Competition for Resources

The Equilibrium Existence Problem in Congestion Games

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Contents

Chapter 1

Introduction

The word “resource” has its linguistic roots in the old French verb “resourdre” which, in turn, is based on the Latin verb “resurgere” expressing an action of recovery or rising [104, p. 1278]. Apparently, when first used in the early 17th century, the word referred to means of unlimited supply that inexhaustibly regenerate. In the past 400 years, the initial meaning of the word has turned quite into the reverse. Today, resources are seen as precious and valuable goods or means. In fact, modern economic theory relies on the assumption that resources are scarce. As put in Robbins [112], “economics is the science which studies human behavior as a relationship between ends and scarce means which have alternative uses”.

One of the earliest reference points of the mathematical treatment of scarcity is the work of Cournot [27] from the first half of the 19th century. He studied a situation in which two owners of mineral springs supply a single market with water. He made the observation that the selling price they can achieve depends on the total quantity offered. If the quantity is small compared to the demand, the resource water is scarce and its selling price is high, while in situations in which the supply is large only a small selling price can be achieved. In brief, the market clearing price is a non-increasing function of the total quantity offered. Given the quantity delivered by her opponent, each producer strives to maximize her revenue, which is defined as her production quantity multiplied by the achievable market price. The optimal production quantity of the first producer, in turn, affects the strategic choice of the production quantity of second producer, which again has influence on the choice of first producer, and so on. Cournot showed that following such trajectory of reasoning, one reaches production quantities that are *in equilibrium*, that is, no producer may increase her profit switching to a different production size while the other sticks with her offered quantity. Cournot presumed that the total state of an economy is determined by the stable points of such equilibrium problems.

A century later, von Neumann and Morgenstern [130, 131], laid the mathematical and conceptional foundations of the analysis of such strategic interactions between individuals with possibly conflicting interests. Their definition of a strategic game contains the essential ingredients of the Cournot model: a finite set of *players* (corresponding to the set of producers in Cournot’s model) with a private set of strategic choices, henceforth called *strategies*, (corresponding to the feasible production quantities of each producer in Cournot’s model) who strive to optimize a private objective function which depends on the combination of the strategic choices of all players. In his famous theorem, Nash [101, 102] showed that, for games in which each player has a finite set of strategies, there is always an equilibrium in *mixed strategies*, that is, an equilibrium in which each player plays according to a fixed probability distribution over her strategies and no player can improve her expected profit by unilaterally altering her probability distribution.

Despite its huge success and its enormous influence on economic theory, the concept of a Nash equilibrium has the drawback that for finite games Nash equilibria are only guaranteed to exist in mixed strategies. Mixed equilibria rely on the critical assumption that the players are *risk-neutral*, that is, they are only interested in maximizing their expected profit regardless of the underlying variances. This hypothesis has been refuted in experiments; see, e.g., the influential critique published by Allais [7]. Risk-neutrality, however, is necessary to obtain Nash's existence result. For risk-averse players, for instance, an equilibrium in mixed strategies need not exist; see Fiat and Papadimitriou [48]. For further critics of the mixed Nash equilibrium concept, see also the discussion by Osborne and Rubinstein in [107, §3.2].

One remedy to overcome these issues related to mixed Nash equilibria is to specifically study relevant classes of games that admit a Nash equilibrium in pure (deterministic) strategies – as, e.g., the Cournot model. Another class of games, called *unweighted congestion games*, that allows for pure Nash equilibria was introduced by Rosenthal [114]. In an unweighted congestion game, we are given a finite set of resources and a finite set of players. Each player is associated with several subsets of resources available to her. A strategy of each player is to choose one such subset of resources. The cost of each resource depends on the number of players choosing it, and each player strives to minimize the sum of the costs incurred on the chosen resources.

Unweighted congestion games provide an elegant model of competition for scarce resources and have a wide range of applications a few of which we mention here. Consider a finite population of consumers demanding for bundles of goods available at different markets. The wholesale price of each product on each market increases as the number of consumers demanding the product on that market increases. The strategic choice of each consumer is which markets to visit to procure her bundle of goods. The geographic locations of the consumers and the markets impose restrictions on the subsets of markets each player can visit. Naturally, every consumer aims to choose a feasible set of markets that minimizes the sum of the wholesale prices of all products bought. This situation can be modeled as a congestion game by introducing one resource for each good on each market and one player for each consumer.

Related to this interpretation are the animal experiments conducted by Milinsky [97] that gave practical foundation to the pure Nash equilibria in congestion games. He observed that Sticklebacks locate themselves around feeding patches so as to maximize their individual feeding rate given the location of their fellows. This situation can be interpreted as an unweighted congestion game in which the feeding patches are the resources whose attractiveness decreases with a growing number of adjacent fishes.

As their name suggests, unweighted congestion games also occur naturally in traffic networks. Here, the resources correspond to the edges of a road network. Each traffic participant chooses a collection of edges that forms a path from her origin to her destination so as to minimize her total travel time. The travel time on each street segment depends on the number of other traffic participants choosing the same road.

Unweighted congestion games can also be applied to data routing in telecommunication networks. Here, the resources are associated with the routers distributed in a network. Each player controls a certain amount of data that she wishes to be routed from a source to a destination node. The delay of each router increases as the amount of data routed over it increases. Thus, the players strive to choose a chain of routers so as to minimize the total delay experienced.

More generally speaking, the class of unweighted congestion games is narrow enough to guarantee the existence of a pure Nash equilibrium, but also large enough to capture the main aspects of many strategic interactions. Interestingly, out of the five finite games mentioned in the introductory chapter of the standard game theory textbook by Osborne and Rubinstein [107, Chapter 2], four are congestion games.[1] The only exception, the Matching-Pennies-Game (see also Example 2.16), does not possess a pure Nash equilibrium.

1.1 Weighted Congestion Games

Despite the wide range of applications that can be modeled as unweighted congestion games, many situations do not exhibit the property that players contribute on equal terms to the cost of the resources. In the examples seen so far, the consumers may have different demands for goods, fish may vary in their consumption of food, different vehicle categories may have a different impact on congestion, or sending rates in telecommunication networks may be heterogeneous.

All of the above strategic interactions are captured more realistically by *weighted congestion games*. In a weighted congestion game, each player has a strictly positive demand that she places on the chosen resources.[2] The cost of each resource is a function of the aggregated demand of all players using that resource. Thus, unweighted congestion games are a special case of weighted congestion games in which all players have unit demands. In contrast to unweighted congestion games, weighted congestion games may fail to admit a pure Nash equilibrium; see the counterexamples given by Fotakis et al. [51], Goemans et al. [59], and Libman and Orda [88]. On the positive side, it is known that for affine resource cost functions (Fotakis et al. [51]) or exponential resource cost functions (Panagopoulou and Spirakis [108], and Harks et al. [67]) a pure Nash equilibrium always exists.

These positive results establish the existence of a pure Nash equilibrium *independent* of the underlying structure of the game, i.e., independent of the number of players, the combinatorial structure of their strategies, and their demands. Such independence is desirable in most applications because the number of players and their types (expressed in terms of their demands and their strategies) are only known to the players and subject to frequent changes. In light of the positive results obtained for affine and exponential cost functions it is a natural open problem to decide which *maximal* sets of cost functions actually guarantee the existence of pure Nash equilibria. This is the main question addressed in Chapter 3.

[1] To see this, note that the games *Bach-or-Stravinsky* and *Mozart-or-Mahler*, the *Prisoner's Dilemma*, and the *Hawk-Dove-Game* are exact potential games and, thus, isomorphic to unweighted congestion games; see Monderer and Shapley [99].

[2] Note that the demand of each a player does *not* depend on the resource; this simplifying assumption will be dropped in the next section when introducing congestion games with resource-dependent demands.

Contributions of Chapter 3

To precisely capture the influence of the cost structure on the existence of equilibria in weighted congestion games, we introduce the notion of consistency. We say that a set $\mathcal{C}$ of cost functions is *consistent* for weighted congestion games if every weighted congestion game with the property that the cost function of each resource is contained in $\mathcal{C}$ possesses at least one pure Nash equilibrium.

As the main result of Chapter 3, we give a complete characterization of consistency for weighted congestion games. Specifically, we show that a set $\mathcal{C}$ of continuous cost functions is consistent for weighted congestion games if and only if at least one of the following two cases holds: *(i)* $\mathcal{C}$ only contains affine functions; *(ii)* $\mathcal{C}$ only contains exponential cost functions with the property that there is a constant $\phi \in \mathbb{R}$ and for each $c \in \mathcal{C}$ two constants $a_c, b_c \in \mathbb{R}$ such that $c(x) = a_c e^{\phi x} + b_c$ for all $x \geq 0$. The necessity of these conditions is even valid for games with three players. This implies that for every non-affine and non-exponential function c there is a three-player weighted congestion game where all resources have cost function c and that does not possess a pure Nash equilibrium.

Our second main result is a similar characterization for two-player weighted congestion games. Specifically, we show that a set $\mathcal{C}$ of continuous cost functions is consistent for two-player weighted congestion games if and only if $\mathcal{C}$ contains only monotonic functions and each two non-constant functions $c_1, c_2 \in \mathcal{C}$ are linear transformations of each other, that is, there are $a, b \in \mathbb{R}$ such that $c_1(x) = a\,c_2(x) + b$ for all $x \geq 0$.

We further show that these results essentially translate (under mild restrictions on the set of cost functions) to the special cases in which the resources are associated with the edges of a directed or undirected graph, players are associated with a source and a sink vertex, and the set of strategies of each player equals the set of all simple paths connecting that player's source and sink vertex.

Finally, we examine *singleton* weighted congestion games. In such a game, every strategy of each player contains a single resource only. We show that for two-player singleton weighted congestion games the set of monotonic functions is the unique maximal set of consistent cost functions. This result does not translate to three-player games. We provide an example of a three-player singleton weighted congestion game with monotonic (both non-increasing and non-decreasing) resource cost functions that does not admit a pure Nash equilibrium.

1.2 Congestion Games with Resource-Dependent Demands

In a weighted congestion game, each player has a unique strictly positive demand that she places on the resources contained in her strategy. Dropping the assumption that the demands are equal for all resources we obtain congestion games with resource-dependent demands.

As a natural generalization of weighted congestion games, congestion games with resource-dependent demands allow to model a much broader scope of applications. For illustration, reconsider the example in which a finite set of consumers visits local markets where the prices react on demands. Allowing for resource-dependent demands, we can model the most intuitive situation in which the players' have different demands for different goods. Congestion games with resource-dependent demands also provide a more general model of data routing in telecommunication networks. As different routers may have different policies on how the traffic is handled, it is a natural assumption that the actual workload (or demand) each user imposes depends on the identity of the router. Moreover, resource-dependent demands may be used to yield a much more accurate model of traffic networks that incorporates the fitness of different vehicles types to the physical properties of road segments, such as slopes, terrain, and so on.

As congestion games with resource-dependent demands contain weighted congestion games as a special case, necessary conditions for the consistency of cost functions persist, that is, any set of cost functions that is consistent for games with resource-dependent demands may only contain affine or exponential functions. In Chapter 4 we explore the equilibrium existence for congestion games with resource-dependent demands, asking which maximal sets of cost functions are in fact consistent.

Contributions of Chapter 4

We first remark that there are two natural ways of defining the players' private cost functions. In the first variant, which we call *proportional games*, the costs of the resources are interpreted as monetary per-unit costs. In this regime, it is natural to assume that each player incurs a cost equal the sum of the costs of the used resources multiplied with her respective demand. We also study a slightly different class of games, which we call *uniform* games. They differ from proportional games solely in the fact that in the definition of the players private costs, the costs of the resources are *not* multiplied with the player's demands. Such cost structure occurs when the resource costs are interpreted as latencies or travel times and thus are the same for each user, regardless of their demands.

The distinction between proportional and uniform games is immaterial for the existence of pure Nash equilibria in weighted congestion games, but matters for games with resource-dependent demands. To characterize the influence of the cost functions on the existence of pure Nash equilibria, we adapt the notion of consistency. A set $\mathcal{C}$ of cost functions is consistent for proportional congestion games with resource-dependent demands (respectively, consistent for uniform congestion games with resource-dependent demands) if each proportional game (respectively, uniform game) with cost functions in $\mathcal{C}$ possesses a pure Nash equilibrium. As our main result of Chapter 4, we give a complete characterization of consistency of cost functions for proportional and uniform congestion games with resource-dependent demands. For a set $\mathcal{C}$ of continuous cost functions, we show that $\mathcal{C}$ is consistent for proportional games if and only if $\mathcal{C}$ only contains affine functions. Furthermore, $\mathcal{C}$ is consistent for uniform games if and only if $\mathcal{C}$ only contains constant functions. This characterization is even valid for three-player games. Assuming that the cost functions are strictly positive, the above characterizations also translate to games in which the players' strategies correspond to directed or undirected paths in a network.

1.3 Congestion Games with Variable Demands

Most previous work on congestion games has the common feature that the players use the resources contained in their strategy with a fixed – resource-dependent or resource-independent – demand. Although these games capture the main features of many interesting applications, they do not take into account the elasticity of the demand due to price changes. Price elasticity of demand is an intrinsic property of many applications, such as consumer markets and the flow control problem in telecommunication networks. In the latter setting, the players strive to establish an unsplittable data stream in a network. The sending rate will be reduced if the latency increases and increased if the latency decreases.

To model elasticity of demands, we study congestion games with variable demands. We assume that each player is associated with an interval of feasible demands and a non-decreasing and concave utility function modeling the utility received from satisfying a certain demand. In each strategy profile, each player chooses both a feasible demand and exactly one feasible subset of resources. The private payoff of each player then is defined as the difference between the utility received from the chosen demand and the costs incurred on the used resources. Such payoff structure is also called *quasi-linear*; see Mas-Colell et al. [91, Chapter 3].

Contributions of Chapter 5

Our study focuses on the existence of pure Nash equilibria with respect to the cost functions on the resources. As in the previous chapter we distinguish between proportional and uniform games. In a proportional game, the private payoff of each player equals her utility received from satisfying her demand minus the product of the demand and the sum of the costs of the resources. Uniform games differ from proportional games in the fact that the resource costs are *not* multiplied with the demand.

Our main result of Chapter 5 is a complete characterization of consistency for congestion games with variable demands in the proportional and uniform cost model, respectively. Specifically, we prove that a set $\mathcal{C}$ of continuous and non-negative cost functions is consistent for proportional congestion games with variable demands if and only if at least one of the following two cases holds: *(i)* there are a constant $\phi > 0$ and for each $c \in \mathcal{C}$ a constant $a_c > 0$ such that $c(x) = a_c e^{\phi x}$ for all $x \geq 0$; *(ii)* for each $c \in \mathcal{C}$ there are constants $a_c > 0$ and $b_c \geq 0$ such that $c(x) = a_c x + b_c$ for all $x \geq 0$. In addition, we prove that $\mathcal{C}$ is consistent for uniform congestion games with variable demands if and only if (i) holds. As in the previous chapters, our result continues to hold for games in which the resources are associated with the edges of a graph and each player establishes a path from her source node to her target node.

1.4 Bottleneck Congestion Games

So far, we assumed that the players strive to minimize the *sum* of the costs on the chosen resources. In many scenarios, however, sum-objectives do not represent the players' incentives correctly. An important example of such a situation is data streaming in telecommunication networks. Here, the delay of a data stream is restricted by the available bandwidth of the links on the chosen path. The total delay experienced by a selfish user is closely related to the performance of the link with least bandwidth; see Banner and Orda [15], Cole et al. [26], Keshav et al. [79], and Qiu et al. [110]. To capture this situation more realistically, Banner and Orda [15] introduced *bottleneck congestion games*. They differ from weighted congestion games solely in the fact that in each strategy profile the private cost of each player is the maximum (instead of the sum) of the costs of all chosen resources. Banner and Orda [15] proved the existence of a pure Nash equilibrium for non-decreasing cost functions on the resources.

Contributions of Chapter 6

As the main result of Chapter 6, we generalize the existence result of Banner and Orda [15]. We weaken the assumptions on the cost functions, assuming that the costs of the resources may even depend on the *set* of players using it. This is more general than the demand-based model studied by Banner and Orda [15]. Even for these more general cost functions, we are able to prove the existence of a *strong equilibrium* for this class of bottleneck congestion games with set-dependent costs. Strong equilibria are a strengthening of the pure Nash equilibrium concept that is even resilient to coalition deviations that decrease the private costs of each of its members. Each strong equilibrium is a pure Nash equilibrium, but not conversely. As a byproduct of our analysis, we further obtain that bottleneck congestion games with set-dependent costs have the strong finite improvement property, that is, every sequence of coalitional deviations that decreases the private costs of each of its members, is finite.

We further study *splittable* bottleneck congestion games. In such a game, each player is associated with a strictly positive demand that she is allowed to split fractionally over the sets of resources available to her. For continuous and non-decreasing cost functions on the resources, we show that splittable bottleneck congestion games admit a strong equilibrium as well.

Contributions of Chapter 7

The existence of strong equilibria in bottleneck congestion games raises some important questions regarding the computability of equilibria in such games. While for unweighted congestion games with sum-objective the complexity of computing pure Nash equilibria is relatively well understood, the complexity status of computing equilibria under bottleneck-objectives remains open. In Section 7 we prove first results in this direction.

First, we propose a generic algorithm that computes a strong equilibrium for any unweighted bottleneck congestion game with non-decreasing costs. Our algorithm crucially relies on a *strategy packing oracle* that decides for a given vector of capacities on the resources whether there exists a strategy profile that obeys the capacity constraint on each resource, and outputs such a strategy profile if it exists. The running time of our algorithm is essentially determined by the running time of the oracle. This implies that the problem of computing a strong equilibrium in an unweighted bottleneck congestion game with non-decreasing costs can be reduced to solving the strategy packing problem. As a characterization, we also prove the reverse direction, i.e., solving a strategy packing problem is reducible to computing a strong equilibrium in an unweighted bottleneck congestion game with non-decreasing costs.

There are a number of important classes of bottleneck congestion games for which a strategy packing oracle can be implemented in polynomial time, including single-commodity networks, branchings, and matroids. In all these cases, a strong equilibrium can be determined efficiently using our generic algorithm. For general games, however, we show that the computation of a strong equilibrium is NP-hard. This holds even for two-commodity networks.

For unweighted bottleneck congestion games with single-commodity network or matroids strategies we present an interesting dichotomy. Although for both classes of games there exists an efficient algorithm calculating a strong equilibrium, the recognition of a strong equilibrium is co-NP-hard.

1.5 Organization of this Thesis

In Chapter 2 we give a short introduction into the most important concepts of game theory and formally introduce the variants of congestion games examined in this thesis. Section 2.4 contains an overview on known results on the existence and computability of equilibria in congestion games. In Chapters 3, 4, and 5 we explore the existence of pure Nash equilibria in weighted congestion games, congestion games with resource-dependent demands, and congestion games with variable demands, respectively. The two latter chapters use in part results from the preceding chapters so that we recommend the three chapters to be read in the predetermined order. Bottleneck congestion games are studied in Chapters 6 and 7. They can be read independently from Chapters 3 to 5. Chapter 8 concludes.

In order to make Chapters 3 to 7 as self-contained as possible the most relevant pieces of related work are reviewed in the introductory section of each chapter. Moreover, most chapters contain a section "Problem Description" which briefly recapitulates the definition of the class of congestion games the chapter deals with.

Chapter 2

Preliminaries

In this chapter, we introduce the central concepts used in this thesis. Section 2.1 covers fundamental elements of game theory. For a comprehensive treatment, see also the textbooks by Fudenberg and Tirole [52] and Osborne and Rubinstein [107]. Section 2.2 is devoted to the most important theoretical tools for proving the existence of equilibria in games, that are, potential functions and fixed points. In Section 2.3, we define the variants of congestion games covered in this thesis. Finally, we review related work on the existence and computational complexity of equilibria in congestion games and variants thereof in Section 2.4.

2.1 Strategic Games and Equilibria

The central notion of game theory is that of a *strategic game*. In a strategic game, we are given a finite set $N = \{1,\dots,n\}$ of $n \in \mathbb{N}_{>0}$ players and, for each player $i \in N$, a nonempty set S_i of *strategies*. The Cartesian product $S = S_1 \times \cdots \times S_n$ of the players' strategies is called the *strategy space* and an element $s \in S$ is called a *strategy profile*. Each player has a *private cost function* $\pi_i : S \to \mathbb{R}$ assigning to each strategy profile s a private cost value $\pi_i(s)$ that player i strives to minimize. The function $\pi : S \to \mathbb{R}^n$ that maps each strategy profile s to the corresponding private cost vector $\pi_1(s) \times \cdots \times \pi_n(s)$ is called the *combined private cost function*. The fact that the private cost player i experiences is defined as a function of S rather than S_i expresses the fact that each player may not only care about her own strategic choice but also about the strategic choices of other players. The set of players, their strategies, and the combined private cost functions define a strategic minimization game.

Definition 2.1 (Strategic minimization game)
A *strategic minimization game* is a tuple $G = (N,S,\pi)$, where $N = \{1,\dots,n\}$ is the nonempty and finite set of players, $S = S_1 \times \cdots \times S_n$ is the nonempty strategy space, and $\pi : S \to \mathbb{R}^n$ is the combined private cost function that assigns a private cost vector $\pi(s)$ to each strategy profile $s \in S$.

In the following we call strategic minimization games simply *minimization games*. We say that a minimization game $G = (N,S,\pi)$ is *finite* if S_i is finite for all $i \in N$, and *infinite* otherwise. A nonempty subset $\emptyset \neq K \subseteq N$ of players is called a *coalition*. For a coalition $K = \{k_1,\dots,k_l\}$ we denote by $-K = N \setminus K$ its complement and by $S_K = S_{k_1} \times \cdots \times S_{k_l}$ the joint strategy space of the players in K. If $K = \{i\}$ for some $i \in N$, we write S_i and S_{-i} instead of $S_{\{i\}}$ and $S_{-\{i\}}$, respectively. With a slight abuse of notation, for a strategy profile s and a coalition K, we write s as (s_K, s_{-K}), which means that s is the strategy profile in which the members of K play $s_K \in S_K$ while all other players play $s_{-K} \in S_{-K}$.

2.1.1 Equilibrium Concepts

The most important equilibrium concept of game theory is that of a *pure Nash equilibrium*. In essence, a strategy profile is called a pure Nash equilibrium if no player can decrease her private cost by unilaterally deviating to another strategy.

Definition 2.2 (Pure Nash equilibrium)
For a minimization game $G = (N, S, \pi)$, a strategy profile $s \in S$ is called a *pure Nash equilibrium* if $\pi_i(s) \leq \pi_i(s'_i, s_{-i})$ for all $i \in N$ and $s'_i \in S_i$.

We use the word "pure" to emphasize that each player deterministically chooses exactly one strategy. This is in contrast to a *mixed equilibrium*, where players are allowed to randomize over their set of strategies.[1] The concept of a mixed equilibrium presumes that players are risk-neutral, that is, they are only interested in minimizing their expected private costs. This assumption has been refuted in many experiments; see e.g. Allais [7]. Also, for many games, mixed strategies have no meaningful physical interpretation; see the discussion by Osborne and Rubinstein in [107, §3.2]. Throughout this thesis, we only consider pure Nash equilibria.

While the pure Nash equilibrium concept excludes the possibility that a single player can unilaterally improve her private cost, it does not necessarily imply that a pure Nash equilibrium is stable against coordinated deviations of a group of players if their joint deviation is profitable for each of its members. So when coordinated actions are possible, the Nash equilibrium concept is not sufficient to analyze stable states of a game. To cope with the issue of coordination, Aumann [11] proposed the solution concept of a strong equilibrium. In a strong equilibrium, no coalition (of any size) can deviate and strictly improve the utility of each of its members.

Definition 2.3 (Strong equilibrium)
For a minimization game $G = (N, S, \pi)$, a strategy profile $s \in S$ is a *strong equilibrium* if for all coalitions $K \subseteq N$ there is a player $i \in K$ such that $\pi_i(s) \leq \pi_i(s'_K, s_{-K})$ for all $s'_K \in S_K$.

Throughout this thesis, strong equilibria are only considered in pure strategies.

Approximate Equilibria

Players might be willing to stick with their strategy, if they can decrease their private cost only slightly when deviating to another strategy. This idea is captured by the notion of a ρ-approximate equilibrium.

[1] A *mixed* Nash equilibrium can be interpreted as the pure Nash equilibrium of another game, called the *mixed extension*, in which each player i's strategy set S_i is replaced by the set of probability distributions over S_i and where the players' new private cost functions are equal to their expected private cost.

Definition 2.4 (**ρ-Approximate pure Nash equilibrium**)
For a minimization game $G = (N,S,\pi)$ and $\rho > 0$, a strategy profile $s \in S$ is a *ρ-approximate pure Nash equilibrium* if $\pi_i(s) - \rho \le \pi_i(s_i', s_{-i})$ for all $i \in N$ and $s_i' \in S_i$.

Accordingly, ρ-approximate strong equilibria are those strategy profiles for which no coalition may decrease the costs of all of its members by at least ρ.

Definition 2.5 (**ρ-Approximate strong equilibrium**)
For a minimization game $G = (N,S,\pi)$ and $\rho > 0$, a strategy profile $s \in S$ is a *ρ-approximate strong equilibrium* if for all $K \subseteq N$ there is $i \in K$ such that $\pi_i(s) - \rho \le \pi_i(s_K', s_{-K})$ for all $s_K' \in S_K$.

For minimization games with non-negative private costs also multiplicative versions of approximate equilibria are used frequently in the literature. In this thesis, we only use approximate equilibria in the additive sense as defined in Definitions 2.4 and 2.5.

Invariance of the Set of Equilibria

It is a useful observation that the sets of pure Nash equilibria and strong equilibria are invariant under strictly increasing transformations of the players' private cost functions and renaming of the players' strategies. Formally, let $G = (N,S,\pi)$ and $G' = (N,S',\pi')$ be two minimization games with the same player set N. We say that G' is a *monotonic transformation* of G, if, for each player i, there are a bijection $\Sigma_i : S_i \to S_i'$ and a strictly increasing function $\Pi_i : \mathbb{R} \to \mathbb{R}$ such that $\pi_i(s) = \Pi_i\big(\pi_i'\big(\Sigma_1(s_1), \dots, \Sigma_n(s_n)\big)\big)$ for all $s \in S$ and $i \in N$. Note that, for each player i, the strictly increasing function Π_i is a bijection (possibly on a subset of $\mathbb{R}$) and hence invertible. This implies that G' is a monotonic transformation of G if and only G is a monotonic transformation of G'.

Proposition 2.6. *Let $G = (N,S,\pi)$ and $G' = (N,S',\pi')$ be two minimization games such that G' is a monotonic transformation of G. Then, G has a pure Nash equilibrium (respectively, a strong equilibrium) if and only if G' has a pure Nash equilibrium (respectively, a strong equilibrium).*

Proof. For each player i, let $\Sigma_i : S_i \to S_i'$ and $\Pi_i : \mathbb{R} \to \mathbb{R}$ be the functions associated with the monotonic transformation. We claim that a strategy profile s is a pure Nash equilibrium of G if and only if the strategy profile $\Sigma(s) = \big(\Sigma_1(s_1), \dots, \Sigma_n(s_n)\big)$ is a pure Nash equilibrium of G'.

To see the first part of the claim, let s be a pure Nash equilibrium of G. Then, $\pi_i(s) \le \pi_i(t_i, s_{-i})$ for all $i \in N$, $t_i \in S_i$. We observe that

$$\pi_i'\big(\Sigma_1(s_1), \dots, \Sigma_n(s_n)\big) = \Pi_i^{-1}(\pi_i(s)) \le \Pi_i^{-1}(\pi_i(t_i, s_{-i})) = \pi_i'\big(\Sigma_1(s_1), \dots, \Sigma_i(t_i), \dots, \Sigma_n(s_n)\big)$$

for all $i \in N$ and $t_i \in S_i$ and derive that $\Sigma(s)$ is a pure Nash equilibrium of G'. The fact that the monotonic transformation relation is symmetric shows the second part of the claim. The proof for strong equilibria is analogous and, thus, omitted. $\square$

If G' is a monotonic transformation of G and $\Pi_i(x) = x$ for all $x \in \mathbb{R}$ and $i \in N$, we say that G and G' are *isomorphic*. Roughly speaking, two games that are isomorphic have the same private costs, but their strategies may have different names.

2.1.2 Convergence to Equilibria

The idea of using learning or adjustment processes in order to substantiate the plausibility of a predicted outcome dates back to Cournot [27] who proposed a process, called *tâtonnement*, that leads the players in a Cournot oligopoly game to play the Nash equilibrium of that game. In the tâtonnement process the play starts in an arbitrary strategy profile and in each turn one of the players chooses a best reply to her opponents' strategy.

In many applications the assumption that the players repeatedly choose a *best* reply might be too restrictive since an optimum strategy might be computationally expensive to find. A slightly weaker assumption is that the players repeatedly choose a better reply (or stick with their strategy if no better reply exists).

Monderer and Shapley [99] explored the convergence of such myopic improvement processes based on repeated better replies. Formally, let $G = (N, S, \pi)$ be a minimization game. We call a tuple $\big(s, (s'_i, s_{-i})\big) \in S \times S$ an *improving move* of player i if $\pi_i(s) > \pi_i(s'_i, s_{-i})$. A sequence of strategy profiles $\gamma = (s^1, s^2, \dots)$ is called an *improvement path* if for every $k = 1, 2, \dots$ the tuple (s^k, s^{k+1}) is an improving move for some player i. For a finite improvement path $\gamma = (s^1, s^2, \dots, s^k)$ with $s^k \neq s^1$ the strategy profile s^k is called the *endpoint* of γ. A closed improvement path $(s^1, s^2, \dots, s^k, s^1)$ is referred to as an *improvement cycle*. Closed paths that may not have the property that neighbored strategy profiles yield an improvement of the deviating player are simply called *cycles*. The following property of a game ensures that every sequence of unilateral improvements converges.

Definition 2.7 (Finite improvement property)
A minimization game G has the *finite improvement property* if all improvement paths of G are finite.

It is a useful observation that every game with the finite improvement property possesses a pure Nash equilibrium.

Proposition 2.8. *Every game with the finite improvement property possesses a pure Nash equilibrium.*

Proof. For an arbitrary strategy profile s^0, let Γ denote the set of improvement paths starting in s^0. We define a partial ordering $\subseteq$ on Γ as follows. For two improvement paths $\gamma = (s^0, s^1, \dots, s^k)$ and $\tilde{\gamma} = (\tilde{s}^0, \tilde{s}^1, \dots, \tilde{s}^{\tilde{k}})$ with $k \leq \tilde{k}$, we say that $\gamma \subseteq \tilde{\gamma}$ if $s^j = \tilde{s}^j$ for all $j \in \{0, \dots, k\}$. We claim that every totally ordered subset $\Gamma' \subseteq \Gamma$ is finite. Suppose not. Let Γ' be an infinite totally ordered subset of Γ. This implies the existence of an infinite sequence $\gamma^0 \subseteq \gamma^1 \subseteq \dots$ of improvement paths, which we write as

$$
\begin{aligned}
\gamma^1 &= (s^{0,0}, s^{0,1}, \dots, s^{0,k_1}) \\
\gamma^1 &= (s^{1,0}, s^{1,1}, \dots, s^{1,k_1}, \dots, s^{1,k_1}) \\
&\vdots \\
\gamma^m &= (s^{m,0}, s^{m,1}, \dots, s^{m,k_0}, \dots, s^{m,k_1}, \dots, s^{m,k_m}) \\
&\vdots
\end{aligned}
$$

By construction, the diagonal sequence $\gamma = (s^{0,0}, s^{1,1}, \dots)$ is an improvement path of infinite length which gives a contradiction to the finite improvement property. We conclude that every totally ordered subset is finite. Applying Zorn's Lemma, we obtain the existence of a finite improvement path $\gamma' \in \Gamma$ with $\gamma' \not\subseteq \gamma$ for all $\gamma \in \Gamma \setminus \{\gamma'\}$. Thus, the endpoint of γ' is a pure Nash equilibrium. □

To get a similar result for ρ-approximate pure Nash equilibria we also consider sequences of improving moves that improve the cost of each deviating player by at least a fixed strictly positive parameter ρ. For a minimization game $G = (N, S, \pi)$, a tuple $\big(s, (s'_i, s_{-i})\big) \in S \times S$ is an *ρ-improving move* of player i if $\pi_i(s) - \rho > \pi_i(s'_i, s_{-i})$. Analogously, a sequence of ρ-improving moves is called a *ρ-improvement path* and closed ρ-improvement path are called *ρ-improvement cycles*. We define the approximate finite improvement property as the approximate analog of the finite improvement property.

Definition 2.9 (Approximate finite improvement property)
A minimization game G has the *approximate finite improvement property* if, for every $\rho > 0$, every ρ-improvement path is finite.

With the same arguments as in the proof of Proposition 2.8, we obtain the following result.

Proposition 2.10. *Every game with the approximate finite improvement property possesses a ρ-approximate pure Nash equilibrium for every $\rho > 0$.*

When coordinated actions of the players are possible, it is natural to consider deviations of coalitions of players. Let $G = (N, S, \pi)$ be a minimization game. For a coalition $K \subseteq N$ we say that $\big(s, (s'_K, s_{-K})\big)$ is a *strong improving move* of coalition K if $\pi_i(s) > \pi_i(s'_K, s_{-K})$ for all $i \in K$. A sequence $\gamma = (s^1, s^2, \dots)$ of strong improving moves will be called a *strong improvement path* and closed strong improvement paths will be referred to as strong improvement cycles. The following property of a game ensures that every sequence of coalitional improvements converges.

Definition 2.11 (Strong finite improvement property)
The game G has the *strong finite improvement property* if every strong improvement path is finite.

We obtain the following analogue to Proposition 2.8.

Proposition 2.12. *Every game with the strong finite improvement property possesses a strong equilibrium.*

To get an equivalent result for approximate strong equilibria, let $\rho > 0$. We call $\big(s, (s'_K, s_{-K})\big)$ a *strong ρ-improving move* of coalition $K \subseteq N$ if $\pi_i(s) - \rho > \pi_i(s'_K, s_{-K})$ for all $i \in K$. Sequences of strong ρ-improving moves are called strong ρ-improvement cycles. The following definition of the approximate strong finite improvement property is straightforward.

Definition 2.13 (Approximate strong finite improvement property)
A game G has the *approximate strong finite improvement property* if, for every $\rho > 0$, every strong ρ-improvement path is finite.

Analogously to Proposition 2.10, we obtain the following result.

Proposition 2.14. *Every game with the approximate strong finite improvement property possesses a ρ-approximate strong equilibrium for every $\rho > 0$.*

2.1.3 Maximization Games

So far, we assume that each player *minimizes* a private cost function. Many interactions can be described more conveniently assuming that the players *maximize* a private *payoff* function $\varpi(s)$ that is associated with each strategy profile s. Exchanging the private cost function π by a private payoff function ϖ in the definition of a strategic minimization game, we obtain a strategic maximization game.

Definition 2.15 (Strategic maximization game)
A *strategic maximization game* is a tuple $G = (N, S, \varpi)$, where $N = \{1, \ldots, n\}$ is the nonempty and finite set of players, $S = S_1 \times \cdots \times S_n$ is the nonempty strategy space, and $\varpi : S \to \mathbb{R}^n$ is the combined *private payoff* function assigning a private payoff vector $\varpi(s)$ to each $s \in S$.

Throughout this thesis, we call strategic maximization games simply *maximization games*. For a maximization game $G = (N, S, \varpi)$, consider the corresponding minimization game $G^- = (N, S, \pi)$ with $\pi(s) = -\varpi(s)$ for all $s \in S$. All notions defined so far translate to maximization games in the following way. A strategy profile s is a pure Nash equilibrium (respectively, a strong equilibrium, a ρ-approximate pure Nash equilibrium, a ρ-approximate strong equilibrium) for G if and only if it is a pure Nash equilibrium (respectively, a strong equilibrium, a ρ-approximate pure Nash equilibrium, a ρ-approximate strong equilibrium) for G^-. A tuple (s, s') is an improving move (respectively, a strong improving move, a ρ-improving move, a strong ρ-improving move) for G if and only it is an improving move (respectively, a strong improving move, a ρ-improving move, a strong ρ-improving move) for G^-. The game G has the finite improvement property, if and only if G^- has the finite improvement property. The strong finite improvement property, the approximate finite improvement property, and the approximate strong finite improvement property are defined analogously.

2.1.4 Examples

In the following, we discuss the existence of equilibria in some exemplary games.

Example 2.16 (Matching pennies game). In the matching pennies game, there are two players who both choose simultaneously either "Heads" or "Tails". If the choices differ, player 1 pays 1 dollar to player 2. If they are the same, player 2 pays 1 dollar to player 1. The private cost of each player equals the amount of money that she pays to her counterpart, negative cost indicating a gain of money. We model this situation as the finite minimization game $G = (N, S, \pi)$, where $N = \{1, 2\}$, $S_1 = S_2 = \{\text{"Heads"}, \text{"Tails"}\}$. The private cost of player 1 is defined as $\pi_1(s_1, s_2) = -1$ if $s_1 = s_2$ and $\pi_1(s_1, s_2) = 1$, otherwise, and the payoff of player 2 equals $\pi_2(s_1, s_2) = 1$ if $s_1 = s_2$ and $\pi_1(s_1, s_2) = -1$, otherwise. We can write this game as the matrix shown in Figure 2.1. Here, the strategies of player 1 are identified with the rows of the matrix. The strategies of player 2 appear as the columns. Thus, each cell of the matrix corresponds to a strategy profile s of the game and shows the corresponding private cost vector. That is, in each cell the first entry equals the private cost of player 1 and the second entry equals the private cost of player 2 in the corresponding strategy profile. The matching pennies game has no pure Nash equilibrium as in each strategy profile the loosing player gains by picking her other strategy.

The following game is a slight modification of a game discussed by Tardos and Vazirani [125].

		Player 2	
		“Heads”	“Tails”
Player 1	“Heads”	−1, 1	1, −1
	“Tails”	1, −1	−1, 1

Figure 2.1: Matrix representation of the Matching-Pennies-Game (Example 2.16). Rows are associated with the strategies of player 1 and columns are associated with the strategies of player 2. Each cell of the matrix is shows the corresponding private cost vector.

		Player 2	
		“Pollute”	“Restrict”
Player 1	“Pollute”	3, 3, 3	2, 4, 2
	“Restrict”	4, 2, 2	3, 3, 1

“Pollute”

	Player 2	
	“Pollute”	“Restrict”
“Pollute”	2, 2, 4	1, 3, 3
“Restrict”	3, 1, 3	2, 2, 2

“Restrict”

Player 3

Figure 2.2: Matrix representation of the pollution game. The two matrices correspond to the two strategies of player 3. In each of the two matrices, rows are identified with the strategies of player 1 and columns are identified with the strategies of player 2. Each cell of a matrix shows the private cost vector of the corresponding strategy profile.

Example 2.17 (Pollution game). A set of three countries may either pass legislation to restrict pollution or not. Restricting pollution has a cost of 2 for the country itself, but each polluting country adds 1 to the cost of all countries. We model this situation as the finite strategic minimization game $G = (N,S,\pi)$, where $N = \{1,2,3\}$ and $S_1 = S_2 = S_3 = \{\text{“Pollute”},\text{“Restrict”}\}$. The private cost of each player $i \in N$ is defined as $\pi_i(s) = 2 + |\{j \in N : s_j = \text{“Pollute”}\}|$ if $s_i = \text{“Restrict”}$, and $\pi_i(s) = |\{j \in N : s_j = \text{“Pollute”}\}|$ otherwise. A convenient way of describing the game is shown in Figure 2.2. Here, we have two matrices, one for each strategy of player 3. In each of them, strategies of players 1 and 2 are identified with the rows and columns of the matrix, respectively. Thus, each strategy profile s corresponds to exactly one cell in one of the two matrices. Each cell shows the private cost vector of the corresponding strategy profile.

Each player i is always better off polluting, no matter what the strategies of the other two players are. Thus, the strategy profile where all three players pollute is the unique pure Nash equilibrium of the game. The pollution game also has the finite improvement property because every improving move involves a player switching from “Restrict” to “Pollute” implying that each improvement path is finite. The unique pure Nash equilibrium of the pollution game is not a strong equilibrium as all three players may deviate jointly from (“Pollute”, “Pollute”, “Pollute”) to

("Restrict","Restrict","Restrict") decreasing the cost for each player. Thus, the pollution game has no strong equilibrium.

To this point all examples have been finite. The following variant of a Cournot competition features two players with an *infinite* set of strategies.

Example 2.18 (Cournot competition). In Cournot's model of duopoly competition [27] there are two firms producing a homogeneous good without production cost. Both firms simultaneously choose their respective production levels. They sell their output on a single market at the market clearing price p, which is a non-increasing function of the total quantity offered. In this example, we assume a linear market reaction that equals $p(x) = \max\{0, 1-x\}$ for all $x \geq 0$. The payoff of each firm is the profit from selling their goods. This situation can be modeled as the infinite maximization game $G = (N, S, \varpi)$, for which $N = \{1,2\}$, $S_1 = S_2 = \mathbb{R}_{\geq 0}$, $\varpi_1(s) = s_1 \cdot p(s_1 + s_2)$ and $\varpi_2(s) = s_2 \cdot p(s_1 + s_2)$. To find a pure Nash equilibrium of that game, we note that the best reply $r_1(s_2) \in S_1$ of player 1 to $s_2 \in S_2$ satisfies $r_1(s_2) \in \arg\max_{s_1 \in S_1} \varpi_1(s_1, s_2)$. Using the first-order conditions for optimality, we obtain $r_1(s_2) = (1-s_2)/2$, if $s_2 < 1$. By symmetry, we also derive $r_2(s_1) = (1-s_1)/2$, if $s_1 < 1$ for the best reply of player 2. We obtain two kinds of pure Nash equilibria. In the first type, we have $s_1 < 1$ and $s_2 < 1$. The best replies then give rise to $s_1 = s_2 = 1/3$, that is, $(1/3, 1/3)$ is a pure Nash equilibrium. Note that this pure Nash equilibrium gives a payoff of $1/9$ to both players. Switching to the strategy profile in which both players choose a demand equal to $1/4$, the payoff of both players increases to $1/8$. There is also a second type of pure Nash equilibrium, in which both players choose a strategy greater or equal 1. Because this type of pure Nash equilibrium gives a payoff of 0 to each player, it is also not a strong equilibrium. Thus, the Cournot oligopoly game has no strong equilibrium.

2.2 Sufficient Conditions for the Existence of Equilibria

As seen in the previous examples there are games that lack the existence of pure Nash equilibria or strong equilibria. In this section, we introduce the most important tools for proving the existence of equilibria in games – potential functions and fixed points.

2.2.1 Potential Functions

One approach to establish the existence of a pure Nash equilibrium is the potential function method. Potential functions were introduced in the game theory literature by Rosenthal [114] and further studied and generalized by Monderer and Shapley [99]. For a minimization game, $G = (N, S, \pi)$, a potential function is a real-valued function $P : S \to \mathbb{R}$ defined on the set of strategy profiles that decreases along any improvement path. That is, every profitable deviation of a single player strictly reduces the value of P. A game G that admits a potential function is called a *potential game*.

Definition 2.19 (Potential function)
For a minimization game $G = (N, S, \pi)$, a function $P : S \to \mathbb{R}$ is a *potential function* if $P(s) > P(s_i', s_{-i})$ for all improving moves $\big(s, (s_i', s_{-i})\big)$ of G.

Monderer and Shapley [99] call a function as defined above a *generalized ordinal potential*. To not overburden terminology, we prefer to call it simply *potential*. There are several more restrictive notions of potential functions. For a vector $w = (w_i)_{i\in N}$ of weights with $w_i \in \mathbb{R}_{>0}$, a function $P : S \to \mathbb{R}$ is called a *w-potential* or *weighted potential*, if $\pi(s) - \pi(s_i', s_{-i}) = w_i\big(P(s) - P(s_i', s_{-i})\big)$ for all $s \in S$, $i \in N$ and $s_i' \in S_i$. It is called an *exact* potential, if it is a weighted potential with $w_i = 1$ for all $i \in N$. A function $P : S \to \mathbb{R}$ is called an ordinal potential if $\operatorname{sgn}(\pi(s) - \pi(s_i', s_{-i})) = \operatorname{sgn}(P(s) - P(s_i', s_{-i}))$ for all $s \in S$, $i \in N$ and $s_i' \in S_i$, where the sign function $\operatorname{sgn} : \mathbb{R} \to \mathbb{R}$ is defined as $\operatorname{sgn}(x) = 1$ if $x > 0$, $\operatorname{sgn}(x) = 0$ if $x = 0$, and $\operatorname{sgn}(x) = -1$ otherwise.

It is a useful observation that every finite potential game has the finite improvement property. In fact, as shown by Monderer and Shapley [99] the existence of a potential function is even necessary for a finite game to have the finite improvement property. To prove this result, we here use the more elegant proof due to Milchtaich [94].

Proposition 2.20. *A finite game has the finite improvement property if and only if it has a potential.*

Proof. We first prove that every finite potential game has the finite improvement property. Suppose not. Let $\gamma = (s^0, s^1, \dots)$ be an infinite improvement path. The finiteness of the strategy space S implies that there are $k, l \in \mathbb{N}$ with $k < l$ such that $s^k = s^l$. This contradicts $P(s^k) > P(s^{k+1}) > \cdots > P(s^{l-1}) > P(s^l)$.

For the only if part, consider the function $P : S \to \mathbb{R}$ defined as

$$P(s) = -|\{t \in S : \text{ there exists an improvement path starting in } t \text{ and ending in } s\}|.$$

We claim that P is a potential function. Let $i \in N$, $s \in S$, $s_i' \in S_i$ with $\pi_i(s) > \pi_i(s_i', s_{-i})$ be arbitrary. Using that $\pi_i(s) > \pi_i(s_i', s_{-i})$, we observe that for every strategy profile t for which there exists an improvement path starting in t and ending in s, there is also one starting in t and ending in (s_i', s_{-i}). In addition, $(s, (s_i', s_{-i}))$ is an improvement path starting in s and ending in (s_i', s_{-i}). As the game has the finite improvement property, there is no improvement path starting in (s_i', s_{-i}) and ending in s. This implies $P(s) > P(s_i', s_{-i})$. $\square$

For infinite games the existence of a potential function is not sufficient for the finite improvement property. For instance, the one-player minimization game, with $S_1 = (0, 1]$ and $\pi_1(s_1) = -1/s_1$ has the exact potential function $P(s) = \pi_i(s)$ but the sequence $\gamma = (1/k)_{k\in\mathbb{N}_{>0}}$ is an infinite improvement path. This game can be turned into a two-player game by adding a second player whose strategic choice has no influence on the private cost of player 1.

Using that every game with the finite improvement property possesses a pure Nash equilibrium (Proposition 2.8), we obtain the following result as an immediate corollary of Proposition 2.20.

Corollary 2.21. *Every finite potential game possesses a pure Nash equilibrium.*

For the pollution game considered in Example 2.17, the function $P : S \to \mathbb{R}$ defined as $P(s) = |\{i \in N : s_i = \text{“Restrict”}\}|$ is an exact potential function implying the existence of a pure Nash equilibrium. The pollution game does not possess a strong equilibrium, however. To obtain a necessary condition for the existence of strong equilibria, Holzman and Law-Yone [72] generalized the potential function concept to *strong potential functions*. Roughly speaking, P is a strong potential function if the value of P decreases along every strong improving move.

Definition 2.22 (Strong potential function)
For a minimization game $G = (N, S, \pi)$, a function $P : S \to \mathbb{R}$ is called a *strong potential function* if $P(s) > P(s'_K, s_{-K})$ for all strong improving moves $\big(s, (s'_K, s_{-K})\big)$ of G.

We obtain the following result analogously to Proposition 2.20.

Proposition 2.23. *A finite game has the strong finite improvement property if and only if it has a strong potential.*

Using that every game with the strong finite improvement property possesses a strong equilibrium (Proposition 2.12), we obtain the following corollary.

Corollary 2.24. *Every finite game with a strong potential possesses a strong equilibrium.*

2.2.2 Fixed Points

Nash's famous theorem [101] for the existence of mixed equilibria in finite games relies on a fixed point theorem due to Kakutani. The main idea is to consider for each player i the so-called *best-reply correspondence* that maps each strategy profile s to the *set* of strategies that maximize player i's payoff when her opponents play s_{-i}. Kakutani's fixed point theorem [76] then implies the existence of a fixed point, which is a mixed Nash equilibrium.[2]

In the following years, many researchers (Debreu [31], Fan [42], Glicksberg [58], Rosen [113]) independently discovered, that Kakutani's fixed point theorem can in fact be applied in a more general context where the strategy set of each player is compact, and the private payoff function ϖ_i of each player i has the property that upper-contour levels are convex, that is, for each $i \in N$, each $s_{-i} \in S_{-i}$ and each $\alpha > 0$ the set $\{s_i \in S_i : \varpi_i(s_i, s_{-i}) \geq \alpha\}$ is convex. Such private payoff unctions are called *quasi-concave*. We here state the theorem as it appears in Fudenberg and Tirole [52].

Theorem 2.25. *Let $G = (N, S, \varpi)$ be a maximization game. If each strategy set S_i is a compact and convex subset of $\mathbb{R}^{k_i}$, $k_i \in \mathbb{N}$ and each payoff function $\varpi_i : S \to \mathbb{R}$ is continuous in s and quasi-concave in s_i, then G has at least one pure Nash equilibrium.*

2.3 Congestion Games

Rosenthal [114] introduced the class of *congestion games* to the game theory literature and proved existence of a pure Nash equilibrium in each such game. Based on his pioneering work, many variants and extensions of congestion games have been discussed and their behavior related to the existence of equilibria has been analyzed intensively. In this section, we will introduce the class of congestion games as defined by Rosenthal and the variants thereof covered in this thesis. We slightly abuse terminology by calling all games introduced in this section simply *congestion games*. When referring explicitly to the class of games introduced by Rosenthal, we refer to them as *unweighted* congestion games.

[2] In his thesis Nash [102] uses an alternative proof relying on Brouwer's fixed point theorem which circumvents dealing with set-valued functions.

Congestion games are an elegant model to investigate the effects of resource usage by selfish users. In such a game, players use several resources from a common pool. The cost that each player experiences while using a resource depends on the set of users of that resource. Formally, we are given a finite set N of players and a finite set R of resources. Each player has a set of subsets of resources $\mathcal{A}_i \subseteq 2^R$ available to her. The set $\mathcal{A}_i$ is called the set of *feasible allocations* of player i and the Cartesian product of the players' feasible allocations $\mathcal{A} = \mathcal{A}_1 \times \cdots \times \mathcal{A}_n$ is called the *allocation space*. Each resource $r \in R$ is endowed with a cost function $c_r : \mathbb{R}_{\geq 0} \to \mathbb{R}$. The tuple $\mathcal{M} = \big(N, R, (\mathcal{A}_i)_{i\in N}, (c_r)_{r\in R}\big)$ is called a *congestion model*.

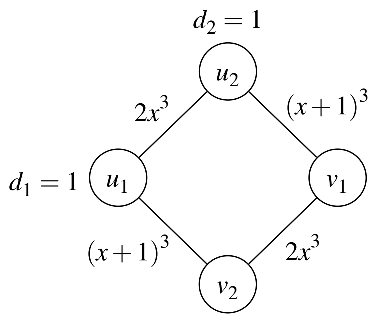
2.3.1 Unweighted Congestion Games

We first define congestion games as they were introduced by Rosenthal [114]. In this class of games, the cost of each resource depends solely on the *number* of players using that resource.

Definition 2.26 (Unweighted congestion game)
Let $\mathcal{M} = \big(N, R, (\mathcal{A}_i)_{i\in N}, (c_r)_{r\in R}\big)$ be a congestion model. The corresponding *unweighted congestion game* is the minimization game $G = (N, S, \pi)$ with $S_i = \mathcal{A}_i$ and $\pi_i(s) = \sum_{r\in s_i} c_r(\ell_r(s))$, where $\ell_r(s) = |\{j \in N : r \in s_j\}|$ for all $i \in N$.

For illustration, consider the following example.

Example 2.27 (Unweighted congestion game). There are two players with unit demand. The resources correspond to the edges of the undirected graph shown in Figure 2.3(a). The cost of each resource as a function of the aggregated demand x is written beneath the edges. Each player i wants to establish a simple (u_i, v_i)-path. That is, $\mathcal{A}_1 = \big\{\{(u_1,u_2),(u_2,v_1)\},\{(u_1,v_2),(v_1,v_2)\}\big\}$ and $\mathcal{A}_2 = \big\{\{(u_1,u_2),(u_1,v_2)\},\{(u_2,v_1),(v_1,v_2)\}\big\}$. Consider the strategy profile in which player 1 chooses her upper path $\{(u_1,u_2),(u_2,v_1)\}$ and player 2 chooses her left path $\{(u_1,u_2),(u_1,v_2)\}$. The upper left edge is congested with 2 units leading to a cost of 16 to each player. In addition, both the upper right and the lower left edge are congested with one unit of demand leading to a cost of 8. Thus, the private cost of each player equals $16+8=24$. Likewise, the private costs of the remaining three strategy profiles can be computed; see Figure 2.3(b). The game has two strong equilibria.



(a) Network

	Player 2: $\{(u_1,u_2),(u_1,v_2)\}$	Player 2: $\{(u_2,v_1),(v_1,v_2)\}$
Player 1: $\{(u_1,u_2),(u_2,v_1)\}$	24, 24	29, 29
Player 1: $\{(u_1,v_2),(v_1,v_2)\}$	29, 29	24, 24

(b) Private costs

Figure 2.3: (a) Network structure and (b) Private costs of the unweighted network congestion game considered in Example 2.27.

2.3.2 Weighted Congestion Games

In a weighted congestion game, each player i has a strictly positive demand $d_i \in \mathbb{R}_{>0}$ that she puts on the chosen resources. The cost of each resource then is a function of the aggregated demand on the resource. In the congestion games literature, two natural variants have been considered. In the first variant, which we term *proportional games*, the cost function of the resources is interpreted as a *per-unit* cost. That is, each resource r that is congested with an aggregated demand of x units produces a cost of $x\,c_r(x)$. These costs are then divided among the users proportionally to their respective demands on that resource. As a result each user pays for each resource she uses the resource cost multiplied by her demand.

Definition 2.28 (Proportional weighted congestion game)
Let $\mathcal{M} = \big(N, R, (\mathcal{A}_i)_{i\in N}, (c_r)_{r\in R}\big)$ be a congestion model and $(d_i)_{i\in N}$ be a vector of strictly positive demands. The corresponding *proportional weighted congestion game* is the minimization game $G = (N, S, \pi)$ with $S_i = \mathcal{A}_i$ and $\pi_i(s) = \sum_{r\in s_i} d_i\, c_r\big(\ell_r(s)\big)$, where $\ell_r(s) = \sum_{j\in N: r\in s_j} d_j$ for all $i \in N$.

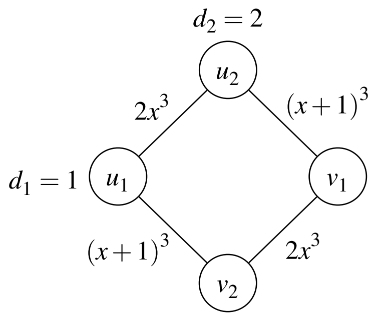
Games as in the above definition are often called simply "weighted congestion games" in the literature; see e.g. Aland et al. [6], Dunkel and Schulz [33], Goemans et al. [59], and Meyers [93].

The second variant which we term *uniform games* differs from proportional games in the definition of the players' payoff functions. In uniform games the cost of a player on a resource is *not* multiplied with the demand of that player. Uniform costs are suitable when the resource costs are interpreted as *latencies* and thus equal for all users, regardless of their demand. This is a common assumption in scheduling applications where the cost function is frequently used to model the achieved makespan that is – under round-robin processing – equal for every job on the same resource.

Definition 2.29 (Uniform weighted congestion game)
Let $\mathcal{M} = \big(N, R, (\mathcal{A}_i)_{i\in N}, (c_r)_{r\in R}\big)$ be a congestion model and $(d_i)_{i\in N}$ be a vector of positive demands. The corresponding *uniform congestion game* is the minimization game $G = (N, S, \pi)$ with $S_i = \mathcal{A}_i$ and $\pi_i(s) = \sum_{r\in s_i} c_r\big(\ell_r(s)\big)$ for all $i \in N$, where $\ell_r(s) = \sum_{j\in N: r\in s_j} d_j$.

Also, uniform games are often simply called "weighted congestion games", e.g., in the works of Ackermann et al. [4], Fotakis et al. [51], and Panagopoulou and Spirakis [108]. We illustrate the idea of a weighted congestion game with the following example.

Example 2.30 (Weighted congestion game). The set of resources, the set of players and their feasible allocations are as in the unweighted congestion game of Example 2.27. The only difference is that the demand of player 2 now equals 2; see Figure 2.4(a). We first consider the corresponding *proportional* weighted congestion game. For illustration, we compute the players' private costs for the strategy profile in which player 1 chooses her upper path $\{(u_1,u_2),(u_2,v_1)\}$ and player 2 chooses her left path $\{(u_1,u_2),(u_1,v_2)\}$. The upper left edge is congested with three units and its cost equals 54. The lower left edge is congested with two units of demand and costs 27. The upper right edge is congested with one unit of demand and costs 8. The private cost of player 1 thus equals $54+8 = 62$ and the private cost of player 2 equals $2\cdot(54+27) = 162$. The private costs experienced in all four strategy profiles are shown in Figure 2.4(b). None of the strategy profiles constitutes a pure Nash equilibrium. In the uniform congestion game, the private cost are the same except that the private



(a) Network

		Player 2	
		$\{(u_1,u_2),(u_1,v_2)\}$	$\{(u_2,v_1),(v_1,v_2)\}$
Player 1	$\{(u_1,u_2),(u_2,v_1)\}$	62, 162	66, 160
	$\{(u_1,v_2),(v_1,v_2)\}$	66, 160	62, 162

(b) Private costs

Figure 2.4: (a) Network structure and (b) Private costs of the weighted undirected-network congestion game considered in Example 2.30.

costs of player 2 are divided by 2. Thus, also in the uniform weighted congestion game no pure Nash equilibrium exists.

For a congestion model $\mathcal{M}$ and a vector of demand $(d_i)_{i\in N}$, the private costs of each player i in the corresponding proportional and uniform weighted congestion games only differ by the strictly positive factor d_i. Thus, the games are monotonic transformations of each other and Proposition 2.6 establishes that both games have the same set of pure Nash equilibria. Since we are mainly interested in the existence of equilibria, it is sufficient to consider only proportional weighted congestion games as a representative. In the following, we refer to them simply as "weighted congestion games".

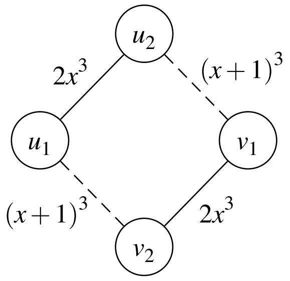
2.3.3 Congestion Games with Resource-Dependent Demands

In a congestion game with resource-dependent demands, we are given for each player i and each resource r a strictly positive demand $d_{i,r}$. The cost of each resource depends on the aggregated (resource-dependent) demand on the resource. We first define proportional games, where the cost of a resource for a player is the product of the resource costs and her demand.

Definition 2.31 (Proportional congestion game with resource-dependent demands)
Let $\mathcal{M} = \big(N,R,(\mathcal{A}_i)_{i\in N},(c_r)_{r\in R}\big)$ be a congestion model and $\big(d_{i,r}\big)_{i\in N, r\in\mathbb{R}}$ be a vector of strictly positive resource-dependent demands. The corresponding *congestion game with resource-dependent demands* is the minimization game $G = (N,S,\pi)$ with $S_i = \mathcal{A}_i$ and $\pi_i(s) = \sum_{r\in s_i} d_{i,r}\, c_r\big(\ell_r(s)\big)$ for all $i \in N$, where $\ell_r(s) = \sum_{j\in N: r\in s_j} d_{j,r}$.

Uniform games differ from proportional games solely in the fact that the cost for a player on a resource is *not* multiplied with the demand of that player.

Definition 2.32 (Uniform congestion game with resource-dependent demands)
Let $\mathcal{M} = \big(N,R,(\mathcal{A}_i)_{i\in N},(c_r)_{r\in R}\big)$ be a congestion model and $\big(d_{i,r}\big)_{i\in N, r\in\mathbb{R}}$ be a vector of strict positive resource-dependent demands. The corresponding *uniform congestion game with resource-dependent demands* is the minimization game $G = (N,S,\pi)$ with $S_i = \mathcal{A}_i$ and $\pi_i(s) = \sum_{r\in s_i} c_r\big(\ell_r(s)\big)$ for all $i \in N$, where $\ell_r(s) = \sum_{j\in N: r\in s_j} d_{j,r}$.



(a) Network

Player 1 \ Player 2	$\{(u_1,u_2),(u_1,v_2)\}$	$\{(u_2,v_1),(v_1,v_2)\}$
$\{(u_1,u_2),(u_2,v_1)\}$	1000, 758	896, 378
$\{(u_1,v_2),(v_1,v_2)\}$	896, 378	1000, 758

(b) Private costs of the proportional game

Player 1 \ Player 2	$\{(u_1,u_2),(u_1,v_2)\}$	$\{(u_2,v_1),(v_1,v_2)\}$
$\{(u_1,u_2),(u_2,v_1)\}$	375, 258	232, 270
$\{(u_1,v_2),(v_1,v_2)\}$	232, 270	375, 258

(c) Private costs of the uniform game

Figure 2.5: (a) Network structure, (b) Private costs of the proportional undirected-network congestion game with resource-dependent demands, (c) Private costs of the uniform undirected-network congestion game with resource-dependent demands considered in Example 2.33.

While the distinction between proportional games and uniform is immaterial for the existence of pure Nash equilibria in weighted congestion games, in the case of resource-dependent demands, both variants have different sets of equilibria, as shown in the following example.

Example 2.33 (Congestion game with resource-dependent demands). Reconsider the network congestion model already seen in Examples 2.27 and 2.30. To give a short representation of the players' resource-dependent demands we call both the upper left and lower right edge a *solid resource* and all other edges a *dashed resource*; see Figure 2.5(a). The demand of player 1 equals 2 for all solid resources and 4 for all dashed resources. The demand of player 2 equals 3 for all solid resources and 1 for all dashed resources. We calculate the private cost of the strategy profile of the proportional game, in which player 1 uses the upper path and player 2 uses the left path. The upper left resource is congested with five units of demand, thereof two units from player 1 and three units from player 2. It produces a cost of $2 \cdot 250$ for player 1 and a cost of $3 \cdot 250$ for player 2. Player 1 additionally uses the upper right resource with cost $4 \cdot 125$ and thus her total private cost equals $2 \cdot 250 + 4 \cdot 125 = 1000$. Player 2 additionally uses the lower left resource with cost 8, resulting in a total private cost of $3 \cdot 250 + 8 = 758$. The private cost of all strategy profiles of the proportional congestion game with resource-dependent demands are given in Figure 2.5(b). The game has two strong equilibria. For comparison, the private cost of the uniform game are given in Figure 2.5(c). Unlike the proportional game, the uniform game does not even admit a pure Nash equilibrium.

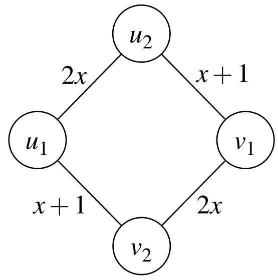


Figure 2.6: Undirected-network congestion game with variable demands as discussed in Example 2.36.

2.3.4 Congestion Games with Variable Demands

In a congestion game with variable demands, the players' demands are not given endogenously but are subject to the strategic decisions of the players. As usual, every player i is associated with a nonempty set $\mathcal{A}_i$ of feasible allocations which is a set of subsets of resources. In addition, we are given for each player i a nonempty closed interval $[\sigma_i, \tau_i]$ of feasible demands with $\sigma_i \in \mathbb{R}_{\geq 0}$, $\tau_i \in \mathbb{R}_{\geq 0} \cup \{\infty\}$, $\sigma_i \leq \tau_i$ and a utility function $U_i : [\sigma_i, \tau_i] \to \mathbb{R}_{\geq 0}$. Technically, is suffices to be given the utility function for each player because its domain already defines the set of feasible demands.

In each strategy profile each player i chooses both a feasible demand $d_i \in [\sigma_i, \tau_i]$ and a feasible allocation $\alpha_i \in \mathcal{A}_i$. The cost of each resource depends as usual on its aggregated demand. Congestion game with variable demands are maximization games and the private payoff of each player equals the utility received from her demand minus the costs of the resources. We first define proportional games.

Definition 2.34 (Proportional congestion game with variable demands)
Let $\mathcal{M} = \big(N, R, (\mathcal{A}_i)_{i\in N}, (c_r)_{r\in R}\big)$ be a congestion model and for all $i \in N$ let $U_i : [\sigma_i, \tau_i] \to \mathbb{R}_{\geq 0}$ be a utility function with $\sigma_i \in \mathbb{R}_{\geq 0}$, $\tau_i \in \mathbb{R}_{\geq 0} \cup \{\infty\}$, $\sigma_i \leq \tau_i$. The corresponding *proportional congestion game with variable demands* is the maximization game $G = (N, S, \varpi)$ with $S_i = \mathcal{A}_i \times [\sigma_i, \tau_i]$ and $\varpi_i(\alpha, d) = U_i(d_i) - \sum_{r\in\alpha_i} d_i\, c_r\big(\ell_r(\alpha, d)\big)$ for all $i \in N$, where $\ell_r(\alpha, d) = \sum_{j\in N: r\in\alpha_j} d_j$.

Uniform games differ from proportional games solely in the fact that the resources costs are *not* multiplied with the demand.

Definition 2.35 (Uniform congestion game with variable demands)
Let $\mathcal{M} = \big(N, R, (\mathcal{A}_i)_{i\in N}, (c_r)_{r\in R}\big)$ be a congestion model and for all $i \in N$ let $U_i : [\sigma_i, \tau_i] \to \mathbb{R}_{\geq 0}$ be a utility function with $\sigma_i \in \mathbb{R}_{\geq 0}$, $\tau_i \in \mathbb{R}_{\geq 0} \cup \{\infty\}$, $\sigma_i \leq \tau_i$. The corresponding *uniform congestion game with variable demands* is the maximization game $G = (N, S, \varpi)$ with $S_i = \mathcal{A}_i \times [\sigma_i, \tau_i]$ and $\varpi_i(\alpha, d) = U_i(d_i) - \sum_{r\in\alpha_i}, c_r\big(\ell_r(\alpha, d)\big)$ for all $i \in N$, where $\ell_r(\alpha, d) = \sum_{j\in N: r\in\alpha_j} d_j$.

For illustration, consider the following example.

Example 2.36 (Proportional congestion game with variable demands). We reconsider the same network topology of the previous examples but use different resource cost functions to simplify our calculations; see Figure 2.6. As in Examples 2.27, 2.30 and 2.33, the set of feasible allocations of each player i is the set of simple (u_i, v_i)-paths, i.e., $\mathcal{A}_1 = \big\{\{(u_1,u_2),(u_2,v_1)\},\{(u_1,v_2),(v_1,v_2)\}\big\}$ and

$\mathcal{A}_2 = \big\{\{(u_1,u_2),(u_1,v_2)\},\{(u_2,v_1),(v_1,v_2)\}\big\}$. The set of feasible demands of each player i equals $[\sigma_i,\tau_i] = \mathbb{R}_{\geq 0}$ and her utility function $U_i : \mathbb{R}_{\geq 0}$ is defined as $U_i(x) = 4x$ for all $x \geq 0$. A strategy of player i is to choose both a (u_i,v_i)-path and a non-negative demands, that is, $S_i = \mathcal{A}_i \times \mathbb{R}_{\geq 0}$. We only consider the proportional congestion game with variable demands G. For a strategy profile $(\alpha_1,d_1,\alpha_2,d_2)$ of G with $\alpha_1 \in \mathcal{A}_1$, $d_1 \in \mathbb{R}_{\geq 0}$, $\alpha_2 \in \mathcal{A}_2$, and $d_2 \in \mathbb{R}_{\geq 0}$, we observe that

$$\varpi_i(\alpha_i,d_i,\alpha_{-i},d_{-i}) =$$
$$\begin{cases} 4d_i - 2d_i(d_i+d_{-i}) - d_i(d_i+1) & \text{if } \alpha_1 = \{(u_1,u_2),(u_2,v_1)\}, \alpha_2 = \{(u_1,u_2),(u_1,v_2)\} \\ & \quad \text{or } \alpha_1 = \{(u_1,v_2),(v_1,v_2)\}, \alpha_2 = \{(u_2,v_1),(v_1,v_2)\}\}, \\ 4d_i - d_i(d_i+d_{-i}+1) - 2d_i^2 & \text{otherwise.} \end{cases}$$

To calculate a pure Nash equilibrium of G, we first assume that player 1 chooses her upper path $\{(u_1,u_2),(u_2,v_1)\}$ while player 2 chooses her left path $\{(u_1,u_2),(u_1,v_2)\}$. If a pure Nash equilibrium exists for this allocation profile, the demands of players 1 and 2 satisfy the first-order conditions on optimality. We obtain the equations $d_i = \frac{1}{2} - \frac{1}{3}d_{-i}$ for $i = 1,2$ which we solve for $d_1 = d_2 = 3/8$. Thus, the strategy profile $\big(\{(u_1,u_2),(u_2,v_1)\},3/8,\{(u_1,u_2),(u_1,v_2)\},3/8\big)$ is the only candidate for a pure Nash equilibrium in which player 1 chooses her upper path and player 2 chooses her lower path. This strategy profile yields a payoff equal to $27/64$. However, switching to the strategy profile $\big(\{(u_1,v_2),(v_1,v_2)\},7/16,\{(u_1,u_2),(u_1,v_2)\},3/8\big)$ player 1 improves her payoff to $147/256 > 27/64$. Thus, $\big(\{(u_1,u_2),(u_2,v_1)\},3/8,\{(u_1,u_2),(u_1,v_2)\},3/8\big)$ is not a pure Nash equilibrium.

Next, let us assume, there is a pure Nash equilibrium in which player 1 chooses her lower path $\{(u_1,v_2),(v_1,v_2)\}$ while player 2 chooses her left path $\{(u_1,u_2),(u_1,v_2)\}$. Again using the first-order conditions, we derive that $s^* = \big(\{(u_1,v_2),(v_1,v_2)\},3/7,\{(u_1,u_2),(u_1,v_2)\},3/7\big)$ is the unique candidate strategy profile for a pure Nash equilibrium and yields a payoff of $27/49$ to both players. Because the players' demands are optimal for s^*, an improving move from s^* (if it exists) necessarily incorporates a change of the allocation. It can be checked that the maximal payoff player i can achieve when switching to her other allocation is $675/1764 < 27/49$. We conclude that s^* is a pure Nash equilibrium.

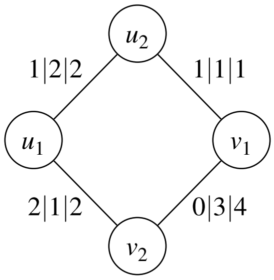
2.3.5 Bottleneck Congestion Games

In the congestion games seen so far the private cost of each player is the *sum* of the costs of all used resources. In a bottleneck congestion game, the private cost each player experiences equals the *maximum* cost among all used resources. We here define them in a more general model in which the cost of a resource may not only depend on the aggregated demand but instead on the *set* of players using it. To this end, let N be a finite set of players and R a finite set of resources. For each player i we are given a set $\mathcal{A}_i \subseteq 2^R$ of feasible allocations. For an allocation profile $\alpha \in \mathcal{A}$ we denote by $\mathcal{N}_r(\alpha) = \{i \in N : r \in \alpha_i\}$ the set of players using r in α. Every resource $r \in R$ has a cost function $c_r : \mathcal{A} \to \mathbb{R}_{\geq 0}$ satisfying the following three properties:

Nonnegativity: $c_r(\alpha) \geq 0$ for all $\alpha \in \mathcal{A}$.

Independence of Irrelevant Choices: $c_r(\alpha) = c_r(\alpha')$ for all $\alpha,\alpha' \in \mathcal{A}$ with $\mathcal{N}_r(\alpha) = \mathcal{N}_r(\alpha')$.

Monotonicity: $c_r(\alpha) \leq c_r(\alpha')$ for all $\alpha,\alpha' \in \mathcal{A}$ with $\mathcal{N}_r(\alpha) \subseteq \mathcal{N}_r(\alpha')$.



(a) Network

Player 1 \ Player 2	$\{(u_1,u_2),(u_1,v_2)\}$	$\{(u_2,v_1),(v_1,v_2)\}$
$\{(u_1,u_2),(u_2,v_1)\}$	2, 2	1, 3
$\{(u_1,v_2),(v_1,v_2)\}$	2, 2	4, 4

(b) Private costs

Figure 2.7: (a) Network structure and (b) Private costs of the bottleneck congestion game with set-dependent costs considered in Example 2.38.

Note that "Monotonicity" implies "Independence of Irrelevant Choices". We call the tuple $\mathcal{M} = \big(N, R, (\mathcal{A}_i)_{i\in N}, (c_r)_{r\in R}\big)$ a *congestion model with set-dependent costs*. We now define bottleneck congestion games with set-dependent costs relative to a congestion model with set-dependent costs.

Definition 2.37 (Bottleneck congestion game with set-dependent costs)
Let $\mathcal{M} = \big(N, R, (\mathcal{A}_i)_{i\in N}, (c_r)_{r\in R}\big)$ be a congestion model with set-dependent costs. The corresponding *bottleneck congestion game with set-dependent costs* is the minimization game $G = (N, S, \pi)$ in which $S_i = \mathcal{A}_i$ and $\pi_i(x) = \max_{r\in s_i} c_r(s)$ for all $i \in \mathbb{N}$.

We illustrate this definition with the following example.

Example 2.38 (Bottleneck congestion game). Consider the network shown in Figure 2.7(a). In contrast to the previous examples, the cost functions of the resources depend on the *set* of players using it. The three entries next to each edge indicate the cost of that edge when used only by player 1, only by player 2, and by both players, respectively. Note that the cost functions are monotonic, in the sense that the cost does not decrease as the set of players using that edge increases. This monotonicity requirement is weaker than the monotonicity of the cost functions in demand-based models since the cost values $1|2|2$ of edge (u_1,u_2) and $2|1|2$ of edge (u_1,v_2) are not representable in a demand-based model with non-decreasing cost functions. For illustration, we calculate the players' private costs for the strategy profile in which the first player uses her upper path and the second player uses her left path. In this profile, edge (u_1,u_2) is used by both players and produces a cost of 2. This is the bottleneck of both players since the edges (u_2,v_1) and (u_1,v_2) produce a cost of 1 each. Thus, the private cost of both players equals 2. Figure 2.7(b) shows the private costs for all remaining strategy profiles. All strategy profiles in which player 2 uses her left path are a strong equilibrium.

2.3.6 Special Strategy Spaces

In the above definitions of the variants of congestion games covered in this thesis we did not impose any particular assumptions on the set $\mathcal{A}_i \subseteq 2^R$ of feasible allocations of each player i. In most applications, however, the set $\mathcal{A}_i$ has a special structure, implicitly given by a problem-specific combinatorial property. For routing applications, for instance, the set of feasible allocations equals the set of all paths connecting that player's source and target node. In this section, we review the most important combinatorial structures imposed on the players' sets of feasible allocations.

Singleton Congestion Games

A congestion game is called a *singleton* game, if $|\alpha_i| = 1$ for each player i and each feasible allocation $\alpha_i \in \mathcal{A}_i$. Singleton games have the common interpretation of selfish load balancing scenarios in which each player controls a job which she wants to be scheduled on exactly one machine out of a set of feasible machines.

Network Congestion Games

In a network congestion game, the set of resources corresponds to the set of edges of a directed or undirected graph $\mathcal{G} = (V,R)$. There is a source-sink pair $(u_i, v_i) \in V \times V$ associated with each player i and the set of her feasible allocations is equal to the set of simple (u_i, v_i)-paths. If there are a common source $u = u_i$ and a common sink $v = v_i$ for all players i, we say that the game is a *single-commodity* network congestion game. Otherwise, we call it a *multi-commodity* network congestion game.

Matroid Congestion Games

In brief, matroid congestion games are games in which for each player i the set $\mathcal{A}_i$ of feasible allocations is the basis of a matroid. Singleton congestion games are matroid congestion games. Another prominent example of matroid games are spanning tree games in which the resources correspond the edges of an undirected graph and the players strive to allocate a spanning tree.

In the following, we give a short introduction to matroids. For a comprehensive treatment we refer to the textbooks of Korte and Vygen [84, Chapter 13] and Schrijver [120, Chapters 39 – 42]. Let R be a finite set. A tuple $M = (R,\mathcal{I})$ where $\mathcal{I} \subset 2^R$ is called a *matroid* if the following three conditions are satisfied:

(1) $\emptyset \in \mathcal{I}$.
(2) If $I \in \mathcal{I}$ and $J \subseteq I$, then $J \in \mathcal{I}$.
(3) If $I,J \in \mathcal{I}$ and $|J| < |I|$, then there exists an $i \in I \setminus J$ with $J \cup \{i\} \in \mathcal{I}$.

A set $A \subseteq R$ is called *independent* if $A \in \mathcal{I}$, and *dependent* otherwise. The set of (inclusion wise) maximal independent subsets of R is called the *basis* of M. For given R, a matroid $(R,\mathcal{I})$ may be of exponential size, thus, one frequently assumes that a matroid comes with an *independence oracle* that returns for all sets $A \subseteq R$ whether $A \in \mathcal{I}$ or not. It shall be noted that for many subclasses of matroids an independence oracle can be implemented in polynomial time.

Another way of representing matroids is via a *rank function* $\mathrm{rk} : 2^R \to \mathbb{N}$. Every sub-cardinal, monotonic and sub-modular function $\mathrm{rk} : 2^R \to \mathbb{N}$ gives rise to a matroid whose independent sets then are defined as $\{A \subseteq R : \mathrm{rk}(A) = |A|\}$. If the independent sets are known a priori via an independence oracle the rank function is defined as $\mathrm{rk}(A) = \max_{I \in \mathcal{I} : I \subseteq A} |I|$.

Symmetric Congestion Games

The allocations space $\mathcal{A}$ is called *symmetric* if $\mathcal{A}_i = \mathcal{A}_j$ for all $i, j \in N$. A prominent example for symmetric allocations spaces are single-commodity network congestion games.

2.4 Equilibria in Congestion Games

In this section, we review related work congestion games. We focus on the existence of equilibria and their computational complexity.

2.4.1 Existence of Equilibria

Below, we summarize known results about the existence of equilibria in several variants of congestion games, such as games with unweighted and weighted players, games with resource-dependent demands, games with bottleneck objectives, and games with player-specific costs.

Unweighted Congestion Games

Rosenthal [114] proved in his seminal work that each unweighted congestion game possesses a pure Nash equilibrium by providing an exact potential function. In fact, every exact potential game is isomorphic to a congestion game as shown by Monderer and Shapley [99]. For a simpler proof of this result, we also refer to the paper by Voorneveld et al. [132].

Although every unweighted congestion games possesses at least one pure Nash equilibrium, a strong equilibrium need not exist, as shown by Holzman and Law-Yone [72]. They further gave a structural characterization of the strategy spaces that give rise to a strong equilibrium in unweighted congestion games with non-decreasing non-positive costs. Their result implies in particular that each singleton game with such cost functions possesses a strong equilibrium. Rozenfeld and Tennenholtz [118] complemented this result showing that also in singleton games with non-increasing non-positive costs a strong equilibrium exists.

Anshelevich et al. [9, 10] proposed to study cost sharing games with fair cost allocation. This class of games corresponds to unweighted multi-commodity network congestion games where the cost function of each resource r is of the form $c_r(x) = b_r/x$, $b_r \in \mathbb{R}_{\geq 0}$. Epstein et al. [36] derive topological properties of the underlying networks that guarantee the existence of strong equilibria in this model.

Feldman and Tennenholtz [45, 46] considered a strengthening of the strong equilibrium concept called *super-strong* equilibrium. In such an equilibrium, no coalition of any size can deviate without increasing the private cost of each of its members while for at least one member the private cost strictly decreases. Under restrictions on the formation of coalitions, they showed the existence of a super-strong equilibrium in various special cases of singleton congestion games with unweighted players.

Weighted Congestion Games

In contrast to unweighted congestion games, games with weighted players need not possess a pure Nash equilibrium. Even two-player games may fail to admit a pure Nash equilibrium; counterexamples were given by Fotakis et al. [51], Goemans et al. [59], and Libman and Orda [88]. On the positive side, Fotakis et al. [51] and Panagopoulou and Spirakis [108] proved the existence of a pure Nash equilibrium in weighted congestion games with affine and exponential costs, respectively. Anshelevich et al. [9, 10] showed that pure Nash equilibria exist in two-player games where all cost functions are of type $c_r(x) = b_r/x$, $b_r \in \mathbb{R}_{\geq 0}$. Milchtaich [95, 96] studied topological properties of the network that guarantee the existence of at least one pure Nash equilibrium.

For singleton games Fotakis et al. [50] showed the existence of a pure Nash equilibrium for linear cost functions (without a constant). Fabrikant et al. [41] gave a short proof for the existence of a pure Nash equilibrium in all singleton congestion games where the cost of each resource is a non-decreasing function depending on the set of its users. This includes singleton weighted congestion games as a special case. Ackermann et al. [4] extended the positive result for singleton weighted congestion games with non-decreasing costs to matroids. They also showed that the matroid property is the maximal property that gives rise to a pure Nash equilibrium for all non-decreasing cost functions, that is, for any strategy space not satisfying the matroid property, there is an instance of a weighted congestion game not having a pure Nash equilibrium.

Kollias and Roughgarden [80] showed that for any game with non-decreasing cost functions it is possible to distribute the costs incurred on each resource in such a way that the existence of a pure Nash equilibrium is always guaranteed. Von Falkenhausen and Harks [129] pursued the question how to distribute the costs incurred in singleton weighted congestion games so as to guarantee the existence of equilibria with good social costs.

A model related to weighted congestion games has been considered by Rosenthal [115]. He showed that in weighted congestion games where players are allowed to split their demand integrally, a pure Nash equilibrium need not exist; see also Tran et al. [126] for further results on this model.

Congestion Games with Resource-Dependent Demands

The above mentioned positive result of Fabrikant et al. [41] for singleton congestion games with set-dependent costs includes singleton uniform congestion games with resource-dependent demands as a special case. Independently, Even-Dar et al. [39, 40] proved the existence of pure Nash equilibria for load balancing games on parallel unrelated machines. This class of games corresponds to uniform congestion games with resource-dependent demands on singletons where the cost function of each resource is the identity. Andelman et al. [8] proved even the existence of a strong Nash equilibrium in scheduling games on unrelated machines. Feldman and Tamir [44] further investigated coalitional deviations in these games. They gave bounds on the ratio by which an arbitrary coalition may improve from any (non-strong) pure Nash equilibrium.

Bottleneck Congestion Games

For bottleneck games with strictly increasing costs the existence of a pure Nash equilibrium was shown by Libman and Orda [88]. Banner and Orda [15] proved the existence of a pure Nash equilibrium both in the unsplittable flow and in the splittable flow setting, respectively. Epstein et al. [35] study the influence of the network topology on the existence of socially optimal pure Nash equilibrium in unsplittable bottleneck congestion games on undirected graphs.

Bottleneck routing with *non-atomic* players and variable demands has been studied by Cole et al. [26]. Among other results, they proved the existence of an equilibrium.

Player-Specific Cost Functions

There is a large body of literature concerning congestion games with player-specific cost functions. In such a game, there are no common cost functions on the resources. Instead, we are given for each player i and each resource r a *player-specific* cost function $c_{i,r} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$.

Milchtaich [94] showed that unweighted singleton congestion games with player-specific cost functions possess at least one pure Nash equilibrium. Ackermann et al. [4] extended this positive result to matroid games. Similar to their result for weighted congestion games, they further showed that the matroid property is the maximal property that guarantees the existence of a pure Nash equilibrium for all non-decreasing cost functions. As shown by Voorneveld et al. [132], the model of Konishi et al. [82] is equivalent to that in [94], which is worth noting as Konishi et al. [82] even proved the existence of strong equilibria in these games.

Gairing et al. [55] showed that best response dynamics do not cycle if the player-specific cost functions are linear without a constant term. Milchtaich [95] further showed that general network games with player-specific costs need not admit a pure Nash equilibrium in general. For the special case where the player-specific cost functions are all affine and differ only by a player-specific additive constant, Mavronicolas et al. [92] gave a weighted potential.

Games with weighted players and player-specific costs need not possess a pure Nash equilibrium even if all strategies are singletons; see the counterexample given by Milchtaich [94].

2.4.2 Computational Complexity of Equilibria

A central challenge in algorithmic game theory is to characterize the computational complexity of equilibria. There are three main computational problems considered in the literature. First, given an instance of a class of congestion games, which is guaranteed to possess a pure Nash equilibrium (respectively, a strong equilibrium), we are interested in calculating such an equilibrium. This computational problem is called the *search problem*. Determining the computational complexity of the search problem is an important challenge for two different reasons. On the one hand, efficient algorithms for the computation of equilibria may help to design systems with desirable equilibria. On the other hand, complexity results yield important indicators for which classes of games, equilibrium concepts are in fact plausible outcomes of strategic play. In order to determine the computational complexity of the search problem it is also important to decide for a given instance of a certain class of congestion games and a given strategy profile whether it is a pure Nash equilibrium (respectively, a strong equilibrium). We refer to this problem as the *recognition problem*. The recognition problem naturally occurs in applications where it is desirable to know whether a certain observed state of the system is stable against unilateral or coalitional deviations. Another question related to the computational complexity of equilibria is the *existence problem*. Given an instance of a class of congestion games, which does not admit a pure Nash equilibrium (respectively, a strong equilibrium) in general, we like to decide whether the game possesses such an equilibrium. Efficient algorithms for the existence problem are important for the design of real-world systems for which oscillating behavior is not desired.

For all three questions, it is important to specify how the instances are encoded. Without imposing additional assumptions on the combinatorial structure, the set of feasible allocations of each player is given explicitly. For network congestion games and matroid congestion games, the set of feasible allocations of each player is given implicitly by a certain combinatorial property.

Below, we summarize known results on the complexity of these three questions for the variants of congestion games considered in this thesis.

Unweighted Congestion Games

The complexity of computing a pure Nash equilibrium in an unweighted congestion game is closely related to the complexity of local search problems. An instance of a local search problem is given by a set of feasible solutions linked by neighboring conditions, and an objective function defined on the set of feasible solutions. Given such an instance, we are interested in calculating a *locally optimal solution*, that is, a feasible solution that does not have a strictly better neighbor. The class PLS (polynomial local search) contains those local search problems for which objective function values can be computed efficiently and the following two problems can be solved in polynomial time: *(i)* compute an initial feasible solution; *(ii)* given an initial feasible solution, compute a strictly better neighbor or decide that no such neighbor exists.

Fabrikant et al. [41] showed that it is PLS-complete to compute a pure Nash equilibrium of a multi-commodity network unweighted congestion game with non-decreasing costs. As shown by Ackermann et al. [3], this holds even if all cost functions are linear (without a constant). The problem is also PLS-complete for non-increasing cost functions; see Syrgkanis [124]. For single-commodity network unweighted congestion games with non-decreasing costs, Fabrikant et al. [41] gave an efficient algorithm computing a pure Nash equilibrium.

Ieong et al. [73] showed that for singleton unweighted congestion games, the length of any every improvement path can be bounded by a polynomial of the input size. Thus, starting with an arbitrary strategy profile and following an arbitrary improvement path yields an efficient algorithm computing a pure Nash equilibrium. Ackermann et al. [3] extended this positive result to matroids. They also showed that the matroid property is the maximal property that ensures polynomially bounded improvement paths for arbitrary non-decreasing cost functions.

Hoefer and Skopalik [71] showed that it is strongly co-NP-hard to decide whether an unweighted congestion games possesses a strong equilibrium, even for games in which the strategy sets are simultaneously matroids and single-commodity network games. They further proved that deciding whether a given strategy profile is a strong equilibrium is strongly co-NP-hard. For two-player games both problems are still weakly co-NP-hard to decide.

Weighted Congestion Games

The negative result of Ackermann [3] implies PLS-completeness of computing a pure Nash equilibrium of a multi-commodity weighted congestion game with linear costs. For the special case of symmetric singleton weighted congestion games in which all cost functions are positive linear functions (without a constant), Fotakis et al. [50] showed that inserting the players in non-increasing order of their demands yields a pure Nash equilibrium. Gairing et al. [54] studied games with asymmetric singleton strategies. They proposed an efficient algorithm computing a pure Nash equilibrium for the special case in which all cost functions are equal and linear (without a constant).

Allowing for general cost functions and arbitrary strategy spaces, Dunkel and Schulz [33] proved that it is strongly NP-hard to decide whether a weighted congestion game possesses a pure Nash equilibrium. As noted in their paper, this negative result translates to the class of integer splittable congestion games introduced by Rosenthal [115].

Congestion Games with Resource-Dependent Demands

For games with non-singleton strategies, all negative results for weighted congestion games persist. For singleton congestion games with resource-dependent demands, the complexity of computing a pure Nash equilibria and strong equilibria are challenging open problems. As the only result is this direction, Feldman and Tamir [44] showed NP-hardness of the problem to decide for a given pure Nash equilibrium whether it is a strong equilibrium.

Congestion Games with Player-Specific Costs

For singleton games with unweighted players, the existence proof of Milchtaich [94] gives rise to an efficient algorithm computing a pure Nash equilibrium. For network congestion games it is NP-complete to decide whether a pure Nash equilibrium exists, as shown by Ackermann and Skopalik [5]. They further proved that for the special case in which the player's cost functions are identical, but each player may not use a certain subset of edges, a pure Nash equilibrium always exists, but is PLS-complete to compute.

Chapter 3

Weighted Congestion Games

In an unweighted congestion game the cost of each resource depends only on the number of players using it. Thus, these games are anonymous in the sense that for every three different players i, j, k the private cost of player i does not alter when players j and k switch their strategies while all other players remain using their strategies. This assumption is limiting for many applications. For selfish flows in telecommunication networks, for instance, the players may control streams with different sending rates that have different impact on the congestion on the resources. As a better model of this and many other situations *weighted* congestion games are studied. In such a game, every player has a strictly positive demand $d_i \in \mathbb{R}_{>0}$ that she places on the chosen resources. The cost of a resource is then a function of the aggregated demand of all players using that resource. An important subclass of weighted congestion games are weighted *network* congestion games. Here, every player is associated with a positive demand that she wants to route from her origin to her destination on a path of minimum cost. In contrast to unweighted congestion games, weighted congestion games do not always admit a pure Nash equilibrium. Fotakis et al. [51] and Libman and Orda [88] each constructed a single-commodity network instance with two players having demands one and two, respectively, and showed that these games do not have a pure Nash equilibrium. Their instances use different non-decreasing cost values per edge that are defined at the three possible loads, $1, 2, 3$. Goemans et al. [59] constructed a two-player single-commodity instance without a pure Nash equilibrium that uses different polynomial cost functions with non-negative coefficients and degree of at most two. Interestingly, Anshelevich et al. [10] showed that for cost functions of the form $c_r(x) = b_r/x$, $b_r \in \mathbb{R}_{\geq 0}$, every two-player game possesses a pure Nash equilibrium. For games with affine cost functions, Fotakis et al. [51] proved that every weighted congestion game possesses a pure Nash equilibrium. Later, Panagopoulou and Spirakis [108] proved that pure Nash equilibria always exist for instances with uniform exponential cost functions ($c_r(x) = e^x$). Harks et al. [67] generalized this existence result to non-uniform exponential cost functions of the form $c_r(x) = a_r e^{\phi x} + b_r$ for some $a_r, b_r, \phi \in \mathbb{R}$, where a_r and b_r may depend on the resource r, while ϕ must be equal for all resources. These positive results in [51, 67, 108] are particularly important as they establish the existence of pure Nash equilibria for the respective sets of cost functions *independent* of the underlying game structure, that is, independent of the underlying strategy set, demand vector, and number of players, respectively.

In this chapter, we further explore the equilibrium existence problem in weighted congestion games. Our goal is to precisely characterize which types of cost functions actually guarantee the existence of a pure Nash equilibrium. To formally capture this issue, we say that a set of cost functions $\mathcal{C}$ is *consistent for weighted congestion games* if every weighted congestion game with cost functions in $\mathcal{C}$ possesses a pure Nash equilibrium.[1] Using this terminology, the results shown in [51, 67, 108] yield that $\mathcal{C}$ is consistent if $\mathcal{C}$ consists of either only affine functions or only certain exponential functions. A natural open question is to decide whether there are further consistent functions, that is, functions guaranteeing the existence of a pure Nash equilibrium. We thus investigate the following question: How large is the set $\mathcal{C}$ of consistent cost functions? We also introduce a stricter notion of consistency which we term *universal consistency*. A set $\mathcal{C}$ of cost functions is called universally consistent, if every weighted congestion game with cost functions in $\mathcal{C}$ has the finite improvement property, that is, every sequence of unilateral improvements is finite. As every game with the finite improvement property possesses a pure Nash equilibrium (Proposition 2.8), universal consistency is a stronger property than consistency.

3.1 Contributions and Chapter Outline

To obtain a complete characterization of the equilibrium existence problem in weighted congestion games, we first derive necessary conditions. For a set $\mathcal{C}$ of continuous functions, we show that if $\mathcal{C}$ is consistent, then $\mathcal{C}$ may only contain monotonic functions. We further show that monotonicity of cost functions is necessary for consistency even in singleton games with only two players, two resources, identical cost functions and symmetric strategies. As our first main result we show that a set of continuous cost functions $\mathcal{C}$ is consistent for two-player games if and only if $\mathcal{C}$ only contains monotonic functions and for all non-constant $c_1, c_2 \in \mathcal{C}$, there are constants $a, b \in \mathbb{R}$ such that $c_1(x) = a\,c_2(x) + b$ for all $x \geq 0$. This characterization precisely explains the seeming dichotomy between the positive result of Anshelevich et al. [10] for two-player games and the two-player instances without a pure Nash equilibrium given in [51, 59, 88]. Our second main result establishes a characterization for the general case. We prove that a set $\mathcal{C}$ of continuous functions is consistent for games with at least three players if and only if exactly one of the following two cases holds: *(i)* $\mathcal{C}$ only contains affine functions; *(ii)* $\mathcal{C}$ only contains exponential functions c such that $c(x) = a_c\, e^{\phi x} + b_c$ for some $a_c, b_c, \phi \in \mathbb{R}$, where a_c and b_c may depend on c, while ϕ must be equal for every $c \in \mathcal{C}$. This characterization is even valid for three-player games. We further show that for both two-player games and games with at least three players, consistency of $\mathcal{C}$ is equivalent to universal consistency.

While the above characterizations hold for *arbitrary* strategy spaces, we also study consistency of cost functions for *restricted* strategy spaces. First, we consider weighted network congestion games. Assuming strictly positive costs, we show that essentially all results translate to directed-network weighted congestion games. For games on undirected networks, we give respective characterizations for games with two players and at least four players leaving a gap for three-player games. For singleton weighted congestion games with two players we show that $\mathcal{C}$ is consistent if and only if $\mathcal{C}$ only contains monotonic functions. This characterization does not extend to games with three

[1] The term "consistency" is due to Holzman and Law-Yone [72]. They call an allocation space *strongly consistent* if every congestion game derived from it possesses a strong equilibrium.

players. We give an instance with three players and monotonic cost functions without a pure Nash equilibrium. For symmetric singleton weighted congestion games, however, we prove that $\mathcal{C}$ is consistent if and only if $\mathcal{C}$ only contains monotonic functions. In contrast to the characterizations for arbitrary strategy spaces, both characterizations do not carry over to universal consistency. We provide corresponding instances with improvement cycles.

The proofs of our main results essentially rely on two ingredients. First, we derive two necessary conditions in Section 3.3 for continuous and consistent cost functions (Monotonicity Lemma (Lemma 3.3) and Extended Monotonicity Lemma (Lemma 3.5)). The Monotonicity Lemma states that any continuous and consistent cost function must be monotonic. The lemma is proved by constructing a generic two-player weighted congestion game in which we identify a unique improvement cycle that contains all strategy profiles. Then, we show that for any non-monotonic cost function, there is a weighted congestion game with a unique improvement cycle. By adding additional players and carefully choosing the players' weights and strategy spaces, we then derive the Extended Monotonicity Lemma, which ensures that the set of cost functions contained in a certain *finite integer linear hull* of the considered cost functions must be monotonic. By analyzing functions contained in the finite integer linear hull corresponding to two-player games, we prove in Section 3.4 that a set of continuous cost functions is consistent for two-player games if and only if all cost functions are monotone and every two non-constant cost functions are affine transformations of each other. In Section 3.5, we consider games with at least three players. We show that the Extended Monotonicity Lemma for games with at least three players implies that consistent and continuous cost functions must be either affine or exponential. In Section 3.6 and Section 3.7, we derive characterizations of consistency and universal consistency of cost functions for games with restricted strategy spaces, such as weighted network congestion games and weighted singleton congestion games, respectively.

Significance. When designing systems for selfishly acting user, there are two fundamental goals: *(i)* the system must be *stabilizable*, that is, there must exist an equilibrium state from which no player wants to unilaterally deviate; *(ii)* myopic play of the players should guide the system to an equilibrium. Because the number of players and their types expressed by the demands and the strategy spaces are only known to the players and not available to the system designer, it is natural to study the above two issues with respect to the used *cost functions*. In fact, for most applications, the cost functions are under control of the system designer since they represent the technology associated with the resources, e.g., queuing disciplines at routers, latency function in transportation networks, etc. Therefore, our results may help to predict and explain unstable traffic distributions in telecommunication networks and road networks. For instance in telecommunication networks, relevant cost functions are the so-called $M/M/1$-delay functions; see Roughgarden and Tardos [117]. These functions are of the form $c_r(x) = 1/(\tau_r - x)$, where τ_r represents the capacity of a resource r. In road networks the most frequently used functions are monomials of degree four put forward by the US Bureau of Public Roads [127]. Our results imply that, for these special types of cost functions, there is *always* a multi-commodity instance (with three players and identical cost functions) that is unstable in the sense that a pure Nash equilibrium does not exist. On the other hand, our characterizations can be used to design a stable system. For instance, uniform $M/M/1$-delay functions are consistent for two-player games.

Our results are also relevant for the large body of work quantifying the worst-case efficiency loss of pure Nash equilibria for different sets of cost functions; see Aland et al. [6], Awerbuch et al. [12], and Christodoulou and Koutsoupias [25]. While mixed Nash equilibria are guaranteed to exist, their use is often unrealistic in practice. On the other hand, our work reveals that for most classes of cost functions pure Nash equilibria may fail to exist in weighted congestion games. Thus, our work provides additional justification to study the worst-case efficiency loss for different solution concepts, such as sink equilibria (Goemans et al. [59]), correlated equilibria, or coarse correlated equilibria (Bhawalkar et al. [19], Roughgarden [116]).

The consistency approach that we pursue in this chapter is orthogonal to that of Ackermann et al. [4]. While they characterize the structure of the strategy space guaranteeing the existence of a pure Nash equilibrium assuming arbitrary positive and non-decreasing costs, we characterize the structure of cost functions guaranteeing the existence of a pure Nash equilibrium assuming arbitrary strategy spaces.

Bibliographic Information. The results presented in this chapter are joint work with Tobias Harks. An extended abstract appeared in the *Proceedings of the 37th International Colloquium on Automata, Languages and Programming, 2010*, see [63]. A more extensive version appeared in *Mathematics of Operations Research*; see [65].

3.2 Problem Description

In this section, we briefly recapitulate the most important concepts used in this chapter. For a detailed treatment, see Chapter 2. Let N be a finite set of players and R a finite set of resources. For each player i, we are given a set $\mathcal{A}_i \subseteq 2^R$ of feasible allocations and a strictly positive demand $d_i \in \mathbb{R}_{>0}$. Each resource r is endowed with a cost function $c_r : \mathbb{R}_{\geq 0} \to \mathbb{R}$. The tuple $\mathcal{M} = \big(N, R, (\mathcal{A}_i)_{i\in N}, (c_r)_{r\in R}\big)$ is called a congestion model. The corresponding weighted congestion game is the minimization game with $S_i = \mathcal{A}_i$ and $\pi_i(s) = \sum_{r\in s_i} d_i\, c_r(\ell_r(s))$ for all $i \in N$, where $\ell_r(s) = \sum_{j\in N: r\in s_j} d_j$ is the load of resource r under strategy profile s. A set $\mathcal{C}$ of functions is called *consistent* for weighted congestion games if every weighted congestion game with cost functions in $\mathcal{C}$ possesses a pure Nash equilibrium. $\mathcal{C}$ is *universally consistent* for weighted congestion games if every weighted congestion game with cost functions in $\mathcal{C}$ has the finite improvement property.

3.3 Necessary Conditions for the Existence of a Pure Nash Equilibrium

We start investigating necessary conditions for the consistency of a set of cost functions $\mathcal{C}$. We first need the following characterizations of continuous monotonic functions.

Lemma 3.1. *For a continuous function $c : \mathbb{R}_{\geq 0} \to \mathbb{R}$ the following three statements are equivalent:*

(1) *c is monotonic on $\mathbb{R}_{\geq 0}$.*

(2) *c is monotonic on each open interval $(a,b) \subset \mathbb{R}_{\geq 0}$ with $c(x) \neq c(0)$ for all $x \in (a,b)$.*

(3) *The following two conditions hold:*

- (3a) *For all $x > 0$ with $c(x) > c(0)$ there is $\varepsilon > 0$ such that $c(y) \geq c(x)$ for all $y \in (x, x+\varepsilon)$.*
- (3b) *For all $x > 0$ with $c(x) < c(0)$ there is $\varepsilon > 0$ such that $c(y) \leq c(x)$ for all $y \in (x, x+\varepsilon)$.*

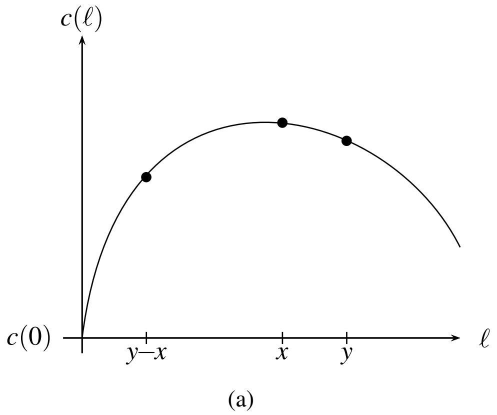


Figure 3.1: For every continuous non-monotonic function there are $x, y \in \mathbb{R}_{>0}$ with $x < y$ such that one of the following cases holds: (a) $c(y-x) < c(y) < c(x)$; (b) $c(y-x) > c(y) > c(x)$; see Lemma 3.2.

Proof. (1) $\Rightarrow$ (3): Trivial.

(3) $\Rightarrow$ (2): Let (a,b) be an open interval with $c(x) \neq c(0)$ for all $x \in (a,b)$. The intermediate value theorem for continuous functions implies that either $c(x) > c(0)$ for all $x \in (a,b)$ or $c(x) < c(0)$ for all $x \in (a,b)$. We prove the result only for the first case because the second follows by the same arguments.

Let $c(x) > c(0)$ for all $x \in (a,b)$. We claim that c is non-decreasing on (a,b). Assume not, for a contradiction. Then, there are $p_1, p_2 \in (a,b)$ with $p_1 < p_2$ and $c(p_1) > c(p_2)$. We define $p_1' = \max\{x \in [p_1, p_2] : c(x) \geq c(p_1)\}$. Note that the set $\{x \in [p_1, p_2] : c(x) \geq c(p_1)\}$ is nonempty because it contains p_1 and closed because c is continuous. Using (3a), there is $\varepsilon = \varepsilon(p_1') > 0$ such that $c(y) \geq c(p_1') \geq c(p_1)$ for all $y \in (p_1', p_1' + \varepsilon)$, contradicting the maximality of p_1'.

(2) $\Rightarrow$ (1): If c is constant, we are done. Otherwise, let $a = \inf\{x > 0 : c(x) \neq c(0)\}$. Roughly speaking, a is the largest element in $\mathbb{R}_{\geq 0}$ such that $c(x) = c(0)$ for all $x \in [0,a]$. We claim that $c(x) \neq c(0)$ for all $x > a$. For a contradiction, assume that there is $x > a$ with $c(x) = c(0)$ and let $b = \min\{x > a : c(x) = c(0)\}$. By construction, $c(x) \neq c(0)$ for all $x \in (a,b)$. Using (2), we derive that c is monotonic on (a,b). The continuity of c implies that c is monotonic on $[a,b]$. This is a contradiction to $c(a) = c(b) = c(0)$ and $c(\frac{a+b}{2}) \neq c(0)$. We conclude that $c(x) \neq c(0)$ for all $x > a$. Using (2), this however implies that c is monotonic on (a,b') for all $b' \geq b$. The fact that $c(x) = c(0)$ for all $x \in [0,a]$ gives the claimed result. □

The following existence result for continuous, non-monotonic functions can be derived directly from Lemma 3.1 and will be very useful in the remainder of this chapter. It states that for each continuous, non-monotonic function c there are $x, y \in \mathbb{R}_{>0}$ with $x < y$ such that either $c(y-x) < c(y) < c(x)$ or $c(y-x) > c(y) > c(x)$, see Figure 3.1 for an illustration.

Lemma 3.2. *For a continuous and non-monotonic function $c : \mathbb{R}_{\geq 0} \to \mathbb{R}$ there are $x, y \in \mathbb{R}_{>0}$ with $x < y$ such that either $c(y-x) < c(y) < c(x)$ or $c(y-x) > c(y) > c(x)$.*

Proof. Using the characterization of monotonic functions of Lemma 3.1, for every continuous non-monotonic function c, there is $x > 0$ such that one of the following holds: $c(x) > c(0)$ and for every

$\varepsilon > 0$ there is $y = y(\varepsilon) \in (x, x+\varepsilon)$ such that $c(y) < c(x)$; or $c(x) < c(0)$ and for every $\varepsilon > 0$ there is $y = y(\varepsilon) \in (x, x+\varepsilon)$ such that $c(y) > c(x)$. Fix such x. Because of the continuity of c, we have $c(y(\varepsilon) - x) \to c(0)$ and $c(y(\varepsilon)) \to c(x)$ for $\varepsilon \to 0$. For sufficiently small ε, x and $y(\varepsilon)$ have the desired property. □

Now consider a resource r with a non-monotonic cost function c and two players with demands $d_1 = y - x$ and $d_2 = x$, where x and y are as in Lemma 3.2. Clearly, in case $c(y-x) < c(y) < c(x)$ player 1 prefers to be alone on r since $c(y-x) < c(y)$ while player 2 would like to share the resource with player 1 since $c(y) < c(x)$. If $c(y-x) > c(y) > c(x)$, player 1 prefers to share the resource with player 2 while player 2 prefers to be alone. This observation is the key for constructing a two-player weighted congestion game with singleton strategies that does not admit a pure Nash equilibrium.

Lemma 3.3 (Monotonicity Lemma). *Let $\mathcal{C}$ be a set of continuous functions. If $\mathcal{C}$ is consistent for weighted congestion games, then every $c \in C$ is monotonic.*

Proof. For a contradiction, suppose that $\mathcal{C}$ is consistent but there is a non-monotonic function $c \in \mathcal{C}$. Consider the congestion model $\mathcal{M} = \big(N, R, (\mathcal{A}_i)_{i \in N}, (c_r)_{r \in R}\big)$ with $N = \{1,2\}$, $R = \{r_1, r_2\}$, $\mathcal{A}_1 = \mathcal{A}_2 = \big\{\{r_1\},\{r_2\}\big\}$, $c_{r_1} = c_{r_2} = c$. Applying Lemma 3.2 we find $x, y \in \mathbb{R}_{>0}$ with $x < y$ such that either $c(y-x) < c(y) < c(x)$ or $c(y-x) > c(y) > c(x)$. For the weighted congestion game G with demands $d_1 = y - x$ respectively $d_2 = x$ the private costs are shown in Figure 3.2. Calculating the differences of the deviating players' private costs along the cycle

$$\gamma = \big((\{r_1\},\{r_1\}), (\{r_2\},\{r_1\}), (\{r_2\},\{r_2\}), (\{r_1\},\{r_2\}), (\{r_1\},\{r_1\})\big),$$

we obtain

$$\pi_1(\{r_2\},\{r_1\}) - \pi_1(\{r_1\},\{r_1\}) = (y-x)\big(c(y-x) - c(y)\big), \tag{3.1}$$
$$\pi_2(\{r_1\},\{r_2\}) - \pi_2(\{r_2\},\{r_1\}) = x\big(c(y) - c(x)\big), \tag{3.2}$$
$$\pi_1(\{r_1\},\{r_2\}) - \pi_1(\{r_2\},\{r_2\}) = (y-x)\big(c(y-x) - c(x)\big), \tag{3.3}$$
$$\pi_2(\{r_1\},\{r_1\}) - \pi_2(\{r_1\},\{r_2\}) = x\big(c(y) - c(x)\big). \tag{3.4}$$

If $c(y-x) < c(y) < c(x)$, the differences (3.1)-(3.4) are negative and γ is an improvement cycle. If $c(y-x) > c(y) > c(x)$, we can reverse the direction of γ and still get an improvement cycle. Using that every strategy profile of G is contained in γ, the claimed result follows. □

Besides the continuity of the functions in $\mathcal{C}$, the proof of Lemma 3.3 relies on rather mild assumptions and thus, this result can be strengthened in the following way.

Corollary 3.4. *Let $\mathcal{C}$ be a set of continuous functions. Let $\mathscr{G}(\mathcal{C})$ be the set of weighted congestion games with cost functions in $\mathcal{C}$ satisfying one or more of the following properties: (i) Each game $G \in \mathscr{G}(\mathcal{C})$ has two players; (ii) Each game $G \in \mathscr{G}(\mathcal{C})$ has two resources; (iii) For each game $G \in \mathscr{G}(\mathcal{C})$ and each player $i \in N$, the set of her strategies S_i contains a single resource only; (iv) Each game $G \in \mathscr{G}(\mathcal{C})$ has symmetric strategies; (v) Cost functions are identical, that is, $c_{r_1} = c_{r_2}$ for all $r_1, r_2 \in R$. If $\mathcal{C}$ is consistent for $\mathscr{G}(\mathcal{C})$, then each $c \in \mathcal{C}$ is monotonic.*

	Player 2: $\{r_1\}$	Player 2: $\{r_2\}$
Player 1: $\{r_1\}$	$(y-x)c(y),\ xc(y)$	$(y-x)c(y-x),\ xc(x)$
Player 1: $\{r_2\}$	$(y-x)c(y-x),\ xc(x)$	$(y-x)c(y),\ xc(y)$

Figure 3.2: Matrix representation of the weighted congestion game constructed in the proof of Lemma 3.3. Each cell of the matrix shows the private cost vector of the corresponding strategy profile.

We now extend the Monotonicity Lemma to obtain an even stronger result by considering more players and more complex strategies. To this end, for $K \in \mathbb{N}$ we consider those functions that can be written as the integral linear combination of K functions in $\mathcal{C}$, possibly with an offset. Formally, we define the *finite integer linear hull* of $\mathcal{C}$ as

$$\mathcal{L}_{\mathbb{Z}}(\mathcal{C}) = \left\{ c : \mathbb{R}_{\geq 0} \to \mathbb{R} : c(x) = \sum_{k=1}^{K} a_k c_k(x+b_k) : K \in \mathbb{N}, a_k \in \mathbb{Z}, b_k \in \mathbb{R}_{\geq 0}, c_k \in \mathcal{C} \right\}, \tag{3.5}$$

and show that consistency of $\mathcal{C}$ implies that $\mathcal{L}_{\mathbb{Z}}(\mathcal{C})$ only contains monotonic functions.

Lemma 3.5 (Extended Monotonicity Lemma). *Let $\mathcal{C}$ be a set of continuous functions. If $\mathcal{C}$ is consistent, then $\mathcal{L}_{\mathbb{Z}}(\mathcal{C})$ only contains monotonic functions.*

Proof. For a contradiction, suppose that $\mathcal{C}$ is consistent but there is a function $c \in \mathcal{L}_{\mathbb{Z}}(\mathcal{C})$ that is not monotonic. Using that $c \in \mathcal{L}_{\mathbb{Z}}(\mathcal{C})$, we can write c as $c(x) = \sum_{k=1}^{m_+} c_k(x+b_k) - \sum_{k=1}^{m_-} \bar{c}_k(x+\bar{b}_k)$ for some $c_k, \bar{c}_k \in \mathcal{C}, m_+, m_- \in \mathbb{N}$, and $b_k, \bar{b}_k \in \mathbb{R}_{\geq 0}$. Note that we can omit the integer coefficients a_k as we allow $c_k = c_l$ for $k \neq l$.

We define the congestion model $\mathcal{M} = \big(N, R, (\mathcal{A}_i)_{i \in N}, (c_r)_{r \in R}\big)$, where $N = \{1,2\} \cup N^+ \cup N^-$ and $R = R^1 \cup R^2 \cup R^3 \cup R^4$. The set of players N^+ contains for each c_k, $1 \leq k \leq m_+$, a player with demand b_k and the set of players N^- contains for each $\bar{c}_k$, $1 \leq k \leq m_-$, a player with demand $\bar{b}_k$. We call the players in $N^- \cup N^+$ *offset* players. Offset players with demand equal to 0 can be removed from the game. For ease of exposition, we assume that all offsets b_k, $k = 1 \ldots, m_+$, and $\bar{b}_k$, $k = 1, \ldots, m_-$, are strictly positive.

We now explain the strategy spaces and the sets R^1, R^2, R^3, R^4. For each function c_k, $1 \leq k \leq m_+$, we introduce two resources r_k^2, r_k^3 with cost function c_k. For each function $\bar{c}_k$, $1 \leq k \leq m_-$, we introduce two resources r_k^1, r_k^4 with cost function $\bar{c}_k$. To model the offsets b_k in (3.5), for each offset player $k \in N^+$, we define $\mathcal{A}_k = \big\{\{r_k^2, r_k^3\}\big\}$. Similarly, for each offset player $k \in N^-$, we set $\mathcal{A}_k = \big\{\{r_k^1, r_k^4\}\big\}$. The nontrivial players 1 and 2 have strategies $\mathcal{A}_1 = \{R^1 \cup R^2, R^3 \cup R^4\}$ and $\mathcal{A}_2 = \{R^1 \cup R^3, R^2 \cup R^4\}$, where $R^1 = \{r_1^1, \ldots, r_{m_-}^1\}$, $R^2 = \{r_1^2, \ldots, r_{m_+}^2\}$, $R^3 = \{r_1^3, \ldots, r_{m_+}^3\}$, and $R^4 = \{r_1^4, \ldots, r_{m_-}^4\}$. As c is assumed to be non-monotonic, by Lemma 3.2, there are $x, y \in \mathbb{R}_{>0}$ with $x < y$ such that either $c(y-x) < c(y) < c(x)$ or $c(y-x) > c(y) > c(x)$. We consider the weighted congestion game G for which the demands of players 1 and 2 equal $d_1 = y - x$ and $d_2 = x$, respectively. The private costs of players 1 and 2 in G are shown in Figure 3.3.

		Player 2	
		$R^1 \cup R^3$	$R^2 \cup R^4$
Player 1	$R^1 \cup R^2$	$(y-x)\big(\sum \bar{c}_k(d_1+d_2+\bar{b}_k)+\sum c_k(d_1+b_k)\big)$, $x\big(\sum \bar{c}_k(d_1+d_2+\bar{b}_k)+\sum c_k(d_2+b_k)\big)$	$(y-x)\big(\sum c_k(d_1+d_2+b_k)+\sum \bar{c}_k(d_1+\bar{b}_k)\big)$, $x\big(\sum c_k(d_1+d_2+b_k)+\sum \bar{c}_k(d_2+\bar{b}_k)\big)$
	$R^3 \cup R^4$	$(y-x)\big(\sum c_k(d_1+d_2+b_k)+\sum \bar{c}_k(d_1+\bar{b}_k)\big)$, $x\big(\sum c_k(d_1+d_2+b_k)+\sum \bar{c}_k(d_2+\bar{b}_k)\big)$	$(y-x)\big(\sum \bar{c}_k(d_1+d_2+\bar{b}_k)+\sum c_k(d_1+b_k)\big)$, $x\big(\sum \bar{c}_k(d_1+d_2+\bar{b}_k)+\sum c_k(d_2+b_k)\big)$

Figure 3.3: Matrix representation of the weighted congestion game constructed in the proof of Lemma 3.5. In each cell of the matrix the upper and lower entry correspond the private cost of player 1 and player 2, respectively. All summations involving c_k are from $k=1$ to m_+; summations involving $\bar{c}_k$ for are from $k=1$ to m_-. The strategies and private costs of all offset players are not shown.

All offset players have a single strategy only. Their unique strategy profile is denoted by $\alpha_{-\{1,2\}}$. For the cycle $\gamma = \big((R^1\cup R^2, R^1\cup R^3, \alpha_{-\{1,2\}}), (R^3\cup R^4, R^1\cup R^3, \alpha_{-\{1,2\}}), (R^3\cup R^4, R^2\cup R^4, \alpha_{-\{1,2\}}), (R^1\cup R^2, R^2\cup R^4, \alpha_{-\{1,2\}}), (R^1\cup R^2, R^1\cup R^3, \alpha_{-\{1,2\}})\big)$, it is straightforward to calculate that

$$\begin{aligned}
&\pi_1(R^3\cup R^4, R^1\cup R^3, \alpha_{-\{1,2\}}) - \pi_1(R^1\cup R^2, R^1\cup R^3, \alpha_{-\{1,2\}})\\
&\quad = (y-x)\Big(\sum_{k=1}^{m_+} c_k(d_1+d_2+b_k) - \sum_{k=1}^{m_-}\bar{c}_k(d_1+d_2+\bar{b}_k) + \sum_{k=1}^{m_-}\bar{c}_k(d_1+\bar{b}_k) - \sum_{k=1}^{m_+} c_k(d_1+b_k)\Big)\\
&\quad = (y-x)\big(c(y)-c(y-x)\big),
\end{aligned}$$

$$\begin{aligned}
&\pi_2(R^3\cup R^4, R^2\cup R^4, \alpha_{-\{1,2\}}) - \pi_2(R^3\cup R^4, R^1\cup R^3, \alpha_{-\{1,2\}})\\
&\quad = x\Big(-\sum_{k=1}^{m_+} c_k(d_1+d_2+b_k) + \sum_{k=1}^{m_-}\bar{c}_k(d_1+d_2+\bar{b}_k) + \sum_{k=1}^{m_+} c_k(d_2+b_k) - \sum_{k=1}^{m_-}\bar{c}_k(d_2+\bar{b}_k)\Big)\\
&\quad = x\big(c(x)-c(y)\big),
\end{aligned}$$

$$\begin{aligned}
&\pi_1(R^1\cup R^2, R^2\cup R^4, \alpha_{-\{1,2\}}) - \pi_1(R^3\cup R^4, R^2\cup R^4, \alpha_{-\{1,2\}})\\
&\quad = (y-x)\Big(\sum_{k=1}^{m_+} c_k(d_1+d_2+b_k) - \sum_{k=1}^{m_-}\bar{c}_k(d_1+d_2+\bar{b}_k) + \sum_{k=1}^{m_-}\bar{c}_k(d_1+\bar{b}_k) - \sum_{k=1}^{m_+} c_k(d_1+b_k)\Big)\\
&\quad = (y-x)\big(c(y)-c(y-x)\big),
\end{aligned}$$

$$\begin{aligned}
&\pi_2(R^1\cup R^2, R^1\cup R^3, \alpha_{-\{1,2\}}) - \pi_2(R^1\cup R^2, R^2\cup R^4, \alpha_{-\{1,2\}})\\
&\quad = x\Big(-\sum_{k=1}^{m_+} c_k(d_1+d_2+b_k) + \sum_{k=1}^{m_-}\bar{c}_k(d_1+d_2+\bar{b}_k) - \sum_{k=1}^{m_-}\bar{c}_k(d_2+\bar{b}_k) + \sum_{k=1}^{m_+} c_k(d_2+b_k)\Big)\\
&\quad = x\big(c(x)-c(y)\big).
\end{aligned}$$

If $c(y-x) > c(y) > c(x)$, all private cost differences are negative and γ is an improvement cycle; if, on the other hand, $c(y-x) < c(y) < c(x)$, the 4-cycle in the other direction is an improvement cycle. Because every strategy combination is contained in γ we get the claimed result. □

3.4 A Characterization for Two-Player Games

We start by analyzing implications of the Extended Monotonicity Lemma (Lemma 3.5) for two-player weighted congestion games. For our analysis, it is sufficient to restrict ourselves to the special case $K = 2$, that is, we only consider those functions that can be written as an integral linear combination of two functions. We define the following set of functions

$$\mathcal{L}^2_{\mathbb{Z}}(\mathcal{C}) = \left\{c : \mathbb{R}_{\geq 0} \to \mathbb{R} : c(x) = a_1\, c_1(x) + a_2\, c_2(x), a_1, a_2 \in \mathbb{Z}, c_1, c_2 \in \mathcal{C}\right\} \subseteq \mathcal{L}_{\mathbb{Z}}(\mathcal{C}).$$

We remark that by setting all offsets b_k in (3.5) equal to zero, the construction in the proof of Lemma 3.5 only involves two players. Thus, we immediately obtain the following result.

Corollary 3.6 (Extended Monotonicity Lemma for Two-Player Games). *Let $\mathcal{C}$ be a set of continuous functions. If $\mathcal{C}$ is consistent for two-player weighted congestion games, then $\mathcal{L}^2_{\mathbb{Z}}(\mathcal{C})$ contains only monotonic functions.*

We proceed to investigate sets of functions $\mathcal{C}$ that guarantee that $L^2_{\mathbb{Z}}(\mathcal{C})$ only contains monotonic functions.

Lemma 3.7. *Let $c_1, c_2 : \mathbb{R}_{\geq 0} \to \mathbb{R}$ be two continuous, monotonic and non-constant functions. Then, the following are equivalent:*

(1) *For all $a_1, a_2 \in \mathbb{Z}$, the function $a_1\, c_1 + a_2\, c_2$ is monotonic on $\mathbb{R}_{\geq 0}$.*

(2) *There are $a, b \in \mathbb{R}$ such that $c_2(x) = a\, c_1(x) + b$ for all $x \geq 0$.*

Proof. (2) ⇒ (1): Calculus.

(1) ⇒ (2): Without loss of generality, we may assume that c_1 and c_2 are non-decreasing because multiplying functions with -1 has no impact on the statements (1) and (2). As c_1 is non-constant and non-decreasing, there is $x_1 \geq 0$ with $c_1(x_1) = c_1(0)$ and $c_1(x) > c_1(0)$ for all $x > x_1$. Fix such $x > x_1$. For all $a_1, a_2 \in \mathbb{Z}$, the function $a_1\, c_1 + a_2\, c_2$ is monotonic. This implies that for every $y > x_1$ and every $\mu \in \mathbb{Q}$ the expressions $\mu\, c_1(x) + c_2(x) - \mu\, c_1(0) - c_2(0)$ and $\mu\, c_1(y) + c_2(y) - \mu\, c_1(0) - c_2(0)$ have identical signs. Thus, for all $y > x_1$ and all $\mu \in \mathbb{Q}$ at least one of the following two cases holds:

$$\mu \geq \max\left\{-\frac{c_2(x)-c_2(0)}{c_1(x)-c_1(0)}, -\frac{c_2(y)-c_2(0)}{c_1(y)-c_1(0)}\right\} \quad \text{or} \quad \mu \leq \min\left\{-\frac{c_2(x)-c_2(0)}{c_1(x)-c_1(0)}, -\frac{c_2(y)-c_2(0)}{c_1(y)-c_1(0)}\right\}.$$

This implies

$$\frac{c_2(y)-c_2(0)}{c_1(y)-c_1(0)} = \frac{c_2(x)-c_2(0)}{c_1(x)-c_1(0)} \tag{3.6}$$

for all $y > x_1$, because otherwise any rational

$$\mu \in \left(\min\left\{-\frac{c_2(y)-c_2(0)}{c_1(y)-c_1(0)}, -\frac{c_2(x)-c_2(0)}{c_1(x)-c_1(0)}\right\}, \max\left\{-\frac{c_2(y)-c_2(0)}{c_1(y)-c_1(0)}, -\frac{c_2(x)-c_2(0)}{c_1(x)-c_1(0)}\right\}\right)$$

would violate both constraints. From (3.6), we obtain

$$c_2(y) = \frac{c_2(x) - c_2(0)}{c_1(x) - c_1(0)} \cdot c_1(y) - \frac{c_2(x) - c_2(0)}{c_1(x) - c_1(0)} \cdot c_1(0) + c_2(0)$$

for all $y > x_1$ and fixed x. This shows the existence of $a, b \in \mathbb{R}$ with $c_2(x) = a\,c_1(x) + b$ for all $x > x_1$. Exchanging the roles of c_1 and c_2, we may also derive the existence of $a', b' \in \mathbb{R}$ such that $c_1(x) = a'\,c_2(x) + b'$ for all $x > x_2$, where x_2 is such that $c_2(x_2) = c_2(0)$ and $c_2(x) > c_2(0)$ for all $x > x_2$. This implies $x_1 = x_2$. Using the fact that c_1 and c_2 are continuous and constant on $[0, x_1]$ finishes the proof. □

We are now ready to prove our first main result.

Theorem 3.8. *For a set $\mathcal{C}$ of continuous functions the following are equivalent:*

(1) *$\mathcal{C}$ is consistent for two-player weighted congestion games.*

(2) *$\mathcal{C}$ is universally consistent for two-player weighted congestion games.*

(3) *$\mathcal{C}$ only contains monotonic functions and for all non-constant $c_1, c_2 \in \mathcal{C}$, there are constants $a, b \in \mathbb{R}$ such that $c_1(x) = a\,c_2(x) + b$ for all $x \geq 0$.*

Proof. (2) ⇒ (1) follows because every game with the finite improvement property has a pure Nash equilibrium (Proposition 2.8).

(1) ⇒ (3): Using Corollary 3.6 we derive that $\mathcal{L}^2_{\mathbb{Z}}(\mathcal{C})$ only contains monotonic functions. As $\mathcal{C} \subseteq \mathcal{L}^2_{\mathbb{Z}}(\mathcal{C})$, this implies in particular that $\mathcal{C}$ only contains monotonic functions. For all non-constant functions $c_1, c_2 \in \mathcal{C}$ and all $a_1, a_2 \in \mathbb{Z}$ the function $a_1 c_1 + a_2 c_2 \in \mathcal{L}^2_{\mathbb{Z}}(\mathcal{C})$ is monotonic. Applying Lemma 3.7 then yields the result.

(3) ⇒ (2): Let $\mathcal{C}$ be as specified in (3). Trivially, the claimed result holds if $\mathcal{C}$ only contains constant functions. If $\mathcal{C}$ contains a non-constant function c consider the set $\bar{\mathcal{C}} = \{a\,c(x) + b : a, b \in \mathbb{R}\} \supseteq \mathcal{C}$. We show that $\bar{\mathcal{C}}$ is consistent for $\mathscr{G}^2(\bar{\mathcal{C}})$. To this end, consider an arbitrary two-player game with costs in $\bar{\mathcal{C}}$ and demands $d_1 < d_2$. We distinguish the following three cases.

First case: $c(d_1) < c(d_2) < c(d_1 + d_2)$, or $c(d_1) > c(d_2) > c(d_1 + d_2)$. Since c is strictly monotonic with respect to the points d_1, d_2 and $d_1 + d_2$, there is a strictly monotonic function $\tilde{c}$ with $\tilde{c}(d_1) = c(d_1)$, $\tilde{c}(d_2) = c(d_2)$ and $\tilde{c}(d_1 + d_2) = c(d_1 + d_2)$. Consequently, we can replace every cost function $c \in \bar{\mathcal{C}} = \{a\,c(x) + b : a, b \in \mathbb{R}\}$ by a cost function $\tilde{c} \in \tilde{\mathcal{C}} = \{a\,\tilde{c}(x) + b : a, b \in \mathbb{R}\}$ without changing the players' private costs. As shown by Harks et al. [67], for any strictly monotonic function $\tilde{c}$, every weighted congestion game G with two players and cost functions in $\bar{\mathcal{C}} = \{a\,\tilde{c}(x) + b : a, b \in \mathbb{R}\}$ admits a potential function and thus, has the finite improvement property.

Second case: $c(d_1) = c(d_2)$. We set $\tilde{d}_1 = \tilde{d}_2 = 1$ and choose for every resource $r \in R$ a new cost function $\tilde{c}_r$ with $\tilde{c}_r(1) = c_r(d_1) = c_r(d_2)$ and $\tilde{c}_r(2) = c_r(d_1 + d_2)$. By construction, the unweighted congestion game with demands $\tilde{d}_1$, $\tilde{d}_2$ and costs $(\tilde{c}_f)_{f \in F}$ has the same private costs as the original game. Rosenthal [114] showed the existence of a potential function in all unweighted congestion games; thus, the game has the finite improvement property.

Third case: $c(d_2) = c(d_1 + d_2)$. We have $\bar{c}(d_2) = \bar{c}(d_1 + d_2)$ for all $\bar{c} \in \bar{\mathcal{C}}$ and thus player 2 is always indifferent whether player 1 shares a resource with her or not. For the finite improvement property and the existence of a pure Nash equilibrium, we argue as follows: Consider the vector-valued function $\phi : S \to \mathbb{R}^2$, $s \mapsto \big(\pi_2(s), \pi_1(s)\big)$ which assigns to every strategy profile the vector which has the private cost of players 2 and 1 in first and second component, respectively. We claim that ϕ decreases lexicographically along any improvement path. This is trivial for improvement moves of player 2. Since player 2 is indifferent whether player 1 shares with her a resource or not, every improvement move of player 1 does not affect the private cost of player 2 but decreases the private cost of player 1. This implies that the lexicographical order of $\phi(s)$ decreases along any improvement path; thus, every such path is cycle-free and thus finite. $\square$

3.5 A Characterization for the General Case

We now consider the case $n \geq 3$; that is, we consider weighted congestion games with at least three players. We will show that a set of continuous cost functions is consistent if and only if this set contains either only linear or only certain exponential functions. For the proof it is sufficient to only consider three-player games. However, the result easily translates to games with more than three players by adding additional players whose allocations do not intersect with the first three players. We proceed to analyze implications of the Extended Monotonicity Lemma (Lemma 3.5) for three-player weighted congestion games. To this end, we will consider a slightly different set of functions that is linked to three-player games. Formally, define

$$\mathcal{L}^3_{\mathbb{Z}}(\mathcal{C}) = \left\{ c : \mathbb{R}_{\geq 0} \to \mathbb{R} : c(x) = a_1\, c_1(x) + a_2\, c_1(x+\delta), a_1, a_2 \in \mathbb{Z}, c_1 \in \mathcal{C}, \delta \in \mathbb{R}_{>0} \right\} \subseteq \mathcal{L}_{\mathbb{Z}}(\mathcal{C}).$$

Note that $\mathcal{L}^3_{\mathbb{Z}}(\mathcal{C})$ involves a single offset $\delta > 0$, which requires only three players in the construction of the proof of the Extended Monotonicity Lemma. However, regarding three-player games in which the strategy available to the third player does not intersect with the strategies of the first two players we still get as a necessary condition that $\mathcal{L}^2_{\mathbb{Z}}(\mathcal{C})$ may only contain monotonic functions. We, thus, obtain the following result.

Corollary 3.9 (Extended Monotonicity Lemma for Three-Player Games). *Let $\mathcal{C}$ be a set of continuous functions. If $\mathcal{C}$ is consistent for three-player weighted congestion games, then both $\mathcal{L}^2_{\mathbb{Z}}(\mathcal{C})$ and $\mathcal{L}^3_{\mathbb{Z}}(\mathcal{C})$ contain only monotonic functions.*

We proceed to characterize the sets of cost functions $\mathcal{C}$ for which $\mathcal{L}^3_{\mathbb{Z}}(\mathcal{C})$ only contains monotonic functions.

Lemma 3.10. *For a set $\mathcal{C}$ of continuous functions the following two statements are equivalent:*

(1) *$\mathcal{L}^3_{\mathbb{Z}}(\mathcal{C})$ only contains monotonic functions.*

(2) *For every $c \in \mathcal{C}$ either $c(x) = a\,e^{\phi x} + b$ for some $a, b, \phi \in \mathbb{R}$, or $c(x) = ax + b$ for some $a, b \in \mathbb{R}$.*

Proof. (2) $\Rightarrow$ (1): Let $c \in \mathcal{C}$ be an exponential or an affine function. By simple calculus one can verify that every function $\tilde{c}(x) = a_1\, c(x) + a_2\, c(x+\delta)$ with $a_1, a_2 \in \mathbb{Z}$, $\delta \in \mathbb{R}_{>0}$ is exponential if c is exponential and affine if c is affine. Both affine functions and exponential functions are monotonic.

(1) $\Rightarrow$ (2): For a contradiction, let us assume that $\mathcal{L}^3_{\mathbb{Z}}(\mathcal{C})$ only contains monotonic functions but that there is a function $c \in \mathcal{C}$ that is neither affine nor exponential. As $\mathcal{L}^3_{\mathbb{Z}}(\mathcal{C})$ only contains monotonic functions, for all $\delta > 0$ and all $a_1, a_2 \in \mathbb{Z}$ the function $\tilde{c} : \mathbb{R}_{\geq 0} \to \mathbb{R}, x \mapsto a_1\, c(x) + a_2\, c(x+\delta)$ is

monotonic. Referring to Lemma 3.7, this implies that for all $\delta > 0$ there are $a, b \in \mathbb{R}$ such that for all $x \geq 0$:

$$c(x+\delta) = a\,c(x) + b. \tag{3.7}$$

As $c \in \mathcal{C}$ is neither affine nor exponential on $\mathbb{R}_{\geq 0}$, there are four points $0 \leq p_1 < p_2 < p_3 < p_4$ following neither an exponential nor an affine law, i.e. there are neither $\alpha, \beta, \phi \in \mathbb{R}$ such that $c(p_i) = \alpha\, e^{\phi p_i} + \beta$ for all $i \in \{1,2,3,4\}$ nor are there $\alpha, \beta \in \mathbb{R}$ such that $c(p_i) = \alpha\, p_i + \beta$ for all $i \in \{1,2,3,4\}$. As c is continuous, we may assume without loss of generality that p_1, p_2, p_3, p_4 are rational and we write them as $p_1 = 2m_1/(2k), \dots, p_4 = 2m_4/(2k)$ for some $m_1, m_2, m_3, m_4, k \in \mathbb{N}$. For $\delta = 1/k$ we derive from (3.7) that there are $a, b \in \mathbb{R}$ such that for all $n \in \mathbb{N}$:

$$c\big((n+1)/k\big) = a\,c\big(n/k\big) + b, \tag{3.8}$$
$$c\big((n+2)/k\big) = a\,c\big((n+1)/k\big) + b. \tag{3.9}$$

Subtracting (3.8) from (3.9) and rearranging terms, we obtain for all $n \in \mathbb{N}$:

$$c\big((n+2)/k\big) - (a+1)\,c\big((n+1)/k\big) + a\,c\big(n/k\big)\big) = 0. \tag{3.10}$$

The above equation defines a second-order linear homogeneous recurrence relation on the sequence $c\big(n/k\big)_{n\in\mathbb{N}}$. Such recurrence relations can be solved with the method of characteristic equations; see the textbook of Balakrishnan [14, §3.2] for more details. The characteristic equation of the recurrence relation equals $x^2 - (a+1)x + a = (x-1)(x-a)$. If $a \neq 1$, then the characteristic equation has two distinct roots and we obtain for even m that

$$c\big(m/k\big) = \beta \cdot 1^m + \alpha \cdot a^m = \beta + \alpha \cdot |a|^m = \alpha \cdot \exp(m \ln |a|) + \beta$$

for some constants $\alpha, \beta \in \mathbb{R}$. If, on the other hand, $a = 1$, we can evaluate $c\big(m/k\big)$ as

$$c\big(m/k\big) = \beta \cdot 1^m + \alpha m \cdot 1^m = \alpha \cdot m + \beta$$

for some constants $\alpha, \beta \in \mathbb{R}$. □

We are now ready to state our second main theorem.

Theorem 3.11. *For a set $\mathcal{C}$ of continuous functions the following three statements are equivalent:*

(1) *$\mathcal{C}$ is consistent for weighted congestion games.*

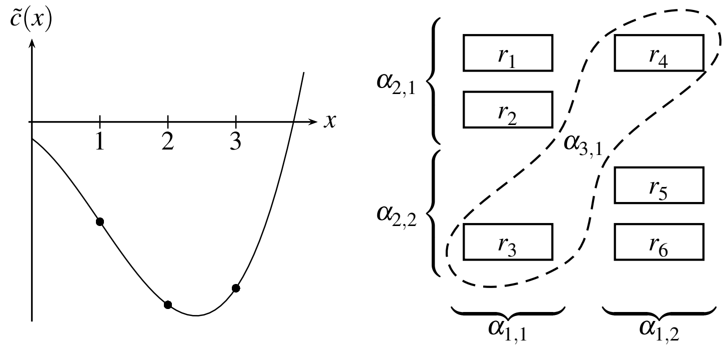
(2) *$\mathcal{C}$ is universally consistent for weighted congestion games.*

(3) *$\mathcal{C}$ only contains affine functions* or *$\mathcal{C}$ only contains functions of type $c(x) = a_c\, e^{\phi x} + b_c$ where $a_c, b_c \in \mathbb{R}$ may depend on c while $\phi \in \mathbb{R}$ is independent of c.*

Proof. (2) $\Rightarrow$ (1) follows because every game with the finite improvement property has a pure Nash equilibrium (Proposition 2.8).

(3) $\Rightarrow$ (2) follows because every weighted congestion game with such cost functions possesses a weighted potential; see Fotakis et al. [51], Harks et al. [67], and Panagopoulou and Spirakis [108].

(1) $\Rightarrow$ (3): By Corollary 3.9 both $\mathcal{L}^2_{\mathbb{Z}}(\mathcal{C})$ and $\mathcal{L}^3_{\mathbb{Z}}(\mathcal{C})$ may only contain monotonic functions. Applying Lemma 3.10 we obtain that every $c \in \mathcal{C}$ is either affine or exponential. In addition, as shown in Lemma 3.7 for each two non-constant functions $c_1, c_2 \in \mathcal{C}$, there are $a, b \in \mathbb{R}$ such that $c_2(x) = a\,c_1(x) + b$ for all $x \geq 0$. Both results together imply (3). □



(a) Non-monotonic function (b) Strategies

		Player 2	
		$\alpha_{2,1}$	$\alpha_{2,2}$
Player 1	$\alpha_{1,1}$	62, 162, 35	66, 160, 65
	$\alpha_{1,2}$	66, 160, 65	62, 162, 35

(c) Private costs

Figure 3.4: Construction of a weighted congestion game without a pure Nash equilibrium in which all resources have the cost function $c(x) = x^3$, as discussed in Example 3.12. (a) shows the function $\tilde{c}(x) = 2x^3 - (x+1)^3$. Note that $\tilde{c}$ is contained in the finite integer linear hull $\mathcal{L}^3_{\mathbb{Z}}(\{c\})$ and non-monotonic. (b) shows how the players' strategies are constructed. (c) shows the players' private costs in the thus constructed game. Note that in (c) only the strategies of the first two players are shown as player 3 has a single strategy only.

This results holds even for three-player weighted congestion games.

We conclude this section by giving an example that illustrates the main ideas presented so far. Recall, that Theorem 3.11 establishes that for each continuous, non-affine and non-exponential cost function c, there is a weighted congestion game G with uniform cost function c on all resources that does not admit a pure Nash equilibrium. In the following example, we show how such a game for $c(x) = x^3$ is constructed.

Example 3.12. As the function $c(x) = x^3$ is neither affine nor exponential, we can find $a_1, a_2 \in \mathbb{Z}$ and $\delta \in \mathbb{R}_{>0}$ such that $\tilde{c}(x) = a_1\, c(x) + a_2\, c(x+\delta)$ has a strict local extreme point. In fact, we can choose $a_1 = 2, a_2 = -1$ and $\delta = 1$, that is, the function

$$\tilde{c}(x) = 2c(x) - c(x+1) = 2x^3 - (x+1)^3$$

has a strict local minimum at $x_0 = 1 + \sqrt{2}$. In particular, we can choose $d_1 = 1$ and $d_2 = 2$ such that $\tilde{c}(d_1) = -6 > \tilde{c}(d_1 + d_2) = -10 > \tilde{c}(d_2) = -11$, see Figure 3.4(a). The weighted congestion game without a pure Nash equilibrium is now constructed as follows. First, we introduce $2(|a_1| + |a_2|)$ resources $r_1, \ldots, r_6$ and the define the following feasible allocations $\alpha_{1,1} = \{r_1, r_2, r_3\}$, $\alpha_{1,2} = \{r_4, r_5, r_6\}$, $\alpha_{2,1} = \{r_1, r_2, r_4\}$, $\alpha_{2,2} = \{r_3, r_5, r_6\}$, and $\alpha_{3,1} = \{r_3, r_4\}$. We then set $\mathcal{A}_1 = \{\alpha_{1,1}, \alpha_{1,2}\}$, $\mathcal{A}_2 = \{\alpha_{2,1}, \alpha_{2,2}\}$, and $\mathcal{A}_{3,1} = \{\alpha_3\}$; see Figure 3.4(b) for an illustration of the strategies. All resources have the cost function $c(x) = x^3$. Player 3 is an offset player, she has a single feasible allocation only and the players' private costs depend only on the strategic choices of players 1 and 2. We conclude that the corresponding weighted congestion game shown in Figure 3.4(c) has no pure Nash equilibrium.

3.6 Weighted Network Congestion Games

In this section, we discuss the implications of our characterizations for the important subclass of weighted network congestion games. In these games, the resources correspond to edges of a directed or undirected graph. Every player is associated with a positive demand that she wants to route from her origin to her destination on a path of minimum cost. We consider directed and undirected networks separately, starting with directed networks.

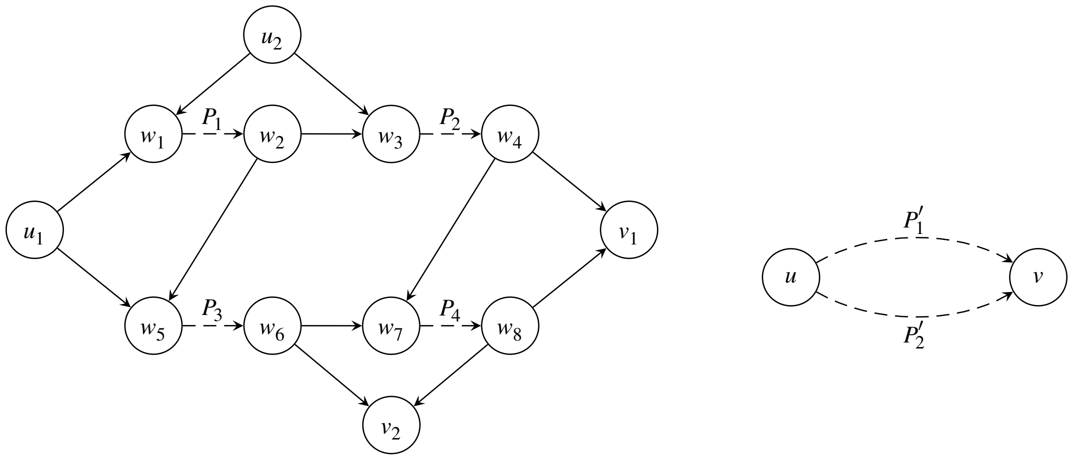
3.6.1 Directed Networks

We first give a version of the Extended Monotonicity Lemma for directed networks with two players and strictly positive costs. Specifically, we prove if a set $\mathcal{C}$ of strictly positive positive and continuous functions is consistent for two-player directed-network congestion games, then $\mathcal{L}^2_{\mathbb{Z}}(\mathcal{C})$ only contains monotonic functions. On the one hand, this result is stronger than the Extended Monotonicity Lemma for Two-Player Games (Corollary 3.6) as it requires only that $\mathcal{C}$ is consistency for two-player directed-network congestion games which is a subclass of two-player congestion games. On the other hand, it is also weaker, since we additionally require that $\mathcal{C}$ only contains positive functions.

Lemma 3.13 (Extended Monotonicity Lemma for Two-Player Directed-Network Games). *Let $\mathcal{C}$ be a set of continuous functions that is strictly positive on $\mathbb{R}_{>0}$. If $\mathcal{C}$ is consistent for two-player directed-network weighted congestion games, then $\mathcal{L}^2_{\mathbb{Z}}(\mathcal{C})$ only contains monotonic functions.*

Proof. Because singleton congestion games are a subclass of directed-network weighted congestion games, Corollary 3.4 implies that every set $\mathcal{C}$ of consistent functions only contains monotonic functions. For a contradiction, assume that there are $a_1, a_2 \in \mathbb{Z}$ and monotonic functions $c_1, c_2 \in \mathcal{C}$ such that the function $c : \mathbb{R}_{\geq 0} \to \mathbb{R}$ defined as $c(x) = a_1 c_1(x) + a_2 c_2(x)$ is not monotonic. By Lemma 3.2 there are $x, y \in \mathbb{R}_{>0}$ with $y > x$ such that either $c(y-x) < c(y) < c(x)$ or $c(y-x) > c(y) > c(x)$. We choose the demands of players 1 and 2 equal to $d_1 = y - x$ and $d_2 = x$, respectively. Note that c is monotonic if and only if $-c$ is monotonic. Thus, it is without loss of generality to assume that $a_2 > 0$. To define the players' strategies we distinguish the following two cases.

First case: $a_1 < 0$. We use a construction similar to the proof of Lemma 3.5. To define the players' strategy spaces, consider the network in Figure 3.5(a). The two players are represented by the two source-terminal pairs $(u_i, v_i), i = 1,2$. The set of strategies available to player i equals the set of directed (u_i, v_i)-paths. The dashed edges in Figure 3.5(a) correspond to directed paths $P_1, \dots, P_4$, which we choose as follows: both the directed path P_1 from w_1 to w_2 and the directed path P_4 from w_7 to w_8 contain $|a_1|$ edges with cost function c_1 each. Both the directed path P_2 from w_3 to w_4 and the directed path P_3 from w_5 to w_6 contain a_2 edges with cost function c_2. All other edges are endowed with the cost function c_1. Because all cost functions are strictly positive $\mathbb{R}_{>0}$, for player 1 all strategies except the upper path $P_{\text{up}} = \{(u_1, w_1) \cup P_1 \cup (w_2, w_3) \cup P_2 \cup (w_4, v_1)\}$ and the lower path $P_{\text{down}} = \{(u_1, w_5) \cup P_3 \cup (w_6, w_7) \cup P_4 \cup (w_8, v_1)\}$ are strictly dominated in the sense that they have strictly higher costs than either P_{up} or P_{down} regardless of the strategy played by player 2. For player 2, all strategies except the left path $P_{\text{left}} = \{(u_2, w_1) \cup P_1 \cup (w_2, w_5) \cup P_3 \cup (w_6, v_2)\}$ and the right path $P_{\text{right}} = \{(u_2, w_3) \cup P_2 \cup (w_4, w_7) \cup P_4 \cup (w_8, v_2)\}$ are strictly dominated. We consider the



(a) Case $a_1 < 0$ (b) Case $a_1 > 0$

Figure 3.5: Directed-network congestion games used in the proof of the Extended Monotonicity Lemma for Two-Player Directed Networks (Lemma 3.13).

cycle $\gamma = \big((P_{\text{up}}, P_{\text{left}}), (P_{\text{down}}, P_{\text{left}}), (P_{\text{down}}, P_{\text{right}}), (P_{\text{up}}, P_{\text{right}}), (P_{\text{up}}, P_{\text{left}})\big)$, and calculate that

$$\begin{aligned}\pi_1(P_{\text{down}}, P_{\text{left}}) - \pi_1(P_{\text{up}}, P_{\text{left}}) &= (y-x)\big(c_1(y-x) + a_2c_2(y) + c_1(y-x) - a_1c_1(y-x) + c_1(y-x)\\ &\quad - c_1(y-x) + a_1c_1(y) - c_1(y-x) - a_2c_2(y-x) - c_1(y-x)\big)\\ &= (y-x)\big(a_1c_1(y) + a_2c_2(y) - a_1c_1(y-x) - a_2c_2(y-x)\big)\\ &= (y-x)\big(c(y) - c(y-x)\big)\end{aligned}$$

Following the same line of argumentation, we obtain that

$$\begin{aligned}\pi_2(P_{\text{down}}, P_{\text{right}}) - \pi_2(P_{\text{down}}, P_{\text{left}}) &= x\big(c(x) - c(y)\big),\\ \pi_1(P_{\text{up}}, P_{\text{right}}) - \pi_1(P_{\text{down}}, P_{\text{right}}) &= (y-x)\big(c(y) - c(y-x)\big),\\ \pi_2(P_{\text{up}}, P_{\text{left}}) - \pi_2(P_{\text{up}}, P_{\text{right}}) &= x\big(c(x) - c(y)\big).\end{aligned}$$

If $c(y-x) > c(y) > c(x)$, then γ is an improvement cycle which gives that none of the strategy profiles contained in γ is a pure Nash equilibrium. If, on the other hand, $c(y-x) < c(y) < c(x)$, we can reverse the direction of γ and get an improvement cycle. Because every strategy profile that uses only non-dominated strategies is contained in γ, the constructed directed network congestion game does not admit a pure Nash equilibrium.

Second case: $a_1 > 0$. Consider the network shown in Figure 3.5(b). Here, both players want to route from u to v, that is, $S_1 = S_2 = \{P'_1, P'_2\}$. The directed paths P'_1 as well as P'_2 each contain a_1 edges with cost function c_1 and a_2 edges with cost function c_2. If $c(y-x) < c(y) < c(x)$, player 1 prefers to be alone on an (u,v)-path while player 2 wants to share the path with player 1. If $c(y-x) > c(y) > c(x)$, the argumentation works the other way round. We conclude that the game does not admit a pure Nash equilibrium. $\square$

Together with Lemma 3.7 and Theorem 3.8, we obtain the following characterization of consistency for two-player network congestion games on directed networks.

Theorem 3.14. *For a set $\mathcal{C}$ of continuous functions that are strictly positive on $\mathbb{R}_{>0}$ the following three statements are equivalent:*

(1) *$\mathcal{C}$ is consistent for two-player directed-network weighted congestion games.*
(2) *$\mathcal{C}$ is universally consistent for two-player directed-network weighted congestion games.*
(3) *$\mathcal{C}$ only contains monotonic functions and for all non-constant $c_1, c_2 \in \mathcal{C}$, there are constants $a, b \in \mathbb{R}$ with $c_1(x) = a\,c_2(x) + b$ for all $x \geq 0$.*

Using similar ideas as in the case of two players, we can also prove a version of the Extended Monotonicity Lemma for directed-network weighted congestion games with three players.

Lemma 3.15 (Extended Monotonicity Lemma for Directed Networks). *Let $\mathcal{C}$ be a set of continuous functions that are strictly positive on $\mathbb{R}_{>0}$. If $\mathcal{C}$ is consistent for three-player directed-network congestion games, then $\mathcal{L}^3_{\mathbb{Z}}(\mathcal{C})$ only contains monotonic functions.*

Proof. For a contradiction, let us assume that $\mathcal{C}$ is consistent but there are $a_1, a_2 \in \mathbb{Z}$, $\delta \in \mathbb{R}_{>0}$ and a function $c_1 \in \mathcal{C}$ such that the function $c : \mathbb{R}_{\geq 0} \to \mathbb{R}$ defined as $c(x) = a_1 c_1(x) + a_2 c_1(x+\delta)$ is not monotonic. Because singleton weighted congestion games are a subclass of directed-network weighted congestion games, Corollary 3.4 implies that c_1 is monotonic. Using that c is monotonic if and only if $-c$ is monotonic, it is without loss of generality to assume that $a_2 > 0$. This implies $a_1 < 0$.

Consider the network in Figure 3.6 where again the directed paths P_1 and P_4 contain $|a_1|$ edges each, and the the directed paths P_2 and P_3 contain a_2 edges each. In addition to the players $i = 1,2$ corresponding to the pairs $(u_i, v_i), i = 1,2$ we now have a third player corresponding to the pair (u_3, v_3) with a single strategy $P_Q = \{P_3 \cup Q \cup P_2\}$ and demand $d_3 = \delta$. Moreover, we define the demands of players 1 and 2 as $d_1 = y - x$ and $d_2 = x$, where $x, y \in \mathbb{R}_{>0}$ with $x < y$ are chosen such that either $c(y-x) < c(y) < c(x)$ or $c(y-x) > c(y) > c(x)$ holds. By Lemma 3.2 such values exist. We design the directed path Q from w_6 to w_3 so as to contain a sufficiently large number of edges, such that for players 1 and 2 all (u_i, v_i)-paths not containing Q are strictly less costly than every path that contains Q. As every (u_i, v_i)-path that does not contain Q has costs less than $2(a_2 - a_1 + 6)\,c_1(y+\delta)$ and every edge in Q has cost at least $c_1(\delta)$, it is sufficient to let Q contain $2(a_2 - a_1 + 6)\lceil \frac{c_1(y+\delta)}{c_1(\delta)} \rceil + 1$ edges. By construction of Q, for player 1, all strategies except the upper path $P_{\text{up}} = \{(u_1, w_1) \cup P_1 \cup (w_2, w_3) \cup P_2 \cup (v_3, v_1)\}$ and the lower path $P_{\text{down}} = \{(u_1, u_3) \cup P_3 \cup (w_6, w_7) \cup P_4 \cup (w_8, v_1)\}$ are strictly dominated in the sense that they have strictly higher costs than either P_{up} or P_{down} regardless of the strategies played by players 2 and 3. For player 2, all strategies except the left path $P_{\text{left}} = \{(u_2, w_1) \cup P_1 \cup (w_2, u_3) \cup P_3 \cup (w_6, v_2)\}$ and the right path $P_{\text{right}} = \{(u_2, w_3) \cup P_2 \cup (v_3, w_7) \cup P_4 \cup (w_8, v_2)\}$ are strictly dominated. With the same calculations as in Lemma 3.13 one can show that the cycle $\gamma = \big((P_{\text{up}}, P_{\text{left}}, P_Q), (P_{\text{down}}, P_{\text{left}}, P_Q), (P_{\text{down}}, P_{\text{right}}, P_Q), (P_{\text{up}}, P_{\text{right}}, P_Q), (P_{\text{up}}, P_{\text{left}}, P_Q)\big)$ is an improvement cycle when traversed in the right direction. Because every strategy profile that uses only non-dominated strategies is contained in γ, we conclude that the thus constructed network congestion game does not admit a pure Nash equilibrium. □

Using Lemma 3.10, we obtain the following characterization of cost functions that are consistent for weighted directed network congestion games.

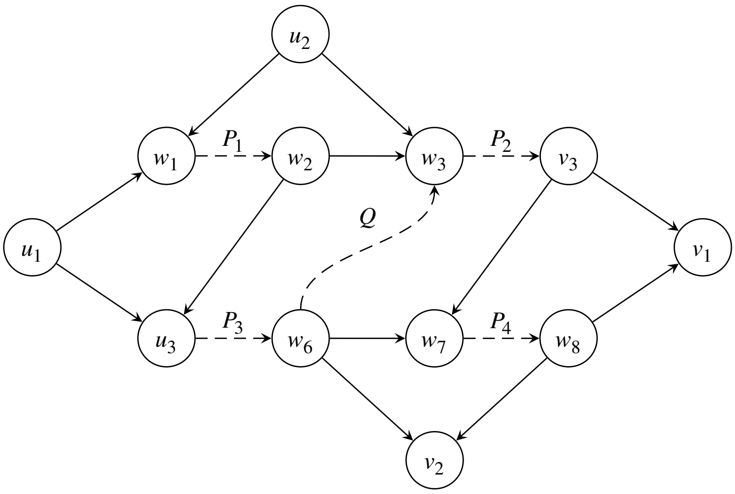


Figure 3.6: Directed-network weighted congestion game used in the proof of the Extended Monotonicity Lemma for Directed Networks (Lemma 3.15).

Theorem 3.16. *For a set $\mathcal{C}$ of continuous functions that are strictly positive on $\mathbb{R}_{>0}$ the following three statements are equivalent:*

(1) *$\mathcal{C}$ is consistent for directed-network weighted congestion games.*
(2) *$\mathcal{C}$ is universally consistent for directed-network weighted congestion games.*
(3) *$\mathcal{C}$ only contains affine functions* or *$\mathcal{C}$ only contains functions of type $c(x) = a_c\, e^{\phi\, x} + b_c$, where $a_c, b_c \in \mathbb{R}$ may depend on c while $\phi \in \mathbb{R}$ is independent of c.*

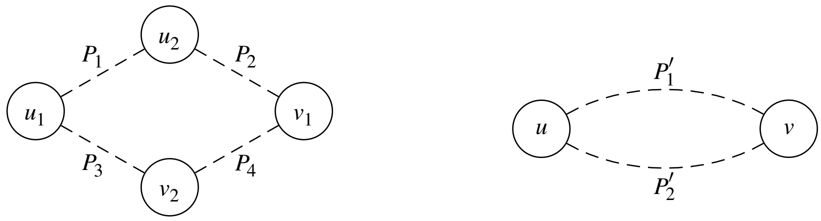
This characterization is even valid for three-player games.

Remark 3.17. In games with strictly negative costs the players strive to establish long paths. In this case, our construction does not work because, e.g., player 2 prefers to take the detour $\{(w_6,w_7),(w_7,w_8),(w_8,v_2)\}$ instead of the edge (w_6,v_2).

3.6.2 Undirected Networks

Turning to undirected-network weighted congestion games, we first show that a version of the Extended Monotonicity Lemma holds also for two-player games on undirected networks. In such a game, we are given an undirected graph and for each player i, two designated vertices u_i and v_i. Resources correspond to the edges of the graph and the strategy set of each player i contains all simple (u_i,v_i)-paths. Each edge can be traversed in any direction and its cost depends on the aggregated flow.

Lemma 3.18 (Extended Monotonicity Lemma, Two-Player Undirected-Network Games). *Let $\mathcal{C}$ be a set of continuous functions that are strictly positive on $\mathbb{R}_{>0}$. If $\mathcal{C}$ is consistent for two-player undirected-network weighted congestion games, then $\mathcal{L}^2_{\mathbb{Z}}(\mathcal{C})$ only contains monotonic functions.*



(a) Case $a_1 < 0$

(b) Case $a_1 > 0$

Figure 3.7: Undirected-network weighted congestion games used in the proof of the Extended Monotonicity Lemma for Two-Player Undirected Networks (Lemma 3.18).

Proof. For a contradiction, let $a_1, a_2 \in \mathbb{Z}$ and $c_1, c_2 \in \mathcal{C}$ be such that the function $c : \mathbb{R}_{\geq 0} \to \mathbb{R}$ defined as $c(x) = a_1\, c_1(x) + a_2\, c_2(x)$ is not monotonic. As argued in the previous proofs it is without loss of generality to assume that $a_2 > 0$. Moreover, let $x, y \in \mathbb{R}_{>0}$ with $x < y$ be such that either $c(y-x) < c(y) < c(x)$ or $c(y-x) > c(y) > c(x)$ holds. We set $d_1 = y - x$, $d_2 = x$ and distinguish the following two cases. If $a_1 < 0$ we consider the network in Figure 3.7(a) where the paths P_1 and P_4 each contain $|a_1|$ edges with cost function c_1 and the paths P_2 and P_3 each contain a_2 edges with cost function c_2. With similar calculations as in the proof of Lemma 3.13 one can verify that the 4-cycle $\gamma = \big((P_1 \cup P_2, P_1 \cup P_3), (P_3 \cup P_4, P_1 \cup P_3), (P_3 \cup P_4, P_2 \cup P_4), (P_1 \cup P_2, P_2 \cup P_4), (P_1 \cup P_2, P_1 \cup P_3)\big)$ is an improvement cycle if traversed in the right sense. If, on the other hand, $a_1 > 0$ we consider the undirected network shown in Figure 3.7(b) and obtain the same contradiction as in Lemma 3.13. $\square$

Likewise, we obtain the following characterization for two-player games on undirected networks.

Theorem 3.19. *For a set $\mathcal{C}$ of continuous functions that are strictly positive on $\mathbb{R}_{>0}$ the following three statements are equivalent:*

(1) *$\mathcal{C}$ is consistent for two-player undirected-network weighted congestion games.*
(2) *$\mathcal{C}$ is universally consistent for two-player undirected-network weighted congestion games.*
(3) *$\mathcal{C}$ only contains monotonic functions and for all non-constant $c_1, c_2 \in \mathcal{C}$, there are constants $a, b \in \mathbb{R}$ with $c_1(x) = a\, c_2(x) + b$ for all $x \geq 0$.*

Turning to games with three players we are not able to characterize the set of consistent cost functions. However, we can still characterize consistency for games with at least four players.

Lemma 3.20 (Extended Monotonicity Lemma for Undirected Networks). *Let $\mathcal{C}$ be a set of continuous functions that are strictly positive on $\mathbb{R}_{>0}$. If $\mathcal{C}$ is consistent for undirected-network weighted congestion games with at least four players, then $\mathcal{L}^3_{\mathbb{Z}}(\mathcal{C})$ only contains monotonic functions.*

Proof. For a contradiction, suppose that there are $a_1, a_2 \in \mathbb{Z}$, $\delta \in \mathbb{R}_{>0}$, and a monotonic function $c_1 \in \mathcal{C}$ such that the function $c : \mathbb{R}_{\geq 0} \to \mathbb{R}$ defined as $c(x) = a_1\, c_1(x) + a_2\, c_1(x+\delta)$ is not monotonic. As argued in the previous proofs it is without loss of generality to assume that c_1 is monotonic, $a_2 > 0$ and $a_1 < 0$.

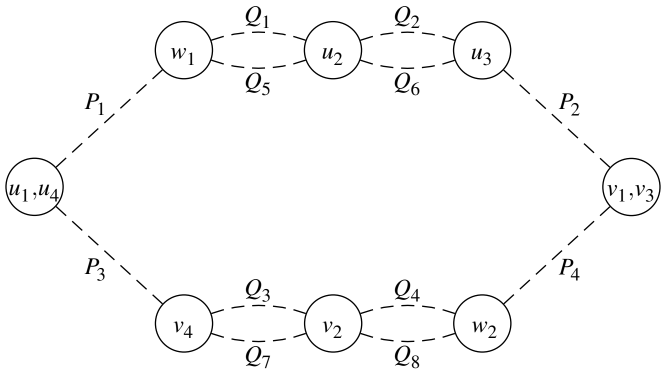


Figure 3.8: Undirected-network weighted congestion games used in the proof of the Extended Monotonicity Lemma for Undirected Networks (Lemma 3.20).

Consider the network in Figure 3.8 where the paths P_1 and P_4 each contain $|a_1|$ edges and the paths P_2 and P_3 each contain a_2 edges. All edges have cost function c_1. The players $i = 1, 2$ correspond to the source sink pairs $(u_i, v_i), i = 1, 2$. Additionally, there are players associated with the source sink pairs (u_i, v_i), $i = 3, 4$, and demand $d_3 = d_4 = \delta$. Moreover, we set the demands of players 1 and 2 equal to $d_1 = y - x$ and $d_2 = x$, where $x, y \in \mathbb{R}_{>0}$ with $x < y$ are chosen such that either $c(y-x) < c(y) < c(x)$ or $c(y-x) > c(y) > c(x)$ holds. By Lemma 3.2 such values exist.

We endow every edge in the paths $Q_1, \ldots, Q_8$ with cost function c_1 and make them sufficiently long such that players 3 and 4 always prefer to choose a strategy not containing any of these paths. As the paths P_2 and P_3 have costs less than $a_2 c_1(y + 2\delta)$ and every edge in Q_i, $i = 1, \ldots, 8$ used by players 3 or 4 has cost at least $c_1(\delta)$, it suffices for all $k = 1, \ldots, 8$ to let Q_k contain $a_2 \left\lceil \frac{c_1(y+2\delta)}{c_1(\delta)} \right\rceil + 1$ edges each. Then, for player 3, all strategies except P_2 are strictly dominated by P_2 and for player 4 all strategies except P_3 are strictly dominated by P_3.

If c_1 is non-decreasing, we may assume that players 1 and 2 will not share any of the Q_k paths in equilibrium. Without loss of generality, we assume that player 1 always uses the paths $Q_1, \ldots, Q_4$ instead of $Q_5, \ldots, Q_8$ while player 2 always uses paths $Q_5, \ldots, Q_8$ instead of $Q_1, \ldots, Q_4$. If, on the other hand, c_1 is non-increasing, players 1 and 2 can only gain sharing the Q_k paths. From there, it is without loss of generality to assume that only the paths $Q_1, \ldots, Q_4$ are used while the paths $Q_5, \ldots, Q_8$ remain unused.

In both cases, the Q_k paths raise the cost of any strategy profile by a constant and with the same calculations as before one can show that there is an improvement cycle γ of the form

$$\gamma = \big((P_{\text{up}}, P_{\text{left}}, P_2, P_3), (P_{\text{down}}, P_{\text{left}}, P_2, P_3), (P_{\text{down}}, P_{\text{right}}, P_2, P_3), (P_{\text{up}}, P_{\text{right}}, P_2, P_3), (P_{\text{up}}, P_{\text{left}}, P_2, P_3)\big),$$

where $P_{\text{up}} = P_1 \cup Q_1 \cup Q_2 \cup P_2$, $P_{\text{down}} = P_3 \cup Q_3 \cup Q_4 \cup P_4$, $P_{\text{left}} = Q_5 \cup P_1 \cup P_3 \cup Q_7$, and $P_{\text{right}} = Q_6 \cup P_2 \cup Q_4 \cup Q_8$. Because every strategy profile that uses only non-dominated strategies is contained in γ the thus constructed network congestion game does not admit a pure Nash equilibrium. $\square$

Using the above Lemma, we obtain the following result.

Theorem 3.21. *For a set $\mathcal{C}$ of continuous functions that are strictly positive on $\mathbb{R}_{>0}$ the following statements are equivalent:*

(1) *$\mathcal{C}$ is consistent for undirected-network weighted congestion.*
(2) *$\mathcal{C}$ is universally consistent for undirected-network weighted congestion.*
(3) *$\mathcal{C}$ only contains affine functions* or *$\mathcal{C}$ only contains functions of type $c(x) = a_c\, e^{\phi\, x} + b_c$, where $a_c, b_c \in \mathbb{R}$ may depend on c while $\phi \in \mathbb{R}$ is independent of c.*

This characterization is even valid for undirected-network games with four players.

For single-commodity network games (directed or undirected) we are not able to characterize consistency of cost functions. However, by introducing a super-source and a super-sink to the network constructions used, it follows that the improvement cycles are preserved; thus, all characterizations for universally consistency obtained in this section continue to hold.

3.7 Weighted Singleton Congestion Games

In this section, we consider the case of singleton weighted congestion games. In this class of games, for every player i, every strategy $s_i \in S_i$ contains a single resource only. As mentioned in Corollary 3.4, the construction of the Monotonicity Lemma (Lemma 3.3) is even valid for singleton games, establishing that every set of continuous cost functions $\mathcal{C}$ that is consistent for singleton games may only contain monotonic functions. Singleton congestion games with weighted players and either only non-decreasing or only non-increasing cost functions admit a pure Nash equilibrium; see Fabrikant [41] for the case on non-decreasing cost functions, and Rozenfeld and Tennenholtz [118] for the case of non-increasing cost functions. Since the positive result for non-decreasing costs is established via a potential function, these games also possess the finite improvement property. With similar arguments it is not difficult to establish the finite improvement property also for the case of non-increasing costs.[2] To the best of our knowledge it was not known before, whether singleton weighted congestion games with both non-decreasing and non-increasing cost functions admit a pure Nash equilibrium or even the finite improvement property. Regarding the existence of pure Nash equilibria, for *two-player* games, we give a positive answer to this question.

Theorem 3.22. *A set $\mathcal{C}$ of continuous functions is consistent for two-player singleton weighted congestion games if and only if $\mathcal{C}$ only contains monotonic functions.*

Proof. The only if part of the statement follows from Corollary 3.4. For the if part let $\mathcal{M} = \big(N, R, (\mathcal{A}_i)_{i\in N}, (c_r)_{r\in R}\big)$ be a congestion model with $|N| = 2$. It is without loss of generality to assume that $d_1 \leq d_2$. We partition the set of resources into sets R_- and R_+, where R_+ contains all resources with non-decreasing cost functions (including all resources with constant functions) and R_- all other resources. It is without loss of generality to assume that both player have access to all resources in R_-, since we can replace the cost function of every resource that is contained in the strategy space of only one player by a constant function. We initialize the players both playing r', where $r' = \arg\min_{r\in R_-} c_r(d_1 + d_2)$. We distinguish two cases.

[2] Consider the function ϕ that assigns to each strategy profile the non-decreasingly sorted vector of the scaled players' private costs $(\pi_i/d_i)_{i\in N}$. Then, ϕ decreases lexicographically along any improvement path, establishing that every such path is finite.

First case: Player 1 has an improving move from $(\{r'\},\{r'\})$. In this case, we let player 1 move to one of her best replies $\{r_1\} \in S_1$. Using that $r' = \arg\min_{r \in R_-} c_r(d_1+d_2)$, we have $r_1 \in R_+$. If player 2 does not have an improving move from $(\{r_1\},\{r'\})$, we are done. So, let $\{r_2\}$ be a best reply of player 2 to $(\{r_1\},\{r'\})$. If $r_1 \neq r_2$, we claim that $(\{r_1\},\{r_2\})$ is a pure Nash equilibrium. To see this, note that if $r_2 \in R_+$, then player 2 switching from $\{r'\}$ to $\{r_2\}$ does not make any of the resources more attractive to player 1. If on the other hand, $r_2 \in R_-$, we obtain

$$\pi_1(\{r_2\},\{r_2\}) \geq \pi_1(\{r'\},\{r'\}) > \pi_1(\{r_1\},\{r'\}) = \pi_1(\{r_1\},\{r_2\}),$$

where the first inequality follows by the choice of r' and the second inequality is due to the fact that player 1 improved switching from $\{r'\}$ to r_1. We derive that player 1 does not want to move to r_2 and that we have reached an equilibrium.

The only interesting case that remains is $r_1 = r_2$. Again, if player 1 does not have an improving move, there is nothing left to show, so let $\{r_1'\} \neq \{r_1\}$ be a best reply of player 1 to $(\{r_1\},\{r_1\})$. Note that $r_1' \notin R_-$ because otherwise we get $\pi_2(\{r_1\},\{r_1'\})/d_2 \leq \pi_1(\{r_1'\},\{r_1\})/d_1 < \pi_1(\{r_1\},\{r_1\})/d_1 = \pi_2(\{r_1\},\{r_1\})/d_2$, where the first inequality follows since $d_2 \geq d_1$. This is a contradiction to the fact that $\{r_1\}$ was a best reply of player 2. As $\pi_2(\{r_1'\},\{r_1\}) \leq \pi_2(\{r_1\},\{r_1\})$, player 2 does not want to deviate from $(\{r_1'\},\{r_2\})$. Also, player 1 will not deviate from $(\{r_1'\},\{r_2\})$ as $\{r_1'\}$ was a best reply.

Second case: Player 1 has no improving move from $(\{r'\},\{r'\})$. If, also, player 2 does not have an improving move from $(\{r'\},\{r'\})$, we are done. Otherwise, let $\{r_2\} \in S_2$ be a best reply of player 2. Note that $\{r_2\} \notin S_1$ because otherwise $\{r_2\}$ would have been an improving move from $(\{r'\},\{r'\})$ of player 1. If player 1 has no improving move from $(\{r'\},\{r_2\})$, we are done. Otherwise, let $\{r_1\}$ be a best reply of player 1 to $(\{r'\},\{r_2\})$. As $\{r_2\} \notin S_1$, we have $r_1 \neq r_2$ and thus $\pi_2(\{r_1\},\{r_2\}) = \pi_2(\{r'\},\{r_2\})$. Since player 2 improved switching from $(\{r'\},\{r'\})$ to $(\{r'\},\{r_2\})$, she has no incentive to switch from $(\{r_1\},\{r_2\})$ and we have reached a pure Nash equilibrium. $\square$

Two-player singleton weighted congestion games with monotonic costs need not possess the finite improvement property as shown in the following example.

Example 3.23. Consider the congestion model $\mathcal{M} = \big(N,R,(\mathcal{A}_i)_{i\in N},(c_r)_{r\in R}\big)$ with two players $N = \{1,2\}$ who have access to all five resources $R = \{r',r_1,r_2,r_3,r_4\}$. The cost functions of the resources are shown in Table 3.1(a). Note that the cost function of resource r' is strictly decreasing while all other cost functions are non-decreasing. The players' demands are given by $d_1 = 1$ and $d_2 = 2$. It is not hard to verify that $\gamma = \big((\{r'\},\{r'\}),(\{r'\},\{r_1\}),(\{r_1\},\{r_1\}),(\{r_1\},\{r_2\}),$ $(\{r_3\},\{r_2\}),(\{r_3\},\{r_3\}),(\{r_4\},\{r_3\}),(\{r_4\},\{r'\}),(\{r'\},\{r'\})\big)$ is an improvement cycle.

We proceed to show that for singleton games with three players monotonicity of cost functions alone is not enough for the existence of a pure Nash equilibrium. This is illustrated in the following example.

Example 3.24. Consider the congestion model $\mathcal{M} = \big(N,R,(\mathcal{A}_i)_{i\in N},(c_r)_{r\in R}\big)$ with $N = \{1,2,3\}$ and $R = \{r_1,r_2,r_3\}$. The used cost functions are given in Table 3.1(b). The private costs of the weighted congestion game $G = (N,S,\pi)$ with $S_1 = \big\{\{r_2\},\{r_3\}\big\}$, $S_2 = \big\{\{r_1\},\{r_2\}\big\}$, $S_3 = \big\{\{r_1\},\{r_3\}\big\}$, and $d_1 = 1$, $d_2 = 2$, $d_3 = 4$ are shown in Figure 3.9. It is easy to verify that G does not possess a pure Nash equilibrium.

(a) Cost functions in the game of Example 3.23

resource	cost $c(x)$ $x=1$	$x=2$	$x=3$
r'	10	5	3
r_1	2	2	9
r_2	8	8	8
r_3	1	7	7
r_4	6	6	6

(b) Cost functions in the game of Example 3.24

resource	cost $c(x)$ $x=1$	$x=2$	$x=3$	$x=4$	$x=5$	$x=6$
r_1	0	0	2	3	3	3
r_2	5	1	1	1	0	0
r_3	2	2	2	2	4	4

Table 3.1: (a) Cost functions of the five resources r', r_1, r_2, r_3, and r_4 in the game of Example 3.23; (b) Cost functions of the three resources r_1, r_2, and r_3 in the game of Example 3.24.

Player 3: $\{r_1\}$

Player 1 \ Player 2	$\{r_1\}$	$\{r_2\}$
$\{r_2\}$	5, 6, 12	1, 2, 12
$\{r_3\}$	2, 6, 12	2, 2, 12

Player 3: $\{r_3\}$

Player 1 \ Player 2	$\{r_1\}$	$\{r_2\}$
$\{r_2\}$	5, 0, 8	1, 2, 8
$\{r_3\}$	4, 0, 16	4, 2, 16

Figure 3.9: Private costs in the singleton weighted congestion game G constructed in Example 3.24. Note that G does not possess a pure Nash equilibrium.

However, we are able to give a positive result for *symmetric* games in which the players have access to all resources.

Theorem 3.25. *A set $\mathcal{C}$ of continuous functions is consistent for symmetric singleton weighted congestion games if and only if $\mathcal{C}$ only contains monotonic functions.*

Note that the only if part also follows from Corollary 3.4. To prove the if part, we give an algorithm that computes a pure Nash equilibrium in such games. In the following, we denote by R_+ and R_- the set of resources with non-decreasing and non-increasing costs, respectively. To obtain a partition of R, we introduce the convention, that resources with constant cost functions are contained in R_+ only. The algorithm that we propose (Algorithm 1) initializes all players on the resource $r' \in R_-$ that minimizes $c_{r'}(\sum_{i\in N} d_i)$. No player has an incentive to switch to another resource $r \in R_-$. The key observation is that, as long as there is at least one player $i \in N$ that wants to switch to a resource $r \in R_+$, the player with smallest demand also wants to. We iteratively take the player with smallest demand on r' and let her move to R_+. Then, we compute a sequence of best replies of the players on R_+ to assure that none of them has an incentive to deviate to another resource in R_+. During this process the players stay in R_+. After that, the players on R_- are placed on the resource minimizing $c_r(\sum_{i\in N: s_i \in R_-} d_i)$. Since we can prove that a player on R_+ never wants to move back to a resource in R_-, this process stops after a finite number of best-reply computations.

Algorithm 1: Computation of a pure Nash equilibrium in symmetric singleton weighted congestion games.

Input: Symmetric singleton weighted congestion game G.
Output: Pure Nash equilibrium s of G.

1 $N_- := N, N_+ := \emptyset$;
2 Compute $r' := \arg\min_{r\in R_-} c_r(\sum_{i\in N_-} d_i)$ and set $s_i := \{r'\}$ for all $i \in N_-$;
3 **if** $k = \arg\min_{i\in N_-} d_i$ can improve switching to $r \in R_+$, **then**
4 $\quad s_k := r, N_- := N_- \setminus \{k\}, N_+ := N_+ \cup \{k\}$;
5 $\quad$ Compute a partial pure Nash equilibrium $(t_i)_{i\in N_+}$ of N_+ on R_+ by best replies and set $(s_i)_{i\in N_+} := (t_i)_{i\in N_+}$;
6 $\quad$ **Goto** line 2;
7 **else**
8 $\quad$ **return** s;
9 **end**

Lemma 3.26. *Algorithm 1 computes a pure Nash equilibrium.*

Proof. Let us first remark that the computation of the partial pure Nash equilibrium of players N_+ on R_+ in line 5 finishes after a finite number of best replies since the cost functions of the resources in R_+ are non-decreasing; see Ackermann et al. [4], and Ieong et al. [73].

Let z denote the outcome of the algorithm. No player $j \in N_+$ can improve switching to another resource $r \in R_+$ since we always recompute a partial pure Nash equilibrium in line 5. Also, no player $j \in N_-$ can improve unilaterally deviating to another resource $r \in R_-$ since $c_r(d_j) \geq c_r(\sum_{i\in N_-} d_i) \geq c_{r'}(\sum_{i\in N_-} d_i)$. In addition, we know that player $k = \arg\min_{i\in N_-} d_i$ does not improve switching from resource r' to another resource $r \in R_+$. In consequence, the same holds for every other player $j \in N_-$ since the cost for her on a resource $r \in R_+$ is not smaller. Finally, it is left to show that in z no player $j \in N_+$ has an interest to switch to some resource $r \in R_-$.

To prove this, let i_t, $t = 1,\ldots,T$, $T \in \mathbb{N}$, denote the player that switches from $r'_t \in R_-$ to $r_t \in R_+$ in the t-th iteration of the algorithm and let $\tilde{z}^t$ and z^t denote the corresponding strategy profiles before and after the re-computation of the partial pure Nash equilibrium on R_+ in line 5, respectively. We claim that

$$\min_{r'\in R_-} c_{r'}(\ell_{r'}(z^t)+d_{i_t}) > \max_{r\in R_+:\ell_r(z^t)>0} c_r\big(\ell_r(z^t)\big) \quad \text{for all } t = 1,\ldots,T, \tag{3.11}$$

where $\ell_{r'}(z^t)$ and $\ell_r(z^t)$ denote the aggregated demand on resource r' (respectively, r) in strategy profile z^t. For $t = 1$, the statement holds, since player i_1 improves switching from R_- to R_+. Now, suppose (3.11) is true for $t-1$. In the t-th iteration, player i_t changes her strategy from $r'_t \in R_-$ to some resource $r_t \in R_+$, that is,

$$\min_{r'\in R_-} c_{r'}(\ell_{r'}(z^t)+d_{i_t}) = c_{r'_t}\big(\ell_{r'_t}(z^t)+d_{i_t}\big) > c_{r_t}\big(\ell_{r_t}(\tilde{z}^t)\big).$$

Using that fact that the cost functions of the resources in R_- are non-increasing we obtain $\min_{r'\in R_-} c_{r'}\big(\ell_{r'}(z^t)+d_{i_t}\big) \geq \min_{r'\in R_-} c_{r'}\big(\ell_{r'}(z^{t-1})+d_{i_{t-1}}\big)$. By the induction hypothesis, this implies

$\min_{r' \in R_-} c_{r'}\big(\ell_{r'}(z^t) + d_{i_t}\big) > c_r(\ell_r(\tilde{z}^t))$ for all $r \in R_+ \setminus \{r_t\}$ with $\ell_r(\tilde{z}^t) > 0$. Thus, we have established $\min_{r' \in R_-} c_{r'}\big(\ell_g(z^t) + d_{i_t}\big) > \max_{r \in R_+ : \ell_r(\tilde{z}^t) > 0} c_r(\ell_r(\tilde{z}^t))$. Since the maximum cost on R_+ cannot increase in the sequence of best-reply steps (c.f. [66]), we obtain

$$\min_{r' \in R_-} c_{r'}\big(\ell_{r'}(z^t) + d_{i_t}\big) > \max_{r \in R_+ : \ell_r(z^t) > 0} c_r(\ell_r(z^t))$$

as claimed.

Because we move the player with the current smallest weight from R_- to R_+ (line 3) it holds that $d_{i_T} = \max_{i \in N_+} d_i$. Thus, $\min_{r' \in R_-} c_{r'}\big(\ell_{r'}(z) + d_i\big) \geq \min_{r' \in R_-} c_{r'}\big(\ell_{r'}(z) + d_{i_T}\big) > \max_{r \in R_+} c_r(\ell_f(\tilde{z}))$ for all $i \in N_+$. We conclude that no player $i \in N_+$ has an incentive to switch to a resource $r' \in R_-$. $\square$

While the above result implies that the set $\mathcal{C}$ of continuous and monotonic cost functions is consistent for symmetric singleton games, Example 3.23 implies that $\mathcal{C}$ is not universally consistent.

3.8 Discussion and Open Problems

In this chapter, we obtained a complete characterization of the existence of pure Nash equilibria in weighted congestion games with respect to the cost functions of the resources. The following issues have not been resolved. All of our results require that cost functions are continuous. It would be interesting to weaken this assumption. For network games we assumed that cost functions are strictly positive on $\mathbb{R}_{>0}$. Moreover, for single-commodity games we were only able to characterize the finite improvement property, not consistency. The single-commodity case, however, behaves completely differently; e.g., every congestion game with positive and non-increasing costs admits a pure Nash equilibrium in which all players use the socially optimal path (see also Anshelevich et al. [10] for a similar result in the context of network design games). Finally, it would be interesting to characterize consistency of cost functions for undirected networks with three players.

Chapter 4

Congestion Games with Resource-Dependent Demands

In the past, the existence of pure Nash equilibria has been analyzed in many variants of congestion games. Examples include scheduling games, routing games, network design games, each variant with weighted and unweighted players. Most of these previous works assumes that each player has a unique demand $d_i \in \mathbb{R}_{>0}$ that she places on the chosen resources. That is, the demand d_i is *independent* of the chosen resource. However, many real-world problems have the property that the demand of a player *depends* on the resource. One prominent example is that of scheduling games on unrelated machines as studied by Andelman et al. [8], Awerbuch et al. [13] and Even-Dar et al. [39, 40]. In such a game, each player controls a job that she wishes to be scheduled on exactly one machine out of a set of feasible machines. The execution time of each job depends on the machine it is scheduled on. Each player is interested in minimizing the *makespan* of the chosen machine which is defined as the sum of execution times of all jobs on the same machine. Even-Dar et al. [39, 40] showed that in such a game a pure Nash equilibrium always exists. Andelman et al. [8] strengthened this result showing that scheduling games on unrelated machines even always possess a strong equilibrium. Feldman and Tamir [44] investigated in how far pure Nash equilibria are good approximate strong equilibria (in the multiplicative sense). They further show that it is NP-hard to decide whether a given pure Nash equilibrium is a strong equilibrium.

As noted by Eve-Dar et al. [39, 40], and Awerbuch et al. [13], one major motivation to study scheduling games on unrelated machines is to model the usage of modern communication infrastructures by selfish users. In the absence of a central authority the users decide individually how to route their traffic. Different routers may have different policies on how the traffic is handled; thus, it is natural to allow that the actual workload (or the demand) that each user imposes on a certain router actually *depends* on the identity of the router. To underline the connection to scheduling on unrelated machines, Awerbuch et al. [13] called this model the “related link model”. They study, however, only networks of parallel links where the related link model is equal to scheduling on unrelated machines. Even though congestion games with resource-dependent demands allow to model a much broader scope of applications than weighted congestion games they not received a similar attention in the literature, in the past. Most previous work concentrated on the important but still limited case of scheduling games. As a model of Internet traffic, for instance, these games have two shortcomings. First, the players' strategies contain a single resource only, while file transmission over the Internet usually incorporates more than one router. Second, in [13, 39, 40, 8] it is assumed that the private cost of each player equals the load on the chosen machine. This may not reflect the actual costs perceived by the players, e.g. when the dropping rate of a router increases exponentially with the load.

Only very recently, Harks et al. [67] studied congestion games with resource-dependent demands with *arbitrary* (non-singleton) strategy sets. Specifically, they investigated which maximal sets of cost functions $\mathcal{C}$ guarantee the existence of a weighted potential function in all congestion games with resource-dependent demands and cost functions in $\mathcal{C}$. It is shown that the set of affine functions is the unique maximal set $\mathcal{C}$ with that property. Finite games with a weighted potential have several desirable properties. First of all, they have the finite improvement property and possess a pure Nash equilibrium (Proposition 2.20 and Corollary 2.21). They also have the fictitious play property (Monderer and Shapley [98]), that is, the process of fictitious play converges to a mixed equilibrium.

In this chapter, we further investigate the equilibrium existence problem in congestion games with resource-dependent demands. Specifically, we relax the strong requirement of a weighted potential function and instead focus on the existence of a pure Nash equilibrium and the finite improvement property. To precisely characterize which maximal sets of cost functions guarantee the existence of pure Nash equilibria, we use the notion of consistency introduced in Chapter 3. We say that a set $\mathcal{C}$ of cost functions is *consistent* for congestion games with resource-dependent demands, if all congestion games with resource-dependent demands and cost functions in $\mathcal{C}$ possess a pure Nash equilibrium. A set $\mathcal{C}$ of cost functions is called *universally consistent* if every congestion game with resource-dependent demands and cost functions in $\mathcal{C}$ has the finite improvement property. The result of Harks et al. [67] implies that the set of affine functions is consistent and universally consistent. In this section, we investigate the question whether there are the further sets cost functions that are consistent for congestion games with resource-dependent demands.

4.1 Contributions and Chapter Outline

First, we observe that there are two natural ways to define the players' private cost functions. In a *proportional* game, the resource costs are interpreted as a *per-unit* cost; that is, each resource that is loaded with x units of demand produces a cost of $x\,c_r(x)$, which is then divided among its users in proportion to their respective demands. In that fashion, each player i pays $d_{i,r}\,c_r(x)$ for each resource r she uses. This is the variant for which Harks et al. [67] proved the existence of a pure Nash equilibrium when all costs are affine. We also consider a slightly different class of games known as *uniform* games. They differ from proportional games solely in the definition of the players' private cost functions. In a uniform game, the resource costs are *not* multiplied with the players' demands. As noted by Koutsoupias and Papadimitriou [85], uniform games are motivated by scheduling applications for which the cost function is used to model the achieved makespan which is (under round-robin processing) the same for every job on a resource. To precisely understand the impact of the different definitions of the private cost functions on the equilibrium behavior of the resulting games, we regard a more general class of games, which we term *g-scaled congestion*

Table 4.1: Pure Nash equilibria (PNE) and the finite improvement property (FIP) in proportional and uniform congestion games with resource-dependent demands. Here, by "exponential", we denote sets $\mathcal{C}$ of cost functions, such that every $c \in \mathcal{C}$ is of type $c(x) = a_c e^{\phi x} + b_c$, where $a_c, b_c \in \mathbb{R}$ may depend on c while ϕ is equal for all $c \in \mathcal{C}$. Note the fundamental structural difference to weighted congestion games (with resource-independent demands) studied in Chapter 3 where games with exponential costs always possess a pure Nash equilibrium.
Reading example: The set of affine functions is consistent for proportional congestion games with resource-dependent demands because in each such game a pure Nash equilibrium (PNE) exists. They are also universally consistent as these games have the finite improvement property (FIP). They are, however, neither universally consistent nor consistent for uniform congestion games with resource-dependent demands as these games do in general not have the finite improvement property or possess a pure Nash equilibrium. For weighted congestion games (where demands are resource-independent) affine functions are universally consistent and consistent.

	res.-dep. demands		res.-dep. demands		indep. demands	
	proportional		uniform		prop. & unif.	
costs	PNE	FIP	PNE	FIP	PNE	FIP
constant	yes	yes	yes	yes	yes	yes
affine	yes	yes	no	no	yes	yes
exponential	no	no	no	no	yes	yes
other	no	no	no	no	no	no

games with resource-dependent demands. For a given *scaling function* $g : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, the private cost function of player i in a g-scaled congestion game with resource-dependent demands is defined as $\pi_i(s) = \sum_{r \in s_i} g(d_{i,r})\, c_r(\ell_r(s))$, where $\ell_r(s) = \sum_{j \in N : r \in s_j} d_{j,r}$. We say that a set $\mathcal{C}$ of cost functions is consistent for g-scaled congestion games with resource-dependent demands if each such game with cost functions in $\mathcal{C}$ possesses a pure Nash equilibrium.

Requiring that scaling functions are continuously differentiable, non-decreasing and strictly positive, our main result is a complete characterization of the consistency of cost functions for g-scaled congestion games. We show that a set $\mathcal{C}$ of continuous cost functions is consistent for g-scaled congestion games with variable demands if and only if at least one of the following three cases holds: *(i)* g is arbitrary and $\mathcal{C}$ contains only constant functions; *(ii)* g is linear and $\mathcal{C}$ contains only affine functions; *(iii)* There is $\phi \in \mathbb{R}$ such that g is of type $g(x) = -\mathrm{sgn}(\phi)\beta(e^{-\phi x} - 1)$, $\beta \in \mathbb{R}_{>0}$ and $\mathcal{C}$ contains only functions of type $c(x) = a_c e^{\phi x} + b_c$ where $a_c, b_c \in \mathbb{R}$ may depend on c. This result implies in particular that the set of affine functions is the unique maximal set of cost functions that is consistent for proportional games. This characterization also reveals that uniform games are only guaranteed to possess a pure Nash equilibrium in the trivial case where all resource cost functions are constant. We further show that this characterization is also valid for universal consistency. Our results are summarized in Table 4.1.

After recalling the most important definitions in Section 4.2, we first study necessary conditions in Section 4.3. As the main result of this section, we provide the Generalized Monotonicity Lemma (Lemma 4.5) which gives necessary conditions for the consistency for g-scaled congestion games with resource-dependent demands in terms of two inequalities depending on a non-negative parameter $\mu \geq 0$. For $\mu = 0$, these conditions imply that every consistent cost function must be monotonic. In that sense this lemma resembles the Monotonicity Lemma (Lemma 3.3) for weighted congestion games proven in Chapter 3.

Building on the results of Chapter 3, we first observe that every set of consistent cost functions contains either only affine functions or only exponential functions. In Section 4.4, we study sets of affine functions and show that they are consistent for g-scaled congestion games with resource-dependent demands if and only g is linear. In Section 4.5, we prove that exponential cost functions are consistent for g-scaled congestion games with variable demands if and only if g is of type $g(x) = -\mathrm{sgn}(\phi)\beta(e^{-\phi x} - 1)$, $\beta \in \mathbb{R}_{>0}$.

While the above results hold for *arbitrary* strategy spaces, in Section 4.7 we study the consistency of cost functions in games on directed and undirected networks. Assuming that all cost functions are strictly positive we are able to translate all results to directed networks. For undirected networks, however, we need to require additionally that every cost function diverges to ∞ as the aggregated demand goes to ∞.

Bibliographic Information. An extended abstract with parts of the results contained in this chapter appeared in the *Proceedings of the 13th Biennial Conference on Theoretical Aspects of Rationality and Knowledge*; see [64].

4.2 Problem Description

First, we recall the most important concepts used in this chapter. For more details, see Chapter 2. Let N be a finite set of players and R a finite set of resources. For each player, we are given a set $\mathcal{A}_i \subseteq 2^R$ of feasible allocations and a vector $(d_{i,r})_{r\in R}$ of strictly positive resource-dependent demands. There is a cost function $c_r : \mathbb{R}_{\geq 0} \to \mathbb{R}$ associated with each resource r that maps the aggregated demands on resource r to a cost value each player perceives. The tuple $\mathcal{M} = \big(N, R, (\mathcal{A}_i)_{i\in N}, (c_r)_{r\in R}\big)$ is called a congestion model. The corresponding *proportional congestion game with resource-dependent demands* is the minimization game $G = (N, S, \pi)$ with $S_i = \mathcal{A}_i$ and $\pi_i(s) = \sum_{r\in s_i} d_{i,r}\, c_r(\ell_r(s))$ for all $i \in N$, where $\ell_r(s) = \sum_{j\in N : r\in s_j} d_{j,r}$. The corresponding *uniform congestion games with resource-dependent demands* has the same strategies but the private cost function of each player i is defined as $\pi_i(s) = \sum_{r\in s_i} c_r(\ell_r(s))$.

To capture both proportional cost games and uniform games simultaneously we define a more general class of games. For a function $g : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ the *g-scaled congestion game with resource-dependent demands* is defined as the congestion game in which the private cost of each player equals the cost of the used resources multiplied with $g(d_{i,r})$.

Definition 4.1 (g-scaled congestion game with resource-dependent demands)
Let $\mathcal{M} = \big(N, R, (\mathcal{A}_i)_{i\in N}, (c_r)_{r\in R}\big)$ be a congestion model, $(d_{i,r})_{i\in N, r\in R}$ a vector of resource-dependent demands with $d_{i,r} \in \mathbb{R}_{>0}$, and $g : \mathbb{R}_{\geq 0} \to \mathbb{R}_{>0}$. The corresponding *g-scaled congestion game with resource-dependent demands* is the minimization game $G = (N, S, \pi)$, where $S_i = \mathcal{A}_i$ and $\pi_i = \sum_{r\in s_i} g(d_{i,r})\, c_r(\ell_r(s))$ for all $i \in N$.

Throughout this chapter, we impose the following assumptions on the scaling function g.

Assumption 4.2. The scaling function $g : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is continuously differentiable and strictly positive on $\mathbb{R}_{>0}$.

For a set $\mathcal{C}$ of cost functions and a scaling function g, we say that $\mathcal{C}$ is *consistent* for g-scaled congestion games with resource-dependent demands if every g-scaled congestion game with resource-dependent demands and cost functions in $\mathcal{C}$ possesses a pure Nash equilibrium. Analogously, $\mathcal{C}$ is *universally consistent* for g-scaled congestion games with resource-dependent demands if every g-scaled congestion game with resource-dependent demands and cost functions has the finite improvement property. For a cost function c, instead of saying that $\{c\}$ is consistent (respectively, universally consistent) for g-scaled congestion games with resource-dependent demands, we simply say that c is consistent (respectively, universally consistent).

4.3 Necessary Conditions for the Existence of a Pure Nash Equilibrium

Our main result of Chapter 3 establishes that every set $\mathcal{C}$ of continuous cost functions that is consistent for weighted congestion games either contains only affine functions or only certain exponential functions as specified in Theorem 3.11. For any scaling function g, this necessary conditions for consistency can be easily translated to g-scaled congestion games with resource-dependent demands,

Proposition 4.3. *Let $\mathcal{C}$ be a set of continuous functions and let g be a scaling function. If $\mathcal{C}$ is consistent for g-scaled congestion games with resource-dependent demands, then $\mathcal{C}$ satisfies at least one of the following two conditions:*
(1) *$\mathcal{C}$ only contains affine functions.*
(2) *$\mathcal{C}$ only contains exponential functions of type $c(x) = a_c\, e^{\phi x} + b_c$, where $a_c, b_c \in \mathbb{R}$ may depend on c, while ϕ is equal for all $c \in \mathcal{C}$.*

Proof. Theorem 3.11 establishes that for every set $\mathcal{C}$ of continuous cost functions that does not satisfy one of the the required conditions, there is a congestion model $\mathcal{M} = \big(N, R, (\mathcal{A}_i)_{i\in N}, (c_r)_{r\in R}\big)$ and a vector of resource-*independent* demands d_i^{w} such that the corresponding weighted congestion game $G^{\mathrm{w}} = (N, S, \pi^{w})$ does not possess a pure Nash equilibrium. For an arbitrary scaling function g, let $G = (N, S, \pi)$ be the corresponding g-scaled congestion game with resource-dependent demands with $d_{i,r} = d_i^{\mathrm{w}}$ for all $r \in R$. We observe that $\pi_i(s) = \frac{g(d_i)}{d_i}\pi_i(s)$ for all $s \in S$ and $i \in N$. Thus, G is a monotonic transformation of G^{w}. Referring to Proposition 2.6 we derive that G does not admit a pure Nash equilibrium. □

To further restrict the set of consistent cost functions, we proceed to prove a stronger version of the Monotonicity Lemma (Lemma 3.3) for games with resource-dependent demands. Recall that the Monotonicity Lemma states that all continuous cost functions that are consistent for weighted congestion games are monotonic. Together with the characterization of non-monotonic continuous functions provided in Lemma 3.2, this implies that for all $x, y \in \mathbb{R}_{>0}$, the following two properties are satisfied: *(i)* If $c(x) > c(x+y)$, then $c(y) \geq c(x+y)$. *(ii)* If $c(x) < c(x+y)$, then $c(y) \leq c(x+y)$. As the first result of this chapter, we show that every cost function that is consistent for g-scaled games with resource-dependent demands satisfies a stronger condition, termed *generalized monotonicity condition*, which is defined below.

Definition 4.4 (Generalized Monotonicity Condition (GMC))
Let g be a scaling function. A differentiable cost function c satisfies the *generalized monotonicity condition (GMC) for* g if for all $x,y \in \mathbb{R}_{>0}$ with $c(x) \neq 0$, $c(y) \neq 0$, and all $\mu \in \mathbb{R}_{\geq 0}$ the following two conditions hold:

(GMC 1) If $c(x) > c(x+y) - \mu c'(x+y)$,
then $\big(1-\mu\frac{g'(y)}{g(y)}\big)c(x+y) - \mu c'(x+y) \leq \big(1-\mu\frac{g'(y)}{g(y)}\big)c(y) - \mu c'(y)$.

(GMC 2) If $c(x) < c(x+y) - \mu c'(x+y)$,
then $\big(1-\mu\frac{g'(y)}{g(y)}\big)c(x+y) - \mu c'(x+y) \geq \big(1-\mu\frac{g'(y)}{g(y)}\big)c(y) - \mu c'(y)$.

Note that for $\mu = 0$, the GMC is independent of the scaling function g and ensures only that c is monotonic. As the first result of this chapter, we show that any differentiable cost function that is consistent for g-scaled congestion games with resource-dependent demands satisfies the GMC for g.

Lemma 4.5 (Generalized Monotonicity Lemma). *Let c be a differentiable function and let g be a scaling function. If c is consistent for g-scaled congestion games with resource-dependent demands, then c satisfies the GMC for g.*

Proof. We start to show that c satisfies (GMC 1). Suppose not. Then, there are $x,y \in \mathbb{R}_{>0}$ with $c(x) \neq 0$ and $c(y) \neq 0$, and $\mu \in \mathbb{R}_{\geq 0}$ such that the following two inequalities hold:

$$c(x) > c(x+y) - \mu c'(x+y), \tag{4.1}$$

$$\Big(\mu\frac{g'(y)}{g(y)} - 1\Big)c(y) + \mu c'(y) > \Big(\mu\frac{g'(y)}{g(y)} - 1\Big)c(x+y) + \mu c'(x+y). \tag{4.2}$$

By continuity, it is without loss of generality to assume that μ is rational and positive; that is, $\mu = p/q$ for some $p,q \in \mathbb{N}$. Let $\varepsilon > 0$ be such that

$$c(x) > c(x+y) - \mu c'(x+y) + \varepsilon,$$

$$\Big(\mu\frac{g'(y)}{g(y)} - 1\Big)c(y) + \mu c'(y) > \Big(\mu\frac{g'(y)}{g(y)} - 1\Big)c(x+y) + \mu c'(x+y) + \varepsilon.$$

Since the functions c and g are differentiable (and thus also continuous), there is $m \in \mathbb{N}$ such that for $\delta = \frac{1}{q\cdot m}$ we have

$$c(y+\delta) \neq 0, \tag{4.3a}$$

$$\Big|\mu \cdot \frac{c(x+y+\delta) - c(x+y)}{\delta} - \mu\, c'(x+y)\Big| \leq \frac{\varepsilon}{4}, \tag{4.3b}$$

$$\Big|\mu \cdot \frac{c(y+\delta) - c(y)}{\delta} - \mu\, c'(y)\Big| \leq \frac{\varepsilon}{4}, \tag{4.3c}$$

$$\Big|\frac{\mu}{g(y)} \cdot \frac{g(y+\delta) - g(y)}{\delta} \cdot c(x+y+\delta) - \mu \cdot \frac{g'(y)}{g(y)} \cdot c(x+y)\Big| \leq \frac{\varepsilon}{4}, \tag{4.3d}$$

$$\Big|\frac{\mu}{g(y)} \cdot \frac{g(y+\delta) - g(y)}{\delta} \cdot c(y+\delta) - \mu \cdot \frac{g'(y)}{g(y)} \cdot c(y)\Big| \leq \frac{\varepsilon}{4}. \tag{4.3e}$$

		Player 2	
		R_1	$R_2 \cup Q_2$
Player 1	$R_1 \cup R_2$	$a\,g(x)\big((p\cdot m{+}1)\,c(x{+}y) + p\cdot m\,c(x)\big)$, $a\,g(y)\big((p\cdot m{+}1)\,c(x{+}y)\big)$	$a\,g(x)\big((p\cdot m{+}1)\,c(x) + p\cdot m\,c(x{+}y{+}\delta)\big)$, $a\,g(y+\delta)p\cdot m\,c(x+y+\delta) + b_2\,g(z)\,c(z)$
	Q_1	$b_1\,g(x)\,c(x)$, $a\,g(y)(p\cdot m{+}1)\,c(y)$	$b_1\,g(x)\,c(x)$, $a\,g(y{+}\delta)p\cdot m\,c(y{+}\delta) + b_2\,g(z)\,c(z)\big)$

Figure 4.1: Matrix representation of the g-scaled congestion game with variable demands constructed in the proof of Lemma 4.5. Each cell shows the private cost of the respective strategy profile.

Our proof proceeds in two steps. In the first step, we construct a g-scaled congestion game with resource-dependent demands parametrized by $b_1, b_2 \in \mathbb{Z}$, and $a \in \mathbb{N}_{>0}$. Then, in the second step, we specify the parameters such that the corresponding game does not possess a pure Nash equilibrium.

For the first step, let the parameters $b_1, b_2 \in \mathbb{Z}$, and $a \in \mathbb{N}$ be fixed. We consider the congestion model $\mathcal{M} = \big(N, R, (\mathcal{A}_i)_{i\in N}, (c_r)_{r\in R}\big)$ with two players $N = \{1,2\}$ and resources $R = R_1 \cup R_2 \cup Q_1 \cup Q_2$. The set R_1 contains $a\,(p\cdot m+1)$ resources, R_2 contains $a\cdot p\cdot m$ resources, Q_1 contains $|b_1|$ resources, and Q_2 contains $|b_2|$ resources. The players' sets of feasible allocations depend on the sign of the parameters b_1 and b_2. If $b_1 \geq 0$, we set $\mathcal{A}_1 = \{R_1 \cup R_2, Q_1\}$, and $\mathcal{A}_1 = \{R_1 \cup R_2 \cup Q_1, \emptyset\}$, otherwise.[1] If $b_2 \geq 0$, we set $\mathcal{A}_2 = \{R_1, R_2 \cup Q_2\}$, and $\mathcal{A}_2 = \{R_1 \cup Q_2, R_2\}$, otherwise. The demand of player 1 equals $d_{1,r} = x$ for all resources $r \in R$ and the demand of player 2 equals $d_{2,r} = y$ for all $r \in R_1$ and $d_{2,r} = y + \delta$ for all $r \in R_2$. The demand of player 2 for the resources in Q_2 depends on the sign of b_2. We set $d_{2,r} = z$ for all $r \in Q_2$, where $z = y + \delta$ if $b_2 \geq 0$, and $z = y$, otherwise. For the $b_1, b_2 \geq 0$, the so-defined game is shown in Figure 4.1. For the case $b_1, b_2 \geq 0$, consider the cycle $\gamma = \big((R_3, R_1), (R_1 \cup R_2, R_1), (R_1 \cup R_2, R_2 \cup Q_2), (Q_1, R_2 \cup Q_2), (Q_1, R_1), (R_3, R_1)\big)$. Calculating the differences in the private costs of the deviating players, we obtain

$$\pi_1(R_1 \cup R_2, R_1) - \pi_1(R_3, R_1) = a\,g(x)\left((p\cdot m+1)\,c(x+y) + p\cdot m\cdot c(x) - \frac{b_1}{a}\,c(x)\right), \tag{4.4a}$$

$$\begin{aligned} &\pi_2(R_1 \cup R_2, R_2 \cup Q_2) - \pi_2(R_1 \cup R_2, R_1) \\ &\quad = a\,g(y)\left(\frac{g(y+\delta)}{g(y)}\Big(p\cdot m\cdot c(x+y+\delta)\Big) + \frac{g(z)}{g(y)}\cdot\frac{b_2}{a}\,c(z) - (p\cdot m+1)\,c(x+y)\right), \end{aligned} \tag{4.4b}$$

[1] To avoid that one strategy of player 1 is empty, we can add two additional resources used with the same demand to both feasible allocations of player 1. This procedure shifts the private cost of player 1 by a constant and has no influence on the existence or nonexistence of pure Nash equilibria.

$$\pi_1(Q_1, R_2 \cup Q_2) - \pi_1(R_1 \cup R_2, R_2 \cup Q_2)$$
$$= a\,g(x)\left(\frac{b_1}{a}\,c(x) - (p\cdot m+1)\,c(x) - p\cdot m\cdot c(x+y+\delta)\right), \tag{4.4c}$$

$$\pi_2(Q_1, R_1) - \pi_2(Q_1, R_2 \cup Q_2)$$
$$= a\,g(y)\left((p\cdot m+1)c(y) - \frac{g(y+\delta)}{g(y)}\Big(p\cdot m\cdot c(y+\delta)\Big) - \frac{g(z)}{g(y)}\cdot\frac{b_2}{a}\,c(z)\right). \tag{4.4d}$$

For the cases in which at least one parameter b_i is negative, the set Q_i is contained in the first allocation of player i instead of the second one. Calculating the private cost differences of player i in the respective sequence of unilateral deviations of the modified game, b_i always appears with the contrary sign. We conclude that for arbitrary signs of b_1 and b_2 there is a cycle for which the differences of the players' private costs have the same values as in (4.4a) – (4.4d).

We proceed to show that there are $b_1, b_2 \in \mathbb{Z}$, and $a \in \mathbb{N}$ such that the expressions in (4.4a)–(4.4d) are negative, implying that the corresponding games does not admit a pure Nash equilibrium. First, we consider the expressions (4.4a) and (4.4c) associated with the private cost differences along the deviations of player 1 in γ. We claim that there is $\beta_1 \in \mathbb{R}$ such that

$$p\cdot m\cdot c(x+y+\delta) + (p\cdot m+1)c(x) > \beta_1\,c(x) > p\cdot m\cdot c(x) + (p\cdot m+1)\,c(x+y).$$

To see this, we use the facts that $c(x) \neq 0$ and $c(x) > c(x+y) - \mu\,c'(x+y) + \varepsilon$ to derive the existence of $\beta_1 \in \mathbb{R}$ with

$$c(x) > \beta_1\,c(x) - p\cdot m\cdot c(x+y+\delta) - p\cdot m\cdot c(x) > c(x+y) - \mu c'(x+y) + \varepsilon.$$

Using $p\cdot m = \mu/\delta$ this implies

$$p\cdot m\cdot c(x+y+\delta) + (p\cdot m+1)c(x) > \beta_1\,c(x)$$
$$> \mu\frac{c(x+y+\delta)-c(x+y)}{\delta} + \mu\frac{c(x+y)}{\delta} + p\cdot m\cdot c(x) + c(x+y) - \mu c'(x+y) + \varepsilon.$$

Rearranging terms and using (4.3b), we obtain

$$p\cdot m\cdot c(x+y+\delta) + (p\cdot m+1)\,c(x) > \beta_1\,c(x) > p\cdot m\cdot c(x) + (p\cdot m+1)\,c(x+y),$$

as claimed. We now turn to the private cost differences (4.4b) and (4.4d) of player 2. Recall that either $z = y$ or $z = y+\delta$. Using $c(y) \neq 0$ and(4.3a), we observe $c(z) \neq 0$. We then use $\big(\mu\frac{g'(y)}{g(y)} - 1\big)\,c(x+y) + \mu c'(x+y) + \varepsilon/2 < \big(\mu\frac{g'(y)}{g(y)} - 1\big)\,c(y) + \mu c'(y) - \varepsilon/2$ to derive the existence of $\beta_2 \in \mathbb{R}$ with

$$\left(\mu\frac{g'(y)}{g(y)} - 1\right)c(x+y) + \mu c'(x+y) + \frac{\varepsilon}{2} < -\beta_2\,c(z)\,\frac{g(z)}{g(y)} < \left(\mu\frac{g'(y)}{g(y)} - 1\right)c(y) + \mu c'(y) - \frac{\varepsilon}{2}.$$

Using (4.3b) and (4.3c), we obtain

$$\mu\frac{g'(y)}{g(y)}c(x+y)+p\cdot m\cdot c(x+y+\delta)-(p\cdot m+1)\,c(x+y)+\frac{\varepsilon}{4}$$
$$<-\beta_2\,c(z)\,\frac{g(z)}{g(y)}<\mu\frac{g'(y)}{g(y)}c(y)+p\cdot m\cdot c(y+\delta)-(p\cdot m+1)\,c(y)-\frac{\varepsilon}{4}.$$

Together with inequalities (4.3d) and (4.3e) this gives rise to

$$\left(p\cdot m\frac{g(y+\delta)}{g(y)}-p\cdot m\right)c(x+y+\delta)+p\cdot m c(x+y+\delta)-(p\cdot m+1)c(x+y)$$
$$<-\beta_2\,c(z)\,\frac{g(z)}{g(y)}<\left(p\cdot m\frac{g(y+\delta)}{g(y+\delta)}-p\cdot m\right)c(y+\delta)+p\cdot m c(y+\delta)-(p\cdot m+1)c(y).$$

Rearranging terms, we finally obtain

$$\frac{g(y+\delta)}{g(y)}p\cdot m\cdot c(x+y+\delta)-(p\cdot m+1)c(x+y)$$
$$<-\beta_2\,c(z)\,\frac{g(z)}{g(y)}<\frac{g(y+\delta)}{g(y)}p\cdot m\cdot c(y+\delta)-(p\cdot m+1)\,c(y).$$

As β_1 and β_2 are rational, we can write them as $\beta_1=b_1/a$ respectively $\beta_2=b_2/a$ for some $a\in\mathbb{N}$ and $b_1,b_2\in\mathbb{Z}$. By construction, the private cost differences (4.4a) – (4.4d) are negative and γ is an improvement cycle. Because every strategy combination is contained in γ, the thus constructed game does not admit a pure Nash equilibrium, which contradicts the consistency of c. To see that c also satisfies (GMC 2) we proceed as above, but traverse the cycle γ in the opposite direction. □

The games constructed to prove the Generalized Monotonicity Lemma (Lemma 4.5) have a quite simple structure; each game has only two players with two strategies each. The first player has a single demand $x\in\mathbb{R}_{>0}$ that she places on all resources. For some $y,\delta\in\mathbb{R}_{>0}$, the second player's demand equals y for all resources contained in her first strategy and $y+\delta$ for all other resources. With these observations, Lemma 4.5 can be strengthened in the following way.

Corollary 4.6. *Let g be a scaling function and let c be a differentiable function not satisfying the GMC for g. Then, there are $x,y,\varepsilon\in\mathbb{R}_{>0}$ such that for each $\delta\in(0,\varepsilon)$, there is a congestion model $\mathcal{M}=\big(N,R,(\mathcal{A}_i)_{i\in N},(c_r)_{r\in R}\big)$ with the following properties:*

(1) $N=\{1,2\}$;
(2) *Each player i has two disjoint allocations, i.e., $\mathcal{A}_i=\{\alpha_{i,1},\alpha_{i,2}\}$ with $\alpha_{i,1},\alpha_{i,2}\subseteq R$;*
(3) *All cost functions are equal to c, i.e., $c_r=c$ for all $r\in R$;*
(4) *The corresponding g-scaled congestion game with the resource-dependent demands $d_{1,r}=x$ for all $r\in R$, $d_{2,r}=y$ for all $r\in\alpha_{2,1}$, and $d_{2,r}=y+\delta$ for all $r\in\alpha_{2,2}$ does not possess a pure Nash equilibrium.*

With other words, for each cost function c not satisfying the GMC for g there is a *threshold* value $\varepsilon>0$ such for all $\delta\in(0,\varepsilon)$ one can construct a quite simple game (two players with two feasible allocations each, all costs equal to c) without a pure Nash equilibrium. Moreover, the players' demands in that game are *almost resource-independent*, that is, only the demand of the second player on the resources contained in her second allocation is increased by δ. This insight will be important in Chapter 5 when characterizing the consistency for congestion games with variable demands.

4.4 Consistency of Affine Functions

As noted in Proposition 4.3, every set of continuous cost functions that is consistent for g-scaled congestion games with resource-dependent demands contains either only affine or only certain exponential functions. In this section, we investigate the question whether affine functions are indeed consistent. While for weighted congestion games the notion of consistency is independent of the scaling function g, for games with resource-dependent demands the scaling function g has a severe impact on the consistency of cost functions. As the main result of this section, we show that for linear scaling functions of type $g(x) = \beta x$ with $\beta > 0$, every set of affine cost functions is consistent. This is particularly interesting since games with linear scaling functions contain the practically relevant case of proportional games as a special case. We also show that linear scaling functions are the maximal class of scaling functions with this property. This implies on the negative side that for every non-linear scaling function g every set of affine functions that contains at least one non-constant function is not consistent. In particular, affine cost functions are *not* consistent for uniform games.

We first show that affine functions are consistent and universally consistent for games with linear scaling functions.

Theorem 4.7. *Let $\mathcal{C}$ be a set of affine functions and let g be a linear scaling function of type $g(x) = \beta x$, $\beta > 0$. Then, $\mathcal{C}$ is consistent and universally consistent for g-scaled congestion games with resource-dependent demands.*

Proof. Let $\mathcal{M} = \big(N, R, (\mathcal{A}_i)_{i\in N}, (c_r)_{r\in R}\big)$ be a congestion model such that for each resource $r \in R$, there are $a_r, b_r \in \mathbb{R}$ with $c_r(x) = a_r x + b_r$ for all $x \in \mathbb{R}_{>0}$. For an arbitrary vector $(d_{i,r})_{i\in N, r\in R}$ of resource-dependent demands, let G be a corresponding congestion game with resource-dependent demands. We claim that the function $P : S \to \mathbb{R}$, $s \mapsto \sum_{i\in N} \sum_{r\in s_i} \beta\, d_{i,r} \big(a_r (\sum_{j\in\{1,\dots,i\}: r\in s_j} d_{j,r}) + b_r\big)$ is an exact potential function for G.

To see this, let $s \in S$ be arbitrary and consider player i with alternative strategy s_i'. Calculating

$$\begin{aligned}
P(s) - P(s_i', s_{-i}) = &\sum_{j\in\{i+1,\dots,n\}} \Big(\sum_{r\in s_i\cap s_j} \beta\, d_{j,r}\, a_r\, d_{i,r} - \sum_{r\in s_i'\cap s_j} \beta\, d_{j,r}\, a_r\, d_{i,r} \Big) \\
&+ \sum_{r\in s_i} \beta\, d_{i,r} \Big(a_r \big(\sum_{j\in\{1,\dots,i\}: r\in s_j} d_{j,r}\big) + b_r \Big) - \sum_{r\in s_i'} \beta\, d_{i,r} \Big(a_r \big(\sum_{j\in\{1,\dots,i\}: r\in x_j} d_{j,r}\big) + b_r \Big) \\
= &\ \pi_i(s) - \pi_i(s_i', s_{-i}),
\end{aligned}$$

we observe that P is an exact potential function. For finite games, the existence of a potential function is sufficient for the finite improvement property and the existence of a pure Nash equilibrium; see Proposition 2.20 and Corollary 2.21. $\square$

The potential function used to prove Theorem 4.7 has already given by Harks et al. [67] for the special case that $g(x) = x$ for all $x \geq 0$. We proceed to show that the linearity of g is in fact necessary for the consistency of affine functions. Specifically, we show that a non-constant affine cost function satisfies the GMC for g if and only if g is linear.

Theorem 4.8. *Let g be a scaling function. A non-constant affine function c satisfies the GMC for g if and only if g is linear.*

Proof. Let $a \in \mathbb{R} \setminus \{0\}$ and $b \in \mathbb{R}$ be such that $c(x) = ax + b$ for all $x \geq 0$. Because c has the GMC for g if and only if $-c$ has the GMC for g it is without loss of generality to assume $a > 0$.

For a contradiction, suppose that g is not linear but c satisfies the GMC for g. Because the differential equation $g(x) = x \cdot g'(x)$ characterizes linear functions uniquely, the non-linearity of g implies the existence of $y \in \mathbb{R}_{>0}$ such that $g(y) \neq y \cdot g'(y)$. Let us first assume that there is $y > 0$ such that $g'(y) = 0$. We choose $x, y, \mu \in \mathbb{R}_{>0}$ such that $c(x) \neq 0$, $c(y) \neq 0$, and $\mu > y$. Checking condition (GMC 1), we observe

$$c(x) - c(x+y) + \mu c'(x+y) = -ay + \mu a > 0, \tag{4.5}$$

$$\left(\mu \frac{g'(y)}{g(y)} - 1\right) c(x+y) + \mu c'(x+y) - \left(\mu \frac{g'(y)}{g(y)} - 1\right) c(y) - \mu c'(y) = c(y) - c(x+y) < 0.$$

Thus, (GMC 1) is violated. We conclude $g'(y) \neq 0$ for all $y > 0$. The non-linearity of g then implies the existence of $y > 0$ such that $g(y)/g'(y) \neq y$. Using the continuity of g/g' and the fact that c has at most one root, we can choose y such that $g(y)/g'(y) \neq y$ and $c(y) \neq 0$. We proceed to distinguish two cases.

First case: $g(y)/g'(y) > y$. We choose $\mu > 0$ such that $g(y)/g'(y) > \mu > y$ and $x > 0$ such that $c(x) \neq 0$. Then, again (4.5) holds because $\mu > y$. Furthermore, we obtain

$$\left(\mu \frac{g'(y)}{g(y)} - 1\right) c(x+y) + \mu c'(x+y) - \left(\mu \frac{g'(y)}{g(y)} - 1\right) c(y) - \mu c'(y) = \left(\mu \frac{g'(y)}{g(y)} - 1\right) ax < 0,$$

which violates (GMC 1).

Second case: $g(y)/g'(y) < y$. Choosing $\mu > 0$ such $g(y)/g'(y) < \mu < y$ and $x > 0$ such that $c(x) \neq 0$, we obtain

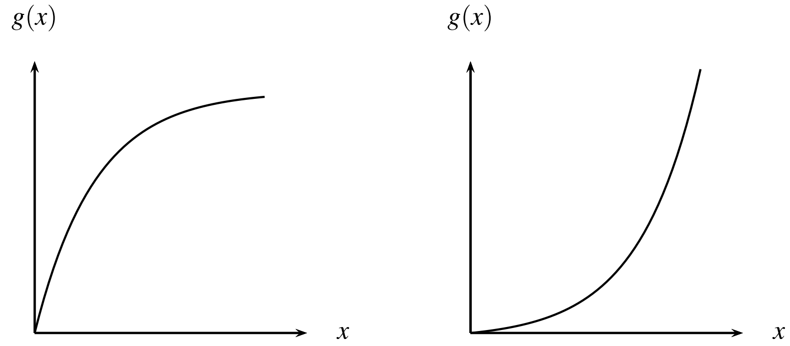
$$c(x) - c(x+y) + \mu c'(x+y) = -ay + \mu a < 0,$$

$$\left(\mu \frac{g'(y)}{g(y)} - 1\right) c(x+y) + \mu c'(x+y) - \left(\mu \frac{g'(y)}{g(y)} - 1\right) c(y) - \mu c'(y) = \left(\mu \frac{g'(y)}{g(y)} - 1\right) ax > 0,$$

violating (GMC 1). $\square$

4.5 Consistency of Exponential Functions

We proceed to characterize the set of scaling functions for which exponential functions are consistent for g-scaled congestion games with resource-dependent demands. As our main result, we show that exponential functions of the form $c(x) = a_c e^{\phi x} + b_c$, $\phi \in \mathbb{R} \setminus \{0\}$ are consistent if and only if g is of the form $g(x) = -\operatorname{sgn}(\phi)\beta(e^{-\phi x} - 1)$ with $\beta \in \mathbb{R}_{>0}$. For illustration, this type of scaling functions is shown in Figure 4.2.



(a) $\phi > 0$ (b) $\phi < 0$

Figure 4.2: Schematic illustration of the scaling function $g(x) = -\mathrm{sgn}(\phi)\beta(e^{-\phi x} - 1)$, $\beta > 0$ that is consistent for exponential cost functions of type $c(x) = a_c e^{\phi x} + b_c$, $a_c, b_c \in \mathbb{R}$, $\phi \in \mathbb{R} \setminus \{0\}$ for positive and negative ϕ, respectively.

Theorem 4.9. *Let $\phi \in \mathbb{R} \setminus \{0\}$ and let $\mathcal{C}$ be a set of cost functions such that for all $c \in \mathcal{C}$ there are $a_c, b_c \in \mathbb{R}$ with $c(x) = a_c e^{\phi x} + b_c$ for all $x \in \mathbb{R}_{\geq 0}$. Let g be a scaling function of type $g(x) = -sgn(\phi)\beta(e^{-\phi x} - 1)$ with $\beta \in \mathbb{R}_{>0}$. Then $\mathcal{C}$ is consistent and universally consistent for g-scaled congestion games with resource-dependent demands.*

Proof. Let $\phi, \beta \neq 0$ be such that $\mathrm{sgn}(\beta) = -\mathrm{sgn}(\phi)$ and let $g(x) = \beta(e^{-\phi x} - 1)$ for all $x \in \mathbb{R}_{>0}$. and let $\mathcal{M} = \big(N, R, (\mathcal{A}_i)_{i \in N}, (c_r)_{r \in R}\big)$ be a congestion model such that for each resource $r \in R$, there are $a_r, b_r \in \mathbb{R}$ with $c(x) = a_r e^{\phi x} + b_r$.

For an arbitrary vector $(d_{i,r})_{i \in N, r \in R}$ of resource-dependent demands let G be a corresponding congestion game. We set $\ell_r^i(s) = \sum_{j \in \{1,\dots,i\} : r \in s_j} d_{j,r}$ and claim that the function $P : S \to \mathbb{R}$ defined as

$$P(s) = \sum_{i \in N} \sum_{r \in s_i} \beta(e^{-\phi d_{i,r}} - 1)\big(a_r e^{\phi \ell_r^i(s)} + b_r\big)$$

is a potential function for G. For arbitrary $s \in S$, $i \in N$, and $s_i' \in S_i$, we calculate that

$$\begin{aligned}
&P(s) - P(s_i', s_i) \\
&= \sum_{j=i+1}^{n} \Big(\sum_{r \in s_i \cap s_j} \beta(e^{-\phi d_{j,r}} - 1)(1 - e^{-\phi d_{i,r}}) a_r e^{\phi \ell_r^j(s)} - \sum_{r \in s_i' \cap s_j} \beta(e^{-\phi d_{j,r}} - 1)(1 - e^{-\phi d_{i,r}}) a_r e^{\phi \ell_r^j(s)} \Big) \\
&\quad + \sum_{r \in s_i} \beta(e^{-\phi d_{i,r}} - 1)(a_r e^{\phi \ell_r^i(s)} + b_r) - \sum_{r \in s_i'} \beta(e^{-\phi d_{i,r}} - 1)(a_r e^{\phi \ell_r^i(s)} + b_r) \\
&= \sum_{r \in s_i} \beta(e^{-\phi d_{i,r}} - 1) a_r \Big(b_r + e^{\phi \ell_r^i(s)} + \sum_{j \in \{i+1,\dots,n\} : r \in s_j} (1 - e^{-\phi d_{r,j}}) e^{\phi \ell_r^j(s)} \Big) \\
&\quad - \sum_{r \in s_i'} \beta(e^{-\phi d_{i,r}} - 1) a_r \Big(b_r + e^{\phi \ell_r^i(s)} + \sum_{j \in \{i+1,\dots,n\} : r \in s_j} (1 - e^{-\phi d_{r,j}}) e^{\phi \ell_r^j(s)} \Big) \\
&= \pi_i(s) - \pi_i(s_i', s_{-i}).
\end{aligned}$$

As every finite potential game has the finite improvement property as well as a pure Nash equilibrium (Proposition 2.20 and Corollary 2.21), the claimed result follows. □

To show that scaling functions of type $g(x) = \beta(e^{-\phi x} - 1)$ are also necessary for the consistency of exponential functions, we first need a characterization of these functions in terms of a differential equation. Such a characterization is provided in the following lemma.

Lemma 4.10. *All solutions to the ordinary differential equation*

$$g'(x) = \frac{\phi}{e^{\phi x} - 1} g(x), \quad x > 0 \tag{4.6}$$

are of the form $g(x) = \beta(e^{-\phi x} - 1)$ *with* $\beta \in \mathbb{R}$.

Proof. We first consider the case $\phi > 0$. Because $\frac{\phi}{e^{\phi x}-1} > 0$, every solution of (4.6) is monotonic. Furthermore, we observe that g solves (4.6) if and only if for all $\lambda \in \mathbb{R}$ the function λg solves (4.6). Let us assume (4.6) admits a non-constant and non-decreasing solution g and let $x_0 = \sup\{x \geq 0 : g(x) = 0\}$. Then, g satisfies the equation $(\ln g(x))' = \frac{\phi}{e^{\phi x}-1}$ on (x_0, ∞). Integrating both sides, we obtain $\ln g(x) = \ln(1 - e^{-\phi x}) + \tilde{\beta}$ for all $x \in (x_0, \infty)$, where $\tilde{\beta} \in \mathbb{R}$ is an arbitrary constant. Setting $\beta = e^{\tilde{\beta}}$ and solving for $g(x)$, we finally obtain $g(x) = -\beta(e^{-\phi x} - 1)$ on (x_0, ∞) for some $\beta > 0$. Using that g is continuous, this implies that $x_0 = 0$.

If $\phi < 0$, we use instead that $(\ln g(x))' = \frac{\phi}{e^{\phi x}-1}$ implies $\ln g(x) = \ln(1 - e^{\phi x}) - \phi x + \tilde{\beta}$ for all (x_0, ∞) and some $\tilde{\beta} \in \mathbb{R}$ and continue as in the case $\phi > 0$.

The observation that g solves (4.6) if and only if for all $\lambda \in \mathbb{R}$ the function λg solves (4.6) finishes the proof. □

We proceed to show that scaling functions of this type are also necessary for the consistency of exponential function in games with resource-dependent demands.

Theorem 4.11. *For* $\phi \neq 0$, *let* g *be a scaling function, and let* c *be a non-constant exponential function of type* $c(x) = a e^{\phi x} + b$, $a \neq 0$, $b \in \mathbb{R}$. *Then,* c *satisfies the GMC for* g *if and only if* g *is of the form* $g(x) = -sgn(\phi)\beta(e^{-\phi x} - 1)$ *with* $\beta \in \mathbb{R}_{>0}$.

Proof. Let c be of the demanded form. For a contradiction, let us assume that c satisfies the GMC for g, but g is not as claimed. The GMC implies that for all $x, y, \mu > 0$ with $c(x) \neq 0$ and $c(y) \neq 0$ the expressions

$$c(x) - c(x+y) + \mu c'(x+y) = a e^{\phi x}\left(1 + e^{\phi y}(\mu\phi - 1)\right) \tag{4.7}$$

and

$$\begin{aligned}&\left(\mu \frac{g'(y)}{g(y)} - 1\right) c(x+y) + \mu c'(x+y) - \left(\mu \frac{g'(y)}{g(y)} - 1\right) c(y) - \mu c'(y)\\ &\quad = a e^{\phi y}\left(e^{\phi x} - 1\right)\left(\mu \frac{g'(y)}{g(y)} + \mu\phi - 1\right)\end{aligned} \tag{4.8}$$

have equal signs.

We first make the useful observation that c has at most one root. Together with the continuity of the expressions in (4.7) and (4.8), this implies that if there are $x,y \in \mathbb{R}_{>0}$ such that (4.7) and (4.8) are nonzero and have different signs, then we can choose x and y such that additional $c(x) \neq 0$ and $c(y) \neq 0$. Furthermore, we observe that altering the sign of a also alters the sign of the right hand sides both of (4.7) and (4.8). Form there, it is without loss of generality to assume that $a > 0$. Regarding the sign of ϕ we distinguish two cases.

First case: $\phi > 0$. If there is $y > 0$ such that $g'(y)/g(y) + \phi \leq 0$, we choose $\mu > 1/\phi$ and $x > 0$ arbitrarily. Then, (4.7) is positive while (4.8) is negative, contradiction! We conclude that $g'(y)/g(y) + \phi > 0$ for all $y > 0$. This implies, that (4.8) is positive if and only if $\mu > \frac{g(y)}{g'(y)+\phi g(y)}$ while (4.7) is positive if and only if $\mu > \frac{e^{\phi y}-1}{\phi e^{\phi y}}$. For every

$$\mu \in \left(\min\left\{ \frac{g(y)}{g'(y)+\phi g(y)}, \frac{e^{\phi y}-1}{\phi e^{\phi y}} \right\}, \max\left\{ \frac{g(y)}{g'(y)+\phi g(y)}, \frac{e^{\phi y}-1}{\phi e^{\phi y}} \right\} \right),$$

the expressions are nonzero and have different signs. We conclude that $\frac{g(y)}{g'(y)+\phi g(y)} = \frac{e^{\phi y}-1}{\phi e^{\phi y}}$ for all $y > 0$. Rearranging the terms, we derive that g satisfies the differential equation (4.6). As shown in Lemma 4.10, this implies that g is of the form $g(x) = -\mathrm{sgn}(\phi)\beta(e^{-\phi x} - 1)$, contradiction!

Second case: $\phi < 0$. If there is $y > 0$ such that $g'(y)/g(y) + \phi \leq 0$, then (4.8) is positive for all $\mu > 0$. Choosing μ large enough such that (4.7) is negative, we obtain a contradiction. We conclude that $g'(y)/g(y) + \phi > 0$ for all $y > 0$ and derive that (4.8) is positive if and only if $\mu < \frac{g(y)}{g'(y)+\phi g(y)}$ while (4.7) is positive if and only if $\mu < \frac{e^{\phi y}-1}{\phi e^{\phi y}}$. As in the first case, we may derive that g solves the differential equation (4.6), a contradiction. □

4.6 A Characterization of Consistency

We are now ready to state the main result of this chapter.

Theorem 4.12. *Let $\mathcal{C}$ be a set of continuous functions and let g be a scaling function. Then, the following are equivalent:*

(1) *$\mathcal{C}$ is consistent for g-scaled congestion games with resource-dependent demands.*

(2) *$\mathcal{C}$ is universally consistent for g-scaled congestion games with resource-dependent demands.*

(3) *At least one of the following three cases holds:*

  (3a) *g is arbitrary and $\mathcal{C}$ contains only constant functions.*

  (3b) *g is linear and $\mathcal{C}$ contains only affine functions.*

  (3c) *There is $\phi \in \mathbb{R}$ such that g is of type $g(x) = -sgn(\phi)\beta(e^{-\phi x} - 1)$, $\beta \in \mathbb{R}_{>0}$ and $\mathcal{C}$ contains only functions of type $c(x) = a_c e^{\phi x} + b_c$ where $a_c, b_c \in \mathbb{R}$ may depend on c.*

Proof. (2) ⇒ (1) follows because every game with the finite improvement property has a pure Nash equilibrium (Proposition 2.8).

(1) ⇒ (3): Referring to Proposition 4.3, consistency of $\mathcal{C}$ implies that one of the following two cases holds: *(i)* $\mathcal{C}$ contains only affine functions; *(ii)* $\mathcal{C}$ contains only exponential functions of type $c(x) = a_c e^{\phi x} + b_c$, where $a_c, b_c \in \mathbb{R}$ may depend on c while $\phi \in \mathbb{R}$ is equal for all $c \in \mathcal{C}$. Let us

first consider the case that $\mathcal{C}$ contains only affine functions. If all functions in $\mathcal{C}$ are constant, then (3a) is satisfied. If $\mathcal{C}$ contains a non-constant affine function, the Generalized Monotonicity Lemma (Lemma 4.5) together with Theorem 4.8 establishes that g is linear and (3b) is satisfied. For the case that $\mathcal{C}$ contains a non-constant exponential function, the Generalized Monotonicity Lemma and Theorem 4.11 imply (3c).

(3) $\Rightarrow$ (2): First, we assume (3a). Let $G = (N, S, \pi)$ be a g-scaled congestion games with resource-dependent demands where all cost functions are constant. Then, the private cost of each player i does not depend on the strategies of all other players. We derive that the function $P : S \to \mathbb{R}$ defined as $P(s) = \sum_{i \in N} \pi_i(s)$ for all $s \in S$ is an exact potential function of G. Finite potential games have the finite improvement property as shown in Proposition 2.20). The implications (3b) $\Rightarrow$ (2) and (3c) $\Rightarrow$ (2) are shown in Theorems 4.7 and 4.9, respectively. $\square$

The characterization of consistency stated in Theorem 4.12 is even valid for three-player games.

4.7 Network Congestion Games with Resource-Dependent Demands

In this section, we consider congestion games with resource-dependent demands on networks. In these games, the resources correspond to the edges of a directed (or undirected) graph $\mathcal{G} = (V, R)$. Every player is associated with a source-sink pair $(u_i, v_i) \in V \times V$ and wants to establish a directed (or undirected) path between u_i and v_i. We first prove a variant of the Generalized Monotonicity Lemma (Lemma 4.5) for games on directed networks. On the one hand, the result we are going to prove is stronger than Lemma 4.5 as it requires only consistency for games on directed networks. On the other hand, it is weaker as we impose the additional assumption that the cost functions are strictly positive on $\mathbb{R}_{>0}$.

Lemma 4.13 (Generalized Monotonicity Lemma for Directed Networks). *Let g be a scaling functions and let c be a differentiable function that is strictly positive on $\mathbb{R}_{>0}$. If c is consistent for g-scaled directed-network congestion games with resource-dependent demands, then c satisfies the GMC for g.*

Proof. We show that if c does not satisfy the GMC for g, then there is a g-scaled directed-network congestion game with resource-dependent demands with costs equal to c that does not admit a pure Nash equilibrium. Let such c be given. Lemma 4.5 implies the existence of a two-player game G with costs equal to c that does not admit a pure Nash equilibrium. The construction of G involves six mutually disjoint sets of resources $R_1, R_2, Q_1, Q_1', Q_2, Q_2' \in 2^R$ such that $\mathcal{A}_1 = \{R_1 \cup R_2 \cup Q_1, Q_1'\}$, and $\mathcal{A}_2 = \{R_1 \cup Q_2, R_2 \cup Q_2'\}$, where for $i \in \{1,2\}$ always one of the two sets Q_i and Q_i' is empty. In addition, there are $x, y, \delta \in \mathbb{R}_{>0}$ such that the players' demands equal $d_{1,r} = x$ for all $r \in R$, $d_{2,r} = y$ for $r \in R_1 \cup Q_2$, and $d_{2,r} = y + \delta$, otherwise.

Because G is finite, there is $\rho > 0$ such that G does not possess a ρ-approximate pure Nash equilibrium. To obtain a game with network structure, we slightly modify G by adding additional resources to each of the players' strategies without changing the equilibrium structure of the game. Let $k, k' \in \mathbb{N}_{>0}$ be such that

$$|(k+1)\, g(y)\, c(y) - (k'+1)\, g(y+\delta)\, c(y+\delta)| < \rho/2. \tag{4.9}$$

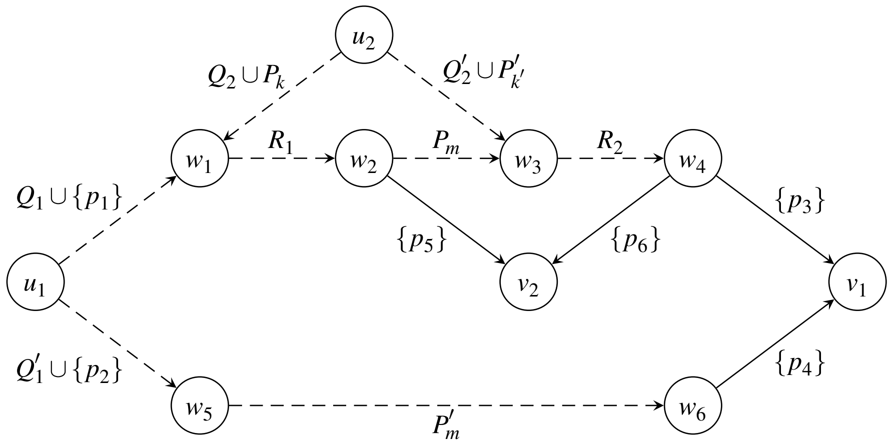


Figure 4.3: Directed-network congestion game with resource-dependent demands as constructed in the proof of the Generalized Monotonicity Lemma for Directed Networks (Lemma 4.13). Solid lines correspond to single edges while dashed lines correspond to directed paths.)

Such k, k' exist since $c(y) > 0$ and $c(y+\delta) > 0$. Let $m \in \mathbb{N}$ be such that

$$m \cdot g(y) \cdot \min\{c(y), c(x+y)\} > g(y)\, c(y). \tag{4.10}$$

We add $k + k' + 2m + 6$ new resources with cost function c. We partition $k + k' + 2m$ of the new resources into subsets $P_k, P'_{k'}, P_m, P'_m$ with cardinalities $|P_k| = k, |P'_{k'}| = k', |P_m| = m, |P'_m| = m$. The additional 6 new resources are called $p_1, \ldots, p_6$. The demands of player 1 equals x for all new resources. Player 2 accesses all new resources with demand y except for the resources contained in $P'_{k'} \cup \{p_6\}$ which she accesses with demand $y + \delta$.

Consider the network game G^{dn} shown in Figure 4.3. The feasible allocations equal the sets of their respective (u_i, v_i)-paths:

$$\mathcal{A}_1 = \Big\{Q_1 \cup \{p_1\} \cup R_1 \cup P_m \cup R_2 \cup \{p_3\},\; Q_1' \cup \{p_2\} \cup P_m' \cup \{p_4\}\Big\},$$
$$\mathcal{A}_2 = \Big\{Q_2 \cup P_k \cup R_1 \cup \{p_5\},\; Q_2' \cup P_{k'}' \cup R_2 \cup \{p_6\},\; Q_2 \cup P_k \cup R_1 \cup P_m \cup R_2 \cup \{p_6\}\Big\}.$$

Using (4.10), we observe that the third strategy $Q_2 \cup P_k \cup R_1 \cup P_m \cup R_2 \cup \{p_6\}$ of player 2 is strictly dominated by her first strategy $Q_2 \cup P_k \cup R_1 \cup \{p_5\}$. This implies that player 2 does not use her third strategy in any pure Nash equilibrium. Compared to G, the private cost of player 1 is raised by the constant $m\, g(x)\, c(x)$. The private cost of player 2 for her first allocation is raised by $(k+1) g(y) c(y)$ while the private cost perceived in her second allocation is raised by $(k'+1) g(y+\delta) c(y+\delta)$. Using (4.9) and the fact that the initial game G does not possess a ρ-approximate pure Nash equilibrium, we conclude that G^{dn} does not possess a pure Nash equilibrium. □

We proceed to translate this result to undirected networks.

Lemma 4.14 (Generalized Monotonicity Lemma for Undirected Networks). *Let c be a differentiable and strictly positive function with $c(x) \to \infty$ as $x \to \infty$ and let g be a scaling function. If c is consistent for g-scaled congestion games with resource-dependent demands on undirected networks, then c satisfies the GMC for g.*

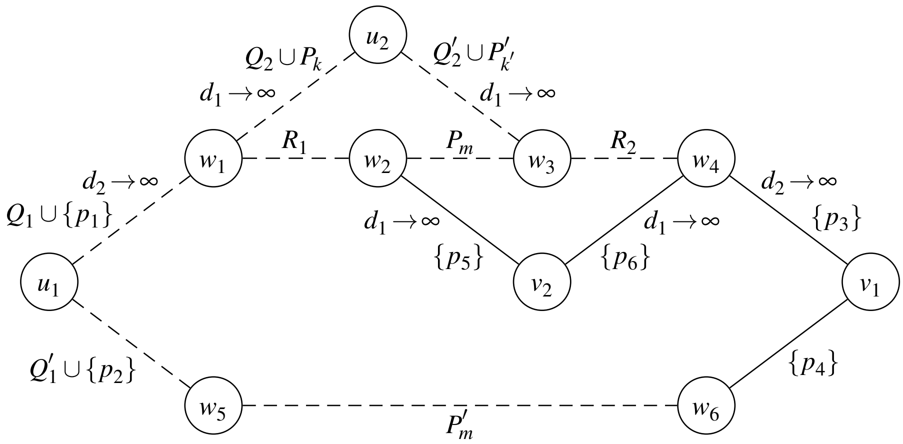


Figure 4.4: Undirected-network congestion game with resource-dependent demands as constructed in the proof of the Generalized Monotonicity Lemma for Directed Networks (Lemma 4.14. Solid lines correspond to single edges while dashed lines correspond to directed paths.)

Proof. We use the same construction as before but replace directed edges by undirected edges. Using that $c(x)$ diverges to ∞ for $x \to \infty$, we can choose the demands for player 1 on the resources contained in $Q_2 \cup P_k \cup Q_2' \cup P_{k'}' \cup \{p_5\} \cup \{p_6\}$ large enough such that all strategies containing one of these resources are strictly dominated. Analogously, we choose the demand of player 2 on the resources contained in $Q_1 \cup \{p_1\} \cup \{p_3\}$ large enough such that all strategies involving one or more of these resources are strictly dominated. The modified network is shown in Figure 4.4. With the same arguments as in the proof of the Generalized Monotonicity Lemma for Directed Networks (Lemma 4.13) no pure Nash equilibrium exists. □

We are ready to state our characterization theorem for network congestion games with resource-dependent demands.

Theorem 4.15. *Let $\mathcal{C}$ be a set of continuous functions that is strictly positive on $\mathbb{R}_{>0}$ and let g be a scaling function. Then, the following are equivalent:*

(1) *$\mathcal{C}$ is consistent for g-scaled directed-network congestion games with resource-dependent demands.*
(2) *$\mathcal{C}$ is universally consistent for g-scaled directed-network congestion games with resource-dependent demands.*
(3) *One of the following three cases holds:*
   (3a) *g is arbitrary and $\mathcal{C}$ contains only constant functions.*
   (3b) *g is linear and $\mathcal{C}$ contains only affine functions.*
   (3c) *There is $\phi \in \mathbb{R}$ such that g is of type $g(x) = -sgn(\phi)\beta(e^{-\phi x} - 1)$, $\beta \in \mathbb{R}_{>0}$ and $\mathcal{C}$ contains only functions of type $c(x) = a_c e^{\phi x} + b_c$ where $a_c, b_c \in \mathbb{R}$ may depend on c.*

If every $c \in C$ satisfies $c(x) \to \infty$ as $x \to \infty$, this equivalence is also valid for undirected networks.

Proof. (2) $\Rightarrow$ (1) follows because every game with the finite improvement property has a pure Nash equilibrium (Proposition 2.8). (3) $\Rightarrow$ (2) follows as in Theorem 4.12.

We proceed to show (1) $\Rightarrow$ (3) for directed networks. Theorem 3.16 implies that every set of cost functions $\mathcal{C}$ that is consistent for directed-network weighted congestion games satisfies one of the following three conditions: *(i)* $\mathcal{C}$ contains only affine functions of type $c(x) = a_c x + b_c$ with $a_c, b_c \in \mathbb{R}_{\geq 0}$; *(ii)* $\mathcal{C}$ contains only exponential functions of type $c(x) = a_c e^{\phi x} + b_c$, where $a_c, b_c \in \mathbb{R}$ may depend on c while $\phi \in \mathbb{R}$ is equal for all $c \in \mathcal{C}$. These necessary conditions are also valid for consistency for g-scaled congestion games with resource-dependent demands, see Proposition 4.3.

Let us first consider the case that $\mathcal{C}$ contains only affine functions. If all functions in $\mathcal{C}$ are constant, then (3a) is satisfied. If $\mathcal{C}$ contains a non-constant affine function $c(x) = a_c x + b_c$ with $a_c \in \mathbb{R}_{>0}$ and $b_c \in \mathbb{R}_{\geq 0}$ we apply the Generalized Monotonicity Lemma for Directed Networks (Lemmas 4.13) and derive that c satisfies the GMC for g. Theorem 4.8 then implies that g is linear and (3b) is satisfied.

For the case that $\mathcal{C}$ contains a non-constant exponential functions, the versions of the Generalized Monotonicity Lemma for directed or undirected networks (Lemmas 4.13) imply (3c).

To see (1) $\Rightarrow$ (3) for games on undirected networks, we proceed as in the directed case but use Theorem 3.21 instead of Theorem 3.16 and Lemma 4.14 instead of Lemma 4.13. Note that Lemma 4.14 requires that every cost function $c \in \mathcal{C}$ diverges to ∞. $\square$

This characterization of consistency is even valid for games with three-players.

4.8 Discussion and Open Problems

In this chapter, we explored the existence of pure Nash equilibria in congestion games with resource-dependent demands with respect to the cost functions of the resources. Our results revealed interesting structural differences to the class of weighted congestion games studied in Chapter 3. While the distinction between proportional games and uniform games (see Section 2.3.2 for the definitions) has no impact on the existence of pure Nash equilibria for weighted congestion games, it matters for the consistency of cost functions in congestion games with resource-dependent demands. Specifically, affine cost functions are consistent for proportional congestion games with resource-dependent demands while they are inconsistent for uniform congestion games with resource-dependent demands. For both classes of games, exponential cost functions are not consistent, contrasting the positive result for weighted congestion games with exponential costs.

Our characterizations of the consistency for proportional and uniform congestion games with resource-dependent demands were proven by carefully constructing generic games that do not admit a pure Nash equilibrium. Interestingly, these games require only two players and have the additional property that the demands of the first player are equal for all resources, and the differences of the demands of the second player can be made arbitrarily small. This fact sheds a new light on the positive results for weighted congestion games. In fact, every weighted congestion game with exponential cost functions and every uniform congestion game with affine costs may lack a pure Nash equilibrium if the players' demands are perceived at the resources with an arbitrary small (resource-dependent) error. On the other hand, the consistency of affine cost functions for proportional weighted congestion games is more stable as even misreported or misconceived demands may not destroy the pure Nash equilibria of the game.

As in the previous chapter we assumed that cost functions are continuous. It would be interesting to weaken this assumption.

While our characterizations of consistency are valid for three-games, we leave it as an open problem to derive a characterization for two-player games. Another interesting research direction would be to investigate the existence of pure Nash equilibria in games with restricted strategy spaces such as symmetric strategies, singletons, or matroids.

Chapter 5

Congestion Games with Variable Demands

Most of the previous work on congestion games – including the two preceding chapters – has a common feature: every player allocates a *fixed* demand to an available subset of resources. While obviously important, such models do not take into account a fundamental property of many real-world applications: the intrinsic coupling between the costs of the resources and the resulting demands. A prominent example of this coupling is the flow control problem in telecommunication networks. In this setting, players receive a non-negative utility from sending data and the perceived costs increases with congestion. In this and other examples, the demands will be reduced if the resources are heavily congested, and increased if the resources are less congested. Allowing for *variable demand* is, thus, a natural prerequisite for modeling the tradeoff between the benefit from satisfying a certain demand, and the costs of the resources.

There is a large body of work addressing the issue of variable demands, e.g. Cole et al. [26], Kelly et al. [78], Low and Lapsley [90], Shenker [121], and Srikant [123] in the context of telecommunication networks and Beckmann et al. [16] and Haurie and Marcotte [69] in the context of traffic networks. Most of these works assume that the (variable) demand may be *fractionally* distributed over the available subsets of resources. This assumption together with convexity assumptions on the cost and utility functions gives a convex game for which standard fixed point arguments apply; see Theorem 2.25. Allowing a fractional distribution of the demand, however, is obviously not possible in many applications. For instance, the standard TCP/IP protocol suite uses single path routing, because splitting the demand comes with several practical complications, e.g., packets arriving out of order, packet jitter due to different paths delays etc.

In this chapter, we study congestion games with *variable* demands where the demand has to be assigned to *exactly one* feasible allocation. Formally, we are given for each player a set of feasible allocations and an interval of feasible demands. In each strategy profile, a strategy of each player is to choose both a feasible allocation and a feasible demand. We assume that every player is associated with a non-decreasing and concave utility function measuring the utility for the demand and that the private payoff of each player is quasi-linear, i.e., the private payoff of each player is defined as the difference between the utility and the associated congestion cost of the used resources.

To avoid oscillating behavior, it is desirable to design telecommunication and transportation systems such that a stable state exists and is reached by myopic play. Because the utility functions are private information and not available to the system designer (cf. Kelly et al. [78]), it is natural to study the existence of equilibria and the convergence of selfish behavior with respect to the *cost functions* which represent the technology associated with the resources, e.g., queuing disciplines at routers, latency functions in transportation networks, etc. To this end, we adopt the consistency approach used in the previous chapters. We say that as set $\mathcal{C}$ of cost functions *consistent* for congestion games with variable demands if each congestion game with variable demands and costs in $\mathcal{C}$ possesses a pure Nash equilibrium.

Because congestion games with variable demands are infinite games they do have the finite improvement property (except for the degenerate case where for each player the private payoff does not depend on her demand). Thus, we resort to the approximate finite improvement property instead. Recall that a maximization game has the approximate finite improvement property if for every $\rho > 0$ all sequences of improvements that add at least ρ to the payoff of the deviating player are finite. A set $\mathcal{C}$ of cost functions is called *approximately universally consistent* for congestion games with variable demands if every congestion game with variable demands has the approximately finite improvement property. Note that the notions of consistency and approximate universal consistency are independent: neither does approximate universal consistency imply consistency, nor conversely.

5.1 Contributions and Chapter Outline

Our main results are complete characterizations of the consistency and approximate universal consistency of cost functions for congestion games with variable demands. As for congestion games with resource-dependent demands it is important to distinguish between *proportional games* and *uniform games*. In a proportional game, the payoff of each player equals her utility for her demand minus the cost on the used resources *multiplied with her demand*.

We show that a set $\mathcal{C}$ of continuous, non-negative and non-decreasing cost functions is consistent for proportional congestion games with variable demands if and only if exactly one of the following cases holds: *(i)* $\mathcal{C}$ only contains affine functions $c(x) = a_c x + b_c$ with $a_c > 0$, $b_c \geq 0$; *or (ii)* $\mathcal{C}$ only contains homogeneously exponential functions such that $c(x) = a_c e^{\phi x}$ for some $a_c, \phi > 0$, where a_c may depend on c, while ϕ must be equal for all $c \in \mathcal{C}$. Moreover, we prove that $\mathcal{C}$ is approximately universally consistent for proportional congestion games with variable demands if and only if $\mathcal{C}$ only contains affine functions.

Uniform congestion games with variable demands differ from the previously studied games solely in that fact that in the definition of the players' payoff functions the cost for a player on a resource is *not* multiplied with the demand of that player. Uniform cost structures play a key role in large-scale telecommunication networks where it is highly desirable to charge every player the *same* cost regardless of the actual resource consumption of every player, because every resource needs only to communicate a single value to the players giving rise to an efficient and scalable implementation; see Johari and Tsitsiklis [75] and Srikant [123].

Our second main result provides a complete characterization of consistency of a set of cost functions for uniform congestion games with variable demands. Under the same assumptions on $\mathcal{C}$ as before, we prove that $\mathcal{C}$ is consistent for uniform games if and only if $\mathcal{C}$ only contains homogeneously exponential functions such that $c(x) = a_c e^{\phi x}$ for some $a_c, \phi > 0$, where a_c may depend on c, while ϕ

Table 5.1: Pure Nash equilibria and the approximate finite improvement property (AFIP) in congestion games with variable demands with proportional costs and uniform costs. Here, by "inhomogeneously exponential", we denote sets $\mathcal{C}$ of cost functions, such that every $c \in \mathcal{C}$ is of type $c(x) = a_c e^{\phi x} + b_c$, where $a_c \in \mathbb{R}_{>0}$ and $b_c \geq -a_c$ may depend on c while $\phi \in \mathbb{R}_{>0}$ is equal for all $c \in \mathcal{C}$. By "homogeneously exponential", we denote sets of functions that have the additional property that $b_c = 0$ for all $c \in \mathcal{C}$. Note the fundamental structural difference to weighted congestion games (with fixed demands) studied in Chapter 3 where games with inhomogeneously exponential costs always possess a pure Nash equilibrium.
Reading example: A set of homogeneously exponential functions is both consistent for proportional and uniform congestion games with variable demands because in each such game a pure Nash equilibrium (PNE) exists. It is, however, neither approximately universally consistent for proportional games nor for uniform games as these games do not have the approximate finite improvement property (AFIP). A weighted congestion games for which all costs are homogeneously exponential has both a pure Nash equilibrium (PNE) and the finite improvement property (FIP).

costs	var. demands proportional PNE	var. demands proportional AFIP	var. demands uniform PNE	var. demands uniform AFIP	fix. demands prop. & unif. PNE	fix. demands prop. & unif. FIP
affine	yes	yes	no	no	yes	yes
homogeneously exponential	yes	no	yes	no	yes	yes
inhomogeneously exponential	no	no	no	no	yes	yes
other	no	no	no	no	no	no

must be equal for all $c \in \mathcal{C}$. Surprisingly, this characterization reveals that uniform games need not possess a pure Nash equilibrium, even if costs are affine. We also show that there is no nonempty set of cost functions that is approximately universally consistent for uniform games; that is $\mathcal{C}$ is approximately universally consistent if and only if $\mathcal{C} = \emptyset$. Our results are summarized in Table 5.1.

In Section 5.2, we recall the basic notions used in this chapter. Section 5.3 is devoted to the case of homogeneously exponential functions. To prove the consistency of homogeneously exponential functions, we introduce a novel concept that we term *essential improving moves*. A subset of improving moves is called *essential*, if for each strategy profile s the fact that there is no essential improving move from s implies that there is no improving move from s. We show that (proportional and uniform) congestion games with variable demands and homogeneously exponential costs exhibit a nontrivial subset of essential improving moves. Specifically, we prove that the set of strategy switches in which either only the demand or only the allocation is adapted has this property. We then use this result to derive the existence of a pure Nash equilibrium in all (proportional or uniform) congestion games with variable demands and homogeneously exponential costs.

We then study necessary conditions for the existence of a pure Nash equilibrium in Section 5.4. First, we observe that by restricting for each player the set of feasible demands to singletons, all necessary conditions for the consistency of cost functions from Chapter 3 translate. To further cut down the set of consistent cost functions, we establish a connection to congestion games with resource-dependent demands. Roughly speaking, we show that if for a set $\mathcal{C}$ of cost functions there is a game with resource-dependent demands with cost functions in $\mathcal{C}$ not possessing a pure Nash equilibrium *and* $\mathcal{C}$ contains a function c such that c'/c is injective on $\mathbb{R}_{>0}$, then there is also a game with variable demands and cost functions in $\mathcal{C}$ not possessing a pure Nash equilibrium. This allows us to translate the results obtained in Chapter 4 in part to the case of variable demands. Note that for any homogeneously exponential function c the function c'/c is constant. Thus, the additional condition that there is $c \in \mathcal{C}$ such that c'/c is injective precisely explains why homogeneously exponential cost functions are consistent for congestion games with *variable* demands, but not for congestion games with *resource-dependent* demands.

The main contributions of this chapter are stated and proven in Section 5.5. Directed-network congestion games with variable demands are studied in Section 5.6. We prove that for cost functions that are strictly positive on $\mathbb{R}_{>0}$, the characterization obtained in the last section carries over.

In Section 5.7, we further show that our results obtained in the previous sections remain valid under the additional assumption, that for each player her set of feasible demands is the set of non-negative reals and all utility functions are infinitely often differentiable functions.

We conclude the paper in Section 5.8 by presenting new research directions.

Bibliographic Information. Parts of the results presented in this chapter are joint work with Tobias Harks. An extended abstract appeared in the *Proceedings of the 13th Biennial Conference on Theoretical Aspects of Rationality and Knowledge*; see [64].

5.2 Problem Description

We first recall the most important definitions from Chapter 2. Let N be a finite set of players and R a finite set of resources. For each player $i \in N$, we are given a set $\mathcal{A}_i \subseteq 2^R \setminus \{\emptyset\}$ of feasible allocations,[1] an interval $[\sigma_i, \tau_i] \subseteq \mathbb{R}_{\geq 0}$ of feasible demands with $\sigma_i \in \mathbb{R}_{\geq 0}$, $\tau_i \in \mathbb{R}_{\geq 0} \cup \{\infty\}$, $\sigma_i \leq \tau_i$ and a utility function $U_i : [\sigma_i, \tau_i] \to \mathbb{R}$.[2] We say that player i has an *unrestricted demand* if $\sigma_i = 0$ and $\tau_i = \infty$. Each resource $r \in R$ is endowed with a cost function $c_r : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ that maps the aggregated demand on r to a cost value each of its users perceives. We call the tuple $\mathcal{M} = \big(N, R, (\mathcal{A}_i)_{i \in N}, (c_r)_{r \in R}\big)$ a congestion model.

As for congestion games with resource-dependent demands studied in Chapter 4, we distinguish between proportional and uniform games. Given a congestion model and the set of utility functions, the corresponding *proportional congestion game with variable demands* is the maximization game $G = (N, S, \varpi)$ with $S_i = \mathcal{A}_i \times [\sigma_i, \tau_i]$ and $\varpi_i(\alpha, d) = U_i(d_i) - \sum_{r \in \alpha_i} d_i \, c_r(\ell_r(\alpha, d))$ for all $i \in N$, where $\ell_r(\alpha, a) = \sum_{j \in N : r \in \alpha_i} d_j$. In the corresponding *uniform congestion game with variable demands* the private payoff is defined as $\varpi_i(\alpha, d) = U_i(d_i) - \sum_{r \in \alpha_i} c_r(\ell_r(\alpha, d))$. As in Chapter 4, we want to treat both classes of games simultaneously. To this end, we define g-scaled congestion games with variable demands.

[1] In contrast to the previous chapters we require that each allocation is nonempty. This is necessary since otherwise players might want to let their demand go to ∞ preventing the existence of a pure Nash equilibrium.

[2] Technically, it is sufficient to be given the utility function of each player i only since its domain already defines the set of feasible demands.

Definition 5.1 (g-scaled congestion game with variable demands)
Let $\mathcal{M} = \big(N, R, (\mathcal{A}_i)_{i\in N}, (c_r)_{r\in R}\big)$ be a congestion model, $g : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ a scaling function and for all $i \in N$ let $U_i : [\sigma_i, \tau_i] \to \mathbb{R}_{\geq 0}$ be a utility function. The corresponding *g-scaled congestion game with variable demands* is the maximization game $G = (N, S, \varpi)$, where $S_i = \mathcal{A}_i \times [\sigma_i, \tau_i]$ and $\varpi_i(\alpha, d) = U_i(d_i) - \sum_{r\in\alpha_i} g(d_i)\, c_r(\ell_r(\alpha, d))$ for all $i \in N$.

Setting $g(x) = x$ for all $x \geq 0$, we obtain proportional games as a special case of g-scaled congestion games with variable demands. For the choice $g(x) = 1$ for all $x \geq 0$, we obtain uniform games. Throughout this chapter, we impose the following assumption on the scaling function g.

Assumption 5.2. The scaling function $g : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is affine and strictly positive on $\mathbb{R}_{>0}$, that is, $g(x) = \beta x + \eta$ with $\beta, \eta \in \mathbb{R}_{\geq 0}$ and $\eta > 0$ if $\beta = 0$.

Note that this assumption on g is stricter than Assumption 4.2 in Chapter 4. Nonetheless, for the two practically relevant cases of proportional games and uniform games the assumption is still satisfied.

Let $\mathcal{C}$ be a class of cost functions and let g be a scaling function. We call $\mathcal{C}$ *consistent for g-congestion games with variable demands* if every g-scaled congestion game with variable demands and cost functions in $\mathcal{C}$ admits a pure Nash equilibrium. We say that $\mathcal{C}$ is *approximately universally consistent* if every congestion game with variable demands and cost functions in $\mathcal{C}$ has the approximate finite improvement property.

The following two assumptions imposed throughout this chapter contain mild restrictions on feasible utility functions and cost functions.

Assumption 5.3. For every resource $r \in R$ the cost function $c_r : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is locally Lipschitz-continuous and non-decreasing.

Assumption 5.4. For each player i the utility function $U_i : [\sigma_i, \tau_i] \to \mathbb{R}_{\geq 0}$ is continuous, non-decreasing and concave.

Remark 5.5. In contrast to most of the works in the area of congestion games with splittable demands (e.g., Haurie and Marcotte [69], Kelly et al. [78], and Orda et al. [106]), we do not assume semi-convexity of cost functions.

For each player i with $\sigma_i < \tau_i$, the concavity of U_i already implies that U_i is continuous on (σ_i, τ_i). Moreover, for each $x \in (\sigma_i, \tau_i)$ the left and right derivatives, denoted by $\frac{\partial^-}{\partial x} U_i(x)$ respectively $\frac{\partial^+}{\partial x} U_i(x)$ exist. We further obtain the inequalities

$$\frac{\partial^-}{\partial x} U_i(x) \geq \frac{\partial^+}{\partial x} U_i(x) \geq \frac{\partial^-}{\partial y} U_i(y) \geq \frac{\partial^+}{\partial y} U_i(y)$$

for all $\sigma_i < x < y < \tau_i$; see Webster [133, Theorem 5.1.3] for a reference.

Our assumption that each cost function c_r is locally Lipschitz continuous is rather weak as, e.g., every continuously differentiable function has this property. Even without requiring that cost functions are differentiable, we will see in Theorem 5.9 that every set of consistent cost functions may only contain infinitely often differentiable cost functions. Hence, the following lemma that states necessary conditions for a pure Nash equilibrium in games with differentiable cost functions will be useful in the remainder of this chapter.

Lemma 5.6. *Let G be a g-scaled congestion game with variable demands and differentiable cost functions. If (α,d) is a pure Nash equilibrium of G, then for all $i\in N$ the following two conditions hold:*

(1) *If $d_i<\tau_i$, then* $\frac{\partial^+}{\partial d_i}U_i(d_i)\le\frac{\partial}{\partial d_i}\big(g(d_i)\sum_{r\in\alpha_i}c_r(\ell_r(\alpha,d))\big)$.

(2) *If $d_i>\sigma_i$, then* $\frac{\partial^-}{\partial d_i}U_i(d_i)\ge\frac{\partial}{\partial d_i}\big(g(d_i)\sum_{r\in\alpha_i}c_r(\ell_r(\alpha,d))\big)$.

5.3 Homogeneously Exponential Cost Functions

In this section, we show that for any scaling function g, homogeneously exponential functions are consistent for g-scaled congestion games with variable demands. To this end, we introduce a novel concept termed *essential improving moves*. A subset of improving moves is called *essential* if every player that has an improving move from a strategy profile s also has an essential improving move from s. Formally, let $G=(N,S,\varpi)$ be a maximization game and let $I=\{\big(s,(s_i',s_{-i})\big)\in S\times S:\varpi_i(s)<\varpi_i(s_i',s_{-i})\}$ denote the set of improving moves of G. A subset $I'\subseteq I$ of improving moves is called *essential* if $\{s':(s,s')\in I'\}=\emptyset$ implies $\{s':(s,s')\in I\}=\emptyset$ for all $s\in S$. Such subsets exist since the set of improving moves I itself is essential.

We proceed to show that for congestion games with variable demands in which all cost functions are homogeneously exponential there is an essential subset of improving moves which is a strict subset of the set of improving moves. In fact, we will show for every strategy profile $s=(\alpha,d)$ that whenever there is a player i who may improve when switching her strategy from $s_i=(\alpha_i,d_i)$ to $s_i'=(\alpha_i',d_i')$, then player i may also improve by only adapting her demand or only changing her allocation. That is, one of the strategies $s_i''=(\alpha_i,d_i')$ or $s_i'''=(\alpha_i',d_i)$ yields also an improvement for player i.

Lemma 5.7. *Let G be a g-scaled congestion game with variable demands such that all cost functions are of type $c(x)=a_c\,e^{\phi x}$ where a_c may depend on c while ϕ is equal for all $c\in\mathcal{C}$. Let I be the set of improving moves of G. Then, $I'=\{\big((\alpha,d),(\alpha',d')\big)\in I:\alpha=\alpha'\}\cup\{\big((\alpha,d),(\alpha',d')\big)\in I:d=d'\}$ is an essential subset of improving moves.*

Proof. For a contradiction, let us assume that $\big((\alpha,d),(\alpha_i',\alpha_{-i},d_i',d_{-i})\big)$ is an improving move of player i but $\big((\alpha,d),(\alpha,d_i',d_{-i})\big)$ and $\big((\alpha,d),(\alpha_i',\alpha_{-i},d)\big)$ are not. We use $\ell_r(\alpha_{-i},d_{-i})$ to denote the aggregated demands of all players $j\in N\setminus\{i\}$ when playing $s_j=(\alpha_j,d_j)$. We obtain

$$\begin{aligned}
&\varpi_i(\alpha,d_i',d_{-i})-\varpi_i(\alpha,d)\\
&\quad=U_i(d_i')-U_i(d_i)-g(d_i')\,e^{\phi d_i'}\sum_{r\in\alpha_i}a_r\,e^{\phi\ell_r(\alpha_{-i},d_{-i})}+g(d_i)\,e^{\phi d_i}\sum_{r\in\alpha_i}a_r\,e^{\phi\ell_r(\alpha_{-i},d_{-i})}\le 0, &&\text{(5.1a)}\\
&\varpi_i\big(\alpha_i',\alpha_{-i},d\big)-\varpi_i(\alpha,d)\\
&\quad=\qquad -g(d_i)\,e^{\phi d_i}\sum_{r\in\alpha_i'}a_r\,e^{\phi\ell_r(d_{-i},d_{-i})}+g(d_i)\,e^{\phi d_i}\sum_{r\in\alpha_i}a_r\,e^{\phi\ell_r(\alpha_{-i},d_{-i})}\le 0, &&\text{(5.1b)}\\
&\varpi_i(\alpha_i',\alpha_{-i},d_i',d_{-i})-\varpi_i(\alpha,d)\\
&\quad=U_i(d_i')-U_i(d_i)-g(d_i')\,e^{\phi d_i'}\sum_{r\in\alpha_i'}a_r\,e^{\phi\ell_r(\alpha_{-i},d_{-i})}+g(d_i)\,e^{\phi d_i}\sum_{r\in\alpha_i}a_r\,e^{\phi\ell_r(\alpha_{-i},d_{-i})}>0, &&\text{(5.1c)}
\end{aligned}$$

The last inequality expresses the fact that $\big((\alpha,d),(\alpha_i',\alpha_{-i},d_i',d_{-i})\big)$ is an improving move for G. Subtracting (5.1a) from (5.1c), we obtain

$$-g(d_i')\,e^{\phi d_i'}\left(\sum_{r\in\alpha_i'} a_r\,e^{\phi \ell_r(\alpha_{-i},d_{-i})}-\sum_{r\in\alpha_i} a_r\,e^{\phi \ell_r(\alpha_{-i},d_{-i})}\right)>0,$$

a contradiction to (5.1b). □

Next, we use the above Lemma to prove that every congestion game with variable demands and homogeneously exponential costs admits a pure Nash equilibrium.

Theorem 5.8. *Let g be a scaling function and $\mathcal{C}$ be a set of functions of type $c(x)=a_c\,e^{\phi x}$, where $a_c\in\mathbb{R}_{>0}$ may depend on c while $\phi\in\mathbb{R}_{>0}$ is equal for all $c\in\mathcal{C}$. Then, $\mathcal{C}$ is consistent for g-scaled congestion games with variable demands.*

Proof. Let $\phi\in\mathbb{R}_{>0}$ and let $\mathcal{M}=\big(N,R,(\mathcal{A}_i)_{i\in N},(c_r)_{r\in R}\big)$ be a congestion model with the property that for all $r\in R$, there is $a_r\in\mathbb{R}_{>0}$ such that $c_r(x)=a_r\,e^{\phi x}$ for all $x\in\mathbb{R}_{\geq 0}$. Let g be a scaling function, $(U_i)_{i\in N}$ a set of utility functions, and G a corresponding g-scaled congestion game with variable demands. Consider the function $\Phi: S\to\mathbb{R}$ defined as

$$\Phi(\alpha,d)=\sum_{i\in N}\int_0^{d_i}\frac{\partial^+ U_i(x)/\partial x}{g(x)+\frac{1}{\phi}g'(x)}\,\mathrm{d}x-\sum_{r\in R}c_r\big(\ell(\alpha,d)\big).$$

We first show that for each player i, there is $\omega_i<\infty$ such that $d_i\leq\omega_i$ for each pure Nash equilibrium $s=(\alpha,d)$ of G. For players with $\tau_i<\infty$, we set $\omega_i=\tau_i$ and we are done.

Let i be a player with $\tau_i=\infty$ and let $a=\min_{r\in R}a_r$. (The minimum exists as R is finite.) The marginal cost of each player i when playing a demand $x>0$ can be bounded from below by $g'(x)\,a\,e^{\phi x}+g(x)\,a\,\phi\,e^{\phi x}\geq g(x)\,a\,\phi\,e^{\phi x}$ because each allocation contains at least one resource. Using that g is non-decreasing, we derive that $g(x)\,a\,\phi\,e^{\phi x}$ diverges to ∞ as x goes to ∞. This implies that for each player i there is $\omega_i>\sigma_i$ such that $g(x)\,a\,\phi\,e^{\phi x}>\frac{\partial^+}{\partial d_i}U_i(0)$ for all $x>\omega_i$. Using Lemma 5.6 together with the fact that utility functions are concave, we obtain that $d_i\leq\omega_i$ for all $i\in N$ and each pure Nash equilibrium $s=(\alpha,d)$ of G.

Let $\bar{S}=\{(\alpha,d)\in S: d_i\in[\sigma_i,\omega_i] \text{ for all } i\in N\}$. As $\bar{S}$ is compact and Φ is continuous, Φ attains its maximum and we may choose $(\alpha^*,d^*)\in\arg\max_{(\alpha,d)\in\bar{S}}:\Phi(\alpha,d)$. We proceed to show that (α^*,d^*) is a pure Nash equilibrium. In light of Lemma 5.7 it suffices to show that there is no improving move from (α^*,d^*) in which exclusively either the demand or the allocation of a single player is adapted.

We first show that there is no improving move from (α^*,d^*) in which a single player only changes her demand. This is trivial for players with $\sigma_i=\tau_i$. For all other players, the optimality conditions of (α^*,d^*) give rise to $\partial\Phi(\alpha^*,d^*)/\partial d_i^*\geq 0$ for all $i\in N$ with $d_i^*>\sigma_i$ and $\partial\Phi(\alpha^*,d^*)/\partial d_i^*\leq 0$ for all $i\in N$ with $d_i^*<\tau_i$. For $i\in N$, we thus obtain the equations

$$\frac{\partial^+}{\partial d_i^*}U_i(d_i^*)\geq\left(g(d_i^*)+\frac{1}{\phi}g'(d_i^*)\right)\sum_{r\in\alpha_i^*}\phi\,a_r\,e^{\phi\ell_r(\alpha^*,d^*)}=\frac{\partial}{\partial d_i^*}\left(g(d_i^*)\sum_{r\in\alpha_i^*}a_r\,e^{\phi\ell_r(\alpha^*,d^*)}\right),$$

if $d_i^* > \sigma_i$ and

$$\frac{\partial^+}{\partial d_i^*}U_i(d_i^*) \leq \left(g(d_i^*) + \frac{1}{\phi}g'(d_i^*)\right)\sum_{r\in\alpha_i^*}\phi\, a_r\, e^{\phi\ell_r(\alpha^*,d^*)} = \frac{\partial}{\partial d_i^*}\left(g(d_i^*)\sum_{r\in\alpha_i^*}a_r\, e^{\phi\ell_r(\alpha^*,d^*)}\right),$$

if $d_i^* < \tau_i$. As the utility functions are concave we further obtain $\partial^- U_i(d_i^*)/\partial d_i^* \geq \partial^+ U_i(d_i^*)/\partial d_i^*$. Using that the private payoff functions of each player are concave in her demand, this implies that the demand d_i^* is optimal for player i when the allocation profile α^* is played. Thus, there is no improving move in which player i solely changes her demand.

Next, we prove that there is no improving move from (α^*,d^*) in which a player only changes her allocation. For a contradiction, suppose there is a player i that deviates profitably from strategy (α_i^*,d_i^*) to strategy $(\alpha_i',d_i^*) \in S_i$. If $g(d_i^*) = 0$, then player i does not improve switching from (α_i^*,d_i^*) to (α_i',d_i^*). Thus, we may assume that $g(d_i^*) > 0$. We obtain

$$\begin{aligned}
&\Phi(\alpha_i',\alpha_{-i}^*,d^*) - \Phi(\alpha^*,d^*)\\
&\quad = \Big(\frac{1}{e^{\phi d_i^*}} - 1\Big)\Big(\sum_{r\in\alpha_i} a_r\, e^{\phi\ell_r(\alpha_i',\alpha_{-i}^*,d^*)}\Big) + \Big(1 - \frac{1}{e^{\phi d_i^*}}\Big)\Big(\sum_{r\in\alpha_i'} a_r\, e^{\phi\ell_r(\alpha^*,d^*)}\Big)\\
&\quad = \frac{1}{g(d_i^*)}\Big(1 - \frac{1}{e^{\phi d_i^*}}\Big)\Big(-g(d_i^*)\sum_{r\in\alpha_i} a_r\, e^{\phi\ell_r(\alpha_i',\alpha_{-i}^*,d^*)} + g(d_i^*)\sum_{r\in\alpha_i^*} a_r\, e^{\phi\ell_r(\alpha^*,d^*)}\Big)\\
&\quad = \frac{1}{g(d_i^*)}\Big(1 - \frac{1}{e^{\phi d_i^*}}\Big)\Big(\varpi_i(\alpha_i',\alpha_{-i}^*,d^*) - \varpi_i(\alpha^*,d^*)\Big) > 0.
\end{aligned}$$

This is a contradiction to the fact that (α^*,d^*) maximizes Φ. We derive that (α^*,d^*) is a pure Nash equilibrium. □

5.4 Necessary Conditions for the Existence of a Pure Nash Equilibrium

In this section, we derive necessary conditions for the existence of pure Nash equilibria in congestion games with variable demands. Our conditions are based on connections to weighted congestion games and congestion games with resource-dependent demands.

5.4.1 A Connection to Weighted Congestion Games

We start with the useful observation that every set of cost functions that is not consistent for weighted congestion games is not consistent for g-scaled congestion games with variable demands as well for any scaling function g.

Theorem 5.9. *Let $\mathcal{M} = \big(N,R,(\mathcal{A}_i)_{i\in N},(c_r)_{r\in R}\big)$ be a congestion model and G^w a corresponding weighted congestion game. If G^w does not possess a pure Nash equilibrium, then, for every scaling function g, there exists a g-scaled congestion game with variable demands to the same congestion model that does not possess a pure Nash equilibrium*

Proof. Let $\mathcal{M} = \big(N, R, (\mathcal{A}_i)_{i\in N}, (c_r)_{r\in R}\big)$ be a congestion model, d_i^{w} a vector of (fixed) demands and $G^{\text{w}} = (N, S^{\text{w}}, \pi^{\text{w}})$ a corresponding weighted congestion game without a pure Nash equilibrium. For each player i, we set $\sigma_i = \tau_i = d_i^{\text{w}}$ and $U_i(d_i^{\text{w}}) = 0$. Let g be an arbitrary scaling function and $G = (N, S, \varpi)$ a corresponding g-scaled congestion game with variable demands. For each player i, we define the bijection $\Sigma_i : S^{\text{w}} \to S$ as $\Sigma_i(\alpha_i) = (\alpha_i, d_i^{\text{w}})$. We observe that $\pi_i^{\text{w}}(s) = -\frac{d_i^{\text{w}}}{g(d_i^{\text{w}})}\varpi_i\big(\Sigma_1(s_1),\ldots,\Sigma_n(s_n)\big)$ and conclude that G is a monotonic transformation of G^{w}. With the same argumentation as in Proposition 2.6 we obtain that G^{w} and G have the same set of pure Nash equilibria. $\square$

Building on our characterization of consistency for weighted congestion games obtained in Chapter 3, we derive the following result.

Theorem 5.10. *For an arbitrary scaling function g, let $\mathcal{C}$ be consistent for g-scaled congestion games with variable demands. Then, one of the following two cases holds:*

(1) *$\mathcal{C}$ only contains affine functions of type $c(x) = a_c x + b_c$ with $a_c \in \mathbb{R}_{>0}$, $b \in \mathbb{R}_{\geq 0}$.*

(2) *$\mathcal{C}$ only contains functions of type $c(x) = a_c e^{\phi x} + b_c$ where $a_c \in \mathbb{R}_{>0}$ and $b_c \geq -a_c$ may depend on c while $\phi \in \mathbb{R}_{>0}$ is independent of c.*

Proof. Fix a scaling function g arbitrarily. Theorem 5.9 establishes that every set $\mathcal{C}$ of cost functions that is consistent for g-scaled congestion games with variable demands is also consistent for weighted congestion games. Theorem 3.11 then implies that $\mathcal{C}$ only contains affine functions *or* $\mathcal{C}$ contains only functions of type $c(x) = a_c e^{\phi x} + b_c$ where $a_c, b_c \in \mathbb{R}$ may depend on c while $\phi \in \mathbb{R}$ is independent of c. In the following, we treat both cases individually.

First case: $\mathcal{C}$ only contains affine functions. As we consider only non-negative functions (cf. Assumption 5.3), we may assume that $\mathcal{C}$ only contains affine functions of type $c(x) = a_c x + b_c$ with $a_c \in \mathbb{R}_{\geq 0}$ and $b \in \mathbb{R}_{\geq 0}$. We proceed to show that $a_c \neq 0$ for all $c \in \mathcal{C}$. For a contradiction, suppose $\mathcal{C}$ contains a function $\tilde{c}$ with $a_{\tilde{c}} = 0$, that is, $\tilde{c}(x) = b_{\tilde{c}}$ for all $x \in \mathbb{R}_{\geq 0}$ and some $b_{\tilde{c}} \in \mathbb{R}_{\geq 0}$. Consider the congestion model $\mathcal{M} = \big(N, R, (\mathcal{A}_i)_{i\in N}, (c_r)_{r\in R}\big)$ with $N = \{1\}$, $R = \{r\}$, $\mathcal{A}_1 = \big\{\{r\}\big\}$, and $c_r = \tilde{c}$. The marginal cost of player 1 when playing a demand of x equals $g'(x)\tilde{c}(x) + g(x)\tilde{c}'(x) = g'(x)b_{\tilde{c}}$ in any corresponding g-scaled congestion game with variable demands. As g is affine (cf. Assumption 5.2) the marginal cost of player 1 is bounded by a constant $\mu > 0$. We derive that the game in which $[\sigma_1, \tau_1] = \mathbb{R}_{\geq 0}$ and $U_1(x) = (\mu+1)x$ for all $x \geq 0$ does not admit a pure Nash equilibrium as player 1 always gains by increasing her demand. We conclude that $\mathcal{C}$ only contains affine functions of type $c(x) = a_c x + b_c$ with $a_c \in \mathbb{R}_{>0}$, $b \in \mathbb{R}_{\geq 0}$ and (1) follows.

Second case: $\mathcal{C}$ only contains functions of type $c(x) = a_c e^{\phi x} + b_c$ where $a_c, b_c \in \mathbb{R}$ may depend on c while $\phi \in \mathbb{R}$ is independent of c. If $\phi = 0$, every function contained in $\mathcal{C}$ is constant and we obtain the same contradiction as in the first case. Next, suppose $\phi < 0$. Let $\tilde{c} \in \mathcal{C}$ with $\tilde{c}(x) = a_{\tilde{c}} e^{\phi x} + b_{\tilde{c}}$ be arbitrary. We distinguish two sub-cases.

First sub-case: $a_{\tilde{c}} < 0$. The non-negativity of $\tilde{c}$ implies that $b_{\tilde{c}} \geq -a_{\tilde{c}}$. We regard the same one-player congestion model as in the first case. The marginal cost of player 1 when playing a demand of x equals $g'(x)\tilde{c}(x) + g(x)\tilde{c}'(x)$ in any corresponding g-scaled congestion game with variable demands. We calculate

$$g'(x)\tilde{c}(x) + g(x)\tilde{c}'(x) \leq g'(x)b_{\tilde{c}} + g(x)a_{\tilde{c}}\,\phi\, e^{\phi x}.$$

Using that g is affine, we derive that the marginal costs of player 1 are bounded by a constant $\mu > 0$ giving the same contradiction as in the first case.

Second sub-case: $a_{\tilde{c}} > 0$. Using that $\tilde{c}$ is non-negative, we derive that $b_{\tilde{c}} \geq 0$. In the same congestion model as before, we obtain for the marginal cost of player 1 that

$$g'(x)\tilde{c}(x) + g(x)\tilde{c}'(x) = g'(x)(a_{\tilde{c}}\, e^{\phi x} + b_{\tilde{c}}) + g(x) a_{\tilde{c}}\, \phi\, e^{\phi x} \leq g'(x)(a_{\tilde{c}}\, e^{\phi x} + b_{\tilde{c}}).$$

Because $a_{\tilde{c}} e^{\phi x}$ vanishes as x goes to ∞ and g is linear, we conclude that the marginal cost of player 1 is bounded which gives the same contradiction as before. We conclude that $\phi > 0$. As argued before, $\mathcal{C}$ may not contain any constant function. The non-negativity imposed on the cost functions implies that $\mathcal{C}$ only contains functions of type $c(x) = a_c\, e^{\phi x} + b_c$ where $a_c \in \mathbb{R}_{>0}$ and $b_c \geq -a_c$ may depend on c while $\phi \in \mathbb{R}_{>0}$ is independent of c. □

5.4.2 A Connection to Congestion Games with Resource-Dependent Demands

In the last section, we built on our results for weighted congestion games obtained in Chapter 3 to prove that every set of consistent cost functions either only consist of affine functions or only consists of exponential functions of type $c(x) = a_c\, e^{\phi} + b_c$ where $a_c \in \mathbb{R}_{>0}$ and $b_c \geq -a_c$ may depend on c while $\phi \in \mathbb{R}_{>0}$ is equal for all $c \in \mathcal{C}$; see Theorem 5.10. Our positive result obtained in Section 5.3, however, holds only for *homogeneously* exponential functions of type $c(x) = a_c\, e^{\phi}$ where $a_c \in \mathbb{R}_{>0}$ may depend on c while $\phi \in \mathbb{R}_{>0}$ is equal for all $c \in \mathcal{C}$. This leaves it open whether inhomogeneously exponential cost functions are constant for (proportional and/or uniform) congestion games with variable demands. In this section, we will give a negative answer to this question.

In fact, we show a more general result. Recall that our characterizations of consistency for congestion games with resource-dependent demands given in Chapter 4 are obtained studying the implications of the generalized monotonicity condition (GMC), which we state again for completeness. For a scaling function g, a cost function c is said to satisfy the generalized monotonicity condition for g, if for all $x, y \in \mathbb{R}_{>0}$ with $c(x) \neq 0$, $c(y) \neq 0$, and all $\mu \in \mathbb{R}_{\geq 0}$ the following two conditions hold:

(GMC 1) If $c(x) > c(x+y) - \mu c'(x+y)$,
then $\left(1 - \mu \frac{g'(y)}{g(y)}\right) c(x+y) - \mu c'(x+y) \leq \left(1 - \mu \frac{g'(y)}{g(y)}\right) c(y) - \mu c'(y)$.

(GMC 2) If $c(x) < c(x+y) - \mu c'(x+y)$,
then $\left(1 - \mu \frac{g'(y)}{g(y)}\right) c(x+y) - \mu c'(x+y) \geq \left(1 - \mu \frac{g'(y)}{g(y)}\right) c(y) - \mu c'(y)$.

The Generalized Monotonicity Lemma (Lemma 4.5) proven in the previous chapter establishes that every cost function c that is consistent for g-scaled congestion games with resource-dependent demands satisfies the GMC for g.

Roughly speaking, as the main result of this section we will show that if a cost function c is consistent for g-scaled congestion games with variable demands *and* has the property that c'/c is injective on $\mathbb{R}_{>0}$, then c satisfies the GMC for g. The additional condition that c'/c must by injective precisely explains why homogeneously exponential cost functions are consistent for uniform congestion games with *variable* demands although they are not consistent for uniform congestion games with *resource-dependent* demands.

Before we give the formal proof of the result we first give some intuition. The proof of the Generalized Monotonicity Lemma (Lemma 4.5) relies on the construction of prototypical game with a quite simple structure. As noted in Corollary 4.6, the GMC is still necessary for games with two players that have two feasible allocations each, that is, $\mathcal{A}_1^{\mathrm{rd}} = \{\alpha_{1,1}^{\mathrm{rd}}, \alpha_{2,1}^{\mathrm{rd}}\}$, $\mathcal{A}_2^{\mathrm{rd}} = \{\alpha_{2,1}^{\mathrm{rd}}, \alpha_{2,2}^{\mathrm{rd}}\}$ with $\alpha_{i,k}^{\mathrm{rd}} \subseteq R$, $i,k \in \{1,2\}$. In addition, there are $x,y,z \in \mathbb{R}_{>0}$ with $y < z$ such that the demand of player 1 equals $d_{1,r} = x$ for all $r \in R$ and the demand of player 2 equals $d_{2,r} = y$ if $r \in \alpha_{2,1}^{\mathrm{rd}}$, and $d_{2,r} = z$ otherwise.

We strive to design the utility function $U_i : [\sigma_i, \tau_i] \to \mathbb{R}_{\geq 0}$ of each player i such that in equilibrium she uses the (fixed) demand of the game with resource-dependent demands G^{rd}. That is, player 1 always plays $d_1 = x$ and player 2 plays $d_2 = y$ when allocated at $\alpha_{2,1}^{\mathrm{rd}}$ and $d_2 = z$ when allocated at $\alpha_{2,2}^{\mathrm{rd}}$. If such construction is possible, then the necessary conditions for consistency in the case of resource-dependent demands translate to the case of variable demands. For player 1, we may simply set $\sigma_1 = \tau_1 = x$ and $U_1(\sigma_1) = 0$, that is, we allow player 1 only to use the demand x. For player 2 the situation is more subtle since we want her to use two distinct demand values depending on which resources she is allocated at.

We show that player 2 can be forced to use the right equilibrium demands, if c'/c is injective on $\mathbb{R}_{>0}$. The main idea of the construction is to add additional resources to each of the allocations of player 2. The key point is to also introduce an additional third player to the game who has only a single feasible demand and whose only feasible allocation contains all of the additional resources added to *one* of the allocations of player 2. Like this, the demand of player 3 artificially increases the aggregated demand on some of the additional resources by a certain *offset*. The condition that c'/c is injective ensures that adding an offset to the functional argument has a different impact on the derivative of the function than scalar multiplication. This can be seen, when considering the extreme case of a homogeneously exponential function of type $\tilde{c}(x) = a_{\tilde{c}}\, e^{\phi x}$ for which $\tilde{c}'/c$ is constant. For such a function, adding an offset q to the argument has the same effect as multiplying the function by the constant $e^{\phi q}$. Roughly speaking, the condition that c'/c is injective ensures that this situation cannot occur. By carefully choosing the number of supplementary resources added to each of the allocations of player 2 and the feasible demand of player 3, we can show that the marginal costs of player 2 can be manipulated as desired.

To illustrate the manipulation of the marginal costs by additional resources, we give a concrete example. Consider the inhomogeneously exponential function $c(x) = e^x + 1$. We calculate that $c'(x)/c(x) = \frac{1}{1+e^{-x}}$ is injective on $\mathbb{R}_{>0}$. When adding two resources with resource-dependent demand 1 and cost function c to the second allocation of player 1 her costs are increased by $2e+2$ and her marginal costs are increased by $4e+2$. Adding also one resource with the same resource-dependent demand and cost function and a trivial player who always plays a demand equal to $\ln(2+e^{-1})$ to the first allocation of player 1 her costs are increased by $2e+2$ as well, but her marginal costs are increased by $4e+3$. Using standard continuity arguments, such manipulation is still feasible when the demand for the resources contained in her second allocation is slightly larger than the demand in the first allocation. In that fashion, we can increase the marginal costs of one allocation more than in some other allocation while leaving their cost differences constant.

To prove that such manipulation can always been done, we first need the following technical lemma.

Lemma 5.11. *Let $c : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be a strictly increasing function that is strictly positive on $\mathbb{R}_{>0}$ and let g be a scaling function. If c'/c is injective on $\mathbb{R}_{>0}$, then for all $y \in \mathbb{R}_{>0}$ and $\delta \in \mathbb{R}_{>0}$ there are $\theta, \mu \in \mathbb{R}_{>0}$ and $z \in (y, y+\delta)$ such that one of the following two cases holds:*

(1) $\frac{\partial}{\partial y}\big(g(y)\,c(y+\theta)\big) > \dfrac{\mu\, g(z)\,c(z) - g(y)\,c(y+\theta)}{z-y} > \frac{\partial}{\partial z}\big(\mu\, g(z)\,c(z)\big);$

(2) $\frac{\partial}{\partial y}\big(\mu\, g(y)\,c(y)\big) > \dfrac{g(z)\,c(z+\theta) - \mu\, g(y)\,c(y)}{z-y} > \frac{\partial}{\partial z}\big(g(z)\,c(z+\theta)\big).$

Proof. Let c with the demanded properties and $y > 0$ be given. As c'/c is injective we can find $\theta \in \mathbb{R}_{>0}$ such that $c'(y+\theta)/c(y+\theta) \neq c'(y)/c(y)$. We distinguish two cases.

First case: $c'(y+\theta)/c(y+\theta) > c'(y)/c(y)$. Multiplying with $g(y)\,c(y+\theta)$ and setting $\mu = c(y+\theta)/c(y)$ we obtain

$$g(y)\,c'(y+\theta) > \mu\, g(y)c'(y)$$

Adding $g'(y)\,c(y+\theta)$ to both sides this gives

$$g'(y)\,c(y+\theta) + g(y)\,c'(y+\theta) > \mu\, g'(y)\,c(y) + \mu\, g(y)\,c'(y). \tag{5.2}$$

As the expression on the left hand side of (5.2) is continuous in θ, there is $\theta' < \theta$ such that the inequality

$$g'(y)\,c(y+\theta') + g(y)\,c'(y+\theta') > \mu\, g'(y)\,c(y) + \mu\, g(y)\,c'(y). \tag{5.3}$$

holds. Using the fact that c is strictly increasing, we observe that $0 = \mu\, g(y)\,c(y) - g(y)\,c(y+\theta) < \mu\, g(y)\,c(y) - g(y)\,c(y+\theta')$ and hence

$$g(y)c(y+\theta') < \mu\, g(y)c(y) \tag{5.4}$$

Because the right hand sides of (5.3) and (5.4) are continuous in y, there is a sequence $(z_n)_{n\in\mathbb{N}}$ with $z_n \in (y, y+\delta)$ for all $n \in \mathbb{N}$ that converges to y and satisfies the inequalities

$$\begin{gathered} g'(y)\,c(y+\theta') + g(y)\,c'(y+\theta') > \mu\, g'(z_n)\,c(z_n) + \mu\, g(z_n)\,c'(z_n), \\ \mu\, g(z_n)c(z_n) > g(y)c(y+\theta') \end{gathered}$$

for all $n \in \mathbb{N}$. This, however, implies the existence of $m \in \mathbb{N}$ such that

$$g'(y)\,c(y+\theta') + g(y)\,c'(y+\theta') - \mu\, g'(z_m)\,c(z_m) - \mu\, g(z_m)\,c'(z_m) > 0, \tag{5.5}$$

$$\frac{\mu\, g(z_m)\,c(z_m) - g(y)\,c(y+\theta')}{z_m - y} - \mu\, g'(z_m)\,c(z_m) - \mu\, g(z_m)\,c'(z_m) > 0 \tag{5.6}$$

are satisfied. Note that the left hand side of (5.5) increases as μ decreases. We set $\varepsilon = g'(y)\,c(y+\theta') + g(y)\,c'(y+\theta') - \mu\, g'(z_m)\,c(z_m) - \mu\, g(z_m)\,c'(z_m)$. The left hand side of (5.6) is continuous as a function of μ and negative for $\mu = g(y)\,c(y+\theta')/\big(g(z_m)\,c(z_m)\big)$. Thus, there is $\mu' \in \mathbb{R}_{>0}$ with $\frac{g(y)\,c(y+\theta')}{g(z_m)\,c(z_m)} \leq \mu' \leq \frac{c(y+\theta)}{c(y)}$ such that

$$0 < \frac{\mu'\,g(z_m)\,c(z_m) - g(y)\,c(y+\theta')}{z_m - y} - \mu'\,g'(z_m)\,c(z_m) - \mu'\,g(z_m)\,c'(z_m) < \varepsilon.$$

We have shown that

$$g'(y)\,c(y+\theta') + g(y)\,c'(y+\theta') > \frac{\mu'\,g(z_m)\,c(z_m) - g(y)\,c(y+\theta')}{z_m - y} > \mu' g'(z_m)\,c(z_m) + \mu'\,g(z_m)\,c'(z_m),$$

which finishes the proof of the first case.

Second case: $c'(y+\theta)/c(y+\theta) < c'(y)/c(y)$. As before, there are $\mu, y, \theta \in \mathbb{R}_{>0}$ such that $g'(y)\,c(y+\theta) + g(y)\,c'(y+\theta) < \mu\,g'(y)\,c(y) + \mu\,g(y)\,c'(y)$. For $z > y$ sufficiently close to y, we further obtain the inequalities

$$g'(z)\,c(z+\theta) + g(z)\,c'(z+\theta) < \mu\,g'(y)\,c(y) + \mu\,g(y)\,c'(y) < \frac{g(z)\,c(z+\theta) - \mu\,g(y)\,c(y)}{z-y}.$$

As we increase μ continuously, we find $\mu' \in (\frac{c(y+\theta)}{c(y)}, \frac{g(z)\,c(z+\theta)}{g(y)c(y)}]$ such that

$$g'(y)\,c(y+\theta) + g(y)\,c'(y+\theta) < \frac{\mu\,g(z)\,c(z) - g(y)\,c(y+\theta)}{z-y} < \mu\,g'(y)\,c(y) + \mu g(y)\,c'(y).$$

□

The next lemma establishes a necessary condition on the consistency of cost functions for g-sclaed congestion games with variable demands.

Lemma 5.12. *Let g be a scaling function and let c be a differentiable and convex cost function that is strictly positive on $\mathbb{R}_{>0}$ and has the property that c'/c is injective on $\mathbb{R}_{>0}$. If c is consistent for g-scaled congestion games with variable demands, then c satisfies the GMC for g.*

Proof. Suppose not, for a contradiction. Let c be a function with the demanded properties that is consistent for g-scaled congestion games with variable demands and let us assume that c does not satisfy the GMC for g. Applying Corollary 4.6 we derive the existence of $x, y \in \mathbb{R}_{>0}$ and $\varepsilon > 0$ such that for each $\delta \in (0, \varepsilon)$ there is a congestion model $\mathcal{M}^{\mathrm{rd}}_{\delta} = (N^{\mathrm{rd}}, R^{\mathrm{rd}}_{\delta}, (\mathcal{A}^{\mathrm{rd}}_{\delta,i})_{i \in N^{\mathrm{rd}}}, (c^{\mathrm{rd}}_r)_{r \in R^{\mathrm{rd}}_{\delta}})$ with two players that have access to two disjoint allocations each (i.e., $N^{\mathrm{rd}} = \{1,2\}$, and $\mathcal{A}^{\mathrm{rd}}_i = \{\alpha^{\mathrm{rd}}_{i,1}, \alpha^{\mathrm{rd}}_{i,2}\}$ for some $\alpha^{\mathrm{rd}}_{i,1}, \alpha^{\mathrm{rd}}_{i,2} \subseteq R^{\mathrm{rd}}$ with $\alpha^{\mathrm{rd}}_{i,1} \cap \alpha^{\mathrm{rd}}_{i,2} = \emptyset$, $i \in \{1,2\}$). Further there is a corresponding g-scaled congestion game with variable demands G^{rd}_{δ} that does not possess a pure Nash equilibrium and for which the players' resource-dependent demands equal $d_{1,r} = x$ for all $r \in R^{\mathrm{rd}}$, $d_{2,r} = y$ if $r \in \alpha^{\mathrm{rd}}_{2,1}$, and $d_{2,r} = y + \delta$ otherwise.

Let $x, y, \varepsilon \in \mathbb{R}_{>0}$ be fixed accordingly. Referring to Lemma 5.11, there are $z \in (y, y+\varepsilon)$ and $\mu, \theta \in \mathbb{R}_{>0}$ such that one of the following two cases holds:

$$\frac{\partial}{\partial x}\big(g(y)\,c(y+\theta)\big) > \frac{\mu\, g(z)\,c(z) - g(y)\,c(y+\theta)}{z-y} > \frac{\partial}{\partial z}\big(\mu\, g(z)\,c(z)\big); \tag{5.7}$$

$$\frac{\partial}{\partial y}\big(\mu\, g(y)\,c(y)\big) > \frac{g(z)\,c(z+\theta) - \mu\, g(y)\,c(y)}{z-y} > \frac{\partial}{\partial z}\big(g(z)\,c(z+\theta)\big). \tag{5.8}$$

We fix such z, μ, and θ and set $\delta = z - y$. In the following, we omit the subscript δ and denote by $\mathcal{M}^{\mathrm{rd}} = (N^{\mathrm{rd}}, R^{\mathrm{rd}}, (\mathcal{A}_i^{\mathrm{rd}})_{i\in N^{\mathrm{rd}}}, (c_r^{\mathrm{rd}})_{r\in R^{\mathrm{rd}}})$ the congestion model and by G^{rd} the corresponding g-scaled congestion game with demands x, y, and z not possessing a pure Nash equilibrium.

We proceed to show the proof for the case (5.7), the other case follows along the same lines.

As all expressions occurring in (5.7) are continuous in μ, it is without loss of generality to assume that μ is positive and rational. Thus, we may write $\mu = p/q$ for some $p, q \in \mathbb{N}$. Multiplying (5.7) with q, we obtain

$$\frac{\partial}{\partial y}\big(q\,g(y)\,c(y+\theta)\big) > \frac{p\,g(z)\,c(z) - q\,g(y)\,c(y+\theta)}{z-y} > \frac{\partial}{\partial z}\big(p\,g(z)\,c(z)\big). \tag{5.9}$$

For $k \in \mathbb{N}$ we define a new congestion model $\mathcal{M}^k = (N, R^k, (\mathcal{A}_i^k)_{i\in N}, (c_r)_{r\in R})$. The set of players $N = N^{\mathrm{rd}} \cup \{3\}$ contains an additional third player, the set of resources R^k contains additional $k(p+q)$ resources partitioned into two subsets R_1^k, R_2^k of cardinality $|R_1^k| = kq$ and $|R_2^k| = kp$, respectively. We obtain $R^k = R^{\mathrm{rd}} \cup R_1^k \cup R_2^k$. Each resource $r \in R^{\mathrm{rd}}$ is endowed with the same cost function as in $\mathcal{M}^{\mathrm{rd}}$. The new resources are endowed with cost function c. Player 1 has the same set of feasible allocations as in G^{rd}, that is, $\mathcal{A}_1^k = \{\alpha_{1,1}^{\mathrm{rd}}, \alpha_{1,2}^{\mathrm{rd}}\}$. For player 2, we add the kq new resources contained in R_1^k to the first, and the kp new resources contained in R_2^k to the second allocation, that is, $\mathcal{A}_2^k = \{\alpha_{2,1}^{\mathrm{rd}} \cup R_1^k, \alpha_{2,2}^{\mathrm{rd}} \cup R_2^k\}$. Player 3 has a single feasible allocation where she uses the kq new resources contained in R_1. As Player 3 has a single allocation only, her only strategic action is to choose the demand.

The players' sets of feasible demands are given by $\sigma_1 = \tau_1 = x$, $\sigma_2 = y$, $\tau_2 = z$, and $\sigma_3 = \tau_3 = \theta$. The utility functions of players 1 and 3 are arbitrary. We may simply define them as $U_1(\sigma_1) = U_3(\sigma_3) = 0$. The utility function $U_2^k : [y,z] \to \mathbb{R}_{\geq 0}$ of player 2 is the linear function with slope $\frac{k}{z-y} \cdot \big(p\,g(z)\,c(z) - q\,g(y)\,c(y+\theta)\big)$ through the point $(y, 0)$. The three utility functions are shown in Figure 5.1.

We claim that there is $m \in \mathbb{N}$ such that $d_2^m = y$ for each pure Nash equilibrium (α^m, d^m) of G^m with $\alpha_2^m = \alpha_{2,1}^{\mathrm{rd}} \cup R_1^k$ and $d_2^m = z$ for each pure Nash equilibrium (α^m, d^m) of G^m with $\alpha_2^m = \alpha_{2,2}^{\mathrm{rd}} \cup R_2^k$. For a contradiction, suppose for each $k \in \mathbb{N}$, there is a pure Nash equilibrium (α^k, d^k) such that one of the following cases holds: *(i)* $\alpha_2^k = \alpha_{2,1}^{\mathrm{rd}} \cup R_1^k$ and $d_2^k \in (y, z]$; *(ii)* $\alpha_2^k = \alpha_{2,2}^{\mathrm{rd}} \cup R_2^k$ and $d_2^k \in [y, z)$. Considering sub-sequences, it is without loss of generality to assume that either *(i)* holds for all $k \in \mathbb{N}$ or *(ii)* holds for all $k \in \mathbb{N}$.

Let us first assume, that *(i)* holds for all $k \in \mathbb{N}$. We calculate

$$\frac{\partial \varpi_2(\alpha^k, d^k)}{\partial d_2^k} = k\Big(\frac{pg(z)\,c(z) - qg(y)c(y+\theta)}{z-y} - q\frac{\partial}{\partial d_2^k} g(d_2^k)c(d_2^k+\theta)\Big) - \frac{\partial}{\partial d_2^k} \sum_{r\in\alpha_{2,1}^{\mathrm{rd}}} c_r\big(\ell_r(\alpha^k, d^k)\big)$$

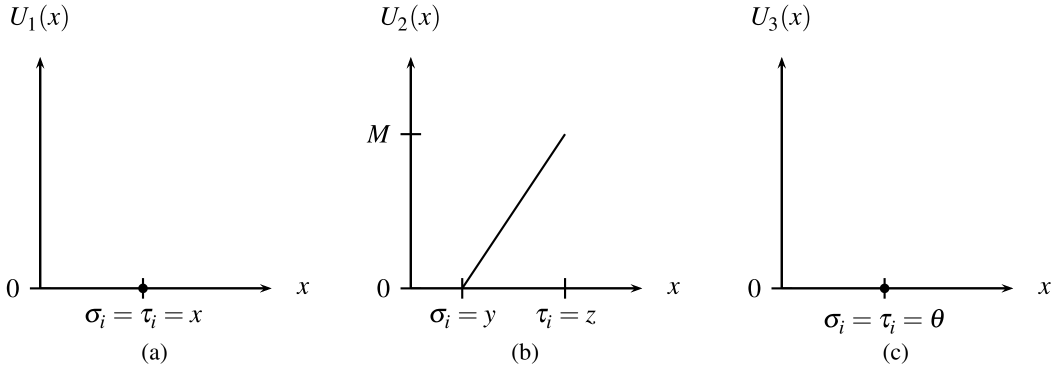


Figure 5.1: The players' utility functions U_i in the three-player g-scaled congestion game with variable demands constructed in the proof of Lemma 5.12. For the second player's utility the parameter M is chosen such that the slope of the function equals $\frac{k}{z-y} \cdot \big(p\,g(z)\,c(z) - q\,g(y)\,c(y+\theta)\big)$, that is, $M = kp\,g(z)\,c(z) - kq\,g(y)\,c(y+\theta)$.

Using that $d_2^k > y$ and that c and g are convex, we obtain

$$\frac{\partial \varpi_2(\alpha^k,d^k)}{\partial d_2^k} \leq k\left(\frac{pg(z)c(z)-qg(y)c(y+\theta)}{z-y} - q\frac{\partial}{\partial y}g(y)\,c(y+\theta) - \frac{1}{k}\cdot\frac{\partial}{\partial d_2^k}\sum_{r\in\alpha_{2,1}^{\mathrm{rd}}} c_r\big(\ell_r(\alpha^k,d^k)\big)\right)$$

Form the left inequality of (5.9) we derive $\lim_{k\to\infty} \partial\varpi_2(\alpha^k,d^k)/\partial d_2^k < 0$. This implies the existence of $m \in \mathbb{N}$ with $\partial\varpi_2(\alpha^k,d^m)/\partial d_2^m < 0$, which contradicts the assumption that (α^m,d^m) is a pure Nash equilibrium of G^m.

If, on the other hand, *(ii)* holds for all $k \in \mathbb{N}$, we calculate

$$\begin{aligned}\frac{\partial \varpi_2(\alpha^k,d^k)}{\partial d_2^k} &= k\Big(\frac{pg(z)c(z)-qg(y)c(y+\theta)}{z-y} - p\frac{\partial}{\partial d_2^k}g(d_2^k)\,c(d_2^k) - \frac{1}{k}\cdot\frac{\partial}{\partial d_2^k}\sum_{r\in\alpha_{2,2}^{\mathrm{rd}}} c_r\big(\ell_r(\alpha^k,d^k)\big)\Big)\\ &\geq k\left(\frac{pg(z)\,c(z)-qg(y)c(y+\theta)}{z-y} - p\frac{\partial}{\partial z}g(z)\,c(z) - \frac{1}{k}\cdot\frac{\partial}{\partial d_2^k}\sum_{r\in\alpha_{2,2}^{\mathrm{rd}}} c_r\big(\ell_r(\alpha^k,d^k)\big)\right),\end{aligned}$$

where we use the convexity of c and g and the fact that $d_2^k < z$. Using the right inequality of (5.9), we derive $\lim_{k\to\infty} \partial\varpi_2(\alpha^k,d^k)/\partial\, d_2^k > 0$. In light of Lemma 5.6, this is a contradiction to the assumption that (α^k,d^k) is a pure Nash equilibrium for all $k \in \mathbb{N}$. We conclude that $d_2^m = y$ for each pure Nash equilibrium (α^m,d^m) of G^m with $\alpha_2^m = \alpha_{2,1}^{\mathrm{rd}} \cup R_1^m$ and $d_2^m = z$ for each pure Nash equilibrium (α^m,d^m) of G^m with $\alpha_2^m = \alpha_{2,2}^{\mathrm{rd}} \cup R_2^m$.

To finish the proof, we show that G^m does not possess a pure Nash equilibrium. For a contradiction, let (α^m,d^m) be a pure Nash equilibrium of G^m. We here show the contradiction only for the case that each player plays her first allocation, that is, $\alpha_1^m = \alpha_{1,1}^{\mathrm{rd}}$ and $\alpha_2^m = \alpha_{2,1}^{\mathrm{rd}} \cup R_1^m$. The other three cases can be treated with the same arguments. Consider the strategy profile $(\alpha_{1,1}^{\mathrm{rd}}, \alpha_{2,1}^{\mathrm{rd}})$ of G^{rd}. Because G^{rd} does not admit a pure Nash equilibrium, one of the players 1 or 2 improves switching to her second allocation. We distinguish two cases.

First case: $\pi_1^{\text{rd}}(\alpha_{1,2}^{\text{rd}}, \alpha_{2,1}^{\text{rd}}) < \pi_1^{\text{rd}}(\alpha_{1,1}^{\text{rd}}, \alpha_{2,1}^{\text{rd}})$. Consider the strategy profile $(\alpha_{1,2}^{\text{rd}}, x) \in S_1^m$. We calculate

$$\begin{aligned}
&\varpi_1^m\big((\alpha_{1,2}^{\text{rd}},x),(\alpha_{2,1}^{\text{rd}} \cup R_1^m, y),(R_1^m,\theta)\big) - \varpi_1^m\big((\alpha_{1,1}^{\text{rd}},x),(\alpha_{2,1}^{\text{rd}} \cup R_1^m, y),(R_1^m,\theta)\big)\\
&\quad = -g(x) \sum_{r\in\alpha_{1,2}^{\text{rd}}} c_r\Big(\ell_r\big((\alpha_{1,2}^{\text{rd}},x),(\alpha_{2,1}^{\text{rd}} \cup R_1^m, y),(R_1^m,\theta)\big)\Big)\\
&\qquad + g(x) \sum_{r\in\alpha_{1,1}^{\text{rd}}} c_r\Big(\ell_r\big((\alpha_{1,1}^{\text{rd}},x),(\alpha_{2,1}^{\text{rd}} \cup R_1^m, y),(R_1^m,\theta)\big)\Big)\\
&\quad = \pi_1^{\text{rd}}(\alpha_{1,1}^{\text{rd}}, \alpha_{2,1}^{\text{rd}}) - \pi_1^{\text{rd}}(\alpha_{1,2}^{\text{rd}}, \alpha_{2,1}^{\text{rd}}) > 0.
\end{aligned}$$

Second case: $\pi_2^{\text{rd}}(\alpha_{1,1}^{\text{rd}}, \alpha_{2,2}^{\text{rd}}) < \pi_2^{\text{rd}}(\alpha_{1,1}^{\text{rd}}, \alpha_{2,1}^{\text{rd}})$. Consider the strategy profile $(\alpha_{2,2}^{\text{rd}} \cup R_2^m, z)$ where player 2 chooses allocation $\alpha_{2,2}^{\text{rd}}$ and her demands equals z. We obtain

$$\begin{aligned}
&\varpi_2^m\big((\alpha_{1,1}^{\text{rd}},x),(\alpha_{2,2}^{\text{rd}} \cup R_2^m, z),(R_1^m,\theta)\big) - \varpi_2^m\big((\alpha_{1,1}^{\text{rd}},x),(\alpha_{2,1}^{\text{rd}} \cup R_1^m, z),(R_1^m,\theta)\big)\\
&\quad = U_2(z) - g(z) \sum_{r\in\alpha_{2,2}^{\text{rd}}} c_r\Big(\ell_r\big((\alpha_{1,1}^{\text{rd}},x),(\alpha_{2,2}^{\text{rd}} \cup R_2^m, z),(R_1^m,\theta)\big)\Big) - kp\,g(z)\,c(z)\\
&\qquad - U_2(y) + g(y) \sum_{r\in\alpha_{2,1}^{\text{rd}}} c_r\Big(\ell_r\big((\alpha_{1,1}^{\text{rd}},x),(\alpha_{2,1}^{\text{rd}} \cup R_1^m, z),(R_1^m,\theta)\big)\Big) + kq\,g(y)c(y+\theta)\\
&\quad = -g(z) \sum_{r\in\alpha_{2,2}^{\text{rd}}} c_r\Big(\ell_r\big((\alpha_{1,1}^{\text{rd}},x),(\alpha_{2,2}^{\text{rd}} \cup R_2^m, z),(R_1^m,\theta)\big)\Big)\\
&\qquad + g(y) \sum_{r\in\alpha_{2,1}^{\text{rd}}} c_r\Big(\ell_r\big((\alpha_{1,1}^{\text{rd}},x),(\alpha_{2,1}^{\text{rd}} \cup R_1^m, z),(R_1^m,\theta)\big)\Big)\\
&\quad = \pi_2^{\text{rd}}(\alpha_{1,1}^{\text{rd}}, \alpha_{1,1}^{\text{rd}}) - \pi_1^{\text{rd}}(\alpha_{1,1}^{\text{rd}}, \alpha_{2,2}^{\text{rd}}) > 0.
\end{aligned}$$

This is a contradiction to the assumption that (α^m, d^m) is a pure Nash equilibrium of G^m. □

5.5 A Characterization of Consistency

This section contains the two main results of this chapter – a complete characterization of the consistency and approximate universal consistency for games with variable demands, respectively. We start with a characterization of the approximate universal consistency.

Theorem 5.13. *Let $\mathcal{C}$ be a set of continuous functions and let g be a scaling function. Then, $\mathcal{C}$ is approximately universally consistent for g-scaled congestion games with variable demands if and only if the following two conditions are satisfied:*

(1) *g is linear of type $g(x) = \beta x$, $\beta \in \mathbb{R}_{>0}$.*

(2) *$\mathcal{C}$ only contains affine functions of type $c(x) = a_c x + b_c$ where $a_c \in \mathbb{R}_{>0}$ and $b_c \in \mathbb{R}_{\geq 0}$.*

If (1) and (2) are satisfied, then $\mathcal{C}$ is even consistent for g-scaled congestion games with variable demands.

Proof. We first show that conditions (1) and (2) imply consistency and approximate universal consistency of $\mathcal{C}$. Let $\mathcal{M} = \big(N, R, (\mathcal{A}_i)_{i\in N}, (c_r)_{r\in R}\big)$ be a congestion model such that for each resource $r \in R$, there are $a_r \in \mathbb{R}_{>0}$ and $b_r \in \mathbb{R}_{\geq 0}$ with $c_r(x) = a_r x + b_r$ for all $x \in \mathbb{R}_{>0}$. For a set $(U_i)_{i\in N}$ of utility functions let G be a corresponding g-scaled congestion game with variable.

We claim that for each player i there is $\omega_i \in \mathbb{R}_{>0}$ such that each demand $d_i > \omega_i$ is strictly dominated by the demand $d_i' = \omega_i$, that is, $\pi_i(\alpha_i, d_i, \alpha_{-i}, d_{-i}) < \pi_i(\alpha_i, d_i', \alpha_{-i}, d_{-i})$ for all $\alpha_i \in \mathcal{A}_i$ and $(\alpha_{-i}, d_{-i}) \in S_{-i}$. For players with $\tau_i < \infty$, we simply set $\omega_i = \tau_i$. Let i be a player with $\tau_i = \infty$. We set $a = \min_{r\in R} a_r$. (The minimum exists as R is finite.) For each $d_i \in [\sigma_i, \infty)$, $\alpha_i \in \mathcal{A}_i$ and $(\alpha_{-i}, d_{-i}) \in S_{-i}$ the marginal cost of player i can be bounded from below by $a\,g'(d_i)d_i + a\,g(d_i)$ which diverges monotonically to ∞ as d_i goes to ∞. We choose ω_i such that $a\,g'(d_i)d_i + a\,g(d_i) > \partial^+ U_i(0)/\partial d_i$ for all $d_i \geq \omega_i$ and conclude that the marginal payoff of player i for demands $d_i \in [\omega_i, \infty)$ is negative. This implies that demands larger than ω_i are strictly dominated for player i.

We proceed to show that the function $P : S \to \mathbb{R}$ defined as

$$P(\alpha, d) = \sum_{i\in N}\Big(U_i(d_i) - \beta\, d_i \sum_{r\in\alpha_i} a_r \big(\sum_{j\in\{1,\dots,i\}: r\in s_j} d_j\big) + b_r\Big)$$

is an exact potential function for G. To prove this claim, let the strategy $(\alpha, d) \in S$ and the player i with alternative strategy (α_i', d_i') be arbitrary. We calculate

$$\begin{aligned}
&P(\alpha, d) - P(\alpha_i', d_i', \alpha_{-i}, d_{-i}) \\
&\quad = U_i(d_i) - U_i(d_i') - \sum_{j=i+1}^{n}\Big(\sum_{r\in\alpha_i\cap\alpha_j} \beta\, d_j\, a_r\, d_i - \sum_{r\in\alpha_i'\cap\alpha_j} \beta\, d_j\, a_r\, d_i\Big) \\
&\quad\quad - \sum_{r\in\alpha_i} \beta\, d_i \Big(a_r \big(\sum_{j\in\{1,\dots,i\}: r\in\alpha_j} d_j\big) + b_r\Big) + \sum_{r\in\alpha_i'} \beta\, d_i \Big(a_r \big(\sum_{j\in\{1,\dots,i\}: r\in\alpha_j} d_j\big) + b_r\Big) \\
&\quad = \varpi_i(\alpha, d) - \varpi_i(\alpha_i', d_i', \alpha_{-i}, d_{-i}).
\end{aligned}$$

Let $\bar{S} = \{(\alpha, d) \in S : d_i \in [\sigma_i, \omega_i] \text{ for all } i \in N\}$. As $\bar{S}$ is compact and P is continuous, we may choose $(\alpha^*, d^*) \in \arg\max_{(\alpha,d)\in\bar{S}} P(\alpha, d)$. Using that strategies with demands larger than ω_i are strictly dominated we derive that (α^*, d^*) also maximizes P over S. This implies that (α^*, d^*) is a pure Nash equilibrium. To see that $\mathcal{C}$ is also approximately universally consistent, let $\rho > 0$ and $(\alpha^0, d^0) \in S$ be arbitrary. As $P(\alpha^0, d^0)$ is finite and every ρ-improving move increases the value of P by at least ρ, we conclude that every ρ-improvement path is finite.

We proceed to prove that if $\mathcal{C}$ is approximately universally consistent for g-scaled congestion games with variable demands, then (1) and (2) hold. If $\mathcal{C}$ contains a constant function we can construct a one-player game where player 1 can always improve her payoff by an arbitrary constant raising her demand.

If $\mathcal{C}$ contains a non-affine function $\tilde{c}$ *or* g is not linear, Theorem 4.12 establishes the existence of a g-scaled congestion game with resource-dependent demands G^{rd} that does not possess a pure Nash equilibrium and has the additional property that $c_r = \tilde{c}$ for all $r \in R$. In light of Corollary 4.6 the game G^{rd} can be chosen such that for each player all resources contained in one allocation are accessed with the same resource-dependent demand, that is for all $i \in N$ and $\alpha_i \in \mathcal{A}_i$, there is $d_i(\alpha_i)$ such that

$d_{i,r} = d_i(\alpha_i)$ for all $r \in \alpha_i$. As G^{rd} has no pure Nash equilibrium G^{rd} there is an improvement cycle $\gamma^{\mathrm{rd}} = (\alpha^0, \dots, \alpha^k, \alpha^0)$. As γ^{rd} is finite there is $\varepsilon > 0$ such that γ^{rd} is an ε-improvement cycle.

We consider the g-scaled congestion game with variable demands G with the same set of players, resources, and feasible allocations as G^{rd} where for all players i we have $[\sigma_i, \tau_i] = \mathbb{R}_{\geq 0}$ and $U_i(x) = 0$ for all $x \geq 0$. By construction the cycle

$$\gamma = \Big(\big(\alpha_1^0, d_1(\alpha_1^0), \dots, \alpha_n^0, d_1(\alpha_n^0)\big), \dots,$$
$$\big(\alpha_1^k, d_1(\alpha_1^k), \dots, \alpha_n^k, d_1(\alpha_n^k)\big), \big(\alpha_1^0, d_1(\alpha_1^0), \dots, \alpha_n^0, d_1(\alpha_n^0)\big) \Big)$$

in which each player chooses the demand specified in G^{rd} is an ε-improvement cycle of G. We derive that $\mathcal{C}$ is not approximately universally consistent. $\square$

The following theorem provides a complete characterization of the consistency of cost functions in g-scaled congestion games with variable demands.

Theorem 5.14. *Let $\mathcal{C}$ be a set of continuous functions and g a scaling function. Then, the following are equivalent:*

(1) *$\mathcal{C}$ is consistent for g-scaled congestion games with variable demands;*

(2) *At least one of the following two holds:*

  (2a) *$\mathcal{C}$ only contains homogeneously exponential functions of type $c(x) = a_c e^{\phi x}$, where $a_c \in \mathbb{R}_{>0}$ may depend on c, while $\phi \in \mathbb{R}_{>0}$ must be equal for all $c \in \mathcal{C}$.*

  (2b) *g is linear and $\mathcal{C}$ only contains affine functions of type $c(x) = a_c x + b_c$, where $a_c \in \mathbb{R}_{>0}$ and $b_c \in \mathbb{R}_{\geq 0}$.*

Proof. (1) $\Rightarrow$ (2): For a contradiction, suppose there are a scaling function g and a set of cost functions $\mathcal{C}$ that is consistent for g-scaled congestion games with variable demands, but neither (2a) nor (2b) are satisfied.

Referring to Theorem 5.10, the consistency of $\mathcal{C}$ implies that one of the following two cases holds: *(i)* $\mathcal{C}$ only contains affine functions of type $c(x) = a_c x + b_c$ with $a_c \in \mathbb{R}_{>0}$ and $b_c \in \mathbb{R}_{\geq 0}$; or *(ii)* $\mathcal{C}$ only contains functions of type $c(x) = a_c e^{\phi x} + b_c$, where $a_c \in \mathbb{R}_{>0}$ and $b_c \geq -a_c$ may depend on c while $\phi \in \mathbb{R}_{>0}$ is independent of c.

Let us first assume that $\mathcal{C}$ only contains affine functions. Because (2a) is not satisfied, at least one affine function in $c \in \mathcal{C}$ is not constant. Furthermore, g is not linear since (2b) is violated. Theorem 4.8 then implies that c does not satisfy the GMC for g. Furthermore, as c is linear but not constant, we derive that c'/c is injective on $\mathbb{R}_{>0}$. Applying Lemma 5.12 we derive the existence of a g-scaled congestion game with variable demands not possessing a pure Nash equilibrium. This is contradiction to the consistency of $\mathcal{C}$.

For the second case, let us assume that $\mathcal{C}$ only contains functions of type $c(x) = a_c e^{\phi x} + b_c$, where $a_c \in \mathbb{R}_{>0}$ and $b_c \geq -a_c$ may depend on c while $\phi \in \mathbb{R}_{>0}$ is independent of c. Because (2a) is violated, $\mathcal{C}$ contains at least one inhomogeneously exponential function c, that is, a function with $b_c \neq 0$. Theorem 4.11 implies that c does not satisfy the GMC for g. Using that c is inhomogeneously exponential, we further see that $c'(x)/c(x) = \phi / \big(1 + \frac{b_c}{a_c} e^{-\phi x}\big)$ for all $x \in \mathbb{R}_{>0}$. Thus, c'/c is injective on $\mathbb{R}_{>0}$. Applying Lemma 5.12, the existence of a g-scaled congestion game with variable demands not possessing a pure Nash equilibrium follows.

(2a) $\Rightarrow$ (1) and (2b) $\Rightarrow$ (1) are shown in Theorems 5.8 and 5.13, respectively. $\square$

5.6 Network Congestion Games with Variable Demands

In this section, we explore the consistency of cost functions for congestion games with variable demands on directed networks. In these games, the resources are associated with the edges of a directed graph $\mathcal{G} = (V,R)$. For every player i, we are given a source-sink pair $(u_i, v_i) \in V \times V$. The set of feasible allocations of player i then equals the set of all simple directed (u_i, v_i)-paths.

Lemma 5.15. *Let g be a scaling function and c be a differentiable and convex cost function that is strictly positive on $\mathbb{R}_{>0}$ and has the property that c'/c is injective on $\mathbb{R}_{>0}$. If c is consistent for g-scaled directed-network congestion game with variable demands, then c satisfies the GMC for g.*

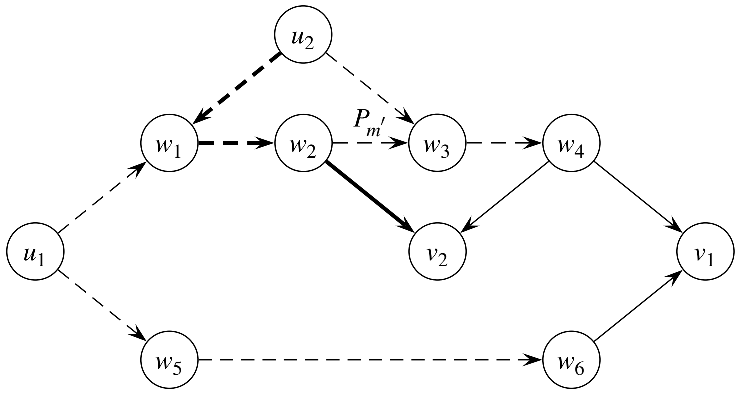
Proof. Let g be an arbitrary scaling function and let c be a cost function with the demanded properties that is consistent for g-scaled congestion games with variable demands. For a contradiction, let us assume that c does not satisfy the GMC for g. The Generalized Monotonicity Lemma for Directed Networks (Lemma 4.13) implies the existence of a g-scaled network congestion game with resource-dependent demands and costs equal c on all resources not possessing a pure Nash equilibrium. We first briefly recapitulate the construction used in the proof of Lemma 4.13.

Consider the network in Figure 5.2(a). The two players are represented by the two source-sink pairs (u_i, v_i), $i = 1,2$. The set of strategies available to each player i equals the set of directed (u_i, v_i)-paths. The dashed lines in Figure 5.2(a) correspond to paths while solid lines correspond to single edges. For carefully chose $x, y, \delta \in \mathbb{R}_{>0}$, the demand of player 1 equals x on all edges. The demand of player 2 equals y for the path connecting u_2 with w_1, the path connecting w_1 with w_2, and the edge connecting w_2 with v_2 (all drawn with bold lines in Figure 5.2(a)). All other edges are accessed with demand $y+\delta$ by player 2. The directed path P_m connecting w_2 and w_3 contains m edges with cost function c where m is chosen large enough such that for player 2 the unique strategy containing P_m is strictly dominated.

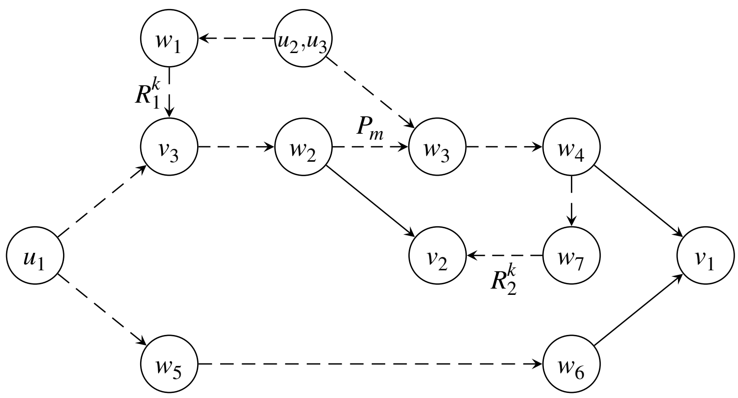
We proceed to modify this construction to obtain a directed network congestion game with variable demands not possessing a pure Nash equilibrium. First, we define the player's sets of feasible demands setting $\sigma_1 = \tau_1 = x$, $\sigma_2 = y$, and $\tau_2 = y+\delta$. Next, consider the network game shown in Figure 5.2(b). Because the cost player 2 experiences on the edge (w_2, v_2) is bounded by $\max_{z \in [y, y+\delta]} g(z)c(z)$ and the cost functions are strictly positive, we can make the path $P_{m'}$ sufficiently long such that the unique strategy of player 2 containing $P_{m'}$ is strictly dominated.

Our goal is to enforce player 2 to use the demand y when on her left path $u_2 \to w_1 \to w_2 \to v_2$ and the demand $y+\delta$ when on her right path $u_2 \to w_3 \to w_4 \to v_2$. As in Lemma 5.12 this can be achieved by adding additional resources to each strategy of player 2, where the additional resources contained in the left path are used by an additional third player. Thus, for $k \in \mathbb{N}$, we add additional paths R_1^k, R_2^k containing k additional edges. We also add a third player associated with the source-sink pair (u_3, v_3) whose only strategy is to follow the paths $u_3 \to w_1 \to v_3$.

Along the same chain of reasoning as in Lemma 5.12, we can choose k large enough and an appropriate utility function of player 2 such that player 2 always uses the demand y when allocated



(a) Directed-network congestion game with resource-dependent demands used in the proof of the Generalized Monotonicity Lemma for Directed Networks (Lemma 4.13)



(b) Directed-network congestion game with variable demands used in the proof of Lemma 5.15

Figure 5.2: Directed-network congestion games used in the proofs of (a) Lemma 4.13 and (b) Lemma 5.15. Dashed lines correspond to paths while solid edges correspond to single edges.

on her left path and the demand $y+\delta$ when allocated on her right path. Then, using that the network congestion game with resource-dependent demands has no pure Nash equilibrium implies that also the network congestion game with variable demands does not possess a pure Nash equilibrium. $\square$

We obtain the following result.

Theorem 5.16. *Let $\mathcal{C}$ be a set of continuous functions that are strictly positive on $\mathbb{R}_{>0}$ and let g be a scaling function. Then, the following are equivalent:*

(1) *$\mathcal{C}$ is consistent for g-scaled directed-network congestion games with variable demands.*
(2) *At least one of the following two statements holds:*
  - (2a) *$\mathcal{C}$ only contains homogeneously exponential functions of type $c(x) = a_c e^{\phi x}$, where $a_c \in \mathbb{R}_{>0}$ may depend on c, while $\phi \in \mathbb{R}_{>0}$ must be equal for all $c \in \mathcal{C}$;*
  - (2b) *g is linear and $\mathcal{C}$ only contains affine functions of type $c(x) = a_c x + b_c$, where $a_c \in \mathbb{R}_{>0}$, $b_c \in \mathbb{R}_{\geq 0}$.*

5.7 A Characterization for Unrestricted Demands

In the last sections, we characterized the sets of cost functions that are consistent for g-scaled congestion games with variable demands, both with arbitrary strategy spaces and network strategy spaces. We assumed that for each player i the set of feasible demands is restricted to an interval $[\sigma_i, \tau_i] \subseteq \mathbb{R}_{\geq 0}$, with $\sigma_i \in \mathbb{R}_{\geq 0}$, $\tau_i \in \mathbb{R}_{\geq 0} \cup \{\infty\}$, and $\sigma_i \leq \tau_i$. In particular, we allowed the degenerated case $\sigma_i = \tau_i$. This assumption might be too restrictive and to artificially narrow the set of consistent cost functions. This section is devoted to the case of unrestricted demands, that is, we consider the case $[\sigma_i, \tau_i] = \mathbb{R}_{\geq 0}$ for all $i \in N$. As the main result, we show that our characterizations of consistency for the case of restricted demands translate to the case of unrestricted demands.

Theorem 5.17. *Let $\mathcal{M} = \big(N, R, (\mathcal{A}_i)_{i\in N}, (c_r)_{r\in R}\big)$ be a congestion model and for each player i let $U_i^r : [\sigma_i^r, \tau_i^r]$ be a utility function with $\tau_i^r < \infty$. If the corresponding g-scaled congestion game with variable demands G^r does not possess a pure Nash equilibrium, then there is for each player i a utility function $U_i : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ and a corresponding g-scaled congestion game with unrestricted variable demands that does not admit a pure Nash equilibrium as well.*

Proof. We set $T = \sum_{i\in N} \tau_i$. For every $r \in R$, the cost function $c_r : \mathbb{R}_{\geq 0} \to \mathbb{R}$ is locally Lipschitz-continuous on $\mathbb{R}_{\geq 0}$ and hence globally Lipschitz-continuous with Lipschitz constant L_r on the compact $[0, T]$. We set $\sigma_{\max} = \max_{i\in N} \sigma_i$ and $L = \sum_{r\in R} L_r$. As g is continuously differentiable, it is globally Lipschitz-continuous on $[0, \sigma_{\max}]$ and we denote the Lipschitz constant by M. We define $C_{\max} = \sum_{r\in R} \max_{x\in[0,T]} c_r(x)$. For $\mu > \max\big\{L \cdot g(\sigma_{\max}) + M \cdot C_{\max}, \frac{\partial^+ U_i^{\mathrm{r}}(\sigma_i^{\mathrm{r}})}{\partial \sigma_i^{\mathrm{r}}} \cdot \sigma_i^{\mathrm{r}}\big\}$, we define the utility function of player i as the function $U_i : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ where

$$U_i(x) = \begin{cases} \frac{U_i^{\mathrm{r}}(\sigma_i^{\mathrm{r}})+\mu}{\sigma_i^{\mathrm{r}}} \cdot x, & \text{if } x \in [0, \sigma_i^{\mathrm{r}}) \\ U_i^{\mathrm{r}}(x) + \mu, & \text{if } x \in [\sigma_i^{\mathrm{r}}, \tau_i^{\mathrm{r}}] \\ U_i^{\mathrm{r}}(\tau_i^{\mathrm{r}}) + \mu, & \text{otherwise.} \end{cases}$$

On $[\sigma_i^{\mathrm{r}}, \tau_i^{\mathrm{r}}]$, the new utility function equals the old utility function raised by μ, on $[\tau_i^{\mathrm{r}}, \infty)$ it is constant and on $[0, \sigma_i^{\mathrm{r}}]$ it equals the linear function through the origin and the point $(\sigma_i^{\mathrm{r}}, U_i(\sigma_i^{\mathrm{r}}) + \mu)$. Note that U_i is concave as U_i^{r} is concave and $\mu > \frac{\partial^+ U_i^{\mathrm{r}}(\sigma_i^{\mathrm{r}})}{\partial \sigma_i^{\mathrm{r}}} \cdot \sigma_i^{\mathrm{r}}$; see also Figure 5.3 for an illustration.

The new set $(U_i)_{i\in N}$ of utility functions defines a new g-scaled congestion game with variable demands G. We claim that G does not admit a pure Nash equilibrium. For a contradiction, suppose (α, d) is a pure Nash equilibrium of G.

We first show that it is without loss of generality to assume that $d_i \leq \tau_i^{\mathrm{r}}$ for all $i \in N$. Specifically, we show that if there is a pure Nash equilibrium (α, d) of G, then there is also a pure Nash equilibrium (α, d') with $d_i' \leq \tau_i^{\mathrm{r}}$ for all $i \in N$. To see this, note that $U_i(d_i) = U_i(\tau_i^{\mathrm{r}})$ for all $d_i \geq \tau_i^{\mathrm{r}}$ and all $i \in N$. If there is a resource $r \in \alpha_i$ with $c_r\big(\ell_r(\alpha, d)\big) > c_r\big(\ell_r(\alpha, d) - d_i + \tau_i^{\mathrm{r}}\big)$, we derive that player i improves lowering her demand from d_i to τ_i^{r}, which contradicts the fact that (α, d) is a pure Nash equilibrium. This implies that $c_r\big(\ell_r(\alpha, d)\big) = c_r\big(\ell_r(\alpha, d) - d_i + \tau_i^{\mathrm{r}}\big)$ for all $r \in \alpha_i$ and setting $d_i' = \tau_i^{\mathrm{r}}$ we derive that $\varpi_j(\alpha, d_i', d_{-i}) = \varpi_j(\alpha, d)$ for all $j \in N$. From there, (α, d_{-i}, d_i') is also a pure Nash equilibrium. Iterating this argument, we obtain a pure Nash equilibrium (α, d') with $d_i' \leq \tau_i^{\mathrm{r}}$ for all $i \in N$.

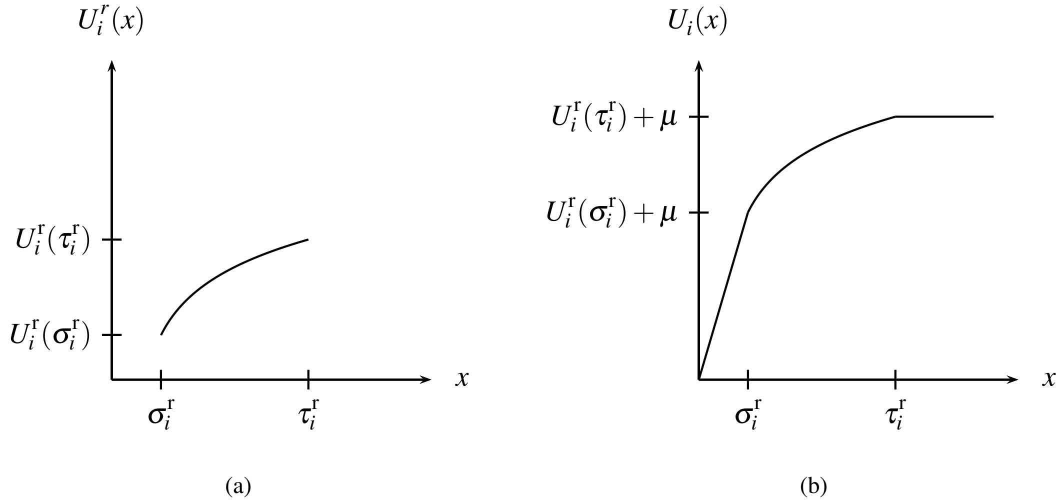


Figure 5.3: Lifting of the players' utility functions used in the proof of Theorem 5.17. (a) the utility function U_i^{r} of player i in the game with restricted demands; (b) the lifted utility function U_i of player i in the game with unrestricted demands.

Let (α, d') be such pure Nash equilibrium. We claim that $d_i' \geq \sigma_i^{\mathrm{r}}$ for all $i \in N$. Suppose there is $i \in N$ with $d_i' < \sigma_i^{\mathrm{r}}$ and consider the strategy where player i plays a demand of $d_i'' = \sigma_i^{\mathrm{r}}$ instead. We calculate

$$\begin{aligned}
&\varpi_i(\alpha, d_i'', d_{-i}') - \varpi_i(\alpha, d_i') \\
&\quad = \mu(d_i'' - d_i') - g(d_i'')\sum_{r\in\alpha_i} c_r\big(\ell_r(\alpha, d') - d_i' + d_i''\big) + g(d_i')\sum_{r\in\alpha_i} c_r\big(\ell_r(\alpha, d')\big) \\
&\quad \geq \mu(d_i'' - d_i') - g(d_i'')L(d_i'' - d_i') - (g(d_i'') - g(d_i'))\sum_{r\in\alpha_i} c_r\big(\ell_r(\alpha, d')\big) \\
&\quad \geq \big(d_i'' - d_i\big)\big(\mu - g(\sigma_{\max})L - M \cdot C_{\max}\big) > 0.
\end{aligned}$$

Hence, player i improves contradicting the fact that (α, d') is a pure Nash equilibrium. To finish the proof, assume there is a pure Nash equilibrium (α, d') with $d_i \in [\sigma_i^{\mathrm{r}}, \tau_i^{\mathrm{r}}]$ for all $i \in N$. Using the fact that the new utility function on $[\sigma_i^{\mathrm{r}}, \tau_i^{\mathrm{r}}]$ equals the old utility function raised by the constant μ, we derive that (α, d') is also a pure Nash equilibrium of G^{r}, contradiction! ☐

Smooth Utilities. The utility functions constructed in the proof of Theorem 5.17 are not differentiable. We proceed to show that the result continues to hold if we assume that all utility functions are infinitely often differentiable.

Theorem 5.18. *Let $\mathcal{M} = \big(N, R, (\mathcal{A}_i)_{i\in N}, (c_r)_{r\in R}\big)$ be a congestion model and for each player i let $U_i^r : [\sigma_i^r, \tau_i^r]$ be a utility function with $\tau_i^r < \infty$. If the corresponding g-scaled congestion game with variable demands G^r does not possessing a pure Nash equilibrium, then there is for each player i an infinitely often differentiable utility function $U_i : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ and a corresponding g-scaled congestion game with unrestricted variable demands that does not admit a pure Nash equilibrium as well.*

Proof. Let U_i denote the piece-wise linear utility function of player i constructed in the proof of Theorem 5.17. For $i \in N$, we define the improvement function $\zeta_i : S \to \mathbb{R}_{\geq 0}$ as the function that maps every strategy profile to the value by with each player can maximally improve her utility when switching to her best reply. Formally,

$$\zeta_i(s) = \max_{s'_i \in S_i} \varpi_i(s'_i, s_{-i}) - \varpi_i(s).$$

Note that for each player, there is $\omega_i \in \mathbb{R}_{>0}$ such that U_i is constant on $[\omega_i, \infty]$. From there, we may effectively restrict the demand of each player i to $[0, \omega_i]$. Setting $\bar{S}_i = \{s_i = (\alpha_i, d_i) \in S_i : d_i \leq \omega_i\}$, we observe that $\max_{s'_i \in S_i} \varpi_i(s_i, s_{-i}) = \max_{s'_i = \in \bar{S}_i} \varpi_i(s_i, s_{-i})$. Using that ϖ_i is continuous and $\bar{S}$ is compact, the maximum is attained and thus, ζ is well-defined.

Writing the strategy profile s as $s = (\alpha, d)$ and using that all private payoff functions are continuous in d, we observe that $\zeta_i(\alpha, d)$ is continuous in d as well. Next, we define $\zeta : S \to \mathbb{R}_{\geq 0}$ as $\zeta(s) = \max_{i \in N} \zeta_i(s)$. As the maximum of finitely many continuous functions, ζ is continuous in d as well. As a consequence, $\varepsilon = \min_{s \in \bar{S}} \zeta(s) = \min_{s \in S} \zeta(s)$ is attained and, since G does not admit a pure Nash equilibrium, $\varepsilon > 0$.

For any $\delta > 0$, the utility function U_i of each player i is infinitely often differentiable except on δ-balls around σ_i and τ_i. Ghomi [57] investigated the problem of approximating convex functions by smooth convex functions such that both functions comply in all regions where the original function is already smooth. He shows that such function can always be found if the boundaries of the smooth regions of the original function are compact. Applying this result, we can replace U_i by an approximation $\tilde{U}_i : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ that is concave, infinitely often differentiable and satisfies $u_i(x) = \tilde{U}_i$ for all $x \in [0, \sigma_i - \delta) \cup (\sigma_i + \delta, \tau_i - \delta) \cup (\tau_i + \delta, \infty)$. Using that U_i is continuous, we may choose $\delta > 0$ such that $|U_i(x) - \tilde{U}_i(x)| < \varepsilon/2$ for all $x \in \mathbb{R}_{\geq 0}$. The new set of utility functions $(\tilde{U}_i)_{i \in N}$ defines a new g-scaled congestion game with variable demands $\tilde{G} = (N, S, \tilde{\varpi})$. Because $\zeta(s) \geq \varepsilon$ for all $s \in S$, for all $s \in S$ there is a player $i(s)$ and an alternative strategy $s'_{i(s)}(s) \in S_{i(s)}$ such that $\varpi_{i(s)}(s) \leq \varpi_{i(s)}(s'_{i(s)}(s), s) - \varepsilon$. Using that $|\varpi_i(s) - \tilde{\varpi}_i(s)| < \varepsilon/2$ for all $s \in S$ and $i \in N$, we derive that $\tilde{\varpi}_{i(s)}(s) < \tilde{\varpi}_{i(s)}(s'_i(s), s_{-i})$. Hence, the game G^{ε} does not admit a pure Nash equilibrium. □

We have obtained the following characterization of consistency for congestion games with variable demands.

Theorem 5.19. *Let $\mathcal{C}$ be a set of continuous functions and g a scaling function. Then, the following are equivalent:*

(1) *$\mathcal{C}$ is consistent for g-scaled congestion games with unrestricted variable demands and infinitely often differentiable utility functions;*

(2) *At least one of the following two holds:*

  (2a) *$\mathcal{C}$ only contains homogeneously exponential functions of type $c(x) = a_c e^{\phi x}$, where $a_c \in \mathbb{R}_{>0}$ may depend on c, while $\phi \in \mathbb{R}_{>0}$ must be equal for all $c \in \mathcal{C}$.*

  (2b) *g is linear and $\mathcal{C}$ only contains affine functions of type $c(x) = a_c x + b_c$, where $a_c \in \mathbb{R}_{>0}$ and $b_c \in \mathbb{R}_{\geq 0}$.*

5.8 Discussion and Open Problems

We considered the fundamental problem of the existence of pure Nash equilibria and the approximate finite improvement property in congestion games with variable demands. Several characterizations of the cost structure with respect to the existence of pure Nash equilibria and the approximate finite improvement property have been obtained. Since games with variable demands are general enough to closely capture many elements of practical applications, we are confident that our results help to understand the behavior of myopic play in real systems.

While this chapter addressed the existence of pure Nash equilibria and the approximate finite improvement property with respect to the cost structure (without constraints on the strategy spaces and the utility functions), it is natural to ask for combinatorial properties of the strategy spaces that ensure the existence of pure Nash equilibria for general cost functions. In light of the positive result of Ackermann et al. [4] for matroid weighted congestion games, particularly congestion games with variable demands where the set of feasible allocations of each player form the basis of a matroid are a promising avenue for future work. Alternatively, one can restrict the set of feasible utility functions (e.g., assume linear functions) and ask for the existence of pure Nash equilibria. Also, as for weighted congestion games, the case of symmetric strategy spaces is not understood.

Another interesting research direction is to investigate the prices of anarchy and stability in congestion games with variable demands. In particular, it would be very interesting to compare the prices of anarchy and stability of congestion games with variable demands and unrestricted demands with known results for weighted congestion games.

As shown in this chapter, the concept of essential improving moves may help to show the existence of pure Nash equilibria in games that do not admit a potential functions. It would be interesting to see this technique being applied to further classes of games for which the finite improvement property is refuted but it is conjectured that a pure Nash equilibrium exists (as, e.g., in weighted singleton congestion games with player-specific linear cost functions; see Gairing et al. [55], and Georgiou et al. [56]).

Chapter 6

Bottleneck Congestion Games

Most of the congestion game literature including Chapters 3, 4, and 5 of this thesis, focuses on *additive* private cost structures, i.e., it is assumed that the players strive to minimize the *sum* of the costs of the resources contained in her strategy. For such games, the existence of pure Nash equilibria and strong equilibria in terms of the strategy space and the resource cost functions is relatively well understood.

In many scenarios, however, sum-objectives capture the incentive structure of the players only partially. A prominent example of such a situation occurs when selfish users route data in a telecommunication network. Here, the delay of a stream of packets is usually restricted by the available bandwidth (or capacity) of the links on the chosen path. Hence, the total delay experienced by the player is closely related to the performance of the weakest (or most congested) link; see Banner and Orda [15], Cole et al. [26], Keshav et al. [79], and Qiu et al. [110]. Another important application is that of file transmission in mobile wireless networks; see Banner and Orda [15]. In this setting, the resources have a limited battery lifetime which decreases with the total load. The resource costs are used to model the loss of battery life due to outgoing traffic. Each user is interested in maintaining a connection as long as possible and, thus, minimizes the maximum cost among the chosen resources.

A class of games that captures such situations more realistically are bottleneck congestion games as introduced by Banner and Orda [15]. In a bottleneck congestion game the private cost of a player is the maximum (instead of the sum) of the resource costs in her strategy. A related class of games are splittable bottleneck congestion games. They differ from ordinary bottleneck congestion games solely in the fact that each player may distribute her demand fractionally among her feasible allocations. Banner and Orda [15] investigated the existence of pure Nash equilibria both in bottleneck congestion games and splittable bottleneck congestion games with weighted players. They observed that for splittable bottleneck congestion games standard techniques (such as Kakutani's fixed-point theorem) for proving existence of a pure Nash equilibrium do not apply as the players' private cost functions may be discontinuous. They proved existence of a pure Nash equilibrium by showing that bottleneck games are better reply secure, quasi-convex, and compact. Under these conditions, they recalled Reny's existence theorem [111] to prove the existence of a pure Nash equilibrium in splittable bottleneck congestion games with weighted players. Banner and Orda [15] also showed the existence of a pure Nash equilibrium in unsplittable bottleneck congestion games with weighted players. For the special case that the players are unweighted and the cost function of each resource is the identity, Busch and Magdon-Ismail [20] gave a shorter proof for the existence of a pure Nash equilibrium.

6.1 Contributions and Chapter Outline

In this chapter, we further pursue the equilibrium existence problem for bottleneck congestion games. We weaken the assumptions on the resource cost functions in that we assume that the resource costs may even depend on the *set* of players using it. Set-dependent cost functions are more general than the *load-based* models usually used in the congestion games literature and allow to model, e.g., interferences among the players. Yet, we are able to prove even the existence of a strong equilibrium in this (more general) class of games. As a byproduct of our analysis, we further derive that bottleneck congestion games with set-dependent costs have the strong finite improvement property, that is, every sequence of coalitional deviations that decreases the private costs of each of its members is finite.

Our proofs rely on a novel potential function concept, which we term *strong vector-valued potential*. For a minimization game $G = (N, S, \pi)$, we call a function $\phi : S \to \mathbb{R}^q_{\geq 0}$ with $q \in \mathbb{N}_{>0}$, a strong vector-valued potential if every strong improving move[1] from $s \in S$ strictly reduces $\phi(s)$ with respect to a certain lexicographic order defined on $\mathbb{R}^q_{\geq 0}$. Every strong potential is a strong vector-valued potential, but not conversely.

The main contribution of this chapter is twofold. First, we study *general* finite games with a strong vector-valued potential. In Section 6.2, we show that a finite game has a strong vector-valued potential if and only if it has a strong potential. The proof is constructive, that is, given a game G with a strong vector-valued potential $\phi : S \to \mathbb{R}^q_{\geq 0}$ for some $q \in \mathbb{N}_{>0}$, we explicitly construct a strong potential $P : S \to \mathbb{R}$. We derive that finite games with a strong vector-valued potential have the strong finite improvement property and possess at least one strong equilibrium. In Section 6.3, we investigate games for which the vector of the players' payoff is a strong vector-valued potential. Specifically, we show that such games always possess a strong equilibrium that is strictly Pareto efficient and min-max fair. Moreover, tight bounds on the strong price of anarchy and strong price of stability are given.

As the second main contribution of this chapter, we show in Section 6.4 that bottleneck congestion games have the property that the private cost vector is a strong vector-valued potential and hence possess strong equilibria with the above mentioned properties.

Note that for congestion games with singleton strategies (where the concepts of standard congestion games and bottleneck congestion games coincide), Even-Dar et al. [39, 40], and Fabrikant et al. [41] have already proved existence of a pure Nash equilibrium by arguing that the vector of resource costs decreases lexicographically for every improving move. Andelman et al. [8] used the same argument to even establish existence of strong equilibria in this case. Our work generalizes these results to arbitrary strategy spaces and more general resource cost functions. In contrast to most congestion games considered so far, we require only that the resource cost functions sat-

[1] A strong improving move is a coalitional deviation that is profitable to each of its members; see Section 2.1.2 for a formal definition.

isfy three properties: “Non-negativity”, “Independence of Irrelevant Choices”, and “Monotonicity”. Roughly speaking, the latter two conditions require that the cost of a resource solely depends on the set of players using that resource and decreases if some players leave that resource. Thus, this framework extends classical *load-based* models in which the cost of a resource depends on the number or aggregated demands of players using it.

In Section 6.5, we study *infinite* games, that is, games with infinite strategy spaces that can be described by compact subsets of $\mathbb{R}^p$, $p \in \mathbb{N}_{>0}$. We slightly generalize the concept of a strong vector-valued potential by introducing the notion of a pairwise strong vector-valued potential function $\phi : S \to \mathbb{R}^q_{\geq 0} \times \mathbb{R}^q_{\geq 0}$ with $q \in \mathbb{N}_{>0}$. Here, we require that every strong improving move from $s \in S$ strictly reduces a certain lexicographical order of $\phi(s)$. We prove that continuity of the pairwise strong vector-valued potential ϕ is sufficient for the existence of a strong equilibrium. We then introduce *splittable bottleneck congestion games*. A splittable bottleneck congestion game arises from a bottleneck congestion game G by allowing players to fractionally distribute a certain *demand* over the set of her feasible allocations. We prove that these games have the have a strong pairwise vector-valued potential provided that the resource cost functions satisfy the three properties of “Non-negativity”, “Independence of Irrelevant Choices”, and “Monotonicity”. If the resource cost functions are also continuous, we obtain a pairwise strong continuous vector-valued potential ϕ and, thus, the existence of a strong equilibrium for splittable bottleneck congestion games. For *bounded* cost functions on the resources (that may be discontinuous), we show that a ρ-approximate strong equilibrium exist for every $\rho > 0$.

Further Related Work. Bottleneck congestion games can be seen in the more general framework of *generalized congestion games* studied by Kukushkin [86]. A generalized congestion game differs from an unweighted congestion game with sum-objective solely in the fact that the additive aggregation of the resource costs in the definition of each player’s private cost is replaced by an arbitrary aggregation function (that takes the vector of the costs of the chosen resources as input).[2] Kukushkin [86] proved that (up to monotonic transformations) additive aggregation is the only strictly increasing aggregation rule that guarantees the existence of a pure Nash equilibrium in all games with unweighted players. This characterization, however, requires that the aggregation function is strictly decreasing in the sense that if the cost of one resource is strictly decreased then the private cost of each of its users must strictly decrease. This property is violated by bottleneck-objectives. Moreover, our positive results only holds for monotonic cost functions while in [86] it is required that a pure Nash equilibrium exists for all (not necessarily monotonic) cost functions.

Bibliographic Information. The results presented in this chapter are joint work with Tobias Harks and Rolf H. Möhring. An extended abstract appeared in the *Proceedings of the 5th International Workshop on Internet and Network Economics*; see [66]. A more extensive version is accepted for publication in the *International Journal of Game Theory*; see [68].

[2] Technically, one has to be given one such aggregation function for each natural number that appears as the cardinality of a feasible allocation.

6.2 Strong Vector-Valued Potentials

Before we introduce strong vector-valued potentials, we define the sorted lexicographical order on non-negative vectors. To this end, let $q \in \mathbb{N}_{>0}$ and let $a, b \in \mathbb{R}^q_{\geq 0}$. Denote by $\tilde{a}, \tilde{b} \in \mathbb{R}^q_{\geq 0}$ the sorted vectors derived from a, b by sorting the entries in non-increasing order, that is, $\tilde{a}_1 \geq \cdots \geq \tilde{a}_q$ and $\tilde{b}_1 \geq \cdots \geq \tilde{b}_q$. Then, a is *strictly sorted lexicographically smaller* than b (written $a \prec b$) if there exists an index m such that $\tilde{a}_i = \tilde{b}_i$ for all $i < m$, and $\tilde{a}_m < \tilde{b}_m$. The vector a is *sorted lexicographically smaller* than b (written $a \preceq b$) if either $a \prec b$ or $\tilde{a} = \tilde{b}$. A strong vector-valued potential is a function $\phi : S \to \mathbb{R}^q_{\geq 0}$ that strictly decreases with respect to the sorted lexicographical order along any strong improving move.

Definition 6.1 (Strong vector-valued potential)
For a minimization game $G = (N, S, \pi)$, a function $\phi : S \to \mathbb{R}^q_{\geq 0}$ is called a *strong vector-valued potential* if $\phi(s) \succ \phi(s'_K, s_{-K})$ for all strong improving moves $\big(s, (s'_K, s_{-K})\big)$ of G.

A strong vector-valued potential ϕ is a strong potential if $q = 1$. The next proposition states that in finite games the existence of a strong vector-valued strong potential is equivalent to the existence of a strong potential.

Proposition 6.2. *For a finite minimization game $G = (N, S, \pi)$ the following two statements are equivalent:*

(1) *G has a strong vector-valued potential $\phi : S \to \mathbb{R}^q_{\geq 0}$ for some $q \in \mathbb{N}_{>0}$.*
(2) *G has a strong potential function $P : S \to \mathbb{R}_{\geq 0}$.*

Proof. We only prove (1) $\Rightarrow$ (2) as the reverse direction is trivial. Let $q \in \mathbb{N}$ and $\phi : S \to \mathbb{R}^q_{\geq 0}$ be a strong vector-valued potential. We claim that there is $\mu \in \mathbb{N}$ such that that $P_\mu(s) = \sum_{i=1}^q \phi_i(s)^\mu$ is a strong potential. To see this, let $\big(s, (s'_K, s_{-K})\big)$ be an arbitrary strong improving move of coalition $K \subseteq N$. We denote by $\tilde{\phi}(s)$ and $\tilde{\phi}(s'_K, s_{-K})$ the vectors that arise by sorting $\phi(s)$ and $\phi(s'_K, s_{-K})$ in non-increasing order. As $\phi(s'_K, s_{-K}) \prec \phi(s)$, there is an index $m \in \{1, \ldots, q\}$ such that $\tilde{\phi}_i(s) = \tilde{\phi}_i(s'_K, s_{-K})$ for all $i < m$ and $\tilde{\phi}_m(s) < \tilde{\phi}_m(s'_K, s_{-K})$. We obtain

$$
\begin{aligned}
P_\mu(s) - P_\mu(s'_K, s_{-K}) &= \sum_{i=1}^{q} \phi_i(s)^\mu - \sum_{i=1}^{q} \phi_i(s'_K, s_{-K})^\mu \\
&= \tilde{\phi}_m(s)^\mu - \tilde{\phi}_m(s'_K, s_{-K})^\mu + \sum_{i=m+1}^{q} \tilde{\phi}_i(s)^\mu - \sum_{i=m+1}^{q} \tilde{\phi}_i(s'_K, s_{-K})^\mu \\
&\geq \tilde{\phi}_m(s)^\mu - \tilde{\phi}_m(s'_K, s_{-K})^\mu - (q-m)\tilde{\phi}_m(s'_K, s_{-K})^{M'} \geq \tilde{\phi}_m(s)^\mu - q\,\tilde{\phi}_m(s'_K, s_{-K})^\mu. \qquad (6.1)
\end{aligned}
$$

Standard calculus shows that the expression on the right hand side of (6.1) is positive if

$$
\mu > \frac{\log(q)}{\log\big(\tilde{\phi}_m(s)\big) - \log\big(\tilde{\phi}_m(s'_K, s_{-K})\big)} > 0.
$$

As the number of improvement steps is finite, we can chose μ sufficiently large to obtain the claimed result. $\square$

6.3 Efficiency and Fairness of Equilibria

As games with a strong vector-valued potential are guaranteed to possess strong equilibria, it is natural to investigate efficiency and fairness properties of these strong equilibria. We focus on games that have the special property that the private cost vector π is a strong vector-valued potential, that is, the sorted lexicographic order of the private cost vector decreases along any strong improvement path. We consider strict Pareto efficiency, min-max fairness, strong price of anarchy, and strong price of stability.

6.3.1 Pareto Efficiency

Pareto efficiency is one of the fundamental concepts studied in the economics literature. For a minimization game $G = (N, S, \pi)$, a strategy profile s is called *weakly Pareto efficient* if there is no $s' \in S$ such that $\pi_i(s') < \pi_i(s)$ for all $i \in N$. A strategy profile s is *strictly Pareto efficient* if there is no $s' \in S$ such that $\pi_i(s') \leq \pi_i(s)$ for all $i \in N$, where at least one inequality is strict. So strictly Pareto efficient strategy profiles are those for which every improvement of a player is at the expense of at least one other player. Pareto efficiency has also been studied in the context of standard congestion games (with sum-objective). Holzman and Law-Yone [72] gave sufficient conditions on the strategy spaces of congestion games that guarantee the existence of a strong equilibrium which is strictly Pareto efficient. Chien and Sinclair [23] quantified the social welfare achieved in weakly Pareto efficient pure Nash equilibria of unweighted congestion games.

Every strong equilibrium is weakly Pareto optimal as it is resilient against a profitable deviation of the whole player set N. In games with the property that the private cost vector π is a strong vector-valued potential this result can be strengthened in the sense that there always is a strong equilibrium, that is even *strictly* Pareto efficient.

Theorem 6.3. *Let G be a finite minimization game with the property that the private cost vector π is a strong vector-valued potential. Then, there exists a strong equilibrium that is strictly Pareto optimal.*

Proof. The sorted lexicographical minimum s of π is a strong equilibrium. To see that it is also strictly Pareto efficient, assume by contradiction that there is $s' \in S$ and a player i such that $\pi_i(s') < \pi_i(s)$ and $\pi_j(s') \leq \pi_j(s)$ for all $j \in N \setminus \{i\}$. Then, $s' \prec s$, contradicting the minimality of s. $\square$

6.3.2 Min-Max Fairness

Min-max fairness is a central topic in the theory of resource allocation in communication networks; see Srikant [123] for an overview and pointers to the large body of research in this area. While strict Pareto efficiency requires that there is no alternative profile that improves the cost for a single player without strictly deteriorating the other players' costs, the notion of min-max-fairness is stronger. A profile s is called *min-max fair* if for any other strategy profile s' with $\pi_i(s') < \pi_i(s)$ for some $i \in N$, there either exists $j \in N \setminus \{i\}$ such that $\pi_j(s) \geq \pi_i(s)$ and $\pi_j(s') > \pi_j(s)$, or there exists $j \in N \setminus \{i\}$ such that $\pi_j(s) < \pi_i(s)$ and $\pi_j(s') \geq \pi_i(s')$. Note that in contrast to Pareto efficiency, an improvement that increases the cost of a player with smaller original cost is allowed (up to the threshold $\pi_i(s)$).

It is easy to see that every min-max fair strategy profile is a strictly Pareto efficient state, but not conversely.

Theorem 6.4. *Let G be a finite minimization game with the property that the private cost vector π is a strong vector-valued potential. Then, there exists a strong equilibrium that is min-max fair.*

Proof. We show that the strategy profile s minimizing π with respect to the sorted lexicographical order is min-max fair. Assume by contradiction that there is another strategy profile s' such that $\pi_i(s') < \pi_i(s)$ for some $i \in N$ and the following two statements hold:

(1) $\pi_j(s') \leq \pi_j(s)$ for all $j \in N \setminus \{i\}$ with $\pi_j(s) \geq \pi_i(s)$,

(2) $\pi_j(s') < \pi_i(s)$ for all $j \in N \setminus \{i\}$ with $\pi_j(s) < \pi_i(s)$.

We observe that every entry of $\pi(s)$ that is larger than $\pi_i(s)$ only decreases under s', while every entry strictly smaller than $\pi_i(s)$ may only increase to a value strictly smaller than the threshold $\pi_i(s)$. Since the value $\pi_i(s)$ strictly decreases under s', we obtain $\pi(s') \prec \pi(s)$, contradicting the minimality of s. □

6.3.3 Price of Stability and Price of Anarchy

To quantify the efficiency loss of selfish behavior with respect to a predefined social cost function, two notions have evolved. The *price of anarchy* has been introduced by Koutsoupias and Papadimitriou [85] in the context of congestion games and is defined as the ratio of the cost of the worst pure Nash equilibrium and that of a social optimum. A more optimistic performance index, termed the *price of stability*, measures the ratio of the cost of the best pure Nash equilibrium and that of a social optimum; see Anshelevich et al. [9, 10]. Both concepts have been studied extensively in computer science and operations research; see Nisan et al. [105, Part III] for a survey. More recently, they have also been studied in economics; see e.g. Johari and Tsitsiklis [74] and Moulin [100].

Andelman et al. [8] propose to study also the worst case ratio of the cost of a strong equilibrium and that of a social optimum, which they term the *strong* price of anarchy. Clearly, the strong price of anarchy is not larger than the price of anarchy. For some classes of games this inequality is strict; see e.g. the results of Czumaj and Vöcking [29] and Fiat et al. [47] on the price of anarchy and strong price of anarchy of scheduling games on related machines, respectively. The strong price of anarchy has been studied recently for standard congestion games by Chien and Sinclair [23]. Andelman et al. [8] also define the *strong price of stability* in the obvious way as the ratio of the cost of a best strong equilibrium and that of a social optimum.

Formally, given a minimization game $G = (N, S, \pi)$ and a social cost function $\Pi : S \to \mathbb{R}_{>0}$, whose minimum is attained in a strategy profile $s_{\min} \in S$, let $E \subseteq S$ denote the set of strong equilibria. Then, the strong price of anarchy for G with respect to Π is defined as $\sup_{s\in E} \Pi(s)/\Pi(s_{\min})$ and the strong price of stability for G with respect to Π is defined as $\inf_{s\in E} \Pi(s)/\Pi(s_{\min})$. In the following, we consider the following natural social cost functions: *(i)* the sum of the players' private costs defined as $L_1(s) = \sum_{i\in N} \pi_i(s)$, *(ii)* the $||\cdot||_p$-norm of the players' private costs defined as $L_p(s) = \left(\sum_{i\in N} \pi_i(s)^p\right)^{1/p}$, $p \in \mathbb{N}$, *(iii)* the maximum norm of the players' private costs defined as $L_\infty(s) = \max_{i\in N}\{\pi_i(s)\}$.

		Player 2	
		“left"	“right"
Player 1	“up"	$1-\varepsilon,\ 1-\varepsilon,\ 1-\varepsilon,\ \dots,\ 1-\varepsilon$	$1,\quad 1,\quad 1,\ \dots,\quad 1$
	“down"	$1,\quad 0,\quad 0,\ \dots,\quad 0$	$1,\quad \varepsilon,\quad 1,\ \dots,\quad 1$

Figure 6.1: Matrix representation of the game considered in Example 6.6. The game has n players. Player 1 has the strategies “up” and “down” identified with the rows of the matrix, player 2 has the strategies “left” and “right” identified with the columns of the matrix, all other players have a single strategy called “zero” not shown in the figure. Every cell of the matrix shows the private cost vector of the corresponding strategy profile. The price of stability w.r.t. any L_p-norm approaches $n^{1/p}$ as ε goes to 0.

Theorem 6.5. *For a finite minimization game for which the private cost vector π is a strong vector-valued potential, the strong price of stability w.r.t. L_∞ is 1, and, for any $p \in \mathbb{N}$, the strong price of stability w.r.t. L_p is at most $n^{1/p}$.*

Proof. To see that the strong price of stability w.r.t. L_∞ is 1, note that a sorted lexicographical minimum s^* of π is a strong equilibrium. By construction, s^* minimizes L_∞. To prove the claimed bound for L_p, we first show that for arbitrary $p,q \in \mathbb{N}$ with $p < q$ and $s \in S$ we have $L_p(s) \le n^{1/p-1/q} L_q(s)$ and $L_p(s) \le n^{1/p} L_\infty(s)$. To see the first inequality, let $a = \frac{q}{p} > 1$ and $b > 0$ be such that $\frac{1}{a} + \frac{1}{b} = 1$. With Hölder’s inequality we obtain

$$L_p(s) = \Big(\sum_{i=1}^{n} \pi_i(s)^p\Big)^{\frac{1}{p}} \le \left(\Big(\sum_{i=1}^{n} \pi_i(s)^{p\cdot a}\Big)^{\frac{1}{a}} \Big(\sum_{i=1}^{n} 1\Big)^{\frac{1}{b}}\right)^{\frac{1}{p}} = n^{\frac{1}{pb}} L_q(s) = n^{\frac{1}{p}-\frac{1}{q}} L_q(s).$$

For the L_∞-norm we have $a = \infty$ and $b = 1$, thus, we obtain $L_p(s) \le n^{1/p} L_\infty(s)$.

Next, let s^* be a sorted lexicographical minimum of π. Fix $p \in N$ and let $s_{\min}$ be a strategy profile minimizing L_p. We derive $L_p(s^*) \le n^{1/p} L_\infty(s^*) \le n^{1/p} L_\infty(s_{\min}) \le n^{1/p} L_p(s_{\min})$, where for the second inequality we use that s^* minimizes L_∞ and for the third inequality we use that the L_p-norm is decreasing in p. □

We now provide an example of a class of games with the property that π is a strong vector-valued potential and whose parameters can be chosen in such a way that the strong price of stability w.r.t. L_p is arbitrarily close to $n^{1/p}$, implying that the result of Theorem 6.5 is tight.

Example 6.6 (Strong price of stability). For a fixed $\varepsilon \in (0,1)$, let us consider the minimization game $G = (N,S,\pi)$ with $N = \{1,\dots,n\}$, $S_1 = \{$“up”, “down”$\}$, $S_2 = \{$“left”, “right”$\}$ and $S_i = \{$“zero”$\}$ for all $i \in \{3,\dots,n\}$. The private costs are shown in Figure 6.1. It is straightforward to check π is a strong vector-valued potential. The unique strong equilibrium is the strategy profile (“up”, “left”, “zero”,…, “zero”) realizing a private cost vector of $(1-\varepsilon,\dots,1-\varepsilon)$. For

		Player 2	
		“left"	“right"
Player 1	“up"	0, 0	0, k
	“down"	k, k	0, k

Figure 6.2: Matrix representation of the minimization game G with the property that the private cost vector π is a strong vector-valued potential; see Example 6.7. The price of anarchy w.r.t. any L_p-norm is unbounded as considered.

any $p \in \mathbb{N}$, we can choose $\varepsilon > 0$ sufficiently small such that $L_p(\cdot)$ is minimized in strategy profile (“down”, “left”, “zero”,…, “zero”) realizing a cost vector of $(1,0,\dots,0)$. Hence the price of stability approaches $n^{1/p}$.

So far, our results concern the strong price of stability only. The next example shows that games for which π is a strong vector-valued potential may have an unbounded strong price of anarchy.

Example 6.7 (Strong price of anarchy). For $k > 0$, consider the minimization game $G = (N, S, \pi)$ with $N = \{1,2\}$, $S_1 = \{\text{“up”}, \text{“down”}\}$, $S_2 = \{\text{“left”}, \text{“right”}\}$ and private costs as given in Figure 6.2. It is straightforward to check that π is a strong vector-valued potential and that both (“up”, “left”) and (“down”, “right”) are strong equilibria. Hence, the price of anarchy w.r.t. any L_p norm is unbounded from above.

6.4 Bottleneck Congestion Games

We now present a rich class of finite games with the property that the private cost vector π is a strong vector-valued potential. We call these games *bottleneck congestion games with set-dependent costs*. They are natural generalizations of variants of congestion games. In contrast to standard congestion games, we focus on *bottleneck-objectives*, that is, the cost of a player only depends on the highest cost of the resources she uses. For the sake of a clean mathematical definition, we introduce the notion of a congestion model with set-dependent costs.

Definition 6.8 (Congestion model with set-dependent costs)
A tuple $\mathcal{M} = \big(N, R, (\mathcal{A}_i)_{i\in N}, (c_r)_{r\in R}\big)$ is called a *congestion model with set-dependent costs* if $N = \{1,\dots,n\}$ is a nonempty, finite set of players, $R = \{1,\dots,m\}$ is a nonempty set of resources, and $\mathcal{A}_i \subseteq 2^R$ is the set of allocations available to player i. We set $\mathcal{A} = \mathcal{A}_1 \times \cdots \times \mathcal{A}_n$ and for each allocation profile $\alpha \in \mathcal{A}$, we define $\mathcal{N}_r(\alpha) = \{i \in N : r \in \alpha_i\}$ for all $r \in R$. Every resource $r \in R$ has a cost function $c_r : \mathcal{A} \to \mathbb{R}_{\geq 0}$ satisfying the following three properties:

Nonnegativity: $c_r(\alpha) \geq 0$ for all $\alpha \in \mathcal{A}$,
Independence of Irrelevant Choices: $c_r(\alpha) = c_r(\alpha')$ for all $\alpha, \alpha' \in \mathcal{A}$ with $\mathcal{N}_r(\alpha) = \mathcal{N}_r(\alpha')$,
Monotonicity: $c_r(\alpha) \leq c_r(\alpha')$ for all $\alpha, \alpha' \in \mathcal{A}$ with $\mathcal{N}_r(\alpha) \subseteq \mathcal{N}_r(\alpha')$.

Note that ”monotonicity“ implies ”independence of irrelevant choices“. We now define bottleneck congestion games relative to a congestion model with set-dependent costs.

Definition 6.9 (Bottleneck congestion game with set-dependent costs)
Let $\mathcal{M} = \big(N, R, (\mathcal{A}_i)_{i\in N}, (c_r)_{r\in R}\big)$ be a congestion model with set-dependent costs. The corresponding *bottleneck congestion game* is the minimization game $G = (N, S, \pi)$ in which $S_i = \mathcal{A}_i$ and $\pi_i(s) = \max_{r\in s_i} c_r(s)$ for all $i \in N$.

Note that for singleton strategies, congestion games with bottleneck-objective and congestion games with sum-objective coincide.

Our assumptions on the cost functions are weaker than in the load-based models considered in Chapters 3, 4 and 5 and frequently used in the congestion games literature as, e.g., in Banner and Orda [15]. We only require that the cost function $c_r(s)$ of resource r for strategy profile s depends on the *set* of players using r in s and that costs are increasing with larger sets. Note that this may cover, e.g., dependencies on the identities of players using r.

The condition “Independence of Irrelevant Choices” already appeared in Konishi et al. [82].[3] The authors of [82] impose three additional assumptions that ensure the existence of a pure Nash equilibrium. Specifically, they require that strategy spaces are symmetric and, given a strategy profile $s = (s_1, \dots, s_n)$, the utility of a player i depends only on her own choice s_i and the *cardinality* of the set of other players who also choose s_i. On the one hand, our model is more general than that discussed in [82] as it does neither require symmetry of strategies, nor that the utility of player i only depends on the set-cardinality of other players who also choose s_i. On the other hand, the model of Konishi et al. allows for *player-specific* resource cost functions, which our model does not.

For the relation between games considered by Konishi et al. [82] and congestion games see the discussion in Voorneveld et al. [132].

Before we prove that bottleneck congestion games have the property that the private cost vector π is a strong vector-valued potential and, thus, possess a strong equilibrium with the efficiency and fairness properties shown in the last section, we give a series of examples of games that fit into the rich class of bottleneck congestion games and show how they are related to the literature.

Scheduling Games. Scheduling games model situations in which each player controls a task that needs to be processed by one machine out of a finite number of available machines; see Vöcking [128] for a survey. In each strategy profile every player $i \in N$ selects a single machine on which her job is processed. In the most general machine model of unrelated machines each job is associated with a machine-dependent weight $w_{i,r} \in \mathbb{R}_{>0}$. Scheduling games are singleton bottleneck congestion games where the cost function of machine r is defined as $c_r(s) = \sum_{i\in N: s_i = \{r\}} w_{i,r}$. This function satisfies non-negativity, independence of irrelevant choices and monotonicity. The existence of strong equilibria in scheduling games has been established before by Andelman et al. [8] by arguing that the lexicographically minimal schedule is a strong equilibrium. They also showed that the strong price of stability w.r.t. L_∞ is 1. Note that our general framework of bottleneck congestion games allows more complex cost structures on the machines than in these classical load-based models. One such example are dependencies between the weights of jobs on the same machine.

[3] In Konishi et al. [81, 83], the very same condition is called “No Spillovers”.

Resource Allocation in Wireless Networks. The problem of resource allocation in wireless networks motivates the study of so-called interference games. Consider a set of n terminals that want to connect to one out of m available base stations. Terminals assigned to the same base station impose interferences among each other as they use the same frequency band. We model the interference relations by an undirected interference graph $\mathcal{G} = (V, E)$, where $V = \{1, \ldots, n\}$ is the set of vertices/terminals and an edge $e = (v, w)$ between terminals v, w has a non-negative weight $w_e \geq 0$ representing the level of pair-wise interference. We assume that the service quality of a base station j is proportional to the total interference $w(j)$, which is defined as $w(j) = \sum_{(v,w) \in E: s_v = s_w = j} w_{(v,w)}$.

We now obtain an interference game as follows. The nodes of the graph are the players, the set of strategies is given by $S_i = \{\{1\}, \ldots, \{m\}\}, i = 1, \ldots, n$, that is, the set of base stations, and the private cost function for every player is defined as $\pi_i(s) = w(s_i)$, $i = 1, \ldots, n$. Interference games fit into the framework of singleton bottleneck congestion games with m resources.

Note that in interference games, we crucially exploit the property that resource cost functions depend on the *set* of players using the resource, that is, their *identity* determines the resulting cost. The existence of a strong equilibrium in all interference games follows from our main theorem, while most previous game-theoretic works addressing wireless networks only considered Nash equilibria; see for instance Liu and Wu [89] and Etkin et al. [38].

Bottleneck Routing in Networks. A special case of bottleneck congestion games are bottleneck routing games. Here, the set of resources are the edges of a directed or undirected graph $\mathcal{G} = (V, R)$. Every player is associated with a pair of vertices $(u_i, v_i) \in V \times V$ and a fixed demand $d_i > 0$ that she wishes to send along a path in $\mathcal{G}$ connecting u_i to v_i. Every edge $r \in R$ has a cost function c_r depending on the aggregated demand on r. The private cost for every player is the maximum arc cost along the path, which is a common assumption for data routing in computer networks; see [79, 15, 26, 110]. The existence of pure Nash equilibria in bottleneck routing games has been studied before by Banner and Orda [15]. They, however, did not study the existence of strong equilibria. To the best of our knowledge, our main result (Theorem 6.10) is the first to establish that bottleneck routing games have the strong finite improvement property, and, thus, also the finite improvement property. In Banner and Orda [15] it is only proven that best-response dynamics converge.

6.4.1 Existence of Strong Equilibria

We are now ready to state our main result for bottleneck congestion games.

Theorem 6.10. *For every bottleneck congestion game $G = (N, S, \pi)$ the private cost vector π is a strong vector-valued potential.*

Proof. For an arbitrary strong improving move $\big(s, (s'_K, s_{-K})\big)$, let $j \in K$ be a player with highest cost before the strong improving move among all players in the coalition, i.e., $j \in \arg\max_{i \in K} \pi_i(s)$. We set $N^+ = \{i \in -K : \pi_i(s) \geq \pi_j(s)\}$ and claim that $\pi_i(s) \geq \pi_i(s'_K, s_{-K})$ for all $i \in N^+$. For a contradiction, let us suppose that there is $i \in N^+$ such that $\pi_i(s) < \pi_i(s'_K, s_{-K})$. The independence of irrelevant choices and the monotonicity of the cost functions imply that there is a member $k \in K$ of the coalition and a resource $r \in s'_k \cap s_i \neq \emptyset$ with $c_r(s'_K, s_{-K}) = \pi_i(s'_K, s_{-K})$. As $r \in s_k$, we derive that

$\pi_k(s'_K, s_{-K}) \geq \pi_i(s'_K, s_{-K})$. We calculate

$$\pi_j(s) \geq \pi_k(s) > \pi_k(s'_K, s_{-K}) \geq \pi_i(s'_K, s_{-K}) > \pi_i(s),$$

where the first inequality stems from the definition of j and the second inequality is due to the fact that player k improves. We obtain $\pi_j(s) > \pi_i(s)$, which contradicts $i \in N^+$.

Next, we set $N^- = \{i \in -K : \pi_i(s) < \pi_j(s)\}$ and claim that $\pi_i(s'_K, s_{-K}) < \pi_j(s)$ for all $i \in N^-$. For a contradiction, let us suppose that there is $i \in N^-$ such that $\pi_i(s'_K, s_{-K}) \geq \pi_j(s)$. Because $\pi_j(s) \geq \pi_i(s)$, the independence of irrelevant choices and the monotonicity of the cost functions, there is a member $k \in S$ of the coalition with $s'_k \cap s_i \neq \emptyset$ and $\pi_k(s'_K, s_{-K}) \geq \pi_i(s'_K, s_{-K})$. With the same arguments as above we obtain

$$\pi_j(s) \geq \pi_k(s) > \pi_k(s'_K, s_{-K}) \geq \pi_i(s'_K, s_{-K}) \geq \pi_j(s),$$

a contradiction. Note that $N = N^+ \cup N^- \cup K$ and that we have shown $\pi_i(s) \geq \pi_i(s'_K, s_{-K})$ for all $i \in N^+$ and $\pi_i(s'_K, s_{-K}) < \pi_j(s)$ for all $i \in N^-$. This means that the private cost of the players with costs larger than $\pi_j(s)$ does not increase, the private cost of player j strictly decreases, and the private costs of all other players may only increase up to a value strictly smaller than $\pi_j(s)$. Thus, we have $\pi(s) \succ \pi(s'_K, s_{-K})$ as claimed. □

As a corollary of Theorem 6.10 we obtain that bottleneck congestion games possess a strong equilibrium with the efficiency and fairness properties shown in Section 6.3. Note that our existence result holds for *arbitrary* strategy spaces. This contrasts a result of Holzman and Law-Yone [72] who have shown that, for standard congestion games (with sum-objective), a certain combinatorial property of the players' strategy spaces (called *good configuration*) is necessary and sufficient for the existence of a strong equilibrium.

In bottleneck congestion games, the vector-valued potential function need not be unique. In fact, one can prove with similar arguments as in the proof of Theorem 6.10 that the function $\psi : S \to \mathbb{R}^{m \cdot n}_{\geq 0}$ defined as $\psi_{i,r}(s) = c_r(s)$ if $r \in s_i$, and $\psi_{i,r} = 0$ otherwise, decreases lexicographically along any improvement path. Moreover, if cost functions are strictly monotonic, one can show along the same lines that also the function $\upsilon : S \to \mathbb{R}^m_{\geq 0}$ defined as $\upsilon(s) = (c_r(s))_{r \in R}$ has this property. Interestingly, the sorted lexicographical minima of the functions π, ψ, and υ need not coincide, as illustrated in the following example.

Example 6.11. Consider the symmetric bottleneck routing game with two players $N = \{1,2\}$ depicted in Figure 6.3. Edges correspond to resources; the cost of each edge depends only on the number of players using it and is given explicitly for the two possible values. The strategy set S_i of each player $i \in N$ comprises all paths from u to v, that are $P_1 = \{(u,w_1),(w_1,v)\}, P_2 = \{(u,w_2),(w_2,w_3),(w_3,w_4),(w_4,v)\}$ and $P_3 = \{(u,w_2),(w_2,w_1),(w_1,v)\}$. There are three types of strong equilibria. In the first type, one player plays P_1 and the other player plays P_2. Here, the player on P_1 experiences a cost of 0 while the player on P_2 experiences a cost of 1. It is easy to see, that (up to permutation of the two players) this strategy profile is the unique sorted lexicographical minimum of π. In the second type, one player chooses P_1 while the other player chooses P_3. Here, both players experience a cost of 1, thus this strong equilibrium is not strictly Pareto efficient. It is easy to see that

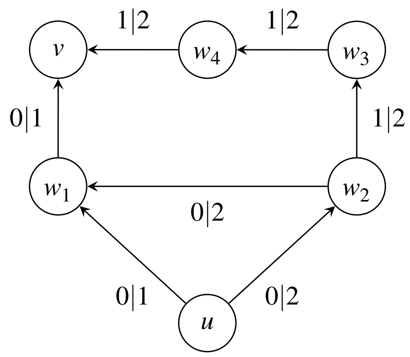


Figure 6.3: Bottleneck routing game with multiple strong equilibria.

this equilibrium minimizes sorted lexicographically both ψ and ν. There is a third strong equilibrium where both players choose P_1. This profile minimizes none of the functions π, ψ, and υ. These different strong equilibria also have different efficiency properties. While the sorted lexicographical minimum s^{π} of π is strictly Pareto efficient and min-max fair (as show in Theorems 6.3 and 6.4), the lexicographical minimum s^{υ} of υ has the property that it is strictly Pareto efficient with respect to using the resources, i.e., there is no strategy profile $s' \in S$ such that $c_r(s') \leq c_r(s^{\upsilon})$ for all $r \in R$ where at least one of these inequalities is strict.

6.5 Infinite Strategic Games

We now consider *infinite* strategic games in which the players' strategy sets are topological spaces and the private cost functions are defined on the product topology. Formally, an infinite game is a tuple $G = (N, S, \pi)$, where $N = \{1, \dots, n\}$ is the set of players, and $S = S_1 \times \cdots \times S_n$ is the set of strategy profiles. For each player i, we assume that her set of strategies S_i is a compact subsets of $\mathbb{R}^{n_i}$ for some $n_i \in \mathbb{N}_{>0}$. The cost function of player i is defined by a non-negative real-valued function $\pi_i : S \to \mathbb{R}_{\geq 0}$, $i \in N$. Turning from finite games to infinite games, it becomes more complicated to characterize structural properties of games with a strong vector-valued potential. First, Proposition 6.2 is no longer valid, that is, infinite games with a strong vector-valued potential need not possess a strong potential.[4] Also the existence of a strong equilibrium does not follow immediately. The global minimum of the function strong vector valued potential ϕ need not exist as the strategy space is not finite. We will show that continuity of ϕ is sufficient for the existence of a strong equilibrium. However, this assumption may be too strong for many classes of games. For instance, the splittable version of bottleneck congestion games (formally defined in Section 6.5.1) has the property that the private cost vector π is a strong vector-valued potential but the function π may be discontinuous in general.

To proof the existence of a strong equilibrium in splittable bottleneck congestion games as well, we slightly generalize the lexicographical improvement property. Let $G = (N, S, \pi)$ be an infinite game and let $\phi : S \to \mathbb{R}^q_{\geq 0} \times \mathbb{R}^q_{\geq 0}$ be a function that associates with each strategy profile s a pair

[4] This observation resembles Debreu's result [32] showing that the lexicographical ordering on an uncountable subset of $\mathbb{R}^2$ cannot be represented by a real-valued function.

$\phi(s) = \big(\phi^{(1)}(s), \phi^{(2)}(s)\big)$. Note that for each $s \in S$ both $\phi^{(1)}(s)$ and $\phi^{(2)}(s)$ are q-dimensional non-negative vectors. For fixed $s \in S$ we want to sort the q entries $\big(\phi^{(1)}(s), \phi^{(2)}(s)\big)$ with respect to the ordinary lexicographic order on $\mathbb{R}^2_{\geq 0}$ (not involving any sorting of the entries), which we denote by $\trianglelefteq$. Formally, for two indices $i, j \in \{1, \ldots, q\}$ and $s \in S$, let $\phi_i(s) \trianglelefteq \phi_j(s)$ if and only if $\phi_i^{(1)}(s) < \phi_j^{(1)}(s)$ or $\phi_i^{(1)}(s) = \phi_j^{(1)}(s)$ and $\phi_i^{(2)}(s) \leq \phi_j^{(2)}(s)$. Let $\phi_i(s) \triangleleft \phi_j(s)$ if and only if $\phi_i(s) \trianglelefteq \phi_j(s)$ and $\phi_i(s) \neq \phi_j(s)$. Moreover, let $\preceq$ denote the sorted lexicographical order, where $\phi_i(s)$ is sorted according to $\trianglelefteq$ in descending order. Then, we say that ϕ is a *pairwise strong vector-valued potential* if $\phi(s') \prec \phi(s)$ for all strong improving moves (s, s'). G has the *pairwise lexicographical improvement property* if it admits a pairwise strong vector-valued potential.

Every game with a strong vector-valued potential has a pairwise strong vector-valued potential as well, since we may simply set the second component of the pairwise strong vector-valued potential equal to the first component (or, alternatively, equal to zero). We show below that every game with a continuous pairwise strong vector-valued potential admits a strong equilibrium.

Theorem 6.12. *Every infinite game G with a continuous pairwise strong vector-valued potential ϕ possesses a strong equilibrium.*

Proof. To get the desired result, we show by induction over $q \in \mathbb{N}$ that for each $q \in \mathbb{N}$, each compact $S \neq \emptyset$ and each continuous function $\phi : S \to \mathbb{R}^q_{\geq 0} \times \mathbb{R}^q_{\geq 0}$ there is a strategy profile $s_{\min} \in S$ with $\phi(s_{\min}) \preceq \phi(s)$ for all $s \in S$.

For the base case $q = 1$, let $T = \{s \in S : \phi^{(1)}(s) = \min_{s \in S} \phi^{(1)}(s)\}$ be the subset of those $s \in S$ for which the first component $\phi^{(1)}$ is minimized. Note that T is nonempty and compact as ϕ is continuous and S is compact. Next, let $T' = \{s \in T : \phi^{(2)}(s) = \min_{s \in T} \phi^{(2)}(s)\}$. With the same arguments, $T' \neq \emptyset$ and by construction, T' contains all vectors that minimize ϕ.

For the inductive step, fix $q \geq 2$ and suppose that the statement holds for all continuous functions $\phi' : S' \to \mathbb{R}^{q'}_{\geq 0} \times \mathbb{R}^{q'}_{\geq 0}$ with $q' \leq q - 1$. We consider an arbitrary compact S and an arbitrary continuous function $\phi : S \to \mathbb{R}^q_{\geq 0} \times \mathbb{R}^q_{\geq 0}$. To construct a lexicographical minimum of ϕ, we set $Q = \{1, \ldots, q\}$ and solve the minimization problem

$$\min_{s \in S} \max_{i \in Q} \phi_i^{(1)}(s) \tag{6.2}$$

of minimizing the maximum value within the first component of ϕ. Let μ be the optimal value of (6.2). For arbitrary $\emptyset \neq J \subseteq Q$, we set

$$T^J = \{s \in S : \phi_i^{(1)}(s) \leq \mu \;\forall i \in Q \setminus J,\, \phi_j^{(1)}(s) = \mu \;\forall j \in J\}.$$

Then, we define $\mathcal{J} = \{J \subseteq Q : J \neq \emptyset,\, T^J \neq \emptyset\}$. Note that because ϕ is continuous and S is compact, the optimal value of (6.2) is attained, and thus $\mathcal{J}$ is nonempty. For each $J \in \mathcal{J}$, we solve the minimization problem

$$\mu^J = \min_{s \in T^J} \max_{j \in J} \phi_j^{(2)}(s). \tag{6.3}$$

For each $J \in \mathcal{J}$ and $j \in J$, we set

$$Y^{J,j} = \{s \in T^J : \phi_i^{(2)}(s) \leq \mu^J \;\forall i \in J \setminus \{j\},\, \phi_j^{(2)}(s) = \mu^J\}.$$

We define $\mathcal{J}' = \{(J,j) \in \mathcal{J} \times Q : j \in J, T^{J,j} \neq \emptyset\}$. Again, $\mathcal{J}'$ is nonempty as ϕ is continuous and S is compact. For each pair $(J,j) \in \mathcal{J}'$, we consider the function $\phi^{J,j} : Y^{J,j} \to \mathbb{R}^{q-1}_{\geq 0}$ that arises from ϕ by deleting the j-th index, i.e., $\phi_i^{J,j}(t) = (\phi_i^{(1)}(t), \phi_i^{(2)}(t))$ for all $i < j$ and $\phi_i^{J,j}(t) = (\phi_{i+1}^{(1)}(t), \phi_{i+1}^{(2)}(t))$ for all $i \in \{j,\dots,q-1\}$. For all $(J,j) \in \mathcal{J}'$, the function $\phi^{J,j}$ is continuous and its domain $T^{J,j}$ is compact and nonempty. For each $(J,j) \in \mathcal{J}'$, we apply the induction hypothesis and obtain $|\mathcal{J}'|$ vectors $t_{\min}^{J,j}$ minimizing $\phi^{J,j}$ on $T^{J,j}$. We claim that the sorted lexicographically minimal vector among the vectors $\big((\mu,\mu^J), \phi^{J,j}(t_{\min}^{J,j})\big) \in \mathbb{R}^q_{\geq 0} \times \mathbb{R}^q_{\geq 0}$ for each pair $(J,j) \in \mathcal{J}'$ is also a sorted lexicographical minimum of the original function ϕ on S. For a contradiction, suppose that there is a vector $z \in S$ with $\phi(z) \prec \big((\mu,\mu^J), \phi^{J,j}(t_{\min}^{J,j})\big)$ for all $(J,j) \in \mathcal{J}'$. First, we observe that there is a set $\emptyset \neq J^* \subseteq Q$ such that $\phi_i^{(1)}(z) = \mu$ for all $i \in J^*$ and $\phi_i^{(1)}(z) < \mu$ for all $i \in Q \setminus J^*$ as otherwise we obtain a contradiction to the fact that μ is the optimal value of (6.2). This implies in particular that $J^* \in \mathcal{J}$. Because μ^{J^*} is the optimal value of (6.3), for at least one index $j^* \in J^*$, we have $\phi_{j^*}^{(2)}(z) = \mu^{J^*}$. This fact together with the induction hypothesis that $t_{\min}^{J^*,j^*}$ minimizes ϕ among the vectors with $\phi_i^{(1)}(z) = \mu$ for all $i \in J^*$ and $\phi_{j^*}^{(2)} = \mu^{J^*}$ leads to a contradiction. □

6.5.1 Splittable Bottleneck Congestion Games

In this section, we introduce the splittable counterpart of bottleneck congestion games. Let N be a finite set of players and R be a finite set of resources. Each player $i \in N$ is associated with a strictly positive demand $d_i \in \mathbb{R}_{>0}$ and a set $\mathcal{A}_i = \{\alpha_{i,1},\dots,\alpha_{i,n_i}\}$ of $n_i \in \mathbb{N}_{>0}$ feasible allocations, where as usual for each $j \in \{1,\dots,n_i\}$ the allocation $\alpha_{i,j}$ is a subset of resources of R. For $i \in N$, we define

$$\Delta_i = \Big\{ \xi_i = (\xi_{i,1},\dots,\xi_{i,n_i}) : \ \xi_{i,k} \geq 0 \ \forall k \in \{1,\dots,n_i\}, \ \sum_{k=1}^{n_i} \xi_{i,k} = d_i \Big\}.$$

For each player i, the set Δ_i can be interpreted as the set of feasible distributions of her demand d_i over her set of feasible allocations $\mathcal{A}_i$. Note that Δ_i is a compact subset of $\mathbb{R}^{n_i}_{\geq 0}$ for all $i \in N$. We set $\Delta = \Delta_1 \times \dots \times \Delta_n$ and for a vector $\xi = (\xi_1,\dots,\xi_n) \in \Delta$, we define $\ell_{i,r}(\xi) = \sum_{k \in \{1,\dots,n_i : r \in \alpha_{i,k}\}} \xi_{i,k}$ as the total demand put on resource r by player i. The set of used resources of player i is defined as $R_i(\xi) = \{r \in R : \ell_{i,r}(\xi) > 0\}$. The tuple $\tilde{\mathcal{M}} = (N,R,(\mathcal{A}_i)_{i\in N},(\tilde{c}_r)_{r\in R})$ is called the corresponding *splittable congestion model with set-dependent costs* if for all $r \in R$ the cost function $\tilde{c}_r : \Delta \to \mathbb{R}_{\geq 0}$ satisfies the assumptions:

Nonnegativity: $\tilde{c}_r(\xi) \geq 0$ for all $\xi \in \Delta$,

Independence of Irrelevant Choices:
$\tilde{c}_r(\xi) = \tilde{c}_r(\xi')$ for all $\xi,\xi' \in \Delta$ with $\ell_{i,r}(\xi) = \ell_{i,r}(\xi')$ for all $i \in N$,

Monotonicity: $\tilde{c}_r(\xi) \leq \tilde{c}_r(\xi')$ for all $\xi,\xi' \in \Delta$ with $\ell_{i,r}(\xi) \leq \ell_{i,r}(\xi')$ for all $i \in N$,

Note that “Monotonicity” implies “Independence of Irrelevant Choices”. We basically impose the same assumptions as in the case of finite bottleneck congestion games.

We say that a cost function c_r is continuous if $\tilde{c}_r(\xi)$ is continuous in ξ.

Definition 6.13 (Splittable bottleneck congestion game with set-dependent costs)
For a splittable congestion model $\tilde{\mathcal{M}} = (N,R,(\mathcal{A}_i)_{i\in N},(\tilde{c}_r)_{r\in R})$, the corresponding *splittable bottleneck congestion game with set-dependent costs* is the infinite minimization game $G = (N,S,\pi)$, where $S_i = \Delta_i$ and $\pi_i(s) = \max_{r \in R_i(s)} \tilde{c}_r(s)$ for all $i \in N$.

The following examples fit into this model.

Bottleneck routing games with splittable demands. The resources correspond to the edges of a directed or undirected graph graph $\mathcal{G} = (V,R)$. Each player i is associated with a source-sink pair $(u_i, v_i) \in V \times V$ and a positive demand d_i that she wishes to route from u_i to v_i. The private cost of each player equals the maximum cost over all resources she uses with positive demand. The fundamental difference to unsplittable bottleneck routing games is that each player i is allowed to distribute her demand among all paths connecting u_i and v_i. Thus, bottleneck routing games with splittable demands serve as a model of multi-path routing protocols in telecommunication networks; see Banner and Orda [15]. Banner and Orda, however, study only existence of pure Nash equilibria. In addition to being more general, our result also gives an alternative proof for the existence of pure Nash equilibrium in bottleneck routing games with splittable demands which is rather concise compared to the involved proof by Banner and Orda [15] and additionally constructive.

Scheduling of malleable jobs. In the scheduling literature jobs are called *malleable* if they can be distributed among multiple machines (see, e.g., Feitelson and Rudolph [43] and Carroll and Grosu [21]). In a scheduling game with malleable jobs, each player i controls a job with weight w_i that she distributes over an arbitrary *subset* of allowable machines. The private cost is determined by the makespan, which is a non-decreasing function of the total load of the machine that finishes latest among the chosen machines. To the best of our knowledge, our work is the first to investigate the existence of equilibria (pure Nash equilibria or strong equilibria) in such games.

6.5.2 Existence of Strong Equilibria

As mentioned earlier, using similar arguments as in the proof of Theorem 6.10 one can prove that for a splittable bottleneck congestion games $G = (N,S,\pi)$ the private cost vector π is a strong vector-valued potential. However, the function π may be discontinuous even if cost functions are continuous. To see this, consider the bottleneck congestion game with one player having access to two resources $\mathcal{A}_1 = \{\{r_1\},\{r_2\}\}$ over which she has to assign a demand of size 1. The resource r_1 has a cost function equal to the aggregated demand of all players using r_1, while resource r_2 has a constant cost function equal to 2. Let $s_{1,2}(\varepsilon) = \varepsilon > 0$ be the demand assigned to resource r_2 and the remaining demand $s_{1,1}(\varepsilon) = 1-\varepsilon$ be assigned to r_1. Then, for *any* $\varepsilon > 0$ we have $\pi\big(s(\varepsilon)\big) = 2$, while $\pi\big(s(0)\big) = 1$.

To resolve this difficulty, we first define the *load* of resource r under strategy profile s as the total demand put on r by all players, that is, $\ell_r(s) = \sum_{i\in N} \ell_{i,r}(s)$. We processed to show that the function $\nu : S \to \mathbb{R}^m_{\geq 0} \times \mathbb{R}^m_{\geq 0}, s \mapsto \big(\tilde{c}_r(s), \ell_r(s)\big)_{r\in R}$ is a continuous pairwise strong vector-valued potential.

Theorem 6.14. *Every splittable bottleneck congestion game G with continuous cost functions possesses a strong equilibrium.*

Proof. We show that the function $\nu : S \to \mathbb{R}^m_{\geq 0} \times \mathbb{R}^m_{\geq 0}, s \mapsto \big(\tilde{c}_r(s), \ell_r(s)\big)_{r\in R}$ is a pairwise strong vector-valued potential. Because ν is continuous, Theorem 6.12 then gives the desired result. Let $K \subseteq N$ be an arbitrary coalition and let $\big(s,(s'_K, s_{-K})\big)$ be a strong improving move of coalition K. Choose a deviating player $j \in \arg\max_{i\in K} \pi_i(s)$ with highest cost before the strong improving move and one of the resources $r' \in \arg\max_{r \in R_j(s)} \tilde{c}_r(s)$ at which $\pi_j(s)$ is attained. Decompose R into R^+ and R^- defined as $R^+ = \{r \in R : \tilde{c}_r(s) \geq \tilde{c}_{r'}(s)\}$ and $R^- = \{r \in R : \tilde{c}_r(s) < \tilde{c}_{r'}(s)\}$.

We first claim that $\tilde{c}_r(s'_K, s_{-K}) \leq \tilde{c}_r(s)$ for all $r \in R^+$. Assume by contradiction that there is $r \in R^+$ with $\tilde{c}_r(s'_K, s_{-K}) > \tilde{c}_r(s)$. The independence of irrelevant choices and the monotonicity of the cost functions imply that there is a player $k \in K$ with $\ell_{k,r}(s'_K, s_{-K}) > 0$. We obtain $\pi_k(s'_K, s_{-K}) \geq \tilde{c}_r(s'_K, s_{-K}) > \tilde{c}_r(s) \geq \tilde{c}_{r'}(s) = \pi_j(s) \geq \pi_k(s)$, which contradicts that k must improve.

Next we show that $\ell_r(s'_K, s_{-K}) \leq \ell_r(s)$ for all resources $r \in R^+$ with $\tilde{c}_r(s'_K, s_{-K}) = \tilde{c}_r(s)$. For a contradiction, assume that there is $r \in R^+$ with $\ell_r(s'_K, s_{-K}) > \ell_r(s)$ and $\tilde{c}_r(s'_K, s_{-K}) = \tilde{c}_r(s)$. Again this implies the existence of a player $k \in K$ with $\ell_{k,r}(s'_K, s_{-K}) > 0$. Using $\tilde{c}_r(s'_K, s_{-K}) = \tilde{c}_r(s)$, we obtain the same contradiction to the fact that k improves as before.

Finally, we claim that $\tilde{c}_r(s'_K, s_{-K}) < \tilde{c}_{r'}(s)$ for all $r \in R^-$. To see this, assume that there is $r \in R^-$ with $\tilde{c}_r(s'_K, s_{-K}) \geq \tilde{c}_{r'}(s)$. This again implies that there is a player $k \in K$ with $\ell_{k,r}(s'_K, s_{-K}) > 0$, thus, $\pi_k(s'_K, s_{-K}) \geq \tilde{c}_r(s'_K, s_{-K}) \geq \tilde{c}_{r'}(s) = \pi_j(s) \geq \pi_k(s)$, and player k did not improve, contradiction!

To complete the proof, we show that $\big(\tilde{c}_{r'}(s'_K, s_{-K}), \ell_{r'}(s'_K, s_{-K})\big) < \big(\tilde{c}_{r'}(s), \ell_{r'}(s)\big)$. We distinguish two cases. If $\ell_{j,r'}(s'_K, s_{-k}) > 0$, we obtain $\tilde{c}_{r'}(s'_K, s_{-K}) < \tilde{c}_{r'}(s)$ using the fact that player j improves. For the second case, let $s'_{j,r'} = 0$ and assume for a contradiction that $\big(\tilde{c}_{r'}(s'_K, s_{-K}), \ell_{r'}(s'_K, s_{-K})\big) \geq \big(\tilde{c}_{r'}(s), \ell_{r'}(s)\big)$. If $\tilde{c}_{r'}(s'_K, s_{-K}) > \tilde{c}_{r'}(s)$, we immediately derive the existence of a player $k \in K$ with $s'_{k,r'} > 0$. On the other hand, if $\tilde{c}_{r'}(s'_K, s_{-K}) = \tilde{c}_{r'}(s)$ and $\ell_{r'}(s'_K, s_{-K}) \geq \ell_{r'}(s)$, we obtain the existence of $k \in K$ with $s'_{k,r'} > 0$ using that $\ell'_{j,r'}(s'_K, s_{-K}) = 0$. In both cases, we calculate $\pi_k(s'_K, s_{-K}) \geq \tilde{c}_{r'}(s'_K, s_{-K}) \geq \tilde{c}_{r'}(s) = \pi_j(s) \geq \pi_k(s)$, a contradiction to the fact that k improves. $\square$

6.5.3 Existence of Approximate Strong Equilibria

We now relax the continuity assumption on the resource cost functions by assuming that they are only bounded from above. We first prove that bottleneck congestion games with bounded cost functions have the approximate strong finite improvement property. That is, for each $\rho > 0$, each sequence of coalitional deviations that increase the private cost of each of member of the coalition by at least ρ is finite.

Theorem 6.15. *Every splittable bottleneck congestion game G with bounded cost functions has the approximate strong finite improvement property.*

Proof. Consider the function $\psi : S \to \mathbb{R}_{\geq 0}^{m \cdot n}$ defined as

$$\psi_{i,r}(s) = \begin{cases} \tilde{c}_r(s), & \text{if } r \in R_i(s) \\ 0, & \text{else} \end{cases} \qquad \text{for all } i \in N, r \in R.$$

Fix $\rho > 0$ arbitrarily. For $\mu \geq \big(\frac{2}{\rho} \cdot \sup_{s \in S, r \in R} \tilde{c}_r(s) + 1\big) \log(n \cdot m)$, define $P_\mu(s) = \sum_{r \in R, i \in N} \psi_{i,r}(s)^\mu$. We claim that P_μ satisfies $P_\mu(s) - P_\mu(s'_K, s_{-K}) \geq (\frac{\rho}{2})^\mu$ for all strong ρ-improving moves $\big(s, (s'_K, s_{-K})\big)$. To see this, let $\big(s, (s'_K, s_{-K})\big)$ be an arbitrary strong ρ-improving move of a coalition $K \subseteq N$. We choose a player $j \in \arg\max_{i \in K} \pi_i(s'_K, s_{-K})$ with highest costs after the strong ρ-improving move among all players in the coalition and define the sets $\Psi^+ = \{(i,r) \in -K \times R : \psi_{i,r}(s) \geq \pi_j(s'_K, s_{-K})\}$

and $\Psi^- = \{(i,r) \in -K \times R : \psi_{i,r}(s) < \pi_j(s'_K, s_{-K})\}$. We continue to prove that

$$\psi_{i,r}(s'_K, s_{-K}) \le \psi_{i,r}(s) \qquad \text{for all } (i,r) \in \Psi^+, \tag{6.4}$$
$$\psi_{i,r}(s'_K, s_{-K}) \le \pi_j(s'_K, s_{-K}) \qquad \text{for all } (i,r) \in \Psi^-. \tag{6.5}$$

To prove (6.4), suppose there is $(i,r') \in \Psi^+$ such that $\psi_{i,r'}(s) < \psi_{i,r'}(s'_K, s_{-K})$. Because of the independence of irrelevant choices and the monotonicity of cost functions there exists $k \in K$ with $r' \in R_k(s'_K, s_{-K})$ implying

$$\pi_j(s'_K, s_{-K}) \le \psi_{i,r'}(s) < \psi_{i,r'}(s'_K, s_{-K}) \le \pi_k(s'_K, s_{-K}) \le \pi_j(s'_K, s_{-K}),$$

which is a contradiction. For proving (6.5), suppose there is $(i,r') \in \Psi^-$ such that $\psi_{i,r'}(s'_K, s_{-K}) > \pi_j(s'_K, s_{-K})$. Again, the independence of irrelevant choices and monotonicity of cost functions imply that there is $k \in K$ with $r' \in R_k(s'_K, s_{-K})$ giving rise to

$$\pi_k(s'_K, s_{-K}) \ge \psi_{i,r'}(s'_K, s_{-K}) > \pi_j(s'_K, s_{-K}) \ge \pi_k(s'_K, s_{-K}),$$

which is a contradiction. To complete the proof, we observe that $N \times R = \Psi^+ \cup \Psi^- \cup (K \times R)$. Then,

$$P_\mu(s) - P_\mu(s'_K, s_{-K}) = \sum_{(i,r) \in N \times R} \psi_{i,r}(s)^\mu - \psi_{i,r}(s'_K, s_{-K})^\mu \ge \sum_{(i,r) \in \Psi^- \cup (K \times R)} \psi_{i,r}(s)^\mu - \psi_{i,r}(s'_K, s_{-K})^\mu.$$

The inequality follows from (6.4). We further derive

$$\begin{aligned} \sum_{(i,r) \in \Psi^- \cup (K \times R)} \psi_{i,r}(s)^\mu - \psi_{i,r}(s'_K, s_{-K})^\mu &\ge \sum_{(i,r) \in (K \times R)} \psi_{i,r}(s)^\mu - \sum_{(i,r) \in \Psi^- \cup (K \times R)} \psi_{i,r}(s'_K, s_{-K})^\mu \\ &\ge \big(\pi_j(s'_K, s_{-K}) + \rho\big)^\mu - n \cdot m \cdot \pi_j(s'_K, s_{-K})^\mu, \end{aligned}$$

where the first inequality follows from the non-negativity of ψ. The second inequality follows from $\pi_j(s) \ge \pi_j(s'_S, s_{-S}) + \rho$ and (6.5). Finally

$$P_\mu(s) - P_\mu(s'_K, s_{-K}) \ge \left(\frac{\rho}{2}\right)^\mu + \left(\pi_j(s'_K, s_{-K}) + \frac{\rho}{2}\right)^\mu - n \cdot m \cdot \pi_j(s'_K, s_{-K})^\mu \ge \left(\frac{\rho}{2}\right)^\mu,$$

where the last inequality follows from the choice of μ. Using that P_μ is bounded on S and that the value of P_μ decreases along any strong ρ-improving move, we conclude that every strong ρ-improvement path is finite. □

Using that every minimization game with the approximate strong finite improvement property has a ρ-approximate strong equilibrium for every $\rho > 0$ (Proposition 2.14), we obtain the following immediate corollary.

Corollary 6.16. *Every splittable bottleneck congestion game with set-dependent and bounded costs possesses a ρ-approximate strong equilibrium for every $\rho > 0$.*

6.6 Discussion and Open Problems

As the main result of this section, we proved that bottleneck congestion games with set-dependent costs possess a strong equilibrium. Set-dependent cost functions are a natural generalization of the load-based models considered in the previous chapters that allows even to model interferences between different players. While our result for bottleneck congestion games with set-dependent costs implies that every singleton congestion game with set-dependent costs admits a pure Nash equilibrium, it is an interesting open problem whether equilibria exist in standard congestion games with set-dependent costs and arbitrary strategy spaces.

We proved the existence of a strong equilibrium in bottleneck congestion games by showing that they admit a strong *vector-valued potential* – a novel potential function concept, that requires that a certain lexicographic order of a vector attached to each strategy profile decreases. An important and fascinating open problem is to find other interesting classes of games for which a strong vector-valued potential exists.

Chapter 7

Computing Equilibria in Unweighted Bottleneck Congestion Games

One of the central challenges in algorithmic game theory is to characterize the computational complexity of equilibria. Results in this direction yield important indicators whether game-theoretic solution concepts are plausible outcomes of competitive environments in practice. Historically, the focus has been put on the computation of mixed equilibria as their existence is granted in each finite game. Lemke and Howson [87] proposed an algorithm that finds a mixed Nash equilibrium in any finite two-player game. However, Savani and von Stengel [119] constructed a class of games for which the runtime of the Lemke-Howson-Algorithm is exponential in the input size of the game. Moreover, the problem of computing a mixed Nash equilibrium in finite games is known to be PPAD-complete, even when payoffs are restricted to be binary; see Abbott et al. [1], Chen and Deng [22], and Daskalakis et al. [30]. As shown by Etessami and Yannakakis [37], for games with three or more players, the problem is FIXP-complete, even for zero-sum games. In contrast to these negative results for mixed equilibria, all computational problems related to pure Nash equilibria are trivial for general two-player bi-matrix games. For a general $|S_1| \times |S_2|$ two-player game, $2 \cdot |S_1| \cdot |S_2|$ numbers are needed to describe the private cost of all players in all strategy profiles. This huge input size trivializes both the decision problem whether a pure Nash equilibrium exists as well as the computation of a pure Nash equilibrium – a trivial algorithm for computing all pure Nash equilibria of such a game simply checks for each strategy profile whether a unilateral deviation exists and outputs it if this is not the case.

There are, however, important classes of games for which the players' private costs can be represented *succinctly*. That is, the players' private costs are given implicitly rather than explicitly for every strategy profile. For such a game, the trivial algorithm sketched above is not efficient as its runtime might be exponential in then input size. Generally speaking, due to their condensed input size, deciding the existence of equilibria tends to be harder in games with succinct representation than in arbitrary games. For graphical games, for instance, deciding the existence of a pure Nash equilibrium is NP-complete, and deciding the existence of a strong equilibrium in pure strategies is even Σ_2^P-complete; see Gottlob et al. [60]. A key challenge of algorithmic game theory is to identify those classes of games for which equilibria can be computed efficiently. One important class of succinctly representable games for which the computation of pure Nash equilibria has been under increased scrutiny is the class of congestion games. For references to further classes of games with a succinct representation, we refer to Papadimitriou [109, §2.5].

For standard congestion games, the complexity of computing exact and approximate pure Nash equilibria is now relatively well-understood. A detailed characterization in terms of, e.g., the structure of strategy spaces (Fabrikant et al. [41], Ackermann et al. [3]) or the cost functions (Chien and Sinclair [24], Bhalgat et al. [18], Skopalik and Vöcking [122]) has been derived. However, as discussed in Chapter 6, standard congestion games (with sum-objective) have shortcomings, especially as models for the prominent application of routing in computer networks. The incentive structure in selfish routing scenarios is captured more realistically by *bottleneck congestion games*, in which the private cost of a player is the maximum (instead of the sum) of the costs of the resources in her strategy.

The central result of Chapter 6 establishes that bottleneck congestion games always admit a strong equilibrium – a strengthening of the pure Nash equilibrium concept that is resilient against coordinated deviations of coalitions of players. The existence of pure Nash equilibria and strong equilibria in bottleneck congestion games raises a variety of important questions regarding their computational complexity. In which cases can pure Nash equilibria and strong equilibria be computed efficiently? As the games have the strong finite improvement property, another important issue is the duration of natural (coalitional) improvement dynamics. More fundamentally, it is not obvious that even a single such coalitional improving move can be found efficiently. These are the main questions that we address in this chapter.

7.1 Contributions and Chapter Outline

We examine the computational complexity of pure Nash equilibria and strong equilibria in unweighted bottleneck congestion games. In such a game, each player chooses a subset of resources available to her. The cost of a resource depends on the cardinality of the set of players using it, and the private cost of each player is defined as the maximum cost among all chosen resources. In Section 7.3 we focus on computing pure Nash equilibria and strong equilibria using centralized algorithms. We provide a generic algorithm that computes a strong equilibrium for any unweighted bottleneck congestion game. The algorithm iteratively decreases capacities on the resources and relies on a *strategy packing oracle*. The oracle decides if a given set of capacities allows to pack a collection of feasible strategies for all players and outputs a feasible packing if one exists. The running time of the algorithm is essentially determined by the running time of this oracle, i.e., the problem of computing a strong equilibrium can be reduced to the strategy packing problem. As a characterization we also prove the reverse direction: the class of set packing problems addressed by strategy packing oracles can be solved efficiently if we can efficiently compute strong equilibria in bottleneck congestion games. As a slight drawback, the games constructed in this reduction exhibit a slightly different combinatorial structure than the packing problem. For the case of two players we can circumvent this problem and show polynomial equivalence between packing and strong equilibrium computation even when we fix the underlying combinatorial structure.

In terms of complexity, we prove a number of upper and lower bounds for specific classes of games. For upper bounds we focus on three classes of games: single-commodity networks, branchings, and matroids. Single-commodity network games represent a natural and frequently studied class of network routing. Branchings model a natural scenario when players strive to implement a broadcast from a set of source nodes to all other nodes in the network. In all three cases, there are strategy packing oracles that can be implemented in polynomial time. Thus, our generic algorithm yields an *efficient algorithm* to compute a strong equilibrium. For general games, however, we show that the problem of computing a strong equilibrium is NP-hard, even in two-commodity networks.

In Section 7.4 we study the duration and complexity of sequential improvement dynamics that converge to pure Nash equilibria and strong equilibria. Note that quick convergence (i.e., in a polynomial number of rounds) implies efficient computation if the improving move can be computed efficiently. Therefore, we focus particularly on these classes of games, for which we found positive results in terms of computation. We first observe that for every matroid bottleneck congestion game a variant of best response dynamics presented in Ackermann et al. [3] called “lazy best response” converges to a pure Nash equilibrium in polynomial time. In contrast to this positive result for unilateral dynamics, we show that it is NP-hard to decide if a coalitional improving move exists, even for matroid and single-commodity network games, and even if the deviating coalition is fixed a priori. This highlights an interesting contrast for these two classes of games: While there are polynomial-time algorithms to compute a strong equilibrium, it is impossible to decide efficiently if a given strategy profile is a strong equilibrium – the decision problem is co-NP-hard.

We conclude in Section 7.5 by outlining some interesting open problems regarding the convergence to approximate equilibria.

Significance. Our work shows an interesting dichotomy for bottleneck congestion games with matroid and single-commodity strategies, respectively. While in both cases there are polynomial centralized algorithms that compute a strong equilibrium, it is co-NP-hard to decide in polynomial time if a given NE is a strong equilibrium. These contrasting results obviously stem from the fact that for the aforementioned classes of games some strong equilibria are easy to compute while others are even hard to recognize. As this property is inherent to the strong equilibria themselves and independent from the utilized algorithms, this suggests to use computational hardness as an *equilibrium selection* tool.

Bibliographic Information. The results presented in this chapter are joint work with Tobias Harks, Martin Hoefer, and Alexander Skopalik. An extended abstract with parts of the results appeared in the *Proceedings of the 18th Annual European Symposium on Algorithms*; see [61]. A more extensive version is accepted for publication in *Mathematical Programming*; see [62]. The latter work also contains more results concerning the hardness of computing a pure Nash equilibrium in bottleneck congestion games. Specifically, it is shown, that that the constructions of Skopalik and Vöcking [122] regarding the hardness of computing a pure Nash equilibrium in ordinary games can be adjusted to yield similar results for bottleneck games. In particular, in *(i)* symmetric games with arbitrary cost functions; and *(ii)* asymmetric games with bounded-jump cost functions computing a pure Nash equilibrium is PLS-complete.

7.2 Problem Description

In contrast to the relatively general class of bottleneck congestion games considered in Chapter 6, in this chapter we restrict ourselves to unweighted bottleneck congestion games. Also unlike in the previous chapters we assume that all cost values are non-negative integers. Formally, let $N = \{1,\ldots,n\}$ be a finite set of players and $R = \{1,\ldots,m\}$ a finite set of resources. Every player i is associated with a set $\mathcal{A}_i \subseteq 2^R$ of feasible allocations, given implicitly by a certain combinatorial property. Each resource r has a cost functions $c_r : \mathbb{N} \to \mathbb{N}$. We call the tuple $\mathcal{M} = \big(N,R,(\mathcal{A}_i)_{i\in N},(c_r)_{r\in R}\big)$ a congestion model. The corresponding *unweighted bottleneck congestion game* is the minimization game $G = (N,S,\pi)$ with $S_i = \mathcal{A}_i$ and $\pi_i(s) = \max_{r\in s_i} c_r\big(\ell_r(s)\big)$, where the load is defined as $\ell_r(s) = |\{j \in N : r \in s_j\}|$ for all $i \in N$. Note that due to our previous assumptions an unweighted bottleneck congestion game can be described by specifying the combinatorial properties of the players' strategy spaces and the $|R|\cdot|N|$ possible cost values of the resources. Throughout this chapter, we call unweighted bottleneck games congestion simply *bottleneck congestion games*.

7.3 Computing Strong Equilibria

In this section, we investigate the complexity of computing a strong equilibrium in bottleneck congestion games. We first present a generic algorithm that computes a strong equilibrium for an arbitrary bottleneck congestion game. It uses an oracle that solves a strategy packing problem (see Definition 7.1), which we term *strategy packing oracle*. For games in which the strategy packing oracle can be implemented in polynomial time, we obtain an efficient algorithm computing a strong equilibrium. We then examine games for which this is the case. On the other hand, we prove that in general computing a strong equilibrium is NP-hard, even for two-commodity bottleneck congestion games.

7.3.1 The Dual Greedy

The general approach of our algorithm is to introduce capacities τ_r on each resource r. The idea is to iteratively reduce capacities of costly resources as long as the residual capacities admit a feasible strategy packing; see Definition 7.1 below.

Definition 7.1 (Strategy packing oracle)
INPUT: Finite set of resources R with capacities $(\tau_r)_{r\in R}$, and n sets $S_1,\ldots,S_n \subseteq 2^R$ of subsets of R, given implicitly by a certain combinatorial property.
OUTPUT: Sets $s_1 \in S_1,\ldots,s_n \in S_n$ such that $|\{i \in \{1,\ldots,n\} : r \in s_i\}| \leq \tau_r$ for all $r \in R$, or the information that no such sets exist.

More specifically, when the algorithm starts, no strategy has been assigned to any player and each resource can be used by n players, thus, $\tau_r = n$. If r is used by n players, its cost equals $c_r(n)$. The algorithm now iteratively reduces the maximum resource cost by picking a resource r' that maximizes $c_r(\tau_r)$ among all resources with $\tau_r > 0$. The number of players allowed on r' is reduced by one, and the strategy packing oracle checks if there is a feasible strategy profile obeying the capacity constraints. If the strategy packing oracle outputs such a feasible strategy profile s, the

Algorithm 2: Dual Greedy

Input: Bottleneck congestion game G to the model $\mathcal{M} = \big(N, R, (\mathcal{A}_i)_{i\in N}, (c_r)_{r\in R}\big)$, pointer to corresponding strategy packing oracle $\mathfrak{O}$
Output: Strong equilibrium of G

1 **set** $N' = N$, $\tau_r = n$, $\sigma_r = 0$ for all $r \in R$, **and** $s' = \mathfrak{O}(R, S_{N'}, \tau_r)$;
2 **while** $\{r \in R : \tau_r > 0\} \neq \emptyset$ **do**
3   **choose** $r' \in \arg\max_{r\in R:\tau_r>0}\{c_r(\tau_r + \sigma_r)\}$;
4     $\tau_{r'} := \tau_{r'} - 1$;
5   **if** $\mathfrak{O}(R, S_{N'}, \tau_r) = \emptyset$ **then**
6         $\tau_{r'} := \tau_{r'} + 1$;
7   **foreach** $j \in N'$ **with** $r' \in s'_j$ **do**
8             $s_j := s'_j$;
9   **set** $\sigma_r := \sigma_r + 1$, $\tau_r := \tau_r - 1$ for all $r \in s'_j$;
10             $N' := N' \setminus \{j\}$;
11   **end**
12   **end**
13     $s' = \mathfrak{O}(R, S_{N'}, \tau_r)$;
14 **end**
15 **return** s ;

algorithm reiterates by choosing a (possibly different) resource that has currently maximum cost. If the strategy packing oracle returns $\emptyset$ after the capacity of some $r' \in R$ was reduced to $\tau_{r'} - 1$, we fix the strategies of those $\tau_{r'}$ many players that used r' in the strategy profile the strategy packing oracle computed in the previous iteration and decrease the bounds τ_r of all resources used in the strategies accordingly. This ensures that r' is *frozen*, i.e., there is no residual capacity on r' for allocating this resource in future iterations of the algorithm. The algorithm terminates after at most $n \cdot m$ calls of the oracle. For a formal description of the algorithm see Algorithm 2.

Theorem 7.2. *Dual Greedy computes a strong equilibrium.*

Proof. Let s denote the output of the algorithm. In addition, we denote by N_p, $p = 1\dots,P$, the sets of players whose strategies are determined after the strategy packing oracle $\mathfrak{O}$ returned $\emptyset$ for the p-th time. Clearly, $\pi_i(s) \leq \pi_j(s)$ for all $i \in N_p$, $j \in N_q$, with $p \geq q$. We show by induction over p that the players in $N_1 \cup \dots \cup N_p$ will not participate in any strong improving move of any coalition.

We start with the base case $p = 1$. Let $(\tau_r)_{r\in R}$ be the vector of capacities in the algorithm after the strategy packing oracle returned $\emptyset$ in line 5 for the first time and $\tau_{r'}$ is updated in line 6. Suppose there is a coalition $K \subseteq N$ with $K \cap N_1 \neq \emptyset$ that deviates profitably from s to $t = (s'_K, s_{-K})$. We distinguish two cases.

First case: $\ell_r(t) \leq \tau_r$ for all $r \in R$. Let $\tilde{\tau}_r = \tau_r - 1$ if $r = r'$, and $\tilde{\tau}_r = \tau_r$, else. Since $\mathfrak{O}(R, S, \tilde{\tau}) = \emptyset$, at least $|N_1|$ players use r' in t. Using $c_{r'}\big(\ell_r(t)\big) \geq c_r\big(\ell_r(s)\big)$ for all $r \in R$, we obtain a contradiction to the fact that every member of K must strictly improve.

Second case: There is $\tilde{r} \in R$ such that $\ell_{\tilde{r}}(t) > \tau_{\tilde{r}}$. Using that Dual Greedy iteratively reduces the capacity of those resources with maximum cost (line 3), we derive that $c_{\tilde{r}}(\ell_{\tilde{r}}(t)) \geq c_r\big(\ell_r(s)\big)$ for all

$r \in R$. Using $\ell_{\tilde{r}}(t) > \tau_{\tilde{r}}$ and $\ell_r(s) \leq \tau_r$ for all $r \in R$, there is at least one player $i \in K$ with $\tilde{r} \in s_i'$, hence, this player does not strictly improve.

For the induction step $p \to p+1$, suppose the players in $N_1 \cup \dots \cup N_p$ stick to their strategies and consider the players in N_{p+1}. As the strategies of the players in $N_1 \cup \dots \cup N_p$ are fixed, the same arguments as above imply that no subset of N_{p+1} will participate in a profitable deviation from s. $\square$

It is worth noting that the dual greedy algorithm applies to arbitrary strategy spaces. If the strategy packing problem can be solved in polynomial time, this algorithm computes a strong equilibrium in polynomial time.

Corollary 7.3. *For bottleneck congestion games in which the strategy packing problem is solvable in polynomial time, Dual Greedy computes a strong equilibrium in polynomial time.*

While the problem of computing a strong equilibrium is polynomial-time reducible to the strategy packing problem, for general bottleneck congestion games the converse is also true.

Theorem 7.4. *The strategy packing problem is polynomial-time reducible to the problem of computing a strong equilibrium in a bottleneck congestion game.*

Proof. Given an instance of the strategy packing problem Π we construct a bottleneck congestion game G_Π. Let Π be given as set of resources R with capacities $(\tau_r)_{r\in R}$, and n sets $S_1,\dots,S_n \subseteq 2^R$ of subsets of R. The game G_Π consists of the resources $R \cup \{r_1,\dots,r_n\}$ and the players $1,\dots,n+1$. The set of strategies of player $i \in \{1,\dots,n\}$ is $\{s_i \cup \{r_i\} : s_i \in S_i\}$. Player $n+1$ has the strategies R and $\{r_1,\dots,r_n\}$. For each resource $r \in R$ the cost is 0 if used by at most $\tau_r + 1$ players, and 2 otherwise. For each resource $r \in \{r_1,\dots,r_n\}$ the cost is 0 if used by at most one player and 1 otherwise.

If a strategy profile of players $1,\dots,n$ violates a capacity τ_r on a resource $r \in R$, player $n+1$ has cost of 2 if she plays strategy R. If she plays $\{r_1,\dots,r_n\}$, she and all other players have cost of 1. We conclude that if there is a feasible strategy packing, every strong equilibrium of the game yields cost 0 for every player. Otherwise, every strong equilibrium yields cost 1 for every player. Therefore, the strategy profile of the players $1,\dots,n$ in a strong equilibrium of G_Π corresponds to a solution for the strategy packing problem Π, if such a solution exists. On the other hand, if there is no solution for Π, every player in every strong equilibrium in G_Π has cost of 1. $\square$

Note that while the previous theorem establishes a general reduction, the game G_Π constructed from the instance Π of the packing problem has a different combinatorial structure than Π. More precisely, G_Π is based on a larger set of resources and different strategy sets than the ones used in Π. The next theorem shows that for games with two players, we can obtain a stronger equivalence result without changing the underlying combinatorial structure. It remains an open problem to extend this to games with an arbitrary number of players and more general packing problems.

Theorem 7.5. *Let R be a finite set and $S_1, S_2 \subseteq 2^R$ two sets of subsets of R. Then the following two problems are polynomially equivalent:*

(1) *For an arbitrary set of non-decreasing cost functions $(c_r)_{r\in R}$ compute a strong equilibrium of a bottleneck congestion game G to the congestion model $\mathcal{M} = (\{1,2\}, R, (S_i)_{i\in N}, (c_r)_{r\in R})$.*

(2) *For an arbitrary vector $(\tau_r)_{r\in R} \in \{1,2\}^m$ of capacities compute $s_1 \in S_1, s_2 \in S_2$ such that $|i \in \{1,2\} : r \in s_i\}| \leq \tau_r$ or decide that no such strategies exist.*

Proof. (2) $\Rightarrow$ (1): The dual greedy algorithm computes a strong equilibrium calling the strategy packing oracle polynomially often. Thus, the first problem is polynomially reducible to the second one.

(1) $\Rightarrow$ (2): Suppose we are given an instance $\big(R, S, (\tau_r)_{r\in R}\big)$ of the second problem. We consider the congestion model $\mathcal{M} = \big(\{1,2\}, R, S, (c_r)_{r\in R}\big)$ where c_r is defined as

$$c_r(\ell) = \begin{cases} 0, & \text{if } \ell \leq \tau_r \\ 1, & \text{otherwise.} \end{cases}$$

Now, let G be a corresponding bottleneck congestion game and let s^* be a strong equilibrium of G. We claim that $\pi_1(s^*) = \pi_2(s^*) = 0$ and s^* is a solution of the strategy packing problem if such a solution exists, and $\pi_1(s^*) = \pi_2(s^*) = 1$ otherwise. At first, note that $\tau_r \in \{1,2\}$, and therefore a player gets a cost of 1 if and only if there is $r \in s_1^* \cap s_2^*$ with $\tau_r = 1$. In this case, however, both players have a cost of 1. Therefore, we have either $\pi_1(s^*) = \pi_2(s^*) = 1$ or $\pi_1(s^*) = \pi_2(s^*) = 0$.

Suppose that $\pi_1(s^*) = \pi_2(s^*) = 1$ and assume for a contradiction that there is y solution $s' = (s'_1, s'_2)$ to the strategy packing problem. Then, by the definition of c_r we get that $\pi_1(s') = \pi_2(s') = 0$ and thus, the deviation from s^* to s' is profitable both for player 1 and 2. This is a contradiction to the fact that s^* is a strong equilibrium. Hence, no such strategy profile s' exists. On the other hand, $\pi_1(s^*) = \pi_2(s^*) = 0$ only holds if the strategies s_1^* and s_2^* obey the capacities on each resource. □

7.3.2 Complexity of Strategy Packing

In the previous section, we have characterized the computation of strong equilibria in terms of a set packing problem. In this section, we examine the computational complexity of strategy packing and strong equilibrium computation. In particular, we consider three classes of games, in which strategy packing can be done efficiently. For the general case, we show that computation becomes NP-hard. A more detailed characterization as to which structural properties are crucial for efficient strategy packing or hardness is an interesting avenue for future work.

Our first result is for matroid games, which represent a natural extension of singleton games.[1] In a singleton game we have $|s_i| = 1$ for every strategy $s_i \in S_i$ of every player i. In such games, strong equilibria are exactly the pure Nash equilibria and computation of a strong equilibrium can trivially be done in polynomial time. Also, strategy packing reduces to finding a maximal matching in a bipartite graph.[2] For matroid games, we have to resort to more advanced algorithmic techniques using matroid unions. This concept has been introduced by Nash-Williams [103] and Edmonds [34].

Definition 7.6 (Matroid union)
Let $k \in \mathbb{N}_{>0}$ and let $M_1 = (R_1, \mathcal{I}_1), \ldots, M_k = (R_k, \mathcal{I}_k)$ be k matroids. The *matroid union* $M_1 \vee \cdots \vee M_k$ of these matroids is defined as $M_1 \vee \cdots \vee M_k = (R_1 \cup \cdots \cup R_k, \mathcal{I}_1 \vee \cdots \vee \mathcal{I}_k)$ where

$$\mathcal{I}_1 \vee \cdots \vee \mathcal{I}_k = \{I_1 \cup \cdots \cup I_k : I_1 \in \mathcal{I}_1, \ldots, I_k \in \mathcal{I}_k\}.$$

[1] Readers not familiar with matroids may consider consulting the short introduction to matroids in Section 2.3.6 first.

[2] We construct the graph as follows: the first partition contains a node for each player, the second partition contains τ_r nodes for each $r \in R$. The node of player i is connected to all nodes r for which $\{r\} \in S_i$.

Nash-Williams proved that for k matroids $M_1 = (R_1, \mathcal{I}_1), \dots, M_k = (R_k, \mathcal{I}_k)$ their matroid union $M_1 \vee \cdots \vee M_k$ is a matroid again. The maximum cardinality of an independent set in $\mathcal{I}_1 \vee \cdots \vee \mathcal{I}_k$ equals the maximum cardinality of a common independent set of two suitably constructed matroids. This observation reduces the problem of finding a maximum-size set in $\mathcal{I}_1 \vee \cdots \vee \mathcal{I}_k$ to the intersection problem of two matroids, which can be solved in polynomial time; see Cunningham [28].

Theorem 7.7. *The strategy packing problem can be solved in polynomial time for matroid bottleneck congestion games where the strategy set of player i equals the set of bases of a matroid $M_i = (R, \mathcal{I}_i)$ given by a polynomial independence oracle.*

Proof. For each matroid $M_i = (R, \mathcal{I}_i)$, we construct a matroid $M_i' = (R', \mathcal{I}_i')$ as follows. For each resource $r \in R$, we introduce τ_r resources $r^1, \dots, r^{\tau_r}$ to R'. We say that r is the *representative* of $r^1, \dots, r^{\tau_r}$. Then, a set $I' \subset R'$ is independent in M_i' if the set I that arises from I' by replacing resources by their representatives is independent in M_i. This construction gives rise to a polynomial independence oracle for M_i'.

Now, we consider the matroid union $M' = M_1' \vee \cdots \vee M_n'$, which is again a matroid. Using the algorithm proposed by Cunningham [28] we can compute a maximum-size set $\mathcal{B}$ in $I_1' \vee \cdots \vee I_n'$ in time polynomial in n, m, $rk(\mathcal{M})$, and the maximum complexity of the n independence oracles.

If $|\mathcal{B}| < \sum_{i \in N} \mathrm{rk}(M_i)$, there is no feasible packing of the bases of $M_1, \dots, M_n$. If, in contrast, $|\mathcal{B}| = \sum_{i \in N} \mathrm{rk}(M_i)$, we obtain the corresponding strategies $(s_1, \dots, s_n)$ using Cunningham's algorithm. □

Let us now consider the case where the strategy spaces are *a-arborescences*, which are in general not matroids. Let $\mathcal{D} = (V, R)$ be a directed graph with $|R| = m$. For a distinguished node $a \in V$, we define an *a-arborescence* as a directed spanning tree, where a has in-degree zero and every other vertex has in-degree one. In this case, we can regard $a \in V$ as a common source, and each player strives to make a broadcast with source a by allocating a tree.

Theorem 7.8. *The strategy packing problem can be solved in time $\mathcal{O}(m^2 n^4)$ for a-arborescence games in which the set of strategies of each player equals the set of a-arborescences in a directed graph $\mathcal{D} = (V, R)$.*

Proof. The problem of finding k disjoint a-arborescences in $\mathcal{D}$ can be solved in polynomial time of order $\mathcal{O}(m^2 k^2)$; see Gabow [53, Theorem 3.1]. Introducing τ_r copies for each edge $r \in R$, the problem of finding admissible strategies in the original problem is equivalent to finding n disjoint a-arborescences. As the modified graph has $m \cdot n$ edges, the total running time of Gabow's algorithm is $\mathcal{O}(m^2 n^4)$. □

Recently, the polynomial packing algorithm for a-arborescences has been extended to branchings. Formally, for each player i we are given a root set $R_i \subseteq V$ and a convex[3] set $U_i \subseteq V$ with $R_i \subseteq U_i$. For any vector of capacities $(\tau_r)_{r \in R}$, the polynomial algorithm of Bérczi and Frank [17] computes for every player a branching which is rooted in R_i and spans U_i, that is, the in-degree of every vertex $v \in R_i$ is zero and the in-degree of every vertex $v \in U_i \setminus R_i$ is one, such that the capacity restriction of every edge is satisfied. This more general framework allows to model situations in

[3] In this context, a subset of vertices $U \subseteq V$ is called *convex* if there is no vertex $v \in V \setminus U$ such that there is both a directed path from v to some vertex $u \in U$ and a directed path from some node $u' \in U$ to v.

which the players wish to broadcast from multiple broadcasting stations and where the broadcasts need not cover all vertices. It is worth mentioning that the convexity of U_i is necessary for efficient computation because otherwise, the corresponding decision problem turns out to be NP-complete.

When we turn to single-commodity networks, efficient computation of a strong equilibrium is possible using well-known flow algorithms to implement the oracle. For more general cases with two commodities, however, a variety of problems concerning strong equilibria become NP-hard by a simple construction.

Theorem 7.9. *The strategy packing problem can be solved in time $\mathcal{O}(m^3)$ for single-commodity network bottleneck congestion games.*

Proof. Assigning a capacity of τ_r to each edge and using the algorithm of Edmonds and Karp we obtain a maximum flow within $\mathcal{O}(m^3)$. Clearly, if the value of the flow is smaller than n, no admissible strategies exist and we can return $\emptyset$. If the flow is n or larger we can decompose it in at least n unit flows and return n of them. □

Theorem 7.10. *In two-commodity network bottleneck congestion games the following problems are strongly* NP*-hard: (i) compute a strong equilibrium; (ii) decide for a given strategy profile whether any coalition has a strong improving move; (iii) decide for a given strategy profile and a given coalition if it has a strong improving move.*

Proof. We reduce from the 2 Directed Arc-Disjoint Paths (2DADP) problem, which is strongly NP-hard; see Fortune et al. [49]. The problem is to decide if for a given directed graph $\mathcal{D} = (V,A)$ and two node pairs (u_1,v_1), (u_2,v_2) there exist two arc-disjoint (u_1,v_1)- and (u_2,v_2)-paths. For the reduction, we define a corresponding two-commodity bottleneck game by introducing non-decreasing cost functions on every arc r by $c_r(x) = 0$ if $x \leq 1$, and $c_r(x) = 1$ otherwise. We associate every commodity with a player. For proving *(i)*, we observe that 2DADP is a Yes-instance if and only if every strong equilibrium provides a payoff of zero to every player. For proving *(ii)* and *(iii)*, we greedily construct a solution in which the strategies for both players are not arc-disjoint and give it as input to the problems. □

7.4 Convergence of Improvement Dynamics

In the previous section, we have outlined some prominent classes of games for which a strong equilibrium can be computed in polynomial time, most notably single-commodity network bottleneck congestion games and matroid bottleneck congestion games.

Theorem 6.10 together with Proposition 6.2 from the last chapter implies that improvement dynamics converge to pure Nash equilibria and strong equilibria. In this section, we consider the duration of such improvement dynamics in these games. We focus on those classes of games for which we have shown efficient computation, i.e., matroid and single-commodity network games. For matroid games we show polynomial-time convergence to a pure Nash equilibrium using unilateral improving moves. For the convergence to a strong equilibrium we have to consider strong improving moves, but we show that deciding if such a move exists is NP-hard even in matroid games or single-commodity network games. This implies that even in these specialized classes of games with efficient computation of a strong equilibrium, recognition of a strategy profile as a strong equilibrium is co-NP-hard.

We first observe that bottleneck congestion games can be transformed into ordinary congestion games while preserving useful properties regarding the convergence to a pure Nash equilibrium. This allows to show fast convergence to a pure Nash equilibrium in matroid bottleneck games and mirrors a prominent result for ordinary matroid games of Ackermann et al. [3].

7.4.1 Convergence to Pure Nash Equilibria

The following lemma establishes a connection between bottleneck and ordinary congestion games. For a bottleneck congestion game G we denote by G^{sum} the ordinary congestion game with the same congestion model as G except that we choose $c'_r(S) = m^{c_r(\cdot)}$ for all $r \in R$.

Lemma 7.11. *Every pure Nash equilibrium of G^{sum} is a pure Nash equilibrium of G.*

Proof. Suppose s is a pure Nash equilibrium of G^{sum} but not of G. Thus, there is a player $i \in N$ and an alternative strategy $s'_i \in S_i$, such that $\max_{r\in s_i} c_r\big(\ell_r(s)\big) > \max_{r\in s'_i} c_r\big(\ell_r(s'_i, s_{-i})\big)$. We define $\bar{c} = \max_{r\in s'_i} c_r(\ell_r(s'_i, s_{-i}))$. This implies $\max_{r\in s_i} c_r(\ell_r(s)) \geq \bar{c}+1$. We observe

$$\sum_{r\in s_i} c'_r\big(\ell_r(s)\big) \geq \max_{r\in s_i} c'_r\big(\ell_r(s)\big) \geq m^{\bar{c}+1} > (m-1)\,m^{\bar{c}} \geq \sum_{r\in s'_i} c'_r\big(\ell_r(s'_i, s_{-i})\big),$$

a contradiction. □

We now analyze the lazy best response dynamics considered for ordinary matroid congestion games presented in Ackermann et al [3]. Note that in matroid games, a player's strategy is a basis of a matroid. A *lazy* best response means that a player only exchanges a minimum number of resources that is needed to arrive at a basis representing a best response strategy (for details see [3]). Our analysis here is quite simple and does not explicitly rely on these details. In particular, we transform the game to an ordinary game as outlined in Lemma 7.11. Then we use the lazy best response dynamics in the ordinary game and the convergence result of [3] as a "black box" with the slight adjustment that we only execute moves yielding a strict improvement in the bottleneck resource of the moving player. This allows to establish the following result.

Theorem 7.12. *Let G be a matroid bottleneck congestion game. Then the lazy best response dynamics converges to a pure Nash equilibrium in at most $n^2 \cdot m \cdot \max_{i\in N} rk(M_i)$ steps.*

Proof. We consider the lazy best response dynamics in the corresponding game G^{sum}. In addition, we suppose that a player accepts a deviation only if his bottleneck value is strictly reduced. This might lead to even earlier termination of the dynamics. Thus, the number of moves is still bounded from above by $n^2 \cdot m \cdot \max_{i\in N} \mathrm{rk}(M_i)$ as shown in Ackermann et al. [3]. □

7.4.2 Convergence to Strong Equilibria

For matroid bottleneck congestion games we have shown above that there are polynomially long sequences of unilateral improving moves to a pure Nash equilibrium from every starting state. While our results obtained in Chapter 6 also establish convergence to a strong equilibrium for every sequence of coalitional improving moves, it may already be hard to find one such move. In fact, we show that even a strong ρ-improving move can be strongly NP-hard to find, for any polynomially encodable ρ, even if strategy spaces have simple matroid structures. This implies that deciding whether a given strategy profile is a ρ-approximate strong equilibrium is strongly co-NP-hard.

Theorem 7.13. *In matroid bottleneck congestion games for every polynomially encodable ρ it is strongly* NP*-hard to decide for a given strategy profile s if there is some coalition $K \subseteq N$ that has a strong ρ-improving move.*

Proof. We reduce from SET PACKING. An instance of SET PACKING is given by a set of elements E, a set $\mathcal{U} \subseteq 2^E$ of subsets of E, and a number k. The goal is to decide if there are k mutually disjoint sets in $\mathcal{U}$. SET PACKING is strongly NP-complete; see Karp [77]. Given an instance of SET PACKING we construct a matroid game G and a strategy profile s such that there is a strong ρ-improving move for some coalition of players K if and only if the instance of SET PACKING has a solution.

The game will include $|N| = 1 + |\mathcal{U}| + |E| + \sum_{U\in\mathcal{U}} |U|$ many players. First, we introduce a master player p_1, which has two possible strategies. She can either pick a *coordination resource* r_{co} or the *trigger resource* r_{tr}. For each set $U \in \mathcal{U}$, there is a *set player* p_U. Player p_U can either choose r_{tr} or a *set resource* r_U. For each set U and each element $e \in U$, there is an *inclusion player* $p_{U,e}$. Player $p_{U,e}$ can either use the set resource r_U or an *element resource* r_e. Finally, for each element e, there is an *element player* p_e that has strategies $\{r_{\text{co}}, r_e\}$ and $\{r_{\text{co}}, r_{\text{ab}}\}$ for some absorbing resource r_{ab}. For an overview, see Table 7.1(a).

Consider the strategy profile s in which each player chooses the first strategy in Table 7.1(a). That is, player p_1 uses r_{co}, all set players use r_{tr}, all inclusion players the corresponding set resources r_U, and all element players the strategies $\{r_{\text{co}}, r_e\}$. The coordination resource r_{co} is a bottleneck for the master player and all element players. The costs are $c_{r_{\text{co}}}(x) = 1$ if $x \leq |E|$, and $c_{r_{\text{co}}}(x) = \rho + 2$ otherwise. The trigger resource has cost $c_{r_{\text{tr}}}(x) = 1$ if $x \leq |\mathcal{U}| - k + 1$, and $c_{r_{\text{tr}}}(x) = \rho + 2$ otherwise. For the set resources r_U the cost is $c_{r_U}(x) = 1$ if $x \leq 1$, and $\rho + 2$ otherwise. Finally, for the element resources the cost is $c_{r_e}(x) = 1$ if $x \leq 1$, and $c_{r_e}(x) = \rho + 2$ otherwise. The cost of the absorbing resource r_{ab} is constantly 1. Note that for all resource cost functions c_r, there is a *threshold value* $\theta_r \in \mathbb{N} \cup \{\infty\}$ such that c_r is of the form $c_r(x) = 1$ if $x < \theta_r$, and $c_r(x) = \rho + 2$ otherwise. On overview of the different threshold values can be found in Table 7.1 (b).

Suppose that the underlying SET PACKING instance is a Yes-instance. Then a strong ρ-improving move is as follows. The master player moves to r_{tr}, the k set players corresponding to a solution choose their set resources, the respective inclusion players move to the element resources, and all element players move to r_{ab}. Both the cost of r_{co} and that of r_{tr} reduce from $\rho + 2$ to 1. Thus, the master player, all set players, and all element players improve their bottleneck by $\rho + 1$. The migrating inclusion players do not interfere with each other on the element resources. Thus, they also improve the cost of their bottleneck resource by $\rho + 1$, and we have constructed a strong ρ-improving move for the coalition of all migrating players, all set players, and all element players.

Suppose that the underlying SET PACKING instance is a No-instance. For a contradiction, let us assume that there is a coalition K that has a strong ρ-improving move $\big(s, (s'_K, s_{-K})\big)$. We distinguish two cases.

(a) Player types

player type		1st strat.	2nd strat.
master player p_1		$\{r_{\text{co}}\}$	$\{r_{\text{tr}}\}$
set player p_U,	$U \in \mathcal{U}$	$\{r_{\text{tr}}\}$	$\{r_U\}$
inclusion player $p_{U,e}$,	$U \in \mathcal{U}, e \in E$	$\{r_U\}$	$\{r_e\}$
element player p_e,	$e \in E$	$\{r_{\text{co}}, r_e\}$	$\{r_{\text{co}}, r_{\text{ab}}\}$

(b) Resource types

resource type		threshold θ
coordination resource r_{co}		$\lvert E\rvert+1$
trigger resource r_{tr}		$\lvert\mathcal{U}\rvert-k+2$
set resource r_U,	$U \in \mathcal{U}$	2
element resource r_e,	$e \in E$	2
absorbing resource r_{ab}		∞

Table 7.1: (a) Player types and (b) resource types used in the hardness proof of Theorem 7.13. All cost functions are of the form $c(x) = 1$ if $x < \theta$, and $c(x) = \rho + 2$ otherwise. Table (b) shows this threshold value.

First case: $p_1 \notin K$. Because the master player stays at the coordination resource r_{co} and all element players use r_{co} in both strategies available to them, they have a private cost of $\rho + 2$ in both their strategies and, thus, will not participate in the deviation. As the set players stay on their set resources no inclusion player has an incentive to switch from her respective set resource to her respective element resource. This finally implies that no set player will move from the coordination resource r_{co}. We derive $K = \emptyset$, a contradiction.

Second case: $p_1 \in K$. A move from the coordination resource r_{co} to the trigger resource r_{tr} is an improvement for player p_1 if and only if at least k set players drop r_{tr}. These players must switch to the corresponding set resources. However, for a set player p_U such a move is an improvement if and only if all inclusion players on r_U drop this resource from their strategy. These inclusion players must switch to the element resources. An inclusion player $p_{U,e}$ improves by such a move if and only if the element player drops the resource and $p_{U,e}$ is the only inclusion player moving to r_e. This implies that the moving set players must correspond to sets that are mutually disjoint. This is a contradiction to the fact that we have a No-instance.

Finally, we can add the absorbing resource r_{ab} to every strategy of the master, set, and inclusion players. In this way, the combinatorial structure of all strategy spaces is the same – a partition matroid M with $\text{rk}(M) = 2$ and partitions of size 1 and 2 – only the mapping to resources is different for each player. □

The previous theorem shows hardness of the problem of finding a suitable coalition and a corresponding strong improving move. Even if we specify the coalition in advance and search only for strategies corresponding to a strong improving move, the problem remains strongly NP-hard.

Corollary 7.14. *In matroid bottleneck congestion games, for every polynomially encodable ρ it is strongly* NP*-hard to decide for a given strategy profile s and a given coalition $K \subseteq N$ if there is a strong ρ-improving move for K.*

Proof. We will show this corollary using the games constructed in the previous proof by fixing the coalition $K = N$. Consider the construction in the previous proof. The coalition that has a strong improving move for a Yes-instance consists of the master player, all set players, all element players and the inclusion players that correspond to the sets of the solution to SET PACKING. However, the inclusion players are only needed to transfer the chain of dependencies to the element players. We can set the strategy space of the inclusion player $p_{U,e}$ to

$$S_{p_{U,e}} = \{r_{\text{hi}}, r_{\text{lo}}\} \times \{r_U, r_e\} = \Big\{\{r_{\text{hi}}, r_U\}, \{r_{\text{hi}}, r_e\}, \{r_{\text{lo}}, r_U\}, \{r_{\text{lo}}, r_e\}\Big\}.$$

for two additional resources r_{hi} and r_{lo} with constant costs $c_{r_{\text{hi}}}(x) = \rho + 2$ and $c_{r_{\text{lo}}}(x) = 0$ for all $x \in \mathbb{N}$. In s we assign the inclusion players to strategies $\{r_{\text{hi}}, r_U\}$. Then a strong improving move for the inclusion players that remain on r_U is to exchange r_{hi} by r_{lo}. Thus, the problem of finding an arbitrary coalition with a strong improving move becomes trivial. However, we strive to obtain a strong improving move for $K = N$, and this must generate improvements for the master player and the set players. Thus, we still must reassign some inclusion players from the resources r_U to the element resources r_e. Here we need to resolve conflicts as before, because otherwise inclusion players end up with a cost of $\rho + 2$ on r_e and do not improve. Following the previous reasoning we have a strong ρ-improving move if and only if the underlying SET PACKING instance is solvable. Finally, by appropriately adding dummy resources, we can again ensure that the combinatorial structure of all strategy spaces is the same partition matroid. □

We can adjust the previous two hardness results on matroid games to hold for single-commodity network bottleneck congestion games as well.

Theorem 7.15. *in single-commodity network bottleneck congestion games, for every polynomially computable $\rho > 0$ the following problems are strongly* NP*-hard: (i) decide for a given strategy profile s if there is some coalition $K \subseteq N$ that has a strong ρ-improving move, (ii) decide for a given strategy profile s if a given coalition $K \subseteq N$ has a strong ρ-improving move.*

Proof. We transform the construction of Theorem 7.13 into a symmetric network bottleneck congestion game; see Figure 7.1 for an example. First, we introduce an edge for each resource r_{co}, r_{tr}, $(r_U)_{U \in \mathcal{U}}$ and $(r_e)_{e \in E}$ with the corresponding cost function as before. Additionally, we identify players with the different player types used in the proof of Theorem 7.13 and their strategies by routing them through a set of gadgets composed of edges which have capacities. The capacities are implemented by cost functions that are 1 up to a certain capacity bound $\bar{\theta}$ and $\rho + 10$ above.

The first gadget is to separate the players into groups. An edge with capacity 1 identifies the master player, an edge with capacity $|\mathcal{U}|$ the set players, an edge with capacity $\sum_{U \in \mathcal{U}} |U|$ the inclusion players, and an edge with capacity $|E|$ the element players. The set and inclusion players are then further divided into their particular identities by edges of capacity 1. The element players route all over r_{co}. In addition, the master player has the alternative to route over r_{co} or r_{tr}. After the players have passed r_{co} they again split into specific element players using edges of capacity 1. One of these player is allowed to route directly to the source v. This is meant to be the master player, but our argument still works if this is not the case.

After the players have routed through the capacitated gadgets, they reach an identification point (indicated by gray nodes in Figure 7.1) and can be identified with a player type. They then decide on a strategy from the previous game by routing over one of two allowed paths. In particular, we can allow the set players to either route over r_{tr} or the corresponding r_U, the inclusion players over r_U or r_e, and the element players over r_e or directly to the sink v.

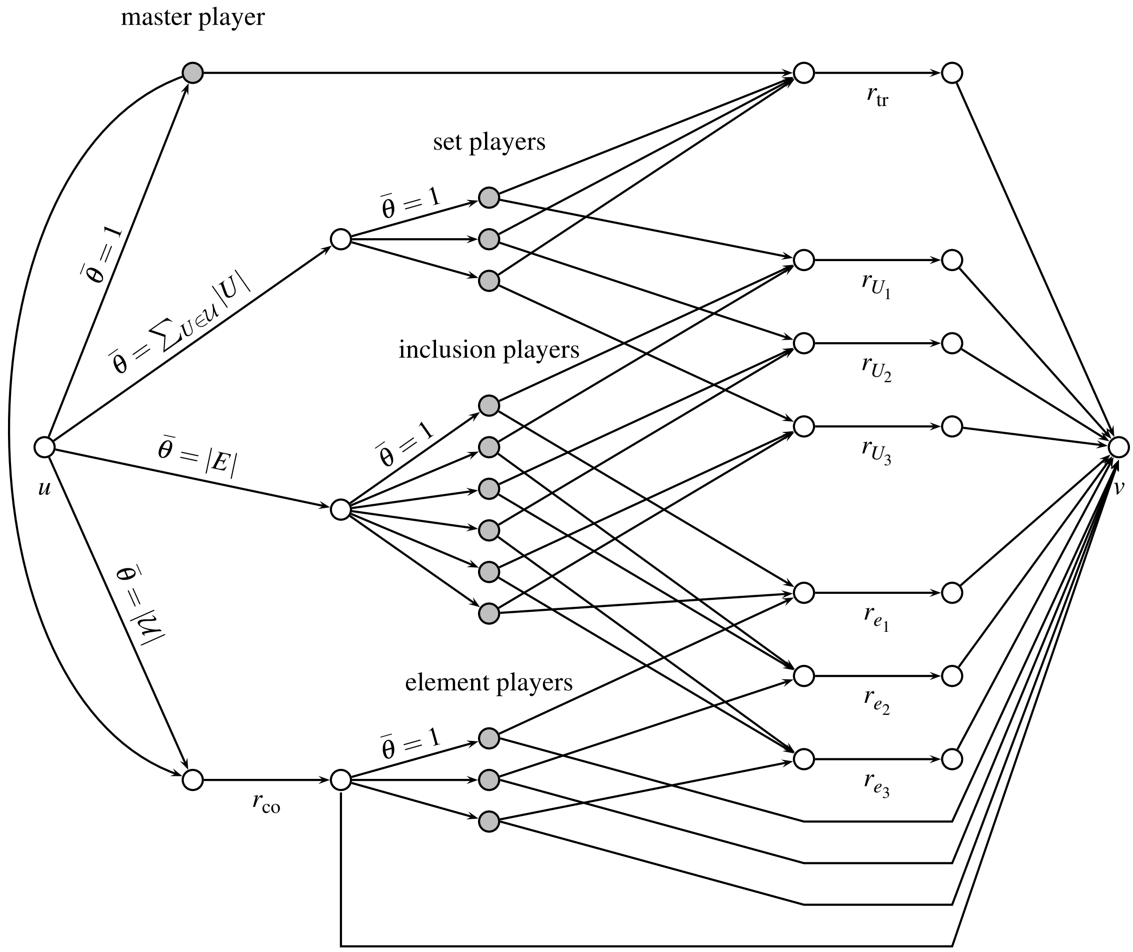


Figure 7.1: Network construction for a SET PACKING instance with $\mathcal{U} = \{\{e_1, e_2\}, \{e_2, e_3\}, \{e_3, e_1\}\}$. Gray nodes serve as identification of the players with the respective player types (master player, set players, inclusion players, and element players) as discussed in the text.

We can create the corresponding state s as before by letting the master player route over r_{co} directly to the sink, the set players over r_{tr}, the inclusion players over r_U and the element players over r_e. This assignment is such that every player receives one identity (i.e., routes over exactly one gray node) and every identity is taken (i.e., every gray node is reached by exactly one player). This property also holds for every strong improving move – with the exception of one element player, who might route directly from r_{co} to the sink, but as noted before our argument still works if this is not the case.

Our network structure allows to reconstruct the reasoning as before. Any strong improving move must include the master player, who improves if and only if she moves together with players corresponding to a solution to the Set Packing instance. Note that even by switching player identities, we cannot create a strong improving move when the underlying Set Packing instance is unsolvable. This proves the first part of the theorem.

For the second part, we use the same adjustment as in Corollary 7.14 to ensure that inclusion players can always improve. Directly before the middle fan out (see Figure 7.1) that results in identification of inclusion players we simply insert a small gadget with 2 parallel edges r_{lo} and r_{hi}. In this way, all inclusion players must route over one of r_{lo} or r_{hi} and one of their corresponding r_U or r_e. This resembles the strategy choices in the matroid game and yields hardness of computing a strong improving move for the coalition $K = N$. □

Remark 7.16. Theorem 7.13, Corollary 7.14, and Theorem 7.15 show that for bottleneck congestion games with matroid or single-commodity strategy spaces, for every polynomially encodable ρ, the following problems are NP-hard: *(i)* given a strategy profile s decide whether there is a strong ρ-improving move from s; *(ii)* given a strategy profile s and a coalition $K \subseteq N$ decide whether K has a strong ρ-improving move from s. The bottleneck congestion games constructed to prove these result have the property that the strong ρ-improving move decreases the private costs of all members of the coalition from $\rho + 2$ to 1. Thus, all hardness results continue to hold for multiplicative approximations. That is, in these games, for every polynomially encodable $\alpha > 1$, the following problems are strongly NP-hard to decide: *(i)* given a strategy profile s decide whether there is a strong improving move from s that decreases the private cost of each member of the coalition by a factor larger than α; *(ii)* given a strategy profile s and a coalition $K \subseteq N$ decide whether K has a strong improving move from s that decreases the private cost of each member of the coalition by a factor larger than α.

7.5 Discussion and Open Problems

In this section, we have provided a detailed study of the computational complexity of pure Nash and strong equilibria in bottleneck congestion games. For more results on the complexity of approximate pure Nash equilibria, see [62]. Still some important open problems remain. A major open problem is to find other interesting classes of games, for which efficient computation of and/or fast convergence to a strong equilibrium can be shown.

In light of the hardness of recognizing a strong equilibrium for games with matroid or single-commodity structure, it is a major open problem how to augment the concept of a pure Nash equilibrium with resilience to coalitional deviations and avoid the hardness results we have observed. It would be interesting to consider computation and convergence characteristics of, e.g., k-strong equilibria, for $1 < k < n$, or equilibrium notions based on player partitions (Feldman and Tennenholtz [45, 46]) or social networks (Hoefer et al. [70]).

Chapter 8

Summary and Conclusion

In this thesis we provided a systematic study of the existence of pure Nash equilibria in multiple variants of congestion games, including games with weighted players, games with resource-dependent demands, and games with variable demands. Our main focus was to precisely characterize the impact of the resource cost functions on the existence of pure Nash equilibria in these games. Specifically, we were interested in the question which maximal sets of cost function we can allow on the resources such that the existence of a pure Nash equilibrium is guaranteed. To formalize this question, we introduced the notion of consistency. A set $\mathcal{C}$ of cost functions is consistent for a certain class of congestion games if every congestion game of that class with cost functions in $\mathcal{C}$ possesses a pure Nash equilibrium. In Chapters 3, 4, and 5 we gave a complete characterization of the sets of consistent cost functions for weighted congestion games, congestion games with resource dependent demands, and congestion games with variable demands, respectively. A characterization of the consistency of cost functions imposes an "if and only if" condition on the existence of pure Nash equilibria, i.e., for every set of cost functions that is not consistent there is a respective congestion game without a pure Nash equilibrium.

Building upon previous work we showed in Chapter 3 that for weighted congestion games every set of consistent cost functions consists either only of affine functions or only of certain exponential functions. This implies that for *any* non-affine and non-exponential function c there is a weighted congestion game with costs identical to c on all resources that does not possess a pure Nash equilibrium. This negative result includes practical relevant functions used to model congestion effects in road or telecommunication networks, such as the so-called BPR and $M/M/1$ functions, and thus may help to predict oscillating behavior in such systems. We further showed that our characterization is even valid for weighted congestion games with three players. For weighted congestion games with two players, we proved that every set of consistent cost functions only contains monotonic functions that are linear transformations of each other. The latter result reveals the common ground of the multitude of counterexamples given in the literature: Fotakis et al. [51], Goemans et al. [59], and Libman and Orda [88] each constructed a two-player congestion game without a pure Nash equilibrium. Their counterexamples share the property that the cost functions are not linear transformations of each other. Our result also explains the seeming dichotomy of the above counterexamples to a result of Anshelevich et al. [10] who showed that for two-player games for which each cost function is of the form $c_r(x) = b_r/x$ with $b_r \in \mathbb{R}_{\geq 0}$ a pure Nash equilibrium always exists. Our characterization answers an open question posed by Orda et al. [106] in the final section of their paper.

In Chapter 4 we studied a slightly more general class of games termed congestion games with resource-dependent demands. They generalize weighed congestion games in the fact that the assumption that each player's demand is equals for all resources is dropped. Interestingly, the equilibrium behavior of these games crucially depends on whether the costs on the resources are interpreted as per-unit costs (and thus multiplied with the demand), or not. This is in contrast to weighted congestion games for which this distinction is immaterial for the existence of pure Nash equilibria. For the case of per-unit costs we showed that the set of affine functions is the unique maximal set of consistent cost functions. When costs are not multiplied with the demand only constant functions are consistent. This characterization is even valid for games with three players whose demands for the distinct resources differs by at most ε where ε can be made arbitrarily small. This result puts the results for weighted congestion games obtained in Chapter 3 in a new focus. In weighted congestion game with exponential cost even though a pure Nash equilibrium exists arbitrarily small errors in the quantification of the demands at the resources may lead to unstable behavior of the system. Moreover, in the presence of errors, games with affine cost functions are stable against those errors only if the cost functions on the resources are taken as per-unit costs.

In many situations players will react on high congestion by lowering their demand and vice versa. To model this effect we initiated the systematic study of congestion games with variable demands where the variable demand has to be assigned to *exactly one* subset of resources in Chapter 5. In such a game each player both chooses her demand out of an interval of feasible demands and a subset of resources. Congestion games with variable demands thus combine the characteristics of finite games and infinite games which complicates their analysis. Yet, based on the result in the two preceding chapters, we gave a complete characterization of consistency. For games in which the costs of the resources are interpreted as per-unit costs we showed that every set of consistent cost functions either contains only affine functions or only homogeneously exponential functions. For games in which the costs of the resources are not multiplied with the players' demands only homogeneously exponential functions are consistent.

For all our characterizations of consistency obtained in Chapters 3, 4 and 5, we required that cost functions are continuous. Although most important classes of cost functions used in the literature have this property it would be very interesting to obtain a characterization for non-continuous cost functions as well. Moreover, our characterizations relied on the fact that strategies are non-symmetric. Symmetric games, however, behave differently as for instance every weighted congestion game with positive and non-increasing costs admits a pure Nash equilibrium (in which all players choose the same strategy). More generally speaking, the impact of how the single players' sets of strategies are interweaved on the existence of pure Nash equilibria is not well understood and desires more attention; see also the discussion in the final section of Ackermann [2].

In Chapters 6 bottleneck congestion games were studied. In a bottleneck congestion game the private cost of a player is the maximum (rather the sum) of the costs of resources used. For this class of games we allowed that the costs of a resource may even depend on the *set* of players using it which is far more general than the demand-based models studied in Chapters 3, 4, and 5. Nonetheless, they always possess a strong equilibrium as proven in Chapter 6. Our proof is constructive and uses the notion of a vector-valued potential. It would be very interesting to see this technique applied to further classes of games.

Because congestion games with bottleneck-objectives and congestion games with sum-objective coincide for singleton strategies, our result of Chapter 6 implies in particular that every singleton congestion game with set-dependent costs admits a strong equilibrium. As a natural generalization of the classical demand-based models it would be interesting to better understand congestion games with set-dependent costs for different (non-singleton) strategy spaces.

The existence of strong equilibria in bottleneck congestion games with set-dependent costs raises a variety of question related to the complexity of computing equilibria in these games. In Chapter 7 we gave first results in this direction. We showed that the problem of computing a strong equilibrium is reducible to solving a certain strategy packing problem. For matroids, single-commodity networks, and branching we further showed that a solution to the packing problem and, thus, a strong equilibrium can be found in polynomial time. A major open problem is to find further classes of games for which this is the case. Furthermore, the complexity of computing equilibria in games with weighted players is still open.

Furthermore, our results for single-commodity bottleneck congestion games and matroid bottleneck congestion games show an interesting dichotomy. While there is an efficient algorithm computing a strong equilibrium in these games, it is co-NP-hard to decide whether a given strategy profile is a strong equilibrium. Thus, these classes of bottleneck congestion games possess strong equilibria with different computational complexity. This interesting effect suggest to use computational complexity as an equilibrium selection tool, that is, to select those equilibria of a game that are easier to compute. It is an interesting question whether this is also possible for further classs of games.

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