Lattice points in convex bodies: counting and approximating

vorgelegt von
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Von der Fakultät II - Mathematik und Naturwissenschaften
der Technischen Universität Berlin
zur Erlangung des akademisches Grades

Doktor der Naturwissenschaften
-Dr. rer. nat.-

genehmtigte Dissertation

Prüfungsausschuss:

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Tag der wissenschaftlichen Aussprache: 21. September 2018

Berlin 2018

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Zusammenfassung

Diese Dissertation befasst sich mit klassischen Gitterpunktproblemen aus der diskreten und konvexen Geometrie. Ganzzahlige Punkte in konvexen Körpern sind die zentralen Objekte unserer Untersuchungen.

Im zweiten Kapitel werden wir Abschätzungen für die Anzahl der Gitterpunkten in *zentrierten* konvexen Körpern herleiten. Das zugrunde liegende Problem ist dabei motiviert von klassischen Resultaten aus der *Geometrie der Zahlen.* Wir werden zeigen, dass die Annahme der zentralen Lage eines konvexen Körpers Abschätzungen liefert, welche asymptotisch vergleichbar sind zu denen, die für *symmetrische* konvexe Körper bekannt sind. Indem ein Ansatz entwickelt wird, der ganzzahlige Punkte zu deren *baryzentrischen Koordinaten* in Verbindung setzt, werden bestmögliche Schranken für zentrierte Simplizes bewiesen.

Das dritte Kapitel ist der Erweiterung der *Ehrhart Theorie* zu tensorwertiger Ehrhart Theorie gewidmet. Dies wird sich als natürliche Verallgemeinerung der klassischen Theorie herausstellen. Wir werden die Koeffizienten der *Ehrhart Tensoren* und der *h^r-Tensor Polynome* untersuchen und, ausgehend vom Kantengraphen eines Polygons, tensorwertige Varianten von *Picks Theorem* vorstellen. Ein bemerkenswertes Problem ist die Verallgemeinerung des Begriffes der Nichtnegativität für tensorwertige Koeffizienten in Hinblick auf *Stanleys Nichtnegativitätstheorem.* Eine entsprechende Variante von Stanleys Theorem für Tensoren von Rang 2 in der Ebene wird erarbeitet, für welche der Begriff der Nichtnegativität mit dem der positiven Semidefinitheit für Matrizen zusammenfällt. Es wird zudem vermutet, dass dieses Resultat auch in höheren Dimensionen gültig ist. Schließlich wird eine neue Charakterisierung *reflexiver* Polytope bezüglich der *palindromischen* Eigenschaft von h^r-Tensoren vorgestellt, wodurch ein Resultat von Hibi verallgemeinert wird.

Kapitel 4 behandelt diskrete *John Theoreme*, welche aussagen, dass die ganzzahligen Punkte in einem symmetrischen konvexen Körper durch *symmetrische verallgemeinerte arithmetische Progressionen* (im Folgenden kurz: *arithmetische Progression*) approximiert werden können. Dies bildet eine intuitive Diskretisierung des bekannten Satzes von John, welcher aussagt dass ein symmetrischer konvexer Körper in $\mathbb{R}^d$ zwischen einem Ellipsoid und dessen Vielfachen zum Faktor $\sqrt{d}$ eingeschlossen werden kann. Tao und Vu haben gezeigt, dass dies ebenso mittels

einer arithmetischen Progression, und einem Faktor abhängig von d, möglich ist. Zudem präsentierten Tao und Vu obere Schranken für die Anzahl ganzzahliger Punkte in einem symmetrischen konvexen Körper bezüglich einer in diesem enthaltenen arithmetischen Progression. In diesem Kapitel werden wir beide Resultate wesentlich verbessern. Dazu werden wir arithmetische Progression und deren Eigenschaften ausführlich untersuchen. Desweiteren werden mögliche Extremfälle diskutiert, welche ebenfalls untere Schranken liefern.

Kapitel 5 konzentriert sich auf *discrete slicing Ungleichungen*. Allgemein beinhalten diese das Abschätzen ganzzahliger Punkte in einem konvexen Körper K in Abhängigkeit zu der maximalen Gitterpunktanzahl in dem Schnitt von K mit einem linearen oder affinen Unterraum. Es wird erörtert, wie die bekannte *Brunns Ungleichung* diskretisiert werden kann. Wir werden ebenso eine *vollständig diskretisierte slicing Ungleichung* beweisen, in der lediglich das diskrete Volumen vorkommt. Desweiteren wird insbesondere die Frage nach Gitterpunktabschätzungen bezüglich eindimensionaler Unterräume vollständig beantwortet werden können. Daraus ergibt sich zudem eine diskrete Furstenberg-Tzkoni Ungleichung.

Die Kapitel 2, 4 und 5 sind Kollaborationen mit Martin Henk. Die Ergebisse aus Kapitel 2 erschienen in [20]. Kapitel 3 basiert auf der Zusammenarbeit mit Katharina Jochemko und Laura Silverstein, welche in [19] erschienen ist.

Lattice points in convex bodies: counting and approximating

Abstract

This thesis addresses classical lattice point problems in discrete and convex geometry. Integer points in convex bodies are the central objects of our studies.

In the second chapter, we will prove bounds on the number of lattice points in *centered convex bodies*. The underlying problems are motivated by classical results in *geometry of numbers*. We will show that the assumption of centricity yields asymptotically comparable bounds to those which are known and well-studied for *symmetric* convex bodies. Moreover, by developing an approach which links lattice points in a simplex to its *barycentric coordinates*, the best possible upper bound for the number of lattice points in centered simplices is deduced.

The third chapter is devoted to expanding *Ehrhart theory* to tensor-valued Ehrhart theory. This will become apparent to be a natural generalization of the classical theory. We will examine the coefficients of the *Ehrhart tensor* and h^r*-tensor polynomials* and deduce *Pick-type formulas* in the plane, which depend on the edge graph of the given polygon. It is an intriguing problem to extend the notion of nonnegativity to tensor-valued coefficients with regard to *Stanley's nonnegativity theorem*. Therefore, a variant of Stanley's result for rank 2 tensors in the plane will be shown; here, the notion of nonnegativity is aligned with positive semidefiniteness of matrices. It is conjectured that this also holds in higher dimensions as well. Finally, a new characterization of *reflexive polytopes* depending on the *palindromicity* of h^r-tensors is given, extending a result due to Hibi.

Chapter 4 deals with *discrete John-type theorems*, which say that the integer points in a symmetric convex body can be approximated by a *symmetric generalized arithmetic progression*, or *symmetric GAP* for short. This represents an intuitive discretization of the famous result due to John that a symmetric convex body in $\mathbb{R}^d$ can be enclosed between an ellipsoid and its multiple by a factor of $\sqrt{d}$. Tao and Vu showed that the same can indeed be done in the discrete case with a factor depending only on d. Moreover, Tao and Vu provided upper bounds for the number of lattice points in a given symmetric convex body in terms of the contained symmetric GAP. In this chapter, we will improve both results significantly. To this end, we study symmetric GAPs and their properties elaborately. Furthermore, we will discuss possible extreme cases leading to lower bounds.

Chapter 5 focuses on *discrete slicing inequalities*. In general, these involve bounding the discrete volume of a convex body K concerning the maximal number of lattice points in the intersection of K with linear or affine subspaces. It is discussed how the well-known *Brunn's inequality* can be discretized. We also prove a fully-discretized slicing inequality exclusively involving the discrete volume. In particular, we will establish lattice point bounds for symmetric convex bodies regarding their maximal one-dimensional slices. This yields a discrete, one-dimensional Furstenberg-Tzkoni inequality.

The Chapters 2, 4 and 5 are joint work with Martin Henk. The results from Chapter 2 appeared in [20]. Chapter 3 is based on joint work with Katharina Jochemko and Laura Silverstein which appeared in [19].

Contents

Acknowledgments

I would like to express my sincere gratitude to my advisor Martin Henk for the continuous support of my studies, for his encouragement, motivation, knowledge, guidance and numerous inspiring discussions. Besides my advisor, I would like to thank the rest of my thesis committee: María A. Hernández Cifre, Raman Sanyal and Wilhelm Stannat for refereeing this thesis.

I would like to thank Monika Ludwig for the incredible hospitality in Vienna, helpful conversations and support. Thank you to Raman Sanyal, who supported my research with mathematical discussions and hospitality.

Moreover, I am deeply grateful to my other co-authors Katharina Jochemko and Laura Silverstein for the joyous collaboration. Thank you for your generous support during my stay in Vienna.

Many thanks go to the members and former members of the "Discrete Geometry" workgroup at TU Berlin. Special thanks to Hannes Pollehn for being my office neighbor and friend. Many thanks to Romanos Malikiosis and Matthias Schymura for insightful discussions regarding my work.

Finally, I want to express my very profound gratitude to my mother, the rest of my family and Lisa Taute for providing me with unfailing support and continuous encouragement throughout the years of writing this thesis. Additional thanks to Lisa also, for proofreading this thesis. This accomplishment would not have been possible without all of you. Thank you.

Introduction

The present thesis is to be classified in *geometry of numbers*, *discrete geometry* and *additive combinatorics*. It discusses problems from the aforementioned fields, often located in the intersection between them. Studying integer points in *convex bodies* is the key focus and core concern throughout this dissertation. Whenever we seek to examine the problem of counting integer points in *polytopes* with respect to a dilation factor, *Ehrhart theory* provides a powerful framework of theory and tools. The field of *additive combinatorics* is concerned with *additive structures* from a broader viewpoint and combines a variety of disciplines, such as *number theory*, *convex geometry* and others. Although all of said branches of mathematics have been developed for many decades, they are still very active fields of research today; particularly due to its connection to other areas.

Basic concepts, definitions and results which provide the theoretical background for the studies of this thesis, are introduced in Chapter 1. Concepts and terms which are more specific, demand an extensive presentation or whose significance for this thesis is limited to certain sections, will be introduced in later chapters as they are needed. We will use some of them throughout this chapter without giving rigorous definitions, knowing that those can be found in later chapters.

Chapter 2 deals with problems which are motivated by classical results in the geometry of numbers. In his celebrated book “Geometrie der Zahlen”, which is the constitutional work in this field, Hermann Minkowski presented two fundamental results of the geometry of numbers. Firstly, he argued that the *number of integer points* $|K|_{\mathbb{Z}^d} := |K \cap \mathbb{Z}^d|$ in a *symmetric convex body* $K \subseteq \mathbb{R}^d$, which contains no other integer point in its interior except for the origin, admits the upper bound

$$|K|_{\mathbb{Z}^d} \leq 3^d. \tag{1}$$

Secondly, under the same assumptions on K, Minkowski showed that the *Lebesgue measure* or *volume* $\mathrm{vol}\,(K)$ of K satisfies

$$\mathrm{vol}\,(K) \leq 2^d. \tag{2}$$

In both cases the upper bound is exponential in the dimension d and it is easily verified that the *unit cube* $C_d = [-1,1]^d$ attains equality in (1) and (2). Originating from Minkowski's results are remarkable generalizations. Known as *Minkowski's second theorem* is the statement that

$$\frac{2^d}{d!} \leq \lambda_1(K) \dots \lambda_d(K) \mathrm{vol}\,(K) \leq 2^d, \tag{3}$$

where $\lambda_i(K)$ denotes the *i-th successive minima* of the symmetric convex body K. The second inequality (3) then is a direct generalizaton of (2). Moreover, Betke et al. proved that

$$|K|_{\mathbb{Z}^d} \leq \left(\left\lfloor \frac{2}{\lambda_1(K)} + 1 \right\rfloor \right)^d, \tag{4}$$

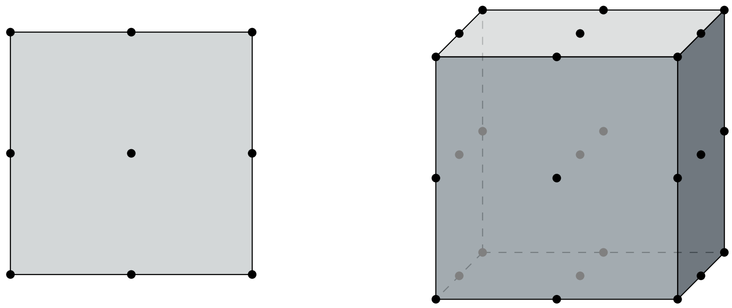

Figure 1: Cubes $C_2 = [-1,1]^2$ and $C_3 = [-1,1]^3$

from which follows (1) as a special case. Furthermore, (2) can be deduced from (4) by applying a limit argument. This exemplifies that in many cases a result in a *discrete formulation*, i.e., concerning integer points, is more general than a continuous variant dealing with the volume of a convex body, see also Section 2.2. We will study convex bodies which are not required to be symmetric, yet having their *centroid* at the origin. This describes a proper superclass of symmetric convex bodies called *centered* convex body. This setting has been investigated by Ehrhart [45] in 1964 who conjectured that for a d-dimensional centered convex body K with a single interior integer point, one has the bound

$$\mathrm{vol}\,(K) \leq \frac{(d+1)^d}{d!},$$

and verified his conjecture for simplices. However, not much progress has been made on this setting afterwards. Furthermore, we need to ask whether the number of integer points in a centered convex body in $\mathbb{R}^d$, which contains no further integral point in its interior except for the origin, can be bounded exponentially in the dimension d. There are convex bodies with a single interior integer point which contain double exponentially many lattice points, cf. Perles, Wills and Zaks [128]. Nevertheless, we will give an affirmative answer to this question in Theorem 2.5. Moreover, we prove the best possible upper bound for the number of lattice points in a centered simplex, see Theorem 2.11, thus providing a discrete version of Ehrhart's result. Chapter 2 is based on joint work with Martin Henk [20].

In Chapter 3, we study the notion of Ehrhart tensor polynomials, a natural generalization of the Ehrhart polynomial of a lattice polytope. Ehrhart discovered in 1962 that for a lattice polytope P and a positive integer n the quantity

$$\mathrm{L}^0(nP) := \sum_{\boldsymbol{x} \in nP \cap \mathbb{Z}^d} 1 = |nP|_{\mathbb{Z}^d} = \sum_{i=0}^{d} g_i(P) n^i, \tag{5}$$

is a polynomial of n with coefficients $g_i(P)$, $0 \leq i \leq d$, depending only on P. In our studies, we examine the natural generalization of (5)

$$\mathrm{L}^r(nP) := \sum_{\boldsymbol{x}\in nP} \boldsymbol{x}^r = \sum_{i=0}^{d+r} \mathrm{L}_i^r(P)n^i, \tag{6}$$

where $\boldsymbol{x}^r$, $r \in \mathbb{N}$, denotes the r-fold symmetric tensor product. (6) is called the *Ehrhart tensor polynomial*, which was introduced by Ludwig and Silverstein [85]. Therefore, the case $r = 0$ corresponds to the classical Ehrhart theory. We initiate a study of the involved coefficients and determine some of them, in particular the leading, second leading and constant coefficient. These identities collapse naturally to known results for $r = 0$. Moreover, we consider the Ehrhart tensor polynomial of a lattice polytope with the following change of basis:

$$\mathrm{L}^r(nP) = h_0^r(P)\binom{n+d+r}{d+r} + h_1^r(P)\binom{n+d+r-1}{d+r} + \cdots + h_{d+r}^r(P)\binom{n}{d+r}$$

For the case that $r = 0$, the respective coefficients $h_i^0(P)$, $0 \le i \le d$, are nonnegative integers, which is a well-known result known as *Stanley's nonnegativity theorem* [113]. We propose a natural extension of nonnegativity for tensors of positive rank, which corresponds to positive semidefiniteness of matrices for $r = 2$. Stanley [114] also showed that the coefficients are monotone with respect to inclusion, that is, $P \subseteq Q$ implies $h_i^0(P) \le h_i^0(Q)$ for two lattice polytopes P and Q. However, we will prove exemplarily that this is not true for tensors of rank 2. Consequently, this indicates that known approaches for proving nonnegativity of the coefficients are likely to be inapplicable to tensor coefficients, because they respect the monotonicity property in general. Nevertheless, we will manage to prove nonnegativity of the tensor-valued coefficients $h_i^r(P)$ in the plane. An extensive computer testing procedure suggests that this holds in higher dimensions as well, which we therefore conjecture. A classical theorem by Pick [99] states that the number of integer points in a lattice polygon can be expressed in terms of its area and number of integer points on its boundary. We introduce similar results, called *Pick-type formulas*, for tensors of rank 1 (vectors) and 2 (matrices), respectively. Since (6) is not invariant under translation by an integral vector for $r > 0$, these formulae involve the *edge graph* of the given lattice polygon. Furthermore, we generalize *Hibi's palindromic theorem* for *reflexive* polytopes to h^r-tensor polynomials.

Chapter 4 is devoted to the study of lattice points in convex bodies from the viewpoint of additive combinatorics. In classical convex geometry, it is a famous result due to John that a symmetric convex body can be approached by an ellipsoid. More precisely, John showed that for every symmetric convex body $K \subseteq \mathbb{R}^d$ there exists an ellipsoid $\mathcal{E} \subseteq \mathbb{R}^d$, such that

$$\mathcal{E} \subseteq K \subseteq \sqrt{d}\mathcal{E}.$$

John also proved that $\mathcal{E}$ can be chosen to be of maximal volume among all ellipsoids contained in K. The factor $\sqrt{d}$ cannot be improved, and is indeed needed if K is the cube $[-1,1]^d$. In 2008 Tao and Vu [121] presented a discrete analogue of John's theorem and showed that the lattice points in a convex body can be approached by a symmetric generalized arithmetic progression.

The latter is a set of the form

$$\mathrm{P}\left(A, \boldsymbol{u}\right) := \left\{ \sum_{i=1}^{d} z_i \boldsymbol{a}_i \;:\; -u_i \le z_i \le u_i,\; z_i \in \mathbb{Z},\; i \in \{1, \dots, d\} \right\},$$

where the vectors $\boldsymbol{a}_i$, $i \in \{1, \dots, d\}$, are linearly independent vectors of a lattice Λ, and each u_i is a nonnegative numbers. A central problem of Chapter 4 is, thus, to find a small constant τ_d depending only on d, such that the inclusions

$$\mathrm{P}\left(A, \boldsymbol{u}\right) \subseteq K \cap \mathbb{Z}^d \subseteq \mathrm{P}\left(A, \tau_d\, \boldsymbol{u}\right)$$

hold for every d-dimensional convex body K and some symmetric generalized arithmetic progression $\mathrm{P}\left(A, \boldsymbol{u}\right)$. Tao and Vu not only showed that such a constant τ_d does indeed exist, but also

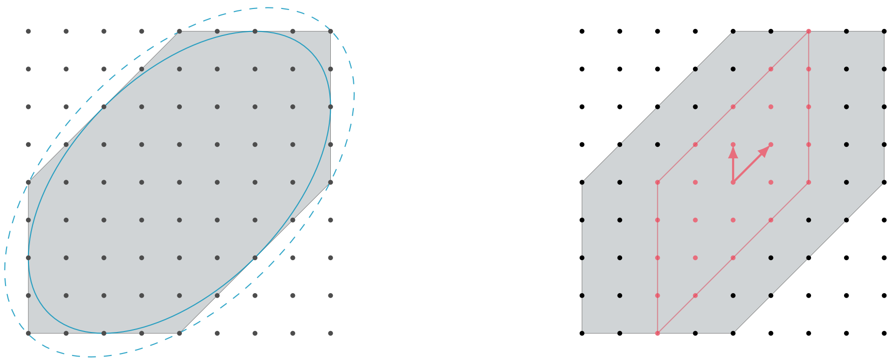

Figure 2: Approaching a symmetric convex body (shaded) by an ellipsoid (left) and by a symmetric generalized arithmetic progression (right)

that it admits the upper bound $\tau_d = O\left(d\right)^{3d/2}$. We will improve this to $\tau_d = d^{O(\ln d)}$. Another related problem is to estimate $|K|_{\mathbb{Z}^d}$ in terms of a symmetric generalized arithmetic progression $\mathrm{P}\left(A, \boldsymbol{u}\right)$ which is contained in $K \cap \mathbb{Z}^d$. That is, finding the smallest constant $\nu_d > 0$ such that for every symmetric convex body K there exists $\mathrm{P}\left(A, \boldsymbol{u}\right)$ such that

$$\mathrm{P}\left(A, \boldsymbol{u}\right) \subseteq K \cap \mathbb{Z}^d, \quad \text{and} \quad |K|_{\mathbb{Z}^d} \le \nu_d |\mathrm{P}\left(A, \boldsymbol{u}\right)|.$$

Tao and Vu showed that $\nu_d = O\left(d\right)^{7d/2}$, which we will improve to $\nu_d = O\left(d\right)^d$ in Theorem 4.15. In two dimensions, we will especially show that $\tau_2 \le 3$ and $\nu_2 \le 27$.

Chapter 5 examines *discrete slicing inequalities.* Classical, non-discrete slicing inequalities are an essential topic in convex geometry. They deal with describing a convex body, or its volume, in terms of lower dimensional slices. *Bourgain's slicing problem* [28, 29, 92], for instance, is a notable, unsolved, prestigious and central problem. It asks for the existence of an absolute constant C such that for every symmetric convex body $K \subseteq \mathbb{R}^d$, it holds

$$\mathrm{vol}_d\left(K\right)^{\frac{d-1}{d}} \le C \max_{H_{d-1}} \mathrm{vol}_{d-1}\left(K \cap H_{d-1}\right), \tag{7}$$

where the maximum in the right-hand side ranges over all $(d-1)$-dimensional linear subspaces $H_{d-1} \subseteq \mathbb{R}^d$. It is known that $C = O\left(d^{1/4}\right)$, see Klartag [76], and there is a vast amount of recent research about Bourgain's slicing problem; for a survey we refer the reader to Koldobsky [77]. It is obvious to investigate whether there are *discrete slicing inequalities*, for which the volume is replaced by the discrete volume. Recently, Koldobsky asked if a positive constant $c = c(d)$, which possibly depends on the dimension d, exists such that for every symmetric convex body $K \subseteq \mathbb{R}^d$ with $\dim K \cap \mathbb{Z}^d = d$ it holds

$$|K|_{\mathbb{Z}^d} \;\leq\; c \max_{H_{d-1}} |K \cap H_{d-1}|_{\mathbb{Z}^d} \operatorname{vol}(K)^{1/d}.$$

Alexander et al. [3] showed that one can choose $c = O(1)^d$. Moreover, they showed the more general result that

$$|K|_{\mathbb{Z}^d} \leq O(1)^d d^{d-n} \max_{H_n} |K \cap H_n|_{\mathbb{Z}^d} \operatorname{vol}(K)^{\frac{d-n}{d}},$$

where the maximum in the right-hand side is taken over all n-dimensional subspaces $H_n \subseteq \mathbb{R}^d$. We will prove a fully discrete slicing inequality in Section 5.3; that is, we present an upper bound of $|K|_{\mathbb{Z}^d}$ in terms of the lower dimensional sections of K. Furthermore, it is a well-known consequence of *Brunn's inequality* that for every symmetric convex body K, every $(d-1)$-dimensional subspace $H_{d-1} \subseteq \mathbb{R}^d$ and every $\boldsymbol{x} \in \mathbb{R}^d$, we have

$$\operatorname{vol}_{d-1}(K \cap (\boldsymbol{x} + H_{d-1})) \leq \operatorname{vol}_{d-1}(H_{d-1} \cap K). \tag{8}$$

Motivated by (8), we show a discrete analogue in Section 5.2 and argue that under the same assumptions it holds

$$|K \cap (\boldsymbol{x} + H_{d-1})|_{\mathbb{Z}^d} \leq 2^{d-1} |K \cap H_{d-1}|_{\mathbb{Z}^d}.$$

Another important finding of our investigations in Section 5.4 is the 1-dimensional slicing inequality

$$|K|_{\mathbb{Z}^d} < \left(\frac{4}{3}\right)^d |K \cap H_1|_{\mathbb{Z}^d}^d,$$

for every 1-dimensional subspace $H_1 \subseteq \mathbb{R}^d$. This inequality is best possible and we will take considerable effort in the construction of a class of polytopes for which this inequality cannot be improved. The chapter is closed by giving a brief discussion of convex bodies whose non-zero integer points do not contain a sum of the form $\boldsymbol{x} + \boldsymbol{y} = \boldsymbol{z}$; hence, they are called *symmetric sum-free*. We will prove a lattice point bound for symmetric sum-free convex bodies and see that a convex body K is symmetric sum-free if and only if no 2-dimensional section of K contains more than 5 integer points.

1
Basics

This thesis seeks to address topics located in discrete and convex geometry, geometry of numbers as well as additive and enumerative combinatorics. In this chapter, we will introduce the main objects and notations of our studies combined with basic properties and results. These can be considered to be common knowledge in the aforementioned mathematical branches. There is an extensive range of books in which these subjects are thoroughly discussed. For studies on discrete and convex geometry, we refer to the books by Gruber [55] and Barvinok [13], to the books by Cassels [36] and Gruber and Lekkerkerker [54] on the geometry of numbers, to the book of Beck and Robins [16] on the counting lattice points in polyhedra and to the book [123] by Tao and Vu on additive combinatorics. Moreover, the books by Ziegler [129] on polytopes and Stanley [115] on enumerative combinatorics are recommended.

Convex bodies

We consider the d-dimensional Euclidean space $\mathbb{R}^d$ equipped with the standard topology, which is induced by the Euclidean norm $\|\cdot\|$. Our notion of openness, closedness, compactness, boundary $\mathrm{bd}\,(A)$ or interior $\mathrm{int}\,(A)$ of a subset $A \subseteq \mathbb{R}^d$ result from this topology. By $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \boldsymbol{x}^T \boldsymbol{y}$ we denote the natural inner product on $\mathbb{R}^d$, and by $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_d$ the **standard unit basis** of $\mathbb{R}^d$. Moreover, we abbreviate $\mathbf{0} := (0, \ldots, 0)^T \in \mathbb{R}^d$, and $\mathbf{1} := (1, \ldots, 1)^T \in \mathbb{R}^d$ the **zero vector** and the **all-ones vector**, respectively. Whenever necessary, in order to avoid any confusion, we will write $\mathbf{0}_d$ or $\mathbf{1}_d$ to clarify that the meant vectors are elements in $\mathbb{R}^d$. Throughout this thesis, $\mathbb{N}$ will signify the nonnegative integers, and $[n] := \{1, \ldots, n\}$ will denote the first n positive integers. For $A \subseteq \mathbb{R}^d$, the linear subspace $A^\perp := \{\boldsymbol{y} \in \mathbb{R}^d \ : \ \langle \boldsymbol{x}, \boldsymbol{y} \rangle = 0 \ \forall \boldsymbol{x} \in A\}$ is the **orthogonal complement** of A, and we write $\boldsymbol{x}^\perp := \{\boldsymbol{x}\}^\perp$, $\boldsymbol{x} \in \mathbb{R}^d$, for short. By $\mathrm{lin}\,(A)$ and $\mathrm{aff}(A)$, we denote the **linear hull** and **affine hull** of A, respectively. We abbreviate $\mathrm{lin}\,(\boldsymbol{x}) := \mathrm{lin}\,(\{\boldsymbol{x}\})$ and $\mathrm{aff}(\boldsymbol{x}) := \mathrm{aff}(\{\boldsymbol{x}\})$ for a single vector $\boldsymbol{x}$. Moreover, for two vectors $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^d$ we write $\boldsymbol{x} \leq \boldsymbol{y}$ if $x_i \leq y_i$ for all $i \in [d]$. The expressions $\boldsymbol{x} \geq \boldsymbol{y}$, $\boldsymbol{x} < \boldsymbol{y}$ and $\boldsymbol{x} > \boldsymbol{y}$ are defined accordingly. For $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{Z}^d$ and $n \in \mathbb{N}$, we also write $\boldsymbol{x} \equiv \boldsymbol{y} \bmod n$ if $x_i \equiv y_i \bmod n$ for all $i \in [d]$, and say that in this case $\boldsymbol{x}$ and $\boldsymbol{y}$ are **equivalent** $\bmod\ n$.

A set $C \subseteq \mathbb{R}^d$ is **convex** if $\mu \boldsymbol{x} + (1 - \mu)\boldsymbol{y} \in C$ for all $\boldsymbol{x}, \boldsymbol{y} \in C$ and $\mu \in [0, 1]$. A sum of the form $\sum_{i=1}^{n} \mu_i \boldsymbol{v}_i$ with $\sum_{i=1}^{n} \mu_i = 1$ and $\mu_i \geq 0$, $i \in [n]$, is called a **convex combination** of $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$. For a non-empty set $A \subseteq \mathbb{R}^d$, we define its **convex hull** as the set of all convex combinations of elements in A, that is,

$$\operatorname{conv}(A) := \left\{ \sum_{i=1}^{n} \mu_i \boldsymbol{x}_i \ : \mu_i \geq 0, \boldsymbol{x}_i \in A, i \in [n] \,,\ \sum_{i=1}^{n} \mu_i = 1,\ n \geq 1 \right\}.$$

The convex hull of A is inclusion-wise the smallest convex set which contains A. It is a famous result due to Carathéodory that a point $\boldsymbol{x}$ in the convex hull of A can be written as a convex combination of at most $d+1$ points in A.

Theorem 1.1 (Carathéodory's Theorem, e.g., Theorem 3.1 in [55])**.** *If $A \subseteq \mathbb{R}^d$, then*

$$\operatorname{conv}(A) = \left\{ \sum_{i=1}^{d+1} \mu_i \boldsymbol{x}_i \ : \mu_i \geq 0, \boldsymbol{x}_i \in A, i \in [d+1] \,,\ \sum_{i=1}^{d+1} \mu_i = 1 \right\}.$$

Perhaps the most fundamental mathematical object this thesis deals with is the notion of a convex body. A **convex body** in $\mathbb{R}^d$ is a non-empty convex compact set $K \subseteq \mathbb{R}^d$, and $\mathcal{K}^d$ denotes the family of all such sets. The subfamily of $\mathcal{K}^d$ consisting of all symmetric convex bodies is denoted by $\mathcal{K}_o^d$. Here, we call a subset A of $\mathbb{R}^d$ **symmetric** if it satisfies that $\boldsymbol{x} \in A$ if and only if $-\boldsymbol{x} \in A$. For $K \in \mathcal{K}^d$ its **polar body** is defined by $K^* = \{\boldsymbol{y} \in \mathbb{R}^d \ : \ \langle \boldsymbol{x}, \boldsymbol{y} \rangle \leq 1 \ \forall \boldsymbol{x} \in K\}$. Besides, $\mathcal{G}(k,d)$ denotes the Grassmanian of k-dimensional linear subspaces of $\mathbb{R}^d$.

The **dimension** of $A \subseteq \mathbb{R}^d$, denoted by $\dim(A)$, is the dimension of its affine hull $\operatorname{aff}(A)$. The **relative interior** $\operatorname{relint}(A)$ is the interior of A with respect to its affine hull $\operatorname{aff} A$. For a Lebesgue measurable set $A \subseteq \mathbb{R}^d$, we will refer to its *d-dimensional Lebesgue measure* as the **volume** of A, or $\operatorname{vol}(A)$ for short. If $A \subseteq \mathbb{R}^d$ is contained in a n-dimensional affine subspace U of $\mathbb{R}^d$, we denote its n-dimensional Lebesgue measure by $\operatorname{vol}_n(A)$. Thus, $\operatorname{vol}(A) = \operatorname{vol}_d(A)$, and we will write $\operatorname{vol}_d(A)$ instead of $\operatorname{vol}(A)$ whenever it increases readability and avoids confusion.

If $A, B \subseteq \mathbb{R}^d$, we call $A + B := \{\boldsymbol{a} + \boldsymbol{b} \ : \ \boldsymbol{a} \in A,\ \boldsymbol{b} \in B\}$ the **Minkowski sum** of A and B. Moreover, for $\alpha \in \mathbb{R}$, $\boldsymbol{x} \in \mathbb{R}^d$ and $M \in \mathbb{R}^{m \times d}$ we will agree on the common notation $\alpha A := \{\alpha \boldsymbol{a} \ : \ \boldsymbol{a} \in A\}$, $A + \boldsymbol{x} := \{\boldsymbol{a} + \boldsymbol{x} \ : \ \boldsymbol{a} \in A\}$ and $MA := \{M\boldsymbol{a} \ : \ \boldsymbol{a} \in A\}$.

For the **unit ball** $B_d = \{\boldsymbol{x} \in \mathbb{R}^d \ : \ \|\boldsymbol{x}\| \leq 1\}$ in $\mathbb{R}^d$ it holds ${B_d}^* = B_d$, and we set

$$\kappa_d := \operatorname{vol}(B_d) = \frac{\pi^{d/2}}{\Gamma(d/2+2)}.$$

An affine image $\mathcal{E} := TB_d + \boldsymbol{t}$, $T \in \mathbb{R}^{d \times d}$, $\det T \neq 0$, $\boldsymbol{t} \in \mathbb{R}^d$, of the unit ball is called an **ellipsoid**. Clearly, $\operatorname{vol}(\mathcal{E}) = |\det T| \kappa_d$. The boundary of B_d is the $(d-1)$**-dimensional unit sphere** $\mathbb{S}^{d-1} := \operatorname{bd}(B_d) = \left\{\boldsymbol{x} \in \mathbb{R}^d \ : \ \|\boldsymbol{x}\| = 1\right\}$.

Lattices

A **lattice** in the d-dimensional Euclidean space is a d-dimensional, discrete, additive subgroup of $\mathbb{R}^d$. More precisely, a lattice is an additive subgroup of $\mathbb{R}^d$ whose linear span equals $\mathbb{R}^d$ and intersects any given bounded set in only finitely many points. The family of all lattices in $\mathbb{R}^d$ is denoted by $\mathcal{L}^d$. The set

$$\mathrm{GL}\,(d,\mathbb{Z}) = \left\{U \in \mathbb{Z}^{d\times d} \ : \ |\det U| = 1\right\}$$

is the **general linear group over the integers**, and a $U \in \mathrm{GL}\,(d,\mathbb{Z})$ is referred to as a **unimodular matrix**. Every lattice $\Lambda \in \mathcal{L}^d$ can be written in the form $\Lambda = B\mathbb{Z}^d$, where B is an invertible matrix in $\mathbb{R}^{d\times d}$. The column vectors of B are then called a **(lattice) basis** of Λ. For the sake of brevity, we will also refer to just B as a basis of Λ. Equivalently, the vectors $\boldsymbol{b}_1, \ldots, \boldsymbol{b}_d \in \Lambda$ are a lattice basis of $\Lambda \in \mathcal{L}^d$, if

$$\Lambda = \left\{\sum_{i=1}^{d} z_i \boldsymbol{b}_i \ : \ z_i \in \mathbb{Z},\ i \in [d]\right\}.$$

In general, a lattice Λ has more than one basis. In fact, we have $B\mathbb{Z}^d = B'\mathbb{Z}^d$ if and only if $B^{-1}B' \in \mathrm{GL}\,(d,\mathbb{Z})$. Therefore, $|\det B|$ is equal for every matrix B whose columns are a basis of Λ, and in this case we call $\det \Lambda := |\det B|$ the **determinant of** Λ. The lattice $\mathbb{Z}^d$ is called the **integer lattice** or **standard lattice**. We shall mean the integer points in $\mathbb{Z}^d$ if we speak of **lattice points** whenever no further lattice is specified. For $A \subseteq \mathbb{R}^d$ and $\Lambda \in \mathcal{L}^d$, we denote by $|A|_\Lambda := |A \cap \Lambda|$. The quantity $|A|_{\mathbb{Z}^d}$ in A is also known as the **discrete volume** or the **lattice point enumerator** of A. If the set A is *Jordan measurable*, we have that

$$\lim_{m\to\infty} \frac{|mA|_{\mathbb{Z}^d}}{m^d} = \lim_{m\to\infty} \frac{|A|_{\frac{1}{m}\mathbb{Z}^d}}{m^d} = \mathrm{vol}\,(A)\ , \tag{1.1}$$

see e.g. [55, Section 7.2]. We emphasize that convex bodies are Jordan measurable, cf. [55, Theorem 7.4].

The quantity $|A|_{\mathbb{Z}^d}$ is invariant under **unimodular transformations**. A unimodular transformation is a map $\varphi : \mathbb{R}^d \to \mathbb{R}^d$, $\varphi(\boldsymbol{x}) = U\boldsymbol{x} + \boldsymbol{t}$ with $U \in \mathrm{GL}\,(d,\mathbb{Z})$ and $\boldsymbol{t} \in \mathbb{Z}^d$. Thus, $|\varphi(A)|_{\mathbb{Z}^d} = |A|_{\mathbb{Z}^d}$ for every unimodular transformation φ. Two sets $A, B \subseteq \mathbb{R}^d$ are **unimodularly equivalent** if $A = \varphi(B)$ for some unimodular transformation φ. Note that two symmetric or centered convex bodies K and L are unimodularly equivalent if and only if $K = UL$ for some $U \in \mathrm{GL}\,(d,\mathbb{Z})$, in particular.

For a lattice $\Lambda \in \mathcal{L}^d$, we define its **dual lattice** as $\Lambda^\star = \{\boldsymbol{y} \in \mathbb{R}^d \ : \ \langle \boldsymbol{x}, \boldsymbol{y}\rangle \in \mathbb{Z}\ \forall \boldsymbol{x} \in \Lambda\}$. Indeed, we have that $\Lambda^\star \in \mathcal{L}^d$, and if $B = (\boldsymbol{b}_1, \ldots, \boldsymbol{b}_d)$ is a basis of Λ, its **dual basis**, a lattice basis of $\Lambda^\star$, is $B^\star = (\boldsymbol{b}_1{}^\star, \ldots, \boldsymbol{b}_d{}^\star)$ with $\langle \boldsymbol{b}_i, \boldsymbol{b}_j{}^\star\rangle$ being equal to 1 if $i = j$ and equal to 0 otherwise, cf. [55, Section 21.4]. Thus, if the columns of B are basis of Λ, the columns of the matrix B^{-T} are a basis of $\Lambda^\star$.

Polyhedra and polytopes

A **polyhedron** $P \subseteq \mathbb{R}^d$ is a set described by finitely many linear inequalities, that is,

$$P = \{\boldsymbol{x} \in \mathbb{R}^d \ : \ \langle \boldsymbol{x}, \boldsymbol{a}_i\rangle \le b_i,\ i \in [m]\},$$

where $\boldsymbol{a}_i \in \mathbb{R}^d$, $i \in [m]$. If $\langle \boldsymbol{x}, \boldsymbol{a} \rangle \leq b$ for all $\boldsymbol{x} \in P$ and some $\boldsymbol{a} \in \mathbb{R}^d$, $b \in \mathbb{R}$, then we call

$$F := P \cap \{\boldsymbol{x} \in \mathbb{R}^d \ : \ \langle \boldsymbol{x}, \boldsymbol{a} \rangle = b\}$$

a **face** of the polyhedron P. The **dimension** $\dim F$ of F is the dimension of its affine hull. Faces of dimension 0 are called **vertices** and the set of all vertices of P is denoted by $\operatorname{vert}(P)$. Moreover, the faces of P of dimension 1 and $\dim P - 1$ are called **edges** and **facets**, respectively.

A bounded polyhedron is called a **polytope**. $P \subseteq \mathbb{R}^d$ is a polytope if and only if P is the convex hull of finitely many points in $\mathbb{R}^d$, cf. [129, Theorem 2.15]. In particular, a polytope is the convex hull of its vertices. The family of all polytopes in $\mathbb{R}^d$ is denoted by $\mathcal{P}^d$. A polytope $P \in \mathcal{P}^d$ with $\operatorname{vert}(P) \subseteq \Lambda$, for a lattice $\Lambda \in \mathcal{L}^d$, is called a **$\Lambda$-lattice polytope** and if $\Lambda = \mathbb{Z}^d$ we will call P just a **lattice polytope**. The subfamily of $\mathcal{P}^d$ of all lattice polytopes in $\mathbb{R}^d$ is denoted by $\mathcal{P}^d_{\mathbb{Z}}$. Polytopes of dimension 2 are called **polygons**.

There is a collection of important polytopes, which we will recurrently refer to throughout this thesis and introduce in this paragraph. The convex hull of $d+1$ affinely independent points in $\mathbb{R}^d$ is called a **d-simplex**. By $\Delta_d := \operatorname{conv}(\boldsymbol{0}, \boldsymbol{e}_1, \ldots, \boldsymbol{e}_{d+1}) \subseteq \mathbb{R}^d$, we denote the **standard simplex** in $\mathbb{R}^d$. $\mathcal{S}^d$ is the class of all d-dimensional simplices. The volume of a d-simplex $S = \operatorname{conv}(\boldsymbol{v}_0, \ldots, \boldsymbol{v}_d)$ is $\operatorname{vol}(S) = d!^{-1}|\det(\boldsymbol{v}_1 - \boldsymbol{v}_0, \ldots, \boldsymbol{v}_d - \boldsymbol{v}_0)|$. Consequently, $\operatorname{vol}(\Delta_d) = 1/d!$. In general, a lattice simplex $S \in \mathcal{P}^d_{\mathbb{Z}}$ with $\operatorname{vol}(S) = 1/d!$ is called **unimodular**. Note that $1/d!$ is the minimal volume for a lattice simplex of dimension d. $C_d := [-1,1]^d \subseteq \mathbb{R}^d$ is called the **(unit) cube** in $\mathbb{R}^d$. If Π is an non-degenerate affine image of a cube, we call Π a **parallelepiped**. Equivalently, Π can be described as the Minkowski sum of d line segments, i.e.,

$$\Pi = [\boldsymbol{v}_1, \boldsymbol{w}_1] + \cdots + [\boldsymbol{v}_d, \boldsymbol{w}_d] = \left\{ \sum_{i=1}^{d} (\mu_i \boldsymbol{v}_i + (1-\mu_i)\boldsymbol{w}_i) \ : \ \mu_i \in [-1,1], i \in [d] \right\},$$

where we use the abbreviation $[\boldsymbol{v}_i, \boldsymbol{w}_i] := \operatorname{conv}(\boldsymbol{v}_i, \boldsymbol{w}_i)$ for the **(straight) line segment** joining $\boldsymbol{v}_i$ and $\boldsymbol{w}_i$, $i \in [d]$. Similarly, we define the **half-open line segment** $(\boldsymbol{x}, \boldsymbol{y}] := \{\mu \boldsymbol{x} + (1-\mu)\boldsymbol{y} \ : \ \mu \in (0,1]\}$ for $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^d$. A **cross-polytope** is a polytope of the form $\operatorname{conv}(\pm \boldsymbol{v}_1, \ldots, \pm \boldsymbol{v}_d)$, for linearly independent $\boldsymbol{v}_i \in \mathbb{R}^d$, $i \in [d]$. The cross-polytope $C_d{}^* = \operatorname{conv}(\pm \boldsymbol{e}_1, \ldots, \pm \boldsymbol{e}_d)$, henceforth called the **standard cross-polytope**, is the polar body of the cube C_d. Furthermore, $\operatorname{vol}(C_d) = 2^d$ and $\operatorname{vol}(C_d{}^*) = 2^d/d!$.

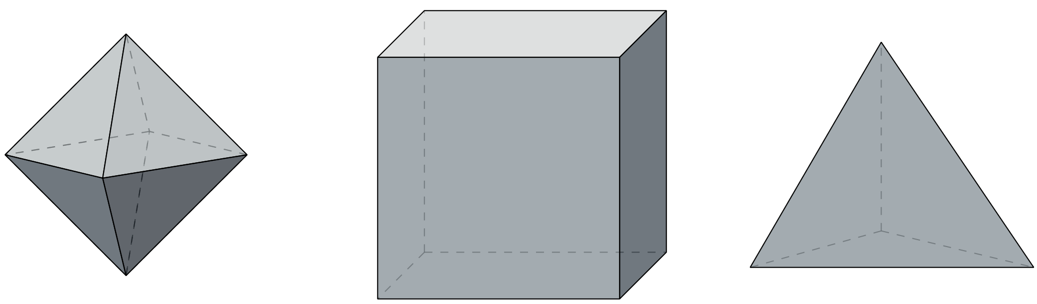

Figure 1.1: Cross-polytope, cube and simplex in $\mathbb{R}^3$

Triangulations

A **polytopal complex** $\mathcal{C}$ is a finite collection of polytopes in $\mathbb{R}^d$ such that

i) $\emptyset \in \mathcal{C}$,
ii) for $A, B \in \mathcal{C}$ we have $A \cap B \in \mathcal{C}$ and $A \cap B$ is a face of A and B,
iii) if $A \in \mathcal{C}$ and F is a face of A, then $F \in \mathcal{C}$.

$A \in \mathcal{C}$ is then called a **cell** of $\mathcal{C}$ and said to be **maximal** if the dimension of A equals the maximal dimension of all polytopes in $\mathcal{C}$. If $P \in \mathcal{P}^d$ is a polytope, then a **triangulation** of P is a polytopal subdivision $\mathcal{C}$ if every polytope in $\mathcal{C}$ is a simplex and $\bigcup_{A \in \mathcal{C}} A = P$.

The centroid

If $A \subseteq \mathbb{R}^d$ is of positive volume, we define its **centroid** as

$$\mathrm{c}(A) := \frac{1}{\mathrm{vol}\,(A)} \int_A \boldsymbol{x}\,\mathrm{d}\boldsymbol{x},$$

where $\mathrm{d}\boldsymbol{x}$ is the integration with respect to the d-dimensional Lebesgue measure. If $A \subseteq \mathbb{R}^d$ is convex in addition to having positive volume, its centroid lies in its interior. A convex body $K \in \mathcal{K}^d$ hereinafter called **centered** has its centroid at the origin, that is, it satisfies $\mathrm{c}(K) = \mathbf{0}$. The family of all centered convex bodies in $\mathbb{R}^d$ is denoted by $\mathcal{K}_c^d$ and the family of all centered d-simplices by $\mathcal{S}_c^d$. For the d-simplex $S = \mathrm{conv}\,(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{d+1})$ and $\boldsymbol{x} \in \mathbb{R}^d$ there are unique coefficients $\beta_S(\boldsymbol{x}) = (\beta_S(\boldsymbol{x})_1, \ldots, \beta_S(\boldsymbol{x})_{d+1})$, henceforth referred to as the **barycentric coordinates** of $\boldsymbol{x}$ with respect to S, such that

$$\boldsymbol{x} = \sum_{i=1}^{d+1} \beta_S(\boldsymbol{x})_i \boldsymbol{v}_i \quad \text{and} \quad \sum_{i=1}^{d+1} \beta_S(\boldsymbol{x})_i = 1.$$

Moreover, we have $\boldsymbol{x} \in S$ if and only if $\beta_S(\boldsymbol{x}) \in \mathbb{R}^{d+1}_{\geq 0}$, i.e., all entries are non-negative, as well as $\boldsymbol{x} \in \mathrm{int}\,(S)$ if and only if $\beta_S(\boldsymbol{x}) \in \mathbb{R}^{d+1}_{>0}$. For $\boldsymbol{x}, \boldsymbol{y}$ and $\lambda, \mu \in \mathbb{R}$ we have

$$\beta_S(\mu\,\boldsymbol{x} + \lambda\,\boldsymbol{y}) = \mu\beta_S(\boldsymbol{x}) + \lambda\beta_S(\boldsymbol{y}) + (1 - (\mu + \lambda))\,\beta_S(\mathbf{0}). \tag{1.2}$$

For any $S \in \mathcal{S}^d$ we have

$$\beta_S\left(\mathrm{c}(S)\right)_i = \frac{1}{d+1}, \quad i \in [d+1]\,,$$

by the following argument. Since $\mathrm{c}(f(S)) = f(\mathrm{c}(S))$ for every affine regular transformation $f : \mathbb{R}^d \to \mathbb{R}^d$, it suffices to consider the barycentric coordinates of the standard simplex Δ_d. By definition of the centroid and Fubini's Theorem, we find

$$\begin{aligned}\mathrm{c}(\Delta_d)_i &= \frac{1}{\mathrm{vol}\,(\Delta_d)} \int_{\Delta_d} x_i \,\mathrm{d}\boldsymbol{x} = d! \int_0^1 t \,\mathrm{vol}\,((1-t)\Delta_{d-1})\,\mathrm{d}t \\ &= \frac{d!}{(d-1)!} \int_0^1 t\,(1-t)^{d-1}\,\mathrm{d}t = \frac{1}{d+1},\end{aligned}$$

for the i-th coordinate $\mathrm{c}(\Delta_d)_i$ of the centroid of Δ_d, $i \in [d+1]$. Hence, $\beta_{\Delta_d}(\mathrm{c}(\Delta_d)) = \frac{1}{d+1}\mathbf{1}$. In other words, the centroid of a d-simplex $S = \mathrm{conv}\,(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{d+1})$ is the arithmetic mean of its vertices, i.e.,

$$\mathrm{c}(S) = \frac{1}{d+1} \sum_{i=1}^{d+1} \boldsymbol{v}_i.$$

This fact notwithstanding, for an arbitrary polytope the centroid is not the arithmetic mean of the vertices in general. Nevertheless, if $\mathcal{C}$ is a triangulation of the d-dimensional polytope P and $\mathcal{M}$ denotes the maximal cells of $\mathcal{C}$ of dimension d, we may write

$$\mathrm{c}(P) = \sum_{S \in \mathcal{M}} \frac{\mathrm{vol}\,(S)}{\mathrm{vol}\,(P)}\,\mathrm{c}(S), \tag{1.3}$$

by additivity of the Lebesgue measure. Note that (1.3) is a convex combination of the individual centroids $\mathrm{c}(S)$, $S \in \mathcal{M}$.

Successive minima

For a convex body $K \in \mathcal{K}^d$ with $\mathbf{0} \in \mathrm{int}\,(K)$ we define the ***i*-th successive minimum of *K*** as

$$\lambda_i\,(K, \Lambda) = \min\{\lambda > 0 \;:\; \lambda > 0,\; \dim\,(\lambda K \cap \Lambda) \geq i\}.$$

We write $\lambda_i(K) = \lambda_i(K, \mathbb{Z}^d)$ for short and say that a set of linearly independent vectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_d \in \Lambda$ is **associated with the successive minima of K with respect to Λ** if $\boldsymbol{v}_i \in \lambda_i(K, \Lambda)K \cap \Lambda$. By definition we have

$$0 < \lambda_1(K, \Lambda) \leq \cdots \leq \lambda_d(K, \Lambda).$$

Note that for $d \leq 2$ it is well-known that there are vectors associated with the successive minima of a symmetric convex body $K \in \mathcal{K}_o^d$ which form a lattice basis of the underlying lattice Λ. This can be concluded by Theorem 4 on page 20 in [54], since in two dimensions the convex hull of the two linearly independent vectors associated to first and second successive minima and the origin contains no further lattice points. However, this is not true for $d \geq 3$, e.g., consider the convex hull of $(1,0,0)^T$, $(0,2,0)^T$, $(1,1,2)^T$ and their negatives with $\Lambda = \mathbb{Z}^3$ [111, Chapter X, §5], cf. Figure 1.2. For an additional study on successive minima we refer to [36, 54].

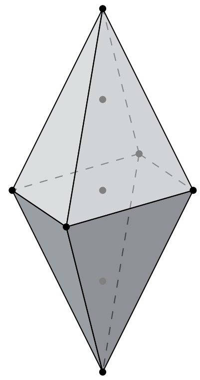

Figure 1.2: A lattice polytope whose vectors associated to its successive minima are not a lattice basis.

Generating functions

If $f : \mathbb{N} \to \mathbb{R}$, we can associate a **generating function** F to f by

$$F(t) = \sum_{n \geq 0} f(n) t^n.$$

Generating functions are often able to provide surprising insights on the sequence $f(n)$. For our investigations, the following result is indispensable.

Theorem 1.2 ([115], Corollary 4.3.1)**.** *Let $f : \mathbb{N} \to \mathbb{R}$, and let $d \in \mathbb{N}$. The following two conditions are equivalent.*

i)

$$\sum_{n \geq 0} f(n) t^n = \frac{p(t)}{(1-t)^{d+1}},$$

where $p(t)$ is a polynomial of t of degree at most d.

ii) $f(n)$ is a polynomial in n of degree at most d. Moreover, $f(n)$ has degree exactly d if and only if $p(1) \neq 0$.

Counting lattice points

Pick [99] showed that the number of lattice points in a lattice polygon can be precisely described regarding its area, i.e. its two dimensional volume, as well as the number of lattice points in its boundary.

Theorem 1.3 (Pick's Formula, 1899)**.** *Let $P \in \mathcal{P}^2_{\mathbb{Z}}$ be a lattice polygon, we have*

$$|P|_{\mathbb{Z}^2} = \mathrm{vol}\,(P) + \frac{1}{2} |\mathrm{bd}\,(P)\,|_{\mathbb{Z}^2} + 1.$$

In fact, the identity in Pick's Formula can be generalized to non-convex polygonal regions, whose vertices are lattice points, if the constant term 1 is replaced by the Euler characteristic of the region. Nonetheless, a version of Theorem 1.3 for lattice polytopes of dimension higher than 2 cannot hold true. A counterexample is given by the family of **Reeve tetrahedra**, where a Reeve tetrahedron is the convex hull of $(0,0,0)^T$, $(1,0,0)^T$, $(0,1,0)^T$ and $(1,1,n)^T$, $n \in \mathbb{N} \setminus \{0\}$, which contains 4 lattice points yet has an arbitrarily large volume depending on the variable n. Theorem 1.3 does, however, reveal a intriguing insight. If n is a positive integer, then the area of nP is n^2 the area of P by the homogeneity of the area, and $|\mathrm{bd}\,(nP)\,|_{\mathbb{Z}^2} = n|\mathrm{bd}\,(P)\,|_{\mathbb{Z}^2}$ by a simple observation. Consequently, the number of lattice points in the dilate nP admits the following polynomial identity.

$$|nP|_{\mathbb{Z}^2} = \mathrm{vol}\,(P)\; n^2 + \frac{1}{2}|\mathrm{bd}\,(P)\,|_{\mathbb{Z}^2}\; n + 1, \quad n \in \mathbb{N} \setminus \{0\}.$$

The conception of a higher dimensional generalization of this observation is due to Ehrhart. Namely, Ehrhart proved that the number of lattice points in the dilate of a lattice polytope by a positive, integral factor is a polynomial in the factor of dilation.

Theorem 1.4 (Ehrhart, 1962 [44])**.** *Let $P \in \mathcal{P}^d_{\mathbb{Z}}$ be a lattice polytope. Then*

$$|nP|_{\mathbb{Z}^d} = \sum_{i=0}^{d} g_i(P) n^i,$$

for every positive integer n, where $g_i(P)$, $i \in [d]$, are rational numbers depending only on P.

The polynomial $|nP|_{\mathbb{Z}^d}$ of n is called the **Ehrhart polynomial** of P. Observe that even though the Ehrhart polynomial represents a combinatorial quantity, its coefficients can be negative, e.g. for the Reeve tetrahedron with vertices $(0,0,0)^T$, $(1,0,0)^T$, $(0,1,0)^T$ and $(1,1,13)^T$, see [69]. There are considerable generalizations by Ehrhart of Theorem 1.4 [41, 46].

In view of Theorem 1.2, we may examine the generating function of the Ehrhart polynomial of a d-dimensional $P \in \mathcal{P}^d_{\mathbb{Z}}$, and define

$$\mathrm{Ehr}_P(t) := \sum_{n \geq 0} |nP|_{\mathbb{Z}^d} t^n = \frac{\sum_{i=0}^{d} h_i(P) t^i}{(1-t)^{d+1}}. \tag{1.4}$$

$\mathrm{Ehr}_P(t)$ is referred to as the **Ehrhart series** of P and the polynomial $\sum_{i=0}^{d} h_i(P) t^i$ is the **h^*-polynomial** of P. The tuple $(h_0(P), \ldots, h_d(P))$ is called the **h^*-vector** of P. One can obtain the Ehrhart polynomial of a d-dimensional lattice polytope $P \subseteq \mathbb{R}^d$ from the h^*-polynomial of P by performing a change of basis. More precisely, it holds for every positive integer n, see e.g. [16, Lemma 3.14], that

$$|nP|_{\mathbb{Z}^d} = h_0^* \binom{n+d}{d} + h_1^* \binom{n+d-1}{d} + \cdots + h_{d-1}^* \binom{n+1}{d} + h_d^* \binom{n}{d}. \tag{1.5}$$

In particular, (1.5) implies immediately that the constant term $g_0(P)$ of the polynomial $|nP|_{\mathbb{Z}^d}$ is 1. More generally, some coefficients of the Ehrhart and the h^*-polynomials are known, e.g.,

$$
\begin{aligned}
g_0(P) &= 1, & h_0^*(P) &= 1,\\
g_d(P) &= \mathrm{vol}\,(P)\,, & h_1^*(P) &= |P|_{\mathbb{Z}^d} - (d+1),\\
g_{d-1}(P) &= \frac{1}{2}\sum_{F \text{ is a facet of } P} \frac{\mathrm{vol}_{d-1}\,(F)}{|\det(\mathrm{aff}(F)\cap\mathbb{Z}^d)|}, & h_d^*(P) &= |\mathrm{int}\,(P)\,|_{\mathbb{Z}^d},
\end{aligned}
$$

cf. [16]. Here,

$$
\frac{\mathrm{vol}_{d-1}\,(F)}{|\det(\mathrm{aff}(F)\cap\mathbb{Z}^d)|} = \lim_{m\to\infty}\frac{|mF|_{\mathbb{Z}^d}}{m^{d-1}},
$$

denotes the volume of the $d-1$-dimensional facet F *relative to* $\mathrm{aff}(F)\cap\mathbb{Z}^d$, if $\mathrm{aff}(F)\cap\mathbb{Z}^d$ is considered as a lattice in the subspace $\mathrm{aff}(F)$. (1.4) also yields the following identity:

$$
h_0^*(P) + h_1^*(P) + \cdots + h_d^*(P) = d!\ \mathrm{vol}\,(P)\ .
$$

The h^*-vector of a lattice polytope has several interesting properties. Firstly, it was shown by Stanley that its entries are nonnegative integers.

Theorem 1.5 (Stanley's Nonnegativity Theorem [113])**.** *The coefficients of the h^*-polynomial of a lattice polytope $P \in \mathcal{P}^d_{\mathbb{Z}}$ are nonnegative integers. That is, $h_i^*(P) \in \mathbb{N}$ for $0 \le i \le d$.*

Secondly, h^*-vectors have a monotocity property regarding inclusions.

Theorem 1.6 (Stanley's Monotonicity Theorem [114])**.** *If $P \subseteq Q$ for two lattice polytopes $P, Q \in \mathcal{P}^d_{\mathbb{Z}}$, then $h_i^*(P) \le h_i^*(Q)$ for $0 \le i \le d$.*

Moreover, the following reciprocity identity was discovered by Ehrhart, in the particular case $d = 3$, and more generally by Macdonald.

Theorem 1.7 (Ehrhart-Macdonald reciprocity [46, 86])**.** *If $P \in \mathcal{P}^d_{\mathbb{Z}}$ and n is a nonnegative integer, it holds that*

$$
\sum_{i=0}^{d} g_i(P)(-n)^i = (-1)^d |\mathrm{int}\,(nP)\,|_{\mathbb{Z}}. \tag{1.6}
$$

Note that the left-hand side of (1.6) corresponds to plugging in $-n$ into the Ehrhart polynomial of P.

Tensors

By $\mathbb{T}^r$ we denote the real vector space of symmetric tensors on $\mathbb{R}^d$ of rank r. $\mathbb{T}^r$ is then canonically isomorphic to the space of multi-linear functionals from $(\mathbb{R}^d)^r$ to $\mathbb{R}$ that are invariant with respect to permutations of the arguments. More precisely, a tensor $T \in \mathbb{T}^r$ is a multilinear map $T : \left(\mathbb{R}^d\right)^r \to \mathbb{R}$ such that $T(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_r) = T(\boldsymbol{v}_{\pi(1)}, \ldots, \boldsymbol{v}_{\pi(r)})$ for every permutation π on $[r] = \{1, \ldots, r\}$ and $\boldsymbol{v}_i \in \mathbb{R}^d$, $i \in [r]$. For $\boldsymbol{x} \in \mathbb{R}^d$, we write $\boldsymbol{x}\boldsymbol{x}$ instead of the more widely spread $\boldsymbol{x} \otimes \boldsymbol{x}$ for the sake of brevity, and more generally we will write $\boldsymbol{x}^r = \boldsymbol{x} \otimes \cdots \otimes \boldsymbol{x}$ and define $\boldsymbol{x}^0 := 1$. The r-fold tensor product $\boldsymbol{x}^r \in \mathbb{T}^r$ of $\boldsymbol{x} \in \mathbb{R}^d$ then describes the multilinear map $\boldsymbol{x}^r : \left(\mathbb{R}^d\right)^r \to \mathbb{R}$, $\boldsymbol{x}^r(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_r) = \prod_{i=1}^r \langle \boldsymbol{x}, \boldsymbol{v}_i\rangle$. In particular, $T \in \mathbb{T}^r$ is completely determined by the values $T(\boldsymbol{e}_{i_1}, \ldots, \boldsymbol{e}_{i_r})$, where the indices i_k, $k \in [r]$, range over $\{1, \ldots, d\}$. The vector spaces $\mathbb{T}^0$, $\mathbb{T}^1$ and $\mathbb{T}^2$ can naturally be identified by the dual vector spaces $\mathbb{R}^* \cong \mathbb{R}$, $\left(\mathbb{R}^d\right)^* \cong \mathbb{R}^d$ and $\left(\mathbb{R}^{d\times d}\right)^* \cong \mathbb{R}^{d\times d}$, respectively. Depending on what is more convenient and lucid, we will switch freely between the representation of a tensor of rank ≤ 2 as a multilinear map and the representation as a number, vector or matrix.

In Chapter 3, we will consider polynomials having tensor valued coefficients. However, we are solely investigating the additive structure of the space of tensors $\mathbb{T}^r$. Formally, if $T_d, \ldots, T_0 \in \mathbb{T}^r$ are tensors of rank r, we will abuse the notation slightly and use the polynomial expression $T_d x^d + \cdots + T_1 x + T_0$ for the formally defined tensor

$$T(\boldsymbol{e}_1, \ldots, \boldsymbol{e}_r) := \sum_{i=0}^{d} T_i(\boldsymbol{e}_1, \ldots, \boldsymbol{e}_r) x^i,$$

where x is considered to be an arbitrary real number. For example, we will write

$$\begin{pmatrix} 2 \\ 1 \end{pmatrix} x^2 + \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 2x^2 + 3 \\ x^2 + x + 1 \end{pmatrix}.$$

Since we then can apply Theorem 1.2 component-wise, it is clear that Theorem 1.2 also holds if we replace $\mathbb{R}$ with the more general $\mathbb{T}^r$.

Eulerian numbers

The **Eulerian number** $\mathrm{A}\,(d, k)$ is defined by

$$\mathrm{A}\,(d, k) = \sum_{i=0}^{k} (-1)^i \binom{d+1}{i} (k-i)^d.$$

Equivalently, one can obtain the Eulerian numbers by the expansion of the following generating function:

$$\sum_{n\geq 0} n^d t^n = \frac{\sum_{k=0}^{d} \mathrm{A}\,(d, k) t^k}{(1-t)^{d+1}}, \tag{1.7}$$

cf. Theorem 1.2. In particular, it holds $\sum_{k=0}^{d} \mathrm{A}\,(d, k) = d!$. The numerator in the right-hand side of (1.7) is the **d-Eulerian polynomial** $A_d(t) := \sum_{k=0}^{d} \mathrm{A}\,(d, k) t^k$. For further studies on Eulerian numbers and proofs of the stated identities, we refer the reader to [115] or [16, Section 2.2].

Landau asymptotic notation

Let f and g be two real-valued functions. We write $f(x) = O\left(g(x)\right)$ if g is nonnegative and there exists a constant $C > 0$ such that $f(x) \leq C g(x)$ for all sufficiently large $x > 0$. We write $f(x) = \Omega\left(g(x)\right)$ if g is non-negative and there exists a constant $C > 0$ such that $f(x) \geq C g(x)$ for all sufficiently large $x > 0$.

2

Lattice points in centered convex bodies

2.1 Introduction

Exploring the discrete volume of convex bodies with respect to further properties or quantities such as symmetry, continuous volume or successive minima has been a fundamental interest of geometry of numbers since its very beginning. Hermann Minkowski showed in [94, p. 79] that a symmetric convex body $K \subseteq \mathbb{R}^d$ of which the origin is the sole lattice point in its interior admits the following bound for the discrete volume

$$|K|_{\mathbb{Z}^d} \leq 3^d. \tag{2.1}$$

Moreover, the right-hand side can be improved to $2^{d+1} - 1$ if, in addition, K is strictly convex. The bound in (2.1) is known as *Minkowski's* 3^d*-Theorem* and will be revisited in Section 5.5. The equality case in (2.1) has been characterized by Groemer [53]. Furthermore, it was shown by Betke, Henk and Wills [22] that Minkowski's bounds can be extended to symmetric convex bodies with more than one interior lattice point through the notion of the first successive minimum $\lambda_1(K)$. The mentioned authors proved the following generalizaton of (2.1).

Theorem 2.1 (Betke, Henk, Wills)**.** *If* $K \in \mathcal{K}_o^d$*, then*

$$|K|_{\mathbb{Z}^d} \leq \left\lfloor \frac{2}{\lambda_1(K)} + 1 \right\rfloor^d. \tag{2.2}$$

In order to deduce bounds on the number of lattice points of further and possibly larger classes of convex bodies, the underlying assumptions have to be chosen carefully. For instance, even for

The results from this chapter are joint work with Martin Henk and appeared in [20], cf. https://doi.org/10.1137/15M1031369. First Published in "Lattice Point Inequalities for Centered Convex Bodies" in SIAM Journal on Discrete Mathematics, Volume 30, Issue 2, 2018, published by the Society for Industrial and Applied Mathematics (SIAM).

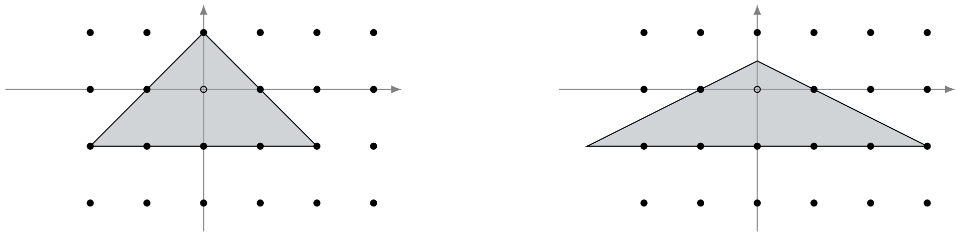

Figure 2.1: Triangles and D_2 and D_3

simplices with one interior lattice point there is no general upper bound as the family of triangles D_m with vertices $(-m,-1)$, $(m,-1)$ and $(0,1/(m-1))$, $m>1$, shows, since $|K|_{\mathbb{Z}^2}=2m+4$ for every integral $m>2$, cf. Figure 2.1.

If we are dealing, however, only with lattice polytopes, there are bounds on the number of lattice points in terms of the (nonzero) number of interior lattice points, see., e.g., Hensley [65], Lagarias & Ziegler [80], Pikhurko [100, 101], Averkov [6]. These results are actually bounds on the volume and the following popular result by Blichfeldt [25] is applied to gain bounds on the discrete volume.

Theorem 2.2 (Blichfeldt)**.** *If* $K\in\mathcal{K}_o^d$ *with* $\dim(K\cap\mathbb{Z}^d)=d$*, then*

$$|K|_{\mathbb{Z}^d}\leq d!\mathrm{vol}\,(K)\,+d.$$

Note that the studies on derivations of Blichfeldt's Theorem have received considerable attention in recent years, e.g. [23, 62, 64, 66].

Even in the case of lattice simplices having only one interior lattice point, an upper bound on its discrete volume has to be double exponential in d as shown by Perles, Wills and Zaks [128]. They presented a lattice simplex $T_d\subseteq\mathbb{R}^d$ with a single interior lattice point and

$$|T_d|_{\mathbb{Z}^d}\geq\frac{2}{6(d-2)!}2^{2^{d-a}},$$

where $a=0.5856\ldots$ is a constant. Averkov, Krümpelmann and Nill [7] proved that T_d has maximum volume among all lattice simplices having one interior lattice point. It remains an open question if a similar result is true considering the number of lattice points instead of the volume as conjectured by Hensley [65].

In Section 2.2, we will present that inequalities of the same type as (2.1) and (2.2) do exist for centered convex body as well. Following a result of Milman and Pajor it is shown that the number of lattice points in a centered convex body with one interior lattice point can indeed be bounded exponentially in the dimension d. This gives an affirmative answer to the principal question as to whether the conditions of symmetry and centricity yield asymptotically similar upper bounds for the discrete volume.

Section 2.3 examines lattice point bounds for simplices. We will follow a new approach involving the barycentric coordinates of a simplex. The bounds obtained are best possible.

In Section 2.4, we discuss the particular case of centered convex bodies in two dimensions. Using classical results from discrete geometry, we show that a centered convex body in the plane with one interior lattice point cannot contain more than 10 integer points. This bound cannot be improved and all instances for which equality is attained are classified.

2.2 Centered convex bodies

Considering a convex body K, it appears plausible that its centroid $\mathrm{c}(K)$ is located deep inside K, i.e., not too close to the boundary of K. In conclusion, one would expect that the volume of the intersection of K with its reflection at $\mathrm{c}(K)$ cannot be too small with respect to the volume of K. Indeed Milman and Pajor proved the following result.

Theorem 2.3 ([93, Corollary 3])**.** *Let $K \in \mathcal{K}_c^d$ with $\mathrm{c}(K) = \mathbf{0}$. Then*

$$\mathrm{vol}\,(K \cap -K) \geq 2^{-d}\mathrm{vol}\,(K)\ . \tag{2.3}$$

Remark 2.4. *It is not clear whether 2^{-d} is the best factor which one can achieve in (2.3). To the best of the author's knowledge, this is an open question to date. Moreover, it seems to be widely believed that a centered d-simplex $S \subseteq \mathbb{R}^d$ attains the minimum for the functional $\rho(S) := \mathrm{vol}\,(S \cap -S)\ /\mathrm{vol}\,(S)$ for which it is exponentially greater than 2^{-d}, namely roughly of order $(2/\mathrm{e})^d$. For $d = 2$, this problem has indeed been settled by Stewart, c.f.* [120]*. Note, that ρ is invariant under linear transformation, and hence, $\rho(S) = \rho(S')$ for every pair of centered d-simplices S and S'. Prior to the work of Milman and Pajor, it was known due to Stein* [119] *that for every convex body $K \subseteq \mathbb{R}^d$ there exists at least a translate of K satisfying the inequality (2.3). A bound of $\rho(K) \geq 2/(1+d^d)$ for a centered K was shown by Levi* [83]*. For recent results regarding this problem see* [124]*.*

Theorem 2.3 is the central component for the proof of the following theorem, which is the main result of this section.

Theorem 2.5. *If $K \in \mathcal{K}_c^d$, then*

$$|K|_{\mathbb{Z}^d} < 2^d \left(\frac{2}{\lambda_1(K)} + 1 \right)^d . \tag{2.4}$$

In particular, if $\lambda_1(K) \geq 1$, i.e., $\mathrm{int}\,(K) \cap \mathbb{Z}^d = \{\mathbf{0}\}$,

$$|K|_{\mathbb{Z}^d} < 6^d.$$

Proof of Theorem 2.5. Let $K \in \mathcal{K}_c^d$ and for short we set $\lambda_1 = \lambda_1(K)$. First we observe that $\frac{\lambda_1}{2}(K \cap -K)$ is a packing set with respect to the integer lattice, i.e., we have

$$\mathrm{int}\left(\boldsymbol{u} + \frac{\lambda_1}{2}(K \cap -K)\right) \cap \mathrm{int}\left(\boldsymbol{v} + \frac{\lambda_1}{2}(K \cap -K)\right) = \emptyset,$$

for $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{Z}^d$, $\boldsymbol{u} \neq \boldsymbol{v}$; otherwise, by the symmetry and convexity of $K \cap -K$ we get

$$\boldsymbol{u} - \boldsymbol{v} \in \frac{\lambda_1}{2}\,\mathrm{int}\,(K \cap -K)\, + \frac{\lambda_1}{2}\,\mathrm{int}\,(K \cap -K)\, = \lambda_1\ \mathrm{int}\,(K \cap -K)\, \subseteq \lambda_1\,\mathrm{int}\,(K)\ ,$$

contradicting the minimality of λ_1. We also have, cf. Figure 2.2,

$$(\mathbb{Z}^d \cap K) + \tfrac{\lambda_1}{2}(K \cap -K) \subseteq K + \tfrac{\lambda_1}{2}(K \cap -K) \subseteq \left(1 + \frac{\lambda_1}{2}\right) K. \tag{2.5}$$

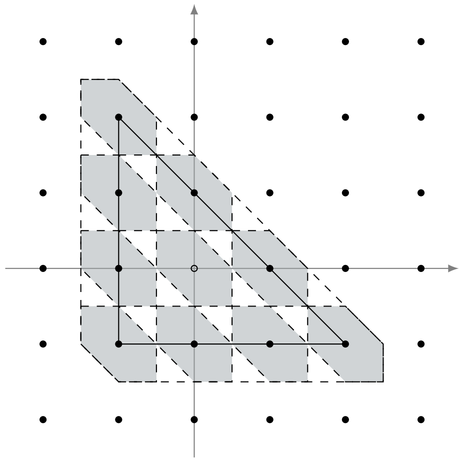

Figure 2.2: K (triangle), packing $(K \cap \mathbb{Z}^d) + (\frac{\lambda_1}{2}(K \cap -K))$ (shaded) and $(1 + \frac{\lambda_1}{2})K$ (dashed)

Hence, in view of Theorem 2.3 and since $\left(K \cap \mathbb{Z}^d\right) + \frac{\lambda_1}{2}(K \cap -K)$ is a packing, we find that

$$|K|_{\mathbb{Z}^d} \leq \frac{\operatorname{vol}\left(\left(1 + \frac{\lambda_1}{2}\right) K\right)}{\operatorname{vol}\left(\frac{\lambda_1}{2}(K \cap -K)\right)} = \left(\frac{2}{\lambda_1} + 1\right)^d \frac{\operatorname{vol}(K)}{\operatorname{vol}(K \cap -K)} \leq 2^d \left(\frac{2}{\lambda_1} + 1\right)^d. \tag{2.6}$$

If K is symmetric, the last inequality is strict, and if K is not symmetric with respect to $\mathbf{0}$, the set $K + \frac{\lambda_1}{2}(K \cap -K)$ does not cover $\left(1 + \frac{\lambda_1}{2}\right) K$, cf. (2.5), which by compactness implies the strict inequality of the theorem. □

Remark 2.6. *In the case that the convex body K is symmetric, the proof above gives essentially the result (2.2) since then $K \cap -K = K$ in (2.6). In particular, it also recovers Minkowski's result (2.1) which he proved by a simple residue class argument. Actually, by such an argument Minkowski also showed that in the case of strictly symmetric convex bodies K with $\lambda_1(K) = 1$ the stronger bound holds true*

$$|K|_{\mathbb{Z}^d} \leq 2^{d+1} - 1.$$

In fact, this bound is even true without symmetry solely under the assumption $\operatorname{int}(K) \cap \mathbb{Z}^d = \{\mathbf{0}\}$.

The bound in Theorem 2.5 is quite likely asymptotically not sharp and we believe that the extreme case is attained by d-simplices. We introduce the following simplex hereinafter called the d-dimensional **Ehrhart simplex**

$$\begin{aligned} S_d &= (d+1)\operatorname{conv}\{\mathbf{0}, \boldsymbol{e}_1, \ldots, \boldsymbol{e}_d\} - \mathbf{1}_d \\ &= \{\boldsymbol{x} \in \mathbb{R}^d : x_i \geq -1, i \in [d], x_1 + x_2 + \cdots + x_d \leq 1\}. \end{aligned} \tag{2.7}$$

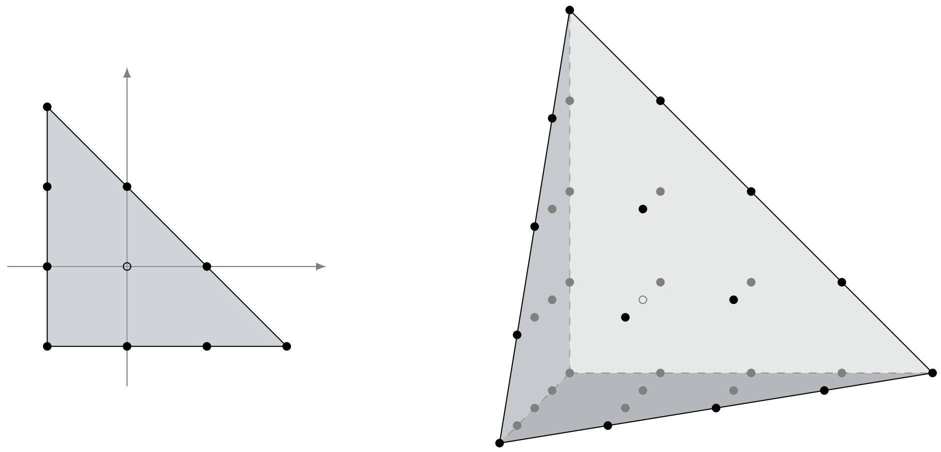

Figure 2.3: Ehrhart simplices S_2 and S_3

For an integer $m \in \mathbb{N}$ it is easily verified that $|m\, S_d|_{\mathbb{Z}^d} = \binom{d+m(d+1)}{d}$ and we believe that this is the correct upper bound in Theorem 2.5, i.e.,

Conjecture 2.7. *If $K \in \mathcal{K}_c^d$, then*

$$|K|_{\mathbb{Z}^d} \leq \binom{d + \lceil \lambda_1(K)^{-1}(d+1)\rceil}{d}.$$

Observe that compared to (2.4), this bound is asymptotically smaller by a factor of $(\mathrm{e}/4)^d$. We will verify this conjecture for arbitrary simplices in Theorem 2.11 in the next section.

It is noteworthy that Conjecture 2.7 is strongly linked to a prestigious conjecture by Ehrhart.

Conjecture 2.8 (Ehrhart, [45])**.** *Let $K \subseteq \mathbb{R}^d$ be a convex body with* $\mathrm{c}(K) = \mathbf{0}$*, and* $\mathrm{int}\,(K) \cap \mathbb{Z}^d = \{\mathbf{0}\}$*. Then*

$$\mathrm{vol}\,(K) \;\leq \mathrm{vol}\,(S_d) \;=\; \frac{(d+1)^d}{d!}.$$

In fact, due to the Jordan measurability of convex bodies, Conjecture 2.7 implies Ehrhart's conjecture:

$$\begin{aligned}
\mathrm{vol}\,(K) \;&= \lim_{m\to\infty} m^{-d} \left| K \cap \frac{1}{m}\mathbb{Z}^d \right| = \lim_{m\to\infty} m^{-d} \left| m\, K \cap \mathbb{Z}^d \right| \\
&\leq \lim_{m\to\infty} m^{-d} \binom{d + \lceil \lambda_1\,(m\,K)^{-1}\,(d+1)\rceil}{d} = \lim_{m\to\infty} m^{-d} \binom{d + \lceil m\lambda_1\,(K)^{-1}\,(d+1)\rceil}{d} \\
&= \lim_{m\to\infty} m^{-d} \frac{\left(d + \lceil m\lambda_1\,(K)^{-1}\,(d+1)\rceil\right)!}{d! \left\lceil m\lambda_1\,(K)^{-1}\,(d+1)\right\rceil!} = \lim_{m\to\infty} \frac{1}{d!} \prod_{i=1}^{d} \frac{\lceil m\lambda_1\,(K)^{-1}\,(d+1)\rceil + i}{m} \\
&\leq \lim_{m\to\infty} \frac{1}{d!} \prod_{i=1}^{d} \frac{m\lambda_1\,(K)^{-1}\,(d+1) + i + 1}{m} = \frac{1}{d!}(\lambda_1\,(K)^{-1}\,(d+1))^d = \lambda_1\,(K)^{-d}\,\frac{(d+1)^d}{d!}.
\end{aligned}$$

Hence, if $\operatorname{int}(K) \cap \mathbb{Z}^d = \{\mathbf{0}\}$ we have $\lambda_1(K) \geq 1$ and Conjecture 2.7 gives Ehrhart's conjecture. For recent progress regarding the latter, we refer to [21] and [97]. Ehrhart proved his conjecture in the plane [43,45] and for simplices in any dimension[42]. Observe that Theorem 2.11 also implies Ehrhart's result for simplices, cf. [61, Proposition 2.15]. The problem is also briefly discussed in [38, p. 147]. It is worth mentioning that the Ehrhart simplex S_d has maximal volume among all lattice simplices having the centroid as their unique interior lattice point [42]. Moreover, every such lattice simplex has also constant Mahler volume [7, Theorem 2.4, Proposition 6.1]. Here, the **Mahler volume** of a convex body K containing the origin in its interior is the product of $\operatorname{vol}(K)$ and $\operatorname{vol}(K^*)$.

2.3 Centered simplices

The first step to the proof of our lattice point bounds on centered simplices is to show that the pairwise difference of the barycentric coordinates of two distinct integer points in a centered simplex must be large in at least one coordinate regarding the first successive minimum.

Lemma 2.9. *Let $S \in \mathcal{S}_c^d$, and let $\boldsymbol{u}, \boldsymbol{w} \in S \cap \mathbb{Z}^d$, $\boldsymbol{u} \neq \boldsymbol{w}$. Then there exists an index $k \in [d+1]$ with*

$$\beta_S(\boldsymbol{u})_k - \beta_S(\boldsymbol{w})_k \geq \lambda_1(S)\,\frac{1}{d+1}.$$

Proof. Suppose the opposite, i.e., $\beta_S(\boldsymbol{u})_i - \beta_S(\boldsymbol{w})_i < \lambda_1(S)/(d+1)$ for all $i \in [d+1]$, and let $\boldsymbol{v} = \boldsymbol{w} - \boldsymbol{u}$. Then with $\lambda_1 = \lambda_1(S)$ we get in view of (1.2)

$$\beta_S\left(\frac{1}{\lambda_1}\boldsymbol{v}\right) = \frac{1}{\lambda_1}\left(\beta_S(\boldsymbol{w}) - \beta_S(\boldsymbol{u})\right) + \beta_S(\mathbf{0}) \in \mathbb{R}^{d+1}_{>0}.$$

Hence, the non-trivial lattice point $\boldsymbol{v}$ belongs to $\operatorname{int}(\lambda_1 S)$, which contradicts the minimality of λ_1. □

The proof of Lemma 2.9 says that if two lattice points $\boldsymbol{u}$ and $\boldsymbol{w}$ in a given simplex are located close to each other, that is, the difference of their barycentric coordinates is small, then $\boldsymbol{u} - \boldsymbol{w}$ will lie inside the simplex. We introduce some more notation. Let

$$B = \left\{\boldsymbol{x} \in \mathbb{R}^{d+1}_{\geq 0} \ : \ \sum_{i=1}^{d+1} x_i = 1\right\},$$

which we regard as the d-dimensional simplex in $\mathbb{R}^{d+1}$ of all feasible barycentric coordinates of points contained in a d-dimensional simplex. For a given real number $\rho > 0$, let

$$n(\rho) = \left\lceil \rho^{-1}(d+1) \right\rceil \tag{2.8}$$

and

$$R_\rho = \frac{1}{n(\rho)} \left\{ \boldsymbol{a} \in \mathbb{Z}^{d+1}_{\geq 0} \ : \ \sum_{i=1}^{d+1} a_i = n(\rho) \right\}.$$

Note that $|R_\rho| = \binom{d+n(\rho)}{d}$ and $R_\rho \subseteq B$. Let $Z_\rho = \left[0, \frac{1}{n(\rho)}\right)^d \times \mathbb{R} \subseteq \mathbb{R}^{d+1}$ be the cylinder over the half-open d-dimensional cube of edge length $\frac{1}{n(\rho)}$. For $\boldsymbol{r} \in \mathbb{R}^{d+1}$, the intersection of $\boldsymbol{r} + Z_\rho$ with the affine space $\{\boldsymbol{x} \in \mathbb{R}^{d+1} : \sum x_i = 1\}$, which contains B, yields a half-open d-dimensional parallelepiped.

Next, we claim the following inclusion.

Lemma 2.10. *With the notation above, we have $B \subseteq R_\rho + Z_\rho$.*

Proof. We may write

$$R_\rho + Z_\rho = \bigcup_{\boldsymbol{r} \in R_\rho} \left\{ \boldsymbol{x} \in \mathbb{R}^{d+1} \ : \ r_i \leq x_i < r_i + \frac{1}{n(\rho)}, \ i \in [d] \right\}. \tag{2.9}$$

For a given $\boldsymbol{x} \in B$, let $\boldsymbol{r} \in \mathbb{R}^{d+1}$ be defined as

$$\begin{aligned} r_i &= \frac{\lfloor n(\rho) x_i \rfloor}{n(\rho)}, \quad \text{for } i \in [d], \\ r_{d+1} &= 1 - \sum_{i=1}^{d} r_i = \frac{n(\rho) - \sum_{i=1}^{d} \lfloor n(\rho) x_i \rfloor}{n(\rho)}. \end{aligned} \tag{2.10}$$

Obviously, $r_i \geq 0$, $i \in [d]$, $\sum_{i=1}^{d+1} r_i = 1$ and

$$r_{d+1} = \frac{n(\rho) - \sum_{i=1}^{d} \lfloor n(\rho) x_i \rfloor}{n(\rho)} \geq 1 - \sum_{i=1}^{d} x_i = x_{d+1} \geq 0.$$

Hence, $\boldsymbol{r} \in R_\rho$. We also have $r_i \leq x_i$, $i \in [d]$, as well as

$$x_i - r_i = \frac{n(\rho) x_i - \lfloor n(\rho) x_i \rfloor}{n(\rho)} < \frac{1}{n(\rho)}.$$

Thus, we have shown $\boldsymbol{x} \in \boldsymbol{r} + Z_\rho$. □

Although not needed further, we remark that the union in (2.9) is disjoint.

We will now show that the bound from Conjecture 2.7 holds for simplices, which is the main result of this section.

Theorem 2.11. *Let $S \in \mathcal{K}^d_c$ be a d-simplex, then*

$$|S|_{\mathbb{Z}^d} \leq \binom{d + \lceil \lambda_1(S)^{-1}(d+1) \rceil}{d}.$$

Furthermore, for $\lambda_1(S)^{-1} \in \mathbb{N}$ equality holds if and only if S is unimodularly equivalent to $\lambda_1(S)^{-1}\, S_d$.

Proof of Theorem 2.11. Let $S \in \mathcal{S}_c^d$ be a d-simplex and for short we write $\lambda_1 = \lambda_1(S)$. Suppose

$$|S|_{\mathbb{Z}^d} > \binom{d + n(\lambda_1)}{d} = |R_{\lambda_1}|.$$

According to Lemma 2.10, the set of all barycentric coordinates B of points in S is covered by the union of cylinders $R_{\lambda_1} + Z_{\lambda_1}$. Using the pigeonhole principle, there exist an $\boldsymbol{r} \in R_{\lambda_1}$ and two lattice points $\boldsymbol{u}, \boldsymbol{w} \in S$ such that

$$\beta_S(\boldsymbol{u}), \beta_S(\boldsymbol{w}) \in \boldsymbol{r} + Z_{\lambda_1}.$$

We may assume $\beta_S(\boldsymbol{u})_{d+1} - \beta_S(\boldsymbol{w})_{d+1} < 0 < 1/n(\lambda_1)$ and due to the definition of Z_{λ_1} we also have $|\beta_S(\boldsymbol{u})_i - \beta_S(\boldsymbol{w})_i| < 1/n(\lambda_1)$, $i \in [d]$. Thus

$$\beta_S(\boldsymbol{u})_i - \beta_S(\boldsymbol{w})_i < \frac{1}{n(\lambda_1)} \leq \lambda_1 \frac{1}{d+1},$$

contradicting Lemma 2.9.

Next, we discuss the equality case. Let $\lambda_1^{-1} \in \mathbb{N}$ and let $|S|_{\mathbb{Z}^d} = \binom{d+n(\lambda_1)}{d}$. We first show that the set $\beta_S(S \cap \mathbb{Z}^d)$, consisting of the barycentric coordinates of all lattice points in S, equals R_{λ_1}. Since $|\beta_S(S \cap \mathbb{Z}^d)| = |R_{\lambda_1}|$, it suffices to show $\beta_S(S \cap \mathbb{Z}^d) \subseteq R_{\lambda_1}$. To this end, suppose there exists $\boldsymbol{x} \in X = \beta_S(S \cap \mathbb{Z}^d) \setminus R_{\lambda_1}$. Thus, there exists an index ℓ such that

$$x_\ell \neq \frac{k}{n(\lambda_1)} \quad \text{for every integer } 0 \leq k \leq n(\lambda_1). \tag{2.11}$$

Note that (2.11) must hold for at least one further index $\ell' \neq \ell$. Consequently, we may assume $\ell \neq d+1$ and that for $\boldsymbol{x} \in X$ the ℓ-th coordinate x_ℓ is maximal among all points in X for which (2.11) holds. We define $\boldsymbol{r}$ as in (2.10), where $\rho = \lambda_1$. Accordingly, it holds $\boldsymbol{r} \in R_{\lambda_1}$. Since $r_\ell < x_\ell$, we have $r_{d+1} > 0$ and thus $r_{d+1} \geq \frac{1}{n(\lambda_1)}$. Let

$$\begin{aligned}
t_i &= r_i \quad \text{for } i \in [d] \setminus \{\ell\}, \\
t_\ell &= r_\ell + \frac{1}{n(\lambda_1)} = \frac{\lfloor n(\rho) x_i \rfloor + 1}{n(\lambda_1)}, \\
t_{d+1} &= r_{d+1} - \frac{1}{n(\lambda_1)}.
\end{aligned}$$

Then $\boldsymbol{t} \in R_{\lambda_1}$. Let $\boldsymbol{v} \in S \cap \mathbb{Z}^d$ be the unique lattice point in S such that $\beta_S(\boldsymbol{v}) \in \boldsymbol{t} + Z_{\lambda_1}$. Then $\beta_S(\boldsymbol{v})_\ell = t_\ell$ by the assumption on $\boldsymbol{x}$. We now conclude

$$x_i - \beta_S(\boldsymbol{v})_i \leq x_i - t_i \leq x_i - r_i < \frac{1}{n(\lambda_1)}, \quad i \in [d]\,,$$

$$\beta_S(\boldsymbol{v})_i - x_i < t_i + \frac{1}{n(\lambda_1)} - x_i = r_i + \frac{1}{n(\lambda_1)} - x_i \le \frac{1}{n(\lambda_1)}, \quad i \notin \{\ell, d+1\},$$
$$\beta_S(\boldsymbol{v})_\ell - x_\ell = t_\ell - x_\ell = r_\ell + \frac{1}{n(\lambda_1)} - x_\ell < \frac{1}{n(\lambda_1)}.$$

Clearly, $x_{d+1} - \beta_S(\boldsymbol{v})_{d+1} < \frac{1}{n(\lambda_1)}$ or $\beta_S(\boldsymbol{v})_{d+1} - x_{d+1} < \frac{1}{n(\lambda_1)}$. Let $\boldsymbol{z}$ be the lattice point in S such that $\beta_S(\boldsymbol{z}) = \boldsymbol{x}$. Then $\boldsymbol{z}$ and $\boldsymbol{v}$ contradict Lemma 2.9, and thus we have $\beta_S(S \cap \mathbb{Z}^d) = R_{\lambda_1}$.

We now show that S and $\lambda_1^{-1} S_d$ are unimodularly equivalent. Let $S = \mathrm{conv}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_{d+1}\}$. Since $\beta_S(S \cap \mathbb{Z}^d) = R_{\lambda_1}$ there are exactly $n(\lambda_1)+1$ lattice points on every edge of S. Therefore $\boldsymbol{w}_i = \frac{1}{n(\lambda_1)}(\boldsymbol{v}_i - \boldsymbol{v}_{d+1}) \in \mathbb{Z}^d$ for all i. Moreover, by Lemma 5 in [101] it holds $\mathrm{vol}\,(\lambda_1 S) \le \mathrm{vol}\,(S_d)$ and letting M be the $d \times d$ matrix having columns $\boldsymbol{w}_1, \boldsymbol{w}_2, \dots, \boldsymbol{w}_d$, we conclude

$$\det(M) = \frac{d!}{n(\lambda_1)^d}\mathrm{vol}\,(S) \le \frac{d!}{n(\lambda_1)^d}\mathrm{vol}\left(\lambda_1^{-1} S_d\right) = \frac{d!}{(d+1)^d}\mathrm{vol}\,(S_d) = 1.$$

Thus, M is unimodular and the equation

$$M(\lambda_1^{-1} S_d + \lambda_1^{-1}\mathbf{1}) + \boldsymbol{v}_{d+1} = S,$$

implies that S and $\lambda_1^{-1} S_d$ are indeed unimodularly equivalent. □

Remark 2.12. *The results in this section, including the proof of Theorem 2.11, can be generalized verbatim to a simplex S having its unique interior lattice point $\boldsymbol{v}$ not necessarily at the origin. In this case, one obtains the bound*

$$|S|_{\mathbb{Z}^d} \le \binom{d + \lceil \lambda_1(S)^{-1} \beta_S(\boldsymbol{v})_{min}^{-1} \rceil}{d},$$

where $\beta_S(\boldsymbol{v})_{min} := \min_{1 \le i \le d+1} \beta_S(\boldsymbol{v})_i$ is the smallest barycentric coordinate of $\boldsymbol{v}$. However, by applying the best possible bound for $\beta_S(\boldsymbol{v})_{min}$ (cf. [7, Theorem 2.1]*) this does not lead to "good" upper bounds or to any new insights into Hensley's conjecture mentioned in Section 2.1. For the currently best known bound regarding this problem, we refer to* [6]*.*

2.4 The planar case

We recall Pick's Theorem (Theorem 1.3) which states that

$$|P|_{\mathbb{Z}^2} = \mathrm{vol}\,(P) + \frac{1}{2}|\mathrm{bd}\,(P)\,|_{\mathbb{Z}^2} + 1.$$

Scott [109] stated the following result having a very similar flavor.

Theorem 2.13 (Scott, [109])**.** *Let P be a convex lattice polygon with at least one interior point, then*

$$|\mathrm{bd}\,(P)\,|_{\mathbb{Z}^2} - 2|\mathrm{int}\,(P)\,|_{\mathbb{Z}^2} \le 7,$$

and equality is attained if and only if P is unimodularly equivalent to the Ehrhart simplex S_2.

One of the most fascinating theorems regarding convex bodies and their centroids was presented by Grünbaum. It provides an interesting property of the centroid in terms of mass-distribution of the given convex body.

Theorem 2.14 (Grünbaum, [57])**.** *Let $K \in \mathcal{K}_c^d$ be a centered convex body and let $H \subseteq \mathbb{R}^d$ be a half-space containing the centroid $c(K)$, then*

$$\mathrm{vol}\,(K \cap H)\, \geq \left(\frac{d}{d+1}\right)^d \mathrm{vol}\,(K)\, .$$

It is noteworthy that equality holds in Grünbaum's Theorem if K is a simplex.

In order to prove the following Theorem 2.16, our main result of this section, we will also use the following theorem by Ehrhart, which verifies Conjecture 2.8 in dimension two, as discussed already in Section 2.2.

Theorem 2.15 (Ehrhart, [43, 45])**.** *Let $K \in \mathcal{K}_c^2$ with $\mathrm{vol}\,(K)\, \geq 9/2$, then K contains at least two lattice points distinct from the origin.*

Finally, by using classical results of planar geometry, our last result verifies Conjecture 2.7 for planar convex bodies whose only lattice point is the origin.

Theorem 2.16. *Let $K \in \mathcal{K}_c^2$ with $|\mathrm{int}\,(K)\,|_{\mathbb{Z}^d} = 1$, then*

$$|K|_{\mathbb{Z}^2} \leq 10.$$

Furthermore, equality holds if and only if K is unimodularly equivalent to the Ehrhart simplex S_2.

Proof of Theorem 2.16. Let $P = \mathrm{conv}(K \cap \mathbb{Z}^2)$. If $\mathbf{0} \notin \mathrm{int}\,(P)$, then a half-space H_+ containing P and containing $\mathbf{0}$ in its boundary such that $K \cap H_+ \cap \mathbb{Z}^2 = K \cap \mathbb{Z}^2$ exists. Letting $H_- = -H_+$ denote the opposite half-space of H_+, Grünbaum's Theorem and Ehrhart's Theorem imply that

$$\mathrm{vol}\,(P)\, \leq \mathrm{vol}\,(K \cap H_+)\, = \mathrm{vol}\,(K)\, - \mathrm{vol}\,(K \cap H_-)\, \leq \mathrm{vol}\,(K)\, - \left(\frac{2}{3}\right)^2 \mathrm{vol}\,(K)\, \leq \frac{5}{2}.$$

In turn, applying Pick's Theorem yields

$$|K|_{\mathbb{Z}^2} = |K \cap H_+|_{\mathbb{Z}^2} \; = \; |P|_{\mathbb{Z}^2} = \mathrm{vol}\,(P)\, + \frac{1}{2}|\mathrm{bd}\,(P)\,|_{\mathbb{Z}^2} + 1 \leq \frac{7}{2} + \frac{1}{2}|K|_{\mathbb{Z}^2}.$$

Thus, $|K|_{\mathbb{Z}^2} \leq 7$.

Next, we assume that $\mathbf{0} \in \mathrm{int}\,(P)$. Applying the theorems of Ehrhart and Pick again gives that

$$\begin{aligned} \frac{9}{2} &\geq \mathrm{vol}\,(K)\, \geq \mathrm{vol}\,(P)\, = |P|_{\mathbb{Z}^2} - \frac{1}{2}|\mathrm{bd}\,(P)\,|_{\mathbb{Z}^2} - 1 \\ &= |K|_{\mathbb{Z}^2} - \frac{1}{2}(|K|_{\mathbb{Z}^2} - 1) - 1 = \frac{1}{2}|K|_{\mathbb{Z}^2} - \frac{1}{2}, \end{aligned} \tag{2.12}$$

and thus, $|K|_{\mathbb{Z}^2} \leq 10$.

Now, let $|K|_{\mathbb{Z}^2} = 10$. Then, (2.12) implies that $\mathrm{vol}(K) = \mathrm{vol}(P)$ and thus, $P = K$ by compactness. Furthermore, we know that P contains a lattice point in its interior as $|P|_{\mathbb{Z}^2} = 10$. Scott's Theorem implicates that $P = K$ is unimodularly equivalent to S_2. $\square$

3
Tensor Ehrhart theory

3.1 Introduction

The Ehrhart polynomial of a lattice polytope counts the number of lattice points in its integer dilates and is arguably the most fundamental arithmetic invariant of a lattice polytope. It is a cornerstone of geometric combinatorics and takes on various forms in other areas of mathematics, such as commutative algebra, optimization, representation theory, or voting theory (see, e.g., [14, 18, 39, 82, 91]). Concepts from Ehrhart theory have been generalized in various directions; for example, q-analogs of Ehrhart polynomials [37], equivariant versions [117], multivariate extensions [15, 24, 58], and generalizations to valuations [73, 74, 89].

Ludwig and Silverstein [85] introduced Ehrhart tensor polynomials based on discrete moment tensors that were defined by Böröczky and Ludwig [27]. The **discrete moment tensor of rank** r of a polytope $P \in \mathcal{P}^d_{\mathbb{Z}}$ is

$$\mathrm{L}^r(P) \;=\; \sum_{\boldsymbol{x} \in P \cap \mathbb{Z}^d} \boldsymbol{x}^r, \tag{3.1}$$

where r is a nonnegative integer. Note that, for our convenience, this definition differs by a scalar from the original definition given in [27]. A version of $\mathrm{L}^r(P)$, the discrete directional moment, was studied in [108]. For $r = 0$, the usual discrete volume or lattice point enumerator $\mathrm{L}^0(P) = |P|_{\mathbb{Z}^d} = |P \cap \mathbb{Z}^d|$ is recovered. In the context of discrete moment tensors, we will deviate from our notation and we will write $\mathrm{L}^0(P)$ rather than $|P|_{\mathbb{Z}^d}$ for the sake of consistency with the notation of the discrete moment tensors of higher ranks. For $r = 1$, $\mathrm{L}^1(P)$ equals the discrete moment vector defined in [26], which is the sum of all integer points contained in P. Based on results by Khovanskiĭ and Pukhlikov [102] and Alesker [2], it was identified in [85] that $\mathrm{L}^r(nP)$

The results of this chapter are joint work with Katharina Jochemko and Laura Silverstein and appeared in [19], cf. `https://doi.org/10.1016/j.laa.2017.10.021`. First Published in "Ehrhart tensor polynomials" in Linear Algebra and its Applications, Volume 539, 2018, published by Elsevier.

is given by a polynomial, for any positive $n \in \mathbb{N}$, extending Ehrhart's celebrated result for the lattice point enumerator, see Theorem 1.4.

Theorem 3.1 ([85, Theorem 1])**.** *There exist* $\mathrm{L}_i^r \colon \mathcal{P}_{\mathbb{Z}}^d \to \mathbb{T}^r$ *for all* $0 \leq i \leq d+r$ *such that*

$$\mathrm{L}^r(nP) \;=\; \sum_{i=0}^{d+r} \mathrm{L}_i^r(P) n^i$$

for any $n \in \mathbb{N} \setminus \{0\}$ *and* $P \in \mathcal{P}_{\mathbb{Z}}^d$.

The expansion of $\mathrm{L}^r(nP)$ will be denoted as $\mathrm{L}_P^r(n)$ and is called the **Ehrhart tensor polynomial** of P in commemoration of this result. Furthermore, the coefficients $\mathrm{L}_0^r, \ldots, \mathrm{L}_{d+r}^r$ are the **Ehrhart tensor coefficients** or **Ehrhart tensors**. For a positive integer $n > 0$, $\mathrm{L}^r(nP)$ and $\mathrm{L}_P^r(n)$ describe the same quantity, and we may use both interchangeably. However, for $-n \leq 0$, $\mathrm{L}^r(-nP)$ describes the discrete moment tensor $-nP$, whereas $\mathrm{L}^r(-n)$ corresponds to plugging in $-n$ into the Ehrhart tensor polynomial of P.

A fundamental and intensively studied question in Ehrhart theory is the characterization of Ehrhart polynomials and their coefficients. The only coefficients that are known to have explicit geometric descriptions are the leading, second-highest, and constant coefficients for the classic Ehrhart polynomial, see Chapter 1, page 15. For the Ehrhart tensor polynomial, the leading and constant coefficients were given in [85] and we give an interpretation for the second-highest coefficient (Proposition 3.3) as the weighted sum of moment tensors over the facets of the polytope; the descriptions of all are given in Section 3.2.

The principal tool we use to study Ehrhart tensor polynomials are *h^r-tensor polynomials* which encode the Ehrhart tensor polynomial in a certain binomial basis. Extending the notion of the usual Ehrhart h^*-polynomial, we consider

$$\mathrm{L}^r(nP) \;=\; h_0^r(P)\binom{n+d+r}{d+r} + h_1^r(P)\binom{n+d+r-1}{d+r} + \;\ldots\; + h_{d+r}^r(P)\binom{n}{d+r}$$

for a d-dimensional lattice polytope P and define the **h^r-tensor polynomial** of P to be

$$h_P^r(t) \;=\; \sum_{i=0}^{d+r} h_i^r(P) t^i.$$

Stanley's Nonnegativity Theorem [113] is a foundational result which states that all coefficients of the h^*-polynomial of a lattice polytope are nonnegative. Stanley, moreover, proved that the coefficients are monotone with respect to inclusion; that is, for all lattice polytopes $Q \subseteq P$ and all $0 \leq i \leq d$, it holds that $h_i^*(Q) \leq h_i^*(P)$. Using half-open decompositions, it was proven in [74] that, with regard to translation invariant valuations, monotonicity and nonnegativity are equivalent.

Section 3.3 discusses the coefficients of the h^r-tensors. More importantly, we determine a formula for the h^r-tensor polynomial of half-open simplices (Theorem 3.6) by using **half-open decompositions of polytopes**; an important tool which was introduced by Köppe and Verdoolaege [78]. From this formula and the existence of a unimodular triangulation, we deduce an interpretation of all Ehrhart vectors and matrices of lattice polygons in the subsequent section.

In Section 3.4, we determine Pick-type formulas for the discrete moment vector and matrix. For lattice polygons, the coefficients of the (classical) Ehrhart polynomial are positive and well-understood. They are given by Pick's Formula (Theorem 1.3), which shows that for $P \in \mathcal{P}^2_{\mathbb{Z}}$ it holds $\mathrm{L}^0_0(P) = 1$, $\mathrm{L}^0_1(P) = \frac{1}{2}\,\mathrm{L}^0(\mathrm{bd}\,(P)\,) = \frac{1}{2}|\mathrm{bd}\,(P)\,|_{\mathbb{Z}^2}$, and $\mathrm{L}^0_2(P)$ equals the area of P. Our interpretation of the coefficients of the discrete moment vector and discrete moment matrix is given with respect to a triangulation of the respective polygon.

In Section 3.5, we discuss notions of positivity for Ehrhart tensors and investigate Ehrhart tensor polynomials and h^2-tensor polynomials with respect to *positive semidefiniteness*. In contrast to the usual Ehrhart polynomial, Ehrhart tensor coefficients can even be negative definite for lattice polygons (Example 3.11). Moreover, the coefficients of h^2-tensor polynomials are not monotone which is demonstrated by Example 3.13. Therefore, techniques such as irrational decompositions and half-open decompositions that have been used to prove Stanley's Nonnegativity Theorem (see [16, 74]) can not immediately be applied to h^2-tensor coefficients. Nevertheless, considering an intricate decomposition of lattice points inside a polygon, we are able to prove positive semi-definiteness of the coefficients of h^2-tensor polynomial in dimension two (Theorem 3.12). Here, we want to remark that the theorem holds true for lattice polygons in a higher dimensional ambient space. Furthermore, all of the results given in this chapter are independent of the ambient space. Based on computational results, we further conjecture positive-semidefiniteness of the h^2-tensor coefficients in higher dimensions (Conjecture 3.27).

Section 3.6 presents some results and technical methods on how h^r-tensors and Ehrhart tensors polynomials can be calculated. These are linked to analogous results from the classical Ehrhart theory. Particularly, we will discuss pyramids, bipyramids and joins of lattice polytopes and their respective tensor coefficients. This provides a powerful theoretical framework, since the approach can be applied inductively on the dimension. It will enable us to discuss the tensor coefficients of the standard simplex and the standard cross-polytope.

In Section 3.7, we prove a generalization of Hibi's Palindromic Theorem [70] characterizing reflexive polytopes as having palindromic h^r-tensor polynomials for $r \in \mathbb{N}$ of even rank and conclude by discussing possible future research directions.

3.2 Discrete moment tensors

As discussed in Chapter 1, we have $\mathbb{T}^0 = \mathbb{R}$ and identify $\mathbb{T}^1$ with $\mathbb{R}^d$, and for $r = 2$, the bilinear form $T \in \mathbb{T}^2$ can then be identified with a symmetric $d \times d$ matrix $T = (T_{ij})$. To that end, we will call the discrete moment tensor (3.1) of ranks 1 and 2 the **discrete moment vector** and **discrete moment matrix**, respectively. We will also regard their associated coefficients, their Ehrhart tensors, as **Ehrhart vectors** and **Ehrhart matrices**.

Prior to describing the known Ehrhart tensors, we provide some properties of the discrete moment tensor that we will need. For a lattice polytope $P \in \mathcal{P}^d_{\mathbb{Z}}$ it holds that

$$\mathrm{L}^r(P)(\boldsymbol{e}_{i_1}, \ldots, \boldsymbol{e}_{i_r}) \;=\; \sum_{\boldsymbol{x} \in P \cap \mathbb{Z}^d} \langle \boldsymbol{x}, \boldsymbol{e}_{i_1} \rangle \ldots \langle \boldsymbol{x}, \boldsymbol{e}_{i_r} \rangle.$$

Hence, the action of $\mathrm{GL}\,(d,\mathbb{Z})$, the general linear group over the integers, on L^r is observed to be
$$\mathrm{L}^r(\phi P)(\boldsymbol{e}_{i_1},\ldots,\boldsymbol{e}_{i_r}) \;=\; \mathrm{L}^r(P)(\phi^T\boldsymbol{e}_{i_1},\ldots,\phi^T\boldsymbol{e}_{i_r})$$
for any $P\in\mathcal{P}^d_{\mathbb{Z}}$ and $\phi\in\mathrm{GL}\,(d,\mathbb{Z})$; we say that L^r is $\mathrm{GL}\,(d,\mathbb{Z})$**-equivariant**.

For any $P\in\mathcal{P}^d_{\mathbb{Z}}$ and $r\in\mathbb{N}$, we set
$$\mathrm{L}^r(\mathrm{relint}\,(P)\,) \;:=\; \sum_{\boldsymbol{x}\in\mathrm{relint}(P)}\boldsymbol{x}^r.$$

The Ehrhart-Macdonald reciprocity (Theorem 1.7) was a fundamental result in Ehrhart theory that was established by Ehrhart [44] for $d=3$ and first proven for all dimensions d by Macdonald [86]. It states that for a d-dimensional lattice polytope P, it holds
$$\mathrm{L}^0(\mathrm{relint}\,(nP)\,) \;=\; (-1)^d\,\mathrm{L}^0_P(-n).$$

A general version of this result was given for translation-invariant valuations by McMullen [89]. Unlike the discrete volume, the discrete moment tensor varies under translations by elements in $\mathbb{Z}^d$. More precisely, for all $r\in\mathbb{N}$, the discrete moment tensor of a translated polytope is
$$\mathrm{L}^r(P+t) \;=\; \sum_{j=0}^{r}\binom{r}{j}\,\mathrm{L}^{r-j}(P)t^j$$

and we say that the discrete moment tensor is covariant with respect to translations or **translation covariant**. Similar to McMullen's result, a reciprocity theorem was given for translation covariant valuations in [85]. Extending the classical Ehrhart-Macdonald reciprocity, the following reciprocity theorem gives the special case of the discrete moment tensor.

Theorem 3.2. [85, Theorem 2] *Let P be lattice polytope. Then*
$$\mathrm{L}^r_P(-n) \;=\; (-1)^{\dim(P)+r}\,\mathrm{L}^r(\mathrm{relint}\,(nP)\,)\,.$$

We use this theorem in our characterization of the second-highest Ehrhart tensor.

A complete characterization of the Ehrhart coefficients has been inaccessible up to this point. The coefficients can even be negative and, therefore, are difficult to describe combinatorially. However, it is known that the leading coefficient equals the volume, the second highest coefficient is related to the normalized surface area, and the constant coefficient is always 1.

More generally, for Ehrhart tensors, it has been proven [85, Lemma 26] that the leading coefficient of the discrete moment tensor equals the **moment tensor of rank** r which is defined as
$$\mathrm{M}^r(P) \;=\; \int_P \boldsymbol{x}^r\;\mathrm{d}\boldsymbol{x}\,,$$
where $\mathrm{d}\boldsymbol{x}$ denotes the Lebesgue measure with respect to the affine hull of P. It is also clear that, for $r\geq 1$, the constant coefficient vanishes identically by its $\mathrm{GL}\,(d,\mathbb{Z})$ equivariance; that is, $\mathrm{L}^r_0(P)=\mathrm{L}^r(0P)=0$ for any $P\in\mathcal{P}^d_{\mathbb{Z}}$ [85].

We give an interpretation for the second coefficient (Proposition 3.3) as the weighted sum of moment tensors over the facets of the polytope. The coefficient $\mathrm{L}^0_{d-1}(P)$, specifically, was shown to be equal to one half of the sum over the normalized volumes of the facets of P by Ehrhart [41]. We extend this statement to Ehrhart tensor polynomials by proving the following.

Proposition 3.3. *Let P be a lattice polytope, then*

$$\mathrm{L}^r_{\dim(P)+r-1}(P) \;=\; \sum_F \frac{1}{|\det(\operatorname{aff}(F)\cap\mathbb{Z}^d)|}\int_F \boldsymbol{x}^r \;\mathrm{d}\boldsymbol{x}\,,$$

where the sum is over all facets $F \subseteq P$ and $\mathrm{d}\boldsymbol{x}$ denotes the $(d-1)$-dimensional Lebesgue measure on the affine hull of the respective facet F.

Proof. Theorem 3.2, on the one hand, implies

$$\sum_{\boldsymbol{x}\in\operatorname{bd}(nP)} \boldsymbol{x}^r \;=\; \sum_{F\subsetneq P}\;\sum_{\boldsymbol{x}\in\operatorname{relint}(nF)} \boldsymbol{x}^r \;=\; \sum_{F\subsetneq P}(-1)^{\dim(F)+r}\,\mathrm{L}^r_F(-n)\,,$$

where the sum is taken over all proper faces $F \subsetneq P$. On the other hand, we have

$$\begin{aligned}\sum_{\boldsymbol{x}\in\operatorname{bd}(nP)} \boldsymbol{x}^r \;&=\; \mathrm{L}^r(nP) - \mathrm{L}^r(\operatorname{relint}(nP)) \;=\; \mathrm{L}^r(nP) - (-1)^{\dim(P)+r}\,\mathrm{L}^r_P(-n)\\ &=\; 2\sum_{i\geq 0}\mathrm{L}^r_{\dim(P)+r-1-2i}(nP)\end{aligned}$$

where we set $\mathrm{L}^r_i = 0$ for all $i<0$. Using both equations, we obtain

$$\begin{aligned}\mathrm{L}^r_{\dim(P)+r-1}(P) \;&=\; \lim_{n\to\infty}\frac{1}{n^{\dim(P)+r-1}}\sum_{i\geq 0}\mathrm{L}^r_{\dim(P)+r-1-2i}(nP)\\ &=\; \frac{1}{2}\sum_{F\subsetneq P}(-1)^{\dim(F)+r}\lim_{n\to\infty}\frac{1}{n^{\dim(P)+r-1}}\,\mathrm{L}^r_F(-n)\\ &=\; \frac{1}{2}\sum_{F\text{ facet}}\frac{1}{|\det(\operatorname{aff}(F)\cap\mathbb{Z}^d)|}\int_F \boldsymbol{x}^r\;\mathrm{d}\boldsymbol{x}\,,\end{aligned}$$

where the last equality follows from [85, Lemma 26]. $\square$

3.3 h^r-tensor polynomials

Let P be a d-dimensional lattice polytope. Since $\mathrm{L}^r(nP)$ is a polynomial of degree at most $d+r$, it can be written as a linear combination of the polynomials $\binom{n+d+r}{d+r}, \binom{n+d+r-1}{d+r}, \ldots, \binom{n}{d+r}$, that is,

$$\mathrm{L}^r(nP) \;=\; h^r_0(P)\binom{n+d+r}{d+r} + h^r_1(P)\binom{n+d+r-1}{d+r} + \;\ldots\; + h^r_{d+r}(P)\binom{n}{d+r} \qquad (3.2)$$

for some $h_0^r(P), \dots, h_{d+r}^r(P) \in \mathbb{T}^r$. Equivalently, in terms of generating functions,

$$\mathrm{Ehr}_P^r(t) \;:=\; \sum_{n\geq 0} \mathrm{L}^r(nP)t^n \;=\; \frac{h_0^r(P) + h_1^r(P)t + \cdots + h_{d+r}^r(P)t^{d+r}}{(1-t)^{d+r+1}}\,. \tag{3.3}$$

We call $\mathrm{Ehr}_P^r(t)$ the **Ehrhart tensor series**, $h^r(P) = (h_0^r(P), h_1^r(P), \dots, h_{d+r}^r(P))$ the h^r**-vector**, its entries the h^r**-tensor coefficients** or h^r**-tensors** of P, and

$$h_P^r(t) \;=\; \sum_{i=0}^{d+r} h_i^r t^i$$

the h^r**-tensor polynomial** of P. Observe that for $r = 0$ we obtain the usual h^*-polynomial and h^*-vector of an Ehrhart polynomial. By evaluating equation (3.2) at $n = 0, 1$, we obtain $h_0^r = 0$ for $r \geq 1$ and $h_1^r = \mathrm{L}^r(P)$ for $r \geq 0$. Inspecting the leading coefficient, we obtain

$$h_1^r(P) + h_2^r(P) + \cdots + h_{d+r}^r(P) \;=\; (d+r)! \int_P \boldsymbol{x}^r \,\mathrm{d}\boldsymbol{x}\,.$$

Applying Theorem 3.2 and evaluating at $n = 1$, we obtain

$$h_{d+r}^r(P) \;=\; \mathrm{L}^r(\mathrm{relint}\,(P)\,)\,.$$

An important oberservation is that the theory of tensor polynomials and h^r-tensors of higher rank implicitely contains the theory of tensors of lesser rank. More precisely, we consider the embedding $P^+ := P \times \{1\} \subseteq \mathbb{R}^{d+1}$ of $P \in \mathcal{P}_{\mathbb{Z}}^d$. If $m \in \{0, \dots, r\}$ and if $i_1, \dots, i_m \in \{1, \dots, d\}$, then we find

$$\begin{aligned}
\mathrm{L}^r\left(nP^+\right)(\boldsymbol{e}_{i_1}, \dots, \boldsymbol{e}_{i_m}, \boldsymbol{e}_{d+1}, \dots, \boldsymbol{e}_{d+1}) &= \sum_{\boldsymbol{x}\in nP} \binom{\boldsymbol{x}}{n}^r (\boldsymbol{e}_{i_1}, \dots, \boldsymbol{e}_{i_m}, \boldsymbol{e}_{d+1}, \dots, \boldsymbol{e}_{d+1}) \\
&= \sum_{\boldsymbol{x}\in nP} \boldsymbol{x}^m(\boldsymbol{e}_{i_1}, \dots, \boldsymbol{e}_{i_m})n^{r-m} = n^{r-m}\,\mathrm{L}^m(nP).
\end{aligned}$$

Consequently, the Ehrhart tensors polynomial of rank $m \leq r$ can be recovered from the Ehrhart tensors polynomial of rank r. Likewise, the same holds true for the h^r-tensor polynomials. In Example 3.4 we will give an example of how tensor coefficients can be reclaimed from tensors coefficients of higher rank.

Example 3.4. We assume that the Ehrhart tensor polynomial of the embedding ${\Delta_2}^+$ of the standard simplex is given:

$$24\,\mathrm{L}^2(n{\Delta_2}^+) = \begin{pmatrix} 4 & -2 & 0 \\ -2 & 4 & 0 \\ 0 & 0 & 0 \end{pmatrix} n + \begin{pmatrix} 10 & -1 & 8 \\ -1 & 10 & 8 \\ 8 & 8 & 24 \end{pmatrix} n^2 + \begin{pmatrix} 8 & 2 & 12 \\ 2 & 8 & 12 \\ 12 & 12 & 36 \end{pmatrix} n^3 + \begin{pmatrix} 2 & 1 & 4 \\ 1 & 2 & 4 \\ 4 & 4 & 12 \end{pmatrix} n^4.$$

And thus, $24n^2 + 36n^3 + 12n^4 = 24n^2\,\mathrm{L}^0(n\Delta_2)$, which gives the Ehrhart polynomial $\mathrm{L}^0(n\Delta_2) =$

$1+3/2n+1/2n^2$. Similarly, $(8,8)^T n^2+(12,12)^T n^3+(4,4)^T n^4 = 24n\,\mathrm{L}^1_{\Delta_2}(n)$, and thus, $\mathrm{L}^1(n\Delta_2) = (1/3,1/3)^T n + (1/2,1/2)^T n^2 + (1/6,1/6)^T n^3$.

Half-open polytopes

We will not only consider relatively open polytopes, but also **half-open polytopes**. Let P be a polytope with facets $F_1,\ldots,F_k$ and let q be a generic point, i.e., q is not contained in the affine hull of any facet of P, in its affine span $\mathrm{aff}(P)$, then a facet F_i is **visible** from $\boldsymbol{q}$ if $(\boldsymbol{p},\boldsymbol{q}]\cap P = \emptyset$ for all $\boldsymbol{p}\in F_i$. Moreover, if $I_{\boldsymbol{q}}(P) = \{i\in[k]\colon F_i \text{ is visible from } \boldsymbol{q}\}$ then the point set

$$H_{\boldsymbol{q}}(P) \;=\; P\setminus \bigcup_{i\in I_{\boldsymbol{q}}(P)} F_i$$

defines a half-open polytope. In particular, $H_q(P) = P$ for all $q\in P$. The following result by Köppe and Verdoolaege [78] shows that every polytope can be decomposed into half-open polytopes, and is implicitely also contained in works by Stanley and Ehrhart (see [112]). A decomposition of a polytope into pairwise disjoint half-open polytopes is called a **partition**.

Theorem 3.5 ([78]). *Let P be a polytope and let $P_1,\ldots,P_m$ be the maximal cells of a triangulation of P. Let $\boldsymbol{q}\in\mathrm{aff}(P)$ be a generic point, then*

$$H_q(P) \;=\; H_q(P_1)\sqcup H_q(P_2)\sqcup\cdots\sqcup H_q(P_m)$$

is a partition.

The discrete moment tensor naturally can be defined for half-open polytopes by setting

$$\mathrm{L}^r(H_q(P)) \;:=\; \mathrm{L}^r(P) - \sum_{\emptyset\neq J\subseteq I_q(P)} (-1)^{\dim P-\dim F_J}\,\mathrm{L}^r(F_J)$$

where $F_J := \bigcap_{i\in J} F_i$. Then, from Theorem 3.5 and the inclusion-exclusion principle, we obtain that

$$\mathrm{L}^r(P) \;=\; \mathrm{L}^r(H_q(P_1)) \;+\; \mathrm{L}^r(H_q(P_2)) \;+\;\ldots\;+\; \mathrm{L}^r(H_q(P_m)), \tag{3.4}$$

cf. [74, Corollary 3.2].

Half-open simplices

Let $S\in\mathcal{P}^d_{\mathbb{Z}}$ be a d-dimensional lattice simplex with vertices $\boldsymbol{v}_1,\ldots,\boldsymbol{v}_{d+1}$ and let $F_1,\ldots,F_{d+1}$ denote the facets of S such that $\boldsymbol{v}_i\notin F_i$ for all $i\in[d+1]$. Let $S^* = H_{\boldsymbol{q}}(S)$ be a d-dimensional half-open simplex and let $I = I_{\boldsymbol{q}}(S)$. We define the half-open polyhedral cone

$$C_{S^*} \;=\; \left\{\sum_{i=1}^{d+1}\lambda_i\bar{\boldsymbol{v}}_i \;:\; \lambda_i\geq 0 \text{ for } i\in[d+1],\ \lambda_i\neq 0 \text{ if } i\in I\right\}\subseteq\mathbb{R}^{d+1}$$

where $\bar{\boldsymbol{v}}_i := (\boldsymbol{v}_i,1)\in\mathbb{R}^{d+1}$ for all $i\in[d+1]$. Then, by identifying hyperplanes of the form $\{\boldsymbol{x}\in\mathbb{R}^{d+1}\colon x_{d+1} = n\}$ with $\mathbb{R}^d$ via $p\colon\mathbb{R}^{d+1}\to\mathbb{R}^d$ which maps $\boldsymbol{x}\mapsto(x_1,\ldots,x_d)$, we have

$C_{S^*} \cap \{x_{d+1} = n\} = nS^*$. We consider the half-open parallelepiped

$$\Pi_{S^*} \;=\; \left\{ \sum_{i=1}^{d+1} \lambda_i \bar{\boldsymbol{v}}_i \;:\; 0 < \lambda_i \leq 1 \text{ if } i \in I,\; 0 \leq \lambda_i < 1 \text{ if } i \notin I \right\}.$$

Then

$$C_{S^*} \;=\; \bigsqcup_{u \in \mathbb{Z}^{d+1}} \Pi_{S^*} + u_1\bar{\boldsymbol{v}}_1 + \cdots + u_{d+1}\bar{\boldsymbol{v}}_{d+1}\,.$$

Let $S_i = \Pi_{S^*} \cap \{x_{d+1} = i\}$. Then S_i is a *partially open* hypersimplex; that is, a hypersimplex with certain facets removed.

Our next result shows that $\mathrm{L}^r(nS^*)$ is given by a polynomial in n by determining its generating series. We follow the line of argumentation in [74, Proposition 3.3]. Observe that, together with equation (3.4), this reproves the polynomiality result of $\mathrm{L}^r(nP)$.

Theorem 3.6. *With the notation given above, the equation*

$$\begin{aligned}
&\sum_{n\geq 0} \mathrm{L}^r(nS^*)t^n \\
&= \sum_{\substack{k_0,\dots,k_{d+1}\geq 0\\ \sum k_j = r}} \binom{r}{k_0,\dots,k_{d+1}} \boldsymbol{v}_1^{k_1}\dots\boldsymbol{v}_{d+1}^{k_{d+1}} \frac{(1-t)^{k_0}A_{k_1}(t)\dots A_{k_{d+1}}(t)}{(1-t)^{d+r+1}} \sum_{i=0}^{d} \mathrm{L}^{k_0}(S_i)t^i\,,
\end{aligned}$$

holds true where $A_j(t)$ is the j-th Eulerian polynomial, see Chapter 1, page 16.

Proof. The generating function of the discrete moment tensor allows us to consider the discrete moment tensor of nS^* by cutting the cone C_{S^*} with the hyperplane $\{x_{d+1} = n\}$. The geometric interpretation of the half-open parallelepipeds tiling the cone, the translation covariance of the discrete moment tensor, and the binomial theorem together yield the equation

$$\begin{aligned}
\sum_{n\geq 0} \mathrm{L}^r(nS^*)t^n &= \sum_{i=0}^{d} t^i \sum_{u_1,\dots,u_{d+1}\geq 0} \mathrm{L}^r(S_i + u_1\bar{\boldsymbol{v}}_1 + \cdots + u_{d+1}\bar{\boldsymbol{v}}_{d+1})t^{\sum u_m} \\
&= \sum_{i=0}^{d} t^i \sum_{u_1,\dots,u_{d+1}\geq 0} \sum_{j=0}^{r} \binom{r}{j} \mathrm{L}^{r-j}(S_i)(u_1\bar{\boldsymbol{v}}_1 + \cdots + u_{d+1}\bar{\boldsymbol{v}}_{d+1})^j t^{\sum u_m} \\
&= \sum_{i=0}^{d} t^i \sum_{u_1,\dots,u_{d+1}\geq 0} \sum_{\substack{k_0,\dots,k_{d+1}\geq 0\\ \sum k_j = r}} \binom{r}{k_0,\dots,k_{d+1}} \mathrm{L}^{k_0}(S_i)(u_1\bar{\boldsymbol{v}}_1)^{k_1}\dots(u_{d+1}\bar{\boldsymbol{v}}_{d+1})^{k_{d+1}} t^{\sum u_m} \\
&= \sum_{i=0}^{d} t^i \sum_{\substack{k_0,\dots,k_{d+1}\geq 0\\ \sum k_j = r}} \binom{r}{k_0,\dots,k_{d+1}} \mathrm{L}^{k_0}(S_i)\bar{\boldsymbol{v}}_1^{k_1}\dots\bar{\boldsymbol{v}}_{d+1}^{k_{d+1}} \sum_{u_1,\dots,u_{d+1}\geq 0} u_1^{k_1}\dots u_{d+1}^{k_{d+1}} t^{\sum u_m},
\end{aligned}$$

where we abbreviated $\sum k_j := \sum_{j=0}^{d+1} k_j$ and $\sum u_m := \sum_{m=1}^{d+1} u_m$. The result then follows by the identity (1.7). □

We remark that the results and proofs of this section immediately carry over to general translative polynomial valuations (see [85] for a definition). In particular, Theorem 3.6 can be generalized to give a new proof of [102, Corollary 5].

3.4 Pick-type formulas

Pick's Theorem [99] gives an interpretation for the coefficients of the Ehrhart polynomial of a lattice polygon which establishes a relationship between the area of the polygon, the number of lattice points in the polygon and on its boundary. An analogue in higher dimensions can not exist (see, e.g., [55]) as it is crucial that every polygon in dimension two has a unimodular triangulation; that is, a triangulation into lattice simplices of minimal possible area $1/d!$. We offer interpretations for the coefficients of the Ehrhart tensor polynomial in the vector and the matrix cases by taking the route over the h^r-tensor polynomial.

Given a polygon $P \in \mathcal{P}^2_{\mathbb{Z}}$, we will consider unimodular triangulations of P where such a triangulation will always be denoted by $\mathcal{T}$. The triangulation will be described by the **edge graph** $G = (V, E)$ of $\mathcal{T}$ where V are the lattice points contained in P and E the edges of $\mathcal{T}$. Furthermore, the notation $\boldsymbol{x}$ will be reserved for elements of V and $\boldsymbol{y}, \boldsymbol{z}$ for endpoints of the edge $\{\boldsymbol{y}, \boldsymbol{z}\} \in E$. We define $\operatorname{relint}(V) := \operatorname{relint}(P) \cap V$, $\operatorname{bd}(V) := \operatorname{bd}(P) \cap V$, $\operatorname{relint}(E) := \{\{\boldsymbol{y}, \boldsymbol{z}\} \in E : (\boldsymbol{y}, \boldsymbol{z}) \not\subseteq \operatorname{bd}(P)\}$, and $\operatorname{bd}(E) := \{\{\boldsymbol{y}, \boldsymbol{z}\} \in E : (\boldsymbol{y}, \boldsymbol{z}) \subseteq \operatorname{bd}(P)\}$.

Up to unimodular transformations, there are three types of half-open unimodular simplices in $\mathbb{R}^2$ that we will consider: these are T_0, T_1, and T_2 as given in Figure 3.1.

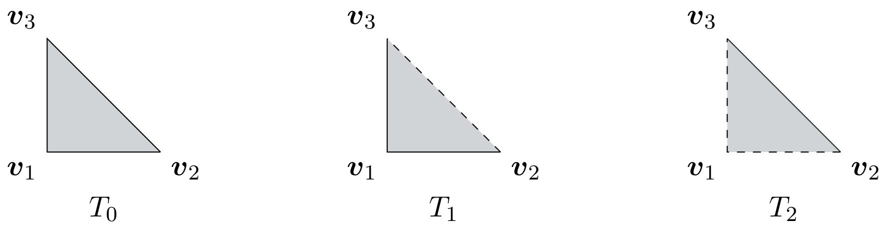

Figure 3.1: Types of half-open unimodular simplices in $\mathbb{R}^2$

A Pick-type vector formula

To determine the h^1-tensors from Theorem 3.6, note that a closed form of the Eulerian polynomial is

$$A_j(t) \;=\; \sum_{n=0}^{j}\sum_{i=0}^{n}(-1)^i\binom{j+1}{i}(n-i)^j t^n \tag{3.5}$$

(see, e.g., [16]). We then observe that $A_0(t) = 1$, $A_1(t) = t$, and $A_2(t) = t^2 + t$.

A comparison of coefficients of the numerator of (3.3) and that in Theorem 3.6 yields the formula

$$h^1_{S^*}(t) \;=\; \sum_{i=0}^{2} \mathrm{L}^1(S_i)t^i(1-t) + \mathrm{L}^0(S_i)t^{i+1}(\boldsymbol{v}_1+\boldsymbol{v}_2+\boldsymbol{v}_3)$$

implying that

$$h^1_i(S^*) \;=\; \mathrm{L}^1(S_i) - \mathrm{L}^1(S_{i-1}) + \mathrm{L}^0(S_{i-1})(\boldsymbol{v}_1+\boldsymbol{v}_2+\boldsymbol{v}_3) \tag{3.6}$$

for a half-open simplex S^* where S_i are defined as in Section 3.3.

By Theorem 3.5, any lattice polygon can be partitioned into unimodular transformations of half-open simplices. Therefore, to calculate h^r-tensors, we will need to understand the half-open parallelepipeds Π_{T_0}, Π_{T_1}, and Π_{T_2}. For ease, we provide skeletal descriptions of these here. By setting S^* to T_0, T_1, and T_2 with the vertices given in Figure 3.1, we obtain:

$$\begin{aligned}
&\Pi_{T_0} \cap \mathbb{Z}^3 = \{\boldsymbol{0}\},\\
&\Pi_{T_1} \cap \mathbb{Z}^3 = \{\boldsymbol{v}_1\} \times \{1\},\\
&\Pi_{T_2} \cap \mathbb{Z}^3 = \{\boldsymbol{v}_2+\boldsymbol{v}_3\} \times \{2\}.
\end{aligned} \tag{3.7}$$

Proposition 3.7. *For any lattice polygon, we have*

$$h^1_P(t) \;=\; t\sum_{\boldsymbol{x}\in V}\boldsymbol{x} + t^2\left(\sum_{\{\boldsymbol{y},\boldsymbol{z}\}\in \mathrm{relint}(E)}(\boldsymbol{y}+\boldsymbol{z}) - 2\sum_{\boldsymbol{x}\in\mathrm{relint}(V)}\boldsymbol{x}\right) + t^3\sum_{\boldsymbol{x}\in\mathrm{relint}(V)}\boldsymbol{x}.$$

Proof. We determine the h^1-tensor polynomial of all half-open unimodular simplices with vertices $\boldsymbol{v}_1,\boldsymbol{v}_2,\boldsymbol{v}_3$ up to a unimodular transformation. Using formula (3.6) together with the values given in (3.7), we obtain the following h^1-tensor polynomials for each T_i:

$$\begin{aligned}
h^1_{T_0}(t) \;&=\; t(\boldsymbol{v}_1+\boldsymbol{v}_2+\boldsymbol{v}_3)\\
h^1_{T_1}(t) \;&=\; t\boldsymbol{v}_1 + t^2(\boldsymbol{v}_2+\boldsymbol{v}_3)\\
h^1_{T_2}(t) \;&=\; t^2((\boldsymbol{v}_1+\boldsymbol{v}_2)+(\boldsymbol{v}_1+\boldsymbol{v}_3)-2\boldsymbol{v}_1) + t^3\boldsymbol{v}_1
\end{aligned}$$

Theorem 3.5 together with a careful inspection of the h^1-tensor polynomials of the half-open simplices yield the result. □

From Proposition 3.7, we can deduce formulas for the Ehrhart vectors.

Proposition 3.8. *For any lattice polygon P, we have*

$$\begin{aligned}
\mathrm{L}^1(nP) \;=\;& \frac{n}{6}\left(2\sum_{\boldsymbol{x}\in V}\boldsymbol{x} + 4\sum_{\boldsymbol{x}\in\mathrm{relint}(V)}\boldsymbol{x} - \sum_{\{\boldsymbol{y},\boldsymbol{z}\}\in\mathrm{relint}(E)}(\boldsymbol{y}+\boldsymbol{z})\right)\\
&+\frac{n^2}{2}\sum_{\boldsymbol{x}\in\mathrm{bd}(V)}\boldsymbol{x} + \frac{n^3}{6}\left(\sum_{\boldsymbol{x}\in\mathrm{bd}(V)}\boldsymbol{x} + \sum_{\{\boldsymbol{y},\boldsymbol{z}\}\in\mathrm{relint}(E)}(\boldsymbol{y}+\boldsymbol{z})\right).
\end{aligned} \tag{3.8}$$

Proof. By definition, the Ehrhart vector polynomial equals

$$\mathrm{L}^1(nP) \;=\; h_0^1(P)\binom{n+3}{3} + h_1^1(P)\binom{n+2}{3} + h_2^1(P)\binom{n+1}{3} + h_3^1(P)\binom{n}{3}.$$

A substitution of values from Proposition 3.7 yields

$$\mathrm{L}^1(nP) = \frac{n^3+3n^2+2n}{6}\sum_V \boldsymbol{x} + \frac{n^3-n}{6}\left(\sum_{\mathrm{relint}(E)}(\boldsymbol{y}+\boldsymbol{z}) - 2\sum_{\mathrm{relint}(V)}\boldsymbol{x}\right) + \frac{n^3-3n^2+2n}{6}\sum_{\mathrm{int}(V)}\boldsymbol{x}\,.$$

The result now follows from a quick comparison of coefficients. □

Note that the cubic term in the polynomial in the right-hand side of (3.8) is the moment vector $\int_P \boldsymbol{x}\,\mathrm{d}\boldsymbol{x}$. This can alternatively be derived as follows. We triangulate P into unimodular simplices $S_1,\dots,S_m$. If S_i, $i\in[m]$, has vertices $\boldsymbol{v}_1,\boldsymbol{v}_2$ and $\boldsymbol{v}_3$, then $\int_{S_i}\boldsymbol{x}\,\mathrm{d}\boldsymbol{x} = \frac{1}{6}\sum_{k=1}^3 \boldsymbol{v}_k$ since the mean of the vertices of a simplex equals its centroid, cf. Chapter 1. The equation $\int_P \boldsymbol{x}\,\mathrm{d}\boldsymbol{x} = \sum_{i=1}^m \int_{S_i}\boldsymbol{x}\,\mathrm{d}\boldsymbol{x}$ then yields the same formula for the moment vector of P.

A Pick-type matrix formula

We now determine the h^2-tensors in order to find a Pick-type formula for the discrete moment matrix.

Similar to the vector case, by comparing coefficients of the numerator of (3.3) and that in Theorem 3.6, we obtain the formula

$$\begin{aligned} h^2_{S^*}(t) \;&=\; \sum_{i=0}^{2}\mathrm{L}^2(S_i)t^i(1-t)^2 + 2(\boldsymbol{v}_1+\boldsymbol{v}_2+\boldsymbol{v}_3)\,\mathrm{L}^1(S_i)t^{i+1}(1-t)\\ &\quad+(\boldsymbol{v}_1^2+\boldsymbol{v}_2^2+\boldsymbol{v}_3^2)\,\mathrm{L}^0(S_i)t^{i+1} + (\boldsymbol{v}_1+\boldsymbol{v}_2+\boldsymbol{v}_3)^2\,\mathrm{L}^0(S_i)t^{i+2}\end{aligned}$$

for a half-open simplex S^* where S_i are defined as in Section 3.3. The h^2-tensors of a half-open simplex are then found to be

$$\begin{aligned} h_i^2(S^*) \;&=\; \mathrm{L}^2(S_i) - 2\,\mathrm{L}^2(S_{i-1}) + \mathrm{L}^2(S_{i-2}) + 2(\boldsymbol{v}_1+\boldsymbol{v}_2+\boldsymbol{v}_3)\left(\mathrm{L}^1(S_{i-1}) - \mathrm{L}^1(S_{i-2})\right)\\ &\quad+(\boldsymbol{v}_1^2+\boldsymbol{v}_2^2+\boldsymbol{v}_3^2)\,\mathrm{L}^0(S_{i-1}) + (\boldsymbol{v}_1+\boldsymbol{v}_2+\boldsymbol{v}_3)^2\,\mathrm{L}^0(S_{i-2})\,.\end{aligned} \tag{3.9}$$

Proposition 3.9. *If P is a lattice polygon, then*

$$h_P^2(t) \;=\; t\sum_V \boldsymbol{x}^2 + t^2\left(\sum_E(\boldsymbol{y}+\boldsymbol{z})^2 - \sum_V \boldsymbol{x}^2\right) + t^3\left(\sum_{\mathrm{relint}(E)}(\boldsymbol{y}+\boldsymbol{z})^2 - \sum_{\mathrm{relint}(V)}\boldsymbol{x}^2\right) + t^4\sum_{\mathrm{relint}(V)}\boldsymbol{x}^2.$$

Proof. Similar to the h^1-tensor polynomial, we determine the h^2-tensor polynomial of all half-open unimodular simplices, up to unimodular transformation. Formula (3.9) for each T_i with the values from (3.7) yields the following:

$$\begin{aligned}
h^2_{T_0}(t) &= t(\boldsymbol{v}_1^2+\boldsymbol{v}_2^2+\boldsymbol{v}_3^2)+t^2((\boldsymbol{v}_1+\boldsymbol{v}_2)^2+(\boldsymbol{v}_2+\boldsymbol{v}_3)^2+(\boldsymbol{v}_3+\boldsymbol{v}_1)^2-\boldsymbol{v}_1^2-\boldsymbol{v}_2^2-\boldsymbol{v}_3^2)\\
h^2_{T_1}(t) &= t\boldsymbol{v}_1^2+t^2((\boldsymbol{v}_1+\boldsymbol{v}_2)^2+(\boldsymbol{v}_1+\boldsymbol{v}_3)^2-\boldsymbol{v}_1^2)+t^3(\boldsymbol{v}_2+\boldsymbol{v}_3)^2\\
h^2_{T_2}(t) &= t^2(\boldsymbol{v}_2+\boldsymbol{v}_3)^2+t^3((\boldsymbol{v}_1+\boldsymbol{v}_2)^2+(\boldsymbol{v}_1+\boldsymbol{v}_3)^2-\boldsymbol{v}_1^2)+t^4\boldsymbol{v}_1^2
\end{aligned}$$

The claim now follows from Theorem 3.5. □

From Proposition 3.9, we can now deduce formulas for the Ehrhart matrices.

Proposition 3.10. *Given a lattice polygon P, we have*

$$\begin{aligned}
\mathrm{L}^2(nP) &= \frac{n}{12}\sum_{\mathrm{bd}(E)}(\boldsymbol{y}-\boldsymbol{z})^2+\frac{n^2}{24}\left(12\sum_{V}\boldsymbol{x}^2+12\sum_{\mathrm{relint}(V)}\boldsymbol{x}^2-\sum_{E}(\boldsymbol{y}+\boldsymbol{z})^2-\sum_{\mathrm{relint}(E)}(\boldsymbol{y}+\boldsymbol{z})^2\right)\\
&\quad+\frac{n^3}{12}\left(2\sum_{\mathrm{bd}(V)}\boldsymbol{x}^2+\sum_{\mathrm{bd}(E)}(\boldsymbol{y}+\boldsymbol{z})^2\right)+\frac{n^4}{24}\left(\sum_{E}(\boldsymbol{y}+\boldsymbol{z})^2+\sum_{\mathrm{relint}(E)}(\boldsymbol{y}+\boldsymbol{z})^2\right).
\end{aligned}$$

Proof. By definition, the Ehrhart matrix polynomial equals

$$\mathrm{L}^2(nP) = h_0^2(P)\binom{n+4}{4}+h_1^2(P)\binom{n+3}{4}+h_2^2(P)\binom{n+2}{4}+h_3^2(P)\binom{n+1}{4}+h_4^2(P)\binom{n}{4}.$$

Proposition 3.9 and comparing coefficients now imply the result. For $\mathrm{L}_1^2(P)$, we further observe that

$$\mathrm{L}_1^2(P) = \frac{1}{12}\left(4\sum_{\mathrm{bd}(V)}\boldsymbol{x}^2-\sum_{\mathrm{bd}(E)}(\boldsymbol{y}+\boldsymbol{z})^2\right) = \frac{1}{12}\sum_{\mathrm{bd}(E)}(\boldsymbol{y}-\boldsymbol{z})^2.$$ □

3.5 Positivity for h^2-vectors

A fundamental theorem in Ehrhart theory is Stanley's Nonnegativity Theorem (Theorem 1.5) that states that the h^*-vector of every lattice polytope has nonnegative entries. While positivity of real numbers is canonically defined up to sign change, there are many different choices for higher dimensional vector spaces such as $\mathbb{T}^r$; one for every pointed cone (compare, e.g., [74, Section 6]). An important and well-studied cone inside the vector space of symmetric matrices is the cone of positive semidefinite matrices.

A symmetric matrix $M \in \mathbb{R}^{d\times d}$ is called **positive semidefinite** if $\boldsymbol{x}^T M\boldsymbol{x} \geq 0$ for all $\boldsymbol{x} \in \mathbb{R}^d$. By the identification of $\mathbb{T}^2$ with $\mathbb{R}^{d\times d}$, we call a tensor $T \in \mathbb{T}^2$ **positive semidefinite** if its corresponding symmetric matrix (T_{ij}) is positive semidefinite. By the spectral theorem, T is then a **sum of squares**; more precisely, if T has eigenvalues $\lambda_1,\dots,\lambda_d \geq 0$ and corresponding normalized eigenvectors $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_d$, then

$$(T_{ij}) \;=\; \sum_{k=1}^{d} \lambda_k \boldsymbol{u}_k \boldsymbol{u}_k^T$$

which is equivalent to $T = \sum_{k=1}^{d} \lambda_k \boldsymbol{u}_k^2 \in \mathbb{T}^2$. Therefore, a tensor is positive semidefinite if and only if it is a sum of squares.

As is the case for usual Ehrhart polynomials, the coefficients of Ehrhart tensor polynomials can be negative. However, in contrast to Ehrhart polynomials, this phenomenon already appears in dimension 2. For segments, it can be seen that the linear coefficient of the Ehrhart tensor polynomial is $\sum_E (\boldsymbol{y} - \boldsymbol{z})^2$. Furthermore, by [85, Lemma 26] and Proposition 3.3, all coefficients for line segments are positive semidefinite. The following example demonstrates negative definiteness in the plane.

Example 3.11. Let P be the triangle spanned by vertices $\boldsymbol{v}_1 = (0,1)^T$, $\boldsymbol{v}_2 = (-1,-7)^T$ and $\boldsymbol{v}_3 = (1,-4)^T$. The Ehrhart tensor polynomial of P can be calculated by Proposition 3.10 to be

$$\mathrm{L}^2(nP) = \begin{pmatrix} \frac{1}{2} & \frac{3}{4} \\ \frac{3}{4} & \frac{49}{6} \end{pmatrix} n + \begin{pmatrix} -\frac{1}{12} & -\frac{1}{8} \\ -\frac{1}{8} & -\frac{23}{12} \end{pmatrix} n^2 + \begin{pmatrix} \frac{1}{2} & \frac{3}{4} \\ \frac{3}{4} & \frac{149}{6} \end{pmatrix} n^3 + \begin{pmatrix} \frac{13}{12} & \frac{13}{8} \\ \frac{13}{8} & \frac{1079}{12} \end{pmatrix} n^4.$$

We observe that the coefficient of n^2 is negative definite. Lattice triangles for which this coefficient is indefinite also exist; for example, the triangle with vertices at $(0,-4)^T$, $(0,4)^T$, and $(-1,0)^T$.

Our main result is the following analogue to Stanley's Nonnegativity Theorem for the h^2-tensor polynomial of a lattice polygon.

Theorem 3.12. *The h^2-tensors of any lattice polygon are positive semidefinite.*

Before proving Theorem 3.12, we make a few more observations. Positive semidefiniteness of h^2-tensors is preserved under unimodular transformations since, from Equation (3.2) and comparing coefficients, we have

$$h_i^r(\phi P)(\boldsymbol{v}, \boldsymbol{v}) = h_i^r(\phi^T \boldsymbol{v}, \phi^T \boldsymbol{v})$$

for all $P \in \mathcal{P}_{\mathbb{Z}}^d$, $\phi \in \mathrm{GL}\,(d, \mathbb{Z})$, and $\boldsymbol{v} \in \mathbb{R}^d$. However, as the next example shows, positive semidefiniteness of the h^2-vector is in general not preserved under translation.

Example 3.13. Let $S = \mathrm{conv}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\} \setminus \mathrm{conv}\{\boldsymbol{v}_2, \boldsymbol{v}_3\}$ be the half-open simplex with vertices $\boldsymbol{v}_1 = (3,-2)^T$, $\boldsymbol{v}_2 = (2,-2)^T$, and $\boldsymbol{v}_3 = (2,-1)^T$. From the formula of the h^2-vector of a half-open simplex which can be found in the proof of Proposition 3.9, we obtain that

$$h_S^2(t) \;=\; \begin{pmatrix} 4 & -4 \\ -4 & 4 \end{pmatrix} t + \begin{pmatrix} 37 & -28 \\ -28 & 21 \end{pmatrix} t^2 + \begin{pmatrix} 25 & -15 \\ -15 & 9 \end{pmatrix} t^3 .$$

That is, with a determinant of -7, the matrix $h_2^2(S)$ is not positive semidefinite. However, it can be seen that the positive semidefiniteness of h^2-tensors is not preserved under translations.

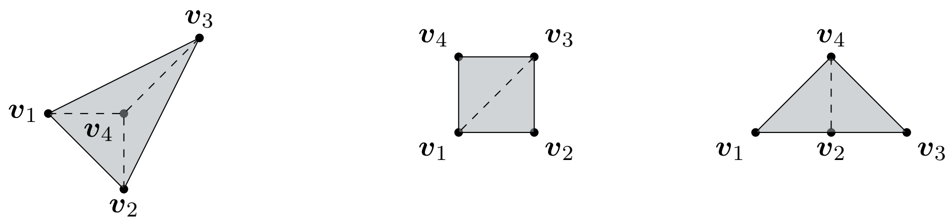

Figure 3.2: Lattice polygons with 4 lattice points and their unimodular triangulations

To illustrate, consider the translate $S - \boldsymbol{v}_2$. The h^2-vector of the translated simplex

$$h^2_{S-\boldsymbol{v}_2}(t) \;=\; \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} t^2 + \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} t^3$$

has positive semidefinite coefficients.

Since Example 3.13 shows that h^2-tensors of half-open polytopes can be negative, it follows that h^2-tensors are not monotone with respect to inclusion in contrast to the coefficients of the h^*-polynomial [114]. Therefore, techniques such as irrational decomposition or half-open decomposition that succesfully helped prove Stanley's Nonnegativity Theorem (see [16, 74]) cannot immediately be applied along with Theorem 3.5; we need to approach this issue differently.

To prove Theorem 3.12, we decompose a lattice polygon into lattice polygons with few vertices for which the h^2-vectors can easily be calculated. For the remainder of this chapter, allow a lattice polygon to always mean a full-dimensional lattice polygon in $\mathbb{R}^2$ although the argument is independent from the chosen ambient space. A **sparse decomposition** of $P \in \mathcal{P}^d_{\mathbb{Z}}$ is a finite set $\mathcal{D} = \{P_1, P_2, \dots, P_m\}$ of lattice polygons such that

i) $\mathrm{L}^0(P_i) \in \{3, 4\}$ for each $i \in [m]$,
ii) $P_i \cap P_j = \emptyset$ or is a common vertex of P_i and P_j for all $i \neq j$, and
iii) $P \cap \mathbb{Z}^2 = \bigcup_{i=1}^m P_i \cap \mathbb{Z}^2$.

Lemma 3.14. [84, Section 4] *Up to unimodular transformation, there are three different lattice polygons containing exactly four lattice points. They are given in Figure 3.2.*

The following lemma ensures that every lattice polygon has a sparse decomposition.

Lemma 3.15. *Every lattice polygon has a sparse decomposition.*

Proof. We proceed by induction on $\mathrm{L}^0(P)$. The statement is trivially true if $\mathrm{L}^0(P) \in \{3, 4\}$. Hence, we may assume that $\mathrm{L}^0(P) > 4$ and choose a vector $\boldsymbol{a} \in \mathbb{R}^2 \setminus \{0\}$ such that $\langle \boldsymbol{a}, \boldsymbol{v} \rangle \neq \langle \boldsymbol{a}, \boldsymbol{w} \rangle$

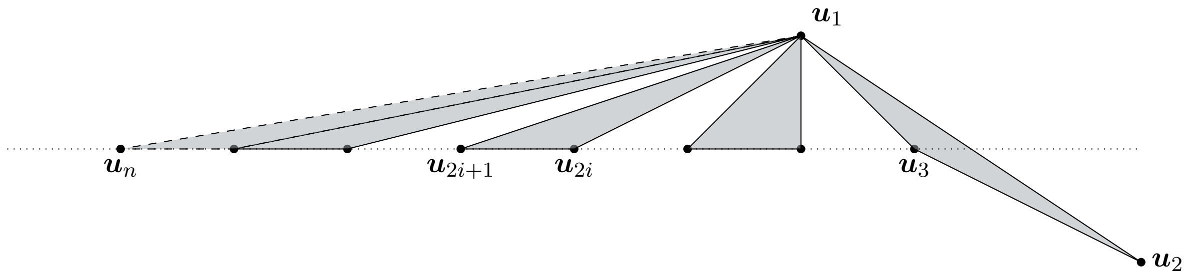

Figure 3.3: Sparse decomposition of P for the case of a collinear Q

for each $\boldsymbol{v}, \boldsymbol{w} \in P \cap \mathbb{Z}^2$ where $\boldsymbol{v} \neq \boldsymbol{w}$. Note that such an $\boldsymbol{a}$ exists since $\mathrm{L}^0(P)$ is finite. Let $P \cap \mathbb{Z}^2 = \{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$ be such that

$$\langle \boldsymbol{a}, \boldsymbol{v}_1 \rangle \;>\; \langle \boldsymbol{a}, \boldsymbol{v}_2 \rangle \;>\; \ldots \;>\; \langle \boldsymbol{a}, \boldsymbol{v}_n \rangle,$$

and set $Q = \operatorname{conv}\{\boldsymbol{v}_3, \boldsymbol{v}_4, \ldots, \boldsymbol{v}_n\}$. Then, by convexity, we obtain $Q \cap \mathbb{Z}^2 = P \cap \mathbb{Z}^2 \setminus \{\boldsymbol{v}_1, \boldsymbol{v}_2\}$.

If Q is not full-dimensional and all lattice points of Q lie on a single line, then a sparse decomposition of P can easily be constructed in the following way. If $\boldsymbol{v}_1, \boldsymbol{v}_2$, and $\boldsymbol{v}_3$ are not collinear, then we can construct a sparse decomposition which is illustrated in Figure 3.3. Let $P_1 = \operatorname{conv}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$. Then, by design, the triangle P_1 does not contain any other lattice point and at least one of $\boldsymbol{v}_1$ or $\boldsymbol{v}_2$ are visible from all points $\boldsymbol{v}_4, \ldots, \boldsymbol{v}_n$. Without loss of generality, assume $\boldsymbol{v}_1$ is visible. Then for all $2 \leq i \leq \lfloor \frac{n}{2} \rfloor - 1$ define $P_i = \operatorname{conv}\{\boldsymbol{v}_1, \boldsymbol{v}_{2i}, \boldsymbol{v}_{2i+1}\}$, $P_{\lfloor \frac{n}{2} \rfloor} = \operatorname{conv}\{\boldsymbol{v}_1, \boldsymbol{v}_{n-2}, \boldsymbol{v}_n\}$ if n is even, and $P_{\lfloor \frac{n}{2} \rfloor} = \operatorname{conv}\{\boldsymbol{v}_1, \boldsymbol{v}_{n-1}, \boldsymbol{v}_n\}$ if n is odd. Then $\{P_1, \ldots, P_{\lfloor \frac{n}{2} \rfloor}\}$ is a sparse decomposition. If $\boldsymbol{v}_1, \boldsymbol{v}_2$, and $\boldsymbol{v}_3$ are collinear, then a sparse decomposition can be obtained by instead setting $P_1 = \operatorname{conv}\{\boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4\}$.

Suppose Q is full-dimensional. Then, by the induction hypothesis, there is a sparse decomposition $\mathcal{D}_Q$ of Q. Let i be the smallest index such that the points $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_i$ do not lie on a common straight line. By construction, the simplex $S = \operatorname{conv}(\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_i)$ contains no other lattice points, and thus $\mathcal{D}_Q \cup \{S\}$ is a sparse decomposition of P. □

Lemma 3.16. *If $P \in \mathcal{P}^2_{\mathbb{Z}}$ is a lattice polygon containing exactly three or four lattice points, then $h^2_2(P)$ is positive semidefinite.*

Proof. If $\mathrm{L}^0(P) = 3$, then $P = \operatorname{conv}(\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3)$ is a unimodular lattice simplex and the statement follows from Proposition 3.9 as

$$h^2_2(P) = (\boldsymbol{v}_1 + \boldsymbol{v}_2)^2 + (\boldsymbol{v}_1 + \boldsymbol{v}_3)^2 + (\boldsymbol{v}_2 + \boldsymbol{v}_3)^2 - \boldsymbol{v}_1^2 - \boldsymbol{v}_2^2 - \boldsymbol{v}_3^2 = (\boldsymbol{v}_1 + \boldsymbol{v}_2 + \boldsymbol{v}_3)^2.$$

Suppose $\mathrm{L}^0(P) = 4$. We have to distinguish between the three possible cases, up to unimodular transformation, given in Figure 3.2. First, if P contains one interior lattice point $\boldsymbol{v}_4$ and vertices $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$, then we have $\boldsymbol{v}_4 = \frac{1}{3}(\boldsymbol{v}_1 + \boldsymbol{v}_2 + \boldsymbol{v}_3)$, cf. Lemma 3.17 *ii)*, and Proposition 3.9 implies that

$$\begin{aligned}h_2^2(P) &= (\boldsymbol{v}_1+\boldsymbol{v}_2)^2+(\boldsymbol{v}_1+\boldsymbol{v}_3)^2+(\boldsymbol{v}_2+\boldsymbol{v}_3)^2+(\boldsymbol{v}_1+\boldsymbol{v}_4)^2+(\boldsymbol{v}_2+\boldsymbol{v}_4)^2+(\boldsymbol{v}_3+\boldsymbol{v}_4)^2\\ &\quad -\boldsymbol{v}_1^2-\boldsymbol{v}_2^2-\boldsymbol{v}_3^2-\boldsymbol{v}_4^2\\ &= (\boldsymbol{v}_1+\boldsymbol{v}_2)^2+(\boldsymbol{v}_1+\boldsymbol{v}_3)^2+(\boldsymbol{v}_2+\boldsymbol{v}_3)^2+2\boldsymbol{v}_4^2+2\boldsymbol{v}_4(\boldsymbol{v}_1+\boldsymbol{v}_2+\boldsymbol{v}_3)\\ &= (\boldsymbol{v}_1+\boldsymbol{v}_2)^2+(\boldsymbol{v}_1+\boldsymbol{v}_3)^2+(\boldsymbol{v}_2+\boldsymbol{v}_3)^2+\tfrac{8}{9}(\boldsymbol{v}_1+\boldsymbol{v}_2+\boldsymbol{v}_3)^2.\end{aligned}$$

Next, if P is a parallelepiped, then $\boldsymbol{v}_1+\boldsymbol{v}_3=\boldsymbol{v}_2+\boldsymbol{v}_4$, and thus

$$\begin{aligned}h_2^2(P) &= (\boldsymbol{v}_1+\boldsymbol{v}_2)^2+(\boldsymbol{v}_2+\boldsymbol{v}_3)^2+(\boldsymbol{v}_3+\boldsymbol{v}_4)^2+(\boldsymbol{v}_1+\boldsymbol{v}_4)^2\\ &\quad +\tfrac{1}{2}(\boldsymbol{v}_1+\boldsymbol{v}_3)^2+\tfrac{1}{2}(\boldsymbol{v}_2+\boldsymbol{v}_4)^2-\boldsymbol{v}_1^2-\boldsymbol{v}_2^2-\boldsymbol{v}_3^2-\boldsymbol{v}_4^2\\ &= \tfrac{1}{2}(\boldsymbol{v}_1+\boldsymbol{v}_2+\boldsymbol{v}_3+\boldsymbol{v}_4)^2+\tfrac{1}{2}(\boldsymbol{v}_1+\boldsymbol{v}_2)^2+\tfrac{1}{2}(\boldsymbol{v}_2+\boldsymbol{v}_3)^2+\tfrac{1}{2}(\boldsymbol{v}_3+\boldsymbol{v}_4)^2+\tfrac{1}{2}(\boldsymbol{v}_1+\boldsymbol{v}_4)^2.\end{aligned}$$

Finally, if P has three vertices and no interior lattice point, then one lattice point of P, say $\boldsymbol{v}_2$ as in Figure 3.2, lies in the relative interior of the edge given by the vertices $\boldsymbol{v}_1$ and $\boldsymbol{v}_3$ implying that $\boldsymbol{v}_2=\frac{1}{2}(\boldsymbol{v}_1+\boldsymbol{v}_3)$, cf. Lemma 3.17 *i)*. In this case, we obtain

$$\begin{aligned}h_2^2(P) &= (\boldsymbol{v}_1+\boldsymbol{v}_2)^2+(\boldsymbol{v}_2+\boldsymbol{v}_3)^2+(\boldsymbol{v}_3+\boldsymbol{v}_4)^2+(\boldsymbol{v}_1+\boldsymbol{v}_4)^2+(\boldsymbol{v}_2+\boldsymbol{v}_4)^2-\boldsymbol{v}_1^2-\boldsymbol{v}_2^2-\boldsymbol{v}_3^2-\boldsymbol{v}_4^2\\ &= \tfrac{5}{2}\boldsymbol{v}_1^2+\tfrac{5}{2}\boldsymbol{v}_3^2+2\boldsymbol{v}_4^2+3\boldsymbol{v}_1\boldsymbol{v}_4+3\boldsymbol{v}_3\boldsymbol{v}_4+3\boldsymbol{v}_1\boldsymbol{v}_3\\ &= \tfrac{3}{2}(\boldsymbol{v}_1+\boldsymbol{v}_3+\boldsymbol{v}_4)^2+\boldsymbol{v}_1^2+\boldsymbol{v}_3^2+\tfrac{1}{2}\boldsymbol{v}_4^2.\end{aligned}$$ □

We will need the following geometric observation in our proof of Theorem 3.12.

Lemma 3.17. *Let $P\in\mathcal{P}^2_{\mathbb{Z}}$ and $\boldsymbol{v}$ be a lattice point in the relative interior of P. Then at least one of the following two statements is true:*

i) $\boldsymbol{v}=\frac{1}{2}(\boldsymbol{v}_1+\boldsymbol{v}_2)$ *for lattice points $\boldsymbol{v}_1,\boldsymbol{v}_2\in P$ such that $\boldsymbol{v}_1\neq\boldsymbol{v}_2$;*

ii) $\boldsymbol{v}=\frac{1}{3}(\boldsymbol{v}_1+\boldsymbol{v}_2+\boldsymbol{v}_3)$ *for pairwise distinct lattice points $\boldsymbol{v}_1,\boldsymbol{v}_2,\boldsymbol{v}_3\in P$.*

Proof. If $\boldsymbol{v}$ is contained in a segment formed by two lattice points in P, then $\boldsymbol{v}$ is easily seen to be of the form given in *i)*.

Therefore, we may assume that $\boldsymbol{v}$ is not contained in any line segment formed by lattice points in P. By Carathéodory's Theorem (see Theorem 1.1), there are lattice points $\boldsymbol{v}_1,\boldsymbol{v}_2,\boldsymbol{v}_3\in P$ such that $\boldsymbol{v}$ is contained in the simplex formed by $\boldsymbol{v}_1,\boldsymbol{v}_2$, and $\boldsymbol{v}_3$. If $\boldsymbol{v},\boldsymbol{v}_1,\boldsymbol{v}_2,\boldsymbol{v}_3$ are the only lattice points in the simplex, then condition *ii)* follows from Lemma 3.14. Otherwise, there is a lattice point $u\in\operatorname{conv}\{\boldsymbol{v}_1,\boldsymbol{v}_2,\boldsymbol{v}_3\}\setminus\{\boldsymbol{v},\boldsymbol{v}_1,\boldsymbol{v}_2,\boldsymbol{v}_3\}$ and, consequently, $\boldsymbol{v}$ must be contained in one of the three lattice simplices

$$S_1=\operatorname{conv}\{\boldsymbol{v}_2,\boldsymbol{v}_3,u\},\quad S_2=\operatorname{conv}\{\boldsymbol{v}_1,\boldsymbol{v}_3,u\},\quad S_3=\operatorname{conv}\{\boldsymbol{v}_1,\boldsymbol{v}_2,u\}.$$

Without loss of generality, let $\boldsymbol{v}\in S_1\subsetneq\operatorname{conv}\{\boldsymbol{v}_1,\boldsymbol{v}_2,\boldsymbol{v}_3\}$. By reiteration of the above procedure, each time with a replacement of $\boldsymbol{v}_1$ by $\boldsymbol{u}$, we eventually find affinely independent $\boldsymbol{v}_1,\boldsymbol{v}_2,\boldsymbol{v}_3$ such that $\{\boldsymbol{v},\boldsymbol{v}_1,\boldsymbol{v}_2,\boldsymbol{v}_3\}=\operatorname{conv}\{\boldsymbol{v}_1,\boldsymbol{v}_2,\boldsymbol{v}_3\}\cap\mathbb{Z}^2$ and condition *ii)* follows again from Lemma 3.14. □

We are now equipped to give the proof of our nonnegativity theorem.

Proof of Theorem 3.12. From Proposition 3.9, it immediately follows that $h_0^2(P), h_1^2(P)$, and $h_4^2(P)$ are sums of squares.

Let $\mathcal{D} = \{P_1, P_2, \ldots, P_m\}$ be a sparse decomposition of P which exists by Lemma 3.15 and let $\mathcal{S}$ be some triangulation of $\cup_{i=1}^m P_i$. Observe that the closure of $P \setminus (P_1 \cup \cdots \cup P_m)$ is a union of not necessarily convex lattice polygons and any triangulation of $\cup_{i=1}^m P_i$ can be extended to a triangulation in P. Let $\mathcal{T}$ be a triangulation of P such that $\mathcal{S} \subseteq \mathcal{T}$. Let $G = (V, E)$ be the edge graph of $\mathcal{T}$ and $G' = (V', E')$ be the edge graph of $\mathcal{S}$. For every $x \in V$, we define $\alpha_x = |\{i \in [m] : \boldsymbol{x} \in P_i\}|$. Note that $\alpha_{\boldsymbol{x}} \geq 1$ for all $\boldsymbol{x} \in V$ since $\mathcal{D}$ is a sparse decomposition. Proposition 3.9 then implies that

$$\begin{aligned} h_2^2(P) &= \sum_E (\boldsymbol{y}+\boldsymbol{z})^2 - \sum_V \boldsymbol{x}^2 \\ &= \sum_{E'} (\boldsymbol{y}+\boldsymbol{z})^2 - \sum_V \alpha_{\boldsymbol{x}}\, \boldsymbol{x}^2 + \sum_{E\setminus E'} (\boldsymbol{y}+\boldsymbol{z})^2 - \sum_V (1-\alpha_{\boldsymbol{x}})\, \boldsymbol{x}^2 \\ &= \sum_{i=1}^m h_2^2(P_i) + \sum_{E\setminus E'} (\boldsymbol{y}+\boldsymbol{z})^2 + \sum_V (\alpha_{\boldsymbol{x}} - 1)\, \boldsymbol{x}^2 \end{aligned}$$

and, therefore, by Lemma 3.16, $h_2^2(P)$ is a sum of squares.

We still need to show that $h_3^2(P)$ is a sum of squares as well. For every $\boldsymbol{v} \in V$, we define $N(\boldsymbol{v}) = \{\boldsymbol{u} \in V : \{\boldsymbol{u}, \boldsymbol{v}\} \in E\}$ to be the set of vertices adjacent to $\boldsymbol{v}$ in G. Let $E_1 \subseteq \operatorname{int}(E)$ be the set of edges that have exactly one endpoint on the boundary of P and $E_2 \subseteq \operatorname{int}(E)$ be the set of edges with both endpoints on the boundary of P but relative interior in $\operatorname{int}(P)$. By Proposition 3.9, we obtain

$$\begin{aligned} h_3^2(P) &= \sum_{\operatorname{int}(E)} (\boldsymbol{y}+\boldsymbol{z})^2 - \sum_{\operatorname{int}(V)} \boldsymbol{x}^2 \\ &= \sum_{\boldsymbol{v}\in\operatorname{int}(V)} \left(\sum_{\boldsymbol{u}\in N(\boldsymbol{v})} \left(\frac{1}{2}(\boldsymbol{v}+\boldsymbol{u})^2 \right) - \boldsymbol{v}^2 \right) + \sum_{E_1} \frac{1}{2}(\boldsymbol{y}+\boldsymbol{z})^2 + \sum_{E_2} (\boldsymbol{y}+\boldsymbol{z})^2 . \end{aligned}$$

Thus, it is sufficient to show that

$$a(\boldsymbol{v}) := \sum_{\boldsymbol{u}\in N(\boldsymbol{v})} \left(\frac{1}{2}(\boldsymbol{v}+\boldsymbol{u})^2 - \boldsymbol{v}^2 \right)$$

is a sum of squares for all $\boldsymbol{v} \in \operatorname{int}(V)$. In view of Lemma 3.17, we distinguish two cases. First, suppose that there are $\boldsymbol{v}_1, \boldsymbol{v}_2 \in V \setminus \{\boldsymbol{v}\}$ such that $\boldsymbol{v} = \frac{1}{2}(\boldsymbol{v}_1 + \boldsymbol{v}_2)$. Then

$$a(\boldsymbol{v}) = \tfrac{1}{2}(\boldsymbol{v}+\boldsymbol{v}_1)^2 + \tfrac{1}{2}(\boldsymbol{v}+\boldsymbol{v}_2)^2 - \boldsymbol{v}^2 + \sum_{u\in N(\boldsymbol{v})\setminus\{\boldsymbol{v}_1,\boldsymbol{v}_2\}} \tfrac{1}{2}(\boldsymbol{v}+\boldsymbol{u})^2$$

$$= \tfrac{1}{2}(\boldsymbol{v}_1+\boldsymbol{v}_2)^2 + \tfrac{1}{2}\boldsymbol{v}_1^2 + \tfrac{1}{2}\boldsymbol{v}_2^2 + \sum_{\boldsymbol{u}\in N(\boldsymbol{v})\setminus\{\boldsymbol{v}_1,\boldsymbol{v}_2\}} \tfrac{1}{2}(\boldsymbol{v}+\boldsymbol{u})^2\,.$$

In the second case, a pairwise disjoint $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3 \in V \setminus \{\boldsymbol{v}\}$ exists, such that $\boldsymbol{v} = \frac{1}{3}(\boldsymbol{v}_1+\boldsymbol{v}_2+\boldsymbol{v}_3)$. Therefore,

$$\begin{aligned} a(\boldsymbol{v}) &= \tfrac{1}{2}(\boldsymbol{v}+\boldsymbol{v}_1)^2 + \tfrac{1}{2}(\boldsymbol{v}+\boldsymbol{v}_2)^2 + \tfrac{1}{2}(\boldsymbol{v}+\boldsymbol{v}_3)^2 - \boldsymbol{v}^2 + \sum_{\boldsymbol{u}\in N(\boldsymbol{v})\setminus\{\boldsymbol{v}_1,\boldsymbol{v}_2,\boldsymbol{v}_3\}} \tfrac{1}{2}(\boldsymbol{v}+\boldsymbol{u})^2 \\ &= \tfrac{7}{18}(\boldsymbol{v}_1+\boldsymbol{v}_2+\boldsymbol{v}_3)^2 + \tfrac{1}{2}\boldsymbol{v}_1^2 + \tfrac{1}{2}\boldsymbol{v}_2^2 + \tfrac{1}{2}\boldsymbol{v}_3^2 + \sum_{\boldsymbol{u}\in N(\boldsymbol{v})\setminus\{\boldsymbol{v}_1,\boldsymbol{v}_2,\boldsymbol{v}_3\}} \tfrac{1}{2}(\boldsymbol{v}+\boldsymbol{u})^2\,. \end{aligned}$$ □

3.6 Determining h^r-tensors

In this section, we will discuss some methods motivated from the classical Ehrhart theory on how we can calculate h^r-tensors of polytopes. We will present techniques for joins, pyramids and bipyramids of lattice polytopes, which we will demonstrate on the standard simplex and the standard cross-polytope.

h^r-tensors of pyramids and bipyramids

It is well-known that the classical Ehrhart series of a *pyramid* over a lattice polytope P admits a particular nice formula. Namely, for a lattice polytope $P \subseteq \mathbb{R}^{d-1}$ we define the **pyramid** over P as

$$\operatorname{Pyr}(P) = \operatorname{conv}\left(P \times \{0\}, \boldsymbol{e}_{d+1}\right)\,.$$

Then, for the classical Ehrhart series of $\operatorname{Pyr}(P)$ it holds [16, Theorem 2.4]

$$\operatorname{Ehr}^0_{\operatorname{Pyr}(P)}(t) = \frac{\operatorname{Ehr}^0_P(t)}{1-t}. \tag{3.10}$$

Similarly, we define the **bipyramid** over the lattice polytope $P \subseteq \mathbb{R}^{d-1}$ by

$$\operatorname{BiPyr}(P) = \operatorname{conv}\left(P \times \{0\}, \pm\boldsymbol{e}_{d+1}\right)\,,$$

and for the classical Ehrhart series of $\operatorname{BiPyr}(P)$ it holds [16, Theorem 2.6]

$$\operatorname{Ehr}^0_{\operatorname{BiPyr}(P)}(t) = \frac{1+t}{1-t}\operatorname{Ehr}^0_P(t). \tag{3.11}$$

In this section, we will follow the approach presented in Section 2.4 and Section 2.5 in [16] and apply it to the Ehrhart tensor series of positive rank to derive a generalizaton of (3.10) and (3.11).

First, we formalize a simple formula for the Ehrhart tensor polynomial of a pyramid over a polytope.

Lemma 3.18. *Let $r > 0$, $i_1, \dots, i_r \in [d+1]$ be indices, such that for some $0 \le m \le r$ we have that $i_1, \dots, i_m < d+1$, and $i_{m+1} = \cdots = i_r = d+1$, then*

$$\mathrm{L}^r(n\mathrm{Pyr}\,(P))(\boldsymbol{e}_{i_1}, \dots, \boldsymbol{e}_{i_r}) = \sum_{j=0}^{n} (n-j)^{r-m}\, \mathrm{L}^m(jP)(\boldsymbol{e}_{i_1}, \dots, \boldsymbol{e}_{i_m}).$$

Proof.

$$\begin{aligned}
&\mathrm{L}^r(n\mathrm{Pyr}\,(P))(\boldsymbol{e}_{i_1}, \dots, \boldsymbol{e}_{i_r}) = \sum_{\boldsymbol{x} \in n\mathrm{Pyr}(P)} \boldsymbol{x}^r(\boldsymbol{e}_{i_1}, \dots, \boldsymbol{e}_{i_r}) \\
&= \sum_{j=0}^{n} \sum_{\boldsymbol{y} \in (n-j)P} \begin{pmatrix} \boldsymbol{y} \\ j \end{pmatrix}^r (\boldsymbol{e}_{i_1}, \dots, \boldsymbol{e}_{i_r}) = \sum_{j=0}^{n} \sum_{\boldsymbol{y} \in jP} \begin{pmatrix} \boldsymbol{y} \\ n-j \end{pmatrix}^r (\boldsymbol{e}_{i_1}, \dots, \boldsymbol{e}_{i_r}) \\
&= \sum_{j=0}^{n} \sum_{\boldsymbol{y} \in jP} (n-j)^{r-m} \boldsymbol{y}^m(\boldsymbol{e}_{i_1}, \dots, \boldsymbol{e}_{i_m}) = \sum_{j=0}^{n} (n-j)^{r-m}\, \mathrm{L}^m(jP)(\boldsymbol{e}_{i_1}, \dots, \boldsymbol{e}_{i_m}).
\end{aligned}$$

□

Next, we will present the analog of (3.10) for Ehrhart tensor polynomials of positive rank.

Theorem 3.19. *Let $r > 0$, $i_1, \dots, i_r \in [d+1]$ be indices, such that for some $0 \le m \le r$ we have that $i_1, \dots, i_m < d+1$, and $i_{m+1} = \cdots = i_r = d+1$, then*

$$\mathrm{Ehr}^r_{\mathrm{Pyr}(P)}(t)(\boldsymbol{e}_{i_1}, \dots, \boldsymbol{e}_{i_r}) = p_{r-m}(t)\, \mathrm{Ehr}^m_P(t)(\boldsymbol{e}_{i_1}, \dots, \boldsymbol{e}_{i_m}), \tag{3.12}$$

where

$$p_\ell(t) = \frac{\sum_{k=0}^{\ell} \mathrm{A}\,(\ell, k) t^k}{(1-t)^{\ell+1}} = (1-t)^{-(l+1)} A_\ell(t),$$

where $\mathrm{A}\,(\ell, k)$ are the Eulerian numbers, and $A_\ell(t)$ is the ℓ-th Eulerian polynomial, see Chapter 1, page 16.

Proof.

$$\begin{aligned}
&\mathrm{Ehr}^r_{\mathrm{Pyr}(P)}(t)(\boldsymbol{e}_{i_1}, \dots, \boldsymbol{e}_{i_r}) = \sum_{n \ge 0} \mathrm{L}^r(n\mathrm{Pyr}\,(P)) t^n (\boldsymbol{e}_{i_1}, \dots, \boldsymbol{e}_{i_r}) \\
&= \sum_{n \ge 0} \sum_{j=0}^{n} (n-j)^{r-m}\, \mathrm{L}^m(jP)(\boldsymbol{e}_{i_1}, \dots, \boldsymbol{e}_{i_m}) t^n \\
&= \sum_{j \ge 0} \sum_{n \ge j} (n-j)^{r-m}\, \mathrm{L}^m(jP)(\boldsymbol{e}_{i_1}, \dots, \boldsymbol{e}_{i_m}) t^n \\
&= \sum_{j \ge 0} \mathrm{L}^m(jP)(\boldsymbol{e}_{i_1}, \dots, \boldsymbol{e}_{i_m}) \sum_{n \ge j} (n-j)^{r-m} t^n \\
&= \sum_{j \ge 0} \mathrm{L}^m(jP)(\boldsymbol{e}_{i_1}, \dots, \boldsymbol{e}_{i_m}) t^j \sum_{n \ge 0} n^{r-m} t^n \\
&= \mathrm{Ehr}^m_P(t)(\boldsymbol{e}_{i_1}, \dots, \boldsymbol{e}_{i_m}) \frac{\sum_{k=0}^{r-m} \mathrm{A}\,(r-m, k) t^k}{(1-t)^{r-m+1}}.
\end{aligned}$$

□

Remark 3.20. *The rational function* $(1/t)p_d(t)$ *is the (classical) Ehrhart series of the standard cube* $C_d = [0,1]^d$*, i.e.,* $\mathrm{Ehr}^0_{C_d}(t) = \frac{1}{t}p_d(t)$*, see* [16, Theorem 2.1].

With basic calculations one can deduce that

$$\begin{aligned} p_0(t) &= \frac{1}{1-t}, \\ p_1(t) &= \frac{t}{(1-t)^2}, \\ p_2(t) &= \frac{t^2+t}{(1-t)^3}. \end{aligned}$$

Therefore, for the tensor Ehrhart series of rank 1 and 2 of a pyramid over the lattice polytope P (3.12) there are the following identities in vector and matrix form, respectively.

$$\begin{aligned} \mathrm{Ehr}^1_{\mathrm{Pyr}(P)}(t) &= \begin{pmatrix} \frac{1}{1-t}\,\mathrm{Ehr}^1_P(t) \\ \frac{t}{(1-t)^2}\,\mathrm{Ehr}^0_P(t) \end{pmatrix}, \\ \mathrm{Ehr}^2_{\mathrm{Pyr}(P)}(t) &= \begin{pmatrix} \frac{1}{1-t}\,\mathrm{Ehr}^2_P(t) & \frac{t}{(1-t)^2}\,\mathrm{Ehr}^1_P(t) \\ \frac{t}{(1-t)^2}\left(\mathrm{Ehr}^1_P(t)\right)^T & \frac{t^2+t}{(1-t)^3}\,\mathrm{Ehr}^0_P(t) \end{pmatrix}. \end{aligned} \tag{3.13}$$

Example 3.21. We consider the d-dimensional standard simplex Δ_d in $\mathbb{R}^d$. For the Δ_1 we have the well-known formulas $\mathrm{L}^1(n\Delta_1) = \sum_{j=0}^n j = n(n+1)/2$ and $\mathrm{L}^2(n\Delta_1) = \sum_{j=0}^n j^2 = n(n+1)(2n+1)/6$, and thus

$$\begin{aligned} \mathrm{Ehr}^1_{\Delta_1}(t) &= \sum_{n\geq 0} \frac{n(n+1)}{2} t^n = \frac{t}{(1-t)^3}, \\ \mathrm{Ehr}^2_{\Delta_1}(t) &= \sum_{n\geq 0} \frac{n(n+1)(2n+1)}{6} t^n = \frac{t^2+t}{(1-t)^4}. \end{aligned}$$

Since Δ_{d+1} is the pyramid over Δ_d, we conclude inductively over the dimension d by (3.13) and using that $\mathrm{Ehr}^0_{\Delta_d}(t) = 1/(1-t)^{d+1}$ [16, Theorem 2.2]

$$\mathrm{Ehr}^1_{\Delta_d}(t) = \frac{t}{(1-t)^{d+2}}\mathbf{1}_d. \tag{3.14}$$

Moreover, using (3.13) and (3.14) it holds

$$\mathrm{Ehr}^2_{\Delta_d}(t) = \frac{\left(\mathbf{1}_{d\times d}t^2 + \mathrm{I}_d t\right)}{(1-t)^{d+3}},$$

by induction, where $\mathbf{1}_{d\times d}$ denotes the $d\times d$ all-ones matrix.

Now, we will follow a similar method for the bipyramid over a lattice polytope.

Lemma 3.22. *Let* $r > 0$*,* $i_1,\dots,i_r \in [d]$ *be indices, such that* $i_1,\dots,i_m < d+1$ *and* $i_{m+1} = \cdots = i_r = d+1$ *for some* $0 \leq m \leq r$*. Then, the following three statements hold true.*

- *If $m = r$, we have that*

$$\mathrm{L}^r(n\mathrm{BiPyr}\,(P))(\boldsymbol{e}_{i_1}, \ldots, \boldsymbol{e}_{i_r}) = \left(2\sum_{j=0}^{n} \mathrm{L}^r(jP) - \mathrm{L}^r(nP)\right)(\boldsymbol{e}_{i_1}, \ldots, \boldsymbol{e}_{i_r}).$$

- *If $m < r$ and $r \equiv m \bmod 2$, we have*

$$\mathrm{L}^r(n\mathrm{BiPyr}\,(P))(\boldsymbol{e}_{i_1}, \ldots, \boldsymbol{e}_{i_r}) = 2\sum_{j=0}^{n}(n-j)^{r-m}\,\mathrm{L}^m(jP)(\boldsymbol{e}_{i_1}, \ldots, \boldsymbol{e}_{i_m}),$$

- *If $m < r$ and $r \not\equiv m \bmod 2$, we have*

$$\mathrm{L}^r(n\mathrm{BiPyr}\,(P))(\boldsymbol{e}_{i_1}, \ldots, \boldsymbol{e}_{i_r}) = 0.$$

Proof. We calculate

$$\begin{aligned}
&\mathrm{L}^r(n\mathrm{BiPyr}\,(P))(\boldsymbol{e}_{i_1}, \ldots, \boldsymbol{e}_{i_r}) = \sum_{\boldsymbol{x}\in n\mathrm{BiPyr}(P)} \boldsymbol{x}^r(\boldsymbol{e}_{i_1}, \ldots, \boldsymbol{e}_{i_r}) \\
&= \sum_{j=0}^{n}\left(\sum_{\boldsymbol{y}\in(n-j)P}\binom{\boldsymbol{y}}{j}^r + \sum_{\boldsymbol{y}\in(n-j)P}\binom{\boldsymbol{y}}{-j}^r\right)(\boldsymbol{e}_{i_1}, \ldots, \boldsymbol{e}_{i_r}) - \binom{\mathrm{L}^r(nP)}{0}(\boldsymbol{e}_{i_1}, \ldots, \boldsymbol{e}_{i_r}) \\
&= \sum_{j=0}^{n}\sum_{\boldsymbol{y}\in jP}\left(\binom{\boldsymbol{y}}{n-j}^r + \binom{\boldsymbol{y}}{j-n}^r\right)(\boldsymbol{e}_{i_1}, \ldots, \boldsymbol{e}_{i_r}) - \binom{\mathrm{L}^r(nP)}{0}(\boldsymbol{e}_{i_1}, \ldots, \boldsymbol{e}_{i_r}) \\
&= \sum_{j=0}^{n}\sum_{\boldsymbol{y}\in jP}\left((n-j)^{r-m} + (j-n)^{r-m}\right)\boldsymbol{y}^m(\boldsymbol{e}_{i_1}, \ldots, \boldsymbol{e}_{i_m}) - \binom{\mathrm{L}^r(nP)}{0}(\boldsymbol{e}_{i_1}, \ldots, \boldsymbol{e}_{i_r}).
\end{aligned}$$

Observing that $\binom{\mathrm{L}^r(nP)}{0}(\boldsymbol{e}_{i_1}, \ldots, \boldsymbol{e}_{i_r})$ is zero if $m < r$ closes the proof. □

Theorem 3.23. *Let $r > 0$, $i_1, \ldots, i_r \in [d]$ be indices, such that for some $0 \le m \le r$ we have that $i_1, \ldots, i_m < d+1$, and $i_{m+1} = \cdots = i_r = d+1$. If $m = r$, we have that*

$$\mathrm{Ehr}^r_{\mathrm{BiPyr}(P)}(t)(\boldsymbol{e}_{i_1}, \ldots, \boldsymbol{e}_{i_r}) = \frac{1+t}{1-t}\,\mathrm{Ehr}^r_P(t)(\boldsymbol{e}_{i_1}, \ldots, \boldsymbol{e}_{i_r})$$

If $m \neq r$ and $r \equiv m \bmod 2$ we have

$$\mathrm{Ehr}^r_{\mathrm{BiPyr}(P)}(t)(\boldsymbol{e}_{i_1}, \ldots, \boldsymbol{e}_{i_r}) = 2p_{r-m}(t)\,\mathrm{Ehr}^m_P(t).$$

and $\mathrm{Ehr}^r_{\mathrm{BiPyr}(P)}(t)(\boldsymbol{e}_{i_1}, \ldots, \boldsymbol{e}_{i_r}) = 0$ *otherwise.*

Proof. For $r \not\equiv m \bmod 2$ the statement is trivial, and for $r \neq m$, $r \equiv m \bmod 2$ the proof is analogous to the proof of Theorem 3.19. Thus, we assume $r = m$.

$$\begin{aligned}
\mathrm{Ehr}^r_{\mathrm{BiPyr}P}(t)(\boldsymbol{e}_{i_1}, \ldots, \boldsymbol{e}_{i_r}) &= \sum_{n\geq 0} \mathrm{L}^r(n\mathrm{BiPyr}\,(P))(\boldsymbol{e}_{i_1}, \ldots, \boldsymbol{e}_{i_r})t^n \\
&= \sum_{n\geq 0} \left(2\sum_{j=0}^{n} \mathrm{L}^r(jP) - \mathrm{L}^r(nP) \right) (\boldsymbol{e}_{i_1}, \ldots, \boldsymbol{e}_{i_r})t^n \\
&= 2\sum_{n\geq 0}\sum_{j=0}^{n} \mathrm{L}^r(jP)(\boldsymbol{e}_{i_1}, \ldots, \boldsymbol{e}_{i_r})t^n - \mathrm{Ehr}^r_P(t)(\boldsymbol{e}_{i_1}, \ldots, \boldsymbol{e}_{i_r}).
\end{aligned}$$

And since

$$\begin{aligned}
&\sum_{n\geq 0}\sum_{j=0}^{n} \mathrm{L}^r(jP)(\boldsymbol{e}_{i_1}, \ldots, \boldsymbol{e}_{i_r})t^n = \sum_{j\geq 0}\sum_{n\geq j} \mathrm{L}^r(jP)(\boldsymbol{e}_{i_1}, \ldots, \boldsymbol{e}_{i_r})t^n \\
&= \sum_{j\geq 0} \mathrm{L}^r(jP)(\boldsymbol{e}_{i_1}, \ldots, \boldsymbol{e}_{i_r}) \sum_{n\geq 0} t^{n+j} = \frac{1}{1-t}\sum_{j\geq 0} \mathrm{L}^r(jP)(\boldsymbol{e}_{i_1}, \ldots, \boldsymbol{e}_{i_r})t^j = \frac{1}{1-t}\,\mathrm{Ehr}^r_P(t),
\end{aligned}$$

we conclude that

$$\begin{aligned}
&\mathrm{Ehr}^r_{\mathrm{BiPyr}P}(t)(\boldsymbol{e}_{i_1}, \ldots, \boldsymbol{e}_{i_r}) = 2\frac{1}{1-t}\,\mathrm{Ehr}^r_P(t)(\boldsymbol{e}_{i_1}, \ldots, \boldsymbol{e}_{i_r}) - \mathrm{Ehr}^r_P(t)(\boldsymbol{e}_{i_1}, \ldots, \boldsymbol{e}_{i_r}) \\
&= \frac{1+t}{1-t}\,\mathrm{Ehr}^r_P(t)(\boldsymbol{e}_{i_1}, \ldots, \boldsymbol{e}_{i_r}).
\end{aligned}$$

□

We will use Theorem 3.23 to calculate the h^r-tensors of the standard cross-polytope in the following example.

Example 3.24. If $r = 2$, Theorem 3.23 gives the following matrix form for the Ehrhart tensor series of a bipyramid over a lattice polytope $P \subseteq \mathbb{R}^d$.

$$\mathrm{Ehr}^r_{\mathrm{BiPyr}P}(t)(\boldsymbol{e}_{i_1}, \ldots, \boldsymbol{e}_{i_r}) = \begin{pmatrix} \frac{1+t}{1-t}\,\mathrm{Ehr}^r_P(t)(\boldsymbol{e}_{i_1}, \ldots, \boldsymbol{e}_{i_r}) & \mathbf{0}_{d-1} \\ \mathbf{0}^T_{d-1} & 2\frac{t^2+t}{(1-t)^3}\,\mathrm{Ehr}^0_P(t) \end{pmatrix}. \tag{3.15}$$

For the Ehrhart tensor series of rank 2 the one-dimensional cross-polytope ${C_1}^*$ it holds

$$\mathrm{Ehr}^2_{{C_1}^*}(t) = \sum_{n\geq 0} 2\sum_{j=0}^{n} j^2t^n = \sum_{n\geq 0} n(n+1)t^n = \frac{2(t^2+t)}{(1-t)^4} = \frac{2t(1+t)}{(1-t)^4},$$

and thus by applying (3.15) inductively and using that $\mathrm{Ehr}^0_{{C_d}^*}(t) = (1+t)^d/(1-t)^{d+1}$ [16, Theorem 2.6], we see that

$$\mathrm{Ehr}^2_{{C_d}^*}(t) = \frac{2t(1+t)^d}{(1-t)^{d+2}}\mathrm{I}_d.$$

h^r-TENSORS OF THE JOIN OF TWO POLYTOPES

The **join** of two polytopes $P \subseteq \mathbb{R}^m$ and $Q \subseteq \mathbb{R}^n$ is defined as

$$P \star Q = \operatorname{conv}\left(P \times \{\mathbf{0}_n\} \times \{0\}, \{\mathbf{0}_m\} \times Q \times \{1\}\right) \subseteq \mathbb{R}^{m+n+1},$$

and since $P \star Q = \operatorname{conv}\left(\{(\boldsymbol{v}, \mathbf{0}_n, 0), (\mathbf{0}_m, \boldsymbol{w}, 1) \ : \boldsymbol{v} \in \operatorname{vert}(P), \boldsymbol{w} \in \operatorname{vert}(Q)\}\right)$ the join of two lattice polytopes is a lattice polytope itself. For the Ehrhart polynomial of a join of $P, Q \in \mathcal{P}^d_{\mathbb{Z}}$ one has [16, Exercise 3.38], [63, Lemma 1.3].

$$\operatorname{Ehr}^0_{P\star Q}(t) = \operatorname{Ehr}^0_P(t)\operatorname{Ehr}^0_Q(t). \tag{3.16}$$

We will now prove a variant of (3.16).

Proposition 3.25. *If $P \in \mathcal{P}^m_{\mathbb{Z}}$ and $Q \in \mathcal{P}^n_{\mathbb{Z}}$ are lattice polytopes, then the Ehrhart tensor series of the join of P and Q has the form*

$$\operatorname{Ehr}^1_{P\star Q}(t) = \left(\operatorname{Ehr}^0_Q(t)\operatorname{Ehr}^1_P(t),\ \operatorname{Ehr}^0_P(t)\operatorname{Ehr}^1_Q(t),\ \operatorname{Ehr}^0_P(t)\sum_{\ell\geq 0}\mathrm{L}^0(\ell Q)\ell t^\ell\right)^T,$$

Proof. Let s be a positive integer. We observe that the lattice points of the dilate of the join of P and Q can be described as

$$\begin{aligned}
&s(P\star Q)\cap\mathbb{Z}^{m+n+1}\\
&= \operatorname{conv}\left(\{\mu(\boldsymbol{x},\mathbf{0}_n,0)^T + (s-\mu)(\mathbf{0}_m,\boldsymbol{y},1)^T \ : \boldsymbol{x}\in P,\ \boldsymbol{y}\in Q,\ 0\leq\mu\leq s\}\right)\cap\mathbb{Z}^{m+n+1}\\
&= \bigcup_{\substack{k,\ell\geq 0\\ k+\ell=s}}\{(kP\cap\mathbb{Z}^m)\times\{\mathbf{0}_n\}\times\{0\} + \{\mathbf{0}_m\}\times(\ell\,(Q\cap\mathbb{Z}^n\times\{1\}))\},
\end{aligned}$$

and therefore,

$$\mathrm{L}^1\left(s(P\star Q)\right) = \sum_{\substack{k,\ell\geq 0\\ k+\ell=s}}\left(\mathrm{L}^0(\ell Q)\,\mathrm{L}^1(kP),\quad \mathrm{L}^0(kP)\,\mathrm{L}^1(\ell Q),\quad \ell\,\mathrm{L}^0(kP)\,\mathrm{L}^0(\ell Q)\right)^T,$$

which shows

$$\begin{aligned}
\operatorname{Ehr}^1_{P\star Q}(t) &= \sum_{s\geq 0}\mathrm{L}^1\left(s(P\star Q)\right)t^s\\
&= \left(\sum_{s\geq 0}\sum_{\substack{k,\ell\geq 0\\ k+\ell=s}}\mathrm{L}^0(\ell Q)\,\mathrm{L}^1(kP)t^s, \sum_{s\geq 0}\sum_{\substack{k,\ell\geq 0\\ k+\ell=s}}\mathrm{L}^0(kP)\,\mathrm{L}^1(\ell Q)t^s, \sum_{s\geq 0}\sum_{\substack{k,\ell\geq 0\\ k+\ell=s}}\ell\,\mathrm{L}^0(kP)\,\mathrm{L}^0(\ell Q)t^s\right)^T\\
&= \left(\operatorname{Ehr}^0_Q(t)\operatorname{Ehr}^1_P(t),\ \operatorname{Ehr}^0_P(t)\operatorname{Ehr}^1_Q(t),\ \operatorname{Ehr}^0_P(t)\sum_{\ell\geq 0}\mathrm{L}^0(\ell Q)\ell t^\ell\right)^T. \qquad \square
\end{aligned}$$

Remark 3.26. *It is possible to follow the same approach as for Proposition 3.25 for the Ehrhart tensor series of rank $r > 1$. However, it becomes increasingly difficult in terms of notation as the tensor rank increases. For instance, $\operatorname{Ehr}^2_{P\star Q}(t)$ can be described as*

$$\begin{pmatrix} \operatorname{Ehr}^2_P(t)\operatorname{Ehr}^0_Q(t) & \operatorname{Ehr}^1_P(t)(\operatorname{Ehr}^1_Q(t))^\top & \operatorname{Ehr}^1_P(t)\left(\sum_{s\geq 0} s\,\mathrm{L}^0(sQ)t^s\right) \\ \operatorname{Ehr}^1_Q(t)(\operatorname{Ehr}^1_P(t))^\top & \operatorname{Ehr}^0_P(t)\operatorname{Ehr}^2_Q(t) & \left(\sum_{s\geq 0} s\,\mathrm{L}^1(sQ)t^s\right)\operatorname{Ehr}^0_P(t) \\ \operatorname{Ehr}^1_P(t)\left(\sum_{s\geq 0} s\,\mathrm{L}^0(sQ)t^s\right) & \operatorname{Ehr}^0_P(t)\left(\sum_{s\geq 0} s\,\mathrm{L}^1(sQ)t^s\right) & \operatorname{Ehr}^0_P(t)\left(\sum_{s\geq 0} s^2\,\mathrm{L}^0(sQ)t^s\right) \end{pmatrix}.$$

3.7 Further results and outlook

It is natural to ask whether Theorem 3.12 holds true in higher dimensions. Using the software package polymake [4, 51] we have calculated the h^2-tensor polynomials of several hundred randomly generated polytopes in dimension 3 and 4. Based on these computational results, we offer the following conjecture.

Conjecture 3.27. *For $d \geq 1$, the coefficients of the h^2-tensor polynomial of a lattice polytope in $\mathbb{R}^d$ are positive semidefinite.*

For our proof of Theorem 3.12, it was crucial that every lattice polygon has a unimodular triangulation. Since this no longer holds true in general for higher dimensional polytopes, a proof of Conjecture 3.27 would need to be conceptually different. Although it might be possible to extend the idea to polytopes of higher dimensions which can be triangulated into unimodular simplices, this will very likely include extensive calculations of the individual coefficients.

Finding inequalities among the coefficients of the h^*-polynomial of a lattice polytope, beyond Stanley's Nonnegativity Theorem, is currently of great interest in Ehrhart theory. The ultimate goal is a classification of all possible h^*-polynomials, and a classification of all h^*-polynomials of degree 2 can be found in [63, Proposition 1.10]. Another fundamental inequality is due to Hibi [71] who proved, for $1 \leq i < d$, that $h_i(P) - h_1(P) \geq 0$ for all full-dimensional lattice polytopes that have an interior lattice point. Calculations with polymake again suggest that there might be a version for matrices motivating the following conjecture.

Conjecture 3.28. *Let P be a lattice polytope containing a lattice point in its interior. Then the matrices $h^2_i(P) - h^2_1(P)$ for $1 \leq i < \dim(P) + 2$ are positive semidefinite.*

In recent years, additional inequalities for the coefficients of the h^*-polynomial have been given (see e.g. [5, 116, 118]) which raises the question as to whether there are analogous results for Ehrhart tensors.

Question 3.29. *Which known inequalities among the coefficients of the h^*-polynomial of a lattice polytope can be generalized to h^r-tensor polynomials of higher rank?*

An answer would depend on the notion of positivity that is chosen. A natural choice for higher rank h^r-tensors, extending positive semidefiniteness of matrices, is to define $T \in \mathbb{T}^r$ to be positive semidefinite if and only if $T(\boldsymbol{v},\dots,\boldsymbol{v}) \geq 0$ for all $\boldsymbol{v} \in \mathbb{R}^d$. However, assuming this definition of positivity, there can not be any inequalities that are valid for all polytopes if the rank r is odd since $T(\boldsymbol{v},\dots,\boldsymbol{v}) = (-1)^r T(-\boldsymbol{v},\dots,-\boldsymbol{v})$.

In the case that the rank r is even, we are able to extend another classical result—namely, Hibi's Palindromic Theorem [70] characterizing reflexive polytopes. A lattice polytope $P \in \mathcal{P}^d_{\mathbb{Z}}$ is called **reflexive** if

$$P = \{\boldsymbol{x} \in \mathbb{R}^d \colon A\boldsymbol{x} \leq \mathbf{1}_m\},$$

where $A \in \mathbb{Z}^{m\times d}$ is an integral matrix. Equivalently, $P \in \mathcal{P}^d_{\mathbb{Z}}$ is reflexive if its polar body P^* is a lattice polytope, too.

Theorem 3.30 (Hibi [70])**.** *A polytope $P \in \mathcal{P}^d_{\mathbb{Z}}$ is reflexive if and only if $h^*_i(P) = h^*_{d-i}(P)$ for all $0 \leq i \leq d$.*

A crucial step in the proof of Theorem 3.30 is to observe that a polytope P is reflexive if and only if

$$nP \cap \mathbb{Z}^d = (n+1)\,\mathrm{int}\,(P)\, \cap \mathbb{Z}^d,$$

for all $n \in \mathbb{N}$ (see [16]). We use this fact to give the following generalization.

Proposition 3.31. *Let $r \in \mathbb{N}$ be even and $P \in \mathcal{P}^d_{\mathbb{Z}}$ be a lattice polytope that contains the origin in its relative interior. The polytope P is reflexive if and only if $h^r_i = h^r_{d+r-i}$ for all $0 \leq i \leq d+r$.*

Proof. By Theorem 3.2 and comparing coefficients in equation (3.2), it follows that the assertion $h^r_i(P) = h^r_{d+r-i}(P)$ is equivalent to $\mathrm{L}^r((n-1)P) = \mathrm{L}^r(n\,\mathrm{int}\,(P)\,)$ for all integers n.

If P is a reflexive polytope, then $\mathrm{L}^r((n-1)P) = \mathrm{L}^r(n\,\mathrm{int}\,(P)\,)$ for all integers n since, as given above, we have $(n-1)P \cap \mathbb{Z}^d = n\,\mathrm{int}\,(P)\, \cap \mathbb{Z}^d$.

Now assume that P is not reflexive. Then there exists an $n \in \mathbb{N}$ such that

$$(n-1)P \cap \mathbb{Z}^d \subsetneq n\ \mathrm{int}\,(P)\, \cap \mathbb{Z}^d\,.$$

Therefore, for any $\boldsymbol{v} \in \mathbb{R}^d \setminus \{0\}$, we obtain

$$\sum_{\boldsymbol{x}\in(n-1)P\cap\mathbb{Z}^d} \langle \boldsymbol{x},\boldsymbol{v}\rangle^r \ < \sum_{\boldsymbol{x}\in n\,\mathrm{int}(P)\,\cap\mathbb{Z}^d} \langle \boldsymbol{x},\boldsymbol{v}\rangle^r,$$

and, in particular, $\mathrm{L}^r((n-1)P) \neq \mathrm{L}^r(n\,\mathrm{int}\,(P)\,)$ completing the proof. □

Note that the proof of Proposition 3.31 shows that for odd rank r palindromicity of the h^r-tensor polynomial of a reflexive polynomial is still necessary, but not sufficient, since all centrally symmetric polytopes have a palindromic h^r-tensor polynomial—namely, the constant zero polynomial.

4

Discrete John-type theorems

4.1 Introduction

A popular discipline in convex geometry is to approximate a convex body, or its properties, in terms of simpler structures. This may include enclosing a convex body between two ellipsoids which differ only in a specified scaling factor or estimating its volume in terms of its lower dimensional sections. More precisely, a classical theorem in convex geometry due to Fritz John states that a convex body K can be approximated by two concentric ellipsoids.

Theorem 4.1 (John [75])**.** *If $K \in \mathcal{K}^d$ is a convex body, there exist a regular matrix $T \in \mathbb{R}^{d\times d}$ and a vector $\boldsymbol{t} \in \mathbb{R}^d$ such that*

$$\boldsymbol{t} + TB_d \subseteq K \subseteq \boldsymbol{t} + d\,TB_d. \tag{4.1}$$

Moreover, if K is symmetric, (4.1) can be improved such that

$$TB_d \subseteq K \subseteq \sqrt{d}\,TB_d. \tag{4.2}$$

Note that in both cases of (4.1) and (4.2) the ellipsoids $\boldsymbol{t} + TB_d$ and TB_d can be chosen to be the ellipsoids of maximal volume among all ellipsoids contained in K, respectively. For a continuative survey on such ellipsoids called **Löwner-John ellipsoids** see e.g. [56, 60], and [9] for further characterizations. Ellipsoids themselves do have a variety of characterizations, see [1, 32–34, 81, 95, 96] and for a rather surveying article [98], which make them well studied geometric objects in many ways. Following recents developments in the field of convex and discrete geometry, Tao and Vu presented a discretized version of John's Theorem in 2008, in which they replaced the role of the enclosing ellipsoids by symmetric generalized arithmetic progressions (Definition 4.3), which seem to be natural discretizations of ellipsoids in several ways. We will discuss the theorem of Tao and Vu in Theorem 4.4. In comparison to the volume, the discrete volume is not homogeneous. However, the discrete volume of scalations of symmetric

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The results from this chapter are joint work with Martin Henk.

Figure 4.1: Triangle and rectangle with their respective Löwner-John ellipsoids

generalized arithmetic progressions is easily controlled, see (4.3). Therefore, enclosing a given convex body between two symmetric GAPs means approaching its discrete structures, i.e., the set of lattice points it contains, by discrete sets which are significantly easier to handle. Note that ellipsoids are not suitable substitutes in this sense because the number of lattice points in an ellipsoid and its scalations is difficult to examine. In particular, even the number of integer points inside a circle of radius r is hard to determine.

Remark 4.2. *The problem of estimating the number of integer points in a circle of radius r was suggested by Gauss, and hence is called **Gauss' circle problem**. Gauss also showed the estimate $\pi r^2 + O(r)$. The currently best known result is due to Huxley* [72]*. Note that there are higher dimensional and primitive analogues as well, e.g.,* [127]*.*

We will now introduce the terminology and objects of study for this chapter thoroughly. Afterwards, we will study symmetric generalized arithmetic progressions in Section 4.2 and present the main results of this chapter in Section 4.3. The chapter closes with a close examination of the previous discussed problems in two dimensions in Section 4.4.

Definition 4.3 (Symmetric generalized arithmetic progression)**.** Let $\boldsymbol{a}_i \in \mathbb{R}^d$, $i \in [d]$, be d linearly independent vectors. For $A = (\boldsymbol{a}_1, \ldots, \boldsymbol{a}_d)$, and $\boldsymbol{u} \in \mathbb{R}^d_{\geq 0}$ we define the **symmetric generalized arithmetic progression**, or **symmetric GAP** for short, as

$$\mathrm{P}(A, \boldsymbol{u}) = \left\{ \sum_{i=1}^{d} z_i \boldsymbol{a}_i \ : \ z_i \in [-u_i, u_i] \cap \mathbb{Z},\ i \in [d] \right\} = \Big\{ A\boldsymbol{z} \ : \ \boldsymbol{z} \in \mathbb{Z}^d,\ -\boldsymbol{u} \leq \boldsymbol{z} \leq \boldsymbol{u} \Big\}.$$

The vectors $\boldsymbol{a}_i$, $i \in [d]$, are called **steps** and the numbers u_i, $i \in [d]$, are called the **dimensions** of $\mathrm{P}(A, \boldsymbol{u})$. Moreover, if $\Lambda \in \mathcal{L}^d$ is a lattice, we say that $\mathrm{P}(A, \boldsymbol{u})$ is a **symmetric generalized arithmetic progression (GAP) in** Λ if $\boldsymbol{a}_i \in \Lambda$ for all $i \in [d]$.

In a more general context, such a symmetric GAP is additionally called *infinitely proper* of *rank* d since its steps $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_d$ are linearly independent and there are d of them, cf. [121, Definition 1.3].

We will consider the steps A of a symmetric GAP as a $d \times d$-matrix or alternatively as an ordered set of vectors, which correspond to the column vectors of the former, depending on the given context and on what is clearer notation-wise in the given situation.

For the size of a symmetric GAP, we have

$$|\mathrm{P}(A,\boldsymbol{u})| = \prod_{i=1}^{d} (2\lfloor u_i \rfloor + 1)\,. \tag{4.3}$$

The continuous counterpart of a symmetric GAP is a parallelepiped. Accordingly, for linearly independent $\boldsymbol{a}_1,\dots,\boldsymbol{a}_d \in \mathbb{R}^d$, $A = (\boldsymbol{a}_1,\dots,\boldsymbol{a}_d)$ and $\boldsymbol{u} \in \mathbb{R}^d_{\geq 0}$, we define the **parallelepiped induced by A and $\boldsymbol{u}$** as

$$\mathrm{P}_{\mathbb{R}}(A,\boldsymbol{u}) = \left\{ \sum_{i=1}^{d} z_i \boldsymbol{a}_i \;:\; z_i \in [-u_i, u_i],\; i \in [d] \right\} = \{A\boldsymbol{z} \;:\; -\boldsymbol{u} \leq \boldsymbol{z} \leq \boldsymbol{u}\}.$$

Naturally, we have the obvious inclusion $\mathrm{P}(A,\boldsymbol{u}) \subseteq \mathrm{P}_{\mathbb{R}}(A,\boldsymbol{u})$ and that $\mathrm{vol}(\mathrm{P}_{\mathbb{R}}(A,\boldsymbol{u})) =$

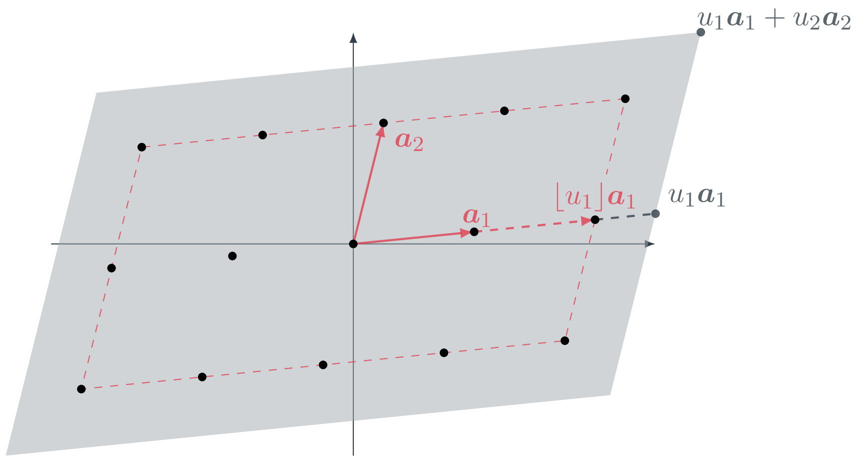

Figure 4.2: A symmetric GAP $\mathrm{P}(A,\boldsymbol{u})$ (black dots) with steps $A = (\boldsymbol{a}_1,\boldsymbol{a}_2)$ and the parallelepiped $\mathrm{P}_{\mathbb{R}}(A,\boldsymbol{u})$ (shaded)

$2^d\ |\det(A)|\ \prod_{i=1}^{d} u_i$. In this section, we will consider the situation in which the columns of $A = (\boldsymbol{a}_1,\dots,\boldsymbol{a}_d)$ form a lattice basis of Λ repeatedly. In particular, this implies that $\mathrm{P}(A,\boldsymbol{u}) = \mathrm{P}_{\mathbb{R}}(A,\boldsymbol{u}) \cap \Lambda$.

One central problem of this chapter is to find a small constant τ_d which possibly depends on the dimension d such that the set of lattice points $K \in \mathcal{K}^d_o$ in $\Lambda \in \mathcal{L}^d$ is enclosed between a symmetric generalized arithmetic progression and its scalation by a factor of τ_d. More precisely, for $\Lambda \in \mathcal{L}^d$ we define $\tau_d(\Lambda)$ as the infimum of all $\tau > 0$ for which the following holds: For every symmetric convex body $K \in \mathcal{K}^d_o$ exists a symmetric generalized arithmetic progression $\mathrm{P}(A,\boldsymbol{u})$ in Λ such that

$$\mathrm{P}(A,\boldsymbol{u}) \subseteq K \cap \Lambda \subseteq \mathrm{P}(A,\tau\boldsymbol{u}). \tag{4.4}$$

Note that $\tau_d(\Lambda)$ is independent of Λ, since (4.4) ranges over every symmetric convex body K, $\mathcal{K}^d_o$ is invariant under linear transformations and two lattices are linearly equivalent. Therefore, we write $\tau_d := \tau_d(\Lambda)$. We also observe that for the matter of (4.4) it is insignificant if we consider a symmetric convex body $K \in \mathcal{K}_o^d$ or the Λ-lattice polytope $\operatorname{conv}(K \cap \Lambda)$ it contains. Moreover, without loss of generality, we may assume that such a Λ-lattice polytope is d-dimensional, since we could concentrate our study to the lower dimensional subspace spanned by its lattice points otherwise.

Similarly to τ_d, we define $\nu_d(\Lambda)$, $\Lambda \in \mathcal{L}^d$, as the infimum of all $\nu > 0$ for which the following holds: For every symmetric convex body $K \in \mathcal{K}_o^d$ exists a symmetric generalized arithmetic progression $\mathrm{P}(A, \boldsymbol{u})$ in Λ such that

$$\mathrm{P}(A, \boldsymbol{u}) \subseteq K \cap \Lambda, \text{ and } |\mathrm{P}(A, \boldsymbol{u})| \leq |K|_\Lambda \leq \nu |\mathrm{P}(A, \boldsymbol{u})|. \tag{4.5}$$

$\nu_d(\Lambda)$ is independent of the choice of Λ, and we write $\nu_d := \nu_d(\Lambda)$. Again, for the purpose of (4.5) we may as well consider the Λ-lattice polytope $\operatorname{conv}(K \cap \Lambda)$ and we may also assume that it is d-dimensional. Tao and Vu indeed showed that the constants τ_d and ν_d do exist.

Theorem 4.4 (Tao, Vu; discrete John theorem [121, 123])**.** *Let $K \in \mathcal{K}_o^d$ and let $\Lambda \in \mathcal{L}^d$. Then there exists a symmetric GAP $\mathrm{P}(A, \boldsymbol{u})$ in Λ such that we have the inclusions*

$$\mathrm{P}(A, \boldsymbol{u}) \subseteq K \cap \Lambda \subseteq \mathrm{P}\left(A, O(d)^{3d/2} \boldsymbol{u}\right). \tag{4.6}$$

Furthermore, for the symmetric GAP $\mathrm{P}(A, \boldsymbol{u})$ we have the size bounds

$$|\mathrm{P}(A, \boldsymbol{u})| \leq |K|_\Lambda \leq O(d)^{7d/2} |\mathrm{P}(A, \boldsymbol{u})|. \tag{4.7}$$

In particular, $\tau_d = O(d)^{3d/2}$ and $\nu_d = O(d)^{7d/2}$. Note that the symmetric GAP $\mathrm{P}(A, \boldsymbol{u})$ in (4.6) and (4.7) is the same.

However, as Tao and Vu state [123, p. 142], it remains of interest whether the involved expressions $O(d)^{3d/2}$ can be significantly improved, e.g., to $e^{O(d)}$ or even $d^{O(1)}$. We will present improvements of (4.6) and (4.7) in Section 4.3, cf. Theorem 4.18 and Theorem 4.15.

For a given symmetric convex body $K \in \mathcal{K}_o^d$ and $\Lambda \in \mathcal{L}^d$, let $\tau_d(K, \Lambda)$ be the smallest $\tau > 0$ for which (4.4) holds. Similarly, let $\nu_d(K, \Lambda)$ be the smallest $\nu > 0$ for which (4.5) holds. It is important to notice, and conceivably contrary to what one initially might expect, that for a given convex body $K \in \mathcal{K}_o^d$ a symmetric GAP which attains $\tau_d(K, \Lambda)$, does not need to attain $\nu_d(K, \Lambda)$ as well. We give an example of this occurence in Example 4.5.

We do not know if τ_d or ν_d are actually attained for convex bodies, i.e., if $\tau_d = \tau_d(K, \Lambda)$ or $\nu_d = \nu_d(K, \Lambda)$ for some $K \in \mathcal{K}_o^d$. In view of the proof of Proposition 4.13, where we deduce that $\tau_d = \Omega(d)$ by considering a limit process, it is plausible that this actually might not be the case, which is possibly a significant difference to the classical theorem of John. Accordingly, we will regularly write $c > \tau_d$ throughout this paper to signify that c is a positive constant such that for every convex body $K \in \mathcal{K}_o^d$ and every $\Lambda \in \mathcal{L}^d$, there exists a symmetric GAP, $\mathrm{P}(A, \boldsymbol{u})$ in Λ such that

$$\mathrm{P}(A, \boldsymbol{u}) \subseteq K \cap \Lambda \subseteq \mathrm{P}(A, c\boldsymbol{u}).$$

Example 4.5. Let

$$Q = \operatorname{conv}\left(\pm \begin{pmatrix} 3 \\ 0 \end{pmatrix}, \pm \begin{pmatrix} -3 \\ 1 \end{pmatrix}, \pm \begin{pmatrix} -1 \\ 1 \end{pmatrix}\right),$$

cf. Figure 4.3. The lattice polytope Q contains 13 integer points. On the one hand, to find the largest symmetric GAP $\mathrm{P}(A,\boldsymbol{u})$ in $\mathbb{Z}^2$ contained in $Q\cap\mathbb{Z}^2$, we observe that $|\mathrm{P}(A,\boldsymbol{u})| = (2\lfloor u_1\rfloor+1)(2\lfloor u_2\rfloor+1)$ and the only line through the origin containing more than 3 points of $Q\cap\mathbb{Z}^d$ is the x_1-axis, which contains 7 lattice points of Q. Therefore, without loss of generality we assume $u_1\geq u_2$ and conclude that $\lfloor u_1\rfloor\leq 2$ and $\lfloor u_2\rfloor\leq 1$. If $\lfloor u_1\rfloor = 2$ and $\lfloor u_2\rfloor = 1$, then $|\mathrm{P}(A,\boldsymbol{u})| = 15 > |Q|_{\mathbb{Z}^2}$, which shows that the largest symmetric GAP in $Q\cap\mathbb{Z}^2$ cannot be larger than 9. Except for sign changes and ordering there are 6 possible choices for $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$, $A=(\boldsymbol{a}_1,\boldsymbol{a}_2)$; however, only $A=((1,0)^T,(-2,1)^T)$ yields a symmetric GAP of size 9 in $Q\cap\mathbb{Z}^2$. Choosing $\boldsymbol{u}=(2-\varepsilon,2-\varepsilon)$ even for an arbitrarily small $\varepsilon>0$, shows that

$$\mathrm{P}(A,\boldsymbol{u})\subseteq Q\cap\mathbb{Z}^2\not\subseteq\mathrm{P}(A,C\boldsymbol{u}),$$

for every factor $C\leq 3/2$.

On the other hand, we consider the symmetric GAP $\mathrm{P}(A',\boldsymbol{u}')$ with $A'=((1,0)^T,(0,1)^T)$, $\boldsymbol{u}'=(3,1-\varepsilon)$, where $\varepsilon>0$ is again arbitrarily small. We observe that

$$\mathrm{P}\left(A',\boldsymbol{u}'\right)\subseteq Q\cap\mathbb{Z}^2\subseteq\mathrm{P}\left(A',\frac{1}{1-\varepsilon}\boldsymbol{u}'\right).$$

This shows that the largest symmetric GAP in $Q\cap\mathbb{Z}^d$ is not the symmetric GAP which encloses $Q\cap\mathbb{Z}^d$ with respect to the smallest possible scaling factor.

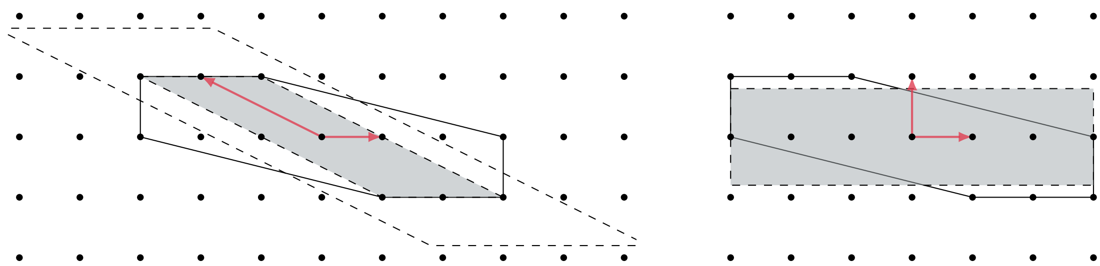

Figure 4.3: Q of Example 4.5

4.2 Symmetric GAPs in convex bodies

In this section, we will develop our tools for proving the main result of this chapter in the subsequent sections. We investigate how inclusions of the form

$$\mathrm{P}(A,\boldsymbol{u})\subseteq K\cap\Lambda\subseteq\mathrm{P}(A,c\boldsymbol{u})$$

behave when the symmetric GAPs $\mathrm{P}(A,\boldsymbol{u})$, $\mathrm{P}(A,c\boldsymbol{u})$ and the convex body $K\in\mathcal{K}_o^d$ are multiplied by a factor; in particular, under what circumstances the inclusions are preserved. Moreover, we will show that it is sufficient to examine symmetric GAPs $\mathrm{P}(A,\boldsymbol{u})$ in Λ for which the columns of A are a lattice basis of Λ when it comes to study τ_d. Simple size bounds for symmetric GAPs and the relation between τ_d and ν_d are explored as well.

We will start by discussing upper bounds for the size of an enclosing symmetric GAP. This will enable us to bound ν_d in terms of τ_d, in particular.

Proposition 4.6. *Let $K \in \mathcal{K}_o^d$, $\tau \geq 1$ and let $\mathrm{P}\,(A, \boldsymbol{u})$ be a symmetric generalized arithmetic progression in $\Lambda \in \mathcal{L}^d$, such that*

$$\mathrm{P}\,(A, \boldsymbol{u}) \subseteq K \cap \Lambda \subseteq \mathrm{P}\,(A, \tau\boldsymbol{u}). \tag{4.8}$$

Then,

$$|K|_\Lambda \leq 3^d \tau^d |\mathrm{P}\,(A, \boldsymbol{u})|.$$

In particular, $\nu_d \leq 3^d {\tau_d}^d$ and thus $\nu_d = O\,(\tau_d)^d$.

Proof. Since the inclusions in (4.8) still hold if we replace $\tau\boldsymbol{u}$ by the vector $(\lfloor \tau u_1 \rfloor, \ldots, \lfloor \tau u_d \rfloor)$, we assume without loss of generality that $\tau\boldsymbol{u} \in \mathbb{Z}^d$. Therefore,

$$\begin{aligned}
|K|_\Lambda \leq |\mathrm{P}\,(A, \tau\boldsymbol{u})| &= \prod_{i=1}^d (2\lfloor \tau u_i \rfloor + 1) \leq \prod_{i=1}^d (2\tau(\lfloor u_i \rfloor + 1) + 1) \\
&= \tau^d \prod_{i=1}^d \left(2\lfloor u_i \rfloor + 2 + \frac{1}{\tau}\right) \leq \tau^d \prod_{i=1}^d (2\lfloor u_i \rfloor + 3) \leq \tau^d \prod_{i=1}^d (6\lfloor u_i \rfloor + 3) \\
&\leq 3^d \tau^d \prod_{i=1}^d (2\lfloor u_i \rfloor + 1) \leq 3^d \tau^d |\mathrm{P}\,(A, \boldsymbol{u})|,
\end{aligned}$$

and we have $\nu_d \leq 3^d {\tau_d}^d$. $\square$

The following lemma will discuss under what circumstances the inclusions of (4.4) are preserved if we scale both the convex body K and the involved symmetric generalized arithmetic progressions. Note that if we speak of scaling a symmetric GAP $\mathrm{P}\,(A, \boldsymbol{u})$, we really refer to scaling its dimensions $\boldsymbol{u}$.

Lemma 4.7. *Let $K \in \mathcal{K}_o^d$, $\Lambda \in \mathcal{L}^d$, m be a positive integer and let $\mathrm{P}\,(A, \boldsymbol{u})$ be a symmetric GAP in Λ. The following two statements hold.*

i) If $\mathrm{P}\,(A, \boldsymbol{u}) \subseteq K \cap \Lambda$ then $\mathrm{P}\left(A, m^{-1}\boldsymbol{u}\right) \subseteq \left(m^{-1}K\right) \cap \Lambda$.

ii) If $K \cap \Lambda \subseteq \mathrm{P}\,(A, \boldsymbol{u})$ and the columns of A form a lattice basis of Λ, then $\left(m^{-1}K\right) \cap \Lambda \subseteq \mathrm{P}\left(A, m^{-1}\boldsymbol{u}\right)$.

Proof. Throughout this proof we assume without loss of generality that $\Lambda = \mathbb{Z}^d$.

Suppose that $\mathrm{P}\,(A, \boldsymbol{u}) \subseteq K \cap \mathbb{Z}^d$. Let $\boldsymbol{x} \in \mathrm{P}\left(A, m^{-1}\boldsymbol{u}\right)$. Then $\boldsymbol{x} \in \mathbb{Z}^d$ and there exists $\boldsymbol{z} \in \mathbb{Z}^d$ with $-m^{-1}\boldsymbol{u} \leq \boldsymbol{z} \leq m^{-1}\boldsymbol{u}$ and $\boldsymbol{x} = A\boldsymbol{z}$. In addition, we have $m\boldsymbol{z} \in \mathbb{Z}^d$, $-\boldsymbol{u} \leq m\boldsymbol{z} \leq \boldsymbol{u}$ and $m\boldsymbol{x} = A(m\boldsymbol{z}) \in \mathrm{P}\,(A, \boldsymbol{u}) \subseteq K \cap \mathbb{Z}^d$ and thus $\boldsymbol{x} \in \left(m^{-1}K\right) \cap \mathbb{Z}^d$.

Now, assume that $K \cap \mathbb{Z}^d \subseteq \mathrm{P}\,(A, \boldsymbol{u})$ and that the columns of A form a lattice basis of $\mathbb{Z}^d$. If $\boldsymbol{y} \in \left(m^{-1}K\right) \cap \mathbb{Z}^d$ then $m\boldsymbol{y} \in K \cap \mathbb{Z}^d \subseteq \mathrm{P}\,(A, \boldsymbol{u})$ and there exists $\boldsymbol{z}$, $-\boldsymbol{u} \leq \boldsymbol{z} \leq \boldsymbol{u}$ with $m\boldsymbol{y} = A\boldsymbol{z}$. Since the columns of A form a lattice basis, the matrix A is unimodular and therfore $m^{-1}\boldsymbol{z} = A^{-1}\boldsymbol{y} \in \mathbb{Z}^d$ with $-m^{-1}\boldsymbol{u} \leq (m^{-1}\boldsymbol{z}) \leq m^{-1}\boldsymbol{u}$. In other words, $\boldsymbol{y} = A(m^{-1}\boldsymbol{z}) \in \mathrm{P}\left(A, m^{-1}\boldsymbol{u}\right)$. $\square$

We will now show that for $\Lambda \in \mathcal{L}^d$ the constant τ_d is the infimum of all $\tau > 0$ such that for every symmetric convex body $K \in \mathcal{K}_o^d$ there exists a symmetric GAP $\mathrm{P}\,(A, \boldsymbol{u})$ in Λ for which its steps $A = (\boldsymbol{a}_1, \ldots, \boldsymbol{a}_d)$ represent a lattice basis Λ and the inclusions $\mathrm{P}\,(A, \boldsymbol{u}) \subseteq K \cap \Lambda \subseteq \mathrm{P}\,(A, \tau\boldsymbol{u})$ hold. This allows us to assume that the columns of the matrix A in our initial problem (4.4) are a lattice basis of Λ.

Proposition 4.8. *Let $c > \tau_d$ and $\Lambda \in \mathcal{L}^d$, then for every $K \in \mathcal{K}_o^d$ there exists a symmetric GAP $\mathrm{P}\,(A, \boldsymbol{u})$ in Λ for which $A = (\boldsymbol{a}_1, \ldots, \boldsymbol{a}_d)$ is a basis of Λ, such that*

$$\mathrm{P}\,(A, \boldsymbol{u}) \subseteq K \cap \mathbb{Z}^d \subseteq \mathrm{P}\,(A, c\boldsymbol{u}). \tag{4.9}$$

Proof. Let ℓ be a positive integer such that the dilation ℓK contains a lattice basis of Λ. Then, by assumption, there exists a symmetric arithmetic progression $\mathrm{P}\,(A, \boldsymbol{u})$ such that

$$\mathrm{P}\,(A, \boldsymbol{u}) \subseteq \ell K \cap \Lambda \subseteq \mathrm{P}\,(A, c\boldsymbol{u}).$$

Hence, $\mathrm{P}\,(A, c\boldsymbol{u})$ contains a lattice basis of Λ, whose vectors are linear combinations of the columns of A with integer coefficients. Therefore, the columns of A have to be a lattice basis, too. By Lemma 4.7 we have $\mathrm{P}\left(A, \ell^{-1}\boldsymbol{u}\right) \subseteq K \cap \Lambda$ and $K \cap \Lambda \subseteq \mathrm{P}\left(A, c\ell^{-1}\boldsymbol{u}\right)$. □

Despite Proposition 4.8, it can actually be the case that $K \in \mathcal{K}_o^d$ contains a symmetric GAP $\mathrm{P}\,(A, \boldsymbol{u})$ in Λ, but contains no symmetric GAP $\mathrm{P}\,(B, \boldsymbol{w})$ in Λ for which the columns of B are a lattice basis with $|\mathrm{P}\,(B, \boldsymbol{w})| \geq |\mathrm{P}\,(A, \boldsymbol{u})|$. For instance, if $\boldsymbol{a}_1 = (1,0,0)^T, \boldsymbol{a}_2 = (0,1,0)^T, \boldsymbol{a}_3 = (1,1,2)^T$, and $K := \mathrm{conv}\,(\mathrm{P}\,(A, \boldsymbol{u}))$ with $\boldsymbol{u} = (1,1,1)^T$, then K clearly contains the symmetric GAP $\mathrm{P}\,(A, \boldsymbol{u})$ of size 27 but no symmetric GAP $\mathrm{P}\,(B, \boldsymbol{w})$ for size at least 27 for which $B = (\boldsymbol{b}_1, \boldsymbol{b}_2, \boldsymbol{b}_3)^T$ is a lattice basis of $\mathbb{Z}^d$. This can be verified by simple calculations.

We will now provide a link between symmetric GAPs approximating the lattice points of convex bodies and parallelepipeds, their continous counterparts.

Proposition 4.9. *If we consider the following three statements, we have the implications i) $\Rightarrow$ ii) and ii) $\Rightarrow$ iii).*

i) $c > \tau_d$.

ii) *Let $\Lambda \in \mathcal{L}^d$. For every Λ-lattice polytope $Q \in \mathcal{K}_o^d$ there exists a symmetric generalized arithmetic progression $\mathrm{P}\,(A, \boldsymbol{u})$ in Λ, for which the columns of $A = (\boldsymbol{a}_1, \ldots, \boldsymbol{a}_d)$ form a lattice basis of Λ, such that*

$$\mathrm{P}_{\mathbb{R}}\,(A, \boldsymbol{u}) \subseteq Q \subseteq \mathrm{P}_{\mathbb{R}}\,(A, c\boldsymbol{u}). \tag{4.10}$$

iii) $c \geq \tau_d$.

Proof. We first show that the assumption i) implies ii). Without loss of generality, we assume that $\Lambda = \mathbb{Z}^d$ throughout this proof. Let $\varepsilon > 0$ be a positive real and let Q be a lattice polytope in $\mathcal{K}_o^d$. Since Q is d-dimensional, for a sufficiently large integral factor $m > 0$ the lattice polytope mQ contains $c(1+c/\varepsilon)\boldsymbol{b}_1, \ldots, c(1+c/\varepsilon)\boldsymbol{b}_d$, where the $\boldsymbol{b}_i$s are a lattice basis of $\mathbb{Z}^d$.

Thus, there exists a symmetric GAP $\mathrm{P}(A, \boldsymbol{u}')$, such that $\mathrm{P}(A, \boldsymbol{u}') \subseteq mQ \cap \mathbb{Z}^d \subseteq \mathrm{P}(A, c\boldsymbol{u}')$. By Proposition 4.8 we may assume that the columns of $A = (\boldsymbol{a}_1, \dots, \boldsymbol{a}_d)$ make up a lattice basis of $\mathbb{Z}^d$. We know that the $c(1+c/\varepsilon)\boldsymbol{b}_i$s lie in mQ and by Proposition 4.10 this implies that $cu'_k \geq c(1+c/\varepsilon)$ for all $k \in [d]$, and thus $\boldsymbol{a}_i \in mQ \cap \mathbb{Z}^d$ for $i \in [d]$, since $u'_k \geq 1 + c/\varepsilon > 1$ for all $k \in [d]$. Now, let $\lfloor \boldsymbol{u}' \rfloor = (\lfloor u'_1 \rfloor, \dots, \lfloor u'_d \rfloor)$. We then conclude that $\mathrm{P}_{\mathbb{R}}(A, \lfloor \boldsymbol{u}' \rfloor) = \mathrm{conv}(\mathrm{P}(A, \lfloor \boldsymbol{u}' \rfloor)) = \mathrm{conv}\left(\mathrm{P}(A, \boldsymbol{u}') \cap \mathbb{Z}^d\right) \subseteq \mathrm{conv}\left(mQ \cap \mathbb{Z}^d\right) = mQ$, using that the columns of A form a lattice basis of $\mathbb{Z}^d$ and that mQ is a lattice polytope. Moreover, we have that

$$\frac{u'_i}{\lfloor u'_i \rfloor} \leq \frac{u'_i}{u'_i - 1} \leq 1 + \varepsilon/c,$$

for all $i \in [d]$. This shows that $cu'_i \leq (c+\varepsilon)\lfloor u'_i \rfloor$, and thus

$$mQ = \mathrm{conv}\left(mQ \cap \mathbb{Z}^d\right) \subseteq \mathrm{conv}\left(\mathrm{P}\left(A, c\boldsymbol{u}'\right)\right) \subseteq \mathrm{P}_{\mathbb{R}}\left(A, c\boldsymbol{u}'\right) \subseteq \mathrm{P}_{\mathbb{R}}\left(A, (c+\varepsilon)\lfloor \boldsymbol{u}' \rfloor\right).$$

In summary, we have shown $\mathrm{P}_{\mathbb{R}}(A, \lfloor \boldsymbol{u}' \rfloor) \subseteq mQ \subseteq \mathrm{P}_{\mathbb{R}}(A, (c+\varepsilon)\lfloor \boldsymbol{u}' \rfloor)$. Since these inclusions are invariant under scalation, we deduce $\mathrm{P}_{\mathbb{R}}(A, \boldsymbol{u}) \subseteq Q \subseteq \mathrm{P}_{\mathbb{R}}(A, (c+\varepsilon)\boldsymbol{u})$ with $\boldsymbol{u} = m^{-1}\lfloor \boldsymbol{u}' \rfloor$. However,

$$Q \subseteq \mathrm{P}_{\mathbb{R}}(A, (c+\varepsilon)\boldsymbol{u}) = (c+\varepsilon)\mathrm{P}_{\mathbb{R}}(A, \boldsymbol{u}), \tag{4.11}$$

and we have already shown that the columns of A are lattice points contained in mQ, which means there are only finitely many choices for A. (4.11) holds for all $\varepsilon > 0$ and since Q and $\mathrm{P}_{\mathbb{R}}(A, (c+\varepsilon)\boldsymbol{u})$ are both compact sets, among all the possible yet finite choices for A there must exist one, such that $Q \subseteq \mathrm{P}_{\mathbb{R}}(A, c\boldsymbol{u})$.

We now show that ii) implies iii). Again, we assume that $\Lambda = \mathbb{Z}^d$. Let $Q \in \mathcal{K}_o^d$ be a lattice polytope. Then, by assumption ii), there exists a symmetric GAP $\mathrm{P}(A, \boldsymbol{u})$, where the columns of A are a lattice basis of $\mathbb{Z}^d$, such that (4.10) holds. This implies that

$$\mathrm{P}_{\mathbb{R}}(A, \boldsymbol{u}) \cap \mathbb{Z}^d \subseteq Q \cap \mathbb{Z}^d \subseteq \mathrm{P}_{\mathbb{R}}(A, c\boldsymbol{u}) \cap \mathbb{Z}^d. \tag{4.12}$$

Now, $\mathrm{P}(A, \boldsymbol{u}) = \{A\boldsymbol{z} \ : \ -\boldsymbol{u} \leq \boldsymbol{z} \leq \boldsymbol{u}, \ \boldsymbol{z} \in \mathbb{Z}^d\} \subseteq \{A\boldsymbol{z} \in \mathbb{Z}^d \ : \ -\boldsymbol{u} \leq \boldsymbol{z} \leq \boldsymbol{u}\} = \mathrm{P}_{\mathbb{R}}(A, \boldsymbol{u}) \cap \mathbb{Z}^d$, and since the the columns of A form a lattice basis of $\mathbb{Z}^d$, we find that $A\boldsymbol{z} \in \mathbb{Z}^d$ if and only if $\boldsymbol{z} \in \mathbb{Z}^d$, and thus $Q \cap \mathbb{Z}^d \subseteq \mathrm{P}_{\mathbb{R}}(A, c\boldsymbol{u}) \cap \mathbb{Z}^d = \{A\boldsymbol{z} \in \mathbb{Z}^d \ : \ -c\boldsymbol{u} \leq \boldsymbol{z} \leq c\boldsymbol{u}\} = \{A\boldsymbol{z} \ : \ -c\boldsymbol{u} \leq \boldsymbol{z} \leq c\boldsymbol{u}, \ \boldsymbol{z} \in \mathbb{Z}^d\} = \mathrm{P}(A, c\boldsymbol{u})$, hence, $\mathrm{P}(A, \boldsymbol{u}) \subseteq Q \cap \mathbb{Z}^d \subseteq \mathrm{P}(A, c\boldsymbol{u})$. □

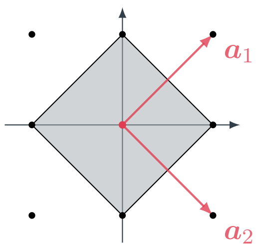

Figure 4.4: $\mathrm{P}_{\mathbb{R}}(A, \boldsymbol{u}) = {C_2}^*$

In the last part of the proof of Proposition 4.9 it is important that the columns of A are indeed a lattice basis of $\mathbb{Z}^d$, or otherwise the inclusion $Q \cap \mathbb{Z}^d \subseteq \mathrm{P}\,(A, c\boldsymbol{u})$ may not hold in general. For instance, we consider the two dimensional cross-polytope ${C_2}^* = \operatorname{conv}(\pm\boldsymbol{e}_1, \pm\boldsymbol{e}_2) \in \mathcal{K}_o^2$ and the symmetric GAP $\mathrm{P}\,(A, \boldsymbol{u})$ with $A = (\boldsymbol{a}_1, \boldsymbol{a}_2)$, $\boldsymbol{a}_1 = (1,1)^T$, $\boldsymbol{a}_2 = (1,-1)^T$ and $\boldsymbol{u} = (1/2, 1/2)^T$, see Figure 4.4. Then we see that $\mathrm{P}_{\mathbb{R}}\,(A, \boldsymbol{u}) = {C_2}^*$, although $\mathrm{P}\,(A, \boldsymbol{u}) = \{\boldsymbol{0}\}$, and therefore ${C_2}^* \cap \mathbb{Z}^2 = \{\pm\boldsymbol{e}_1, \pm\boldsymbol{e}_2, \boldsymbol{0}\} \not\subseteq \mathrm{P}\,(A, \boldsymbol{u})$, cf. Figure 4.4.

In the following proposition and the subsequent corollary, we will deduce some size bounds for symmetric GAPs which cover the lattice points of a convex body.

Proposition 4.10. *Let $\Lambda \in \mathcal{L}^d$, $K \in \mathcal{K}_o^d$ and let $r_1\boldsymbol{x}_1, \dots, r_d\boldsymbol{x}_d \in K \cap \Lambda$ linearly independent, where for each i we have $\boldsymbol{x}_i \in \Lambda$ and $r_i \in \mathbb{Z}$. Moreover, let $\mathrm{P}\,(A, \boldsymbol{u})$ be a symmetric GAP in Λ, where $A = (\boldsymbol{a}_1, \dots, \boldsymbol{a}_d)$ forms a lattice basis of Λ, such that*

$$K \cap \Lambda \subseteq \mathrm{P}\,(A, \boldsymbol{u}).$$

Then

$$\prod_{i=1}^{d} (2|r_i| + 1) \leq |\mathrm{P}\,(A, \boldsymbol{u})|,$$

and

$$\lfloor u_j \rfloor \geq \min\{r_i \ : \ i \in [d]\} \quad \forall\, j \in [d]\,.$$

Proof. Without loss of generality, we assume that $\Lambda = \mathbb{Z}^d$. By the assumption $K \cap \mathbb{Z}^d \subseteq \mathrm{P}\,(A, \boldsymbol{u})$ there are $\boldsymbol{z}_i \in \mathbb{Z}^d$, with $-\boldsymbol{u} \leq \boldsymbol{z}_i \leq \boldsymbol{u}$ and $r_i\boldsymbol{x}_i = A\boldsymbol{z}_i$, $i \in [d]$. Since A is unimodular, we have $\boldsymbol{z}_i = r_i(A^{-1}\boldsymbol{x}_i) \in r_i\mathbb{Z}^d$. In other words, the axis-oriented parallelepiped $Q = \mathrm{P}_{\mathbb{R}}\,(\mathrm{I}_d, \boldsymbol{u}) = \{\boldsymbol{y} \in \mathbb{R}^d \ : \ -\boldsymbol{u} \leq \boldsymbol{y} \leq \boldsymbol{u}\}$ contains the linear independent lattice points $\boldsymbol{z}_i \in r_i\mathbb{Z}^d$. Therefore, a permutation $\sigma : [d] \to [d]$ exists such that $(z_i)_{\sigma(i)} \neq 0$, and thus $\lfloor u_{\sigma(i)} \rfloor \geq |r_i (z_i)_{\sigma(i)}| \geq |r_i|$ $i \in [d]$. We conclude that $|\mathrm{P}\,(A, \boldsymbol{u})| = |Q|_{\mathbb{Z}^d} = \prod_{i=1}^d (2\lfloor u_i \rfloor + 1) \geq \prod_{i=1}^d (2r_i + 1)$, and $\min\{r_i \ : \ i \in [d]\} \leq \lfloor u_j \rfloor$ for all $j \in [d]$. □

Corollary 4.11. *Let $\Lambda \in \mathcal{L}^d$, $K \in \mathcal{K}_o^d$ and let $\mathrm{P}\,(A, \boldsymbol{u})$ be a symmetric GAP in Λ, where $A = (\boldsymbol{a}_1, \dots, \boldsymbol{a}_d)$ forms a lattice basis of Λ, such that*

$$K \cap \Lambda \subseteq \mathrm{P}\,(A, \boldsymbol{u}).$$

Then

$$\prod_{i=1}^{d} \left(2 \left\lfloor \frac{1}{\lambda_i(K, \Lambda)} \right\rfloor + 1\right) \leq |\mathrm{P}\,(A, \boldsymbol{u})|. \tag{4.13}$$

Proof. If $r_i := \lfloor \lambda_i(K, \Lambda)^{-1} \rfloor$ and $\boldsymbol{x}_i$, $i \in [d]$, are the vectors associated to the successive minima $\lambda_i(K, \Lambda)$, the corollary follows immediately by applying Proposition 4.10. □

Remark 4.12. *Equation* (4.13) *might be related to a conjecture of Betke, Henk and Wills, who conjectured in* [22, Conjecture 2.1] *that for* $K \in \mathcal{K}_o^d$

$$|K|_\Lambda \leq \prod_{i=1}^{d} \left\lfloor \frac{2}{\lambda_i(K,\Lambda)} + 1 \right\rfloor .$$

Note that

$$\prod_{i=1}^{d} \left\lfloor \frac{2}{\lambda_i(K,\Lambda)} + 1 \right\rfloor = \prod_{i=1}^{d} \left(2 \left\lfloor \frac{1}{\lambda_i(K)} \right\rfloor + 1 \right),$$

if $\left\{\lambda_i(K)^{-1}\right\} \in [0, 1/2)$ *for all* $i \in [d]$.

4.3 Discrete John-type theorems

Proposition 4.13. *It holds* $\tau_d = \Omega\,(d)$, *that is, the constant* τ_d *has to be at least linear in the dimension d. Moreover,* $\nu_d = \Omega\left(2^d\right)$, *that is, the constant* ν_d *has to be at least exponential in d.*

Proof. We choose a $c > \tau_d$ and show that then $c = \Omega\,(d)$, and since c can be chosen arbitrarily close to τ_d this will entail that $\tau_d = \Omega\,(d)$. To this end, let k be an arbitrary positive integer and $\lfloor \boldsymbol{u} \rfloor = (\lfloor u_1 \rfloor, \ldots, \lfloor u_d \rfloor)$. We consider the scalation of the d-dimensional cross-polytope ${C_d}^*$, namely $k{C_d}^* = \{\boldsymbol{x} \in \mathbb{R}^d \ : \ \sum_{i=1}^d |x_i| \leq k\}$, and assume that there exists a symmetric GAP $\mathrm{P}\,(A, \boldsymbol{u})$, such that

$$\mathrm{P}\,(A, \boldsymbol{u}) \subseteq k{C_d}^* \cap \mathbb{Z}^d \subseteq \mathrm{P}\,(A, c\boldsymbol{u}). \tag{4.14}$$

By Proposition 4.8, we may assume that the columns of A form a lattice basis of $\mathbb{Z}^d$. According to the first inclusion of (4.14), we deduce by convexity that $\mathrm{P}_{\mathbb{R}}\,(A, \lfloor \boldsymbol{u} \rfloor) = \operatorname{conv}(\mathrm{P}\,(A, \boldsymbol{u})) \subseteq \operatorname{conv}\left(k{C_d}^* \cap \mathbb{Z}^d\right) = k{C_d}^*$, and thus, in view of the volumes, we conclude that $2^d \prod_{i=1}^d \lfloor u_i \rfloor = \operatorname{vol}(\mathrm{P}_{\mathbb{R}}\,(A, \lfloor \boldsymbol{u} \rfloor)) \leq \operatorname{vol}(k{C_d}^*) = k^d 2^d / d!$. In addition, by the second inclusion of (4.14) it holds for $i \in [d]$, that $k\boldsymbol{e}_i \in k{C_d}^* \cap \mathbb{Z}^d \subseteq \mathrm{P}\,(A, c\boldsymbol{u})$, and therefore there exist $\boldsymbol{z}_i \in \mathbb{Z}^d$, $-c\boldsymbol{u} \leq \boldsymbol{z}_i \leq c\boldsymbol{u}$, with $k\boldsymbol{e}_i = A\boldsymbol{z}_i$ for all $i \in [d]$. Using that $\boldsymbol{z}_i = kA^{-1}\boldsymbol{e}_i \in k\mathbb{Z}^d$, $i \in [d]$, are linearly independent vectors, we get $k \leq cu_i$ for all $i \in [d]$. Thus, $\operatorname{vol}(k{C_d}^*) = k^d 2^d / d! \leq \left(c^d 2^d \prod_{i=1}^d u_i\right)/d!$, and therefore

$$\frac{d}{e} \left(\frac{\prod_{i=1}^d u_i}{\prod_{i=1}^d \lfloor u_i \rfloor} \right)^{1/d} \leq (d!)^{1/d} \left(\frac{\prod_{i=1}^d u_i}{\prod_{i=1}^d \lfloor u_i \rfloor} \right)^{1/d} \leq c, \tag{4.15}$$

since $d/(d!)^{1/d} \leq e$ for all $d \in \mathbb{N}$. By means of Proposition 4.10, we find that $u_i \geq k/c$ for $i \in [d]$, because the linear independent vectors $k\boldsymbol{e}_i$, $i \in [d]$, are contained in $k{C_d}^*$. Hence, if $k \to \infty$ we have $u_i \to \infty$ and thus the left-hand side in (4.15) tends to d/e, eventually showing that $d/e \leq c$.

It is left to show that ν_d needs to be exponential in d. Let $Q = \operatorname{conv}\left(\pm\left([0,1]^{d-1} \times \{1\}\right)\right)$. Then it is easy to see that $Q \cap \mathbb{Z}^d = \pm\left(\{0,1\}^{d-1} \times \{1\}\right) \cup \{\boldsymbol{0}\}$. Moreover, this means that $Q \cap \mathbb{Z}^d$ does not contain any nonzero points $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{Z}^d$ with $\boldsymbol{x} + \boldsymbol{y} \in Q \cap \mathbb{Z}^d$ and $\boldsymbol{x} \neq -\boldsymbol{y}$. As a consequence, the set $Q \cap \mathbb{Z}^d$ cannot contain a symmetric GAP of size larger than 3, and thus $2^d + 1 = |Q|_{\mathbb{Z}^d} \leq \nu_d |\mathrm{P}\,(A, \boldsymbol{u})|_{\mathbb{Z}^d} \leq 3\nu_d$ for every symmetric GAP $\mathrm{P}\,(A, \boldsymbol{u}) \subseteq Q \cap \mathbb{Z}^d$. $\square$

A BOUND FOR ν_d

We will now prove that $\nu_d = O(d)^d$, which improves the bound $\nu_d = O(d)^{7d/2}$ from Theorem 4.4. First, we need the following inequality.

Lemma 4.14. *For a real number $r > 0$ and an integer $d > 1$, we have*

$$\frac{1}{d}\lfloor 2r+1\rfloor \leq 3\left(2\left\lfloor\frac{r}{d}\right\rfloor + 1\right).$$

Proof. It holds

$$\frac{\frac{1}{d}\lfloor 2r+1\rfloor}{\left(2\left\lfloor\frac{r}{d}\right\rfloor + 1\right)} = \frac{\lfloor 2r+1\rfloor}{\left(2d\left\lfloor\frac{r}{d}\right\rfloor + d\right)} \leq \frac{2r+1}{\left(2d\left\lfloor\frac{r}{d}\right\rfloor + d\right)}. \tag{4.16}$$

We distinguish two cases. Firstly, if $r < d$, then the right-hand side of (4.16) is

$$\frac{2r+1}{d} \leq 2 + \frac{1}{d} \leq 3.$$

Secondly, if $r \geq d$, then the right-hand side of (4.16) is not larger than

$$\frac{2r+1}{\left(2d\left(\frac{r}{d} - 1\right) + d\right)} = \frac{2r+1}{2r-d} \leq \frac{2r+1}{r} \leq 3.$$ □

Now, we can proceed with the actual proof for our bound on ν_d.

Theorem 4.15. *For the constant ν_d holds the upper bound $\nu_d \leq 6^d d^d$. In particular, $\nu_d = O(d)^d$.*

Proof. Let $K \in \mathcal{K}_o^d$ and let $\boldsymbol{a}_i$, $i \in [d]$, be the vectors associated with the successive minima $\lambda_i := \lambda_i(K, \Lambda)$, $i \in [d]$, of K. Since $\lambda_i^{-1}\boldsymbol{a}_i \in K$, it follows that

$$\left\{\sum_{i=1}^{d} \mu_i \frac{1}{d\lambda_i}\boldsymbol{a}_i \;:\; -1 \leq \mu_i \leq 1,\ i \in [d]\right\} \subseteq \operatorname{conv}\left(\pm\lambda_i^{-1}\boldsymbol{a}_i \;:\; i \in [d]\right) \subseteq K,$$

and thus $\mathrm{P}(A, \boldsymbol{u}) \subseteq K \cap \Lambda$, where the $\boldsymbol{a}_i$s are the column vectors of A, and $\boldsymbol{u} \in \mathbb{R}^d_{\geq 0}$, $u_i = (d\lambda_i)^{-1}$, $i \in [d]$. By [59, Theorem 1.5], we have that

$$|K|_\Lambda \leq 2^d \prod_{i=1}^{d} \left\lfloor \frac{2}{\lambda_i} + 1 \right\rfloor = 2^d d^d \prod_{i=1}^{d} \left(\frac{1}{d}\left\lfloor \frac{2}{\lambda_i} + 1 \right\rfloor\right).$$

Now, Lemma 4.14 shows that the right-hand side is smaller or equal to

$$6^d d^d \prod_{i=1}^{d} \left(2\left\lfloor \frac{1}{d\lambda_i} \right\rfloor + 1\right) = 6^d d^d |\mathrm{P}(A, \boldsymbol{u})|.$$ □

Note that the steps of the symmetric GAP in the proof of Theorem 4.15 are the vectors associated to the successive minima of K.

A BOUND FOR τ_d

We will now prove a discrete John-type theorem, which improves the estimate $\tau_d = O(d)^{3d/2}$ of Theorem 4.4 to $\tau_d = d^{O(\ln d)}$. To this end, we will utilize the existence of a certain kind of lattice bases due to Seysen.

Theorem 4.16 (Seysen [110])**.** *Let $\Lambda \in \mathcal{L}^d$ be a lattice. Then there exists a basis $B = (\boldsymbol{b}_1, \ldots, \boldsymbol{b}_d)$ of Λ, called **Seysen reduced basis of** Λ with dual basis $B^\star = (\boldsymbol{b}_1{}^\star, \ldots, \boldsymbol{b}_d{}^\star)$ such that*

$$|\boldsymbol{b}_i| \cdot |\boldsymbol{b}_i{}^\star| \leq d^{O(\ln(d))}, \tag{4.17}$$

for all $i \in [d]$.

Remark 4.17. *Although not included in Seysen's original proof, the constant hidden in (4.17) can be explicitely bounded. That is, Maze* [88, Proposition 2.2.] *showed that*

$$\sum_{i=1}^{d} |\boldsymbol{b}_i|^2 |\boldsymbol{b}_i{}^\star|^2 \leq d^{\left(\frac{2}{\ln 2}+1\right)\ln d \;+\; 4}.$$

We are now able to prove the following discrete John-type theorem.

Theorem 4.18. *Let $K \in \mathcal{K}_o^d$ and $\Lambda \in \mathcal{L}^d$. There exists a symmetric generalized arithmetic progression $\mathrm{P}(A, \boldsymbol{u})$ in Λ such that*

$$\mathrm{P}(A, \boldsymbol{u}) \subseteq K \cap \Lambda \subseteq \mathrm{P}\left(A, d^{O(\ln d)}\ \boldsymbol{u}\right), \tag{4.18}$$

which shows $\tau_d = d^{O(\ln d)}$. Furthermore, for the symmetric GAP $\mathrm{P}(A, \boldsymbol{u})$ we have the size bound

$$|\mathrm{P}(A, \boldsymbol{u})| \leq |K|_\Lambda \leq d^{O(d \ln d)} |\mathrm{P}(A, \boldsymbol{u})|. \tag{4.19}$$

Note that the symmetric GAP $\mathrm{P}(A, \boldsymbol{u})$ in (4.18) and (4.19) is the same.

Proof. By applying a linear transformation to both K and Λ according to John's Theorem (Theorem 4.1), we may assume that

$$B_d \subseteq K \subseteq \sqrt{d} B_d. \tag{4.20}$$

Let $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_d$ be a Seysen reduced basis of Λ according to Theorem 4.16. Moreover, let A be the matrix with columns $\boldsymbol{a}_i$ and let $\boldsymbol{u} = (u_1, \ldots, u_d)^T$ be the vector with $u_i = \frac{1}{d}\frac{1}{|\boldsymbol{a}_i|}$, $i \in [d]$. For $\boldsymbol{x} \in K$ let $\beta_i \in \mathbb{R}$, $i \in [d]$, such that $\boldsymbol{x} = \sum_{i=1}^{d} \beta_i \boldsymbol{a}_i$. According to Cramer's rule and (4.20), we have

$$|\beta_i| = \frac{|\det(\boldsymbol{x}, \boldsymbol{a}_1, \ldots, \boldsymbol{a}_{i-1}, \boldsymbol{a}_{i+1} \ldots, \boldsymbol{a}_d)|}{\det \Lambda} \leq \sqrt{d} \frac{\mathrm{vol}_{d-1}(\boldsymbol{a}_1, \ldots, \boldsymbol{a}_{i-1}, \boldsymbol{a}_{i+1} \ldots, \boldsymbol{a}_d)}{\mathrm{vol}_d(\boldsymbol{a}_1, \ldots, \boldsymbol{a}_d)},$$

where $\mathrm{vol}_k(\boldsymbol{a}_1, \ldots, \boldsymbol{a}_k) = \mathrm{vol}_k\left(\left\{\sum_{i=1}^{k} \mu_i \boldsymbol{a}_i : \mu_i \in [0,1]\right\}\right)$ denotes the k-dimensional volume of a the parallelepiped generated by the linearly independent vectors $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_k$. Since $\boldsymbol{a}_i^\star \in$

$\{\boldsymbol{a}_1, \ldots, \boldsymbol{a}_{i-1}, \boldsymbol{a}_{i+1}, \ldots, \boldsymbol{a}_d\}^\perp$, we find that

$$\begin{aligned}\mathrm{vol}_d\left(\boldsymbol{a}_1, \ldots, \boldsymbol{a}_d\right) &= \mathrm{vol}_{d-1}\left(\boldsymbol{a}_1, \ldots, \boldsymbol{a}_{i-1}, \boldsymbol{a}_{i+1}, \ldots, \boldsymbol{a}_d\right)\frac{\langle \boldsymbol{a}_i^\star, \boldsymbol{a}_i\rangle}{|\boldsymbol{a}_i^\star|}\\ &=\mathrm{vol}_{d-1}\left(\boldsymbol{a}_1, \ldots, \boldsymbol{a}_{i-1}, \boldsymbol{a}_{i+1}, \ldots, \boldsymbol{a}_d\right)\frac{1}{|\boldsymbol{a}_i^\star|},\end{aligned}$$

and thus,

$$|\beta_i| \le \sqrt{d}|\boldsymbol{a}_i^\star|.$$

Hence, in view of (4.17) we conclude

$$|\beta_i| \le d^{3/2} d^{O(\ln d)}\, u_i.$$

Therefore, we get

$$\mathrm{P}\left(A, \boldsymbol{u}\right) \subseteq K \cap \Lambda \subseteq \mathrm{P}\left(A, d^{O(\ln d)}\ \boldsymbol{u}\right). \tag{4.21}$$

The lower bound in (4.19) follows directly from the first inclusion of (4.21), and the upper bound from applying Proposition 4.6, which yields

$$|\mathrm{P}\left(B, \boldsymbol{u}\right)| \le 3^d \left(d^{O(\ln d)}\right)^d |K|_\Lambda = d^{O(d \ln d)}|K|_\Lambda. \qquad \square$$

Symmetric GAPs in unconditional convex bodies

For a particular class of convex bodies, the so called unconditional convex bodies, a discrete John-type theorem with a scaling factor linear in the dimension can be deduced for symmetric GAPs in $\mathbb{Z}^d$, cf. Proposition 4.13 where we already showed that we cannot expect the scaling factor to be of smaller order than linear in the dimension d.

Definition 4.19. A convex body $K \in \mathcal{K}^d$ is called **unconditional** if for every $\boldsymbol{x} = (x_1, \ldots, x_d) \in K$ it holds $(\pm x_1, \ldots, \pm x_d) \in K$. Equivalently, K is unconditional if it is symmetric with respect to every coordinate hyperplane $\{\boldsymbol{x} \in \mathbb{R}^d \ : \ \langle \boldsymbol{x}, \boldsymbol{e}_i\rangle = 0\}$, $i \in [d]$.

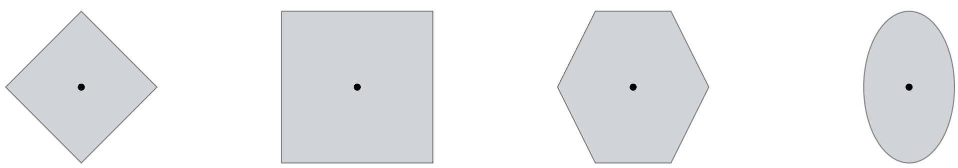

Figure 4.5: Unconditional convex bodies in $\mathbb{R}^2$

Theorem 4.20. *Let $K \subseteq \mathbb{R}^d$ be an unconditional convex body. Then there exists a lattice basis $A = (\boldsymbol{a}_1, \ldots, \boldsymbol{a}_d)$ of $\mathbb{Z}^d$ and $\boldsymbol{u} \in \mathbb{Q}^d_{>0}$, such that*

$$\mathrm{P}\left(A, \boldsymbol{u}\right) \subseteq K \cap \mathbb{Z}^d \subseteq \mathrm{P}\left(A, d\boldsymbol{u}\right).$$

Moreover, we have the size bound

$$|K|_{\mathbb{Z}^d} \leq (2d-1)^d |\mathrm{P}\,(A, \boldsymbol{u})| = O\,(d)^d |\mathrm{P}\,(A, \boldsymbol{u})|. \tag{4.22}$$

In fact, the basis A can be chosen to be the standard lattice basis.

Proof. We consider the lattice polytope $Q = \operatorname{conv}\left(K \cap \mathbb{Z}^d\right)$ and note that it suffices to show the theorem for Q. Let $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n \in \mathbb{Z}^d$ be the vertices of Q, and let

$$m_j = \max_{1 \leq i \leq n} |(\boldsymbol{v}_i)_j|, \quad j \in [d]\,.$$

Then it is clear that $m_j \boldsymbol{e}_j \in Q$ for every $j \in [d]$: For each $j \in [d]$ let $k \in [n]$ such that $m_j = |(\boldsymbol{v}_k)_j|$. Negating the j-th coordinate of $\boldsymbol{v}_k$ yields a point still lying in Q by unconditionality and we may assume that $m_j = (v_k)_j$. Let $\boldsymbol{v}_k' \in Q$ be the point we get from $\boldsymbol{v}_k$ by negating every but the j-th coordinate. Then, $m_j e_j = 1/2(\boldsymbol{v}_k) + 1/2(\boldsymbol{v}_k')$. We conclude

$$\frac{1}{d}\begin{pmatrix} \pm m_1 \\ \vdots \\ \pm m_d \end{pmatrix} = \frac{1}{d}(\pm m_1 \boldsymbol{e}_1) + \cdots + \frac{1}{d}(\pm m_d \boldsymbol{e}_d) \in Q,$$

which shows

$$\left[-\frac{1}{d}m_1, \frac{1}{d}m_1\right] \times \cdots \times \left[-\frac{1}{d}m_d, \frac{1}{d}m_d\right] \subseteq Q,$$

and thus

$$\mathrm{P}\left(\mathrm{I}_d, d^{-1}\boldsymbol{m}\right) \subseteq Q \cap \mathbb{Z}^d, \quad \boldsymbol{m} = (m_1, \ldots, m_d).$$

Here, I_d denotes the $d \times d$-unit matrix. It also holds

$$Q \subseteq [-m_1, m_1] \times \cdots \times [-m_d, m_d]$$

by the maximality of the m_js. Thus,

$$Q \cap \mathbb{Z}^d \subseteq \mathrm{P}\,(\mathrm{I}_d, \boldsymbol{m}). \tag{4.23}$$

The size bound (4.22) is left to show. We denote by $\{r\} := r - \lfloor r \rfloor$ the **fractional part** of $r \geq 0$ and notice that for $i \in [d]$ we have

$$\begin{aligned}
&\left(2\left\lfloor \frac{m_i}{d} \right\rfloor + 1\right)(2d-1) = \left(2d + 2d\left\lfloor \frac{m_i}{d} \right\rfloor - 1\right) + 2\left\lfloor \frac{m_i}{d} \right\rfloor (d-1) \\
&\geq \left(2d + 2d\left\lfloor \frac{m_i}{d} \right\rfloor - 1\right) = 2\left((d-1) + d\left\lfloor \frac{m_i}{d} \right\rfloor\right) + 1 \\
&\geq 2\left(d\left\{\frac{m_i}{d}\right\} + d\left\lfloor \frac{m_i}{d} \right\rfloor\right) + 1 = 2m_i + 1.
\end{aligned} \tag{4.24}$$

Note that the last inequality in (4.24) holds because m_i is a nonnegative integer. By (4.23) and

(4.24) we conclude

$$|Q|_{\mathbb{Z}^d} \leq |\mathrm{P}\,(\mathrm{I}_d, \boldsymbol{m})| = \prod_{i=1}^{d} (2m_i + 1) \leq (2d-1)^d \prod_{i=1}^{d} \left(2\left\lfloor \frac{m_i}{d} \right\rfloor + 1\right) = (2d-1)^d \left|\mathrm{P}\left(\mathrm{I}_d, d^{-1}\boldsymbol{m}\right)\right|.$$

This shows (4.24). □

4.4 Symmetric GAPs in planar convex bodies

In this section, we will show that describing a symmetric convex body K in terms of a symmetric generalized arithmetic progression in two dimensions is possible with a scaling factor of 3. In particular, Theorem 4.21 whose proof is presented below shows that the underlying symmetric GAP can be linked to the vectors associated to the successive minima of K.

Theorem 4.21. *Let $\Lambda \in \mathcal{L}^2$. For a planar symmetric convex body K, there exists a symmetric generalized arithmetic progression $\mathrm{P}\,(A, \boldsymbol{u})$ in Λ such that*

$$\mathrm{P}\,(A, \boldsymbol{u}) \subseteq K \cap \Lambda \subseteq \mathrm{P}\,(A, 3\,\boldsymbol{u}),$$

and

$$|K|_\Lambda \leq 27 |\mathrm{P}\,(A, \boldsymbol{u})|. \tag{4.25}$$

Therefore, $\tau_2 \leq 3$ and $\nu_2 \leq 27$. Moreover, the columns of A can be chosen to be a lattice basis associated with the successive minima of K in which case the dilation factor of 3 is best possible.

Proof. We assume that $\Lambda = \mathbb{Z}^2$ without loss of generality and we write $\lambda_i = \lambda_i(K, \mathbb{Z}^2)$, $i = 1, 2$. If $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are the associated vectors with the successive minima λ_1 and λ_2 it follows from their definition that $\mathrm{conv}\,(\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{0})$ contains no further lattice point. By [54, Theorem 4, p. 20] this implies that $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are a lattice basis of $\mathbb{Z}^2$. Thus, after applying a unimodular transformation, we may assume $\frac{1}{\lambda_i}\boldsymbol{e}_i \in \mathrm{bd}K$, $i = 1, 2$. Hence, K has empty intersection with the four open cones

$$\begin{aligned}
&\{\boldsymbol{x} \in \mathbb{R}^2 : \lambda_1 x_1 + \lambda_2 x_2 > 1 \text{ and } \lambda_1 x_1 - \lambda_2 x_2 > 1\},\\
&\{\boldsymbol{x} \in \mathbb{R}^2 : -\lambda_1 x_1 + \lambda_2 x_2 > 1 \text{ and } -\lambda_1 x_1 - \lambda_2 x_2 > 1\},\\
&\{\boldsymbol{x} \in \mathbb{R}^2 : \lambda_1 x_1 + \lambda_2 x_2 > 1 \text{ and } -\lambda_1 x_1 + \lambda_2 x_2 > 1\},\\
&\{\boldsymbol{x} \in \mathbb{R}^2 : \lambda_1 x_1 - \lambda_2 x_2 > 1 \text{ and } -\lambda_1 x_1 - \lambda_2 x_2 > 1\},
\end{aligned} \tag{4.26}$$

with apices $\pm\frac{1}{\lambda_i}\boldsymbol{e}_i$, $i = 1, 2$. Moreover, by the definition of λ_2, we also know that no point $\boldsymbol{y} \in \lambda_2 K$ can satisfy

$$|y_1| > 1 \text{ and } |y_2| > 1, \tag{4.27}$$

by the following argument. For if $\boldsymbol{y} > 0$, then

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{y_1 + y_2 - 1}\boldsymbol{y} + \frac{y_2 - 1}{y_1 + y_2 - 1}\boldsymbol{e}_1 + \frac{y_1 - 1}{y_1 + y_2 - 1}\boldsymbol{e}_2,$$

which shows that $(1,1)^T$ is an interior point of $\lambda_2 K$ contradicting the definition of λ_2. Hence, for $\boldsymbol{x} \in K$, we know that either $|x_1| \leq \frac{1}{\lambda_2}$ or $|x_2| \leq \frac{1}{\lambda_2}$. Together with (4.26), we find for $\boldsymbol{x} \in K$

$$\begin{aligned} &|x_1| \leq \frac{1}{\lambda_2} \text{ and } |x_2| \leq \frac{2}{\lambda_2}, \text{ or} \\ &|x_2| \leq \frac{1}{\lambda_2} \text{ and } |x_1| \leq \frac{2}{\lambda_1}, \end{aligned} \tag{4.28}$$

see Figure 4.6. We let $A = (\boldsymbol{e}_1, \boldsymbol{e}_2)$ and show that there exist a $\mu \in [0,1]$ such that by taking $\boldsymbol{u} = \left(\mu\lambda_1^{-1}, (1-\mu)\lambda_2^{-1}\right)^T$ it holds

$$\mathrm{P}_{\mathbb{R}}\,(A, \boldsymbol{u}) \subseteq K \subseteq \mathrm{P}_{\mathbb{R}}\,(A, 3\boldsymbol{u}). \tag{4.29}$$

The first inclusion of (4.29) is true for every $\mu \in [0,1]$ since $(\pm\mu\lambda_1^{-1}, \pm(1-\mu)\lambda_2^{-1})^T = \mu\left(\pm\lambda_1^{-1}\boldsymbol{e}_1\right) + (1-\mu)\left(\pm\lambda_2^{-1}\boldsymbol{e}_2\right) \in K$, and thus

$$\mathrm{P}_{\mathbb{R}}\,(A, \boldsymbol{u}) \subseteq \operatorname{conv}\left(\pm\lambda_1^{-1}\boldsymbol{e}_1, \pm\lambda_2^{-1}\boldsymbol{e}_2\right) \subseteq K.$$

Therefore, we now show the second inclusion of (4.29). To this end, let $\boldsymbol{t} \in K$, such that

$$t_1 = \max\{x_1 \ : \boldsymbol{x} \in K\}, \qquad \text{and let} \qquad \mu = \frac{\lambda_1}{3} t_1.$$

If $t_1 < \lambda_2^{-1}$ then we know that for every $\boldsymbol{y} \in K$ we have $y_1 < \lambda_2^{-1}$ by the definition of $\boldsymbol{t}$ and $y_2 \leq 2\lambda_2^{-1}$ by (4.28). Since $3(1-\mu)\lambda_2^{-1} = (3 - \lambda_1 t_1)\lambda_2^{-1} > 2\lambda_2^{-1}$ the second inclusion of (4.29) follows. Thus, we assume that $t_1 > \lambda_2^{-1}$. Let $\boldsymbol{y} \in K$ and we show that $\boldsymbol{y} \in \mathrm{P}_{\mathbb{R}}\,(A, 3\boldsymbol{u})$. Clearly, $|y_1| \leq 3u_1 = t_1$ by the definition of $\boldsymbol{z}$ and we assume that $|y_2| > 3u_2$. This implies

$$|y_2| > 3\lambda_2^{-1}(1-\mu) = 3\lambda_2^{-1}\left(1 - \frac{\lambda_1}{3}t_1\right) = 3\lambda_2^{-1} - \lambda_1\lambda_2^{-1}t_1 \geq \lambda_2^{-1},$$

using that $t_1 \leq 2\lambda_2^{-1} \leq 2\lambda_1^{-1}$ according to (4.28). Thus, $|y_1| \leq \lambda_2^{-1}$. By the symmetry of K, we assume that $y_2 > 0$ without loss of generality.

We will distinguish two cases.

Firstly, we will consider the case that $0 \leq y_1 \leq \lambda_2^{-1}$. Then, we conclude that

$$\begin{aligned} &y_2 - y_1 > \lambda_2^{-1} - \lambda_2^{-1} = 0, \\ &t_1 - t_2 > \lambda_2^{-1} - \lambda_2^{-1} = 0, \\ &t_1 y_2 - t_2 y_1 > \left(\frac{1}{\lambda_2}\right)^2 - \left(\frac{1}{\lambda_2}\right)^2 = 0. \end{aligned} \tag{4.30}$$

Moreover, since $\lambda_1 y_1 + \lambda_2 y_2 \geq \lambda_2 y_2 > 1$ and $\boldsymbol{y} \in K$ must lie in the third cone of (4.26), it must hold $-\lambda_1 y_1 + \lambda_2 y_2 \leq 1$, and consequently

$$y_1 \geq \lambda_1^{-1}(\lambda_2 y_2 - 1). \tag{4.31}$$

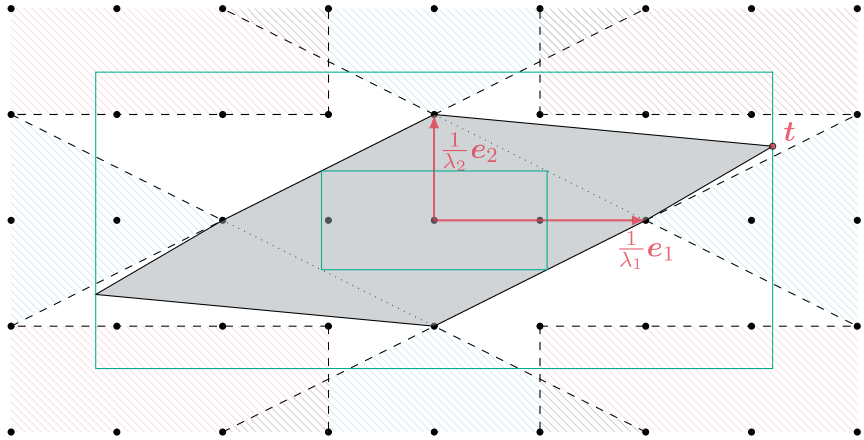

Figure 4.6: K (gray), cones from (4.26) (blue) and parallelepipeds $\mathrm{P}_{\mathbb{R}}(A, \boldsymbol{u})$, $\mathrm{P}_{\mathbb{R}}(A, 3\boldsymbol{u})$ (green)

Now, $\lambda_2^{-1} - t_2, t_1 - \lambda_2^{-1}, y_1, y_2 \geq 0$ and we estimate

$$\begin{aligned}
&\begin{pmatrix}\lambda_2^{-1} - t_2 \\ t_1 - \lambda_2^{-1}\end{pmatrix}^T \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \geq \begin{pmatrix}\lambda_2^{-1} - t_2 \\ t_1 - \lambda_2^{-1}\end{pmatrix}^T \begin{pmatrix} \lambda_1^{-1}(\lambda_2 y_2 - 1) \\ y_2 \end{pmatrix} \\
&\geq \begin{pmatrix}\lambda_2^{-1} - t_2 \\ t_1 - \lambda_2^{-1}\end{pmatrix}^T \begin{pmatrix} \lambda_1^{-1}\lambda_2 y_2 \\ y_2 \end{pmatrix} \geq \begin{pmatrix}\lambda_2^{-1} - t_2 \\ t_1 - \lambda_2^{-1}\end{pmatrix}^T \begin{pmatrix} y_2 \\ y_2 \end{pmatrix} \\
&= y_2(t_1 - t_2) > \lambda_2^{-1}(t_1 - t_2),
\end{aligned} \tag{4.32}$$

using (4.31). The estimate (4.32) implies that

$$\frac{\lambda_2^{-1}(t_1 - t_2 + y_2 - y_1)}{t_1 y_2 - t_2 y_1} < 1. \tag{4.33}$$

Now, we find that

$$\begin{aligned}
\lambda_2^{-1}\begin{pmatrix}1\\1\end{pmatrix} &= \frac{\lambda_2^{-1}(y_2 - y_1)}{t_1 y_2 - t_2 y_1}\begin{pmatrix}t_1\\t_2\end{pmatrix} + \frac{\lambda_2^{-1}(t_1 - t_2)}{t_1 y_2 - t_2 y_1}\begin{pmatrix}y_1\\y_2\end{pmatrix} \\
&+ \left(1 - \frac{\lambda_2^{-1}(t_1 - t_2 + y_2 - y_1)}{t_1 y_2 - t_2 y_1}\right)\begin{pmatrix}0\\0\end{pmatrix}.
\end{aligned}$$

By (4.30) and (4.33) this shows that $(1,1)^T$ lies in the interior of $\lambda_2 K$; a contradiction.

Secondly, we assume that $-\lambda_2^{-1} \leq y_1 < 0$. Again, we find that

$$\begin{aligned}
&-y_1\lambda_2^{-1} > 0, \\
&t_1\lambda_2^{-1} > 0, \\
&t_1 y_2 - t_2 y_1 \geq t_1 y_2 > 0.
\end{aligned} \tag{4.34}$$

Furthermore, $\lambda_2^{-1}(t_1 - y_1) < t_1y_2 - t_2y_1$, since $y_2 > \lambda_2^{-1}$ and $t_2 < \lambda_2^{-1}$ by (4.28), and thus

$$\frac{\lambda_2^{-1}(t_1 - y_1)}{t_1y_2 - t_2y_1} < 1. \tag{4.35}$$

Now,

$$\lambda_2^{-1}\begin{pmatrix}0\\1\end{pmatrix} = \frac{-y_1\lambda_2^{-1}}{t_1y_2 - t_2y_1}\begin{pmatrix}t_1\\t_2\end{pmatrix} + \frac{t_1\lambda_2^{-1}}{t_1y_2 - t_2y_1}\begin{pmatrix}y_1\\y_2\end{pmatrix} + \left(1 - \frac{\lambda_2^{-1}(t_1 - y_1)}{t_1y_2 - t_2y_1}\right)\begin{pmatrix}0\\0\end{pmatrix}.$$

Using (4.34) and (4.35) this shows that $(0,1)^T$ lies in the interior of λ_2K. In summary, we showed that for an arbitrary $\boldsymbol{y} \in K$, we have that $y_2 \leq 3u_2$, and therefore (4.29) holds, which implies

$$\mathrm{P}\,(A, \boldsymbol{u}) \subseteq K \cap \mathbb{Z}^2 \subseteq \mathrm{P}\,(A, 3\boldsymbol{u}).$$

As demonstrated by Example 4.22, the dilation factor of 3 is best possible if the columns of A are the vectors associated with the successive minima of K. The size bound (4.25) follows from Proposition 4.6. □

Since generally the vectors associated with the successive minima of a symmetric convex body $K \in \mathcal{K}_o^d$ do not span a lattice basis, the proof of Theorem 4.21 cannot be generalized to higher dimensions. Moreover, for a given $K \in \mathcal{K}_o^d$, the symmetric GAP induced by a lattice basis associated with the successive minima of $K \in \mathcal{K}_o^2$ does not have to attain $\tau_d\,(K, \Lambda)$ as demonstrated in Example 4.22.

Example 4.22. Let $k \geq 1$ an integer and

$$Q = \mathrm{conv}\left(\pm(k,0)^T, \pm(k-1, 2k-1)^T\right),$$

cf. Figure 4.7. Then, $k^{-1}Q$ has the outer description

$$k^{-1}Q = \left\{\boldsymbol{x} \in \mathbb{R}^2 \;:\; |x_1 - x_2| \leq 1,\; |x_1 + (2k-1)^{-1}| \leq 1\right\},$$

and it is easy to see that $k^{-1}Q \cap \mathbb{Z}^2 = \{\pm\boldsymbol{e}_1, \pm\boldsymbol{e}_2, \boldsymbol{0}\}$. Therefore, $\lambda_1(Q) = \lambda_2(Q) = k^{-1}$ and the successive minima are attained at $\pm\boldsymbol{e}_1, \pm\boldsymbol{e}_2$. Now, let $A = (\boldsymbol{e}_1, \boldsymbol{e}_2)$, and we assume that

$$\mathrm{P}\,(A, \boldsymbol{u}) \subseteq Q \cap \mathbb{Z}^2 \subseteq \mathrm{P}\,(A, c\boldsymbol{u}), \tag{4.36}$$

for a $\boldsymbol{u} \in \mathbb{R}^2_{\geq 0}$ and a factor $c > 0$. We will show that $c \geq 3$ if k is sufficiently large. By (4.36), we find that

$$cu_1 \geq k, \quad \text{and} \quad cu_2 \geq 2k - 1, \tag{4.37}$$

since $(k,0)^T, (k-1, 2k-1)^T \in K \cap \mathbb{Z}^d \subseteq \mathrm{P}\,(A, c\boldsymbol{u})$, and likewise

$$\lfloor u_1 \rfloor + \lfloor u_2 \rfloor \leq k,$$

because $(\lfloor u_1 \rfloor, \lfloor u_2 \rfloor)^T \in \mathrm{P}\,(A, \boldsymbol{u}) \subseteq Q \cap \mathbb{Z}^d$. We conclude that

$$c \geq \frac{k}{u_1} > \frac{k}{\lfloor u_1 \rfloor + 1} \geq \frac{k}{k - \lfloor u_2 \rfloor + 1} =: f(\lfloor u_2 \rfloor), \quad \text{and} \quad c \geq \frac{2k-1}{u_2} > \frac{2k-1}{\lfloor u_2 \rfloor + 1} =: g(\lfloor u_2 \rfloor),$$

and thus

$$c \geq \min_{\lfloor u_2 \rfloor \in [0,k]} \max \{ f(\lfloor u_2 \rfloor), g(\lfloor u_2 \rfloor) \} \geq \min_{x \in [0,k]} \max \{ f(x), g(x) \} . \tag{4.38}$$

Since for $x \in [0, k]$ f is strictly increasing and g is strictly decreasing, respectively, we find that the minimum in the right-hand side of (4.38) is attained if $f(x^*) = g(x^*)$ provided such an x^* exist. Indeed, $f(x^*) = g(x^*)$ for

$$x^* = \frac{(2k-1)(k+1) - k}{3k-1} = \frac{2k^2 - 1}{3k - 1}.$$

Clearly, $x^* \in [0, k]$, since $k \geq 1$. This shows that

$$c > g(x^*) = f(x^*) = \frac{3k^2 - k}{k^2 - k - 2},$$

which in turn implies that $c \geq 3$ if $k \to \infty$.

We observe that for $B = (\boldsymbol{b}_1, \boldsymbol{b}_2)$, $\boldsymbol{b}_1 = (1, 1)^T$, $\boldsymbol{b}_2 = (0, 1)^T$ and $\boldsymbol{w} = (k - k\varepsilon/(1+\varepsilon), k - k\varepsilon/(1+\varepsilon))^T$, where $\varepsilon > 0$ is arbitrarily small, it holds

$$\mathrm{P}\,(B, \boldsymbol{w}) \subseteq Q \cap \mathbb{Z}^d \subseteq \mathrm{P}\,(B, (1+\varepsilon)\boldsymbol{w}).$$

In particular, $\mathrm{P}\,(B, \boldsymbol{w}) = \left(Q \cap \mathbb{Z}^d\right) \setminus \{(\pm k, 0)^T\}$, meaning that the symmetric GAP $\mathrm{P}\,(B, \boldsymbol{w})$ covers all of $Q \cap \mathbb{Z}^d$ but two points. However, the vector $\boldsymbol{b}_2$ is not associated to one of the successive minima of Q.

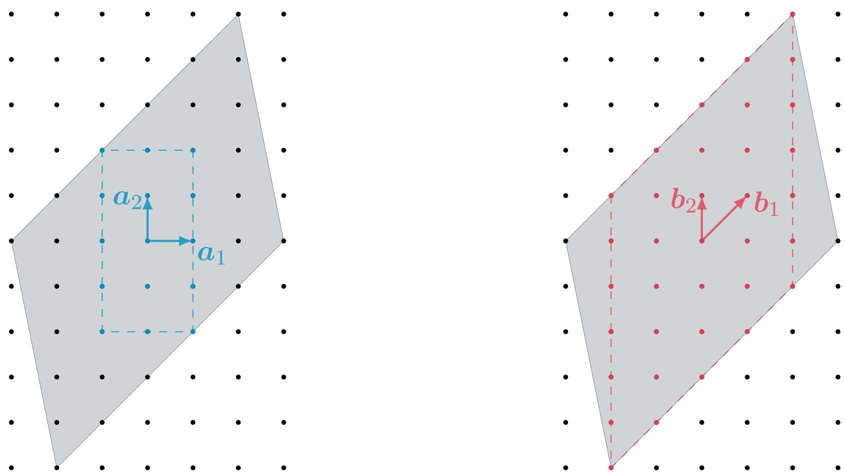

Figure 4.7: Q of Example 4.22 for $k = 3$, symmetric GAPs $\mathrm{P}\,(A, \boldsymbol{u})$ (left) and $\mathrm{P}\,(B, \boldsymbol{w})$ (right)

5 Discrete slicing inequalities

5.1 Introduction

Quantifying the volume of a symmetric convex body K in terms of its lower dimensional sections is an active field of research within convex and discrete geometry. Most notably, the question whether there is an absolute constant C such that for every symmetric convex body $K \in \mathcal{K}_o^d$ it holds

$$\mathrm{vol}_d\,(K)^{\frac{d-1}{d}} \leq C \max_{\theta \in \mathbb{S}^{d-1}} \mathrm{vol}_{d-1}\left(K \cap \theta^{\perp}\right), \tag{5.1}$$

known as the **slicing problem of Bourgain** [28, 29, 92], is easily one of the most prestigious in the field. The best known result to date is due to Klartag [76] who showed that $C = O\left(d^{1/4}\right)$ by improving an earlier bound of Bourgain [30].

Moreover, it is well-known that the $(d-1)$-dimensional section of a symmetric convex body K containing the origin has maximal volume among all other parallel sections. More precisely, for every $H \in \mathcal{G}(d-1, d)$ and every $\boldsymbol{x} \in \mathbb{R}^d$, we have

$$\mathrm{vol}_{d-1}\,((H + \boldsymbol{x}) \cap K) \leq \mathrm{vol}_{d-1}\,(H \cap K), \tag{5.2}$$

a consequence of *Brunn's inequality*, which will be discussed in Section 5.2. There, we will show how (5.2) can be discretized by replacing the volume with the discrete volume.

In recent years there has been generally growing interest in studying sections of lattice point sets, e.g. [50]. In 2013 Koldobsky suggested the following *semi-discretized* version of the slicing conjecture at the AIM workshop "Sections of Convex Bodies". *Semi-discrete* means that a statement contains the volume as well as the discrete volume. Here, we denote by $H_\sigma(K, k)$ the **maximal (k-dimensional) discrete section** of K containing the origin; more precisely,

$$|K \cap H_\sigma(K, k)|_{\mathbb{Z}^d} := \max\left\{|K \cap H|_{\mathbb{Z}^d} : H \in \mathcal{G}(k, d)\right\}.$$

The results from this chapter are joint work with Martin Henk.

Question 5.1 (Koldobsky)**.** *Does a constant $c := c(d)$, which possibly depends on the dimension d, exist, such that for every $K \in \mathcal{K}_o^d$ with $\operatorname{lin}\left(K \cap \mathbb{Z}^d\right) = \mathbb{R}^d$, it holds*

$$|K|_{\mathbb{Z}^d} \leq c\ |K \cap H_\sigma(K, d-1)|_{\mathbb{Z}^d}\ \operatorname{vol}(K)^{1/d}\,? \tag{5.3}$$

Alexander, Henk and Zvavitch [3] gave a confirmative answer to Koldobsky's question.

Theorem 5.2 (Alexander, Henk, Zvavitch [3])**.** *If $K \in \mathcal{K}_o^d$, then*

$$|K|_{\mathbb{Z}^d} \leq O(1)^d\, d^{d-k} |K \cap H_\sigma(K, k)|_{\mathbb{Z}^d} \operatorname{vol}(K)^{\frac{d-k}{d}}. \tag{5.4}$$

This shows for $k = d-1$ that one can choose $c = O(1)^d$ in (5.3). Moreover, for $d = 2$ the constant c can be chosen to be 4 [3, Theorem. 1] and if $K \in \mathcal{K}_o^d$ is unconditional then $c = O(d)$ [3, Theorem. 6]. It is unclear, however, if $c = O(d)$ also holds for arbitrary symmetric convex bodies. Regev [105] showed that this is indeed the case for convex bodies with volume at most C^{d^2} where $C > 0$ is a constant. The d-dimensional cross-polytope ${C_d}^* = \operatorname{conv}(\pm\boldsymbol{e}_1, \ldots, \pm\boldsymbol{e}_d)$ shows that the factor in (5.4) has to be at least $O(d)^{d-k}$. In particular, the constant c in (5.3) has to be at least linear in the dimension d. We present a fully discretized slicing inequality in Theorem 5.7 of Section 5.3. Furthermore, for 1-dimensional sections containing the origin, we prove in Section 5.4 the following inequality including an explicit constant:

$$|K|_{\mathbb{Z}^d} < \left(\frac{4}{3}\right)^d |K \cap H_\sigma(K, 1)|_{\mathbb{Z}^d}^d,$$

cf. Theorem 5.14. This inequality cannot be improved.

In Section 5.5, we will study nonzero integer points in symmetric convex bodies which do not contain a sum. To this end, we will introduce the notion of *symmetric sum-free* convex bodies, which is derived from the notion of *sum-free sets* in abelian groups. It will be revealed that the discrete volume of symmetric sum-free convex bodies can be bounded satisfyingly and that a convex body is symmetric sum-free if and only if none of its 2-dimensional slices contains more than 5 integer points.

5.2 A discrete, symmetric Brunn inequality

For a convex body $K \in \mathcal{K}^d$ and a hyperplane $H \in \mathcal{G}(d-1, d)$, the function $f_H : \mathbb{R}^d \to \mathbb{R}_{\geq 0}$, $f(\boldsymbol{x}) = \operatorname{vol}_{d-1}((H + \boldsymbol{x}) \cap K)$ is log-concave, i.e.,

$$f(\boldsymbol{x})^{1-\lambda} f(\boldsymbol{y})^\lambda \leq f((1-\lambda)\boldsymbol{x} + \lambda\boldsymbol{y}), \tag{5.5}$$

which is known as **Brunn's inequality**, see [31, Theorem 1.2.2]. By plugging in $\boldsymbol{y} = -\boldsymbol{x}$ and $\lambda = 1/2$ and considering symmetric $K \in \mathcal{K}_o^d$, we deduce (5.2). It is however clear, that there is no immediate discrete analogue of (5.5) since, on the one hand, the function $\bar{f}_H(\boldsymbol{x}) = |(\boldsymbol{x} + H) \cap K|_{\mathbb{Z}^d}$ is nonzero only for $\boldsymbol{x}$ lying in a finite union of proper subspaces of $\mathbb{R}^d$ and, on the other hand, $\bar{f}$ is not log-concave (or even concave) even if $\boldsymbol{x}$ is restricted to the support of $\bar{f}$, see Figure 5.1. The following theorem shows that (5.2), however, can in fact be discretized by

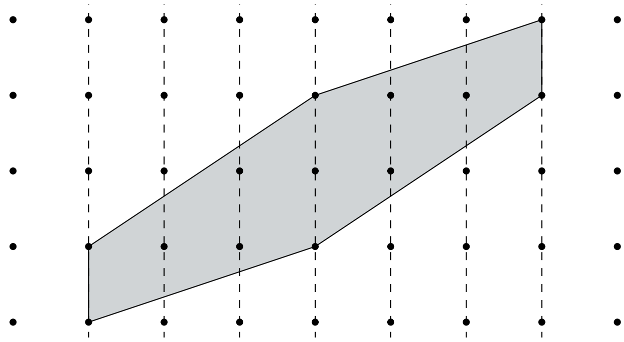

Figure 5.1: Illustration of $\bar{f}_H$ not being (log-)concave

replacing the $d-1$-dimensional volume by the lattice points enumerator, although the right-hand side might be larger by an additional factor exponential in the dimension of the linear subspace H.

Theorem 5.3. *Let $K \in \mathcal{K}_o^d$ and $H \in \mathcal{G}(k,d)$. Then,*

$$|K \cap (\boldsymbol{z} + H)|_{\mathbb{Z}^d} \leq 2^k |K \cap H|_{\mathbb{Z}^d},$$

for all $\boldsymbol{z} \in \mathbb{Z}^d$, and this inequality is best possible.

Proof. Let $\boldsymbol{v}_1, \dots, \boldsymbol{v}_k \in \mathbb{Z}^d$ be a lattice basis of $H \cap \mathbb{Z}^d$, and for a given $\boldsymbol{z} \in \mathbb{Z}^d$ let m be the largest integer such that $m\ 2^k < |(\boldsymbol{z} + H) \cap K|_{\mathbb{Z}^d} =: r$. Let $\boldsymbol{g}_1, \dots, \boldsymbol{g}_r$ be the lattice points in $(\boldsymbol{z} + H) \cap K$, and let $\boldsymbol{z}_1, \dots, \boldsymbol{z}_r \in \mathbb{Z}^k$ be their unique coordinate vectors with respect to the basis $\boldsymbol{v}_1, \dots, \boldsymbol{v}_k$, i.e., $\boldsymbol{g}_j = \boldsymbol{z} + \sum_{i=1}^k z_{j,i} \boldsymbol{v}_i$; here $\boldsymbol{z}_j = (z_{j,1}, \dots, z_{j,k})^T$. By the pigeonhole principle, there exist distinct $\boldsymbol{z}_1, \dots, \boldsymbol{z}_m, \boldsymbol{z}_{m+1}$, say, with $\boldsymbol{z}_1 \equiv \boldsymbol{z}_2 \equiv \cdots \equiv \boldsymbol{z}_{m+1} \bmod 2$. Since, $-\boldsymbol{z}_1, \dots, -\boldsymbol{z}_{m+1} \in (-\boldsymbol{z} + H) \cap K$, we have

$$\frac{1}{2}(\boldsymbol{g}_{m+1} - \boldsymbol{g}_j) = \sum_{i=1}^{k} \boldsymbol{v}_i \frac{1}{2}(z_{m+1,i} - z_{j,i}) \in K \cap H \cap \mathbb{Z}^d, \quad j \in [m]\,.$$

These are pairwise different and, taking into account the origin $\mathbf{0}$, we obtain $|K \cap H|_{\mathbb{Z}^d} \geq m+1$. Since $2^k(m+1) \geq |(\boldsymbol{z} + H) \cap K|_{\mathbb{Z}^d}$ by the choice of m, the bound follows.

In order to see that the bound is best possible, let $Q = \operatorname{conv}(C_{d-1} \times \{1\}, -C_{d-1} \times \{-1\})$, i.e., we embed the $(d-1)$-cube C_{d-1} into $\mathbb{R}^n$ with last coordinate 1, $-C_{d-1}$ into $\mathbb{R}^d$ with last coordinate -1 and then we consider their convex hull. The only lattice points of Q except for the origin are its vertices, i.e., the lattice points of $C_{d-1} \times \{1\}$ and $-C_{d-1} \times \{-1\}$. If H is parallel to an k-face of C_{d-1}, then $\max_{\boldsymbol{z} \in \mathbb{Z}^d} |(\boldsymbol{z} + H) \cap Q|_{\mathbb{Z}^d} = 2^k$, whereas $|H \cap Q|_{\mathbb{Z}^d} = 1$. □

It has not escaped our notice that there are related discrete problems dealing with lattice points in affine subspaces omitting convexity, e.g. [47].

5.3 A discrete slicing inequality

In this section, we will prove a lower bound on $|K \cap H_\sigma(K,i)|_{\mathbb{Z}^d}$ in terms of $|K \cap H_\sigma(K,j)|_{\mathbb{Z}^d}$, where $j > i$, in Theorem 5.7. Moreover, in Section 5.4, we will settle the case $i = 1$.

In order to prove Theorem 5.7, we will first deduce a simple lower bound for the i-th successive minimum of a convex body in terms of its number of lattice points based on a result by Betke, Henk and Wills.

Theorem 5.4 (Betke, Henk, Wills[22]). *Let $K \in \mathcal{K}_o^d$ be a symmetric convex body, and $\lambda_d(K) \le 2$. Then,*

$$|K|_{\mathbb{Z}^d} \ge \frac{1}{d!} \prod_{i=1}^{d} \left(\frac{2}{\lambda_i(K)} - 1 \right).$$

Corollary 5.5. *Let $K \in \mathcal{K}_o^d$ be a symmetric convex body with $\lambda_d(K) \le 2$. Then, for every $i \in [d]$,*

$$\lambda_i(K) \ge (i! \ |K|_{\mathbb{Z}^d})^{-1/i}.$$

Proof. For $k \in [i]$, let $\boldsymbol{a}_k \in \lambda_k(K)K \cap \mathbb{Z}^d$ such that $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_i$ are linearly independent. If A denotes the i-dimensional linear span of $\{\boldsymbol{a}_1, \ldots, \boldsymbol{a}_i\}$, then $\lambda_k(K) = \lambda_k(K \cap A)$ for $k \in [i]$ and by Theorem 5.4

$$|K|_{\mathbb{Z}^d} \ge |K \cap A|_{\mathbb{Z}^d} \ge \frac{1}{i!} \prod_{k=1}^{i} \left(\frac{2}{\lambda_k(K)} - 1 \right) \ge \frac{1}{i!} \left(\frac{2}{\lambda_i(K)} - 1 \right)^i.$$

And since $|K|_{\mathbb{Z}^d} \ge 1$,

$$\lambda_i(K) \ge \frac{2}{(i!|K|_{\mathbb{Z}^d})^{1/i} + 1} \ge \frac{1}{(i!|K|_{\mathbb{Z}^d})^{1/i}}. \qquad \square$$

Banaszczyk presented the following relation between the successive minima of $K \in \mathcal{K}_o^d$ and its polar body K^*.

Theorem 5.6 (Banaszczyk [10]). *If $K \in \mathcal{K}_o^d$, then*

$$\lambda_i(K)\lambda_{d-i+1}(K^*) \ \le \ O\,(1) \ d \ \ln d.$$

Theorem 5.7. *Let $K \in \mathcal{K}_o^d$ with $\lambda_d(K) \le 1$. Then, for $k \in [d]$ there exist a linear subspace $U \in \mathcal{G}(d-k,d)$ and $A_i \in \mathcal{G}(i,d)$, $i = d-k+1, \ldots, d$, with $A_{d-k+1} \subseteq \cdots \subseteq A_d$ such that*

$$O\,(1)^k \ |U \cap K|_{\mathbb{Z}^d} \ge \left(\frac{1}{d \ \ln d + 1} \right)^k \left(\frac{1}{d!} \right)^{k/d} \prod_{i=d-k+1}^{d} \frac{|K|_{\mathbb{Z}^d}^{1/k}}{|K \cap A_i|_{\mathbb{Z}^d}^{1/i}}.$$

In particular,

$$O\,(1)^k \ |K \cap H_\sigma(K,d-k)|_{\mathbb{Z}^d} \ge \left(\frac{1}{d \ \ln d + 1} \right)^k \left(\frac{1}{d!} \right)^{k/d} \prod_{i=d-k+1}^{d} \left(\frac{|K|_{\mathbb{Z}^d}^{(1/k)}}{|K \cap H_\sigma(K,i)|_{\mathbb{Z}^d}^{(1/i)}} \right),$$

and

$$O(1)^k\ |K \cap H_\sigma(K, d-k)|_{\mathbb{Z}^d} \geq \left(\frac{1}{d\ \ln d + 1}\right)^k \left(\frac{1}{d!}\right)^{k/d} |K|_{\mathbb{Z}^d}^{1-\left(\frac{1}{d-k+1}+\cdots+\frac{1}{d}\right)}.$$

Proof. For $i \in [k]$, let $\boldsymbol{a}_i{}^* \in \lambda_i(K^*)K^* \cap \mathbb{Z}^d$ with $\boldsymbol{a}_1{}^*, \ldots, \boldsymbol{a}_k{}^*$ linearly independent. Similarly, for $i \in [d]$ let $\boldsymbol{a}_i \in \lambda_i(K)K \cap \mathbb{Z}^d$ with $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_d$ linearly independent. If A_i denotes the i-dimensional linear span of $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_i$, then $\lambda_i(K) = \lambda_i(K \cap A_i)$, and by Theorem 5.4

$$|K \cap A_i|_{\mathbb{Z}^d} \geq \frac{1}{i!} \prod_{k=1}^{i} \left(\frac{2}{\lambda_k(K)} - 1\right) \geq \frac{1}{i!} \left(\frac{2}{\lambda_i(K)} - 1\right)^i.$$

Since $|K \cap A_i|_{\mathbb{Z}^d} \geq 1$,

$$\lambda_i(K) \geq \frac{2}{(i!|K \cap A_i|_{\mathbb{Z}^d})^{1/i} + 1} \geq \frac{1}{(i!|K \cap A_i|_{\mathbb{Z}^d})^{1/i}}$$

Moreover, for every $\boldsymbol{x} \in K$

$$\langle \boldsymbol{a}_i{}^*, \boldsymbol{x} \rangle \in [-\lambda_i(K^*), \lambda_i(K^*)] \cap \mathbb{Z},$$

and by Theorem 5.6

$$\langle \boldsymbol{a}_i{}^*, \boldsymbol{x} \rangle \in \left[-O(1)\frac{d\ \ln d}{\lambda_{d-i+1}(K)}, O(1)\frac{d\ \ln d}{\lambda_{d-i+1}(K)}\right] \cap \mathbb{Z} =: I_i.$$

Therefore, for $i \in [d]$ there exist $b_i \in I_i$ such that

$$\left|\bigcap_{i=1}^{k} \left\{\boldsymbol{x} \in \mathbb{R}^d\ :\ \langle \boldsymbol{a}_i{}^*, \boldsymbol{x} \rangle = b_i\right\} \cap K\right|_{\mathbb{Z}^d} \geq \frac{|K|_{\mathbb{Z}^d}}{|I_1|\ |I_2|\ \ldots\ |I_k|} = \frac{|K|_{\mathbb{Z}^d}}{\prod_{i=1}^k \left(2\left\lfloor O(1)\frac{d\ \ln d}{\lambda_{d-i+1}(K)}\right\rfloor + 1\right)}$$

$$\geq \frac{|K|_{\mathbb{Z}^d}}{\prod_{i=1}^k \left(2\ O(1)\frac{d\ \ln d}{\lambda_{d-i+1}(K)} + \frac{1}{\lambda_{d-1+1}(K)}\right)} = O(1)^{-k}\ |K|_{\mathbb{Z}^d}\ \frac{\prod_{i=1}^k \lambda_{d-i+1}(K)}{\prod_{i=1}^k d\ \ln d + 1}$$

$$\geq O(1)^{-k}|K|_{\mathbb{Z}^d}\left(\frac{1}{d\ \ln d + 1}\right)^k \prod_{i=d-k+1}^{d} \lambda_i(K) \geq O(1)^{-k}\left(\frac{1}{d\ \ln d + 1}\right)^k \prod_{i=d-k+1}^{d} \frac{|K|_{\mathbb{Z}^d}^{1/k}}{(i!|K \cap A_i|_{\mathbb{Z}^d})^{1/i}}$$

$$\geq O(1)^{-k}\left(\frac{1}{d\ \ln d + 1}\right)^k \left(\frac{1}{d!}\right)^{k/d} \prod_{i=d-k+1}^{d} \frac{|K|_{\mathbb{Z}^d}^{1/k}}{|K \cap A_i|_{\mathbb{Z}^d}^{1/i}}.$$

This implies the inequality of the theorem. □

5.4 Lattice points in 1-dimensional slices

In this section, we discuss the question how the number of lattice points in a symmetric convex body can be bounded by the number of its maximal 1-dimensional discrete section. More precisely, we will seek to bound $|K|_{\mathbb{Z}^d}$ in terms of $|K \cap H_\sigma(K, 1)|_{\mathbb{Z}^d}$. Naturally, the question arises if

$$|K|_{\mathbb{Z}^d} \leq O(1)\ |K \cap H_\sigma(K,1)|^d_{\mathbb{Z}^d}, \tag{5.6}$$

and we will present an thorough answer in Theorem 5.10. Studying collinear lattice points in convex sets has been of interest before, for instance in the works of Bell [17] regarding the outer description of a lattice polytope, Bárány, Füredi [12] and Averkov, Wagner [8] regarding the lattice diameter of a lattice polygon. Furthermore, the question in what way a discrete set can be reconstructed from a small number of one-dimensional projections or *(discrete) x-rays* leads to the field of discrete tomography, see [67, 68, 79] and [49, p.88-91, p.228-229]. In view of (5.6), there is a related result due to Rabinowitz, which examines lattice points on lines, albeit not lines through the origin.

Theorem 5.8 (Rabinowitz [103])**.** *Let $K \in \mathcal{K}^d$ such that there are no $m+1$ collinear points in $K \cap \mathbb{Z}^d$, then*

$$|K|_{\mathbb{Z}^d} \leq m^d.$$

As mentioned above, we are only interested in collinear lattice points such that the origin is contained in the line segment joining those points. As a consequence, we restrict our study to symmetric convex bodies to exclude degenerate case such as $\operatorname{conv}\left([-m,m]^{d-1} \times \{1\}\right)$ for m large. We start by observing a very basic relation between $H_\sigma(K,1)$ and the first successive minimum of K.

Proposition 5.9. *Let $K \in \mathcal{K}^d_o$ and let $\boldsymbol{v}_1 \in \lambda_1(K)K \cap \mathbb{Z}^d \setminus \{\mathbf{0}\}$. Then,*

$$|K \cap H_\sigma(K,1)|_{\mathbb{Z}^d} = |K \cap \operatorname{lin}\,(\boldsymbol{v}_1)\,|_{\mathbb{Z}^d} = 2\left\lfloor \frac{1}{\lambda_1(K)} \right\rfloor + 1. \tag{5.7}$$

Proof. If $|K \cap H_\sigma(K,1)|_{\mathbb{Z}^d} = 1$, we clearly have $|K \cap H_\sigma(K,1)|_{\mathbb{Z}^d} \leq |K \cap \operatorname{lin}\,(\boldsymbol{v}_1)\,|_{\mathbb{Z}^d}$, and hence we assume $|K \cap H_\sigma(K,1)|_{\mathbb{Z}^d} > 1$. Since $H_\sigma(K,1)$ is a one-dimensional linear subspace, there exist $\boldsymbol{w} \in \mathbb{Z}^d \setminus \{\mathbf{0}\}$ and an integer $m > 0$ such that $K \cap H_\sigma(K,1) = \{k\boldsymbol{w} \ : \ k \in [-m,m]\}$. We conclude $m\boldsymbol{w} \in K$, and thus $\boldsymbol{w} \in m^{-1}K$ we have $m^{-1} \geq \lambda_1(K)$. Furthermore, $m \leq \lfloor \lambda_1(K)^{-1} \rfloor$, because m is an integer. Finally, this yields

$$|K \cap H_\sigma(K,1)|_{\mathbb{Z}^d} = 2m+1 \leq 2\lfloor \lambda_1(K)^{-1} \rfloor + 1 = |K \cap \operatorname{lin}\,(\boldsymbol{v}_1)\,|_{\mathbb{Z}^d}. \tag{5.8}$$

The inequality $|K \cap H_\sigma(K,1)|_{\mathbb{Z}^d} \geq |K \cap \operatorname{lin}\,(\boldsymbol{v}_1)\,|_{\mathbb{Z}^d}$ holds by the definition of $H_\sigma(K,1)$. $\square$

Theorem 5.10. *Let $K \in \mathcal{K}^d_o$ and $\rho = |K \cap H_\sigma(K,1)|_{\mathbb{Z}^d}$. Then,*

$$|K|_{\mathbb{Z}^d} \leq (\rho+1)^d - 2^d + 1. \tag{5.9}$$

Moreover, this inequality is best possible, see Proposition 5.13.

Proof. Let $\rho = 2k+1$, $k \in \mathbb{N}$. If $\boldsymbol{x} \in \{0, k+1\}^d$ and $\boldsymbol{v} \in K \cap \mathbb{Z}^d \setminus \{\mathbf{0}\}$, then

$$\boldsymbol{v} \not\equiv \boldsymbol{x} \mod 2(k+1),$$

since otherwise $\boldsymbol{v} \in (k+1)\mathbb{Z}^d$ which implied $|K \cap H_\sigma(K,1)|_{\mathbb{Z}^d} \geq 2(k+1)+1 > \rho$. Therefore, the points $K \cap \mathbb{Z}^d \setminus \{\mathbf{0}\}$ are contained in at most $(2(k+1))^d - 2^d = (\rho+1)^d - 2^d$ equivalent classes $\bmod 2(k+1)$.

Suppose $|K|_{\mathbb{Z}^d} > (\rho+1)^d - 2^d + 1 = (2(k+1))^d - 2^d + 1$, then there are $\boldsymbol{y}, \boldsymbol{z} \in K \cap \mathbb{Z}^d \setminus \{\mathbf{0}\}$, $\boldsymbol{y} \neq \boldsymbol{z}$ with

$$\begin{aligned}
&\boldsymbol{y} \equiv \boldsymbol{z} \mod 2(k+1)\\
&\Leftrightarrow \boldsymbol{y} - \boldsymbol{z} \in 2(k+1)\mathbb{Z}^d\\
&\Leftrightarrow \tfrac{1}{2}(\boldsymbol{y} - \boldsymbol{z}) \in K \cap (k+1)\mathbb{Z}^d.
\end{aligned}$$

This contradicts $\rho = 2k+1$. We will verify that the bound (5.9) is best possible in Proposition 5.13. $\square$

Remark 5.11. *In view of Proposition 5.9, we may write (5.9) equivalently as*

$$|K|_{\mathbb{Z}^d} \leq \left(2\left\lfloor \frac{1}{\lambda_1(K)} \right\rfloor + 2\right)^d - 2^d + 1. \tag{5.10}$$

We compare this inequality to the known bound by Betke, Henk and Wills, cf. Theorem 2.1, and note that it depends on the fractional part of $\lambda_1(K)^{-1}$ which inequality yields the smaller upper bound. For instance, if $\{\lambda_1(K)\} \geq 1/2$, then Theorem 2.1 gives $|K|_{\mathbb{Z}^d} \leq (2\lfloor\lambda_1(K)^{-1}\rfloor + 2)^d$, which is larger than (5.10) by a summand of $2^d - 1$. In contrast, if $\{\lambda_1(K)\} < 1/2$, then Theorem 2.1 gives the bound $|K|_{\mathbb{Z}^d} \leq \left(2\lfloor\lambda_1(K)^{-1}\rfloor + 1\right)^d$ which is substantially better in this case in comparison to (5.10).

We will continue by showing that the inequality (5.9) in Theorem 5.10 is sharp. In order to do so, we will give a constructive proof by presenting a way to construct polytopes which attain equality in (5.9). First, we need the following technical lemma. It will ensure the existence of a half-space which contains half of the nonzero lattice points of a cube $[-k,k]^d$ such that the same holds true if we consider the coordinate projections of the cube and the half-spaces induced by the coordinate projection of the normal vector of the half-space, cf. Figure 5.2. We also introduce the following notation: Let $\boldsymbol{x} = (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d$ and let $I = \left\{i_1, \ldots, i_{|I|}\right\} \subseteq [d]$ be non-empty with $i_k < i_{k+1}$, $k \in [|I|-1]$. Then, we denote by $\boldsymbol{x}_I := (x_{i_1}, x_{i_2}, \ldots, x_{i_{|I|}})$ the **projection of x onto its coordinates with indices in I**.

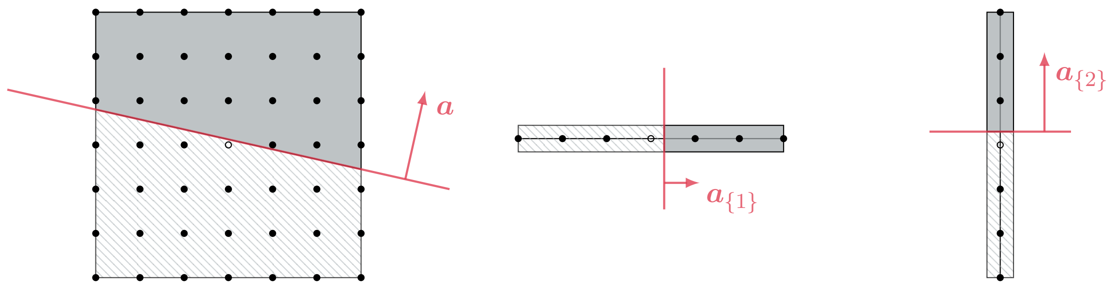

Figure 5.2: Illustration of Lemma 5.12

Lemma 5.12. *Let $d, k \in \mathbb{N}$. Then there exists $\varepsilon > 0$ and a vector $\boldsymbol{a} \in \mathbb{R}^d_{>0}$, such that*

$$\left|[-k,k]^{|I|} \cap \left\{x \in \mathbb{R}^{|I|} \ : \ \langle x, \boldsymbol{a}_I \rangle \ > \ \varepsilon\right\}\right|_{\mathbb{Z}^d} = \frac{1}{2}\left((2k+1)^{|I|} - 1\right),$$

for every non-empty $I \subseteq [d]$.

Proof. Let $\boldsymbol{a} \in \mathbb{R}^d$ be a vector with $\{\boldsymbol{x} \in \mathbb{R}^d : \langle \boldsymbol{x}, \boldsymbol{a} \rangle = 0\} \cap [-k,k]^d = \{\boldsymbol{0}\}$. Such an $\boldsymbol{a}$ certainly exists, since every normal vector of a hyperplane containing more than one lattice point of the cube $[-k,k]^d$ lies in a finite union of lower dimensional subspaces. By our assumption, it must hold $a_i \neq 0$ for every $i \in [d]$ and by changing the signs of the a_is if necessary, we may assume that $\boldsymbol{a} \in \mathbb{R}^d_{>0}$. Now, let $\emptyset \neq I \subseteq [d]$. Without loss of generality, we may assume that $I = [m]$ for some $m \in [d]$. We have $\{\boldsymbol{y} \in \mathbb{R}^{|I|} : \langle \boldsymbol{y}, \boldsymbol{a}_I \rangle = 0\} \cap [-k,k]^{|I|} = \{\boldsymbol{0}\}$, and for a sufficiently small $\varepsilon_I > 0$ the (open) half-space $H_I^+ = \{\boldsymbol{y} \in \mathbb{R}^{|I|} : \langle \boldsymbol{y}, \boldsymbol{a}_I \rangle > \varepsilon_I\}$ contains exactly half of the nonzero lattice points of the cube $[-k,k]^{|I|}$, i.e.,

$$\left|[-k,k]^{|I|} \cap \{\boldsymbol{x} \in \mathbb{R}^{|I|} : \langle \boldsymbol{x}, \boldsymbol{a}_I \rangle > \varepsilon_I\}\right|_{\mathbb{Z}^d} = \frac{1}{2}\left((2k+1)^{|I|} - 1\right).$$

The lemma follows for $\boldsymbol{a} \in \mathbb{R}^d$ and $\varepsilon = \min\{\varepsilon_I : \emptyset \neq I \subseteq [d]\}$. □

Proposition 5.13. *Let $d \in \mathbb{N}$ and let $\rho > 0$ be an odd integer, then there exists a lattice polytope $P_\rho \in \mathcal{K}^d_o$ with*

$$|P_\rho|_{\mathbb{Z}^d} = (\rho+1)^d - 2^d + 1, \qquad \textit{and} \qquad |P_\rho \cap H_\sigma(P_\rho, 1)|_{\mathbb{Z}^d} = \rho. \tag{5.11}$$

Proof. Let $\rho = 2k+1$ and let $\boldsymbol{a} \in \mathbb{R}^d_{>0}$ and $\varepsilon > 0$ be according to Lemma 5.12. More precisely, for any $\emptyset \neq I \subseteq [d]$, we have that if $H_I^+ = \left\{\boldsymbol{y} \in \mathbb{R}^{|I|} : \langle \boldsymbol{y}, \boldsymbol{a}_I \rangle > \varepsilon\right\}$ then the set $A^{(I)} = [-k,k]^{|I|} \cap H_I^+ \cap \mathbb{Z}^{|I|}$ satisfies

$$\left|A^{(I)}\right| = \frac{1}{2}\left((2k+1)^{|I|} - 1\right) = \frac{1}{2}\left(\rho^{|I|} - 1\right).$$

We now consider the following embeddings of $A^{(I)}$, $I \subseteq [d]$, into $\mathbb{R}^d$.

$$B^{(I)} = \left\{\boldsymbol{x} \in \mathbb{R}^d : \boldsymbol{x}_I \in A^{(I)},\ x_j = k+1 \ \ \forall j \in [d] \setminus I\right\}.$$

By construction $B^{(I)} \cap (\pm B^{(J)}) = \emptyset$ for $I \neq J$, and $B^{(I)} \cap (-B^{(I)}) = \emptyset$. For

$$M = \{\boldsymbol{0}\} \cup \bigcup_{I \subsetneq [d]} \left(B^{(I)} \cup -A^{(I)}\right),$$

we have

$$\begin{aligned} |M| &= 1 + \sum_{\emptyset \neq I \subseteq [d]} 2|A^{(I)}| = 1 + \sum_{\emptyset \neq I \subseteq [d]} \left(\rho^{|I|} - 1\right) = 1 + \sum_{\emptyset \neq I \subseteq [d]} \rho^{|I|} - (2^d - 1) \\ &= \sum_{I \subseteq [d]} \rho^{|I|} - (2^d - 1) = (\rho+1)^d - 2^d + 1. \end{aligned}$$

We claim that $P_\rho = \operatorname{conv}(M)$ is a polytope with the desired properties (5.11). Since $M \subseteq P_\rho$, we have $|P_\rho|_{\mathbb{Z}^d} \geq (\rho+1)^d - 2^d + 1$. It is thus left to show that P_ρ indeed does not contain any nonzero lattice point $\boldsymbol{x}$ with $\frac{1}{m}\boldsymbol{x} \in \mathbb{Z}^d$, $k+1 \leq m$. Since $M \subseteq [-(k+1), k+1]^d$, it clearly suffices to show that there is no $\boldsymbol{y} \in P_\rho \cap (k+1)\mathbb{Z}^d \setminus \{\boldsymbol{0}\}$. We assume the contrary and let $\boldsymbol{y} \in P_\rho \cap (k+1)\mathbb{Z}^d \setminus \{\boldsymbol{0}\}$. In conlusion, the coordinates of $\boldsymbol{y}$ either have to be zero or $\pm(k+1)$. Moreover, if $\boldsymbol{z} \in M \subseteq [-(k+1), k+1]^d$ then, by construction, all coordinates of $\boldsymbol{z}$ which are $\pm(k+1)$ must all either be $k+1$ or all $-(k+1)$. Being a convex combination of such points, the same holds for $\boldsymbol{y}$, and without loss of generality, we assume that $\boldsymbol{y}$ has no coordinates equal to $-(k+1)$, and therefore all remaining coordinates of $\boldsymbol{y}$ are either 0 or $k+1$. We conclude that $\boldsymbol{y}$ is a convex combination of points from sets A_I where $j \notin I$. In other words, $\boldsymbol{y}$ is a convex combination of vectors of the set

$$U = \bigcup_{\emptyset \neq J \subseteq I} A^{(J)},$$

where $I = \{i : y_i \neq k+1\} = \{i : y_i = 0\}$. To this end, let

$$\boldsymbol{y} = \sum_{i=1}^{m} \lambda_i \boldsymbol{v}^i, \quad \sum_{i=1}^{m} \lambda_i = 1, \quad \boldsymbol{v}^i \in U,$$

for some $m \in \mathbb{N}$. By definition of I, it holds

$$\boldsymbol{0} = \boldsymbol{y}_I = \sum_{i=1}^{m} \lambda_i \boldsymbol{v}_I^i.$$

If $\boldsymbol{w} \in A^{(J)}$ for some $J \subseteq I$, then we have

$$\langle \boldsymbol{w}_I, \boldsymbol{a}_I \rangle = \langle \boldsymbol{w}_J, \boldsymbol{a}_J \rangle + \sum_{\ell \in I \setminus J} \langle \boldsymbol{w}_\ell, \boldsymbol{a}_\ell \rangle = \langle \boldsymbol{w}_J, \boldsymbol{a}_J \rangle + (k+1) \sum_{\ell \in I \setminus J} a_\ell \geq \langle \boldsymbol{w}_J, \boldsymbol{a}_J \rangle > \varepsilon > 0.$$

In conclusion, $0 = \langle \boldsymbol{y}_I, \boldsymbol{a}_I \rangle > m\varepsilon > 0$; a contradiction. □

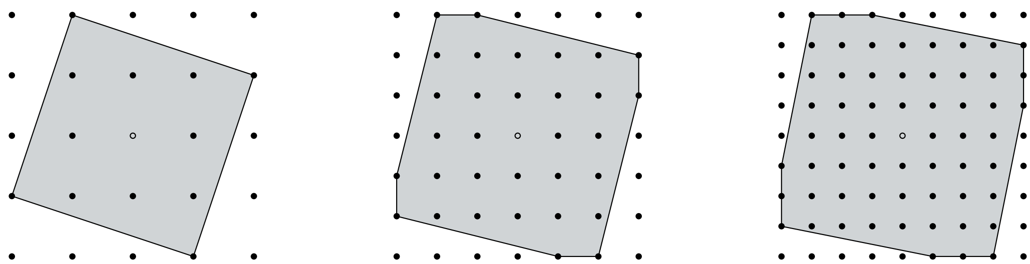

Figure 5.3: Lattice polytopes P_3, P_5 and P_7 from the proof of Proposition 5.13

Theorem 5.14. *Let $K \in \mathcal{K}_o^d$, $d \geq 2$, then it holds*

$$|K|_{\mathbb{Z}^d} < \left(\frac{4}{3}\right)^d |K \cap H_\sigma(K,1)|_{\mathbb{Z}^d}^d. \tag{5.12}$$

Moreover, the constant $(4/3)^d$ *is best possible.*

Proof. Let $\rho = |K \cap H_\sigma(K,1)|_{\mathbb{Z}^d}$. The theorem is trivial for $\rho = 1$, because then $|K|_{\mathbb{Z}^d} = 1$. Hence, we assume that the odd, positive integer ρ is at least 3. By (5.9), it is sufficient to show that

$$(\rho+1)^d - 2^d + 1 \leq \left(\frac{4}{3}\right)^d \rho^d.$$

To this end, let

$$f(\rho, d) = \frac{(\rho+1)^d - 2^d + 1}{\rho^d}.$$

The derivative

$$\frac{\partial}{\partial \rho} f(\rho, d) = \frac{d((\rho+1)2^d - (\rho+1)^d - 1)}{(\rho+1)\rho^{d-1}},$$

is negative for $d \leq 2$ and $\rho \geq 5$. Since it is immediately verified that $f(3,d) \geq f(5,\rho)$, we are done by noticing $f(3,d) < (4/3)^d$ and that $\lim_{d\to\infty} f(3,d) = (4/3)^d$. Since the bound in Theorem 5.10 is best possible, it follows that the factor $(4/3)^d$ in (5.12) is as well, i.e., considering K according to Proposition 5.13 with $d \to \infty$ shows that we cannot expect a smaller factor than $(4/3)^d$. □

Theorem 5.14 can be roughly considered as a one-dimensional discretized Furstenberg-Tzkoni-type inequality. The **Furstenberg-Tzkoni formula** states that for a centered ellipsoid $\mathcal{E} \in \mathcal{K}_c^d$ and $i \in [d-1]$,

$$\mathrm{vol}\,(\mathcal{E})^i = \frac{\kappa_d^i}{\kappa_i^d} \int_{\mathcal{G}(d,i)} \mathrm{vol}_i\,(\mathcal{E} \cap H)^d \, \mathrm{d}H,$$

see [48, 90, 106] and [49, p. 373].

Lower bounds

Lower bounds on the number of lattice points in a symmetric convex body in terms of its 1-dimensional sections, which can be regarded as reverse inequalities of (5.9) or (5.12), are generally trivial. For instance, if $K \in \mathcal{K}_o^d$, then $|K \cap H_\sigma(K,1)|_{\mathbb{Z}^d} \leq |K|_{\mathbb{Z}^d}$ and equality is attained for $\mathrm{conv}\left(\pm(n-1)/2\boldsymbol{e}_1, \pm\frac{1}{2}\boldsymbol{e}_2, \ldots, \pm\frac{1}{2}\boldsymbol{e}_d\right) \in \mathcal{K}_o^d$, where $n > 0$ is an odd integer. Even under the additional constraint that $\dim\left(K \cap \mathbb{Z}^d\right) = d$, we cannot expect any better bound than $|K \cap H_\sigma(K,1)|_{\mathbb{Z}^d} + 2d - 2 \leq |K|_{\mathbb{Z}^d}$, since equality is attained for the polytope $Q = \mathrm{conv}\,(\pm(n-1)/2\boldsymbol{e}_1, \pm\boldsymbol{e}_2, \ldots, \pm\boldsymbol{e}_d) \in \mathcal{K}_o^d$, $n > 3$ odd. Moreover, for large n, this shows that for a constant $c > 0$ with

$$|Q|_{\mathbb{Z}^d} \geq c\,|Q \cap H_\sigma(Q,1)|_{\mathbb{Z}^d},$$

it must hold $c \leq 1$, cf. (5.12). Nevertheless, we will present a non-trivial lower bound for $|K|_{\mathbb{Z}^d}$ in terms of the quantity $|\lambda_1(K)K|$ involving the first successive minimum of K; this will imply that $|K|_{\mathbb{Z}^d} = |\lambda_1(K)K|_{\mathbb{Z}^d}^{\Omega\left(\ln\left(\lambda_1(K)^{-1}\right)\right)}$.

Theorem 5.15. *If $K \in \mathcal{K}_o^d$, then*

$$|K|_{\mathbb{Z}^d} \geq |\lambda_1(K)K|_{\mathbb{Z}^d}^{\left\lfloor \ln_3\left(\frac{2}{\lambda_1(K)}+1\right)\right\rfloor}.$$

Furthermore, equality is attained for $K = k[-1,1]^d$, where k is an integer such that $2k+1 = 3^m$ is a power of 3 for some m.

Proof. Let

$$f(s) = \frac{3^s - 1}{2},$$

and let $k \geq 0$ be an integer and $\boldsymbol{x} \in \lambda_1(K)K \cap \mathbb{Z}^d$. Then,

$$3^k\boldsymbol{x} + f(k)\lambda_1(K)K \subseteq (3^k + f(k))\lambda_1(K)K \subseteq f(k+1)\lambda_1(K)K. \tag{5.13}$$

Furthermore, if $\boldsymbol{y} \in \lambda_1(K)K \cap \mathbb{Z}^d$ and if $\left(3^k\boldsymbol{x} + f(k)\lambda_1(K)K\right) \cap \left(3^k\boldsymbol{y} + f(k)\lambda_1(K)K\right) \neq \emptyset$, then

$$\begin{aligned}\boldsymbol{x} - \boldsymbol{y} \in\ & \frac{f(k)\lambda_1(K)}{3^k}K - \frac{f(k)\lambda_1(K)}{3^k}K = 2\frac{f(k)}{3^k}\lambda_1(K)K \\ =\ & \left(1 - \frac{1}{3^k}\right)\lambda_1(K)K \subseteq \operatorname{int}\left(\lambda_1(K)K\right),\end{aligned}$$

and therefore $\boldsymbol{x} = \boldsymbol{y}$. Together with (5.13), we conclude

$$\bigcup_{\boldsymbol{x} \in \lambda_1(K)K \cap \mathbb{Z}^d} \left(3^k\boldsymbol{x} + f(k)\lambda_1(K)K\right) \subseteq f(k+1)\lambda_1(K)K,$$

where the union in the left-hand side is disjoint. Inductively, we have

$$|\lambda_1(K)K|_{\mathbb{Z}^d}^k \leq |f(k)\lambda_1(K)K|_{\mathbb{Z}^d},$$

using that $\left|3^k\boldsymbol{x} + f(k)\lambda_1(K)K\right|_{\mathbb{Z}^d} = |f(k)\lambda_1(K)K|_{\mathbb{Z}^d}$. For $k = \left\lfloor \ln_3\left(\frac{2}{\lambda_1} + 1\right)\right\rfloor$, we have $f(k) \leq 1/\lambda_1(K)$ and thus $|\lambda_1(K)K|_{\mathbb{Z}^d}^k \leq |K|_{\mathbb{Z}^d}$. □

5.5 Further results

Primitive lattice points in symmetric convex bodies

We will present two variants involving additional assumptions of the following theorem due to Minkowski known as **Minkowski's 3^d-Theorem**, already mentioned in Section 2.1.

Theorem 5.16 (Minkowski [94]). *Let $K \in \mathcal{K}_o^d$ such that $\mathbf{0}$ is the sole lattice point in its interior, then*

$$|K|_{\mathbb{Z}^d} \leq 3^d. \tag{5.14}$$

Equality in (5.14) is attained for the cube $C_d = [-1,1]^d$ and its images under unimodular transformations, cf. [40, 53]. Starting from Minkowski's 3^d-Theorem, there are numerous works on derivations and related results dealing with lattice points in convex bodies and lattice polytopes on a variety of assumptions, e.g. [6, 7, 22, 35, 40, 52, 65, 80, 87, 101, 104, 107, 125, 126].

Our variants of Theorem 5.16 deal with *primitive* lattice points.

Definition 5.17. A lattice point $\boldsymbol{z} \in \mathbb{Z}^d \setminus \{\mathbf{0}\}$ is called **primitive** if the line segment joining $\boldsymbol{z}$ and $\mathbf{0}$ does not contain any point of $\mathbb{Z}^d$ except for $\boldsymbol{z}$ and $\mathbf{0}$. Equivalently, $\boldsymbol{z}$ is primitive if the greatest common divisor of its coordinates in 1.

Clearly, every nonzero lattice point in a given symmetric convex body $K \in \mathcal{K}_o^d$ is primitive if and only if $|K \cap H_\sigma(K,1)|_{\mathbb{Z}^d} \leq 3$. Hence, in view of Theorem 5.10, we immediately deduce the following statement.

Proposition 5.18. *If all lattice points in $K \in \mathcal{K}_o^d$ are primitive except for the origin, it holds*

$$|K|_{\mathbb{Z}^d} \leq 4^d - 2^d + 1.$$

This bound is best possible.

We also deduce a further, slightly different variant of Minkowski's Theorem.

Proposition 5.19. *If all lattice points in the interior of $K \in \mathcal{K}_o^d$ are primitive except for the origin, it holds*

$$|K|_{\mathbb{Z}^d} \leq 5^d,$$

and this bound is best possible.

Proof. If K contained $5^d + 1$ lattice points there were two distinct points $\boldsymbol{v}, \boldsymbol{u}$ equivalent mod 5. Thus there are 6 lattice points on a line ℓ in K. If the line ℓ contains the origin, $\ell \cap K$ must contain at least 7 lattice points by the symmetry of K, and thus the interior of K contains a nonprimitive point in $2\mathbb{Z}^d$. We therefore assume that ℓ does not intersect the origin and that the lattice points in $\ell \cap K$ are $\boldsymbol{v}, \boldsymbol{v}+\boldsymbol{z}, \boldsymbol{v}+2\boldsymbol{z}, \ldots, \boldsymbol{v}+5\boldsymbol{z} = \boldsymbol{u}$, where $\boldsymbol{z} = \frac{1}{5}(\boldsymbol{u}-\boldsymbol{v}) \in \mathbb{Z}^d \setminus \{\mathbf{0}\}$. By the symmetry of K we have

$$\boldsymbol{w} := \frac{5}{2}\boldsymbol{z} = \frac{1}{2}(-\boldsymbol{v}) + \frac{1}{2}(\boldsymbol{v}+5\boldsymbol{z}) \in K,$$

which shows that the lattice point $2\boldsymbol{z} = (4/5)\boldsymbol{w} \in 2\mathbb{Z}^d$ lies in the interior of K.

Choosing $K = [-2,2]^d$ shows that the bound is best possible. ◻

Symmetric sum-free convex bodies

In this last section of the present chapter we will investigate how the study of sum-free sets can be extended to integer points in convex bodies.

Definition 5.20. Let A be a subset of an additive group G. The set A is **sum-free** if $a + b \notin A$ for all $a, b \in A$.

Investigating sum-free subsets in finite abelian groups or $\mathbb{Z}$ is a very active field of research. We refer the reader to the survey by Tao and Vu [122]. Nonetheless, there is only a narrowing selection of work regarding the d-dimensional integer lattice $\mathbb{Z}^d$, particularly involving convex bodies.

Definition 5.21. We say that a symmetric convex body $K \in \mathcal{K}_o^d$ is **symmetric sum-free** if $K \cap \mathbb{Z}^d \setminus \{\mathbf{0}\}$ is sum-free.

Equivalently, a convex body $K \in \mathcal{K}_o^d$ is symmetric sum-free if and only if the equation $\boldsymbol{u} + \boldsymbol{v} = \boldsymbol{w}$ for $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in K \cap \mathbb{Z}^d$ implies $\{\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}\} \cap \{\mathbf{0}\} \neq \emptyset$. We observe that then every point in $K \cap \mathbb{Z}^d \setminus \{\mathbf{0}\}$ must be primitive. Therefore, on account of Proposition 5.18 we have that $|K|_{\mathbb{Z}^d} \leq 4^d - 2^d + 1$ for every symmetric sum-free $K \in \mathcal{K}_o^d$. However, our next result shows that this bound can be drastically improved.

Proposition 5.22. *If $K \in \mathcal{K}_o^d$ is symmetric sum-free, then*

$$|K|_{\mathbb{Z}^d} \leq 2^{d+1} + 1.$$

Proof. Let $A := K \cap \mathbb{Z}^d \setminus \{\mathbf{0}\}$ and assume that $|K \setminus \{\mathbf{0}\}|_{\mathbb{Z}^d} > 2^{d+1}$. Then, there two points $\boldsymbol{u}, \boldsymbol{v} \in A$ with $\boldsymbol{u} \neq \pm\boldsymbol{v}$ and $\boldsymbol{u} \equiv \boldsymbol{v} \bmod 2$. Clearly, it holds $\boldsymbol{u} \equiv -\boldsymbol{v} \bmod 2$ as well, which implies $\boldsymbol{x} := \frac{1}{2}(\boldsymbol{u} - \boldsymbol{v}) \in A$ and $\boldsymbol{y} = \frac{1}{2}(\boldsymbol{u} + \boldsymbol{v})$. Observing that $\boldsymbol{x} + \boldsymbol{y} = \boldsymbol{u} \in A$ shows that A contains a sum. $\square$

It remains unclear whether Proposition 5.22 represents the best possible upper bound for the number of lattice points in a symmetric sum-free convex body. However, there are symmetric sum-free convex bodies in $\mathbb{R}^d$ containing $2^d + 1$ lattices points such as

$$SF_d := \operatorname{conv}\left(\pm\left([0,1]^{d-1} \times \{1\}\right)\right).$$

Consequently, we conjecture the following bound.

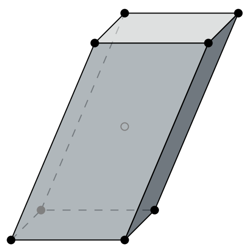

Figure 5.4: Symmetric sum-free convex body SF_d in $\mathbb{R}^3$ containing 9 lattice points

Conjecture 5.23. *For every symmetric sum-free $K \in \mathcal{K}_o^d$ it holds $|K|_{\mathbb{Z}^d} \leq 2^d + 1$.*

Indeed, we confirm Conjecture 5.23 in dimension two. While it is possible to verify this conjecture in the plane by basic calculations, we will follow a different approach to avoid any lengthy computations and utilize Bárány's colorful variant of Carathéodory's theorem (Theorem 1.1).

Theorem 5.24 (Colorful Carathéodory's Theorem, [11])**.** *Let* $A_1, \ldots, A_{d+1}$ *be* $d+1$ *sets in* $\mathbb{R}^d$. *For a point* $\boldsymbol{x} \in \cap_{i=1}^{d+1} \operatorname{conv}(A_i)$, *there are* $\boldsymbol{a}_i \in A_i$, $i \in [d+1]$, *such that* $\boldsymbol{x} \in \operatorname{conv}(\boldsymbol{a}_1, \ldots, \boldsymbol{a}_{d+1})$.

Theorem 5.25. *If* $K \in \mathcal{K}_o^2$ *is symmetric sum-free, then* $|K|_{\mathbb{Z}^2} \leq 5$.

Proof. It is sufficient to show that every $K \in \mathcal{K}_o^2$ with $|K|_{\mathbb{Z}^2} = 7$ is not symmetric sum-free, since $|K|_{\mathbb{Z}^2}$ is odd by symmetry of K and a convex body L is symmetric sum-free if and only if every convex body contained in L is symmetric sum-free. We assume that $|K|_{\mathbb{Z}^2} = 7$ and that $K \cap \mathbb{Z}^d \setminus \{\mathbf{0}\} = \{\pm \boldsymbol{v}_1, \pm \boldsymbol{v}_2, \pm \boldsymbol{v}_3\}$. For $A_i := \{\pm \boldsymbol{v}_i\}$, $i \in [3]$, we have that $\mathbf{0} \in \operatorname{conv}(A_i)$ for every $i \in [3]$. Therefore, according to Theorem 5.24 there are pairwise distinct lattice points $\boldsymbol{a}_i \in A_i$, $i \in [3]$, with $\mathbf{0} \in \operatorname{conv}(\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3)$. According to Lemma 3.17 *ii)*, it holds that $\mathbf{0} = \frac{1}{3}(\boldsymbol{a}_1 + \boldsymbol{a}_2 + \boldsymbol{a}_3)$, but this implies $\boldsymbol{a}_1 + \boldsymbol{a}_2 = -\boldsymbol{a}_3$ which is a sum in $K \cap \mathbb{Z}^2 \setminus \{\mathbf{0}\}$ by symmetry of K. $\square$

From Theorem 5.25, we can deduce that a convex body $K \in \mathcal{K}_o^d$ is symmetric sum-free if and only if no 2-dimensional section of K contains more than 5 lattice points.

Corollary 5.26. $K \in \mathcal{K}_o^d$ *is symmetric sum-free if and only if* $|K \cap H_\sigma(K, 2)|_{\mathbb{Z}^d} \leq 5$.

Proof. Suppose $|K \cap H_\sigma(K, 2)|_{\mathbb{Z}^d} > 5$. Thus, by Theorem 5.25, the 2-dimensional polytope $\operatorname{conv}(K \cap H_\sigma(K, 2))$, and therefore K itself, are not symmetric sum-free. Otherwise, if K contains a sum $\boldsymbol{x} + \boldsymbol{y} = \boldsymbol{z}$ with $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in K \cap \mathbb{Z}^d \setminus \{\mathbf{0}\}$, then the intersection of K and the 2-dimensional subspace spanned by $\boldsymbol{x}$ and $\boldsymbol{y}$ contains the lattice points $\pm\boldsymbol{x}, \pm\boldsymbol{y}, \pm\boldsymbol{z}, \mathbf{0}$; thus, $|K \cap H_\sigma(K, 2)|_{\mathbb{Z}^d} \geq 7$. $\square$

Conclusion

This dissertation has investigated lattice point problems in discrete and convex geometry. Within this area, we approached these problems from different perspectives.

In Chapter 2, we investigated lattice point bounds in centered convex bodies. The problem of bounding the number of integer points in convex bodies, which have their centroid at the origin, is a natural and canonical extension of some of the most influential problems and results in *geometry of numbers*, cf. Section 2.1. In this regard, we are the first to present nontrivial bounds under the assumption of centricity, see the results Theorem 2.5 and Theorem 2.11. The former, in particular, shows that the number of lattice points in a centered convex body with a single lattice point in its interior is at most exponential in the dimension. This allowed us to conclude that the assumption of centricity yields upper bounds which are asymptotically comparable to the bounds for symmetric convex bodies. Therefore, centricity is a remarkable restrictive condition for lattice point counting. Moreover, we contributed a thorough discussion of simplices in this regard. This provided new insights to a well-known yet unsettled conjecture due to Ehrhart as well. There are indeed many open problems relating this topic, for instance, what the best possible bound could be in Theorem 2.5. We suggested a possible outcome in Conjecture 2.7. In two dimensions, this might be a challenging yet manageable task, cf. Theorem 2.16. Another intriguing problem is if there is a generalization of Theorem 2.11 involving all successive minima.

We extended the tensor valued Ehrhart theory in Chapter 3. To this end, we examined coefficients of the underlying Ehrhart tensor polynomials and discussed their generating functions for the first time. This also involved deducing vector- and matrix-valued Pick-type formulas for lattice polygons. A major finding was that nonnegativity results from the classical theory on h^*-vectors can be extended to rank 2 tensors, i.e., matrices, and the notion of nonnegativity surprisingly agrees with positive semidefiniteness. Although we argued that these problems are most likely difficult to approach since the known techniques for h^*-vectors fail, we managed to settle the planar case by presenting a new approach, cf. the proof of Theorem 3.12. In Section 3.6, we also evolved a practical machinery on how h^r-tensors can be determined; this involves fundamental constructions of polytopes such as pyramids, bipyramids and joins. By considering palindromicity of h^r-tensors and extending a famous result due to Hibi, we also managed to derive a new characterization of reflexive polytopes. Finally, we discussed future research directions including open problems and conjectures. The presented new approach of considering tensors in Ehrhart theory provides particularly captivating open questions. We discussed in Section 3.7 that many classical results could actually be generalized to tensors-valued Ehrhart theory. These, however, are inclined to be very challenging problems as discussed earlier. Nevertheless, during the research of the presented work, comprehensive computer-oriented testing of some hypotheses of this kind has been performed, verifying that these hold for a few thousand polytopes.

Chapter 4 has given an extensive account of symmetric generalized arithmetic progressions, symmetric GAPs for short, in convex bodies. We first studied how these can serve to approach the lattice points in a convex body in two different ways. Firstly, in terms of containment by enclosing the integral points of a symmetric convex body between two scalations of a generalized arithmetic progression. Secondly, by considering *large* symmetric GAPs contained in the given convex body. In this regard, we outlined properties and managed to simplify some of the underlying problems to symmetric generalized arithmetic progressions, which are induced by lattice bases. This enabled us to approach — and consequently improve — a recent result by Tao and Vu from a different, more structural starting point. Moreover, we could demonstrate that the problems of approaching the aforementioned size- and containment-estimates regarding lattice points in convex bodies, do not necessarily align; therefore, possibly resulting in two individual problems. Moreover, in the plane we accomplished to show notable results including explicit constants for the discussed problems. The corresponding proofs and discussions also outlined the relationship between symmetric generalized arithmetic progressions enclosing the lattice points of a convex body and its successive minima, also drawing possible limitations in dimensions higher than two. On account of the young age and recent developments in the theory of blending additive combinatorics and convex geometry, this field offers a vast amount of open questions. Thus, possible directions of research in this field could particularly include trying to determine the involved constants — at least asymptotically. Potential generalizations of the presented work may involve results for a broader class of convex bodies than symmetric ones, e.g., centered or even arbitrary bodies even though this probably requires additional constraints on the respective class of convex bodies.

In Chapter 5, we discussed discrete slicing theorems. Due to the absence of homogeneity for the discrete volume these kind of problems tend to be hard to approach. Nonetheless, we have obtained accurate results for one-dimensional slices in Section 5.3. We notably presented a way of how to construct lattice polytopes which attain equality for these inequalities, which interestingly turned out to be significantly more ambitious, see Lemma 5.12 and Proposition 5.13, than to prove the initial upper bounds. To some extend, our findings present a topical bridge to the very classic results in the geometry of numbers as well, cf. Section 5.5. This also enabled us to present an accurate discretized Furstenberg-Tzkoni formula for one-dimensional slices in Theorem 5.14. In Theorem 5.7, we showed a fully discretized slicing inequality, which does only involve the discrete volume of the given convex body. Future studies could naturally target discrete Furstenberg-Tzkoni formulas for higher-dimensional slices.

References

[1] P. W. Aitchison, C. M. Petty, and C. A. Rogers, *A convex body with a false centre is an ellipsoid*, Mathematika **18** (1971), 50–59. MR0293495

[2] S. Alesker, *Integrals of smooth and analytic functions over Minkowski's sums of convex sets*, Convex geometric analysis (Berkeley, CA, 1996), 1999, pp. 1–15. MR1665573

[3] M. Alexander, M. Henk, and A. Zvavitch, *A discrete version of Koldobsky's slicing inequality*, Israel J. Math. **222** (2017), no. 1, 261–278. MR3736506

[4] B. Assarf, E. Gawrilow, K. Herr, M. Joswig, B. Lorenz, A. Paffenholz, and T. Rehn, *Computing convex hulls and counting integer points with* `polymake`, Math. Program. Comput. **9** (2017), no. 1, 1–38. MR3613012

[5] C. A. Athanasiadis, *h^*-vectors, Eulerian polynomials and stable polytopes of graphs*, Electron. J. Combin. **11** (2004/06), no. 2, Research Paper 6, 13. MR2120101

[6] G. Averkov, *On the size of lattice simplices with a single interior lattice point*, SIAM J. Discrete Math. **26** (2012), no. 2, 515–526. MR2967480

[7] G. Averkov, J. Krümpelmann, and B. Nill, *Largest integral simplices with one interior integral point: solution of Hensley's conjecture and related results*, Adv. Math. **274** (2015), 118–166. MR3318147

[8] G. Averkov and C. Wagner, *Inequalities for the lattice width of lattice-free convex sets in the plane*, Beitr. Algebra Geom. **53** (2012), no. 1, 1–23. MR2890359

[9] K. Ball, *Ellipsoids of maximal volume in convex bodies*, Geom. Dedicata **41** (1992), no. 2, 241–250. MR1153987

[10] W. Banaszczyk, *New bounds in some transference theorems in the geometry of numbers*, Math. Ann. **296** (1993), no. 4, 625–635. MR1233487

[11] I. Bárány, *A generalization of Carathéodory's theorem*, Discrete Math. **40** (1982), no. 2-3, 141–152. MR676720

[12] I. Bárány and Z. Füredi, *On the lattice diameter of a convex polygon*, Discrete Math. **241** (2001), no. 1-3, 41–50. Selected papers in honor of Helge Tverberg. MR1861406

[13] A. Barvinok, *A course in convexity*, Graduate Studies in Mathematics, vol. 54, American Mathematical Society, Providence, RI, 2002. MR1940576

[14] A. I. Barvinok, *A polynomial time algorithm for counting integral points in polyhedra when the dimension is fixed*, Math. Oper. Res. **19** (1994), no. 4, 769–779. MR1304623

[15] M. Beck, *Multidimensional Ehrhart reciprocity*, J. Combin. Theory Ser. A **97** (2002), no. 1, 187–194. MR1879135

[16] M. Beck and S. Robins, *Computing the continuous discretely*, Second Edition, Undergraduate Texts in Mathematics, Springer, New York, 2015. MR3410115

[17] D. E. Bell, *A theorem concerning the integer lattice*, Studies in Appl. Math. **56** (1976/77), no. 2, 187–188. MR0462617

[18] A. D. Berenstein and A. V. Zelevinsky, *Tensor product multiplicities and convex polytopes in partition space*, J. Geom. Phys. **5** (1988), no. 3, 453–472. MR1048510

[19] S. Berg, K. Jochemko, and L. Silverstein, *Ehrhart tensor polynomials*, Linear Algebra Appl. **539** (2018), 72–93. MR3739398

[20] S. L. Berg and M. Henk, *Lattice point inequalities for centered convex bodies*, SIAM J. Discrete Math. **30** (2016), no. 2, 1148–1158. MR3504986

[21] R. J. Berman and B. Berndtsson, *The volume of Kähler-Einstein Fano varieties and convex bodies*, J. Reine Angew. Math. **723** (2017), 127–152. MR3604979

[22] U. Betke, M. Henk, and J. M. Wills, *Successive-minima-type inequalities*, Discrete Comput. Geom. **9** (1993), no. 2, 165–175. MR1194034

[23] C. Bey, M. Henk, and J. M. Wills, *Notes on the roots of Ehrhart polynomials*, Discrete Comput. Geom. **38** (2007), no. 1, 81–98. MR2322117

[24] F. Bihan, *Irrational mixed decomposition and sharp fewnomial bounds for tropical polynomial systems*, Discrete Comput. Geom. **55** (2016), no. 4, 907–933. MR3505334

[25] H. F. Blichfeldt, *A new principle in the geometry of numbers, with some applications*, Trans. Amer. Math. Soc. **15** (1914), no. 3, 227–235. MR1500976

[26] K. J. Böröczky and M. Ludwig, *Minkowski valuations on lattice polytopes*, J. Eur. Math. Soc.(JEMS), in press.

[27] K. J. Böröczky and M. Ludwig, *Valuations on lattice polytopes*, Tensor valuations and their applications in stochastic geometry and imaging, 2017, pp. 213–234. MR3702374

[28] J. Bourgain, *On high-dimensional maximal functions associated to convex bodies*, Amer. J. Math. **108** (1986), no. 6, 1467–1476. MR868898

[29] J. Bourgain, *Geometry of Banach spaces and harmonic analysis*, Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Berkeley, Calif., 1986), 1987, pp. 871–878. MR934289

[30] J. Bourgain, *On the distribution of polynomials on high-dimensional convex sets*, Geometric aspects of functional analysis (1989–90), 1991, pp. 127–137. MR1122617

[31] S. Brazitikos, A. Giannopoulos, P. Valettas, and B.-H. Vritsiou, *Geometry of isotropic convex bodies*, Mathematical Surveys and Monographs, vol. 196, American Mathematical Society, Providence, RI, 2014. MR3185453

[32] G. R. Burton, *Sections of convex bodies*, J. London Math. Soc. (2) **12** (1975/76), no. 3, 331–336. MR0402605

[33] G. R. Burton, *Some characterisations of the ellipsoid*, Israel J. Math. **28** (1977), no. 4, 339–349. MR0513965

[34] H. Busemann, *The geometry of geodesics*, Academic Press Inc., New York, N. Y., 1955. MR0075623

[35] J. W. S. Cassels, *On a theorem of Rado in the geometry of numbers*, J. London Math. Soc. **22** (1947), 196–200 (1948). MR0025518

[36] J. W. S. Cassels, *An introduction to the geometry of numbers*, Springer-Verlag, Berlin-New York, 1971. Second printing, corrected, Die Grundlehren der mathematischen Wissenschaften, Band 99. MR0306130

[37] F. Chapoton, *q-analogues of Ehrhart polynomials*, Proc. Edinb. Math. Soc. (2) **59** (2016), no. 2, 339–358. MR3509233

[38] H. T. Croft, K. J. Falconer, and R. K. Guy, *Unsolved problems in geometry*, Problem Books in Mathematics, Springer-Verlag, New York, 1994. Corrected reprint of the 1991 original [MR1107516 (92c:52001)], Unsolved Problems in Intuitive Mathematics, II. MR1316393

[39] J. A. De Loera, R. Hemmecke, J. Tauzer, and R. Yoshida, *Effective lattice point counting in rational convex polytopes*, J. Symbolic Comput. **38** (2004), no. 4, 1273–1302. MR2094541

[40] J. Draisma, T. B. McAllister, and B. Nill, *Lattice-width directions and Minkowski's* 3^d*-theorem*, SIAM J. Discrete Math. **26** (2012), no. 3, 1104–1107. MR3022127

[41] E. Ehrhart, *Sur un problème de géométrie diophantienne linéaire. I. Polyèdres et réseaux*, J. Reine Angew. Math. **226** (1967), 1–29. MR0213320

[42] E. Ehrhart, *Volume réticulaire critique d'un simplexe*, J. Reine Angew. Math. **305** (1979), 218–220. MR518864

[43] E. Ehrhart, *Une généralisation du théorème de Minkowski*, C. R. Acad. Sci. Paris **240** (1955), 483–485. MR0066430

[44] E. Ehrhart, *Sur les polyèdres rationnels homothétiques à n dimensions*, C. R. Acad. Sci. Paris **254** (1962), 616–618. MR0130860

[45] E. Ehrhart, *Une généralisation probable du théorème fondamental de Minkowski*, C. R. Acad. Sci. Paris **258** (1964), 4885–4887. MR0163219

[46] E. Ehrhart, *Sur la loi de réciprocité des polyèdres rationnels*, C. R. Acad. Sci. Paris Sér. A-B **266** (1968), A696–A697. MR0234919

[47] P. C. Fishburn, J. C. Lagarias, J. A. Reeds, and L. A. Shepp, *Sets uniquely determined by projections on axes. II. Discrete case*, Discrete Math. **91** (1991), no. 2, 149–159. MR1124762

[48] H. Furstenberg and I. Tzkoni, *Spherical functions and integral geometry*, Israel J. Math. **10** (1971), 327–338. MR0301682

[49] R. J. Gardner, *Geometric tomography*, Second Edition, Encyclopedia of Mathematics and its Applications, vol. 58, Cambridge University Press, New York, 2006. MR2251886

[50] R. J. Gardner, P. Gronchi, and C. Zong, *Sums, projections, and sections of lattice sets, and the discrete covariogram*, Discrete Comput. Geom. **34** (2005), no. 3, 391–409. MR2160045

[51] E. Gawrilow and M. Joswig, *polymake: a framework for analyzing convex polytopes*, Polytopes—combinatorics and computation (Oberwolfach, 1997), 2000, pp. 43–73. MR1785292

[52] B. González Merino and M. Henze, *A generalization of the discrete version of Minkowski's fundamental theorem*, Mathematika **62** (2016), no. 3, 637–652. MR3480976

[53] H. Groemer, *Abschätzungen für die Anzahl der konvexen Körper, die einen konvexen Körper berühren*, Monatsh. Math. **65** (1961), 74–81. MR0124819

[54] P. M. Gruber and C. G. Lekkerkerker, *Geometry of numbers*, Second Edition, North-Holland Mathematical Library, vol. 37, North-Holland Publishing Co., Amsterdam, 1987. MR893813

[55] P. M. Gruber, *Convex and discrete geometry*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 336, Springer, Berlin, 2007. MR2335496

[56] P. M. Gruber, *John and Loewner ellipsoids*, Discrete Comput. Geom. **46** (2011), no. 4, 776–788. MR2846178

[57] B. Grünbaum, *Partitions of mass-distributions and of convex bodies by hyperplanes.*, Pacific J. Math. **10** (1960), 1257–1261. MR0124818

[58] C. Haase, M. Juhnke-Kubitzke, R. Sanyal, and T. Theobald, *Mixed Ehrhart polynomials*, Electron. J. Combin. **24** (2017), no. 1, Paper 1.10, 14. MR3609180

[59] M. Henk, *Successive minima and lattice points*, Rend. Circ. Mat. Palermo (2) Suppl. **70, part I** (2002), 377–384. IV International Conference in "Stochastic Geometry, Convex Bodies, Empirical Measures & Applications to Engineering Science", Vol. I (Tropea, 2001). MR1962579

[60] M. Henk, *Löwner-John ellipsoids*, Doc. Math. Extra vol.: Optimization stories (2012), 95–106. MR2991474

[61] M. Henk, M. Henze, and M. A. Hernández Cifre, *Variations of Minkowski's theorem on successive minima*, Forum Math. **28** (2016), no. 2, 311–325. MR3466571

[62] M. Henk, M. Henze, and J. M. Wills, *Blichfeldt-type inequalities and central symmetry*, Adv. Geom. **11** (2011), no. 4, 731–744. MR2852929

[63] M. Henk and M. Tagami, *Lower bounds on the coefficients of Ehrhart polynomials*, European J. Combin. **30** (2009), no. 1, 70–83. MR2460218

[64] M. Henk and J. M. Wills, *A Blichfeldt-type inequality for the surface area*, Monatsh. Math. **154** (2008), no. 2, 135–144. MR2419059

[65] D. Hensley, *Lattice vertex polytopes with interior lattice points*, Pacific J. Math. **105** (1983), no. 1, 183–191. MR688412

[66] M. Henze, *A Blichfeldt-type inequality for centrally symmetric convex bodies*, Monatsh. Math. **170** (2013), no. 3-4, 371–379. MR3055793

[67] G. T. Herman and A. Kuba, *Discrete tomography*, Applied and Numerical Harmonic Analysis, Birkhäuser Boston, Inc., Boston, MA, 1999. Foundations, algorithms, and applications. MR1722457

[68] G. T. Herman and A. Kuba, *Advances in discrete tomography and its applications*, Applied and Numerical Harmonic Analysis, Birkhäuser Boston, Inc., Boston, MA, 2007. MR2452005

[69] T. Hibi, A. Higashitani, A. Tsuchiya, and K. Yoshida, *Ehrhart polynomials with negative coefficients*, ArXiv e-prints (2015jun), available at `1506.00467`.

[70] T. Hibi, *Ehrhart polynomials of convex polytopes, h-vectors of simplicial complexes, and nonsingular projective toric varieties*, Discrete and computational geometry (New Brunswick, NJ, 1989/1990), 1991, pp. 165–177. MR1143294

[71] T. Hibi, *A lower bound theorem for Ehrhart polynomials of convex polytopes*, Adv. Math. **105** (1994), no. 2, 162–165. MR1275662

[72] M. N. Huxley, *Exponential sums and lattice points. III*, Proc. London Math. Soc. (3) **87** (2003), no. 3, 591–609. MR2005876

[73] K. Jochemko and R. Sanyal, *Combinatorial mixed valuations*, Adv. Math. **319** (2017), 630–652. MR3695886

[74] K. Jochemko and R. Sanyal, *Combinatorial positivity of translation-invariant valuations and a discrete Hadwiger theorem*, J. Eur. Math. Soc. (JEMS) **20** (2018), no. 9, 2181–2208. MR3836844

[75] F. John, *Extremum problems with inequalities as subsidiary conditions*, Studies and Essays Presented to R. Courant on his 60th Birthday, January 8, 1948, 1948, pp. 187–204. MR0030135

[76] B. Klartag, *On convex perturbations with a bounded isotropic constant*, Geom. Funct. Anal. **16** (2006), no. 6, 1274–1290. MR2276540

[77] A. Koldobsky, *Measures of sections of convex bodies*, ArXiv e-prints (November 2015), available at `1511.05525`.

[78] M. Köppe and S. Verdoolaege, *Computing parametric rational generating functions with a primal Barvinok algorithm*, Electron. J. Combin. **15** (2008), no. 1, Research Paper 16, 19. MR2383436

[79] A. Kuba and G. T. Herman, *Discrete tomography: a historical overview*, Discrete tomography, 1999, pp. 3–34. MR1722458

[80] J. C. Lagarias and G. M. Ziegler, *Bounds for lattice polytopes containing a fixed number of interior points in a sublattice*, Canad. J. Math. **43** (1991), no. 5, 1022–1035. MR1138580

[81] D. Larman, L. Montejano, and E. Morales-Amaya, *Characterization of ellipsoids by means of parallel translated sections*, Mathematika **56** (2010), no. 2, 363–378. MR2678035

[82] D. Lepelley, A. Louichi, and H. Smaoui, *On Ehrhart polynomials and probability calculations in voting theory*, Soc. Choice Welf. **30** (2008), no. 3, 363–383. MR2377347

[83] F. W. Levi, *über zwei Sätze von Herrn Besicovitch*, Arch. Math. **3** (1952), 125–129. MR0050295

[84] H. Liu and C. Zong, *On the classification of convex lattice polytopes*, Adv. Geom. **11** (2011), no. 4, 711–729. MR2852928

[85] M. Ludwig and L. Silverstein, *Tensor valuations on lattice polytopes*, Adv. Math. **319** (2017), 76–110. MR3695869

[86] I. G. Macdonald, *Polynomials associated with finite cell-complexes*, J. London Math. Soc. (2) **4** (1971), 181–192. MR0298542

[87] R.-D. Malikiosis, *A discrete analogue for Minkowski's second theorem on successive minima*, Adv. Geom. **12** (2012), no. 2, 365–380. MR2911155

[88] G. Maze, *Some inequalities related to the Seysen measure of a lattice*, Linear Algebra Appl. **433** (2010), no. 8-10, 1659–1665. MR2718227

[89] P. McMullen, *Valuations and Euler-type relations on certain classes of convex polytopes*, Proc. London Math. Soc. (3) **35** (1977), no. 1, 113–135. MR0448239

[90] R. E. Miles, *A simple derivation of a formula of Furstenberg and Tzkoni*, Israel J. Math. **14** (1973), 278–280. MR0328843

[91] E. Miller and B. Sturmfels, *Combinatorial commutative algebra*, Graduate Texts in Mathematics, vol. 227, Springer-Verlag, New York, 2005. MR2110098

[92] V. D. Milman and A. Pajor, *Isotropic position and inertia ellipsoids and zonoids of the unit ball of a normed n-dimensional space*, Geometric aspects of functional analysis (1987–88), 1989, pp. 64–104. MR1008717

[93] V. D. Milman and A. Pajor, *Entropy and asymptotic geometry of non-symmetric convex bodies*, Adv. Math. **152** (2000), no. 2, 314–335. MR1764107

[94] H. Minkowski, *Geometrie der Zahlen*, Bibliotheca Mathematica Teubneriana, Band 40, Johnson Reprint Corp., New York-London, 1968. MR0249269

[95] L. Montejano and E. Morales-Amaya, *Variations of classic characterizations of ellipsoids and a short proof of the false centre theorem*, Mathematika **54** (2007), no. 1-2, 35–40. MR2428204

[96] L. Montejano and E. Morales, *Characterization of ellipsoids and polarity in convex sets*, Mathematika **50** (2003), no. 1-2, 63–72 (2005). MR2136352

[97] B. Nill and A. Paffenholz, *On the equality case in Ehrhart's volume conjecture*, Adv. Geom. **14** (2014), no. 4, 579–586. MR3276123

[98] C. M. Petty, *Ellipsoids*, Convexity and its applications, 1983, pp. 264–276. MR731114

[99] G. A. Pick, *Geometrisches zur Zahlenlehre.*, Sitzungsber. Lotos (Prag) **19** (1899), 311–319.

[100] O. Pikhurko, *Lattice Points inside Lattice Polytopes*, ArXiv e-prints (August 2000), available at `math/0008028`.

[101] O. Pikhurko, *Lattice points in lattice polytopes*, Mathematika **48** (2001), no. 1-2, 15–24 (2003). MR1996360

[102] A. V. Pukhlikov and A. G. Khovanskiĭ, *Finitely additive measures of virtual polyhedra*, Algebra i Analiz **4** (1992), no. 2, 161–185. MR1182399

[103] S. Rabinowitz, *A theorem about collinear lattice points*, Utilitas Math. **36** (1989), 93–95. MR1030772

[104] R. Rado, *A theorem on the geometry of numbers*, J. London Math. Soc. **21** (1946), 34–47. MR0019659

[105] O. Regev, *A Note on Koldobsky's Lattice Slicing Inequality*, ArXiv e-prints (August 2016), available at `1608.04945`.

[106] L. A. Santaló, *Integral geometry and geometric probability*, Second Edition, Cambridge Mathematical Library, Cambridge University Press, Cambridge, 2004. With a foreword by Mark Kac. MR2162874

[107] U. Schnell, *A Minkowski-type theorem for covering minima in the plane*, Geom. Dedicata **55** (1995), no. 3, 247–255. MR1334448

[108] C. Schulz, *A discrete moment problem on lattice polytopes*, Master's Thesis, 2015.

[109] P. R. Scott, *On convex lattice polygons*, Bull. Austral. Math. Soc. **15** (1976), no. 3, 395–399. MR0430960

[110] M. Seysen, *Simultaneous reduction of a lattice basis and its reciprocal basis*, Combinatorica **13** (1993), no. 3, 363–376. MR1238824

[111] C. L. Siegel, *Lectures on the geometry of numbers*, Springer-Verlag, Berlin, 1989. Notes by B. Friedman, Rewritten by Komaravolu Chandrasekharan with the assistance of Rudolf Suter, With a preface by Chandrasekharan. MR1020761

[112] R. P. Stanley, *Combinatorial reciprocity theorems*, Advances in Math. **14** (1974), 194–253. MR0411982

[113] R. P. Stanley, *Decompositions of rational convex polytopes*, Ann. Discrete Math. **6** (1980), 333–342. Combinatorial mathematics, optimal designs and their applications (Proc. Sympos. Combin. Math. and Optimal Design, Colorado State Univ., Fort Collins, Colo., 1978). MR593545

[114] R. P. Stanley, *A monotonicity property of h-vectors and h^*-vectors*, European J. Combin. **14** (1993), no. 3, 251–258. MR1215335

[115] R. P. Stanley, *Enumerative combinatorics. Volume 1*, Second Edition, Cambridge Studies in Advanced Mathematics, vol. 49, Cambridge University Press, Cambridge, 2012. MR2868112

[116] A. Stapledon, *Inequalities and Ehrhart δ-vectors*, Trans. Amer. Math. Soc. **361** (2009), no. 10, 5615–5626. MR2515826

[117] A. Stapledon, *Equivariant Ehrhart theory*, Adv. Math. **226** (2011), no. 4, 3622–3654. MR2764900

[118] A. Stapledon, *Additive number theory and inequalities in Ehrhart theory*, Int. Math. Res. Not. IMRN **5** (2016), 1497–1540. MR3509934

[119] S. Stein, *The symmetry function in a convex body*, Pacific J. Math. **6** (1956), 145–148. MR0080321

[120] B. M. Stewart, *Asymmetry of a plane convex set with respect to its centroid*, Pacific J. Math. **8** (1958), 335–337. MR0097771 (20 #4238)

[121] T. Tao and V. Vu, *John-type theorems for generalized arithmetic progressions and iterated sumsets*, Adv. Math. **219** (2008), no. 2, 428–449. MR2435645

[122] T. Tao and V. Vu, *Sum-free sets in groups: a survey*, J. Comb. **8** (2017), no. 3, 541–552. MR3668880

[123] T. Tao and V. H. Vu, *Additive combinatorics*, Cambridge Studies in Advanced Mathematics, vol. 105, Cambridge University Press, Cambridge, 2010. Paperback edition [of MR2289012]. MR2573797

[124] S. Taschuk, *The harmonic mean measure of symmetry for convex bodies*, Adv. Geom. **15** (2015), no. 1, 109–120. MR3300714

[125] B. Uhrin, *On a generalization of Minkowski's convex body theorem*, J. Number Theory **13** (1981), no. 2, 192–209. MR612682

[126] J. van der Corput, *Verallgemeinerung einer mordellschen beweismethode in der geometrie der zahlen, zweite mitteilung*, Acta Arithmetica **2** (1936), no. 1, 145–146 (ger).

[127] J. Wu, *On the primitive circle problem*, Monatsh. Math. **135** (2002), no. 1, 69–81. MR1894296

[128] J. Zaks, M. A. Perles, and J. M. Wills, *On lattice polytopes having interior lattice points*, Elem. Math. **37** (1982), no. 2, 44–46. MR651251

[129] G. M. Ziegler, *Lectures on polytopes*, Graduate Texts in Mathematics, vol. 152, Springer-Verlag, New York, 1995. MR1311028

Index

List of symbols

$\mathbf{0}$	zero vector, p. 7
$\mathbf{1}$	all-ones vector, p. 7
$A^{\perp}$	orthogonal complement of $A \subseteq \mathbb{R}^d$, p. 7
$\mathrm{A}\,(d,k)$	Eulerian number, p. 16
$\mathrm{aff}(\cdot)$	affine hull, p. 7
$\mathrm{bd}\,(\cdot)$	boundary of a set, p. 7
B_d	unit ball, p. 8
C_{S^*}	half-open polyhedral cone, p. 35
$\mathrm{c}(\cdot)$	centroid, p. 11
$\mathrm{conv}\,(\cdot)$	convex hull, p. 8
$\det\Lambda$	determinant of the lattice Λ, p. 9
$\dim(\cdot)$	dimension, p. 8
$\mathrm{Ehr}^r_P(t)$	Ehrhart tensor series of P, p. 34
$\mathrm{Ehr}_P(t)$	Ehrhart series of P, p. 14
$\boldsymbol{e}_i$	i-th standard unit vector, p. 7
$\mathrm{GL}\,(d,\mathbb{Z})$	Unimodular $d \times d$ matrices, p. 9
$\mathcal{G}(k,d)$	Grassmanian of k-dimensional linear subspaces in $\mathbb{R}^d$, p. 8
$H_\sigma(K,k)$	maximal k-dimensional discrete section of K, p. 75
$H_{\boldsymbol{q}}(P)$	half-open polytope, p. 35
$h^r_P(t)$	h^r-tensor polynomial, p. 30
$I_{\boldsymbol{q}}(P)$	indices of facets of P visible from $\boldsymbol{q}$, p. 35

$[n]$	$\{1, \ldots, n\}$, p. 7
$\operatorname{int}(\cdot)$	interior of a set, p. 7
$\mathcal{K}^d$	family of all convex bodies in $\mathbb{R}^d$, p. 8
$\mathcal{K}_o^d$	family of all symmetric convex bodies in $\mathbb{R}^d$, p. 8
κ_d	volume of B_d, p. 8
$\lvert K\rvert_{\mathbb{Z}^d}$	number of lattice points, p. 9
$\mathcal{L}^d$	family of lattices in $\mathbb{R}^d$, p. 9
$\mathrm{L}^0(P)$	number of lattice points, p. 29
$\mathrm{L}_P^0(n)$	Ehrhart polynomial, p. 30
$\mathrm{L}^r(P)$	discrete moment tensor, p. 29
$\mathrm{L}_P^r(n)$	Ehrhart tensor polynomial, p. 30
$\mathrm{L}_i^r(P)$	coefficients of the Ehrhart tensor polynomial, p. 30
$\lambda_i(\cdot)$	i-th successive minimum, p. 12
$\operatorname{lin}(\cdot)$	linear hull, p. 7
$\mathrm{M}^r(P)$	moment tensor, p. 32
$\mathbb{N}$	nonnegative integers, p. 7
$\mathbb{Q}$	rational numbers
$\mathbb{R}$	real numbers
$\mathbb{R}_{>0}$	positive real numbers
$\mathbb{R}_{\geq 0}$	nonnegative real numbers
$O(\cdot)$	Landau O, p. 16
$\Omega(\cdot)$	Landau Ω, p. 16
$\mathcal{P}_{\mathbb{Z}}^d$	lattice polytopes in $\mathbb{R}^d$, p. 10
$\mathcal{P}^d$	polytopes in $\mathbb{R}^d$, p. 10

$\mathrm{P}_{\mathbb{R}}(A, \boldsymbol{u})$	parallelepiped induced by A and $\boldsymbol{u}$, p. 57
$\mathrm{P}(A, \boldsymbol{u})$	symmetric generalized arithmetic progression (GAP), p. 56
$\operatorname{relint}(\cdot)$	relative interior of a set, p. 8
$\mathbb{S}^{d-1}$	$d-1$-dimensional unit sphere in $\mathbb{R}^d$, p. 8
$\mathcal{S}_c^d$	class of centered d-simplices, p. 11
S_d	Ehrhart simplex, p. 20
$\mathcal{S}^d$	class of d-simplices, p. 10
$\mathbb{T}^r$	tensors on $\mathbb{R}^d$ of rank r, p. 15
$\operatorname{vol}(\cdot)$	volume, p. 8
$\boldsymbol{x} \equiv \boldsymbol{y} \bmod n$	$x_i \equiv y_i$ for all $i \in [d]$, p. 7
$\boldsymbol{x} \leq \boldsymbol{y}$	$x_i \leq y_i$ for all $i \in [d]$, p. 7
$\boldsymbol{x}_I$	projection of $\boldsymbol{x} \in \mathbb{R}^d$ to coordinates with indices in $I \subseteq [d]$, p. 81
$P \star Q$	join of two polytopes, p. 51
$\lvert A\rvert$	size/cardinality of the set A
$\{r\}$	fractional part of the number r, p. 68
$\langle \cdot, \cdot \rangle$	inner product, p. 7
$\lVert\cdot\rVert$	Euclidean norm, p. 7
$[\boldsymbol{x}, \boldsymbol{y}]$	line segment, p. 10
$(\boldsymbol{x}, \boldsymbol{y}]$	half-open line segment, p. 10
$\tau_d(K, \Lambda)$	enclosing constant, p. 58
τ_d	enclosing constant, p. 57
$\nu_d(K, \Lambda)$	size approximating constant, p. 58
ν_d	size approximating constant, p. 58