Condensation in an inclined tube with small diameter. Modeling, analysis and numerical simulation of a moving boundary problem with phase change and surface tension

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Ich erkläre hiermit, dass ich die vorliegende Arbeit selbständig verfasst und keine anderen als die angegebenen Quellen und Hilfsmittel verwendet habe.
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Zusammenfassung

In dieser Arbeit wird der Einfluss der Oberflächenspannung auf die Kondensation in Strömungskanälen mit geringem Durchmesser untersucht. Solche Strömungskanäle kommen in der Prozessindustrie in kompakten Wärmetauschern vor. Um die Physik des Prozesses besser zu verstehen und damit den Wärmeübergang zu verbessern war zunächst geplant, das Problem mit kommerzieller CFD-Software zu simulieren. Es stellte sich jedoch heraus, dass die Modellgleichungen für das Problem in den verwendeten Programmen nicht richtig implementiert waren.

Stattdessen werden die vollständigen kontinuumsmechanischen Modellgleichungen für solche „Moving Boundary“ Probleme mit Phasenübergang und Oberflächenspannung hergeleitet und analysiert. Die Oberflächenspannung ist eine physikalische Eigenschaft der Phasengrenzfläche und erfordert daher eine eigene Bilanzgleichung. Außerdem ist sie eine Funktion der mittleren Krümmung, und damit der Geometrie der Phasengrenzfläche. Diese beiden Tatsachen erhöhen die Komplexität der Modellgleichungen wesentlich.

Die Modellgleichungen werden mit verallgemeinerter Dimensionsanalyse für ein vertikales Rohr vereinfacht und die wesentlichen Phänomene des Problems bestimmt. Das Ergebnis wird mit einer experimentellen Untersuchung verglichen, es erklärt den besseren Wärmeübergang bei Fluiden hoher Oberflächenspannung in geneigten Rohren. Verallgemeinerte Dimensionsanalyse ist eine (in Vergessenheit geratene) Weiterentwicklung der klassischen Dimensionsanalyse, bei der zusätzlich auch die Modellgleichungen ausgenutzt werden, um die dimensionslosen Kennzahlen des Problems zu erhalten. Damit ist es möglich die Modellgleichungen auf Basis der beiden unterschiedlichen Längen (Filmdicke und Rohrlänge) zu analysieren. Es werden Phasengrenzflächenbedingungen („Jump conditions“) hergeleitet, die auf den gleichen Annahmen beruhen wie die Prandtlschen Grenzschichtgleichungen.

Anschließend wird auf Basis der vereinfachten Modellgleichungen ein Ein-Gleichungs-Modell hergeleitet und numerisch berechnet. Das entwickelte Modell ist eine Erweiterung der Nußelt Theorie. Der Wärmeübergang wird im vertikalen Rohr für sehr kleine Rohrdurchmesser unabhängig von der Oberflächenspannung schlechter (der Film wird dicker) und für extrem hohe Oberflächenspannung besser (der Film wird dünner). Das entwickelte Modell wird mit parametrischen Modellen verglichen, es ist besser als Nußelts Modell und etwas schlechter als Chens Modell.

Die diskretisierten Modellgleichungen bilden ein System von differentiellen und algebraischen Gleichungen (DAE). Der Differentiationsindex des Systems wird untersucht. Es wird gezeigt, das der Index durch die Navier-Stokes Gleichungen bestimmt ist und nicht durch das (instationäre) Moving Boundary Problem verändert wird. Verschiedene Methoden der Indexreduktion werden verglichen. Das Hauptproblem von Moving Boundary Problemen ist die Nichtlinearität der Gleichungen. Auf Basis der Arbeit werden die Vor-und Nachteile von verschiedenen numerischen Methoden für Moving-Boundary Probleme diskutiert.

Die Transformation von dimensionsbehafteten Gleichungen in dimensionslose Gleichungen ist eine Symmetrietransformation. Es wird ausgeführt wie man mit der Lie-Gruppen Theorie analytische Lösungen für Differentialgleichungen entwickelt. Die Lie Theorie wird mit der verallgemeinerten Dimensionsanalyse verglichen und die Symmetrien der Modellgleichungen werden bestimmt.

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Nomenclature

Symbol	SI-Unit	Definition
A	m^2	area
C	m	curve
$\boldsymbol{C}$	$\mathrm{kg\,m^{-3}}$	mass matrix (energy equation)
D	m	film thickness
$\boldsymbol{D}$	$\mathrm{s^{-1}}$	discretized gradient operator
$\boldsymbol{D}^T$	$\mathrm{s^{-1}}$	discretized divergence operator
$\boldsymbol{G}^T$		Jacobian of algebraic constraints
H	$\mathrm{m^{-1}}$	mean curvature
$\boldsymbol{I}$		identity tensor
J		Jacobian determinant
K	$\mathrm{m^{-2}}$	Gauß curvature
$\boldsymbol{K}$	$\mathrm{kg\,s^{-1}\,m^{-1}}$	conductivity matrix
L	m	tube length
L		Weingarten map
$\boldsymbol{M}$	$\mathrm{kg\,m^{-3}}$	mass matrix
$\boldsymbol{N}$	$\mathrm{kg\,s^{-1}\,m^{-3}}$	convection matrix
$\boldsymbol{P}$		projector matrix
R	m	tube radius
$\boldsymbol{S}$	$\mathrm{N\ m^{-2}}$	stress tensor ($\mathrm{J = kg\,m\,s^{-2}}$)
$\boldsymbol{S}$	$\mathrm{N\,s\ m^{-4}}$	stiffness matrix
$\boldsymbol{T}$	$\mathrm{N\ m^{-2}}$	viscous stress tensor
T	K	temperature (°C = K - 273,15)
U	$\mathrm{m\,s^{-1}}$	velocity
V	$\mathrm{m^3}$	volume
$\boldsymbol{Z}$		selector matrix

$\boldsymbol{a}_\alpha$	m	surface base vectors
$a_{\alpha\beta}$	m	covariant metric tensor, first fundamental tensor
$b_{\alpha\beta}$	$\mathrm{m^{-1}}$	second fundamental tensor
c	$\mathrm{J\,kg^{-1}\,K^{-1}}$	spezific heat capacity ($\mathrm{J = N\,m}$)
d	m	tube diameter
e	$\mathrm{J\,kg^{-1}}$	spezific internal energy
$\boldsymbol{e}_i$	m	orthogonal base vectors
$\boldsymbol{g}$	$\mathrm{m\,s^{-2}}$	gravity vector ($g = 9,81\ \mathrm{m\,s^{-2}}$)
h	m	tube radius minus film thickness
α	$\mathrm{W\,m^{-2}\,K^{-1}}$	heat transfer coefficient ($\mathrm{W = J\,s^{-2}}$)

Symbol	SI-Unit	Definition
Δh	$\mathrm{J\,kg^{-1}}$	latent heat of vaporization
j		surface Jacobian determinant
$\dot{m}$	$\mathrm{kg\,s^{-1}\,m^{3}}$	volume specific mass flux
$\boldsymbol{m}$	m	vector normal to a boundary curve
$\boldsymbol{n}$	m	normal vector
p	$\mathrm{N\,m^{-2}}$	pressure
$\boldsymbol{p}$		vector of the algebraic variables
$\boldsymbol{q}$	$\mathrm{J\,m^{-2}\,s^{-1}}$	heat flux vector
$\boldsymbol{t}$	m	tangential vector
$\boldsymbol{u}$	$\mathrm{m\,s^{-1}}$	interface velocity
$\boldsymbol{x}$		vector of the differential variables
$\boldsymbol{y}$		vector of the unknowns
u^{α}	m	surface coordinate
$\boldsymbol{v}$	$\mathrm{m\,s^{-1}}$	velocity vector
$\boldsymbol{w}$	$\mathrm{m^{-1}}$	intrinsic surface velocity $\boldsymbol{v}-\boldsymbol{u}$
x_i	m	Cartesian coordinates
$\boldsymbol{x}$	m	position vector
δ_{ij}		Kronecker-delta
δ	m	film thickness
ε		film thickness divided by tube length $\varepsilon = \frac{D}{L}$
$\boldsymbol{\zeta}$		arbitrary conductive flux vector
μ	$\mathrm{kg\,m^{-1}\,s^{-1}}$	dynamical viscosity
λ	$\mathrm{W\,m^{-1}\,K^{-1}}$	thermal conductivity
$\boldsymbol{\xi}$	m	position vector of reference configuration
π		arbitrary volume specific production density
ρ	$\mathrm{kg\,m^{-3}}$	density
σ	$\mathrm{N\ m^{-1}}$	surface tension
φ		arbitrary volume specific balance property
ϑ		rotation angle
ψ		arbitrary mass specific balance property $\psi = \frac{\varphi}{\rho}$
$\mathcal{B}$		arbitrary extensive balance property
$\mathcal{S}$		arbitrary extensive surface balance property

Chapter 1

Introduction

1.1 Motivation of the thesis

Condensation is important in the refrigeration, automotive and process industries [Car92]. Higher energy efficiency requirements and the move to more environmentally friendly refrigerants increased the need for highly efficient heat transfer for in-tube condensation (and evaporation) processes [KSD99]. Improved heat transfer technologies are nowadays not only used to save energy but rather to save space. Over the last decades experimental studies show that the heat transfer is significantly higher in compact heat exchangers than in classical tube condensers (i.e. 1.5 to 2 times greater), which made compact heat exchangers popular [CT94]. The diameters of hydrodynamic flow channels in such condensers are in the millimeter range. This implies that surface tension plays a crucial role in the heat transfer. Increasingly, numerical simulations are used to reduce the costs of laborious experimental studies. Efficient computational fluid dynamics (CFD) software packages offer a great deal in flexibility in geometry and material properties. However, two-phase flow problems with moving boundaries still represents a major challenge to the current state of engineering in computational fluid dynamics [Li06].

At the beginning of this thesis, the plan was to simulate the condensation process numerically using a computational fluid dynamics (CFD)-program. The condensation process has to be modeled three-dimensional, or two-and-a-half-dimensional, if the problem is assumed to have rotational symmetry. It turned out that the necessary equations were not implemented correctly in this (and other) CFD-programs. Expecting no further difficulties we looked for the correct equations. But what we were looking for did not exist. Physics of fluids and heat transfer has been well established during the last century, see the reference work [TT60], and has been intensively in-

vestigated for a wide range of hydrodynamical problems [BSL60]. However, the formulation of the governing equations for flows involving moving surfaces is more difficult.

Balance equations at moving boundaries are called jump conditions. A moving surface is described mathematically either using tensor calculus or modern differential geometry.[1] The equations of continuum physics are tensor equations, so that the jump conditions are naturally expressed in tensor notation. Anyway, jump conditions for moving boundary problems with phase change and surface tension are not covered by the existing literature. It is possible to find literature for one of them (phase change) without the other (surface tension) or vice versa, but not both together [Gre03] [Hut03] [Sch70] [Spu93]; with the exception of three references which deal with this problem, but not in a sufficient manner [Ari89] [Sla90] [EBW91].[2] However, these references are beyond comprehension for most engineers and mathematicians. To bridge this gap we derive in this thesis jump conditions mostly in symbolic tensor notation, and give a straightforward description of the geometry of the surface. By this we make available the necessary equations for the solution of moving boundary problems with phase change and surface tension in a way that they are comprehensible by the mathematician, the physicist and the engineer.

The next step was to simplify the model equations with the aim to derive model equations that can be solved either analytically or numerically (within feasible limits). In literature simplifications of the complete model equations exist for special cases, but not for condensation in a cylindrical tube when surface tension has to be considered. Interestingly sometimes for the same process different jump conditions are used. Often the model equations are simplified intuitively or the method of simplification remains unclear. What was needed was a reliable method to find the physically relevant terms in the model equations. Classical dimensional analysis [Bri31] can not be applied because the problem depends on two different length scales (film thickness and tube length). Perturbation method [Hol95] overcomes this problem by rescaling the variables. However, we finally found a better method: generalized dimensional analysis [Lon63], a generalization of dimensional analysis which fell into oblivion over the last decades, a highly algorithmically method which

[1] The notation in differential geometry varies from author to author (which is makes it not easier for someone not close to this discipline).

[2] [Ari89] added jump conditions at the end of his book, based on the paper of [Scr60], who first described interfacial dynamics for Newtonian surface fluids, but this last chapter is not an organic part of the book. [Sla90] and [EBW91] also cover both but lack a convincing description of curvature (for surface tension).

is based on the physical dimensions of the problem. By this it serves as a less error-prone method which made apparent the major phenomena of the analyzed problem. Here we reintroduce generalized dimensional analysis in a more mathematical rigor and bring the method on a level with classical dimensional analysis, so that it can be applied by researchers of different disciplines to other problems.

In appendix A we make a digression to Lie groups [BK89] [Olv93] and extend our analysis further. A dimensionless equation is the result of a symmetry transformation of the original equation (with dimensions). Symmetry transformations can be used in certain cases to successively transform a differential equation to a simpler equation and to finally solve it analytically. We analyze the Lie groups of the simplified model equations using a Mathematica program [Bau00] and demonstrate for a sub-problem how to construct an analytical solution from the symmetry groups. The intention is to point out a way towards an analytical solution, which can be used in a next step to derive a parametrical model by including additional phenomena.

Next, it was planned to solve the simplified model equations numerically. But the equations are still too complicated so that the work to do this would have been beyond the scope of this thesis. Instead, we simplify the equations further and derive an ordinary differential equation for condensation in a vertical tube under rotational symmetry where we take surface tension into account. We solve it numerically and compare the results for two different fluids with experimental results. Yet by this we also close another gap. Two kinds of models are used by engineers: parametrical models (correlations) based on experimental data, and models from continuum physics. However, in the various handbooks of heat transfer [Cro06] [RHC98] [Lie87] there can be found analytical models only for the simple case of condensation along a flat plate (as derived by [Nuß16]). Our model is an extension of Nußelt's theory for condensation in a tube of rotational symmetry by taking surface tension into account.

For a numerical simulation it is essential to know the characteristics of the model equations. After spatial discretization the model equations form a system of differential algebraic equations (DAE). We analyze the complexity of the discretized model equations from a DAE point of view and analyze the "index" of the moving boundary problem. We show the main problems in the numerical solution of moving boundary problems and discuss appropriate solution methods.

Our interest is to give an analytical foundation for researchers of both disciplines, mathematics and engineering, who are working with moving boundary problems and by this to contribute to improved heat exchangers.

1.2 Literature related to condensation in tubes with small diameter

In process industries compact plate heat exchangers are increasingly used for reflux condensation applications [JC00]. In reflux condensation a vapor enters a vertical or inclined mounted condenser at the bottom and flows upward. The condensate stream flows gravity controlled downward countercurrently to the vapor flow. Typical applications of reflux condensation are in overhead condensers of distillation columns and vent condensers of reactors or stirred vessels. Other applications are in the vent cooling section of air-cooled steam condensers and in two-phase closed thermosyphons. In such condensers the hydraulic diameter of the flow channels formed between two plates is 5-10 mm, and the flow channels are inclined to the vertical, as depicted in figure 1.1. The fundamental mechanisms of heat and mass transfer as well as of two phase flow in these small channels are not well understood.

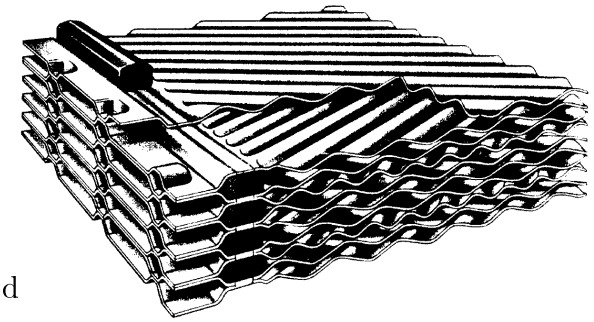


Figure 1.1: *Compact plate heat exchanger*

Experimental studies on compact heat exchangers

The vast majority of studies about condensation in compact heat exchangers are studies on compact plate heat exchangers (see e.g. the 5th International Conference on Enhanced, Compact and Ultra-Compact Heat Exchangers [Sha05] and its predecessors). In an experimental study at the Institute for Energy Engineering, Area of Momentum, Heat and Mass Transfer of the University of Technology of Berlin, the heat transfer in a single representative sub-channel of a compact condenser was investigated [Fie03]. The inner diameter of the tube is 7 mm and its length is 500 mm. As condensate the refrigerant R134a was used. In figure 1.2 the Nußelt number (dimensionless heat transfer number) as a function of the Reynolds number with the inclination angle as

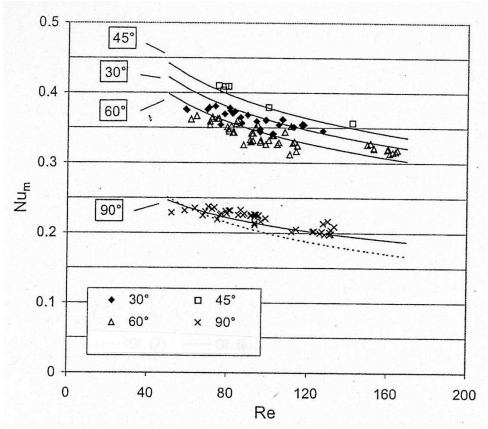


Figure 1.2: *Experimental results [Fie03]*

parameter is shown. The main result is that for an inclination angle of 45° against the horizontal the heat transfer is twice times better compared to the vertical.

Analytical and numerical studies on condensation in tubes with small diameters

Literature on condensation in tubes (or channels) with small diameters is on inclined, vertical or horizontal tubes (or channels), and the effect of surface tension is either taken into account or not. Three authors studied condensation in inclined tubes. [FR93] derived an analytical solution for condensation in/on elliptical cylinders but neglected variations of the film thickness with the tube radius. [Mos99] studied interfacial shear stress but without considering surface tension. [WD00] compared heat transfer in a horizontal and an inclined tube by energy considerations. He considered surface tension. However, he studied tubes with diameters of 1.94 mm and 4.98 mm, which are smaller than diameters of compact heat exchangers. Condensation in vertical tubes are investigated by the following authors, where the first two authors considered surface tension but the last three authors do not. According to [WD03] small surface waves enhance the heat transfer mainly due to film thinning effect. [ZL02] investigated condensation in vertical triangular channels with a diameter between 0.2 mm and 0.3 mm. [Pan01] showed that the effect of interfacial shear stress on the heat transfer depend on the vapor velocity and on the mass transfer (operating Temperature). [Pan03] investigated a tube with a diameter of 24 mm, where he showed that turbulent flow enhances the heat transfer. [FBMB01] studied evaporative cooling. Further there are two studies on horizontal tubes. [SOS02] studied condensation in horizontal parallel plate channels but he neglected surface tension. [WHN02] investigated horizontal micro-fin tubes by dividing the flow in two flow regimes. However, these papers are based on simplified model equations and show a great variety especially in the interface equations.

Stability of thin films

Heat exchanger often operate at moderate Reynolds numbers ($Re \lessapprox 100$), with a film thickness in the millimeter range and small surface waves with wave length in the centimeter range. Stability of thin films was first investigated of [Ben57] and [Yih63] by a linear stability analysis with the result that thin films are instable for all Reynolds numbers and that surface tension has a stabilizing effect on the film. [Ban70] [ML72] [ÜT78] and [Spi81] extended linear stability analysis on phase change problems and showed that condensation in opposite to evaporation tends to stabilize the film. However, critical Reynolds numbers predicted with linear stability

analysis are too small for technical applications ($Re < 5$). Later nonlinear stability analysis as derived by [Ben66] (Benney equation), where small surface waves are modeled by partial differential equations which are then perturbed, was applied on small films with phase change. [BBD88], [JDB90] (evaporation) and [HW87] (condensation) demonstrated the dependency of stability on the frequency of the initial perturbation. However, only flows along plates are studied in Cartesian coordinates, so that the effect of surface tension due to the small tube diameter is not modeled. With Benney's equation experimental observations of two and three-dimensional waves studied by [ANP94] and [YNN96] were simulated numerically by [MNT02].

Studies on numerical methods for moving boundary problems

Numerical methods for moving boundary problems with continuous interfaces are of two kinds: Tracking methods, such as Marker and Cell methods (MAC), first developed by [HW96] and later considerably improved by [Kot98] and [TBaWT98] and capturing methods, such as Volume of Fluid methods (VOF) [HN75] [HN81] [KR98], or level set methods [Set99]. [BKZ92] extended VOF methods by a continuous surface tension force (CSF) model, [JT98] enhanced tracking methods for phase change problems, [WW00] proposed a VOF method with a CSF model together with the phase change model of [JT98], [SD02] extended level set methods on phase change problems. However, in these studies surface tension and phase change are not implemented exactly but are treated as source terms. A comparison of both methods by means of the underlying equations can not be found in literature.

1.3 Organization of the thesis

In **part I** we derive the model equations for the condensation problem. The model equations for the condensation problem are the mass, momentum, and energy balance equations for the liquid film and the vapor phase and for the interface between both phases. All equations in the first part are valid for both condensation and evaporation.

The balance equations at the interface (jump conditions) contain geometrical quantities, such as normal and tangential vectors on the interface and mean curvature of the interface. For these quantities we need a geometrical description of the interface. This is given in **chapter 2**. For the derivation of the balance equations and the jump conditions we also need kinematical relations, such as the Reynolds

transport theorem for a material body, but also a Reynolds transport theorem for a material body with an internal interface. They are given in **chapter 3**. After this preliminaries we are able to derive generic balance equations. This is done in **chapter 4**. Here we derive generic balance equations for the two phases and generic jump conditions for the interface, where we assume both phases to be incompressible Newtonian fluids. In **chapter 5**, we apply the generic equations to mass momentum and energy and derive the desired model equations for the condensation problem. Here also the boundary conditions are given. By this we conclude the first part with the model equations for the condensation problem in symbolic notation.

Modeling the two-phase problem becomes more laborious when the effect of surface tension is taken into account. In all chapters of part I surface tension increases the complexity. The normal and tangential vectors on the interface can be derived by a simple geometric demonstration, as described at the beginnings of section 2.1 and in section 2.3. However, for the description of the mean curvature (to model the effect surface tension) we need tensor notation. Similarly the derivation of a jump condition including the effect of a curved surface is not a trivial task. In section 4.2 in the first attempt we derived the generic jump condition starting from the generic balance equation. However, the resulting jump condition does not contain mean curvature. We need additionally a generic balance equation for the interface itself, as described in section 4.3. This in turn involves the need of kinematical relations of the interface and the Reynolds transport theorem for surfaces, given in section 3.3 and 3.4.

After the model equations are formulated the next step is to find out as much as possible about the problem. This is what we do in **part II**. We analyze the model equations by means of a generalized dimensional analysis and simplify the model equations according to the main relevant physical phenomena of the condensation problem.

Based on the experimental results, that the flow regime of the condensate in the inclined tube is an almost circular film along the inner tube wall [Fie03], we derive in **chapter 6** the equations for condensation in a tube with small diameter under the assumption of rotational symmetry. Then we reduce the complexity of this model equations further by a generalized dimensional analysis in **chapter 7**.[3] We derive equations of boundary layer type for the bulk flow equations and find comparable

[3] Dimensional analysis should not be confused with a nondimensionalization of the terms of an equation, which is often useful for a numerical simulation.

simplifications for the jump conditions. By this the main physical phenomena in the condensation process are determined.

Generalized dimensional analysis allows us to analyze the equations in terms of the characteristic length scales of the problem, that is the film thickness (≈ 1 mm) and the tube length (0.5 m). In section 7.1 we present the method and compare it to dimensional analysis. In the following sections, the method is applied to the model equations for the vertical tube. As a result of this analysis we get nondimensional equations containing the main physical phenomena of condensation in a tube with small diameter. We refer to section 7.4 for the equations of the bulk flow and section 7.6 for the jump conditions. In section 7.7 we evaluate the nondimensional equations for two different fluids and discuss the effect of surface tension for water and R134a. Finally we consider the effect of an inclination of the tube on the results.

A general definition of symmetry due to [Wey52] is, that a thing is symmetrical if there is something we can do to it so that after we have done it, it looks the same as it did before. From this definition we understand that the transformation of a differential equation into a nondimensional differential equation by means of a dimensional analysis is a symmetry transformation. The first who systematically investigated symmetry transformations of differential equations was [Lie22]. In the last years there has been intense research on solving differential equations using their symmetries. In **appendix A** we discuss the symmetries of the simplified model equations by a Lie group analysis. We present the method of finding the Lie groups of point transformations which let a given system of differential equations invariant and discuss its relation to generalized dimensional analysis. We find the symmetry groups of transformations admitted by the model equations of the condensation problem using a *Mathematica*® software package and discuss them.

In **part III** we deal with the numerical simulation of the moving boundary problem from two different angles. In **chapter 8** we simplify the model equations which we derived in part II further and derive a single nonlinear ordinary differential equation (ODE) for the film thickness and in **chapter 9** we analyze the complete model equations from a DAE point of view.

In section 8.1 we demonstrate how the single model equation is derived and we compare it with Nußelt's condensation theory. Then we extend this equation in section 8.2 for the case of surface tension and solve it by a standard Runge-Kutta method. In section 8.3 we compare the effect of surface tension for two different fluids with experimental results of [Fie03].

The complete model equations consist of the transient Navier-Stokes equations, the energy equation and the jump conditions, as derived in part I. The numerical solution of Navier-Stokes problems is not a trivial task. The main problems are the nonlinearity due to the convective terms in the momentum equation and that the pressure cannot be computed directly, but is determined by the continuity equation. With conventional methods in every time iteration step some further iterations need to be done to compute the pressure. As a result only lower order time integration methods can be used. The DAE approach is to solve the pressure problem by reducing the index of the differential algebraic equations (which are the result of the spatial discretization). This has the advantage that higher order methods can be used for the solution of the transient problem and is especially useful if the problem is highly transient and/or nonlinear. However, it has also disadvantages: In section 9.2 we compare this approach with the conventional methods for the numerical solution of Navier-Stokes problems, and in section 9.4 we analyze how the jump conditions change the index of the DAEs. Finally we discuss some aspects of numerical algorithms for the solution of moving boundary problems based on our analysis and on our work with two CFD-software packages in section 9.5.[4]

[4] We worked with SEPRAN, a semi-commercial program distributed by G. Segal, Delft, and with FIDAP, a trademark of FLUENT, both of which were not able to perform the task satisfactory.

Part I

Modelling

Chapter 2

Geometry of the moving interface

In this chapter we derive the geometrical properties of the moving interface between condensate and vapor phase, such as normal and tangential vectors and mean curvature. For the description of three-dimensional flows a Cartesian frame of reference really suffice. However, a surface is a two-dimensional non Euclidean space and demands a tensorial treatment. First, the formulas for normal and tangential vectors, mean curvature and interface velocity are given. Then, by an implicit surface representation the geometrical properties for the interface of the condensation problem are computed. Two formulas to compute the normal vector and two formulas to compute the mean curvature are discussed.

The material in this chapter is mainly based on [EBW91] [Ari89] [TT60] and [Eri71]. They present classical continuum physics intrinsically tied with tensor calculus. For an introduction in tensor calculus we refer to [Sch97]. [Sla90] [Sla99] and [Dee98] present transport phenomena including interphase transport phenomena using moderate tensor notation. [Küh99] is a differential geometry book designed for the mathematician [Opr00] has (for the engineer) a far easier notation but also lack some derivations.

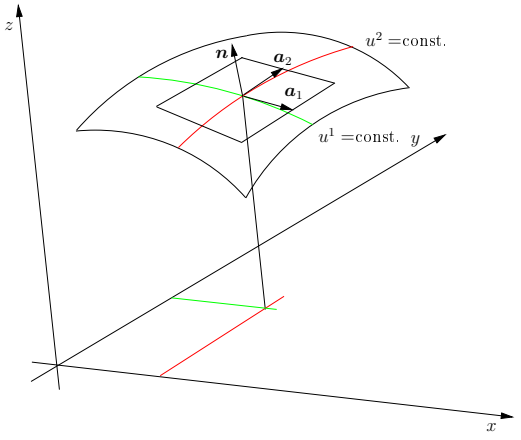


Figure 2.1: *Tangent space on a surface*

2.1 Curvilinear coordinate systems

A two-dimensional surface can be best analyzed by covering the surface with a grid, see figure 2.1. The grid is obtained by the curves defined by u^1 and u^2 held constant. The position of a point on the surface can be given intrinsically in terms of the two curvilinear surface coordinates (or parameters) u^1 and u^2, or extrinsically by a position vector to that point This defines a curvilinear coordinate system which is not orthogonal in general. In this thesis cartesian coordinates are denoted by x_i and indexed by a subscript. Curvilinear coordinates are denoted by u^α and are indexed by a superscript. When the index takes only the values $1, 2$ greek letters are used and when the index takes the values $1, 2, 3$ latin letters are used. Einstein summation convention is used, which states that if an index is repeated in a term that implies a sum over all possible values for that index.

Tangential and normal vectors

In our case the interface between condensate and vapor is not stationary, so that the position vector to a point on the interface is given in cartesian coordinates by

$$\boldsymbol{x}(u^1, u^2, t) = x_1(u^1, u^2, t)\, \boldsymbol{e}_1 + x_2(u^1, u^2, t)\, \boldsymbol{e}_2 + x_3(u^1, u^2, t)\, \boldsymbol{e}_3 \,, \tag{2.1}$$

or in index notation by

$$\boldsymbol{x}(u^\alpha, t) = x_i(u^\alpha, t)\, \boldsymbol{e}_i \,, \qquad \text{where } u^\alpha = u^1, u^2. \tag{2.2}$$

Hence, all geometrical properties are functions of time.

A Taylor series expansion of $\boldsymbol{x}$ in the surface variables u^α up to the linear term yields the total derivative

$$\mathrm{d}\boldsymbol{x} = \frac{\partial \boldsymbol{x}}{\partial u^\alpha}\, \mathrm{d}u^\alpha \,.$$

If $u^2 = \text{constant}$ ($\mathrm{d}u^2 = 0$), only the component along the u^1 curve remains and $\frac{\partial \boldsymbol{x}}{\partial u^1}$ defines the tangent vector along this curve; similarly $\frac{\partial \boldsymbol{x}}{\partial u^2}$ defines the tangent vector along the u^2 curve [Sch97]. The derivatives with respect to the curvilinear coordinates u^α are called covariant derivatives. The covariant derivatives of a position vector

$$\boldsymbol{a}_\alpha = \frac{\partial \boldsymbol{x}}{\partial u^\alpha} = \boldsymbol{e}_i \frac{\partial x_i}{\partial u^\alpha} \tag{2.3}$$

form the base vectors of a local surface coordinate system. In terms of the covariant base vectors the surface metric tensor is defined as

$$a_{\alpha\beta} = \boldsymbol{a}_\alpha \cdot \boldsymbol{a}_\beta \ , \tag{2.4}$$

[Ari89]. The metric tensor is also called first fundamental tensor. The local unit normal vector at a point (u^1, u^2) normal to the surface is defined by

$$\boldsymbol{n} = \frac{\boldsymbol{a}_1 \times \boldsymbol{a}_2}{|\boldsymbol{a}_1 \times \boldsymbol{a}_2|} \ . \tag{2.5}$$

Dual basis

Another set of base vectors $\boldsymbol{a}^\beta$ is defined by the surface Kronecker delta $\delta_{\alpha\beta}$

$$\boldsymbol{a}_\alpha \cdot \boldsymbol{a}^\beta = \delta_{\alpha\beta} \ ,$$

they are called dual base vectors or reciprocal or contravariant base vectors respectively [Sch97]. This condition is called orthogonality relation. It defines a vector $\boldsymbol{a}^1$ that lies in the plane formed by the vectors $\boldsymbol{a}_1, \boldsymbol{a}_2$, is perpendicular to $\boldsymbol{a}_2$, forms an acute angle with $\boldsymbol{a}_1$ and its length is given by $\boldsymbol{a}_1 \cdot \boldsymbol{a}^1$. This is also the definition of the gradient $\boldsymbol{\nabla} u^1$ of the surface coordinate, which is perpendicular to the level surface defined by $u^1(x_1, x_2, x_3, t) =$ constant. Similarly the orthogonality relation defines the vector $\boldsymbol{a}^2$. Then the dual or contravariant base vectors are given by

$$\boldsymbol{a}^\alpha = \boldsymbol{\nabla} u^\alpha = \boldsymbol{e}_i \frac{\partial u^\alpha}{\partial x_i} \ . \tag{2.6}$$

With the contravariant base vectors the covariant surface metric tensor is

$$a^{\alpha\beta} = \boldsymbol{a}^\alpha \cdot \boldsymbol{a}^\beta \ . \tag{2.7}$$

However, the dual basis is more convenient calculated by means of the local unit normal vector

$$\boldsymbol{a}^1 = \frac{\boldsymbol{a}_2 \times \boldsymbol{n}}{[\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{n}]} \ , \qquad \boldsymbol{a}^2 = \frac{\boldsymbol{n} \times \boldsymbol{a}_1}{[\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{n}]} \ . \tag{2.8}$$

$[\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{n}] = [\boldsymbol{a}_1 \times \boldsymbol{a}_2] \cdot \boldsymbol{n}$ is the scalar triple product [Dee98].

Orthogonal curvilinear coordinate systems

If the base vectors of a curvilinear coordinate system are orthogonal ($\boldsymbol{a}_1 \cdot \boldsymbol{a}_2 = 0$), the covariant and contravariant metric tensor reduce to

$$a_{\alpha\beta} = \begin{bmatrix} a_{11} & 0 \\ 0 & a_{22} \end{bmatrix} , \qquad a^{\alpha\beta} = \begin{bmatrix} a^{11} & 0 \\ 0 & a^{22} \end{bmatrix} .$$

For such orthogonal systems the normalized surface vectors are called self reciprocal, in the sense that $\frac{\boldsymbol{a}_1}{\sqrt{a_{11}}} = \frac{\boldsymbol{a}^1}{\sqrt{a^{11}}}$ and $\frac{\boldsymbol{a}_2}{\sqrt{a_{22}}} = \frac{\boldsymbol{a}^2}{\sqrt{a^{22}}}$. It is convenient to introduce unit vectors

$$\boldsymbol{e}_1 = \frac{\boldsymbol{a}_1}{\sqrt{a_{11}}} , \qquad \boldsymbol{e}_2 = \frac{\boldsymbol{a}_2}{\sqrt{a_{22}}} ,$$

[EBW91].

Surface gradient

The identity tensor is defined by

$$\boldsymbol{I} = \boldsymbol{a}^1 \, \boldsymbol{a}_1 + \boldsymbol{a}^2 \, \boldsymbol{a}_2 + \boldsymbol{n} \, \boldsymbol{n} ,$$

and possesses the property $\boldsymbol{I} \cdot \boldsymbol{x} = \boldsymbol{x}$ for any $\boldsymbol{x}$ [EBW91]. This relation is also called orthogonality relation. By subtracting the part related to the normal vector from $\boldsymbol{I}$ the surface identity tensor is defined as

$$\boldsymbol{I}_S = \boldsymbol{I} - \boldsymbol{n} \, \boldsymbol{n} = \boldsymbol{a}^\alpha \, \boldsymbol{a}_\alpha , \tag{2.9}$$

[EBW91]. Similarly the surface gradient is defined by the projection in normal direction subtracted from the gradient

$$\boldsymbol{\nabla}_S = \boldsymbol{\nabla} - \boldsymbol{n} \, \boldsymbol{n} \, \boldsymbol{\nabla} . \tag{2.10}$$

By this we get (with $\boldsymbol{e}_i \cdot \boldsymbol{e}_j = \delta_{ij}$)

$$\boldsymbol{\nabla}_S = \boldsymbol{I}_S \cdot \boldsymbol{\nabla} = \boldsymbol{a}^\alpha \, \boldsymbol{a}_\alpha \cdot \left(\boldsymbol{e}_i \, \frac{\partial}{\partial x_i} \right) = \boldsymbol{a}^\alpha \, \left(\boldsymbol{e}_j \, \frac{\partial x_j}{\partial u^\alpha} \right) \cdot \left(\boldsymbol{e}_i \, \frac{\partial}{\partial x_i} \right) = \boldsymbol{a}^\alpha \, \frac{\partial}{\partial u^\alpha} . \tag{2.11}$$

2.2 Mean curvature

The mean curvature is defined as

$$H = -\frac{1}{2}\,\boldsymbol{\nabla}_S \cdot \boldsymbol{n} = -\frac{1}{2}\left(\boldsymbol{a}^\alpha \frac{\partial}{\partial u^\alpha}\right) \cdot \boldsymbol{n}\ , \tag{2.12}$$

[Dee98]. The mean curvature is proportional to the rate of change with respect to the surface coordinates of the local normal vector.

2.3 Implicit parameterized surface

To compute the normal and tangential vectors and the mean curvature of the moving interface it is necessary to choose a parametrization of the interface. For example, if a vertical tube is assumed and the condensate flows down along the inner walls without showing waves, then the problem has rotational symmetry and the interface between the condensate and the vapor can be parameterized with the surface coordinates $u^1 = z$ and $u^2 = \vartheta$, see section 6. Here we derive the geometrical surface properties by a more general parametrization.

Normal and tangential vectors of an implicit parametrized surface

Every moving surface can be locally described by a real-valued function of two variables and time

$$z = h(u, v, t) \qquad \text{or implicitly} \qquad F(u, v, z, t) = z - h(u, v, t) = 0\ ,$$

[Ari89] [Opr00]. For convenience we write $u^1 = u$, $u^2 = v$. By this parametrization the position vector becomes

$$\boldsymbol{x}(u, v, t) = u\,\boldsymbol{e}_1 + v\,\boldsymbol{e}_2 + h(u, v, t)\,\boldsymbol{e}_3\ .$$

The tangential vectors are given by the covariant base vectors (2.3) as

$$\boldsymbol{a}_1 = \frac{\partial \boldsymbol{x}}{\partial u} = \begin{bmatrix} 1 \\ 0 \\ \frac{\partial h}{\partial u} \end{bmatrix}\ , \qquad \boldsymbol{a}_2 = \frac{\partial \boldsymbol{x}}{\partial v} = \begin{bmatrix} 0 \\ 1 \\ \frac{\partial h}{\partial v} \end{bmatrix}\ . \tag{2.13}$$

The local unit normal vector is the cross product of the tangent vectors, scaled by its length. Alternatively, if the surface is given by $z = h(u, v, t)$, the unit normal vector can be obtained from the gradient of the implicit function $F(u, v, z, t) = 0$. Expanding $F(u, v, z, t)$ in a Taylor series up to the linear term for the variables u, v, z leads to the total derivative

$$\mathrm{d}F = \frac{\partial F}{\partial u}\,\mathrm{d}u + \frac{\partial F}{\partial v}\,\mathrm{d}v + \frac{\partial F}{\partial z}\,\mathrm{d}z\ ,$$

which is zero because $F = 0$. Writing the total derivative as $\boldsymbol{\nabla} F \cdot \mathrm{d}\boldsymbol{u} = 0$ where $\mathrm{d}\boldsymbol{u} = [\mathrm{d}u, \mathrm{d}v, \mathrm{d}z]$ shows that the gradient $\boldsymbol{\nabla} F$ is perpendicular to the level surface defined by $F(u, v, z, t) = 0$. The local unit normal vector reads then

$$\boldsymbol{n} = \frac{\boldsymbol{\nabla} F}{|\boldsymbol{\nabla} F|} = \begin{bmatrix} -\frac{\partial h}{\partial u} \\ -\frac{\partial h}{\partial v} \\ 1 \end{bmatrix} \frac{1}{\sqrt{\left(\frac{\partial h}{\partial u}\right)^2 + \left(\frac{\partial h}{\partial v}\right)^2 + 1}}\ . \tag{2.14}$$

Mean curvature of an implicit parametrized surface

The mean curvature is given by (2.12) as being proportional to the rate of change of the surface normal with respect to the surface coordinates

$$H = -\frac{1}{2}\left(\boldsymbol{a}^1 \frac{\partial}{\partial u} + \boldsymbol{a}^2 \frac{\partial}{\partial v}\right) \cdot \boldsymbol{n} = -\frac{1}{2}\left(\boldsymbol{a}^1 \cdot \frac{\partial \boldsymbol{n}}{\partial u} + \boldsymbol{a}^2 \cdot \frac{\partial \boldsymbol{n}}{\partial v}\right)\ .$$

Alternatively the mean curvature is often computed more conveniently by means of the first and second fundamental tensor as explained below [Ari89] [Küh99]. For the implicit surface parametrization the first fundamental tensor, or covariant metric tensor (2.4), reads in matrix form

$$a_{\alpha\beta} \;=\; \begin{bmatrix} \boldsymbol{a}_1 \cdot \boldsymbol{a}_1 & \boldsymbol{a}_1 \cdot \boldsymbol{a}_2 \\ \boldsymbol{a}_2 \cdot \boldsymbol{a}_1 & \boldsymbol{a}_2 \cdot \boldsymbol{a}_2 \end{bmatrix} = \begin{bmatrix} 1 + (\frac{\partial h}{\partial u})^2 & \frac{\partial h}{\partial u}\,\frac{\partial h}{\partial v} \\ \frac{\partial h}{\partial v}\,\frac{\partial h}{\partial u} & 1 + (\frac{\partial h}{\partial v})^2 \end{bmatrix}\ .$$

Between the covariant and the contravariant metric tensor (2.7) the relation

$$a^{\alpha\gamma}\, a_{\gamma\beta} = (\boldsymbol{a}^\alpha \cdot \boldsymbol{a}^\gamma)\,(\boldsymbol{a}_\gamma \cdot \boldsymbol{a}_\beta) = \boldsymbol{a}^\alpha \cdot (\boldsymbol{a}^\gamma\, \boldsymbol{a}_\gamma) \cdot \boldsymbol{a}_\beta = \boldsymbol{a}^\alpha \cdot \boldsymbol{I}_S \cdot \boldsymbol{a}_\beta = \boldsymbol{a}^\alpha \cdot \boldsymbol{a}_\beta = \delta_{\alpha\beta}$$

holds, so that the contravariant metric tensor $a^{\alpha\beta}$ is the inverse of the covariant metric tensor $a_{\alpha\beta}$ [EBW91]. The inverse of a matrix is given by the transpose of

the cofactor matrix (denoted by a tilde), divided by the determinant of the matrix.[1] This yields

$$a^{\alpha\beta} = (a_{\alpha\beta})^{-1} = \frac{\tilde{a}_{\alpha\beta}}{\det a_{\alpha\beta}} = \frac{1}{a_{11}\,a_{22} - a_{12}\,a_{21}} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix} ,$$

and for the implicit surface parametrization

$$a^{\alpha\beta} = \frac{1}{1 + (\frac{\partial h}{\partial v})^2 + (\frac{\partial h}{\partial v})^2} \begin{bmatrix} 1 + (\frac{\partial h}{\partial v})^2 & -\frac{\partial h}{\partial u}\frac{\partial h}{\partial v} \\ -\frac{\partial h}{\partial u}\frac{\partial h}{\partial v} & 1 + (\frac{\partial h}{\partial u})^2 \end{bmatrix} .$$

Both metric tensors are evidently symmetric. They are also positive definite.[2]

Next, the second fundamental tensor is defined as

$$b_{\alpha\beta} = \frac{\partial \boldsymbol{a}_\alpha}{\partial u^\beta} \cdot \boldsymbol{n} \qquad \text{or alternatively} \qquad b_{\alpha\beta} = -\boldsymbol{a}_\alpha \cdot \frac{\partial \boldsymbol{n}}{\partial u_\beta} . \tag{2.15}$$

[Ari89] [Küh99]. The first equation of (2.15) yields for our parametrization

$$b_{\alpha\beta} = \frac{\partial \boldsymbol{a}_\alpha}{\partial \beta} \cdot \boldsymbol{n} = -\begin{bmatrix} \frac{\partial \boldsymbol{a}_1}{\partial u} \cdot \boldsymbol{n} & \frac{\partial \boldsymbol{a}_1}{\partial v} \cdot \boldsymbol{n} \\ \frac{\partial \boldsymbol{a}_2}{\partial u} \cdot \boldsymbol{n} & \frac{\partial \boldsymbol{a}_2}{\partial v} \cdot \boldsymbol{n} \end{bmatrix} = \frac{1}{1 + (\frac{\partial h}{\partial u})^2 + (\frac{\partial h}{\partial v})^2} \begin{bmatrix} \frac{\partial^2 h}{\partial u^2} & \frac{\partial^2 h}{\partial u \partial v} \\ \frac{\partial^2 h}{\partial v \partial u} & \frac{\partial^2 h}{\partial v^2} \end{bmatrix} .$$

The second fundamental tensor is also symmetric but not necessarily positive definite. From the first and second fundamental tensor the shape operator or Weingarten map is defined

$$L \;=\; b_{\alpha\gamma}\, a^{\gamma\beta} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \frac{1}{\det a_{\alpha\beta}} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix} , \tag{2.16}$$

[Küh99]. It becomes for the implicit surface parametrization

$$L = \frac{1}{\sqrt{o}} \begin{bmatrix} \frac{\partial^2 h}{\partial u^2}\left(1 + (\frac{\partial h}{\partial v})^2\right) - \frac{\partial^2 h}{\partial u \partial v}\frac{\partial h}{\partial u}\frac{\partial h}{\partial v} & \frac{\partial^2 h}{\partial v \partial u}\left(1 + (\frac{\partial h}{\partial u})^2\right) - \frac{\partial^2 h}{\partial u^2}\frac{\partial h}{\partial u}\frac{\partial h}{\partial v} \\ \frac{\partial^2 h}{\partial u \partial v}\left(1 + (\frac{\partial h}{\partial v})^2\right) - \frac{\partial^2 h}{\partial v^2}\frac{\partial h}{\partial u}\frac{\partial h}{\partial v} & \frac{\partial^2 h}{\partial v^2}\left(1 + (\frac{\partial h}{\partial u})^2\right) - \frac{\partial^2 h}{\partial u \partial v}\frac{\partial h}{\partial u}\frac{\partial h}{\partial v} \end{bmatrix} ,$$

with $\sqrt{o} = \sqrt{1 + (\frac{\partial h}{\partial u})^2 + (\frac{\partial h}{\partial v})^2}$. The eigenvalues of a product of a symmetric matrix with a symmetric positive definite matrix are all real, Zurmühl [Zur64].

[1] For the computation of the inverse of a matrix by its cofactor matrix [Apo69].

[2] The condition that a matrix is positive definite is that all upper left determinants are positive, Apostol [Apo69] $a^{11} = 1 + (\frac{\partial h}{\partial u})^2 > 0$ and
$a^{11}a^{22} - (a^{12})^2 = (1 + (\frac{\partial h}{\partial u})^2)(1 + (\frac{\partial h}{\partial v})^2) - \frac{\partial h}{\partial u}\frac{\partial h}{\partial v} = 1 + (\frac{\partial h}{\partial u})^2 + (\frac{\partial h}{\partial u})^2 + \frac{\partial h}{\partial u}\frac{\partial h}{\partial v} - \frac{\partial h}{\partial u}\frac{\partial h}{\partial v} > 0.$

The two eigenvalues κ_1 and κ_2 of L are called principal curvatures. The mean curvature and the Gauß curvature K are defined by

$$H = \frac{1}{2}\,\mathrm{trace}\,L = \frac{1}{2}\,b_{\alpha\beta}\,a^{\alpha\beta} = \kappa_1 + \kappa_2 \quad \text{and} \quad K = \det L = \frac{\det b_{\alpha\beta}}{\det a_{\alpha\beta}} = \kappa_1\,\kappa_2, \tag{2.17}$$

[Sla90]. So that finally the mean curvature becomes for our parametrization

$$H = \frac{1}{2}\left(\frac{\frac{\partial^2 h}{\partial u^2}\left(1+(\frac{\partial h}{\partial v})^2\right) - 2\,\frac{\partial^2 h}{\partial u \partial v}\,\frac{\partial h}{\partial u}\,\frac{\partial h}{\partial v} + \frac{\partial^2 h}{\partial v^2}\left(1+(\frac{\partial h}{\partial u})^2\right)}{\left(\frac{\partial h}{\partial u}\right)^2 + \left(\frac{\partial h}{\partial v}\right)^2 + 1\Big)^{\frac{3}{2}}}\right). \tag{2.18}$$

For the computation of the mean curvature with (2.17) only the covariant base vectors and the derivatives of the covariant base vectors need to be computed. If (2.12) is used to compute the mean curvature also the contravariant base vectors and their derivatives need to be computed. That shows that using the shape operator to compute the mean curvature often simplifies the computations, especially in the case of orthogonal coordinate systems. However, the definition of the mean curvature with (2.12) is more physically intuitive. We will use this formulas again in chapter 6.

Chapter 3

Kinematics of bulk fluids and of the moving interface

In this chapter we discuss the kinematical relations that are necessary to formulate the balance equations of the condensation problem.[1] Experiments show that a fluid interface is in fact a three-dimensional region with a thickness on the micro-scale level [EBW91] [Sla99]. Following [Gib28] such an interface can be regarded as a two-dimensional dividing surface where the effects of the interface on the adjoining bulk phases are represented by surface excess mass, momentum and energy [Sla90]. By this the effect of surface tension is included in the balance equations of the condensation problem (see section 4.3).

First we give the kinematical relations for the bulk fluids, then a material volume with an internal interface is considered. Next we discuss the kinematical relations of the two-dimensional moving interface and in the last section we deal with kinematical relations related to curvature of the interface.

The first part of this chapter is mainly based on [Ari89] [Sch99] [BB75] and [TT60]. The interface related sections are further based on [Sla90] [Sla99] and on [EBW91].

3.1 Kinematics of a material volume

To clarify terminology we first recall some kinematics of bulk fluids and the Reynolds transport theorem for a material volume.

[1] "Kinematics is the description of motion per se" [Ari89].

Basic kinematics

From the basic assumption of continuum theory, a body consist at every moment of infinitely many particles without dimension and no space between them, it follows that every particle correspond to a position in space. A particle is represented at a given initial time, by a position vector $\boldsymbol{\xi}$, as shown in figure 3.1. The coordinates of $\boldsymbol{\xi}$ are called material coordinates. At another time the same particle is represented by another position vector as a function of the initial position of the particle and time

$$\boldsymbol{x} = \boldsymbol{x}(\boldsymbol{\xi}, t) \tag{3.1}$$

The coordinates of $\boldsymbol{x}$ are called spatial coordinates. The initial position of the particle is taken as a reference configuration. Equation (3.1) defines the motion of a particle [TT60]. Assuming continuous motion and that a particle can not occupy two places at the same time the relation is a one-to-one mapping and we can also write conversely

$$\boldsymbol{\xi} = \boldsymbol{\xi}(\boldsymbol{x}, t) \ .$$

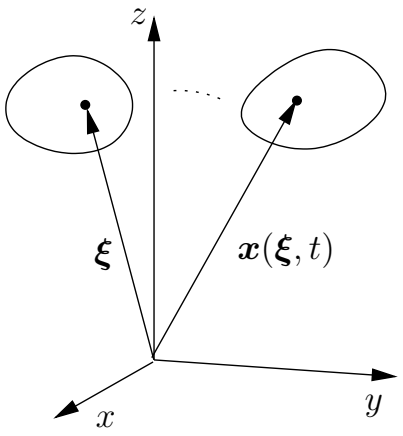


Figure 3.1: *Moving particle*

Physical quantities like density, velocity and temperature, which are functions of space and time, are called field variables and they are here denoted by φ. A field variable can also be given as a function of particle and time. The representation of a field variable as a function of space and time is called spatial (or Euler) representation, that is

$$\varphi = \varphi(\boldsymbol{x}, t) \qquad \text{or} \qquad \varphi = \varphi(\boldsymbol{\xi}(\boldsymbol{x}, t), t) \ .$$

The representation of a field variable as a function of particle and time is called material (or Lagrange) representation, that is

$$\varphi = \varphi(\boldsymbol{\xi}, t) \qquad \text{or} \qquad \varphi = \varphi(\boldsymbol{x}(\boldsymbol{\xi}, t), t) \ .$$

Balance equations of mass, momentum and energy are appropriately represented in spatial variables [BB75].

Material derivative and velocity

Field variables are functions of several variables, so that their derivatives are partial derivatives. Partial derivatives where spatial coordinates are held constant are denoted by ∂. Partial derivatives where material coordinates are held constant are denoted with an uppercase D [Sch99].[2] The partial derivative with respect to time

$$\frac{\mathrm{D}\varphi}{\mathrm{D}t} = \frac{\partial\varphi(\boldsymbol{\xi},t)}{\partial t} = \left(\frac{\partial\varphi}{\partial t}\right)_{\boldsymbol{\xi}} .$$

is called material (or convected) derivative and gives the rate of change an observer moving with the particle would see. The material derivative of a position vector is the velocity of a given particle

$$\boldsymbol{v} = \frac{\mathrm{D}\boldsymbol{x}}{\mathrm{D}t} . \tag{3.2}$$

Balance equations are given in spatial coordinates. To obtain the material derivative of a field variable $\varphi(\boldsymbol{x}(\boldsymbol{\xi},t),t)$ in spatial variables the chain rule has to be applied

$$\begin{aligned} \frac{\mathrm{D}\varphi}{\mathrm{D}t} &= \frac{\partial\varphi}{\partial t} + \frac{\partial\varphi}{\partial x_i}\frac{\mathrm{D}x_i}{\mathrm{D}t} , \\ &= \frac{\partial\varphi}{\partial t} + \boldsymbol{v}\cdot\boldsymbol{\nabla}\varphi , \end{aligned} \tag{3.3}$$

where we used Einstein summation convention. The material derivative is the local rate of change of a given particle at a given position and at a given time plus the convective rate of change related to the moving volume [Sla99].

Reynolds transport theorem for a material volume

For the derivation of the balance equations we need the Reynolds transport theorem for a material volume. A mass conserving volume is called material volume (or material body) and here denoted by V_0. It is moving with time and deforming in general. A quantity $\mathcal{B}_0$ continuously defined over a material volume V_0 is given by $\mathcal{B}_0 = \int_{V_0}\varphi\,\mathrm{d}V$. The rate of change of $\mathcal{B}_0$ with respect to time is given by

$$\frac{\mathrm{d}\mathcal{B}_0(t)}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{V_0(\boldsymbol{x},t)} \varphi(\boldsymbol{x},t)\,\mathrm{d}V .$$

[2] Another common notation for the material derivative is a dot on the variable $\dot{x}$.

In a spatial representation $V_0(\boldsymbol{x},t)$ depends on time, so that integration and differentiation can not be interchanged. With the Jacobian $J = \det\left(\frac{\partial x_i}{\partial \xi_j}\right)$ the volume element can be transformed from spatial coordinates into material coordinates $\mathrm{d}V = J\,\mathrm{d}V_0$. The material volume element $\mathrm{d}V_0$ does not depend on time, so that then integration and differentiation can be interchanged. The time derivative becomes the material derivative

$$\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{V_0(\boldsymbol{x},t)} \varphi(\boldsymbol{x},t)\,\mathrm{d}V = \int\limits_{V_0(\boldsymbol{\xi},t)} \frac{\mathrm{D}}{\mathrm{D}t}\Big(\varphi(\boldsymbol{\xi})\,J\Big)\,\mathrm{d}V_0\;.$$

Using the material derivative of the Jacobian $\frac{\mathrm{D}J}{\mathrm{D}t} = J\,\boldsymbol{\nabla}\cdot\boldsymbol{v}$ we get

$$\int \frac{\mathrm{D}}{\mathrm{D}t}\Big(\varphi\,J\Big)\,\mathrm{d}V_0 = \int\left(\frac{\mathrm{D}\varphi}{\mathrm{D}t}\,J + \varphi\,\frac{\mathrm{D}J}{\mathrm{D}t}\right)\,\mathrm{d}V_0 = \int\left(\frac{\mathrm{D}\varphi}{\mathrm{D}t} + \varphi\,\boldsymbol{\nabla}\cdot\boldsymbol{v}\right)J\,\mathrm{d}V_0\;,$$

where we dropped the integration limits for simplicity [Ari89]. After transforming the volume element back to the spatial volume element and by using the material derivative (3.3) of a field variable we get for the rate of change with time of $\mathcal{B}_0$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{V_0}\varphi\,\mathrm{d}V &= \int\left(\frac{\mathrm{D}\varphi}{\mathrm{D}t} + \varphi\,\boldsymbol{\nabla}\cdot\boldsymbol{v}\right)\,\mathrm{d}V\;, \\ &= \int\left(\frac{\partial\varphi}{\partial t} + \boldsymbol{\nabla}\cdot[\varphi\,\boldsymbol{v}]\right)\,\mathrm{d}V\;.\end{aligned} \tag{3.4}$$

Note that when the derivative of the integral is taken the integration domain has to be indicated. Using Gauß theorem[3] the divergence term in the volume integral can be changed into a surface integral

$$\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{V_0}\varphi\,\mathrm{d}V = \int\frac{\partial\varphi}{\partial t}\,\mathrm{d}V + \oint\varphi\,\boldsymbol{v}\cdot\boldsymbol{n}\,\mathrm{d}A\;, \tag{3.5}$$

where the normal vector is directed outwards on the surface. The velocity $\boldsymbol{v}$ is the velocity of mass while moving across the surface. Equations (3.4) – (3.5) are called Reynolds transport theorem. In the form of (3.5) the Reynolds transport theorem has a physical meaning: The rate of accumulation of a quantity in a material volume can be interpreted as the rate of accumulation of the quantity in a volume that equals the material volume at a given time plus convective flux (connected to mass) leaving the volume through the surface at that time [Sch99]. In Dziubek [Dzi04] a more detailed discussion of the Reynolds transport theorem can be found.

[3] The Gauß theorem or divergence theorem for a vector $\boldsymbol{f}$ is given as: $\int\boldsymbol{\nabla}\cdot\boldsymbol{f}\,\mathrm{d}V = \oint\boldsymbol{f}\cdot\boldsymbol{n}\,\mathrm{d}A$.

3.2 Reynolds transport theorem for a material volume with an interface

In this section we consider a material volume $V_0 = V_l + V_g$, where the field variable has the value φ_l (l for liquid phase) in the volume V_l and the value φ_g (g for gas phase) in the volume V_g, as shown in figure 3.2. An interface between two immiscible fluids is called a material interface, if is formed by the same material elements or particles at all times. If phase change occurs at an interface between two aggregate states of a fluid, as is the case in the condensation problem, the surface velocity $\boldsymbol{u}$ of the interface differs from the velocity $\boldsymbol{v}$ of the mass, and the interface is called singular interface, Greve [Gre03].

Jump

The difference between the two values at the surface is denoted by

$$\Big[\!\Big[\varphi\Big]\!\Big] := \varphi_g - \varphi_l$$

and called the jump of φ [TT60].

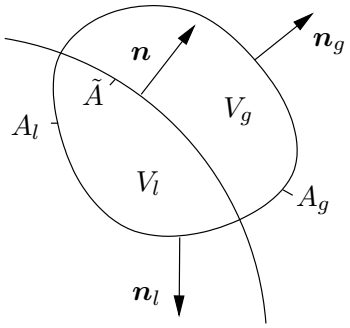


Figure 3.2: *Material volume with interface*

The rate of change of $\mathcal{B}_0$ with respect to time is the sum of the rate of change of $\mathcal{B}_l$ and $\mathcal{B}_g$ with respect to time

$$\frac{\mathrm{d}\mathcal{B}_0(t)}{\mathrm{d}t} = \frac{\mathrm{d}\mathcal{B}_l(t)}{\mathrm{d}t} + \frac{\mathrm{d}\mathcal{B}_g(t)}{\mathrm{d}t} \,,$$

that is

$$\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{V_0} \varphi \,\mathrm{d}V = \frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{V_l} \varphi_l \,\mathrm{d}V + \frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{V_g} \varphi_g \,\mathrm{d}V \,.$$

The volumes V_l and V_g are not material, so that we need the Reynolds transport theorem in a modified version for an arbitrary volume.

Reynolds transport theorem for two arbitrary volumes

A quantity $\mathcal{B}_u$ which is continuously defined over an arbitrary volume V_u is given by $\mathcal{B}_u = \int_{V_u} \varphi \, \mathrm{d}V$. The volume V_u is assumed to consist of fictive mass and shall be material (conserving the fictive mass). Then the rate of change with time of $\mathcal{B}_u$ is according to the Reynolds transport theorem (3.5) given as

$$\frac{\mathrm{d}}{\mathrm{d}t} \int\limits_{V_u} \varphi \, \mathrm{d}V = \int \frac{\partial \varphi}{\partial t} \, \mathrm{d}V + \oint \varphi \, \boldsymbol{u} \cdot \boldsymbol{n} \, \mathrm{d}A \,, \tag{3.6}$$

though here the velocity $\boldsymbol{u}$ is the velocity of the boundary of the considered volume.

Now we apply the general formula to our control volumes. With (3.6) the Reynolds transport theorems for the two volumes V_l and V_g are given as

$$\frac{\mathrm{d}}{\mathrm{d}t} \int\limits_{V_l} \varphi \, \mathrm{d}V \;=\; \int\limits_{V_l} \frac{\partial \varphi}{\partial t} \, \mathrm{d}V + \int\limits_{A_l} \varphi \, \boldsymbol{v} \cdot \mathrm{d}\boldsymbol{A} + \int\limits_{\tilde{A}} \varphi_l \, \boldsymbol{u} \cdot \boldsymbol{n} \, \mathrm{d}\tilde{A} \,, \tag{3.7}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t} \int\limits_{V_g} \varphi \, \mathrm{d}V \;=\; \int\limits_{V_g} \frac{\partial \varphi}{\partial t} \, \mathrm{d}V + \int\limits_{A_g} \varphi \, \boldsymbol{v} \cdot \mathrm{d}\boldsymbol{A} + \int\limits_{\tilde{A}} \varphi_g \, \boldsymbol{u} \cdot (-\boldsymbol{n}) \, \mathrm{d}\tilde{A} \,, \tag{3.8}$$

where we choose the normal vector $\boldsymbol{n}$ at the interface such that it shows from the liquid to the vapor. By adding (3.7) and (3.8) we get the Reynolds transport theorem for the entire volume $V_0 = V_l + V_g$ as

$$\frac{\mathrm{d}}{\mathrm{d}t} \int\limits_{V_0} \varphi \, \mathrm{d}V = \int \frac{\partial \varphi}{\partial t} \, \mathrm{d}V + \oint \varphi \, \boldsymbol{v} \cdot \mathrm{d}\boldsymbol{A} - \int\limits_{\tilde{A}} \Big[\varphi\Big] \, \boldsymbol{u} \cdot \boldsymbol{n} \, \mathrm{d}\tilde{A} \,, \tag{3.9}$$

[Sch99]. Equation (3.9) is the Reynolds transport theorem for a material volume with a singular interface. It states that the rate of accumulation of a quantity in a material volume, where φ undergoes a jump on an interface can be interpreted as the rate of accumulation of the quantity in a volume that equals the material volume at a given time plus convective flow of the quantities φ_l and φ_g leaving the volume through the outer surface and the interface at that time. Here again the integration limits are dropped where the integrals are evaluated at a given time. Only the integration domain of the integral along the interface $\tilde{A}$ has to be indicated.

Using Gauß theorem[4] the Reynolds transport theorem (3.9) for a material volume with a singular interface can be rewritten as

$$\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{V_0}\varphi\,\mathrm{d}V=\int\left(\frac{\partial\varphi}{\partial t}+\boldsymbol{\nabla}\cdot[\varphi\,\boldsymbol{v}]\right)\mathrm{d}V+\int\limits_{\tilde{A}}\Big[\varphi\,(\boldsymbol{v}-\boldsymbol{u})\Big]\cdot\boldsymbol{n}\,\mathrm{d}\tilde{A}\,, \tag{3.10}$$

[Sla90].

3.3 Kinematics of the moving interface and velocities

As explained in the introduction of this chapter, the interface between the vapor and the condensate is assumed to be a two-dimensional surface. In this section the kinematics of the moving interface are discussed and the material (or convected) surface derivative, the fluid velocity and the velocity of the interface are given.

Kinematics of the moving interface

The interface is not composed of a fixed set of particles, there will be mass transfer between the interface and the two adjoining phases [EBW91]. According to the basic assumption of continuum theory also a surface consist at every moment of infinitely many particles. In particular, a particle joining the interface coincides with the particle that was at that position before. For a particle that is leaving the interface instantaneously another particle emerges. So although there is a many-to-one mapping between particles and the region in the surface that is occupied by them, we can assign one representing particle for all possible particles at one point [TT60].

The position vector to a point on the surface was given in chapter 2 as a function of surface coordinates and time (2.1) and is here denoted by a lower index S

$$\boldsymbol{x}_{_S}=\boldsymbol{x}_{_S}(u^\alpha,t)\qquad\text{with }\alpha=1,2\,. \tag{3.11}$$

At a given time this particle on the surface is represented by a position vector, which is here also denoted by a lower index S

$$\boldsymbol{\xi}_{_S}=\boldsymbol{\xi}_{_S}(u_0^\alpha)\,. \tag{3.12}$$

[4] For a material volume with an internal interface Gauß theorem becomes (compare footnote 3) $\int\boldsymbol{\nabla}\cdot[\varphi\,\boldsymbol{v}]\,\mathrm{d}V=\oint\varphi\,\boldsymbol{v}\cdot\mathrm{d}\boldsymbol{A}-\int_{\tilde{A}}[\![\varphi]\!]\boldsymbol{v}\cdot\boldsymbol{n}\,\mathrm{d}\tilde{A}$.

We take this as a initial position and call it intrinsic surface reference configuration. Conversely every position in the surface corresponds to a particle

$$u_0^\alpha = u_0^\alpha(\boldsymbol{\xi}_S) \; . \tag{3.13}$$

Obviously the initial position of a particle located at the surface can be identified either by (3.12) or by (3.13). At another time the particle is represented by another set of coordinates as a function of the reference configuration of the surface particle and time

$$u^\alpha = u^\alpha(u_0^\alpha, t) \; . \tag{3.14}$$

With the assumption of a representing surface particle we established a one-to-one mapping between the coordinates of a surface particle and the surface coordinates, so that we can write reversely

$$u_0^\alpha = u_0^\alpha(u^\alpha, t) \; . \tag{3.15}$$

Equations (3.14) and (3.15) describe the intrinsic motion of a surface particle within the surface, without knowing how the surface itself is moving.

The motion of a surface particle in space we get from the motion of the surface (3.11) and the intrinsic motion of the surface particles on the surface (3.14), (3.15) as

$$\boldsymbol{x}_S = \boldsymbol{x}_S(u_0^\alpha, t) \qquad \text{or} \qquad \boldsymbol{x}_S = \boldsymbol{x}_S(u^\alpha(u_0^\alpha, t), t) \; . \tag{3.16}$$

Equation (3.16) is not reversible. A position in space is corresponding to every surface particle, but the converse is not true [Sla90].

A surface field variable is here denoted by φ_S. It can be given as a function of space and time

$$\varphi_S = \varphi_S(\boldsymbol{x}_S, t) \; , \qquad \text{that is with (3.11)} \qquad \varphi_S = \varphi_S(u^\alpha, t) \; .$$

Or it can be given as a function of particle and time

$$\varphi_S = \varphi_S(u_0^\alpha, t) \; , \quad \varphi_S = \varphi_S(u^\alpha(u_0^\alpha, t), t) \; .$$

Material surface derivative

The partial derivative of a surface field variable with respect to time where material surface coordinates are held constant is called material (or convected) surface derivative and it is here denoted with D_S

$$\frac{\mathrm{D}_S \varphi_S}{\mathrm{D}t} = \frac{\partial \varphi_S(u_0^\alpha, t)}{\partial t} = \left(\frac{\partial \varphi_S}{\partial t}\right)_{u_0^\alpha} .$$

It is the rate of change of a surface field variable with respect to time an observer moving with a surface particle would see. The material surface derivative of a position vector is the velocity of a given surface particle

$$\boldsymbol{v} = \frac{\mathrm{D}_S \boldsymbol{x}_S}{\mathrm{D}t} . \tag{3.17}$$

Here we did not denote $\boldsymbol{v}$ with a lower index S to be consistent with the balance equations as they are given later. Surface balance equations are given in spatial surface coordinates. To obtain the surface material derivative of a surface field variable $\varphi_S(u^\alpha(u_0^\alpha, t), t)$ in spatial surface coordinates the chain rule has to be applied

$$\frac{\mathrm{D}_S \varphi_S}{\mathrm{D}t} = \frac{\partial \varphi_S}{\partial t} + \frac{\partial \varphi_S}{\partial u^\alpha}\frac{\mathrm{D}u^\alpha}{\mathrm{D}t} ,$$

With the surface gradient (2.11)

$$\boldsymbol{\nabla}_S = \frac{\partial}{\partial u^\alpha}\boldsymbol{a}^\alpha$$

and the intrinsic surface velocity

$$\boldsymbol{w} = \frac{\mathrm{D}u^\alpha}{\mathrm{D}t}\boldsymbol{a}_\alpha \tag{3.18}$$

the material surface derivative becomes

$$\frac{\mathrm{D}_S \varphi_S}{\mathrm{D}t} = \frac{\partial \varphi_S}{\partial t} + \boldsymbol{w} \cdot \boldsymbol{\nabla}_S \varphi_S , \tag{3.19}$$

where we used $\boldsymbol{a}^\alpha \boldsymbol{a}_\alpha = \boldsymbol{I}_S$, according to (2.9). The material surface derivative is the local rate of change at a position of a given surface particle at a given time plus the convective rate of change related to the moving surface.

Velocity of an interface particle relative to velocity of the moving interface

The material surface derivative of the surface position vector $\boldsymbol{x}_S(u^\alpha(u_0^\alpha, t), t)$ is with (3.19) given as

$$\frac{\mathrm{D}_S \boldsymbol{x}_S}{\mathrm{D}t} = \frac{\partial \boldsymbol{x}_S}{\partial t} + \boldsymbol{w} \cdot \boldsymbol{\nabla}_S \boldsymbol{x}_S \ . \tag{3.20}$$

The partial derivative of the surface position vector with respect to time (where u^α held constant) is the velocity of the moving interface

$$\boldsymbol{u} = \frac{\partial \boldsymbol{x}_S}{\partial t} \tag{3.21}$$

The surface gradient of the surface position vector is with (2.3) the surface identity tensor

$$\boldsymbol{\nabla}_S \boldsymbol{x}_S = \frac{\partial \boldsymbol{x}_S}{\partial u^\alpha} \boldsymbol{a}^\alpha = \boldsymbol{a}_\alpha \boldsymbol{a}^\alpha = \boldsymbol{I}_S \ . \tag{3.22}$$

Using (3.21), (3.22) and (3.17) the material derivative of the surface position vector becomes an equation with three velocities

$$\begin{aligned} \frac{\mathrm{D}_S \boldsymbol{x}_S}{\mathrm{D}t} &= \frac{\partial \boldsymbol{x}_S}{\partial t} + \boldsymbol{w} \cdot \boldsymbol{\nabla}_S \boldsymbol{x}_S \ , \\ \boldsymbol{v} &= \boldsymbol{u} + \boldsymbol{w} \ . \end{aligned} \tag{3.23}$$

By this we see that the intrinsic surface velocity is the velocity of a surface particle relative to the velocity of the surface

$$\boldsymbol{w} = \boldsymbol{v} - \boldsymbol{u} \ . \tag{3.24}$$

Note that in general $\boldsymbol{u}$ has a normal and a tangential part, so that $\boldsymbol{w}$ is not necessarily the tangential part of $\boldsymbol{v}_S$.

Surface velocity for an implicit surface parametrization

In section 2.3 we discussed the geometrical properties of a surface defined by an implicit function $F(\boldsymbol{x}_S(u^\alpha, t), t) = 0$. Differentiating $F = 0$ with respect to time gives

$$\frac{\partial F}{\partial t} + \frac{\partial F}{\partial x_{Si}} \frac{\partial x_{Si}}{\partial t} = 0 \qquad \text{and equivalently} \qquad \frac{\partial F}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} F = 0 \ . \tag{3.25}$$

With the normal vector $\boldsymbol{n} = \frac{\boldsymbol{\nabla} F}{|\boldsymbol{\nabla} F|}$ as derived in (2.14) we then can write either

$$\boldsymbol{u} \cdot \boldsymbol{n} = -\frac{\frac{\partial F}{\partial t}}{|\boldsymbol{\nabla} F|} \qquad \text{or} \qquad \boldsymbol{u} \cdot \boldsymbol{n} = \boldsymbol{u} \cdot \frac{\boldsymbol{\nabla} F}{|\boldsymbol{\nabla} F|} \; . \tag{3.26}$$

The first equation is independent of the parametrization, so that all possible surface velocities have the same normal component $\boldsymbol{u} \cdot \boldsymbol{n}$, which is called speed of displacement [TT60] [Sla90]. It is convenient to choose a parametrization such that the surface velocity becomes the surface normal velocity

$$\boldsymbol{u} = \boldsymbol{u} \cdot \boldsymbol{n} \, \boldsymbol{n} \; .$$

The surface defined by $F(u, v, z, t) = z - h(u, v, t)$ has the surface position vector

$$\boldsymbol{x}_S(u, v, t) = u \, \boldsymbol{e}_x + v \, \boldsymbol{e}_y + h(u, v, t) \, \boldsymbol{e}_z \; .$$

For an implicit surface parametrization the surface velocity is the surface normal velocity and is given by

$$\boldsymbol{u} = \frac{\partial \boldsymbol{x}_S}{\partial t} = \begin{bmatrix} 0 \\ 0 \\ \frac{\partial h}{\partial t} \end{bmatrix} \; . \tag{3.27}$$

Multiplying (3.23) with $\boldsymbol{n}$ gives

$$\boldsymbol{v}_s \cdot \boldsymbol{n} = \boldsymbol{u} \cdot \boldsymbol{n} + \frac{\mathrm{D} u^\alpha}{\mathrm{D} t} \, \boldsymbol{a}_\alpha \cdot \boldsymbol{n} \qquad \text{that is} \qquad \boldsymbol{v}_s \cdot \boldsymbol{n} = \boldsymbol{u} \cdot \boldsymbol{n} \; .$$

3.4 Reynolds transport theorem and divergence theorem for a surface

In the balance equations we also need the Reynolds transport theorem for the interface between condensate and vapor. Because of the mass transfer due to condensation the interface is not material. However, we can always assume the interface is composed of a fixed set of particles, as explained in section 3.3. For a quantity $\mathcal{S}_0$ continuously defined over such an interface $\tilde{A}_0$ we write $\mathcal{S}_0 = \int_{\tilde{A}_0} \varphi_S \mathrm{d}\tilde{A}$. The rate of change of $\mathcal{S}_0$ with respect to time is given by

$$\frac{\mathrm{d}\mathcal{S}_0(t)}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t} \int\limits_{\tilde{A}_0(\boldsymbol{x}_S, t)} \varphi_S(\boldsymbol{x}_S, t) \, \mathrm{d}\tilde{A} \; .$$

In a spatial representation $\tilde{A}_0(\boldsymbol{x}_S, t)$ depends on time. Analog to the Reynolds transport theorem for a material volume we transform the area element with the surface Jacobian determinant $j = \det\left(\frac{\partial x_{S_i}}{\partial \xi_{S_j}}\right)$ from spatial coordinates into material coordinates $\mathrm{d}\tilde{A} = j\,\mathrm{d}\tilde{A}_0$ [Sla90]. Then $\tilde{A}_0(\boldsymbol{\xi}_S)$ does not depend on time and integration and differentiation can be interchanged. The time derivative becomes the material surface derivative (3.19)

$$\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{A_0} \varphi_S\,\mathrm{d}\tilde{A} = \int \frac{\mathrm{D}_S}{\mathrm{D}t}\Big(\varphi_S\, j\Big)\,\mathrm{d}\tilde{A}_0\ .$$

The material surface derivative of the surface Jacobian determinant is $\frac{\mathrm{D}_S j}{\mathrm{D}t} = j\,\boldsymbol{\nabla}_S \boldsymbol{v}_S$, so that we get

$$\int \frac{\mathrm{D}_S}{\mathrm{D}t}\Big(\varphi_S\, j\Big)\,\mathrm{d}\tilde{A}_0 = \int \left(\frac{\mathrm{D}_S\varphi_S}{\mathrm{D}t}\, j + \varphi_S\,\frac{\mathrm{D}_S j}{\mathrm{D}t}\right)\,\mathrm{d}\tilde{A}_0 = \int \left(\frac{\mathrm{D}_S\varphi_S}{\mathrm{D}t} + \varphi_S\,\boldsymbol{\nabla}\cdot\boldsymbol{v}_S\right) j\,\mathrm{d}\tilde{A}_0,$$

where we can transform the area element back into spatial coordinates [EBW91]. Using the material surface derivative (3.19) with the relative velocity (3.24) we get for the rate of change of $\mathcal{S}_0$ with respect to time

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{\tilde{A}_0} \varphi_S\,\mathrm{d}\tilde{A} &= \int \left(\frac{\mathrm{D}_S\varphi_S}{\mathrm{D}t} + \varphi_S\,\boldsymbol{\nabla}\cdot\boldsymbol{v}_S\right)\,\mathrm{d}\tilde{A}\,, \\
&= \int \left(\frac{\partial\varphi_S}{\partial t} + [\boldsymbol{v}_S - \boldsymbol{u}]\cdot\boldsymbol{\nabla}_S\varphi_S + \varphi_S\,\boldsymbol{\nabla}\cdot\boldsymbol{v}_S\right)\,\mathrm{d}\tilde{A}\,, \qquad (3.28)\\
&= \int \left(\frac{\partial\varphi_S}{\partial t} + \boldsymbol{\nabla}_S\cdot[\varphi_S\,\boldsymbol{v}_S] - \boldsymbol{u}\cdot\boldsymbol{\nabla}_S\varphi_S\right)\,\mathrm{d}\tilde{A}\,.
\end{aligned}$$

This equations are called Reynolds transport theorem for surfaces. Comparing the last equation with the Reynolds transport theorem for material volumes (3.4) here we have an additional term related to the moving surface.

Divergence theorem for surfaces

Finally the surface integral with the surface divergence term in the last equation of (3.28) is transformed further using the so called divergence theorem for surfaces. The divergence theorem for surfaces will be used here and again in section 4.3. Therefore we derive it using the abbreviation $\boldsymbol{f} = \varphi_S\,\boldsymbol{v}_S$. Then, splitting $\boldsymbol{f}$ in

normal and tangential part we get for the surface integral of the surface divergence of $\boldsymbol{f}$

$$\int \boldsymbol{\nabla}_S \cdot \boldsymbol{f}\, \mathrm{d}\tilde{A} = \int \boldsymbol{\nabla}_S \cdot [(\boldsymbol{f}\cdot\boldsymbol{n})\,\boldsymbol{n}]\, \mathrm{d}\tilde{A} + \int \boldsymbol{\nabla}_S \cdot [(\boldsymbol{f}\cdot\boldsymbol{m})\,\boldsymbol{m}]\, \mathrm{d}\tilde{A}\,.$$

Firstly, the first integral on the right hand side is considered. Applying product rule on the integrand result in

$$\boldsymbol{\nabla}_S \cdot [(\boldsymbol{f}\cdot\boldsymbol{n})\,\boldsymbol{n}] = \underbrace{\boldsymbol{n}\cdot\boldsymbol{\nabla}_S}_{=0}(\boldsymbol{f}\cdot\boldsymbol{n}) + (\boldsymbol{f}\cdot\boldsymbol{n})\,\boldsymbol{\nabla}_S\cdot\boldsymbol{n} = -2\,H\,\boldsymbol{f}\cdot\boldsymbol{n}\,, \tag{3.29}$$

where we used mean curvature (2.12) as defined in section 2.2, and that the surface gradient (2.9) is perpendicular to the normal vector on the surface.

The second integral is an intrinsic surface integral. The vector $\boldsymbol{m}$ is directed outwards normal on the boundary curve. It is perpendicular to the tangential vector $\boldsymbol{t}$ along the curve and to the surface normal vector $\boldsymbol{m} = \boldsymbol{t}\times\boldsymbol{n}$ as shown in figure 3.3.[5] We use Stokes theorem[6] to transform the surface integral in a curve integral

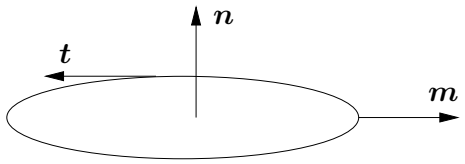


Figure 3.3: *Base vectors on a bounded surface*

$$\int \boldsymbol{\nabla}_S \cdot [(\boldsymbol{f}\cdot\boldsymbol{m})\,\boldsymbol{m}]\, \mathrm{d}\tilde{A} = \oint (\boldsymbol{f}\cdot\boldsymbol{m})\, \mathrm{d}\tilde{C}\,. \tag{3.30}$$

With (3.29) and (3.30) the surface integral of the surface divergence of $\boldsymbol{f}$ becomes

$$\int \boldsymbol{\nabla}_S \cdot \boldsymbol{f}\, \mathrm{d}\tilde{A} = -\int 2\,H\,\boldsymbol{f}\cdot\boldsymbol{n}\, \mathrm{d}\tilde{A} + \oint \boldsymbol{f}\cdot\boldsymbol{m}\, \mathrm{d}\tilde{C}\,. \tag{3.31}$$

This equation is called the divergence theorem for surfaces, (it should not be confused with the divergence theorem in footnote 6).

Alternative version of the Reynolds transport theorem for surfaces

If we substitute in the last equation of (3.28) the surface integral of the divergence term with (3.31) an alternative version of the Reynolds transport theorem for surfaces can be given as

$$\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{\tilde{A}_0} \varphi_S\, \mathrm{d}\tilde{A} = \int \left(\frac{\partial\varphi_S}{\partial t} - \boldsymbol{u}\cdot\boldsymbol{\nabla}_S\varphi_S - 2\,H\,\varphi_S\,\boldsymbol{v}_S\cdot\boldsymbol{n}\right) \mathrm{d}\tilde{A} + \oint \varphi_S\,\boldsymbol{v}_S\cdot\boldsymbol{m}\, \mathrm{d}\tilde{C}. \tag{3.32}$$

[5] See [Sla99] for the relation between $\boldsymbol{m}$, $\boldsymbol{t}$ and the local base vectors at the interface as defined in chapter 2.

[6] Stokes theorem entirely defined in surface vectors is given as: $\int \boldsymbol{\nabla}_S\cdot\boldsymbol{f}\, \mathrm{d}A = \oint \boldsymbol{f}\cdot\boldsymbol{m}\, \mathrm{d}C$.

The rate of accumulation of a quantity in a surface can be interpreted as the rate of accumulation of the quantity in a material surface that equals the surface at a given time plus flux arising from the moving surface, plus convective flux normal to the surface (curvature term) and convective flux through the boundary curve of the area.

Chapter 4

Generic model equations for two-phase flows with surface tension

In this chapter we derive the generic model equations for the moving surface problem. First we give the generic differential balance equation for incompressible fluids and then we derive a generic differential balance equation at a moving interface between two fluids, the so called jump condition. This interface balance equation includes phase change but does not yet include surface tension. To include surface tension a balance equation for the interface itself has to be formulated, which is then added to the generic balance equation for the bulk phases. By this we derive the desired generic jump condition.

The material about balance equations in general is mainly based on [BSL60] [Sch99] and [Sla99]. Further on [Dee98] [Gre03] and [Hut03]. The material about interface balance equations is mainly based on [TT60] [Sch99] [Sla90] and [EBW91].

4.1 Balance equations for bulk fluids

Balance equations are formulated for physical quantities that are continuously defined over a spatial region (for instances a volume), such as mass, momentum or energy. We denote those quantities by $\mathcal{B} = \int_V \varphi \, \mathrm{d}V$.

Generic balance equation for a material volume

A material (mass conserving) volume V_0 is in general moving with time. A generic balance equation for a physical quantity $\mathcal{B}_0 = \int_{V_0} \varphi \, \mathrm{d}V$ in a material volume states

that the rate of accumulation of quantity in the material volume is given by conductive flux of the quantity (not connected to mass) that enters the volume across the surface plus supply of the quantity to the material volume[1]

$$\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{V_0} \varphi \,\mathrm{d}V = -\oint \boldsymbol{\zeta}\cdot \mathrm{d}\boldsymbol{A} + \int \pi \,\mathrm{d}V \;, \tag{4.1}$$

where $\boldsymbol{\zeta}$ is the flux density and π is the supply density [Sch99]. The surface element vector $\mathrm{d}\boldsymbol{A} = \boldsymbol{n}\,\mathrm{d}A$ is directed outwards normal on the surface. As before we drop the integration limits except in the case the derivative of the integral is taken. With the Reynolds transport theorem (3.4) the generic balance equation (4.1) for a material volume becomes

$$\int \left(\frac{\partial \varphi}{\partial t} + \boldsymbol{\nabla}\cdot[\varphi\,\boldsymbol{v}]\right) \mathrm{d}V = -\oint \boldsymbol{\zeta}\cdot \mathrm{d}\boldsymbol{A} + \int \pi \,\mathrm{d}V \;. \tag{4.2}$$

Generic balance equation for a stationary volume

To derive a generic balance equation for a stationary volume the Reynolds transport theorem in the form of (3.5) is substituted for the right hand side of (4.1)

$$\int \frac{\partial \varphi}{\partial t}\,\mathrm{d}V + \oint \varphi\,\boldsymbol{v}\cdot \mathrm{d}\boldsymbol{A} \;= -\oint \boldsymbol{\zeta}\cdot \mathrm{d}\boldsymbol{A} + \int \pi \,\mathrm{d}V \;.$$

Now the integration domain of the volume integral on the left hand side is constant and derivation and integration can be interchanged. By doing so we get the generic balance equation for a stationary volume

$$\frac{\mathrm{d}}{\mathrm{d}t}\int \varphi \,\mathrm{d}V = -\oint \varphi\,\boldsymbol{v}\cdot \mathrm{d}\boldsymbol{A} - \oint \boldsymbol{\zeta}\cdot \mathrm{d}\boldsymbol{A} + \int \pi \,\mathrm{d}V \;. \tag{4.3}$$

This equation has again a physical meaning: The accumulation of $\mathcal{B} = \int \varphi \,\mathrm{d}V$ in a stationary volume is given by convective and conductive flux of quantity across the surface to the volume plus supply of quantity to the volume [BSL60].

[1] Some authors distinguish between supply of quantity to the volume and production of quantity within the volume. Then conservation equations can be defined as balance equations without a production term. However, it is more intuitive to distinguish only between surface terms and volume terms.

Generic differential balance equation

Next, we need a differential version of the generic balance equation for the numerical computation. The starting point is (4.2)

$$\int \left(\frac{\partial \varphi}{\partial t} + \boldsymbol{\nabla} \cdot [\varphi \, \boldsymbol{v}] \right) \mathrm{d}V = - \oint \boldsymbol{\zeta} \cdot \mathrm{d}\boldsymbol{A} + \int \pi \, \mathrm{d}V \; .$$

Using divergence theorem (see footnote 3 in section 3.1) we transform the surface integral in a volume integral and get

$$\int \left(\frac{\partial \varphi}{\partial t} + \boldsymbol{\nabla} \cdot \left[\varphi \, \boldsymbol{v} + \boldsymbol{\zeta} \right] - \pi \right) \mathrm{d}V = 0 \; .$$

This equation must hold for any arbitrary volume. By this we get the differential balance equation we were looking for

$$\frac{\partial \varphi}{\partial t} + \boldsymbol{\nabla} \cdot (\varphi \, \boldsymbol{v}) = - \boldsymbol{\nabla} \cdot \boldsymbol{\zeta} + \pi \; . \tag{4.4}$$

Although this equation represent the same physical phenomenon as before (accumulation, flux, supply) the various terms can not be interpreted in the same way as the integral balance equations [Sch99].

In section 5.1 the generic differential balance equation will be applied on mass, momentum and energy of the condensate and the vapor.

4.2 Jump conditions at an interface between two fluids

In the condensation process we have two homogeneous bulk phases, the vapor and the condensate, and we have the interface between both phases. At the interface the phase change from the gas phase to the liquid phase occur. Here the generic balance equations of the interface are derived, particulary under the criterion of phase change.

Generic balance equation for a material volume with a singular interface

The generic balance equation (4.1) for a quantity $\mathcal{B}_0$ in a material volume

$$\frac{\mathrm{d}}{\mathrm{d}t} \int\limits_{V_0} \varphi \, \mathrm{d}V = - \oint \boldsymbol{\zeta} \cdot \mathrm{d}\boldsymbol{A} + \int \pi \, \mathrm{d}V$$

holds for a material volume whether or not there is a singular surface within it [TT60]. However, the rate of accumulation in V_0 is the sum of the rate of accumulation in the volumes V_l and V_g

$$\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{V_0}\varphi\,\mathrm{d}V = \frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{V_l}\varphi\,\mathrm{d}V + \frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{V_g}\varphi\,\mathrm{d}V\ .$$

The volumes V_l and V_g are not material, so we need the Reynolds transport theorem for an arbitrary volume with a singular interface. Substituting (3.10) for the left hand side of (4.1) gives us the generic balance equation for a material volume with a singular interface

$$\int\left(\frac{\partial\varphi}{\partial t} + \boldsymbol{\nabla}\cdot[\varphi\,\boldsymbol{v}]\right)\mathrm{d}V + \int\limits_{\tilde{A}}\Big[\!\Big[\varphi\,(\boldsymbol{v}-\boldsymbol{u})\Big]\!\Big]\cdot\boldsymbol{n}\,\mathrm{d}\tilde{A} = -\oint\boldsymbol{\zeta}\cdot\mathrm{d}\boldsymbol{A} + \int\pi\,\mathrm{d}V \quad (4.5)$$

The double bracket denotes the difference between the two values at the surface as discussed in section 3.2.

Generic jump condition

To derive a differential form of (4.5) a special volume in a form of a small box is considered, which is moving together with the interface as shown in figure 4.1. Two faces of the box are parallel to the interface. By taking the limit of the shorter side faces $A_h \to 0$ the volume integrals vanish and A_l and A_g merge with $\tilde{A}$. For the volume integrals to vanish their integrands must be limited (but not necessary continuously) [TT60]. Then the normal vectors of the two outer faces of the box transform into either $\boldsymbol{n}$ or $-\boldsymbol{n}$ and only the surface integral over the interface $\tilde{A}$ remains

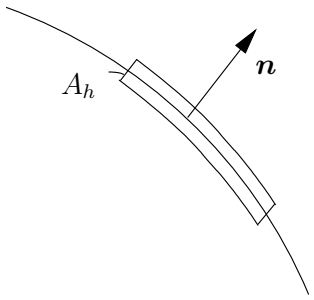


Figure 4.1: *Volume in form of a box*

$$\int\limits_{\tilde{A}}\left(\Big[\!\Big[\varphi\,(\boldsymbol{v}-\boldsymbol{u})\Big]\!\Big]\cdot\boldsymbol{n} + \Big[\!\Big[\boldsymbol{\zeta}_l\cdot(-\boldsymbol{n}) - \boldsymbol{\zeta}_g\cdot\boldsymbol{n}\Big]\!\Big]\right)\mathrm{d}\tilde{A} = 0\ .$$

The integral must hold for any arbitrary surface so that the integrand must be zero

$$\Big[\!\Big[\varphi\,[\boldsymbol{v}-\boldsymbol{u}]\cdot\boldsymbol{n}\Big]\!\Big] + \Big[\!\Big[\boldsymbol{\zeta}\cdot\boldsymbol{n}\Big]\!\Big] = 0\ . \quad (4.6)$$

Equation (4.6) is called generic jump condition and describes the phase change at the interface and the conductive flux across the interface [Sla90].

With this jump condition we can describe phase change and conductive flux across the surface, but it does not allow to model intrinsic surface properties like surface tension, compare chapter 3. In analogy to the external surfaces an interface of infinitesimal thickness was assumed (we took the limit of $A_h \to 0$).

4.3 Balance equation and jump condition including surface tension

Based on the assumption that a fluid interface is actually a three-dimensional region with a thickness of maybe one or more molecule diameter, the effect of the interface on the adjoining bulk fluids is best represented by assuming a two-dimensional interface consisting of surface excess mass. Surface mass is assumed to have similar properties as three-dimensional mass, such as surface density, surface viscosity, surface tension and so on [EBW91]. Then, analog to the generic balance equation of three dimensional continua, a generic balance equation for the interface can be given. By adding the so derived interface balance equation to the generic balance equation for a material volume with a singular interface we finally get a generic balance equation including surface tension.

Generic balance equation for a surface

The interface between condensate and vapor is not material, the fluid velocity differs from the velocity of the surface. Nevertheless a balance equation similar to the generic balance equation (4.1) for a material volume can ge given, as explained in section 3.3. For a quantity $\mathcal{S}_0$ continuously defined over a surface $\tilde{A}$ we write $\mathcal{S}_0 = \int_{\tilde{A}} \varphi_S \, \mathrm{d}\tilde{A}$. Then a generic balance equation for $\mathcal{S}_0$ states that the rate of accumulation of surface quantity in the surface $\tilde{A}$ is given by conductive flux of surface quantity across the boundary curve of the surface plus supply of surface quantity at the surface

$$\frac{\mathrm{d}}{\mathrm{d}t} \int\limits_{\tilde{A}} \varphi_S \, \mathrm{d}\tilde{A} = - \oint \boldsymbol{\zeta}_S \cdot \mathrm{d}\tilde{\boldsymbol{C}} + \int \pi_S \, \mathrm{d}\tilde{A} \,, \tag{4.7}$$

where $\boldsymbol{\zeta}_S$ is the surface flux density and π_S is the surface supply density [Sla90]. The curve element vector $\mathrm{d}\tilde{\boldsymbol{C}} = \boldsymbol{m} \, \mathrm{d}\tilde{C}$ is directed outwards normal on the boundary

curve, see figure 3.3 in section 3.4. With the Reynolds theorem for a surface (3.28) equation (4.7) becomes

$$\int \left(\frac{\partial \varphi_S}{\partial t} + \boldsymbol{\nabla}_S \cdot [\varphi_S \, \boldsymbol{v}_S] - \boldsymbol{u} \cdot \boldsymbol{\nabla}_S \varphi_S \right) \mathrm{d}\tilde{A} = - \oint \boldsymbol{\zeta}_S \cdot \mathrm{d}\tilde{\boldsymbol{C}} + \int \pi_S \, \mathrm{d}\tilde{A} \, . \tag{4.8}$$

Generic differential balance equation for a surface

To get a differential version of the generic surface balance equation (4.8) we transform the curve integral on the right hand side in an area integral using the surface divergence theorem (3.31)

$$\int \left(\frac{\partial \varphi_S}{\partial t} + \boldsymbol{\nabla}_S \cdot [\varphi_S \, \boldsymbol{v}_S] - \boldsymbol{u} \cdot \boldsymbol{\nabla}_S \varphi_S + \left[\boldsymbol{\nabla}_S \cdot \boldsymbol{\zeta}_S + 2 \, H \, \boldsymbol{\zeta}_S \cdot \boldsymbol{n} \right] - \pi_S \right) \mathrm{d}\tilde{A} = 0 \, ,$$

where H is the mean curvature. This equation must hold for any arbitrary area so that we get the differential surface balance equation

$$\frac{\partial \varphi_S}{\partial t} + \boldsymbol{\nabla}_S \cdot [\varphi_S \, \boldsymbol{v}_S] - \boldsymbol{u} \cdot \boldsymbol{\nabla}_S \varphi_S = - \left[\boldsymbol{\nabla}_S \cdot \boldsymbol{\zeta}_S + 2 \, H \, \boldsymbol{\zeta}_S \cdot \boldsymbol{n} \right] + \pi_S \, . \tag{4.9}$$

Generic balance equation including phase change and surface tension

Now we are able to state a generic balance equation that includes phase chance and surface tension. For that we add the generic surface balance equation (4.8) to the generic balance equation for a material volume with a singular interface (4.5) and get

$$\begin{aligned} \int \left(\frac{\partial \varphi}{\partial t} + \boldsymbol{\nabla} \cdot [\varphi \, \boldsymbol{v}] \right) \mathrm{d}V + \int_{\tilde{A}} \Big[\varphi \, (\boldsymbol{v} - \boldsymbol{u}) \Big] \cdot \boldsymbol{n} \, \mathrm{d}\tilde{A} \\ + \int \left(\frac{\partial \varphi_S}{\partial t} + \boldsymbol{\nabla}_S \cdot [\varphi_S \, \boldsymbol{v}_S] - \boldsymbol{u} \cdot \boldsymbol{\nabla}_S \varphi_S \right) \mathrm{d}\tilde{A} \\ = - \oint \boldsymbol{\zeta} \cdot \mathrm{d}\boldsymbol{A} - \oint \boldsymbol{\zeta}_S \cdot \mathrm{d}\tilde{\boldsymbol{C}} + \int \pi \, \mathrm{d}V + \int \pi_S \, \mathrm{d}\tilde{A} \, . \end{aligned} \tag{4.10}$$

Generic jump condition including phase change and surface tension

From (4.10) we derive a generic jump condition in the same way as discussed in the last section. For that we transform the curve integral in an area integral using

the surface divergence theorem (3.31) and consider a small volume enclosing the interface that we let shrink into a surface. By this we get

$$\Big[\varphi\,[\boldsymbol{v}-\boldsymbol{u}]\cdot\boldsymbol{n}\Big]+\Big[\boldsymbol{\zeta}\cdot\boldsymbol{n}\Big]=-\frac{\partial\varphi_S}{\partial t}-\boldsymbol{\nabla}_S\cdot[\varphi_S\,\boldsymbol{v}_S]+\boldsymbol{u}\cdot\boldsymbol{\nabla}_S\varphi_S-\underline{\big[\boldsymbol{\nabla}_S\cdot\boldsymbol{\zeta}_S+2\,H\,\boldsymbol{\zeta}_S\cdot\boldsymbol{n}\big]}+\pi_S. \tag{4.11}$$

If there is no material accumulation in the surface, the surface density variables φ_S and π_S are zero and the surface is called a clean surface and from the right hand side only the (underlined) terms remain [EBW91]. For the condensation problem material accumulation in the surface can be neglected. So we finally get

$$\Big[\varphi\,[\boldsymbol{v}-\boldsymbol{u}]\cdot\boldsymbol{n}\Big]+\Big[\boldsymbol{\zeta}\cdot\boldsymbol{n}\Big]=-\boldsymbol{\nabla}_S\cdot\boldsymbol{\zeta}_S-2\,H\,\boldsymbol{n}\cdot\boldsymbol{\zeta}_S\,. \tag{4.12}$$

This jump condition covers phase change, conductive flux across the interface, and surface tension. In section 5.2 we apply (4.12) on mass, momentum and energy and derive jump conditions at the moving interface between condensate and the vapor. For the jump condition we made no additional assumption than to assume a continuous surface, in particular we do not allow the interface to break off.

Chapter 5

Model equations for condensation in a tube with small diameter

In this chapter we obtain the balance equations for mass, momentum and energy for the bulk flow of the both phases and for the interface between them from the generic balance equations derived in the last chapter. Further we discuss appropriate simplifications for the condensate problem. The last section we summarize the up to then deduced system of partial differential equations plus the jump conditions and discuss the boundary conditions. The references for the equations used in this chapter are given in the last chapter unless otherwise noted.

5.1 Mass, momentum and energy equation

Mass

The mass balance equation is given by (4.4) with $\varphi = \rho$ and $\zeta = \pi = 0$

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \, \boldsymbol{v}) = 0 \, , \tag{5.1}$$

which gives for the condensate film with the assumption of constant density

$$\boldsymbol{\nabla} \cdot \boldsymbol{v} = 0 \, . \tag{5.2}$$

Also the vapor can be treated as an incompressible fluid.[1]

[1] If the Mach number of a fluid is small compared to unity the fluid can be considered as an incompressible fluid [BSL60].

For momentum and energy another balance equation which makes use of (5.1) is more preferable. For that we substitute φ in (4.4) by $\varphi = \rho\,\psi$, apply the product rule on both terms on the left side and recieve

$$\frac{\partial(\rho\,\psi)}{\partial t} + \boldsymbol{\nabla}\cdot[(\rho\,\psi)\,\boldsymbol{v}] \;=\; \psi\,\underbrace{\left[\frac{\partial\rho}{\partial t} + \boldsymbol{\nabla}\cdot(\rho\,\boldsymbol{v})\right]}_{=0} + \rho\,\left[\frac{\partial\psi}{\partial t} + \boldsymbol{v}\cdot\boldsymbol{\nabla}\psi\right]\,.$$

The first bracket is zero according to (5.1). The second bracket is the material derivative of ψ as derived in section 3.1.

$$\frac{\mathrm{D}\psi}{\mathrm{D}t} = \frac{\partial\psi}{\partial t} + \boldsymbol{v}\cdot\boldsymbol{\nabla}\psi\,.$$

Thus we get the generic balance equation (4.4) in an equivalent form [Dee98]

$$\rho\,\frac{\mathrm{D}\psi}{\mathrm{D}t} \;=\; -\boldsymbol{\nabla}\cdot\boldsymbol{\zeta} + \pi\,, \tag{5.3}$$

Momentum

The momentum equation we get by substituting in (5.3) $\boldsymbol{\psi} = \boldsymbol{v}$, $\boldsymbol{\zeta} = -\boldsymbol{S}$ and $\boldsymbol{\pi} = \rho\,\boldsymbol{g}$ (note that here $\boldsymbol{\psi}$ and $\boldsymbol{\pi}$ are vectors and $\boldsymbol{\zeta}$ is a second rank tensor)

$$\rho\,\left[\frac{\partial\boldsymbol{v}}{\partial t} + \boldsymbol{v}\cdot\boldsymbol{\nabla}\boldsymbol{v}\right] \;=\; \boldsymbol{\nabla}\cdot\boldsymbol{S} + \rho\,\boldsymbol{g}\,. \tag{5.4}$$

The stress tensor $\boldsymbol{S} = -p\,\boldsymbol{I} + \boldsymbol{T}$ can be divided into a contribution of the fluid at rest and the fluid in motion [BB75].[2] $\boldsymbol{g}$ is the vector of body force, in our case the gravity vector. The condensate and the vapor are Newtonian fluids, so the viscous stress tensor for both phases is given by $\boldsymbol{T} = \mu\,\left[\boldsymbol{\nabla}\boldsymbol{v} + (\boldsymbol{\nabla}\boldsymbol{v})^T\right] + \frac{\mu'}{3}\,(\boldsymbol{\nabla}\cdot\boldsymbol{v})$, with viscosity μ and modified bulk viscosity μ' [BSL60]. Together with the incompressibility condition the momentum equation

$$\rho\,\left[\frac{\partial\boldsymbol{v}}{\partial t} + \boldsymbol{v}\cdot\boldsymbol{\nabla}\boldsymbol{v}\right] \;=\; -\boldsymbol{\nabla}p + \mu\,\nabla^2\boldsymbol{v} + \rho\,\boldsymbol{g}$$

forms then the well known Navier-Stokes equations.

[2] If we had derived the momentum equation from (4.4), the divergence term in the convective term would be nonlinear. The advantage of deriving the momentum equation from (5.3) is that the divergence term is then linear, that makes the numerical discretization easier.

Energy

Lastly, we consider the heat transfer. According to the first law of thermodynamics the increase of internal and kinetic energy in a material control volume is given by heat supply plus power due to work acting on the fluid. The differential energy equation is

$$\rho \frac{\mathrm{D}}{\mathrm{D}t}(e + \frac{v^2}{2}) \quad = \quad (\boldsymbol{\nabla} \cdot \boldsymbol{v} \cdot \boldsymbol{S} + \rho\, \boldsymbol{v} \cdot \boldsymbol{g}) + (-\boldsymbol{\nabla} \cdot \boldsymbol{q} + \rho\, z) \ , \tag{5.5}$$

where e is the internal energy per unit mass, $\boldsymbol{v} \cdot \boldsymbol{S}$ is the power due to surface forces per unit area, $\rho\, \boldsymbol{v} \cdot \boldsymbol{g}$ is the gravity power per unit volume, $\boldsymbol{q}$ the heat flux per unit area and z the heat production per unit volume, which is zero in our case [BSL60]. Here $\psi = e + \frac{v^2}{2}$, $\boldsymbol{\zeta} = \boldsymbol{-v \cdot S + q}$ and $\pi = \rho\,(\boldsymbol{v} \cdot \boldsymbol{g} + z)$. To get the energy in a more commonly used form, we subtract the mechanical energy equation from (5.5). The mechanical energy equation is formed by the scalar product of momentum equation and velocity [Dee98]. By this we get[3]

$$\rho \left[\frac{\partial e}{\partial t} + \boldsymbol{v} \cdot \boldsymbol{\nabla} e\right] \quad = \quad -\boldsymbol{\nabla} \cdot \boldsymbol{q} + \boldsymbol{S} \colon \boldsymbol{\nabla v} \ .$$

The heating effect of friction can be neglected for the condensation problem, so that the dissipative term $\boldsymbol{T} \colon \boldsymbol{\nabla v}$ is zero. Moreover the whole term vanishes $\boldsymbol{S} \colon \boldsymbol{\nabla v} = \boldsymbol{T} \colon \boldsymbol{\nabla v} - p\,(\boldsymbol{I} \colon \boldsymbol{\nabla v}) = -p\,(\boldsymbol{\nabla} \cdot \boldsymbol{v}) = 0$.

Constitutive equations for internal energy and heat flux complete our equations. For small temperature differences internal energy can be described by a linear function $e = c\,(T - T_0) + e(T_0)$, where c is the specific heat capacity. The heat flux is given by Fourier's law $\boldsymbol{q} = -\lambda \boldsymbol{\nabla} T$, where λ is the heat conductivity, Hutter [Hut03]. By this we finally get the energy equation for the condensate and the vapor

$$\rho\, c \left[\frac{\partial T}{\partial t} + \boldsymbol{v} \cdot \boldsymbol{\nabla} T\right] \quad = \quad \lambda\, \nabla^2 T \ .$$

The material properties viscosity, heat capacity and heat conductivity are in general functions of density, pressure and temperature, but for incompressible fluids only temperature dependency need to be considered. The temperature interval between wall temperature and vapor temperature is small ($T_{wall} - T_{vapor} < 5\,\mathrm{K}$), so that we can assume the material properties to be constant.

[3] Note that we used hereby the identity $\boldsymbol{\nabla} \cdot (\boldsymbol{v} \cdot \boldsymbol{S}) = \boldsymbol{v} \cdot (\boldsymbol{\nabla} \cdot \boldsymbol{S}) + \boldsymbol{S} \colon \boldsymbol{\nabla v}$

By this we derived three partial differential equations for the three unknowns velocity, pressure and temperature. For a solution of this system of partial differential equations we need boundary and initial conditions[4] and especially we need to know appropriate balance equations at the interface between the condensate and the vapor phase.

5.2 Mass, momentum and energy jump conditions

Mass

We start with (4.12) and set the bulk variables $\varphi = \rho$, $\zeta = 0$ and the surface variable $\boldsymbol{\zeta}_S = \mathbf{0}$ to get the mass jump condition at the surface between condensate and vapor

$$\Big[\rho\,[\boldsymbol{v} - \boldsymbol{u}] \cdot \boldsymbol{n}\Big] \;=\; 0\,. \tag{5.6}$$

Equation (5.6) states that the same amount of mass flux that enters the surface as vapor phase leaves the surface as condensate phase $\dot{m}_l = \dot{m}_g$ (so we can omit the index).

Momentum

To get the momentum jump condition at the surface between condensate and vapor we set in (4.12) $\boldsymbol{\varphi} = \rho\,\boldsymbol{v}$, $\boldsymbol{\zeta} = -\boldsymbol{S}$ and $\boldsymbol{\zeta}_S = -\boldsymbol{S}_S$. Analog to the stress tensor the surface stress tensor can be divided into two components $\boldsymbol{S}_S = \sigma \boldsymbol{I}_S + \boldsymbol{T}_S$, where $\boldsymbol{I}_S$ is the surface identity tensor (2.9). Assuming a clean surface $\boldsymbol{T}_S = 0$ from the surface stress tensor simply the interfacial tension remains $\boldsymbol{S}_S = \sigma\,\boldsymbol{I}_s$, where σ is the surface tension coefficient, see section 4.3. The surface tension can be understood as the counterpart of the pressure in the bulk fluid [EBW91]. With this substitutions we get for the momentum jump condition

$$\Big[\rho\boldsymbol{v}\,[\boldsymbol{v} - \boldsymbol{u}] \cdot \boldsymbol{n}\Big] - \Big[\boldsymbol{S} \cdot \boldsymbol{n}\Big] \;=\; \boldsymbol{\nabla}_S\,\sigma + 2\,H\,\sigma\,\boldsymbol{n}\,. \tag{5.7}$$

In the condensation problem we neglect temperature dependency of the liquid-vapor surface tension coefficient (Marangony effects) but assume σ to be constant within the surface.

[4] We need as many boundary conditions for each coordinate of an unknown as the equation has derivatives of that unknown.

We split the vector equation (5.7) into three scalar equations by multiplying it first with the normal vector and then with the two tangential vectors. The tangential equations are equal, so we skip the third equation and use the symbol $\boldsymbol{t}$ to denote the tangential vectors. Further we make use of (5.6) and the assumption of no-slip at the surface $\Big[\!\Big[\boldsymbol{v}\cdot\boldsymbol{t} \Big]\!\Big] = 0$ and get

$$\begin{aligned} \Big[\!\Big[\dot{m}\,\boldsymbol{v}\cdot\boldsymbol{n} \Big]\!\Big] + \Big[\!\Big[p \Big]\!\Big] - \Big[\!\Big[\boldsymbol{n}\cdot\boldsymbol{T}\cdot\boldsymbol{n} \Big]\!\Big] &= 2\,H\,\sigma\ , \\ \Big[\!\Big[\boldsymbol{t}\cdot\boldsymbol{T}\cdot\boldsymbol{n} \Big]\!\Big] &= 0\ . \end{aligned} \tag{5.8}$$

Energy

The energy jump condition is with $\varphi = \rho\,(e + \frac{v^2}{2})$, $\boldsymbol{\zeta} = -\boldsymbol{v}\cdot\boldsymbol{S} + \boldsymbol{q}$ and with $\boldsymbol{\zeta}_S = -\sigma\,\boldsymbol{I}_s\cdot\boldsymbol{u} = -\sigma\,\boldsymbol{u}$ (with the same assumptions as for the momentum jump condition) given by

$$\Big[\!\Big[\rho\,(e+\frac{v^2}{2})\,[\boldsymbol{v}-\boldsymbol{u}]\cdot\boldsymbol{n} \Big]\!\Big] - \Big[\!\Big[\boldsymbol{v}\cdot\boldsymbol{S}\cdot\boldsymbol{n} \Big]\!\Big] + \Big[\!\Big[\boldsymbol{q}\cdot\boldsymbol{n} \Big]\!\Big] = \boldsymbol{\nabla}_S\cdot\big(\sigma\,\boldsymbol{u}\big) + 2\,H\,\sigma\,\boldsymbol{n}\cdot\boldsymbol{u}. \tag{5.9}$$

Instead of a balance equation for the internal energy we need a formulation with the enthalpy $h = e + \frac{p}{\rho}$, because enthalpy is a measurable quantity, whereas the internal energy is not easy to measure. To get this equation we find it convenient to split the vectors on the left side of (5.9) into their normal and tangential components according to $\boldsymbol{a}\cdot\boldsymbol{b} = (\boldsymbol{a}\cdot\boldsymbol{n})(\boldsymbol{b}\cdot\boldsymbol{n}) + (\boldsymbol{a}\cdot\boldsymbol{t})(\boldsymbol{b}\cdot\boldsymbol{t})$. Applying chain rule on the surface gradient term gives $\boldsymbol{\nabla}_S\cdot\big(\sigma\,\boldsymbol{u}\big) = \sigma\,\boldsymbol{\nabla}_S\cdot\boldsymbol{u} + \boldsymbol{u}\cdot\boldsymbol{\nabla}_S\,\sigma$. By this we get

$$\begin{aligned} \Big[\!\Big[\dot{m}\left(u + \frac{(\boldsymbol{v}\cdot\boldsymbol{n})^2}{2} + \frac{(\boldsymbol{v}\cdot\boldsymbol{t})^2}{2}\right) \Big]\!\Big] - \Big[\!\Big[(\boldsymbol{v}\cdot\boldsymbol{n})\,(\boldsymbol{n}\cdot\boldsymbol{S}\cdot\boldsymbol{n}) + (\boldsymbol{v}\cdot\boldsymbol{t})\,(\boldsymbol{t}\cdot\boldsymbol{S}\cdot\boldsymbol{n}) \Big]\!\Big] + \Big[\!\Big[\boldsymbol{q}\cdot\boldsymbol{n} \Big]\!\Big] \\ = \sigma\,\boldsymbol{\nabla}_S\cdot\boldsymbol{u} + \boldsymbol{u}\cdot\boldsymbol{\nabla}_S\,\sigma + 2\,H\,\sigma\,\boldsymbol{n}\cdot\boldsymbol{u}\ , \end{aligned} \tag{5.10}$$

We then subtract the scalar product of momentum jump condition (5.7) and surface velocity

$$\Big[\!\Big[\dot{m}\,(\boldsymbol{u}\cdot\boldsymbol{n})(\boldsymbol{v}\cdot\boldsymbol{n}) \Big]\!\Big] - \Big[\!\Big[(\boldsymbol{u}\cdot\boldsymbol{n})\,(\boldsymbol{n}\cdot\boldsymbol{S}\cdot\boldsymbol{n}) \Big]\!\Big] = \boldsymbol{u}\cdot\left[\boldsymbol{\nabla}_S\,\sigma + 2\,H\,\sigma\,\boldsymbol{u}\cdot\boldsymbol{n}\right]\ ,$$

from the energy jump condition, of which after splitting the vectors on the left side in normal and tangential components only the contribution in normal direction remains, because the surface velocity has no tangential component [Sch70]. Together

with the no-slip condition, the tangential momentum jump condition and after re-arranging the pressure term we get

$$\left[\!\left[\dot{m} \left(u + \frac{p}{\rho} + \frac{(\boldsymbol{v}\cdot\boldsymbol{n})^2}{2} - (\boldsymbol{u}\cdot\boldsymbol{n})(\boldsymbol{v}\cdot\boldsymbol{n}) \right) \right]\!\right] - \left[\!\left[[\boldsymbol{v}-\boldsymbol{u}]\cdot\boldsymbol{n}\,(\boldsymbol{n}\cdot\boldsymbol{T}\cdot\boldsymbol{n}) \right]\!\right] + \left[\!\left[\boldsymbol{q}\cdot\boldsymbol{n} \right]\!\right] = \sigma\,\boldsymbol{\nabla}_S\cdot\boldsymbol{u}.$$

We expand the kinetic energy term

$$\left[\!\left[\frac{(\boldsymbol{v}\cdot\boldsymbol{n})^2}{2} \right]\!\right] = \left[\!\left[\frac{(\boldsymbol{v}\cdot\boldsymbol{n}-\boldsymbol{u}\cdot\boldsymbol{n})^2}{2} + (\boldsymbol{v}\cdot\boldsymbol{n})(\boldsymbol{u}\cdot\boldsymbol{n}) - \frac{(\boldsymbol{u}\cdot\boldsymbol{n})^2}{2} \right]\!\right]$$

and use that the surface velocity jump is zero. Then the energy jump condition at a surface between two fluids becomes

$$\left[\!\left[\dot{m}\,h \right]\!\right] + \left[\!\left[\dot{m}\,\frac{([\boldsymbol{v}-\boldsymbol{u}]\cdot\boldsymbol{n})^2}{2} \right]\!\right] - \left[\!\left[[\boldsymbol{v}-\boldsymbol{u}]\cdot\boldsymbol{n}\,(\boldsymbol{n}\cdot\boldsymbol{T}\cdot\boldsymbol{n}) \right]\!\right] + \left[\!\left[\boldsymbol{q}\cdot\boldsymbol{n} \right]\!\right] = \sigma\,\boldsymbol{\nabla}_S\cdot\boldsymbol{u}\ . \quad (5.11)$$

Similar as for the energy equation we assume that kinetic energy and viscous energy can be neglected in the energy jump condition. Certainly the surface gradient of the surface velocity is small. by this the energy jump condition finally becomes

$$\dot{m}\,\Delta h + \left[\!\left[\boldsymbol{q}\cdot\boldsymbol{n} \right]\!\right] = 0\ , \quad (5.12)$$

where $\Delta h = \left[\!\left[h \right]\!\right]$ is the latent heat of vaporization.

5.3 Governing equations

We finish this chapter with a summary of the derived system of equations for the condensation problem. The condensate (index l) and the vapor (index g) bulk phase are described by the continuity equation for an incompressible fluid, the momentum equation and the energy equation (For clarification the assumptions involved are shown again on the right):

Balance equations for condensate and gas phase $(i = l, g)$

Continuity equation — const. density

$$\boldsymbol{\nabla} \cdot \boldsymbol{v}_i = 0$$

Momentum equation — Newtonian fluid, const. viscosity

$$\rho_i \underbrace{\left(\frac{\partial \boldsymbol{v}_i}{\partial t} + \boldsymbol{v}_i \cdot \boldsymbol{\nabla} \boldsymbol{v}_i \right)}_{\text{inertia}} = \underbrace{-\boldsymbol{\nabla} p_i}_{\text{pressure}} + \underbrace{\mu_i \nabla^2 \boldsymbol{v}_i}_{\text{friction}} + \underbrace{\rho_i \boldsymbol{g}}_{\text{gravity}}$$

Energy equation — no dissipation, const. heat conductivity, const. thermal capacity

$$\underbrace{\rho_i c_i \left(\frac{\partial T_i}{\partial t} + \boldsymbol{v}_i \cdot \boldsymbol{\nabla} T_i \right)}_{\text{transient + convection}} = \underbrace{\lambda_i \nabla^2 T_i}_{\text{heat conduction}}$$

At the interface between condensate and vapor we have jump conditions for mass, momentum and energy (from now on we leave out the tilde over the normal and tangential vectors):[5]

Jump conditions at the interface

Mass

$$\left[\!\left[\rho \left[\boldsymbol{v} - \boldsymbol{u} \right] \cdot \boldsymbol{n} \right]\!\right] = \underbrace{\left[\!\left[\dot{m} \right]\!\right]}_{\text{mass flux}}$$

Momentum — const. surface tension coefficient, no slip between both phases

$$\underbrace{\left[\!\left[\dot{m}\, \boldsymbol{v} \cdot \boldsymbol{n} \right]\!\right]}_{\substack{\text{momentum} \\ \text{due to} \\ \text{condensation}}} + \underbrace{\left[\!\left[p \right]\!\right]}_{\text{surface pressure}} - \left[\!\left[\boldsymbol{n} \cdot \boldsymbol{T} \cdot \boldsymbol{n} \right]\!\right] = \underbrace{2\, H\, \sigma}_{\text{surface tension}}$$

$$\underbrace{\left[\!\left[\boldsymbol{t} \cdot \boldsymbol{T} \cdot \boldsymbol{n} \right]\!\right]}_{\text{shear stress}} = 0$$

Energy — no kinetic energy, no dissipation

$$\underbrace{\dot{m}\, \Delta h}_{\text{condensation}} = \underbrace{\left[\!\left[\boldsymbol{q} \cdot \boldsymbol{n} \right]\!\right]}_{\text{heat flux}}$$

with $\boldsymbol{T} = \mu \left[\boldsymbol{\nabla} \boldsymbol{v} + (\boldsymbol{\nabla} \boldsymbol{v})^T \right]$ and $\boldsymbol{q} = -\lambda \boldsymbol{\nabla} T$.

[5] with $\left[\!\left[\varphi \right]\!\right] := \varphi_g - \varphi_l$.

From the five jump conditions we need one equation to calculate the mass flux, so that four equations remain to calculate the boundary conditions for three velocity components, pressure and temperature. We need one more equation. As the missing equation we take the condition of thermodynamic equilibrium, according to which the condensate and vapor temperatures are equal at the moving surface

$$T_{\text{condensate}} = T_{\text{vapor}} \ . \tag{5.13}$$

Theoretically we can use either the mass jump condition or the energy jump condition to calculate the mass flux.[6] From the remaining equations we can use either the mass or the normal momentum jump condition to calculate one velocity boundary condition, depending on which equation we use to compute the mass flux. Which one of both equations is more suitable we discuss in part III, where we present the numerical algorithms for the computational solution of the condensation problem. The second and third velocity boundary condition we take from the tangential momentum jump conditions.

[6] In chapter 6 we derive some more simplifications, one of which is that we can drop the term with the mass flux in the normal momentum jump condition, so that this equation will not be available to calculate the mass flux.

Part II

Analysis

Chapter 6

Model equations under the assumption of rotational symmetry

Experimental results show that for condensation in an inclined tube the flow regimes are almost cylindrical [Fie03]. Based on this we begin our analysis of the model equations by assuming a rotational film as depicted in figure 6.1. First the equations for the bulk fluids of the condensate problem and the jump conditions for the interface between condensate and vapor defined in part I are evaluated for this case. The references for this equations are given in part I unless otherwise noted.

6.1 Bulk flow equations and outer boundary conditions

If the tube is in vertical position as shown in figure 6.1, then gravity is acting in the direction of the tube axis. A laminar flow can be assumed. As velocity increases surface becomes wavy with at first two-dimensional surface waves [ANP94] [YNN96]. For small surface waves the flow is still laminar [BS98]. Then the condensation problem has rotational symmetry, so that all derivatives of velocity, temperature and pressure with respect to the rotation angle are zero. The velocity component in the plane of rotation is zero. Further vapor velocity is assumed to be small, so that shear stress exerted by the vapor on the condensate film can be neglected.

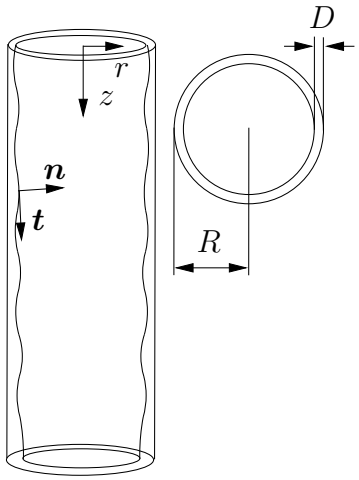


Figure 6.1: *Vertical tube*

The vapor temperature is constant, so that heat flux in the vapor phase is zero. The pressure in the vapor phase is determined mainly by the hydrostatic pressure [Fie03].

Under this conditions the problem reduce to a one-fluid problem and only the bulk flow equations for the condensate phase have to be solved. The balance equations for the condensate flow, given in section 5.3, are in cylindrical coordinates r, ϑ, z and in the case of rotational symmetry reads

the continuity equation:

$$\frac{1}{r}\,\frac{\partial}{\partial r}\,(r\,v_r) + \frac{\partial v_z}{\partial z} = 0\;, \tag{6.1}$$

the momentum equations:

$$\begin{aligned} \rho\left(\frac{\partial v_r}{\partial t} + v_r\,\frac{\partial v_r}{\partial r} + v_z\frac{\partial v_r}{\partial z}\right) &= -\frac{\partial p}{\partial r} + \mu\left(\frac{\partial}{\partial r}\left[\frac{1}{r}\,\frac{\partial}{\partial r}(r\,v_r)\right] + \frac{\partial^2 v_r}{\partial z^2}\right)\;, \\ \rho\left(\frac{\partial v_z}{\partial t} + v_r\,\frac{\partial v_z}{\partial r} + v_z\,\frac{\partial v_z}{\partial z}\right) &= -\frac{\partial p}{\partial z} + \mu\left(\frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial v_z}{\partial r}\right] + \frac{\partial^2 v_z}{\partial z^2}\right) + \rho\,g \end{aligned} \tag{6.2}$$

the energy equation:

$$\rho\,c\left(\frac{\partial T}{\partial t} + v_r\,\frac{\partial T}{\partial r} + v_z\,\frac{\partial T}{\partial z}\right) = \lambda\left(\frac{1}{r}\,\frac{\partial}{\partial r}\left[r\,\frac{\partial T}{\partial r}\right] + \frac{\partial^2 T}{\partial z^2}\right)\;. \tag{6.3}$$

[BSL60]. From the balance equations of the vapor phase only the hydrostatic pressure terms in the second momentum equation remain

$$\frac{\partial p_g}{\partial z} = \rho_g\,g\;. \tag{6.4}$$

Outer boundary conditions

For the condensation problem in a vertical tube the outer boundary conditions of the momentum equations are
a given parabolic velocity profile at the tube inlet:

$$v_z\big|_{r,z=0} = U(r)\;, \qquad v_r\big|_{r,z=0} = 0\;, \tag{6.5}$$

zero velocity at the tube wall:

$$v_r\big|_{r=R,z} = 0\;, \qquad v_z\big|_{r=R,z} = 0\;, \tag{6.6}$$

the outflow condition at the outlet:

$$\left.\frac{\partial v_r}{\partial z}\right|_{r,z=L} = 0 \,, \qquad v_z\big|_{r,z=L} = 0 \,, \tag{6.7}$$

where R is the tube diameter and L is the tube length. The outer boundary condition of the energy equation is that
the temperature at the tube wall is known:

$$T\big|_{r=R,z} = T_w \,, \tag{6.8}$$

Finally the temperature at the interface is given by the vapor temperature:

$$T\big|_{r=R-D,z} = T_v \,, \tag{6.9}$$

the temperature derivative is zero at the tube inlet and at the tube outlet:

$$\left.\frac{\partial T}{\partial z}\right|_{r,z=0} = 0 \,, \qquad \left.\frac{\partial T}{\partial z}\right|_{r,z=L} = 0 \,, \tag{6.10}$$

where $D = R - h$ is the film thickness (compare figure 6.1).

The question of the correct pressure boundary conditions is discussed later after the equations are simplified, compare section 7.3.

Cylindrical basis

The equations (6.1) – (6.10) are given in cylindrical coordinates and are related to a cylindrical coordinate system. A cylindrical coordinate system is orthogonal and the corresponding covariant and contravariant cylindrical base vectors are self reciprocal (they only differ in scale factors), so that conveniently unit vectors are used, see section 2.1. The balance equations above are related to the unit base vectors

$$\boldsymbol{e}_r = \begin{bmatrix} \cos\vartheta \\ \sin\vartheta \\ 0 \end{bmatrix} \,, \qquad \boldsymbol{e}_\vartheta = \begin{bmatrix} -\sin\vartheta \\ \cos\vartheta \\ 0 \end{bmatrix} \,, \qquad \boldsymbol{e}_z = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \,. \tag{6.11}$$

The velocity vector is given in this unit vectors by $\boldsymbol{v} = v_r\,\boldsymbol{e}_r + v_\vartheta\,\boldsymbol{e}_\vartheta + v_z\,\boldsymbol{e}_z$, with Einstein summation convention we write shortly $\boldsymbol{v} = v_i\,\boldsymbol{e}_i$. Equations (6.1) – (6.3) for the condensate flow in a vertical tube are found in [BSL60].

The procedure of deriving them from the symbolic equations given in the last section of part I by computing the derivatives of the base vectors (which are not constant but functions of r, ϑ, z) and the metric tensor and finally relating the variables to the unit base vectors (6.11) is here already done. However, for the interface jump conditions there exist no such reference.

6.2 Geometrical properties and velocity of the interface

To derive the interface balance equations in cylindrical coordinates we need a parametrization of the surface, so that the normal and tangential vectors, the surface velocity and the mean curvature can be computed (compare chapter 2).

The position vector of a surface with rotational symmetry is given in a cartesian coordinate system as

$$\boldsymbol{x}(\vartheta, z, t) = h(z,t)\, \cos\vartheta\, \boldsymbol{e}_x + h(z,t)\, \sin\vartheta\, \boldsymbol{e}_y + z\, \boldsymbol{e}_z \ ,$$

where radius h is a function of time and tube length and the surface variables are z and ϑ. Related to a cylindrical coordinate system the position vector becomes

$$\boldsymbol{x}(\vartheta, z, t) = h(z,t)\, \boldsymbol{e}_r + 0\, \boldsymbol{e}_\vartheta + z\, \boldsymbol{e}_z \ , \tag{6.12}$$

[Dee98].

Tangential and normal vectors

The tangential vectors are obtained by (2.3) as

$$\boldsymbol{t}_\vartheta = \frac{\partial \boldsymbol{x}}{\partial \vartheta} = h\, \frac{\partial \boldsymbol{e}_r}{\partial \vartheta} + 0\, \boldsymbol{e}_\vartheta + 0\, \boldsymbol{e}_z = \begin{bmatrix} 0 \\ h \\ 0 \end{bmatrix} \ , \ \boldsymbol{t}_z = \frac{\partial \boldsymbol{x}}{\partial z} = \frac{\partial h}{\partial z}\, \boldsymbol{e}_r + 0\, \boldsymbol{e}_\vartheta + 1\, \boldsymbol{e}_z = \begin{bmatrix} \frac{\partial h}{\partial z} \\ 0 \\ 1 \end{bmatrix} ,$$

and the normal vector is obtained by (2.5) as

$$\boldsymbol{n}^- = \frac{\boldsymbol{t}_\vartheta \times \boldsymbol{t}_z}{|\boldsymbol{t}_\vartheta \times \boldsymbol{t}_z|} = \begin{bmatrix} 1 \\ 0 \\ -\frac{\partial h}{\partial z} \end{bmatrix} \frac{1}{\sqrt{1 + \left(\frac{\partial h}{\partial z}\right)^2}} \ .$$

Note that this normal vector points inwards on the condensate film. The normal vector that points outwards on the condensate film is given by

$$\boldsymbol{n} = -\boldsymbol{n}^- \ .$$

Mean curvature

The mean curvature we defined in (2.12) as

$$H = -\frac{1}{2}\nabla_s \cdot \boldsymbol{n} = -\frac{1}{2}\left(\boldsymbol{t}^z \frac{\partial}{\partial z} + \boldsymbol{t}^\vartheta \frac{\partial}{\partial \vartheta}\right) \cdot \boldsymbol{n}$$

and alternatively by (2.17) using the first and second fundamental form. Because the basis is orthogonal, here it is more convenient to compute H by means of (2.17). With the first fundamental form (2.4)

$$I = \begin{bmatrix} \boldsymbol{t}_z \cdot \boldsymbol{t}_z & \boldsymbol{t}_z \cdot \boldsymbol{t}_\vartheta \\ \boldsymbol{t}_\vartheta \cdot \boldsymbol{t}_z & \boldsymbol{t}_\vartheta \cdot \boldsymbol{t}_\vartheta \end{bmatrix} = \begin{bmatrix} 1 + \left(\frac{\partial h}{\partial z}\right)^2 & 0 \\ 0 & h^2 \end{bmatrix}$$

and the second fundamental form (2.15)

$$II = \begin{bmatrix} \boldsymbol{t}_{zz} \cdot \boldsymbol{n} & \boldsymbol{t}_{z\vartheta} \cdot \boldsymbol{n} \\ \boldsymbol{t}_{\vartheta z} \cdot \boldsymbol{n} & \boldsymbol{t}_{\vartheta\vartheta} \cdot \boldsymbol{n} \end{bmatrix} = \begin{bmatrix} -\frac{\frac{\partial^2 h}{\partial z^2}}{\sqrt{1+\left(\frac{\partial h}{\partial z}\right)^2}} & 0 \\ 0 & \frac{h}{\sqrt{1+\left(\frac{\partial h}{\partial z}\right)^2}} \end{bmatrix}$$

the shape operator (2.16) for a rotational surface becomes

$$L = \begin{bmatrix} \boldsymbol{t}_{zz} \cdot \boldsymbol{n} & \boldsymbol{t}_{z\vartheta} \cdot \boldsymbol{n} \\ \boldsymbol{t}_{\vartheta z} \cdot \boldsymbol{n} & \boldsymbol{t}_{\vartheta\vartheta} \cdot \boldsymbol{n} \end{bmatrix} \frac{1}{\det I} \begin{bmatrix} \boldsymbol{t}_\vartheta \cdot \boldsymbol{t}_\vartheta & -\boldsymbol{t}_z \cdot \boldsymbol{t}_\vartheta \\ -\boldsymbol{t}_\vartheta \cdot \boldsymbol{t}_z & \boldsymbol{t}_z \cdot \boldsymbol{t}_z \end{bmatrix} = \begin{bmatrix} -\frac{\frac{\partial^2 h}{\partial z^2}}{\left(1+\left(\frac{\partial h}{\partial z}\right)^2\right)^{3/2}} & 0 \\ 0 & +\frac{1}{h\sqrt{1+\left(\frac{\partial h}{\partial z}\right)^2}} \end{bmatrix}.$$

The shape operator is already in diagonal form, so that the diagonal entries are the principal curvatures κ_1, κ_2. The mean curvature of the liquid-vapor surface is then according to (2.17) given by

$$H = \frac{1}{2}\operatorname{trace} L = \frac{1}{2}\left(-\frac{\frac{\partial^2 h}{\partial z^2}}{\left(1+\left(\frac{\partial h}{\partial z}\right)^2\right)^{3/2}} + \frac{1}{h\sqrt{1+\left(\frac{\partial h}{\partial z}\right)^2}}\right) = \frac{1}{2}(\kappa_1 + \kappa_2)\,.$$

The first term is zero if the surface is flat. It describes the curvature due to surface waves in the r, z plane. The second term describes the curvature due to the tube radius.

Surface velocity

The surface velocity is according to (3.21) given by the derivative of the position vector (6.12) with respect to time

$$\boldsymbol{u} = \left.\frac{\partial \boldsymbol{x}}{\partial t}\right|_{\vartheta,z} = \begin{bmatrix} \frac{\partial h}{\partial t} \\ 0 \\ 0 \end{bmatrix} .$$

6.3 Jump conditions

After we discussed the geometrical properties of the interface between the condensate and the vapor phase, now we can compute the jump conditions.

Mass flux

The mass flux (5.6) across the surface of rotational symmetry is obtained in cylindrical coordinates as

$$\begin{aligned} \dot{m} &= \rho\,[\boldsymbol{v}-\boldsymbol{u}]\cdot\boldsymbol{n} \\ &= \rho\,[(v_i-u_i)\,\boldsymbol{e}_i]\cdot(n_j\,\boldsymbol{e}_j) = \rho\,(v_i-u_i)\,n_j\,\underbrace{\boldsymbol{e}_i\cdot\boldsymbol{e}_j}_{\delta_{ij}} = \rho\,(v_i-u_i)\,n_i \\ &= \frac{\rho}{\sqrt{1+\left(\frac{\partial h}{\partial z}\right)^2}}\left(-v_r+\frac{\partial h}{\partial t}+v_z\,\frac{\partial h}{\partial z}\right) , \end{aligned} \tag{6.13}$$

where δ_{ij} is the Kronecker delta.

Momentum jump conditions

In the momentum jump conditions the stress tensor appear. In cylindrical coordinates and under the assumption of rotational symmetry the stress tensor is

$$\boldsymbol{T} = 2\,\mu \begin{bmatrix} \frac{1}{2}\frac{\partial v_r}{\partial r} & 0 & \left(\frac{\partial v_r}{\partial z}+\frac{\partial v_z}{\partial r}\right) \\ 0 & 0 & 0 \\ \left(\frac{\partial v_r}{\partial z}+\frac{\partial v_z}{\partial r}\right) & 0 & \frac{1}{2}\frac{\partial v_z}{\partial z} \end{bmatrix} ,$$

[BSL60]. From the momentum jump condition in normal direction (5.8), with the assumption of small vapor velocity, only the velocity terms of the condensate remain

$$\dot{m}\,\boldsymbol{v}\cdot\boldsymbol{n} + (p-p_g) - \boldsymbol{n}\cdot\boldsymbol{T}\cdot\boldsymbol{n} = -2\,H\,\sigma ,$$

We compute the first and the third terms separately. With

$$\dot{m}\,\boldsymbol{v}\cdot\boldsymbol{n} \;=\; \dot{m}\,(v_i\,\boldsymbol{e}_i)\cdot(n_j\,\boldsymbol{e}_j) = \dot{m}\,v_i\,n_i\,\underbrace{\boldsymbol{e}_i\cdot\boldsymbol{e}_j}_{\delta_{ij}} = \dot{m}\,v_i\,n_i$$

and

$$\boldsymbol{n}\cdot\boldsymbol{T}\cdot\boldsymbol{n} \;=\; (n_i\,\boldsymbol{e}_i)\cdot(T_{kl}\,\boldsymbol{e}_k\,\boldsymbol{e}_l)\cdot(n_j\,\boldsymbol{e}_j) = n_i\,T_{kl}\,n_j\,\underbrace{\boldsymbol{e}_i\cdot\boldsymbol{e}_k}_{\delta_{ik}}\,\underbrace{\boldsymbol{e}_l\cdot\boldsymbol{e}_j}_{\delta_{lj}} = n_i\,T_{ij}\,n_j$$

the momentum jump condition in normal direction is given as

$$\begin{aligned}\frac{\rho}{1+\left(\frac{\partial h}{\partial z}\right)^2}\left(-v_r+\frac{\partial h}{\partial t}+v_z\,\frac{\partial h}{\partial z}\right)\left(-v_r+v_z\,\frac{\partial h}{\partial z}\right)+p-p_g\\ -\frac{\mu}{1+\left(\frac{\partial h}{\partial z}\right)^2}\left(\frac{\partial v_r}{\partial r}-4\left[\frac{\partial v_r}{\partial z}+\frac{\partial v_z}{\partial r}\right]\frac{\partial h}{\partial z}+\frac{\partial v_z}{\partial z}\left[\frac{\partial h}{\partial z}\right]^2\right)\\ =-\sigma\left(-\frac{\frac{\partial^2 h}{\partial z^2}}{(1+\left(\frac{\partial h}{\partial z}\right)^2)^{3/2}}+\frac{1}{h\sqrt{1+\left(\frac{\partial h}{\partial z}\right)^2}}\right)\;. \qquad (6.14)\end{aligned}$$

The momentum jump condition in tangential direction (5.8) becomes

$$\begin{aligned}\boldsymbol{t}\cdot\boldsymbol{T}\cdot\boldsymbol{n} = (t_i\,\boldsymbol{e}_i)\cdot(T_{kl}\,\boldsymbol{e}_k\,\boldsymbol{e}_l)\cdot(n_j\,\boldsymbol{e}_j) = t_i\,T_{kl}\,n_j\,\underbrace{\boldsymbol{e}_i\cdot\boldsymbol{e}_k}_{\delta_{ik}}\,\underbrace{\boldsymbol{e}_l\cdot\boldsymbol{e}_j}_{\delta_{lj}} = t_i\,T_{ij}\,n_j\\ =\frac{\mu}{\sqrt{1+\left(\frac{\partial h}{\partial z}\right)^2}}\left(-\frac{\partial v_r}{\partial r}\frac{\partial h}{\partial z}-2\left[\frac{\partial v_r}{\partial z}+\frac{\partial v_z}{\partial r}\right]\left[1-\left(\frac{\partial r}{\partial z}\right)^2\right]+\frac{\partial v_z}{\partial z}\frac{\partial h}{\partial z}\right)=0\;.\end{aligned} \qquad (6.15)$$

Energy jump condition

Finally, we consider the energy jump condition (5.12) at the interface between condensate and vapor

$$\dot{m}\,\Delta h = -\boldsymbol{q}\cdot\boldsymbol{n}\;.$$

With $\boldsymbol{q} = -\lambda\,[\frac{\partial T}{\partial r}, 0, \frac{\partial T}{\partial z}]^T$ we get

$$\frac{\rho\,\Delta h}{\sqrt{1+\left(\frac{\partial h}{\partial z}\right)^2}}\left(-v_r+\frac{\partial h}{\partial t}+v_z\,\frac{\partial h}{\partial z}\right)=-\frac{\lambda}{\sqrt{1+\left(\frac{\partial h}{\partial z}\right)^2}}\left(-\frac{\partial T}{\partial r}+\frac{\partial T}{\partial z}\,\frac{\partial h}{\partial z}\right)\;. \qquad (6.16)$$

Equations (6.1) – (6.16) form the system of model equations for the condensation problem in the case of a vertical position of the tube where rotational symmetry is assumed. In the next chapter we simplify this model equations further by means of a dimensional analysis.

Chapter 7

Dimensional analysis of the model equations/rotational symmetry

In this chapter the condensation equations for the vertical position of the tube derived in chapter 6 are analyzed by a generalized dimensional analysis.

For the vertical tube position the average thickness D of the condensate film is much smaller than the length L of the condensate film, compare figure 6.1. With classical dimensional analysis the variables of a problem are reduced to a fewer amount of nondimensional variables, but all spatial variables are scaled by the same length scale [Bri31]. Generalized dimensional analysis is based on different length scales for each spatial variable and therefore allows an analysis in terms of the slenderness $\varepsilon = \frac{D}{L}$ of the balance region. We get equations of boundary layer type for the condensate phase and comparably great simplifications are achieved for the jump conditions. In the last section the derived equations are evaluated for water and for R134a.

Generalized dimensional analysis is not very well documented in literature, so that we elaborate more on this method here.

The benefit of a generalized dimensional analysis is twofold. It gives the main relevant terms of the model equations, but also reduces the number of variables by combining them to a fewer number of variables. This does not change the equations besides a scaling and such a transformation is called symmetry transformation.

The first section of this chapter is mainly based on the lecture notes of [NS01]. Further on [Gör75] [Spu92] [BK89], and [Bar79] [Bar96]. From this authors only [NS01] describe generalized dimensional analysis. [Gör75] and [Spu93] present mainly dimensional analysis for hydrodynamical problems. [Bar96] also starts with dimensional analysis but eventually differs by using symmetry transformations to solve

problems which contain a small dimensionless parameter such as ε. [BK89] present the theory of solving differential equations using symmetry methods in a more mathematical rigor.

7.1 Dimensional analysis and generalized dimensional analysis

In this section some terminology is given. Generalized dimensional analysis is compared to dimensional analysis.

Units and Dimensions

Length, mass and time are called dimensions. Meter, kilogram, second (MKS) are called units and centimeter, gram, second (c.g.s) are other units. Length, mass and time (LTM) are fundamental dimensions of the MKS unit system and the c.g.s unit system [Gör75]. They have the same physical properties but differ by scale factors. Another set of fundamental dimensions are length, force and time (LFT). An equation in which the units balance on both sides of the equal sign is called coherent. An equation in which the dimensions balance on both sides of the equal sign is called homogeneous. The equation $3\,\mathrm{m} + 3\,\mathrm{cm} = 3\,\mathrm{m} + 3 \times 0.1\,\mathrm{m} = 3.3\,\mathrm{m}$ is homogeneous but not coherent. It is nevertheless a meaningful physical equation. The usage of two different unit systems in one equation makes it necessary to convert all units to the same system of units by scaling them appropriately before calculating the result, and is therefore laborious, but permissible. As afore mentioned balance equations are tensor equations and have to be invariant under a change of a coordinate system. Similarly physically meaningful equations have to be invariant under a change of system of units.[1] Yet, a dimensionless equation is invariant under a change of unit system, the values of the dimensionless quantities do not change.

Buckingham Π-Theorem

According to the Buckingham Π-Theorem every physically meaningful equation with n variables

$$f(a_1, a_2, \ldots, a_n) = 0\ ,$$

[1] It can be understood from the example that dimensional homogeneity is a necessary but not sufficient condition for an equation being invariant under a change of system of units.

where the n variables are expressed in terms of r fundamental dimensions, can be rewritten as an equation of $n - r = k$ dimensionless variables constructed from the original variables [Spu93].

Here we denote the value of a variable by a hat and the dimension by a tilde: $a_i = \hat{a}_i \, \tilde{a}_i$.[2] Each tilded variable is a power monomial function of fundamental dimensions. For instance the dimension of the streamwise velocity is in the LMT system $\tilde{v}_z = L\,T^{-1}$. We denote the fundamental dimensions by $\tilde{b}_i$, that is $\tilde{b}_1 = L$, $\tilde{b}_2 = M$, $\tilde{b}_3 = T$. The dimensions of the variables a_i are in terms of power monomials of fundamental dimensions

$$\begin{aligned} \tilde{a}_1 &= \tilde{b}_1^{b_{11}} \, \tilde{b}_2^{b_{21}} \, \ldots \tilde{b}_r^{b_{r1}} \, , \\ \tilde{a}_2 &= \tilde{b}_1^{b_{12}} \, \tilde{b}_2^{b_{22}} \, \ldots \tilde{b}_r^{b_{r2}} \, , \\ &\vdots \\ \tilde{a}_n &= \tilde{b}_1^{b_{1n}} \, \tilde{b}_2^{b_{2n}} \, \ldots \tilde{b}_r^{b_{rn}} \, . \end{aligned} \tag{7.1}$$

The vector of the powers $\boldsymbol{b}_i = [b_{1i}, b_{2i}, \ldots, b_{ri}]^T$ is called dimension vector of a_i [BK89]. For instance the dimension vector of the streamwise velocity we get from $\tilde{v}_z = L^1 \, M^0 \, T^{-1}$ as $\boldsymbol{b}_{v_z} = [1, 0, -1]^T$. The dimension vectors form the dimension matrix of the problem

$$\boldsymbol{B} = \begin{bmatrix} b_{11} & b_{12} & \ldots & b_{1n} \\ b_{21} & b_{22} & \ldots & b_{2n} \\ \vdots & \vdots & & \vdots \\ b_{r1} & b_{r2} & \ldots & b_{rn} \end{bmatrix} \, .$$

The rank of the dimension matrix $r = \operatorname{rank} \boldsymbol{B}$ is the number of the fundamental dimensions. To construct the $n - r = k$ nondimensional variables we make an ansatz in form of power monomials of the dimensions of the original variables

$$\pi_j = \tilde{a}_1^{x_{1j}} \, \tilde{a}_2^{x_{2j}} \, \ldots \, \tilde{a}_n^{x_{nj}} \, , \qquad j = 1, \ldots, k \, , \tag{7.2}$$

where the dimensions are substituted with (7.1) by their power monomials of fundamental dimensions

$$\pi_j = \left(\tilde{b}_1^{b_{11}} \, \tilde{b}_2^{b_{21}} \, \ldots \tilde{b}_r^{b_{r1}} \right)^{x_{1i}} \left(\tilde{b}_1^{b_{12}} \, \tilde{b}_2^{b_{22}} \, \ldots \tilde{b}_r^{b_{r2}} \right)^{x_{2i}} \, \ldots \, \left(\tilde{b}_1^{b_{1n}} \, \tilde{b}_2^{b_{2n}} \, \ldots \tilde{b}_r^{b_{rn}} \right)^{x_{ni}} \, ,$$

[2] According to DIN 1313 the value of a variable is set in curly brackets and the dimension in square brackets $a = \{a\}\,[a]$. However, for a dimensional analysis this notation seems a bit cumbersome.

[NS01]. By sorting the right hand side in terms of fundamental dimensions

$$\pi_j = \tilde{b}_1^{b_{11}\,x_{1j}+b_{12}\,x_{2j}+\cdots+b_{1n}\,x_{nj}}\ \tilde{b}_2^{b_{21}\,x_{1j}+b_{22}\,x_{2j}+\cdots+b_{2n}\,x_{nj}}\ \cdots\ \tilde{b}_r^{b_{r1}\,x_{1j}+b_{r2}\,x_{2j}+\cdots+b_{rn}\,x_{nj}}\ ,$$

we see that the exponents of the fundamental dimensions have to be zero because π_j has no dimension. The equations defined by this form a linear equation system

$$\boldsymbol{B}\,\boldsymbol{x}_j = \boldsymbol{0}\ ,$$

with $n - r = k$ linearly independent solutions. The nondimensional variables are thus constructed and f can be written in dimensionless form

$$F(\pi_1, \pi_2, \ldots, \pi_k) = 0\ . \tag{7.3}$$

With dimensional analysis a given problem can be analyzed without knowing the model equations of the problem explicitly. As a result the relation (7.3) of nondimensional combinations of the original dependent and independent variables is obtained. However, the decision what are the relevant variables of the problem and which unit system is appropriate is sometimes not trivial, see [Gör75] [Spu93] and also [Bar96].

Note that in dimensional analysis the nondimensional variables are constructed from the coherent dimension equations (7.1) of the problem variables. By this all dimensions have the same fundamental dimensions, for instance the spatial dimensions are $\tilde{r} = \tilde{z} = L$.

Generalized dimensional analysis

From the last section we know that in dimensional analysis only one fundamental dimension is assigned for all spatial dimensions. The generalized dimensional analysis allows different fundamental dimensions for each dimension, so that different lengths for each spatial dimension are permissible (i.e. $\tilde{r} = L_x$, $\tilde{z} = L_y$). It can be applied to the model equations, e.g. the ordinary or partial differential equations, including the necessary initial and boundary conditions [NS01].

For a generalized dimensional analysis the variables in the differential equations are substituted by their product of value and dimension. Because every equation must be of dimensional homogeneity this result in equations for the dimensions only. For example, in the definition of the velocity $v = \frac{\partial x}{\partial t}$ we substitute $v = \tilde{v}\,\hat{v}$, $x = \tilde{x}\,\hat{x}$ and

$t = \tilde{t}\,\hat{t}$. The derivatives are not carried out over the dimensions. With the condition of dimensional homogeneity we get

$$\tilde{v}\,\hat{v} = \frac{\partial(\tilde{x}\,\hat{x})}{\partial(\tilde{t}\,\hat{t})} = \frac{\tilde{x}}{\tilde{t}}\,\frac{\partial\hat{x}}{\partial\hat{t}}\,, \qquad \text{which gives} \qquad \tilde{v} = \frac{\tilde{x}}{\tilde{t}}\,.$$

If the dimension equation is written as: $\tilde{x}^{-1}\,\tilde{t}\,\tilde{v} = 1$, it is called normalized dimension equation.

Thus, using generalized dimensional analysis we derive the coherent equations for the construction of the nondimensional variables from the differential equations and their boundary conditions. Depending on this equations eventually more than only one length scale appears in the dimensionless variables of the problem.

A system of i physically meaningful ordinary or partial differential equations with n variables, where the n variables are expressed in terms of r fundamental dimensions, result in l normalized dimension equations

$$\begin{aligned}
\tilde{a}_1^{b_{11}}\,\tilde{a}_2^{b_{12}}\,\ldots\tilde{a}_n^{b_{1n}} &= 1\,,\\
\tilde{a}_1^{b_{21}}\,\tilde{a}_2^{b_{22}}\,\ldots\tilde{a}_n^{b_{2n}} &= 1\,,\\
&\vdots\\
\tilde{a}_1^{b_{l1}}\,\tilde{a}_2^{b_{l2}}\,\ldots\tilde{a}_n^{b_{ln}} &= 1\,.
\end{aligned} \tag{7.4}$$

Note that here the sequence of numbering is the other way round then in (7.1). The powers b_{ij} form the matrix

$$\boldsymbol{B} = \begin{bmatrix} b_{11} & b_{12} & \ldots & b_{1n}\\ b_{21} & b_{22} & \ldots & b_{2n}\\ \vdots & \vdots & & \vdots\\ b_{l1} & b_{l2} & \ldots & b_{ln} \end{bmatrix}\,,$$

with $k = \operatorname{rank}\boldsymbol{B}$. From the l coherent equations k are linear independent, such that k nondimensional variables can be constructed.

Surely the dimension of the variable a_j is a function of power monomials of the dimensions of all variables, so that we make the following ansatz:

$$\begin{aligned}
\tilde{a}_1 &= \tilde{a}_1^{y_{11}}\,\tilde{a}_2^{y_{12}}\,\ldots\tilde{a}_n^{y_{1n}}\,,\\
\tilde{a}_2 &= \tilde{a}_1^{y_{21}}\,\tilde{a}_2^{y_{22}}\,\ldots\tilde{a}_n^{y_{2n}}\,,\\
&\vdots\\
\tilde{a}_n &= \tilde{a}_1^{y_{n1}}\,\tilde{a}_2^{y_{n2}}\,\ldots\tilde{a}_n^{y_{nn}}\,,
\end{aligned} \tag{7.5}$$

where the coefficients y_{ij} are unknown [NS01]. Next, in the dimension equations (7.4) the dimensions $\tilde{a}_j$ are substituted with (7.5), which is here exemplarily done for the i'th equation:

$$(\tilde{a}_1^{y_{11}}\,\tilde{a}_2^{y_{12}}\,\ldots\tilde{a}_n^{y_{1n}})^{b_{i1}}\;(\tilde{a}_1^{y_{21}}\,\tilde{a}_2^{y_{22}}\,\ldots\tilde{a}_n^{y_{2n}})^{b_{i2}}\;\ldots\;(\tilde{a}_1^{y_{n1}}\,\tilde{a}_2^{y_{n2}}\,\ldots\tilde{a}_n^{y_{nn}})^{b_{in}} \;=\; 1\,.$$

By reordering the exponents in terms of dimensions

$$\tilde{a}_1^{b_{i1}\,y_{11}+b_{i2}\,y_{21}+\cdots+b_{in}\,y_{n1}}\;\tilde{a}_2^{b_{i1}\,y_{12}+b_{i2}\,y_{22}+\cdots+b_{in}\,y_{n2}}\;\ldots\;\tilde{a}_n^{b_{i1}\,y_{1n}+b_{i2}\,y_{2n}+\cdots+b_{in}\,y_{nn}} = 1\,,$$

we see that the exponents have to be zero. Doing so for all l unit equations result in the equation system

$$\boldsymbol{B}\,\boldsymbol{y}_j = \boldsymbol{0}\,.$$

We said that from the $k = \operatorname{rank}\boldsymbol{B}$ linear independent equations, k dimensionless variables can be constructed. Then the number of dimensions which we can choose as fundamental dimensions is given by the Buckingham Π-Theorem as the number of variables minus the number of nondimensional variables $n - k = r$.

To construct the dimensionless variables the columns of the matrix $\boldsymbol{B}$ are reordered such that the r column vectors corresponding to the fundamental dimensions are collected at the right hand side and those column vectors form the matrix $\boldsymbol{R}$. The remaining k column vectors form the matrix $\boldsymbol{K}$. We write the matrix $\boldsymbol{B}$ as

$$\boldsymbol{B} = [\boldsymbol{K}\,\big|\,\boldsymbol{R}]\,,$$

Note that the r base units are chosen under the condition that $\operatorname{rank}\boldsymbol{K} = \operatorname{rank}\boldsymbol{B}$ [NS01]. The nondimensional variables are then given by

$$\pi_j \;=\; \tilde{a}_1^{y_{1j}}\,\tilde{a}_2^{y_{2j}}\,\ldots\,\tilde{a}_n^{y_{nj}}\,, \qquad\qquad j = 1,\ldots,k\,.$$

Generalized dimensional analysis extends dimensional analysis

Generalized dimensional analysis removes the restriction of only one fundamental dimension for every dimension. Therefore it is an extension of dimensional analysis.

Generalized dimensional analysis is poorly documented in the literature, most authors focus on dimensional analysis. Instead of using generalized dimensional analysis [Spu93] and [Gör75] receive some of the results that can be obtained with

a generalized dimensional analysis by first using a dimensional analysis and then applying some further treatment. [Bar79] archives the result of a generalized dimensional analysis also by first using dimensional analysis but then transforming the gained nondimensional equations with a transformation group. We will come back to the group aspect in appendix A.

In the next sections the generalized dimensional analysis is applied to the bulk flow equations and the jump conditions of the condensate problem.

7.2 Bulk flow equations and outer boundaries

In this section we use generalized dimensional analysis to reduce the amount of variables to a fewer amount of nondimensional variables of the bulk equations (6.1) – (6.3) and the outer boundary conditions (6.5) – (6.10) of condensation in a vertical tube as derived in chapter 6.

Variables in terms of value and dimension

We start by writing all variables as a product of two new variables, one that represent the value (hatted variables) while the other represent the dimension (tilded variables). The variables of the moving surface problem are the independent variables r, z, t and the dependent variables v_r, v_z, p, the material properties ρ, μ, gravity g, and the variables of the boundary conditions

$$\begin{aligned}
r &= \hat{r}\,\tilde{r}\,, & v_r &= \hat{v}_r\,\tilde{v}_r\,, & \rho &= \hat{\rho}\,\tilde{\rho}\,, & U &= \hat{U}\,\tilde{U}\,,\\
z &= \hat{z}\,\tilde{z}\,, & v_z &= \hat{v}_z\,\tilde{v}_z\,, & \mu &= \hat{\mu}\,\tilde{\mu}\,, & R &= \hat{R}\,\tilde{R}\,,\\
t &= \hat{t}\,\tilde{t}\,, & p &= \hat{p}\,\tilde{p}\,, & g &= \hat{g}\,\tilde{g}\,, & D &= \hat{D}\,\tilde{D}\,,\\
& & & & & & L &= \hat{L}\,\tilde{L}\,.
\end{aligned}$$

For the condensation problem we also have to consider the variables of the energy equation

$$\begin{aligned}
T &= \hat{T}\,\tilde{T}\,, & c &= \hat{c}\,\tilde{c}\,, & T_w &= \hat{T}_w\,\tilde{T}_w\,,\\
& & \lambda &= \hat{\lambda}\,\tilde{\lambda}\,, & T_v &= \hat{T}_v\,\tilde{T}_v\,,
\end{aligned}$$

and the variables of the momentum equation for the vapor phase

$$p_g = \hat{p}_g\,\tilde{p}_g\,, \qquad \rho_g = \hat{\rho}_g\,\tilde{\rho}_g\,.$$

Next, the variables in the equations of the condensation problem in the vertical tube are substituted by these expressions.

Dimension equations of the boundary conditions

We begin with the boundary conditions. The first boundary condition (6.5), which states a given velocity profile at the inlet, becomes in terms of value and dimension

$$\hat{v}_z\,\tilde{v}_z\big|_{\hat{r}\,\tilde{r},\hat{z}\,\tilde{z}=0} = \hat{U}\,\tilde{U}(\hat{r}\,\tilde{r})\ , \quad \hat{v}_r\,\tilde{v}_r\big|_{\hat{r}\,\tilde{r},\hat{z}\,\tilde{z}=0} = 0\ .$$

Each term of an equation must be of dimensional homogeneity, so the streamwise velocity must have the same dimension as the velocity at the inlet

$$\tilde{v}_z = \tilde{U}\ . \tag{7.6}$$

The other dimension equations of (6.5) are homogeneous and provide no further information.

The outflow condition (6.7) becomes in terms of value and dimension and by keeping in mind that derivatives are not carried over dimensions

$$\frac{\tilde{v}_z}{\tilde{z}}\,\frac{\partial\hat{v}_z}{\partial\hat{z}}\bigg|_{\hat{r}\,\tilde{r},\hat{z}\,\tilde{z}=\hat{L}\,\tilde{L}} = 0\ , \quad \hat{v}_r\,\tilde{v}_r\big|_{\hat{r}\,\tilde{r},\hat{z}\,\tilde{z}=\hat{L}\,\tilde{L}} = 0\ ,$$

from which we found the dimension equation

$$\tilde{z} = \tilde{L}\ . \tag{7.7}$$

From the first boundary condition of the energy equation (6.8) and from the second boundary condition of the energy equation (6.9) we get the dimension equations

$$\tilde{T} = \tilde{T}_w, \ (7.8) \qquad \tilde{r} = \tilde{R}, \ (7.9) \qquad \tilde{R} = \tilde{D}, \ (7.10) \qquad \tilde{T} = \tilde{T}_S. \ (7.11)$$

The last two boundary equations of the energy equation provide no new information. We continue with the bulk flow equations.

Dimension equation of the continuity equation

Substituting the variables of the continuity equation (6.1) by the product of value and dimension gives

$$\frac{\tilde{r}\,\tilde{v}_r}{\tilde{r}^2}\,\frac{1}{\hat{r}}\frac{\partial}{\partial\hat{r}}\,(\hat{r}\,\hat{v}_r)+\frac{\tilde{v}_z}{\tilde{z}}\,\frac{\partial\hat{v}_z}{\partial\hat{z}}=0\,,$$

from which we get the dimension equation

$$\frac{\tilde{v}_r}{\tilde{r}}=\frac{\tilde{v}_z}{\tilde{z}}\qquad\text{which is in normal form}\qquad \tilde{r}^{-1}\,\tilde{z}\,\tilde{v}_r\,\tilde{v}_z^{-1}=1\,. \tag{7.12}$$

The normal dimension equations of the remaining model equations we write without derivation.

Remaining dimension equations in normal form

The normalized dimension equations of the momentum equations are

$$\tilde{r}\,\tilde{t}^{-1}\,\tilde{v}_r^{-1} = 1\,, \tag{7.13}$$
$$\tilde{r}^{-1}\,\tilde{z}\,\tilde{v}_r\,\tilde{v}_z^{-1} = 1\,, \tag{7.14}$$
$$\tilde{r}\,\tilde{z}^{-1}\,\tilde{v}_r\,\tilde{v}_z\,\tilde{p}^{-1}\,\tilde{\rho} = 1\,, \tag{7.15}$$
$$\tilde{r}\,\tilde{v}_r^{-1}\,\tilde{p}\,\tilde{\mu}^{-1} = 1\,, \tag{7.16}$$
$$\tilde{r}^{-2}\,\tilde{z}^2 = 1\,, \tag{7.17}$$
$$\tilde{r}\,\tilde{t}^{-1}\,\tilde{v}_r^{-1} = 1\,, \tag{7.18}$$
$$\tilde{r}^{-1}\,\tilde{z}\,\tilde{v}_r\,\tilde{v}_z^{-1} = 1\,, \tag{7.19}$$
$$\tilde{v}_z^2\,\tilde{p}^{-1}\,\tilde{\rho} = 1\,, \tag{7.20}$$
$$\tilde{r}^2\,\tilde{z}^{-1}\,\tilde{v}_z^{-1}\,\tilde{p}\,\tilde{\mu}^{-1} = 1\,, \tag{7.21}$$
$$\tilde{r}^{-2}\,\tilde{z}^2 = 1\,, \tag{7.22}$$
$$\tilde{z}^{-2}\,\tilde{v}_z\,\tilde{\rho}^{-1}\,\tilde{\mu}\,\tilde{g}^{-1} = 1\,. \tag{7.23}$$

The normalized dimension equations of the energy equation are

$$\tilde{r}\,\tilde{t}^{-1}\,\tilde{v}_r^{-1} = 1\,, \tag{7.24}$$
$$\tilde{r}^{-1}\,\tilde{z}\,\tilde{v}_r\,\tilde{v}_z^{-1} = 1\,, \tag{7.25}$$
$$\tilde{r}^2\,\tilde{z}^{-1}\,\tilde{v}_z\,\tilde{\rho}\,\tilde{c}\,\tilde{\lambda}^{-1} = 1\,, \tag{7.26}$$
$$\tilde{r}^{-2}\,\tilde{z}^2 = 1\,. \tag{7.27}$$

The momentum equation of the vapor phase completes the system of dimension equations for the bulk flows. From that we get the dimension equation in normal form

$$\tilde{z}^{-1}\,\tilde{p}_g\,\tilde{\rho}_g^{-1}\,\tilde{g}^{-1} = 1\,. \tag{7.28}$$

Before we continue deriving the nondimensional variables of the jump conditions the derived dimension equations are checked for linear dependence.

Linear dependent dimension equations

The equations (7.7) – (7.28) form a homogeneous system of 23 linear equations for $n = 20$ unknowns

$$\boldsymbol{B}\,\boldsymbol{x} = \boldsymbol{0} \qquad \text{with} \qquad k = \operatorname{rank}\boldsymbol{B} = 14\,. \tag{7.29}$$

The linear equation system is solved by Gaußian elimination. After applying Gaußian elimination to $\boldsymbol{B} = [\boldsymbol{K}\big|\boldsymbol{R}]$ the matrix $\boldsymbol{K}$ becomes essentially the identity matrix (plus some zero rows for the linear dependent equations) and the nondimensional variables can read from the row-reduced echelon form of $\boldsymbol{B}$. Clearly the rank deficiency is the result of the fact that some equations appear as linear combinations of others.

- The dimension equations of the convective terms (7.14) and (7.19) are obviously equal. Furthermore they are also equal to the dimension equation (7.12) of the continuity equation and to the dimension equation (7.25) of the energy equation.
- The dimension equation (7.13) describing the transient-convection relation in the radial momentum equation, is equal to (7.18) of the streamwise momentum equation and to (7.24) of the energy equation.
- The dimension equation (7.17) describing the second derivatives of the viscous term is equal to (7.22) of the streamwise momentum equation and to (7.27) of the energy equation.
- From the four pressure related equations (7.15), (7.16), (7.20), (7.21) two are linearly dependent.

If the above $23 - k = 9$ linear dependent equations are eliminated, then $\boldsymbol{B}$ will be a 14×20 matrix with rank $k = 14$, so that $\boldsymbol{B}$ would have maximum rank. However, before we do this the equations we found are analyzed in terms of the two different length scales.

7.3 Simplifications in terms of the two different length scales

If we apply Gaußian elimination to our linear equation system this result in nondimensional variables where the dimensions of all length scales are equal: $\tilde{r} = \tilde{z} =$

$\tilde{R} = \tilde{L} = \tilde{D}$. This is not the solution we are looking for. The problem arises from the fact that until now no values are assigned for any of these variables. By setting $\tilde{L} = L$ and $\tilde{D} = D$ we get with (7.7) and (7.9)

$$\tilde{z} = L\,, \qquad \tilde{r} = D\,.$$

But after the length scales are assigned as dimensions for the spatial variables, the linear equation system are inconsistent. There are some equations that are contradictory to our assignment. To get back a consistent linear equation system the equations we have to investigate the equations further.

Viscous terms

From the dimension equations of the conductive terms of the momentum equations (7.17), (7.22) and the heat equation (7.27) we get

$$\tilde{r}^{-2}\,\tilde{z}^2 = 1\,.$$

Then either $\tilde{r}$ and $\tilde{z}$ have to be equal, or, if they are different as in our case, one of the second derivatives in the conductive terms in each equation must vanish. We multiply the conductive terms in the momentum equations (6.2) and in the energy equation with $\tilde{r}^2$. For the conductive terms of the first momentum equation we get with $\varepsilon = \frac{D}{L}$ (=0.2 e-3, see appendix B)

$$\frac{\tilde{v}_r}{\tilde{r}^2}\frac{\partial}{\partial\hat{r}}\left[\frac{1}{\hat{r}}\frac{\partial}{\partial\hat{r}}\left(\hat{r}\,\hat{v}_r\right)\right] + \frac{\tilde{v}_r}{\tilde{z}^2}\frac{\partial^2 v_r}{\partial z^2} \quad \text{we get} \quad \tilde{v}_r\,\frac{\partial}{\partial\hat{r}}\left[\frac{1}{\hat{r}}\frac{\partial}{\partial\hat{r}}\left(\hat{r}\,\hat{v}_r\right)\right] + \underbrace{\frac{\tilde{r}^2}{\tilde{z}^2}}_{\varepsilon}\,\tilde{v}_r\,\frac{\partial^2 v_r}{\partial z^2}\,.$$

The second derivatives in streamwise direction are of order ε^2 smaller than the second derivatives in radial direction

$$\frac{\partial^2 v_r}{\partial z^2} \ll \frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\,v_r\right)\right], \quad \frac{\partial^2 v_z}{\partial z^2} \ll \frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\,v_z\right)\right], \quad \frac{\partial^2 T}{\partial z^2} \ll \frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\,T\right)\right].$$

So we drop also the remaining equation of (7.17), (7.22), and (7.27). Note that eliminating (7.22) changes (7.23) to

$$\tilde{r}^{-2}\,\tilde{v}_z\,\tilde{\rho}^{-1}\,\tilde{\mu}\,\tilde{g}^{-1} = 1\,. \tag{7.30}$$

The gravity term refers now to the second derivative of v_z with respect to r (instead of z).

Radial pressure derivative and radial velocity

With two different length scales also the pressure related dimension equations (7.15) and (7.20), or (7.16) and (7.21)

$$\begin{aligned} \tilde{r}\,\tilde{z}^{-1}\,\tilde{v}_r\,\tilde{v}_z\,\tilde{p}^{-1}\,\tilde{\rho} &= 1\,, & \tilde{r}\,\tilde{v}_r^{-1}\,\tilde{p}\,\tilde{\mu}^{-1} &= 1\,, \\ \tilde{v}_z^2\,\tilde{p}^{-1}\,\tilde{\rho} &= 1\,, & \tilde{r}^2\,\tilde{z}^{-1}\,\tilde{v}_z^{-1}\,\tilde{p}\,\tilde{\mu}^{-1} &= 1\,, \end{aligned}$$

are inconsistent. They result in the dimension equation $\tilde{r}\,\tilde{z}^{-1}\,\tilde{v}_r\,\tilde{v}_z^{-1} = 1$, which is contradictive to the dimension equation of the continuity equation $\tilde{r}^{-1}\,\tilde{z}\,\tilde{v}_r\,\tilde{v}_z^{-1} = 1$. A zero pressure derivative in either radial or streamwise direction would solve the problem. In streamwise direction acts the hydrostatic pressure. Hence, the pressure derivative in radial direction must be zero and the condensate pressure is a function of z and t

$$\frac{\partial p}{\partial r} = 0\,, \qquad \text{so} \qquad p = p(z,t)\,. \tag{7.31}$$

To neglect the inertial terms and the viscous terms in the radial momentum equation would also solve the problem, since it would involve a zero radial pressure derivative in the condensate phase. To investigate this further we evaluate the continuity equation (7.12) with $\tilde{D} = D$ and $\tilde{L} = L$ and get that the dimension of the radial velocity is of order ε smaller compared to the streamwise velocity

$$\tilde{v}_r = \varepsilon\,\tilde{v}_z = U\,. \tag{7.32}$$

Substituting this in (6.2) we see that the convective and viscous terms of the radial momentum dimension equation are of order ε smaller compared to the inertial and viscous terms of the streamwise momentum dimension equation. With the assumption of no sudden changes (so that transient and convective terms are of the same order) this suggest to neglect the radial pressure derivative because the inertial and viscous terms are small. Hence we omit the dimension equations (7.13) – (7.16) of the momentum equation in radial direction.

With the assumption that the pressure is a function of z (and t), the pressure derivative in the condensate equals the pressure derivative in the vapor flow

$$\frac{\partial p}{\partial z} = \frac{\partial p_g}{\partial z} \qquad \text{which gives with (6.4)} \qquad \frac{\partial p}{\partial z} = \rho_g\,g\,. \tag{7.33}$$

Note that the results of our analysis in terms of the two different length scales (i.e. second order derivatives in streamwise directions are of order two smaller than

second order derivatives in radial direction, zero pressure derivative in radial direction, radial velocity is of order one smaller compare to sreamwise velocity) are the assumptions that Prandtl made when he derived the boundary layer equations, see Schlichting [Sch79]. Prandtl's boundary layer equations are based on classical dimensional analysis, plus that he had a good intuition. With generalized dimensional analysis we prove that the boundary layer equations can also be obtained by a method of more mathematical rigor.

7.4 Dimensionless variables and dimensionless equations of the bulk flows

Next we solve the system of the dimension equations to find the dimensionless variables, as discussed in the first section of this chapter. Then the model equations of the condensate are given in dimensionless form.

The simplifications discussed above reduces the system of linear equations further $\boldsymbol{B}\,\boldsymbol{x} = \boldsymbol{0}$. After removing the linearly dependent equations from the system of linear equations, we have a system of 13 homogeneous linear equations and $n = 19$ unknowns with $k = \operatorname{rank}\boldsymbol{B} = 13$, such that the matrix $\boldsymbol{B}$ formed by the coefficients of the normalized dimension equations becomes

	$\tilde{r}$	$\tilde{z}$	$\tilde{t}$	$\tilde{v}_r$	$\tilde{v}_z$	$\tilde{p}$	$\tilde{T}$	$\tilde{\rho}$	$\tilde{\mu}$	$\tilde{g}$	$\tilde{c}$	$\tilde{\lambda}$	$\tilde{\rho}_g$	$\tilde{U}$	$\tilde{L}$	$\tilde{D}$	$\tilde{R}$	$\tilde{T}_w$	$\tilde{T}_v$
(7.6)					1									-1					
(7.7)		1													-1				
(7.8)							1											-1	
(7.9)	1																-1		
(7.10)																-1	1		
(7.11)							1												-1
(7.12)	-1	1		1	-1														
(7.18)	1		-1	-1															
(7.20)					2	-1		1											
(7.21)	2	-1			-1	1			-1										
(7.23)	-2				1			-1	1	-1									
(7.24)	2	-1			1			1			1	-1							
(7.28)		-1				1				-1			-1						

According to the Buckingham Π-Theorem from this equation system $k = 13$ nondimensional variables can be constructed from $n - k = 6$ fundamental dimensions. We choose the characteristic quantities of the condensation problem $\tilde{D}$, $\tilde{L}$, $\tilde{U}$, $\tilde{T}_v$ and the material properties $\tilde{\rho}$, $\tilde{c}$ as fundamental dimensions, they fulfill the condition that $\operatorname{rank} \boldsymbol{K} = \operatorname{rank} \boldsymbol{B}$. After applying Gaußian elimination to the matrix $\boldsymbol{B} = [\boldsymbol{K} | \boldsymbol{R}]$, we obtain the row-reduced echelon form of $\boldsymbol{B}$ as:

$$
\operatorname{rref} \boldsymbol{B} = \left[\begin{array}{cccccccccccc|cccccc}
\tilde{r} & \tilde{z} & \tilde{t} & \tilde{v}_r & \tilde{v}_z & \tilde{p} & \tilde{T} & \tilde{\mu} & \tilde{g} & \tilde{\lambda} & \tilde{\rho}_g & \tilde{R} & \tilde{T}_w & \tilde{\rho} & \tilde{c} & \tilde{U} & \tilde{D} & \tilde{L} & \tilde{T}_v \\
\hline
1 & & & & & & & & & & & & & & & & -1 & & \\
 & 1 & & & & & & & & & & & & & & & & -1 & \\
 & & 1 & & & & & & & & & & & & & 1 & & -1 & \\
 & & & 1 & & & & & & & & & & & & -1 & -1 & 1 & \\
 & & & & 1 & & & & & & & & & & & -1 & & & \\
 & & & & & 1 & & & & & & & & -1 & & -2 & & & \\
 & & & & & & 1 & & & & & & & & & & & & -1 \\
 & & & & & & & 1 & & & & & & -1 & & -1 & -2 & 1 & \\
 & & & & & & & & 1 & & & & & & & -2 & & 1 & \\
 & & & & & & & & & 1 & & & & -1 & -1 & -1 & -2 & 1 & \\
 & & & & & & & & & & 1 & & & -1 & & & & & \\
 & & & & & & & & & & & 1 & & & & & -1 & & \\
 & & & & & & & & & & & & 1 & & & & & & -1
\end{array}\right].
$$

By assigning the values $\tilde{D} = D$, $\tilde{L} = L$, $\tilde{U} = U$, $\tilde{T}_v = T_v$ and $\tilde{\rho} = \rho$, $\tilde{g} = g$, $\tilde{c} = c$ we get the nondimensional variables:[3]

$$
\pi_1 = \frac{\tilde{r}}{D}\,, \quad \pi_2 = \frac{\tilde{z}}{L}\,, \quad \pi_3 = \frac{\tilde{t}\,U}{L}\,, \quad \pi_4 = \frac{\tilde{v}_r\,L}{U\,D}\,, \quad \pi_5 = \frac{\tilde{v}_z}{U}\,, \quad \pi_6 = \frac{\tilde{p}}{\rho\,U^2}\,,
$$

$$
\pi_7 = \frac{\tilde{T}}{T_v}\,, \quad \pi_8 = \frac{\tilde{\mu}\,L}{\rho\,U\,D^2}\,, \quad \pi_9 = \frac{\tilde{g}\,L}{\tilde{U}^2}\,, \quad \pi_{10} = \frac{\tilde{\lambda}\,L}{\rho\,c\,U\,D^2}\,, \quad \pi_{11} = \frac{\tilde{\rho}_g}{\rho}\,,
$$

$$
\pi_{12} = \frac{\tilde{R}}{D}\,, \quad \pi_{13} = \frac{\tilde{T}_w}{T_v}\,.
$$

The variables π_1 – π_7 define the dimensions of the dependent and the independent variables, such as $\tilde{r} = D$, $\tilde{z} = L$, $\tilde{t} = \frac{L}{U}$, etc.

[3] Note that for a numerical simulation the temperature dimension is set to $\tilde{T} = T_v - T_w = \Delta T$, compare chapter 8.

Simplified bulk flow equations in dimensionless form

To write the bulk flow equations in dimensionless form we define the Reynolds number, Froude number, Prandtl number and Peclet number of the condensation problem as follows

$$\mathrm{Re} = \frac{\rho\, U\, D}{\mu}\,, \qquad \mathrm{Fr} = \frac{U^2}{g\, D}\,, \qquad \mathrm{Pr} = \frac{\mu\, c}{\lambda}\,, \qquad \mathrm{Pe} = \frac{U\, D\, \rho\, c}{\lambda}\,.$$

Setting $\tilde{\mu} = \mu$, $\tilde{g} = g$ and $\tilde{\lambda} = \lambda$, we write the dimensionless numbers $\pi_8 - \pi_{10}$ as

$$\pi_8 = \frac{1}{\varepsilon\,\mathrm{Re}}\,, \qquad \pi_9 = \frac{1}{\varepsilon\,\mathrm{Fr}}\,, \qquad \pi_{10} = \frac{1}{\varepsilon\,\mathrm{Re}\,\mathrm{Pr}} = \frac{1}{\varepsilon\,\mathrm{Pe}}\,.$$

Then we substitute the dimensions of the independent and dependent variables in (6.1) –(6.4) with $\pi_1 - \pi_7$. We do not use the dimensionless variables $\pi_8 - \pi_{11}$ but let the material properties remain in the equations. By this we get
the continuity equation:

$$\frac{1}{\hat{r}}\,\frac{\partial}{\partial \hat{r}}\,(\hat{r}\,\hat{v}_r) + \frac{\partial \hat{v}_z}{\partial \hat{z}} = 0\,, \tag{7.34}$$

the momentum equations:

$$\begin{aligned}\left[\frac{\partial \hat{v}_z}{\partial \hat{t}} + \hat{v}_r\,\frac{\partial \hat{v}_z}{\partial \hat{r}} + \hat{v}_z\,\frac{\partial \hat{v}_z}{\partial \hat{z}}\right] &= -\frac{\partial \hat{p}}{\partial \hat{z}} + \frac{1}{\varepsilon\,\mathrm{Re}}\,\frac{1}{\hat{r}}\,\frac{\partial}{\partial \hat{r}}\left(\hat{r}\,\frac{\partial \hat{v}_z}{\partial \hat{r}}\right) + \frac{1}{\varepsilon\,\mathrm{Fr}}\,, \qquad (7.35)\\ \frac{\partial \hat{p}}{\partial \hat{r}} &= 0\,,\end{aligned}$$

the energy equation:

$$\left[\frac{\partial \hat{T}}{\partial \hat{t}} + \hat{v}_r\,\frac{\partial \hat{T}}{\partial \hat{r}} + \hat{v}_z\,\frac{\partial \hat{T}}{\partial \hat{z}}\right] = \frac{1}{\varepsilon\,\mathrm{Pe}}\,\left(\frac{1}{\hat{r}}\,\frac{\partial}{\partial \hat{r}}\left[\hat{r}\,\frac{\partial \hat{T}}{\partial \hat{r}}\right]\right)\,, \tag{7.36}$$

and the momentum equation of the vapor flow:

$$\frac{\partial \hat{p}_g}{\partial \hat{z}} = \rho_g\, g\,. \tag{7.37}$$

The pressure derivative in the momentum equation (7.35) of the condensate may be substituted with the momentum equation (7.37) of the vapor flow, compare (7.33).

First and zero order equations

We conclude this section with a discussion of the first and zero order equations. In the continuity equation all terms are of the same order. In the momentum equations the inertial terms, and in the energy equation the transient and convective terms are of order ε. For zero order momentum equations only viscous and gravity terms remain and the radial pressure derivative vanishes [Sch79]. The zero order energy equation (conductive terms) is the Laplace equation in cylindrical coordinates. Nußelt used zero order equations for the condensate flow in his theory of condensation along a flat plate [Nuß16] [BS98].

With a generalized dimensional analysis of the equations of the flow problem (yet without considering the condensation) we confirmed the equations of Nußelt and the boundary layer equations of Prandtl [Spu93]. The advantage of using generalized dimensional analysis is that we now have a tool to investigate the jump conditions, where the result is not known.

7.5 Jump conditions

In this section we analyze the jump conditions (6.13) – (6.16) for condensation in a vertical tube by a generalized dimensional analysis in terms of $\varepsilon = \frac{D}{L}$. From the normalized dimension equations of the jump conditions we find two more nondimensional variables.

Dimension equations from the mass jump condition

The mass jump condition (6.13) becomes in terms of value and dimension variables

$$\dot{m} = \frac{\tilde{\rho}\hat{\rho}}{\sqrt{1 + \frac{\tilde{h}^2}{\tilde{z}^2}\left(\frac{\partial \hat{h}}{\partial \hat{z}}\right)^2}} \left(-\tilde{v}_r\, \hat{v}_r + \frac{\tilde{r}}{\tilde{t}} \left[\frac{\partial \hat{h}}{\partial \hat{t}}\right] + \tilde{v}_z \frac{\tilde{h}}{\tilde{z}} \left[\hat{v}_z \frac{\partial \hat{h}}{\partial \hat{z}}\right] \right) \quad .$$

From the square root we get the dimension equation

$$1 = \frac{\tilde{h}^2}{\tilde{z}^2} \,, \qquad \text{which is equivalent to} \qquad \tilde{h}^2\, \tilde{z}^{-2} = 1 \,.$$

From $R - D = h$ we know that $\tilde{h} = D$. But then again either the length scales are equal or the square of the derivative term has to vanish. With $\frac{\tilde{h}^2}{\tilde{z}^2} = \frac{D^2}{L^2} = \varepsilon^2$ the

square of the derivative of h with respect to the tube axis z is of order two smaller than one and is neglected

$$\left(\frac{\partial h}{\partial z}\right)^2 \ll 1\,. \tag{7.38}$$

This reflects the assumption of small surface waves. The remaining dimension equations of the mass jump condition are linearly dependent on the dimension equation of the continuity equation (7.12). So the mass jump condition provides no additional dimension equation.

Dimension equations from the normal momentum jump condition

The momentum jump condition in normal direction (6.14) becomes

$$\frac{\tilde\rho\tilde\rho}{1+\frac{\tilde h^2}{\tilde z^2}\left(\frac{\partial\hat h}{\partial\hat z}\right)^2}\left(-\tilde v_r\,\hat v_r+\frac{\tilde h}{\tilde t}\left[\frac{\partial\hat h}{\partial\hat t}\right]+\tilde v_z\frac{\tilde h}{\tilde z}\left[\hat v_z\frac{\partial\hat h}{\partial\hat z}\right]\right)\left(-\tilde v_r\hat v_r+\tilde v_z\frac{\tilde h}{\tilde z}\left[\hat v_z\frac{\partial\hat h}{\partial\hat z}\right]\right)+\tilde p\hat p-\tilde p_g\hat p_g$$

$$-\frac{\tilde\mu\,\hat\mu}{1+\frac{\tilde h^2}{\tilde z^2}\left(\frac{\partial\hat h}{\partial\hat z}\right)^2}\left(\frac{\tilde v_r}{\tilde r}\left[\frac{\partial\hat v_r}{\partial\hat r}\right]-4\left(\frac{\tilde v_r}{\tilde z}\left[\frac{\partial\hat v_r}{\partial\hat z}\right]+\frac{\tilde v_z}{\tilde r}\left[\frac{\partial\hat v_z}{\partial\hat r}\right]\right)\frac{\tilde h}{\tilde z}\left[\frac{\partial\hat h}{\partial\tilde z}\right]+\frac{\tilde v_z}{\tilde z}\left[\frac{\partial\hat v_z}{\partial\hat z}\right]\frac{\tilde h^2}{\tilde z^2}\left(\frac{\partial\hat h}{\partial\hat z}\right)^2\right)$$

$$=-\tilde\sigma\,\hat\sigma\left(-\frac{\frac{\tilde h}{\tilde z^2}\left(\frac{\partial^2\hat h}{\partial\hat z^2}\right)}{\left[1+\frac{\tilde h^2}{\tilde z^2}\left(\frac{\partial\hat h}{\partial\hat z}\right)^2\right]^{3/2}}+\frac{1}{\tilde h\,\hat h\sqrt{1+\frac{\tilde h^2}{\tilde z^2}\left(\frac{\partial\hat h}{\partial\hat z}\right)^2}}\right)$$

and is analyzed term by term:

1. The first term describes the momentum due to the phase transition ($\dot m\,\boldsymbol{v}\cdot\boldsymbol{n}$). The square of $\tilde h$ with respect to $\tilde z$ in the denominator of the first term is of order ε^2. From the dimensions $\frac{\tilde h}{\tilde z}=\varepsilon$, $\tilde t=\frac{L}{U}$, $\tilde v_r=\varepsilon\,U$, $\tilde v_z=U$, we see that all terms in the brackets are of order ε

$$\tilde\rho\,\hat\rho\left(-\underbrace{\tilde v_r}_{\varepsilon U}\hat v_r+\underbrace{\frac{\tilde h}{\tilde t}}_{\varepsilon U}\left[\frac{\partial\hat h}{\partial\hat t}\right]+\underbrace{\tilde v_z\frac{\tilde h}{\tilde z}}_{\varepsilon U}\left[\hat v_z\frac{\partial\hat h}{\partial\hat z}\right]\right)\left(-\underbrace{\tilde v_r}_{\varepsilon U}\hat v_r+\underbrace{\tilde v_z\frac{\tilde h}{\tilde z}}_{\varepsilon U}\left[\hat v_z\frac{\partial\hat h}{\partial\hat z}\right]\right),$$

   so that after the multiplication is carried out all terms are of order ε^2. That is the momentum transport due to the condensation can be neglected in the zero and first order normal momentum jump condition. In the case of evaporation this term describes the recoil of vapor particles from the interface after

evaporation. In the case of condensation the vapor particles slow down after the phase change.

2. The pressure terms remain.

3. The third term describes the viscous normal stress exerted on the interface ($\boldsymbol{n} \cdot \boldsymbol{T} \cdot \boldsymbol{n}$). The orders of the terms in the brackets are

$$\underbrace{\underline{\underline{\underbrace{\frac{\tilde{v}_r}{\tilde{r}}}_{\frac{U}{L}} \left[\frac{\partial \hat{v}_r}{\partial \hat{r}}\right]}}}_{} - 4 \left(\underbrace{\underbrace{\frac{\tilde{v}_r}{\tilde{z}}}_{\varepsilon \frac{U}{L}} \left[\frac{\partial \hat{v}_r}{\partial \hat{z}}\right]}_{} + \underbrace{\underline{\underline{\underbrace{\frac{\tilde{v}_z}{\tilde{r}}}_{\frac{U}{D}} \left[\frac{\partial \hat{v}_z}{\partial \hat{r}}\right]}}}_{} \right) \underbrace{\frac{\tilde{h}}{\tilde{z}}}_{\varepsilon} \left[\frac{\partial \hat{h}}{\partial \hat{z}}\right] + \underbrace{\frac{\tilde{v}_z}{\tilde{z}}}_{\frac{U}{L}} \left[\frac{\partial \hat{v}_z}{\partial \hat{z}}\right] \underbrace{\frac{\tilde{h}^2}{\tilde{z}^2}}_{\varepsilon^2} \left(\frac{\partial \hat{h}}{\partial \hat{z}}\right)^2 .$$

Terms of different order in one term are not permissible, the dimension equation of the terms in the inner brackets is $\tilde{r}\, \tilde{z}^{-1}\, \tilde{v}_r\, \tilde{v}_z^{-1} = 1$ and hence a singular equation. After the multiplication is carried out the underlined terms are of the same order and the other two terms are of order two smaller than the underlined terms. In the first instance the underlined terms remain in the normal momentum jump condition.

4. The surface tension terms (after dropping second order terms) are

$$-\tilde{\sigma}\, \hat{\sigma} \left(- \underbrace{\frac{\tilde{h}}{\tilde{z}^2}}_{\varepsilon \frac{1}{D}} \left(\frac{\partial^2 \hat{h}}{\partial \hat{z}^2}\right) + \underbrace{\underline{\underline{\underbrace{\frac{1}{\tilde{h}}}_{\frac{1}{D}} \frac{1}{\hat{h}}}}}_{} \right) ,$$

where again the terms of different order cause the singular dimension equation: $\tilde{h}^2\, \tilde{z}^{-2} = 1$, so that one of both terms has to be dropped. The curvature term related to the surface waves is of order ε smaller than the (underlined) curvature term related to the small diameter of the tube, so that the surface tension is determined by the small diameter of the tube. In the case of a flow along a flat plate only the curvature term related to surface waves would remain in the equation.

For higher velocities (higher Reynolds numbers) surface waves may become more wavy and cause instabilities, so that the laminar flow breaks down [DR04]. However, the dimension equation of the curvature terms suggest that for a small tube diameter the influence of surface waves is comparably small.

The dimension equations of the normal momentum jump condition are

$$\tilde{p}\,\tilde{p}_g^{-1} = 1\,, \quad (7.39) \qquad\qquad \tilde{h}^{-1}\,\tilde{z}\,\tilde{v}_r\,\tilde{v}_z^{-1} = 1\,, \quad (7.41)$$

$$\tilde{r}\,\tilde{v}_r^{-1}\,\tilde{p}_g\,\tilde{\mu}^{-1} = 1\,, \quad (7.40) \qquad\qquad \tilde{r}^{-1}\,\tilde{h}^2\,\tilde{z}^{-1}\,\tilde{v}_z\,\mu\,\tilde{\sigma}^{-1} = 1\,. \quad (7.42)$$

Equation (7.41) depends linearly on the continuity equation (7.12). Equation (7.40) depends linearly on the pressure dimension equation (7.16) of the radial momentum equation. However, this equation was contradictive to the pressure dimension equations of the streamwise momentum equation. So we have to eliminate either the pressure term or the normal stress term. Because the normal stresses are of order $\varepsilon\frac{U}{L}$ smaller than the pressure terms we drop the normal stress terms. This cause a change in (7.42)

$$\tilde{h}\,\tilde{p}_g\,\tilde{\sigma}^{-1} = 1\,. \quad (7.43)$$

The surface tension term refers now to the pressure.

By this, two dimension equations (7.39) and (7.43) are derived from the normal momentum jump condition for condensation in a vertical tube.

Dimension equations from the tangential momentum jump condition

The interface momentum equation in tangential direction becomes

$$\frac{\tilde{\mu}\,\hat{\mu}}{\sqrt{1+\frac{\tilde{h}^2}{\tilde{z}^2}\left(\frac{\partial\hat{h}}{\partial\hat{z}}\right)^2}}\left(-\frac{\tilde{v}_r\,\tilde{h}}{\tilde{r}\,\tilde{z}}\left[\frac{\partial v_r}{\partial r}\frac{\partial h}{\partial z}\right]\right.$$
$$\left.-2\left(\frac{\tilde{v}_r}{\tilde{z}}\left[\frac{\partial\hat{v}_r}{\partial\hat{z}}\right]+\frac{\tilde{v}_z}{\tilde{r}}\left[\frac{\partial\hat{v}_z}{\partial\hat{r}}\right]\right)\left(1-\frac{\tilde{h}^2}{\tilde{z}^2}\left[\frac{\partial h}{\partial z}\right]^2\right)+\frac{\tilde{v}_z\,\tilde{h}}{\tilde{z}^2}\left[\frac{\partial\hat{v}_z}{\partial\hat{z}}\frac{\partial\hat{h}}{\partial\hat{z}}\right]\right)=0\,.$$

After dropping second order terms we get

$$\tilde{\mu}\,\hat{\mu}\left(-\underbrace{\frac{\tilde{v}_r\,\tilde{h}}{\tilde{r}\,\tilde{z}}}_{\varepsilon\frac{U}{L}}\left[\frac{\partial v_r}{\partial r}\frac{\partial h}{\partial z}\right]-2\left(\underbrace{\frac{\tilde{v}_r}{\tilde{z}}}_{\varepsilon\frac{U}{L}}\left[\frac{\partial\hat{v}_r}{\partial\hat{z}}\right]+\underbrace{\frac{\tilde{v}_z}{\tilde{r}}}_{\frac{U}{D}}\left[\frac{\partial\hat{v}_z}{\partial\hat{r}}\right]\right)+\underbrace{\frac{\tilde{v}_z\,\tilde{h}}{\tilde{z}^2}}_{\varepsilon\frac{U}{L}}\left[\frac{\partial\hat{v}_z}{\partial\hat{z}}\frac{\partial\hat{h}}{\partial\hat{z}}\right]\right)=0\,.$$

The order of the terms in the inner round brackets are different, so that the term of order $\varepsilon\frac{U}{L}$ vanishes. By this the tangential momentum jump condition reduces to the condition of no shear stress at the interface. This yields a homogeneous dimension equation which is useless.

Dimension equations from the energy jump condition

Finally the interface energy equation becomes in terms of value and dimension variables

$$\frac{\tilde{\rho}\,\hat{\rho}\,\widetilde{\Delta h}\,\widehat{\Delta h}}{\sqrt{1+\frac{\tilde{h}^2}{\tilde{z}^2}\,\left[\frac{\partial h}{\partial z}\right]^2}}\left(-\tilde{v}_r\,\hat{v}_r+\frac{\tilde{h}}{\tilde{t}}\left[\frac{\partial \hat{h}}{\partial \hat{t}}\right]+\tilde{v}_z\,\frac{\tilde{h}}{\tilde{z}}\left[\hat{v}_z\,\frac{\partial \hat{h}}{\partial \hat{z}}\right]\right)$$
$$=-\frac{\tilde{\lambda}\,\hat{\lambda}}{\sqrt{1+\frac{\tilde{h}^2}{\tilde{z}^2}\left(\frac{\partial \hat{h}}{\partial \hat{z}}\right)^2}}\left(-\frac{\tilde{T}}{\tilde{r}}\,\frac{\partial \hat{T}}{\partial \hat{r}}+\frac{\tilde{T}\,\tilde{h}}{\tilde{z}^2}\left[\frac{\partial \hat{T}}{\partial \hat{z}}\,\frac{\partial \hat{h}}{\partial \hat{z}}\right]\right)\,.$$

After dropping the second order terms we get

$$\tilde{\rho}\,\hat{\rho}\,\widetilde{\Delta h}\,\widehat{\Delta h}\underline{\underline{\left(-\underbrace{\tilde{v}_r}_{\varepsilon\,U}\,\hat{v}_r+\underbrace{\frac{\tilde{h}}{\tilde{t}}}_{\varepsilon\,U}\left[\frac{\partial \hat{h}}{\partial \hat{t}}\right]+\underbrace{\tilde{v}_z\,\frac{\tilde{h}}{\tilde{z}}}_{\varepsilon\,U}\left[\hat{v}_z\,\frac{\partial \hat{h}}{\partial \hat{z}}\right]\right)}}$$
$$=\lambda\,\hat{\lambda}\left(\underbrace{\underline{\underline{\frac{\tilde{T}}{\tilde{r}}}}}_{\frac{T_v}{D}}\,\frac{\partial \hat{T}}{\partial \hat{r}}-\underbrace{\frac{\tilde{T}\,\tilde{h}}{\tilde{z}^2}}_{\varepsilon\,\frac{T_v}{L}}\left[\frac{\partial \hat{T}}{\partial \hat{z}}\,\frac{\partial \hat{h}}{\partial \hat{z}}\right]\right)\,.$$

The mass flux terms remain. On the right hand side the second term is of order two smaller than the first term, so it vanishes. By this the energy jump condition of the condensation problem under rotational symmetry yields the dimension equation

$$\tilde{h}\,\tilde{r}\,\tilde{z}^{-1}\,\tilde{v}_z\,\tilde{T}^{-1}\,\tilde{\rho}\,\tilde{\lambda}^{-1}\,\widetilde{\Delta h}=1 \tag{7.44}$$

7.6 Dimensionless variables and dimensionless equations of the interface

Introducing (7.39), (7.43) and (7.44) into the equation system $\boldsymbol{B}\,\boldsymbol{x}=\boldsymbol{0}$ results in three additional nondimensional variables

$$\pi_{14}=\frac{\tilde{p}}{\tilde{p}_g}\qquad \pi_{15}=\frac{\sigma}{\rho\,U^2\,D}\,,\qquad \pi_{16}=\frac{\Delta h}{c\,T_v}$$

for the condensation problem. We define Weber number and Stefan number as follows:[4]

$$\mathrm{We} = \frac{\rho\, U^2\, D}{\sigma}\,, \qquad\qquad \mathrm{St} = \frac{c\, T_v}{\Delta h}\,,$$

Then the nondimensional numbers π_{15} and π_{16} becomes

$$\pi_{14} = \frac{1}{\mathrm{We}}\,, \qquad\qquad \pi_{15} = \frac{1}{\mathrm{St}}\,.$$

Simplified jump conditions in nondimensional form

Now we write the analyzed and simplified jump conditions at the interface between the condensate and the vapor in nondimensional form. For that we use the dimensions of the independent and dependent variables defined by π_1 – π_7 in section 7.4. Again we do not use the dimensions of the material properties defined by π_8 – π_{11} and π_{15} – π_{16} but the material properties remain in the equations. This gives the mass jump condition:

$$\dot{m} = \rho\, \varepsilon\, U \left(-\hat{v}_r + \frac{\partial \hat{h}}{\partial \hat{t}} + \hat{v}_z \frac{\partial \hat{h}}{\partial \hat{z}} \right)\,, \tag{7.45}$$

the normal momentum jump condition:

$$\hat{p} - \hat{p}_g = -\frac{1}{\mathrm{We}} \frac{1}{\hat{h}}\,, \tag{7.46}$$

the tangential momentum jump condition:

$$\frac{\partial \hat{v}_z}{\partial \hat{r}} = 0\,, \tag{7.47}$$

the energy jump condition:

$$\frac{\varepsilon}{\mathrm{St}} \left(-\hat{v}_r + \frac{\partial \hat{h}}{\partial \hat{t}} + \hat{v}_z \frac{\partial \hat{h}}{\partial \hat{z}} \right) = \frac{1}{\mathrm{Pe}} \frac{\partial \hat{T}}{\partial \hat{r}}\,. \tag{7.48}$$

[4]Note that in chapter 8 the temperature dimension is set to $\tilde{T} = T_v - T_w = \Delta T$, so that there the Stefan number is defined as $\mathrm{St} = \frac{c\,\Delta T}{\Delta h}$.

Main physical relevant terms of condensation in a vertical tube

All mass flux terms are of the same order. In the normal momentum jump condition the momentum transport due to phase change and some of the normal viscous stress terms were of order ε^2 and were dropped. The remaining normal viscous stress terms were of order ε but contradictive to the pressure dimension equation. So the pressure difference at the interface is mainly determined by surface tension stress. The tangential momentum jump condition is the condition of no shear stress at the interface. The energy jump condition states that the heat flux of condensation is proportional to the heat flux at the interface exerted by the vapor. The jump conditions (7.45) – (7.48) are based on the same assumptions as the boundary layer conditions. However, note that also dimension matrix has to be consistent (compare normal momentum jump condition).

Zero order equations

All mass flux terms are of order ε, which means that the condensation process cannot be modeled with zero order jump conditions. The zero order normal momentum jump condition is the Young-Laplace equation [Dee98]. The condition of no shear stress is a zero order equation. A zero order energy jump condition would state that the heat flux at the interface vanishes. This comes from the assumption that the heat flux in the vapor vanishes, see section 6.1.

With a generalized dimensional analysis based on the slenderness of the condensate film we determined the main relevant physical effects of condensation in a vertical tube.

7.7 Water and R134a

We conclude this chapter by evaluating equations (7.34) – (7.37) and (7.45) – (7.48) derived in the last sections for water and for the refrigerant R134a. These fluids were also used in [Fie03]. The dimensionless numbers of water and R134a are given in appendix B, where also material properties and process properties (such as film thickness and velocity) are be found. For both fluids the quotient of the two length scales is $\varepsilon = 0.2\,\text{e-}3$.

In the nondimensional continuity equation (7.34) all terms have the same magnitude. For the nondimensional momentum equation (7.35) and the nondimensional heat equation (7.36) we get

$$\left[\frac{\partial \hat{v}_z}{\partial \hat{t}} + \hat{v}_r \frac{\partial \hat{v}_z}{\partial \hat{r}} + \hat{v}_z \frac{\partial \hat{v}_z}{\partial \hat{z}}\right] = -\frac{\partial \hat{p}}{\partial \hat{z}} + \underset{\substack{=1263 \text{ Water} \\ =58 \text{ R134a}}}{\frac{1}{\varepsilon \,\mathrm{Re}}} \frac{1}{\hat{r}} \frac{\partial}{\partial \hat{r}} \left(\hat{r} \frac{\partial \hat{v}_z}{\partial \hat{r}}\right) + \underset{\substack{=9034 \text{ Water} \\ =247 \text{ R134a}}}{\frac{1}{\varepsilon \,\mathrm{Fr}}} , \tag{7.49}$$

$$\left[\frac{\partial \hat{T}}{\partial \hat{t}} + \hat{v}_r \frac{\partial \hat{T}}{\partial \hat{r}} + \hat{v}_z \frac{\partial \hat{T}}{\partial \hat{z}}\right] = \underset{\substack{=321 \text{ Water} \\ =17 \text{ R134a}}}{\frac{1}{\varepsilon \,\mathrm{Pe}}} \left(\frac{1}{\hat{r}} \frac{\partial}{\partial \hat{r}} \left[\hat{r} \frac{\partial \hat{T}}{\partial \hat{r}}\right]\right) . \tag{7.50}$$

In both fluids viscous forces and gravity forces dominate the inertial terms. Conductive heat transport dominates convective heat transport, but the difference is more distinct for water than for R134a.

The terms of the nondimensional mass jump condition (7.45) are again of same order. For the nondimensional normal momentum jump condition (7.46) and the nondimensional energy jump condition (7.48) we get

$$\hat{p} - \hat{p}_g = -\underset{\substack{=1261 \text{ Water} \\ =3.46 \text{ R134a}}}{\frac{1}{\mathrm{We}}} \frac{1}{\hat{h}} , \tag{7.51}$$

$$\underset{\substack{=1.8 \text{ Water} \\ =0.4 \text{ R134a}}}{\frac{1}{\mathrm{St}}} \left(\hat{v}_r + \frac{\partial \hat{h}}{\partial \hat{t}} + \hat{v}_z \frac{\partial \hat{h}}{\partial \hat{z}}\right) = \underset{\substack{=321 \text{ Water} \\ =17 \text{ R134a}}}{\frac{1}{\varepsilon \,\mathrm{Pe}}} \frac{\partial \hat{T}}{\partial \hat{r}} . \tag{7.52}$$

Clearly for water surface tension is the dominant force. However, if the condensate fluid is R134a, surface tension has almost the same order of magnitude as the pressure terms. In the energy jump condition the temperature deriative dominates. This shows that the flow is the dominating process, the position of the moving surface is mainly determined by the solution of the free surface problem. Again this is more true for water than for R134a.

The inclined tube

From what we found out about condensation in a vertical tube we can also explain the results given in [Fie03] about condensation in a tube inclined to the vertical.

There it was shown that for R134a the heat transfer due to condensation is approximately two times better for an inclination angle of 45° degree compared to the vertical tube, but this effect could not be observed with water (compare section 1.2).

For a horizontal position of the tube the interface is most probably not any more rotational. We showed that the main forces in the condensation process are gravity force and surface tension force. In an inclined tube gravity force can be divided into two components. One component acts in the plane of rotation and the other acts in the direction of the tube axis. Because of gravity the condensate will flow down along the tube walls and continue flowing in a gathered stream at the bottom of the tube. Surface tension force acts normal to the interface. The effect of surface tension can be described by the tendency to minimize the surface, and the minimal surface is the circular surface. The balance of the two forces acting in the plane of rotation result for both fluids in a cross section between the two extremal positions of the moving surface as shown in figure 7.1.

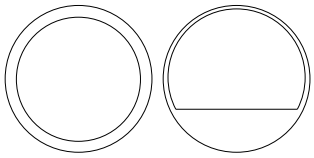


Figure 7.1: *Condensate film cross sections*

Actually the effect of surface tension is to minimize potential energy. Surface tension is caused by molecular forces. In the bulk of the condensate the molecules are surrounded by other molecules and are attracted equally in all directions. At the surface of the condensate the attraction forces result in a force that is directed inwards the condensate film such that the condensate film is pulled into circular shape [WIK].

We showed that surface tension force is the dominant force for water, but is comparably small for R134a. By this, we conclude that for a given inclination angle of the tube the cross section for water will be more like the figure on the left and the cross section for R134a will be more like the figure on the right.

We showed that the heat transfer through the condensate is mainly conductive. According to Fourier's law the conductive heat flux through a fluid is for a given film thickness and a constant temperature difference better if the film is thinner. That is the reason why the cross section of the right figure is the better one for a better heat transfer. So a better heat transfer can be expected for R134a, which is in agreement with the experimental results.

Part III

Numerical simulations and related issues

Chapter 8

Single ordinary differential equation (ODE)/rotational symmetry

In this chapter, we consider again the model equations for condensation in a vertical tube with small diameter, as derived in part II, chapter 7. In section 7.7 we showed that, if the condensate is water, surface tension is the most dominant force beside gravity. For R134a the effect of surface tension was considerably small. Based on the results of our evaluation of the model equations for water and R134a we simplify the model equations further and derive a nonlinear ordinary differential equation for the film thickness. In a first attempt we neglect surface tension, then we include surface tension into the single model equation, finally we compute numerical solutions of these ODEs and discuss the effect of surface tension for water and R134a.

For numerical methods for the solution of ODEs we refer to [SW95].

Dimensionless variables

In a numerical simulation, dimensionless variables are used to reduce the amount of variables to a fewer amount of nondimensional variables, whereas in a numerical analysis the main purpose is to normalize the value of the variables such that all variables range between 0 and a positive number a, say 10, to avoid roundoff and cancelation errors.

For a better normalization of the temperature we revise the dimension of the temperature, which we assigned in chapter 7.4 to $\tilde{T} = T_v$. The energy equation is a linear differential equation and is therefore invariant under translation of the de-

pendent variable (compare appendix A.2), for example a translation with the wall temperature

$$T = \hat{T}\,\tilde{T} + T_w \; . \tag{8.1}$$

The dimension of the temperature we set to $\tilde{T} = T_v - T_w = \Delta T$. Then the dimensionless temperature is given as

$$\hat{T} = \frac{T - T_w}{\Delta T} \; , \qquad \text{so that} \qquad \hat{T}_w = 0 \; , \qquad \hat{T}_v = 1 \; .$$

In section 7.4 the Stefan number was defined as $\mathrm{St} = \frac{c\tilde{T}}{\Delta h}$. With the temperature dimension $\tilde{T} = \Delta T$ the Stefan number is

$$\mathrm{St} = \frac{c\,\Delta T}{\Delta h} \; .$$

This definition is more appropriate for the numerical simulations, compare section 8.3. The other nondimensional variables and numbers remain as discussed in chapter 7 and as given in appendix B.

8.1 Derivation of a single ODE for the film thickness

We derive a single ODE for the film thickness, in a first attempt without taking surface tension into account. We will use this equation in section 8.3 as a test equation for the numerical computations. We show that our model equation is an extension of Nußelt's theory of condensation along a flat plate.

Further reduction of the model equations to ODEs

Based on the evaluation of the model equations for water and R134a in section 7.7 we neglect transient and convective terms in the condensate equations (7.49), (7.50), (7.51), and we neglect the pressure derivative in (7.49). For the moment we also neglect the normal momentum jump condition (7.51). Then we get

$$\frac{1}{\hat{r}}\frac{\partial}{\partial \hat{r}}\left(\hat{r}\frac{\partial \hat{v}_z}{\partial \hat{r}}\right) = -\frac{\mathrm{Re}}{\mathrm{Fr}} \; , \tag{8.2}$$

$$\frac{\partial}{\partial \hat{r}}\left(\hat{r}\,\frac{\partial \hat{T}}{\partial \hat{r}}\right) = 0 \; . \tag{8.3}$$

$$\hat{v}_z\,\frac{\partial \hat{h}}{\partial \hat{z}} = \frac{\mathrm{St}}{\varepsilon\,\mathrm{Pe}}\,\frac{\partial \hat{T}}{\partial \hat{r}} \qquad \text{at} \qquad \hat{r} = \hat{h} \; , \tag{8.4}$$

together with the boundary conditions

$$\hat{v}_z\big|_{\hat{r}=\hat{R}} = 0 \,, \quad \left.\frac{\partial \hat{v}_z}{\partial \hat{r}}\right|_{\hat{r}=\hat{h}} = 0 \,, \qquad \text{and} \qquad \hat{T}\big|_{\hat{r}=\hat{R}} = \hat{T}_w \,, \quad \hat{T}\big|_{\hat{r}=\hat{h}} = \hat{T}_v \,. \tag{8.5}$$

By this we reduced the partial differential model equations to ordinary differential equations for the variables $\hat{v}_z$, $\hat{T}$, and $\hat{h}$, which are much easier to solve.

The boundary condition for the last ODE (8.4) is that the film thickness is given at the inlet

$$\hat{h}\big|_{\hat{z}=0} = \hat{h}_0 \,. \tag{8.6}$$

Velocity and temperature profile

Equations (8.2) and (8.3) can be solved analytically. Integrating (8.2) once gives

$$\frac{\partial \hat{v}_z}{\partial \hat{r}} = -\frac{\mathrm{Re}}{2\,\mathrm{Fr}}\,\hat{r} + \frac{C_1}{\hat{r}} \,,$$

where the first integration constant is determined by the condition of no shear stress at the interface, $C_1 = \frac{\mathrm{Re}}{2\,\mathrm{Fr}}\,\hat{h}^2$. Integrating again gives

$$\hat{v}_z = -\frac{\mathrm{Re}}{2\,\mathrm{Fr}}\left(\frac{\hat{r}^2}{2} - \hat{h}^2\,\ln\hat{r}\right) + C_2 \,,$$

where the second integration constant is determined by the condition of no-slip at the wall, $C_2 = \frac{\mathrm{Re}}{2\,\mathrm{Fr}}\left(\frac{\hat{R}^2}{2} - \hat{h}^2 \ln\hat{R}\right)$. Then the dimensionless velocity is a function of the radius depending on the film thickness

$$\hat{v}_z(\hat{r},\hat{h}) = \frac{\mathrm{Re}}{2\,\mathrm{Fr}}\left(\frac{\hat{R}^2 - \hat{r}^2}{2} + \hat{h}^2\,\ln\frac{\hat{r}}{\hat{R}}\right) \,. \tag{8.7}$$

To get the temperature we integrate the energy equation (8.3) twice. This gives first

$$\frac{\partial \hat{T}}{\partial \hat{r}} = \frac{C_3}{\hat{r}} \,, \qquad \text{integrating again gives} \qquad \hat{T} = C_3\,\ln\hat{r} + C_4 \,.$$

The integration constants are determined by the boundary conditions, $C_3 = \frac{\hat{T}_v - \hat{T}_w}{\ln\hat{h} - \ln\hat{R}}$, $C_4 = -\frac{\hat{T}_v\,\ln\hat{R} - \hat{T}_w\,\ln\hat{h}}{\ln\hat{h} - \ln\hat{R}}$. Then the dimensionless temperature is given by

$$\hat{T}(\hat{r},\hat{h}) = \frac{\hat{T}_v - \hat{T}_w}{\ln\hat{h} - \ln\hat{R}}\,\ln\hat{r} - \frac{\hat{T}_v\,\ln\hat{R} - \hat{T}_w\,\ln\hat{h}}{\ln\hat{h} - \ln\hat{R}} \,. \tag{8.8}$$

Single ODE for the film thickness

Now we evaluate $\hat{v}$ and $\frac{\partial \hat{T}}{\partial \hat{r}}$ at $\hat{r} = \hat{h}$ and substitute the results into the mass-energy jump condition (8.4). This gives

$$\frac{\mathrm{Re}}{2\,\mathrm{Fr}} \left(\frac{\hat{R}^2 - \hat{h}^2}{2} + \hat{h}^2 \ln \frac{\hat{h}}{\hat{R}} \right) \frac{\partial \hat{h}}{\partial \hat{z}} = \frac{\mathrm{St}}{\varepsilon\,\mathrm{Pe}} \frac{\hat{T}_v - \hat{T}_w}{\left(\ln \hat{h} - \ln \hat{R} \right) \hat{h}}$$

or equivalently

$$\frac{\partial \hat{h}}{\partial \hat{z}} = \frac{2\,\mathrm{Fr}\,\mathrm{St}\,(\hat{T}_v - \hat{T}_w)}{\varepsilon\,\mathrm{Re}\,\mathrm{Pe}} \frac{1}{\left(\frac{\hat{R}^2 - \hat{h}^2}{2} + \hat{h}^2 \ln \frac{\hat{h}}{\hat{R}} \right) \ln \frac{\hat{h}}{\hat{R}}\, \hat{h}} \,. \tag{8.9}$$

This equation is a first order homogeneous nonlinear ordinary differential equation for the film thickness in autonomous form[1]. Obviously the condensation process causes the nonlinearity, without condensation we would have $\frac{\partial \hat{h}}{\partial \hat{z}} = 0$ and hence a film of constant thickness. Equation (8.9) is quasilinear[2], and can be solved numerically together with the boundary condition (8.6). We come back to the numerical solution of (8.9) later in this section.

Comparison with Nußelt's theory of condensation along a flat plate

The equation we derived in the last section for condensation in a vertical tube is as an extension of Nußelt's theory of condensation along a flat plate [Nuß16]. In Cartesian coordinates the momentum equation, energy equation and mass-energy jump condition are given as follows

$$\frac{\partial^2 \hat{u}}{\partial \hat{y}^2} = -\frac{\mathrm{Re}}{\mathrm{Fr}} \,, \qquad \frac{\partial^2 \hat{T}}{\partial \hat{y}^2} = 0 \,, \qquad \hat{u}\, \frac{\partial \hat{\delta}}{\partial \hat{x}} = \frac{\mathrm{St}}{\mathrm{Pe}} \frac{\partial \hat{T}}{\partial \hat{y}} \,.$$

Here $\hat{x}$ is the dimensionless streamwise coordinate, $\hat{y}$ denotes the dimensionless coordinate perpendicular to $\hat{x}$ in counter clockwise direction, $\hat{u}$ is the dimensionless streamwise velocity, and $\hat{\delta}$ is the dimensionless film thickness [BS98]. The boundary conditions are again the condition of no-slip at the wall and of no shear-stress at the interface and the given temperature at the wall and at the interface. By using the

[1] implicit in the independent variable $y(x)' = f(y(x))$

[2] linear in the highest derivative

boundary conditions the dimensionless velocity and the dimensionless temperature derivative become as follows

$$\hat{u} = \frac{\mathrm{Re}}{\mathrm{Fr}} \left(\hat{\delta}\, \hat{y} - \frac{\hat{y}^2}{2} \right) , \qquad \frac{\partial \hat{T}}{\partial \hat{y}} = \frac{\hat{T}_v - \hat{T}_w}{\hat{\delta}} .$$

Evaluating the velocity and the temperature derivative at $\hat{y} = \hat{\delta}$ we get the following ordinary differential equation for the dimensionless film thickness

$$\frac{\mathrm{Re}}{\mathrm{Fr}} \frac{\hat{\delta}^2}{2} \frac{\partial \hat{\delta}}{\partial \hat{x}} = \frac{\mathrm{St}}{\mathrm{Pe}} \frac{\hat{T}_v - \hat{T}_w}{\hat{\delta}} , \quad \text{or equivalently} \quad \frac{\partial \hat{\delta}}{\partial \hat{x}} = \frac{2\,\mathrm{Fr}\,\mathrm{St}\,(\hat{T}_v - \hat{T}_w)}{\mathrm{Re}\,\mathrm{Pe}} \frac{1}{\hat{\delta}^3} .$$

This equation is a Bernoulli differential equation of type $y' = k\, y^{-3}$ and can be solved analytically after linearization [WIK] [MW]. With $\hat{\delta}(x = 0) = 0$ we obtain

$$\hat{\delta} = \sqrt{2} \left(\frac{2\,\mathrm{Fr}\,\mathrm{St}\,(\hat{T}_v - \hat{T}_w)}{\mathrm{Re}\,\mathrm{Pe}}\, \hat{x} \right)^{\frac{1}{4}} . \tag{8.10}$$

The film thickness is a function of the streamwise coordinate to the power of one fourth. This result is in exact agreement with Nußelt's well known equation for the film thickness of a condensate flow along a flat plate [BS98].

8.2 ODE for the film thickness including surface tension

Now we include surface tension in the single model equation (8.9) which we derived in the last section.

System of ordinary differential equations including surface tension

To include the effect of surface tension we again neglect transient and convective terms in (7.49), (7.50), (7.51), but let remain the pressure gradient in (7.49)

$$\frac{1}{\varepsilon\,\mathrm{Re}} \frac{1}{\hat{r}} \frac{\partial}{\partial \hat{r}} \left(\hat{r} \frac{\partial \hat{v}_z}{\partial \hat{r}} \right) = \frac{\partial \hat{p}}{\partial \hat{z}} - \frac{1}{\varepsilon\,\mathrm{Fr}} , \qquad \frac{\partial \hat{p}}{\partial \hat{r}} = 0 . \tag{8.11}$$

Then we use the normal momentum jump condition (7.51)

$$\hat{p} = -\frac{1}{\mathrm{We}} \frac{1}{\hat{h}} \tag{8.12}$$

to get the pressure gradient by differentiating $\hat{p}$ with respect to $\hat{z}$

$$\frac{\partial \hat{p}}{\partial \hat{z}} = \frac{1}{\mathrm{We}} \frac{1}{\hat{h}^2} \frac{\partial \hat{h}}{\partial \hat{z}} . \tag{8.13}$$

With the assumption of a steady condensate flow the pressure is only a function of $\hat{z}$ (compare section 7.3), and we can substitute the pressure gradient in (8.11) with (8.13). This gives the following system of ODEs

$$\frac{1}{\hat{r}} \frac{\partial}{\partial \hat{r}} \left(\hat{r} \frac{\partial \hat{v}_z}{\partial \hat{r}} \right) = \frac{\varepsilon \,\mathrm{Re}}{\mathrm{We}} \frac{1}{\hat{h}^2} \frac{\partial \hat{h}}{\partial \hat{z}} - \frac{\mathrm{Re}}{\mathrm{Fr}} . \tag{8.14}$$

$$\frac{\partial}{\partial \hat{r}} \left(\hat{r} \frac{\partial \hat{T}}{\partial \hat{r}} \right) = 0 , \tag{8.15}$$

$$\hat{v}_z \frac{\partial \hat{h}}{\partial \hat{z}} = \frac{\mathrm{St}}{\varepsilon \,\mathrm{Pe}} \frac{\partial \hat{T}}{\partial \hat{r}} \qquad \text{at } \hat{r} = \hat{h} . \tag{8.16}$$

The boundary conditions are the same as before, see (8.5) and (8.6).

Velocity (including surface tension) and temperature gradient

Integrating (8.14) and (8.15) and evaluating the boundary conditions gives the dimensionless velocity and the dimensionless temperature gradient with respect to $\hat{r}$ as follows

$$\hat{v}_z = \left(-\frac{\varepsilon \mathrm{Re}}{2\,\mathrm{We}} \frac{1}{\hat{h}^2} \frac{\partial \hat{h}}{\partial \hat{z}} + \frac{\mathrm{Re}}{2\,\mathrm{Fr}} \right) \left(\frac{\hat{R}^2 - \hat{r}^2}{2} + \hat{h}^2 \ln \frac{\hat{r}}{\hat{R}} \right) , \tag{8.17}$$

$$\frac{\partial \hat{T}}{\partial \hat{r}} = \frac{\hat{T}_v - \hat{T}_w}{\ln \frac{\hat{h}}{\hat{R}}} \frac{1}{\hat{r}} . \tag{8.18}$$

Single ordinary differential equation for the film thickness including surface tension

Evaluating (8.17) and (8.18) at $\hat{r} = \hat{h}$ and substituting the results into the mass-energy jump condition (8.16) gives

$$\left(-\frac{\varepsilon \,\mathrm{Re}}{2\,\mathrm{We}} \frac{1}{\hat{h}^2} \frac{\partial \hat{h}}{\partial \hat{z}} + \frac{\mathrm{Re}}{2\,\mathrm{Fr}} \right) \left(\frac{\hat{R}^2 - \hat{h}^2}{2} + \hat{h}^2 \ln \frac{\hat{h}}{\hat{R}} \right) \frac{\partial \hat{h}}{\partial \hat{z}} = \frac{\mathrm{St}}{\varepsilon \,\mathrm{Pe}} \frac{\hat{T}_v - \hat{T}_w}{\ln \frac{\hat{h}}{\hat{R}}\, h} ,$$

or equivalently

$$\left(\frac{\partial \hat{h}}{\partial \hat{z}} \right)^2 - \frac{\mathrm{We}}{\varepsilon \mathrm{Fr}} \hat{h}^2 \frac{\partial \hat{h}}{\partial \hat{z}} = -\frac{2\,\mathrm{We}\,\mathrm{St}\,(\hat{T}_v - \hat{T}_w)}{\varepsilon^2\,\mathrm{Re}\,\mathrm{Pe}} \frac{\hat{h}}{\left(\frac{\hat{R}^2 - \hat{h}^2}{2} + \hat{h}^2 \ln \frac{\hat{h}}{\hat{R}} \right) \ln \frac{\hat{h}}{\hat{R}}} . \tag{8.19}$$

This equation is a fully nonlinear first order differential equation for the film thickness. However, we can transform (8.19) into a quasilinear ODE. For that we write more conveniently

$$\left(\frac{\partial \hat{h}}{\partial \hat{z}}\right)^2 - a\,\hat{h}^2\,\frac{\partial \hat{h}}{\partial \hat{z}} = -b\,f(\hat{h})\;.$$

with $a = \frac{\mathrm{We}}{\varepsilon\,\mathrm{Fr}}$ and $b = \frac{2\,\mathrm{We}\,\mathrm{St}\,(\hat{T}_v - \hat{T}_w)}{\varepsilon^2\,\mathrm{Re}\,\mathrm{Pe}}$. The left hand side can be transformed in an quadratic expression

$$\left(\frac{\partial \hat{h}}{\partial \hat{z}} - a\,\frac{\hat{h}^2}{2}\right)^2 - a^2\,\frac{\hat{h}^4}{4} = b\,f(\hat{h})\;,$$

so that

$$\frac{\partial \hat{h}}{\partial \hat{z}} = a\,\frac{\hat{h}^2}{2} \pm \sqrt{a^2\,\frac{\hat{h}^4}{4} - b\,f(\hat{h})}\;.$$

Substituting back the parameters a and b we finally receive the following quasilinear ODE for condensation in a vertical tube with small diameter

$$\frac{\partial \hat{h}}{\partial \hat{z}} = \frac{\mathrm{We}}{\varepsilon\,\mathrm{Fr}}\,\frac{\hat{h}^2}{2} + \sqrt{\left(\frac{\mathrm{We}}{\varepsilon\,\mathrm{Fr}}\right)^2\,\frac{\hat{h}^4}{4} - \frac{2\,\mathrm{We}\,\mathrm{St}\,(\hat{T}_v - \hat{T}_w)}{\varepsilon^2\,\mathrm{Re}\,\mathrm{Pe}}\,\frac{\hat{h}}{\left(\frac{\hat{R}^2 - \hat{h}^2}{2} + \hat{h}^2\,\ln\frac{\hat{h}}{\hat{R}}\right)\ln\frac{\hat{h}}{\hat{R}}}}\;. \tag{8.20}$$

We are seeking for the solution of the ODE with the negative signed root.

8.3 Numerical simulations of condensation including surface tension

To investigate the effect of surface for condensation in a vertical tube for Water and R134a we solve equation (8.20) numerically.

Water

First we compare the model equations without surface tension (8.9) and with surface tension (8.20) with Nußelt's solution (8.10). We assume a tube diameter of 7 mm and a tube length of 500 mm and take water as condensate. The initial film thickness is 0.1 mm. All calculations are dimensionless. Further details, such as

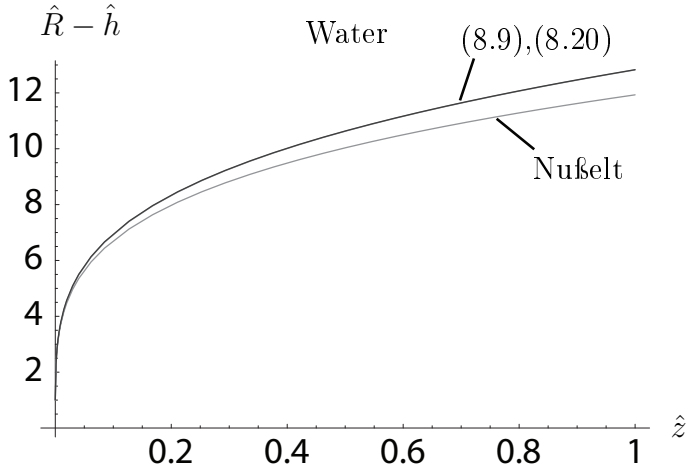


Figure 8.1: *Water:* (8.10) *Nußelt,* (8.9) *without and* (8.20) *with surface tension*

wall temperature and dimensionaless numbers of water, are given in appendix B. The results of our computations using Euler implicit and Runge-Kutta(4) are done with *Mathematica* and are shown in figure 8.1. The numerical solutions of the model equations without surface tension and with surface tension are not distinguishable. If the condensing fluid is water an effect of surface tension cannot be observed. The film thickness predicted by (8.9) and (8.20) is slightly above the film thickness predicted by Nußelt. The reason is that because of the circular tube the condensing mass result in a thicker condensate film.

For a larger diameter of about 60 mm Nußelt's solution is almost identical with the numerical solution. Figure 8.2 shows the film thickness for two different tube diameter.[3] The numerical solution of nonlinear ODEs often requires a higher order

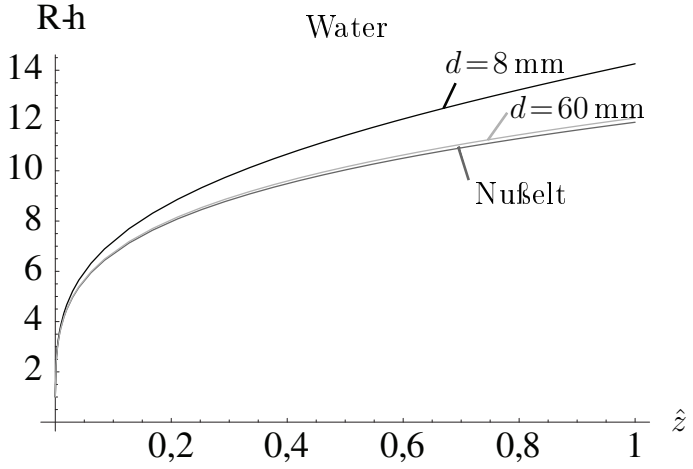


Figure 8.2: *Water:* (8.10) *Nußelt,* (8.9) *and* (8.20) *for* $d = 60$ *mm,* $d = 8$ *mm*

[3]Note that the characteristic velocity and length are based on measured values with a fixed diameter.

accuracy method or a more stable implicit method, or a combination of both, an implicit method of higher order [SW95]. However, the nonlinearity of (8.9) and (8.20) is smooth, so that the results of Euler implicit and Runge-Kutta are not distinguishable and a standard Runge-Kutta(4) is sufficient.

R134a

For R134a we assume the same tube geometry, the same initial film thickness, and the same temperature interval as before. The data and the dimensionless numbers of R134 are given in appendix B. The numerical solutions of (8.9) and (8.20) for R134a using Runge-Kutta are shown in figure 8.3. The R134a film is thinner than the

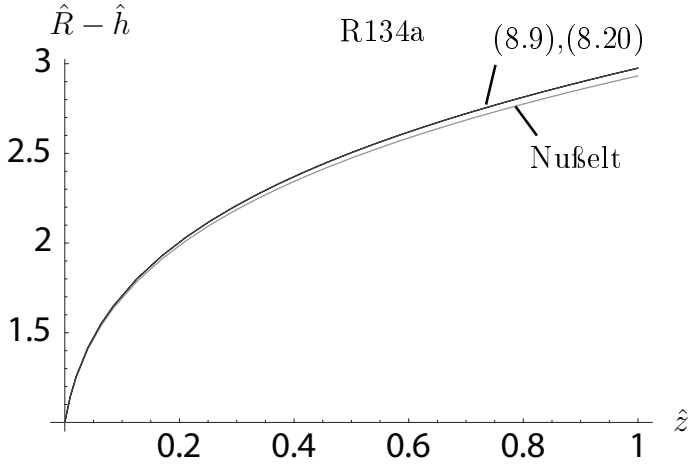


Figure 8.3: *R134a:* (8.10) *Nußelt,* (8.9) *without and* (8.20) *with surface tension*

water film, which is in agreement with [Fie03]. Beside this the results are similar as before. The difference between Nußelt's solution and the numerical solutions of (8.9) and (8.20) is smaller, because of the thinner film thickness of the R134a film.

Higher surface tension coefficient

For the vertical tube we could not observe an effect of surface tension on the film thickness for water and R134a. Now we increase the surface tension coefficient (i.e., reduce the Weber number) by two powers of ten. This is an unrealistic value but serves for an estimation. The result is shown in figure 8.4. In a vertical tube cross sections of the film are always circular, but especially at the tube inlet the film thickness varies substantial over the tube length (compare section 7.7). In the vertical tube the effect of surface tension to minimize the surface results in a more evenly distributed film thickness along the tube length.

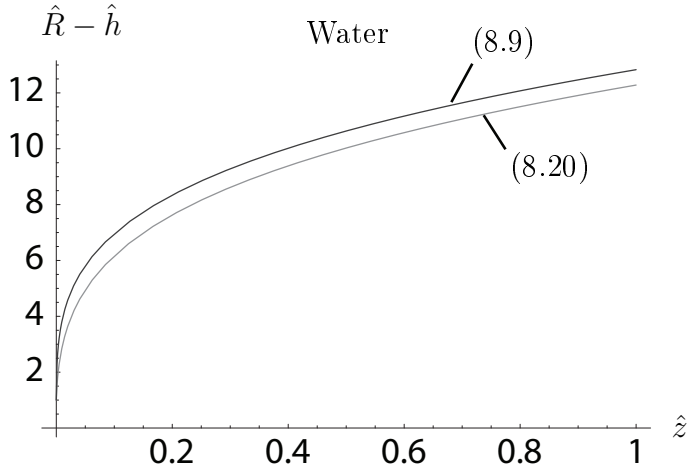


Figure 8.4: *R134a:* (8.9) *without surface tension,* (8.20) *with surface tension*

Nußelt number – dimensionless heat transfer

The heat transfer in the condensate film is mostly conductive so that the temperature profile in the film is almost linear and we can write

$$q'' = \alpha\,\Delta T\ .$$

Equating q'' with the energy jump condition at the interface (where the heat transfer process occurs) gives

$$\alpha\,\Delta T = \lambda\,\boldsymbol{\nabla} T\cdot\boldsymbol{n} = \dot{m}\,\Delta h\ ,$$

which becomes in terms of dimensionless variables,[4]

$$\alpha\,\Delta T = \frac{\lambda\,\Delta T}{D}\,\frac{\partial\hat{T}}{\partial\hat{r}} = \frac{\Delta h\,\rho\,U\,D}{L}\,\hat{u}\,\frac{\partial\hat{h}}{\partial\hat{z}}\ .$$

Multiplying this equation with $\frac{D}{\lambda\,\Delta T}$ gives the local Nußelt number, which is defined as the dimensionless temperature gradient at the interface

$$\mathrm{Nu}_{\mathrm{DGL}} = \frac{\alpha\,D}{\lambda} = \frac{\partial\hat{T}}{\partial\hat{r}} = \underbrace{\frac{\Delta h\,\rho\,U\,D^2}{\lambda\,\Delta T\,L}}_{\frac{\varepsilon\,\mathrm{Pe}}{\mathrm{St}}}\,\hat{u}\,\frac{\partial\hat{h}}{\partial\hat{z}}\ .$$

Figure 8.5 shows the local Nußelt number of Water and R134a as a function of tube length. Note that the local Nußelt number is almost inversely proportional to the film thickness.

[4] With the dimensions $\tilde{r}=\tilde{h}=D$, $\tilde{u}=U$, $\tilde{z}=L$, and $\tilde{T}=\Delta T$, compare chapter 7 and (8.1).

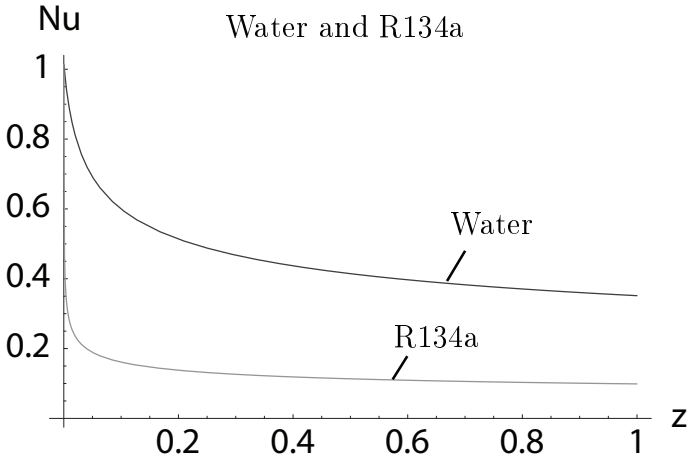


Figure 8.5: *Nußelt number of Water and R134a*

We compare our results with the mean Nußelt number $\overline{\mathrm{Nu}}_{\mathrm{Nu}} = (3\,\mathrm{Re})^{1/3}$ for condensation along a flat plate according to Nusselt's theory [BS98], and with the mean Nußelt number given by Chen for condensation in tubes [Bej04]

$$\overline{\mathrm{Nu}}_{\mathrm{Chen}} = \left(\mathrm{Re}^{-0.44} + 5.82\,10^{-6}\mathrm{Re}^{0.8}\mathrm{Pr}^{1/3} + 3.27\,10^{-4}\frac{\mathrm{Pr}^{1.3}}{2\,R^2}\left(\frac{\nu_w^2}{g}\right)^{2/3}\left(\frac{\eta_{wg}}{\eta_w}\right)^{0.156}\left(\frac{\rho_w^2}{\rho_w}\right)^{0.788}\frac{\mathrm{Re}^{1.8}}{4.121}\right)^{1/2}. \tag{8.21}$$

The mean Nußelt numbers for water and R134a are given in table 8.1. It shows

	Water	R134a
$\overline{\mathrm{Nu}}_{\mathrm{Nu}}$	0.44	0.15
$\overline{\mathrm{Nu}}_{\mathrm{DGL}}$	0.52	0.23
$\overline{\mathrm{Nu}}_{\mathrm{Chen}}$	0.74	0.37

Table 8.1: *Comparison of mean Nußelt numbers*

that our model is better then Nußelt's model, but Chen's model is better then our model. Chen reviewed available experimental information for co-current condensation inside vertical tubes. His model is more accurate for Reynolds numbers greater than Re=30. It takes into account that at higher Reynolds numbers the film is thinner due to the co-current vapor flow. This explains the difference between our model and Chen's model. However, we derived an ordinary differential equation that is easy to solve by standard numerical methods, and can be extended in a next step to include the effect of an inclination angle.

Chapter 9

Model equations as differential algebraic equations (DAEs)

In this chapter we analyze the model equations for condensation including surface tension from a DAE point of view and discuss the requirements for a numerical solution of this moving boundary problem.

For a three-dimensional numerical simulation of the condensation problem where surface tension occurs at the interface, the Navier-Stokes equations and the energy equation have to be solved, together with the jump conditions at the interface (compare section 5.3). We discretize the spatial terms of the bulk flow equations, which lead to a system of differential and algebraic equations (DAEs). We discuss the basic ideas of DAEs with Navier-Stokes equations for which the results are known. Then we analyze the complete system of DAEs including the jump conditions, and we relate the DAE approach with conventional methods. Finally we discuss some aspects of numerical solution schemes for the moving boundary problem based on our analysis.

For the spatial discretization of the model equations with finite element method (FEM) and and finite difference method (FDM) for Navier-Stokes equations we refer to [KA00] [Bra97]. For the numerical solution of differential algebraic equations we refer to [BCP96] [HW02a] [AP98]. For the numerical solution of the linear equation systems (LES) which result from the discretization we refer to [Mei05] [Saa00] [GL96] [QSS00].

9.1 Spatial discretization of Navier-Stokes equations and energy equation

In figure 9.1 the numerical methods for the solution of the bulk equations of the condensation problem are depicted.

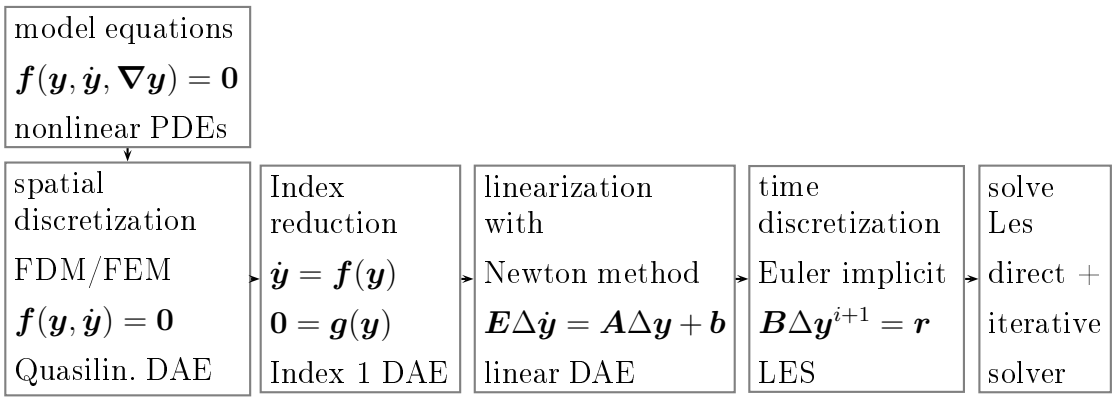


Figure 9.1: *Numerical methods for the solution of the bulk equations*

Both programs we worked with[1], use finite element method (FEM) for the spatial discretization of the unknowns. The main advantage of FEM for the solution of moving boundary problems over finite difference method (FDM) and finite volume method (FVM) is that FEM allows greater flexibility to model complex geometries. The matrices resulting from all three methods have similar characteristics. However, in this chaper we use finite difference method for the discretization of the spatial terms, because this is more instructive and allows us to perform some necessary calculations using *Matlab*.

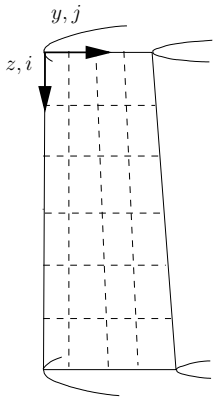


Figure 9.2: *Rectangular grid*

Discretization of the spatial terms – FDM

For the three-dimensional simulation we can write the equations in cartesian coordinates, there is no need to use cylindrical coordinates as long as we not assume rotational symmetry. We first consider a two-dimensional condensate slice as shown in figure 9.2. The complexity of a three-dimensional simulation arises on the level of

[1]SEPRAN and FIDAP

implementation due to the geometrical information that has to be handled and due to the immense increase of the number of unknowns and therefore the dimension of the equation systems.

The equations for the bulk flow equations as given at the end of part I in section 5.3 form a system of nonlinear partial differential equations, where the nonlinearity comes from the convective terms in the momentum equations. Assume a fluid slice as shown in figure 9.2 covered by a quadrilateral grid, with $i = 1, \ldots, n$ points in z direction and $j = 1, \ldots, m$ points in y direction. Using central difference formulas for the second order spatial derivatives and backward difference formulas for the first order spatial derivatives, see e.g. [QSS00], we get for the fluid unknowns at an inner grid point i, j

the equation of continuity

$$\frac{v_{z_{i,j}} - v_{z_{i-1,j}}}{\Delta z} + \frac{v_{y_{i,j}} - v_{y_{i,j-1}}}{\Delta y} = 0 \,, \tag{9.1}$$

the momentum equation

$$\begin{aligned} \rho \left(\dot{v}_{z_{i,j}} + v_{z_{i,j}} \frac{v_{z_{i,j}} - v_{z_{i-1,j}}}{\Delta z} + v_{y_{i,j}} \frac{v_{z_{i,j}} - v_{z_{i,j-1}}}{\Delta y} \right) &= -\frac{p_{i,j} - p_{i-1,j}}{\Delta z} \\ + \mu \left(\frac{v_{z_{i+1,j}} - 2\, v_{z_{i,j}} + v_{z_{i-1,j}}}{\Delta z^2} + \frac{v_{z_{i,j+1}} - 2\, v_{z_{i,j}} + v_{z_{i,j-1}}}{\Delta y^2} \right) &+ \rho\, g\, \cos\vartheta \,, \end{aligned} \tag{9.2}$$

$$\begin{aligned} \rho \left(\dot{v}_{y_{i,j}} + v_{z_{i,j}} \frac{v_{y_{i,j}} - v_{y_{i-1,j}}}{\Delta z} + v_{y_{i,j}} \frac{v_{y_{i,j}} - v_{y_{i,j-1}}}{\Delta y} \right) &= -\frac{p_{i,j} - p_{i,j-1}}{\Delta y} \\ + \mu \left(\frac{v_{y_{i+1,j}} - 2\, v_{y_{i,j}} + v_{y_{i-1,j}}}{\Delta z^2} + \frac{v_{y_{i,j+1}} - 2\, v_{y_{i,j}} + v_{y_{i,j-1}}}{\Delta y^2} \right) &+ \rho\, g\, \sin\vartheta \,, \end{aligned}$$

the energy equation

$$\begin{aligned} \rho \left(\dot{T}_{i,j} + v_{z_{i,j}} \frac{T_{i,j} - T_{i-1,j}}{\Delta z} + v_{y_{i,j}} \frac{T_{i,j} - T_{i,j-1}}{\Delta y} \right) & \\ = \frac{\lambda}{c} \left(\frac{T_{i+1,j} - 2\, T_{i,j} + T_{i-1,j}}{\Delta z^2} + \frac{T_{i,j+1} - 2\, T_{i,j} + T_{i,j-1}}{\Delta y^2} \right) \,, & \end{aligned} \tag{9.3}$$

where $\Delta z = z_{i,j} - z_{i-1,j}$ and $\Delta y = y_{i,j} - y_{i,j-1}$.

If we let run the indices over the entire domain, equations (9.1) – (9.3) define the matrix equations

$$\begin{aligned} \boldsymbol{D}_z^T \boldsymbol{v}_z + \boldsymbol{D}_y^T \boldsymbol{v}_y &= 0 \,, \\ \boldsymbol{M}_z \dot{\boldsymbol{v}}_z + \left[\boldsymbol{N}_z(\boldsymbol{v}_z) + \boldsymbol{N}_y(\boldsymbol{v}_y)\right] \boldsymbol{v}_z &= -\boldsymbol{D}_z \boldsymbol{p} + \left[\boldsymbol{S}_z + \boldsymbol{S}_y\right] \boldsymbol{v}_z + \boldsymbol{g}_z \,, \\ \boldsymbol{M}_y \dot{\boldsymbol{v}}_y + \left[\boldsymbol{N}_z(\boldsymbol{v}_z) + \boldsymbol{N}_y(\boldsymbol{v}_y)\right] \boldsymbol{v}_y &= -\boldsymbol{D}_y \boldsymbol{p} + \left[\boldsymbol{S}_z + \boldsymbol{S}_y\right] \boldsymbol{v}_y + \boldsymbol{g}_y \,, \\ \boldsymbol{C} \dot{\boldsymbol{T}} + \left[\boldsymbol{N}_z(\boldsymbol{v}_z) + \boldsymbol{N}_y(\boldsymbol{v}_y)\right] \boldsymbol{T} &= \left[\boldsymbol{K}_z + \boldsymbol{K}_y\right] \boldsymbol{T} \,, \end{aligned} \tag{9.4}$$

where the vectors are of length $n\,m$ and the matrices are of dimension $(n\,m) \times (n\,m)$.

Quasi-linear system of differential algebraic equations (DAE)

With $\boldsymbol{v} = [\boldsymbol{v}_z, \boldsymbol{v}_y]^T$, $\boldsymbol{D} = [\boldsymbol{D}_z, \boldsymbol{D}_y]^T$, $\boldsymbol{g} = [\boldsymbol{g}_z, \boldsymbol{g}_y]^T$, and

$$\boldsymbol{M} = \begin{bmatrix} \boldsymbol{M}_z & \\ & \boldsymbol{M}_y \end{bmatrix}, \quad \boldsymbol{N} = \begin{bmatrix} \boldsymbol{N}_z & \\ & \boldsymbol{N}_y \end{bmatrix}, \quad \boldsymbol{S} = \begin{bmatrix} \boldsymbol{S}_z & \\ & \boldsymbol{S}_y \end{bmatrix}, \quad \boldsymbol{K} = \begin{bmatrix} \boldsymbol{K}_z & \\ & \boldsymbol{K}_y \end{bmatrix},$$

the four matrix equations (9.4) can be written more conveniently as

$$\begin{aligned} \boldsymbol{M}\,\dot{\boldsymbol{v}} &= \left[\boldsymbol{S} - \boldsymbol{N}(\boldsymbol{v})\right] \boldsymbol{v} - \boldsymbol{D}\,\boldsymbol{p} + \boldsymbol{g} \,, \\ \boldsymbol{C}\,\dot{\boldsymbol{T}} &= \left[\boldsymbol{K} - \boldsymbol{N}(\boldsymbol{v})\right] \boldsymbol{T} \,, \\ \boldsymbol{0} &= \boldsymbol{D}^T \boldsymbol{v} \,. \end{aligned} \tag{9.5}$$

$\boldsymbol{M}$, $\boldsymbol{C}$, $\boldsymbol{S}$, $\boldsymbol{K}$, $\boldsymbol{N}$ are quadratic matrices with dimension $(2\,n\,m) \times (2\,n\,m)$, and $\boldsymbol{D}$ is a rectangular matrix with dimension $(2\,n\,m) \times (n\,m)$. The matrices $\boldsymbol{M}$, $\boldsymbol{C}$, $\boldsymbol{S}$, and $\boldsymbol{K}$ are symmetric. They are also strictly diagonally dominant and therefore positive definite [Saa00]. $\boldsymbol{N}$ is not symmetric. Because the convective terms are discretized with backward difference formulas (upwind discretization) the sums $\boldsymbol{S} - \boldsymbol{N}$ and $\boldsymbol{K} - \boldsymbol{N}$ respectively are diagonal dominant.[2] $\boldsymbol{D}$ has full rank. Equation (9.5) is a quasi-linear system of differential algebraic equations.

[2] Discretization with second order difference formulas would result in a non-diagonally dominant matrix. A discretization that makes the matrix diagonal dominant is called upwind discretization and is preferred, although it decreases the order of the discretization [Saa00].

9.2 Basic definitons of DAEs

In this section we analyze the quasi-linear DAE (9.5) which we derived by spatial discretization of the model equations. The differential equations and the algebraic constraints in (9.5) are separated. The differential variables are velocity and temperature and the algebraic variable is the pressure. The DAE (9.5) is a semi-explicit DAE in autonomous form and can be written as

$$\begin{aligned} \dot{\boldsymbol{x}} &= \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{p}) \,, \\ \mathbf{0} &= \boldsymbol{g}_1(\boldsymbol{x}) \,. \end{aligned} \tag{9.6}$$

Index of a DAE

The complexity of a DAE is characterized by the smallest number of times that the algebraic equations must be differentiated with respect to time in order to determine an explicit ODE. This number is called the (differential) "index" of the DAE [BCP96].[3] Thus ODEs have index zero. If the algebraic equations depend on the algebraic variables $\boldsymbol{g} = \boldsymbol{g}(\boldsymbol{x}, \boldsymbol{p})$, then one derivative with respect to time[4]

$$\frac{\mathrm{d}\boldsymbol{g}_1}{\mathrm{d}t} = \frac{\partial \boldsymbol{g}_1}{\partial \boldsymbol{x}} \, \dot{\boldsymbol{x}} + \frac{\partial \boldsymbol{g}_1}{\partial \boldsymbol{p}} \, \dot{\boldsymbol{p}} \,.$$

gives $\dot{\boldsymbol{p}}$ in principle. For a DAE of index one the matrix of the derivatives $\frac{\partial \boldsymbol{g}_1}{\partial \boldsymbol{p}}$ must be non-singular. However, in our case $\boldsymbol{g}_1 = \boldsymbol{g}_1(\boldsymbol{x})$ and hence $\frac{\partial \boldsymbol{g}_1}{\partial \boldsymbol{p}} = \mathbf{0}$ singular. Differentiating $\boldsymbol{g}_1(\boldsymbol{x})$ twice with respect to time using chain rule gives with (9.6)

$$\begin{aligned} \frac{\mathrm{d}\boldsymbol{g}_1}{\mathrm{d}t} &= \frac{\partial \boldsymbol{g}_1}{\partial \boldsymbol{x}} \, \dot{\boldsymbol{x}} = \frac{\partial \boldsymbol{g}_1}{\partial \boldsymbol{x}} \, \boldsymbol{f} \,, \\ \frac{\mathrm{d}^2\boldsymbol{g}_1}{\mathrm{d}t^2} &= \frac{\partial^2 \boldsymbol{g}_1}{\partial \boldsymbol{x}^2} \, \dot{\boldsymbol{x}} \, \boldsymbol{f} + \frac{\partial \boldsymbol{g}_1}{\partial \boldsymbol{x}} \left(\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}} \, \dot{\boldsymbol{x}} + \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{p}} \, \dot{\boldsymbol{p}} \right) \,. \end{aligned} \tag{9.7}$$

By this $\dot{\boldsymbol{p}}$ is determined in principle. For a semi-explicit DAE of index two $\frac{\partial \boldsymbol{g}_1}{\partial \boldsymbol{x}} \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{p}}$ must be non-singular [AP98]. So obviously (9.6) is an index two DAE.

[3] In literature different index definitions exist, from which the differential index is most commonly used [SW95].

[4] Here we follow the notation commonly used in DAE literature, where $\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{y}} = \begin{bmatrix} \frac{\partial f_1}{\partial y_1} & \cdots & \frac{\partial f_1}{\partial y_n} \\ \vdots & & \vdots \\ \frac{\partial f_n}{\partial y_1} & \cdots & \frac{\partial f_n}{\partial y_n} \end{bmatrix}$. Note that this notation confuses with tensor notation when the gradient is defined as $\boldsymbol{\nabla} = \boldsymbol{e}_i \frac{\partial}{\partial y_i}$. Then $\boldsymbol{\nabla} \boldsymbol{f} = \boldsymbol{e}_i \frac{\partial}{\partial y_i} (f_j \, \boldsymbol{e}_j)$, which is the transposed of $\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{y}}$.

A DAE of index one can be solved by an implicit ODE method or a higher order method (e.g. Runge-Kutta method) [HW02a]. However, from (9.6) we see that an explicit time discretization of $\dot{\boldsymbol{x}}$ does not work for the constraint, so that at least an semi-explicit method must be used, with an explicit discretization of the ODE and an implicit discretization of the constraints.

Consistent initial conditions

The constraints of a DAE must be satisfied at any time. An index two DAE must satisfy the constraint of the DAE

$$\boldsymbol{g}_1(\boldsymbol{x}) = \boldsymbol{0} \qquad \text{and} \qquad \boldsymbol{g}_2(\boldsymbol{x}, \boldsymbol{p}) = \frac{\partial \boldsymbol{g}_1}{\partial \boldsymbol{x}} \dot{\boldsymbol{x}} = \boldsymbol{0} \tag{9.8}$$

at any time. The second constraint is called hidden constraint [AP98]. Initial conditions of a DAE that satisfy the constraints of the DAE are called consistent.

Index reduction and drift

The first idea to reduce the index of (9.6) is to perturb the algebraic constraint with the algebraic variable multiplied by a small penalty parameter

$$\boldsymbol{0} = \boldsymbol{g}_1(\boldsymbol{x}) + \varepsilon \, \boldsymbol{p} \; .$$

Then the algebraic constraint is a function of differential and algebraic variables and the DAE has index one.[5] The algebraic variable can be eliminated from the momentum equation by substituting $\boldsymbol{p} = \varepsilon^{-1} \boldsymbol{g}_1$. However, the penalty method may lead to inaccurate results.

The next idea is to reduce the index by differentiating the constraints. If we substitute the constraint $\boldsymbol{g}_1(\boldsymbol{x}) = \boldsymbol{0}$ in (9.6) by its derivative (9.7) with respect to time we get the DAE

$$\begin{aligned} \dot{\boldsymbol{x}} &= \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{p}) \; , \\ \boldsymbol{0} &= \boldsymbol{g}_2(\boldsymbol{x}, \boldsymbol{p}) \; , \end{aligned} \tag{9.9}$$

This DAE has index one. However, with the differentiation we lost the integration constant.[6] Even if the initial conditions satisfies $\boldsymbol{g}_1 = \boldsymbol{0}$, the numerical solution

[5] The penalty method is easy to implement and a standard method in CFD-software packages.

[6] The integration constant is in our case zero, integrating $\boldsymbol{g}_2$ gives $\int \boldsymbol{g}_2 \, \mathrm{d}t = \boldsymbol{g}_1 + \boldsymbol{c}$, with $\boldsymbol{c} = \boldsymbol{0}$.

not any longer satisfy the constraint exactly due to roundoff errors in every iteration step, and the solution may drift away from the constraints. This is why DAEs require more stable methods, so that implicit methods are preferred over explicit methods. But for an DAE of index two or higher an implicit method is not sufficient [HW02a].

Overdetermined system and constraint stabilization

The DAE (9.6) together with the second constraint

$$\begin{aligned} \dot{\boldsymbol{x}} &= \boldsymbol{f}(\boldsymbol{x},\boldsymbol{p}) \,, \\ \boldsymbol{0} &= \boldsymbol{g}_1(\boldsymbol{x}) \,, \\ \boldsymbol{0} &= \boldsymbol{g}_2(\boldsymbol{x},\boldsymbol{p}) \,, \end{aligned} \tag{9.10}$$

is a DAE with index one. However, (9.10) is an overdetermined system, which is contradictive after time discretization [BCP96]. The ODE has dimension $n_v = 2\,n\,m$ and the constraints have dimension $n_p = n\,m$, so that we have $n_v + 2\,n_p$ equations for $n_v + n_p$ unknowns. The hidden constraints are still present in the ODE. The idea is to eliminate the n_p superfluous equations by multiplying the ODE with a matrix $\boldsymbol{Z}$ whose column vectors are perpendicular to the row vectors of

$$\boldsymbol{G}^T = \frac{\partial \boldsymbol{g}_1}{\partial \boldsymbol{x}} \,,$$

so that

$$\boldsymbol{G}^T\boldsymbol{Z} = \boldsymbol{0} \,, \qquad \text{or equivalent} \qquad \boldsymbol{Z}^T\boldsymbol{G} = \boldsymbol{0} \tag{9.11}$$

[BCP96]. $\boldsymbol{G}^T$ has n_p rows and n_v columns and has full rank. $\boldsymbol{Z}$ has n_v rows and $n_v - n_p$ columns and must have full rank. Then any vector $\boldsymbol{v}$ which satisfies $\boldsymbol{G}^T\boldsymbol{v} = \boldsymbol{0}$ can be written as a linear combination of the $n_v - n_p$ base vectors of $\boldsymbol{Z}$. The matrix $\boldsymbol{Z}$ is computed by means of an orthogonal projector.[7] Multiplying the ODE in (9.10)

[7] The projector $\tilde{\boldsymbol{P}} = \boldsymbol{G}\left(\boldsymbol{G}^T\boldsymbol{G}\right)^{-1}\boldsymbol{G}^T$ project an arbitrary vector onto the vector space spanned by all vectors which are the result of $\boldsymbol{y} = \boldsymbol{G}\,\boldsymbol{x}$ for any $\boldsymbol{x}$; that is, the span (or range) of $\boldsymbol{G}$. From $\boldsymbol{Z}^T(\boldsymbol{G}\,\boldsymbol{x}) = \boldsymbol{0}$ and $\boldsymbol{G}^T(\boldsymbol{Z}\,\boldsymbol{x}) = \boldsymbol{0}$ (for any $\boldsymbol{x}$) we know that the span of $\boldsymbol{G}$ is the orthogonal complement of the span of $\boldsymbol{Z}$. $\boldsymbol{P} = \boldsymbol{I} - \tilde{\boldsymbol{P}}$ project an arbitrary vector onto the span of $\boldsymbol{Z}$ (or onto the null space of $\boldsymbol{G}^T$). Multiplying $\boldsymbol{P}$ with $n_v - n_p$ arbitrary linearly independent vectors $\boldsymbol{A} = [\boldsymbol{a}_1, ..., \boldsymbol{a}_{n_v - n_p}]$ (such that $\boldsymbol{Z}$ has full rank) gives $\boldsymbol{Z} = \boldsymbol{P}\,\boldsymbol{A}$ [Saa00] [WIK]. Then (9.11) is satisfied, $\boldsymbol{G}^T\boldsymbol{Z} = \boldsymbol{G}^T\boldsymbol{P}\,\boldsymbol{A} = \boldsymbol{G}^T\boldsymbol{A} - \boldsymbol{G}^T\left(\boldsymbol{G}\left(\boldsymbol{G}^T\boldsymbol{G}\right)^{-1}\boldsymbol{G}^T\right)\boldsymbol{A} = \boldsymbol{0}$.

with $\boldsymbol{Z}^T$ eliminates the redundant $n_v - n_p$ equations, so that the reduced index one DAE

$$\begin{aligned} \boldsymbol{Z}^T \dot{\boldsymbol{x}} &= \boldsymbol{Z}^T \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{p}) \,, \\ \boldsymbol{0} &= \boldsymbol{g}_1(\boldsymbol{x}) \,, \\ \boldsymbol{0} &= \boldsymbol{g}_2(\boldsymbol{x}, \boldsymbol{p}) \,, \end{aligned}$$

can then be solved numerically with an implicit method or an semi-implicit method of higher order [HW02a]. See [Ste06] for a concise index-reduction procedure for any arbitrary semi-explicit DAE.

Transformation of an index two DAE of special structure into an ODE with invariant

If the index two DAE (9.6) can be written as

$$\begin{aligned} \dot{\boldsymbol{x}} &= \tilde{\boldsymbol{f}}(\boldsymbol{x}) + \boldsymbol{G}\,\boldsymbol{p} \,, \\ \boldsymbol{0} &= \boldsymbol{G}^T \boldsymbol{x} \,, \end{aligned} \tag{9.12}$$

then by multiplication the ODE with $\boldsymbol{Z}^T$ the DAE is transformed to an ODE with invariant

$$\begin{aligned} \dot{\boldsymbol{x}} &= \tilde{\boldsymbol{f}}(\boldsymbol{x}) \,, \\ \boldsymbol{0} &= \boldsymbol{g}_1(\boldsymbol{x}) \,, \end{aligned}$$

[AP98].

9.3 Index of Navier-Stokes equations and energy equation

Now we analyze the index of the discretized bulk equations (9.5) of the condensation problem

$$\begin{aligned} \boldsymbol{M}\,\dot{\boldsymbol{v}} &= [\boldsymbol{S} - \boldsymbol{N}(\boldsymbol{v})]\;\boldsymbol{v} - \boldsymbol{D}\,\boldsymbol{p} + \boldsymbol{g} \,, \\ \boldsymbol{C}\,\dot{\boldsymbol{T}} &= [\boldsymbol{K} - \boldsymbol{N}(\boldsymbol{v})]\;\boldsymbol{T} \,, \\ \boldsymbol{0} &= \boldsymbol{D}^T \boldsymbol{v} \,. \end{aligned} \tag{9.13}$$

The Navier-Stokes equations have index two [BCP96]. Equation (9.13) is a semi-explicit DAE of form (9.6). Consequently we obtain the following result: The Navier-Stokes equations together with the energy equations have index two. The energy equation does not change the index.

Index reduction

The hidden constraint of the Navier-Stokes equations we get by differentiating the discretized equation of continuity with respect to time

$$\frac{\mathrm{d}\left(\boldsymbol{D}^T\boldsymbol{v}\right)}{\mathrm{d}t} = \boldsymbol{D}^T\dot{\boldsymbol{v}} = \boldsymbol{0}\,. \tag{9.14}$$

Multiplying the ODE in (9.5) from left with $\boldsymbol{M}^{-1}$

$$\dot{\boldsymbol{v}} = \boldsymbol{M}^{-1}\left[\boldsymbol{S} - \boldsymbol{N}(\boldsymbol{v})\right]\boldsymbol{v} - \boldsymbol{M}^{-1}\boldsymbol{D}\,\boldsymbol{p} + \boldsymbol{M}^{-1}\boldsymbol{g}$$

and substituting this in (9.14) gives the hidden constraint as follows

$$\boldsymbol{0} = \boldsymbol{D}^T\boldsymbol{M}^{-1}\left[\boldsymbol{S} - \boldsymbol{N}(\boldsymbol{v})\right]\boldsymbol{v} - \boldsymbol{D}^T\boldsymbol{M}^{-1}\boldsymbol{D}\,\boldsymbol{p} + \boldsymbol{D}^T\boldsymbol{M}^{-1}\boldsymbol{g}\,. \tag{9.15}$$

$\boldsymbol{D}^T\boldsymbol{M}^{-1}\boldsymbol{D}$ must be non-singular. $\boldsymbol{D}^T$ is the discretized divergence operator, $\boldsymbol{D}$ is the discretized gradient operator, and $\boldsymbol{M} = \rho\,\boldsymbol{I}$. Then $\boldsymbol{D}^T\boldsymbol{M}\,\boldsymbol{D}$ is the discretized Laplace operator $\nabla^2 = \boldsymbol{\nabla}\cdot\boldsymbol{\nabla}$ times density, so that (9.15) defines a Poisson equation for the pressure, where the right hand side is a function of velocity

$$\nabla^2\,\boldsymbol{p} = \boldsymbol{f}(\boldsymbol{v})\,. \tag{9.16}$$

The Navier-Stokes equations have the same structure as (9.12). So the index can be reduced by transforming the Navier-Stokes equations in an ODE plus invariant. Multiplying the momentum equation in (9.13) with a matrix $\boldsymbol{Z}^T$ (constructed as described in section 9.2) such that

$$\boldsymbol{D}^T\boldsymbol{Z} = \boldsymbol{0} \qquad \text{or equivalent} \qquad \boldsymbol{Z}^T\boldsymbol{D} = \boldsymbol{0}\,, \tag{9.17}$$

gives an ODE for the velocity only, with the discretized equation of continuity as invariant

$$\begin{aligned} \boldsymbol{Z}^T\boldsymbol{M}\,\dot{\boldsymbol{v}} &= \boldsymbol{Z}^T\left[\boldsymbol{S} - \boldsymbol{N}(\boldsymbol{v})\right]\boldsymbol{v} - \underbrace{\boldsymbol{Z}^T\boldsymbol{D}}_{=\boldsymbol{0}}\,\boldsymbol{p} + \boldsymbol{Z}^T\boldsymbol{g}\,, \\ \boldsymbol{0} &= \boldsymbol{D}^T\boldsymbol{v}\,. \end{aligned} \tag{9.18}$$

By this the Navier-Stokes equations are decoupled in an ordinary differential equation for velocity and a Poisson equation (9.15) for pressure. See [Sch07] for a more general version of this decoupling for the Navier-Stokes equation[8]

[8] If e.g. finite element method is used, then $\boldsymbol{M}$ is not anymore the identity matrix times a constant. To apply the above transformation, then $\boldsymbol{M}$ must be decomposed by means of a Cholesky factorization $\boldsymbol{M} = \boldsymbol{U}^T\boldsymbol{U}$ and the velocity $\boldsymbol{v}$ must be transformed by $\boldsymbol{u} = \boldsymbol{U}\,\boldsymbol{v}$.

However, multiplication with $\boldsymbol{Z}^T$ destroys the sparse structure of the matrices resulting from spatial discretization, so that this method is only practicable for systems with small matrix dimensions. To avoid this, the projection step in index reduction of Navier-Stokes equations is usually implemented as follows.

Pressure projection schemes

CFD-software packages usually solve the Navier-Stokes equations not in DAE form, but solve in every iteration the momentum equation and project the velocity (in every iteration or once a while) back to the subspace of divergence free velocities [Wes00]. A different Poisson equation is used, based on that any velocity $\boldsymbol{w}$ can be decomposed into $\boldsymbol{w} = \boldsymbol{v} + \boldsymbol{\nabla} p$, where $\boldsymbol{v}$ has zero divergence and is parallel to the boundary $\boldsymbol{v} \cdot \boldsymbol{n} = \boldsymbol{0}$ [CM92]. Then

$$\boldsymbol{\nabla} \cdot \boldsymbol{w} = \nabla^2 p \,. \tag{9.19}$$

Pressure projection schemes (as e.g. used by [FTM] [STM]) are summarized by the following scheme:

- solve momentum equation (9.13) with a pressure guess, get an intermediate velocity $\boldsymbol{w}$,
- solve the Poisson equation (9.19) with the intermediate velocity, get an intermediate pressure,
- project the intermediate velocity on its divergence free part.[9]

Linearization with Newton method

For the sake of completeness we outline the remaining steps in the numerical solution of the bulk equations as depicted in figure 9.1. The momentum equation has to be linearized. This is done using Newton method. If we summarize the DAE (9.13) by $\boldsymbol{f}(\boldsymbol{y}, \dot{\boldsymbol{y}}) = \boldsymbol{0}$, then a Taylor series expansion for

$$\boldsymbol{f}(\boldsymbol{y}_0 + \Delta\boldsymbol{y}, \dot{\boldsymbol{y}}_0 + \Delta\dot{\boldsymbol{y}}) = \boldsymbol{0}$$

gives

$$\boldsymbol{0} = \boldsymbol{f}(\boldsymbol{y}_0, \dot{\boldsymbol{y}}_0) + \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{y}}\Big|_{\boldsymbol{y}_0} \Delta\boldsymbol{y} \;+\; \frac{\partial \boldsymbol{f}}{\partial \dot{\boldsymbol{y}}}\Big|_{\dot{\boldsymbol{y}}_0} \Delta\dot{\boldsymbol{y}} \;+\; \text{higher order terms} \,.$$

[9] By means of an orthogonal projector $\boldsymbol{P}$ such that $\boldsymbol{P}\,\boldsymbol{u} = \boldsymbol{u}$ and $\boldsymbol{P}\,\boldsymbol{\nabla} p = \boldsymbol{0}$ the intermediate velocity can be projected on its divergence free part $\boldsymbol{P}\,\boldsymbol{w} = \boldsymbol{P}(\boldsymbol{u} + \boldsymbol{\nabla} p) = \boldsymbol{u}$.

This a quasi-linear DAE, which we can write as

$$\boldsymbol{E}\,\Delta\dot{\boldsymbol{y}} = \boldsymbol{A}(\boldsymbol{y}_0)\,\Delta\boldsymbol{y} + \boldsymbol{f}(\boldsymbol{y}_0, \dot{\boldsymbol{y}}_0)\ , \tag{9.20}$$

where $\boldsymbol{E}$ is a constant singular matrix and $\boldsymbol{A}$ is a non-constant singular matrix. Linearization did not change the index, (9.20) has still index two. The linearized DAE (9.20) has again structure (9.12), so that it can be transformed into an index one DAE consisting of an ODE with constraint as before. The so derived linear index one DAE we denote by

$$\tilde{\boldsymbol{E}}\,\Delta\dot{\boldsymbol{y}} = \tilde{\boldsymbol{A}}(\boldsymbol{y}_0)\,\Delta\boldsymbol{y} + \tilde{\boldsymbol{f}}(\boldsymbol{y}_0, \dot{\boldsymbol{y}}_0)\ . \tag{9.21}$$

The Jacobi matrix $\tilde{\boldsymbol{A}}(\boldsymbol{y}_0) = \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{y}}\Big|_{\boldsymbol{y}_0}$ is not updated in every iteration (modified Newton) [QSS00]. Note that Newton method has a small convergence radius and therefore needs a good starting guess, so that it is important to provide a good initial mesh configuration and initial velocity field.

Time discretization

The linearized index one DAE (9.21) is then solved by an implicit ODE method, for example Euler implicit (or any other higher order implicit time integration method). Using Euler implicit we get

$$\tilde{\boldsymbol{E}}\,\frac{1}{\Delta t}\Big(\Delta\boldsymbol{y}_{i+1} - \Delta\boldsymbol{y}_i\Big) = \tilde{\boldsymbol{A}}(\boldsymbol{y}_i)\,\Delta\boldsymbol{y}_{i+1} + \tilde{\boldsymbol{f}}(\boldsymbol{y}_i)\ ,$$

where $\Delta\boldsymbol{y}_{i+1} = \boldsymbol{y}_{i+1} - \boldsymbol{y}_i$ and $\Delta\boldsymbol{y}_i = \boldsymbol{y}_i - \boldsymbol{y}_{i-1}$. Rearranging the terms such that the unknowns are on the left gives

$$\Big(\tilde{\boldsymbol{E}} - \Delta t\,\tilde{\boldsymbol{A}}\Big)\,\Delta\boldsymbol{y}_{i+1} = \boldsymbol{E}\,\Delta\boldsymbol{y}_i + \Delta t\,\tilde{\boldsymbol{f}}\ . \tag{9.22}$$

This linear equation system (LES) is solved for $\Delta\boldsymbol{y}_{i+1}$ in every time iteration by means of direct (Gauß) or iterative (CGS – Conjugate gradient square, GMRES – Generalized minimal residual) methods [Saa00] [Mei05], so that the solution at the next iteration is given by

$$\boldsymbol{y}_{i+1} = \boldsymbol{y}_i + \Delta\boldsymbol{y}_{i+1}\ .$$

By this we summarized the steps in the numerical solution of the index two DAE which was the result of the spatial discretization of the bulk equations. We discussed the Navier-Stokes equations from a DAE point of view. The next step is to investigate the system of DAEs defined by the complete system of equations including the jump conditions.

9.4 Index of the condensation problem

To analyze the system of differential algebraic equations formed by the discretized bulk equations (9.5) and (9.13) respectively, together with the discretized jump conditions we first discretize the spatial terms of the jump conditions. For that we simplify the jump conditions such that they are more easy to analyze but still have the main features of the moving boundary problem with phase change.

Two-dimensional jump conditions for the one-fluid condensation problem

We neglect shear stress exerted by the vapor, assume zero heat flux in the vapor phase and constant vapor pressure; so that the problem reduces to to a one-fluid-problem. Further we neglect momentum due to phase transition. Then the jump conditions (from section 5.3) are given as

$$\begin{aligned} \dot{m} &= \rho\,[\boldsymbol{v}-\boldsymbol{u}]\cdot\boldsymbol{n}\,, \\ p_g - p + \boldsymbol{n}\cdot\boldsymbol{T}\cdot\boldsymbol{n} &= 2\,H\,\sigma\,, \\ \boldsymbol{t}\cdot\boldsymbol{T}\cdot\boldsymbol{n} &= 0\,, \\ \dot{m}\,\Delta h &= \boldsymbol{q}\cdot\boldsymbol{n}\,. \end{aligned} \tag{9.23}$$

We discretize this equations again for a two-dimensional fluid-slice as depicted in figure 9.3. With normal and tangent vectors (2.14), (2.13), mean curvature (2.18), $\boldsymbol{q} = -\lambda\boldsymbol{\nabla}T$ and $\boldsymbol{T} = \mu\,[\boldsymbol{\nabla}\boldsymbol{v} + (\boldsymbol{\nabla}\boldsymbol{v})^T]$, the jump conditions become (compare chapter 5 and 6)

$$\frac{\rho}{\sqrt{1+\left(\frac{\partial h}{\partial z}\right)^2}}\left(v_y - \frac{\partial h}{\partial t} - v_z\,\frac{\partial h}{\partial z}\right) = \dot{m}, \tag{9.24}$$

$$p - p_g - \frac{\mu}{\sqrt{1+\left(\frac{\partial h}{\partial z}\right)^2}}\left(\frac{\partial v_y}{\partial y} - 4\left[\frac{\partial v_y}{\partial z}+\frac{\partial v_z}{\partial y}\right]\frac{\partial h}{\partial z} + \frac{\partial v_z}{\partial z}\left[\frac{\partial h}{\partial z}\right]^2\right) = \sigma\frac{\frac{\partial^2 h}{\partial z^2}}{\left(1+\left[\frac{\partial h}{\partial z}\right]^2\right)^{3/2}},$$

$$\frac{\mu}{\sqrt{1+\left(\frac{\partial h}{\partial z}\right)^2}}\left(\frac{\partial h}{\partial z}\,\frac{\partial v_y}{\partial y} + 2\left[\frac{\partial v_y}{\partial z}+\frac{\partial v_z}{\partial y}\right]\left[1-\left(\frac{\partial h}{\partial z}\right)^2\right] - \frac{\partial h}{\partial z}\frac{\partial v_z}{\partial z}\right) = 0,$$

$$\dot{m}\,\Delta h = \frac{\lambda}{\sqrt{1+\left(\frac{\partial h}{\partial z}\right)^2}}\left(\frac{\partial T}{\partial y} - \frac{\partial T}{\partial z}\frac{\partial h}{\partial z}\right).$$

Note that here surface tension is related to surface waves (and not to the tube radius). Further we assume small surface waves so that the square of rate of change of film thickness with respect to the length coordinate $\left(\frac{\partial h}{\partial z}\right)^2$ can be neglected.

Spatial disctretization of the jump conditions

If we assume that the mesh nodes on lines of constant i are free to move along this lines (the last node, some nodes, or all nodes), then the grid size $\Delta y_{i,j}$ in y-direction changes with time, but the grid size Δz in z-direction is fixed (method of spines [FTM]), see figure 9.3. Film thickness and grid size in y direction are coupled by

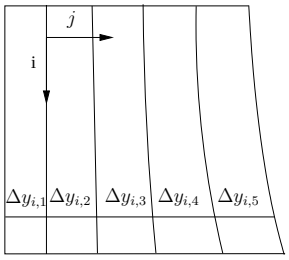


Figure 9.3: *Grid size variable in y-direction*

$$h_i = \sum_{j=1}^{m} \Delta y_{i,j} \ . \tag{9.25}$$

The jump conditions are defined at the moving boundary, so that we discretize the equations at $j = m$. Doing so we get

$$p_{g_{i,m}} - p_{i,m} + \mu \left(\frac{v_{y_{i,m}} - v_{y_{i,m-1}}}{\Delta y_{i,m}} - 4 \left[\frac{v_{y_{i,m}} - v_{y_{i-1,m}}}{\Delta z} + \frac{v_{z_{i,m}} - v_{z_{i,m-1}}}{\Delta y_{i,m}} \right] \frac{h_i - h_{i-1}}{\Delta z} \right) = \sigma \, \frac{h_{i+1} - 2\, h_i + h_{i-1}}{\Delta z^2},$$

$$\frac{h_i - h_{i-1}}{\Delta z} \frac{v_{y_{i,m}} - v_{y_{i,m-1}}}{\Delta y_{i,m}} + 2 \left[\frac{v_{y_{i,m}} - v_{y_{i-1,m}}}{\Delta z} + \frac{v_{z_{i,m}} - v_{z_{i,m-1}}}{\Delta y_{i,m}} \right] - \frac{h_i - h_{i-1}}{\Delta z} \frac{v_{z_{i,m}} - v_{z_{i-1,m}}}{\Delta z} = 0,$$

$$v_{y_{i,m}} - \frac{\partial h_i}{\partial t} - v_{z_{i,m}} \frac{h_i - h_{i-1}}{\Delta z} = \frac{\lambda}{\rho \, \Delta h} \left(\frac{T_{i,m} - T_{i,m-1}}{\Delta y_{i,m}} - \frac{T_{i,m} - T_{i-1,m}}{\Delta z} \frac{h_i - h_{i-1}}{\Delta z} \right).$$

Discretized jump conditions are differential algebraic equations

With the assumption of equidistant step size in y-direction, so that all nodes move equally, we get $h_i = m \, \Delta y_i$. Then these equations define for $i = 1, \ldots, n$ the following matrix equations

$$\begin{aligned} \boldsymbol{p}_m - \boldsymbol{p}_{g_m} + \boldsymbol{a}_1(\boldsymbol{v}_m, \boldsymbol{v}_{m-1}, \boldsymbol{h}) + \boldsymbol{B}_1(\boldsymbol{v}_m, \boldsymbol{v}_{m-1}, \boldsymbol{h})\, \boldsymbol{h} &= \boldsymbol{B}_2\, \boldsymbol{h} \ , \\ \boldsymbol{B}_3(\boldsymbol{v}_m, \boldsymbol{v}_{m-1}, \boldsymbol{h})\, \boldsymbol{h} + \boldsymbol{a}_2(\boldsymbol{v}_m, \boldsymbol{v}_{m-1}, \boldsymbol{h}) + \boldsymbol{B}_4(\boldsymbol{v}_m)\, \boldsymbol{h} &= \boldsymbol{0} \ , \\ \boldsymbol{B}_5\, \boldsymbol{v}_m - \dot{\boldsymbol{h}} + \boldsymbol{B}_6(\boldsymbol{v}_m)\, \boldsymbol{h} &= \boldsymbol{a}_3(\boldsymbol{T}_m, \boldsymbol{T}_{m-1}, \boldsymbol{h}) + \boldsymbol{B}_7(\boldsymbol{T}_m, \boldsymbol{T}_{m-1})\, \boldsymbol{h} \ , \end{aligned} \tag{9.26}$$

where the velocities $\boldsymbol{v}_m$, $\boldsymbol{v}_{m-1}$ and the temperatures $\boldsymbol{T}_m$, $\boldsymbol{T}_{m-1}$ are the velocities and the temperatures at the grid points on the moving boundary and next to the moving boundary. All equations are nonlinear in velocity and film thickness, and the last equation is also nonlinear in temperature.

The discretized momentum jump conditions are algebraic equations. The discretized combined mass-energy jump condition is a quasi-linear ordinary differential equation for the film thickness. We can summarize (9.26) by the following nonlinear differential algebraic equations

$$\begin{aligned} \mathbf{0} &= \boldsymbol{g}_{n-m}(\boldsymbol{p}, \boldsymbol{v}, \boldsymbol{h}) \,, \\ \mathbf{0} &= \boldsymbol{g}_{t-m}(\boldsymbol{v}, \boldsymbol{h}) \,, \\ \dot{\boldsymbol{h}} &= \boldsymbol{f}_{m-e}(\boldsymbol{v}, \boldsymbol{T}, \boldsymbol{h}) \,. \end{aligned}$$

Index of the condensation problem

In the moving boundary problem the film thickness is an additional variable to the variables of the bulk equations, so that now the vector of differential variables is given by $\boldsymbol{x} = [\boldsymbol{v}, \boldsymbol{T}, \boldsymbol{h}]^T$; and the pressure is still an algebraic variable. The discretized bulk equations (9.13), together with the discretized jump conditions (9.26), form a DAE which we write as

$$\begin{aligned} \dot{\boldsymbol{x}} &= \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{p}) \,, \\ \mathbf{0} &= \boldsymbol{g}_1(\boldsymbol{x}) \,, \\ \mathbf{0} &= \boldsymbol{g}_2(\boldsymbol{x}, \boldsymbol{p}) \,, \end{aligned} \tag{9.27}$$

Here $\boldsymbol{f}$ represents the right hand sides of the momentum equation, the energy equation, and the combined mass-energy jump condition. $\boldsymbol{g}_1$ represents the continuity equation and the tangential momentum jump condition, and $\boldsymbol{g}_2$ represents the normal momentum jump condition. Differentiating $\boldsymbol{g}_2$ with respect to time gives $\boldsymbol{p}$ in principle, so that $\boldsymbol{g}_2$ is an index one constraint (compare section 9.2).

By this we obtained the following result: The discretized jump conditions have index one. The discretized condensation problem is an index two DAE, where the index is the index of the Navier-Stokes equations. The jump conditions do not change the index.

9.5 Algorithms for the moving boundary problem

We finish this chapter with a short summery of the main difficulties in solving moving boundary problems and discuss the two different approaches used by numerical methods for such problems.

Moving boundary problems are highly nonlinear

From (9.26) we see that the discretized jump conditions are highly nonlinear DAEs, depending on how many nodes are free to move. If only the step size of the last interval near the boundary is variable (and the other intervals are fixed), then only the discretized jump conditions are affected. If all nodes are free to move, then the discretized momentum equation and energy equation are also nonlinear in the film thickness ($\Delta y \neq$ const.) [CS90].

Tracking methods and capturing methods

Numerical algorithms for the solution of moving boundary problems fall basically in two categories, depending on the coordinate system which is used to describe the interface. In part I we derived the jump conditions in the local coordinate system of the moving surface (Lagrangian representation). Later we related the jump conditions to a fixed coordinate system (Eulerian representation) by means of a parametrizion of the surface.[10] For example, for a moving boundary problem without phase change $\dot{m} = 0$ the mass jump condition in Lagrangian representation is given as

$$\rho\,[\boldsymbol{v} - \boldsymbol{u}] \cdot \boldsymbol{n} \;\; = \;\; 0\,,$$

(compare (9.23)). For a two-dimensional moving boundary problem without phase change the mass jump condition is given in Eulerian representation as[11]

$$v_y - \frac{\partial h}{\partial t} - v_z \frac{\partial h}{\partial z} = 0\,,$$

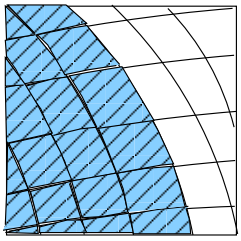
or, using $F(y,z,t) = y - h(z,t)$, equivalently

$$\frac{\partial F}{\partial t} - \boldsymbol{v} \cdot \boldsymbol{\nabla} F = 0\,, \tag{9.28}$$

[10] In part II we used a cylindrical parametrization, here we used an implict parametrization.
[11] Note that the negative sign in (9.28) comes from the definition of the normal vector.

(compare (9.24)). The last equation is also called level-set equation.

Methods based on a Lagrangian representation of the interface are called tracking methods, and methods based on an Eulerian representation of the interface are called capturing methods (e.g. Marker and Cell (MAC), Volume of Fluid (VOF), [Kot98].



1	0.5	0	0	0
1	0.9	0.7	0	0
1	1	0.7	0.9	0
1	1	1	0.6	0
1	1	1	0.6	0

Figure 9.4: *Tracking methods and capturing methods*

In tracking methods the surface normal and tangential vectors and the mean curvature are computed from the geometry of the surface [FP96]. In every iteration the position of the interface is computed. After some iterations during the simulation the mesh has to be adopted to avoid a distorted mesh topology. In capturing methods usually the mesh is fixed and the interface is reconstructed by means of equation (9.28), but sometimes with a slightly different meaning. Then F describes the volume fraction of one phase, e.g. $F = 1$ for a cell filled with condensate and $F = 0$ for a cell filled with vapor [Kot98], as shown in the right side of figure 9.4.

Although for the later methods the mesh is inherently robust, the reconstruction of the interface remains inaccurately. This qualifies tracking methods for moving boundary problems with interface phenomena, such as phase change and surface tension. However, the robustness of capturing methods can not be overestimated.

Implicit versus explicit method for the update of the moving boundary

If the film thickness is treated as a new unknown, additionally to velocity, pressure and temperature, so that all variables are equally part of the solution vector (as in section 9.4), then the moving boundary is treated implicitly. Implicit methods are more stable and therefore preferred [HW02b].

Often the numerical solution of a moving boundary problem is done as follows: First the Navier-Stokes equations and the energy equation are solved using some of the jump conditions as boundary conditions, and then the remaining jump condition is used to update the position of the moving boundary, so that the moving boundary is treated explicitly.

Chapter 10

Summary and Outlook

In this thesis we investigated condensation in a tube with small diameter where surface tension is important. Such hydrodynamical channels are found in compact heat exchangers. The goal was to establish a better understanding of the physical process and to enhance the heat transfer. First, the plan was to simulate the problem numerically using a CFD-program. But it turned out that the equations were not implemented correctly, so that we decided to analyze the equations of continuum mechanics of such moving boundary problems and to derive a suitable model. For this we worked out the complete model equations for moving boundary problems with phase change and surface tension. Surface tension is both a characteristic of geometry, and physics of the interface between vapor and liquid. This increase the complexity of the interface model equations significantly. To the best of our knowledge such a complete derivation can not be found in literature.

Then we analyzed these equations using generalized dimensional analysis and derived a simplified model for the vertical tube. By this we reintroduced and reformulated generalized dimensional analysis, a very algorithmically method which fell occasionally into oblivion. Generalized dimensional analysis is an extension of classical dimensional analysis, where additionally the model equations are evaluated to find the dimensionless numbers of the process. It allows an analysis based on the two length sales of the process (film thickness and tube length). The results are compared with experimental results and explain the better heat transfer in an inclined tube in the case of low surface tension. The derived interface model equations (jump conditions) are equivalent to boundary layer equations in the sense that they are based on the same conditions.

Next we derived an ordinary differential equation for condensation in a vertical tube and by this we extended Nußelt's theory to condensation in a tube under rotational

symmetry where surface tension is taken into account. The heat transfer decrease for very small tube diameters independent of surface tension (thicker condensate film) and increase for extreme high surface tension. The derived model is compared to parametrical models from literature. It is better then Nußelt's model. With Chen's model higher heat transfer rates are predicted. Chen's model is based on available experimental information for co-current condensation inside vertical tubes and takes shear stress exerted by co-current vapor flow into account (which result in a thinner condensate film).

After spatial discretization the model equations form a system of ordinary differential equations and algebraic equations (DAE). We analyzed the complexity (the index) of this DAE system and showed that the index of moving boundary problems is determined by the Navier-Stokes equations and not by the (transient) moving boundary problem. We compared different index reduction methods. Based on this thesis we discussed some aspects of numerical methods for moving boundary problems. Decisive for a moving boundary problem is not the index but the nonlinearity introduced by the film thickness as a new unknown.

The transformation of an equation with dimension into a dimensionless equation is a symmetry transformation. We extended the idea of symmetry analysis to Lie group analysis and determined the symmetry groups of the model equations, and we showed how to construct analytical solutions for differential equations using the symmetries of an equation.

The computational development provides us with powerful simulation possibilities. To use them efficiently we need to know the equations which we want to compute and the characteristics of the numerical methods which we use to solve them. The development in computer-aided simulation forces towards more and more interdisciplinary work. Mathematicians are more and more requested to show results that can be used in applied technologies. The demand on engineers is more and more to use and implement simulation software. In every interdisciplinary work at first a common language has to be defined. This thesis also intends to make a contribution to this.

The next projects would be: to extend the derived model equations for the case of an inclined tube; to implement the equations derived for the vertical tube which we analyzed further in part III; to work out the potentialities of Lie group analysis further and to derive an analytical model equation based on the symmetries of the condensation problem.

Appendix A

Symmetry groups of the model equations/rotational symmetry

In this chapter we analyze the symmetry groups (Lie groups) of transformations under which the model equations of the condensation problem from part II are invariant. In section A.1 we briefly outline the method of finding symmetry groups of a given differential equation by a Lie group analysis. In section A.2 we discuss generalized dimensional analysis in the wider framework of a Lie group analysis, and in section A.3 we investigate the Lie groups admitted by the equations of the bulk flow of the condensation problem and take the outer boundary conditions into account. In section A.4 we also consider the jump conditions and analyze the Lie groups of the condensation problem. This chapter is also intended to be useful for future research. For example, based on a Lie group analysis one may construct (using canonical variables) an equivalent system of equations for the condensation problem, representing the same physical phenomena as the original model equations, but easier to solve (it still may need to be solved numerically).

The first section of this chapter is mainly based on [BK89], [Olv93] and [Ste94]. Further on [Bau00] [Can02] and [Ibr99]. From an engineering point of view the approach of [Can02] and [Bau00] is very appropriate.

A.1 Lie groups of transformations of a given differential equation

The main idea of a Lie analysis is to construct a linear operator, based on the local action of the group of transformations in the neighborhood of the original variables,

apply it to a given differential equation and demand whether this operation let the differential equation invariant. The result is an over-determined system of linear partial differential equations, whose solution gives the symmetry transformations. We first introduce the method and then we demonstrate how to find the Lie groups of a given ordinary differential equation exemplarily.

Lie groups of transformations

A change of variables of a physical problem depending on (at least) one parameter

$$\hat{\boldsymbol{x}} \;=\; \hat{\boldsymbol{x}}(\boldsymbol{x}, a)\;, \tag{A.1}$$

where $\boldsymbol{x} = [x_1, x_2, ...x_n]^T$ denotes the vector of the dependent and independent variables, is called a point transformation if the transformation mapping the variables into variables is invertible, if repeated transformations yield again a transformation of this kind (associative law), and if a transformation exist that maps a point to itself, e.g. for $a = 0$,

$$\hat{\boldsymbol{x}} = \hat{\boldsymbol{x}}(\boldsymbol{x}, 0) = \boldsymbol{x}\;, \qquad \text{(identity element)}$$

[Ste94]. Then the transformation satisfies the four axioms of a group and we say (A.1) defines a one-parameter group of transformations.[1]

If we further assume that a is a continuous parameter, that $\hat{\boldsymbol{x}}$ is infinitely differentiable with respect to $\boldsymbol{x}$ and an analytic function (representable in a power series of a), and that $\hat{a}$ is also an analytic function, then (A.1) defines a one-parameter Lie group of point transformations [BK89]. For the rest of this section we write $\boldsymbol{x} = [x, y]^T$.

Two examples for a one-parameter Lie group are a group of translation transformations parallel to the x-axis

$$\hat{x} = x + a\;, \qquad \hat{y} = y\;, \tag{A.2}$$

and a nonlinear group of scaling (dilatation) transformations

$$\hat{x} = (1 + b)\, x\;, \qquad \hat{y} = (1 + b)^2\, y\;. \tag{A.3}$$

[1] The four group axioms are closure (the result of the transformation of a variable in $\mathbb{R}^n$ is again a variable in $\mathbb{R}^n$), associative law, identity element, and inverse element.

The effect of the group of transformations $\hat{\boldsymbol{x}}(\boldsymbol{x}, a)$ on a point is shown in figure A.1. Consider the point $\hat{\boldsymbol{x}}(\boldsymbol{x}, 0) = \boldsymbol{x}$. Varying the group parameter a will move the point along the curve $\hat{\boldsymbol{x}}(a)$ in the xy–plane [Ste94]. Different initial points are transformed into different points on the curve.

Local coordinates and tangent vector

The Lie group of transformations $\hat{\boldsymbol{x}}(\boldsymbol{x}, a)$ is a function of $\boldsymbol{x}$ and a, so that $\boldsymbol{x} = [x, y]^T$ can be considered as local coordinates. Then the two base vectors of the local coordinate system are the derivatives of $\hat{\boldsymbol{x}}$ with respect to the local coordinates $\frac{\partial \hat{\boldsymbol{x}}}{\partial x}$ and $\frac{\partial \hat{\boldsymbol{x}}}{\partial y}$ (compare section 2.1 and 2.3), and the tangent vector $\boldsymbol{X}$ attached at the point $\hat{\boldsymbol{x}}$ reads in the local coordinate system

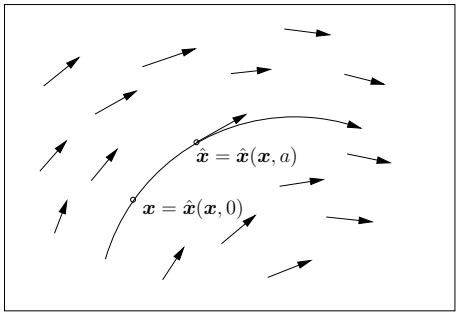


Figure A.1: *Integral curve of a vector field*

$$\boldsymbol{X}(\hat{\boldsymbol{x}}) = \xi(\hat{\boldsymbol{x}}) \frac{\partial \hat{\boldsymbol{x}}}{\partial x} + \eta(\hat{\boldsymbol{x}}) \frac{\partial \hat{\boldsymbol{x}}}{\partial y} , \tag{A.4}$$

where ξ and η are the local coordinates of $\boldsymbol{X}$ [Olv93].

Tangent vector field and integral curve

On the other hand $\hat{\boldsymbol{x}}(\boldsymbol{x}, a)$ represent a vector field depending on a, and such a vector field can be associated with a system of first order differential equations (one may think of a as a time variable). For a fixed initial point $\boldsymbol{x}$ at $a = 0$ the integral curve $\hat{\boldsymbol{x}}(a)$ that coincides at any point with the tangent vector $\boldsymbol{X}$ along the curve at that point is then given by

$$\frac{\mathrm{d}\hat{\boldsymbol{x}}}{\mathrm{d}a} = \boldsymbol{X}(\hat{\boldsymbol{x}}) \qquad \text{with} \qquad \hat{\boldsymbol{x}}(\boldsymbol{x}, 0) = \boldsymbol{x}$$

and in local coordinates

$$\frac{\mathrm{d}\hat{\boldsymbol{x}}}{\mathrm{d}a} = \boldsymbol{\xi}(\hat{\boldsymbol{x}}) \qquad \text{with} \qquad \hat{\boldsymbol{x}}(\boldsymbol{x}, 0) = \boldsymbol{x} , \tag{A.5}$$

where $\boldsymbol{\xi} = [\xi, \eta]^T$ is called vector of infinitesimals [Olv93]. Equations (A.5) are called Lie equations. That is, once we found the infinitesimals for a given differential equation, the Lie group of transformations can be reconstructed by integrating the Lie equations [BK89].

Infinitesimal transformations

A local group of transformations we get by expanding $\hat{\boldsymbol{x}}(\boldsymbol{x}, a)$ in a Taylor series about $a = 0$

$$\hat{\boldsymbol{x}}(\boldsymbol{x}, a) = \boldsymbol{x} + a \left.\frac{\partial \hat{\boldsymbol{x}}}{\partial a}\right|_{a=0} + O(a^2) \,, \tag{A.6}$$

where

$$\boldsymbol{\xi}(\boldsymbol{x}) = \left.\frac{\partial \hat{\boldsymbol{x}}(\boldsymbol{x}, a)}{\partial a}\right|_{a=0} \tag{A.7}$$

is the vector $\boldsymbol{\xi}$ of the infinitesimals at $a = 0$. By this we get the Lie group of transformations in terms of infinitesimal transformations

$$\hat{\boldsymbol{x}}(\boldsymbol{x}, a) = \boldsymbol{x} + a\, \boldsymbol{\xi}(\boldsymbol{x}) + O(a^2) \tag{A.8}$$

[Can02]. The infinitesimals for the group of scaling transformations (A.3) are

$$\xi = \left.\frac{\partial \hat{x}}{\partial a}\right|_{a=0} = x \,, \qquad\qquad \eta = \left.\frac{\partial \hat{y}}{\partial a}\right|_{a=0} = 2\, y \,.$$

Infinitesimal generator

Now we construct a linear generator X based on the tangent vector in local coordinates, such that $X\, \boldsymbol{x} = \boldsymbol{\xi}$. We define the infinitesimal generator X with the gradient vector as

$$X(\boldsymbol{x}) = \boldsymbol{\xi}(\boldsymbol{x}) \cdot \boldsymbol{\nabla} = \xi(\boldsymbol{x}) \frac{\partial}{\partial x} + \eta(\boldsymbol{x}) \frac{\partial}{\partial y} \,, \tag{A.9}$$

so that we get for the Lie group of transformations $\hat{\boldsymbol{x}}(\boldsymbol{x}, a)$

$$\hat{\boldsymbol{x}} = \boldsymbol{x} + a\, X\, \boldsymbol{x} + \frac{a}{2!}\, X\,(X\, \boldsymbol{x}) + \frac{a^2}{3!}\, X\,(X\,(X\, \boldsymbol{x})) + \dots \,, \tag{A.10}$$

This series is called a Lie series [BK89].

The infinitesimal generator for the group of scaling transformations (A.3) is

$$X = x \frac{\partial}{\partial x} + 2\, y \frac{\partial}{\partial y} \,.$$

A Lie series of x with this generator gives

$$\hat{x} = x + a\,X\,x + \frac{a}{2!}X\,(X\,x) + \ldots = x + a\,x + \frac{a}{2!}\,x + \ldots = x\left(1 + a + \frac{a^2}{2!} + \ldots\right),$$

which is the exponential function. By this we find a parametrization of the group (A.3) of scaling transformations such that the identity element is $\hat{\boldsymbol{x}}(\boldsymbol{x}, 0) = \boldsymbol{x}$

$$\begin{aligned} \hat{x} &= e^a\,x\,, \\ \hat{y} &= e^{2a}\,y\,. \end{aligned} \tag{A.11}$$

Invariant functions

With the infinitesimal generator we are now in the position to analyze if a function is invariant under a Lie group of transformations. $F(\boldsymbol{x})$ is said to be invariant under a group of transformations $\hat{\boldsymbol{x}}(\boldsymbol{x}, a)$ if

$$F(\hat{\boldsymbol{x}}) = F(\boldsymbol{x})$$

holds true for every value of a. Expanding $F(\hat{\boldsymbol{x}})$ in a Lie series gives

$$F(\hat{\boldsymbol{x}}) \;=\; F(\boldsymbol{x}) + a\,X\,F(\boldsymbol{x}) + O(a^2)\,.$$

[Ste94]. This gives us the invariance condition we are seeking:

$$X\,F = 0 \qquad \text{or with (A.9)} \qquad \xi\,\frac{\partial F}{\partial x} + \eta\,\frac{\partial F}{\partial y} = 0\,, \tag{A.12}$$

A function is invariant under a group of transformations if the infinitesimal generator X applied to the function equals zero. It is the key point of Lie's theory that the infinitesimal generator is a linear operator, although the groups of transformations admitted by a given function may be nonlinear (e.g. the scaling transformation) [BK89]. To derive an invariance condition for differential equations, the invariance conditions (A.12) has to be extended further.

Extended transformations

We want to apply the point transformation (A.1) to a differential equation, so we need to know how to transform the derivatives. We consider a single ordinary differential equation, with one independent variable x and one dependent variable y. It turns out that the transformations of derivatives are functions not only of the dependent and the independent variables, but also of the derivatives of the dependent variables [BK89]. So we write

$$\begin{aligned} \hat{x} &= \hat{x}(x, y, a) \,, \\ \hat{y} &= \hat{y}(x, y, a) \,, \\ \hat{y}_{\hat{x}} &= \hat{y}_{\hat{x}}(x, y, y_x, a) \,, \\ \hat{y}_{\hat{x}\hat{x}} &= \hat{y}_{\hat{x}\hat{x}}(x, y, y_x, y_{xx}, a) \,, \qquad \dots \,. \end{aligned} \tag{A.13}$$

The transformations of the derivatives are the derivatives of the transformed variables with respect to the transformed variables

$$\hat{y}_{\hat{x}} = \frac{\mathrm{d}\hat{y}}{\mathrm{d}\hat{x}} \,, \qquad \hat{y}_{\hat{x}\hat{x}} = \frac{\mathrm{d}\hat{y}_{\hat{x}}}{\mathrm{d}\hat{x}} \,, \qquad \dots \,.$$

Using total derivatives

$$\begin{aligned} \mathrm{d}\hat{x} &= \frac{\partial \hat{x}}{\partial x}\,\mathrm{d}x + \frac{\partial \hat{x}}{\partial y}\,\mathrm{d}y \,, \\ \mathrm{d}\hat{y} &= \frac{\partial \hat{y}}{\partial x}\,\mathrm{d}x + \frac{\partial \hat{y}}{\partial y}\,\mathrm{d}y \,, \\ \mathrm{d}\hat{y}_{\hat{x}} &= \frac{\partial \hat{y}_{\hat{x}}}{\partial x}\,\mathrm{d}x + \frac{\partial \hat{y}_{\hat{x}}}{\partial y}\,\mathrm{d}y \,, + \frac{\partial \hat{y}_{\hat{x}}}{\partial y_x}\,\mathrm{d}y_x \,, \end{aligned}$$

we get for the first and second extension of the group of transformations

$$\hat{y}_{\hat{x}} = \frac{\frac{\partial \hat{y}}{\partial x}\,\mathrm{d}x + \frac{\partial \hat{y}}{\partial y}\,\mathrm{d}y}{\frac{\partial \hat{x}}{\partial x}\,\mathrm{d}x + \frac{\partial \hat{x}}{\partial y}\,\mathrm{d}y} = \frac{\frac{\partial \hat{y}}{\partial x} + \frac{\partial \hat{y}}{\partial y}\,y_x}{\frac{\partial \hat{x}}{\partial x} + \frac{\partial \hat{x}}{\partial y}\,y_x} = \hat{y}_{\hat{x}}(x, y, y_x, a) \,, \tag{A.14}$$

$$\hat{y}_{\hat{x}\hat{x}} = \frac{\frac{\partial \hat{y}_{\hat{x}}}{\partial x} + \frac{\partial \hat{y}_{\hat{x}}}{\partial y}\,y_x + \frac{\partial \hat{y}_{\hat{x}}}{\partial y_x}\,y_{xx}}{\frac{\partial \hat{x}}{\partial x} + \frac{\partial \hat{x}}{\partial y}\,y_x} = \hat{y}_{\hat{x}\hat{x}}(x, y, y_x, y_{xx}, a) \,, \tag{A.15}$$

[Ste94].

Extended infinitesimal transformations

For the invariance condition we need the extended group of transformations (A.13) in terms of infinitesimal transformations, which means we need the transformations of the derivatives in terms of infinitesimals. So we are seeking extended infinitesimals $\eta_{[x]}, \eta_{[xx]}, \ldots$, that we can write

$$\begin{aligned} \hat{x} &= x + a\,\xi(\hat{x}) + O(a^2)\,, \\ \hat{y} &= y + a\,\eta(\hat{y}) + O(a^2)\,, \\ \hat{y}_{\hat{x}} &= y_x + a\,\eta_{[x]}(\hat{y}_{\hat{x}}) + O(a^2)\,, \\ \hat{y}_{\hat{x}\hat{x}} &= y_{xx} + a\,\eta_{[xx]}(\hat{y}_{\hat{x}\hat{x}}) + O(a^2)\,, \qquad \ldots\,. \end{aligned} \tag{A.16}$$

The once extended infinitesimal $\eta_{[x]}$ we get by substituting the infinitesimal transformations (A.8) into the first extension of the group of transformations (A.14)

$$\hat{y}_{\hat{x}} = \frac{\mathrm{d}\hat{y}}{\mathrm{d}\hat{x}} = \frac{\mathrm{d}(y + a\,\eta + O(a^2))}{\mathrm{d}(x + a\xi + O(a^2))} = \frac{y_x + a\,\frac{\mathrm{d}\eta}{\mathrm{d}x} + O(a^2)}{1 + a\,\frac{\mathrm{d}\xi}{\mathrm{d}x} + O(a^2)} = y_x + a\,\left[\frac{\mathrm{d}\eta}{\mathrm{d}x} - y_x\,\frac{\mathrm{d}\xi}{\mathrm{d}x}\right] + O(a^2),$$

[Ste94]. By this we get the once extended infinitesimal as

$$\eta_{[x]} = \left[\frac{\mathrm{d}\eta}{\mathrm{d}x} - y_x\,\frac{\mathrm{d}\xi}{\mathrm{d}x}\right]\,. \tag{A.17}$$

The twice extended infinitesimal $\eta_{[xx]}$ we get by substituting the infinitesimal transformations (A.8) and the first extended infinitesimal (A.17) into the second extension of the group of transformations (A.15)

$$\hat{y}_{\hat{x}\hat{x}} = \frac{\mathrm{d}\hat{y}_{\hat{x}}}{\mathrm{d}\hat{x}} = \frac{y_{xx} + a\,\frac{\mathrm{d}\eta_{[x]}}{\mathrm{d}x} + O(a^2)}{1 + a\,\frac{\mathrm{d}\xi}{\mathrm{d}x} + O(a^2)} = y_{xx} + a\,\left[\frac{\mathrm{d}\eta_{[x]}}{\mathrm{d}x} - y_{xx}\,\frac{\mathrm{d}\xi}{\mathrm{d}x}\right] + O(a^2)\,,$$

[Ste94]. This gives the twice extended infinitesimal as

$$\eta_{[xx]} = \left[\frac{\mathrm{d}\eta_{[x]}}{\mathrm{d}x} - y_{xx}\,\frac{\mathrm{d}\xi}{\mathrm{d}x}\right]\,. \tag{A.18}$$

Higher order extended infinitesimals are found similar, we refer to [BK89], or one of the other references cited at the beginning of this chapter. From (A.17) and (A.18) we get the explicit formulas

$$\begin{aligned} \eta_{[x]} &= \eta_x + (\eta_y + \xi_x)\;y_x - \xi_y\,y_x^2\,, \\ \eta_{[xx]} &= \eta_{xx} + (2\,\eta_{xy} - \xi_{xx})\;y_x + (\eta_{yy} - 2\,\xi_{xy})\;y_x^2 - \xi_{yy}\,y_x^3 \\ &\quad + (\eta_y - 2\,\xi_x)\;y_{xx} - 3\,\xi_y\,y_x\,y_{xx}\,. \end{aligned} \tag{A.19}$$

The extended infinitesimals are linear in the highest derivative of the dependent variable (y_{xx}) and polynomial in the other derivatives of the dependent variable, where the coefficients are linear in the non-extended infinitesimals.

Since the derivatives have to be taken with respect to all variables, explicit formulas for higher order extended infinitesimals contain more and more terms. However, this laborious work is generally left to computers.

Extended infinitesimal generator

The twice extended infinitesimal generator is then given by

$$X_{[xx]} = \xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y} + \eta_{[x]} \frac{\partial}{\partial y_x} + \eta_{[xx]} \frac{\partial}{\partial y_{xx}} , \tag{A.20}$$

[Can02]. With the extended infinitesimal generator we are able to find the Lie groups of transformations which leave a given ordinary differential equation invariant. The invariance condition for partial differential equations is derived similar and we refer to the references given at the beginning of this chapter for it. In the last section of this chapter we analyze exemplarily a second order ordinary differential equation.

Multi-parameter Lie groups and Lie algebras

In general a Lie group of transformations admitted by a differential equation depends on more than one parameter

$$\hat{\boldsymbol{x}} \;=\; \hat{\boldsymbol{x}}(\boldsymbol{x}, \boldsymbol{a}) \quad \text{where} \quad \boldsymbol{x} = [x_1, x_2, ..., x_n]^T, \quad \boldsymbol{a} = [a_1, a_2, ..., a_k]^T. \tag{A.21}$$

If the parameters are independent of each other and if (A.21) satisfies the four group axioms, then (A.21) defines a k-parameter Lie group of transformations [BK89].

Here we restrict ourselves to one dependent and one independent variable $\boldsymbol{x} = [x, y]^T$. To each parameter a_r a non-extended infinitesimal generator X_r can be associated

$$X_r = \xi_r \frac{\partial}{\partial x} + \eta_r \frac{\partial}{\partial y} \qquad \text{with} \qquad \xi_r = \left.\frac{\partial \hat{x}}{\partial a_r}\right|_{\boldsymbol{a}=0}, \qquad \eta_r = \left.\frac{\partial \hat{y}}{\partial a_r}\right|_{\boldsymbol{a}=0}. \tag{A.22}$$

The vector space spanned by the k infinitesimal generators is called a Lie algebra, if the infinitesimal generators have an additional structure which is called commutator [BK89]. The commutator of two infinitesimal generators is defined as

$$[X_1, X_2] = X_1 (X_2) - X_2 (X_1) . \tag{A.23}$$

Its result is again an infinitesimal generator and can be written as a linear combination of the k basic infinitesimal generators

$$[X_1, X_2] = c_r \, X_r \, , \tag{A.24}$$

where the coefficients c_r are called structural constants [Olv93]. Commutators are conveniently displayed in a commutator table, such as the commutator table in figure A.2, and for that it is more convenient to write (A.24) as $[X_1, X_2] = C^r_{12} \, X_r$, so that the structural constants can be directly read off the commutator table.

As an example we consider the three dimensional Lie algebra where the infinitesimal generators

$$X_1 = \frac{\partial}{\partial x} \, , \qquad X_2 = x \, \frac{\partial}{\partial x} \, , \qquad X_3 = x^2 \, \frac{\partial}{\partial x}$$

span the basis of a vector space [Can02]. We compute with (A.23) the commutator $[X_1, X_2]$ and get

$$[X_1, X_2] = \frac{\partial}{\partial x}\left(x \, \frac{\partial}{\partial x}\right) - x \, \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\right) = \left(\frac{\partial}{\partial x} + x \, \frac{\partial^2}{\partial x^2}\right) - x \, \frac{\partial^2}{\partial x^2} = \frac{\partial}{\partial x} = X_1, \tag{A.25}$$

where the structure constants are $C^1_{12} = 1$ and $C^2_{12} = C^3_{12} = 0$. The other commutators are calculated similarly. By this we get the commutator table shown in figure A.2. Note that second order derivatives are canceled out, so that the result is again an infinitesimal operator.

A subspace of infinitesimal generators is called a subalgebra if the commutators of the subspace are the infinitesimal generators of the subspace [BK89]. That is $[X_1, X_2] = X_1$ and $[X_2, X_3] = X_3$ are subalgebras, but $[X_1, X_3] = 2\,X_2$ is not. A Lie algebra is called solvable if there exist an inclusive chain of subalgebras [Can02]. The chain of subalgebras $[[X_1, X_2], X_3] = [X_1, X_3] = 2\,X_2$ is not solvable, so the Lie algebra of our example is non-solvable. Any two-dimensional Lie algebra is solvable. A zero subalgebra is always solvable.

	X_1	X_2	X_3
X_1	0	X_1	$2\,X_2$
X_2	$-X_1$	0	X_3
X_3	$-2\,X_2$	$-X_3$	0

Figure A.2: *Commutator table of* (A.25)

Canonical variables

Lie group analysis is a method to solve differential equations analytically. Here we discuss the method of so called canonical variables [Ibr99]. By a suitable change of

variables $u(x,y)$, $v(x,y)$ any one-parameter Lie group of transformations admitted by a given equation can be simplified to a group of translation transformation

$$\begin{aligned} \hat{u} &= u + a \,, \\ \hat{v} &= v \,, \end{aligned}$$

[BK89]. The new variables u and v follow from the solution of the system of linear partial differential equations defined by

$$\begin{aligned} X\, u &= 1 \,, \\ X\, v &= 0 \,. \end{aligned}$$

Once the canonical variables are found they can be used to simplify the original equation. The procedure is as follows: Find the infinitesimals, calculate the canonical variables, and then transform the original equations to a simpler form by changing the variables [Bau00].

In case of an ordinary differential equation of order two or higher this procedure will reduce the order of the ordinary differential equation. In case of a partial differential equation with more then two independent variables the result will still be a system of nonlinear partial differential equations but in less independent variables. Repeated application (if possible) then leads to an ordinary differential equation. The success of the method depends on the symmetry groups (i.e. the infinitesimals) admited by the differential equations. For further reading we refer to [BK89] [Can02] [Bau00].

How to find the symmetry groups of transformations of a differential equation

Now we show exemplarily how to find the symmetry groups (Lie groups) of transformations under which a given ordinary differential equation is invariant. Using this example we demonstrate the main steps of a Lie group analysis in the way it is implemented in the software package of [Bau00], which we use in the next section to investigate the symmetry groups of the condensation problem. The example is taken from [Can02].

For the ordinary differential equation

$$F(x, y, y_x, y_{xx}) = y_{xx} = 0$$

to be invariant, the invariance condition (A.12) with the twice extended infinitesimal generator (A.20) has to be satisfied

$$X_{[xx]}\, y_{xx} = \eta_{[xx]} = 0\ .$$

This gives with (A.19)

$$\eta_{xx} + (2\,\eta_{xy} - \xi_{xx})\ y_x + (\eta_{yy} - 2\,\xi_{xy})\ y_x^2 - \xi_{yy}\, y_x^3 + (\eta_y - 2\,\xi_x)\ y_{xx} - 3\,\xi_y\, y_x\, y_{xx} = 0.$$

The last two terms are zero with $y_{xx} = 0$. The remaining terms have to be zero independently, because y_x is an arbitrary function. This result in the following over-determined system of linear partial differential equations

$$\begin{aligned} \eta_{xx} &= 0\ , \\ 2\,\eta_{xy} - \xi_{xx} &= 0\ , \\ \eta_{yy} - 2\,\xi_{xy} &= 0\ , \\ \xi_{yy} &= 0\ , \end{aligned} \tag{A.26}$$

which has to be solved for the infinitesimals to find the groups of infinitesimal transformations under which the differential equation is invariant. One possibility is to try an ansatz in the form of a third-order series

$$\begin{aligned} \xi &= a_1 + a_2\, x + a_3\, y + a_4\, x^2 + a_5\, x\, y + a_6\, y^2 + a_7\, x^3 + a_8\, x^2\, y + a_9\, x\, y^2 + a_{10}\, y^3, \\ \eta &= b_1 + b_2\, x + b_3\, y + b_4\, y^2 + b_5\, x\, y + b_6\, y^2 + b_7\, x^3 + b_8\, x^2\, y + b_9\, x\, y^2 + b_{10}\, y^3. \end{aligned}$$

Substituting this ansatz into the determining equations (A.26) and solving them for ξ and η by comparing the coefficients of x and y gives the infinitesimals as

$$\begin{aligned} \xi &= a_1 + a_2\, x + a_3\, y + a_4\, x^2 + a_5\, x\, y\ , \\ \eta &= b_1 + b_2\, x + b_3\, y + a_4\, x\, y + a_5\, y^2\ . \end{aligned}$$

The symmetry group of transformations defined by this infinitesimals is a projective group and its Lie algebra is spanned by the one-parameter infinitesimal generators

$$X_1 = \frac{\partial}{\partial x}\ , \qquad X_2 = x\,\frac{\partial}{\partial x}\ , \qquad X_3 = y\,\frac{\partial}{\partial x}\ , \qquad X_4 = x^2\,\frac{\partial}{\partial x} + x\,y\,\frac{\partial}{\partial y}\ ,$$

$$X_5 = x\,y\,\frac{\partial}{\partial x} + y^2\,\frac{\partial}{\partial y}\ , \qquad X_6 = \frac{\partial}{\partial y}\ , \qquad X_7 = x\,\frac{\partial}{\partial y}\ , \qquad X_8 = y\,\frac{\partial}{\partial y}\ .$$

Amongst others this Lie algebra has the solvable subalgebra $[X_2, X_6] = 0$, which defines the Lie group of translation transformations (A.2)

$$\xi = x\ , \qquad \eta = 1\ .$$

A.2 Scaling transformations and Lie groups of bulk equations

In this section first we discuss the similarities and differences between the generalized dimensional analysis and the Lie group analysis. Then we make a Lie group analysis of the bulk flow equations for condensation in a vertical tube, using a software package implemented in *Mathematica*® [Bau00].

Scaling transformations of the dependent and the independent variables

Dimensional analysis is based on the invariance of an equation under a change of system of units. Therefore any physical equation is invariant under a group of scaling (dilatation) transformation applied to the dimensions of the variables. In chapter 7 we denoted the value of a variable by a hat $\hat{x}$ and its dimension by a tilde $\tilde{x}$, $x = \hat{x}\,\tilde{x}$. Here we the notation commonly used in Lie group analysis and denote a scaling transformation by an exponential term $\hat{x} = e^{\bar{x}}\,x$ (with the identity element $\hat{x} = e^0\,x = x$).

Then scaling transformations of the dependent and the independent variables of the bulk equations for condensation in a vertical tube are given by

$$\begin{aligned}
\hat{r} &= e^{\bar{r}}\,r\,, \qquad \hat{v_r} = e^{\bar{v}_r}\,v_r\,,\\
\hat{z} &= e^{\bar{z}}\,z\,, \qquad \hat{v_z} = e^{\bar{v}_z}\,v_z\,,\\
\hat{t} &= e^{\bar{t}}\,t\,, \qquad \hat{p} = e^{\bar{p}}\,p\,, \qquad \hat{T} = e^{\bar{T}}\,T\,.
\end{aligned}$$

This scaling transformations we substitute in the simplified model equations of the condensation problem for the vertical position of the tube in dimensional form, compare (6.1) – (6.3) and (7.34) – (7.36). This gives us the following equations. The continuity equation:

$$e^{(\bar{v}_r-\bar{r})}\,\frac{1}{r}\,\frac{\partial}{\partial r}\,(r\,v_r) + e^{(\bar{v}_z-\bar{z})}\,\frac{\partial v_z}{\partial z} = 0\,, \tag{A.27}$$

the momentum equations:

$$e^{(\bar{v}_z-\bar{t})}\frac{\partial v_z}{\partial t}+e^{(\bar{v}_r+\bar{v}_z-\bar{r})}v_r\frac{\partial v_z}{\partial r}+e^{(2\bar{v}_z-\bar{z})}v_z\frac{\partial v_z}{\partial z}=-e^{(\bar{p}-\bar{z})}\frac{1}{\rho}\frac{\partial p}{\partial z}+e^{(\bar{v}_z-2\bar{r})}\frac{\eta}{\rho}\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)+g,$$

$$e^{(\bar{p}-\bar{r})}\frac{\partial p}{\partial r} = 0\,, \tag{A.28}$$

and the energy equation:

$$e^{(\bar{T}-\bar{t})}\,\frac{\partial T}{\partial t} + e^{(\bar{v}_r+\bar{T}-\bar{r})}\,v_r\,\frac{\partial T}{\partial r} + e^{(\bar{v}_z+\bar{T}-\bar{z})}\,v_z\,\frac{\partial T}{\partial z} = e^{(\bar{T}-2\bar{r})}\,\frac{\lambda}{\rho\,c}\,\frac{1}{r}\,\frac{\partial}{\partial r}\left(r\,\frac{\partial T}{\partial r}\right)\,. \tag{A.29}$$

If the terms with an exponential factor in each equation are equal we can group them together. For example, for the continuity to be invariant under a scaling transformation $(\bar{v}_r - \bar{r}) = (\bar{v}_z - \bar{z})$ has to hold. Then we can write

$$\underbrace{e^{(\bar{v}_r-\bar{r})-(\bar{v}_z-\bar{z})}}_{=1} \left(\frac{1}{r} \frac{\partial}{\partial r} [r\, v_r] + \frac{\partial v_z}{\partial z} \right) = 0 \, .$$

Obviously the scaling transformations of the variables let the equation invariant if the exponential term vanishes.

In a first attempt we assume zero gravity, that makes it easier to compare generalized dimensional analysis with Lie group analysis. Then in (A.28) the sum of the exponents in each term has to be zero to let the equation invariant. The system of linear equations we get from (A.27) – (A.29) is then given by

$$\begin{array}{ll}
\text{(A.27)} & \left[\begin{array}{cccccc} -1 & 1 & & 1 & -1 & \\ 1 & & -1 & -1 & & \\ & & & & 2 & -1 \\ 2 & -1 & & & -1 & 1 \\ 2 & -1 & & & 1 & \end{array}\right] \left[\begin{array}{c} \bar{r} \\ \bar{z} \\ \bar{t} \\ \bar{v}_r \\ \bar{v}_z \\ \bar{p} \end{array}\right] = \mathbf{0} \, .
\end{array}$$

(A.28)
(A.28)
(A.28)
(A.29)

The matrix is the same matrix as the lower right sub-matrix of $\boldsymbol{B}$ in section 7.4 except that the scaling variable for the temperature and the two gravity related rows do not appear here. To solve this system of linear equations we proceed in the same way as for a generalized dimensional analysis (as described in chapter 7). We choose some of the scaling variables as base variables and collect the corresponding columns at the right side of the matrix $\bar{\boldsymbol{B}} = [\bar{\boldsymbol{K}}|\bar{\boldsymbol{R}}]$ such that $\operatorname{rank} \bar{\boldsymbol{K}} = \operatorname{rank} \bar{\boldsymbol{B}}$. We choose $\bar{v}_z = a$ as base scaling variable. After applying Gaußian elimination to $\bar{\boldsymbol{B}}$ we get the row-reduced echelon form of $\bar{\boldsymbol{B}}$ as

$$\operatorname{rref} \boldsymbol{B} = \begin{array}{c} \begin{array}{ccccccc} \bar{r} & \bar{z} & \bar{t} & \bar{v}_r & \bar{p} & & \bar{v}_z \end{array} \\ \left[\begin{array}{ccccc|c} 1 & & & & & -\frac{a}{2} \\ & 1 & & & & -2a \\ & & 1 & & & -a \\ & & & 1 & & \frac{a}{2} \\ & & & & 1 & -2a \end{array}\right] \end{array} .$$

By this we found that the bulk equations of the condensation problem are invariant under a two-parameter group of scaling transformations given by:

$$\begin{aligned}
\hat{r} &= e^{\frac{a}{2}} r\,, \qquad \hat{v_r} &= e^{-\frac{a}{2}} v_r\,, \\
\hat{z} &= e^{2a} z\,, \qquad \hat{v_z} &= e^{a} v_z\,, \\
\hat{t} &= e^{a} t\,, \qquad \hat{p} &= e^{2a} p\,, \qquad \hat{T} &= e^{b} T\,.
\end{aligned} \tag{A.30}$$

Note that the scaling transformations (A.30) are valid also if we consider gravity in the momentum equations. Then each exponent has to be zero itself, because the gravity remains unscaled in the equations, $g = e^0 g$.

Lie groups of transformations of bulk equations without gravity

We again consider equations (A.27) – (A.29) with the assumption of vanishing gravity, but now we analyze these equations by the method of Lie as described in section A.1. For that we use the software package *MathLie* [Bau00]. By this we discuss some of the relevant groups of the condensation problem. The result of our Lie group analysis gives the following infinitesimals

$$\begin{aligned}
\xi_r &= \tfrac{a}{2} r\,, & \eta_{v_r} &= -\tfrac{a}{2} v_r\,, \\
\xi_z &= 2\,(a+b)\, z + F_1(t)\,, & \eta_{v_z} &= (a+b)\, v_z + \tfrac{\partial F_1}{\partial t}\,, \\
\xi_t &= d + a\, t\,, & \eta_p &= 2\,(a+b)\, p + F_2(t) - \rho\, z\, \tfrac{\partial^2 F_1}{\partial t^2}\,, \\
& & \eta_T &= e + c\, T\,.
\end{aligned} \tag{A.31}$$

This infinitesimals correspond to seven infinitesimal generators

$$\begin{aligned}
X_a &= \frac{r}{2}\frac{\partial}{\partial r} + 2\, z \frac{\partial}{\partial z} + t\frac{\partial}{\partial t} - \frac{v_r}{2}\frac{\partial}{\partial v_r} + v_z \frac{\partial}{\partial v_z} + 2\, p \frac{\partial}{\partial p}\,, \\
X_b &= z\frac{\partial}{\partial z} + v_z \frac{\partial}{\partial v_z} + 2\, p \frac{\partial}{\partial p}\,, \\
X_c &= T\frac{\partial}{\partial T}\,, \\
X_d &= \frac{\partial}{\partial t}\,, \\
X_e &= \frac{\partial}{\partial T}\,, \\
X_{F_1} &= F_1(t)\frac{\partial}{\partial z} + \frac{\partial F_1}{\partial t}\frac{\partial}{\partial v_z} - \rho\, z\, \frac{\partial^2 F_1}{\partial t^2}\frac{\partial}{\partial p}\,, \\
X_{F_2} &= F_2(t)\frac{\partial}{\partial p}\,.
\end{aligned}$$

The infinitesimal generators X_a, X_b, X_c define a three-parameter group of scaling transformations, which is solvable, because $[[X_a, X_b], X_c] = 0$. X_d and X_e define

a translation transformation in time and in temperature. The translation group of the dependent variable temperature represents the superposition principle which holds for the energy equation [Can02]. The superposition principle states that for a linear (differential) equation a linear combination of solutions is again a solution of the equation. X_{F1} defines an arbitrary time dependent translation in streamwise direction, showing that the model equations are invariant for an observer at rest or an observer translating or accelerating in streamwise direction [Can02]. X_{F_2} defines an arbitrary time dependent translation of pressure.

The three-parameter group of scaling transformations defined by X_a, X_b, X_c are

$$\begin{aligned} \hat{r} &= e^{\frac{a}{2}} r\,, & \hat{v_r} &= e^{-\frac{a}{2}} v_r\,, \\ \hat{z} &= e^{(2a+b)} z\,, & \hat{v_z} &= e^{(a+b)} v_z\,, \\ \hat{t} &= e^{a} t\,, & \hat{p} &= e^{2(a+b)} p\,, \qquad \hat{T} = e^{c} T\,. \end{aligned}$$

For $b = 0$ (and with $c = b$) this gives the scaling transformations (A.30) we derived by generalized dimensional analysis, which shows the close connection between generalized dimensional analysis and Lie group analysis.

A.3 Lie groups of bulk equations together with outer boundary conditions

In this section analyze the Lie groups of the bulk equations together with the outer boundary conditions. First we find a similar three-parameter group of scaling transformations that leaves the bulk equations invariant, and then we fix one parameter such that also the boundary conditions admit this group of transformations.

Gravity

The infinitesimals which we get from a Lie group analysis for the model equations (A.27) – (A.29) of the condensate problem, differ from (A.31) (where we neglected gravity) in the infinitesimals

The corresponding group of scaling transformations is called a special scaling group. We come back to special scaling groups later.

(A.32)

$$\begin{aligned}
\xi_z &= \left(\tfrac{a}{2} + \tfrac{b}{g}\right) z + F_1(t) \,, \\
\eta_{v_z} &= \left(\tfrac{b}{g} - \tfrac{a}{2}\right) v_z + \tfrac{\partial F_1}{\partial t} \,, \\
\eta_p &= \left(\tfrac{2b}{g} - a\right) p + \left(\tfrac{3}{2}\, a\, g - b - \tfrac{\partial^2 F_1}{\partial t^2}\right) \rho\, z + F_2(t) \,.
\end{aligned}$$

Boundary value problem

Until now we did not mention how to treat boundary conditions in a Lie group analysis. For a given boundary value problem to be invariant under a group of transformations, not only the system of partial differential equations has to be invariant under the group of transformations, but also the boundary conditions have to admit the group of transformations [BK89].

For the inflow boundary condition (6.5)

$$v_z\big|_{r,z=0} = U(r) \tag{A.33}$$

to be invariant under the group of scaling transformations $\hat{v}_z = e^{\bar{v}_z}\, v_z$, $\hat{r} = e^{\bar{r}}\, r$, and $\hat{z} = e^{\bar{z}}\, z$,

$$\hat{v}_z\big|_{\hat{r},\hat{z}=0} = U(\hat{r})$$

must hold. The boundary itself has to be invariant under the group of transformations (i.e. $\hat{z} = z$ must hold at the boundary), and the boundary condition has to be invariant under the group of transformations (i.e. $\hat{v}_z = v_z$ must hold at the boundary) [BK89]. For a homogeneous boundary condition, such as the no-slip condition (6.6) at the wall and the outflow condition (6.7)

$$v_r\big|_{r=R,z} = 0 \,, \qquad v_z\big|_{r=R,z} = 0 \,, \qquad \text{and} \qquad \left.\frac{\partial v_z}{\partial z}\right|_{r,z=L} = 0 \,, \tag{A.34}$$

to be invariant under the same group of scaling transformations the corresponding dimensionless equations has to hold. In a homogeneous boundary condition the exponential factors of the scaling group of transformations cancel out by division, so that it is sufficient if the boundary is invariant.

The remaining non-homogeneous boundary conditions of the model equations of the condensation problem are the temperature boundary conditions (6.8) and (6.9) at the wall and at the interface

$$T\big|_{r=R,z} = T_w \,, \qquad T\big|_{r=h,z} = T_v \,. \tag{A.35}$$

Invariance of the outer boundaries

The outer boundaries are invariant under the group of scaling transformations defined by (A.31) and modified by (A.32) if the invariance condition (A.12) applied to the boundary is zero locally at the boundary. The invariance condition applied to a variable gives the corresponding infinitesimal, e.g. $Xr = \xi_r$, or $Xz = \xi_z$, so that the infinitesimal at the boundary must be zero at this boundary.

We have for the inlet, the outlet, the wall, and the interface

$$\begin{aligned}
\xi_z(z=0) &= F_1(t)\ , \\
\xi_z(z=L) &= \left(\frac{a}{2}+\frac{b}{g}\right) L + F_1(t)\ . \\
\xi_r(r=R) &= \frac{a}{2}\, R\ , \\
\xi_r(r=h) &= \frac{a}{2}\, h\ .
\end{aligned}$$

The inlet is invariant under (A.32) if $F_1(t) = 0$. The outlet is invariant under (A.32) if $F_1 = -\left(\frac{a}{2}+\frac{b}{g}\right) L$. If we locate the coordinate system at the outlet boundary, such that the outlet boundary is at $x = 0$ and the inlet boundary is at $x = -L$ the result is vice versa. That is, by taking the inlet and outlet boundary conditions into account we restrict the invariance of the model equations under a change of frame of reference to a simple translation, so that F_1 is determined by a constant $F_1 = j$.

If we locate the coordinate system at the interface, then the interface is invariant under (A.32). If we locate the coordinate system at the wall, then the wall is invariant under this group of transformations. Because the model equations are given in cylindrical coordinates the bulk equations are not invariant under a translation in radial direction, only one boundary is invariant under (A.32) and we decide that it is the wall boundary. We come back to the interface in section A.4.

Invariance of the outer boundary conditions

The inflow boundary condition (A.33) is invariant under the group of transformations if the invariance condition applied to the boundary condition is zero at $z = 0$, and if the inlet boundary is invariant under the group of transformation. That is the infinitesimal η_{v_z} must vanish at $z = 0$ under the conditions imposed by $\xi_z(z = 0) = 0$. Then the infinitesimal η_{v_z} given by (A.32)

$$\eta_{v_z}(z=0) = \left(\frac{b}{g} - \frac{a}{2}\right) v_z\ ,$$

must be zero, which is true for $b = \frac{a\,g}{2}$.

The temperature boundary condition (A.35) at the wall is invariant under the group of transformations if

$$\eta_T(r = R) = e + c\,T$$

vanishes. This gives $e = c = 0$.

By this the group of transformations defined by the infinitesimals (A.31) and modified by the infinitesimals (A.32) of the condensation problem given by the bulk equations (A.27) – (A.29) and the boundary conditions (A.33) – (A.35) becomes with $F_1 = j$, $e = c = 0$, and $b = \frac{a\,g}{2}$

$$\begin{aligned} \xi_r &= \tfrac{a}{2}\,r\,, & \eta_{v_r} &= -\tfrac{a}{2}\,v_r\,, \\ \xi_z &= a\,z + j\,, & \eta_{v_z} &= 0\,, \\ \xi_t &= d + a\,t\,, & \eta_p &= a\rho\,g\,z + F_2(t)\,, \qquad \eta_T = 0\,. \end{aligned} \tag{A.36}$$

We see that the three-parameter group of scaling transformations reduced to a one-parameter group. The infinitesimal generator of the scaling group is given by

$$X_a = \frac{r}{2}\,\frac{\partial}{\partial r} + z\,\frac{\partial}{\partial z} + t\,\frac{\partial}{\partial t} - \frac{v_r}{2}\,\frac{\partial}{\partial v_r} + \rho\,g\,z\,\frac{\partial}{\partial p}\,. \tag{A.37}$$

Special scaling transformations

To get the group of scaling transformations defined by (A.37) we compute $\hat{p}$ by a Lie series (A.10)

$$\hat{p} = p + a\,X_a\,p + \frac{a^2}{2!}\,X_a\,(X_a\,p) + ... = p + \rho\,g\,z\left(a + \frac{a^2}{2!} + ...\right) = p + (e^a - 1)\,\rho\,g\,z\,.$$

This gives the group of scaling transformations as

$$\begin{aligned} \hat{r} &= e^{\frac{a}{2}}\,r\,, & \hat{v_r} &= e^{-\frac{a}{2}}\,v_r\,, \\ \hat{z} &= e^a\,z\,, & \hat{v_z} &= v_z\,, \\ \hat{t} &= e^a\,t\,, & \hat{p} &= p + (e^a - 1)\,\rho\,g\,z \qquad \hat{T} = T\,. \end{aligned} \tag{A.38}$$

The scaling transformations defined by $\hat{p}$ are called special scaling transformations. The additional term in $\hat{p}$ cancels out the gravity term in the momentum equation.[2]

[2] For the sum of pressure and gravity terms in (A.28) we get
$-\frac{1}{\rho}\frac{\partial \hat{p}}{\partial \hat{z}} + g = -\frac{1}{\rho}\frac{\partial (p + (e^a - 1)\,\rho\,g\,z)}{e^a\,\partial z} + g = -\frac{1}{\rho\,e^a}\frac{\partial p}{\partial z} + \frac{g}{e^a}$.

A.4 Lie groups of the condensation problem including surface tension

Finally we analyze the condensation problem defined by the model equations (A.27) – (A.29) together with the jump conditions at the interface. Recall the jump conditions (compare (6.1) – (6.16) and (7.45) – (7.48)). That is,
the mass jump condition:

$$\dot{m} = \rho \left(-v_r + \frac{\partial h}{\partial t} + v_z \frac{\partial h}{\partial z} \right) , \tag{A.39}$$

the normal momentum jump condition:

$$p - p_g = -\frac{\sigma}{h} , \tag{A.40}$$

the condition of free shear-stress:

$$\frac{\partial v_z}{\partial r} = 0 , \tag{A.41}$$

and the energy jump condition:

$$\dot{m}\, \Delta h = \lambda \frac{\partial T}{\partial r} . \tag{A.42}$$

Here we consider only the model equations of the condensate, so that we can assume $p_g = 0$.

Lie groups of transformations of the condensation problem (without surface tension)

At first we investigate the problem defined by the bulk equations (A.27) – (A.29) plus the mass jump condition (A.39) and the energy jump condition (A.42), but ignore the momentum jump condition. A Lie group analysis with *MathLie* gives the infinitesimals

(A.43)

$$
\begin{aligned}
\xi_r &= \tfrac{a}{2}\, r \;, & \eta_{v_r} &= -\tfrac{a}{2}\, v_r \;, \\
\xi_z &= b\, z + F_1(t) \;, & \eta_{v_z} &= (b-a)\, v_z + \tfrac{\partial F_1}{\partial t} \;, \\
\xi_t &= d + a\, t \;, & \eta_p &= 2\,(b-a)\, p + (2\,a-b)\,\rho\, g\, z + F_2(t) - \rho\, z\, \tfrac{\partial^2 F_1}{\partial t^2} \;, \\
& & \eta_T &= e \;, \\
& & \eta_h &= \tfrac{a}{2}\, h + F_3(r) \;.
\end{aligned}
$$

Invariance of the interface

The interface is defined by $F = r - h(z,t) = 0$ (compare section 2.3 and section 6.2). That is, the interface is invariant under the group (A.43) of transformations, if $\hat{r} = \hat{h}$ at $r = h$. From this we get

$$
\begin{aligned}
\xi_r(r=h) &= \eta_h(r=h) \;, \\
\frac{a}{2}\, h &= \frac{a}{2}\, h + F_3(h) \;,
\end{aligned}
$$

which gives $F_3 = 0$.

Surface tension

Next we consider the normal momentum jump condition (A.40). For the normal momentum jump condition to be invariant under the group (A.43) of transformations the invariance condition (A.12) applied to (A.40)

$$
X\,(p + \frac{\sigma}{h}) = \eta_p + \eta_h\, \frac{\sigma}{h^2}
$$

has to be zero at the interface. This gives

$$
\eta_p + \eta_h\, \frac{\sigma}{h^2} = \left(a\, g - \frac{\partial^2 F_1}{\partial t^2} \right) \rho\, z + F_2(t) + \frac{a\,\sigma}{2\,h} = 0 \;,
$$

which is zero if $F_2 = 0$ and

$$
\frac{\partial^2 F_1}{\partial t^2} = a\, g + \frac{a\,\sigma}{2\,h\,\rho\, z} \;. \tag{A.44}
$$

The second quotient on the right hand side is small ($< 1e-4$ for R134a and $< 1e-3$ for water, see appendix B) and can be neglected. Then we get F_1 by integrating (A.44) twice as

$$
F_1(t) = \frac{a\, g\, t^2}{2} + f\, t + j \;, \tag{A.45}
$$

which defines a non-uniform motion in streamwise direction and is called a special Galilean boost [Ibr99].

Not both

Remember that the boundary conditions were invariant under the group of transformations for constant F_1. This means that we can find a group of transformations under which the condensation problem is invariant where either the momentum jump condition or the inlet and outlet boundary conditions are invariant under the same group of transformations, but not both. However, we can assume a homogeneous inflow condition (i.e. $\frac{\partial v_z}{\partial z} = 0$ at $z = 0$) to proceed. A Lie group analysis for the model equations (A.27) – (A.29) plus the jump conditions (A.39) – (A.42) including the momentum jump conditions gives the infinitesimals

$$\begin{aligned} \xi_r &= \tfrac{4a}{5g}\, r\ , & \eta_{v_r} &= -\tfrac{4a}{5g}\, v_r\ , \\ \xi_z &= \tfrac{6a}{5g}\, a\, z + a\, t^2 + (f\, t + j)\ , & \eta_{v_z} &= -\tfrac{2a}{5g}\, v_z + 2\, a\, t + f\ , \\ \xi_t &= d + \tfrac{8a}{5g}\, t\ , & \eta_p &= -\tfrac{4a}{5g}\, p\ , \\ & & \eta_T &= e\ , \\ & & \eta_h &= \tfrac{4a}{5g}\, h\ . \end{aligned}$$

Obviously here the coefficient b suppress the gravity terms in $\hat{p}$. The corresponding Lie groups of scaling transformations are with $s = 5\, a\, g$ given as follows

$$\begin{aligned} \hat{r} &= e^{4\,s}\, r\ , & \hat{v_r} &= e^{-4\,s}\, v_r\ , \\ \hat{z} &= e^{6\,s}\, z + \tfrac{g\,t^2}{2}\, (e^{16\,s} - e^{6\,s})\ , & \hat{v_z} &= e^{-2\,s}\, v_z + e^{-2\,s}\, (e^{2\,s} - 1)\, g\, t\ , \\ \hat{t} &= e^{8\,s}\, t\ , & \hat{p} &= e^{-4\,s}\, p\ , \\ & & \hat{T} &= T\ , \\ & & \hat{h} &= e^{4\,s}\, h\ . \end{aligned} \tag{A.46}$$

By this we conclude our chapter about Lie group analysis. We related Lie group analysis to generalized dimensional analysis and analyzed the groups of transformations admitted by the model equations of the condensation problem. We discussed the groups of scaling transformations admitted by our model equations, which are special scaling transformations. We demonstrated how to deal with the boundary conditions and the interface conditions in terms of a Lie group analysis. The next step is to construct an analytical solution of the condensation problem using the symmetry groups of the condensation problem by first transforming the PDEs into ODEs and than into algebraic equations, as outlined in section A.1. This is beyond the scope of this thesis. However, by this appendix we accomplish the necessary basis for this.

Appendix B

Material properties and dimensionless numbers

In this chapter the material properties, measured values, and dimensionless numbers of water and of R134a are given. We use the SI unit system.

B.1 Water

[Fie03] measured the following quantities (vapor has saturation temperature):

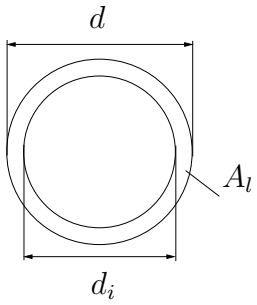
vapor temperature	T_v	=	318.98		K	(=45.83 °C)
temperature difference	$T_s - T_w$	$\approx$	5		K	T_w not measured
pressure	p	=	10	k	$\mathrm{N\,m^{-2}}$	(= 0.10 bar)
mass flux	$\dot{M}_l$	=	0.05	m	$\mathrm{kg\,s^{-1}}$	($= 3.0\ \mathrm{g\,min^{-1}}$)
	$\dot{M}_g$	=	0.67	m	$\mathrm{kg\,s^{-1}}$	($= 4.0\ \mathrm{g\,min^{-1}}$)
film thickness	δ_l	$\leq$	0.1	m	m	

The material properties of water are at T_v and p:

density	ρ_l	=	989.9		$\mathrm{kg\,m^{-3}}$	
	ρ_g	=	0.068		$\mathrm{kg\,m^{-3}}$	
dynamical viscosity	μ_l	=	0.60	m	$\mathrm{N\,s\,m^{-2}}$	(N=$\mathrm{kg\,m\,s^{-2}}$)
$\mu = \nu\,\rho$	μ_g	=	162.9	m	$\mathrm{N\,s\,m^{-2}}$	
kinematical viscosity	ν_l	=	0.606	μ	$\mathrm{m^2\,s^{-1}}$	
	ν_g	=	164.6	μ	$\mathrm{m^2\,s^{-1}}$	
thermal conductivity	λ_l	=	0.637		$\mathrm{W\,K^{-1}\,m^{-1}}$	(W=$\mathrm{J\,s^{-1}}$=$\mathrm{kg\,m^2\,s^{-3}}$)
	λ_g	=	19.98	m	$\mathrm{W\,K^{-1}\,m^{-1}}$	

specific heat capacity	c_l	=	4.179	k	$\mathrm{J\,kg^{-1}\,K^{-1}}$
latent heat of evaporation	Δh	=	2.393	k	$\mathrm{J\,kg^{-1}}$
surface tension	σ	=	68.78	m	$\mathrm{N\,m^{-1}}$

With $d_i = d - 2\,\delta_l = (7 - 0.2)\,\mathrm{mm}$ the characteristic length and the characteristic velocity are:



length	$D = \frac{A_l}{\pi\, d_i}$	=	0.1	m	m
velocity	$U = \frac{\dot{M}_l}{\rho\, A_l}$	=	0.023		$\mathrm{m\,s^{-1}}$

The dimensionless numbers used in chapter 7 are then:

Reynolds	$\mathrm{Re} = \frac{\rho\, U\, D}{\mu}$	=	3.90	$(= \frac{\dot{M}_l}{\mu\, \pi\, d_i})$	
Froude	$\mathrm{Fr} = \frac{U^2}{g\, D}$	=	0.545		
Prandtl	$\mathrm{Pr} = \frac{\mu\, c_P}{\lambda}$	=	3.94		
Peclet	$\mathrm{Pe} = \mathrm{Re}\,\mathrm{Pr}$	=	15.4		
Stefan	$\mathrm{St}(T_v) = \frac{c_P\, T_v}{\Delta h}$	=	55.6	$\mathrm{St}(\Delta T) = \frac{c_P\, \Delta T}{\Delta h}$	= 8.373
Weber	$\mathrm{We} = \frac{\rho\, D\, U^2}{\sigma}$	=	0.793 e-3		
Nußelt	$\mathrm{Nu} = \frac{\alpha\, D}{\lambda}$				

The quotient of film thickness and tube length is

$$\varepsilon = \frac{D}{L} = \frac{0.1\,\text{e-3}\,\mathrm{m}}{0.5\,\mathrm{m}} = 0.2\,\text{e-3}\ .$$

B.2 R134a (1,1,1,2-Tetrafluorethan)

[Fie03] measured the following quantities (vapor has saturation temperature):

vapor temperature	T_v	=	297.15		K	(=273.15 + 24 °C)
wall temperature	T_w	=	294.15		K	$(T_v - T_w = 5\,\mathrm{K})$
pressure	p	=	0.65	M	$\mathrm{kg\,m^{-2}}$	(=6.5 bar)
condensate mass flux	$\dot{M}_l$	=	0.367	m	$\mathrm{kg\,s^{-1}}$	$(=22\ \mathrm{g\,min^{-1}})$
film thickness (vertical tube)	δ_l	=	0.1 .. 0.2	m	m	

According to Tillner-Roth the material properties are at T_v and p:

density	ρ_l	=	1210		$\mathrm{kg\,m^{-3}}$
	ρ_g	=	31.39		$\mathrm{kg\,m^{-3}}$
dynamical viscosity	μ_l	=	198.70	μ	$\mathrm{kg\,s\,m^{-2}}$
	μ_g	=	12.10	μ	$\mathrm{kg\,s\,m^{-2}}$
kinematical viscosity	ν_l	=	0.164	μ	$\mathrm{m^2\,s^{-1}}$
	ν_g	=	0.385	μ	$\mathrm{m^2\,s^{-1}}$
thermal conductivity	λ_l	=	82.98	m	$\mathrm{W\,K^{-1}\,m^{-1}}$
	λ_g	=	14.35	m	$\mathrm{W\,K^{-1}\,m^{-1}}$
specific heat capacity	c_l	=	1.421	k	$\mathrm{J\,kg^{-1}\,K^{-1}}$
	c_g	=	1.025	k	$\mathrm{J\,kg^{-1}\,K^{-1}}$
thermal diffusity	a_l	=	48.26	n	$\mathrm{m^2\,s^{-1}}$
	a_g	=	445.9	n	$\mathrm{m^2\,s^{-1}}$
latent heat of evaporation	h_l	=	233.1	k	$\mathrm{J\,kg^{-1}}$
	h_g	=	411.8	k	$\mathrm{J\,kg^{-1}}$
	Δh	=	178.72	k	$\mathrm{J\,kg^{-1}}$
surface tension	σ	=	8.21	m	$\mathrm{N\,m^{-1}}$

With $d_i = d - 2\,\delta_l = (7 - 0.2)\,\mathrm{mm}$ the characteristic length and the characteristic velocity are:

length	$D = \frac{A_l}{D\pi\, d_i}$	=	0.1	m	m
velocity	$U = \frac{\dot{M}_l}{\rho\, A_l}$	=	0.170		$\mathrm{m\,s^{-1}}$

The dimensionless numbers are then:

Reynolds	$\mathrm{Re} = \frac{\rho\, U\, D}{\mu}$	=	85.25	$(= \frac{\dot{M}_l}{\mu\,\pi\, d_i})$			
Froude	$\mathrm{Fr} = \frac{U^2}{g\, D}$	=	19.9				
Prandtl	$\mathrm{Pr} = \frac{\mu\, C_p}{\lambda}$	=	3.40				
Peclet	$\mathrm{Pe} = \mathrm{Re}\,\mathrm{Pr}$	=	290.1				
Stefan	$\mathrm{St}(T_v) = \frac{c_P\, T_v}{\Delta h}$	=	2.36	$\mathrm{St}(\Delta T)$	=	$\frac{c_P\, \Delta T}{\Delta h}$	= 0.024
Weber	$\mathrm{We} = \frac{\rho\, D\, U^2}{2\,\sigma}$	=	0.289				

The quotient of film thickness and tube length is again

$$\varepsilon = \frac{D}{L} = \frac{0.1\,\text{e-3}\,\mathrm{m}}{0.5\,\mathrm{m}} = 0.2\,\text{e-3}\ .$$

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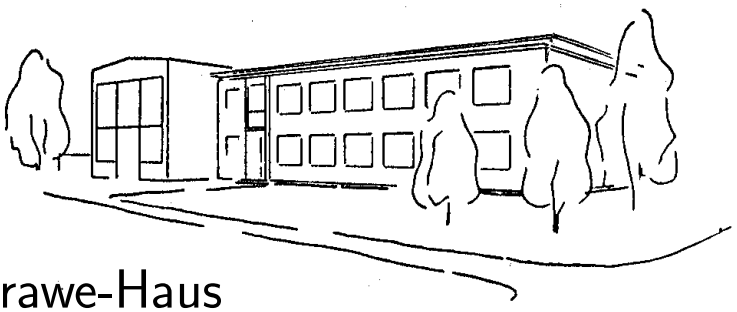

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