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# ON INSTANTANEOUS CONTROL FOR A NONLINEAR PARABOLIC BOUNDARY CONTROL PROBLEM

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A method of instantaneous control type is considered for a nonlinear parabolic boundary control problem with box constraints on the control. It is shown that the method exhibits fixed points. In numerical examples, convergence towards a fixed point occurs, which is not the best possible one. Consequently, a new hybrid method is suggested, which behaves essentially better as the standard method.

**Key words.** Optimal boundary control, parabolic equation, control constraints, instantaneous control, receding horizon

**AMS subject classifications.** 49M30, 49K20

# 1 Introduction

In this paper we study instantaneous control of the following parabolic boundary control problem:

$$\min J(u,y) = \frac{1}{2}\int_\Omega (y(x,T) - y_d(x))^2 dx + \frac{\gamma}{2}\int_\Sigma u(x,t)^2 dt dS_x \tag{$P$}$$

subject to the control constraint

$$u \in U_{ad}$$

and the state equation

$$\begin{aligned} y_t - \Delta y + y &= 0 && \text{in } Q,\\ \partial_n y &= b(y) + u && \text{on } \Sigma,\\ y(0) &= y_0. \end{aligned} \tag{1.1}$$

The set of admissible controls describes box constraints:

$$U_{ad} = \{u \in L^\infty(\Sigma) : u_a(x) \le u(x,t) \le u_b(x) \text{ for almost all } (x,t) \in \Sigma\}. \tag{1.2}$$

Here, $u_a$ and $u_b$ are given functions of $L^\infty(\Gamma)$ satisfying

$$u_a(x) \le 0 \le u_b(x) \tag{1.3}$$

almost everywhere on $\Omega$. Here, $\Omega$ is an open domain of $\mathbb{R}^n$ with $C^{2,1}$-boundary $\Gamma$. The set $Q$ and $\Sigma$ are products of $\Omega$ resp. $\Gamma$ with the time axis $(0,T)$. The required properties of the nonlinearity $b$ will be specified later.

The control problem $(P)$ is well-understood. Necessary conditions were derived for instance in [6, 22, 24], also sufficient conditions were established [10, 21]. In consequence, a large variety of numerical methods are known, for instance methods of SQP-type to cope with the nonlinearity and primal-dual active-set strategies to solve the constrained programming problem, see [3, 11, 12, 15].

Suboptimal methods of instantaneous control type were first introduced in the control of the stochastic Burgers equation, [8]. However, such methods are known in the literature under different names as for example model-predictive control or receding horizon control. They require less computational resources than the solution of the optimal control problem. So these methods are very attractive for the control of — maybe turbulent — fluid flows, where the numerical solution of the optimal control problem with adequate accuracy can take days

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or weeks. See for instance [4, 7]. In the control theory of finite-dimensional dynamical systems many articles are concerned with stabilizing properties of suboptimal controlled systems, see [20] and the references in [2, 19].

However, for infinite-dimensional system i.e. systems governed by partial differential equations at present only a few papers are dealing with the stability analysis of the instantaneous control method, cf. [13, 14]. There the control is brought in the system as distributed control, which is an essential pre-requisite of the underlying analysis. In [26], instantaneous control for the one-dimensional heat equation was investigated. Due to the simplicity of the problem, box-constrained boundary control could be studied. It was reported, that this method yields to suboptimal controls with a weak performance of the cost functional of the control problem $(P)$. Hence, a improved hybrid method was suggested. One concern of the present paper is to generalize this hybrid method to higher dimensional and nonlinear parabolic systems.

## 1.1 The method of instantaneous control

We explain briefly, how the instantaneous control method works. At first, the time interval $[0,T]$ is divided in $N$ subintervals of length $\tau = T/N$, $I_j = [j\tau, (j+1)\tau]$, $j = 1 \dots N$. The aim of minimizing $(P)$ is to approximate $y_d$ in the $L^2$-norm at the final time $T$. Therefore, it seems to be natural to choose the control on the first time interval $[0, t_1]$ such that $|y(t_1) - y_d|_2$ is minimized. Starting from $y(t_1)$, the control on the next time interval $[t_1, t_2]$ is selected such that $|y(t_2) - y_d|_2$ is minimal. These steps are repeated until the suboptimal control on the whole interval $[0,T]$ is computed. To be more formal, the instantaneous method minimizes for $j = 1 \dots N$

$$\min J_\tau(u_j, y_{j-1}) = \frac{1}{2}\int_\Omega (y_j(x) - y_d(x))^2 dx + \frac{\gamma}{2}\int_{\Sigma_\tau} u_j(x,t)^2 dt dS_x \tag{$P_\tau$}$$

subject to the control constraint

$$u_j \in U_{ad,\tau} = \{u \in L^\infty(\Sigma_\tau) : u_a(x) \le u(x,t) \le u_b(x) \text{ for almost all } (x,t) \in \Sigma_\tau\},$$

and $y_j = y(x,\tau)$, where $y$ solves

$$\begin{aligned} y_t - \Delta y + y &= 0 && \text{in } Q_\tau, \\ \partial_n y &= b(y) + u_j && \text{on } \Sigma_\tau, \\ y(0) &= y_{j-1}. \end{aligned} \tag{1.4}$$

Here, we used the notation $Q_\tau = \Omega \times [0,\tau]$ and $\Sigma_\tau = \Gamma \times [0,\tau]$. The numerical results show, that this method converges to a fixed point. We give an explanation, why this could happen at all. Behind each step there is the following mapping $\Phi$: to each initial value $y_{j-1}$ is associated the optimal control of $(P_\tau)$, denoted by $u_j := \bar{u}(y_{j-1})$, which gives in turn a new state $y_j$, i.e. $y_j = \Phi(y_{j-1})$. By simple induction it follows $y_j = \Phi(y_{j-1}) = \cdots = \Phi^j(y_0)$, so that this method realizes a fixed point iteration. In the linear case $\Phi$ was shown to be contractive, [26]. Thus, the instantaneous control method converges to a fixed point. However, in the nonlinear case $\Phi$ is not even well-defined. Since the optimization problem $(P_\tau)$ is not uniquely solvable in general, the control $u_j$ might not be uniquely determined and so the new state $y_j$. Therefore, we will define a fixed point $y_f$ of $\Phi$ in the following sense:

**Definition 1.1.** A function $y_f$ is called a generalized fixed point of $\Phi$ if the following conditions are satisfied:

(i) There exists an admissible control $u_f \in U_{ad,\tau}$, which satisfies the necessary optimality conditions of $(P_\tau)$ with initial condition $y(0) = y_f$.

(ii) It holds $y_f = y(\tau)$, where $y$ solves (1.4) with initial condition $y(0) = y_f$ and control $u = u_f$.

This definition is made from a practical point of view. In real computations, the optimal control of the nonlinear control problem is determined by methods solving the optimality system. The so-obtained control satisfies the first-order necessary optimality conditions. Therefore, it is natural to require the fixed point control $u_f$ to fulfill only these necessary conditions. In the following, we will not make use of any second-order necessary or sufficient optimality conditions to ensure local optimality.

The main result of this paper is that under suitable conditions a generalized fixed point of $\Phi$ exists. The rest of this section is devoted to a formal investigation of the properties of such a fixed point. For the detailed analysis we refer to the Sections 2 and 3.

Let us assume that a generalized fixed point of $\Phi$ exists and denote it by $y_f$ with associated control $u_f$. Then it holds $y_f = \bar{y}(0) = \bar{y}(\tau)$, where $\bar{y}$ solves

$$\begin{aligned} y_t - \Delta y + y &= 0 && \text{in } Q_\tau, \\ \partial_n y &= b(y) + u_f && \text{on } \Sigma_\tau, \\ y(0) &= y(\tau), \end{aligned} \tag{1.5}$$

which is a problem with time-periodicity condition. The solvability of (1.5) follows from the solvability of the original initial value problem since $y(0) = y_f$ is given. The necessary condition for $u_f$ to be locally optimal of $(P_\tau)$ is

$$(\bar{y}(\tau) - y_d,\, V(u - u_f)(\tau)) + \gamma(u_f, u - u_f) \geq 0 \qquad \forall u \in U_{ad,\tau},$$

where $V$ is defined as $Vu = y$, and $y$ solves the linearized equation

$$\begin{aligned} y_t - \Delta y + y &= 0 && \text{in } Q_\tau, \\ \partial_n y &= b'(\bar{y})y + u && \text{on } \Sigma_\tau, \\ y(0) &= 0. \end{aligned} \tag{1.6}$$

In order to prove the existence of a fixed point, we will regard linearized systems. The state $\bar{y}$ is also a solution of system (1.5) with changed boundary condition

$$\partial_n y = b(\bar{y}) + b'(\bar{y})(y - \bar{y}) + u_f \quad \text{on } \Sigma_\tau.$$

Therefore, we will consider systems with the linear boundary condition

$$\partial_n y + \alpha y = f + u \quad \text{on } \Sigma_\tau,$$

for general $\alpha$, $f$. Here, we have in mind to set $\alpha := -b'(\bar{y})$ and $f := b(\bar{y}) - b'(\bar{y})\bar{y}$, where the state $\bar{y}$ is given. We will show that the instantaneous control method for systems with linear boundary condition will converge to a fixed point $(u^*, y^*)$.

We prove that a generalized fixed point of $\Phi$ is associated with a fixed point of the mapping $\Psi$ defined in the following. Let be given a state $\bar{y}$. Set the coefficients of the linear boundary condition $\alpha$, $f$ as explained above. Then we consider instantaneous control of the linear parabolic system: Minimize $(P_\tau)$ subject to $u \in U_{ad,\tau}$ and

$$\begin{aligned} y_t - \Delta y + y &= 0 && \text{in } Q_\tau, \\ \partial_n y + \alpha y &= f + u && \text{on } \Sigma_\tau, \\ y(0) &= y_0. \end{aligned}$$

The instantaneous method applied to this linear system will converge to a uniquely determined fixed point consisting of a state $y^*$ and a control $u^*$. The state $y^*$ fulfills the conditions of Definition 1.1 adapted to the linear case. Hence, it holds $y^* = \tilde{y}(0) = \tilde{y}(\tau)$, where $\tilde{y}$ solves the time-periodic system

$$\begin{aligned} y_t - \Delta y + y &= 0 && \text{in } Q_\tau, \\ \partial_n y + \alpha y &= f + u^* && \text{on } \Sigma_\tau, \\ y(0) &= y(\tau), \end{aligned}$$

Putting this together, we get the following chain of mappings defining $\Psi$:

$$\bar{y} \mapsto \big( -b'(\bar{y}),\ b(\bar{y}) - b'(\bar{y})\bar{y} \big) =: (\alpha, f) \mapsto (y^*, u^*) \mapsto \tilde{y} =: \Psi(\bar{y}).$$

We will develop sufficient conditions for the existence of a fixed point of $\Psi$. This will be done by showing that $\Psi$ is a compact mapping of a set $Y$ on itself. Thus, Schauder's fixed point theorem will apply.

The plan of the paper is as follows: Its first part, contained in Section 2, deals with the linearized parabolic equation, where dependence of solutions on the data $(\alpha,\ f)$ is studied. Time-periodic linear systems are investigated in Section 2.4. Secondly, the instantaneous control method for linear parabolic systems is analyzed in Section 3. The existence theorem for fixed points of the mapping $\Psi$ is given in Section 3.2. Section 4 is devoted to a brief discussion of improved suboptimal mehtods based on the standard instantaneous control method. Finally, numerical results confirming the theory are given in Section 5.

**Remark 1.1.** The instantaneous control method we are considering here is a bit different to common approaches in the literature. Frequently in the literature, only one gradient step in direction $-\nabla J_\tau(u_j^0, y_{j-1})$ is applied to obtain the control instead of solving the optimization problem, confer [4, 7, 13].

# 2 The linearized equation

In the following, we will investigate linear parabolic systems. At first, we define function spaces which we will need in the analysis of the problem. Then, the properties of solutions and the associated solution operators are studied. Here, we utilize the theory of semigroups. Further, we deal with time-periodic linear systems.

## 2.1 Notations and Preliminary results

Once and for all, we fix $1 < p < \infty$ and

$$n/p < 2\sigma < 2\sigma' < 1 + 1/p. \tag{A1}$$

The left-hand side of (A1) implies that $W_p^{2\sigma}(\Omega)$ is continuously imbedded in $C(\bar{\Omega})$. Here $W_p^{2\sigma}(\Omega)$ denotes the standard Sobolev-Slobodeskii space. To begin with, let us define a linear operator $A$ in $L_p(\Omega)$ by

$$D(A) = \{w \in W_p^2(\Omega) : \partial_n w = 0 \text{ on } \Gamma\},$$

where $\partial_n w$ is regarded as an element of $W_p^{1-1/p}(\Gamma)$, and

$$Aw = -\Delta w + w, \quad w \in D(A).$$

$-A$ is the infinitesimal generator of an analytic semigroup $S(t)$, $t \geq 0$, of linear continuous operators in $L_p(\Omega)$, [9, 17, 18]. Furthermore, we introduce the linear continuous *"Neumann operator"* $N : L_p(\Gamma) \to W_p^s(\Omega)$, $s < 1 + 1/p$, by $Nf = w$, where $w$ is the solution of

$$-\Delta w + w = 0, \quad \partial_n w = f, \quad f \in L_p(\Gamma).$$

Let us define an abstract Nemytski operator $B : W_p^{2\sigma}(\Omega) \to L_p(\Gamma)$ by

$$(B(w))(x) = b(w(x)) \qquad x \in \Gamma.$$

If the functions $b$, $u$, and $w_0$ in (1.1) are sufficiently smooth, then the classical solution $w$ of (1.1) satisfies the following integral equation,

$$w(\cdot, t) = S(t)w_0 + \int_0^t AS(t-s)N(B(w) + u)ds. \tag{2.1}$$

This is the well-known *Variation of constants formula*, confer [1, 17]. In [1], the following useful estimate to work with the integral in (2.1) was proven:

$$|AS(t)N|_{L^p(\Gamma)\mapsto W_p^{2\sigma}(\Omega)} \le c\,t^{-(1-(\sigma'-\sigma))}, \quad 2\sigma < 2\sigma' < 1 + 1/p. \tag{2.2}$$

Moreover, it holds that $S(t)$ restricts to a strongly continuous semigroup on $W_p^{2s}(\Omega)$, $2s < 1 + 1/p$. Therefore, the function $w(t) = S(t)w_0$ belongs to $C\left([0,T], W_p^{2\sigma}\right)$ provided that $w_0 \in W_p^{2\sigma}(\Omega)$.

To make clear in what sense (2.1) has to be solved, we generalize (2.1) in an obvious way.

**Definition 2.1 (Mild solution).** A function $w \in C\left([0,T], W_p^{2\sigma}\right)$ is called a mild solution of (1.1) if it satisfies the nonlinear integral equation (2.1) in $W_p^{2\sigma}(\Omega)$.

## 2.2 Existence and uniqueness of solutions

Here we study solvability of the linear system

$$\begin{aligned} y_t - \Delta y + y &= 0 \quad \text{in } Q,\\ \partial_n y + \alpha y &= g \quad \text{on } \Sigma,\\ y(0) &= y_0, \end{aligned} \tag{2.3}$$

where we suppose $y_0 \in W_p^{2\sigma}(\Omega)$,

$$\alpha \in L^\infty(\Sigma), \quad \alpha(x,t) \ge \alpha_1 > 0 \text{ a.e. on } \Sigma, \tag{A2}$$

and $g \in L^\nu(0,T;L^p(\Gamma))$ with $\nu$ chosen such that

$$(\sigma' - \sigma)^{-1} < \nu \le \infty. \tag{A3}$$

Henceforth, we will assume that the assumptions (A1), (A2), (A3) are satisfied. This is necessary to prove existence and uniqueness of solutions of (2.3). According to Definition 2.1, we say that $w \in C\left([0,T], W_p^{2\sigma}\right)$ is a mild solution of (2.3) if it satisfies

$$w(\cdot,t) = S(t)w_0 + \int_0^t AS(t-s)N(g(s) - \alpha w(s))ds. \tag{2.4}$$

It is known, that (2.3) admits a unique mild solution.

**Theorem 2.1.** *Let be $\alpha \in L^\infty(\Sigma)$ and $g \in L^\nu(0,T;L^p(\Gamma))$ given. Then (2.3) has a unique mild solution $y \in C\left([0,T], W_p^{2\sigma}\right)$.*

A proof is given for instance [25]. For further regularity results confer also [21, 22].

For convenience, we introduce the following integral operators $S_\alpha(t)$ and $V_\alpha$. We define $S_\alpha(t)y_0 = w(t)$ to be the solution of

$$w(t) = S(t)y_0 - \int_0^t AS(t-s)N\alpha(s)w(s)ds,$$

and $(V_\alpha g)(t) = v(t)$ to solve

$$v(t) = \int_0^t AS(t-s)N\big(g(s) - \alpha(s)v(s)\big)ds.$$

Then the mild solution $w$ of (2.3) can be splitted in the following way:

$$w(t) = S_\alpha(t)y_0 + (V_\alpha g)(t) \tag{2.5}$$

The mapping properties of these operators are investigated in the next lemma.

**Lemma 2.2.** *Let $\alpha \in L^\infty(\Sigma)$ be given. Then it holds*

$$\begin{aligned}
|V_\alpha|_{L^\nu(0,T;L^p(\Gamma)) \mapsto C\left([0,T],W_p^{2\sigma}\right)} &\leq c\,\phi\left(|\alpha|_{L^\infty(\Sigma)}\right), \\
|S_\alpha|_{W_p^{2\sigma}(\Omega) \mapsto C\left([0,T],W_p^{2\sigma}\right)} &\leq c\,\phi\left(|\alpha|_{L^\infty(\Sigma)}\right),
\end{aligned}$$

*with a positive and monotone increasing function $\phi$ and constants $c > 0$ independent of $\alpha$. Moreover, $V_\alpha$ is a compact mapping from $L^\nu(0,T;L^p(\Gamma))$ to $C\left([0,T],W_p^{2\sigma}\right)$.*

Before proving this lemma, we will cite a result from [25].

**Proposition 2.3.** *The integral operator $L$ given as*

$$(Lf)(t) = \int_0^t AS(t-s)Nf(s)ds$$

*is a compact operator from $L^\nu(0,T;L^p(\Gamma))$ to $C\left([0,T],W_p^{2\sigma}\right)$, provided $\nu, p, \sigma$ satisfy* (A1) *and* (A3).

*Proof of Lemma 2.2.* For given $b \in C\left([0,T],W_p^{2\sigma}\right)$ define the operator $T_\alpha$ as $w = T_\alpha b$, where $w$ solves the integral equation

$$w(t) = b(t) - \int_0^t AS(t-s)N\alpha(s)w(s)ds.$$

The unique solvability of this equation follows from the solvability of the nonlinear integral equation (2.1) proven in [25]. We then estimate,

$$|w(t)|_{W_p^{2\sigma}} \leq |b(t)|_{W_p^{2\sigma}} + \left|\int_0^t AS(t-s)N\alpha(s)w(s)ds\right|_{W_p^{2\sigma}}.$$

The second addend we treat using (2.2) as follows

$$\begin{aligned}
\left|\int_0^t AS(t-s)N\alpha(s)w(s)ds\right|_{W_p^{2\sigma}} &\leq c\int_0^t (t-s)^{-\beta}|\alpha(s)w(s)|_{L^p(\Gamma)}ds \\
&\leq c|\alpha|_{L^\infty(\Sigma)}\int_0^t (t-s)^{-\beta}|w(s)|_{W_p^{2\sigma}}ds
\end{aligned}$$

where $\beta = 1-(\sigma'-\sigma)$. The exponent $\beta$ is smaller than 1, thus the kernel $(t-s)^{-\beta}$ is at least integrable. Lemma 2.4, which can be found below, yields

$$|(T_\alpha b)(t)|_{W_p^{2\sigma}} \leq |w(t)|_{W_p^{2\sigma}} \leq \phi\left(|\alpha|_{L^\infty(\Sigma)}\right)|b|_{C([0,T],W_p^{2\sigma})}, \tag{2.6}$$

where $\phi$ is a positive and monotone function. Hence we proved

$$|T_\alpha|_{C\left([0,T],W_p^{2\sigma}\right)\mapsto C\left([0,T],W_p^{2\sigma}\right)} \leq \phi\left(|\alpha|_{L^\infty(\Sigma)}\right).$$

The operators $S_\alpha$ and $V_\alpha$ we can rewrite using $T_\alpha$ as

$$\begin{aligned}
S_\alpha(t)y_0 &= (T_\alpha S(\cdot)y_0)(t) \\
(V_\alpha g)(t) &= (T_\alpha Lg)(t)
\end{aligned} \tag{2.7}$$

The norm estimate of $S_\alpha$ follows immediately from the fact that $S(t)$ is a strongly continuous semigroup on $W_p^{2\sigma}(\Omega)$, cf. [1]. The claims concerning $V_\alpha$ are a consequence of Proposition 2.3 and the linearity and boundedness of $T_\alpha$. ∎

Here, we will provide an estimate on the solution of the integral inequality

$$z(t) \leq a(t) + \omega \int_0^t (t-s)^{-\beta} z(s) ds, \tag{2.8}$$

with $\beta = 1 - (\sigma' - \sigma) < 1$. In the sequel we make frequently use of the following well-known, [23, 25], result.

**Lemma 2.4.** *Let $\omega \geq 0$ and $a \in C[0,T]$ be given. If the solution $z$ of the inequality (2.8) is in $C[0,T]$, then we have*

$$\|z\|_{C[0,T]} \leq \phi(\omega) \, \|a\|_{C[0,T]},$$

*with a positive and monotone increasing function $\phi$.*

*Proof.* It is convenient to introduce a new norm $\|\cdot\|_\lambda$ by

$$\|g\|_\lambda := \|h\|_{C[0,T]} \qquad \text{with} \qquad g(t) = e^{-\lambda t} h(t),$$

where $\lambda$ is a nonnegative real number to be specified later. This norm is equivalent to $\|\cdot\|_{C[0,T]}$. Multiplying (2.8) by $e^{-\lambda t}$, we proceed

$$\begin{aligned} e^{-\lambda t} z(t) &\leq e^{-\lambda t} a(t) + \omega \int_0^t (t-s)^{-\beta} e^{-\lambda t} z(s) ds \\ &\leq e^{-\lambda t} a(t) + \omega \int_0^t (t-s)^{-\beta} e^{-\lambda(t-s)} e^{-\lambda s} z(s) ds \\ &\leq \|a\|_\lambda + \omega \|z\|_\lambda \int_0^t (t-s)^{-\beta} e^{-\lambda(t-s)} ds. \end{aligned} \tag{2.9}$$

By assumption (A3), the kernel $(t-s)^{-\beta}$ is in $L^q(0,t)$ with $1/q + 1/\nu = 1$. We can apply Hölders inequality with $q$ and $\nu$,

$$\begin{aligned} \int_0^t (t-s)^{-\beta} e^{-\lambda(t-s)} ds &\leq \left( \int_0^t (t-s)^{-\beta q} ds \right)^{1/q} \left( \int_0^t e^{-\lambda\nu(t-s)} ds \right)^{1/\nu} \\ &\leq \left( \frac{1}{1-\beta q} t^{1-\beta q} \right)^{1/q} \left( \frac{1}{\lambda\nu} \left(1 - e^{-\lambda\nu t}\right) \right)^{1/\nu} \\ &\leq \left( \frac{1}{1-\beta q} T^{1-\beta q} \right)^{1/q} \left( \frac{1}{\lambda\nu} \right)^{1/\nu}. \end{aligned}$$

The latter estimate is independent of $t$, so we can take the maximum over $t \in [0,T]$ of the left-hand side in (2.9). Hence, we arrive at the inequality

$$\|z\|_\lambda \leq \|a\|_\lambda + c\,\omega \|z\|_\lambda \lambda^{-1/\nu},$$

with a constant $c$ independent of $\lambda$. Choosing $\lambda = (2c\omega)^\nu$ we find

$$\|z\|_{C[0,T]} \leq 2\exp(\lambda T) \|a\|_{C[0,T]} \leq 2\exp((2c\omega)^\nu T) \|a\|_{C[0,T]} \leq \phi(\omega) \|a\|_{C[0,T]},$$

with $\phi(\omega) = 2\exp((2c\omega)^\nu T)$, and the claim is proven. □

**Remark 2.1.** Let $g \in L^\infty(\Sigma)$ be given. The associated solution $y$ of (2.3) obeys the maximum principle. Thus, the maximum of $y(x,t)$ is attained on the set $\Gamma_T = \Sigma \cup (\Omega \times \{0\})$. By assumption (A2) we conclude that

$$\vartheta_1 \leq y(x,t) \leq \vartheta_2$$

holds for all $(x,t) \in Q$, where

$$\vartheta_1 = \min\left( \operatorname*{ess\,inf}_{x\in\Omega} y_0(x), \operatorname*{ess\,inf}_{(x,t)\in\Sigma} \frac{g(x,t)}{\alpha(x,t)} \right), \quad \vartheta_2 = \max\left( \operatorname*{ess\,sup}_{x\in\Omega} y_0(x), \operatorname*{ess\,sup}_{(x,t)\in\Sigma} \frac{g(x,t)}{\alpha(x,t)} \right).$$

## 2.3 Continuous dependency on the data

In the following, we want to prove continuity of the mappings $\alpha \mapsto S_\alpha$ and $\alpha \mapsto V_\alpha$. To do so, we have to show that the mapping $\alpha \mapsto T_\alpha$ is continuous. Then, continuity follows from (2.7), since $L$ and $S$ are independent of $\alpha$.

**Proposition 2.5.** *The mapping $\alpha \mapsto T_\alpha$ is continuous from $L^\infty(\Sigma)$ to $\mathcal{L}\left(C\left([0,T], W_p^{2\sigma}\right)\right)$.*

*Proof.* Let a sequence $\{\alpha_n\} \subset L^\infty(\Sigma)$ satisfying (A2) be given, which converges in $L^\infty(\Sigma)$ to $\alpha$. Let $b \in C([0,T], W_p^{2\sigma}(\Omega)$. We have to show that $w_n := T_{\alpha_n} b$ converges to $w := T_\alpha b$. We substract the corresponding integral equations defining $w$ and $w_n$ to obtain

$$(w - w_n)(t) = \int_0^t AS(t-s)N\big\{(\alpha(s) - \alpha_n(s))w(s) + \alpha_n(s)(w(s) - w_n(s))\big\}ds.$$

Using the uniform bound of $w$ given by (2.6), we get analogously as above

$$|(w - w_n)(t)|_{W_p^{2\sigma}} \le c\left(|\alpha - \alpha_n|_{L^\infty(\Sigma)} + \int_0^t (t-s)^{-\beta}|w(s) - w_n(s)|_{W_p^{2\sigma}} ds\right).$$

By Lemma 2.4, we estimate

$$|(w - w_n)(t)|_{W_p^{2\sigma}} \le c|\alpha - \alpha_n|_{L^\infty(\Sigma)}.$$

Hence we proved, $w_n \to w$ in $C\left([0,T], W_p^{2\sigma}\right)$ as $\alpha_n \to \alpha$ in $L^\infty(\Sigma)$, which gives the convergence of $T_{\alpha_n}$ to $T_\alpha$ in $\mathcal{L}\left(C\left([0,T], W_p^{2\sigma}\right)\right)$. □

Using the representation (2.7), we can establish the following lemma.

**Lemma 2.6.** *The mappings $\alpha \mapsto S_\alpha$ and $\alpha \mapsto V_\alpha$ are continuous from $L^\infty(\Sigma)$ to $\mathcal{L}\big(W_p^{2\sigma}(\Omega)$, $C\left([0,T], W_p^{2\sigma}\right)\big)$ and $\mathcal{L}\left(L^\nu(0,T; L^p(\Gamma)), C\left([0,T], W_p^{2\sigma}\right)\right)$, respectively.*

The next step is to prove some compactness results needed later on for the existence of fixed points.

**Lemma 2.7.** *The mapping $(\alpha, g) \mapsto V_\alpha g$ is compact in the following sense: the image of every bounded subset of $L^\infty(\Sigma) \times L^\nu(0,T; L^p(\Gamma))$ is precompact in $C\left([0,T], W_p^{2\sigma}\right)$.*

*Proof.* Let $\mathcal{A} \subset L^\infty(\Sigma)$ and $\mathcal{G} \subset L^\nu(0,T; L^p(\Gamma))$ be bounded sets. Without loss of generality, we assume that $\mathcal{A}$ and $\mathcal{M}$ are not empty. We choose an arbitrary but fixed element $\bar\alpha \in \mathcal{A}$. Denote by $\mathcal{M}_\mathcal{A}$ and $\mathcal{M}_\mathcal{G}$ the bounds of $\mathcal{A}$ and $G$, respectively,

$$|\alpha|_{L^\infty(\Sigma)} \le \mathcal{M}_\mathcal{A} \quad \forall \alpha \in \mathcal{A}, \qquad |g|_{L^\nu(0,T;L^p(\Gamma))} \le \mathcal{M}_\mathcal{G} \quad \forall g \in \mathcal{G}.$$

Take $\alpha \in \mathcal{A}$ and $g \in \mathcal{G}$, and set $y := V_\alpha g$. Then $y$ solves

$$y(t) = \int_0^t AS(t-s)N\big(g(s) - \alpha(s)y(s)\big)ds = \int_0^t AS(t-s)N\big(\bar g(s) - \bar\alpha(s)y(s)\big),$$

where we set $\bar g = g - (\alpha - \bar\alpha)y$. Therefore, it holds $y = V_\alpha g = V_{\bar\alpha}\bar g$. Since $y \in C\left([0,T], W_p^{2\sigma}\right) \hookrightarrow C(\bar Q)$, $\bar g$ is an element of $L^\nu(0,T; L^p(\Gamma))$, which norm we can estimate, confer Lemma 2.2, as

$$|\bar g|_{L^\nu(0,T;L^p(\Gamma))} \le |g|_{L^\nu(0,T;L^p(\Gamma))} + c(|\alpha|_{L^\infty(\Sigma)} + |\bar\alpha|_{L^\infty(\Sigma)})\phi\left(|\alpha|_{L^\infty(\Sigma)}\right) \le \mathcal{G} + c\,\mathcal{A}\,\phi(\mathcal{A}). \quad (2.10)$$

We define the set $\bar{\mathcal{G}}$ of all those functions $\bar g$ as

$$\bar{\mathcal{G}} = \left\{\bar g \in L^\nu(0,T; L^p(\Gamma)) \,\middle|\, \exists \alpha \in \mathcal{A}, g \in \mathcal{G} : y = V_\alpha g,\ \bar g = g - (\alpha - \bar\alpha)y\right\}.$$

By construction, we know that the following sets are equal:

$$\begin{aligned} I_1 &= \left\{y \in C\left([0,T], W_p^{2\sigma}\right) \,\middle|\, \exists \alpha \in \mathcal{A}, g \in \mathcal{G} : y = V_\alpha g\right\}, \\ I_2 &= \left\{y \in C\left([0,T], W_p^{2\sigma}\right) \,\middle|\, \exists \bar g \in \bar{\mathcal{G}} : y = V_{\bar\alpha}\bar g\right\}. \end{aligned}$$

The set $\bar{\mathcal{G}}$ is bounded by (2.10), so $I_2 = V_{\bar\alpha}\bar{\mathcal{G}}$ is precompact in $C\left([0,T], W_p^{2\sigma}\right)$ by Lemma 2.2. Thus, the set $I_1$ is precompact, too. □

**Lemma 2.8.** *Let $Y_0 \subset W_p^{2\sigma}(\Omega)$ be a precompact set and $\mathcal{A} \subset L^\infty(\Sigma)$ be bounded. Then the following set is precompact in $C\left([0,T], W_p^{2\sigma}\right)$, too:*

$$Y = \{y : \; y(t) = S_\alpha(t)y_0, \, \alpha \in \mathcal{A}, \, y \in Y_0\}$$

*Proof.* Analogous to the proof of the previous Lemma 2.7. □

## 2.4 The time-periodic equation

Here, we consider the following time-periodic problem:

$$\begin{aligned} y_t - \Delta y + y &= 0 \qquad \text{in } Q, \\ \partial_n y + \alpha y &= g \qquad \text{on } \Sigma, \\ y(0) &= y(T). \end{aligned} \tag{2.11}$$

We want to show, that (2.11) admits a unique solution $y \in C\left([0,T], W_p^{2\sigma}\right)$, provided that $\alpha$, $g$ satisfy the conditions given in Section 2.2. According to the previous sections, we call $y$ a mild solution of (2.11) if it satisfies the system of equations

$$\begin{aligned} y(t) &= S(t)y(0) + \textstyle\int_0^t AS(t-s)N(g(s) - \alpha(s)y(s))ds, \\ y(0) &= y(T). \end{aligned}$$

or equivalently using (2.5),

$$\begin{aligned} y(t) &= S_\alpha(t)y(0) + (V_\alpha g)(t), \\ y(0) &= y(T). \end{aligned}$$

Defining $z = y(0)$, we get one equation with unknown $z$,

$$z = S_\alpha(T)z + (V_\alpha g)(T). \tag{2.12}$$

This equation would be uniquely solvable if $I - S_\alpha(T)$ has a continuous inverse. The next lemma ensures that property.

**Lemma 2.9.** *For $1 < p < \infty$ the operator family $\{S_\alpha(t)\}$ is an analytic semigroup in $L^p(\Omega)$. $\{S_\alpha(t)\}$ is a semigroup of compact operators in $L^2(\Omega)$. Moreover, there is a constant $\lambda > 0$, independent of $\alpha$, so that*

$$|S_\alpha(t)|_{\mathcal{L}(L^2(\Omega))} \le e^{-\lambda t}. \tag{2.13}$$

*Proof.* We define the differential operator $A_\alpha$ by $A_\alpha y = -\Delta y + y$ for $y \in D(A_\alpha)$ and

$$D(A_\alpha) = \{w \in W_p^2(\Omega) : \, \partial_n w + \alpha w = 0 \text{ on } \Gamma\}.$$

Then $-A_\alpha$ is the generator of an analytic semigroup $U_\alpha(t)$ in $L^p(\Omega)$, cf. [18]. Since $U_\alpha$ and $S_\alpha$ are solution operators of the same uniquely solvable evolution equation they are in fact equal, thus $A_\alpha$ is the infinitesimal generator of $S_\alpha$. Compactness of $S_\alpha(t)$ follows from [18, Theorem 2.3.3] and the fact that the resolvent $(\lambda I - A_\alpha)^{-1}$ is compact in $L^2(\Omega)$ for all $\lambda$ belonging to the resolvent set $\rho(A_\alpha)$. The estimate of the $L^2$-norm of $S_\alpha(t)$ follows by standard techniques using Gronwall's Lemma to obtain the exponential decay. □

So, we are able to prove solvability of (2.11).

**Lemma 2.10.** *For every $g \in L^\nu(0,T; L^p(\Gamma))$ the equation (2.12) has a unique solution $z \in W_p^{2\sigma}(\Omega)$. Moreover, the operator $(I - S_\alpha(T))^{-1}$ is linear continuous from $W_p^{2\sigma}(\Omega)$ to $W_p^{2\sigma}(\Omega)$.*

*Proof.* First observe, that by Lemma 2.9 the equation

$$z = S_\alpha(T)z + (V_\alpha g)(T) \tag{2.14}$$

has a unique solution $z \in L^2(\Omega)$, since $(V_\alpha g)(T) \in W_p^{2\sigma}(\Omega) \hookrightarrow L^2(\Omega)$. It satisfies the estimate

$$|z|_{L^2(\Omega)} \le \left(1 - e^{-\lambda T}\right)^{-1} |(V_\alpha g)(T)|_{L^2(\Omega)} \le c|(V_\alpha g)(T)|_{W_p^{2\sigma}} \le c|g|_{L^\nu(0,T;L^p(\Gamma))}.$$

Now, we improve the regularity result for $z$. Since $S_\alpha$ is an analytic semigroup, it holds $S_\alpha(t)z \in D(A_\alpha) \hookrightarrow W_p^{2\sigma}(\Omega)$, confer [18]. The right-hand side of (2.14) is an element of $W_p^{2\sigma}(\Omega)$, thus $z \in W_p^{2\sigma}(\Omega)$. By a bootstrapping argument we estimate the $W_p^{2\sigma}$-norm of $z$ as follows

$$\begin{aligned} |z|_{W_p^{2\sigma}} &\le |S_\alpha(T)z|_{W_p^{2\sigma}} + |(V_\alpha g)(T)|_{W_p^{2\sigma}} \le |S_\alpha(T)|_{L^2(\Omega)\to W_p^{2\sigma}} |z|_{L^2(\Omega)} + |(V_\alpha g)(T)|_{W_p^{2\sigma}} \\ &\le c|(V_\alpha g)(T)|_{W_p^{2\sigma}} \le c|g|_{L^\nu(0,T;L^p(\Gamma))}. \end{aligned} \tag{2.15}$$

Here we used a result from [1, 18], that $S(t)$ is for $t > 0$ a linear continuous operator from $L^2(\Omega)$ to $W_p^{2\sigma}$. Therefore, the second claim is proven. $\square$

**Corollary 2.11.** *For every $g \in L^\nu(0,T;L^p(\Gamma))$ the time-periodic system* (2.11) *admits a unique solution $y \in C\left([0,T], W_p^{2\sigma}\right)$.*

*Proof.* By the previous Lemma 2.10, the equation (2.12) admits a unique solution $z \in W_p^{2\sigma}(\Omega)$. If we replace in the system (2.11) the time-periodicity condition by the initial condition $y(0) = z$, it has a unique solution $y$ by Theorem 2.1. This solution satisfies

$$y(T) = S_\alpha(T)y_0 + (V_\alpha g)(T) = S_\alpha(T)z + (V_\alpha g)(T) = z = y(0),$$

i.e. $y$ fulfills the time-periodicity condition, and is in fact the unique solution of (2.11). $\square$

Next, continuous dependence of $(I - S_\alpha(T))^{-1}$ on $\alpha$ is investigated.

**Lemma 2.12.** *The mapping $\alpha \mapsto (I - S_\alpha(T))^{-1}$ is continuous and bounded from $L^\infty(\Sigma)$ to $\mathcal{L}(W_p^{2\sigma}(\Omega))$.*

*Proof.* Let $w \in W_p^{2\sigma}(\Omega)$ and $\{\alpha_n\} \subset L^\infty(\Sigma)$ be given, with $\alpha_n \to \alpha$ in $L^\infty(\Sigma)$. We denote $z_n := (I - S_{\alpha_n}(T))^{-1}w$ and $z := (I - S_\alpha(T))^{-1}w$, respectively. Then by Lemma 2.10 $z_n, z \in W_p^{2\sigma}(\Omega)$ holds. We have by definition

$$z - z_n = S_\alpha(T)z - S_{\alpha_n}(T)z_n = S_\alpha(T)(z - z_n) + (S_\alpha(T) - S_{\alpha_n}(T))z_n.$$

Thus, the difference $z - z_n$ can be written as

$$z - z_n = (I - S_\alpha(T))^{-1}(S_\alpha(T) - S_{\alpha_n}(T))z_n.$$

The set $\{z_n\}$ is uniformly bounded in $W_p^{2\sigma}(\Omega)$, confer Lemma 2.10. By Lemma 2.6 we conclude that the right-hand side tends to zero as $\alpha_n \to \alpha$. Therefore, it follows $z \to z_n$ in $W_p^{2\sigma}(\Omega)$ and $(I - S_{\alpha_n}(T))^{-1}w \to (I - S_\alpha(T))^{-1}w$ for all $w \in W_p^{2\sigma}(\Omega)$, and continuity is proven. Boundedness follows by Lemma 2.2 and the estimate (2.15). $\square$

**Lemma 2.13.** *The mapping $(\alpha, g) \mapsto (I - S_\alpha(T))^{-1}V_\alpha g$ is compact in the following sense: the image of every bounded subset of $L^\infty(\Sigma) \times L^\nu(0,T;L^p(\Gamma))$ is precompact in $C\left([0,T], W_p^{2\sigma}\right)$.*

*Proof.* The proof can be done using the same ideas used already in the proof of Lemma 2.7. $\square$

**Remark 2.2.** In the case of the time-periodic equation, one can establish a maximum principle. Let $y$ be the solution of (2.11) for the inhomogeneity $g \in L^\infty(\Sigma)$. It can be shown that the maximum of $y(x,t)$ is attained on the set $\Gamma_T = \Sigma$. Therefore, it follows with assumption (A2) that

$$\vartheta_1 \le y(x,t) \le \vartheta_2$$

holds for all $(x,t) \in Q$, where

$$\vartheta_1 = \operatorname*{ess\,inf}_{(x,t)\in\Sigma} \frac{f(x,t)}{\alpha(x,t)}, \quad \vartheta_2 = \operatorname*{ess\,sup}_{(x,t)\in\Sigma} \frac{f(x,t)}{\alpha(x,t)},$$

cf. also Remark 2.1.

**Remark 2.3.** In this section we have treated linear time-periodic equations. The solvability of the nonlinear time-periodic equation (1.5) is a auxiliary product of the theory we will present below. It can be proven using the same compactness arguments as in Section 3.2. Uniqueness of solutions can be shown requiring $-b'(y) \ge \alpha_1$, confer Assumption 3.7(i), see Section 3.2 below.

# 3 Instantaneous control

## 3.1 Suboptimal control of the linearized equation

Now, we can start with the investigation of the instantaneous control method for the linearized problem (2.3). We want to minimize the functional $(P)$. As already explained above, the instantaneous method minimizes the objective over small time interval of length $\tau$, i.e. it solves

$$\min J_\tau(u,y) = \frac{1}{2}\int_\Omega (y(x,\tau) - y_d(x))^2 dx + \frac{\gamma}{2}\int_{\Sigma_\tau} u(x,t)^2 dt dS_x \qquad (P_{\tau,\alpha,f})$$

subject to $u \in U_{ad,\tau}$ and the differential equation

$$\begin{aligned} y_t - \Delta y + y &= 0 && \text{in } Q_\tau, \\ \partial_n y + \alpha y &= f + u && \text{on } \Sigma_\tau, \\ y(0) &= y_0. \end{aligned} \tag{3.1}$$

Here, $U_{ad,\tau}$ denotes the adaption of $U_{ad}$ to the time interval $[0,\tau]$. The investigation of linear systems is advantageous for two reasons: first, we have an explicite formula for the solution of (3.1). And second, the objective functional $J_\tau$ is strong convex with respect to the control. Hence, the problem $(P_{\tau,\alpha,f})$ has a unique global minimizer $\bar u(y_0)$, and the first-order necessary optimality conditions are also sufficient for global optimality.

The method realizes a mapping $\Phi_{\alpha,f} : y(0) \mapsto y(\tau)$ in the following sense: The initial value $y_0$ uniquely determines an optimal control $u_0$ of $(P_{\tau,\alpha,f})$. The control $u_0$ together with $y_0$ yields the state at time $\tau$, $y(\tau) =: y_1$. So, $\Phi_{\alpha,f}$ is defined as $\Phi_{\alpha,f}(y_0) := y_1$. This mapping is a contraction, as the next Proposition shows.

**Proposition 3.1.** *$\Phi_{\alpha,f}$ is a contraction in $L^2(\Omega)$.*

*Proof.* The objective functional can be rewritten as,

$$J_\tau(u) = \frac{1}{2}\left|(V_\alpha u)(\tau) - (y_d - S_\alpha(\tau)y_0 - (V_\alpha f)(\tau))\right|^2 + \frac{\gamma}{2}|u|^2.$$

The necessary condition for $\bar u \in U_{ad,\tau}$ to be optimal for $(P_{\tau,\alpha,f})$ is therefore

$$((V_\alpha u)(\tau) - (y_d - S_\alpha(\tau)y_0 - (V_\alpha f)(\tau))\,,\ (V_\alpha(\bar u - u))(\tau)) + \gamma(\bar u, u - \bar u) \ge 0 \quad \forall u \in U_{ad,\tau}. \tag{3.2}$$

Now, let $y_{01}$ and $y_{02}$ be two initial states in $L^2(\Omega)$. Define $y_i := \Phi_{\alpha,f}(y_{0i})$. Denote the associated optimal control of $(P_{\tau,\alpha,f})$ by $u_i$, $i = 1, 2$. Then the controls $u_i$ fulfill the variational inequality (3.2). We test the corresponding inequalities $u_i$ by $u_j$, $j \neq i$, add them to get

$$-\left((V_\alpha(u_1 - u_2),\ S_\alpha(\tau)(y_{01} - y_{02})\right) - |(V_\alpha(u_1 - u_2))(\tau)|^2 - \gamma|u_1 - u_2|^2 \geq 0.$$

Hence it holds

$$((V_\alpha(u_1 - u_2),\ S_\alpha(\tau)(y_{01} - y_{02})) \leq -|(V_\alpha(u_1 - u_2))(\tau)|^2. \tag{3.3}$$

Using (3.3), a final estimation yields contractivity of $\Phi_{\alpha,f}$:

$$\begin{aligned}
|\Phi_{\alpha,f}(y_{01}) - \Phi_{\alpha,f}(y_{02})|^2 &= |y_1 - y_2|^2 = |S_\alpha(\tau)(y_{01} - y_{02}) + (V_\alpha(u_1 - u_2))(\tau)|^2 \\
&= |S_\alpha(\tau)(y_{01} - y_{02})|^2 + 2(S_\alpha(\tau)(y_{01} - y_{02}),\ V_\alpha(u_1 - u_2))(\tau)) + |V_\alpha(u_1 - u_2))(\tau)|^2 \\
&\leq |S_\alpha(\tau)(y_{01} - y_{02})|^2 - |V_\alpha(u_1 - u_2))(\tau)|^2 \leq |S_\alpha(\tau)|^2\, |(y_{01} - y_{02})|^2.
\end{aligned}$$

We conclude that $\Phi_{\alpha,f}$ is contractive, since $S_\alpha(T)$ is a contraction in $L^2(\Omega)$. □

Therefore, $\Phi_{\alpha,f}$ has a uniquely determined fixed point $y^* \in L^2(\Omega)$. With the same arguments as used in the proof of Lemma 2.10, one shows $y^* \in W^{2\sigma}_p(\Omega)$. Since $\Phi_{\alpha,f}$ is contractive, state and control in the instantaneous control method will converge to the fixed point $(y^*, u^*)$, confer [26]. The fixed point control $u^*$ and state $y^*$ are connected in two ways: First, $y^*$ satisfies $y^* = y(\tau)$, where $y$ is the solution of the following boundary value problem with time-periodic condition

$$\begin{aligned}
y_t - \Delta y + y &= 0 && \text{in } Q_\tau, \\
\partial_n y + \alpha y &= f + u^* && \text{on } \Sigma_\tau, \\
y(0) &= y(\tau).
\end{aligned} \tag{3.4}$$

With the notation of Section 2, we can write

$$y^* = (I - S_\alpha(\tau))^{-1} \left(V_\alpha(f + u^*)\right)(\tau). \tag{3.5}$$

Second, the control $u^*$ is the unique optimal control of $(P_{\tau,\alpha,f})$ subject to the initial condition $y_0 := y^*$. Hence, it satisfies necessarily the variational inequality

$$(y(\tau) - y_d,\, V_\alpha(u - u^*)(\tau)) + \gamma(u^*, u - u^*) \geq 0 \qquad \forall u \in U_{ad,\tau},$$

or equivalently, confer (3.5),

$$\left((I - S_\alpha(\tau))^{-1} V_\alpha(f + u^*)(\tau) - y_d,\, V_\alpha(u - u^*)(\tau)\right) + \gamma(u^*, u - u^*) \geq 0 \qquad \forall u \in U_{ad,\tau}. \tag{3.6}$$

Here, we introduce a bilinear form $H_\alpha$ and a functional $q_{\alpha,f}$ for $u, v \in U_{ad,\tau}$ as

$$H_\alpha(u, v) := \left((I - S_\alpha(\tau))^{-1}(V_\alpha u)(\tau),\, (V_\alpha v)(\tau)\right) + \gamma(u, v)$$

and

$$q_{\alpha,f}(v) := \left((I - S_\alpha(\tau))^{-1}(V_\alpha f)(\tau) - y_d,\, (V_\alpha v)(\tau)\right).$$

Then the variational inequality (3.6) reads

$$H_\alpha(u^*, u - u^*) + q_{\alpha,f}(u - u^*) \geq 0 \qquad \forall u \in U_{ad,\tau}. \tag{3.7}$$

Since $H_\alpha$ is coercive, this variational inequality determines uniquely the fixed point control $u^*$.
Next, we want to show continuous dependence of $(u^*, y^*)$ on $\alpha$. To begin with, we construct a space $L^{\tilde{s}}(0, T; L^s(\Gamma))$ such that $L^\nu(0, T; L^p(\Gamma))$ is an intermediate space between $L^2(\Sigma)$ and $L^{\tilde{s}}(0, T; L^s(\Gamma))$. The exponents $\tilde{s}$, $s$ are given by Lemma A.2, provided

$$2 \leq \nu < 2p \tag{A4}$$

holds, which we assume to be fulfilled in the sequel. Then the set of admissible controls $U_{ad}$ is bounded in $L^{\tilde{s}}(0, T; L^s(\Gamma))$:

$$|u|_{L^{\tilde{s}}(0,T;L^s(\Gamma))} \leq \mathcal{M} \qquad \forall u \in U_{ad}. \tag{3.8}$$

**Lemma 3.2.** *Let $u^*$ denote the solution of the variational inequality* (3.6) *for given* $\alpha$, $f$. *Then it holds: the mapping* $(\alpha, f) \mapsto u^*$ *is continuous and bounded from* $L^\infty(\Sigma_\tau) \times L^\nu(0,\tau; L^p(\Gamma))$ *to* $L^\nu(0,\tau; L^p(\Gamma))$.

*Proof.* First, observe that the variational inequality (3.6) admits a unique solution for every given $\alpha$, $f$. This follows from the above consideration. Namely, the instantaneous method converges to a unique fixed point $(y^*, u^*)$, where $u^*$ satisfies the variational inequality

$$H_\alpha(u^*, v - u^*) + q_{\alpha,f}(v - u^*) \geq 0 \qquad \forall v \in U_{ad,\tau},$$

cf. (3.7). Since $0 \in U_{ad,\tau}$ by (1.3), we can test the variational inequality by $v = u^*/2$. Thus, we obtain

$$-\frac{1}{2} q_{\alpha,f}(u^*) \geq \frac{1}{2} H_\alpha(u^*, u^*) \geq \frac{\gamma}{2} |u^*|^2_{L^2(\Sigma_\tau)}. \tag{3.9}$$

Abbreviating $\mathcal{A} := \phi\left(|\alpha|_{L^\infty(\Sigma_\tau)}\right)$, we estimate the left-hand side as follows,

$$|q_{\alpha,f}(u^*)| \leq c_1 \left(\mathcal{A}|f|_{L^\nu(0,\tau;L^p(\Gamma))} + 1\right) \mathcal{A}|u^*|_{L^\nu(0,\tau;L^p(\Gamma))}, \tag{3.10}$$

where we used heavily the estimates given by Lemma 2.2 and Lemma 2.10. By $c_1$ a constant independent of $\alpha$, $f$ is denoted. The bound in $L^\nu(0,\tau; L^p(\Gamma))$ we get with the aid of Lemma A.2,

$$|u^*|_{L^2(\Sigma_\tau)} \geq \mathcal{M}^{-(p-1)} |u^*|^{p-1}_{L^{\tilde{s}}(0,\tau;L^s(\Gamma))} |u^*|_{L^2(\Sigma_\tau)} \geq \mathcal{M}^{-(p-1)} |u^*|^p_{L^\nu(0,\tau;L^p(\Gamma))}, \tag{3.11}$$

where we used the boundedness of the set of admissible controls in $L^{\tilde{s}}(0,\tau; L^s(\Gamma))$, cf. (3.8). Putting the estimates (3.9)-(3.11) together yields

$$c_1 \left(\mathcal{A}|f|_{L^\nu(0,\tau;L^p(\Gamma))} + 1\right) \mathcal{A}|u^*|_{L^\nu(0,\tau;L^p(\Gamma))} \geq \gamma |u^*|^2_{L^2(\Sigma_\tau)} \geq c_2 |u^*|^{2p}_{L^\nu(0,\tau;L^p(\Gamma))},$$

which gives in turn the desired bound

$$|u^*|^{2p-1}_{L^\nu(0,\tau;L^p(\Gamma))} \leq c\,\mathcal{A}\left(\mathcal{A}|f|_{L^\nu(0,\tau;L^p(\Gamma))} + 1\right).$$

Let be given two sequences $\{\alpha_n\} \subset L^\infty(\Sigma_\tau)$ and $\{f_n\} \subset L^\nu(0,\tau; L^p(\Gamma))$ converging to $\alpha$ and $f$ in $L^\infty(\Sigma_\tau)$ and $L^\nu(0,\tau; L^p(\Gamma))$ as $n \to \infty$, respectively. Then for $u, v \in U_{ad,\tau}$ it holds

$$|H_\alpha(u,v) - H_{\alpha_n}(u,v)| \to 0, \qquad |q_{\alpha,f}(v) - q_{\alpha_n,f_n}(v)| \to 0 \tag{3.12}$$

as $n$ tends to infinity, since $H$ and $q$ are composed of operators which depends continuously on $\alpha$. Denote by $u_n$ and $u$ the solutions of (3.6) associated with $(\alpha_n, f_n)$ and $(\alpha, f)$. Then we test the corresponding variational inequalities with $u$ and $u_n$, add them, and obtain

$$-H_\alpha(u, u - u_n) + H_{\alpha_n}(u_n, u - u_n) + (q_{\alpha,f} - q_{\alpha_n,f_n})(u - u_n) \geq 0.$$

This is equivalent to

$$-H_\alpha(u_n, u - u_n) + H_{\alpha_n}(u_n, u - u_n) - H_\alpha(u - u_n, u - u_n) + (q_{\alpha,f} - q_{\alpha_n,f_n})(u - u_n) \geq 0.$$

Using the coercivity of $H_\alpha$ we arrive at

$$-\left(H_{\alpha_n} - H_\alpha\right)(u_n, u - u_n) + (q_{\alpha,f} - q_{\alpha_n,f_n})(u - u_n) \geq \gamma |u - u_n|^2_{L^2(\Sigma_\tau)},$$

which gives convergence of $u_n \to u$ as $n \to \infty$ in $L^2(\Sigma_\tau)$, since the left-hand side tends to zero. Convergence in $L^\nu(0,\tau; L^p(\Gamma))$ we get analogous to (3.11). □

**Lemma 3.3.** *The mapping* $(\alpha, f) \mapsto y^*$, *where* $y^*$ *denotes the fixed point state given by* (3.4), *is compact from* $L^\infty(\Sigma_\tau) \times L^\nu(0,\tau; L^p(\Gamma))$ *to* $W^{2\sigma}_p(\Omega)$.

*Proof.* By the previous Lemma 3.2 we conclude the mapping $(\alpha, f) \mapsto f + u^*$ transforms bounded sets in bounded sets. Lemma 2.13 yields compactness of the mapping $(\alpha, f + u^*) \mapsto (I - S_\alpha(\tau))^{-1}\left(V_\alpha(f + u^*)\right)(\tau) = y^*$, which is the claim. □

### 3.2 Existence of fixed points

In Section 1.1 we introduced a mapping $\Psi$ as

$$\bar{y} \mapsto \big(-b'(\bar{y}),\ b(\bar{y}) - b'(\bar{y})\bar{y}\big) =: (\alpha, f) \mapsto (y^*, u^*) \mapsto y =: \Psi(\bar{y}),$$

where $(y^*, u^*) \in W_p^{2\sigma}(\Omega) \times L^{\tilde{s}}(0,\tau; L^s(\Gamma))$ denotes the fixed point of the instantaneous control method for the linear problem investigated in Section 3.1, and $y \in C\left([0,\tau], W_p^{2\sigma}\right)$ is the solution of the associated time-periodic system. Here, we will show that a fixed point of $\Psi$ is connected with a generalized fixed point of $\Phi$. To do so, let $\tilde{y}$ be a fixed point of $\Psi$ with associated $(y^*, u^*)$. Hence, $y = \tilde{y}$ solves,

$$\begin{aligned} y_t - \Delta y + y &= 0 && \text{in } Q_\tau,\\ \partial_n y &= b(\tilde{y}) + b'(\tilde{y})(y - \tilde{y}) + u^* && \text{on } \Sigma_\tau,\\ y(0) &= y(\tau), \end{aligned} \tag{3.13}$$

where the boundary condition is equivalent to

$$\partial_n y = b(y) + u^* \quad \text{on } \Sigma_\tau.$$

Therefore, the fixed point satisfies $y^* = y$, where $y$ solves the nonlinear time-periodic equation

$$\begin{aligned} y_t - \Delta y + y &= 0 && \text{in } Q_\tau,\\ \partial_n y &= b(y) + u^* && \text{on } \Sigma_\tau,\\ y(0) &= y(\tau), \end{aligned} \tag{3.14}$$

compare with (1.5). Further, the control $u^*$ fulfills the variational inequality (3.7), where we set

$$\alpha := -b'(\tilde{y}), \qquad f := b(\tilde{y}) - b'(\tilde{y})\tilde{y},$$

which is in fact the necessary condition of $(P_\tau)$ in Section 1.1. Thus, we proved

**Lemma 3.4.** *Let $\tilde{y}$ be a fixed point of $\Psi$ with associated state $y^*$ and control $u^*$. Then $y^*$ is a generalized fixd point of $\Phi$.*

The rest of this section is devoted to the question of existence of fixed points of $\Psi$. We have to impose some restrictions on the nonlinearity $b$.

**Assumption 3.5.** *Let $b$ a function of class $C^1(\mathbb{R})$. The first derivative is bounded from above,*

$$-b'(x) \geq \alpha_1 \qquad \forall x \in \mathbb{R},$$

*to meet the requirement* (A2).

This allows us to prove

**Lemma 3.6.** *The mapping $\Psi$ is compact from $C(\bar{Q}_\tau)$ to $C(\bar{Q}_\tau)$.*

*Proof.* Let $Y$ be a bounded subset of $C(\bar{Q}_\tau)$. From the form of the assumptions on $b$ it follows that the sets of all possible $\alpha$ and $f$,

$$\mathcal{A} = \{-b'(\bar{y}) : \bar{y} \in Y\}, \qquad \mathcal{F} = \{b(\bar{y}) - b'(\bar{y})\bar{y} : \bar{y} \in Y\},$$

are bounded in $L^\infty(\Sigma_\tau)$ and $L^{\tilde{s}}(0,\tau; L^s(\Gamma))$, respectively. Let $\bar{y} \in Y$ with associated $\alpha$ and $f$. The mapping from $(\alpha, f) \in \mathcal{A} \times \mathcal{F}$ to $(u^*, y^*) \in L^{\tilde{s}}(0,\tau; L^s(\Gamma)) \times W_p^{2\sigma}(\Omega)$ is continuous and bounded by Lemma 3.2 and 3.3. The latter one gives additionally compactness of the mapping $(\alpha, f) \mapsto y^*$. Therefore, for all $\bar{y} \in Y$ the fixed point states $y^*$ belongs to a precompact set. The state $y = \Psi(\bar{y})$ is the solution of an evolution problem depending on $\alpha$, $u^*$, $y^*$, $y = S_\alpha y^* + V_\alpha u^*$. From Lemma 2.7 and 2.8 we conclude that $y$ belongs to a precompact set in $C\left([0,\tau], W_p^{2\sigma}\right)$. That means, $\Psi(Y)$ is precompact in $C\left([0,\tau], W_p^{2\sigma}\right)$. Since $W_p^{2\sigma}(\Omega) \hookrightarrow C(\bar{\Omega})$, the imbedding $C\left([0,\tau], W_p^{2\sigma}\right) \hookrightarrow C\left([0,\tau], C(\bar{\Omega})\right) = C(\bar{Q}_\tau)$ is continuous. Therefore, $\Psi(Y)$ is precompact in $C(\bar{Q}_\tau)$, too. $\square$

At last, we are looking for a bounded set $Y \subset C(\bar{Q}_\tau)$ which is mapped by $\Psi$ on itself. To this aim, the maximum principle comes into play. Let $\bar{y} \in C(\bar{Q}_\tau)$ be given, with associated $y^* \in W_p^{2\sigma}(\Omega)$ and $u^* \in U_{ad,\tau}$. Then $y = \Psi(\bar{y})$ is the solution of the time-periodic system

$$\begin{aligned} y_t - \Delta y + y &= 0 && \text{in } Q_\tau, \\ \partial_n y - b'(\bar{y})y &= b(\bar{y}) - b'(\bar{y})\bar{y} + u^* && \text{on } \Sigma_\tau, \\ y(0) &= y(\tau), \end{aligned} \tag{3.15}$$

The maximum principle, cf. Remark 2.2, yields then

$$\vartheta_1 \le y(x,t) \le \vartheta_2 \qquad \forall (x,t) \in \bar{Q}_\tau, \tag{3.16}$$

where $\vartheta_2$ is given by

$$\vartheta_2 = \operatorname*{ess\,sup}_{(x,t)\in\Sigma_\tau} \left( \bar{y}(x,t) - \frac{b(\bar{y}(x,t))}{b'(\bar{y}(x,t))} - \frac{u^*(x,t)}{b'(\bar{y}(x,t))} \right),$$

and $\vartheta_1$ by an analogous formula. We use the following ansatz for the set $Y$. We set

$$Y = \{ y \in C(\bar{Q}_\tau) : \, y_1 \le y(x,t) \le y_2 \ \forall (x,t) \in \bar{Q}_\tau \}, \tag{3.17}$$

where $y_1$, $y_2$ are constants satisfying $y_1 \le 0 \le y_2$. Clearly, $Y$ is non-empty, convex, bounded, and closed in $C(\bar{Q}_\tau)$. It remains to develop conditions on the nonlinearity $b$ and the bounds of $Y$ and $U_{ad,\tau}$ to ensure that $\Psi$ maps $Y$ onto itself. For convenience, we define

$$u_1 := \operatorname*{ess\,inf}_{x\in\Gamma} u_a(x), \qquad u_2 := \operatorname*{ess\,sup}_{x\in\Gamma} u_b(x).$$

Henceforth, we suppose, that the following conditions are fulfilled.

**Assumption 3.7.** *The nonlinearity $b$ satisfies together with the bounds $y_1$, $y_2$ and $u_1$, $u_2$:*

*(i)* $-\alpha_2 \le b'(x) \le -\alpha_1 < 0$ *for all* $x \in [y_1, y_2] \subset \mathbb{R}$,

*(ii)* $b'(x)x - b(x) \le 0 \quad \forall x > 0$, $b(0) = 0$, *respectively* $b'(x)x - b(x) \ge 0 \quad \forall x < 0$.

*(iii)* $y_1 \le 0 \le y_2$,

*(iv)* $\frac{\alpha_1^2}{\alpha_2} y_1 \le u_1 \le 0 \le u_2 \le \frac{\alpha_1^2}{\alpha_2} y_2$.

Assuming this allows us to prove

**Proposition 3.8.** *Let $y \in [y_1, y_2] \subset \mathbb{R}$ be given. Then $\tilde{y}$, defined by*

$$\tilde{y} = y - \frac{b(y)}{b'(y)} - \frac{u}{b'(y)},$$

*satisfies $\tilde{y} \in [y_1, y_2]$ for all $u \in [u_1, u_2]$.*

*Proof.* At the beginning, notice that $b$ is monotone decreasing, thus $b(y_1)$, $-b(y_2)$, and $-b'(y)$ are positive by *(i), (ii), (iii)*. We regard first the case $y_2 \ge y \ge 0$. Then it follows

$$\tilde{y} = y - \frac{b(y)}{b'(y)} - \frac{u}{b'(y)} \overset{(ii)}{\ge} -\frac{u}{b'(y)} = \frac{u}{-b'(y)} \ge \frac{u_1}{-b'(y)} \overset{(i)}{\ge} \frac{u_1}{\alpha_1} \overset{(iv)}{\ge} y_1.$$

By integrating, *(i)* yields $b(y) \le -y\alpha_1$ for all $y > 0$. The other direction is estimated using *(i), (ii), (iv)* as

$$\tilde{y} = y - \frac{b(y)}{b'(y)} - \frac{u}{b'(y)} \le y + \frac{y\alpha_1}{b'(y)} + \frac{u_2}{-b'(y)} \le y\left(1 - \frac{\alpha_1}{\alpha_2}\right) + \frac{u_2}{\alpha_1} \le y_2\left(1 - \frac{\alpha_1}{\alpha_2}\right) + \frac{u_2}{\alpha_1} \le y_2.$$

The proof of the claim for $y_1 \le y \le 0$ can be done similarly. ∎

With the help of the previous lemma, we can conclude that the set $Y$ is mapped by $\Psi$ on itself.

**Lemma 3.9.** *Let $Y \subset C(\bar{Q}_\tau)$ be given by* (3.17). *Suppose, that Assumption 3.7 is satisfied. Then for all $\bar{y} \in Y$ and $u \in U_{ad,\tau}$ the solution $y$ of the time-periodic system* (3.15) *belongs to $Y \cap C\left([0,\tau], W_p^{2\sigma}\right)$.*

*Proof.* Since $\alpha = -b'(\bar{y})$ satisfy (A2), the existence of a unique solution $y \in C\left([0,\tau], W_p^{2\sigma}\right)$ of (3.15) follows from Corollary 2.11. This solution belongs to $C(\bar{Q}_\tau)$, because $C\left([0,\tau], W_p^{2\sigma}\right) \hookrightarrow C(\bar{Q}_\tau)$. The maximum principle (3.16) together with Proposition 3.8 yields

$$y(x,t) \leq \operatorname*{ess\,sup}_{(x,t)\in\Sigma_\tau} \left( \bar{y}(x,t) - \frac{b(\bar{y}(x,t))}{b'(\bar{y}(x,t))} - \frac{u(x,t)}{b'(\bar{y}(x,t))} \right) \leq y_2 \qquad \forall (x,t) \in \bar{Q}_\tau.$$

In the same way, $y(x,t) \geq y_1$ can be shown. Thus, it holds $y \in Y$. □

Finally, we can conclude the existence of at least one fixed point of $\Psi$.

**Theorem 3.10.** *Let $Y$ be as in* (3.17). *Further suppose, Assumption 3.7 holds. Then the mapping $\Psi$ has a fixed point $\tilde{y} \in Y \cap C\left([0,\tau], W_p^{2\sigma}\right)$.*

*Proof.* By Lemma 3.6, the mapping $\Psi$ is compact from $C(\bar{Q}_\tau)$ to $C(\bar{Q}_\tau)$. The set $Y$ is bounded, closed, and convex in $C(\bar{Q}_\tau)$. Thus, $\Psi(Y)$ is precompact in $C(\bar{Q}_\tau)$. Lemma 3.9 yields $\Psi(Y) \subset Y$. Hence, Schauder's fixed point theorem ensures the existence of at least one fixed point $\tilde{y}$ of $\Psi$. Since $\tilde{y}$ is the solution of a time-periodic system, it holds $y \in C\left([0,\tau], W_p^{2\sigma}\right)$ by Corollary 2.11. □

This results together with Lemma 3.4 gives the existence of generalized fixed points of the instantaneous mapping $\Phi$.

**Theorem 3.11.** *The mapping $\Phi$, introduced in Section 1.1, has at least one generalized fixed point $y_f$.*

This theorem claims only existence of generalized fixed points and says nothing about convergence of the instantaneous method. One can establish some analysis to show local convergence of the method to such a generalized fixed point under further assumptions. However, the numerical results show that control and state in the instantaneous method will converge even if it was started far away from the fixed point.

# 4 Improved suboptimal methods

## 4.1 Optimal fixed point approach

In this section, we want to deal with some improvements of the instantaneous control method. We will describe why this is necessary. Let $y_f$ be a generalized fixed point of $\Phi$ with associated fixed point control $u_f$. Then $y_f = y(x,\tau)$, where $y$ solves the time-periodic equation (3.14). But this state is not the best possible among the class of functions satisfying the system (3.14) for given control $u \in U_{ad,\tau}$. To show this, consider the optimal control problem

$$\min J_\tau(u,y) = \frac{1}{2}\int_\Omega (y(x,\tau) - y_d(x))^2 dx + \frac{\gamma}{2}\int_{\Sigma_\tau} u(x,t)^2 dt dS_x \tag{$P_{tp}$}$$

subject to $u \in U_{ad,\tau}$ and the time-periodic nonlinear system

$$\begin{aligned} y_t - \Delta y + y &= 0 && \text{in } Q_\tau, \\ \partial_n y &= b(y) + u && \text{on } \Sigma_\tau, \\ y(0) &= y(\tau). \end{aligned} \tag{4.1}$$

By Remark 2.3, this equation admits for every admissible control $u$ a unique solution $y \in C\left([0,T], W_p^{2\sigma}\right)$, so that the optimal control problem $(P_{tp})$ is well-defined. One can show that the solution mapping $u \mapsto y$ is even Lipschitz.

Let $\hat{u}$ be the optimal control of $(P_{tp})$ with associated optimal state $\hat{y}$. Set $\alpha := -b'(\hat{y})$ and $f := b(\hat{y}) - b'(\hat{y})\hat{y}$. Then it holds, confer (3.5),

$$\hat{y}(\tau) = (I - S_\alpha(\tau))^{-1} \left(V_\alpha(f + \hat{u})\right)(\tau).$$

Moreover, $\hat{u}$ satisfies necessarily the variational inequality

$$\left((I - S_\alpha(\tau))^{-1} V_\alpha(f + \hat{u})(\tau) - y_d,\, (I - S_\alpha(\tau))^{-1} V_\alpha(u - \hat{u})(\tau)\right) + \gamma(\hat{u}, u - \hat{u}) \geq 0,$$

for all $u \in U_{ad,\tau}$. The fixed point control $u_f$ solves a variational inequality of different type, compare (3.6). Thus $u_f$ and $\hat{u}$, respectively $y_f$ and $\hat{y}$ can barely be the same functions. That means, if the instantaneous method converges to a generalized fixed point, then the fixed point state is not the best approximation of $y_d$ in the class of all possible solutions of the nonlinear time-periodic equation (3.14). As a consequence, one can expect poor results of the standard instantaneous control method. This is what the numerical experiments indicate.

So we are led to improve the method in the following way: First, compute the optimal control $\hat{u}$ and state $\hat{y}$ of $(P_{tp})$. Then, apply the standard instantaneous control method for the changed desired function $\hat{y}_d := \hat{y}$ to obtain the suboptimal control. One can verify easily that $\hat{y}$ is a generalized fixed point of the instantaneous mapping $\Phi$ in the case $\gamma = 0$. In our numerical experience, this improved method works as expected, and convergence to $\hat{y}$ occurs.

## 4.2 Receding horizon method

A second way to better the standard instantaneous control method is to give it more insight in the future. So far, the suboptimal approaches minimizes in step $j$ a functional over a time horizon $[t_{j-1}, t_j]$ of length $\tau$ to compute the control on the same time interval. The state at time $t_j$ is the starting point for the next step. The idea of the well-known method of receding horizon is to enlarge the optimization horizon by an additional time $\tau_2$ to $[t_{j-1}, t_j + \tau_2]$. Again, the so-obtained control gives the suboptimal control on $[t_{j-1}, t_j]$, and the state of the system at time $t_j$ is used as the initial state for the next step. This means, the goal functional is minimized over a larger time interval, but the procedure to define the control is the same as in the standard instantaneous control method. In each step $j$, $j = 1 \ldots M$, the following optimization problem is solved:

$$\min J_{rh}(u,y) = \frac{1}{2}\int_\Omega (y(x,\delta) - y_d(x))^2 dx + \frac{\gamma}{2}\int_{\Sigma_\delta} u(x,t)^2 dt dS_x \qquad (P_{rh})$$

subject to $u \in U_{ad,\delta}$ and the nonlinear system

$$\begin{aligned} y_t - \Delta y + y &= 0 && \text{in } Q_\delta, \\ \partial_n y &= b(y) + u && \text{on } \Sigma_\delta, \\ y(0) &= y_{j-1}, \end{aligned} \tag{4.2}$$

where $\delta = \tau + \tau_2$ is the total length of the considered time interval. The state $y_j$ is defined as $y_j = y(\tau)$, where $y$ is the optimal state of the problem $(P_{rh})$. Clearly, if in step $M$ the horizon $t_M + \tau_2$ reaches the final time $T$, the optimal control of $(P_{rh})$ is taken to define the suboptimal control on the last time interval $[t_M, \min(t_M + \tau_2, T)]$. If $\tau_2 = 0$ this method reduces to the standard instantaneous control method.

As in the previous sections, we can find behind each step of the method a mapping $\Phi_{rh}$ defined by $y_j := \Phi_{rh}(y_{j-1})$. The numerical results indicate, that this mapping has a fixed point. Due to the same reasons as in Section 1.1, we introduce the notation of a generalized fixed point.

**Definition 4.1.** A function $y_r$ is said to be a generalized fixed point of $\Phi_{rh}$, if

(i) There exists an admissible control $u_r \in U_{ad,\delta}$, which satisfies the necessary optimality conditions of $(P_{rh})$ for $y(0) = y_r$.

(ii) It holds $y_r = y(\tau)$, where $y$ solves (4.2) with initial condition $y(0) = y_r$ and control $u = u_r$.

Following the lines of Section 3, one can show under some assumptions on $\tau$, $\tau_2$ that generalized fixed points of $\Phi_{rh}$ exists. Let $y$ be the state associated with a generalized fixed point $y_r$. Then it solves

$$\begin{aligned} y_t - \Delta y + y &= 0 && \text{in } Q_\delta, \\ \partial_n y &= b(y) + u_r && \text{on } \Sigma_\delta, \\ y(0) &= y(\tau), \end{aligned} \tag{4.3}$$

which is time-periodic only on the first part of the time interval $[0, \tau] \subset [0, \delta]$. We see that the generalized fixed point $y_r$ of $\Phi_{rh}$ belongs to the same class of functions like the fixed points of the standard instantaneous control method.

Suppose, the receding horizon method converges to a fixed point $y_r$ with associated control $u_r$ after $M - 1$ steps. Denote by $\bar{y}(t)$ the suboptimal state and by $\bar{u}(t)$ the suboptimal control at time $t \in [0, T]$. Then we have $\bar{y}(t_{M-1}) = y_r$. Suppose further, the time lengths $\tau$, $\tau_2$ are chosen such that $t_{M-1} + \tau + \tau_2 = t_M + \tau_2 = T$. Hence, the control $u_M$ computed in the last step is taken to define the suboptimal on the time interval $[t_{M-1}, T]$. Since $y_r$ is a generalized fixed point, we can conclude $u_M = u_r$ and $\bar{y}(t_M) = y_r$. The final state $\bar{y}(T)$ we are interested in is equal to $y(\delta)$, where $y$ solves (4.3). As in the previous section this state is not the best approximation of $y_d$ in the class of all possible solutions of (4.3). This will be the starting point of a improved approach in the next section.

The numerical experiments show a further disadvantage of this method. It tends to produce high control costs. Additionally, the numerical effort increases with $\tau_2$. The larger $\tau_2$ is, the larger optimization problems has to be solved in each step.

## 4.3 Hybrid approach

Here, we propose a hybrid method which can cope with the drawbacks of the receding horizon method. At first, the new method chooses an optimal fixed point, as was done in Section 4.1 for the standard instantaneous method. In other words, it will look for the optimal control and state of the system (4.3).

Secondly, instead of performing in each step an optimization over the enlarged time horizon, the method will optimize over the small time horizon as the standard instantaneous method to reduce the computational effort. And finally, the cost of the control will be adapted to the method.

To begin with, consider the minimization problem

$$\min J_\delta(u, y) = \frac{1}{2} \int_\Omega (y(x, \delta) - y_d(x))^2 dx + \frac{1}{2} \int_{\Sigma_\delta} \hat{\gamma}(t) u(x, t)^2 dt dS_x \tag{$P_{hy}$}$$

subject to $u \in U_{ad,\delta}$ and the hybrid nonlinear system

$$\begin{aligned} y_t - \Delta y + y &= 0 && \text{in } Q_\delta, \\ \partial_n y &= b(y) + u && \text{on } \Sigma_\delta, \\ y(0) &= y(\tau), \end{aligned} \tag{4.4}$$

where $\tau_2$, $\delta$ are as in the previous section. The function $\hat{\gamma}(t)$ measures the control cost and will be specified later.

Hybrid means that system (4.4) is the interlinking of a time-periodic part and a 'normal' parabolic part with interface at time $t = \tau$. Its unique solvability is a consequence of the solvability of the time-periodic part, cf. Section 4.1, and the parabolic part.

This optimization problem is used to establish a further suboptimal method. We want to compute a suboptimal control for the optimization problem $(P)$. First, choose $\tau$ and $\tau_2$, such that with $M \in \mathbb{N}$ we have $T = M\tau + \tau_2$. $\tau$ plays the role of the stepsize in the standard instantaneous control method. Secondly, we solve the problem $(P_{hy})$ and get the optimal control $\hat{u}$ and state $\hat{y}$. Denote by $\hat{y}_1$ the optimal state at time $\tau$, $\hat{y}_1 := \hat{y}(\tau)$. Then $\hat{y}_1$ is the solution of a time-periodic problem and belongs therefore to the same function class as the generalized fixed points of the standard instantaneous method. Hence, we set $\hat{y}_d := \hat{y}_1$ and use the standard instantaneous control method with changed desired function $\hat{y}_d$ to compute the control on the time interval $[0, T - \tau_2]$. On the remaining interval $(T - \tau_2, T]$, the control is defined by performing one optimization with goal function $y_d$.

Now, we can define the function $\hat{\gamma}$. The hybrid method will perform $M$ instantaneous steps. The control in each step will result approximately the same costs as the fixed point control $\hat{u}$ on $[0, \tau]$. Thus, the cost parameter on this time interval is multiplied by the number of steps.Therefore, we define

$$\hat{\gamma}(t) = \begin{cases} M\,\gamma & \text{for } 0 \le t < \tau \\ \gamma & \text{for } \tau \le t \le \delta. \end{cases}$$

The numerical results shows the priority of this method over all other suboptimal approaches studied in this paper.

# 5 Numerical results

## 5.1 Example 1

Here, we present results confirming the theory developed above. We consider the following simple optimal control problem for the one-dimensional heat equation:

$$\min J(u, y) = \frac{1}{2}\int_0^1 (y(x, T) - y_d(x))^2 dx + \frac{\gamma}{2}\int_0^T u(t)^2 dt \tag{$P_1$}$$

subject to

$$\begin{aligned} y_t(x,t) - \Delta y(x,t) + y(x,t) &= 0 \qquad\qquad \text{for } x \in (0,1), \\ \partial_n y(0,t) &= 0, \\ \partial_n y(1,t) &= b(y(1,t)) + u(t), \\ y(x,0) &= y_0(x), \end{aligned} \tag{5.1}$$

and

$$-1 \le u(t) \le 1,$$

$t \in [0, T]$, $\Omega = (0, 1)$. In order to solve this problem numerically, it has to be discretized. We used finite-differences for spatial discretization and the implicit-Euler method for time-integration. The domain $\Omega = (0, 1)$ was divided in $N_x$ subintervals. The time grid is an equidistant mesh of $N_t$ points. The control is defined on a coarser grid of $U_t$ points, were we used piecewise constant controls. All results presented here were calculated with $N_x = 100$, $N_t = 5000$, $U_t = 1000$.

The parameters to set up the optimization problem were chosen as

$$\nu = 0.01, \quad u_a = -1, \quad u_b = +1, \quad y_0(x) = 0, \quad y_d(x) = (1 - x^2)/2.$$

The final time $T$ we fix as $T = 10$. The nonlinearity $b$ is given as

$$b(y) = -|y|y.$$

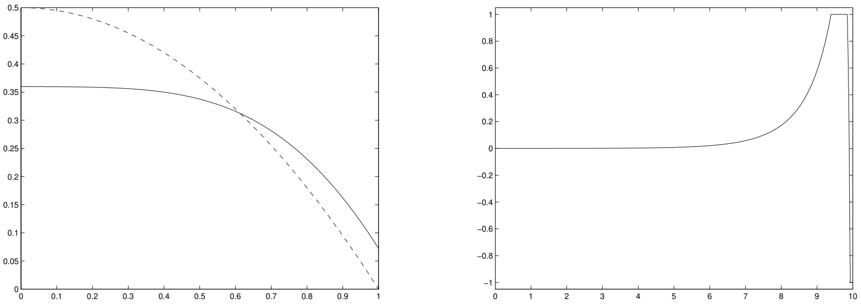

Figure 1: Optimal control and final state

At first, we present the optimal control for the problem (4.1). It was computed using the SQP-method to cope with the nonlinearity of the problem. The linear-quadratic subproblems arising in this method were solved by a primal-dual active set algorithm, cf. [15]. Optimal control and final state are shown in Figure 1. The optimal value of the objective is $J(u^*, y^*) = 0.00815$.

Next, we demonstrate how the instantaneous method fails. We choose the length of the short horizon $\tau = T/U_t = 0.01$. Thus, the control is taken to be constant on the small subintervals, on which the instantaneous method works. So, in each step one-dimensional optimization problems has to be solved. It turns out, that this method could be implemented very efficiently. The results obtained this way are shown in Figure 2. They confirm convergence

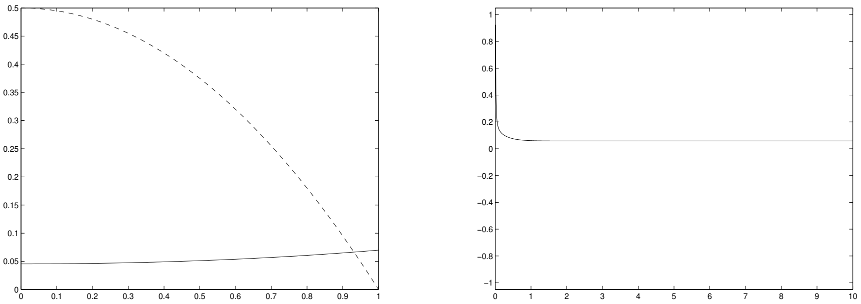

Figure 2: Instantaneous control: Suboptimal control and final state

of this method toward a fixed point. The value of the objective is $J = 0.05166$, which is far away from the optimum.

As mentioned above, the computed fixed point is not the best in the class of all functions satisfying the time-periodic equation (3.14), cf. Section 4.1. There, the first improved method was suggested. Its first step is the computation of the optimal fixed point pair $(\hat{u}, \hat{y})$ as solution of the time-periodic optimal control problem $(P_{tp})$. Then, the desired state is changed to $\hat{y}_d := \hat{y}$. The suboptimal control is computed using the standard instantaneous method. The result of this approach can be found in Figure 3. The value of the objective is reduced significantly to $J = 0.02755$ compared with the result of the instantaneous method.

The results of the receding horizon method are shown in Figure 4. The additional time $\tau_2$ was set to $\tau_2 = 9\,\tau$. So, in each step a 10-dimensional optimization has to be done. In the right-hand figure the control on the time interval $[9.85, 10]$ can be seen, where in the last step of the method all 10 components of the computed control are taken to define the suboptimal control. This results in a good value of the objective of $J = 0.02784$. But the computational effort is even in the order of the effort to compute the optimal control. So, this method is not practicable to use.

The hybrid method proposed in 4.3 gives the best results among the suboptimal ways

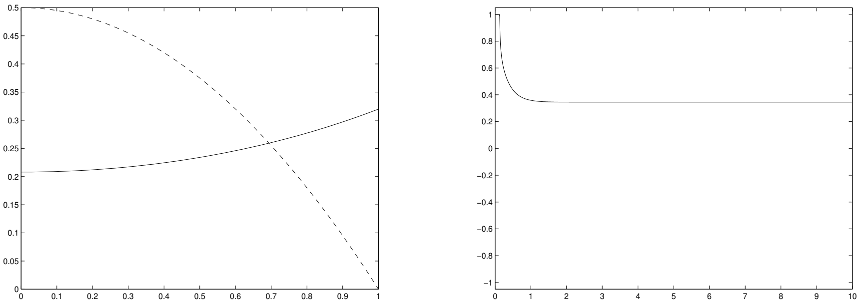

Figure 3: Optimal fixed point approach: Suboptimal control and final state

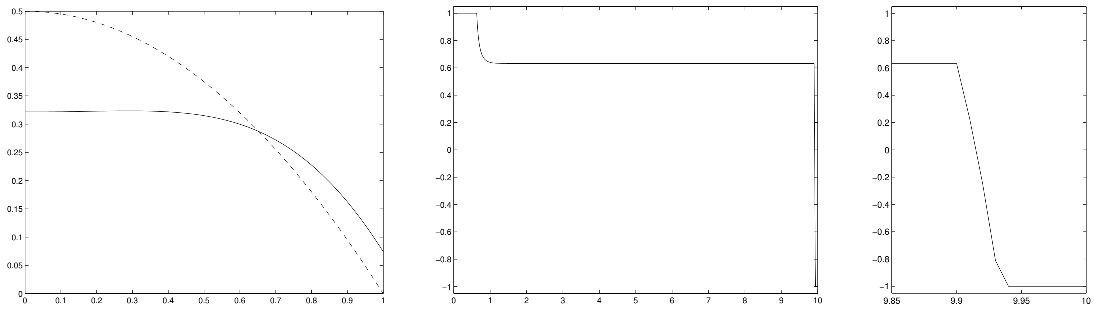

Figure 4: Receding horizon method: Suboptimal control and final state

presented here. First, the hybrid problem $(P_{hy})$ is solved, which gives an optimal fixed point $(u_f, y_f)$. Then, the goal function is changed to $y_f$. The control on the first part of the time interval is computed by the standard instantaneous method, i.e. in each step only a one-dimensional problem is solved. We set again $\tau_2 = 9\,\tau$, so on the last interval, we got a 10-dimensional problem to solve. Its results in the control seen in the right-hand plot of Figure 5. The value of the goal functional was $J = 0.02241$. This is not a significant reduction in comparison with the result of the receding horizon method. But the computational cost are lesser thanks to the fact that in each step only a one-dimensional optimization problem has to be solved.

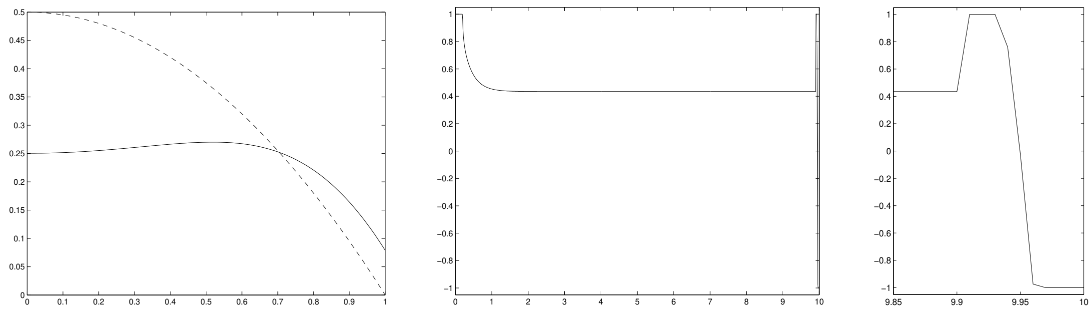

Figure 5: Hybrid method: Suboptimal control and final state

At last, we will present a summary of the results of all the suboptimal methods in Table 1. The columns labelled with NCC contain the normed computational costs. They count how often a linear parabolic problem over $[0, T]$ was solved. As one can see, the hybrid method yields the best results compared with the other suboptimal methods, and its computational effort is much smaller than the effort to solve the optimal control problem.

In this table, also results for an optimal control problem with another nonlinearity,

$$b(y) = -|y|y^3$$

are shown, and refered to as Example 2. All other parameters remain unchanged.

| | Example 1 | | Example 2 | |
|---|---|---|---|---|
| Method | $J(u)$ | NCC | $J(u)$ | NCC |
| Optimal | 0.00815 | 344.0 | 0.00636 | 616.0 |
| Instantaneous | 0.05166 | 8.0 | 0.05162 | 7.9 |
| Optimal fixed point | 0.02755 | 8.9 | 0.02433 | 8.8 |
| Receding horizon | 0.02784 | 160.3 | 0.01795 | 151.5 |
| Hybrid | 0.02241 | 21.8 | 0.01687 | 21.8 |

Table 1: Comparison of the suboptimal methods

# Appendix

The following two Lemmata are consequences of the well-known interpolation inequality of $L^p$-spaces, confer [5]. The proofs are repeated for the convenience of the reader.

**Lemma A.1.** *Let a function $u \in L^s(\Omega)$ be given. Then it holds for $p \geq 2$*

$$|u|_{L^p(\Omega)} \leq |u|_{L^2(\Omega)}^{\frac{1}{p}} |u|_{L^s(\Omega)}^{\frac{p-1}{p}},$$

*where $s$ is defined as $2(p-1)$.*

*Proof.* We use Hölders inequality to show

$$|u|_p^p = \int_\Omega |u|^p dx = \int_\Omega |u|^{p-1}|u| dx \leq \left(\int_\Omega |u|^2 dx\right)^{\frac{1}{2}} \left(\int_\Omega |u|^{2(p-1)} dx\right)^{\frac{1}{2}} = |u|_2 |u|_{2(p-1)}^{p-1}.$$

The claim follows immediately. $\square$

**Lemma A.2.** *Let a function $u \in L^{\tilde{s}}(0,T;L^s(\Gamma))$ be given, where $s = 2(p-1)$ and $\tilde{s} = \frac{2\nu(p-1)}{2p-\nu}$. Then it holds for $2p > \nu \geq 2$*

$$|u|_{L^\nu(0,T;L^p(\Gamma))} \leq |u|_{L^2(\Sigma)}^{\frac{1}{p}} |u|_{L^{\tilde{s}}(0,T;L^s(\Gamma))}^{\frac{p-1}{p}}$$

*Proof.* For convenience we abbreviate $|u|_q := |u|_{L^q(\Gamma)}$ and $|u|_{\tilde{q},q} := |u|_{L^{\tilde{q}}(0,T;L^q(\Gamma))}$. By the previous Lemma A.1 we have

$$|u|_{\nu,p}^\nu = \int_0^T |u|_p^\nu dt \leq \int_0^T |u|_2^{\frac{\nu}{p}} |u|_s^{\frac{\nu(p-1)}{p}} dt.$$

Using Hölder's inequality with $q = 2p/\nu > 1$ and $q' = 2p/(2p-\nu)$, so that $1/q + 1/q' = 1$, we obtain

$$|u|_{\nu,p}^\nu \leq \int_0^T |u|_2^{\frac{\nu}{p}} |u|_s^{\frac{\nu(p-1)}{p}} dt \leq \left(\int_0^T |u|_2^2 dt\right)^{\frac{\nu}{2p}} \left(\int_0^T |u|_s^{\frac{2\nu(p-1)}{2p-\nu}} dt\right)^{\frac{2p-\nu}{2p}} = |u|_{2,2}^{\frac{\nu}{p}} |u|_{\tilde{s},s}^{\frac{\nu(p-1)}{p}},$$

which is the claim. $\square$

With obvious modifications of the proof, this result remains true for $2p = \nu$ and $\tilde{s} = \infty$.

**Remark A.1.** One should add, that there exists exponents $m$, $\nu$, $p$, $\sigma$, $\sigma'$ such that the assumptions (A1), (A3), (A4) are satisfied. One can show, that

$$m \leq 3, \quad \nu = 6, \quad p = 4, \quad \sigma = \frac{19}{48}, \quad \sigma' = \frac{7}{12}$$

meet those conditions. The exponents $\tilde{s}$, $s$ defining the control space are in this case

$$\tilde{s} = 18, \quad s = 6.$$


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