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Analysis and Reformulation of Linear Delay Differential-Algebraic Equations*

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Abstract

In this paper, we study general linear systems of delay differential-algebraic equations (DDAEs) of arbitrary order. We show that under some consistency conditions, every linear high-order DAE can be reformulated as an underlying high-order ordinary differential equation (ODE) and that every linear DDAE with single delay can be reformulated as a high-order delay differential equation (DDE). We derive condensed forms for DDAEs based on the algebraic structure of the system coefficients, and use these forms to reformulate DDAEs as strangeness-free systems, where all constraints are explicitly available. The condensed forms are also used to investigate structural properties of the system like solvability, regularity, consistency and smoothness requirements.

Keywords: Delay differential-algebraic equation, differential-algebraic equation, strangeness-index, regularization, index reduction.
AMS Subject Classification: 34A09, 34A12, 65L05, 65H10

1 Preliminary and notations

In this paper we study general *linear delay differential-algebraic equations (DDAEs)* of the form

$$A_k x^{(k)}(t) + \cdots + A_0 x(t) + A_{-1} x(t-\tau) + \cdots + A_{-\kappa} x^{(\kappa)}(t-\tau) = f(t), \tag{1.1}$$

where the coefficients satisfy $A_i \in \mathbb{C}^{\ell,n}$, $i = -k, \ldots, \kappa$, $A_k \neq 0$, $f : [0, \infty) \to \mathbb{C}^\ell$, and where $\tau > 0$ is a single constant delay. We consider the time interval $t \in [0, \infty)$. Note that most of our analysis also carries over to multiple and nonconstant delays but here we restrict ourselves to the constant single delay case.

An important special case of (1.1) is the initial value problem for a first order linear delay differential-algebraic equation with single delay

$$A_1 \dot{x}(t) + A_0 x(t) + A_{-1} x(t-\tau) = f(t), \tag{1.2}$$

where A_1, A_0, $A_{-1} \in \mathbb{C}^{\ell,n}$, $f : [0, \infty) \to \mathbb{C}^\ell$.

To achieve uniqueness of solutions, for DDAEs one typically has to prescribe initial functions, which for the special case (1.2) take the form

$$x|_{[-\tau,0]} = \phi : [-\tau, 0] \to \mathbb{C}^n, \tag{1.3}$$

Ordinary delay differential equations (DDEs) of the form (1.2), with A_1 being the identity matrix, arise in various applications, see [1, 2, 6, 7] and the references therein. If the states of

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the physical system are constrained, e. g., by conservation laws or interface conditions, then algebraic equations have to be included and one has to analyze delay differential-algebraic equations (DDAEs). DDAEs may be considered from two different perspectives. On the one hand, they are differential-algebraic equations (DAEs) that involve delayed terms. On the other hand, DDAEs are ordinary delay differential equations (DDEs) subject to constraints that also may involve time-delayed variables. Of course, DDAEs inherit all the difficulties that are associated with both DAEs and DDEs. Their interaction, however, leads to new effects that do not arise in either DAEs or DDEs, as has been pointed out in [1, 6].

Although DDEs are well studied analytically and numerically, see e. g. [2, 7], and a similar maturity has been reached for the simulation and control of DAEs, see e. g. [3, 9, 10], the theoretical understanding and the development of appropriate numerical methods for DDAEs, however, is far from complete even for the case of linear systems with constant coefficients. Only very few results are available, see e. g., [1, 4, 5], and these are mainly for the special case of DAEs, where the delay component is nothing else than an additional part of the inhomogeneity.

The main difficulty so far is the lack of a suitable regularity analysis (via the concept of an index) and a canonical form which allows to investigate structural properties like existence, uniqueness of solutions, consistency and smoothness requirements for the initial function.

In this paper, we derive such a canonical form for the linear constant coefficient case by extending the algebraic approach introduced in [10, 14] and combining it with the behavior approach [13]. Surprisingly, already in order to deal with (1.2), it is necessary to study linear high-order differential-algebraic equations of the form

$$A_k x^{(k)}(t) + \cdots + A_1 \dot{x}(t) + A_0 x(t) = f(t), \tag{1.4}$$

with associated initial conditions of the form

$$x^{(k)}(0) = x_0^k, \ldots, \dot{x}(0) = x_0^1,\ x(0) = x_0^0. \tag{1.5}$$

We study the theoretical aspects of (1.4)-(1.5) in Section 3 and then use these to study the general case of DDAEs in Section 4. The analysis is based on reformulation procedures which bring the systems into a *strangeness-free* form and allows also to study theoretical aspects like existence and uniqueness of solutions, as well as the consistency and smoothness requirements for the initial functions.

2 Notation and Preliminaries

In the following, we denote by $I_n \in \mathbb{C}^{n,n}$ (or I) the identity matrix and by A^T the transpose of a matrix A. For an interval $\mathbb{I} \subset [0, \infty)$, by $C^k(\mathbb{I}, \mathbb{C}^n)$ we denote the space of k-times continuously differentiable functions from $\mathbb{I}$ to $\mathbb{C}^n$.

We use the following solution concept.

Definition 2.1. A function $x : [0, \infty) \to \mathbb{C}^n$ is called *(classical) solution* to (1.2) (resp. (1.4)) if $x \in C^1([0, \infty), \mathbb{C}^n)$ and x satisfies (1.2) (resp., (1.4)) pointwise. An initial function ϕ is called *consistent* with system (1.2) if the associated initial value problem (1.2)–(1.3) has at least one classical solution. System (1.2) is called *solvable* if for every sufficiently smooth f and every consistent initial function ϕ, the associated initial value problem (1.2)–(1.3) has a solution. It is called *regular* if it is solvable and the solution is unique.

Introducing as $X_0 := \begin{bmatrix} x_0^{(k)} \\ \vdots \\ x_0 \end{bmatrix}$ *the initial vector* of the initial value problem consisting of (1.4)–(1.5), Definition 2.1 extends to higher order systems, i. e., an initial vector $X_0 \in \mathbb{C}^{(k+1)\ell}$ is called *consistent* for system (1.4) if the initial value problem (1.4)–(1.5) has a solution, and system (1.4) is called *solvable* if for every sufficiently smooth f and every consistent initial vector X_0, the associated initial value problem (1.4)–(1.5) has a solution.

For matrices $Q \in \mathbb{C}^{q,n}$, $P \in \mathbb{C}^{p,n}$, the matrix pair (Q, P) is said to *have no hidden redundancy* if

$$\operatorname{rank}\left(\begin{bmatrix} Q \\ P \end{bmatrix}\right) = \operatorname{rank(Q)} + \operatorname{rank(P)}.$$

Lemma 2.2. *Suppose that for $Q \in \mathbb{C}^{q,n}$, $P \in \mathbb{C}^{p,n}$, the pair (Q, P) has no hidden redundancy. Then, for any matrix $U \in C^{q,q}$ and any $V \in C^{p,p}$, the pair (UQ, VP) has no hidden redundancy.*

Proof. The proof follows from the observation that a matrix pair has no hidden redundancy if and only if the intersection of the two vector spaces spanned by the rows of the two matrices contains only the vector 0. □

If $\begin{bmatrix} Q \\ P \end{bmatrix}$ is of full row rank for two matrices $Q \in \mathbb{C}^{q,n}$, $P \in \mathbb{C}^{p,n}$, then obviously, the pair (Q, P) has no hidden redundancy. However, the converse is not true as is obvious for $Q = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $P = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, since (Q, P) has no hidden redundancy, but $\begin{bmatrix} Q \\ P \end{bmatrix}$ does not have full row rank.

Lemma 2.3. *For $Q \in \mathbb{C}^{q,n}$, $P \in \mathbb{C}^{p,n}$, there exists*

$$\begin{bmatrix} S & 0 \\ Z_1 & Z_2 \end{bmatrix} \in \mathbb{C}^{q,q+p},$$

where $\begin{bmatrix} S \\ Z_1 \end{bmatrix} \in \mathbb{C}^{q,q}$ is nonsingular and the rows of $S \in \mathbb{C}^{p,q}$ are the rows of a permutation matrix such that

$$\begin{bmatrix} S & 0 \\ Z_1 & Z_2 \end{bmatrix} \begin{bmatrix} Q \\ P \end{bmatrix} = \begin{bmatrix} SQ \\ 0 \end{bmatrix},$$

and (SQ, P) has no hidden redundancy.

Proof. The proof follows by taking $[Z_1, Z_2]$ to be a full rank matrix spanning the left nullspace of $\begin{bmatrix} Q \\ P \end{bmatrix}$ and completing it to a full rank matrix by rows of a permutation matrix so that $\begin{bmatrix} S \\ Z_1 \end{bmatrix}$ is invertible and (SQ, P) has no hidden redundancy. □

Lemma 2.3 will be used later to recursively remove hidden redundancy in the coefficients of linear DAEs and DDAEs.

Lemma 2.4. *Consider $k + 1$ full row rank matrices $R_0 \in \mathbb{C}^{r_0,n}, \ldots, R_k \in \mathbb{C}^{r_k,n}$, such that none of the matrix pairs*

$$\left(R_j, \begin{bmatrix} R_{j-1} \\ \vdots \\ R_0 \end{bmatrix} \right), \ j = k, \ldots, 1 \tag{2.1}$$

has a hidden redundancy. Then, $\begin{bmatrix} R_k \\ \vdots \\ R_0 \end{bmatrix}$ has full row rank.

Proof. Since none of the matrix pairs in (2.1) has a hidden redundancy, it follows that

$$\begin{aligned}
\operatorname{rank}\left(\begin{bmatrix} R_k \\ \vdots \\ R_0 \end{bmatrix}\right) &= \operatorname{rank}(R_k) + \operatorname{rank}\left(\begin{bmatrix} R_{k-1} \\ \vdots \\ R_0 \end{bmatrix}\right), \\
&= \operatorname{rank}(R_k) + \operatorname{rank}(R_{k-1}) + \operatorname{rank}\left(\begin{bmatrix} R_{k-2} \\ \vdots \\ R_0 \end{bmatrix}\right), \\
&= \cdots \\
&= \operatorname{rank}(R_k) + \operatorname{rank}(R_{k-1}) + \cdots + \operatorname{rank}(R_0),
\end{aligned}$$

and since $R_k, \ldots, R_0$ have full row rank, also $\begin{bmatrix} R_k \\ \vdots \\ R_0 \end{bmatrix}$ has full row rank. □

3 Analysis and reformulations of high-order DAEs

In this section, we study the analysis of high-order DAEs of the form (1.4) and of the initial value problem (1.4)–(1.5), see also [11, 12, 15] and the references therein for previous work on this topic. We will extend these results by combining it with the regularization procedure for DAEs proposed in [14] in a behavior setting [13]. Let

$$M := [A_k, \ldots, A_0] \in \mathbb{C}^{\ell,(k+1)n} \text{ and } X(t) := \begin{bmatrix} x^{(k)}(t) \\ \vdots \\ x(t) \end{bmatrix}. \tag{3.1}$$

Then M (resp., $X(t)$) is called the *behavior matrix (resp., behavior vector)* of system (1.4), which can be written as

$$MX(t) = f(t). \tag{3.2}$$

Scaling (1.4) with a nonsingular matrix $P \in \mathbb{C}^{\ell,\ell}$, we obtain

$$PMX(t) = P\sum_{i=0}^{k} A_i x^{(i)}(t) = Pf(t). \tag{3.3}$$

For notational convenience, in the following we omit the argument t in X, x, f and their derivatives.

Since the systems (1.4) and (3.3) have the same solution spaces, we introduce the following definition.

Definition 3.1. Two behavior matrices $M = [A_k, \ldots, A_0]$ and $\tilde{M} = [\tilde{A}_k, \ldots, \tilde{A}_0]$ in $\mathbb{C}^{\ell,(k+1)n}$ are called *(strongly) left equivalent* (denoted by $\overset{\ell}{\sim}$) if there exists a nonsingular matrix $P \in \mathbb{C}^{\ell,\ell}$ such that $\tilde{M} = PM$ or equivalently,

$$\tilde{A}_j = PA_j, \; j = k, \ldots, 0.$$

Lemma 3.2. *Consider the behavior matrix M of system* (1.4)*. Then, M is left equivalent to a matrix*

$$\tilde{M} := \left[\begin{array}{c|c|c|c} A_{k,1} & A_{k-1,1} & \cdots & A_{0,1} \\ & A_{k-1,2} & \cdots & A_{0,2} \\ & & \ddots & \vdots \\ & & & A_{0,k+1} \\ \hline 0 & 0 & \cdots & 0 \end{array}\right], \quad \begin{array}{c} r_1 \\ r_2 \\ \vdots \\ r_{k+1} \\ \hline v \end{array} \tag{3.4}$$

where all the matrices $A_{k-j,j+1}$, $j = k, \ldots, 0$ on the main diagonal have full row rank.

Proof. We first compress the first block column of M via a QR-decomposition, see [8], to

$$\begin{array}{rcl} M & = & [A_k, \ldots, A_0], \\ & \stackrel{\ell}{\sim} & \left[\begin{array}{c|ccc} A_{k,1} & A_{k-1,1} & \ldots & A_{0,1} \\ 0 & A_{k-1,2} & \ldots & A_{0,2} \end{array}\right], \end{array}$$

such that $A_{k,1}$ has full row rank. Continuing, by compressing the 2nd block column from the second block row and then inductively the other columns of M, we finally arrive at (3.4). □

We call the number

$$r_u := (k+1)r_1 + kr_2 + \cdots + 2r_k + r_{k+1}.$$

the *upper rank* of the behavior matrix M. Note, that some of the r_i may vanish and obviously, the upper rank is invariant under left equivalence transformations.

In the following, without loss of generality, we assume that the behavior matrix M is already in the form $\tilde{M}$. Rewriting system (3.2) block row-wise, we obtain the system

$$\begin{array}{rcl} A_{k,1}x^{(k)} + A_{k-1,1}x^{(k-1)} + \cdots + A_{1,1}\dot{x} + A_{0,1}x & = & f_1, \\ A_{k-1,2}x^{(k-1)} + \cdots + A_{1,2}\dot{x} + A_{0,2}x & = & f_2, \\ & \cdots & \\ A_{0,k+1}x & = & f_{k+1}, \\ 0 & = & f_{k+2}. \end{array} \tag{3.5}$$

Recall that the diagonal blocks $A_{k,1}, A_{k-1,2}, \ldots, A_{0,k+1}$ have full row rank, therefore in system (3.5), for every J with $k \geq j \geq 0$, the $(k+1-j)$-th block row

$$A_{j,k+1-j}x^{(j)} + \cdots + A_{0,k+1-j}x = f_{k+1-j},$$

represents a number of scalar differential equations of order j. The idea now is to use differential equations of order smaller than j and their derivatives to reduce the number of scalar differential equations of order j. Let us illustrate this idea for the case $j = k$.

If the pair $\left(A_{k,1}, \begin{bmatrix} A_{k-1,2} \\ \vdots \\ A_{0,k+1} \end{bmatrix}\right)$ has hidden redundancy, then Lemma 2.3 implies that there exist a matrix

$$\left[\begin{array}{c|ccc} S_k & 0 & \ldots & 0 \\ \hline Z_{k,k} & Z_{k,k-1} & \ldots & Z_{k,0} \end{array}\right]$$

such that $\begin{bmatrix} S_k \\ Z_{k,k} \end{bmatrix} \in \mathbb{C}^{r_1,r_1}$ is nonsingular,

$$Z_{k,k}A_{k,1} + [Z_{k,k-1} \ldots Z_{k,0}] \begin{bmatrix} A_{k-1,2} \\ \vdots \\ A_{0,k+1} \end{bmatrix} = 0, \tag{3.6}$$

and the matrix pair $\left(S_kA_{k,1}, \begin{bmatrix} A_{k-1,2} \\ \vdots \\ A_{0,k+1} \end{bmatrix}\right)$ has no hidden redundancy.

Scaling the first equation of (3.5) with $\begin{bmatrix} S_k \\ Z_{k,k} \end{bmatrix}$ from the left we get

$$S_kA_{k,1}x^{(k)} + S_kA_{k-1,1}x^{(k-1)} + \cdots + S_kA_{1,1}\dot{x} + S_kA_{0,1}x = S_kf_1,$$

and

$$Z_{k,k}A_{k,1}x^{(k)} + Z_{k,k}\left(A_{k-1,1}x^{(k-1)} + \cdots + A_{0,1}x\right) = Z_{k,k}f_1.$$

>From (3.6), we deduce

$$
\begin{aligned}
Z_{k,k}A_{k,1}x^{(k)} &= -[Z_{k,k-1}\dots Z_{k,0}]\begin{bmatrix}A_{k-1,2}\\ \vdots \\ A_{0,k+1}\end{bmatrix}x^{(k)} \\
&= -\sum_{i=0}^{k-1} Z_{k,i}A_{i,k+1-i}x^{(k)} \\
&= -\sum_{i=0}^{k-1} Z_{k,i}\left(\frac{d}{dt}\right)^{k-i} A_{i,k+1-i}x^{(i)}, \\
&= -\sum_{i=0}^{k-1} \left(\frac{d}{dt}\right)^{k-i} Z_{k,i}\left(-\sum_{j=0}^{i-1} A_{j,j+1-i}x^{(j)} + f_i\right).
\end{aligned}
$$

This leads to the systems

$$
S_kA_{k,1}x^{(k)} + S_kA_{k-1,1}x^{(k-1)} + \cdots + S_kA_{1,1}\dot{x} + S^kA_1^0x + S_kf_1 = 0,
$$

and

$$
\begin{aligned}
&\sum_{i=0}^{k-1} Z_{k,i}\left(\frac{d}{dt}\right)^{k-i}\left(\sum_{j=0}^{i-1} A_{j,j+1-i}x^{(j)} - f_i\right) \\
&+Z_{k,k}\left(A_{k-1,1}x^{(k-1)} + \cdots + A_{0,1}x\right) = Z_{k,k}f_1.
\end{aligned}
\tag{3.7}
$$

Note that (3.7) is a set of differential equations of degree at most $k-1$. Hence, we have reduced the number of scalar differential equations of order k from $\operatorname{rank}(A_{k,1})$ to $\operatorname{rank}(S_kA_{k,1})$.

Applying the same argument to the block rows numbered $j = k-1, \dots, 1$, we obtain the following two lemmas. For notational convenience, we denote by $*$ unspecified matrices.

Lemma 3.3. *Consider the DAE* (1.4) *in its behavior form* (3.2). *Moreover, assume that the behavior matrix M is in the form* (3.4). *Then, there exist matrices S_j, $Z_{j,i}$, $j = m, \dots, 1$, $i = j, \dots, 0$ of appropriate size such that*

i) the matrices $\begin{bmatrix}S_j\\ Z_{j,j}\end{bmatrix} \in \mathbb{C}^{r_j,r_j}$, $k \geq j \geq 1$ *are nonsingular,*

ii) for each j with $k \geq j \geq 1$,

$$
Z_{j,j}A_{j,k+1-j} + [Z_{j,j-1}\dots Z_{j,0}]\begin{bmatrix}A_{j-1,k+2-j}\\ \vdots \\ A_{0,k+1}\end{bmatrix} = 0,
$$

iii) for each j with $k \geq j \geq 1$, the matrix pair

$$
\left(S_jA_{j,k+1-j}, \begin{bmatrix}A_{j-1,k+2-j}\\ \vdots \\ A_{0,k+1}\end{bmatrix}\right)
$$

has no hidden redundancy.

Proof. For each j with $k \geq j \geq 1$, by applying Lemma 2.3 to the matrix pair

$$
\left(A_{j,k+1-j}, \begin{bmatrix}A_{j-1,k+2-j}\\ \vdots \\ A_{0,k+1}\end{bmatrix}\right)
$$

we obtain matrices S_j, $Z_{j,i}$, $i = j, \dots, 0$ that satisfy conditions i)-iii). □

Setting

$$\tilde{P} := \operatorname{diag}\left(\begin{bmatrix} S_k \\ Z_{k,k} \end{bmatrix}, \ldots, \begin{bmatrix} S_1 \\ Z_{1,1} \end{bmatrix}, I_{r_{k+1}+v} \right) \in \mathbb{C}^{\ell,\ell},$$

and scaling system (1.4) with $\tilde{P}$ from the left we obtain

$$\left[\begin{array}{c|c|c|c} S_k A_{k,1} & * & \ldots & * \\ Z_{k,k} A_{k,1} & * & \ldots & * \\ \hline & S_{k-1} A_{k-1,2} & \ldots & * \\ & Z_{k-1,k-1} A_{k-1,2} & \ldots & * \\ \hline & & \ddots & \vdots \\ & & & A_{0,k+1} \\ \hline 0 & 0 & \ldots & 0 \end{array}\right] \begin{bmatrix} x^{(k)} \\ x^{(k-1)} \\ \vdots \\ x \end{bmatrix} = \begin{bmatrix} S_k f_1 \\ Z_{k,k} f_1 \\ S_{k-1} f_2 \\ Z_{k-1,k-1} f_2 \\ \vdots \\ f_{k+1} \\ f_{k+2} \end{bmatrix} \tag{3.8}$$

For each j with $k \geq j \geq 1$, we then reduce the number of differential equations of order j by eliminating the block $Z_{j,j} A_{j,k+1-j}$ of (3.8), as in the following lemma.

Lemma 3.4. *Let matrices S_j, $Z_{j,i}$, $j = k, \ldots, 1$, $i = j, \ldots, 0$, be defined as in Lemma 3.3. Then, the DAE* (3.8) *has the same solution set as the DAE*

$$\begin{array}{c} d_1 \\ s_1 \\ \hline d_2 \\ s_2 \\ \hline \vdots \\ d_{k+1} \\ \hline v \end{array} \left[\begin{array}{c|c|c|c} S_k A_{k,1} & * & \ldots & * \\ 0 & * & \ldots & * \\ \hline & S_{k-1} A_{k-1,2} & \ldots & * \\ & 0 & \ldots & * \\ \hline & & \ddots & \vdots \\ & & & A_{0,k+1} \\ \hline 0 & 0 & \ldots & 0 \end{array}\right] \begin{bmatrix} x^{(k)} \\ x^{(k-1)} \\ \vdots \\ x \end{bmatrix} = \begin{bmatrix} S_k f_1 \\ g_{2k} \\ S_{k-1} f_2 \\ g_{2(k-1)} \\ \vdots \\ f_{k+1} \\ f_{k+2} \end{bmatrix}, \tag{3.9}$$

where

$$g_{2j} := \sum_{i=0}^{j-1} Z_{j,i} \left(\frac{d}{dt}\right)^{j-i} f_{k+1-i} + Z_{j,j} f_{k+1-j}, \;\; j = k, \ldots, 1.$$

Proof. For each j with $k \geq j \geq 1$, by inserting

$$Z_{j,j} A_{j,k+1-j} = -\sum_{i=0}^{j-1} Z_{j,i} A_{i,k+1-i},$$

into the equation

$$Z_{j,j} A_{j,k+1-j} x^{(j)} + Z_{j,j} \sum_{i=0}^{j-1} A_{i,k+1-j} x^{(i)} = Z_{j,j} f_{k+1-j},$$

we have

$$\left(-\sum_{i=0}^{j-1} Z_{j,i} A_{i,k+1-i}\right) x^{(j)} + Z_{j,j} \sum_{i=0}^{j-1} A_{i,k+1-j} x^{(i)} = Z_{j,j} f_{k+1-j}$$

or equivalently

$$-\sum_{i=0}^{j-1} Z_{j,i} \left(\frac{d}{dt}\right)^{j-i} A_{i,k+1-i} x^{(i)} + Z_{j,j} \sum_{i=0}^{j-1} A_{i,k+1-j} x^{(i)} = Z_{j,j} f_{k+1-j}.$$

The $(k+1-i)$-th equation of (3.5) implies that

$$A_{i,k+1-i} x^{(i)} = -\sum_{\ell=0}^{i-1} Z_{i,\ell} A_{\ell,k+1-i} x^{(\ell)} + f_{k+1-i}.$$

Multiplying this equation from the left by $Z_{j,j}$, this can be rewritten as

$$
\begin{aligned}
- \quad & \sum_{i=0}^{j-1} Z_{j,i} \left(\frac{d}{dt}\right)^{j-i} \left(-\sum_{\ell=0}^{i-1} Z_{i,\ell} A_{\ell,k+1-i} x^{(\ell)} - f_{k+1-i} \right) + Z_{j,j} \sum_{i=0}^{j-1} A_{i,k+1-j} x^{(i)} \\
& = Z_{j,j} f_{k+1-j},
\end{aligned}
$$

or equivalently

$$
\begin{aligned}
& \sum_{i=0}^{j-1} Z_{j,i} \left(\frac{d}{dt}\right)^{j-i} \left(\sum_{\ell=0}^{i-1} Z_{i,\ell} A_{\ell,k+1-i} x^{(\ell)} \right) + Z_{j,j} \sum_{i=0}^{j-1} A_{i,k+1-j} x^{(i)} \\
& \quad = \sum_{i=0}^{j-1} Z_{j,i} \left(\frac{d}{dt}\right)^{j-i} f_{k+1-i} + Z_{j,j} f_{k+1-j} =: g_{2j},
\end{aligned}
$$

which is a set of differential equations of order at most $j-1$.

Continuing like this inductively, we obtain (3.9). □

>From (3.9), we deduce that $r_j = d_j + s_j$, $j = 1, \ldots, k+1$, $s_{k+1} = 0$ and therefore the upper rank of the behavior matrix of system (3.9) can be estimated via

$$
\begin{aligned}
\check{r}_u & \leqslant (k+1)d_1 + k(s_1 + d_2) + \cdots + (s_k + d_{k+1}), \\
& = (k+1)(d_1 + s_1) + k(d_2 + s_2) + \cdots + (d_{k+1} + s_{k+1}) - \sum_{i=0}^{k} s_i, \\
& = (k+1)r_1 + kr_2 + \cdots + r_{k+1} - \sum_{i=0}^{k} s_i = r_u - \sum_{i=0}^{k} s_i,
\end{aligned}
$$

and thus this procedure of passing the system (3.4) to (3.9) has reduced the upper rank.

This reduction of the upper rank leads to the following procedure.

Procedure 3.5. Input: The DAE (1.4) and its behavior form (3.2).

Begin: Set $\alpha = 0$ and let $M^0 = M$, $f^0 = f$,

Step 1. Determine a nonsingular matrix $P \in \mathbb{C}^{\ell,\ell}$ (as in Lemma 3.2) such that

$$
PM^\alpha = \left[\begin{array}{c|c|c|c}
A_{k,1} & A_{k-1,1} & \cdots & A_{0,1} \\
 & A_{k-1,2} & \cdots & A_{0,2} \\
 & & \ddots & \vdots \\
 & & & A_{0,k+1} \\
\hline
0 & 0 & \cdots & 0
\end{array}\right], \quad
\begin{array}{c}
r_1 \\ r_2 \\ \vdots \\ r_{k+1} \\ \hline v
\end{array}
$$

where all the matrices on the main diagonal have full row rank, and let

$$
r_u^\alpha := (k+1)r_1 + mr_2 + \cdots + 2r_k + r_{k+1},
$$

be the upper rank of the behavior matrix M^α in the α-th iteration.

Step 2. Determine matrices S_j, $Z_{j,i}$, $j = k, \ldots, 1$, $i = j, \ldots, 0$ of appropriate size such that

i) matrices $\begin{bmatrix} S_j \\ Z_{j,j} \end{bmatrix} \in \mathbb{C}^{r_j, r_j}$, $k \geq j \geq 1$ are nonsingular,

ii) for each j with $k \geq j \geq 1$,

$$
Z_{j,j} A_{j,k+1-j} + [Z_{j,j-1} \ldots Z_{j,0}] \begin{bmatrix} A_{j-1,k+2-j} \\ \vdots \\ A_{0,k+1} \end{bmatrix} = 0,
$$

iii) for each j with $k \geq j \geq 1$, the matrix pair

$$\left(S_j A_{j,k+1-j}, \begin{bmatrix} A_{j-1,k+2-j} \\ \vdots \\ A_{0,k+1} \end{bmatrix} \right)$$

has no hidden redundancy.

Step 3. Setting

$$\tilde{P} := \operatorname{diag}\left(\begin{bmatrix} S_k \\ Z_{k,k} \end{bmatrix}, \ldots, \begin{bmatrix} S_1 \\ Z_{1,1} \end{bmatrix}, I_{r_{k+1}+v} \right) \in \mathbb{C}^{\ell,\ell},$$

and scaling system (1.4) with $\tilde{P}$ from the left we obtain

$$\left[\begin{array}{c|c|c|c} S_k A_{k,1} & * & \ldots & * \\ Z_{k,k} A_{k,1} & * & \ldots & * \\ \hline & S_{k-1} A_{k-1,2} & \ldots & * \\ & Z_{k-1,k-1} A_{k-1,2} & \ldots & * \\ \hline & & \ddots & \vdots \\ & & & A_{0,k+1} \\ \hline 0 & 0 & \ldots & 0 \end{array}\right] \begin{bmatrix} x^{(k)} \\ x^{(k-1)} \\ \vdots \\ x \end{bmatrix} = \begin{bmatrix} S_k f_1 \\ Z_{k,k} f_1 \\ S_{k-1} f_2 \\ Z_{k-1,k-1} f_2 \\ \vdots \\ f_{k+1} \\ f_{k+2} \end{bmatrix}. \tag{3.10}$$

Step 4. For each j with $k \geq j \geq 1$, we then reduce the number of differential equations of order j by eliminating the block $Z_{j,j} A_{j,k+1-j}$ of (3.10), as in Lemma 3.3. In this way, we obtain the system

$$\begin{array}{c} d_1 \\ s_1 \\ \hline d_2 \\ s_2 \\ \hline \vdots \\ d_{k+1} \\ \hline v \end{array} \underbrace{\left[\begin{array}{c|c|c|c} S_k A_{k,1} & * & \ldots & * \\ 0 & * & \ldots & * \\ \hline & S_{k-1} A_{k-1,2} & \ldots & * \\ & 0 & \ldots & * \\ \hline & & \ddots & \vdots \\ & & & A_{0,k+1} \\ \hline 0 & 0 & \ldots & 0 \end{array}\right]}_{\breve{M}} \begin{bmatrix} x^{(k)} \\ x^{(k-1)} \\ \vdots \\ x \end{bmatrix} = \underbrace{\begin{bmatrix} S_k f_1 \\ g_{2k} \\ S_{k-1} f_2 \\ g_{2(k-1)} \\ \vdots \\ f_{k+1} \\ f_{k+2} \end{bmatrix}}_{\breve{f}},$$

with

$$g_{2j} := \sum_{i=0}^{j-1} Z_{j,i} \left(\frac{d}{dt} \right)^{j-i} f_{k+1-i} + Z_j^j f_{k+1-j}, \; j = k, \ldots, 1.$$

Let $s^\alpha := \sum_{i=0}^{k} s_i$, we then increase α by 1, set $M^\alpha = \breve{M}$, $f^\alpha = \breve{f}$, and repeat the process from Step 1.

End.

Since $r_u^{\alpha+1} \leqslant r_u^\alpha - s^\alpha$, Procedure 3.5 terminates after a finite number of iterations, and thus we have the following theorem.

Theorem 3.6. *The DAE* (1.4) *has the same solution set as the DAE*

$$\left[\begin{array}{c|c|c|c} \hat{A}_{k,1} & \hat{A}_{k-1,1} & \ldots & \hat{A}_{0,1} \\ & \hat{A}_{k-1,2} & \ldots & \hat{A}_{0,2} \\ & & \ddots & \vdots \\ & & & \hat{A}_{0,k+1} \\ \hline 0 & 0 & \ldots & 0 \end{array}\right] \begin{bmatrix} x^{(k)} \\ x^{(k-1)} \\ \vdots \\ x \end{bmatrix} = \begin{bmatrix} \hat{f}_1 \\ \hat{f}_2 \\ \vdots \\ \hat{f}_{k+1} \\ \hat{f}_{k+2} \end{bmatrix} \tag{3.11}$$

where $\begin{bmatrix} \hat{A}_{k,1} \\ \vdots \\ \hat{A}_{0,k+1} \end{bmatrix}$ *has full row rank.*

Proof. Clearly, after carrying out Procedure 3.5, we obtain a system of the form (3.11), where $\hat{A}_{k,1}, \ldots, \hat{A}_{0,k+1}$ have full row rank and none of the matrix pairs

$$\left(\hat{A}_{i,k+1-i}, \begin{bmatrix} \hat{A}_{i-1,k+2-i} \\ \vdots \\ \hat{A}_{0,k+1} \end{bmatrix} \right)$$

$i = k, \ldots, 1$ has a hidden redundancy.

Applying Lemma 2.4 to the matrices $\hat{A}_{i,k+1-i}$, $i = 0, \ldots, k$, it follows that $\begin{bmatrix} \hat{A}_{k,1} \\ \vdots \\ \hat{A}_{0,k+1} \end{bmatrix}$ has full row rank. □

Following the notation in [10] we call (3.11) the *strangeness-free reformulation* of the DAE (1.4). Obviously, if at $t = 0$ the consistency assumptions

$$\begin{aligned}
&\left(\frac{d}{dt}\right)^i \left(\hat{A}_{k-1,2}x^{(k-1)}(t) + \cdots + \hat{A}_{1,2}x^{(1)}(t) + \hat{A}_{0,2}x(t) - \hat{f}_2(t)\right) = 0, \; i = 0, 1, \\
&\ldots \\
&\left(\frac{d}{dt}\right)^i \left(\hat{A}_{0,k+1}x(t) - \hat{f}_{k+1}(t)\right) = 0, \; i = 0, \ldots, k, \\
&\hat{f}_{k+2}(t) = 0,
\end{aligned} \tag{3.12}$$

hold, then we can differentiate all but the first equation of system (3.11) to obtain an underlying ODE as in the following theorem.

Theorem 3.7. *Consider the DAE* (1.4) *and assume that the consistency condition* (3.12) *is satisfied. Then,* (1.4) *has the same solution set as the* underlying ODE

$$\left[\begin{array}{c|c|c|c|c} \hat{A}_{k,1} & \hat{A}^{k-1,1} & \ldots & \hat{A}_{1,1} & \hat{A}_{0,1} \\ \hat{A}_{k-1,2} & \hat{A}_{k-1,3} & \ldots & \hat{A}_{0,2} & \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \hat{A}_{1,k} & \hat{A}_{0,k} & & 0 & 0 \\ \hat{A}_{0,k+1} & 0 & \ldots & 0 & 0 \end{array}\right] \begin{bmatrix} x^{(k)} \\ x^{(k-1)} \\ \vdots \\ x^{(1)} \\ x \end{bmatrix} = \begin{bmatrix} \hat{f}_1 \\ \hat{f}_2^{(1)} \\ \vdots \\ \hat{f}_k^{(k-1)} \\ \hat{f}_{k+1}^{(k)} \end{bmatrix}, \tag{3.13}$$

where the first column $\begin{bmatrix} \hat{A}_{k,1} \\ \vdots \\ \hat{A}_{0,k+1} \end{bmatrix}$ *has full row rank.*

The following corollary is a direct consequence of Theorem 3.7.

Corollary 3.8. Consider the initial value-problem (1.4)–(1.5), and assume that the function f is sufficiently smooth. Then

i) consistency conditions for f and the initial vector $X_0 := \begin{bmatrix} x_0^{(k)} \\ \vdots \\ x_0 \end{bmatrix}$ are given by system (3.12).

ii) system (1.4) is uniquely solvable if and only if in addition, the matrix $\begin{bmatrix} \hat{A}_{k,1} \\ \vdots \\ \hat{A}_{0,k+1} \end{bmatrix}$ is square.

Motivated by the strangeness-free reformulation (3.11) of the DAE (1.4), we introduce the following definition.

Definition 3.9. Consider the behavior matrix

$$M = [A_k, \ldots, A_0] \in \mathbb{C}^{\ell,(k+1)n},$$

associated with the DAE

$$A_k x^{(k)}(t) + \cdots + A_0 x(t) + h(t) = 0,$$

where $A_i \in \mathbb{C}^{\ell,n}$, $i = k, \ldots, 0$, and $h : [0, \infty) \to \mathbb{C}^\ell$.

The matrix M is called *strangeness-free* if there exists a nonsingular matrix $P \in \mathbb{C}^{\ell,\ell}$ such that

$$PM = \left[\begin{array}{c|c|c|c} A_{k,1} & A_{k-1,1} & \cdots & A_{0,1} \\ & A_{k-1,2} & \cdots & A_{0,2} \\ & & \ddots & \vdots \\ & & & A_{0,k+1} \\ \hline 0 & 0 & \cdots & 0 \end{array}\right],$$

where

i) each block column contains exactly n columns.

ii) the matrix $\begin{bmatrix} A_{k,1} \\ A_{k-1,2} \\ \vdots \\ A_{0,k+1} \end{bmatrix}$ has full row rank.

In this section we have derived a procedure to transform a linear DAE of arbitrary order to a strangeness free form. In the following section we use this procedure to reformulate DDAEs.

4 Analysis and reformulation of DDAEs

This section is devoted to DDAEs with single delay of the form (1.2) and the initial value problem (1.2)–(1.3). Analogous to Section 3, the behavior approach and the algebraic approach will be combined. Consider a behavior formulation of (1.2) as

$$N^0 X^0 = f_0, \tag{4.1}$$

with

$$N^0 = [A_1 \ A_0 \ A_{-1}] \text{ and } X^0(t) = \begin{bmatrix} \dot{x}(t) \\ x(t) \\ x(t-\tau) \end{bmatrix}, \ f_0(t) := f(t).$$

A first remarkable difference between DAEs and DDAEs is that for the DAE (1.4) of order k, after applying the strangeness-free reformulation (Procedure 3.5) the resulting system is still a DAE of order at most k. However, when applying a similar procedure for the DDAE (1.2) then the order may even increase, as is illustrated in the following example.

Example 4.1. Consider the system

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}(t) \\ \dot{y}(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x(t-\tau) \\ y(t-\tau) \end{bmatrix} + \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}.$$

In behavior form, we have

$$N^0 = \left[\begin{array}{cc|cc|cc} 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & -1 \end{array}\right], \ X^0 = \left[\begin{array}{c} \dot{x}(t) \\ \dot{y}(t) \\ \hline x(t) \\ y(t) \\ \hline x(t-\tau) \\ y(t-\tau) \end{array}\right].$$

Differentiating the second equation and inserting it into the first, we get

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}(t) \\ \dot{y}(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x(t-\tau) \\ y(t-\tau) \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}(t-\tau) \\ \dot{y}(t-\tau) \end{bmatrix} + \begin{bmatrix} f_1 + \dot{f}_2 \\ f_2 \end{bmatrix}.$$

In behavior form we have $N^1 X^1 = \begin{bmatrix} f_1 + \dot{f}_2 \\ f_2 \end{bmatrix}$ with

$$N^1 = \left[\begin{array}{cc|cc|cc|cc} 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 \end{array}\right], \ X^1 = \left[\begin{array}{c} \dot{x}(t) \\ \dot{y}(t) \\ \hline x(t) \\ y(t) \\ \hline x(t-\tau) \\ y(t-\tau) \\ \hline \dot{x}(t-\tau) \\ \dot{y}(t-\tau) \end{array}\right].$$

Thus, the size of the behavior matrix is increased.

The second important difference between DAEs and DDAEs is the strangeness-free reformulation procedure. Let us illustrate this by considering the following example.

Example 4.2. Consider the system

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \dot{x}(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} x(t-\tau) = \begin{bmatrix} f(t) \\ g(t) \end{bmatrix}. \tag{4.2}$$

The associated non-delayed system is

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \dot{x}(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} x(t) = \begin{bmatrix} f(t) \\ g(t) \end{bmatrix}.$$

Using the strangeness-free reformulation in [10] for the non-delayed system, we differentiate the first equation and insert it into the second equation to obtain

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} \dot{x}(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} x(t) = \begin{bmatrix} f(t) \\ g(t) - \dot{f}(t) \end{bmatrix}.$$

Clearly, if $g(t) + \dot{f}(t) = 0$ holds, then we obtain a unique solution $x(t) = f(t)$.

Performing the same steps for system (4.2), we obtain that

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} \dot{x}(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} x(t-\tau) + \begin{bmatrix} 0 \\ -1 \end{bmatrix} \dot{x}(t-\tau) = \begin{bmatrix} f(t) \\ g(t) - \dot{f}(t) \end{bmatrix}. \tag{4.3}$$

The second equation of (4.3) not only gives the consistency condition

$$\dot{x}(t-\tau) + g(t) - \dot{f}(t) = 0, \ t \in [0, \tau],$$

but also the constraint

$$\dot{x}(t) + g(t+\tau) - \dot{f}(t+\tau) = 0,\ t \geq 0.$$

Therefore, one obtains the system

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \dot{x}(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} x(t-\tau) = \begin{bmatrix} f(t) \\ -g(t+\tau) + \dot{f}(t+\tau) \end{bmatrix}. \tag{4.4}$$

Thus, the step that passes system (4.3) to (4.4) changes nothing but the inhomogeneity and we can proceed like this without ever terminating. Thus, DDAEs require another reformulation procedure, which should terminate after a finite number of steps.

Considering system (4.2) again, we can proceed as follows. Replacing the first equation of (4.2) by its derivative gives

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \dot{x}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix} x(t-\tau) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \dot{x}(t-\tau) = \begin{bmatrix} \dot{f}(t) \\ g(t) \end{bmatrix}.$$

Subtracting the first equation from the second we get

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \dot{x}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix} x(t-\tau) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \dot{x}(t-\tau) = \begin{bmatrix} \dot{f}(t) - g(t) \\ g(t) \end{bmatrix}.$$

Shifting the time in the first equation by τ, we obtain

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \dot{x}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix} x(t-\tau) = \begin{bmatrix} \dot{f}(t+\tau) - g(t+\tau) \\ g(t) \end{bmatrix},$$

and subtracting the second equation from the first yields

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \dot{x}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix} x(t-\tau) = \begin{bmatrix} \dot{f}(t+\tau) - g(t+\tau) - g(t) \\ g(t) \end{bmatrix}. \tag{4.5}$$

If the consistency condition in the first equation is satisfied, then we have a unique solution $x(t)$.

Motivated by Example 4.2, we propose a new procedure to treat the system (1.2) in the behavior form (4.1). The idea is to replace nontrivial scalar DDEs in system (1.2) by (appropriately chosen) derivatives. Since in this way the order of the system may be increased, we study directly general DDAEs of the form (1.1). Set

$$\begin{aligned} N &:= \begin{bmatrix} A_k & \ldots A_0 & \mid & A_{-\kappa} & \ldots & A_{-1} \end{bmatrix} =: [N_+ \quad N_-], \\ X_+(t) &:= \begin{bmatrix} x^{(k)}(t) \\ \vdots \\ x(t) \end{bmatrix},\ X_-(t-\tau) := \begin{bmatrix} x^{(\kappa)}(t-\tau) \\ \vdots \\ x(t-\tau) \end{bmatrix}, \end{aligned}$$

then we have the behavior form of (1.1)

$$[N_+ \quad N_-] \begin{bmatrix} X_+(t) \\ X_-(t-\tau) \end{bmatrix} = f(t). \tag{4.6}$$

Set $r := \operatorname{rank}(N_-)$ and $d := \ell - r$ and perform a column compression of N_- as in the following lemma.

Lemma 4.3. *Consider the DDAE* (1.1) *in its behavior form* (4.6)*. Then there exists a nonsingular matrix $P \in \mathbb{C}^{\ell,\ell}$ such that by scaling system* (4.6) *with P from the left, we obtain the system*

$$\begin{bmatrix} F & G \\ H & 0 \end{bmatrix} \begin{bmatrix} X_+(t) \\ X_-(t-\tau) \end{bmatrix} = \begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix}, \quad \begin{matrix} r \\ d \end{matrix} \tag{4.7}$$

where G has full row rank.

Proof. First we determine a matrix $P_2 \in \mathbb{C}^{d,\ell}$ whose rows span the left nullspace of N_-, i. e., $P_2 N_- = 0$ and then we complement P_2 as $P := \begin{bmatrix} P_1 \\ P_2 \end{bmatrix}$ to a nonsingular matrix. Then

$$P[N_+ \quad N_-] = \begin{bmatrix} P_1 N_+ & P_1 N_- \\ P_2 N_+ & 0 \end{bmatrix} =: \begin{bmatrix} F & G \\ H & 0 \end{bmatrix},$$

and $G = P_1 N_-$ has full row rank. □

Since in (4.7) G has full row rank, we see that the behavior system (4.6) has r nontrivial scalar delay differential equations, and d scalar differential equations.

Since typically the matrix $\begin{bmatrix} F \\ H \end{bmatrix}$ is not strangeness-free, then a first idea would be carry out the strangeness-free formulation (Procedure 3.5) for the DAE

$$\begin{bmatrix} F \\ H \end{bmatrix} X_+(t) = \begin{bmatrix} g_1(t) \\ g_2(t) \end{bmatrix}, \tag{4.8}$$

where

$$g(t) := \begin{bmatrix} g_1(t) \\ g_2(t) \end{bmatrix} := -\begin{bmatrix} G \\ 0 \end{bmatrix} X_-(t-\tau) + \begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix}.$$

However, as pointed out in Example 4.2, this may not lead to a procedure that terminates in a finite number of steps.

In order to overcome this difficulty, we propose the following approach. Since the order of the DAE $HX_+(t) = f_2(t)$ is at most k, we replace the first equation of system (4.7) by its $(k+1)$-st derivative and obtain the system

$$\begin{bmatrix} F & 0 \\ 0 & H \end{bmatrix} \begin{bmatrix} \left(\frac{d}{dt}\right)^{k+1} X_+(t) \\ X_+(t) \end{bmatrix} + \begin{bmatrix} G \\ 0 \end{bmatrix} \left(\frac{d}{dt}\right)^{k+1} X_-(t-\tau) = \begin{bmatrix} \left(\frac{d}{dt}\right)^{k+1} f_1(t) \\ f_2(t) \end{bmatrix}. \tag{4.9}$$

To guarantee that system (4.9) has the same solution set as (4.8), we must require that the following consistency condition holds at $t = 0$

$$\left(\frac{d}{dt}\right)^j \Big(FX_+(t) + GX_-(t-\tau) - f_1(t)\Big) = 0, \; j = 0, \ldots, k+1. \tag{4.10}$$

Thus, we have shown the following lemma.

Lemma 4.4. *Consider system* (4.8) *and assume that the consistency condition* (4.10) *is satisfied at* $t = 0$*. Then system* (4.8) *has the same solution set as the DDAE* (4.9)*.*

Setting

$$\begin{bmatrix} \breve{f}_1(t) \\ \breve{f}_2(t) \end{bmatrix} := -\begin{bmatrix} G \\ 0 \end{bmatrix} \left(\frac{d}{dt}\right)^{k+1} X_-(t-\tau) + \begin{bmatrix} \left(\frac{d}{dt}\right)^{k+1} f_1(t) \\ f_2(t) \end{bmatrix},$$

we can apply Procedure 3.5 to the DAE

$$\begin{bmatrix} F & 0 \\ 0 & H \end{bmatrix} \begin{bmatrix} \left(\frac{d}{dt}\right)^{k+1} X_+(t) \\ X_+(t) \end{bmatrix} = \begin{bmatrix} \breve{f}_1(t) \\ \breve{f}_2(t) \end{bmatrix}$$

and have shown the following lemma.

Lemma 4.5. *Consider the DDAE* (1.1) *in its behavior form* (4.6) *and assume that the consistency condition* (4.10) *is satisfied at* $t = 0$*. Then, system* (1.1) *has the same solution set as the DDAE*

$$\begin{array}{c} r-s \\ s \\ d-v \\ v \end{array} \begin{bmatrix} \tilde{F}_1 & \tilde{F}_2 \\ 0 & 0 \\ 0 & \tilde{H} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \left(\frac{d}{dt}\right)^{k+1} X_+(t) \\ X_+(t) \end{bmatrix} + \begin{bmatrix} \tilde{G}_1 \\ \tilde{G}_2 \\ 0 \\ 0 \end{bmatrix} \tilde{X}_-(t-\tau) = \begin{bmatrix} \tilde{f}_1(t) \\ \tilde{f}_2(t) \\ \tilde{f}_3(t) \\ \tilde{f}_4(t) \end{bmatrix}, \tag{4.11}$$

where $\tilde{X}_-(t-\tau) = \begin{bmatrix} x^{(\tilde{\kappa})}(t-\tau) \\ \vdots \\ x(t-\tau) \end{bmatrix}$ *for some* $\tilde{\kappa} \in \mathbb{N}$ *and where the matrix* $\begin{bmatrix} \tilde{F}_1 & \tilde{F}_2 \\ 0 & \tilde{H} \end{bmatrix}$ *is strangeness-free and of full row rank.*

Since the second equation in system (4.11) is a DAE of the variable $x(t-\tau)$, we can shift it to obtain a DAE for $x(t)$. We summarize this and Lemmas 4.3, 4.4, 4.5, in the following theorem.

Theorem 4.6. *Consider the DDAE* (1.1) *in its behavior form* (4.6)*. Moreover, assume that the consistency condition* (4.10) *is satisfied at* $t = 0$ *and that*

$$\left(\tilde{G}_2 \tilde{X}_-(t-\tau) - \tilde{f}_2(t)\right)\Big|_{t\in[0,\tau]} = 0, \tag{4.12}$$

holds. Then, system (1.1) *has the same solution set as the DDAE*

$$\begin{aligned} &\begin{bmatrix} \tilde{F}_1 & \tilde{F}_2 \\ 0 & \tilde{H} \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \left(\frac{d}{dt}\right)^{k+1} X_+(t) \\ X_+(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \tilde{G}_2 \\ 0 \end{bmatrix} \tilde{X}_+(t) + \begin{bmatrix} \tilde{G}_1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tilde{X}_-(t-\tau) + \\ &= \begin{bmatrix} \tilde{f}_1(t) \\ \tilde{f}_3(t) \\ \tilde{f}_2(t+\tau) \\ \tilde{f}_4(t) \end{bmatrix}, \quad \begin{matrix} r-s \\ d-v \\ s \\ v \end{matrix} \end{aligned} \tag{4.13}$$

where

$$\tilde{X}_+(t) := \begin{bmatrix} x^{(\tilde{\kappa})}(t) \\ \vdots \\ x(t) \end{bmatrix}.$$

In (4.13)*, the matrix* $\begin{bmatrix} \tilde{F}_1 & \tilde{F}_2 \\ 0 & \tilde{H} \end{bmatrix}$ *is strangeness-free and of full row rank.*

Observe that by passing system (4.7) to (4.13), we reduced the number of scalar delay differential equations from r (in system (4.7)) to $r-s$ (in system (4.13)). However, the number of system equations (number of rows) is still ℓ.

Definition 4.7. The step that passes system (4.7) to (4.13) is called a *reformulation step.* The natural numbers r, d, s, v are called *characteristic invariants* of system (1.1) and of its behavior form (4.7).

Setting $k^{new} := \max\{2k+1, \tilde{\kappa}\}$, $\kappa^{new} := \tilde{\kappa}$, we can bring system (4.11) into behavior form and perform a new reformulation step. Since the number of nontrivial DDEs decreases every time that we perform a reformulation step, this process terminates after finitely many steps.

We summarize the discussion above in the following procedure.

Procedure 4.8. Input: The DDAE (1.1).
Begin
Set

$$\begin{aligned} N^0 &:= \begin{bmatrix} A_k & \dots A_0 & | & A_{-\kappa} & \dots & A_{-1} \end{bmatrix} =: [N^0_+ \quad N^0_-], \\ X^0_+(t) &:= \begin{bmatrix} x^{(k)}(t) \\ \vdots \\ x(t) \end{bmatrix}, \; X^0_-(t-\tau) := \begin{bmatrix} x^{(\kappa)}(t-\tau) \\ \vdots \\ x(t-\tau) \end{bmatrix}, \\ X^0(t) &= \begin{bmatrix} X^0_+(t) \\ X^0_-(t-\tau) \end{bmatrix}, \; f^0(t) = f(t), \end{aligned}$$

and $i = 0$, $k^0 = k$, $\kappa^0 = \kappa$, $r^0 = \operatorname{rank}(N_-^0)$, $d^0 = \ell - r^0$.
Step 1. Determine a nonsingular matrix $P \in \mathbb{C}^{\ell,\ell}$ such that by scaling P from the left of the behavior system

$$N^0 X^0(t) = f^0(t),$$

we obtain

$$\begin{bmatrix} F & G \\ H & 0 \end{bmatrix} \begin{bmatrix} X_+(t) \\ X_-(t-\tau) \end{bmatrix} = \begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix}, \quad \begin{matrix} r^0 \\ d^0 \end{matrix} \tag{4.14}$$

where G has full row rank.
If $\begin{bmatrix} F \\ H \end{bmatrix}$ is strangeness-free and has full row rank **then STOP**
else proceed to **Step 2**.
Step 2. Check the consistency conditions

$$\frac{d^j}{dt^j}\Big(FX_+(t) + GX_-(t-\tau) - f_1(t)\Big) = 0, \ j = 0, \dots, k+1,$$

at $t = 0$. If it is satisfied, then transform the behavior system (4.14) into

$$\begin{bmatrix} F & 0 \\ 0 & H \end{bmatrix} \begin{bmatrix} \left(\frac{d}{dt}\right)^{k+1} X_+(t) \\ X_+(t) \end{bmatrix} + \begin{bmatrix} G \\ 0 \end{bmatrix} \left(\frac{d}{dt}\right)^{k+1} X_-(t-\tau) = \begin{bmatrix} \left(\frac{d}{dt}\right)^{k+1} f_1(t) \\ f_2(t) \end{bmatrix}. \tag{4.15}$$

Step 3. Apply Procedure 3.5 to system (4.15) to obtain

$$\begin{bmatrix} \tilde{F}_1 & \tilde{F}_2 \\ 0 & 0 \\ 0 & \tilde{H} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \left(\frac{d}{dt}\right)^{k+1} X_+(t) \\ X_+(t) \end{bmatrix} + \begin{bmatrix} \tilde{G}_1 \\ \tilde{G}_2 \\ 0 \\ 0 \end{bmatrix} \tilde{X}_-(t-\tau) = \begin{bmatrix} \tilde{f}_1(t) \\ \tilde{f}_2(t) \\ \tilde{f}_3(t) \\ \tilde{f}_4(t) \end{bmatrix}, \quad \begin{matrix} r^0 - s^0 \\ s^0 \\ d^0 - v^0 \\ v^0 \end{matrix} \tag{4.16}$$

where $\tilde{X}_-(t-\tau) = \begin{bmatrix} x^{(\tilde{\kappa})}(t-\tau) \\ \vdots \\ x(t-\tau) \end{bmatrix}$, for some $\tilde{\kappa} \in \mathbb{N}$, and the matrix $\begin{bmatrix} \tilde{F}_1 & \tilde{F}_2 \\ 0 & \tilde{H} \end{bmatrix}$ is strangeness-free and of full row rank.
Step 4. Check the consistency conditions

$$\Big(\tilde{G}_2 \tilde{X}_-(t-\tau) - \tilde{f}_2(t)\Big)\Big|_{t\in[0,\tau]} = 0.$$

If it is satisfied, then shift the second equation of system (4.16) and permute the second and the third block rows to get

$$\begin{bmatrix} \tilde{F}_1 & \tilde{F}_2 \\ 0 & \tilde{H} \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \left(\frac{d}{dt}\right)^{k+1} X_+(t) \\ X_+(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \tilde{G}_2 \\ 0 \end{bmatrix} \tilde{X}_+(t) + \begin{bmatrix} \tilde{G}_1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tilde{X}_-(t-\tau) +$$

$$= \begin{bmatrix} \tilde{f}_1(t) \\ \tilde{f}_3(t) \\ \tilde{f}_2(t+\tau) \\ \tilde{f}_4(t) \end{bmatrix}, \quad \begin{matrix} r^0 - s^0 \\ d^0 - v^0 \\ s^0 \\ v^0 \end{matrix} \tag{4.17}$$

where

$$\tilde{X}_+(t) := \begin{bmatrix} x^{(\tilde{\kappa})}(t) \\ \vdots \\ x(t) \end{bmatrix}, \ \tilde{X}_-(t-\tau) = \begin{bmatrix} x^{(\tilde{\kappa})}(t-\tau) \\ \vdots \\ x(t-\tau) \end{bmatrix}.$$

Step 5. Reorganize system (4.17) in the form

$$\tilde{A}_{k_1}x^{(k_1)}(t)+\cdots+\tilde{A}_0x(t)+\tilde{A}_{-1}x(t-\tau)+\cdots+\tilde{A}_{-\kappa_1}x^{(\kappa_1)}(t-\tau)=\tilde{f}_1(t),$$

where $\tilde{A}_i\in\mathbb{C}^{\ell,n}$, $i=-\kappa_1,\ldots,k_1$, $\tilde{A}_{k_1}\neq 0$, $\tilde{A}_{-\kappa_1}\neq 0$, $k_1\leqslant\max\{2k+1,\tilde{\kappa}\}$, $\kappa_1\leqslant\tilde{\kappa}$.
Increase i by 1 and set

$$\begin{aligned}
N^i &:= \left[\tilde{A}_{k_1}\quad \ldots\tilde{A}_0 \;\middle|\; \tilde{A}_{-\kappa_1}\quad \ldots \quad \tilde{A}_{-1}\right] := [N_+^i \quad N_-^i],\\
X_+^i(t) &:= \begin{bmatrix} x^{(k_1)}(t)\\ \vdots \\ x(t)\end{bmatrix},\; X_-^i(t-\tau):=\begin{bmatrix} x^{(\kappa_1)}(t-\tau)\\ \vdots \\ x(t-\tau)\end{bmatrix},\\
X^i(t) &:= \begin{bmatrix} X_+^i(t)\\ X_-^i(t-\tau)\end{bmatrix},
\end{aligned}$$

where

$$\operatorname{rank}(N_+^i)=\operatorname{rank}(\tilde{G}_i)\leqslant r^{i-1}-s^{i-1}.$$

Set

$$r^i:=\operatorname{rank}(N_+^i)\leqslant r^{i-1}-s^{i-1},\; d^{i-1}=\ell-r^{i-1}\geq d^i+s^i,$$

and repeat the process from **Step 1**.
End

Definition 4.9. Consider the DDAE (1.1) in its behavior form (4.7) and the sequence (r^i,d^i,s^i,v^i), $i\in\mathbb{N}$ of characteristic invariants generated by Procedure 4.8. Then, we call

$$\omega=\min\{i\in\mathbb{N}^0|\; s^i=0\}$$

the *delay index* of (1.1).

Theorem 4.10. *Consider the DDAE* (1.2) *and let* ω *be the delay-index of* (1.1)*. Moreover, suppose that consistency conditions* (4.10) *at* $t=0$*, and* (4.12) *of all reformulation steps* $1,\ldots,\omega$ *are satisfied. Then,* (1.1) *has the same solution set as the following DDAE*

$$N_+^\omega X_+^\omega(t)+N_-^\omega X_-^\omega(t)=f^\omega(t), \tag{4.18}$$

with delay-index 0*, where*

$$N_+^\omega=\begin{bmatrix} F_1^\omega & F_2^\omega\\ 0 & H^\omega\\ 0 & 0\end{bmatrix},\; N_-^\omega=\begin{bmatrix} *\\ 0\\ 0\end{bmatrix},\; f^\omega=\begin{bmatrix} f_1^\omega\\ f_2^\omega\\ f_3^\omega\end{bmatrix},\quad \begin{matrix} r^\omega\\ d^\omega\\ v^\omega\end{matrix} \tag{4.19}$$

$$X^\omega=[x^{(k_\omega)}(t)^T,\ldots,x(t)^T,\; x^{(\kappa_\omega)}(t-\tau)^T,\ldots,x(t-\tau)^T]^T=\begin{bmatrix} X_+^\omega\\ X_-^\omega\end{bmatrix}.$$

In (4.19)*, the matrix* $\begin{bmatrix} F_1^\omega & F_2^\omega\\ 0 & H^\omega\end{bmatrix}$ *is strangeness-free and of full row rank.*

Since (1.2) is a special case of system (1.1), we can apply Theorem 4.10 to study (1.2). Due to Definition 3.9, the fact that matrix $\begin{bmatrix} F_1^\omega & F_2^\omega\\ 0 & H^\omega\end{bmatrix}$ is strangeness-free and of full row rank implies that there exist a nonsingular matrix $P\in\mathbb{C}^{\ell,\ell}$ such that by scaling system (4.18) with P from the left, we obtain

$$\left[\begin{array}{c|c|c|c} \hat{A}_{k_\omega,1} & \hat{A}_{k_\omega-1,1} & \dots & \hat{A}_{0,1} \\ & \hat{A}_{k_\omega-1,2} & \dots & \hat{A}_{0,2} \\ & & \ddots & \vdots \\ & & & \hat{A}_{0,k_\omega+1} \\ \hline 0 & 0 & \dots & 0 \end{array}\right] \begin{bmatrix} x^{(k_\omega)}(t) \\ x^{(k_\omega-1)}(t) \\ \vdots \\ x(t) \end{bmatrix} +$$

$$+ \left[\begin{array}{ccc} * & \dots & * \\ * & \dots & * \\ \vdots & & \vdots \\ * & \dots & * \\ \hline 0 & 0 & 0 \end{array}\right] \begin{bmatrix} x^{(\kappa^\omega)}(t-\tau) \\ \vdots \\ x(t-\tau) \end{bmatrix} = \left[\begin{array}{c} \hat{f}_1(t) \\ \hat{f}_2(t) \\ \vdots \\ \hat{f}_{k_\omega+1}(t) \\ \hline \hat{f}_{k_\omega+2}(t) \end{array}\right] \tag{4.20}$$

and

$$\tilde{A}_{k_\omega} := \begin{bmatrix} \hat{A}_{k_\omega,1} \\ \hat{A}_{k_\omega-1,2} \\ \vdots \\ \hat{A}_{0,k_\omega+1} \end{bmatrix}$$

has full row rank. Rewriting (4.20) block row-wise in behavior form as

$$\left[\begin{array}{c} N_1^\omega \\ N_2^\omega \\ \vdots \\ N_{k_\omega+1}^\omega \\ \hline 0 \end{array}\right] X^\omega = \left[\begin{array}{c} \hat{f}_1(t) \\ \hat{f}_2(t) \\ \vdots \\ \hat{f}_{k_\omega+1}(t) \\ \hline \hat{f}_{k_\omega+2}(t) \end{array}\right], \tag{4.21}$$

we get the following consistency conditions at $t = 0$

$$\begin{aligned} N_1^\omega X^\omega - f_1(t) &= 0, \\ \frac{d^i}{dt^i}\Big(N_2^\omega X^\omega - f_2(t)\Big) &= 0, \ i = 0, 1, \\ & \quad \dots \\ \frac{d^i}{dt^i}\Big(N_{k_\omega+1}^\omega X^\omega - f_{k_\omega+1}(t)\Big) &= 0, \ i = 0, \dots, k_\omega + 1, \\ f_{k_\omega+2}(t) &= 0. \end{aligned} \tag{4.22}$$

Therefore, similar to Theorem 3.7, we obtain the following theorem, which stresses that every DDAE of the form (1.2) contains an underlying high-order DDE.

Theorem 4.11. *Consider the DDAE* (1.2)*. Let ω be its delay-index and assume that* (4.20) *is the delay-index* 0 *formulation of* (1.2)*. Moreover, assume that the consistency conditions* (4.10) *at $t = 0$, and* (4.12) *of all reformulation steps* $1, \dots, \omega$ *are satisfied. Furthermore, suppose that the consistency condition* (4.22) *is also satisfied. Then,* (1.2) *has the same solution set as the following DDE*

$$\tilde{A}_{k_\omega} x^{(k_\omega)}(t) + \dots + \tilde{A}_0 x(t) + \tilde{A}_{-1} x(t-\tau) + \dots + \tilde{A}_{-\kappa_\omega-k_\omega} x^{(\kappa_\omega+k_\omega)}(t-\tau) + \tilde{f}^\omega(t) = 0,$$

where $\tilde{A}_{k_\omega}$ has full row rank.

Proof. The proof follows by differentiating the j-th equation of system (4.21) $j-1$ times for each $2 \leqslant j \leqslant k_\omega$. □

Corollary 4.12. Consider the DDAE (1.2). Moreover, assume that the function f is sufficiently smooth.

i) The DDAE (1.2) is solvable if and only if the consistency conditions (4.10) at $t = 0$, and (4.12) of all reformulation steps $1, \ldots, \omega$ are satisfied and also the consistency condition (4.22) is satisfied.

ii) The initial value problem (1.2)-(1.3) is uniquely solvable if and only if in addition the matrix $\tilde{A}_{k_\omega}$ is square.

5 Conclusion

In this paper, we have investigated the theoretical and numerical analysis of a class of delay differential-algebraic equations (DDAEs). We have proved that under some consistency conditions every DDAE with single delay can be reformulated as a DDE. We also introduced an appropriate delay-index for nontrivial DDAEs and constructed strangeness-free reformulations and used these to investigate solvability, consistency and smoothness requirements. The key tool in the analysis is a combination of the algebraic approach and the behavior approach.

In summary, we have shown that in order to deal with DDAEs in full generality, one needs to handle not only the structure of the matrix coefficients but also some hidden high-order DDE.

References

[1] U. M. Ascher and L. R. Petzold. The numerical solution of delay-differential algebraic equations of retarded and neutral type. *SIAM J. Numer. Anal.*, 32:1635–1657, 1995.

[2] A. Bellen and M. Zennaro. *Numerical Methods for Delay Differential Equations.* Oxford University Press, Oxford, UK, 2003.

[3] K. E. Brenan, S. L. Campbell, and L. R. Petzold. *Numerical Solution of Initial-Value Problems in Differential Algebraic Equations.* SIAM Publications, Philadelphia, PA, 2nd edition, 1996.

[4] S. L. Campbell. Singular linear systems of differential equations with delays. *Appl. Anal.*, 2:129–136, 1980.

[5] S. L. Campbell. Comments on 2-d descriptor systems. *Automatica*, 27(1):189–192, 1991.

[6] S. L. Campbell and V. H. Linh. Stability criteria for differential-algebraic equations with multiple delays and their numerical solutions. *Appl. Math Comput.*, 208(2):397 – 415, 2009.

[7] H. Gluesing-Luerssen. *Linear Delay-Differential Systems with Commensurate Delays: An Algebraic Approach.* Springer, Berlin, 2002.

[8] G. H. Golub and C. F. Van Loan. *Matrix Computations.* The Johns Hopkins University Press, Baltimore, MD, 3rd edition, 1996.

[9] E. Hairer and G. Wanner. *Solving Ordinary Differential Equations II: Stiff and Differential-Algebraic Problems.* Springer-Verlag, Berlin, Germany, 2nd edition, 1996.

[10] P. Kunkel and V. Mehrmann. *Differential-Algebraic Equations. Analysis and Numerical Solution.* EMS Publishing House, Zürich, Switzerland, 2006.

[11] P. Losse and V. Mehrmann. Controllability and observability of second order descriptor systems. *SIAM J. Cont. Optim.*, 47 (3), 2008.

[12] V. Mehrmann and C. Shi. Transformation of high order linear differential-algebraic systems to first order. *Numer. Alg.*, 42:281, 2006.

[13] J. W. Polderman and J. C. Willems. *Introduction to Mathematical Systems Theory: A Behavioural Approach*. Springer-Verlag, New York, NY, 1998.

[14] A. Steinbrecher. Regularization of quasi-linear differential-algebraic equations in multibody dynamics. In *Proceedings of Multibody Dynamics 2011 - ECCOMAS Thematic Conference Brussels (Bruxelles, Belgium, July 4-7, 2011)*, 2011.

[15] L. Wunderlich. Numerical treatment of second order differential-algebraic systems. In *Proc. Appl. Math. and Mech. (GAMM 2006, Berlin, March 27-31, 2006)*, volume 6, pages 775–776, 2006. 23.01.2007.