2 now illustrates the idea. If certain action profiles a and a' are played in equilibrium strategy profiles of the restricted elaborations $E|[0, a]$ and $E|[a, a']$, we can "patch" these profiles together to obtain a strategy profile s in the elaboration E, such that an upperbest reply iteration starting from s is weakly increasing. Hence the attained action profile $\overline{a}$ in E must be weakly higher than a'. By Theorem 1, a' provides a bound on the GGS. We may also do this iteratively:

Lemma 2. *Fix a supermodular game* $\mathbf{g}$ *and noise distribution* f. *An action profile* a^n *is the unique global game selection, if there is a sequence* $0 = a^0 \leq a^1 \leq \cdots \leq a^n \leq a^{n+1} \leq \cdots \leq a^m = m$ *such that*

(i) a^j *is the unique global game selection in* $\mathbf{g}|[a^{j-1}, a^j]$ *for all* $j \leq n$, *and*

(ii) a^{j-1} *is the unique global game selection in* $\mathbf{g}|[a^{j-1}, a^j]$ *for all* $j > n$.

A noise independence result follows as an immediate corollary. If the restricted games in the hypothesis of Lemma 2 are noise independent, we can use the same decomposition of the game $\mathbf{g}$, regardless of the noise distribution f, and $\mathbf{g}$ must be noise independent.

Theorem 4. *Fix a supermodular game* $\mathbf{g}$. *An action profile* a^n *is the unique noise independent global game selection, if there is a sequence* $0 = a^0 \leq a^1 \leq \cdots \leq a^n \leq a^{n+1} \leq \cdots \leq a^m = m$ *such that*

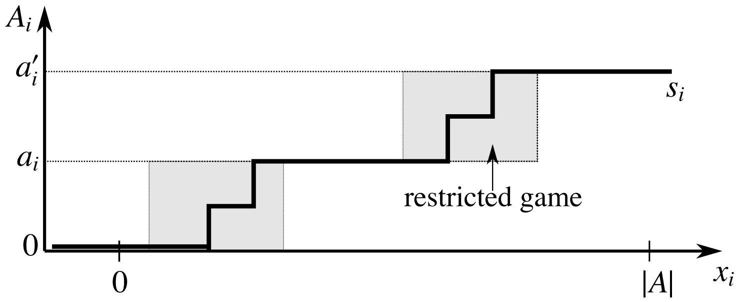


Fig. 2.2: Exploiting the GGS of restricted games

(i) *a^j is the unique noise independent global game selection in $\mathbf{g}|[a^{j-1}, a^j]$ for all $j \leq n$, and*

(ii) *a^{j-1} is the unique noise independent global game selection in $\mathbf{g}|[a^{j-1}, a^j]$ for all $j > n$.*

In Section 2.1.6 we apply these results to derive the noise independent GGS in two examples.

The concept of p-dominance reveals another connection between noise independence and robustness to incomplete information. Together, Proposition 2.7 and 3.8 in Oyama and Tercieux Oyama and Tercieux [2009] imply that if a supermodular game $\mathbf{g}$ can be decomposed – as above – into restricted games, all of which have a strict p-dominant equilibrium with sufficiently small p, uniform across these games, then a^n is also the unique robust equilibrium of $\mathbf{g}$ (and thus the unique GGS).

However, Theorem 4 establishes a more direct result about equilibrium selection in global games. First, its conclusion holds independently of whether the action profile a^n is robust to incomplete information, which is a sufficient, but not a necessary condition for noise independence (see the combined results of Basteck and Daniëls [2011] and Oyama and Takahashi [2011]). Second, it allows application of a range of known criteria for noise independence besides p-dominance; e.g., the fact that symmetric three-player three-action games, symmetric n-player binary-action games, and (as we show below) two-player 2-by-n-action games are noise independent, none of which are equivalent to the p-dominance criterion. Indeed, we can apply a different criterion to each restricted game. Thus, our theorem applies under strictly more general conditions.

2.1.6 *Applications*

It remains to show that our results may be applied in economically interesting settings. We report two examples. In addition, we use Theorem 1 to establish a new noise independence result.

Refinancing Game

Consider a complete information game $\mathbf{g}$ in which 3 lenders decide whether or not to refinance a firm invested in a long term project. Each lender can lend one unit of cash. The firm promises to repay R_h at maturity. However, if the firm cannot fully

		lenders $I - \{i\}$		
		$(0,0)$	$(0,1),(0,2)$ or $(1,1)$	$(1,2)$ or $(2,2)$
	("don't lend") 0	1	1	1
lender i	("secured loan") 1	c	$R_l + c$	R_m
	("unsecured loan") 2	0	R_l	R_h

refinance the project, it may not be able to repay in full. It may repay $R_l < R_h$, or default outright and not repay at all.

The firm has the option to issue collateralised debt that gives lenders an additional payoff of c in both the partial and the complete default scenarios, but yields a lower repayment $R_m < R_h$ when the project succeeds, as the provision of collateral reduces balance sheet flexibility and thus is costly for the firm. The following matrix summarises the possible outcomes of the game, identifying the different lending decisions with actions 0, 1, and 2 as indicated.

For $R_l = \frac{2}{3}$ and $R_h = 2$, and in the absence of collateralised debt, the GGS implies that lenders will not finance the project – recall that in symmetric binary-action games, the GGS is the best reply to the belief assigning equal probability to opponents' profiles $(0,0)$, $(0,2)$ and $(2,2)$.

Introducing collateralised loans may change this result. For example, if $c = \frac{2}{3}$ and $R_m = \frac{3}{2}$, we find that in the restricted game $\mathbf{g}|[0,1]$ (where lenders choose 0 or 1), the GGS is 1,[9] as it is the best reply to the belief assigning equal probability to $(0,0)$, $(0,1)$ and $(1,1)$. Similarly, the GGS in $\mathbf{g}|[1,2]$ is 2. Thus, by Lemma 2, the GGS in $\mathbf{g}$ is 2: lenders are willing to provide *unsecured* loans if we include option 1. Intuitively, this change occurs because the possibility of opponents extending secured loans makes action 2 less risky. Note that all of this holds regardless of the noise distribution, as noise independence is inherited from the two symmetric binary-action games by Theorem 4, and despite the fact that 3-player 3-action games are not in general noise independent.

Asymmetric Minimum Effort Game

Each player $i \in I$ produces an intermediate good that is a necessary input for a final good. Players choose their production level $a_i \in \{0, \ldots, m\}$ and production of the final good is $a_{\min} := \min\{a_i | i \in I\}$. Players' individual payoff functions are given by $g_i(a_i, a_{-i}) = b_i(a_{\min}) - c_i(a_i)$, where b_i and c_i are increasing benefit and cost functions. This game typically has many equilibria. It generalises a model of Bryant [1983] studied by Carlsson and Ganslandt [1998] and Van Huyck et al. [1990].[10]

To solve the game for the GGS, we decompose it into m restricted binary-action games with joint action sets $\{k-1, k\}^{|I|}$, $0 < k \leq m$. We will determine the GGS

[9] In this section, for brevity we use $n \in \mathbb{N}$ to denote the action profile where all players use action n.

[10] Carlsson and Ganslandt [1998] consider a form of trembling hand perfection in a symmetric version of this model; Van Huyck et al. [1990] experimentally test a variant with linear and symmetric payoffs.

in these restricted games. By Theorem 1, the GGS equals k if and only if in the associated elaboration $E|[k-1,k]$ we can construct an increasing strategy profile such that players choose action k for sufficiently high signals, and from which a best reply iteration leads upwards.

Consider a strategy profile for the elaboration $E|[k-1,k]$, given by thresholds $z=(z_i)_{i\in I}$ such that each player i switches from action $k-1$ to k at $z_i \in [0,|A|]$. Let $P_i(z)$ be the probability that player i attaches to all her opponents playing action k, given their thresholds, when she gets signal z_i. The highest GGS equals k if and only if we can adjust the $(z_i)_{i\in I}$ such that each individual player prefers to play k at her threshold; it is unique if they can be made to strictly prefer this, i.e.

$$p_i := \frac{c_i(k)-c_i(k-1)}{b_i(k)-b_i(k-1)} < P_i(z), \tag{2.4}$$

To solve the restricted game, we use the following fact (proved in Appendix A): the beliefs at the thresholds z_i always satisfy the constraint $\sum_{i\in I} P_i(z) = 1$. So, summing (2.4) over all players gives

$$\sum_{i\in I} \frac{c_i(k)-c_i(k-1)}{b_i(k)-b_i(k-1)} < 1. \tag{2.5}$$

Therefore, a necessary condition for k to be the unique GGS is that players' *aggregate* marginal cost-benefit ratios, when everyone switches from $k-1$ to k, are strictly smaller than 1. This is also sufficient, since k is p-dominant with $\sum p_i < 1$ when (2.5) holds, implying noise independent selection. Conversely, if (2.5) holds with the inequality reversed, $k-1$ must (necessarily) be the unique GGS – though in this case the restricted game is generally not p-dominant solvable.

Provided the left hand side of (2.5) crosses 1 at most once and from below for increasing $k=1,2,\ldots,m$, this decomposition yields a generically unique, noise independent GGS. (A sufficient, but not necessary, condition is that the individual c_i are convex and b_i are concave.)

Two-player games with 2 by n actions

To conclude, we will use Theorem 1 to prove a new noise independence result for supermodular 2-by-n-action games. This completes the characterisation of supermodular games for which noise independence follows from the size of individual action sets (Table 2.1). Let $\mathbf{g}$ be a supermodular game with player set $I=\{1,2\}$, and action sets $A_1=\{0,1\}$ and $A_2=\{0,1,\ldots,m_2\}$. E.g., player 1 is a government contemplating an infrastructure project and player 2 is a firm choosing a plant capacity, and their investments are complements.

For an arbitrary noise distribution f, consider the elaboration E of the game $\mathbf{g}$ and its highest equilibrium strategy profile s, in which players jointly play $\overline{a}=(\overline{a}_1,\overline{a}_2)$ at sufficiently high signals. If the attained action $\overline{a}_i=0$ for some player i, she plays 0 for all signals in the elaboration E. Thus the attained action $\overline{a}_j$, $j\neq i$, must be j's highest best reply to 0. Then, the joint strategy profile given by $s_i(x_i) = 0$ for all x_i, and $s_j(x_j) = 0$ if $x_j < 0$ and $s_j(x_j) = \overline{a}_j$ if $x_j \geq 0$ is the highest equilibrium in a lower-f'-elaboration under any other noise distribution f'.

Alternatively, suppose attained action $\bar{a}_1 = 1$ and $\bar{a}_2 = k > 0$. Then the highest strategy profile s may be identified with a threshold z_1^1 and a tuple of thresholds $z_2^1, z_2^2, \ldots, z_2^k$, at which players 1 and 2 switch to higher actions. The opposing action distribution faced by player 1 at signal $x_1 = z_1^1$ is given by the probabilities $Q_j = \mathbb{P}(x_2 < z_2^j | x_1 = z_1^1)$ that player 2 chooses an action below $j = 1, ..., k$ given that $x_1 = z_1^1$. But the opposing action distribution that player 2 faces at each of her thresholds z_2^j is also described by these probabilities, since for all $j = 1, ..., k$ we find

$$\begin{aligned}\mathbb{P}(x_1 > z_1^1 | x_2 = z_2^j) &= \mathbb{P}(x_1 - x_2 > z_1^1 - z_2^j | x_2 = z_2^j) \\ &= \mathbb{P}(x_1 - x_2 > z_1^1 - z_2^j | x_1 = z_1^1) = \mathbb{P}(x_2 < z_2^j | x_1 = z_1^1) = Q_j,\end{aligned}$$

where the second equality follows from the uniform prior distribution of θ. Now, if we consider any other noise distribution f' and associated lower-f'-elaboration E', we can always construct an increasing strategy profile s' in which players jointly play $(1, a_2)$ for sufficiently high signals, by putting $z_1^1 = 1$ and arranging the remaining k thresholds $z_2^1, z_2^2, \ldots, z_2^k$ such that the k independent equations that determine the Q_j, $j = 1, ..., k$, hold under the new noise distribution.

In this way, the action distributions that players face at their thresholds remain unchanged. Since they are willing to switch to higher actions given these beliefs, a best reply iteration in elaboration E' starting at the profile s' is monotonic and leads to an equilibrium strategy profile $s^* \geq s'$ of E'. By Theorem 1, the highest GGS under the noise distribution f', $\overline{a}'$, is weakly higher than $\overline{a}$, the highest GGS under f. Since f and f' were arbitrary, we find that $\overline{a}' = \overline{a}$. A dual argument establishes that $\underline{a}' = \underline{a}$, proving the noise independence of the game $\mathbf{g}$.

2.1.7 Conclusion

We have shown how, for any supermodular complete information game, we may deduce its global game selection directly using solely the complete information game's payoffs and the global game's noise distribution (Theorem 1). For almost all games this gives a unique global game selection (Theorem 2). Our results may be used to establish selection under weakened assumptions on the global game (Theorem 3), which is useful from an applied perspective.

From a practical point of view, our most powerful result is Theorem 4. It implies that the global game selection may be derived by decomposing a many-action game into smaller games, for which existing heuristics and noise independence results can be applied. As we showed in Section 2.1.6, simplified conditions for the global game selection and a manageable heuristic to derive it in many-action games make it easier to apply the theory of global games. That should facilitate new research on topics where strategic complementarities are crucial.

2.2 Every symmetric 3×3 global game of strategic complementarities has noise-independent selection

Section 2.2 has been published as

Basteck, Christian and Daniëls, Tijmen, "Every symmetric 3×3 global game of strategic complementarities has noise-independent selection", *Journal of Mathematical Economics*, 47.6 (2011), pp. 749-754.

2.2.1 Abstract

We prove that the global game selection in all 3×3 payoff-symmetric supermodular games is independent of the noise structure. As far as we know, all other proofs of noise-independent selection in such games rely on the existence of a so-called monotone potential (MP) maximiser. Our result is more general, since some 3×3 symmetric supermodular games do not admit an MP maximiser. As a corollary, noise-independent selection does not imply the existence of an MP maximiser, nor the existence of an equilibrium robust to incomplete information.

2.2.2 Introduction

Global games are used to select a unique equilibrium in models that would typically have multiple equilibria. A global game perturbs a complete information game by supposing that payoffs depend on a state parameter that is only noisily observed by its players. The resulting incomplete information game (generically) has a unique equilibrium profile that may be used to determine a unique equilibrium in the original game. There are many applications, particularly to the theory of financial crises (Morris and Shin [2003] give an overview). In this paper, we prove that for two-player, three-action, supermodular games with symmetric payoffs this *global game selection* is independent of the imposed noise structure when the noise vanishes. (Precise definitions are given in section 2.)

Theorem 5. *Every symmetric 3×3 supermodular game has noise-independent global game selection.*

The significance of this result is in its implication that 3×3 games clarify the connections between noise-independent selection, robustness to incomplete information [Kajii and Morris, 1997], and the existence of a monotone potential (MP) maximiser [Morris and Ui, 2005]. As far as we know, all (subclasses of) supermodular games for which noise-independent selection has been proved so far also admit an MP maximiser. The xistence of an MP maximiser guarantees the existence of an equilibrium robust to incomplete information [Morris and Ui, 2005], and *a fortiori*, noise-independent selection [Oury and Tercieux, 2007, Basteck et al., 2010]. Noise-independent selection in generic 3×3 symmetric supermodular games with three pure Nash equilibria can be proved along these lines; see Oyama and Takahashi [2009]. However, Honda [2011] has found a non-empty open set of 3×3 symmetric supermodular games, with two pure Nash equilibria, that have no MP maximiser. Oyama and Takahashi [2011] show that these games have no equilibrium robust to incomplete information either.

Our proof of noise-independent selection in 3×3 games does not rely on the existence of an MP maximiser. Since it applies to all supermodular 3×3 games with symmetric payoffs, it is necessarily more general. In particular, combined with the results of Honda and Oyama and Takahashi, it shows that noise-independent selection is not equivalent to the existence of an MP maximiser, nor to the existence of an equilibrium robust to incomplete information.

Carlsson and Van Damme [1993], who introduced global games, established noise-independent selection for all 2×2 games. Frankel et al. [2003] ("FMP") examined it for 3×3 symmetric supermodular games. The cases that they formally consider rely on the existence of an MP maximiser.[11] But they also give a heuristic argument that selection is independent of the noise structure in 3×3 games with symmetric payoffs when, in addition, the noise distributions of players' signals are symmetric in the mean. Unfortunately, in general it is not true[12] that if the global game selection is independent of the noise structure for all mean-symmetric noise distributions, the global game selection is noise independent, as we show below by counterexample.

2.2.3 Preliminaries

Let $I = \{1, 2\}$ be a set of two players, both endowed with the same ordered action set $A = \{0, 1, 2\}$ equipped with the usual ordering. Consider a 3×3 game $\mathbf{g}$ with payoff function $g_i : A \times A \to \mathbf{R}$ for $i \in I$, where $g_i(a_i, a_{-i})$ is i's payoff if she chooses a_i and her opponent $-i$ chooses a_{-i}. (For $n \in A$, we will typically denote the action profile $(n, n) \in A \times A$ also by n, economising slightly on notation.)

Let $\Delta_{im}^{n}(a_{-i}) := g_i(n, a_{-i}) - g_i(m, a_{-i})$ denote the payoff difference of playing n instead of m against an opposing action a_{-i} and recall that $\mathbf{g}$ is called (weakly) supermodular[13] if each $\Delta_{im}^{n}(a_{-i})$ is a monotonic function of a_{-i} for all $m < n$. A game $\mathbf{g}$ is called strictly supermodular if each $\Delta_{im}^{n}(a_{-i})$ is strictly monotonic. The *dual* game of $\mathbf{g}$, denoted $\mathbf{g}^{\partial}$, is obtained by reversing the ordering on A. Note that $\mathbf{g}$ is supermodular if and only if $\mathbf{g}^{\partial}$ is supermodular.

Like FMP, we define a global game $G(v)$ in the following way. Extend each $g_i(a_i, a_{-i})$ to a payoff function $u_i(a_i, a_{-i}, \theta)$ that depends continuously on a state parameter $\theta \in \mathbf{R}$. For each fixed θ, let the game given by the $u_i(\cdot, \theta)$ be supermodular. In addition, let the payoff differences $u_i(n, a_{-i}, \theta) - u_i(m, a_{-i}, \theta)$ be strictly monotonic in θ for each a_{-i} and $m < n$.[14] Furthermore, assume there exist $\underline{\theta} \leq 0$ and $\overline{\theta} \geq 0$ such

[11] More specifically, they rely on the existence of a local potential (LP) maximiser, which implies the existence of an MP maximiser in own-action concave games [Morris and Ui, 2005, Oyama and Takahashi, 2009].

[12] Nor, we should add, do FMP claim this is true.

[13] FMP use the terminology "game of strategic complementarities".

[14] Strict monotonicity is slightly weaker than the conditions imposed by FMP. Continuity with respect to θ may be weakened as well, see Basteck et al. [2010].

that for both players, action 0 strictly dominates actions 1 and 2 in the game with payoff functions $u_i(a_i, a_{-i}, \underline{\theta})$ and action 2 dominates actions 0 and 1 in the game with payoff functions $u_i(a_i, a_{-i}, \overline{\theta})$.

Players all share a common prior belief about the true value of θ, given by a continuous density ϕ with connected support that includes $\underline{\theta}$ and $\overline{\theta}$ in its interior. Let $f = (f_1, f_2)$ denote a pair of probability densities, whose supports are subsets of $[-\frac{1}{2}, \frac{1}{2}]$. Each player i also receives a private signal $x_i = \theta + v\eta_i$, where each η_i is drawn independently according to the density f_i, and $v > 0$ is a scaling parameter. We also define the *simplified* global game $G^*(v)$, which differs from $G(v)$ in that players' payoffs are given by $u_i(a_i, a_{-i}, x_i)$ (thus players' payoffs depend directly on their signal), and that the prior density of θ is uniform over a large interval U containing $[\underline{\theta}, \overline{\theta}]$ in its interior.

A strategy profile s is a function that associates an action for each player with each pair of signals. The key result on global games states that in the limit as the scaling parameter $v \to 0$, the equilibrium strategy profiles of the games $G(v)$ and $G^*(v)$ all converge to a limiting strategy profile s^*, which is increasing, and unique up to its finitely many discontinuities (see theorem 1 and lemma A3 in FMP).

For brevity, let us write $s^*(\theta)$ instead of $s^*(\theta, \theta)$. Evaluating s^* at any point of continuity θ selects a unique equilibrium of the complete information game $\mathbf{g}$ given by the $u_i(\cdot, \theta)$, which we refer to as the *global game selection*. To extend the notion of the global game selection to the points of discontinuity of s^*, we work with the left and right continuous version of s^*, denoted $\underline{s}^*$ and $\overline{s}^*$. Then $\underline{s}^*(\theta)$ and $\overline{s}^*(\theta)$ select two equilibria of $\mathbf{g}$ that we refer to as the least and greatest global game selection, respectively. Generically, $\underline{s}^*(\theta)$ and $\overline{s}^*(\theta)$ coincide.

Suppose without loss of generality that $\mathbf{g}$ is such that $g_i(\cdot) = u_i(\cdot, 0)$. The selections $\underline{s}^*(0)$ and $\overline{s}^*(0)$ are always independent of ϕ (FMP) and also of the shape of the $u_i(\cdot, \theta)$ for $\theta \neq 0$ [Basteck et al., 2010]. However, they may depend on the noise structure given by f. If they are independent of f, then the complete information game $\mathbf{g}$ has *noise-independent selection*.[15]

2.2.4 Proof Strategy

Consider a supermodular game $\mathbf{g}$ satisfying the definitions above, and a simplified global game $G^*(v)$ with payoff functions u_i as above, with $\mathbf{g}$ embedded at $\theta = 0$. To avoid having to deal with sequences of increasingly precise noise structures, we will define a new incomplete information game that allows us to establish the global game selection more directly. Following FMP and Basteck et al. [2010], consider an

[15] While FMP do not give an explicit definition of noise-independent selection, like us, they phrase it in terms of the independence between f and the values of the left and right continuous versions of s^* in their theorem 4. A minor difference is that we will take explicit care of the non-generic case where $\underline{s}^*$ and $\overline{s}^*$ differ.

alternative incomplete information game E^v with payoff functions $\tilde{u}_i$ as follows:

$$\tilde{u}_i(a_i, a_{-i}, x_i) := \begin{cases} u_i(a_i, a_{-i}, \underline{\theta}) & \text{if } x_i < 0, \\ u_i(a_i, a_{-i}, 0) & \text{if } x_i \geq 0, \end{cases}$$

where $u_i(\cdot, \underline{\theta})$ are payoff functions such that the least action strictly dominates all others. The prior distribution of θ and the distribution of individual signals remain unchanged. For any x_i, the best reply under the payoff function $\tilde{u}_i(\cdot, x_i)$ is weakly smaller than under $u_i(\cdot, x_i)$—for negative signals because the smallest action dominates under $\tilde{u}_i$, and for positive signals since the payoff difference function increases in x_i under u_i but is constant under $\tilde{u}_i$.

For this reason, E^v can be used to find a lower bound for the global game selection. If for some fixed v and $x > 0$, there exists an equilibrium strategy profile $\tilde{s}^v$ in E^v such that $\tilde{s}^v(vx) = n$, then there must also be an equilibrium strategy profile s^v in the simplified global game $G^*(v)$ such that $s^v(vx) \geq n$. Moreover, this holds for all $v' < v$, as changing v simply scales the equilibrium strategies in E^v. Thus, letting $v \to 0$, we find that for $\overline{s}^*$ (the right continuous limiting strategy profile of all equilibrium strategy profiles in $G^*(v)$), we must have $\overline{s}^*(0) \geq n$. Hence the greatest global game selection in $\mathbf{g}$ must be weakly greater than n.

The above argument to establish a lower bound on the global game selection is used in the proof of theorem 4 in FMP. Basteck et al. [2010] establish a converse result: the greatest equilibrium strategy profile of E^v also determines an *upper* bound for the greatest global game selection. In fact, one can show that the greatest global game selection always equals $s(R)$ for sufficiently large R, where s is the greatest equilibrium strategy profile of E^1 and (for 3×3 games) we may take $R \geq 6$ (see figure 2.3). It is easy to see that the scale factor plays no essential role in the game E^v, so we may as well fix $v = 1$ and drop the index.

In sum, we can completely pin down the greatest global game selection $\overline{s}^*(0)$ by analysing the much simpler game E. This will be our proof strategy in the rest of this text. As a matter of fact, all yet unknown cases of noise independence for symmetric supermodular 3×3 games reduce to either finding an equilibrium strategy profile for E in which both players play action 2 at sufficiently large signals, or to a dual result that we obtain by order-theoretic duality.

2.2.5 *Noise-Independent Selection in* 3×3 *Symmetric Supermodular Games*

Let us now suppose $\mathbf{g}$ is an arbitrary symmetric game, so that we may write $g := g_1 = g_2$ and, for the payoff difference functions, $\Delta := \Delta_1 = \Delta_2$. In the mixed extension of $\mathbf{g}$, a strategy $\mu = (w_0, w_1, w_2)$ is a probability distribution that mixes over the actions $0, 1, 2$ with probabilities ("weights") w_0, w_1, w_2, respectively. By $br(\mu)$ we denote the set of best replies to μ. Now, fix some arbitrary noise structure f and suppose we wish to show that action 2 is a global game selection in $\mathbf{g}$. As argued, our task reduces to finding an equilibrium strategy profile in E where action 2 is

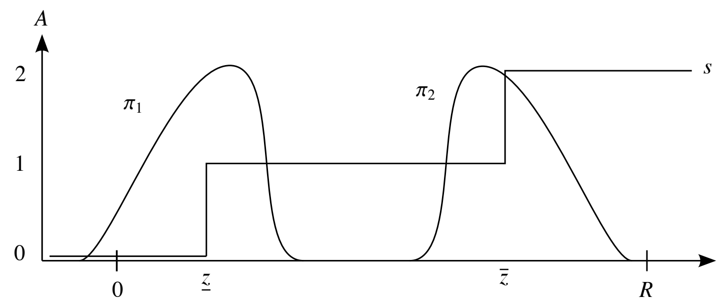


Fig. 2.3: The game E

played. When can we find such a strategy profile?

First, observe that any increasing strategy profile for E may be identified with the threshold signals at which players switch to greater actions. Let $\underline{z}_i$ denote the signal where player i switches from 0 to 1, and $\overline{z}_i$ where she switches from 1 to 2. Given f, let the induced probability density of i's opponent's signal x_{-i}, conditional on i's own signal x_i, be denoted by $\pi_i(x_{-i}|x_i)$. Without loss of generality, we may assume that individual signals are unbiased, in the sense that $\mathbb{P}(x_{-i} \geq x_i) = \frac{1}{2}$.

Denote by μ_m^n the strategy in the mixed extension of $\mathbf{g}$ that puts weight $w_m = w_n = \frac{1}{2}$ on actions $\{m, n\} \subseteq \{0, 1, 2\}$ and zero on the remaining action. Consider a strategy profile s for E in which both players use the same strategy, so that $\underline{z}_i = \underline{z}$ and $\overline{z} = \overline{z}_i$ for each i, and such that they are far apart, as in figure 2.3. At the threshold $\underline{z}$, players face an opponent who plays μ_0^1 and at $\overline{z}$, they face an opponent who plays μ_1^2. Suppose players are willing to switch from action 1 to action 2 when the opponent uses the strategy μ_1^2. If players are also willing to switch from action 0 to action 1 when the opponent uses the strategy μ_0^1, then we are done: by standard results for supermodular games, a greatest best reply iteration starting from s must converge to an equilibrium in which action 2 is played for all signals greater than $\overline{z}$ (see Vives 1990, or Topkis 1998).

But now suppose players are *not* willing to switch from action 0 to 1 when the opponent uses the strategy μ_0^1. We can try and move the thresholds a little closer together. Then, at the upper threshold $\overline{z}$ players will put some probability on the opponent playing 0, and probability $p < \frac{1}{2}$ on the opponent playing 1, but nevertheless action 2 may still be a best reply. By contrast, at the lower threshold $\underline{z}$ players put more probability on the opponent playing 2 and less on 1, so that action 1 may become a best reply.

Figure 2.4 considers the special case where f induces a distribution over signal differences $x_i - x_{-i}$ that is symmetric in the mean,[16] which implies $\pi_1 = \pi_2$. Due to the symmetry, the weight p that players put on their opponent playing 1 at the threshold $\overline{z}$ exactly equals the weight they assign at the threshold $\underline{z}$. In this case, if we may arrange the thresholds as described above, then we may also arrange them under any other mean-symmetric noise structure such that the weight that players put on action 1 at the thresholds equals the same p. This argument suggests:

[16] This is the case if $f_1 = f_2$, or if the individual f_i are symmetric in their mean.

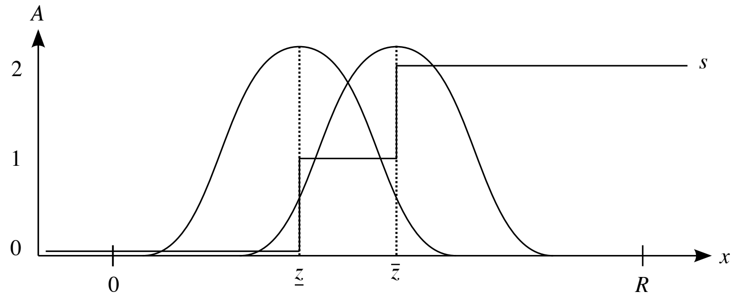


Fig. 2.4: Mean-symmetric noise distributions

Lemma 3. *If there exists a weight $p \in [0, \frac{1}{2}]$ such that:*

(C2) *action 1 or 2 is a best reply when faced with $w_0 = \frac{1}{2}, w_1 = p, w_2 = \frac{1}{2} - p$, and action 2 is a best reply when faced with $w_0 = \frac{1}{2} - p, w_1 = p, w_2 = \frac{1}{2}$,*

then 2 is a global game selection.

For mean-symmetric noise structures, the lemma is almost self-evident, and this is the core of the heuristic argument for noise-independent selection in 3×3 symmetric supermodular games presented in FMP. Unfortunately, it is also easy to see that this argument breaks down when the noise structure is not mean-symmetric. It is in general not possible to adjust the difference $\overline{z} - \underline{z}$ in order to satisfy $\mathbb{P}(\underline{z} < x_{-i} < \overline{z} | x_i = \overline{z}) = p$ and $\mathbb{P}(\underline{z} < x_{-i} < \overline{z} | x_i = \underline{z}) = p$ simultaneously.

Remarkably, the lemma nevertheless holds, but its proof is a non-trivial exercise. Symmetry of $\mathbf{g}$ is essential for the argument, as we show by counterexample in section 2.2.6. Moreover, one has to rely on additional degrees of freedom by allowing the $\underline{z}_i$ and $\overline{z}_i$ to differ, which makes the argument rather technical, and we delegate it to the appendix (B). It is the main ingredient towards the proof of the theorem. From the lemma, we obtain by order-theoretic duality:

Corollary 1. *If there exists a weight $p \in [0, \frac{1}{2}]$ such that:*

(C0) *action 0 is a best reply when faced with $w_0 = \frac{1}{2}, w_1 = p, w_2 = \frac{1}{2} - p$, and action 1 or 0 is a best reply when faced with $w_0 = \frac{1}{2} - p, w_1 = p, w_2 = \frac{1}{2}$,*

then 0 is a global game selection.

Proof. In the dual game of $\mathbf{g}$, the ordering on A is reversed. Replacing all the occurrences of action 0 by action 2 and all occurrences of action 2 by action 0 in the hypotheses and proof of lemma 3, we find that 0 is a global game selection in $\mathbf{g}^\partial$, for any f. Since $\mathbf{g}$ and $\mathbf{g}^\partial$ differ only in their ordering, 0 is a global game selection in $\mathbf{g}$ as well. □

Finally, one can show that if lemma 3 and its corollary fail to hold, 1 must be a Nash equilibrium and $\mathbf{g}$ must be 'decomposable' as in Basteck et al. [2010]. Alternatively, one can show that neither (C0) nor (C2) hold if and only if 1 is a p-dominant equilibrium for some $p < \frac{1}{2}$. Hence, our lemma and its dual deal with all yet unknown cases of noise-independent selection in 3×3 symmetric supermodular games. Nevertheless, in order to keep our exposition self-contained, we give a short explicit proof of the remaining case, in which 1 is always a global game selection.

Lemma 4. *(i) If either:*

(C1) $\quad (a)\;$ *action* 1 *is a best reply to* μ_0^1 *and the greatest best reply to* μ_1^2, *or*

$\qquad\quad (b)\;$ *action* 1 *is a best reply to* μ_1^2 *and the least best reply to* μ_0^1,

then 1 *is a global game selection. Moreover (ii), if hypotheses* (C0) *and* (C2) *do not hold, then* (C1) *necessarily holds.*

Proof. We prove (ii) first. Note that $br(\mu_0^2) = \{1\}$ as any other value would immediately satisfy either lemma 3 or its corollary with $p = 0$. Hence, by supermodularity, $br(\mu_0^1) \subseteq \{0,1\}$ and $br(\mu_1^2) \subseteq \{1,2\}$. We will show that $2 \notin br(\mu_1^2)$. Suppose the contrary. Then $br(\mu_0^1) = \{0\}$, as otherwise the conditions of lemma 3would be satisfied for $p = \frac{1}{2}$. Hence we find that $br(\mu_0^1) = \{0\}$, $br(\mu_0^2) = \{1\}$, and $2 \in br(\mu_1^2)$. The first two equations imply that there exists some $p \in (0, \frac{1}{2})$, such that $br((\frac{1}{2}, p, \frac{1}{2} - p)) = \{0,1\}$. In order for lemma 3 not to hold, it must be that $2 \notin br((\frac{1}{2} - p, p, \frac{1}{2}))$, but that would imply the corollary is satisfied. Thus, we conclude that $2 \notin br(\mu_1^2)$. A similar argument shows $0 \notin br(\mu_0^1)$. So $br(\mu_0^1) = br(\mu_1^2) = \{1\}$ and (C1) is satisfied.

We sketch the proof of (i) for case (a). First note that (C1) implies that 1 is a Nash equilibrium. Let s be the strategy profile where players switch to 1 at $\underline{z}_i = 0$. This is an equilibrium profile of E, and hence 1 is a lower bound for the greatest global game selection. Next, since 2 is not a best reply to μ_1^2, 2 cannot be a best reply for i at $\overline{z}_i \leq \overline{z}_{-i}$ in any equilibrium strategy profile of E. After all, any choice of $\overline{z}_i \leq \overline{z}_{-i}$ will imply that at threshold $\overline{z}_i$, player i attaches probability $\mathbb{P}(x_{-i} \geq \overline{z}_{-i} | x_i = \overline{z}_i) \leq \mathbb{P}(x_{-i} \geq x_i | x_i = \overline{z}_i) = \frac{1}{2}$ to her opponent playing 2. In view of our argument in section 2.2.4, this implies that 1 is also an upper bound for the greatest global game selection. □

Our definition of the global game selection does not entail that it is unique, but that is true generically. As a matter of fact, conditions (C0), (C1), and (C2) also characterise when there is a unique global game selection: it is unique if and only if exactly one of them is satisfied.

To see this, identify the set of 3×3 symmetric supermodular games with a subset $S \subseteq \mathbf{R}^9$ endowed with the Euclidean subspace topology. Let $C_0 \subseteq S$ be the subset where condition (C0) holds, and define C_1 and C_2 analogously. Note that C_0 and C_2 are closed sets, but C_1 is not. Since the global game selection is generically unique, it is easy to argue that if a game $\mathbf{g}$ is in the interior of any of these three sets, the global game selection must be unique.

We will show that, by contrast, if $\mathbf{g}$ is a boundary point of C_0, C_1 or C_2, the global game selection is not unique. Suppose $\mathbf{g}$ is on the boundary of C_0. (If, instead, we choose $\mathbf{g}$ on the boundary of C_1 or C_2, we may apply similar arguments.) As $S = C_0 \cup C_1 \cup C_2$, $\mathbf{g}$ has to be on the boundary of either C_1 or the closed set C_2. If $\mathbf{g} \in C_2$, then 0 is the least and 2 the greatest global game selection. If $\mathbf{g} \in C_1$, then 1 is the greatest global game selection. If $\mathbf{g} \notin C_1$ yet on the boundary of C_1, then $\{1,2\} \subseteq br(\mu_1^2)$ and $\{0,1\} \subseteq br(\mu_0^1)$ so that $\mathbf{g} \in C_2$ after all.

We have now shown conditions (C0), (C1) and (C2) characterise the global game selection, are exhaustive, and none of them depends on the noise structure f. Hence, the theorem is proved.

Remark. *The sets C_0, C_1 and C_2 can also be characterised in terms of the payoff difference functions. For example, $\mathbf{g} \in C_2$ is equivalent to*

$$\begin{array}{rl} (i) & \Delta_1^2(0) + \Delta_1^2(2) \geq 0 \text{ and } \Delta_0^2(0) + \Delta_0^2(2) \geq 0, \\ \text{or} \quad (ii) & \Delta_0^1(0) + \Delta_0^1(2) \geq 0 \text{ and } \Delta_1^2(1) + \Delta_1^2(2) \geq 0 \text{ and} \\ & \dfrac{\Delta_0^1(0) + \Delta_0^1(2)}{\Delta_0^1(2) - \Delta_0^1(1)} \geq \dfrac{\Delta_2^1(2) + \Delta_2^1(0)}{\Delta_2^1(0) - \Delta_2^1(1)}. \end{array}$$

In the subset of games with 3 strict Nash equilibria, these conditions (generically) coincide with the payoff conditions for 2 being a (strict) MP-maximiser given in Oyama and Takahashi [2009]. If one further restricts the set of games by assuming that $br(\mu_0^2) = \{1\}$, (ii) is (generically) equivalent to the payoff conditions for 2 being the global game selection given in FMP. (Note that both papers consider generic games, requiring some inequalities to be strict.)

2.2.6 A Counterexample for Asymmetric 3×3 Games

To conclude our paper, we show that noise-independent selection may fail for aymmetric supermodular 3×3 games. In fact, in such games a heuristic argument based on mean-symmetric noise distributions may fail to provide the right intuition about the global game selection. In an asymmetric 3×3 game, and under asymmetric noise distributions, 0 (or 2) may not be a global game selection even though (C0) (or (C2)) holds. Even if a game has the property that the global game selection is independent of the noise structure for all mean-symmetric f, the game may still not satisfy noise-independent selection.

		player 2		
		0	1	2
	0	2, 1	0, 0	−3, −3
player 1	1	0, −1	0, 0	0, 0
	2	−3, −1	0, 0	2, 2

Tab. 2.2: Asymmetric two-player three-action game

Consider **g** as in the bimatrix in figure 2.2. Both players are indifferent between 0 and 1 when their opponent plays the mixed strategy $\left(\frac{1}{2}, \frac{1}{6}, \frac{1}{3}\right)$, and indifferent between 1 and 2 when their opponent plays the mixed strategy $\left(\frac{1}{3}, \frac{1}{6}, \frac{1}{2}\right)$.

Mean-symmetric noise structures

Suppose f is such that the induced probability densities $\pi_i(x_{-i}|x_i)$ are symmetric in x_i. Note that (C0) and (C2) are satisfied for $p = \frac{1}{6}$. For mean-symmetric noise structures, this is sufficient to establish that 0 and 2 are the least and greatest global game selection under f respectively, as the heuristic argument in section 2.2.5 shows.

Non-mean-symmetric noise structures

Assume the following asymmetric conditional probability density function of player 1 about player 2's signal:

$$\pi_1(x_2|x_1) := \begin{cases} 1 + x_2 - x_1 & \text{if } x_1 - 1 < x_2 < x_1, \\ x_2 - x_1 & \text{if } x_1 \leq x_2 < x_1 + 1 \end{cases}.$$

Player 2 holds a mirrored function, namely

$$\pi_2(x_1|x_2) := \begin{cases} x_2 - x_1 & \text{if } x_2 - 1 < x_1 \leq x_2, \\ 1 + x_2 - x_1 & \text{if } x_2 < x_1 < x_2 + 1 \end{cases}.$$

Set $\underline{z}_1 = \underline{z}_2 = 0$. At $\underline{z}_2$, player 2 expects player 1 to play 0 with probability $\frac{1}{2}$ and is indifferent, irrespective of $\overline{z}_1$. By numerical methods we establish that the smallest $\overline{z}_i$ at which players are indifferent between actions 1 and 2 are $\overline{z}_1 \simeq 0.2214$ and $\overline{z}_2 \simeq 0.5224$. At $\underline{z}_1$, player 1 assigns weight approximately equal to $0.5 - 0.5(0.5224)^2 = 0.3635 > \frac{1}{3}$ to her opponent playing 2, while the weight attached to action 0 is $\frac{1}{2}$. Therefore, player 1 *strictly* prefers to play 1.

Since player 1 has a unique best reply at $\underline{z}_1$, we may slightly increase $\underline{z}_2, \overline{z}_1, \overline{z}_2$, so that both players have unique best replies at each of their thresholds. This leaves us with a strategy profile in which both players use action 2, that must be preserved by slight perturbations of the payoffs. In view of our argument in section 2.2.4, 2 must be a global game selection in **g** and in all games with payoffs close to that of **g**, which implies the global game selection in **g** is unique.[17]

This contrasts with the case of mean-symmetric f, so **g** does not have noise-independent selection. Perturbing the game slightly by setting $g_2(0,1) = g_2(0,2) = -1 + \varepsilon$ for small ε, it still satisfies (C0), but not (C2), so that 0 is the unique

[17] It may be hard to generate the π_i's using two independently distributed error terms with densities f_1, f_2. However, they can be approximated close enough for the numerical result to hold: assume that player 1 receives a very precise signal, while player 2's signal is distributed around θ just like x_2 is distributed around x_1 according to π_1.

global game selection for any mean-symmetric noise structure. Under the non-mean-symmetric f above, 2 still is the unique global game selection.

2.3 New results and own contributions

Section 2.1, Basteck et al. [2013]:

Theorem 1 and 4 were included in my diploma thesis Basteck [2009], supervised by Tijmen Daniëls and Frank Heinemann. Theorem 2 and 3 are new results and extend the previous work as follows.

We know that the global game selection in a complete information game $\mathbf{g}$ is independent of the state dependent payoff functions of the global game embedding, but only depends on the local payoff functions of $\mathbf{g}$ itself as well as the chosen noise distribution f. In many applications, f is chosen as some specific noise distribution – most often as the normal distribution. How often can we then expect the global game selection to be unique for a given f? That was one of the questions that we discussed collaboratively and that I was able to answer in Theorem 2.

Another point that we stumbled upon, was the fact that Frankel et al. [2003] assume continuous global game embeddings while many applications use payoff functions that are discontinuous in the state variable. So we asked, whether the results of Frankel et al. [2003] could be extended to the these cases and found an answer in the form of Theorem 3. Here Tijmen and I were most involved in the proof.

Finally, Basteck et al. [2013], includes two new examples to illustrate the economic relevance of our results, of which I contributed the first one – the Refinancing Game (see Section 2.1.6).

Section 2.2, Basteck and Daniëls [2011]:

In 2010, Jun Honda sent us an email, asking for comments on what became his paper "Noise-independent selection in global games and monotone potential maximizer: A symmetric 3×3 example" [Honda, 2011]. I had a look at the paper and, among other things, noted that he claimed that the global game selection was noise independent in all symmetric 3×3 games. I pointed out to him that to the best of my knowledge this had not been proven, but that we had been able to sketch a proof using our results. When he submitted his paper with the Journal for Mathematical Economics he cited my claim, so that one of the Co-Editors, Atsushi Kajii, contacted me and asked for our proof to be published alongside Jun Honda's paper. Tijmen and I eagerly complied and jointly produced the paper "Every symmetric 3×3 global game of strategic complementarities has noise-independent selection."

3. FAIR SOLUTIONS TO THE RANDOM ASSIGNMENT PROBLEM

This Chapter has been submitted to the BDPEMS working paper series.

3.1 Abstract

We study the problem of assigning indivisible goods to individuals where each is to receive one good. To guarantee fairness in the absence of monetary compensation, we consider random assignments that individuals evaluate according to first order stochastic dominance (sd). In particular, we find that solutions that guarantee sd-no-envy (e.g. the Probabilistic Serial) are incompatible even with the weak sd-core from equal division. Solutions on the other hand that produce assignments in the strong sd-core from equal division (e.g. Hylland and Zeckhauser's Walrasian Equilibria from Equal Incomes) are incompatible with the strong sd-equal-division-lower-bound. As an alternative, we present a Walrasian mechanism, whose outcomes are sd-efficient, lie in the weak sd-core from equal division and satisfy the strong sd-equal-division-lower-bound.

3.2 Introduction

In many allocation problems, we have to assign indivisible objects to individuals where each is to receive at most one. Public housing associations assign apartments to residents, school districts assign seats to students and childcare cooperatives assign chores to its members.

If fairness is understood as equity, the indivisibility of assigned objects will often render any eventual allocation unfair. In order to guarantee fairness at least from an ex-ante perspective, many theorists as well as practitioners have considered lotteries.[1] While the design of such lotteries has received a lot of attention in recent years, most of the work concentrates on their efficiency and incentive properties (i.e. what are the incentives for participants to reveal their true preferences) – see for example [Erdil and Ergin, 2008], [Pathak and Sethuraman, 2011], [Abdulkadiroğlu et al., 2015]. In this Chapter, we try to complement the literature by taking a closer look at the original motivation for applying a lottery mechanism and ask "when is a lottery fair?". For this, we draw on the rich literature on fair allocation[2] and adapt

[1] For example, since 2010 Berlin assigns 30% of seats at overdemanded secondary schools through a lottery, see Basteck et al. [2015].

[2] See Thomson [2011] for an overview.

various equity criteria to random assignments.

Adherence to formal equity criteria may be particularly important when distributing publicly funded (or subsidised) private goods such as school seats, where no individual – or group of individuals – should be discriminated against. In the following, we focus on equity criteria that compare each individual's assignment to the assignments of others or to the average over all assignments. In addition we consider variants of the core from equal division, which can be seen as a group equity criterion. Perhaps surprisingly, we find that all equity criteria are compatible with Pareto-efficiency, while (some) equity criteria for individuals are in conflict with (some) equity criteria for groups.

Since preferences over lotteries are often difficult to elicit, allocation mechanisms typically use individuals' preferences over sure objects. For example school choice mechanisms typically ask students to submit a ranking of schools that they would like to attend. To extend these preferences over sure objects to preferences over lotteries, we will follow Bogomolnaia and Moulin [2001] and rely on first order stochastic dominance (sd). This extension can be seen as the most conservative possible extension, in the sense that an individual will sd-prefer one lottery over another only if she prefers it for any von Neumann Morgenstern utility function compatible with her preferences over sure objects.

The chapter is organised as follows. In Section 3.3, we formally define the set of allocation problems under consideration. Section 3.4 lays out equity criteria; Section 3.5 describes which of these are satisfied by the most prominent existing solutions. Section 3.6 contains our main results – we find that some equity criteria are incompatible with each other so that there exists no solution that is able to satisfy all of them. To bridge this gap, we derive a new solution that satisfies a maximal number of equity criteria while ensuring Pareto efficiency.

3.3 Technicalities

We consider the problem of allocating n objects $a \in A$ among n individuals $i \in I$. Each individual i is to receive one object[3] and holds preferences over objects given by a weak order $\succsim_i$. Let $\succ_i$ and $\sim_i$ denote the associated strict preference and indifference relation, respectively. A preference profile is denoted as $\succsim = (\succsim_i)_{i \in I}$. We restrict preference profiles to cases of objective indifference, i.e. an individual may only be indifferent between objects, if every other individual is indifferent as well.[4] Formally,

$$\forall\, a, b \in A,\ i, j \in I: \qquad a \sim_i b \iff a \sim_j b.$$

We will refer to the tuple $(A, I, \succsim)$ as an *assignment problem (of size n)*. Let $p_{i,a}$ denote the probability that individual i is assigned object a. An *individual (random) assignment* is a probability distribution over A, i.e. a vector $p_i = (p_{i,a})_{a \in A}$ such

[3] possibly a null-object

[4] Objects that everyone is indifferent between may be interpreted as multiple copies of the same object, such as for example multiple seats at a school.

that $\sum_{a \in A} p_{i,a} = 1$. The set of probability distributions over A is denoted $\Delta(A)$. A *random assignment*, $p = (p_i)_{i \in I}$, is a collection of individual assignments such that $\forall a \in A : \sum_{i \in I} p_{i,a} = 1$.[5] A solution S maps assignment problems to (sets of) random assignments.

In order to analyse random assignments and solutions, we extend individuals preferences over objects to preferences over individual assignments,[6] using first order stochastic dominance (sd): define an individual's *weak upper contour set of a* as

$$U_i(a) = \{b \in A \,|\, b \succsim_i a\}$$

and write $p_i \succsim_i^{sd} \tilde{p}_i$ if

$$\forall a \in A : \sum_{b \in U_i(a)} p_{i,a} \geq \sum_{b \in U_i(a)} \tilde{p}_{i,a}.$$

In words, an individual weakly prefers individual assignment if it guarantees her a weakly higher chance of receiving her most preferred object(s) *and* a weakly higher chance of receiving the most or second most preferred object(s) *and* ... so on. If one of the inequalities is strict write $p_i \succ_i^{sd} \tilde{p}_i$. Note that stochastic dominance induces only a partial order over assignments.

At times, we will also evaluate individual assignments according to a vector of weights $w_i = (w_{i,a})_{a \in A} \in \mathbb{R}^n$, where w_i is said to be *compatible with* $\succsim_i$ if

$$\forall a, b \in A : \quad w_{i,a} > w_{i,b} \iff a \succ_i b.$$

Analogously, a collection of weight vectors $w = (w_i)_{i \in I}$ is compatible with preference profile $\succsim$, if the same can be said for each component. The set of all such collections w is denoted $W(\succsim)$. In some contexts – in particular where a social planner is able to elicit them – w_i may be interpreted as von Neumann–Morgenstern (vNM) utilities, associating an expected utility of $w_i \cdot p_i$ with each individual assignment p_i.[7]

Alternatively, the weights might constitute a value judgement on behalf of a social planer, who tries to go beyond a reported preferences profile when choosing between different random assignments. For example, a school board might find that moving to a different random assignment where in expectation some additional k students receive their first rather than their second choice school is preferable even as another k students receive only their third rather than their second most preferred school. Inevitably, such decisions have to be made and making them with respect to fixed weight vectors may increase transparency and accountability.

Finally, a random assignment p is *sd-efficient*[8] unless there exists another as-

[5] The Birkhoff–von Neumann Theorem ensures that any random assignment can be implemented as a convex combination of deterministic assignments where each individual receives one object.

[6] We abstract from consumption externalities, so preferences over random assignments only depend on the individual component.

[7] We use $w_i \cdot p_i$ to denote the inner product of w_i and p_i.

[8] Bogomolnaia and Moulin [2001] introduced this concept as *ordinal efficiency*, to highlight the coarse informational underpinning of the preference relation $\succsim_i^{sd}$.

signment $\tilde{p}$ such that

$$\forall i \in I: \quad \tilde{p}_i \succsim_i^{sd} p_i \qquad \text{and} \qquad \exists i \in I: \quad \tilde{p}_i \succ_i^{sd} p_i.$$

It is *weakly sd-efficient* unless there exists $\tilde{p}$ with $\tilde{p}_i \succ_i^{sd} p_i$, for all $i \in I$. A random assignment p is *efficient with respect to* w unless there exists another assignment $\tilde{p}$ such that

$$\forall i \in I: \quad w_i \cdot \tilde{p}_i \geq w_i \cdot p_i \qquad \text{and} \qquad \exists i \in I: \quad w_i \cdot \tilde{p}_i > w_i \cdot p_i.$$

To familiarize ourself with the above definitions, observe that p is sd-efficient if there exists a collection of compatible weight vectors $w \in W(\succsim)$ such that p is efficient with respect to w.[9]

3.4 Equity Criteria

A minimal fairness requirement on random assignment demands *equal treatment of equals*: two individuals with identical preferences should receive the same amount of all objects that fall in the same indifference class. Formally,

$$\forall\, i, j \in I: \quad \succsim_i = \succsim_j \quad \Longrightarrow \left(\forall a \in A: \quad \sum_{b \sim_i a} p_{i,b} = \sum_{b \sim_i a} p_{j,b} \right).$$

Note that where preferences are strict, this reduces to $\succsim_i = \succsim_j \Rightarrow p_i = p_j$. Equitable treatment of individuals who differ in their preferences is harder to define. If one refrains from interpersonal comparisons of utility (as we do here) envy-freeness is arguably the most prominent such criterion. To check whether an allocation is envy-free, we need to compare individuals' individual assignments - each individual should then prefer her own over anyone else's assignment.

The criterion was introduced to economic theory by Tinbergen [1946] (p. 55 f.)[10] and independently formulated in a dissertation by Foley [1967]. Both Tinbergen and Foley view envy-freeness as a principle of 'macrojustice' and compare individual positions that encompass most (if not all) aspects of individual well-being.[11] In contrast to these two early proponents of the criterion, we will treat envy-freeness as a principle of 'microjustice', applicable to an isolated allocation problem and make no amends for any inequities that originate or persist outside of the model.

Definition 1. Given an assignment problem $(A, I, \succsim)$, a random assignment p is sd-envy-free if for all i, j in I we have $p_i \succsim_i^{sd} p_j$.

[9] The converse holds as well, as proven (non-constructively) by McLennan [2002] and (constructively) by Manea [2008]

[10] Tinbergen credits his professor, Dutch physicist Paul Ehrenfest, to have formulated the criterion in 1925 when they discussed the problem of interpersonal (non-)comparability.

[11] Foley considers "material well-being" and includes not only private consumption goods and leisure but also local public goods. Tinbergen goes further and wants us to consider all of "life's circumstances", including for example health or social status. To make such comparisons viable, Tinbergen suggests to compare representative individuals of different social or occupational groups.

Bogomolnaia and Moulin [2001] are the first to formulate this property in the context of random assignments and refer to it simply as 'envy-free'. Observe that sd-envy-freeness implies equal treatment of equals: if two individuals i, j share the same preferences, sd-envy-freeness implies $p_i \succsim_i^{sd} p_j \succsim_i^{sd} p_i$ which is only possible if $p_i = p_j$.

Sd-envy-freeness is satisfied whenever p is envy-free with respect to *all* compatible weight vectors, i.e. if

$$\forall\, w \in W(\succsim),\ i, j \in I: \qquad w_i \cdot p_i \geq w_i \cdot p_j.$$

From the perspective of a social planer who assumes that individuals evaluate random assignments in an expected utility framework, but who is informed only about their respective rankings over sure objects, sd-envy-freeness allows her to ensure envy-freeness with respect to individuals' expected utilities despite her limited information on the latter.

Another natural yardstick to measure individuals' assignments is equal division, denoted as $\left(\frac{\mathbf{1}}{\mathbf{n}}\right)$, i.e. the individual assignment that grants each object with probability $\frac{1}{n}$.

Definition 2. Given an assignment problem $(A, I, \succsim)$, a random assignment p satisfies the

- *strong sd-equal-division-lower-bound* if $\forall i \in I: \quad p_i \succsim_i^{sd} \left(\frac{\mathbf{1}}{\mathbf{n}}\right)$.
- *weak sd-equal-division-lower-bound* if $\nexists\, i \in I: \quad \left(\frac{\mathbf{1}}{\mathbf{n}}\right) \succ_i^{sd} p_i$.

The weak notion is satisfied if the equal division lower bound is met for some collection of weight vectors $w \in W(\succsim)$,[12] while the strong notion requires that it is met for all such w. Hence, a social planer who only knows individual preferences over objects but chooses a random assignment that meets the strong sd-equal-division-lower-bound is able to ensure that the assignment also satisfies the equal-division-lower-bound with respect to individuals expected utilities, whatever they might be.

Observe that any random assignment that is sd-envy-free also meets the strong sd-equal-division-lower-bound: as each individual's assignment stochastically dominates all other individual assignments, it also dominates the average of all individual assignments that is to say $\left(\frac{\mathbf{1}}{\mathbf{n}}\right)$. Formally:

$$\forall\, i \in I, a \in A: \qquad \sum_{b \in U_i(a)} p_{i,b} \geq \sum_{b \in U_i(a)} \left(\frac{1}{n} \sum_{j \in I} p_{j,b}\right) = \sum_{b \in U_i(a)} \frac{1}{n}.$$

In addition, there are various equity criteria for groups of individuals. Such criteria might be especially important in allocating school seats and other publicly provided goods where we would want to ensure that no group – for example students of a particular neighbourhood or some demographic or ethnic group – is discriminated

[12] I.e. for all $i \in I$, $w_i \cdot p_i \geq w_i \cdot \left(\frac{\mathbf{1}}{\mathbf{n}}\right)$.

against and receives less than their 'fair share'. Perhaps the most notable group equity criterion is the *core from equal division*.

Definition 3. Consider an assignment problem $(A, I, \succsim)$. A group of individuals $G \subset I$ may object to a random assignment $\tilde{p}$ if there is an alternative assignment p such that

- $\forall a \in A: \quad \sum_{i \in G} p_{i,a} = \frac{|G|}{n}$ and
- $\forall\, i \in G: \quad p_i \succ_i^{sd} \tilde{p}_i$.

If there is no such objection that can be raised against a random assignment, the assignment is said to be in the *weak sd-core (from equal division)*.

The core from equal division extends the equal-division-lower-bound, allowing us to asses the assignments that a group of individuals receives with respect to the (aggregate) share that the group receives under equal division. In particular, a random assignment in the weak sd-core will satisfy the weak sd-equal-division-lower-bound, as can be easily verified by restricting attention to cases $G = \{i\}$ in Definition 3.

Observe that any element of the weak sd-core is weakly sd-efficient as any random assignment where everyone could be made strictly better of would be blocked by the grand coalition.

Still, the weak sd-core is comparatively large. Any objection by a blocking coalition would also entail a higher expected utility for each member. Thus, for any compatible weights $w \in W(\succsim)$, the associated weak w-core[13] is included in the weak sd-core. Moreover, the weak sd-core is strictly larger than the union over all weak w-cores – see Appendix, Example 5.

To narrow down the weak sd-core, we consider a prominent subset – the strong sd-core – where coalitions can lean on indifferent members to formulate valid objections.

Definition 4. Consider an assignment problem $(A, I, \succsim)$. A group of individuals $G \subset I$ may object to a random assignment $\tilde{p}$ if there is an alternative assignment p such that

- $\forall a \in A: \quad \sum_{i \in G} p_{i,a} = \frac{|G|}{n}$ and
- $\forall\, i \in G: \quad p_i \succsim_i^{sd} \tilde{p}_i$ and $\exists\, j \in G: \quad p_j \succ_j^{sd} \tilde{p}_j$.

If there is no such objection that can be raised against a random assignment, the assignment is said to be in the *strong sd-core (from equal division)*.

A fortiori, a random assignment in the strong sd-core will satisfy the weak sd-equal-division-lower-bound. However, it may still violate the strong sd-equal-division-lower-bound – and equal treatment of equals:

[13] A random assignment $\tilde{p}$ lies in the w-core, unless there exist $G \subset I$ and assignment p, such that for all $a \in A$ we have $\sum_G p_{i,a} = \frac{|G|}{n}$ and for each member i of G we have $w_i \cdot p_i > w_i \cdot \tilde{p}_i$.

Example 1. Suppose that there are three individuals where preferences of individual 1 and 2 are given as $a \succ b \succ c$ while individual 3 prefers c over a and b. Then the random assignment given by $p_{1,a} = 1$, $p_{2,b} = 1$ and $p_{3,c} = 1$ lies in the strong sd-core – individual 2 does not receive an assignment that sd-dominates equal division but an objection by $G = \{2\}$ is invalid, as $\left(\frac{\mathbf{1}}{\mathbf{n}}\right) \not\succ_2^{sd} p_2$. Also, there is no objection involving either individuals 1 or 3 who both receive their most preferred object, and could not be made as well off by any two-individual coalition. Also, in a blocking coalition involving all three individuals, 1 and 3 would still have to receive their most preferred object, so that 2 could not be made better off. Thus, p is an element of the strong sd-core.

Conversely, equal division necessarily satisfies the strong sd-equal-division-lower-bound and equal treatment of equals, but will typically not be an element of the strong sd-core.[14] In fact, any inefficient random assignment would be blocked by the grand coalition, so that all elements of the strong sd-core are sd-efficient.

Figure 3.4 provides a summary of all equity concepts discussed thus far, including their logical relations. In the center column, there are two independent and comparatively weak equity criteria. The weak sd-equal-division-lower-bound in particular can be strengthened in different ways by either allowing for group comparisons (see the right hand side) or by replacing 'not strictly worse' by a stronger 'weakly better' (left hand side). Also note that sd-envy-freeness implies all other individual equity criteria.

The absence of any connecting arrow(s) between two properties marks their logical independence. To be explicit,

- (strong sd-equal-division-lower-bound + strong sd-core)
  $\not\Longrightarrow$ equal treatment of equals (see Appendix, Example 4).
- equal treatment of equals
  $\not\Longrightarrow$ weak sd-equal-division-lower-bound.[15]
- sd-envy-freeness
  $\not\Longrightarrow$ weak sd-core (follows from Theorem 6).
- (strong sd-core + equal treatment of equals)
  $\not\Longrightarrow$ sd-equal-division-lower-bound (follows from Theorem 7).

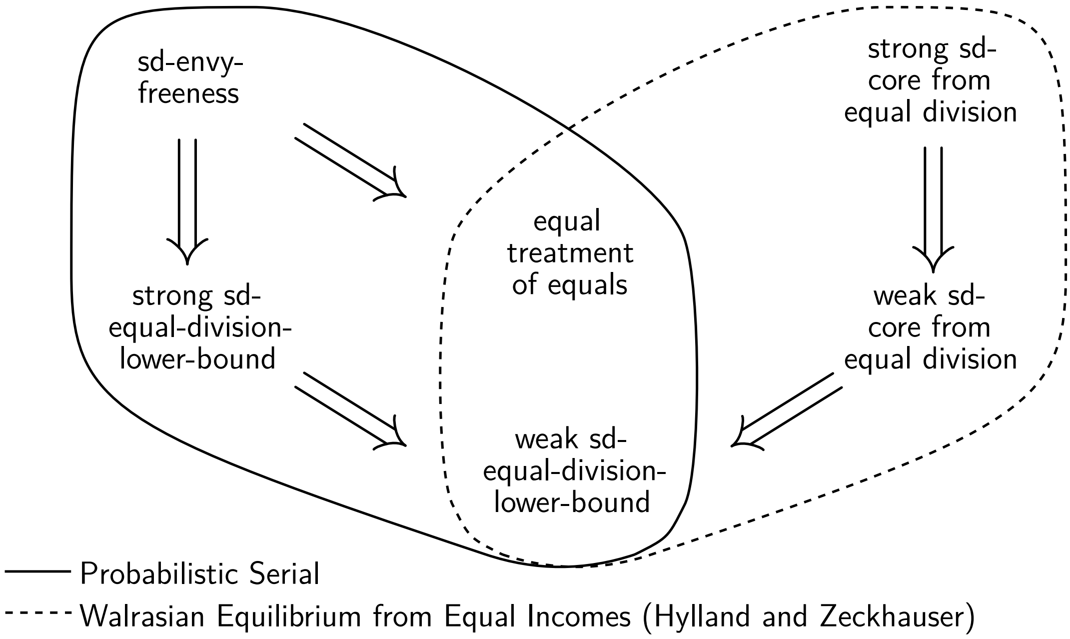
3.5 Prominent Solutions

So far, we have discussed efficiency and equity criteria for particular assignment problems. Let us extend these criteria to solutions, i.e. set valued mappings, defined

[14] Nor would it constitute an element of the weak sd-core – consider the case $n = 2$ where $a \succ_1 b$ and $b \succ_2 a$.

[15] For example, consider the case $n = 3$ where $a \succ_{1,2} b \succ_{1,2} c$ and $b \succ_3 a \succ_3 c$, individuals 1 and 2 receive the same assignment $p_i = (p_{i,a}, p_{i,b}, p_{i,c}) = (1/2, 1/2, 0)$ and 3 receives object c.

Fig. 3.1: Logical relations between equity criteria and 2 prominent solutions



on a domain of assignment problems that map a particular assignment problem to a set of random assignments.

Definition 5. We say that a solution S satisfies criterion X (where X could stand for sd-efficiency, equal treatment of equals, sd-envy-freeness etc.) if for any assignment problem e in the domain of S, all random assignments in $S(e)$ satisfy X.

We now take a look at three existing solutions to the random assignment problem, to see which of the criteria they satisfy. As we will see, all three solutions can be interpreted as taking equal division for a starting point and differ only in the manner in which trades towards the efficiency frontier are conducted.

3.5.1 Random Serial Dictatorship

A frequently used approach towards an equitable solution of assignment problems, is Random Serial Dictatorship (RSD). It requires us to order our n individuals randomly (where all $n!$ orderings are equally likely). The first in line may then choose her most preferred object, while the second chooses the most preferred among the $n-1$ remaining objects.[16] The third individual chooses among $n-2$ available objects and so it continues, until the last in line receives the last object available. From an ex-ante perspective that is before we have decided on a particular ordering of individuals, this procedure generates a random assignment.[17]

[16] If an individual is indifferent between multiple objects, this tie can be broken in any way – since under our assumption of objective indifference all others will similarly be indifferent between the same objects, her choice does not affect any individual that has to choose at a later stage.

[17] Typically, once we have found a random assignment p, we need to construct a Birkhoff–von Neumann decomposition and represent p as a convex combination of deterministic assignments –

With respect to the equity criteria analysed in Section 3.4, let us first point out that a random assignment generated via RSD satisfies equal treatment of equals and meets the strong sd-equal-division-lower-bound: each individual has a chance of $\frac{k}{n}$ to be among the first k individuals to choose, in which case she is guaranteed one of her k-most preferred objects. However, for some preference profiles, RSD falls short of sd-envy-freeness [Bogomolnaia and Moulin, 2001].

The main weakness of RSD lies in the fact that it fails to ensure even weak sd-efficiency [Bogomolnaia and Moulin, 2001]:

Example 2. Consider the case $n = 4$ with a preference profile where $a \succ_{1,2} b \succ_{1,2} c \succ_{1,2} d$ and $b \succ_{3,4} a \succ_{3,4} d \succ_{3,4} c$. Then RSD produces the following random assignment

	a	b	c	d
p_1:	$\frac{5}{12}$	$\frac{1}{12}$	$\frac{5}{12}$	$\frac{1}{12}$
p_2:	$\frac{5}{12}$	$\frac{1}{12}$	$\frac{5}{12}$	$\frac{1}{12}$
p_3:	$\frac{1}{12}$	$\frac{5}{12}$	$\frac{1}{12}$	$\frac{5}{12}$
p_4:	$\frac{1}{12}$	$\frac{5}{12}$	$\frac{1}{12}$	$\frac{5}{12}$

which (from an ex-ante perspective) is Pareto inferior to the random assignment $\tilde{p}_{1,2} = (\frac{1}{2}, 0, \frac{1}{2}, 0), \quad \tilde{p}_{3,4} = (0, \frac{1}{2}, 0, \frac{1}{2})$.

Morover, since RSD may return random assignments that not weakly sd-efficient, it also returns random asisgnment that are not in the weak sd-core from equal division. This contrasts with an alternative description of RSD by Abdulkadiroğlu and Sönmez [1998], who characterize it on the domain of strict preferences as "core from random endowments". More precisely, they consider random initial allocations of goods to individuals where each of the $n!$ deterministic allocations is equally likely. Given their endowments, individuals then trade towards the unique core allocation. From an ex-ante perspective, the convex combination of these core allocations coincides with the convex combination of allocations generated by a fixed pecking order described above.

3.5.2 Probabilistic Serial

To overcome RSD's lack of efficiency from an ex-ante perspective, Bogomolnaia and Moulin [2001] introduce the Probabilistic Serial (PS) mechanism. It generates random assignments via "simultaneous eating" where individuals accumulate probability shares, starting with their most preferred object until it is exhausted, before moving down to their second most preferred object and so on.[18]

only then can we implement p by taking a lottery over all elements of the decomposition. One of the practical advantages of RSD, is that the randomization occurs in the very first step where we choose an ordering of individuals. Once this order is fixed, the algorithm returns a deterministic assignment, obviating any appeal to the Birkhoff–von Neumann–Theorem.

[18] Bogomolnaia and Moulin [2001] consider the case of strict preferences. Their mechanism can be easily generalized to accomodate objective indifferences, i.e. multiple copies of objects – see for example Hashimoto et al. [2014].

Not only does PS generate sd-efficient random assignments, it also ensures sd-envy-freeness [Bogomolnaia and Moulin, 2001]. However, as follows from our Theorem 6, these assignments will not in general lie in the weak sd-core from equal division.

Again, if we think of the core as the set of allocations that might be reached from an initial allocation through trade among individuals, this is in contrast with an alternative description of PS by Kesten [2009], who characterizes PS on the domain of strict preferences as "Top Trading Cycles from Equal Division". For each individual i, her initial assignment $\left(\frac{\mathbf{1}}{\mathbf{n}}\right)$ is managed by n "pseudo-agents" i_a, $a \in A$. Each i_a controls an initial probability share $\frac{1}{n}$ of object a and shares i's preferences over goods. In the first round, pseudo-agents i_a will offer shares of a in exchange for an equal share of the most preferred of $i's$ objects that are still available in the market. Wherever there is a double coincidence of wants, probability shares are exchanged and withdrawn from the market. Over time i_a will have exchanged the whole of her initial share of object a, or she finds that object a is the most preferred among all remaining objects. In both cases, i_a exits the market. After at most n steps, this trading algorithm terminates and the sum of probability shares acquired by i's pseudo-agents is found to coincide with the individual assignment p_i generated by PS.

3.5.3 Walrasian equilibrium from equal incomes

A third prominent solution is offered by Hylland and Zeckhauser [1979], who adapt the familiar concept of a Walrasian equilibrium from equal incomes (WEEI) to assignment problems.[19] In contrast to our setting, individuals report vNM utilities w_i. Nevertheless, as maximisation of expected utilities implies maximisation with respect to stochastic dominance, we find that their solution not only satisfies sd-efficiency, but also many of the equity criteria formulated in Section 3.4.

Formally, define the set of price vectors as $Q = \{q = (q_a)_{a\in A} \in \mathbb{R}^n \mid \forall a \in A : q_a \geq 0\}$. Individuals purchase probability shares, maximizing their expected utility $w_i \cdot p_i$ subject to a constrained budget $B \in \mathbb{R}^+$ and the constraint $\sum_A p_{i,a} = 1$.

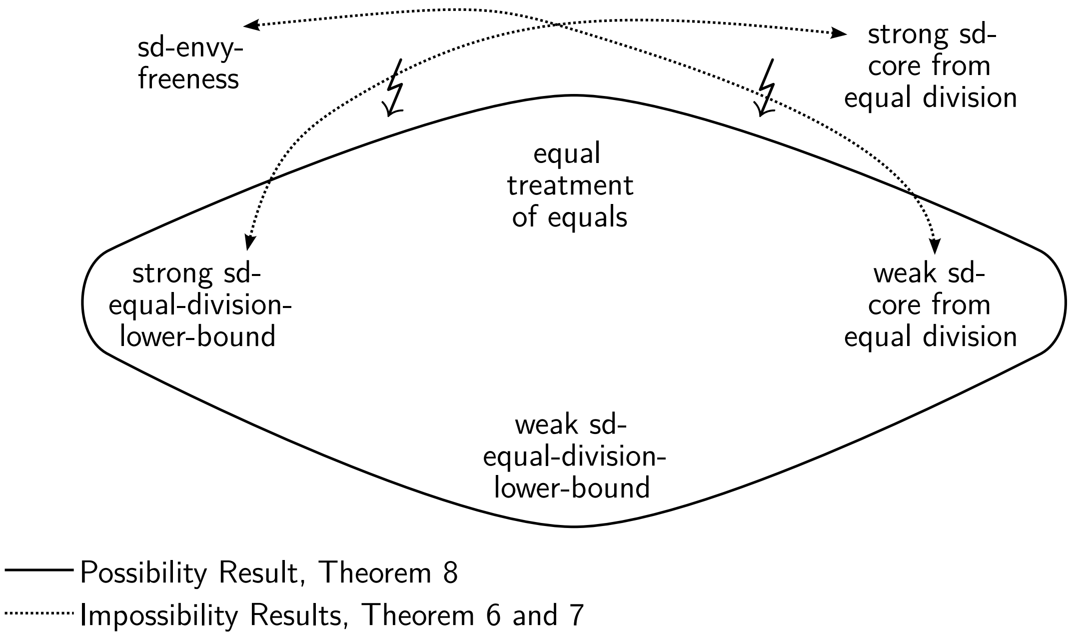
Fact 2. Hylland and Zeckhauser [1979]. *Consider an assignment problem $(A, I, \succ)$. For any collection of compatible weights $w \in W(\succsim)$, their exists a Walrasian equilibrium from equal incomes, i.e. a tupel $(p, q, B) \in \Delta(A)^n \times Q \times \mathbb{R}^+$ such that both*

$$\begin{aligned}
\forall\, i \in I, \tilde{p}^i \in \Delta(A): \quad & q \cdot p^i \leq B \ \textit{and} \ (w^i \cdot \tilde{p}^i > w^i \cdot p^i \Rightarrow q \cdot \tilde{p}^i > B) \\
& \textit{(preference maximisation)}, \\
\textit{and} \quad \forall\, a \in A: \quad & \sum_{i\in I} p^i_a = 1 \qquad \textit{(feasibility)}
\end{aligned}$$

Not surprisingly, such a WEEI will be efficient and in the strong core with re-

[19] Hylland and Zeckhauser [1979] also allow for differences in income, justified for example by the seniority of committee members that need to be assigned to tasks. In the spirit of our equity criteria identified in Section 3.4, we will concentrate on the case of equal incomes.

Fig. 3.2: Three Results on Fair Solutions



spect to w. Moreover, the associated random assignment will also be sd-efficient and an element of the strong sd-core from equal division: any trade (resp. objection) that would make everyone (resp. members of G) weakly better of with respect to first order stochastic dominance would also yield an increase in individuals expected utility. Similarly, preference maximisation and equal budgets guarantee envy-freeness with respect to w, i.e. for all i, j we have $w_i \cdot p_i \geq w_i \cdot p_j$.

One condition that is not automatically satisfied, is equal treatment of equals. If however, we chose $w_i = w_j$ whenever $\succsim_i = \succsim_j$, and constrain these individuals to consume the same probability shares (whenever they are indifferent and might choose different shares), this will guarantee equal treatment without violating preference maximisation or feasibility. Hence, using appropriately chosen weights w, there exists a (sub)solution of WEEI's that selects from the strong sd-core from equal division and satisfies equal treatment of equals.

However, in contrast to both RSD and PS, Hylland and Zeckhauser's solution (and any subsolution) will necessarily violate the strong sd-equal-division-lower-bound, at least for some preference profiles - see Theorem 7.

Figure 3.4 relates the two sd-efficient solutions discussed so far to the equity criteria that they satisfy.

3.6 Main Results

In light of Figure 3.4, we may ask whether there exists a solution that is able to satisfy all of our equity criteria. The following Theorem answers that question in the negative.

Theorem 6. *Consider a solution S whose domain includes all assignment problems of some size n, $n \geq 4$. If S selects from the weak sd-core, it will violate sd-envy-freeness; for every $n \geq 4$ there exist assignment problems of size n, for which no random assignment simultaneously satisfies sd-envy-freeness and lies in the weak sd-core from equal division.*

Proof of Theorem 6. Consider an assignment problem $(A, I, \succ)$ of size $n \geq 4$, label objects as a, b, c, d and $o_5, o_6, \ldots o_n$ and let preferences be given as

- $b \succ_1 a \succ_1 c \succ_1 d \succ_1 o_5 \succ_1 o_6 \succ_1 \cdots \succ_1 o_n$,
- $a \succ_2 c \succ_2 b \succ_2 d \succ_2 o_5 \succ_2 o_6 \succ_2 \cdots \succ_2 o_n$,
- $a \succ_3 b \succ_3 d \succ_3 c \succ_3 o_5 \succ_3 o_6 \succ_3 \cdots \succ_3 o_n$,
- $a \succ_j b \succ_j c \succ_j d \succ_j o_5 \succ_j o_6 \succ_j \cdots \succ_j o_n$, $\forall j = 4, 5 \ldots, n$.

Intuitively, preferences of individuals $j \geq 4$ could be described as 'mainstream preferences' while in the preferences of the first 3 individuals there are reversals in the ranking of objects a, b, c, d that create opportunities for welfare improving trade.

We will proceed by analysing an arbitrary sd-envy-free random assignment p and show that it is no element of the weak sd-core as there exists a valid objection by $G = \{1, 2, 3\}$ who are better of trading only amongst themselves. As p is assumed to be sd-envy-free, and all individuals agree on the ranking of alternatives $o_5, o_6, \ldots o_n$, we know that $p_{i,o_k} = \frac{1}{n}$ for all $i \in I, k \geq 5$. As p also satisfies equal treatment of equals, we can express the individual assignment of individuals $j \geq 4$ can be expressed as

$$p_j = (p_{j,a}, p_{j,b}, p_{j,c}, p_{j,d}, \ldots p_{j,o_k} \ldots) = (1/n + \alpha, 1/n - \alpha + \beta, 1/n - \beta + \gamma, 1/n - \gamma, \ldots 1/n \ldots)$$

with $\alpha, \beta, \gamma \geq 0$. Sd-envy-freeness then implies that p takes the form

	a	b	c	d	o_k
p_1:	$1/n - (n-1)\alpha$	$1/n + (n-1)\alpha + \beta$	$1/n - \beta + \gamma$	$1/n - \gamma$	$1/n$
p_2:	$1/n + \alpha$	$1/n - \alpha - (n-1)\beta$	$1/n + (n-1)\beta + \gamma$	$1/n - \gamma$	$1/n$
p_3:	$1/n + \alpha$	$1/n - \alpha + \beta$	$1/n - \beta - (n-1)\gamma$	$1/4 + (n-1)\gamma$	$1/n$
p_j:	$1/n + \alpha$	$1/n - \alpha + \beta$	$1/n - \beta + \gamma$	$1/n - \gamma$	$1/n$

Individuals 2 and 3 agree with $j \geq 4$ on the most preferred object and hence receive it with probability $p_{2,a} = p_{3,a} = p_{j,a} = 1/n + \alpha$. Individual 1 receives object a with remaining probability $p_{1,a} = 1/n - (n-1)\alpha$. Similarly, individuals $i \neq 2$ agree on the upper contour set $U_i(c) = \{a, b\}$ and hence receive a or b with probability $p_{i,a} + p_{i,b} = 2/n + \beta$ - leaving individual 2 with the remaining probability $p_{2,b} = 1/n - \alpha - (n-1)\beta$. Finally, individuals $i \neq 3$ agree on the upper contour set $U_i(d) = \{a, b, c\}$ and hence receive a, b or c with probability $p_{i,a} + p_{i,b} + p_{i,c} = 3/n + \gamma$ - leaving individual 3 with the remaining probability $p_{3,c} = 1/n - \beta - (n-1)\gamma$. The entries $p_{i,d}$ then follow from the condition $\sum_{x \in A} p_{i,x} = 1$.

As all entries are non-negative, we find three additional constraints on α, β, γ:

$$\begin{array}{lll} \text{(I)} & \alpha \leq \frac{1}{n(n-1)} & (\Leftrightarrow p_{1,a} = 1/n - (n-1)\alpha \geq 0) \\ \text{(II)} & \beta \leq \frac{1}{n(n-1)} - \frac{1}{n-1}\alpha & (\Leftrightarrow p_{2,b} = 1/n - \alpha - (n-1)\beta \geq 0) \\ \text{(III)} & \gamma \leq \frac{1}{n(n-1)} - \frac{1}{n-1}\beta & (\Leftrightarrow p_{3,c} = 1/n - \beta - (n-1)\gamma \geq 0) \end{array}$$

We claim that the following random assignment $\tilde{p}$ constitutes a valid objection by group $G = \{1, 2, 3\}$, who can do better by trading exclusively amongst themselves:

	a	b	c	d	o_k
1:	0	$3/n - \alpha - \beta$	$\alpha + \beta + \gamma$	$1/n - \gamma$	$1/n$
2:	$1/n + \alpha$	0	$3/n - \alpha - \beta - \gamma$	$\beta + \gamma$	$1/n$
3:	$2/n - \alpha$	$\alpha + \beta$	0	$1/2 - \beta$	$1/n$
j:	$1/n$	$1/n$	$1/n$	$1/n$	$1/n$

The random assignment is well defined, as all sums $\sum_{x \in A} \tilde{p}_{i,x} = 1 = \sum_{i \in I} \tilde{p}_{i,x}$ and all entries are non-negative, given that $\alpha, \beta, \gamma \leq 1/n$ - see (I)-(III). Moreover, G's resource constraint is met, as $\sum_{i \in G} \tilde{p}_{i,x} = \frac{3}{n}$, for all $x \in A$.

It remains to show that for all $i \in G$, $\tilde{p}_i \succ_i^{sd} p_i$. First, consider individual 1. Here we find that she receives her most preferred object with strictly greater probability

$$\tilde{p}_{1,b} - p_{1,b} = \frac{2}{n} - n\alpha - 2\beta > \frac{2}{n} - \frac{1}{n-1} - 2\frac{1}{n(n-1)} = \frac{n-4}{n(n-1)} \geq 0$$

where (I) and (II) are used in the inequality. Moreover, she also receives her first or second object with greater probability than before:

$$(\tilde{p}_{1,b} + \tilde{p}_{1,a}) - (p_{1,b} + p_{1,a}) = \frac{1}{n} - \alpha - 2\beta > \frac{1}{n} - \frac{3}{n(n-1)} = \frac{n-4}{n(n-1)} \geq 0.$$

As she receives her least preferred object d with the same probability as before ($\tilde{p}_{1,d} = p_{1,d} = \frac{1}{4} - \gamma$), we conclude that $\tilde{p}_2 \succ_1^{sd} p_2$.

Next, consider individual 2. She receives her most preferred object a with the same probability as before ($\tilde{p}_{2,a} = p_{2,a} = \frac{1}{n} + \alpha$) but receives her second most preferred object with higher probability:

$$\tilde{p}_{2,c} - p_{2,c} = \frac{2}{n} - \alpha - n\beta - 2\gamma \geq \frac{2}{n} - \alpha - \frac{1}{n-1} + \frac{n\alpha}{n-1} - 2\gamma \geq \frac{2}{n} - \frac{1}{n-1} - \frac{2}{n(n-1)} = \frac{n-4}{n(n-1)} \geq 0,$$

where we use (II) in the first and (III) in the second inequality. For the probability of receiving her least preferred object, we find (using (III))

$$\tilde{p}_{2,d} - p_{2,d} = \beta - \frac{1}{n} < 0$$

so that in conclusion $\tilde{p}_2 \succ_2^{sd} p_3$. Finally, consider individual 3. Her most preferred object is a, which she now receives with strictly greater probability:

$$\tilde{p}_{3,a} - p_{3,a} = \frac{1}{n} - 2\alpha \geq \frac{1}{n} - \frac{2}{n(n-1)} = \frac{n-3}{n(n-1)} > 0.$$

As the probability of receiving one of her two most preferred objects remains unchanged ($\tilde{p}_{3,a} + \tilde{p}_{3,b} = p_{3,a} + p_{3,b} = \frac{2}{n} + \beta$) and as she now receives her least preferred object with zero probability, she too strictly prefers $\tilde{p}$ over p, rendering $\tilde{p}$ a valid objection by group $G = \{1, 2, 3\}$. □

According to Theorem 6, the Probabilistic Serial can be seen as a maximally fair solution with respect to our identified equity criteria - no other solution that is similarly sd-envy-free, can in addition select from the weak (or strong) sd-core from equal division.

That raises the question, whether there exist other maximally fair solution – can we satisfy all remaining equity criteria once we give up sd-envy-freeness? Again, the answer is no.

Theorem 7. *Consider a solution S whose domain includes all assignment problems of some size n, $n \geq 3$. If S selects from the strong sd-core it will violate the strong sd-equal-division-lower-bound; for every $n \geq 3$ there exist assignment problems of size n, for which no random assignment simultaneously satisfies the strong sd-equal-division-lower-bound and lies in the strong sd-core from equal division.*

Proof. Consider the assignment problem $(A, I, \succ)$ where $I = \{1, 2, 3\}$ and preferences over A are given as $a \succ_{1,2} b \succ_{1,2} c$ and $b \succ_3 a \succ_3 c$.

Any random assignment p that satisfies the strong sd-equal-division-lower-bound will assign object c with probabilities $p_{i,c} \leq \frac{1}{3}$. But then, $p_{i,c} = \frac{1}{3}$ and p takes the form

	a	b	c
p_1:	$1/3 + \alpha$	$1/3 - \alpha$	$1/3$
p_2:	$1/3 + \beta$	$1/3 - \beta$	$1/3$
p_3:	$1/3 - \alpha - \beta$	$1/3 + \alpha + \beta$	$1/3$

with $\alpha, \beta \geq 0$ and $\alpha + \beta \leq \frac{1}{3}$. For p to lie in the strong sd-core it has to be sd-efficient, i.e. $\alpha + \beta = \frac{1}{3}$. Either α or β will then be less than $\frac{1}{3}$ - assume w.l.o.g. that $\alpha < \frac{1}{3}$. But then, individual 3 could exclusively trade with 1 instead of 2 and arrive at the following alternative random assignment p'

	a	b	c
p_1:	$1/3 + \alpha + \beta$	$1/3 - \alpha - \beta$	$1/3$
p_3:	$1/3$	$1/3$	$1/3$
p_3:	$1/3 - \alpha - \beta$	$1/3 + \alpha + \beta$	$1/3$

While this is a matter of indifference for individual 3, it is strictly preferred by 1. Thus, p' is a valid objection to p by the group of individuals $\{1,3\}$ and goes to show that p is not in the strong sd-core from equal division. As p was chosen as an arbitrary random assignment satisfying the strong sd-equal-division-lower-bound. The counterexample can be extended straightforwardly to the case $n > 3$ by choosing preferences as $a \succ_i b \succ_i c \succ_i o_4 \succ_i \cdots \succ_i o_n$ for $i \in \{1, 2..., n-1\}$ and $b \succ_n a \succ_n c \succ_n o_4 \succ_n \cdots \succ_n o_n$. □

Just as Theorem 6, Theorem 7 can be seen as a result on maximally fair solutions. In particular, (the subsolution of) Hylland and Zeckhauser's Walrasian Equilibria from Equal Incomes, as described in Section 3.5.3, is maximally fair with respect to the identified equity criteria - no other solution that similarly selects from the strong sd-core can in addition satisfy the strong sd-equal-division-lower-bound.

However, our two impossibility results leave space for a potential third class of maximally fair solutions. Can we give up on sd-envy-freeness and the restriction to the strong sd-core but satisfy all remaining equity criteria? Here, the answer is yes.

Theorem 8. *For all n and all assignment problems $(A, I, \succsim)$ of size n, there exist random assignments that satisfy equal treatment of equals, meet the strong sd-equal-division-lower-bound, are in the weak sd-core from equal division and sd-efficient.*

We will proof Theorem 8 by constructing a sequence of Walrasian equilibria with equal incomes. The limit of this sequence will then inherit many desirable properties, even if it is not itself a Walrasian equilibrium.

Our setting raises a number of problems for the existence of Walrasian equilibria. For one, individuals may be satiated and hence leave some of their income unspent, leading to a violation of Walras' law.

Second, if we restrict individuals' consumption sets to random assignments that meet the sd-equal-division-lower-bound, an equal division endowment lies on the boundary of individuals' consumption sets. Then, depending on the price vector, it may be that there is no random assignment that costs less than the initial endowment. This violates the so called strong survival assumption which is typically used to show that any quasi-equilibrium (whose existence may be establish more easily) is in fact a Walrasian equilibrium.

Third, the preference relation given by first order stochastic dominance is not continuous. For example an individual with preference $a \succ_i b \succ_i c$ would (strictly) prefer $p_i = (p_{i,a}, p_{i,b}, p_{i,c}) = (1/3, 2/3, 0)$ over $(1/3, 1/3, 1/3)$ but not over $(1/3 + \varepsilon, 1/3 - \varepsilon, 1/3)$.

To overcome the third problem, we will let individuals act as expected utility maximizers whose vNM utilities are compatible with their strict ordering of objects – preference maximization with respect to these vNM utilities then implies preference maximization with respect to first order stochastic dominance. The second problem can be overcome by relaxing consumption sets to be ε-close to the sd-equal-division-lower-bound – letting ε go to zero will then yield a limit allocation that satisfisfies all our desired criteria. To overcome the problem of satiated individuals, we have to allow for some 'slack' or a 'dividend' that increases the income of unsatiated individuals. In that, we follow Mas-Colell [1992].[20]

Let individuals' consumption sets $X_i \subset \mathbb{R}^n$ be closed, bounded and convex and let each individuals' endowment y_i be in the interior of of X_i. Let the set of possible price vectors be given as $Q = \{q = (q_a)_{a\in A} \in \mathbb{R}^n | \|q\| = \sum|q_a| \le 1\}$ and the state space be denoted as $Z = X_1 \times X_2 \times \ldots \times X_n \times Q$. Individuals' demand is guided by a (set-valued) preference map $P_i\colon X_i \rightrightarrows X_i$ and constrained by a budget $q \cdot y_i + \frac{1-\|q\|}{\|q\|}$ where the term $\frac{1-\|q\|}{\|q\|}$ may be used by the Walrasian auctioneer to increase budgets beyond the value of endowments.

If P_i is irreflexive (i.e. $x_i \notin P_i(x_i)$ for every $x_i \in X_i$), convex-valued (i.e. $P_i(x_i)$ is convex for every $x_i \in X_i$) and has an open graph (i.e. if $x_i \in P_i(x_i')$, the same holds for all $\tilde{v}_i$, $\tilde{w}_i$ in some small neighbourhood of v_i and w_i) we have the following.

Fact 3. Theorem 1 in Mas-Colell [1992].
There exists a Walrasian equilibrium with slack, i.e. a state $z = (x, q)$ such that

- $\forall\, i \in I: \quad q \cdot x_i \le q \cdot y_i + \frac{1-\|q\|}{\|q\|}$ *and* $\left(\tilde{x}_i \in P_i(x_i) \implies q \cdot \tilde{x}_i > q \cdot y_i + \frac{1-\|q\|}{\|q\|}\right)$ *(preference maximisation),*
- $\forall\, a \in A: \quad \sum_{i\in I} x_{i,a} = \sum_{i\in I} y_{i,a}$ *(feasibility).*

Proof of Theorem 8. Consider an assignment problem $(A, I, \succsim)$ and a compatible collection of weight vectors $w \in W(\succsim)$. Define individuals' consumption sets as

$$X_i^\varepsilon = \left\{ x_i = (x_{i,a})_{a\in A} \in \mathbb{R}^n \middle| \sum_{a\in A} x_{i,a} \le 1 + \varepsilon,\ \forall a \in A : x_{i,a} \ge 0 \text{ and } \sum_{b\in U_i(a)} x_{i,b} \ge \frac{|U_i(a)|}{n} - \varepsilon \right\}$$

and endow each individual with a share of $\frac{1}{n}$ of each object, i.e. $y_i = \left(\frac{\mathbf{1}}{\mathbf{n}}\right)$. Note that consumption sets are closed, bounded and convex and that for $\varepsilon > 0$ endowments lie in the interior of the consumption set. Moreover, note that while for positive ε the consumption sets include bundles that cannot be interpreted as lotteries ($\sum_A x_{i,a}$ may not be equal to 1), in the limit as we let ε go to zero, individuals are restricted to consume bundles that can be interpreted in this way and that (weakly) stochastically dominate $\left(\frac{\mathbf{1}}{\mathbf{n}}\right)$. Let the set of possible price vectors be given as

$$Q = \left\{ q = (q_a)_{a\in A} \in \mathbb{R}^n | \|q\| = \sum |q_a| \le 1 \right\},$$

and the state space as $Z^\varepsilon = X_1^\varepsilon \times X_2^\varepsilon \times \cdots \times X_n^\varepsilon \times Q$. To provide individuals with continuous (strict) preferences, we define

$$P_i\colon X_i^\varepsilon \rightrightarrows X_i^\varepsilon : P_i(x_i) = \{\tilde{x}_i \in X_i^\varepsilon | w_i \cdot \tilde{x}_i > w_i \cdot x_i\}.$$

Intuitively, under P_i a consumption bundle is preferred over another if it yields a higher expected utility with respect to vNM utilities w_i – except that bundles

[20] See also Dreze and Müller [1980] and Kajii [1996] for other pioneers of this approach.

only approximate lotteries in that $1 - \varepsilon \leq \sum_A x_{i,a} \leq 1 + \varepsilon$. Clearly, P_i is irreflexive, convex-valued and has an open graph.

By Fact 3, for any ε, there exists a Walrasian equilibrium with slack. Moreover, if we assume that all individuals with the same ordinal preferences, $i \in G$, share the same weight vector w_i, there exists a Walrasian equilibrium with $x_i = x$ for all $i \in G$ – for any other equilibrium, replacing individuals consumption bundles with

$$x_i = \frac{\sum_G x'_j}{|G|}$$

restores equal treatment of equals without violating preference maximisation or feasibility.

Consider a sequence $(\varepsilon_k)_{k\in\mathbb{N}}$ with $\varepsilon_k \searrow 0$ and a sequence of associated equilibria $e^k = (x^k, q^k)$ satisfying equal treatment of equals. As the sequence of equilibria is bounded by $X_1^{\varepsilon_1} \times X_2^{\varepsilon_1} \times \cdots \times X_n^{\varepsilon_1} \times Q$, it has a convergent subsequence – and hence we may assume w.l.o.g. that (e^k) is convergent itself. Denote the limit of that sequence as $e^\star = (x^\star, q^\star)$. Then $x^\star$ satisfies equal treatment of equals and, by construction of our consumption sets, is a random assignment that satisfies the strong sd-equal-division-lower-bound.

Claim 1. *The random assignment $x^\star$ is in the weak sd-core from equal division.*

Proof of Claim: Towards a contradiction, assume there exists a group $G \subset I$ and another random assignment p such that $\sum_{i\in G} p_i = |G| \left(\frac{\mathbf{1}}{\mathbf{n}}\right)$ and, for all $i \in G$, $p_i \succ_i^{sd} x_j^\star$. The latter implies

$$\forall i \in G, \varepsilon > 0: \qquad p_i \in X_i^\varepsilon,$$

and for some sufficiently large $\bar{k}$ we have

$$\forall i \in G, k > \bar{k}: \qquad w_i \cdot p_i > w_i \cdot x_i^{\varepsilon_k}.$$

Then by preference maximization we have

$$\forall i \in G, k > \bar{k}: \qquad q^{\varepsilon_k} \cdot p_i > q^{\varepsilon_k} \cdot \left(\frac{\mathbf{1}}{\mathbf{n}}\right) + \frac{1 - \|q^{\varepsilon_k}\|}{\|q^{\varepsilon_k}\|} \geq q^{\varepsilon_k} \cdot \left(\frac{\mathbf{1}}{\mathbf{n}}\right).$$

But this contradicts $\sum_G q^{\varepsilon_k} \cdot p_i = q^{\varepsilon_k} \sum_G p_i = q^{\varepsilon_k} |G| \left(\frac{\mathbf{1}}{\mathbf{n}}\right)$.

◇

Claim 2. *The random assignment $x^\star$ is sd-efficient.*

Proof of Claim: Towards a contradiction, assume there exists another random assignment p and a group $G \subset I$ such that $p_i \succ_i^{sd} x_i^\star$ for all $i \in G$ and $\sum_G (p_i - x_i^\star) = 0$. As the trade $(p_i - x_i^\star)$ sd-improves individual assignments, we know that

$$\forall i \in G: \qquad w_i \cdot (p_i - x_i^\star) > 0.$$

Nevertheless, the bundle $x_i^\varepsilon + (p_i - x_i^\star)$ may not be in the consumption set X_i^ε if there is a good a such that $x_{i,a}^\varepsilon + (p_{i,a} - x_{i,a}^\star) < 0$. Only as x_i^ε approaches $x_i^\star$, a (scaled down) trade can always be executed:

$$\exists \bar{k} : \quad \forall i \in G, k > \bar{k} : \qquad x_i^{\varepsilon_k} + 1/2(p_i - x_i^\star) \in X_i^{\varepsilon_k}.$$

Then by preference maximization we have

$$\forall i \in G, k > \bar{k} : \qquad q^{\varepsilon_k} \cdot 1/2(p_i - x_i^\star) > 0.$$

But this contradicts $\sum_G (p_i - x_i^\star) = 0$. ◇

This completes the proof. □

3.7 Concluding Remarks

We end this chapter with two remarks.

First, our impossibility results Theorem 6 and 7 illuminate the difference between two of the most prominent efficient solutions to the random assignment problem, namely the Probabilistic Serial (which is sd-envy-free) and Hylland and Zeckhauser's WEEI (which selects from the strong sd-core) and show that no assignment mechanism may satisfy all equity criteria satisfied by either of the two. Such impossibility results may also be of practical importance where an assignment mechanism is challenged in court by individuals who are unsatisfied with their eventual assignment – again, (overly) ambitious parents and the allocation of school seats comes to mind. For example, if the mechanism fails to be sd-envy-free this could be used as an argument by claimants to either void the whole assignment produced by the mechanism or receive special treatment as an individual and be admitted to one's school of choice. In such a situation, impossibility results such as ours would allow the school board to argue that the violation of some equity concept is *unavoidable* and hence, cannot be used as an argument against the assignment mechanism.

Second, we have not considered the issue of strategic reports by participants. Neither the Probabilistic Serial or Hylland and Zeckhauser's WEEI are strategy-proof. However for the Probabilistic Serial Kojima and Manea [2010] show that in large markets, with many participants and sufficiently many copies of each object, reporting truthfully becomes a weakly dominant strategy. Hylland and Zeckhauser argue that individuals can only gain from misrepresenting their preferences if that influences prices and that, in large markets "no individual can have a foreseeable effect on price[s]" [Hylland and Zeckhauser, 1979]. Since our solution also rests on Walrasian equilibria, it is similarly hard for participants to foresee the effect that any misreport would have on prices. Hence, at least in large markets, our solution should not lead to strategic misrepresentation of preferences.

4. SCORING RULES AND IMPLEMENTATION IN ITERATIVELY UNDOMINATED STRATEGIES

This Chapter has been submitted to the BDPEMS working paper series.

4.1 Abstract

We analyse voting games with three candidates and characterize voting procedures according to the solution that they implement in iteratively undominated strategies. Among all positional scoring rules, the Borda Rule is the unique procedure that satisfies (i) *unanimity* (**U**) and is (ii) *majoritarian after eliminating a worst candidate* (**MEW**). In the larger class of direct mechanism scoring rules, Approval Voting is characterised as the only procedure that is *majoritarian after eliminating a Pareto dominated candidate* (**MEPD**). However, it fails a desirable monotonicity property: a candidate that is the unique solution for some preference profile, may lose the election once she gains further in popularity. In contrast, the Borda Rule is the unique direct mechanism scoring rule that satisfies (i) **U**, (ii) **MEW** and satisfies (iii) *monotonicity* (**MON**). Finally, the exist no direct mechanism scoring rules satisfying both **MEPD** and **MON** or *Condorcet consistency*.

4.2 Introduction

Voting procedures allow individual voters to cast ballots that are aggregated to arrive at a collective choice from a set of available alternatives. To compare voting procedures, we ask which alternatives may arise as the outcome of an election when voters cast their ballots strategically, potentially misrepresenting their preferences. Then, for any solution concept that describes voters behaviour in voting games, we can map voters preferences to possible election outcomes and thus arrive at a social choice correspondence said to be implemented by the voting procedure.

Ideally, our voting procedure should implement a normatively appealing social choice correspondence under mild assumptions restricting voters' behaviour. Arguably the mildest such restriction is to assume that voters play undominated strategies. Unfortunately, for all finite voting procedures,[1] we face the following impossibility result: with at least three alternatives, any social choice function[2] that can be implemented in undominated strategies is either dictatorial or rules out the

[1] A finite voting procedure allows each voter to choose from a finite set of admissible ballots, as envisioned by both Gibbard [1973] and Satterthwaite [1975]

[2] i.e. a single valued social choice correspondence

election of some candidate a priori. The original result is due to Gibbard [1973] and Satterthwaite [1975] who are concerned with implementation in dominant strategies; Jackson [1992] shows that if we consider finite voting procedures,[3] any social choice function that can be implemented in dominant strategies can be implemented in undominated strategies and vice versa.

In this chapter, we will focus on the case of three alternatives where these negative results first arise. Moreover, in light of these results, we content ourselves with implementing social choice correspondences and move to a stronger solution concept, considering implementation in *iteratively* undominated strategies.

Here we are able to derive three main characterisation results. First, in the class of positional scoring rules (including among others Plurality-, Antiplurality- and the Borda-Rule), the Borda Rule is the unique voting procedure implementing a social choice correspondence that satisfies *unanimity* **(U)** (i.e., uniquely selects an alternative whenever it is unanimously preferred) and is *majoritarian after eliminating a worst alternative* **(MEW)** (i.e., if there is a unanimously disliked alternative, the majority-preferred alternative among the other two is uniquely selected).

Second, in the larger class of direct mechanism scoring rules (including e.g. all positional scoring rules as well as Approval Voting), Approval Voting is characterized by a single axiom – it is the unique voting procedure that is *majoritarian after eliminating a Pareto-dominated alternative* **(MEPD)** (i.e., if there is a Pareto-dominated alternative, the majority-preferred alternative among the other two is uniquely selected).

Third, in the class of direct mechanism scoring rules, the Borda Rule is the unique voting procedure implementing a social choice correspondence that satisfies **U**, **MEW** and *monotonicity* **(MON)**(i.e., an alternative that is uniquely selected for some preference profile should still be uniquely selected for a preference profile where every voter ranks this alternative weakly higher).

Three recent papers most closely related to our results are [Dhillon and Lockwood, 2004], [Buenrostro et al., 2013] and [Núñez and Courtin, 2013] who all identify conditions for preferences profiles under which particular scoring rules yield a unique solution in iteratively undominated strategies. Dhillon and Lockwood [2004] consider the Plurality Rule with an arbitrary number of alternatives and provide sufficient and necessary conditions. Buenrostro et al. [2013] consider so called general scoring rules – a set that overlaps with the set of direct mechanism scoring rules that we consider – and provide sufficient conditions. Núñez and Courtin [2013] consider Approval Voting and provide sufficient-and-necessary conditions.

The use of iteratively undominated strategies as solution concept has a long tradition in the theory of voting where it was introduced by Farquharson [1969] under the name of *sophisticated voting*. It is particularly well suited to model strategic behaviour in elections where the number of voters is large relative to the number of available alternatives, as under these conditions voters typically find themselves in a position where they are not pivotal. As a result, it is easy to sustain any strategy as a

[3] Jackson's equivalence result is even more general in that he considers bounded mechanism.

best response so that the alternative solution concept of rationalizability has no bite. Similarly, under many intuitive voting procedures, if all voters vote 'in favour' of some arbitrary alternative, it should be elected and an individual deviation should be of no effect. But then, all alternatives are implemented in (some) Nash-equilibrium.

To restrict the set of alternatives implemented in Nash-equilibrium, we could consider refinements, such as undominated [Palfrey and Srivastava, 1991] or trembling-hand perfect equilibrium. However, these refinements leave a second problem of (pure strategy) Nash-implementation unaddressed. To illustrate this, consider two voters who both prefer a over b over c and who have to choose an alternative using the Antiplurality Rule where each voter votes against one alternative and the alternative with the least number of votes is chosen. Then, subject to specifying a tiebreaking procedure, it is easy to see that in any Nash-equilibrium one voter will vote for b while the other votes for c, so that the commonly preferred alternative a is chosen. However, both voters face a coordination problem in that it is unclear who should vote for b and who should vote for c. Hence, while a is the unique outcome in any Nash-equilibrium, it remains doubtful whether miscoordination may not in the end help b or c to arrive at tie with a and hence be potentially chosen.[4]

Many authors have studied the implementation in iteratively undominated strategies. If in each iteration only strictly dominated strategies are removed, Börgers [1995] shows that only dictatorial social choice functions can be implemented, unless we restrict the set of possible preference profiles to exclude cases where voters preferences are identical. If weakly dominated strategies are removed as well (as we will assume throughout this chapter), Moulin [1979] shows that there exist voting procedures that implement anonymous and Pareto efficient social choice functions. Abreu and Matsushima [1994] show that any social choice function may be implemented, when voters can be fined for what are identified as misrepresentations of preferences. For that, they require a large strategy space where each voter reports not just her own preferences and the preferences of some 'neighbour', but also in total K preference profiles, i.e. tupel of all voters' preferences, where K has to be chosen arbitrarily large in order to allow fines to become arbitrarily small. While this allows them to derive a remarkably permissive implementation result, the sheer size of the strategy space (as well as the introduction of fines) rules out the use of their mechanism for elections with many voters.

In order to restrict attention to voting procedures that can be readily applied in practice, we limit our analysis to rules where the size of the strategy space is no larger than the number of possible preference relations that a voter may hold; that is, we consider voting procedures that can be interpreted as direct mechanisms. Moreover, we will consider scoring rules, for which Myerson [1995] provides an axiomatisation based on *reinforcement* and *overwhelming majority*: Consider a voting

[4] To rule out examples such as this, we could consider mixed-strategy equilibria and demand that any outcome sustained by such an equilibrium, is contained in the set of alternatives chosen by the social choice correspondence. However, such an analysis requires that voters' preferences over *lotteries* of alternatives are common knowledge, which constitutes an additional, strong, assumption.

procedure where each voter has access to the same set of strategies, i.e. can cast the same admissible ballots, and where the set of such strategies is independent of the number of voters participating in the election. Reinforcement then demands that if ballots are evaluated for two separate districts and in each district the same alternative is elected, then in a joint district, this alternative should be elected as well. Overwhelming majority demands that if some group of voters, or rather the ballots that they cast, are replicated sufficiently often, the election outcome in the general election has to agree with the outcome of an election where only ballots of the overwhelmingly large, replicated group are considered.[5]

Together with the requirement that a voting procedure be neutral (with respect to a relabelling of alternatives) and anonymous (with respect to a relabelling of voters), these axioms uniquely characterize scoring rules in the class of *all* voting procedures. Hence, unless one is willing to give up on any of these desirable properties, restricting our attention to scoring rules comes at no further loss of generality.

The chapter is organised as follows. Section 4.3 defines voting games and their solution by iterative elimination of dominated strategies. Section 4.4 defines normative criteria for social choice correspondences. Section 4.5 characterizes scoring rules with respect to the social choice correspondences that they implement. Section 4.6 concludes.

4.3 Technicalities

4.3.1 Candidates and voters

Throughout this chapter, we consider a set of three candidates (or alternatives) $A = \{a, b, c\}$ and a finite set of voters I with generic element i. Each voter's preferences are assumed to be given by a strict linear order $\succ_i$ on A. In consequence, there are six distinct sets of voters, characterized by their preferences that we denote $I_{xyz} = \{i \in I | x, y, z \in A, x \succ_i y \succ_i z\}$ and whose generic element we refer to as i_{xyz}. A preference profile is denoted as $\succ_I = (\succ_i)_{i \in I}$.

4.3.2 Scoring rules

Scoring rules allow each voter i to cast a ballot $v_i = (v_i^a, v_i^b, v_i^c)$ from the same set of admissible ballots $V \subset \mathbb{R}^3$. We assume that ballots are *neutral* with respect to a relabelling of candidates; formally, for any admissible ballot $v_i = (k, l, m) \in V$, each permutation of v_i is also an admissible ballot. A ballot is called an abstention if it takes the form $v_i = (k, k, k)$.

Using Cartesian products, we define $V^0 = \prod_{i \in I} V$ and $V_{-i}^0 = \prod_{j \neq i} V$ and denote generic elements as v and v_{-i}. We refer to $v \in V^0$ as a ballot profile and denote the associated score of some candidate x as $|v^x| = \sum v_i^x$. For an opposing ballot profile $v_{-i} \in V_{-i}^0$ we define $|v_{-i}^x| = \sum v_{j \neq i}^x$.

[5] For a detailed description of both axioms, see [Myerson, 1995].

A candidate wins the election if her score is higher than any other candidate's score. To deal with ties, we rely on the report of a tiebreaker, labelled t, who has to chose a strict linear order $\rhd$ on A, where $\mathbb{\rhd}$ denotes the set of such orders.[6] Then, for given v and $\rhd$, candidate x wins the election whenever she has a weakly higher score than all other candidates and, in case of a tie, is ranked first according to $\rhd$. Formally, x wins if and only if

$$\forall y \neq x : |v^x| \geq |v^y| \text{ and } |v^x| = |v^y| x \rhd y.$$

Note that for any reported ballot profile v and a report by the tiebreaker $\rhd$, there exists a unique winner. If we would refrain from breaking ties in a deterministic manner, outcomes would either be set-valued or take the form of a lottery over alternatives. To analyse voting games induced by a scoring rule, we would then have to amend voters preferences, for example to include preferences over sets of candidates[7] or by specifying von Neumann - Morgenstern utility functions. Instead we opt for deterministic tiebreaking which allows us to base our analysis exclusively on ordinal preferences over candidates.

We will consider scoring rules that can be interpreted as direct mechanisms, i.e. rules where the size of voters' strategy space is bounded by the number of voters' types. A scoring rule as described above is a *direct mechanism scoring rule* if, after the removal of abstentions,[8] we have $|V| \leq 6$. For *positional scoring rules*, V is taken to be the set of permutation of $(1, s, 0)$, where $s \in [0, 1]$ is a fixed parameter that characterizes the rule. The most notable positional scoring rules are the *Plurality Rule*, corresponding to $s = 0$, the *Antiplurality Rule* $(s = 1)$ and the *Borda Rule* $(s = \frac{1}{2})$.

Other direct mechanism scoring rules, are rules that allow voters to either vote for one candidate or split their vote between two – we refer to such rules as *vote-splitting scoring rules*. Formally, for a vote-splitting scoring rule, V consists of all permutations of $(s, s, 0)$ and $(1 - s, 0, 0)$, $s \in [0, 1]$. If $s = \frac{1}{3}$, voters have a fixed budget of points that they can award to one candidate or split between two. If $s \neq \frac{1}{3}$, splitting is either rewarded or punished by changes in the budget. The most notable such rule is *Approval Voting*, where $s = \frac{1}{2}$. Note that $s = 1$ is equivalent to the Antiplurality rule, while $s = 0$ corresponds to the Plurality Rule. Hence,

[6] If one objects to the introduction of an additional player, another option would be to break ties by a multiplayer version of "matching pennies": ask each voter to report a number $t_i \in \{0, 1, .., 5\}$, set $t = \sum t_i \bmod 6$ and let each of the 6 possible outcomes $t = \{0, 1, ..., 5\}$ correspond to one of the 6 possible linear orders $\rhd \in \mathbb{\rhd}$. For our purposes, the two approaches are essentially equivalent, as the tiebreaker will be assumed to be indifferent, so that neither the tiebreaker's set of possible reports, nor the voters' set of possible reports t_i can be reduced using elimination of weakly dominated strategies.

[7] Equivalently to the approach followed here, we could refrain from breaking ties and extend each preference relation $\succ_i$ to pairs of subsets of A, by defining for all $A', A'' \subset A$

$$A' \succ_i A'' \quad :\Longleftrightarrow \quad A' \neq A'' \text{ and } \forall x \in A' \backslash A'', y \in A'' : x \succ_i y \text{ and } \forall x \in A', y \in A'' \backslash A' : x \succ_i y.$$

[8] Since abstentions represent dominated strategies, removing them will not affect our analysis.

both Approval Voting and the Borda Rule can be thought of as 'half-way' between the Plurality and Antiplurality Rule. Our first result will show that positional and vote-splitting scoring rules are essentially the only direct mechanism scoring rules.

In a slight abuse of notation, we will at times identify a scoring rule and the set of admissible ballots and denote both by V.

4.3.3 Voting games

Together, the set of candidates, voters' preferences, a scoring rule and a tiebreaker - assumed to be indifferent between candidates - give rise to a complete information voting game $\Gamma(\succ_I, V^0)$ with a set of players $I \cup \{t\}$. In each game $\Gamma(\succ_I, V^0)$, a strategy profile $(v, \triangleright) \in V^0 \times \triangleright$ determines a unique outcome $g(v, \triangleright) \in A$.

We will also consider restricted games $\Gamma(\succ_I, V')$, where each voter's strategies are restricted to some set $V_i' \subseteq V$ and the space of ballot profiles is denoted $V' = \prod_{i \in I} V_i'$. Accordingly, the space of opponents' ballot profiles is denoted $V_{-i}' \prod_{j \neq i} V_j'$. Where all voters $i \in I_{xyz}$ have the same (restricted) strategy set, we denote it $V_{xyz}' = V_i'$.

4.3.4 Iteratively Undominated Strategies

In particular, we will focus on restricted games where weakly dominated strategies have been removed.

Definition 6. A strategy $v_i \in V_i'$ is *weakly dominated* in $\Gamma(\succ_I, V')$ if there exists $w_i \in V_i'$ such that for all $v_{-i} \in V_{-i}'$, $\triangleright \in \triangleright$

$$g(w_i, v_{-i}, \triangleright) \succ_i g(v_i, v_{-i}, \triangleright) \text{ or } g(w_i, v_{-i}, \triangleright) = g(v_i, v_{-i}, \triangleright)$$

with $g(w_i, v_{-i}, \triangleright) \succ_i g(v_i, v_{-i}, \triangleright)$ for at least one $v_{-i} \in V_{-i}'$ and $\triangleright \in \triangleright$.

Strategies $\triangleright \in \triangleright$ are never dominated, as the tiebreaker is assumed to be indifferent between all outcomes $g(v, \triangleright) \in A$. Hence, in iteratively removing dominated strategies, we can focus on voters $i \in I$. First, we define the set of undominated strategies as $V_i^1 = V \backslash \{v_i \in V | v_i \text{ is weakly dominated in } \Gamma(\succ_I, V^0)\}$. We will make use of the following useful fact.

Fact 4. *In approval voting games, the set of undominated strategies V_i^1 for a voter of type i_{xyz} consists of all ballots $v_i \in V$ such that $v_i^x = \frac{1}{2}$ and $v_i^z = 0$ [Brams and Fishburn, 1978]. For positional scoring rule voting games, i_{xyz}'s undominated strategies are all ballots $v_i \in V$, such that $v_i^x \geq s$ and $v_i^z \leq s$ (see Proposition 1 in [Buenrostro et al., 2013]).*

Next, we move to the iterative elimination of dominated strategies and define

$$V_i^{m+1} = V_i^m \backslash \{v_i \in V_i^m | v_i \text{ is weakly dominated in } \Gamma(\succ_I, V^m)\}, \text{ for } m \in \mathbb{N}.$$

Clearly, $V_i^{m+1} \neq \varnothing$, as it is impossible for all strategies in V_i^m to be dominated by one another.[9] Also, as V is finite, there exists some $\overline{m}$, such that no further restrictions are possible; $V^m = V^{\overline{m}}$, for all $m \geq \overline{m}$. This leads us to the following solution of a voting game.

Definition 7. For a voting game $\Gamma(\succ_I, V^0)$ we define its *solution in iteratively undominated strategies* as the set of possible outcomes after iteratively eliminating all weakly dominated strategies, and denote it as

$$S(\succ_I, V) = \{x \in A | \exists v \in V^{\overline{m}} : \forall y \in A : |v^x| \geq |v^y|\}.$$

We say that V implements the *social choice correspondence* $S(\cdot, V)$ that maps preference profiles onto subsets of A.

4.3.5 Order independence and elimination of duplicate strategies

In defining dominance solvability, we followed Moulin [1979] in that we eliminated *all* weakly dominated strategies when moving from V^m to V^{m+1}.[10] This raises the question, whether a different order of elimination, where only some individuals' dominated strategies are eliminated at each step, might yield a different solution.

Fortunately, Marx and Swinkels [1997] assure us that this is not the case. More precisely, their Theorem 1 ensures that once we reach a restricted game $\Gamma(\succ_I, V')$ such that no further strategy can be eliminated based on weak dominance, $\Gamma(\succ_I, V')$ will be equivalent to $\Gamma(\succ_I, V^{\overline{m}})$ up to the elimination of duplicate strategies and the renaming of strategies. In particular, the set of possible outcomes of both games will be the same.

This is because, in our voting games, the elimination of dominated strategies satisfies what Marx and Swinkels [1997] call 'transference of decisionmaker indifference': whenever a voter i, for a given opposing strategy profile, is indifferent between outcomes $g(v_i, v_{-i}, \rhd)$ and $g(v_i', v_{-i}, \rhd)$, then so is every other player. This is of course satisfied, as i will only be indifferent if both outcomes coincide.[11]

Moreover, whether in the process of iterative elimination, we chooses at some point to eliminate a single (of multiple) duplicate strategies, will be of no effect; the game $\Gamma(\succ_I, V')$ that we reach eventually will be equivalent to $\Gamma(\succ_I, V^{\overline{m}})$ up to the elimination of duplicate strategies and the renaming of strategies.

To see this, suppose that in the game $\Gamma(\succ_I, V^m)$ there are two duplicate but undominated strategies $v_i, \tilde{v}_i \in V_i^m$, of which we choose to eliminate only $\tilde{v}_i$ when moving to the next restricted game. If $\tilde{v}_i$ could at some step be instrumental in eliminating another strategy v_j based on weak dominance, the remaining duplicate

[9] Recall that $\succ_i$ is a strict linear order.

[10] Other authors in the context of voting theory, most notably Farquharson [1969], have used the same solution concept under the name of 'sophisticated voting'.

[11] Indifference of the tiebreaker does not transfer to indifference of other voters. However, this is unproblematic, as the principle of 'transference of decisionmaker indifference' is only required to hold for players whose strategies are eliminated (see Definition 2 in [Marx and Swinkels, 1997]).

v_i will suffice to eliminate v_j. If v_i was eliminated based on weak dominance before it becomes instrumental in eliminating v_j, $\tilde{v}_i$ would have been eliminated as well.

4.4 Axioms

We want to compare and characterize scoring rules according to the social choice correspondences that they implement. In particular, we ask for which preference profiles the induced voting games have a unique solution in iteratively undominated strategies – and which outcomes are selected in that case. A minimal and prominent requirement is unanimity.

Definition 8. A scoring rule V is said to satisfy *Unanimity* (**U**), if for any preference profile $\succ_I$ such that $I = I_{abc} \cup I_{acb}$, we have $S(\succ_I, V) = \{a\}$.

Where there is no universal agreement, we have to weigh some voters' preferences against others', in order to choose an alternative. In the case of two alternatives, fairness and efficiency force us to accept simple majority as the guiding principle,[12] but when the number of alternatives grows, it is unclear how this principle should be adjusted.

However, if one of three alternatives is unanimously agreed to be the worst, we are essentially in a situation with just two relevant alternatives, so that a simple majority should suffice to determine the optimal alternative. We can formalize this idea as follows.

Definition 9. Consider an arbitrary preference profile $\succ_I$ such that $I = I_{abc} \cup I_{bac}$. A scoring rule V is said to be *Majoritarian after Eliminating a Worst Alternative* (**MEW**), if $|I_{abc}| > |I_{bac}|$ implies $S(\succ_I, V) = \{a\}$.

A similar situation arises, when one of three alternatives is unanimously agreed to be worse than some other alternative. Again, one might think that the former, Pareto dominated, alternative should be disregarded and the decision between the remaining two alternatives should be made by simple majority.

Definition 10. Consider an arbitrary preference profile $\succ_I$ such that $I = I_{abc} \cup I_{acb} \cup I_{bac}$. A scoring rule V is said to be *Majoritarian after Eliminating a Pareto Dominated Alternative* (**MEPD**), if for $|I_{abc}| + |I_{acb}| > |I_{bac}|$, we have $S(\succ_I, V) = \{a\}$, while for $|I_{abc}| + |I_{acb}| < |I_{bac}|$, we have $S(\succ_I, V) = \{b\}$.

The formal definition reveals what might be a controversial property of **MEPD**: some alternative b might be chosen by the social choice correspondence $S(\cdot, V)$ based on its majority support over another alternative a, even though it may only be a that, according to **MEPD**, forces us to eliminate c, based on Pareto dominance.

[12] See May [1952] who provides an axiomatization of the Majority Rule. His symmetry axioms can be seen as an embodiment of fairness, while the *positive responsiveness* axiom may be seen as a requirement of efficiency.

$$\begin{tikzcd}[column sep=small] & & & \mathbf{U} \\ \mathbf{CON} \arrow[r, Rightarrow] & \mathbf{MEPD} \arrow[urr, Rightarrow] \arrow[drr, Rightarrow] & & \\ & & & \mathbf{MEW} \end{tikzcd}$$

Fig. 4.1: Logical relations between intra-profile axioms

In defence of **MEPD**, observe that it unifies both preceding axioms, i.e. **MEPD** implies both **MEW** and **U**. Moreover, it is implied by another, well known requirement, formulated by the Marquis de Condorcet, according to which an alternative should be chosen whenever it is supported by a majority against any other alternative.[13]

Definition 11. Consider an arbitrary preference profile $\succ_I$. A scoring rule V is said to be *Condorcet consistent* (**CON**), if $S(\succ_I, V) = \{a\}$ whenever

$$|I_{abc}|+|I_{acb}|+|I_{bac}| > |I_{bca}|+|I_{cab}|+|I_{cba}| \quad \text{and} \quad |I_{abc}|+|I_{acb}|+|I_{cab}| > |I_{cba}|+|I_{bac}|+|I_{bca}|.$$

To see that **CON** implies **MEPD**, observe that whenever b is supported by a majority against a, and a Pareto dominates c, b will also be supported by a majority against c and should therefore be chosen according to **CON**.

Figure 4.1 presents the logical relations between the axioms described so far. Note that they are all intra-profile axioms, i.e. they all concern the behaviour of a social choice correspondence within given preference profiles. The next axiom concerns its behaviour across profiles. For that, consider an arbitrary profile $\succ_I = (\succ_i)_{i\in I}$. Another profile $\succ'_I = (\succ'_i)_{i\in I}$ is said to be an a-monotonic transformation of $\succ_I$, iff

$$\forall i \in I: \quad a \succ_i b, c \implies a \succ'_i b, c \quad \text{and} \quad b \succ_i c \iff b \succ'_i c,$$

i.e. such that a is more popular under $\succ'_I$, while the ordering of b and c remains unchanged.[14] Then, if a is the unique solution under $\succ_I$, it should remain so under $\succ'_I$.

Definition 12. A scoring rule V is said to satisfy *Monotonicity* (**MON**), if for any preference profile $\succ_I$ and an a-monotonic transformation $\succ'_I$ we have

$$S(\succ_I, V) = \{a\} \implies S(\succ'_I, V) = \{a\}.$$

Monotonicity is particularly important where candidates are engaged in electoral competition, i.e. where they can choose a policy platform and thereby affect their position in voters' rankings of candidates. A violation of monotonicity could create perverse incentives for candidates – a candidate may then increase her chance of election by adjusting her platform with the *only* effect to hurt some group within

[13] Note that Condorcet famously pointed out that such an alternative may not exist when pairwise majority comparisons yield a cycle, cf. de Condorcet [1785] p. lxi.

[14] The requirement that the ranking between b and c remains unchanged, makes the following notion of monotonicity weaker than Maskin-monotonicity, which is required for Nash-implementation.

the electorate, moving her down in that groups' rankings of candidates (while leaving everyone's ranking of the other candidates unchanged). Then, under a violation of monotonicity, it could be that the candidate is uniquely selected by only after the change in platform, i.e. after she has lost in popularity.

4.5 Results

Our first result maps out the class of scoring rules under consideration, by showing that positional and vote-splitting scoring rules are essentially the only direct mechanism scoring rules; the only other scoring rules are slight variations of the Plurality and Antiplurality Rule. For that, we normalize ballots in a way that exchanges some strategies for duplicate counterparts.

Theorem 9. *Consider a direct mechanism scoring rule V. Then up to the elimination of abstentions and a normalization of ballots, one of the following four cases applies. The set of admissible ballots V consists of*

(1) *all permutations of* $(1, s, 0)$, $s \in [0, 1]$.

(2) *all permutations of* $(s, s, 0)$ *and* $(1 - s, 0, 0)$, $s \in [0, 1]$.

(3) *all permutations of* $(1, 0, 0)$ *and* $(s, 0, 0)$, $s \in [0, 1]$.

(4) *all permutations of* $(1, 1, 0)$ *and* $(s, s, 0)$, $s \in [0, 1]$.

The intuition behind Theorem 9 is straightforward. Suppose V contains an admissible ballot b with three distinct entries. Since V is neutral with respect to a relabelling of candidates, the corresponding 6 permutations of b are also included in, and exhaust, V. Normalizing then yields case (1). If V contains a ballot b with two identical entries, it also contains all 3 of its permutations. This leaves room for another ballot b' which can have only 3 permutations itself, i.e. must contain two identical entries as well. Normalizing b and b', as well as their permutations yields one of the cases (2)-(4). A slightly more formal proof is found in the Appendix.

Within the class of direct mechanism scoring rules, we will show that the Borda Rule, and the social choice correspondence implemented by it, occupy a particularly prominent position. For that, the next two results establish sufficient-and-necessary conditions on preference profiles for the associated Borda Rule voting games to have a unique solution in iteratively undominated strategies.

Theorem 10. *Consider a Borda Rule voting game $\Gamma(\succ_I, V^0)$. A candidate $x \in A$, is the unique solution, i.e. $S(\succ_I, V) = \{x\}$, if we can label candidates so that one of the following three conditions is satisfied:*

(1) $|I_{xyz}| > |I_{xzy}| + |I_{yxz}| + 2|I_{yzx}| + 2|I_{zxy}| + 2|I_{zyx}|$,

(2) *or* $|I_{xyz}| > |I_{xzy}| + |I_{yxz}| + 2|I_{yzx}| + 2|I_{zxy}| + 2|I_{zyx}| - 1$ *and* $|I_{zxy}| > |I_{yxz}|$,

(3) *or* $|I_{xyz}| > |I_{xzy}| + |I_{yxz}| + 2|I_{yzx}| + 2|I_{zxy}| + 2|I_{zyx}| - 2$ *and* $|I_{xzy}| > 0$.

The proof for all three cases proceeds as follows. We first show that either y or z can be ruled out as an element of $S(\succ_I, V)$, as after a few rounds of eliminating dominated strategies we have $|v^x| > |v^y|$ or $|v^x| > |v^z|$. Then, the election is effectively over x and one remaining alternative candidate, and x wins, as it is supported by a majority. We present the proof for case (1) here, and relegate cases (2) and (3) to the Appendix.

Assume (1) holds. After eliminating dominated strategies, we know by Fact 4 that

$$\min_{v \in V^1} |v^x| - |v^z| = 1/2|I_{xyz}| - 1/2|I_{xzy}| - 1/2|I_{yxz}| - |I_{yzx}| - |I_{zxy}| - |I_{zyx}| > 0,$$

so that z is ruled out as an outcome. But then, in the game $\Gamma(\succ_I, V^1)$, for any voter i who prefers x over y, $v_i = (v_i^x, v_i^y, v_i^z) = (1, 0, \frac{1}{2})$ is a best reply for every opposing strategy profile $(v_{-i}, \triangleright) \in V^1_{-i} \times \triangleright$ as it maximizes the impact that i has on $|v^x| - |v^y|$. If another ballot $\tilde{v}_i \neq v_i$ is also a best reply against every $(v_{-i}, \triangleright)$, then $\tilde{v}_i$ is a duplicate strategy. If on the other hand $\tilde{v}_i$ is a worse reply than v_i against some $(v_{-i}, \triangleright)$, it is dominated and hence eliminated as we move to V^2.

To determine the possible outcomes in $\Gamma(\succ_I, V^2)$, we can assume that all $i \in I_{xyz} \cup I_{xzy} \cup I_{zxy}$ cast ballot $v_i = (1, 0, 1/2)$ - any other remaining strategy in V_i^2 would be a duplicate strategy and produce the same outcome. But then, x is the unique outcome after two rounds of eliminating dominated strategies, as by condition (1)

$$|v^x| \geq |I_{xyz}| + |I_{xzy}| + |I_{zxy}| > |I_{yxz}| + |I_{yzx}| + |I_{zyx}| \geq |v^y|.$$

This completes the proof for case (1); the proof for cases (2) and (3) is found in the appendix. The next Theorem shows that the conditions of Theorem 10 are also necessary.

Theorem 11. *Consider a Borda Rule voting game $\Gamma(\succ_I, V^0)$. No candidate can be excluded as a winner, i.e. $S(\succ_I, V) = A$, if for any labelling of candidates $x, y, z \in A$ the following three conditions are satisfied:*

$$\begin{array}{lll}
(1) & |I_{xyz}| \leq |I_{xzy}| + |I_{yxz}| + 2|I_{yzx}| + 2|I_{zxy}| + 2|I_{zyx}| & \\
(2)\ \textit{and} & |I_{xyz}| = |I_{xzy}| + |I_{yxz}| + 2|I_{yzx}| + 2|I_{zxy}| + 2|I_{zyx}| & \Longrightarrow |I_{zxy}| \leq |I_{yxz}| \\
(3)\ \textit{and} & |I_{xyz}| \geq |I_{xzy}| + |I_{yxz}| + 2|I_{yzx}| + 2|I_{zxy}| + 2|I_{zyx}| - 1 & \Longrightarrow |I_{xzy}| = 0.
\end{array}$$

Intuitively, under the conditions of Theorem 11, each group of voters I_{xyz} is small relative to the other groups, bringing us close to a balanced profile where each group is of the same size. For such a balanced profile, it is clear that no outcome can be ruled out.

The proof rests on a Lemma, which shows that if each I_{xyz} is small enough, no strategies beyond the initially dominated ones are eliminated in the process of iterated elimination.

Lemma 5. *Suppose that for any labelling of candidates $x, y, z \in A$ we have*

$$|I_{xyz}| < |I_{xzy}| + |I_{yxz}| + 2|I_{yzx}| + 2|I_{zxy}| + 2|I_{zyx}| - 2.$$

Then for the Borda Rule voting game $\Gamma(\succ_I, V^0)$, the elimination of dominated strategies stops after one round so that $\Gamma(\succ_I, V^1) = \Gamma(\succ_I, V^{\overline{m}})$. Moreover, $S(\succ_I, V) = A$.

The proof of Proposition 11 then delineates the remaining cases where some I_{xyz} may be larger than assumed in Lemma 5 so that some initially undominated strategies are eliminated, yet the process of elimination stops before any outcome can be ruled out. Both the proof of Lemma 5 and remaining proof of Theorem 11 require a large number of case distinctions and are relegated to the appendix.

Corollary 2. *The Borda Rule satisfies both* **U** *and* **MEW**.

Proof. Assume that $I = I_{abc} \cup I_{acb}$. Without loss of generality, we can assume $|I_{abc}| \geq |I_{acb}|$. By Theorem 10, a is the unique solution as $|I_{abc}| > |I_{acb}| - \mathbb{1}_{\{|I_{acb}|>0\}} = |I_{bac}| + |I_{acb}| + 2|I_{bca}| + 2|I_{cab}| + 2|I_{cba}| - \mathbb{1}_{\{|I_{acb}|>0\}}$.

Assume on the other hand that $I = I_{abc} \cup I_{bac}$ and $|I_{abc}| > |I_{bac}|$. Then by Theorem 10, a is the unique solution, as $|I_{abc}| > |I_{bac}| = |I_{bac}| + |I_{acb}| + 2|I_{bca}| + 2|I_{cab}| + 2|I_{cba}|$. □

The above corollary overlaps with results in Buenrostro et al. [2013] who provide sufficient conditions for scoring rule voting games to be dominance solvable, i.e. have a unique solution in iteratively undominated strategies. The corollary extends beyond their Theorem 1 and Theorem 2, in that it includes the case $I = I_{abc} \cup I_{acb}$, $|I_{abc}| = |I_{acb}|$, i.e. we show that a unanimously preferred candidate a is the unique solution even if the electorate is split in half. What might be more remarkable though, is the exceptional position among positional scoring rules that Corollary 2 grants to the Borda Rule:

Theorem 12. *The Borda Rule is the unique positional scoring rule that satisfies* **U** *and* **MEW**. *In particular, positional scoring rules with $s < \frac{1}{2}$ violate* **U**, *while positional scoring rules with $s > \frac{1}{2}$ violate* **MEW**.

The proof can be found in the appendix. To understand the intuition behind Theorem 12, assume that $s > \frac{1}{2}$ and everyone agrees that c is the worst alternative. Furthermore, if the groups I_{abc} and I_{bac} are roughly of the same size, it is possible that a and b receive roughly the same score so that a single voter is pivotal. In such a situation, awarding a score of s to the least preferred alternative c – and a score of zero to the second best alternative a or b – may be undominated, or even a unique best response, as it tips the election in favour of the most preferred alternative. Yet, if awarding a score of s to c cannot be ruled out based on weak dominance, c may win with an average score of $s > \frac{1}{2}$ while a and b are tied with an average score of about $\frac{1}{2}$.

Similarly, assume that that $s < \frac{1}{2}$ and that alternative a is unanimously preferred. If the electorate is split in half between the groups I_{abc} and I_{acb} and every voter supports their second best alternative by awarding it a score of one, a receives an average score of at most $s < \frac{1}{2}$ while b and c will be tied with an average score of $\frac{1}{2}$. An individual who deviates and supports a would then hand the election to their least preferred candidate. Hence, for each voter, supporting their second best alternative is undominated – as long as everyone else may still support their second best alternative. But then supporting the second best alternative can never be eliminated based on weak dominance which establishes both b and c as element of the solution $S(\succ_I, V)$.

In light of Theorem 12, it is natural to ask whether there exist other direct mechanism scoring rules, beyond the Borda Rule that simultaneously satisfy unanimity and are majoritarian after eliminating a worst candidate. The most prominent direct mechanism scoring rule not covered by Theorem 12 is Approval Voting, for which Núñez and Courtin [2013] provide necessary-and-sufficient conditions for the associated voting games to be dominance solvable, i.e. to have a unique solution in iteratively undominated strategies. In fact, we find that Approval Voting satisfies even the stronger axiom of being majoritarian after eliminating a Pareto dominated candidate – and that it is the only direct mechanism scoring rule that satisfies it.

Theorem 13. *Approval Voting is the unique direct mechanism scoring rule that satisfies* **MEPD**. *In particular, vote-splitting scoring rules with $s < \frac{1}{2}$ and scoring rules where V consists of all permutations of $(1,0,0)$ and $(s,0,0)$ violate* **U**, *while vote-splitting scoring rules with $s > \frac{1}{2}$ and scoring rules where V consists of all permutations of $(1,1,0)$ and $(s,s,0)$ violate* **MEW**.

The fact that the Borda Rule, while satisfying **U** and **MEW**, fails to satisfy **MEPD**, follows from Theorem 11. For example, consider a preference profile $\succ_I$ where $I = I_{abc} \cup I_{acb} \cup I_{bac}$ and $|I_{abc}| = |I_{acb}| = |I_{bac}| = n \geq 2$. Then by Theorem 11 $S(\succ_I, V) = A$, while **MEPD** requires a to be the unique solution. All other positional scoring rules violate either **U** or **MEW** and hence also **MEPD**, see Theorem 12.

In order to show that Approval Voting satisfies **MEPD** consider a preference profile where a Pareto dominates c, i.e. such that $I = I_{abc} \cup I_{acb} \cup I_{bac}$. Then after eliminating dominated strategies, no voter awards a higher score to c than to a (see Fact 4), so that for any, $v \in V^1$, the score of a is weakly larger than the score of c.

Moreover, if there exists a voter $i \in I_{abc}$, she will vote either $(\frac{1}{2}, \frac{1}{2}, 0)$ or $(\frac{1}{2}, 0, 0)$, thereby ensuring that $|v^a| > |v^c|$ and ruling out outcome c after one round of elimination. In the next step, each voter will award a score $s = \frac{1}{2}$ to her preferred among the remaining candidates a and b and a score of zero to the other candidate. Then, the candidate supported by a majority is the only remaining outcome after two rounds of elimination of dominated strategies.

If on the other hand $|I_{abc}| = 0$, so that $I = I_{acb} \cup I_{bac}$, we have to consider two cases. First consider $|I_{acb}| > |I_{bac}|$, where a is preferred by a majority over b. Then, for any $v \in V^1$, we have $|v^a| \geq \frac{|I_{acb}|}{2} > \frac{|I_{bac}|}{2} = |v^b|$ so that b is ruled out as an outcome. In the next step, each voter will support a among the two remaining candidates, so that a is the only remaining outcome after two rounds of elimination.

Finally, if $I = I_{acb} \cup I_{bac}$ and $|I_{bac}| > |I_{acb}|$, we know that for any $v \in V^1$, $|v^b| = \frac{|I_{bac}|}{2} > \frac{|I_{acb}|}{2} \geq |v^c|$, such that c is ruled out as an outcome. In the next step, every voter will support either a or b over the other, so that the majority candidate b is the only remaining outcome after two rounds of elimination of dominated strategies.

It remains to show that no other direct mechanism scoring rule satisfies **MEPD**. For that, the reader is referred to the Appendix.

We are now left with only two direct mechanism scoring rules that satisfy **U** and **MEW**, namely the Borda Rule and Approval Voting where only the latter satisfies the even stronger axiom **MEPD**. However, Approval Voting fails monotonicity, as can be seen in the following example.

Example 3. Consider a preference profile $\succ_I$ where $I = I_{abc} \cup I_{bac} \cup I_{cab}$ and

$$|I_{abc}| = 2, \qquad |I_{bac}| = 4, \qquad |I_{cab}| = 3.$$

After eliminating dominated strategies, it is clear that b will have a score of at least $\frac{|I_{bac}|}{2} = 2$, while the score of c is equal to $\frac{|I_{cab}|}{2} < 2$ (see Fact 4). This reduces the game further, to an election between a and b, which a wins with a score of $|v^a| = \frac{|I_{abc}|+|I_{cab}|}{2} = \frac{5}{2} > 2 = \frac{|I_{bac}|}{2} = |v^b|$. Hence a is the unique solution of $\Gamma(\succ_I, V^0)$.

But, if a increases in popularity, so that we now have $\succ'_I$ with $I = I'_{abc} \cup I'_{bac} \cup I'_{cab}$ and $|I'_{abc}| = |I'_{bac}| = |I'_{cab}| = 3$, candidate c is not sure to lose against b so that the game cannot be reduced to an election between a and b. No other candidate is sure to lose either, so that by results in Núñez and Courtin [2013], we know that $\Gamma(\succ'_I, V)$ is not dominance solvable, i.e. has no unique solution in iteratively undominated strategies.[15]

In contrast to Approval Voting, the Borda Rule satisfies monotonicity:

Theorem 14. *The Borda Rule is the unique direct mechanism scoring rule that satisfies* **U**, **MEW** *and* **MON**.

Proof. In light of Theorem 12 and 13 as well as Example 3, it remains to show that the Borda Rule satisfies monotonicity. For that, assume that some candidate, say a, is the unique solution in $\Gamma(\succ_I, V^0)$. Then we know from Theorem 10 and 11 that, up to relabelling of candidates b and c, one of the three conditions are satisfied

$$\begin{array}{llll}
(1) & & |I_{abc}| > |I_{acb}| + |I_{bac}| + 2|I_{bca}| + 2|I_{cab}| + 2|I_{cba}|, & \\
(2) & \text{or} & |I_{abc}| > |I_{acb}| + |I_{bac}| + 2|I_{bca}| + 2|I_{cab}| + 2|I_{cba}| - 1 & \text{and} \quad |I_{cab}| > |I_{bac}|, \\
(3) & \text{or} & |I_{abc}| > |I_{acb}| + |I_{bac}| + 2|I_{bca}| + 2|I_{cab}| + 2|I_{cba}| - 2 & \text{and} \quad |I_{acb}| > 0.
\end{array}$$

Note that as we move to an a-monotonic transformation of $\succ_I$, this

- weakly increase $|I_{abc}|$,

[15] For an Approval Voting game to have a unique solution, there has to be an alternative that is ranked first more often than some other alternative is ranked first or second, see Núñez and Courtin [2013].

- weakly decrease $|I_{acb}| + |I_{bac}| + 2|I_{bca}| + 2|I_{cab}| + 2|I_{cba}|$,
- and weakly relaxes the inequality $|I_{acb}| > 0$.

Hence, if initially conditions (1) or condition (3) were satisfied, they continue to hold, so that a is still the unique solution. If initially only condition (2) was satisfied, the inequality $|I_{cab}| > |I_{bac}|$ could cease to hold when moving to an a-monotonic transformation of $\succ_I$

- as $|I_{bac}|$ increase (some i moves from I_{bac} to I_{abc}),
- or $|I_{cab}|$ shrinks (some i moves from I_{cab} to I_{acb}).

However, then in both cases (1) will be satisfied, as $|I_{acb}|+|I_{bac}|+2|I_{bca}|+2|I_{cab}|+2|I_{cba}|$ is decreased by one. In either case, a remains the unique solution. □

We conclude this section with two impossibility results.

Corollary 3. *No social choice correspondence that satisfies both* **MEPD** *and* **MON** *can be implemented by a direct mechanism scoring rule.*

Corollary 4. *No social choice correspondence that satisfies* **CON** *can be implemented by a direct mechanism scoring rule.*

The first impossibility result is an immediate implication of Theorem 13 and Example 3. The second impossibility follows from Theorem 13 and a result by Peress [2008] who shows that even when a strict Condorcet winner exists, Approval Voting allows for undominated Nash-equilibria where some other alternative is elected – such equilibrium strategies are never eliminated in the process of iterative elimination of dominated strategies.

4.6 Conluding Remarks

While the analysis of social choice correspondences that can be implemented in iteratively undominated strategies has occupied the minds of many social choice theorists, a complete characterization has remained elusive.

This chapter hopes to contribute to such a characterization by a change in perspective. Instead of considering all mechanisms, we begin by concentrating on a limited, yet comparatively large class of voting procedures that includes prominent and intuitive rules. For that class, we are able to characterize voting procedures using a small number of intuitive axioms that are based on simple majority and monotonicity. In particular, Approval Voting and the Borda Rule stand out as optimal voting procedures with respect to our axioms.

For a class of more general mechanisms, our results raise a number of questions. Is Approval Voting still the unique scoring rule that is majoritarian after eliminating a Pareto dominated alternative (**MEPD**), once we drop the direct mechanism restriction? Does there exist a scoring rule or a more general (bounded) mechanism that not only satisfies **MEPD** but is also monotonic? Such a new mechanism could then be seen an improvement over both Approval Voting and the Borda Rule in conducting elections involving three candidates. For elections involving more than three candidates, one may ask whether our axioms, **MEPD** and *Majoritarian after Eliminating a Worst alternative* (**MEW**), can be extended so as to yield analogous characterisations of Approval Voting and the Borda Rule.

We hope that questions such as these will stimulate future research.

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APPENDIX

A Appendix to Section 2.1

Proof of Theorem 1. The lower bound argument used in FMP and given in Section 2.1.4.1 yields $\overline{s}(\theta^*) \geq \overline{a}$. We now prove the converse: $\overline{s}(\theta^*) \leq \overline{a}$. By duality, $\underline{s}(\theta^*) \leq \underline{a}$ and $\underline{s}(\theta^*) \geq \underline{a}$ also hold.

Consider a simplified global game $G^*(v)$ with noise distribution f, and its right continuous, increasing equilibrium strategy profile $\overline{s}^*_v$. As $v \to 0$, $\overline{s}^*_v$ converges to the right continuous version $\overline{s}$ of the limit strategy profile of $G^*(v)$ at all points of continuity (Fact 1). For now, assume $\overline{s}$ is continuous at θ^*. Then there exist $\overline{v}$ and $\delta > 0$ such that for all $v < \overline{v}$ and $x \in [\theta^* - \delta, \theta^* + \delta]$ we have that $\overline{s}^*_v(x)$ equals the (highest) GGS $\overline{s}(\theta^*)$; an equilibrium a^* of the game $\mathbf{g}$. Now fix some $v < \min\{\delta, \overline{v}\}$ and consider $E_{\underline{\theta}}(v)$, the scaled and shifted version of elaboration E as defined at the end of Section 2.1.4.1. The lower dominance regions, scale factors, and noise distributions of $E_{\underline{\theta}}(v)$ and $G^*(v)$ coincide. Assume that in the game $E_{\underline{\theta}}(v)$ players use the strategy profile s' given by:

$$s'(x) = \begin{cases} \overline{s}^*_v(x) & \text{if } x \leq \theta^*, \\ \overline{s}^*_v(\theta^*) & \text{if } x > \theta^*. \end{cases}$$

For any player i, and any signal $x_i < 0$, action 0 is dominant both in $E_{\underline{\theta}}(v)$ and in $G^*(v)$, so for $x < 0$, the upper best reply in $E_{\underline{\theta}}(v)$ is $\beta(s')(x) = 0 = \overline{s}^*_v(x) = s'(x)$. For $x_i \in [\underline{\theta}, \theta^*]$, i's opponents receive signals smaller than $\theta^* + \delta$ and follow $\overline{s}^*_{v,-i}$ in $E_{\underline{\theta}}(v)$. Since the distributions of players' signals are identical in $G^*(v)$ and $E_{\underline{\theta}}(v)$, but i's payoff is given by $g_i(\cdot) = u_i(\cdot, \theta^*)$ in $E_{\underline{\theta}}(v)$ and by $u_i(\cdot, x_i)$ in $G^*(v)$, supermodularity implies that the upper best reply in $E_{\underline{\theta}}(v)$ is $\beta(s')(x) = \beta(\overline{s}^*_v)(x) \geq \overline{s}^*_v(x) = s'(x)$, for $\underline{\theta} \leq x \leq \theta^*$. For $x_i > \theta^*$, player i's opponents receive signals higher than $\theta^* - \delta$ and, following s'_{-i}, play a^*_{-i}. As a^* is a Nash equilibrium of $\mathbf{g}$, $\beta(s')(x) \geq a^* = s'(x)$.

In sum, in the elaboration $E_{\underline{\theta}}(v)$, $\beta(s') \geq s'$. Hence, an upper best reply iteration starting at s' yields a monotonically increasing sequence of strategy profiles that converges to an equilibrium profile $s^* \geq s'$. It follows that $s^*(\theta^*) \geq s'(\theta^*) = \overline{s}^*_v(\theta^*) = \overline{s}(\theta^*)$. As the attained action profile $\overline{a}$ is defined as the highest action profile attained in any equilibrium strategy profile in E, and each equilibrium profile in $E_{\underline{\theta}}(v)$ has a scaled and shifted counterpart in E, we have $\overline{a} \geq s^*(\theta^*) \geq \overline{s}(\theta^*)$.

If the limit strategy profile $\overline{s}$ is not continuous at θ^*, we may choose a decreasing sequence $\theta^0, \theta^1, \theta^2, \ldots$ converging to θ^* such that $\overline{s}$ is continuous at each θ^n. For each game $\mathbf{g}^n$ embedded at θ^n, consider the elaboration E^n, identical to E except that payoffs are given by $u_i(\cdot, \theta_n)$ if $x_i \geq 0$. Let s^n be the highest equilibrium profile of E^n and recall $s^n(|A|)$ is the highest action profile played in s^n. By the first part of the proof, $s^n(|A|) \geq \overline{s}(\theta^n)$. As s^n is an equilibrium of E^n, we have

$$\forall i \in I, \forall a_i \in A_i, \forall x_i \geq 0:$$

$$\int_{\mathbb{R}^{|I|-1}} \left(u_i(s^n_i(x_i), s^n_{-i}(x_{-i}), \theta^n) - u_i(a_i, s^n_{-i}(x_{-i}), \theta^n)\right) \pi_i(x_{-i}|x_i)\, dx_{-i} \geq 0,$$

where $\pi_i(\cdot|x_i)$ is the conditional density over opponents' signals. By state monotonicity (**A3**), the sequence of equilibria s^n converges to $s^* = \inf\{s^n|n \in \mathbb{N}\}$. As payoffs u_i are bounded on the compact set $A \times [\theta^*, \theta^0]$, the dominated convergence theorem ensures expected payoffs also converge:

$$\begin{aligned}
&\forall\, i \in I, a_i \in A_i, x_i \geq 0 : \\
&\qquad \int_{\mathbb{R}^{|I|-1}} \big(u_i(s_i^*(x_i), s_{-i}^*(x_{-i}), \theta^*) - u_i(a_i, s_{-i}^*(x_{-i}), \theta^*)\big)\, \pi_i(x_{-i}|x_i)\, dx_{-i} \\
&= \lim_{n\to\infty} \int_{\mathbb{R}^{|I|-1}} \big(u_i(s_i^n(x_i), s_{-i}^n(x_{-i}), \theta^n) - u_i(a_i, s_{-i}^n(x_{-i}), \theta^n)\big)\, \pi_i(x_{-i}|x_i)\, dx_{-i} \geq 0
\end{aligned}$$

So s^* is an equilibrium strategy profile in the elaboration E of the game $\mathbf{g}$. Finally, since $s^*(|A|) = \inf\{s^n(|A|)|n \in \mathbb{N}\} \geq \inf\{\overline{s}(\theta^n)|n \in \mathbb{N}\} = \overline{s}(\theta^*)$, we conclude that the attained profile $\overline{a} \geq \overline{s}(\theta^*)$. □

Proof of Theorem 2. Fix f. In this proof, for any game $\mathbf{g} \in S$, denote its associated upper- and lower-f-elaborations by $E(\mathbf{g})$ and $E^\partial(\mathbf{g})$ and their respective attained action profiles by $\underline{a}(\mathbf{g})$ and $\overline{a}(\mathbf{g})$. Also, for any game $\mathbf{g} \in S$ (with payoffs g_i) define a global game embedding as in Lemma 1 by setting $u_i^{\mathbf{g}}(a_i, a_{-i}, \theta) = g_i(a_i, a_{-i}) + \theta a_i$. Given this embedding, $\mathbf{g}_\theta$ denotes the complete information game embedded at θ. For $r > 0$, let $B_r(\mathbf{g})$ be the open ball in $\mathbb{R}^{|I \times A|}$ with radius r around $\mathbf{g}$.

To prove that S^f is dense in S we may show that if $\mathbf{g} \in S^{-f}$, there is a game arbitrarily close to $\mathbf{g}$ in which the GGS is unique. But this is always true, as the limit equilibrium strategy profile of $\mathbf{g}$'s embedding given by the payoffs $u_i^{\mathbf{g}}$ is unique up to its finitely many discontinuities (Theorem 0). To prove that S^f is open in S, note that if $\mathbf{g} \in S^f$, the limit equilibrium strategy profile of its embedding given by $u_i^{\mathbf{g}}$ is constant over some interval $(-2\varepsilon, 2\varepsilon)$, as the joint action set A is finite. Then, by the following result, S^f is open in S (and hence S^{-f} is closed and nowhere dense in S):

Claim 3. *If $\mathbf{g} \in S^f$ and, for some $\varepsilon > 0$ and $a^* \in A$, $a^* = \underline{a}(\mathbf{g}_\theta) = \overline{a}(\mathbf{g}_\theta)$ for all $\theta \in (-2\varepsilon, 2\varepsilon)$, then $a^* = \underline{a}(\mathbf{g}') = \overline{a}(\mathbf{g}')$ for all supermodular games $\mathbf{g}'$ in an ε-neighbourhood of $\mathbf{g}$.*

Proof. Let $\mathbf{g}'$ be a supermodular game in an ε-neighbourhood of $\mathbf{g}$. Then for all i, a_{-i}, and $a_i' < a_i$,

$$\begin{aligned}
u_i^{\mathbf{g}}(a_i, a_{-i}, -2\varepsilon) - u_i^{\mathbf{g}}(a_i', a_{-i}, -2\varepsilon) &= g_i(a_i, a_{-i}) - g_i(a_i', a_{-i}) - 2\varepsilon(a_i - a_i') \\
&\leq g_i(a_i, a_{-i}) - g_i(a_i', a_{-i}) - 2\varepsilon \leq g_i'(a_i, a_{-i}) - g_i'(a_i', a_{-i}),
\end{aligned}$$

where g_i' denotes the payoffs of $\mathbf{g}'$. Thus, for any opposing action distribution, the upper best reply in the elaboration $E(\mathbf{g}')$, is weakly higher than in $E(\mathbf{g}_{-2\varepsilon})$. But then the same is true for their highest equilibrium strategy profiles, so that $\overline{a}(\mathbf{g}') \geq \overline{a}(\mathbf{g}_{-2\varepsilon}) = a^*$. Using a symmetric argument, we establish that $\overline{a}(\mathbf{g}') \leq a^*$. Dually, we may show that $\underline{a}(\mathbf{g}') = a^*$, proving the claim. ◇

Now, we may establish measure theoretic genericity. A subset P of $\mathbb{R}^{|I \times A|}$ is called *porous* if there are $\lambda \in (0,1)$ and $k > 0$ such that for any $\mathbf{g} \in P$ and $\varepsilon \in (0,k)$, there exists $\mathbf{g}' \in \mathbb{R}^{|I\times A|}$ such that $B_{\lambda\varepsilon}(\mathbf{g}') \subseteq B_\varepsilon(\mathbf{g}) - P$. Any porous subset of $\mathbb{R}^{|I\times A|}$ is a Lebesgue null set (Lucchetti [2006], p. 220–222). Let:

$$S_k^{-f} := \{\mathbf{g} \in S^{-f} \mid \mathbf{g}_\theta \in S^f, \forall \theta \in (-k,0) \cup (0,k)\}.$$

We will prove that S_k^{-f} is porous. Assume $\mathbf{g} \in S_k^{-f}$ and choose $\varepsilon \in (0,k)$. Setting $\mathbf{g}' := \mathbf{g}_{\frac{\varepsilon}{2}}$, we know that the GGS will be unique and identical to that of $\mathbf{g}'$ for all games $\{\mathbf{g}'_\theta \in S \mid \theta \in (-\frac{\varepsilon}{2}, \frac{\varepsilon}{2})\}$. By Claim 3, we know that the GGS is unique for *all* supermodular games in an $\frac{\varepsilon}{4}$-neighbourhood of $\mathbf{g}'$, thus $B_{\frac{\varepsilon}{4}}(\mathbf{g}') \cap S_k^{-f} = \varnothing$. Setting $\lambda = \frac{1}{4}$, we have for $\varepsilon \in (0,k)$ that $B_{\lambda\varepsilon}(\mathbf{g}') \subseteq B_\varepsilon(\mathbf{g}) - S_k^{-f}$, i.e., S_k^{-f} is porous. Thus $S^{-f} = \bigcup_{\{k\in\mathbb{Q} \mid k>0\}} S_k^{-f}$ is a countable union of Lebesgue null sets and hence a null set itself. To see that, by contrast, S is of infinite Lebesgue measure, pick a game such that the inequalities in (2.1) hold strictly, and note it is contained in an open ball $B \subseteq \mathbb{R}^{|I\times A|}$ of supermodular games. Moreover, for each open ball B in S, we find another ball $B' \subseteq S$ of arbitrarily large measure, if we multiply the payoffs of all games in B with a sufficiently large constant. □

Proof of Theorem 3. For two different games write $\mathbf{g}^1 \prec \mathbf{g}^2$ if for all i, $a_i' < a_i$ and a_{-i} the corresponding payoffs satisfy $g_i^1(a_i,a_{-i}) - g_i^1(a_i',a_{-i}) < g_i^2(a_i,a_{-i}) - g_i^2(a_i',a_{-i})$. Given any opposing action distribution, the lowest best reply in $\mathbf{g}^1$ will then be weakly lower than the lowest best reply in $\mathbf{g}^2$.

Now, consider a generalised global game $\tilde{G}(v)$ with noise distribution f and payoff functions u_i. Write $\mathbf{g}_\theta$ for the complete information game with payoffs $u_i(\cdot,\theta)$. Assume that (i) $\overline{a}_{\theta^*} = \underline{a}_{\theta^*} = a^*$ and (ii) $\overline{a}_\theta$, $\underline{a}_\theta$ continuous at θ^*. Note that $\hat{s} \geq \check{s}$, so it suffices to show that $\check{s}(\theta^*) \geq a^* \geq \hat{s}(\theta^*)$. We will prove the first inequality; the second follows by duality. To do so, we will compare the payoff functions u_i satisfying (**A1**)–(**A3***) to payoff functions u_i' satisfying (**A1**)–(**A4**).

First, by continuity of $\overline{a}_\theta$ and $\underline{a}_\theta$ at θ^*, there is some nearby, game $\mathbf{g}_{\theta^*-\varepsilon}$ embedded at $\theta^* - \varepsilon$ whose GGS is unique and equal to a^*. Next, consider the game $\mathbf{g}'$ given by $g_i'(a_i,a_{-i}) = u_i(a_i,a_{-i},\theta^*-\varepsilon) - ka_i$, $k > 0$. By Claim 3, we can choose k such that the GGS in $\mathbf{g}'$ is unique and equal to a^*. Furthermore, w.l.o.g., assume that there exist extreme values $\check{\theta} < \underline{\theta}$ and $\overline{\theta} < \hat{\theta}$ such that we have a chain of four games satisfying $\mathbf{g}_{\check{\theta}} \prec \mathbf{g}' \prec \mathbf{g}_{\theta^*} \prec \mathbf{g}_{\hat{\theta}}$. Using this chain, we construct u_i':

$$u_i'(a_i,a_{-i},\theta) = \begin{cases} u_i(a_i,a_{-i},\check{\theta}) & \text{if } \theta < \theta^* - \varepsilon, \\ \frac{\theta^*-\theta}{\varepsilon} u_i(a_i,a_{-i},\check{\theta}) + \frac{\theta-(\theta^*-\varepsilon)}{\varepsilon} g_i'(a_i,a_{-i}) & \text{if } \theta^* - \varepsilon \leq \theta < \theta^*, \\ \frac{\hat{\theta}-\theta}{\hat{\theta}-\theta^*} g_i'(a_i,a_{-i}) + \frac{\theta-\theta^*}{\hat{\theta}-\theta^*} u_i(a_i,a_{-i},\theta^*) & \text{if } \theta^* < \theta < \hat{\theta}, \\ (\hat{\theta}+1-\theta) u_i(a_i,a_{-i},\theta^*) + (\theta-\hat{\theta}) u_i(a_i,a_{-i},\hat{\theta}) & \text{if } \hat{\theta} \leq \theta < \hat{\theta}+1, \\ u_i(a_i,a_{-i},\hat{\theta}) & \text{if } \hat{\theta}+1 \leq \theta, \end{cases}$$

Comparing the payoffs u_i' with u_i, we see that under u_i' the dominance regions have been shifted to the right, the game $\mathbf{g}_{\check{\theta}}$ is now embedded at $\theta^* - \varepsilon$, $\mathbf{g}'$ at θ^*, $\mathbf{g}_{\theta^*}$ at $\hat{\theta}$, $\mathbf{g}_{\hat{\theta}}$ at $\hat{\theta}+1$ and the remaining games are linear interpolations. Thus, for any θ, the lowest best reply under u_i is weakly higher than under u'_i. Also, since payoffs are linearly interpolated between $\mathbf{g}_{\check{\theta}} \prec \mathbf{g}' \prec \mathbf{g}_{\theta^*} \prec \mathbf{g}_{\hat{\theta}}$, payoff differences are piecewise linear in θ, thus satisfy **(A3)**. Clearly, **(A4)** is satisfied as well.

Finally, consider the global game $G'(v)$ with the newly constructed payoff function u'_i, and the same noise distribution f and prior as $\tilde{G}(v)$. For any $v > 0$, $G'(v)$ has a lowest equilibrium strategy profile, denoted s'_v. As best replies are higher under u_i than under u'_i, for the lowest equilibrium strategy profile in $\tilde{G}(v)$ we find $\check{s}_v \geq s'_v$. Thus, $\check{s}(\theta^*) = \liminf_{v\to 0} \check{s}_v(\theta^*) \geq \lim_{v\to 0} s'_v(\theta^*) = a^*$, where the last equality follows by Theorem 1 and the fact that a^* is the unique GGS of $\mathbf{g}'$. □

Proof of Lemma 2. Let us first prove two claims about restricted games of $\mathbf{g}$:

Claim 4. *Consider four action profiles a, b, c, d such that $a \leq b \leq d$ and $a \leq c \leq d$. Then the highest GGS in $\mathbf{g}|[a, b]$ is weakly lower than the highest GGS in $\mathbf{g}|[c, d]$.*

Proof. Consider the highest GGS $\overline{a}$ in $\mathbf{g}|[a, b]$. By Theorem 1 there exists an equilibrium strategy profile s in $E|[a, b]$ prescribing the action profile $\overline{a}$ for high signals. Define a strategy profile s' pointwise as $\max\{c, s(x)\}$ for each signal tuple x. Due to supermodularity, an upper best reply iteration in $E|[c, d]$ starting from s' will be increasing. Thus, there exists an equilibrium strategy profile in $E|[c, d]$ that prescribes actions weakly higher than $\overline{a}$ for high enough signals. ◇

Claim 5. *Consider three action profiles $a \leq b \leq c$. If, for fixed f, a is the unique GGS in $\mathbf{g}|[a, b]$ and b is the unique GGS in $\mathbf{g}|[b, c]$, then a is the unique GGS in $\mathbf{g}|[a, c]$.*

Proof. Consider the highest GGS $\overline{a}$ in $\mathbf{g}|[a, c]$. Since b is the highest GGS in $\mathbf{g}|[b, c]$, Claim 4 implies $\overline{a} \leq b$. In addition, $\overline{a}$ is the highest GGS in $\mathbf{g}|[a, \overline{a}]$, as the highest equilibrium strategy profile in $E|[a, c]$ (which attains $\overline{a}$) is also an equilibrium profile in $E|[a, \overline{a}]$. Since $a \leq \overline{a} \leq b$, Claim 4 applied to the games $\mathbf{g}|[a, \overline{a}]$ and $\mathbf{g}|[a, b]$ yields $\overline{a} \leq a$, proving Claim 5. ◇

Now, applying Claim 5 iteratively to the sequence $a^n \leq a^{n+1} \cdots \leq a^m$, we see that a^n is the unique GGS in $\mathbf{g}|[a^n, m]$. Hence, by Claim 4, the highest GGS in $\mathbf{g} = \mathbf{g}|[0, m]$, $\overline{a}$, is weakly lower than a^n. By a dual argument, a^n is the unique GGS in $\mathbf{g}|[0, a^n]$, and the lowest GGS in $\mathbf{g} = \mathbf{g}|[0, m]$, $\underline{a}$, is weakly higher than a^n. Together, this yields $\underline{a} = \overline{a} = a^n$. □

Proof of inequality (2.5). Fix a noise distribution f; let F_i be the c.d.f. of player i's density f_i. Then

$$\sum_{i\in I} P_i = \sum_{i\in I} \prod_{j\in I-\{i\}} \mathbb{P}(x_j \geq z_j | x_i = z_i) = \sum_{i\in I} \int_{-|A|-1}^{|A|+1} f_i(z_i - \theta) \prod_{j\in I-\{i\}} (1 - F_j(z_j - \theta))\, d\theta.$$

Picking some player $t \in I$ and integrating by parts the summand corresponding to

$i = t$ gives

$$\begin{aligned}
&\left[(1 - F_1(z_1 - \theta)) \prod_{j \in I - \{t\}} (1 - F_j(z_j - \theta))\right]_{-|A|-1}^{|A|+1} \\
&- \int_{-|A|-1}^{|A|+1} \sum_{i \in I - \{t\}} \left(f_i(z_i - \theta) \prod_{j \in I - \{i\}} (1 - F_j(z_j - \theta))\right) d\theta \\
&+ \sum_{i \in I - \{t\}} \int_{-|A|-1}^{|A|+1} f_i(z_i - \theta) \prod_{j \in I - \{i\}} (1 - F_j(z_j - \theta))\, d\theta \\
&= \left[\prod_{j \in I} (1 - F_j(z_j - \theta))\right]_{-|A|-1}^{|A|+1} = 1.
\end{aligned}$$

□

B Appendix to Section 2.2

Proof of Lemma 3. We may assume without loss of generality that $\mathbf{g}$ is strictly supermodular.[16] Define $M(w_2)$ to be the number w_0 that solves the equation

$$w_0 g(1,0)+(1-w_0-w_2)g(1,1)+w_2 g(1,2)=w_0 g(2,0)+(1-w_0-w_2)g(2,1)+w_2 g(2,2).$$

Even though $M(w_2)$ is not necessarily in the interval $[0,1]$, we can think of it intuitively as the weight that may be put on the least action, 0, while still leaving the opposing player indifferent between playing the middle action, 1, and the greatest action, 2, when the weight on 2 is w_2. Existence and uniqueness of the solution $M(w_2)$ are guaranteed by strict supermodularity. The function M has derivative

$$\rho_M := \frac{\Delta_1^2(2) - \Delta_1^2(1)}{\Delta_1^2(1) - \Delta_1^2(0)} > 0,$$

thus is linear and (due to supermodularity) increasing. Analogously, define $N(w_0)$ to be the minimal weight that needs to be put on 2 to make the opposing player indifferent between playing 0 and 1 when the weight on 0 is w_0. That is, $N(w_0)$ is the solution w_2 that solves

$$w_0 g(0,0)+(1-w_0-w_2)g(0,1)+w_2 g(0,2)=w_0 g(1,0)+(1-w_0-w_2)g(1,1)+w_2 g(1,2).$$

The function N has derivative

$$\rho_N := \frac{\Delta_0^1(1) - \Delta_0^1(0)}{\Delta_0^1(2) - \Delta_0^1(1)} > 0.$$

We will show that, under the hypothesis of the lemma, there exists an increasing equilibrium strategy profile s in $\mathbf{e}$ such that $s(R) = 2$ for $R \geq 6$. In this case, 2 must be the global game selection. If $2 \in br(\mu_0^2)$, it is easy to verify the existence of such strategy profile s. Simply set $\underline{z}_1 = \underline{z}_2 = \overline{z}_1 = \overline{z}_2 = 0$. For the remainder of the proof, consider $2 \notin br(\mu_0^2)$. By supermodularity, $br((\frac{1}{2}, p, \frac{1}{2} - p)) \subseteq \{0,1\}$, for all $p \in [0, \frac{1}{2}]$. Thus, if (C2) holds for some $p^* \in [0, \frac{1}{2}]$, then $1 \in br((\frac{1}{2}, p^*, \frac{1}{2} - p^*))$, and, in our new notation, $N(\frac{1}{2}) \leq \frac{1}{2} - p^*$. Note that supermodularity implies $1 \in br(\mu_0^2)$. Also, as $2 \in br((\frac{1}{2} - p^*, p^*, \frac{1}{2}))$, it follows that $M(\frac{1}{2}) \geq \frac{1}{2} - p^*$ and in particular $N(\frac{1}{2}) \leq M(\frac{1}{2})$.

[16] If $\mathbf{g}$ is only weakly supermodular, we may embed it in a global game $G(v)$ where the payoff structure is symmetric and strictly supermodular almost everywhere, for example by letting payoffs depend on θ as follows:

$$u_i(a_i, a_{-i}, \theta) := g(a_i, a_{-i}) + \theta a_i (3 + \mathrm{sgn}(\theta) a_{-i}).$$

One may verify that $g(a_i, a_{-i}) = u(a_i, a_{-i}, \theta)$ for $\theta = 0$ and that u satisfies the requirements of a global game as in FMP. If condition (C2) is satisfied for $\theta = 0$, it also holds at all $\theta > 0$ by monotonicity of the payoff difference functions. By results in Basteck et al. [2010], the global game selection at $\theta = 0$ does not depend on the embedding chosen. Since the greatest global game selection is continuous from the right, and equal to 2 at almost all $\theta > 0$ by our proof below, 2 is the greatest global game selection at $\theta = 0$.

As θ is distributed uniformly over a large interval $U = [\underline{U}, \overline{U}]$, the distribution over signal differences $x_1 - x_2$ conditional on the signal x_i received is the same for all $x_i \in [\underline{U} + \frac{1}{2}, \overline{U} - \frac{1}{2}]$. Let H be the cumulative distribution function of this signal difference and, without loss of generality, assume $H(0) = \frac{1}{2}$. We may deduce the following weights from H, which are straightforward to verify. If player 2 receives the signal $x_2 = \overline{z}_2$, she assigns weight

$$w_2(\overline{z}_2|\overline{z}_1) := \mathbb{P}(x_1 \geq \overline{z}_1|x_2 = \overline{z}_2) = \mathbb{P}(x_1 - x_2 \geq \overline{z}_1 - \overline{z}_2|x_2 = \overline{z}_2) = 1 - H(\overline{z}_1 - \overline{z}_2)$$

to player 1 playing 2. Player 1 at $x_1 = \overline{z}_1$ assigns weight $w_2(\overline{z}_1|\overline{z}_2) := H(\overline{z}_1 - \overline{z}_2) = 1 - w_2(\overline{z}_2|\overline{z}_1)$ to player 2 playing 2. In a similar vein, at $\underline{z}_2$, player 2 assigns weight $w_0(\underline{z}_2|\underline{z}_1) := H(\underline{z}_1 - \underline{z}_2)$ to player 1 playing 0. At $\underline{z}_1$, player 1 assigns weight $w_0(\underline{z}_1|\underline{z}_2) := 1 - H(\underline{z}_1 - \underline{z}_2)$ to player 2 playing 0. Also, we will make use of the fact that $w_0(\overline{z}_2|\underline{z}_1) := H(\underline{z}_1 - \overline{z}_2) =: w_2(\underline{z}_1|\overline{z}_2)$ and similarly $w_0(\overline{z}_1|\underline{z}_2) := 1 - H(\overline{z}_1 - \underline{z}_2) =: w_2(\underline{z}_2|\overline{z}_1)$.

Now consider the set $Z \subseteq \mathbb{R}^4$ of all increasing strategy profiles satisfying: (i) at $\underline{z}_2$, 1 or 2 is a best reply for player 2; and (ii) at $\overline{z}_2$, 2 is a best reply for player 2; and (iii) $\overline{z}_1$ player 1 weakly prefers to play 2 over 1 (we make no assumptions about the expected payoff from playing 0); and (iv) $\underline{z}_1 = 1$. Expected payoff in $\mathbf{e}$ is continuous, so the inequalities implied by (i)-(iii) entail Z is a closed set. Note that due to supermodularity, if s satisfies (i)-(iii), decreasing $\underline{z}_2$ preserves (ii) and (iii); decreasing $\overline{z}_2$ preserves (i) and (iii), and decreasing $\overline{z}_1$ preserves (i) and (ii).

We claim the set Z is nonempty. To see this, set $\overline{z}_1 = \overline{z}_2 = 3$ and choose $\underline{z}_2 \in [2, 3]$ such that $w_2(\underline{z}_2|\overline{z}_1) = w_0(\overline{z}_1|\underline{z}_2) = \frac{1}{2} - p^*$. Player 2 at $\underline{z}_2$ faces $\mu = (0, \frac{1}{2} + p^*, \frac{1}{2} - p^*)$ so that supermodularity and (C2) ensure (i). At $\overline{z}_2$, she faces μ_1^2 and by (C2) and supermodularity 2 is best reply, so (ii) holds. Finally, player 1 at $\overline{z}_1$ faces $\mu = (\frac{1}{2} - p^*, p^*, \frac{1}{2})$ so that (iii) is satisfied by (C2).

Since $Z \subseteq \mathbb{R}^4$ is non-empty, closed, and bounded from below, Z has a minimal element $\hat{s}$; that is, there is no other profile $s \in Z$ in which all of the thresholds are weakly smaller than in $\hat{s}$. We will prove that player 1 weakly prefers 1 over 0 at the threshold $\underline{z}_1 = 1$ in $\hat{s}$. Since $\hat{s}$ also satisfies (i)-(iii), standard results for supermodular games imply that a greatest best reply iteration starting from $\hat{s}$ must converge to an equilibrium in which action 2 is played, and the proof is done.

We begin by examining the preferences at the other three thresholds. First, consider the case $\underline{z}_1 = 1 = \overline{z}_1$. By the minimality of $\hat{s}$, player 2 must be indifferent between 0 and 1 at $\underline{z}_2 \leq 1$, as 1 is a best reply to the opposing mixed strategy profile μ_0^2 at signal $x_2 = 1$. Similarly, she is indifferent between 1 and 2 at $\overline{z}_2 \geq 1$. But then

$$N(w_0(\underline{z}_1|\underline{z}_2)) \leq N(\tfrac{1}{2}) \leq M(\tfrac{1}{2}) \leq M(w_2(\overline{z}_2|\overline{z}_1)) = w_0(\overline{z}_2|\underline{z}_1) = w_2(\underline{z}_1|\overline{z}_2).$$

where $M(w_2(\overline{z}_2|\overline{z}_1)) = w_0(\overline{z}_2|\underline{z}_1)$ expresses the indifference at $\overline{z}_2$. But, by the definition of N, this simply says that player 1 weakly prefers 1 over 0 at $\underline{z}_1$, which is what we needed to show.

Next, consider the case $\underline{z}_1 < \overline{z}_1$ and $\underline{z}_2 = \overline{z}_2 =: z_2$ so that by the minimality of $\hat{s}$, both 0 and 2 are best replies for player 2 at z_2. The minimality of $\hat{s}$ also implies that player 1 is indifferent between 1 and 2 at $\overline{z}_1$. As $br(\mu_0^2) \subseteq \{0,1\}$, this implies $w_2(\overline{z}_1|z_2) > \frac{1}{2}$ and thus $z_2 < \overline{z}_1$. Similarly, we have that $\underline{z}_1 < z_2$, as otherwise at z_2 player 2 would face an opposing action distribution that is dominated by μ_0^2, so that her best reply would be strictly smaller than 2. Then we arrive at the following contradiction:

$$w_2(z_2|\overline{z}_1) \leq N(w_0(z_2|\underline{z}_1)) < N(\tfrac{1}{2}) \leq M(\tfrac{1}{2}) < M(w_2(\overline{z}_1|z_2)) = w_0(\overline{z}_1|z_2) = w_2(z_2|\overline{z}_1).$$

Finally, consider $\underline{z}_1 < \overline{z}_1$ and $\underline{z}_2 < \overline{z}_2$ and note that the minimality of $\hat{s}$ implies that each player i is indifferent between 1 and 2 at $\overline{z}_i$ and player 2 is indifferent between 0 and 1 at $\underline{z}_2$. Thus, by definition, $M(w_2(\overline{z}_1|\overline{z}_2)) = w_0(\overline{z}_1|\underline{z}_2)$ and $N(w_0(\underline{z}_2|\underline{z}_1)) = w_2(\underline{z}_2|\overline{z}_1)$. In addition, it is always the case that $w_0(\overline{z}_1|\underline{z}_2) = w_2(\underline{z}_2|\overline{z}_1)$, so we conclude that $M(w_2(\overline{z}_1|\overline{z}_2)) = N(w_0(\underline{z}_2|\underline{z}_1))$. But then, as

$$\begin{aligned} M(w_2(\overline{z}_1|\overline{z}_2)) &= M(\tfrac{1}{2}) + (w_2(\overline{z}_1|\overline{z}_2) - \tfrac{1}{2})\rho_M = M(\tfrac{1}{2}) + \tfrac{1}{2}(w_2(\overline{z}_1|\overline{z}_2) - w_2(\overline{z}_2|\overline{z}_1))\rho_M \\ &= N(\tfrac{1}{2}) + \tfrac{1}{2}(w_0(\underline{z}_2|\underline{z}_1) - w_0(\underline{z}_1|\underline{z}_2))\rho_N = N(w_0(\underline{z}_2|\underline{z}_1)), \end{aligned}$$

and $M(\frac{1}{2}) \geq N(\frac{1}{2})$, this implies

$$(w_2(\overline{z}_1|\overline{z}_2) - w_2(\overline{z}_2|\overline{z}_1))\rho_M \leq (w_0(\underline{z}_2|\underline{z}_1) - w_0(\underline{z}_1|\underline{z}_2))\rho_N.$$

Returning to the situation of player 1 at $\underline{z}_1$, we can now say that

$$\begin{aligned} N(w_0(\underline{z}_1|\underline{z}_2)) &= N(\tfrac{1}{2}) - \tfrac{1}{2}(1 - 2w_0(\underline{z}_1|\underline{z}_2))\rho_N \\ &= N(\tfrac{1}{2}) - \tfrac{1}{2}(w_0(\underline{z}_2|\underline{z}_1) - w_0(\underline{z}_1|\underline{z}_2))\rho_N \\ &\leq M(\tfrac{1}{2}) - \tfrac{1}{2}(w_2(\overline{z}_1|\overline{z}_2) - w_2(\overline{z}_2|\overline{z}_1))\rho_M = M(w_2(\overline{z}_2|\overline{z}_1)). \end{aligned}$$

In this step, symmetry of $\mathbf{g}$ is essential, as otherwise we would have to differentiate between the individual players' ρ_M and ρ_N and hence not be able to appeal to the preceding inequality. Now, since we know that player 2 is indifferent between 1 and 2 at $\overline{z}_2$ and that $w_0(\overline{z}_2|\underline{z}_1) = w_2(\underline{z}_1|\overline{z}_2)$, we conclude that

$$N(w_0(\underline{z}_1|\underline{z}_2)) \leq M(w_2(\overline{z}_2|\overline{z}_1)) = w_0(\overline{z}_2|\underline{z}_1) = w_2(\underline{z}_1|\overline{z}_2).$$

But, by the definition of N, this simply says that player 1 weakly prefers 1 over 0 at $\underline{z}_1$, which is what we needed to show. □

C Appendix to Chapter 3

Example 4. The following example shows that a random assignment in the strong sd-core from equal division that satisfies the sd-equal-division-lower-bound may nevertheless violate equal treatment of equals. Suppose individuals $i \in \{1,2,3\}$ hold preferences $a \succ_i b \succ_i c \succ_i d$ while individual 4 prefers object d. Then in the following random assignment p

	a	b	c	d
p_1:	$1/4$	$1/2$	$1/4$	0
p_2:	$1/4$	$1/2$	$1/4$	0
p_3:	$1/2$	0	$1/2$	0
p_4:	0	0	0	1

each individual receives an assignment p_i that stochastically dominates equal division, *viz.* p satisfies the sd-equal-division-lower-bound.

To see that p lies in the strong sd-core from equal division, observe first that it is sd-efficient - there exists no Pareto improving trade involving 4 (who already receives d with certainty) and no Pareto improving trade among 1,2 and 3 (who hold identical preferences). Thus, the grand coalition will not object to p. Next, consider objections by coalitions of size $k < 4$. Individual 4 cannot be part of such a coalition as it could guarantee only a probability share $p_{4,d} = k/4 < 1$. Nor could the remaining individuals form a blocking coalition where everyone is (weakly) better of in a stochastic dominance sense, as someone would have to accept $p_{i,d} > 0$. Hence, p lies in the strong sd-core.

However, p does not satisfy equal treatment of equals, as $p^1 = p^2 \neq p^3$.

Example 5. The following example shows that the weak sd-core may include allocations that are not member of any weak w-core. Consider the case $n = 4$ and suppose that preferences of individual $i \in \{1,2\}$ are given as $a \succ_i b \succ_i c \succ_i d$. The third individuals preferences are given as $b \succ_3 a \succ_3 c \succ_3 d$ while the fourth individuals holds preferences $a \succ_4 c \succ_4 b \succ_4 d$. Then the following random assignment p

	a	b	c	d
p_1:	$1/4$	$1/4$	$1/4$	$1/4$
p_2:	$1/4$	$1/4$	$1/4$	$1/4$
p_3:	0	$1/2$	$1/4$	$1/4$
p_4:	$1/2$	0	$1/4$	$1/4$

lies in the weak sd-core from equal division: Consider an objection $\tilde{p}$ by a blocking coalition G that includes 4. Since everyone agrees that d is the worst, we have

$\tilde{p}_{i,d} = \frac{1}{4} = p_{i,d}$ for all $i \in G$. Moreover, since for $1, 2 \in G$ they would receive at least $\frac{1}{4}$ of object a under $\tilde{p}$, we have $\tilde{p}_{4,a} = \frac{1}{4}$. But then $\tilde{p}_4 \succ_4^{sd} p_4$ is impossible, as 4 cannot receive less than 0 of b.

Next, consider an objection $\tilde{p}$ by a blocking coalition excluding 4. Since individuals 1, 2 and 3 agree that c is the third- and d is the forth-most preferred object, no one individual may receive more than $\frac{1}{4}$ of d and more than $\frac{1}{2}$ of c and d. Thus we have $\tilde{p}_{i,c} = \tilde{p}_{i,c} = \frac{1}{4} = p_{i,c} = p_{i,d}$ for all $i \in G$. But then $\tilde{p}_3 \succ_3^{sd} p_3$ is impossible, as 3 receives his most preferred object b with maximal probability. Hence, finally, a blocking coalition G may only include individuals 1 and 2 – but since they have identical preferences, there is no scope for a valid objection that improves upon equal division.

However, for any compatible profile of vNM-utility functions the following allocation constitutes a valid objection by individuals 1,2, and 3 - provided ε is chosen sufficiently small:

	a	b	c	d
p_1:	$3/8$	$1/8 - \varepsilon$	$1/4 + \varepsilon$	$1/4$
p_2:	$3/8$	$1/8 - \varepsilon$	$1/4 + \varepsilon$	$1/4$
p_3:	0	$1/2 + 2\varepsilon$	$1/4 - 2\varepsilon$	$1/4$

D Appendix to Chapter 4

Proof of Theorem 9. Consider a ballot $b = (k, l, m) \in V$ and assume w.l.o.g. that $k \geq l \geq m$. Since V is assumed to be neutral, it also includes all permutations of b.

If $k > l > m$, the 6 permutations exhaust V; normalizing all ballots in V by replacing k by $k' = \frac{k-m}{k-m} = 1$, l by $l' = \frac{l-m}{k-m} \in [0,1]$ and m by $m' = \frac{m-m}{k-m} = 0$ yields case (1).

If two entries of b coincide, V contains 3 permutations of b. If those are the only elements of V, we can normalize ballots such that $k' = 1$ and $m' = 0$ which again yields case (1). If V contains another non-abstention ballot $b' = (p, q, r)$, then two of its three entries must coincide - if all were distinct, V would contain not only all permutations of b but also of b', violating $|V| \leq 6$.

W.l.o.g assume $p \geq q \geq r$. If $k = l > m$ and $p > q = r$, normalizing each permutation of b by subtracting m and each permutation of b' by subtracting r before dividing each ballot by $k - m + p - r$ yields $k' = l' = \frac{k-m}{k-m+p-r}$, $m' = 0$, $p' = \frac{p-r}{k-m+p-r}$ and $q' = r' = 0$, which corresponds to case (2).

If $k > l = m$ and $p > q = r$, assume w.l.o.g. that $k - m \geq p - r$. Normalizing each permutation of b by subtracting m and each permutation of b' by subtracting r before dividing each ballot by $k - m$ yields $k' = 1$, $l' = m' = 0$, $p' = \frac{p-r}{k-m} \leq 1$ and $q' = r' = 0$, which corresponds to case (3).

If $k = l > m$ and $p = q > r$, assume w.l.o.g. that $k - m \geq p - r$. Normalizing each permutation of b by subtracting m and each permutation of b' by subtracting r before dividing each ballot by $k - m$ yields $k' = l' = 1$, $m' = 0$, $p' = q' = \frac{p-r}{k-m} \leq 1$ and $r' = 0$, which corresponds to case (4). $\square$

Proof of Theorem 10. In light of the arguments presented in Section 4.5, the two remaining cases are (2) and (3). Assume condition (2) holds, so that

$$\min_{v \in V^1} |v^x| - |v^z| = 1/2|I_{xyz}| - 1/2|I_{xzy}| - 1/2|I_{yxz}| - |I_{yzx}| - |I_{zxy}| - |I_{zyx}| \geq 0.$$

Then, for any i_{xyz} in $\Gamma(\succ_I, V^1)$, ballot $v_i = (v_i^x, v_i^y, v_i^z) = (1, \frac{1}{2}, 0)$ is a weakly better reply than $\tilde{v}_i = (\frac{1}{2}, 1, 0)$ against any $v_{-i} \in V_{-i}^1$:

(i) if for $\tilde{v} = (\tilde{v}_i, v_{-i})$, $|\tilde{v}^y| > |\tilde{v}^x| \geq |\tilde{v}^z|$, then for $v = (v_i, v_{-i})$, we have $|v^y| \geq |v^x| > |v^z|$,

(ii) if for $\tilde{v} = (\tilde{v}_i, v_{-i})$, $|\tilde{v}^x| \geq |\tilde{v}^y|, |\tilde{v}^z|$, then for $v = (v_i, v_{-i})$, we have $|v^x| > |v^y|, |v^z|$.

Hence, $\tilde{v}_i$ is either dominated by $v_i = (1, \frac{1}{2}, 0)$, or it is a duplicate strategy. Eliminating $\tilde{v}_i$ and moving to the restricted game, $\Gamma(\succ_I, V')$, where $V'_{xyz} = V^1_{xyz} \setminus \{(\frac{1}{2}, 1, 0)\} = \{(1, 0, \frac{1}{2}), (1, \frac{1}{2}, 0)\}$ and $V'_j = V^1_j$ for all $j \notin I_{xyz}$ we find that

$$\begin{aligned}\min_{v \in V'} |v^x| - |v^y| &= 1/2|I_{xyz}| + 1/2|I_{xzy}| - |I_{yxz}| - |I_{yzx}| - 1/2|I_{zxy}| - |I_{zyx}| \\ &= \underbrace{1/2|I_{xyz}| - 1/2|I_{xzy}| - 1/2|I_{yxz}| - |I_{yzx}| - |I_{zxy}| - |I_{zyx}|}_{\geq 0} \underbrace{-1/2|I_{yxz}| + 1/2|I_{zxy}|}_{>0} + |I_{xzy}| > 0,\end{aligned}$$

which rules out y as an outcome of $\Gamma(\succ_I, V')$. But then, in the game $\Gamma(\succ_I, V')$, for any voter i who prefers x over z, $v_i = (1, 1/2, 0)$ is a best reply as it maximizes i's impact on $|v^x| - |v^z|$. Eliminating dominated or duplicate strategies and moving to $\Gamma(\succ_I, V'')$, where $V''_i = \{(1, 1/2, 0)\}$ for all $i \in I_{xyz} \cup I_{xzy} \cup I_{yxz}$ and $V''_j = V'_j = V^1_j$ for all $j \notin I_{xyz} \cup I_{xzy} \cup I_{yxz}$, we find that for all $v \in V''$

$$\begin{aligned}|v^x| \geq |I_{xyz}| + |I_{xzy}| + |I_{yxz}| &> 2|I_{xzy}| + 2|I_{yxz}| + 2|I_{yzx}| + 2|I_{zxy}| + 2|I_{zyx}| - 1 \\ &\geq |I_{yzx}| + |I_{zxy}| + |I_{zyx}| \geq |v^z|,\end{aligned}$$

where the strict inequality follows from directly from condition (2), while the next weak inequality follows from the fact that $|I_{zxy}| > 0$ by condition (2). Hence, x is the unique outcome after iteratively eliminating dominated strategies from $\Gamma(\succ_I, V^0)$.

Finally, assume condition (3) holds, so that

$$\min_{v \in V^1} |v^x| - |v^z| = 1/2|I_{xyz}| - 1/2|I_{xzy}| - 1/2|I_{yxz}| - |I_{yzx}| - |I_{zxy}| - |I_{zyx}| \geq -1/2.$$

Then, for any i_{xzy} in $\Gamma(\succ_I, V^1)$, ballot $v_i = (v_i^x, v_i^y, v_i^z) = (1, 0, 1/2)$ is a weakly better reply than $\tilde{v}_i = (1/2, 0, 1)$ against any $v_{-i} \in V^1_{-i}$:

(i) if for $\tilde{v} = (\tilde{v}_i, v_{-i})$, $|\tilde{v}^x| \geq |\tilde{v}^y|$, then for $v = (v_i, v_{-i})$, we have $|v^x| > |v^y|, |v^z|$,

(ii) if for $\tilde{v} = (\tilde{v}_i, v_{-i})$, $|\tilde{v}^y| > |\tilde{v}^x|, |\tilde{v}^z|$, so that $v = (v_i, v_{-i})$ can only yield a weakly better outcome for i_{xzy},

(iii) if for $\tilde{v} = (\tilde{v}_i, v_{-i})$, $|\tilde{v}^z| \geq |\tilde{v}^y| > |\tilde{v}^x|$, then $|\tilde{v}^z| = |\tilde{v}^x| + \frac{1}{2}$ and$|\tilde{v}^z| = |\tilde{v}^y|$. But then $2(|\tilde{v}^x| + |\tilde{v}^y| + |\tilde{v}^z|) = 2(3|\tilde{v}^x| + 1)$. However, as each voter awards score that sum to $\frac{3}{2}$, $2(|\tilde{v}^x| + |\tilde{v}^y| + |\tilde{v}^z|)$ would have to be divisible by three - a contradiction.

Hence, $\tilde{v}_i$ is either dominated by $v_i = (1, 0, 1/2)$, or it is duplicate. Eliminating $\tilde{v}_i$ and moving to the restricted game, $\Gamma(\succ_I, V')$, where $V'_{xzy} = V^1_{xzy} \backslash \{(\frac{1}{2}, 0, 1)\} = \{(1, 0, \frac{1}{2}), (1, \frac{1}{2}, 0)\}$ and $V'_j = V^1_j$ for all $j \notin I_{xyz}$, condition (3) yields

$$\begin{aligned}\min_{v \in V'} |v^x| - |v^z| &= 1/2|I_{xyz}| + 1/2|I_{xzy}| - 1/2|I_{yxz}| - |I_{yzx}| - |I_{zxy}| - |I_{zyx}| \\ &= \underbrace{1/2|I_{xyz}| - 1/2|I_{xzy}| - 1/2|I_{yxz}| - |I_{yzx}| - |I_{zxy}| - |I_{zyx}| + 1}_{>0} \underbrace{-1 + |I_{xzy}|}_{\geq 0} > 0,\end{aligned}$$

which rules out z as an outcome of $\Gamma(\succ_I, V')$. But then, in the game $\Gamma(\succ_I, V')$, for any voter i who prefers x over y, $v_i = (1, 0, 1/2)$ is a best reply as it maximizes i's impact on $|v^x| - |v^y|$. Eliminating dominated or duplicate strategies and moving to $\Gamma(\succ_I, V'')$, where $V''_i = \{(1, 0, 1/2)\}$ for all $i \in I_{xyz} \cup I_{xzy} \cup I_{zxy}$ and $V''_i = V'_i = V^1_i$ for all $i \notin I_{xyz} \cup I_{xzy} \cup I_{zxy}$, we find that for all $v \in V''$

$$\begin{aligned}|v^x| \geq |I_{xyz}| + |I_{xzy}| + |I_{zxy}| &> \underbrace{2|I_{xzy}|}_{\geq 2} + |I_{yxz}| + 2|I_{yzx}| + 3|I_{zxy}| + 2|I_{zyx}| - 2 \\ &\geq |I_{yxz}| + |I_{yzx}| + |I_{zyx}| \geq |v^z|\end{aligned}$$

by condition (3). Hence, x is the unique outcome after iteratively eliminating dominated strategies from $\Gamma(\succ_I, V^0)$. ☐

Proof of Lemma 5. Suppose that for any labelling of candidates, we have

$$(\star) \qquad |I_{xyz}| < |I_{xzy}| + |I_{yxz}| + 2|I_{yzx}| + 2|I_{zxy}| + 2|I_{zyx}| - 2.$$

Now consider a voter of type i_{abc}. We will show that after one round of elimination, no strategy $v_i = (v_i^a, v_i^b, v_i^c)$ in $V_i^1 = \{(1, \frac{1}{2}, 0), (1, 0, \frac{1}{2}), (\frac{1}{2}, 1, 0)\}$ is dominated in the game $\Gamma(\succ_I, V^1)$ and that each outcome $a, b, c \in A$ is possible.

Claim 1. *Neither* $(1, 0, \frac{1}{2})$ *nor* $(1, \frac{1}{2}, 0)$ *is dominated by* $(\frac{1}{2}, 1, 0)$*. Moreover,* $(1, 0, \frac{1}{2})$ *is not dominated by* $(1, \frac{1}{2}, 0)$ *and both* a *and* b *are possible outcomes.*

Proof. We will proof the claim by constructing an opposing strategy profile for which (i) $v_i = (1, 0, \frac{1}{2})$ and $v_i = (1, \frac{1}{2}, 0)$ yield outcome a while $v_i = (\frac{1}{2}, 1, 0)$ yields b and (ii) another opposing profile for which $v_i = (1, 0, \frac{1}{2})$ yields a while $v_i = (1, \frac{1}{2}, 0)$ yields b. To find such profiles, observe that

$$\max_{v \in V^1} |v^a| - |v^b| = |I_{abc}| + |I_{acb}| + \tfrac{1}{2}|I_{bac}| - \tfrac{1}{2}|I_{bca}| + |I_{cab}| + \tfrac{1}{2}|I_{cba}| \geq 1$$

as otherwise $(\star)$ would be violated for $x = b$, $y = c$ and $z = a$. Similarly,

$$\min_{v \in V^1} |v^a| - |v^b| = -\tfrac{1}{2}|I_{abc}| + \tfrac{1}{2}|I_{acb}| - |I_{bac}| - |I_{bca}| - \tfrac{1}{2}|I_{cab}| - |I_{cba}| \leq -1$$

as otherwise $(\star)$ would be violated for $x = a$, $y = c$ and $z = b$. Adjusting opponents' strategies one by one, we can generate a profile v_{-i} such that $|v_{-i}^a| \approx |v_{-i}^b|$. Holding $|v_{-i}^c|$ as small as possible in the process, leads us to the following 5 case distinctions.

Case 1.1 We know that by $(\star)$,

$$\underbrace{2|I_{abc}| + 2|I_{acb}| + |I_{bac}| - |I_{bca}| + 2|I_{cab}| + |I_{cba}|}_{=:n} \geq 2.$$

Suppose

$$\underbrace{|I_{abc}| + |I_{acb}| + |I_{bac}| - |I_{bca}| + 2|I_{cab}| + |I_{cba}|}_{=n - |I_{abc}| - |I_{acb}|} < 0.$$

Construct v_{-i} as follows:

- $n - 1 < |I_{abc}| + |I_{acb}| - 1$ of $I_{abc} \backslash \{i\} \cup I_{acb}$ chose $v_j = (1, \frac{1}{2}, 0)$,
- all remaining $j \in I_{abc} \backslash \{i\} \cup I_{acb}$ chose $v_j = (1, 0, \frac{1}{2})$,
- all $j \in I_{bac}$ chose $v_j = (1, \frac{1}{2}, 0)$,
- all $j \in I_{bca}$ chose $v_j = (\frac{1}{2}, 1, 0)$,
- all $j \in I_{cab}$ chose $v_j = (1, 0, \frac{1}{2})$,

- all $j \in I_{cba}$ chose $v_j = (\frac{1}{2}, 0, 1)$.

Then,

$$\begin{aligned}|v_{-i}^a| - |v_{-i}^b| &= (\tfrac{1}{2}\, n - \tfrac{1}{2}) + (|I_{abc}| - 1 + |I_{acb}| - n + 1) + \tfrac{1}{2}|I_{bac}| - \tfrac{1}{2}|I_{bca}| + |I_{cab}| + \tfrac{1}{2}|I_{cba}| \\ &= -\tfrac{1}{2}\, n + |I_{abc}| + |I_{acb}| + \tfrac{1}{2}|I_{bac}| - \tfrac{1}{2}|I_{bca}| + |I_{cab}| + \tfrac{1}{2}|I_{cba}| - \tfrac{1}{2} = -\tfrac{1}{2}\end{aligned}$$

and

$$\begin{aligned}|v_{-i}^a| - |v_{-i}^c| &= n - 1 + \tfrac{1}{2}(|I_{abc}| - 1 + |I_{acb}| - n + 1) + |I_{bac}| + \tfrac{1}{2}|I_{bca}| + \tfrac{1}{2}|I_{cab}| - \tfrac{1}{2}|I_{cba}| \\ &= \tfrac{1}{2}\, n + \tfrac{1}{2}|I_{abc}| + \tfrac{1}{2}|I_{acb}| + |I_{bac}| + \tfrac{1}{2}|I_{bca}| + |I_{cab}| - \tfrac{1}{2}|I_{cba}| - 1 \\ &\geq {}^3\!/\!_2|I_{abc}| - 1 \geq \tfrac{1}{2}.\end{aligned}$$

Hence for $a \rhd b$, $v_i = (1, 0, \frac{1}{2}), (1, \frac{1}{2}, 0)$ yield a while $v_i = (\frac{1}{2}, 1, 0)$ yields b. Moreover, if $b \rhd a$, $v_i = (1, 0, \frac{1}{2})$ yields a while $v_i = (1, \frac{1}{2}, 0)$ yields b.

Case 1.2 Suppose

$$\underbrace{|I_{abc}| + |I_{acb}| + |I_{bac}| - |I_{bca}| + 2|I_{cab}| + |I_{cba}|}_{=:n} \geq 0$$

but

$$\underbrace{|I_{abc}| + |I_{acb}| + |I_{bac}| - |I_{bca}| + 2|I_{cab}| - 2|I_{cba}|}_{=n-3|I_{cba}|} < 0.$$

Construct v_{-i} as follows:

- all $j \in I_{abc} \backslash \{i\} \cup I_{acb} \cup I_{bac}$ chose $v_j = (1, \frac{1}{2}, 0)$,
- all $j \in I_{bca}$ chose $v_j = (\frac{1}{2}, 1, 0)$,
- all $j \in I_{cab}$ chose $v_j = (1, 0, \frac{1}{2})$,
- $\lfloor \frac{n}{3} \rfloor < |I_{cba}|$ of I_{cba} chose $v_j = (0, 1, \frac{1}{2})$,
- all remaining $j \in I_{cba}$ chose $v_j = (\frac{1}{2}, 0, 1)$.

Then,

$$\begin{aligned}|v_{-i}^a| - |v_{-i}^b| &= \underbrace{\tfrac{1}{2}|I_{abc}| - \tfrac{1}{2} + \tfrac{1}{2}|I_{acb}| + \tfrac{1}{2}|I_{bac}| - \tfrac{1}{2}|I_{bca}| + |I_{cab}| + \tfrac{1}{2}|I_{cba}|}_{=\frac{n}{2}-\frac{1}{2}} - \frac{3}{2}\left\lfloor \frac{n}{3} \right\rfloor \\ &= \begin{cases} -\frac{1}{2} & \text{if } n \bmod 3 = 0 \\ 0 & \text{if } n \bmod 3 = 1 \\ \frac{1}{2} & \text{if } n \bmod 3 = 2 \end{cases}\end{aligned}$$

and

$$|v_{-i}^a| - |v_{-i}^c| = |I_{abc}| - 1 + |I_{acb}| + |I_{bac}| + \tfrac{1}{2}|I_{bca}| + \tfrac{1}{2}|I_{cab}| - \tfrac{1}{2}|I_{cba}| \geq 0,$$

as otherwise, $(\star)$ would be violated for $x = c$, $y = b$ and $z = a$. Hence for $v_i = (1, 0, \frac{1}{2})$, a is elected independent of $\rhd$. If $|v^a_{-i}| - |v^b_{-i}| \in \{-\frac{1}{2}, 0\}$, and $a \rhd b$, then $v_i = (1, \frac{1}{2}, 0)$ yields outcome a while $v_i = (\frac{1}{2}, 1, 0)$ yields b. If $|v^a_{-i}| - |v^b_{-i}| \in \{0, \frac{1}{2}\}$, and $b \rhd a$, then $v_i = (1, \frac{1}{2}, 0)$ yields a while $v_i = (\frac{1}{2}, 1, 0)$ yields b.

If $|v^a_{-i}| - |v^b_{-i}| = -\frac{1}{2}$ and $b \rhd a$, then $v_i = (1, 0, \frac{1}{2})$ yields a while $v_i = (1, \frac{1}{2}, 0)$ yields b. If $|v^a_{-i}| - |v^b_{-i}| \in \{0, \frac{1}{2}\}$, observe that $\lfloor \frac{n}{3} \rfloor < |I_{cba}|$, so that there is some $j \in I_{cba}$ who chooses $v_j = (\frac{1}{2}, 0, 1)$. A switch by j to $\tilde{v}_j = (0, \frac{1}{2}, 1)$ yields $|\tilde{v}^a_{-i}| - |\tilde{v}^b_{-i}| \in \{-1, -\frac{1}{2}\}$ and $|\tilde{v}^a_{-i}| - |\tilde{v}^c_{-i}| \geq -\frac{1}{2}$. If $|\tilde{v}^a_{-i}| - |\tilde{v}^b_{-i}| = -1$ and $a \rhd b, c$, then $v_i = (1, 0, \frac{1}{2})$ yields a while $v_i = (1, \frac{1}{2}, 0)$ yields b. If $|\tilde{v}^a_{-i}| - |\tilde{v}^b_{-i}| = -\frac{1}{2}$ and $b \rhd a \rhd c$, then $(1, 0, \frac{1}{2})$ yields a while $(1, \frac{1}{2}, 0)$ yields b.

Case 1.3 Suppose

$$\underbrace{|I_{abc}| + |I_{acb}| + |I_{bac}| - |I_{bca}| + 2|I_{cab}| - 2|I_{cba}|}_{=:n} \geq 0$$

but

$$\underbrace{-|I_{abc}| + |I_{acb}| - |I_{bac}| - |I_{bca}| + 2|I_{cab}| - 2|I_{cba}|}_{=n-2|I_{abc}|-2|I_{bac}|} < 0.$$

Construct v_{-i} as follows:

- $\lfloor \frac{n}{2} \rfloor \leq |I_{abc}| - 1 + |I_{bac}|$ of $I_{abc} \backslash \{i\} \cup I_{bac}$ chose $v_j = (\frac{1}{2}, 1, 0)$,
- all remaining $j \in I_{abc} \backslash \{i\} \cup I_{bac}$ chose $v_j = (1, \frac{1}{2}, 0)$.
- all $j \in I_{acb}$ chose $v_j = (1, \frac{1}{2}, 0)$,
- all $j \in I_{bca}$ chose $v_j = (\frac{1}{2}, 1, 0)$,
- all $j \in I_{cab}$ chose $v_j = (1, 0, \frac{1}{2})$,
- all $j \in I_{cba}$ chose $v_j = (0, 1, \frac{1}{2})$.

Then,

$$\begin{aligned}
|v^a_{-i}| - |v^b_{-i}| &= -\left\lfloor \frac{n}{2} \right\rfloor + \underbrace{\tfrac{1}{2}(|I_{abc}| - 1 + |I_{bac}|) + \tfrac{1}{2}|I_{acb}| - \tfrac{1}{2}|I_{bca}| + |I_{cab}| - |I_{cba}|}_{\frac{n}{2} - \frac{1}{2}} \\
&= \begin{cases} -\frac{1}{2} & \text{if } n \bmod 2 = 0, \\ 0 & \text{if } n \bmod 2 = 1. \end{cases}
\end{aligned}$$

First, consider the case $n \bmod 2 = 0$, so that $|v^a_{-i}| - |v^b_{-i}| = -\frac{1}{2}$. Towards a contradiction, assume $|v^a_{-i}| < |v^c_{-i}|$. Then $|v^b_{-i}| \leq |v^c_{-i}|$ and $3|v^c_{-i}| > |v^a_{-i}| + |v^b_{-i}| + |v^c_{-i}| = \frac{3}{2}(|I| - 1)$. But $3|v^c_{-i}| \leq \frac{3}{2}(|I| - 1)$ as no $j \in I \backslash \{i\}$ awards more than $v^c_j = \frac{1}{2}$. Hence, $|v^a_{-i}| \geq |v^c_{-i}|$. Then, for $a \rhd b$, $v_i = (1, 0, \frac{1}{2}), (1, \frac{1}{2}, 0)$ yield a while $v_i = (\frac{1}{2}, 1, 0)$ yields b. Moreover, for $b \rhd a$, $v_i = (1, 0, \frac{1}{2})$ yields a while $v_i = (1, \frac{1}{2}, 0)$ yields b.

Next, consider the case $n \bmod 2 = 1$, so that $|v^a_{-i}| = |v^b_{-i}|$. Towards a contradiction, assume $|v^a_{-i}| \leq |v^c_{-i}|$. Then $3|v^c_{-i}| \geq |v^a_{-i}| + |v^b_{-i}| + |v^c_{-i}| = \frac{3}{2}(|I|-1)$. Moreover $3|v^c_{-i}| \leq \frac{3}{2}(|I|-1)$ as no $j \in I\backslash\{i\}$ awards more than $v^c_j = \frac{1}{2}$. Hence, $|v^c_{-i}| = \frac{1}{2}(|I|-1)$ which requires $I\backslash\{i\} = I_{cab} \cup I_{cba}$. Then $(\star)$ requires

$$|I_{cab}| + 1 \leq |I_{cba}| + \underbrace{2|I_{abc}| - 2}_{=0} \qquad \text{and} \qquad |I_{cba}| \leq |I_{cab}| + \underbrace{2|I_{abc}| - 2}_{=0}$$

- a contradiction. Instead we conclude that $|v^a_{-i}| - |v^c_{-i}| \geq \frac{1}{2}$. Then, for $a \rhd b$, $v_i = (1,0,\frac{1}{2}), (1,\frac{1}{2},0)$ yield a while $v_i = (\frac{1}{2},1,0)$ yields b. To see that $v_i = (1,0,\frac{1}{2})$ can be a better reply than $(1,\frac{1}{2},0)$, consider first the case that $\lfloor\frac{n}{2}\rfloor < |I_{abc}| - 1 + |I_{bac}|$. Then there is some $j \in I_{abc}\backslash\{i\} \cup I_{bac}$ who chooses $v_j = (1,\frac{1}{2},0)$. A switch by j to $\tilde{v}_j = (\frac{1}{2},1,0)$ yields $|\tilde{v}^a_{-i}| - |\tilde{v}^b_{-i}| = -1$ and $|\tilde{v}^a_{-i}| - |\tilde{v}^c_{-i}| \geq 0$, so that for $a \rhd b$, $v_i = (1,0,\frac{1}{2})$ yields a while $(1,\frac{1}{2},0)$ yields. If instead $\lfloor\frac{n}{2}\rfloor = |I_{abc}| - 1 + |I_{bac}|$, then, as n is odd,

$$\begin{aligned} 2|I_{abc}| - 2 + 2|I_{bac}| = 2\lfloor\frac{n}{2}\rfloor = n - 1 = |I_{abc}| + |I_{acb}| + |I_{bac}| - |I_{bca}| + 2|I_{cab}| - 2|I_{cba}| - 1 \\ \iff |I_{abc}| + |I_{bac}| = |I_{acb}| - |I_{bca}| + 2|I_{cab}| - 2|I_{cba}| + 1. \end{aligned}$$

If $I_{bac} \cup I_{bca} \cup I_{cab} = \emptyset$, this would yield $|I_{acb}| = |I_{abc}| + 2|I_{cba}| - 1$, contradicting $(\star)$. Hence, there is some $j \in I_{bac} \cup I_{bca} \cup I_{cab}$. A switch by either $j \in I_{bac} \cup I_{bca}$ from $v_j = (\frac{1}{2},1,0)$ to $\tilde{v}_j = (0,1,\frac{1}{2})$ or by $j \in I_{bac}$ from $v_j = (1,0,\frac{1}{2})$ to $\tilde{v}_j = (\frac{1}{2},0,1)$ yields $|\tilde{v}^a_{-i}| - |\tilde{v}^b_{-i}| = -\frac{1}{2}$ and $|\tilde{v}^a_{-i}| - |\tilde{v}^c_{-i}| \geq -\frac{1}{2}$, so that for $b \rhd a \rhd c$, $v_i = (1,0,\frac{1}{2})$ yields a, while $v_i = (1,\frac{1}{2},0)$ yields b.

Case 1.4 Suppose

$$\underbrace{-|I_{abc}| + |I_{acb}| - |I_{bac}| - |I_{bca}| + 2|I_{cab}| - 2|I_{cba}|}_{=:n} \geq 0$$

but

$$\underbrace{-|I_{abc}| + |I_{acb}| - |I_{bac}| - |I_{bca}| - |I_{cab}| - 2|I_{cba}|}_{=n-3|I_{cab}|} < 0.$$

Construct v_{-i} as follows:

- all $j \in I_{abc}\backslash\{i\}$ chose $v_j = (\frac{1}{2},1,0)$.
- all $j \in I_{acb}$ chose $v_j = (1,\frac{1}{2},0)$,
- all $j \in I_{bac} \cup I_{bca}$ chose $v_j = (\frac{1}{2},1,0)$,
- $\lceil\frac{n}{3}\rceil \leq |I_{cab}|$ of I_{cab} chose $v_j = (0,\frac{1}{2},1)$
- all remaining $j \in I_{cab}$ chose $v_j = (1,0,\frac{1}{2})$,
- all $j \in I_{cba}$ chose $v_j = (0,1,\frac{1}{2})$.

Then,

$$|v_{-i}^a| - |v_{-i}^b| = \underbrace{-\tfrac{1}{2}|I_{abc}| + \tfrac{1}{2} - \tfrac{1}{2}|I_{bac}| + \tfrac{1}{2}|I_{acb}| - \tfrac{1}{2}|I_{bca}| + |I_{cab}| - |I_{cba}|}_{=\frac{n}{2}+\frac{1}{2}} - \frac{3}{2}\left\lceil \frac{n}{3} \right\rceil$$
$$= \begin{cases} \frac{1}{2} & \text{if } n \bmod 3 = 0 \\ -\frac{1}{2} & \text{if } n \bmod 3 = 1 \\ 0 & \text{if } n \bmod 3 = 2 \end{cases}$$

and

$$|v_{-i}^a| - |v_{-i}^c| = |v_{-i}^b| - |v_{-i}^c| + |v_{-i}^a| - |v_{-i}^b|$$
$$= \underbrace{|I_{abc}| - 1 + |I_{bac}| + \tfrac{1}{2}|I_{acb}| + |I_{bca}| - \tfrac{1}{2}|I_{cab}| + \tfrac{1}{2}|I_{cba}|}_{=:k} + |v_{-i}^a| - |v_{-i}^b|.$$

Since $\mathbb{1}_{\{|I_{abc}|>0\}}$, $(\star)$ yields $\frac{1}{2}|I_{cab}| \leq \frac{1}{2}|I_{cba}| + \frac{1}{2}|I_{acb}| + |I_{abc}| + |I_{bca}| + |I_{bac}| - \frac{3}{2}$, so that $k \geq \frac{1}{2}$. Hence, $|v_{-i}^a| - |v_{-i}^c| > |v_{-i}^a| - |v_{-i}^b| \geq 0$, so that for $v_i = (1, 0, \frac{1}{2})$, a is elected independent of $\rhd$. If $|v_{-i}^a| - |v_{-i}^b| \in \{-\frac{1}{2}, 0\}$, and $a \rhd b$, then $v_i = (1, \frac{1}{2}, 0)$ yields outcome a while $v_i = (\frac{1}{2}, 1, 0)$ yields b. If $|v_{-i}^a| - |v_{-i}^b| \in \{0, \frac{1}{2}\}$, and $b \rhd a$, then $v_i = (1, \frac{1}{2}, 0)$ yields a while $v_i = (\frac{1}{2}, 1, 0)$ yields$b$.

If $|v_{-i}^a| - |v_{-i}^b| = -\frac{1}{2}$ and $b \rhd a$, then $v_i = (1, 0, \frac{1}{2})$ yields a while $v_i = (1, \frac{1}{2}, 0)$ yields b. If $|v_{-i}^a| - |v_{-i}^b| = \frac{1}{2}$, then $n \bmod 3 = 0$ and hence $\lceil \frac{n}{3} \rceil = \frac{n}{3} < |I_{cab}|$, so that there is some $j \in I_{cab}$ who chooses $v_j = (1, 0, \frac{1}{2})$. A switch by j to $\tilde{v}_j = (0, \frac{1}{2}, 1)$ yields $|\tilde{v}_{-i}^a| - |\tilde{v}_{-i}^b| = -1$ and $|\tilde{v}_{-i}^a| - |\tilde{v}_{-i}^c| \geq -\frac{1}{2}$. Hence, for $a \rhd b$, $v_i = (1, 0, \frac{1}{2})$ yields a while $v_i = (1, \frac{1}{2}, 0)$ yields b.

If $|\tilde{v}_{-i}^a| - |\tilde{v}_{-i}^b| = 0$ then $n \bmod 3 = 2$. If in addition $|I_{bac}| + |I_{bca}| = 0$, then by $(\star)$

$$\underbrace{-|I_{abc}| + |I_{acb}| - |I_{bac}| - |I_{bca}| - |I_{cab}| - 2|I_{cba}|}_{=n-3|I_{cab}|} = -|I_{abc}| + |I_{acb}| - 2|I_{bac}| - 2|I_{bca}| - |I_{cab}| - 2|I_{cba}| \leq -2.$$

Hence $\lceil \frac{n}{3} \rceil = \frac{n}{3} + \frac{1}{3} < \frac{n}{3} + \frac{2}{3} \leq |I_{cab}|$, so that there is some $j \in I_{cab}$ who chooses $v_j = (1, 0, \frac{1}{2})$. A switch by j to $\tilde{v}_j = (\frac{1}{2}, 0, 1)$ yields $|\tilde{v}_{-i}^a| - |\tilde{v}_{-i}^b| = -\frac{1}{2}$ and $|\tilde{v}_{-i}^a| - |\tilde{v}_{-i}^c| \geq -\frac{1}{2}$. Thus, for $b \rhd a \rhd c$, $v_i = (1, 0, \frac{1}{2})$ yields a while $v_i = (1, \frac{1}{2}, 0)$ yields b. If instead $|I_{bac}| + |I_{bca}| > 0$, let some $j \in I_{bac} \cup I_{bca}$ switch from $v_j = (\frac{1}{2}, 1, 0)$ to $\tilde{v}_j = (0, 1, \frac{1}{2})$. Then $|\tilde{v}_{-i}^a| - |\tilde{v}_{-i}^b| = -\frac{1}{2}$ and $|\tilde{v}_{-i}^a| - |\tilde{v}_{-i}^c| \geq -\frac{1}{2}$. Hence, for $b \rhd a \rhd c$, $v_i = (1, 0, \frac{1}{2})$ yields a while $v_i = (1, \frac{1}{2}, 0)$ yields b.

Case 1.5 Suppose

$$\underbrace{-|I_{abc}| + |I_{acb}| - |I_{bac}| - |I_{bca}| - |I_{cab}| - 2|I_{cba}|}_{=:n} \geq 0.$$

We know that

$$\underbrace{-|I_{abc}| + |I_{acb}| - 2|I_{bac}| - 2|I_{bca}| - |I_{cab}| - 2|I_{cba}|}_{=n-|I_{bac}|-|I_{bca}|} \leq -2,$$

as otherwise ($\star$) would be violated for $x = a$, $y = c$ and $z = b$. Construct v_{-i} as follows:

- all $j \in I_{abc} \backslash \{i\}$ chose $v_j = (\frac{1}{2}, 1, 0)$.
- all $j \in I_{acb}$ chose $v_j = (1, \frac{1}{2}, 0)$,
- all $j \in I_{cab}$ chose $v_j = (0, \frac{1}{2}, 1)$,
- all $j \in I_{cba}$ chose $v_j = (0, 1, \frac{1}{2})$,
- $n + 2 \leq |I_{bac}| + |I_{bca}|$ of $I_{bac} \cup I_{bca}$ chose $v_j = (0, 1, \frac{1}{2})$,
- all remaining $j \in I_{bac} \cup I_{bca}$ chose $v_j = (\frac{1}{2}, 1, 0)$.

Then,

$$|v^a_{-i}| - |v^b_{-i}| = -\tfrac{1}{2}|I_{abc}| - \tfrac{1}{2} + \tfrac{1}{2} + \tfrac{1}{2}|I_{acb}| - \tfrac{1}{2}|I_{cab}| - |I_{cba}| - \tfrac{1}{2}|I_{bac}| - \tfrac{1}{2}|I_{bca}| - \tfrac{1}{2}(n + 2) = -\tfrac{1}{2}$$

and

$$\begin{aligned} |v^a_{-i}| - |v^c_{-i}| &= \tfrac{1}{2}|I_{abc}| - \tfrac{1}{2} + |I_{acb}| - |I_{cab}| - \tfrac{1}{2}|I_{cba}| + \tfrac{1}{2}|I_{bac}| + \tfrac{1}{2}|I_{bca}| - (n + 2) \\ &= {}^3\!/_2|I_{abc}| + {}^3\!/_2|I_{bac}| + {}^3\!/_2|I_{bca}| + {}^3\!/_2|I_{cba}| - {}^3\!/_2 \geq 0. \end{aligned}$$

Hence for $a \rhd b$, $v_i = (1, 0, \frac{1}{2}), (1, \frac{1}{2}, 0)$ yields a while $v_i = (\frac{1}{2}, 1, 0)$ yields b. Moreover, for $b \rhd a$, $v_i = (1, 0, \frac{1}{2})$ yields a while $v_i = (1, \frac{1}{2}, 0)$ yields b. *Claim 1* $\diamond$

Claim 2. *Neither $(\frac{1}{2}, 1, 0)$ nor $(1, \frac{1}{2}, 0)$ is dominated by $(1, 0, \frac{1}{2})$. Moreover, $(\frac{1}{2}, 1, 0)$ is not dominated by $(1, \frac{1}{2}, 0)$ and both b and c are possible outcomes.*

Proof. We will proof the claim by constructing an opposing strategy profile for which (i) $v_i = (\frac{1}{2}, 1, 0)$ and $v_i = (1, \frac{1}{2}, 0)$ yield outcome b while $v_i = (1, 0, \frac{1}{2})$ yields c and (ii) another opposing profile for which $v_i = (\frac{1}{2}, 1, 0)$ yields b while $v_i = (1, \frac{1}{2}, 0)$ yields c. To find such profiles, observe that

$$\max_{v \in V^1} |v^b| - |v^c| = |I_{abc}| + \tfrac{1}{2}|I_{acb}| + |I_{bac}| + |I_{bca}| - \tfrac{1}{2}|I_{cab}| + \tfrac{1}{2}|I_{cba}| \geq {}^3\!/_2$$

as otherwise ($\star$) would be violated for $x = c$, $y = a$ and $z = b$. Similarly,

$$\min_{v \in V^1} |v^b| - |v^c| = -\tfrac{1}{2}|I_{abc}| - |I_{acb}| + \tfrac{1}{2}|I_{bac}| - \tfrac{1}{2}|I_{bca}| - |I_{cab}| - |I_{cba}| \leq -1$$

as otherwise ($\star$) would be violated for $x = b$, $y = a$ and $z = c$. Adjusting opponents' strategies one by one, we can generate a profile v_{-i} such that $|v^b_{-i}| \approx |v^c_{-i}|$. Holding $|v^a_{-i}|$ as small as possible in the process, leads us to the following 5 case distinctions.

Case 2.1 We know that by $(\star)$,

$$\underbrace{2|I_{abc}| + |I_{acb}| + 2|I_{bac}| + 2|I_{bca}| - |I_{cab}| + |I_{cba}|}_{=:n} \geq 3.$$

Suppose

$$\underbrace{2|I_{abc}| + |I_{acb}| + |I_{bac}| + |I_{bca}| - |I_{cab}| + |I_{cba}|}_{=n-|I_{bac}|-|I_{bca}|} \leq 0.$$

Construct v_{-i} as follows:

- all $j \in I_{abc} \backslash \{i\}$ chose $v_j = (\frac{1}{2}, 1, 0)$,
- all $j \in I_{acb}$ chose $v_j = (1, \frac{1}{2}, 0)$,
- $n - 1 < |I_{bac}| + |I_{bca}|$ of $I_{bac} \cup I_{bca}$ chose $v_j = (0, 1, \frac{1}{2})$,
- all remaining $j \in I_{bac} \cup I_{bca}$ chose $v_j = (\frac{1}{2}, 1, 0)$,
- all $j \in I_{cab}$ chose $v_j = (0, \frac{1}{2}, 1)$,
- all $j \in I_{cba}$ chose $v_j = (0, 1, \frac{1}{2})$.

Then,

$$|v_{-i}^b| - |v_{-i}^c| = |I_{abc}| - 1 + \frac{1}{2}|I_{acb}| + |I_{bac}| + |I_{bca}| - \frac{1}{2}(n-1) - \frac{1}{2}|I_{cab}| + \frac{1}{2}|I_{cba}| = -\frac{1}{2}$$

and

$$\begin{aligned} |v_{-i}^b| - |v_{-i}^a| &= \frac{1}{2}|I_{abc}| - \frac{1}{2} - \frac{1}{2}|I_{acb}| + \frac{1}{2}|I_{bac}| + \frac{1}{2}|I_{bca}| + \frac{1}{2}(n-1) + \frac{1}{2}|I_{cab}| + |I_{cba}| \\ &= 3/2|I_{abc}| + 3/2|I_{bac}| + 3/2|I_{bca}| + 3/2|I_{cba}| - 3/2 \geq 6, \end{aligned}$$

since by assumption for Case 2.1 we have $|I_{bac}| + |I_{bca}| \geq 4$. Then for $b \rhd c$, $v_i = (\frac{1}{2}, 1, 0), (1, \frac{1}{2}, 0)$, yield b whereas $v_i = (1, 0, \frac{1}{2})$ yields c. Moreover, for $c \rhd b$, $v_i = (\frac{1}{2}, 1, 0)$ yields b while $v_i = (1, \frac{1}{2}, 0)$ yields c.

Case 2.2 Suppose

$$\underbrace{2|I_{abc}| + |I_{acb}| + |I_{bac}| + |I_{bca}| - |I_{cab}| + |I_{cba}|}_{=:n} > 0$$

but

$$\underbrace{2|I_{abc}| - 2|I_{acb}| + |I_{bac}| + |I_{bca}| - |I_{cab}| + |I_{cba}|}_{=n-3|I_{acb}|} \leq 0.$$

Construct v_{-i} as follows:

- all $j \in I_{abc} \backslash \{i\}$ chose $v_j = (\frac{1}{2}, 1, 0)$,
- $\lfloor \frac{n-1}{3} \rfloor < |I_{acb}|$ of I_{acb} chose $v_j = (\frac{1}{2}, 0, 1)$,

- all remaining $j \in I_{acb}$ chose $v_j = (1, \frac{1}{2}, 0)$,
- all $j \in I_{bac} \cup I_{bca}$ chose $v_j = (0, 1, \frac{1}{2})$,
- all $j \in I_{cab}$ chose $v_j = (0, \frac{1}{2}, 1)$,
- all $j \in I_{cba}$ chose $v_j = (0, 1, \frac{1}{2})$.

Then,

$$|v_{-i}^b| - |v_{-i}^c| = \underbrace{|I_{abc}| - 1 + \tfrac{1}{2}|I_{acb}| + \tfrac{1}{2}|I_{bac}| + \tfrac{1}{2}|I_{bca}| - \tfrac{1}{2}|I_{cab}| + \tfrac{1}{2}|I_{cba}|}_{\frac{n}{2}-1} - \frac{3}{2}\left\lfloor \frac{n-1}{3} \right\rfloor$$
$$= \begin{cases} -\frac{1}{2} & \text{if } n-1 \bmod 3 = 0 \\ 0 & \text{if } n-1 \bmod 3 = 1 \\ \frac{1}{2} & \text{if } n-1 \bmod 3 = 2 \end{cases}$$

and

$$|v_{-i}^b| - |v_{-i}^a| = \tfrac{1}{2}|I_{abc}| - \tfrac{1}{2} - \tfrac{1}{2}|I_{acb}| + |I_{bac}| + |I_{bca}| + \tfrac{1}{2}|I_{cab}| + |I_{cba}| \geq \tfrac{1}{2},$$

as otherwise, $(\star)$ would be violated for $x = a$, $y = c$ and $z = b$. If $|v_{-i}^b| - |v_{-i}^c| = -\frac{1}{2}$, and $b \rhd c, a$, then $v_i = (1, \frac{1}{2}, 0), (\frac{1}{2}, 1, 0)$ yields b while $v_i = (1, 0, \frac{1}{2})$ yields c. Moreover, for $c \rhd b, a$, $v_i = (\frac{1}{2}, 1, 0)$ yields b while $v_i = (1, \frac{1}{2}, 0)$ yields c.

Next, consider $|v_{-i}^b| - |v_{-i}^c| = 0$. Again, for $b \rhd a$, then $v_i = (1, \frac{1}{2}, 0), (\frac{1}{2}, 1, 0)$ yields b while $v_i = (1, 0, \frac{1}{2})$ yields c. For a profile where $v_i = (\frac{1}{2}, 1, 0)$ is a better reply than $(1, \frac{1}{2}, 0)$, let some $j \in I_{acb}$ switch from $v_j = (1, \frac{1}{2}, 0)$ to $\tilde{v}_j = (1, 0, \frac{1}{2})$. Then $|\tilde{v}_{-i}^b| - |\tilde{v}_{-i}^c| = -1$ and $|\tilde{v}_{-i}^b| - |\tilde{v}_{-i}^a| \geq 0$. Hence, for $b \rhd c \rhd a$, $v_i = (\frac{1}{2}, 1, 0)$ yields b while $v_i = (1, \frac{1}{2}, 0)$ yields c.

Finally, consider v_{-i} where $|v_{-i}^b| - |v_{-i}^c| = \frac{1}{2}$ or rather the neighbouring profile $\tilde{v}_{-i}$ where some $j \in I_{cab}$ has switched from $v_j = (1, \frac{1}{2}, 0)$ to $\tilde{v}_j = (1, 0, \frac{1}{2})$. Then $|\tilde{v}_{-i}^b| - |\tilde{v}_{-i}^c| = -\frac{1}{2}$ and $|\tilde{v}_{-i}^b| - |\tilde{v}_{-i}^a| \geq 0$. Hence, for $b \rhd c \rhd a$, a or b is elected for $v_i = (1, \frac{1}{2}, 0), (\frac{1}{2}, 1, 0)$ while $v_i = (1, 0, \frac{1}{2})$ yields c. For a profile where $v_i = (\frac{1}{2}, 1, 0)$ is a better reply than $(1, \frac{1}{2}, 0)$, consider $\hat{v}_{-i}$, which differs from v_{-i} in that some $j \in I_{acb}$ switches from $v_j = (1, \frac{1}{2}, 0)$ to $\hat{v}_j = (\frac{1}{2}, 0, 1)$. Then $|\hat{v}_{-i}^b| - |\hat{v}_{-i}^c| = -1$, $|\hat{v}_{-i}^b| - |\hat{v}_{-i}^a| \geq \frac{1}{2}$ so that for $b \rhd c$, $v_i = (\frac{1}{2}, 1, 0)$ yields b while $v_i = (1, \frac{1}{2}, 0)$ yields c.

Case 2.3 Suppose

$$\underbrace{2|I_{abc}| - 2|I_{acb}| + |I_{bac}| + |I_{bca}| - |I_{cab}| + |I_{cba}|}_{=:n} > 0$$

but

$$\underbrace{2|I_{abc}| - 2|I_{acb}| + |I_{bac}| - |I_{bca}| - |I_{cab}| - |I_{cba}|}_{=n-2|I_{bca}|-2|I_{cba}|} \leq 0.$$

Construct v_{-i} as follows:

- all $j \in I_{abc} \backslash \{i\}$ chose $v_j = (\frac{1}{2}, 1, 0)$
- all $j \in I_{acb}$ chose $v_j = (\frac{1}{2}, 0, 1)$,
- all $j \in I_{bac}$ chose $v_j = (0, 1, \frac{1}{2})$,
- $\lfloor \frac{n}{2} \rfloor \leq |I_{bca}| + |I_{cba}|$ of $I_{bca} \cup I_{cba}$ chose $v_j = (0, \frac{1}{2}, 1)$,
- all remaining $j \in I_{bca} \cup I_{cba}$ chose $v_j = (0, 1, \frac{1}{2})$.
- all $j \in I_{cab}$ chose $v_j = (0, \frac{1}{2}, 1)$,

Then,

$$\begin{aligned}|v_{-i}^b| - |v_{-i}^c| &= \underbrace{|I_{abc}| - 1 - |I_{acb}| + \tfrac{1}{2}|I_{bac}| + \tfrac{1}{2}|I_{bca}| - \tfrac{1}{2}|I_{cab}| + \tfrac{1}{2}|I_{cba}|}_{\frac{n}{2} - \frac{1}{2}} - \left\lfloor \frac{n}{2} \right\rfloor \\ &= \begin{cases} -\frac{1}{2} & \text{if } n \bmod 2 = 0, \\ 0 & \text{if } n \bmod 2 = 1, \end{cases}\end{aligned}$$

and

$$\begin{aligned}|v_{-i}^b| - |v_{-i}^a| &= \tfrac{1}{2}|I_{abc}| - \tfrac{1}{2} - \tfrac{1}{2}|I_{acb}| + |I_{bac}| + |I_{bca}| + \tfrac{1}{2}|I_{cab}| + |I_{cba}| - \tfrac{1}{2}\left\lfloor \tfrac{n}{2} \right\rfloor \\ &\geq \tfrac{1}{2}|I_{abc}| - \tfrac{1}{2} - \tfrac{1}{2}|I_{acb}| + |I_{bac}| + |I_{bca}| + \tfrac{1}{2}|I_{cab}| + |I_{cba}| - \tfrac{n}{4} \\ &= \underbrace{3/4|I_{bac}| + 3/4|I_{bca}| + 3/4|I_{cab}| + 3/4|I_{cba}|}_{\geq 3/4, \text{ by } (\star)} - \tfrac{1}{2} > 0.\end{aligned}$$

For $b \rhd c \rhd a$, both $v_i = (1, \frac{1}{2}, 0), (\frac{1}{2}, 1, 0)$ yields b while $v_i = (1, 0, \frac{1}{2})$ yields c. Moreover, if $|v_{-i}^b| - |v_{-i}^c| = -\frac{1}{2}$ and $c \rhd a, b$ then $v_i = (\frac{1}{2}, 1, 0)$ yields b while $v_i = (1, \frac{1}{2}, 0)$ yields c.

If on the other hand $|v_{-i}^b| - |v_{-i}^c| = 0$ then $\lfloor \frac{n}{2} \rfloor = \frac{n-1}{2} < |I_{bca}| + |I_{cba}|$, so that there is some $j \in I_{bca} \cup I_{cba}$ who chooses $v_j = (0, 1, \frac{1}{2})$. Letting her switch to $\tilde{v}_j = (0, \frac{1}{2}, 1)$ gives $|\tilde{v}_{-i}^b| - |\tilde{v}_{-i}^c| = -1$ and $|\tilde{v}_{-i}^b| - |\tilde{v}_{-i}^a| \geq 0$, so that for $b \rhd c \rhd a$, $v_i = (\frac{1}{2}, 1, 0)$ yields b while $v_i = (1, \frac{1}{2}, 0)$ yields c.

Case 2.4 Suppose

$$\underbrace{2|I_{abc}| - 2|I_{acb}| + |I_{bac}| - |I_{bca}| - |I_{cab}| - |I_{cba}|}_{=:n} > 0$$

but

$$\underbrace{-|I_{abc}| - 2|I_{acb}| + |I_{bac}| - |I_{bca}| - |I_{cab}| - |I_{cba}|}_{=n-3|I_{abc}|} \leq 0.$$

Construct v_{-i} as follows:

- $\lceil \frac{n}{3} \rceil - 1 \leq |I_{abc}| - 1$ of $I_{abc} \backslash \{i\}$ chose $v_j = (1, 0, \frac{1}{2})$

- all remaining $j \in I_{abc} \backslash \{i\}$ chose $v_j = (\frac{1}{2}, 1, 0)$
- all $j \in I_{acb}$ chose $v_j = (\frac{1}{2}, 0, 1)$,
- all $j \in I_{bac}$ chose $v_j = (0, 1, \frac{1}{2})$,
- all $j \in I_{bca} \cup I_{cab} \cup I_{cba}$ chose $v_j = (0, \frac{1}{2}, 1)$.

Then,

$$|v_{-i}^b| - |v_{-i}^c| = \underbrace{|I_{abc}| - 1 - |I_{acb}| + \tfrac{1}{2}|I_{bac}| - \tfrac{1}{2}|I_{bca}| - \tfrac{1}{2}|I_{cab}| - \tfrac{1}{2}|I_{cba}|}_{\frac{n}{2}-1} - \frac{3}{2}\left\lceil \frac{n}{3} \right\rceil + \frac{3}{2}$$
$$= \begin{cases} \frac{1}{2} & \text{if } n \bmod 3 = 0, \\ -\frac{1}{2} & \text{if } n \bmod 3 = 1, \\ 0 & \text{if } n \bmod 3 = 2. \end{cases}$$

and

$$|v_{-i}^a| - |v_{-i}^c| = \tfrac{1}{2}|I_{abc}| - \tfrac{1}{2} - \tfrac{1}{2}|I_{acb}| - \tfrac{1}{2}|I_{bac}| - |I_{bca}| - |I_{cab}| - |I_{cba}| \le -3/2,$$

as otherwise ($\star$) would be violated for $x = a$, $y = b$, $z = c$. For $v_i = (\frac{1}{2}, 1, 0)$, b is elected independently of $\triangleright$. If $|v_{-i}^b| - |v_{-i}^c| \in \{-\frac{1}{2}, 0\}$ and $b \triangleright c$, then $v_i = (1, \frac{1}{2}, 0)$ yields b while $v_i = (1, 0, \frac{1}{2})$ yields c. If $|v_{-i}^b| - |v_{-i}^c| \in \{0, \frac{1}{2}\}$ and $c \triangleright b$, then $v_i = (1, \frac{1}{2}, 0)$ yields b while $v_i = (1, 0, \frac{1}{2})$ yields c.

Moreover, if $|v_{-i}^b| - |v_{-i}^c| = -\frac{1}{2}$ and $c \triangleright b$, then $v_i = (\frac{1}{2}, 1, 0)$ yields b while $v_i = (1, \frac{1}{2}, 0)$ yields c. Next, if $|v_{-i}^b| - |v_{-i}^c| = 0$, then $n \bmod 3 = 2$ and hence $\lceil \frac{n}{3} \rceil - 1 = \frac{n+1}{3} - 1$. Towards a contradiction, assume that $\frac{n+1}{3} - 1 = |I_{abc}| - 1$. Then

$$3|I_{abc}| - 1 = n \le 2|I_{abc}| - 2|I_{acb}| + |I_{bac}| - |I_{bca}| \implies |I_{bac}| \ge |I_{bca}| + |I_{abc}| + 2|I_{acb}| - 1,$$

which violates ($\star$). Thus, we know that there exist either some $j \in I_{abc} \backslash \{i\}$ who votes $v_j = (\frac{1}{2}, 1, 0)$. Letting $j \in I_{abc} \backslash \{i\}$ switch to $\tilde{v}_j = (1, \frac{1}{2}, 0)$ gives $|\tilde{v}_{-i}^b| - |\tilde{v}_{-i}^c| = -\frac{1}{2}$ and $|\tilde{v}_{-i}^a| - |\tilde{v}_{-i}^c| \le 1$. Then for $c \triangleright a, b$, $v_i = (\frac{1}{2}, 1, 0)$ yields b while $v_i = (1, \frac{1}{2}, 0)$ yields c.

Finally, if $|v_{-i}^b| - |v_{-i}^c| = \frac{1}{2}$ then $n \bmod 3 = 0$ and hence $\lceil \frac{n}{3} \rceil = \frac{n}{3}$. Towards a contradiction, assume that $\frac{n}{3} - 1 = |I_{abc}| - 1$. Then

$$3|I_{abc}| = n \le 2|I_{abc}| - 2|I_{acb}| + |I_{bac}| - |I_{bca}| \implies |I_{bac}| \ge |I_{bca}| + |I_{abc}| + 2|I_{acb}|,$$

which violates ($\star$). Thus, we know that there exist either some $j \in I_{abc} \backslash \{i\}$ who votes $v_j = (\frac{1}{2}, 1, 0)$. Letting $j \in I_{abc} \backslash \{i\}$ switch to $\tilde{v}_j = (1, 0, \frac{1}{2})$ gives $|\tilde{v}_{-i}^b| - |\tilde{v}_{-i}^c| = -1$ and $|\tilde{v}_{-i}^a| - |\tilde{v}_{-i}^c| \le -3/2$. Then for $b \triangleright c$, $v_i = (\frac{1}{2}, 1, 0)$ yields b while $v_i = (1, \frac{1}{2}, 0)$ yields c.

Case 2.5 Suppose

$$\underbrace{-|I_{abc}| - 2|I_{acb}| + |I_{bac}| - |I_{bca}| - |I_{cab}| - |I_{cba}|}_{=:n} > 0.$$

We know that

$$\underbrace{-|I_{abc}| - 2|I_{acb}| + |I_{bac}| - |I_{bca}| - 2|I_{cab}| - 2|I_{cba}|}_{=n-|I_{cab}|-|I_{cba}|} \leq -2.$$

Construct v_{-i} as follows:

- all $j \in I_{abc} \backslash \{i\}$ chose $v_j = (1, 0, \frac{1}{2})$
- all $j \in I_{acb}$ chose $v_j = (\frac{1}{2}, 0, 1)$,
- all $j \in I_{bac}$ chose $v_j = (0, 1, \frac{1}{2})$,
- all $j \in I_{bca}$ chose $v_j = (0, \frac{1}{2}, 1)$,
- $n + 2$ of $I_{cab} \cup I_{cba}$ chose $v_j = (\frac{1}{2}, 0, 1)$,
- all remaining $j \in I_{cab} \cup I_{cba}$ chose $v_j = (0, \frac{1}{2}, 1)$.

Then,

$$|v^b_{-i}| - |v^c_{-i}| = -\frac{1}{2}|I_{abc}| + \frac{1}{2} - |I_{acb}| + \frac{1}{2}|I_{bac}| - \frac{1}{2}|I_{bca}| - \frac{1}{2}|I_{cab}| - \frac{1}{2}|I_{cba}| - \frac{1}{2}(n + 2) = -\frac{1}{2}$$

and

$$\begin{aligned} |v^a_{-i}| - |v^c_{-i}| &= \frac{1}{2}|I_{abc}| - \frac{1}{2} - \frac{1}{2}|I_{acb}| - \frac{1}{2}|I_{bac}| - |I_{bca}| - |I_{cab}| - |I_{cba}| + \frac{1}{2}(n + 2) \\ &= -3/2|I_{acb}| - 3/2|I_{bca}| - 3/2|I_{cab}| - 3/2|I_{cba}| + \frac{1}{2} \leq -4, \end{aligned}$$

since by assumption for case 2.5, $|I_{cab}|-|I_{cba}| \geq 3$. Then for $b \rhd c$, $v_i = (\frac{1}{2}, 1, 0), (1, \frac{1}{2}, 0)$ yields b while $v_i = (1, 0, \frac{1}{2})$ yields c. Moreover, if $c \rhd b$, $v_i = (\frac{1}{2}, 1, 0)$ yields b and while $v_i = (1, \frac{1}{2}, 0)$ yields c. *Claim 2* ◇

Together, Claim 1 and 2 show that each outcome is possible in $\Gamma(\succ_I, V^1)$ and that for $i \in I_{abc}$, $V^2_i = V^1_i$. In the same way, i.e. just by relabelling candidates in Claim 1 and 2, we find that for any $j \in I_{xyz}$ $V^2_j = V^1_j$. Then by induction $V^m = V^1$, for all $m \geq 1$. This completes the proof. □

Proof of Theorem 11. For $x, y, z \in A$, define $|O_{xyz}| := |I_{xzy}| + |I_{yxz}| + 2|I_{yzx}| + 2|I_{zxy}| + 2|I_{zyx}|$ and to fix labels, assume w.l.o.g. that $|I_{abc}| - |O_{abc}| \geq |I_{xyz}| - |O_{xyz}|$ for all $x, y, z \in A$. We will show that each election outcome is possible under some ballot profile, where each voter i chooses a strategy v_i that is undominated. To guide our construction, we make use of the following fact.

Claim 1. *Consider a ballot profile v such that $|v^x| = |v^y| = |v^y|$ and some voter $i \in I_{xyz}$ such that $v^x_i = 1$. Then v_i is undominated in any Game $\Gamma(\succ_I, V')$ where $v \in V' \subset V^1$.*

Proof. Consider the case $v_i = (v^x_i, v^y_i, v^z_i) = (1, \frac{1}{2}, 0)$. If $x \rhd y, z$, then i's most preferred outcome x is realized. On the other hand, a switch to $\tilde{v}_i = (\frac{1}{2}, 1, 0)$ would yield outcome y and a switch to $\tilde{v}_i = (1, 0, \frac{1}{2})$ would yield z. Hence $v_i = (1, \frac{1}{2}, 0)$ is undominated.

If $v_i = (v_i^x, v_i^y, v_i^z) = (1, 0, \frac{1}{2})$ and $x \rhd y, z$ outcome x is realized, while a switch to $\tilde{v}_i = (\frac{1}{2}, 1, 0)$ or $\tilde{v}_i = (1, \frac{1}{2}, 0)$ would yield y. Hence $v_i = (1, 0, \frac{1}{2})$ is undominated.

Claim 1 $\diamond$

Case 1: $|I_{abc}| = |O_{abc}| = |I_{abc}| + |I_{bac}| + 2|I_{bca}| + 2|I_{cab}| + 2|I_{cba}|$

By the assumptions of Proposition 11 we have $|I_{cab}| \leq |I_{bac}|$ and $|I_{acb}| = 0$. If $|I_{bca}| + |I_{cab}| + |I_{cba}| = 0$, so that $|I_{abc}| = |I_{bac}|$, consider a ballot profile v where all i_{abc} chose $v_i = (1, 0, \frac{1}{2}) \in V_i^1$ while all i_{bac} chose $v_i = (0, 1, \frac{1}{2}) \in V_i^1$. Then $|v^a| = |v^b| = |v^c|$, so that by claim 1 each v_i is undominated and hence no outcome can be ruled out via iterated elimination of dominated strategies.

Next, consider $|I_{bca}| + |I_{cab}| + |I_{cba}| > 0$. To show that no outcome can be eliminated, we will construct two strategy profiles where a, b and c are possible outcomes (depending on $\rhd$) and show that no individual strategy used in the construction can be eliminated based on weak domination.

Profile $v \in V^1$:

- each $i \in I_{abc}$ chooses $v_i = (1, 0, 1/2)$,
- each $i \in I_{bac}$ chooses $v_i = (0, 1, 1/2)$,
- each $i \in I_{bca} \cup I_{zxy} \cup I_{zyx}$ chooses $v_i = (0, 1/2, 1)$.

Then,

$$|v^a| - |v^c| = \tfrac{1}{2}|I_{abc}| \underbrace{-\tfrac{1}{2}|I_{bac}| - |I_{bca}| - |I_{cab}| - |I_{cba}|}_{-\frac{1}{2}|I_{abc}|} = 0,$$

while

$$\begin{aligned} |v^a| - |v^b| &= |I_{abc}| - |I_{bac}| - \tfrac{1}{2}|I_{bca}| - \tfrac{1}{2}|I_{cab}| - \tfrac{1}{2}|I_{cba}| \\ &= 3/2|I_{bca}| + 3/2|I_{cab}| + 3/2|I_{cba}| \geq 3/2, \end{aligned}$$

so that both a and c are possible outcomes, depending on $\rhd$. If $c \rhd a$, then c is elected while any unilateral deviation to some $\tilde{v}_i \in V_i^1$ by some $i \in I_{bca} \cup I_{cab} \cup I_{cba}$ would yield outcome a. Hence, for $i \in I_{bca} \cup I_{cab} \cup I_{cba}$, $(0, \frac{1}{2}, 1)$ is the unique best response and thus undominated in any game $\Gamma(\succ_I, V^n)$ where $v \in V^n \subset V^1$.

Profile $v' \in V^1$:

- let $|I_{bac}| - |I_{cab}|$ of I_{abc} chose $v_i' = (1, 0, \frac{1}{2})$
- let the remaining i_{abc} choose $v_i' = (1, \frac{1}{2}, 0)$
- let each $i \in I_{cab}$ choose $v_i' = (0, \frac{1}{2}, 1)$,
- let each $i \in I_{bac} \cup I_{bca} \cup I_{cba}$ chooses $v_i' = (0, 1, \frac{1}{2})$.

Then,

$$\begin{aligned}|v^a| - |v^b| &= |I_{bac}| - |I_{cab}| + \tfrac{1}{2}(|I_{abc}| - |I_{bac}| + |I_{cab}|) - \tfrac{1}{2}|I_{cab}| - |I_{bac}| - |I_{bca}| - |I_{cba}| \\ &= \tfrac{1}{2}|I_{abc}| \underbrace{-\tfrac{1}{2}|I_{bac}| - |I_{bca}| - |I_{cab}| - |I_{cba}|}_{=\frac{1}{2}|I_{abc}|} = 0,\end{aligned}$$

while

$$\begin{aligned}|v^a| - |v^c| &= \tfrac{1}{2}(|I_{bac}| - |I_{cab}|) + (|I_{abc}| - |I_{bac}| + |I_{cab}|) - |I_{cab}| - \tfrac{1}{2}|I_{bac}| - \tfrac{1}{2}|I_{bca}| - \tfrac{1}{2}|I_{cba}| \\ &= |I_{abc}| - |I_{bac}| - \tfrac{1}{2}|I_{bca}| - \tfrac{1}{2}|I_{cab}| - \tfrac{1}{2}|I_{cba}| \\ &= 3/2|I_{bca}| + 3/2|I_{cab}| + 3/2|I_{cba}| \geq 3/2,\end{aligned}$$

so that both a and b are possible outcomes, depending on $\rhd$. If $b \rhd a$, then b is elected while any unilateral deviation to some $\tilde{v}_i \in V_i^1$ by some $i \in I_{bac} \cup I_{bca} \cup I_{cab}$ would yield outcome a. Hence, for $i \in I_{bac} \cup I_{bca} \cup I_{cab}$, $(0, 1, \frac{1}{2})$ is the unique best response and thus undominated in any game $\Gamma(\succ_I, V^n)$ where $v' \in V^n \subset V^1$.

It remains to check that for i_{abc}, $(1, 0, \frac{1}{2})$ and $(1, \frac{1}{2}, 0)$ are undominated in any game $\Gamma(\succ_I, V^n)$ where $v, v' \in V^n \subset V^1$.

For that, consider again profile v where $v_i = (1, 0, \frac{1}{2})$ and assume that $c \rhd b, a$, so that c is elected. A switch by i to $(\frac{1}{2}, 1, 0)$ would also yield c, as we would now have $|v^a| = |v^c|$ and $|v^a| \geq |v^b|$. On the other hand, a switch to $(1, \frac{1}{2}, 0)$ would yield a, as we would now have $|v^a| > |v^c|$ and $|v^a| > |v^b|$. Hence, for i_{abc}, $(1, \frac{1}{2}, 0)$ is the unique best response and thus undominated in any game $\Gamma(\succ_I, V^n)$ where $v, v' \in V^n \subset V^1$.

Similarly, consider profile v' where some i_{abc} chooses $v'_i = (1, \frac{1}{2}, 0)$ and assume that $b \rhd a, c$, so that b is elected. A switch by i to $(\frac{1}{2}, 1, 0)$ would yield b, as we would now have $|v^b| > |v^a|$ and $|v^a| \geq |v^c|$. On the other hand, a switch to $(1, 0, \frac{1}{2})$ would yield a, as we would now have $|v^a| > |v^b|$ and $|v^a| > |v^c|$. Hence, for i_{abc}, $(1, 0, \frac{1}{2})$ is the unique best response and thus undominated in any game $\Gamma(\succ_I, V^n)$ where $v, v' \in V^n \subset V^1$.

Case 2: $|I_{abc}| = |O_{abc}| - 1 = |I_{abc}| + |I_{bac}| + 2|I_{bca}| + 2|I_{cab}| + 2|I_{cba}| - 1$

By the assumptions of Theorem 11, we have $|I_{acb}| = 0$. Moreover, we know that $|I_{bca}| + |I_{cab}| + |I_{cba}| > 0$ as otherwise $I = I_{abc} \cup I_{bac}$ and $|I_{abc}| = |I_{bac}| - 1$; this would imply $|I_{bac}| > |I_{abc}| = |I_{bca}| + |I_{abc}| + 2|I_{acb}| + 2|I_{cba}| + 2|I_{cab}|$ and hence violate the assumptions of Theorem 11.

First, assume that $|I_{bca}| = 1$ and $|I_{bac}| + |I_{cab}| + |I_{cba}| = 0$ so that $|I_{abc}| = 1$. Let $i \in I_{abc}$ choose $v_i = (1, 0, \frac{1}{2})$ and $j \in I_{bac}$ choose $v_j = (0, 1, \frac{1}{2})$. Then, $|v^a| = |v^b| = |v^c| = 1$ so that by claim 1 each v_i is undominated and hence no outcome can be ruled out via iterated elimination of dominated strategies.

Next, assume that either $|I_{bca}| \neq 1$ or $|I_{bac}| + |I_{cab}| + |I_{cba}| > 0$. We construct a ballot profile v as follows:

- some $j \in I_{abc}$ chooses $v_j = (1, \frac{1}{2}, 0)$
- $|I_{bac}| + |I_{bca}| + |I_{cab}| + |I_{cba}| - 1$ of I_{abc} choose $v_j = (1, 0, \frac{1}{2})$

- $|I_{bca}| + |I_{cab}| + |I_{cba}| - 1$ of I_{abc} choose $v_j = (\frac{1}{2}, 1, 0)$
- all $j \in I_{bac}$ choose $v_j = (0, 1, \frac{1}{2})$,
- all $j \in I_{bca} \cup I_{cab} \cup I_{cba}$ choose $v_j = (0, \frac{1}{2}, 1)$.

Then,

$$|v^a| = |v^b| = |v^c| = |I_{bac}| + 3/2|I_{bca}| + 3/2|I_{cab}| + 3/2|I_{cba}| - \frac{1}{2},$$

and any candidate may win, depending on $\triangleright$.

To see that each strategy used in the construction of v is undominated in $\Gamma(\succ_I, V^n)$ where $v \in V^n \subset V^1$, consider $i \in I_{abc}$ who chooses $v_i = (1, \frac{1}{2}, 0)$. By claim 1, v_i is undominated. Moreover, if $c \triangleright a, b$, then outcome c is realized. Only a switch to $\tilde{v}_i = (\frac{1}{2}, 1, 0)$ would yield b, while a switch to $\tilde{v}_i = (1, 0, \frac{1}{2})$ would yield c as well. Hence, $v_i = (\frac{1}{2}, 1, 0)$ is undominated.

If there is some $i \in I_{abc}$ who votes $v_i = (1, 0, \frac{1}{2})$ (i.e. if $|I_{bac}| + |I_{bca}| + |I_{cab}| + |I_{cba}| - 1 > 0$), then $v_i = (1, 0, \frac{1}{2})$ is undominated by claim 1. Similarly, for each $j \in I_{bac} \cup I_{cab} \cup I_{cba}$, strategy v_j is undominated by claim 1.

Finally, assume that $|I_{bca}| > 0$ so that there is some $j \in I_{bca}$ who chooses $v_j = (0, \frac{1}{2}, 1)$. Then either $|I_{bca}| > 1$, or $|I_{bca}| = 1$ and $|I_{bac}| + |I_{cab}| + |I_{cba}| > 0$, so that in either case $|I_{bac}| + |I_{bca}| + |I_{cab}| + |I_{cba}| - 1 > 0$ and $v_i = (1, 0, \frac{1}{2})$ is undominated for $i \in I_{abc}$. Then, letting voter $i \in I_{abc}$ who chooses $v_i = (1, \frac{1}{2}, 0)$ switch to $\tilde{v}_i = (1, 0, \frac{1}{2})$ yields $|\tilde{v}^a| = |v^a|$, $|\tilde{v}^b| = |v^b| - \frac{1}{2}$ and $|\tilde{v}^c| = |v^c| + \frac{1}{2}$, so that $j \in I_{bca}$'s second most preferred candidate c wins. A switch by j to $(0, 1, \frac{1}{2})$ would again equalize candidates' scores and render j's least preferred candidate a a possible outcome. A switch to $(\frac{1}{2}, 1, 0)$ would even yield a independent of $\triangleright$. Hence, $v_j = (0, \frac{1}{2}, 1)$ is undominated.

Case 3: $|I_{abc}| = |O_{abc}| - 2 = |I_{abc}| + |I_{bac}| + 2|I_{bca}| + 2|I_{cab}| + 2|I_{cba}| - 2$

Assume first that $|I_{acb}| + |I_{bca}| + |I_{cab}| + |I_{cba}| \geq 2$ and $|I_{bac}| + |I_{bca}| + |I_{cab}| + |I_{cba}| \geq 2$. We construct a ballot profile v for which each candidate is a possible outcome as follows:

- 2 of I_{abc} chooses $v_j = (1, \frac{1}{2}, 0)$
- $|I_{acb}| + |I_{bca}| + |I_{cab}| + |I_{cba}| - 2$ of I_{abc} choose $v_j = (\frac{1}{2}, 1, 0)$
- $|I_{bac}| + |I_{bca}| + |I_{cab}| + |I_{cba}| - 2$ of I_{abc} choose $v_j = (1, 0, \frac{1}{2})$
- all $j \in I_{bac}$ choose $v_j = (0, 1, \frac{1}{2})$,
- all $j \in I_{bca} \cup I_{cab} \cup I_{cba}$ choose $v_j = (0, \frac{1}{2}, 1)$.

Then,

$$|v^a| = |v^b| = |v^c| = |I_{bac}| + 3/2|I_{bca}| + 3/2|I_{cab}| + 3/2|I_{cba}| - 1,$$

and any candidate may win, depending on $\triangleright$.

In light of claim 1, we only need to check the undominatedness of strategies $v_i = (\frac{1}{2}, 1, 0)$, $i \in I_{abc}$, and $v_i = (0, \frac{1}{2}, 1)$, $i \in I_{bca}$. For that, note that if $c \triangleright a, b$, then outcome c is realized. A switch by some $i \in I_{abc}$ with $v_i = (1, \frac{1}{2}, 0)$ to $\tilde{v}_i = (\frac{1}{2}, 1, 0)$

would yield b, while a switch to $\tilde{v}_i = (1, 0, \frac{1}{2})$ would yield c as well. Hence, $v_i = (\frac{1}{2}, 1, 0)$ is undominated.

If there is some $i \in I_{bca}$ who votes $v_i = (0, \frac{1}{2}, 1)$, let her switch to $\tilde{v}_i = (0, 1, \frac{1}{2})$. In addition, let some $j \in I_{abc}$ switch from $v_j = (1, \frac{1}{2}, 0)$ to $\tilde{v}_j = (1, 0, \frac{1}{2})$. Then $|\tilde{v}^a| = |\tilde{v}^b| = |\tilde{v}^c|$ and by claim 1, both $\tilde{v}_i$ and $\tilde{v}_j$ are undominated in any game $\Gamma(\succ_i, V')$ where $\tilde{v} \in V'$. Moreover, for ballot profile $\tilde{v}$, if $a \rhd b, c$, then i_{bca}'s least preferred candidate a is elected. Only a switch to $v_i = (0, \frac{1}{2}, 1)$ can prevent this and yields c. Hence $v_i = (0, \frac{1}{2}, 1)$ is undominated.

Now, assume that $|I_{acb}| + |I_{bca}| + |I_{cab}| + |I_{cba}| < 2$ or $|I_{bac}| + |I_{bca}| + |I_{cab}| + |I_{cba}| < 2$. This can be split up further as follows:

(1) $|I_{bca}| + |I_{cab}| + |I_{cba}| = 0$ and $|I_{acb}| < 2$

(2) $|I_{bca}| + |I_{cab}| + |I_{cba}| = 0$ and $|I_{bac}| < 2$

(3) $|I_{bca}| + |I_{cab}| + |I_{cba}| = 1$ and $|I_{acb}| = 0$ and $|I_{bac}| = 0$

(4) $|I_{bca}| + |I_{cab}| + |I_{cba}| = 1$ and $|I_{acb}| = 0$ and $|I_{bac}| > 0$

(5) $|I_{bca}| + |I_{cab}| + |I_{cba}| = 1$ and $|I_{acb}| > 0$ and $|I_{bac}| = 0$

Consider (1): Since $|I_{bca}| + |I_{cab}| + |I_{cba}| = 0$, it follows

$$|I_{abc}| = |I_{acb}| + |I_{bac}| - 2 \le |I_{acb}| + (|I_{abc}| + 2|I_{acb}| - 2) - 2 = |I_{abc}| + 3|I_{acb}| - 4$$

which implies $|I_{acb}| \ge 2$ – a contradiction.

Consider (2): Since $|I_{bca}| + |I_{cab}| + |I_{cba}| = 0$, it follows

$$|I_{abc}| = |I_{acb}| + |I_{bac}| - 2 \le \underbrace{(|I_{abc}| + 2|I_{bac}| - 2)}_{\ge |I_{acb}|} + |I_{bac}| - 2 = |I_{abc}| + 3|I_{bac}| - 4$$

which implies $|I_{bac}| \ge 2$ – a contradiction.

Consider (3): Then $|I_{abc}| = |I_{acb}| + |I_{bac}| + 2(|I_{bca}| + |I_{cab}| + |I_{cba}|) - 2 = 0$ so that I consists of a single voter $i \in I_{bac} \cup I_{cab} \cup I_{cba}$ – a contradiction to the assumptions of Theorem 11.

Consider (4): Then $|I_{abc}| = |I_{bac}| > 0$. Moreover $|I_{bca}| = 0$ as otherwise $|I_{bac}| - |O_{bac}| = |I_{bac}| - |I_{bca}| - |I_{abc}| = -1 > -2 = |I_{abc}| - |I_{bac}| - 2|I_{bca}| = |I_{abc}| - |O_{bca}|$. Construct ballot profile v as follows.

- some $j \in I_{abc}$ chooses $v_j = (1, \frac{1}{2}, 0)$
- remaining $j \in I_{abc}$ choose $v_j = (1, 0, \frac{1}{2})$
- all $j \in I_{bac}$ choose $v_j = (0, 1, \frac{1}{2})$
- $j \in I_{cab} \cup I_{cba}$ chooses $v_j = (\frac{1}{2}, 0, 1)$

Then $|v^a| = |v^b| = |v^c| = |I_{abc}| + \frac{1}{2} = |I_{bac}| + \frac{1}{2}$ and each strategy v_j is undominated by Claim 1.

Consider (5): Then $|I_{abc}| = |I_{acb}| > 0$. Moreover $|I_{cab}| = 0$ as otherwise $|I_{acb}| - |O_{acb}| = |I_{acb}| - |I_{abc}| - |I_{cab}| = -1 > -2 = |I_{abc}| - |I_{bac}| - 2|I_{bca}| = |I_{abc}| - |O_{bca}|$. We will construct tree strategy profiles v, $\tilde{v}$ and $\hat{v}$ and show that each strategy used in the construction is undominated in any game $\Gamma(\succ_I, V')$ where $v, \tilde{v}, \hat{v} \in V'$. First construct ballot profile v as follows.

- some $j \in I_{abc}$ chooses $v_j = (1, \frac{1}{2}, 0)$
- all remaining $j \in I_{abc}$ choose $v_j = (\frac{1}{2}, 1, 0)$
- all $j \in I_{acb}$ choose $v_j = (1, \frac{1}{2}, 0)$
- $j \in I_{bca} \cup I_{cba}$ chooses $v_j = (0, 1, \frac{1}{2})$

Then $|v^a| = {}^3\!/\!{}_2 |I_{abc}| + \frac{1}{2}$, $|v^b| = {}^3\!/\!{}_2 |I_{abc}| + \frac{1}{2}$ and $|v^c| = \frac{1}{2}$ and if $b \rhd a$, b is chosen. A voter $j \in I_{abc} \cup I_{acb}$ who votes $v_j = (1, \frac{1}{2}, 0)$ could change the outcome to a by switching to $(1, 0, \frac{1}{2})$, but not by switching to any other strategy. Hence for $j \in I_{abc} \cup I_{acb}$, $(1, 0, \frac{1}{2})$ is undominated. If voter $j \in I_{bca} \cup I_{cba}$ would switch to any other strategy, the outcome would also be a, so that for her $(0, 1, \frac{1}{2})$ is established to be undominated.

Next construct ballot profile $\tilde{v}$ as follows.

- all $j \in I_{abc}$ choose $\tilde{v}_j = (1, 0, \frac{1}{2})$
- some $j \in I_{acb}$ chooses $\tilde{v}_j = (1, 0, \frac{1}{2})$
- all remaining $j \in I_{acb}$ choose $\tilde{v}_j = (\frac{1}{2}, 0, 1)$
- $j \in I_{bca} \cup I_{cba}$ chooses $\tilde{v}_j = (0, \frac{1}{2}, 1)$

Then $|\tilde{v}^a| = {}^3\!/\!{}_2 |I_{abc}| + \frac{1}{2}$, $|\tilde{v}^b| = \frac{1}{2}$ and $|\tilde{v}^c| = {}^3\!/\!{}_2 |I_{abc}| + \frac{1}{2}$ and if $c \rhd a$, c is chosen. A voter $j \in I_{abc} \cup I_{acb}$ who votes $\tilde{v}_j = (1, 0, \frac{1}{2})$ could change the outcome to a by switching to $(1, \frac{1}{2}, 0)$, but not by switching to any other strategy. Hence for $j \in I_{abc} \cup I_{acb}$, $(1, \frac{1}{2}, 0)$ is undominated. If voter $j \in I_{bca} \cup I_{cba}$ would switch to any other strategy, the outcome would also be a, so that for her $(0, \frac{1}{2}, 1)$ is established to be undominated.

Finally construct ballot profile $\hat{v}$ as follows.

- one $j \in I_{abc}$ chooses $\hat{v}_j = (1, \frac{1}{2}, 0)$
- all remaining $j \in I_{abc}$ choose $\hat{v}_j = (\frac{1}{2}, 1, 0)$
- all $j \in I_{acb}$ choose $\hat{v}_j = (\frac{1}{2}, 0, 1)$
- $j \in I_{bca} \cup I_{cba}$ chooses $\hat{v}_j = (0, 1, \frac{1}{2})$

Then $|\hat{v}^a| = |\hat{v}^b| = |\hat{v}^c| = |I_{abc}| + \frac{1}{2}$, so that for $c \rhd a, b$, outcome c is realized. A switch by $j \in I_{abc}$ to $(1, 0, \frac{1}{2})$ would also yield c, but a switch to $(\frac{1}{2}, 1, 0)$ yields b. Hence, for $j \in I_{abc}$, $(\frac{1}{2}, 1, 0)$ is undominated. A construction symmetric to $\hat{v}$ shows that for $j \in I_{abc}$, $(\frac{1}{2}, 0, 1)$ is undominated, which completes the proof for Case 3.

Case 4: $|I_{abc}| < |O_{abc}| - 2$

Then, $|I_{xyz}| < |O_{xyz}| - 2$ for all $x, y, z \in A$ and Lemma 5 completes the proof.

□

Proof of Theorem 12. We first consider positional scoring rules with $s < \frac{1}{2}$ and show that for any fixed s, there exist preference profiles with $I = I_{abc} \cup I_{acb}$, where the induced voting game fails to elect a after iterated elimination of dominated strategies.

Assume that $|I_{abc}| = |I_{acb}| = n$ with $n > \frac{2-2s}{1-2s} \geq 2$. We will show that the ballot profile v, given by $v_{i_{abc}} = (s, 1, 0)$ and $v_{i_{acb}} = (s, 0, 1)$ respectively, survives the iterative elimination of dominated strategies.

Consider $\Gamma(\succ_I, V^1)$ and assume that all voters $i \in I_{abc}$ chose $v_i = (s, 1, 0)$ while voters $i \in I_{acb}$ chose $v_i = (s, 0, 1)$. Then $|v^b| = |v^c| = n$ while $|v^b| - |v^a| = |v^c| - |v^a| = n - 2ns = n(1 - 2s) > 2 - 2s > 1$. Thus, the winner is either b or c, depending on $\rhd$. If i_{abc} would switch to a different strategy, $(1, s, 0), (1, 0, s) \in V^1_{i_{abc}}$ that awards fewer points to candidate b, c would win the election independent of $\rhd$. Hence, neither $(1, s, 0)$ nor $(1, 0, s)$ dominate $(s, 1, 0)$ for voter i_{abc}, so that $v_{i_{abc}} = (s, 1, 0) \in V^2_{i_{abc}}$.

A symmetric argument applies to i_{acb} for whom $v_i = (s, 0, 1) \in V^2_{acb}$. But then, we can again consider the ballot profile v in $\Gamma(\succ_I, V^2)$ and show that neither strategy is dominated and eliminated as we move to V^3. By induction it follows that the two strategies are never eliminated.

Moreover, we have already seen that for strategy profile v, candidate a does not win the election which concludes the proof for the case $s < \frac{1}{2}$.

Next, we consider the case of Antiplurality, i.e. $s = 1$. Assume that all voters agree on the ranking $a \succ_i b \succ_i c$, so that $V^1_i = \{(1, 1, 0), (1, 0, 1)\}$. If in $\Gamma(\succ_I, V^1)$ all voters $j \neq i$ chose $v_j = (1, 1, 0)$, then i can ensure the election of a by casting the ballot $v_i = (1, 0, 1)$, whereas $v'_i = (1, 1, 0)$ would lead to the election of b whenever $b \rhd a$. Hence, $(1, 0, 1)$ is not dominated. Similarly, if all $j \neq i$ cast ballot $v_j = (1, 0, 1)$ and the tiebreaker chooses $b \rhd a$, i's unique best reply is $v_i = (1, 1, 0)$. But then, voters' strategy sets cannot be narrowed down any further than $V^1_i = \{(1, 1, 0), (1, 0, 1)\}$, so that a is not the unique solution in iteratively undominated strategies.

Last, consider the case $s \in (\frac{1}{2}, 1)$. Assume that $I = I_{abc} \cup I_{bac}$ and $|I_{abc}| = n + 1 > n = |I_{bac}|$ with $n > \frac{2}{(2s-1)(1-s)} > 2$, so that in particular $2sn - n > 2$ and

$$(\star\star) \qquad s > \frac{n+2}{2n} \qquad \text{and} \qquad n > \frac{n+2}{2s}.$$

We will show that in the process of iterative elimination, strategies $(1, s, 0), (1, 0, s) \in V^1_{abc}$ and $(s, 1, 0), (0, 1, s) \in V^1_{bac}$ are never weakly dominated and hence not eliminated. But then, b remains a possible outcome throughout the sequence of restricted games: if all i_{abc} vote $(1,s,0)$ while all i_{bac} vote $(0,1,s)$, candidates scores are

$$|v^a| = n+1, \qquad |v^b| = n + s(n+1), \qquad |v^c| = sn.$$

As $s > \frac{1}{2}$ and $n > 1$, candidate b then wins the election.

First, let us remind ourselves that the sets of undominated strategies are

$$V^1_{abc} = \{(1,s,0),(1,0,s),(s,1,0)\} \text{ and } V^1_{bac} = \{(s,1,0),(0,1,s),(1,s,0)\}.$$

To show that $\{(1,s,0),(1,0,s)\} \subseteq V^{m+1}_{abc} \subseteq V^m_{abc}$ and $\{(s,1,0),(0,1,s)\} \subseteq V^{m+1} \subseteq V^m_{bac}$ for all $m \geq 1$ we consider 6 cases.

Case 1: *For $i \in I_{abc}$, $(1,s,0)$ can be a better reply than $(1,0,s)$ in $\Gamma_s(\succ_I, V^m)$.* Consider the situation of $i \in I_{abc}$ who faces an opposing strategy profile where

- $n-x$ voters $j \in I_{abc}$ vote $v_j = (1,s,0)$,
- x voters $j \in I_{abc}$ vote $v_j = (1,0,s)$,
- all n voters $j \in I_{bac}$ vote $v_j = (0,1,s)$,
- $x = \left\lceil \frac{n}{2s} - \frac{1}{2} \right\rceil$,
- $c \rhd b$.

This profile is well defined, as

$$x = \left\lceil \frac{n}{2s} - \frac{1}{2} \right\rceil < \frac{n}{2s} + \frac{1}{2} < \frac{n}{2s} + \frac{1}{2s} < n+1.$$

If i chooses $v_i = (1,s,0)$, the associated candidates' scores are $|v^a| = n+1$, $|v^b| = s(n-x+1)+n$ and $|v^c| = s(n+x)$. Then, b wins as its score is larger than c's

$$|v^b| - |v^c| = n + s - 2sx > n + s - 2s\left(\frac{n}{2s} + \frac{1}{2}\right) = 0,$$

while c's score is larger than a's:

$$|v^a| - |v^c| = n + 1 - sn - sx \leq n + 1 - sn - s\left(\frac{n}{2s} - \frac{1}{2}\right) = \underbrace{\frac{n+1+s}{2}}_{<sn,\text{ see } (\star\star)} - sn < 0.$$

If on the other hand i chooses $v_i = (1,0,s)$, b's score is at most as high as c's, so that b never wins (ties are broken in favour of c):

$$|v^b| - |v^c| = n - s - 2sx \leq n - s - 2s\left(\frac{n}{2s} - \frac{1}{2}\right) = 0.$$

Instead, c would win as its score has increased an hence is still larger than a's.

Case 2: *For $i \in I_{abc}$, $(1, s, 0)$ can be a better reply than $(s, 1, 0)$ in $\Gamma(\succ_I, V^m)$.* (This case is only relevant if $(s, 1, 0) \in V^m_{abc}$). Consider the situation of $i \in I_{abc}$ who faces an opposing strategy profile where

- n voters $j \in I_{abc}$ vote $v_j = (1, s, 0)$,
- n voters $j \in I_{bac}$ vote $v_j = (s, 1, 0)$.

Ballot $v_i = (1, s, 0)$ would elect a, whereas $(s, 1, 0)$ would elect b.

Together, case 1 and 2 imply that $(1, s, 0) \in V^{m+1}_{abc}$. Next, we show that $(1, 0, s) \in V^{m+1}_{abc}$.

Case 3: *For $i \in I_{abc}$, $(1, 0, s)$ can be the unique best reply in $\Gamma(\succ_I, V^m)$:* Consider the situation of $i \in I_{abc}$ who faces an opposing strategy profile where

- n voters $j \in I_{abc}$ vote $v_j = (1, s, 0)$,
- 1 voter $j \in I_{bac}$ vote $v_j = (0, 1, s)$,
- $n - 1$ voters $j \in I_{bac}$ vote $v_j = (s, 1, 0)$.

Ballot $v_i = (1, 0, s)$ would then elect a, as $|v^a| - |v^b| = 1 - s > 0$. Should i choose $(1, s, 0)$, b would be elected as we would have $|v^a| - |v^b| = 1 - 2s < 0$. Ballot $v_i = (s, 1, 0)$ would only further increase b's lead over a.

Case 4: *For $i \in I_{bac}$, $(0, 1, s)$ can be the unique best reply in $\Gamma(\succ_I, V^m)$:* Consider the situation of $i \in I_{bac}$ who faces an opposing strategy profile where

- $n + 1$ voters $j \in I_{abc}$ vote $v_j = (1, s, 0)$,
- $n - 1$ voters $j \in I_{bac}$ vote $v_j = (s, 1, 0)$.

Ballot $v_i = (0, 1, s)$ would then elect b, as $|v^a| - |v^b| = 1 - 2s < 0$. Should i choose $(s, 1, 0)$, a would be elected, as we would have $|v^a| - |v^b| = 1 - s > 0$. Ballot $v_i = (1, s, 0)$ would only further increase a's lead over b.

From case 4, we learn that $(0, 1, s) \in V^{m+1}_{bac}$. The last two cases establish that $(s, 1, 0) \in V^{m+1}_{bac}$, which concludes the proof.

Case 5: *For $i \in I_{bac}$, $(s, 1, 0)$ can be a better reply than $(1, s, 0)$ in $\Gamma(\succ_I, V^m)$:* (This case is only relevant if $(1, s, 0) \in V^m_{i_{bac}}$). Consider the situation of $i \in I_{bac}$ who faces an opposing strategy profile where

- $n + 1$ voters $j \in I_{abc}$ vote $v_j = (1, s, 0)$,
- x voters $j \in I_{bac}$ vote $v_j = (1, s, 0)$,
- 1 voters $j \in I_{bac}$ vote $v_j = (0, 1, s)$,
- $n - 2 - x$ voters $j \in I_{bac}$ vote $v_j = (s, 1, 0)$,
- $x = \left\lceil \frac{4s-3}{2-2s} \right\rceil \geq 0$,

- $a \rhd b$.

This profile is well defined, as $x < n - 2$:

$$x = \left\lceil \frac{4s-3}{2-2s} \right\rceil < \frac{4s-3}{2-2s} + 1 = \frac{2s-1}{2-2s} < \frac{2s-1}{1-s} = \frac{1}{1-s} - 2 < n - 2.$$

If i chooses $v_i = (s, 1, 0)$, b wins the election as its score is larger than $|v^c| = s$ and larger than a's score $|v^a|$:

$$|v^a| - |v^b| = 1 - 2s + (2-2s)x < 1 - 2s + (2-2s)\left(\frac{4s-3}{2-2s} + 1\right) = 1 - 2s + 4s - 3 + 2 - 2s = 0.$$

If on the other hand, i chooses $v_i = (1, s, 0)$, b's score is weakly less than a's:

$$|v^a| - |v^b| = 3 - 4s + (2-2s)x \geq 3 - 4s + (2-2s)\left(\frac{4s-3}{2-2s}\right) = 0.$$

As ties are broken in favour of a, b would lose the election.

Case 6: *For $i \in I_{bac}$, $(s,1,0)$ can be a better reply than $(0,1,s)$ in $\Gamma(\succ_I, V^m)$:* Consider the situation of $i \in I_{bac}$ who faces an opposing strategy profile where

- $n+1$ voters $j \in I_{abc}$ vote $v_j = (1, 0, s)$,
- $n - x - 1$ voters $j \in I_{bac}$ vote $v_j = (s, 1, 0)$,
- x voters $j \in I_{bac}$ vote $v_j = (0, 1, s)$,
- $x = \left\lceil \frac{n+1}{2s} - \frac{3}{2} \right\rceil$,
- $c \rhd a$.

This profile is well defined, since

$$x = \left\lceil \frac{n+1}{2s} - \frac{3}{2} \right\rceil < \frac{n+1}{2s} - \frac{1}{2} = \frac{n+2}{2s} - \frac{1+s}{2s} \overset{(\star\star)}{<} n - \frac{1+s}{2s} < n$$

and

$$x \geq \frac{\overbrace{n+1}^{>3}}{\underbrace{2s}_{<2}} - \frac{3}{2} > 0.$$

If i chooses $v_i = (s, 1, 0)$, the associated candidates' scores are $|v^a| = n + 1 + s(n-x)$, $|v^b| = n$ and $|v^c| = s(n+1+x)$. Hence, a is elected with a higher score than b and c:

$$|v^a| - |v^c| = n + 1 - s - 2sx > n + 1 - s - 2s\left(\frac{n+1}{2s} - \frac{1}{2}\right) = 0.$$

If on the other hand, i chooses $v_i = (0, 1, s)$, a's score is weakly less than c's:

$$|v^a| - |v^c| = n + 1 - 3s - 2sx \leq n + 1 - 3s - 2s\left(\frac{(n+1)}{2s} - \frac{3}{2}\right) = 0.$$

As ties are broken in favour of c, and $|v^c| \geq |v^a| \geq n+1 > n = |v^b|$, c would be elected. □

Proof of Theorem 13. Let us first analyse scoring rules where V consist of all permutations of $(1,1,0)$ and $(s,s,0)$. For that, consider a preference profile where all voters share the same preferences, $a \succ_i b \succ_i c$.

Claim 1. *For all i, if V_i^m includes at least one of the two ballots $(1,0,1)$ or $(s,0,s)$ as well at least one of the two ballots $(1,1,0)$ or $(s,s,0)$ then after eliminating strategies that are dominated in the game $\Gamma(\succ_I, V^m)$, V_i^{m+1} will contain at least one of the two ballots $(1,0,1)$ or $(s,0,s)$ as well at least one of the two ballots $(1,1,0)$ or $(s,s,0)$.*

Proof. In the game $\Gamma(\succ_I, V^m)$, consider the ballot profile v where all voters choose either $(1,0,1)$ or $(s,0,s)$ so that $|v^a| = |v^c| > |v^b|$ and c is elected if $c \triangleright a \triangleright b$. If an individual voter i switches to $\tilde{v}_i = (1,1,0)$ or $\tilde{v}_i = (s,s,0)$, the outcome is a. If instead she would switch to $(0,1,1)$ or $(0,s,s)$ (provided that these are still included in V_i^m), the outcome would be c as well. Hence, at least one of the ballots $(1,1,0)$, $(s,s,0)$ is undominated and included in V_i^{m+1}.

Analogously, consider the ballot profile v where all voters choose either $(1,1,0)$ or $(s,s,0)$ so that $|v^a| = |v^b| > |v^c|$ and b is elected if $b \triangleright a \triangleright c$. If an individual voter i switches to $\tilde{v}_i = (1,0,1)$ or $\tilde{v}_i = (s,0,s)$, the outcome is a. If instead she would switch to $(0,1,1)$ or $(0,s,s)$ (provided that these are still included in V_i^m), the outcome would be b as well. Hence, at least one of the ballots $(1,0,1)$, $(s,0,s)$ is undominated and included in V_i^{m+1}. *Claim 1* ◇

Since initially they are included in the set of admissible ballots, at least on of $(1,1,0)$ and $(s,s,0)$ survives the process of iterative elimination of dominated strategies. Then, in the game $\Gamma(\succ_I, V^{\overline{m}})$, if all voters choose either $(1,1,0)$ or $(s,s,0)$, b is a possible outcome and hence included in $S(\succ_I, V)$. Thus, such a scoring rule violates **MEW** (as well as **U**).

Next, let us analyse scoring rules where V consist of all permutations of $(1,0,0)$ and $(s,0,0)$. If $s = 1$, the rule is the Plurality rule, for which we know by Theorem 12 that it violates **U**. If $s < 1$, consider a preference profile such that $I = I_{abc} \cup I_{acb}$ and $|I_{abc}| = |I_{acb}| > 1$.

Claim 2. *If V_{abc}^m includes $(0,1,0)$ while V_{acb}^m includes $(0,0,1)$, then both strategies are undominated in the game $\Gamma(\succ_I, V^m)$ and V_{abc}^{m+1} includes $(0,1,0)$ while V_{acb}^{m+1} includes $(0,0,1)$.*

Proof. In the game $\Gamma(\succ_I, V^m)$, consider the ballot profile v where all voters $i \in I_{abc}$ choose $(0,1,0)$ while all $i \in I_{acb}$ choose $(0,0,1)$, so that $|v^b| = |v^c| > |v^a| + 1$. For $b \triangleright c$, the outcome is b. If an individual voter $i \in I_{abc}$ switches to $(1,0,0)$, $(0,0,1)$, $(s,0,0)$, $(0,s,0)$ or $(0,0,s)$ the outcome is c, as it has the highest score. Hence, $v_i = (0,1,0)$ is undominated and included in V_{abc}^{m+1}. By a symmetric argument, $(0,0,1)$ is included in V_{acb}^{m+1}. *Claim 2* ◇

By induction, we know that $(0,1,0) \in V^{\overline{m}}_{abc}$ and $(0,0,1) \in V^{\overline{m}}_{acb}$. Then, in the game $\Gamma(\succ_I, V^{\overline{m}})$, all voters $i \in I_{abc}$ choose $(0,1,0)$ while all $i \in I_{acb}$ choose $(0,0,1)$, the outcome is either b or c. Thus, the scoring rule violates **U**.

Now, let us consider vote-splitting scoring rules, i.e. scoring rules where V consists of all permutations of $(s,s,0)$ and $(1-s,0,0)$. We want to show that such a rule violates unanimity if $s < \frac{1}{2}$. For that, consider a profile such that $I = I_{abc} \cup I_{acb}$ and $|I_{abc}| = |I_{acb}| > 1$.

Claim 3. *If V^m_{abc} includes $(0,1-s,0)$ while V^m_{acb} includes $(0,0,1-s)$, then both strategies are undominated in the game $\Gamma(\succ_I, V^m)$ and V^{m+1}_{abc} includes $(0,1-s,0)$ while V^{m+1}_{acb} includes $(0,0,1)$.*

Proof. In the game $\Gamma(\succ_I, V^m)$, consider the ballot profile v where all voters $i \in I_{abc}$ choose $(0,1-s,0)$ while all $i \in I_{acb}$ choose $(0,0,1-s)$, so that $|v^b| = |v^c| > 1$ while $|v^a| = 0$. For $b \rhd c$, the outcome is b. If an individual voter $i \in I_{abc}$ switches to $(1-s,0,0)$, $(0,0,1-s)$, $(s,s,0)$, $(0,s,s)$ or $(s,0,s)$ the outcome is c, as it has the highest score. Hence, $v_i = (0,1-s,0)$ is undominated and included in V^{m+1}_{abc}. By a symmetric argument, $(0,0,1-s)$ is included in V^{m+1}_{acb}. *Claim 3* ◇

By induction, we know that $(0,1-s,0) \in V^{\overline{m}}_{abc}$ and $(0,0,1-s) \in V^{\overline{m}}_{acb}$. Then, in the game $\Gamma(\succ_I, V^{\overline{m}})$, all voters $i \in I_{abc}$ choose $(0,1-s,0)$ while all $i \in I_{acb}$ choose $(0,0,1-s)$, the outcome is either b or c. Thus, the scoring rule violates **U**.

Finally, we want to show that a vote-splitting scoring rule violates **MEW** if $s \in (\frac{1}{2}, 1)$ ($s = 1$ corresponds to the Antiplurality Rule, for which we know from Theorem 12 that it violates **MEW**). For that, consider a profile such that $I = I_{abc} \cup I_{bac}$, $|I_{abc}| = n+1$ and $|I_{bac}| = n > \frac{1}{(1-s)(2s-1)}$. We will show that strategies $(s,s,0), (s,0,s), (1-s,0,0) \in V^m_{i_{abc}}$ and $(0,s,s), (0,1-s,0) \in V^m_{i_{bac}}$ are undominated in $\Gamma(\succ_I, V^m)$ and hence included in $V^{m+1}_{i_{abc}}$ and $V^{m+1}_{i_{bac}}$ respectively.

(i) *For $i \in I_{abc}$, $(s,s,0)$ is undominated in $\Gamma(\succ_I, V^m)$.*

Consider the the ballot profile v where

- i votes $v_i = (s,s,0)$
- one $j \in I_{abc}$ votes $v_j = (s,0,s)$
- remaining $n-1$ of I_{abc} vote $v_j = (1-s,0,0)$
- all n of I_{bac} vote $v_j = (0,s,s)$

Then $|v^b| = |v^c| = s(n+1)$ and $|v^a| = 2s + (1-s)(n-1)$, so that

$$|v^a| - |v^b| = -sn + 3s + n - 1 - sn - s = (1-n)(2s-1) < 0$$

and b is elected for $b \rhd c$. A switch by i to any other ballot $\tilde{v}_i \in V$ would never raise the score of a and would either reduce the score of b or increase the score of c, thereby changing the outcome to c. Hence $(s,s,0)$ is undominated.

(ii) *For $i \in I_{abc}$, $(s,0,s)$ is undominated.*

Consider the the ballot profile v where

- i votes $v_i = (s, 0, s)$
- n of I_{abc} vote $v_j = (1 - s, 0, 0)$
- $n - \lfloor \frac{s}{2s-1} \rfloor$ of I_{bac} vote $v_j = (0, 1 - s, 0)$
- $\lfloor \frac{s}{2s-1} \rfloor$ of I_{bac} vote $v_j = (0, s, s)$

Then
$$|v^a| - |v^b| = s + (1 - 2s) \left\lfloor \frac{s}{2s-1} \right\rfloor \geq s + (1 - 2s) \frac{s}{2s-1} = 0$$
and
$$|v^a| - |v^b| = s + (1 - 2s) \left\lfloor \frac{s}{2s-1} \right\rfloor < s + (1 - 2s) \left(\frac{s}{2s-1} - 1 \right) = 2s - 1.$$
Moreover, $|v^a| - |v^c| = n(1 - s) - s \lfloor \frac{s}{2s-1} \rfloor > \frac{1}{2s-1} - \frac{s^2}{2s-1} > 0$ so that and a is elected for $a \rhd b$. A switch by i to ballot $(1 - s, 0, 0)$ would change the score difference $|v^a| - |v^b|$ by $-s + (1 - s) = 1 - 2s$ so that b overtakes a. As any other ballot would change the difference $|v^a| - |v^b|$ even more in b's favour, we conclude that $(s, 0, s)$ is undominated.

(iii) *For $i \in I_{abc}$, $(1 - s, 0, 0)$ is not dominated by $(s, 0, s)$ or $(0, 0, 1 - s)$.*

Consider the ballot profile v where all $j \in I_{abc}$ vote $v_j = (1-s, 0, 0)$ while all $j \in I_{bac}$ vote $v_j = (0, s, s)$. Then $|v^b| = |v^c| = sn$ which is larger than $|v^a| = (1 - s)(1 + n)$as n is large. Then for $b \rhd c$, b is elected while a switch by i to $(s, 0, s)$ or $(0, 0, 1 - s)$ would yield c as outcome.

(iv) *For $i \in I_{abc}$, $(1 - s, 0, 0)$ is not dominated by $(s, s, 0)$, $(0, s, s)$, $(0, 1 - s, 0$ or $(0, 0, 1 - s)$.*

Consider the ballot profile v where all $j \in I_{abc}$ vote $v_j = (1 - s, 0, 0)$ while all $j \in I_{bac}$ vote $v_j = (0, 1 - s, 0)$. Then $|v^a| - |v^b| = 1 - s$ and $|v^c| = 0$ and a is elected. A switch by i to $(s, s, 0)$ or $(0, 0, s)$ would yield $|v^a| = |v^b|$, so that for $b \rhd a$, a would no longer be elected. Any other ballot would change the difference $|v^a| - |v^b|$ even more in b's favour, ruling out a as well.

(v) *For $i \in I_{bac}$, $(0, s, s)$ is undominated.*

Consider the ballot profile v where one $j \in I_{abc}$ votes $v_j = (s, s, 0)$ while n of I_{abc} vote $v_j = (1-s, 0, 0)$ and all $j \in I_{bac}$ votes $v_j = (0, 1-s, 0)$. Then $|v^a| = |v^b|$ and $|v^c| = 0$ so that for $a \rhd b$, b is elected. Then, for some i_bac, only a switch to $(0, s, s)$ would increase the difference $|v^b| - |v^a|$ and hence yield outcome b.

(vi) *For $i \in I_{bac}$, $(0, 1-s, 0)$ is undominated by $(s, s, 0)$ (only relevant if $(s, s, 0) \in V_i^m$).*

If $(s, s, 0) \in V_i^m$, consider the ballot profile v where every voter votes $(s, s, 0)$. Then if $a \rhd b$, candidate a is elected. A switch be i to $(0, 1 - s, 0)$ yields outcome b.

(vii) *For $i \in I_{bac}$, $(0, 1 - s, 0)$ is undominated by $(1 - s, 0, 0)$, $(1 - s, 0, 0)$, $(0, s, s)$ and $(s, 0, s)$.*

Consider the the ballot profile v where

- one $j \in I_{abc}$ votes $(s, 0, s)$,

- n of I_{abc} vote $(1-s, 0, 0)$,
- $\lceil \frac{s}{1-s} \rceil > 1$ of I_{bac} vote $(0, 1-s, 0)$,
- $n - \lceil \frac{s}{1-s} \rceil$ of I_{bac} vote $(0, s,)$.

Then

$$|v^b| - |v^c| = (1-s)\left\lceil \frac{s}{1-s} \right\rceil - s \in [0, 1-s)$$

and

$$|v^a| - |v^c| = n(1-s) - \left(n - \left\lceil \frac{s}{1-s} \right\rceil\right) s = n(1-2s) + s\left\lceil \frac{s}{1-s} \right\rceil < -\frac{1}{1-s} + \frac{s^2}{1-s} < 0$$

so the b is elected for $b \rhd c$. If i_{bac} switches from $(0, 1-s, 0)$ to either $(1-s, 0, 0)$, $(1-s, 0, 0)$, $(0, s, s)$ or $(s, 0, s)$, she would reduce the payoff difference $|v^b| - |v^c|$ by at least $1-s$, so that c's score would be higher than the score of b, ruling out b as an outcome.

Together, (i)-(vii) establish that for each $i \in I_{abc}$, $(s, s, 0) \in V_i^{\overline{m}}$ while for each $i \in I_{bac}$, $(0, s, s) \in V_i^{\overline{m}}$. But then b remains a possible outcome in the game $\Gamma(\succ_I, V^{\overline{m}})$, violating **MEW** which requires that a is the only remaining outcome after the iterative elimination of dominated strategies has run its course.

□