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PASSIVITY-PRESERVING BALANCED TRUNCATION FOR ELECTRICAL CIRCUITS

TIMO REIS† AND TATJANA STYKEL‡

Abstract. We present a passivity-preserving balanced truncation model reduction method for differential-algebraic equations arising in circuit simulation. This method is based on balancing the solutions of projected Lur'e equations. By making use of the special structure of circuit equations, we can reduce the numerical effort for balanced truncation significantly. It is shown that the property of reciprocity is also preserved in the reduced-order model. Network topological interpretations of certain circuit effects are given. The presented theory is illustrated by a numerical example.

Key words. balanced truncation, bounded real, electrical circuits, Lur'e equations, model reduction, passivity, positive real, projectors, reciprocity, Riccati equations

1. Introduction.

Design of very large system integrated (VLSI) circuits with distributed elements such as transmission lines and transistors is no longer possible without computer simulations that involve numerical solution of differential-algebraic equations (DAEs). Such equations may have order up to ten millions or even more that makes the analysis and simulations unacceptably time consuming and expensive. In this context, model order reduction is of great importance, especially if simulation is required for different input signals.

A general idea of model order reduction is to approximate the large-scale system by a much smaller model that captures the input-output behavior of the original system to a required accuracy and also preserves essential physical properties such as stability and passivity. Especially, the preservation of passivity allows a back interpretation of the reduced-order model as an electrical circuit which has fewer electrical components than the original one [1, 18].

Krylov subspace based methods are the most used model reduction methods in circuit simulation, e.g., [5, 6, 15]. Although these methods are efficient for very large sparse problems, stability and passivity are not necessarily preserved in the reduced-order model, so that usually a post-processing is needed to guarantee these properties. Passivity-preserving model reduction methods based on Krylov subspaces have been developed for standard state space systems [2, 4, 22] and also for structured generalized state space systems describing interconnect circuits [6, 8, 11, 15]. Despite the successful application of these methods in circuit simulation, they provide only a good local approximation and, so far, there exist no global error bounds.

Balanced truncation is another model reduction approach commonly used in control design. In order to capture specific system properties, different balancing techniques have been developed in the last thirty years [10, 13, 14, 16, 19, 23]. An important property of balancing-related model reduction is the existence of computable error bounds that allow an adaptive choice of the order of the approximate model.

In this paper, we consider a passivity-preserving model reduction method for linear circuit equations obtained via modified nodal analysis (MNA). This method is

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based on bounded real balanced truncation applied to a Moebius-transformed system, see [19]. It requires balancing two Gramians that satisfy the projected Lur'e equations. For a large class of systems, such equations with large-scale matrix coefficients can be solved using Newton's method or a method related to an generalized Hamiltonian eigenvalue problem. The major difficulty in the numerical solution of the projected Lur'e equations is that the spectral projectors onto the deflating subspaces corresponding to the finite and infinite eigenvalues of an underlying pencil are required. Fortunately, the matrix coefficients in MNA circuit equations have some special block structure. We will exploit this structure to construct the required projectors in explicit form. We also present an efficient implementation of the bounded real balanced truncation method for large-scale circuits equations.

The paper is organized as follows. In Section 2, we briefly review the basic framework of linear circuit theory. Section 3 considers the passivity-preserving balanced truncation model reduction method for DAEs. This technique is applied to circuit equations in Section 4. Finally, in Section 5, a numerical example is presented.

Throughout the paper $\mathbb{R}^{n,m}$ and $\mathbb{C}^{n,m}$ denote the spaces of $n\times m$ real and complex matrices, respectively. The open right half-plane is denoted by $\mathbb{C}_+$ and $i=\sqrt{-1}$. The matrices A^T and A^* denote, respectively, the transpose and the conjugate transpose of $A\in\mathbb{C}^{n,m}$, and $A^{-T}=(A^{-1})^T$. An identity matrix of order n is denoted by I_n or simply by I. The zero $n\times m$ matrix is denoted by $0_{n,m}$ or simply by 0. We denote by $\operatorname{im} A$ and $\ker A$ the image and the kernel of A, respectively. Further, for Hermitian matrices $P,Q\in\mathbb{C}^{n,n}$ we write $P>Q$ ($P\geq Q$) if $P-Q$ is positive (semi)definite. The Euclidean vector norm is denoted by $\|\cdot\|$, and the spectral and Frobenius matrix norms are denoted by $\|\cdot\|_2$ and $\|\cdot\|_F$, respectively.

2. Circuit equations. A general electrical circuit can be modelled as a directed graph whose nodes correspond to the nodes of the circuit and whose branches correspond to the circuit elements like capacitors, inductors, resistors, diodes and transistors. Using Kirchhoff's current and voltage laws as well as the branch constitutive relations, linear RLC circuits consisting only of linear resistors, inductors, capacitors and independent current and voltage sources can be described via MNA by the following system of DAEs

$$\begin{aligned} E\dot{x}(t) &= Ax(t)+Bu(t),\\ y(t) &= Cx(t)+Du(t), \end{aligned} \tag{2.1}$$

where

$$E=\begin{bmatrix} A_{\mathcal{C}}\mathcal{C}A_{\mathcal{C}}^T & 0 & 0\\ 0 & \mathcal{L} & 0\\ 0 & 0 & 0\end{bmatrix},\qquad A=\begin{bmatrix} -A_{\mathcal{R}}\mathcal{R}^{-1}A_{\mathcal{R}}^T & -A_{\mathcal{L}} & -A_{\mathcal{V}}\\ A_{\mathcal{L}}^T & 0 & 0\\ A_{\mathcal{V}}^T & 0 & 0\end{bmatrix},$$
$$C=\begin{bmatrix} -A_{\mathcal{I}}^T & 0 & 0\\ 0 & 0 & -I\end{bmatrix}=B^T,\qquad D=0, \tag{2.2}$$

$$x(t)=\begin{bmatrix}\eta(t)\\ i_{\mathcal{L}}(t)\\ i_{\mathcal{V}}(t)\end{bmatrix},\qquad u(t)=\begin{bmatrix} i_{\mathcal{I}}(t)\\ v_{\mathcal{V}}(t)\end{bmatrix},\quad y(t)=\begin{bmatrix} v_{\mathcal{I}}(t)\\ i_{\mathcal{V}}(t)\end{bmatrix}.$$

Here η is a vector of node potentials, $i_{\mathcal{L}}$, $i_{\mathcal{I}}$ and $i_{\mathcal{V}}$ are vectors of currents through inductors, current and voltage sources, respectively, $v_{\mathcal{I}}$ and $v_{\mathcal{V}}$ are vectors of voltages of current and voltage sources, respectively. We denote by n_η the number of nodes excepting the grounding node, by $n_{\mathcal{L}}$ the number of inductors, by $n_{\mathcal{I}}$ the number of current sources and by $n_{\mathcal{V}}$ the number of voltage sources. The number of state variables $n = n_\eta + n_{\mathcal{L}} + n_{\mathcal{V}}$ is called the *order* of system (2.1), and $m = n_{\mathcal{I}} + n_{\mathcal{V}}$ is the number of inputs. Furthermore, $A_{\mathcal{C}}$, $A_{\mathcal{L}}$, $A_{\mathcal{R}}$, $A_{\mathcal{V}}$ and $A_{\mathcal{I}}$ are the incidence matrices describing the circuit topology, and $\mathcal{R}$, $\mathcal{L}$ and $\mathcal{C}$ are the resistance, inductance and capacitance matrices, respectively. Linear RLC circuits are often used to model the interconnects, transmission lines and pin packages in VLSI networks. They arise also in the linearization of nonlinear circuit equations around DC operating point.

We now give our general assumptions on the above defined matrices.

ASSUMPTIONS 2.1.

(A1) *The matrix $A_{\mathcal{V}}$ has full column rank.*

(A2) *The matrix $[\,A_{\mathcal{C}},\, A_{\mathcal{L}},\, A_{\mathcal{R}},\, A_{\mathcal{V}}]$ has full row rank.*

(A3) *The matrices $\mathcal{C}$, $\mathcal{R}$ and $\mathcal{L}$ are symmetric and positive definite.*

Assumption (A1) corresponds to the absence of loops of voltage sources, while (A2) forbids cutsets of current sources. Condition (A3) on the capacitance, resistance and inductance matrices means that all elements of the circuit do not generate energy. These three assumptions together guarantee that the pencil $\lambda E - A$ is *regular* [7], i.e., $\det(\lambda E - A) \neq 0$ for some $\lambda \in \mathbb{C}$, and it is of index at most two [3], see [9] for the definition of index. Moreover, (A1)-(A3) make sure that system (2.1), (2.2) is *passive*, and, hence, it is stable [1]. Note, however, that the asymptotic stability of (2.1), (2.2) is, in general, not guaranteed, since $\lambda E - A$ might have generalized eigenvalues on the imaginary axis. For the asymptotic stability, some further circuit topological conditions have to be fulfilled such as the absence of loops of voltage sources, capacitors and inductors and also the absence of cutsets consisting only of current sources, capacitors and inductors [20]. Note that for the model reduction method considered here, we do not claim that the circuit forms an asymptotically stable system.

Passivity is closely related to positive realness of a transfer function of (2.1) given by $\boldsymbol{G}(s) = C(sE - A)^{-1}B + D$. The transfer function $\boldsymbol{G}$ is *positive real* if it has no poles in $\mathbb{C}_+$ and $\boldsymbol{G}(s) + \boldsymbol{G}(s)^* \geq 0$ for all $s \in \mathbb{C}_+$.

PROPOSITION 2.2. [18] *If assumptions* (A1)-(A3) *are valid, then the transfer function $\boldsymbol{G}(s) = C(sE - A)^{-1}B + D$ with the matrices E, A, B, C and D as in* (2.2) *is positive real.*

Another important property of circuit equations is reciprocity. We call a matrix $S \in \mathbb{R}^{m,m}$ a *signature* if S is diagonal and $S^2 = I_m$. System (2.1) is called *reciprocal* with an external signature $S_{\rm ext} \in \mathbb{R}^{m,m}$ if its transfer function satisfies $\boldsymbol{G}(s) = S_{\rm ext}\boldsymbol{G}(s)^T S_{\rm ext}$ for all $s \in \mathbb{C}$.

PROPOSITION 2.3. [18] *If assumptions* (A1)-(A3) *are valid, then system* (2.1), (2.2) *is reciprocal with the external signature*

$$S_{\rm ext} = \operatorname{diag}(I_{n_{\mathcal{I}}}, -I_{n_{\mathcal{V}}}). \tag{2.3}$$

Note that the transfer function $\boldsymbol{G}$ is positive real if and only if the Moebius-transformed function $\boldsymbol{\mathcal{G}}(s) = (I - \boldsymbol{G}(s))(I + \boldsymbol{G}(s))^{-1}$ is *bounded real*, i.e., $\boldsymbol{\mathcal{G}}$ has no poles in $\mathbb{C}_+$ and $I - \boldsymbol{\mathcal{G}}(s)^*\boldsymbol{\mathcal{G}}(s) \geq 0$ for all $s \in \mathbb{C}_+$, see [1]. Furthemore, if $\boldsymbol{G}$ is reciprocal with the external signature $S_{\rm ext}$, then $\boldsymbol{\mathcal{G}}$ is also reciprocal with the same external signature. For system (2.1), (2.2), a realization of $\boldsymbol{\mathcal{G}}$ is given by $\boldsymbol{\mathcal{G}} = [\,\mathcal{E},\, \mathcal{A},\, \mathcal{B},\, \mathcal{C},\, I\,]$, where

$$\mathcal{E} = E = \begin{bmatrix} A_{\mathcal{C}}\mathcal{C}A_{\mathcal{C}}^T & 0 & 0 \\ 0 & \mathcal{L} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \mathcal{B} = -\sqrt{2}B(I+D)^{-1} = \sqrt{2}\begin{bmatrix} A_{\mathcal{I}} & 0 \\ 0 & 0 \\ 0 & I \end{bmatrix} = -\mathcal{C}^T,$$

$$\mathcal{A} = A - B(I+D)^{-1}C = \begin{bmatrix} -A_{\mathcal{R}}\mathcal{R}^{-1}A_{\mathcal{R}}^T - A_{\mathcal{I}}A_{\mathcal{I}}^T & -A_{\mathcal{L}} & -A_{\mathcal{V}} \\ A_{\mathcal{L}}^T & 0 & 0 \\ A_{\mathcal{V}}^T & 0 & -I \end{bmatrix}. \tag{2.4}$$

For a physical interpretation of the Moebius-transformation of circuit equations, we refer to [1, p. 28].

3. Passivity-preserving balanced truncation. Model order reduction consists in the approximation of the large-scale system (2.1) by a reduced-order model

$$\begin{aligned} \widetilde{E}\,\dot{\widetilde{x}}(t) &= \widetilde{A}\,\widetilde{x}(t) + \widetilde{B}\,u(t), \\ \widetilde{y}(t) &= \widetilde{C}\,\widetilde{x}(t) + \widetilde{D}\,u(t), \end{aligned} \tag{3.1}$$

where $\widetilde{E}$, $\widetilde{A} \in \mathbb{R}^{\ell,\ell}$, $\widetilde{B} \in \mathbb{R}^{\ell,m}$, $\widetilde{C} \in \mathbb{R}^{p,\ell}$, $\widetilde{D} \in \mathbb{R}^{p,m}$ and $\ell \ll n$. It is required that the approximate model (3.1) preserves essential properties of (2.1) like passivity and reciprocity and that the approximation error is small. In this section, we briefly describe a passivity-preserving model reduction method from [19] that is based on bounded real balanced truncation applied to the Moebius-transformed system $\boldsymbol{\mathcal{G}}$.

For simplicity, and since this holds true for electrical circuits, we restrict ourself to system (2.1) with $m = p$ and $D = 0$. Since $\boldsymbol{\mathcal{G}}$ is bounded real, there exist the matrices $\mathcal{M}_0, \mathcal{J}_c, \mathcal{J}_o \in \mathbb{R}^{m,m}$ such that

$$\mathcal{M}_0 = \lim_{s\to\infty} \boldsymbol{\mathcal{G}}(s) = I - 2\lim_{s\to\infty} C(sE - A + BC)^{-1}B, \tag{3.2}$$

$$I - \mathcal{M}_0\mathcal{M}_0^T = \mathcal{J}_c\mathcal{J}_c^T, \qquad I - \mathcal{M}_0^T\mathcal{M}_0 = \mathcal{J}_o^T\mathcal{J}_o.$$

Let $\mathcal{P}_l$ and $\mathcal{P}_r$ be the projectors onto the left and right deflating subspaces of the pencil $\lambda\mathcal{E} - \mathcal{A} = \lambda E - (A - BC)$ corresponding to the finite eigenvalues along the left and right deflating subspaces corresponding to the eigenvalue at infinity. If system (2.1) is R-minimal, i.e.,

$$\operatorname{rank}[\,\lambda E - A\,,\ B\,] = \operatorname{rank}[\,\lambda E^T - A^T,\ C^T\,] = n \ \text{ for all } \ \lambda \in \mathbb{C},$$

then $\boldsymbol{\mathcal{G}} = [\,\mathcal{E},\,\mathcal{A},\,\mathcal{B},\,\mathcal{C},\,I\,]$ as in (2.4) is also R-minimal. This condition together with the bounded realness of $\boldsymbol{\mathcal{G}}$ implies that the *projected Lur'e equations*

$$\begin{aligned} (A-BC)XE^T + EX(A-BC)^T + 2\mathcal{P}_lBB^T\mathcal{P}_l^T &= -2\,\mathcal{K}_c\mathcal{K}_c^T, \\ X = \mathcal{P}_rX\mathcal{P}_r^T \geq 0, \quad EXC^T - \mathcal{P}_lB\mathcal{M}_0^T &= -\mathcal{K}_c\mathcal{J}_c^T, \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} (A-BC)^TYE + E^TY(A-BC) + 2\mathcal{P}_r^TC^TC\mathcal{P}_r &= -2\,\mathcal{K}_o^T\mathcal{K}_o, \\ Y = \mathcal{P}_l^TY\mathcal{P}_l \geq 0, \quad -E^TYB + \mathcal{P}_r^TC^T\mathcal{M}_0 &= -\mathcal{K}_o^T\mathcal{J}_o \end{aligned} \tag{3.4}$$

are solvable for $X \in \mathbb{R}^{n,n}$, $\mathcal{K}_c \in \mathbb{R}^{n,m}$ and $Y \in \mathbb{R}^{n,n}$, $\mathcal{K}_o \in \mathbb{R}^{m,n}$, respectively. Moreover, there exist the extremal solutions that satisfy $0 \leq X_{\min} \leq X \leq X_{\max}$ and

$0 \le Y_{\min} \le Y \le Y_{\max}$ for all symmetric solutions X and Y of (3.3) and (3.4), respectively. The minimal solutions $X_{\min}$ and $Y_{\min}$ are called the *bounded real Gramians* of the Moebius-transformed system $\boldsymbol{\mathcal{G}}$. Note that the R-minimality of $\boldsymbol{\mathcal{G}}$ is violated for a large class of electrical circuits. In general, this may cause that the projected Lur'e equations (3.3) and (3.4) are unsolvable. In the next section we will, however, show that for E, A, B and C as in (2.2) (minimal) solutions of (3.3) and (3.4) exist in any case.

The passivity-preserving balanced truncation model reduction method for $\boldsymbol{G}$ consists in computing the reduced-order system $\widetilde{\boldsymbol{\mathcal{G}}}$ via the bounded real balanced truncation method applied to the Moebius-transformed system $\boldsymbol{\mathcal{G}} = (I - \boldsymbol{G})(I + \boldsymbol{G})^{-1}$ and then the back Moebius transformation $\widetilde{\boldsymbol{G}} = (I - \widetilde{\boldsymbol{\mathcal{G}}})(I + \widetilde{\boldsymbol{\mathcal{G}}})^{-1}$, see [19] for details. This method can be summarized in our notations as follows.

ALGORITHM 3.1 *Passivity-preserving balanced truncation model reduction method. Given a passive system $\boldsymbol{G} = [\,E,\, A,\, B,\, C,\, 0\,]$, compute a passive reduced-order model $\widetilde{\boldsymbol{G}} = [\,\widetilde{E},\, \widetilde{A},\, \widetilde{B},\, \widetilde{C},\, 0\,]$.*

1. *Compute the Cholesky factors $\hat{R}$ and $\hat{L}$ of the solutions $\hat{X} = \hat{R}\hat{R}^T$, $\hat{Y} = \hat{L}\hat{L}^T$ of the projected discrete-time Lyapunov equations*

$$(A-BC)\hat{X}(A-BC)^T - E\hat{X}E^T = 2\mathcal{Q}_l BB^T\mathcal{Q}_l^T, \;\; \hat{X} = \mathcal{Q}_r\hat{X}\mathcal{Q}_r^T, \qquad (3.5)$$

$$(A-BC)^T\hat{Y}(A-BC) - E^T\hat{Y}E = 2\mathcal{Q}_r^T C^T C\mathcal{Q}_r, \;\; \hat{Y} = \mathcal{Q}_l^T\hat{Y}\mathcal{Q}_l, \qquad (3.6)$$

   where $\mathcal{Q}_l = I - \mathcal{P}_l$ and $\mathcal{Q}_r = I - \mathcal{P}_r$.
2. *Compute the singular value decomposition $\hat{L}^T(A - BC)\hat{R} = U\Theta V^T$, where $U^TU = V^TV = I_{\ell_\infty}$ and $\Theta = \operatorname{diag}(\theta_1, \ldots, \theta_{\ell_\infty})$ is nonsingular.*
3. *Compute the matrix $\mathcal{M}_0 = I + 2CT_\infty W_\infty B$ with $W_\infty = \Theta^{-1/2}U^T\hat{L}^T$ and $T_\infty = \hat{R}V\Theta^{-1/2}$.*
4. *Compute the Cholesky factors R and L of $X_{\min} = RR^T$ and $Y_{\min} = LL^T$ that are the minimal solutions of the projected Lur'e equations* (3.3) *and* (3.4), *respectively.*
5. *Compute the singular value decomposition*

$$L^TER = [\,U_1,\, U_2\,]\begin{bmatrix}\Pi_1 & \\ & \Pi_2\end{bmatrix}[\,V_1,\, V_2\,]^T,$$

   where $[U_1,\, U_2]$ and $[V_1,\, V_2]$ have orthonormal columns,

$$\Pi_1 = \operatorname{diag}(\pi_1 I_{l_1}, \ldots, \pi_r I_{l_r}), \quad \Pi_2 = \operatorname{diag}(\pi_{r+1} I_{l_{r+1}}, \ldots, \pi_q I_{l_q})$$

   and $\pi_1 > \ldots > \pi_r > \pi_{r+1} > \ldots > \pi_q$.
6. *Compute the reduced-order system*

$$[\,\widetilde{E},\, \widetilde{A},\, \widetilde{B},\, \widetilde{C},\, 0\,] = [\,WET,\, WAT,\, WB,\, CT,\, 0\,]$$

   where $W = [\,W_f^T,\, W_\infty^T]^T$ and $T = [\,T_f,\, T_\infty\,]$ with

$$W_f = \Pi_1^{-1/2}U_1^TL^T, \qquad T_f = RV_1\Pi_1^{-1/2}.$$

The values $\pi_1, \ldots, \pi_q$ are called the *bounded real characteristic values* of $\boldsymbol{\mathcal{G}}$. They can be used to estimate the $\mathbb{H}_\infty$-*norm* of the error defined as

$$\|\widetilde{\boldsymbol{G}} - \boldsymbol{G}\|_{\mathbb{H}_\infty} = \sup_{s\in\mathbb{C}_+} \|\widetilde{\boldsymbol{G}}(s) - \boldsymbol{G}(s)\|_2.$$

If r is chosen such that $\|I+\boldsymbol{G}\|_{\mathbb{H}_\infty}(\pi_{r+1}+\ldots+\pi_q)<1$, then we have the following error bound

$$\|\widetilde{\boldsymbol{G}}-\boldsymbol{G}\|_{\mathbb{H}_\infty}\leq\frac{\|I+\boldsymbol{G}\|^2_{\mathbb{H}_\infty}(\pi_{r+1}+\ldots+\pi_q)}{1-\|I+\boldsymbol{G}\|_{\mathbb{H}_\infty}(\pi_{r+1}+\ldots+\pi_q)},\tag{3.7}$$

see [19] for details and also for other error bounds.

4. Application to circuit equations. By exploiting the structure of circuit equations, the reduction procedure in Algorithm 3.1 can be made more efficient and accurate. First, we show that the spectral projectors $\mathcal{P}_l$ and $\mathcal{P}_r$ as well as the solutions of the projected Lur'e equations (3.3) and (3.4) are related by a similarity transformation with a signature matrix.

THEOREM 4.1. *For E, A, B and C given in* (2.2), *the spectral projectors $\mathcal{P}_l$ and $\mathcal{P}_r$ onto the left and right deflating subspace of $\lambda E-(A-BC)$ corresponding to the finite eigenvalues satisfy $\mathcal{P}_r=S_{\rm int}\mathcal{P}_l^TS_{\rm int}$, where*

$$S_{\rm int}=\operatorname{diag}(I_{n_\eta},-I_{n_\mathcal{L}},-I_{n_\mathcal{V}}).\tag{4.1}$$

Moreover, Y is a (*minimal*) *solution of* (3.4) *if and only if $X=S_{\rm int}YS_{\rm int}$ is a* (*minimal*) *solution of* (3.3).

Proof. Let $S_{\rm ext}$ be as in (2.3). Then the result follows from $S_{\rm ext}=S_{\rm ext}^{-1}$, $S_{\rm int}=S_{\rm int}^{-1}$ and $A^T=S_{\rm int}AS_{\rm int}$, $E^T=S_{\rm int}ES_{\rm int}$, $C^T=S_{\rm int}BS_{\rm ext}$. □

In the following, we present an explicit expression for the projector $\mathcal{P}_r$.

THEOREM 4.2. *For E, A, B and C as in* (2.2), *the matrix pencil $\lambda E-(A-BC)$ is of index at most two. Furthermore, the projector $\mathcal{P}_r$ onto the right deflating subspace of $\lambda E-(A-BC)$ corresponding to the finite eigenvalues along the right deflating subspace corresponding to the eigenvalue at infinity is given by*

$$\mathcal{P}_r=\begin{bmatrix}H_4(Q_\mathcal{C}H_3^{-1}H_2-I) & H_4Q_\mathcal{C}H_3^{-1}A_\mathcal{L}H_5 & 0\\ 0 & H_5 & 0\\ -A_\mathcal{V}^T(Q_\mathcal{C}H_3^{-1}H_2-I) & -A_\mathcal{V}^TQ_\mathcal{C}H_3^{-1}A_\mathcal{L}H_5 & 0\end{bmatrix},\tag{4.2}$$

where

$$\begin{aligned}
H_1&=P_{\mathcal{CRIV}}^TP_{\mathcal{CRIV}}+Q_{\mathcal{CRIV}}^TA_\mathcal{L}\mathcal{L}^{-1}A_\mathcal{L}^TQ_{\mathcal{CRIV}},\\
H_2&=A_\mathcal{R}\mathcal{R}^{-1}A_\mathcal{R}^T+A_\mathcal{I}A_\mathcal{I}^T+A_\mathcal{V}A_\mathcal{V}^T+A_\mathcal{L}\mathcal{L}^{-1}A_\mathcal{L}^TQ_{\mathcal{CRIV}}H_1^{-1}Q_{\mathcal{CRIV}}^TA_\mathcal{L}\mathcal{L}^{-1}A_\mathcal{L}^T,\\
H_3&=A_\mathcal{C}\mathcal{C}A_\mathcal{C}^T+H_2Q_\mathcal{C},\\
H_4&=Q_{\mathcal{CRIV}}H_1^{-1}Q_{\mathcal{CRIV}}^TA_\mathcal{L}\mathcal{L}^{-1}A_\mathcal{L}^T-I,\\
H_5&=I-\mathcal{L}^{-1}A_\mathcal{L}^TQ_{\mathcal{CRIV}}H_1^{-1}Q_{\mathcal{CRIV}}^TA_\mathcal{L}
\end{aligned}\tag{4.3}$$

with the projectors $Q_\mathcal{C}$ and $Q_{\mathcal{CRIV}}$ onto $\ker A_\mathcal{C}^T$ *and* $\ker[\,A_\mathcal{C},\,A_\mathcal{R},\,A_\mathcal{I},\,A_\mathcal{V}\,]^T$, *respectively, and $P_{\mathcal{CRIV}}=I-Q_{\mathcal{CRIV}}$.*

Proof. See Appendix A. □

The solvability of the projected Lur'e equations and the existence of the minimal solutions requires, in general, the R-minimality of the system. We will now state that for system (2.1) with E, A, B and C as in (2.2) the solvability and, especially, the existence of the minimal solutions is guaranteed nevertheless.

THEOREM 4.3. *Let E, A, B and C be given in* (2.2)*. Under Assumptions 2.1 the projected Lur'e equations* (3.3) *and* (3.4) *are solvable. Furthermore, there exist the minimal solutions $X_{\min}$ and $Y_{\min}$ satisfying $0 \le X_{\min} \le X$ and $0 \le Y_{\min} \le Y$ for all symmetric solutions X and Y of* (3.3) *and* (3.4)*, respectively.*

Proof. See Appendix B. ∎

By Theorem 4.1 it is sufficient to compute only one of the projectors $\mathcal{P}_l$ and $\mathcal{P}_r$ and only one of $X_{\min}$ and $Y_{\min}$. Furthermore, the Cholesky factors R and L of $X_{\min}$ and $Y_{\min}$, respectively, are related as $R = S_{\rm int}L$. Since $ES_{\rm int} = S_{\rm int}E$ is symmetric, this also holds true for $L^TER = L^TES_{\rm int}L$. Thus, to determine the characteristic values π_j we can compute an eigenvalue decomposition of $L^TES_{\rm int}L$ instead of a more expensive singular value decomposition. Let

$$L^TES_{\rm int}L = [\,U_1,\,U_2\,]\begin{bmatrix}\Lambda_1 & \\ & \Lambda_2\end{bmatrix}[\,U_1,\,U_2\,]^T \tag{4.4}$$

be an eigenvalue decomposition, where $[\,U_1,\,U_2\,]$ is orthogonal and

$$\Lambda_1 = \mathrm{diag}(\lambda_1 I,\ldots,\lambda_r I), \qquad \Lambda_2 = \mathrm{diag}(\lambda_{r+1} I,\ldots,\lambda_q I)$$

with $|\lambda_1| > \ldots > |\lambda_r| > |\lambda_{r+1}| > \ldots > |\lambda_q|$. Then $|\lambda_j| = \pi_j$ for $i = 1,\ldots,q$ and the projection matrices W_f and T_f in Algorithm 3.1 can be taken as

$$W_f = |\Lambda_1|^{-1/2}U_1^TL^T, \qquad T_f = S_{\rm int}LU_1S|\Lambda_1|^{-1/2}, \tag{4.5}$$

where $|\Lambda_1| = \mathrm{diag}(|\lambda_1|I,\ldots,|\lambda_r|I)$ and $S = \mathrm{diag}(\mathrm{sign}(\lambda_1)I,\ldots,\mathrm{sign}(\lambda_r)I)$.

We now deliver an explicit expression for the matrix $\mathcal{M}_0$.

THEOREM 4.4. *For E, A, B, C as in* (2.2)*, the matrix $\mathcal{M}_0$ in* (3.2) *is given by*

$$\mathcal{M}_0 = \begin{bmatrix} I - 2A_{\mathcal{I}}^TQ_{\mathcal{C}}H_6^{-1}Q_{\mathcal{C}}^TA_{\mathcal{I}} & 2A_{\mathcal{I}}^TQ_{\mathcal{C}}H_6^{-1}Q_{\mathcal{C}}^TA_{\mathcal{V}} \\ -2A_{\mathcal{V}}^TQ_{\mathcal{C}}H_6^{-1}Q_{\mathcal{C}}^TA_{\mathcal{I}} & -I + 2A_{\mathcal{V}}^TQ_{\mathcal{C}}H_6^{-1}Q_{\mathcal{C}}^TA_{\mathcal{V}} \end{bmatrix}, \tag{4.6}$$

where

$$H_6 = Q_{\mathcal{C}}^T(A_{\mathcal{R}}\mathcal{R}^{-1}A_{\mathcal{R}}^T + A_{\mathcal{I}}A_{\mathcal{I}}^T + A_{\mathcal{V}}A_{\mathcal{V}}^T)Q_{\mathcal{C}} + Q_{\mathcal{RIV}-\mathcal{C}}^TQ_{\mathcal{RIV}-\mathcal{C}} \tag{4.7}$$

and $Q_{\mathcal{RIV}-\mathcal{C}}$ is a projector onto $\ker([\,A_{\mathcal{R}},\,A_{\mathcal{I}},\,A_{\mathcal{V}}\,]^TQ_{\mathcal{C}})$.

Proof. See Appendix C. ∎

Since for circuit equations the matrix $\mathcal{M}_0$ is given in explicit form, we do not need to solve the projected Lyapunov equations (3.5) and (3.6). The bounded real balanced truncation method applied to $\boldsymbol{\mathcal{G}} = [\,E,\,A-BC,\,-\sqrt{2}B,\,\sqrt{2}C,\,I\,]$ provides the system $\widetilde{\boldsymbol{\mathcal{G}}} = [\,\widetilde{\mathcal{E}},\,\widetilde{\mathcal{A}},\,\widetilde{\mathcal{B}},\,\widetilde{\mathcal{C}},\,I\,]$ with

$$\widetilde{\mathcal{E}} = \begin{bmatrix} I & 0 \\ 0 & 0\end{bmatrix}, \qquad \widetilde{\mathcal{A}} = \begin{bmatrix} W_f(A-BC)T_f & 0 \\ 0 & I\end{bmatrix},$$
$$\widetilde{\mathcal{B}} = \begin{bmatrix} -\sqrt{2}W_fB \\ B_\infty\end{bmatrix}, \qquad \widetilde{\mathcal{C}} = \big[\,\sqrt{2}CT_f,\;C_\infty\big],$$

where W_f and T_f are as in (4.5) and $I - \mathcal{M}_0 = C_\infty B_\infty$. From the reciprocity of $\boldsymbol{\mathcal{G}}$, we obtain that $(I-\mathcal{M}_0)S_{\rm ext}$ is symmetric. Let

$$(I-\mathcal{M}_0)S_{\rm ext} = U_0\Lambda_0U_0^T \tag{4.8}$$

be an eigenvalue decomposition, where U_0 is orthogonal and $\Lambda_0 = \operatorname{diag}(\hat{\lambda}_1, \ldots, \hat{\lambda}_m)$. Then the Moebius transformation $\widetilde{\boldsymbol{G}} = (I - \widetilde{\boldsymbol{\mathcal{G}}})(I + \widetilde{\boldsymbol{\mathcal{G}}})^{-1}$ has the realization

$$\begin{aligned} \widetilde{E} = \widetilde{\mathcal{E}} = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, \qquad \widetilde{A} = \widetilde{\mathcal{A}} - \frac{1}{2}\widetilde{\mathcal{B}}\widetilde{\mathcal{C}} = \frac{1}{2}\begin{bmatrix} 2W_f\, A\, T_f & \sqrt{2}W_f\, BC_\infty \\ -\sqrt{2}B_\infty CT_f & 2I - B_\infty C_\infty \end{bmatrix}, \\ \widetilde{B} = -\frac{\sqrt{2}}{2}\widetilde{\mathcal{B}} = \frac{\sqrt{2}}{2}\begin{bmatrix} \sqrt{2}W_f\, B \\ -B_\infty \end{bmatrix} \qquad \widetilde{C} = \frac{\sqrt{2}}{2}\widetilde{\mathcal{C}} = \frac{\sqrt{2}}{2}\Big[\sqrt{2}CT_f,\ C_\infty\Big], \end{aligned} \tag{4.9}$$

where

$$B_\infty = S_0|\Lambda_0|^{1/2}U_0^T S_{\text{ext}}, \qquad C_\infty = U_0|\Lambda_0|^{1/2}, \tag{4.10}$$

with $|\Lambda_0| = \operatorname{diag}(|\hat{\lambda}_1|, \ldots, |\hat{\lambda}_m|)$ and $S_0 = \operatorname{diag}(\operatorname{sign}(\hat{\lambda}_1), \ldots, \operatorname{sign}(\hat{\lambda}_m))$.

We summarize the passivity-preserving balanced truncation method for electrical circuits (PABTEC) in the following algorithm.

Algorithm 4.5. *Passivity-preserving balanced truncation for electrical circuits. Given a passive system* $\boldsymbol{G} = [\,E,\ A,\ B,\ C,\ 0\,]$ *as in* (2.2), *compute a passive reduced-order model* $\widetilde{\boldsymbol{G}} = [\,\widetilde{E},\ \widetilde{A},\ \widetilde{B},\ \widetilde{C},\ 0\,]$.

1. *Compute the matrix* $\mathcal{M}_0$ *given in* (4.6).
2. *Compute the Cholesky factor* L *of the minimal solution* $Y_{\min} = LL^T$ *of the projected Lur'e equation* (3.4), *where* $\mathcal{P}_r$ *is as in* (4.2) *and* $\mathcal{P}_l = S_{\text{ext}}\mathcal{P}_r^T S_{\text{ext}}$ *with* S_{ext} *as in* (2.3).
3. *Compute the eigenvalue decompositions* (4.4) *and* (4.8).
4. *Compute the reduced-order system* $\widetilde{\boldsymbol{G}} = [\,\widetilde{E},\ \widetilde{A},\ \widetilde{B},\ \widetilde{C},\ 0\,]$ *as in* (4.9), *where* W_f, T_f *and* B_∞, C_∞ *are given in* (4.5) *and* (4.10), *respectively.*

The following theorem shows that the reduced-order system computed by this algorithm is reciprocal with the same external signature as the original system.

Theorem 4.6. *Let* E, A, B *and* C *be given in* (2.2). *The reduced-order system* $\widetilde{\boldsymbol{G}} = [\,\widetilde{E},\ \widetilde{A},\ \widetilde{B},\ \widetilde{C},\ 0\,]$ *in* (4.9) *is reciprocal with the external signature* S_{ext} *as in* (2.3).

Proof. For the matrices W_f, T_f as in (4.5) and B_∞, C_∞ as in (4.10), we have

$$\widetilde{E}^T = \widetilde{S}_{\text{int}}\widetilde{E}\widetilde{S}_{\text{int}}, \qquad \widetilde{A}^T = \widetilde{S}_{\text{int}}\widetilde{A}\widetilde{S}_{\text{int}}, \qquad \widetilde{B}^T = S_{\text{ext}}\widetilde{C}\widetilde{S}_{\text{int}}$$

with $\widetilde{S}_{\text{int}} = \operatorname{diag}(S, -S_0)$. Thus, $\widetilde{\boldsymbol{G}}(s) = S_{\text{ext}}\widetilde{\boldsymbol{G}}(s)^T S_{\text{ext}}$. □

If the matrix $I - \mathcal{M}_0^T\mathcal{M}_0$ is nonsingular, then the projected Lur'e equation (3.4) is equivalent to the projected Riccati equation

$$\begin{aligned} &(A - BC)^T YE + E^T Y(A - BC) + 2\mathcal{P}_r^T C^T C\mathcal{P}_r \\ &\qquad + 2(E^T YB - \mathcal{P}_r^T C^T\mathcal{M}_0)(I - \mathcal{M}_0^T\mathcal{M}_0)^{-1}(E^T YB - \mathcal{P}_r^T C^T\mathcal{M}_0)^T = 0, \\ &Y = \mathcal{P}_l^T Y\mathcal{P}_l. \end{aligned} \tag{4.11}$$

The following theorem gives necessary and sufficient topological conditions for the invertibility of $I - \mathcal{M}_0^T\mathcal{M}_0$.

Theorem 4.7. *Let the matrix* $\mathcal{M}_0$ *be as in* (4.6). *Then* $I - \mathcal{M}_0^T\mathcal{M}_0$ *is invertible if and only if*

$$\ker Q_{\mathcal{C}}^T[\,A_{\mathcal{I}},\ A_{\mathcal{V}}\,] = \{0\} \qquad \text{and} \qquad Q_{\mathcal{RC}}^T[\,A_{\mathcal{I}},\ A_{\mathcal{V}}\,] = 0,$$

where $Q_{\mathcal{RC}}$ *is a projector onto* $\ker[\,A_{\mathcal{R}},\ A_{\mathcal{C}}\,]^T$.

Proof. See Appendix D. □

Using [20, Proposition 4.5], it can be seen that the condition that $Q_{\mathcal{C}}^T[\,A_{\mathcal{I}},\, A_{\mathcal{V}}\,]$ has full column rank is equivalent to the absence of loops of branches of capacitances and sources except for loops only consisting of capacitive branches. On the other hand, the matrix $Q_{\mathcal{RC}}^T[\,A_{\mathcal{I}},\, A_{\mathcal{V}}\,]$ vanishes if and only if the circuit does not contain cutsets consisting of branches of inductances and sources except for cutsets only consisting of inductive branches [20, Proposition 4.4]. Equivalently, for each source, its incidence nodes are connected by a path only consisting of resistive and capacitive branches.

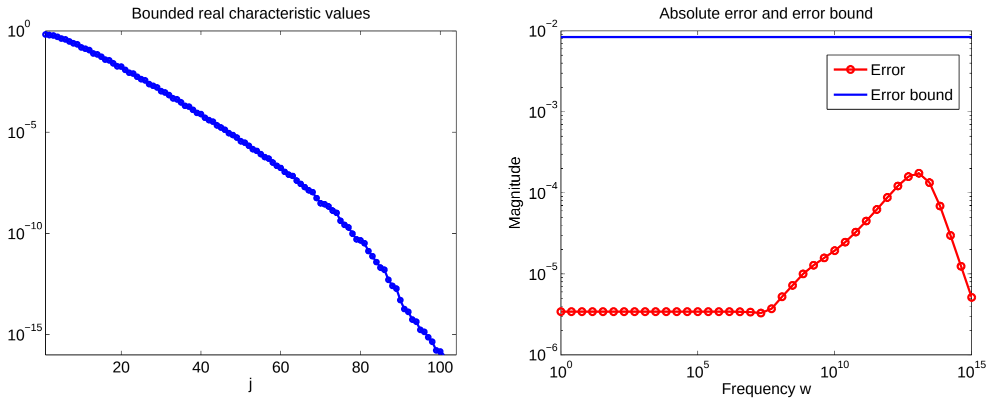
Thus, for systems with nonsingular $I - \mathcal{M}_0^T\mathcal{M}_0$, the Gramian $Y_{\min}$ can be computed by solving the projected Riccati equation (4.11). Such an equation with large-scale matrix coefficients can be solved using Newton's method or a method based on computing the delating subspaces of an extended Hamiltonian pencil, see [19] for details. In case of singular $I - \mathcal{M}_0^T\mathcal{M}_0$, small to medium-sized DAE system can be transformed similarly to the standard state space case [24] to a system of smaller dimension for which we can again write bounded real Riccati equations. However, the computation of the Gramians for large-scale systems with singular $I - \mathcal{M}_0^T\mathcal{M}_0$ remains an open problem.

5. Numerical example. In this section, we present some results of numerical experiments to demonstrate the feasibility of the described model reduction method for circuit equations. The computations were done on IBM RS 6000 44P Model 270 with machine precision $\varepsilon = 2.22 \times 10^{16}$ using MATLAB 7.0.4.

We have tested the PABTEC method on several circuit examples provided by NEC Laboratories Europe, IT Research Division. In all these examples the matrices E and A are badly scaled since the ratio $\|A\|_F/\|E\|_F$ varies from $O(10^{12})$ to $O(10^{18})$. This causes some difficulties in the numerical solution of the projected Riccati equation (4.11). To overcome this difficulties we applied the PABTEC method to the scaled system $\boldsymbol{G}_\alpha = [\,E_\alpha,\, A,\, B,\, C,\, 0\,]$, where $E_\alpha = \alpha E$ and α is chosen such that E_α and A have about the same norm. Note that the pencils $\lambda E-(A-BC)$ and $\lambda E_\alpha-(A-BC)$ have the same right (left) deflating subspaces corresponding to the finite eigenvalues and, hence, the same spectral projectors $\mathcal{P}_r$ and $\mathcal{P}_l$. The computed reduced-order model $\widetilde{\boldsymbol{G}}_\alpha = [\,\widetilde{E}_\alpha, \widetilde{A}, \widetilde{B}, \widetilde{C},\, 0\,]$ was then replaced by $\widetilde{\boldsymbol{G}} = [\,\widetilde{E}_\alpha/\alpha,\, \widetilde{A},\, \widetilde{B},\, \widetilde{C},\, 0\,]$. Next we report the results for one example only.

Example 5.1. This example is a three-port RC circuit described by the passive system of order $n = 2007$. The minimal solution of the projected Riccati equation (4.11) has been approximated by a low-rank matrix $Y_{\min} \approx \widetilde{L}\widetilde{L}^T$ with $\widetilde{L} \in \mathbb{R}^{n,102}$ using Newton's method as presented in [19]. Figure 5.1(a) shows the largest 100 bounded characteristic values of $\boldsymbol{\mathcal{G}} = (I-\boldsymbol{G})(I+\boldsymbol{G})^{-1}$ determined from the eigenvalue decomposition of $\widetilde{L}^T E S_{\rm int}\widetilde{L}$. One can see that the characteristic values decay rapidly. In this case we can expect a good approximation by a reduced-order model. The original system was approximated by a model of order $\ell = 44$. The frequency responses of the full-order and the reduced-order models are not presented, since they were impossible to distinguish. In Figure 5.1(b) we display the absolute error $\|\widetilde{\boldsymbol{G}}(i\omega) - \boldsymbol{G}(i\omega)\|_2$ for a frequency range $\omega \in [1,\, 10^{15}]$ and also the error bound (3.7).

6. Conclusion. In this paper, we have presented the passivity-preserving balanced truncation model reduction method for electrical circuits (PABTEC). This method is based on balancing two Gramians that satisfy the projected Lur'e equations. Exploiting the special structure of circuit equations, we have established the solvability of these equations. We have also shown that their solutions are simply related and thus, only one projected Lur'e equation has to be solved. Moreover, the spectral



(a) (b)

FIG. 5.1. *RC circuit: (a) the bounded real characteristic values of* $\boldsymbol{\mathcal{G}} = (I - \boldsymbol{G})(I + \boldsymbol{G})^{-1}$*; (b) the absolute error* $\|\widetilde{\boldsymbol{G}}(i\omega) - \boldsymbol{G}(i\omega)\|_2$ *and the error bound* (3.7).

projectors and the feedthrough matrix for the Moebius-transformed system required in the Lur'e equation have been derived explicitly. It has also been proved that the PABTEC method preserves the reciprocity in the reduced-order model. A circuit topological characterization has been given for the class of circuits for which the projected Lur'e equations can be rewritten as the projected Riccati equations. The numerical experiments demonstrate the reliability of the presented model reduction method to large-scale circuit equations.

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Appendix A. Spectral projectors for the Moebius-transformed system. Consider a pencil $\lambda\mathcal{E} - \mathcal{A}$ with $\mathcal{E}$ and $\mathcal{A}$ as in (2.4). For computing the spectral projector $\mathcal{P}_r$ onto the right deflating subspace of $\lambda\mathcal{E} - \mathcal{A}$ corresponding to the finite eigenvalues along the right deflating subspace corresponding to the eigenvalue at infinity, we use the canonical projection technique proposed in [12]. Let

$$\begin{aligned} E_0 &= \mathcal{E}, & A_0 &= -\mathcal{A}, \\ E_{k+1} &= E_k + A_k Q_k, & A_{k+1} &= A_k(I - Q_k), \end{aligned} \tag{A.1}$$

where Q_k is a projector onto $\ker E_k$ and $Q_j Q_k = 0$ for $j > k$. We will show that under Assumptions 2.1 the matrix E_2 is nonsingular. Then Q_1 can be chosen such that $Q_1 = Q_1 E_2^{-1} A_1$. In this case, the spectral projector $\mathcal{P}_r$ is computed as

$$\mathcal{P}_r = \big(I - Q_0(I - Q_1)E_2^{-1}A_0\big)\,(I - Q_1), \tag{A.2}$$

see [12] for details. Note that nonsingularity of E_2 implies that the pencil $\lambda\mathcal{E} - \mathcal{A}$ is of index at most two independent of the index of $\lambda E - A$ in (2.2).

A projector Q_0 onto $\ker E_0$ is given by

$$Q_0 = \begin{bmatrix} Q_{\mathcal{C}} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I \end{bmatrix}, \tag{A.3}$$

where Q_C be a projector onto $\ker A_C^T$. Then we get

$$E_1 = E_0 + A_0 Q_0 = \begin{bmatrix} A_C \mathcal{C} A_C^T + (A_\mathcal{R} \mathcal{R}^{-1} A_\mathcal{R}^T + A_\mathcal{I} A_\mathcal{I}^T) Q_C & 0 & A_\mathcal{V} \\ -A_\mathcal{L}^T Q_C & \mathcal{L} & 0 \\ -A_\mathcal{V}^T Q_C & 0 & I \end{bmatrix}, \tag{A.4}$$

$$A_1 = A_0(I - Q_0) = \begin{bmatrix} (A_\mathcal{R} \mathcal{R}^{-1} A_\mathcal{R}^T + A_\mathcal{I} A_\mathcal{I}^T)(I - Q_C) & A_\mathcal{L} & 0 \\ -A_\mathcal{L}^T (I - Q_C) & 0 & 0 \\ -A_\mathcal{V}^T (I - Q_C) & 0 & 0 \end{bmatrix}.$$

Lemma A.1. *Let Q_C be a projector onto $\ker A_C^T$ and let $Q_{\mathcal{RIV}-C}$ be a projector onto $\ker([\, A_\mathcal{R},\, A_\mathcal{I},\, A_\mathcal{V}\,]^T Q_C)$. Then $Q_{C\mathcal{RIV}} = Q_C Q_{\mathcal{RIV}-C}$ is a projector onto $\ker[\, A_C,\, A_\mathcal{R},\, A_\mathcal{I},\, A_\mathcal{V}\,]^T$ and $\ker(A_\mathcal{L}^T Q_{C\mathcal{RIV}}) = \ker Q_{C\mathcal{RIV}}$.*

Proof. We first show that $Q_C Q_{\mathcal{RIV}-C}$ is a projector onto $\ker[\, A_C,\, A_\mathcal{R},\, A_\mathcal{I},\, A_\mathcal{V}\,]^T$. By definition we have $\ker Q_C \subseteq \operatorname{im} Q_{\mathcal{RIV}-C}$ and, hence, $Q_{\mathcal{RIV}-C}(I - Q_C) = (I - Q_C)$. This leads to $Q_{\mathcal{RIV}-C} Q_C = Q_{\mathcal{RIV}-C} - I + Q_C$. Therefore,

$$Q_{C\mathcal{RIV}}^2 = Q_C (Q_{\mathcal{RIV}-C} Q_C) Q_{\mathcal{RIV}-C} = Q_C Q_{\mathcal{RIV}-C} = Q_{C\mathcal{RIV}}.$$

Assume now that $v \in \operatorname{im} Q_{C\mathcal{RIV}}$. This implies that $v \in \operatorname{im} Q_C = \ker A_C^T$. Furthemore,

$$[\, A_\mathcal{R},\, A_\mathcal{I},\, A_\mathcal{V}\,]^T v = [\, A_\mathcal{R},\, A_\mathcal{I},\, A_\mathcal{V}\,]^T Q_C Q_{\mathcal{RIV}-C} v = 0.$$

Thus, $v \in \ker[\, A_C,\, A_\mathcal{R},\, A_\mathcal{I},\, A_\mathcal{V}\,]^T$.

Conversely, assume that $v \in \ker[\, A_C,\, A_\mathcal{R},\, A_\mathcal{I},\, A_\mathcal{V}\,]^T$. Then we have $v = Q_C v$ and $[\, A_\mathcal{R},\, A_\mathcal{I},\, A_\mathcal{V}\,]^T Q_C v = [\, A_\mathcal{R},\, A_\mathcal{I},\, A_\mathcal{V}\,]^T v = 0$, i.e., $v = Q_C v = Q_C Q_{\mathcal{RIV}-C} v$.

It remains to show that $\ker(A_\mathcal{L}^T Q_{C\mathcal{RIV}} = \ker Q_{C\mathcal{RIV}})$. The inclusion $\supseteq$ is trivial. To prove the converse inclusion, we take a vector $v \in \ker(A_\mathcal{L}^T Q_{C\mathcal{RIV}})$. Then

$$Q_{C\mathcal{RIV}} v \in \ker[\, A_\mathcal{L},\, A_C,\, A_\mathcal{R},\, A_\mathcal{I},\, A_\mathcal{V}\,]^T.$$

However, Assumption (A2) implies that $\ker[\, A_\mathcal{L},\, A_C,\, A_\mathcal{R},\, A_\mathcal{I},\, A_\mathcal{V}\,]^T = \{0\}$ and, hence, $v \in \ker Q_{C\mathcal{RIV}}$. ∎

Lemma A.2. *The matrices H_1 and H_3 in (4.3) are nonsingular.*

Proof. The invertibility of $H_1 = P_{C\mathcal{RIV}}^T P_{C\mathcal{RIV}} + Q_{C\mathcal{RIV}}^T A_\mathcal{L} \mathcal{L}^{-1} A_\mathcal{L}^T Q_{C\mathcal{RIV}}$ follows from Lemma A.1 and the general fact that $\ker(NMN^T) = \ker N^T$ for $N \in \mathbb{R}^{m,n}$ and a positive definite $M \in \mathbb{R}^{n,n}$.

In order to show that H_3 is nonsingular, we assume that $H_3 v = 0$. Multiplying this equation from the left by $v^T Q_C^T$ we obtain that

$$v^T Q_C^T (A_\mathcal{R} \mathcal{R}^{-1} A_\mathcal{R}^T + A_\mathcal{I} A_\mathcal{I}^T + A_\mathcal{V} A_\mathcal{V}^T + A_\mathcal{L} \mathcal{L}^{-1} A_\mathcal{L}^T Q_{C\mathcal{RIV}} H_1^{-1} Q_{C\mathcal{RIV}}^T A_\mathcal{L} \mathcal{L}^{-1} A_\mathcal{L}^T) Q_C v = 0.$$

Hence, $[\, A_C,\, A_\mathcal{R},\, A_\mathcal{I},\, A_\mathcal{V}\,]^T Q_C v = 0$ and $Q_{C\mathcal{RIV}}^T A_\mathcal{L} \mathcal{L}^{-1} A_\mathcal{L}^T Q_C v = 0$. Then

$$Q_C v = Q_{C\mathcal{RIV}} v, \qquad A_\mathcal{L}^T Q_{C\mathcal{RIV}} v = 0.$$

By Lemma A.1 we have $Q_{C\mathcal{RIV}} v = 0$. Therefore, $H_3 v = 0$ reduces to $A_C \mathcal{C} A_C^T v = 0$, i.e., $v = Q_C v = 0$. Thus, H_3 is nonsingular. ∎

We now determine $\ker E_1$ and $\operatorname{im} E_1$.

LEMMA A.3. *Let* E_1 *be as in* (A.4). *Then*

$$\ker E_1 = \operatorname{im} \begin{bmatrix} Q_{\mathcal{CRIV}} \\ \mathcal{L}^{-1} A_{\mathcal{L}}^T Q_{\mathcal{CRIV}} \\ 0 \end{bmatrix}, \qquad \operatorname{im} E_1 = \ker \begin{bmatrix} Q_{\mathcal{CRIV}}^T & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{A.5}$$

Proof. Let $[\, v_1^T,\, v_2^T,\, v_3^T\,]^T \in \ker E_1$. Then we have

$$0 = A_{\mathcal{C}} \mathcal{C} A_{\mathcal{C}}^T v_1 + (A_{\mathcal{R}} \mathcal{R}^{-1} A_{\mathcal{R}}^T + A_{\mathcal{I}} A_{\mathcal{I}}^T + A_{\mathcal{V}} A_{\mathcal{V}}^T) Q_{\mathcal{C}} v_1, \tag{A.6}$$
$$v_2 = \mathcal{L}^{-1} A_{\mathcal{L}}^T Q_{\mathcal{C}} v_1,$$
$$v_3 = A_{\mathcal{V}}^T Q_{\mathcal{C}} v_1. \tag{A.7}$$

Multiplication of (A.6) from the left by $v_1^T Q_{\mathcal{C}}^T$ yields

$$v_1^T Q_{\mathcal{C}}^T (A_{\mathcal{R}} \mathcal{R}^{-1} A_{\mathcal{R}}^T + A_{\mathcal{I}} A_{\mathcal{I}}^T + A_{\mathcal{V}} A_{\mathcal{V}}^T) Q_{\mathcal{C}} v_1 = 0$$

and, thus, $v_1 = Q_{\mathcal{RIV}-\mathcal{C}} v_1$. Hence, we get $Q_{\mathcal{C}} v_1 = Q_{\mathcal{C}} Q_{\mathcal{RIV}-\mathcal{C}} v_1 = Q_{\mathcal{CRIV}} v_1$. Then equation (A.6) reduces to $A_{\mathcal{C}} \mathcal{C} A_{\mathcal{C}}^T v_1 = 0$, i.e., $v_1 = Q_{\mathcal{C}} v_1 = Q_{\mathcal{CRIV}} v_1$. Moreover, (A.7) implies $v_3 = A_{\mathcal{V}}^T Q_{\mathcal{CRIV}} v_1 = 0$. Thus, the first equation in (A.5) holds.

Further, we prove the relation

$$\ker E_1^T = \operatorname{im} \begin{bmatrix} Q_{\mathcal{CRIV}} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

that is equivalent to the second equation in (A.5). The inclusion $\supseteq$ is trivial. To prove the converse inclusion assume that $[\, v_1^T,\, v_2^T,\, v_3^T\,]^T \in \ker E_1^T$. Then

$$A_{\mathcal{C}} \mathcal{C} A_{\mathcal{C}}^T v_1 + Q_{\mathcal{C}}^T (A_{\mathcal{R}} \mathcal{R}^{-1} A_{\mathcal{R}}^T + A_{\mathcal{I}} A_{\mathcal{I}}^T + A_{\mathcal{V}} A_{\mathcal{V}}^T) v_1 = 0, \tag{A.8}$$
$$v_2 = 0, \qquad v_3 = -A_{\mathcal{V}}^T v_1.$$

Multiplying (A.8) from the left by $Q_{\mathcal{C}}^T$ we get

$$Q_{\mathcal{C}}^T (A_{\mathcal{R}} \mathcal{R}^{-1} A_{\mathcal{R}}^T + A_{\mathcal{I}} A_{\mathcal{I}}^T + A_{\mathcal{V}} A_{\mathcal{V}}^T) v_1 = 0$$

Then $A_{\mathcal{C}} \mathcal{C} A_{\mathcal{C}}^T v_1 = 0$ and, hence, $v_1 = Q_{\mathcal{C}} v_1$. This yields

$$v_1^T Q_{\mathcal{C}}^T (A_{\mathcal{R}} \mathcal{R}^{-1} A_{\mathcal{R}}^T + A_{\mathcal{I}} A_{\mathcal{I}}^T + A_{\mathcal{V}} A_{\mathcal{V}}^T) Q_{\mathcal{C}} v_1 = 0,$$

i.e., $v_1 = Q_{\mathcal{C}} v_1 \in \ker[\, A_{\mathcal{C}},\, A_{\mathcal{R}},\, A_{\mathcal{I}},\, A_{\mathcal{V}}\,]^T$. Thus, $v_1 \in \operatorname{im} Q_{\mathcal{CRIV}}$ and $v_3 = 0$. □

We now construct a projector Q_1 onto $\ker E_1$ that additionally satisfies $Q_1 Q_0 = 0$.

LEMMA A.4. *The matrix*

$$Q_1 = \begin{bmatrix} 0 & Q_{\mathcal{CRIV}} H_1^{-1} Q_{\mathcal{CRIV}}^T A_{\mathcal{L}} & 0 \\ 0 & \mathcal{L}^{-1} A_{\mathcal{L}}^T Q_{\mathcal{CRIV}} H_1^{-1} Q_{\mathcal{CRIV}}^T A_{\mathcal{L}} & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{A.9}$$

is a projector onto $\ker E_1$ *such that* $Q_1 Q_0 = 0$.

Proof. The equations $Q_1^2 = Q_1$ and $Q_1 Q_0 = 0$ follow by simple calculations. Furthermore, we have

$$\operatorname{im} Q_1 \subseteq \operatorname{im} \begin{bmatrix} Q_{\mathcal{CRIV}} \\ \mathcal{L}^{-1} A_{\mathcal{L}}^T Q_{\mathcal{CRIV}} \\ 0 \end{bmatrix} = \ker E_1.$$

Conversely, let $[\, v_1^T,\, v_2^T,\, v_3^T\,]^T \in \ker E_1$, i.e., $v_1 = Q_{\mathcal{CRIV}} v_1$, $v_2 = \mathcal{L}^{-1} A_{\mathcal{L}}^T Q_{\mathcal{CRIV}} v_1$ and $v_3 = 0$. Then

$$\begin{bmatrix} 0 & Q_{\mathcal{CRIV}} H_1^{-1} Q_{\mathcal{CRIV}}^T A_{\mathcal{L}} & 0 \\ 0 & \mathcal{L}^{-1} A_{\mathcal{L}}^T Q_{\mathcal{CRIV}} H_1^{-1} Q_{\mathcal{CRIV}}^T A_{\mathcal{L}} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} Q_{\mathcal{CRIV}} v_1 \\ \mathcal{L}^{-1} A_{\mathcal{L}}^T Q_{\mathcal{CRIV}} v_1 \\ 0 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}.$$

Thus, $\operatorname{im} Q_1 = \ker E_1$. ∎

Lemma A.5. *The matrix $E_2 = E_1 + A_1 Q_1$ is invertible and its inverse is given by*

$$E_2^{-1} = \begin{bmatrix} H_3^{-1} & H_7 & -H_3^{-1} A_{\mathcal{V}} \\ \mathcal{L}^{-1} A_{\mathcal{L}}^T Q_{\mathcal{C}} H_3^{-1} & \mathcal{L}^{-1} + \mathcal{L}^{-1} A_{\mathcal{L}}^T Q_{\mathcal{C}} H_7 & -\mathcal{L}^{-1} A_{\mathcal{L}}^T Q_{\mathcal{C}} H_3^{-1} A_{\mathcal{V}} \\ A_{\mathcal{V}}^T Q_{\mathcal{C}} H_3^{-1} & A_{\mathcal{V}}^T Q_{\mathcal{C}} H_7 & I - A_{\mathcal{V}}^T Q_{\mathcal{C}} H_3^{-1} A_{\mathcal{V}} \end{bmatrix}, \tag{A.10}$$

where H_3 is given in (4.3) *and* $H_7 = -H_3^{-1} A_{\mathcal{L}} \mathcal{L}^{-1} A_{\mathcal{L}}^T Q_{\mathcal{CRIV}} H_1^{-1} Q_{\mathcal{CRIV}}^T A_{\mathcal{L}} \mathcal{L}^{-1}$.

Proof. The matrix E_2 has the form

$$\begin{aligned} E_2 &= E_1 + A_1 Q_1 = E_1 + \begin{bmatrix} 0 & A_{\mathcal{L}} \mathcal{L}^{-1} A_{\mathcal{L}}^T Q_{\mathcal{CRIV}} H_1^{-1} Q_{\mathcal{CRIV}}^T A_{\mathcal{L}} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} A_{\mathcal{C}} \mathcal{C} A_{\mathcal{C}}^T + (A_{\mathcal{R}} \mathcal{R}^{-1} A_{\mathcal{R}}^T + A_{\mathcal{I}} A_{\mathcal{I}}^T) Q_{\mathcal{C}} & A_{\mathcal{L}} \mathcal{L}^{-1} A_{\mathcal{L}}^T Q_{\mathcal{CRIV}} H_1^{-1} Q_{\mathcal{CRIV}}^T A_{\mathcal{L}} & A_{\mathcal{V}} \\ -A_{\mathcal{L}}^T Q_{\mathcal{C}} & \mathcal{L} & 0 \\ -A_{\mathcal{V}}^T Q_{\mathcal{C}} & 0 & I \end{bmatrix} \\ &= \begin{bmatrix} I & A_{\mathcal{L}} \mathcal{L}^{-1} A_{\mathcal{L}}^T Q_{\mathcal{CRIV}} H_1^{-1} Q_{\mathcal{CRIV}}^T A_{\mathcal{L}} \mathcal{L}^{-1} & A_{\mathcal{V}} \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} H_3 & 0 & 0 \\ -A_{\mathcal{L}}^T Q_{\mathcal{C}} & \mathcal{L} & 0 \\ -A_{\mathcal{V}}^T Q_{\mathcal{C}} & 0 & I \end{bmatrix}. \end{aligned}$$

Since H_3 and $\mathcal{L}$ are nonsingular, it follows from this factorization that E_2 has an inverse given in (A.10). ∎

The equation $Q_1 = Q_1 E_2^{-1} A_1$ can be verified by simple calculations. By the results of [12], the index of the pencil $\lambda\mathcal{E} - \mathcal{A}$ is the smallest number k such that E_k is invertible. Therefore, Lemma A.5 implies that the index of $\lambda\mathcal{E} - \mathcal{A}$ with $\mathcal{E}$ and $\mathcal{A}$ as in (2.4) is at most two. The index is less than two if and only if Q_1 in (A.9) vanishes. Since this is equivalent to $Q_{\mathcal{CRIV}} = 0$, it can be shown analogous to the results in [3] that the index of $\lambda\mathcal{E} - \mathcal{A}$ is less than two if and only if the circuit does not contain cutsets only consisting of inductive branches.

We are now ready to prove Theorem 4.2.

Proof of Theorem 4.2. Taking into account that

$$\begin{aligned} Q_0(I-Q_1) &= \begin{bmatrix} Q_{\mathcal{C}} & -Q_{\mathcal{CRIV}}H_1^{-1}Q_{\mathcal{CRIV}}^T A_{\mathcal{L}} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I \end{bmatrix}, \\ E_2^{-1}A_0 &= \begin{bmatrix} H_3^{-1}H_2 & H_3^{-1}A_{\mathcal{L}} & 0 \\ \mathcal{L}^{-1}A_{\mathcal{L}}^T(Q_{\mathcal{C}}H_3^{-1}H_2-I) & \mathcal{L}^{-1}A_{\mathcal{L}}^T Q_{\mathcal{C}}H_3^{-1}A_{\mathcal{L}} & 0 \\ A_{\mathcal{V}}^T(Q_{\mathcal{C}}H_3^{-1}H_2-I) & A_{\mathcal{V}}^T Q_{\mathcal{C}}H_3^{-1}A_{\mathcal{L}} & I \end{bmatrix}, \end{aligned}$$

we have

$$Q_0(I-Q_1)E_2^{-1}A_0 = \begin{bmatrix} -H_4(Q_{\mathcal{C}}H_3^{-1}H_2-I)+I & -H_4Q_{\mathcal{C}}H_3^{-1}A_{\mathcal{L}} & 0 \\ 0 & 0 & 0 \\ A_{\mathcal{V}}^T(Q_{\mathcal{C}}H_3^{-1}H_2-I) & A_{\mathcal{V}}^T Q_{\mathcal{C}}H_3^{-1}A_{\mathcal{L}} & I \end{bmatrix}$$

with H_4 as in (4.3). Finally, using (A.2) we obtain the relation (4.2). □

Appendix B. Solvability of the projected Lur'e equation.

In this section, we prove the solvability and existence of the minimal solutions of the projected Lur'e equations (3.3) and (3.4).

LEMMA B.1. *Let $\mathcal{E}$, $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ be given in* (2.4) *and let Assumptions* 2.1 *be fulfilled. Then there exist invertible $\mathcal{W},\mathcal{T}\in\mathbb{R}^{n,n}$ such that*

$$\mathcal{W}^T(\lambda\mathcal{E}-\mathcal{A})\mathcal{T} = \begin{bmatrix} \lambda I_{n_i}-\mathcal{A}_i & 0 & 0 \\ 0 & \lambda I_{n_s}-\mathcal{A}_s & 0 \\ 0 & 0 & \lambda\mathcal{E}_\infty - I_{n_\infty} \end{bmatrix}, \tag{B.1}$$

$$\mathcal{W}^T\mathcal{B} = \begin{bmatrix} 0 \\ \mathcal{B}_s \\ \mathcal{B}_\infty \end{bmatrix}, \qquad \mathcal{C}\mathcal{T} = \begin{bmatrix} 0, & \mathcal{C}_s, & \mathcal{C}_\infty \end{bmatrix}, \tag{B.2}$$

where

(i) $\mathcal{C}_s, \mathcal{B}_s^T \in \mathbb{R}^{m,n_s}$ *and* $\mathcal{C}_\infty, \mathcal{B}_\infty^T\in\mathbb{R}^{m,n_\infty}$*;*
(ii) $\mathcal{A}_i\in\mathbb{R}^{n_i,n_i}$ *is diagonalizable and has purely imaginary eigenvalues only;*
(iii) all eigenvalues of $\mathcal{A}_s\in\mathbb{R}^{n_s,n_s}$ *have negative real part;*
(iv) $\mathcal{E}_\infty\in\mathbb{R}^{n_\infty,n_\infty}$ *with* $\mathcal{E}_\infty^2=0$, $\mathcal{E}_\infty\mathcal{B}_\infty=0$ *and* $\mathcal{C}_\infty\mathcal{E}_\infty=0$.

Proof. Due to $\mathcal{A}+\mathcal{A}^T\le 0$ and $\mathcal{E}=\mathcal{E}^T\ge 0$, the function $\boldsymbol{R}(s)=(s\mathcal{E}-\mathcal{A})^{-1}$ is positive real. Then the poles of $\boldsymbol{R}$ on the imaginary axis are simple [1] and thus every purely imaginary eigenvalue of $\lambda\mathcal{E}-\mathcal{A}$ has the same geometric and algebraic multiplicity. We can make use of the Weierstrass canonical form [9] to find invertible $\mathcal{W},\mathcal{T}\in\mathbb{R}^{n,n}$ such that $\mathcal{W}^T(\lambda\mathcal{E}-\mathcal{A})\mathcal{T}$ has the form (B.1) with nilpotent $\mathcal{E}_\infty$. By Theorem 4.2 the index of $\lambda\mathcal{E}-\mathcal{A}$ is at most two and, hence, $\mathcal{E}_\infty^2=0$.

In order to prove that the matrices $\mathcal{W}^T\mathcal{B}$ and $\mathcal{C}\mathcal{T}$ have the block structure as in (B.2), we show that for each eigenvector $v\in\mathbb{C}^n$ of $\lambda\mathcal{E}-\mathcal{A}$ (resp. $\lambda\mathcal{E}^T-\mathcal{A}^T$) corresponding to a generalized eigenvalue $i\omega$ with $\omega\in\mathbb{R}$ holds $\mathcal{C}v=0$ (resp. $v^*\mathcal{B}=0$). Assume that $v=[\,v_1^T,\,v_2^T,\,v_3^T\,]^T$ partitioned according to the block structure of $\mathcal{E}$ satisfies $i\omega\mathcal{E}v=\mathcal{A}v$. Then

$$\begin{aligned} 0 &= v^*(i\omega\mathcal{E}-\mathcal{A})v + v^*(-i\omega\mathcal{E}^T-\mathcal{A}^T)v = -v^*(\mathcal{A}+\mathcal{A}^T)v \\ &= 2v_1^*A_{\mathcal{R}}\mathcal{R}^{-1}A_{\mathcal{R}}^T v_1 + 2v_1^*A_{\mathcal{I}}A_{\mathcal{I}}^T v_1 + 2v_3^*v_3 \end{aligned}$$

and, hence, $A_{\mathcal{R}}^Tv_1=0$, $A_{\mathcal{I}}^Tv_1=0$ and $v_3=0$. The last two relations especially imply that $\mathcal{C}v=0$. The result for $\mathcal{W}^T\mathcal{B}$ can be proven analogously.

It remains to show that $\mathcal{E}_\infty\mathcal{B}_\infty = 0$ and $\mathcal{C}_\infty\mathcal{E}_\infty = 0$. In [12, 17], the following decoupling

$$\lambda\mathcal{E}-\mathcal{A} = \begin{bmatrix} E_2Q_0 & E_2P_0Q_1 & E_2P_0P_1\end{bmatrix}\left[\begin{array}{cc|c} Q_0 & \lambda Q_0Q_1 & Q_0P_1E_2^{-1}A_2 \\ 0 & Q_1 & Q_1E_2^{-1}A_2 \\ \hline 0 & 0 & \lambda P_0P_1 + P_0P_1E_2^{-1}A_2\end{array}\right]\begin{bmatrix} Q_0 \\ P_0Q_1 \\ P_0P_1\end{bmatrix},$$

has been obtained, where E_2 and A_2 are as in (A.1), the projectors Q_0 and Q_1 are given in (A.3) and (A.9), respectively, $P_0 = I - Q_0$ and $P_1 = I - Q_1$. Since P_0P_1 is a projector with $P_0P_1 + P_0Q_1 + Q_0 = I$, see [12, 17], we obtain that

$$\mathcal{C}(s\mathcal{E}-\mathcal{A})^{-1} = -s\mathcal{C}Q_0Q_1E_2^{-1} + \boldsymbol{G}_p(s),$$

where $\boldsymbol{G}_p$ is *proper*, i.e., $\lim\limits_{s\to\infty}\boldsymbol{G}_p(s) < \infty$. Since $\mathcal{C}Q_0Q_1 = 0$, the function $\mathcal{C}(s\mathcal{E}-\mathcal{A})^{-1}$ is proper. This implies that $\mathcal{C}_\infty\mathcal{E}_\infty = 0$. The relation $\mathcal{E}_\infty\mathcal{B}_\infty = 0$ follows analogously from the properness of $(s\mathcal{E}-\mathcal{A})^{-1}\mathcal{B} = -S_{\text{int}}(\mathcal{C}(s\mathcal{E}-\mathcal{A})^{-1})^TS_{\text{ext}}$, where S_{ext} and S_{int} are as in (2.3) and (4.1), respectively. ∎

Similarly to the corresponding results in [20, Theorem 4.6], one can show that the pencil $\lambda\mathcal{E}-\mathcal{A}$ does not have generalized eigenvalues on the imaginary axis if the circuit does not contain cutsets consisting of capacitive and inductive branches only. In this case the block $\lambda I - \mathcal{A}_i$ does not appear in (B.1).

LEMMA B.2. [21] *Let $\hat{A}, \hat{Q} \in \mathbb{R}^{n,n}$ and $\hat{B} \in \mathbb{R}^{n,m}$ be given with $\hat{Q} = \hat{Q}^T$. Assume that*

$$\operatorname{rank}[\,-sI-\hat{A},\ \hat{B}\,] = n \quad \textit{for all } s \in \mathbb{C}_+ \tag{B.3}$$

and that there exist a symmetric matrix $Y \in \mathbb{R}^{n,n}$ satisfying the algebraic Riccati inequality

$$\hat{A}^TY + Y\hat{A} - Y\hat{B}\hat{B}^TY + \hat{Q} \geq 0. \tag{B.4}$$

Then there exists a symmetric matrix $Y_{\min} \in \mathbb{R}^{n,n}$ that solves the algebraic Riccati equation

$$\hat{A}^TY_{\min} + Y_{\min}\hat{A} - Y_{\min}\hat{B}\hat{B}^TY_{\min} + \hat{Q} = 0 \tag{B.5}$$

and satisfies $Y_{\min} \leq Y$ for all symmetric Y fulfilling (B.4).

LEMMA B.3. *Let $A \in \mathbb{R}^{n,n}$, $B \in \mathbb{R}^{n,m}$, $C \in \mathbb{R}^{m,n}$ and $D \in \mathbb{R}^{m,m}$ be given and assume that all eigenvalues of A have negative real part. Moreover, let*

$$\begin{aligned} F:\ \ & \mathbb{R}^{n,n} \to \mathbb{R}^{n+m,n+m} \\ & Y \mapsto \begin{bmatrix} A^TY + YA + C^TC & YB + C^TD \\ B^TY + D^TC & D^TD - I\end{bmatrix}\end{aligned}$$

and assume that

$$\mathcal{L} = \left\{ Y \in \mathbb{R}^{n,n} \mid Y = Y^T \ \textit{and}\ F(Y) \leq 0\right\} \neq \emptyset.$$

Then there exists $Y_{\min} \in \mathcal{L}$ such that $0 \leq Y_{\min} \leq Y$ for all $Y \in \mathcal{L}$ and $\operatorname{rank} F(Y_{\min}) \leq m$.

Proof. For $\varepsilon > 0$, define

$$\begin{aligned} F_\varepsilon:\ \ & \mathbb{R}^{n,n} \to \mathbb{R}^{n+m,n+m} \\ & Y \mapsto \begin{bmatrix} A^TY + YA + C^TC & YB + C^TD \\ B^TY + D^TC & D^TD - (1+\varepsilon)I\end{bmatrix}\end{aligned}$$

and

$$\mathcal{L}_\varepsilon = \left\{ Y \in \mathbb{R}^{n,n} \mid Y = Y^T \text{ and } F_\varepsilon(Y) \le 0 \right\}.$$

It can be seen that for all $\varepsilon_1 > \varepsilon_2 > 0$ holds $\mathcal{L}_{\varepsilon_1} \supset \mathcal{L}_{\varepsilon_2} \supset \mathcal{L}$. Moreover, the matrix $M_\varepsilon = (1+\varepsilon)I - D^T D$ satisfies $M_\varepsilon > 0$ for $\varepsilon > 0$. Thus, there exists invertible $L_\varepsilon \in \mathbb{R}^{m,m}$ such that $M_\varepsilon = L_\varepsilon^T L_\varepsilon$. By using the Schur complement [9], we obtain that $Y \in \mathcal{L}_\varepsilon$ if and only if the algebraic Riccati inequality (B.4) is fulfilled for

$$\hat{A} = -A - BM_\varepsilon^{-1}D^T C, \quad \hat{B} = BL_\varepsilon^{-1}, \quad \hat{Q} = -C^T(I + DM_\varepsilon^{-1}D^T)C.$$

Since all eigenvalues of A have negative real part, relation (B.3) holds true for all $s \in \mathbb{C}_+$. Then Lemma B.2 implies that there exists a symmetric matrix $Y_{\varepsilon,\min}$ that satisfies the algebraic Riccati equation (B.5) and $Y_{\varepsilon,\min} \le Y$ for all $Y \in \mathcal{L}_\varepsilon$. Moreover, we have $\operatorname{rank} F_\varepsilon(Y_{\varepsilon,\min}) \le m$.

Let now $\{\varepsilon_k\}_{k\in\mathbb{N}}$ be a monotonically decreasing sequence tending to 0. Due to $\mathcal{L}_{\varepsilon_k} \supset \mathcal{L}_{\varepsilon_{k+1}} \supset \mathcal{L}$, we have $Y_{\varepsilon_k,\min} \le Y_{\varepsilon_{k+1},\min} \le Y$ for all $Y \in \mathcal{L}$. Therefore, the matrix sequence $\{Y_{\varepsilon_k,\min}\}_{k\in\mathbb{N}}$ converges to some $Y_{\min} \in \mathbb{R}^{n,n}$. We now show that $Y_{\min}$ indeed has the desired properties. Since for all $k \in \mathbb{N}$ holds $Y_{\varepsilon_k,\min} \le Y$, we have that $Y_{\min} \le Y$ for all $Y \in \mathcal{L}$ and, moreover, $F(Y_{\min}) \le 0$. This inequality implies that $A^T Y_{\min} + Y_{\min} A + C^T C = -H$ for some $H \ge 0$. Since all eigenvalues of A have negative real part and $H + C^T C \ge 0$, this Lyapunov equation has a unique symmetric solution $Y_{\min} \ge 0$, see [9]. Furthermore, we obtain that

$$\operatorname{rank} F(Y_{\min}) = \operatorname{rank} \lim_{k\to\infty} F_{\varepsilon_k}(Y_{\varepsilon_k,\min}) \le \max_{k\in\mathbb{N}} \operatorname{rank} F_{\varepsilon_k}(Y_{\varepsilon_k,\min}) \le m. \qquad \square$$

Proof of Theorem 4.3. Due to Theorem 4.1, it suffices to show the statement for the projected Lur'e equation (3.4) only. Regarding the matrices in (2.4), we have that

$$\mathcal{E} = \mathcal{E}^T \ge 0, \qquad \mathcal{A} + \mathcal{A}^T + \mathcal{C}^T\mathcal{C} \le 0, \qquad \mathcal{C}^T = -\mathcal{B}. \tag{B.6}$$

By Lemma B.1 there exist invertible $\mathcal{W}, \mathcal{T} \in \mathbb{R}^{n,n}$ such that $\lambda\hat{\mathcal{E}} - \hat{\mathcal{A}} = \mathcal{W}^T(\lambda\mathcal{E} - \mathcal{A})\mathcal{T}$ and $\hat{\mathcal{B}} = \mathcal{W}^T\mathcal{B}$, $\hat{\mathcal{C}} = \mathcal{C}\mathcal{T}$ are as in (B.1) and (B.2). Defining $\mathcal{Y} = \mathcal{W}^{-1}\mathcal{T}$, relations (B.6) imply that

$$\mathcal{Y}^T\hat{\mathcal{E}} = \hat{\mathcal{E}}^T\mathcal{Y} \ge 0, \qquad \mathcal{Y}^T\hat{\mathcal{A}} + \hat{\mathcal{A}}^T\mathcal{Y} + \hat{\mathcal{C}}^T\hat{\mathcal{C}} \le 0, \qquad \hat{\mathcal{C}}^T = -\mathcal{Y}^T\hat{\mathcal{B}}. \tag{B.7}$$

Let

$$\mathcal{Y} = \begin{bmatrix} \mathcal{Y}_i & \mathcal{Y}_{i,s} & \mathcal{Y}_{i,\infty} \\ \mathcal{Y}_{s,i} & \mathcal{Y}_s & \mathcal{Y}_{s,\infty} \\ \mathcal{Y}_{\infty,i} & \mathcal{Y}_{\infty,s} & \mathcal{Y}_\infty \end{bmatrix}$$

be partitioned according to the block structure in (B.1). Then relations (B.7) imply that

$$\begin{aligned}
&\mathcal{Y}_s = \mathcal{Y}_s^T \ge 0, \quad \mathcal{Y}_{s,\infty} = \mathcal{Y}_{\infty,s}^T\mathcal{E}_\infty, \\
&\mathcal{C}_s^T = -\mathcal{Y}_s^T\mathcal{B}_s - \mathcal{Y}_{\infty,s}^T\mathcal{B}_\infty, \quad \mathcal{C}_\infty^T = -\mathcal{Y}_{s,\infty}\mathcal{B}_s - \mathcal{Y}_\infty^T\mathcal{B}_\infty, \\
&0 \ge \begin{bmatrix} \mathcal{Y}_s\mathcal{A}_s + \mathcal{A}_s^T\mathcal{Y}_s + \mathcal{C}_s^T\mathcal{C}_s & \mathcal{A}_s^T\mathcal{Y}_{s,\infty} + \mathcal{Y}_{\infty,s}^T + \mathcal{C}_s^T\mathcal{C}_\infty \\ \mathcal{Y}_{s,\infty}^T\mathcal{A}_s + \mathcal{Y}_{\infty,s} + \mathcal{C}_\infty^T\mathcal{C}_s & \mathcal{Y}_\infty + \mathcal{Y}_\infty^T + \mathcal{C}_\infty^T\mathcal{C}_\infty \end{bmatrix}.
\end{aligned}$$

Using these relations together with $\mathcal{M}_0 = I - \mathcal{C}_\infty \mathcal{B}_\infty$, we obtain that

$$
\begin{aligned}
0 &\geq \begin{bmatrix} I & 0 \\ 0 & -\mathcal{B}_\infty^T \end{bmatrix} \begin{bmatrix} \mathcal{Y}_s\mathcal{A}_s + \mathcal{A}_s^T\mathcal{Y}_s + \mathcal{C}_s^T\mathcal{C}_s & \mathcal{A}_s^T\mathcal{Y}_{s,\infty} + \mathcal{Y}_{\infty,s}^T + \mathcal{C}_s^T\mathcal{C}_\infty \\ \mathcal{Y}_{s,\infty}^T\mathcal{A}_s + \mathcal{Y}_{\infty,s} + \mathcal{B}_\infty^T\mathcal{C}_s & \mathcal{Y}_\infty + \mathcal{Y}_\infty^T + \mathcal{C}_\infty^T\mathcal{C}_\infty \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & -\mathcal{B}_\infty \end{bmatrix} \\
&= \begin{bmatrix} \mathcal{A}_s^T\mathcal{Y}_s + \mathcal{Y}_s\mathcal{A}_s + \mathcal{C}_s^T\mathcal{C}_s & \mathcal{Y}_s\mathcal{B}_s + \mathcal{C}_s^T(I - \mathcal{C}_\infty\mathcal{B}_\infty) \\ \mathcal{B}_s^T\mathcal{Y}_s + (I - \mathcal{B}_\infty^T\mathcal{C}_\infty^T)\mathcal{C}_s & \mathcal{B}_\infty^T\mathcal{C}_\infty^T\mathcal{C}_\infty\mathcal{B}_\infty - \mathcal{B}_\infty^T\mathcal{C}_\infty^T - \mathcal{C}_\infty\mathcal{B}_\infty \end{bmatrix} \\
&= \begin{bmatrix} \mathcal{A}_s^T\mathcal{Y}_s + \mathcal{Y}_s\mathcal{A}_s + \mathcal{C}_s^T\mathcal{C}_s & \mathcal{Y}_s\mathcal{B}_s + \mathcal{C}_s^T\mathcal{M}_0 \\ \mathcal{B}_s^T\mathcal{Y}_s + \mathcal{M}_0^T\mathcal{C}_s & \mathcal{M}_0^T\mathcal{M}_0 - I \end{bmatrix}.
\end{aligned}
$$

Lemma B.3 yields the existence of some $\mathcal{Y}_{s,\min} \geq 0$ which solves the above matrix inequality and is minimal with this property. Moreover, we have

$$
\operatorname{rank} \begin{bmatrix} \mathcal{A}_s^T\mathcal{Y}_{s,\min} + \mathcal{Y}_{s,\min}\mathcal{A}_s + \mathcal{C}_s^T\mathcal{C}_s & \mathcal{Y}_{s,\min}\mathcal{B}_s + \mathcal{C}_s^T\mathcal{M}_0 \\ \mathcal{B}_s^T\mathcal{Y}_{s,\min} + \mathcal{M}_0^T\mathcal{C}_s & \mathcal{M}_0^T\mathcal{M}_0 - I \end{bmatrix} \leq m. \tag{B.8}
$$

We now define

$$
Y_{\min} = \mathcal{W} \begin{bmatrix} 0 & 0 & 0 \\ 0 & \mathcal{Y}_{s,\min} & 0 \\ 0 & 0 & 0 \end{bmatrix} \mathcal{W}^T.
$$

Taking into account (B.1), the spectral projectors $\mathcal{P}_l$ and $\mathcal{P}_r$ are given by

$$
\mathcal{P}_l = \mathcal{W}^{-T} \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \end{bmatrix} \mathcal{W}^T, \quad \mathcal{P}_r = \mathcal{T} \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \end{bmatrix} \mathcal{T}^{-1}.
$$

Then we have $Y_{\min} = \mathcal{P}_l^T Y_{\min} \mathcal{P}_l \geq 0$. Furthermore, the matrix

$$
\mathcal{F}(Y_{\min}) = \begin{bmatrix} \mathcal{A}^T Y_{\min}\mathcal{E} + \mathcal{E}^T Y_{\min}\mathcal{A} + \mathcal{P}_r^T\mathcal{C}^T\mathcal{C}\mathcal{P}_r & \mathcal{E}^T Y_{\min}\mathcal{B} + \mathcal{P}_r^T\mathcal{C}^T\mathcal{M}_0 \\ \mathcal{B}^T Y_{\min}\mathcal{E} + \mathcal{M}_0^T\mathcal{C}\mathcal{P}_r & \mathcal{M}_0^T\mathcal{M}_0 - I \end{bmatrix}
$$

fulfils $\mathcal{F}(Y_{\min}) \leq 0$ and $\operatorname{rank} \mathcal{F}(Y_{\min}) \leq m$ due to (B.8). Hence, there exist $\mathcal{K}_o \in \mathbb{R}^{m,n}$ and $\mathcal{J}_o \in \mathbb{R}^{m,m}$ such that

$$
\mathcal{F}(Y_{\min}) = -[\,\sqrt{2}\mathcal{K}_o,\ \mathcal{J}_o\,]^T[\,\sqrt{2}\mathcal{K}_o,\ \mathcal{J}_o\,].
$$

Now using $\mathcal{E} = E$, $\mathcal{A} = A - BC$, $\mathcal{B} = -\sqrt{2}B$ and $\mathcal{C} = \sqrt{2}C$, we obtain that

$$
\begin{aligned}
(A - BC)^T Y_{\min} E + E^T Y_{\min}(A - BC) + 2\mathcal{P}_r^T C^T C\mathcal{P}_r &= -2\,\mathcal{K}_o^T\mathcal{K}_o, \\
-E^T Y_{\min} B + \mathcal{P}_r^T C^T \mathcal{M}_0 &= -\mathcal{K}_o^T\mathcal{J}_o.
\end{aligned}
$$

By construction, one can further see that $Y_{\min}$ is minimal with this property. ◻

Appendix C. The feedthrough matrix $\mathcal{M}_0$.

In order to prove Theorem 4.4 we need the following results.

LEMMA C.1. *Let $\mathcal{G}(s) = \mathcal{C}(s\mathcal{E} - \mathcal{A})^{-1}\mathcal{B} + I$ with $\mathcal{E}$, $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ as in* (2.4). *Then the transfer function $\mathcal{G}_r(s) = \mathcal{G}(\frac{1}{s})$ has a realization $\mathcal{G}_r = [\,\mathcal{E}_r,\, \mathcal{A}_r,\, \mathcal{B}_r,\, \mathcal{C}_r,\, I\,]$, where*

$$
\mathcal{E}_r = \begin{bmatrix} A_\mathcal{L}\mathcal{L}^{-1}A_\mathcal{L}^T & 0 & 0 \\ 0 & \mathcal{C}^{-1} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \mathcal{B}_r = \sqrt{2} \begin{bmatrix} A_\mathcal{I} & 0 \\ 0 & 0 \\ 0 & I \end{bmatrix} = -\mathcal{C}^T,
$$

$$
\mathcal{A}_r = \begin{bmatrix} -A_\mathcal{R}\mathcal{R}^{-1}A_\mathcal{R}^T - A_\mathcal{I}A_\mathcal{I}^T & -A_\mathcal{C} & -A_\mathcal{V} \\ A_\mathcal{C}^T & 0 & 0 \\ A_\mathcal{V}^T & 0 & -I \end{bmatrix}.
$$

Proof. We have

$$\begin{aligned}
\boldsymbol{\mathcal{G}}(\tfrac{1}{s}) &= I-2\begin{bmatrix}A_{\mathcal{I}} & 0\\ 0 & 0\\ 0 & I\end{bmatrix}^T\begin{bmatrix}\frac{1}{s}A_{\mathcal{C}}\mathcal{C}A_{\mathcal{C}}^T+A_{\mathcal{R}}\mathcal{R}^{-1}A_{\mathcal{R}}^T+A_{\mathcal{I}}A_{\mathcal{I}}^T & A_{\mathcal{L}} & A_{\mathcal{V}}\\ -A_{\mathcal{L}}^T & \frac{1}{s}\mathcal{L} & 0\\ -A_{\mathcal{V}}^T & 0 & I\end{bmatrix}^{-1}\begin{bmatrix}A_{\mathcal{I}} & 0\\ 0 & 0\\ 0 & I\end{bmatrix}\\
&= I-2\begin{bmatrix}A_{\mathcal{I}} & 0\\ 0 & I\end{bmatrix}^T\begin{bmatrix}\frac{1}{s}A_{\mathcal{C}}\mathcal{C}A_{\mathcal{C}}^T+A_{\mathcal{R}}\mathcal{R}^{-1}A_{\mathcal{R}}^T+A_{\mathcal{I}}A_{\mathcal{I}}^T+sA_{\mathcal{L}}\mathcal{L}^{-1}A_{\mathcal{L}}^T & A_{\mathcal{V}}\\ -A_{\mathcal{V}}^T & I\end{bmatrix}^{-1}\begin{bmatrix}A_{\mathcal{I}} & 0\\ 0 & I\end{bmatrix}\\
&= I+\mathcal{C}_r(s\mathcal{E}_r-\mathcal{A}_r)^{-1}\mathcal{B}_r=\boldsymbol{\mathcal{G}}_r(s). \qquad \square
\end{aligned}$$

LEMMA C.2. *The matrix H_6 in (4.7) is nonsingular.*

Proof. Assume that $H_6v = 0$. Then $[\,A_{\mathcal{R}},\ A_{\mathcal{I}},\ A_{\mathcal{V}}\,]^TQ_{\mathcal{C}}v = 0$ and $Q_{\mathcal{RIV}-\mathcal{C}}v = 0$. It follows from the first equation that $v = Q_{\mathcal{RIV}-\mathcal{C}}v$, whereas the second one implies that $v = 0$. Thus, H_6 is nonsingular. $\square$

Proof of Theorem 4.4. The bounded realness of $\boldsymbol{\mathcal{G}}$ implies the existence of the limit and from the reciprocity we obtain that

$$\mathcal{M}_0=\begin{bmatrix}\mathcal{M}_{11} & -\mathcal{M}_{21}^T\\ \mathcal{M}_{21} & \mathcal{M}_{22}\end{bmatrix},$$

where $\mathcal{M}_{11}\in\mathbb{R}^{n_{\mathcal{I}},n_{\mathcal{I}}}$, $\mathcal{M}_{21}\in\mathbb{R}^{n_{\mathcal{V}},n_{\mathcal{I}}}$, $\mathcal{M}_{22}\in\mathbb{R}^{n_{\mathcal{V}},n_{\mathcal{V}}}$ and $\mathcal{M}_{11}=\mathcal{M}_{11}^T$, $\mathcal{M}_{22}=\mathcal{M}_{22}^T$. Since $\mathcal{M}_0=\boldsymbol{\mathcal{G}}_r(0)$, where $\boldsymbol{\mathcal{G}}_r$ is as in Lemma C.1, the matrices $\mathcal{M}_{11}$, $\mathcal{M}_{21}$ and $\mathcal{M}_{22}$ satisfy

$$\begin{aligned}
\mathcal{M}_{11} &= I-2\left[A_{\mathcal{I}}^T,\quad 0,\quad 0\right]X_1,\\
\mathcal{M}_{21} &= -2\left[0,\quad 0,\quad I\right]X_1,\\
\mathcal{M}_{22} &= I-2\left[0,\quad 0,\quad I\right]X_2,
\end{aligned}$$

where X_1 and X_2 are the solutions of

$$-\mathcal{A}_rX_1=\left[A_{\mathcal{I}}^T,\quad 0,\quad 0\right]^T,\quad -\mathcal{A}_rX_2=\left[0,\quad 0,\quad I\right]^T. \tag{C.1}$$

Let us first solve the system for X_1. Assume that $X_1=[\,X_{11}^T,\ X_{21}^T,\ X_{31}^T\,]^T$. Then we have

$$\begin{aligned}
(A_{\mathcal{R}}\mathcal{R}^{-1}A_{\mathcal{R}}^T+A_{\mathcal{I}}A_{\mathcal{I}}^T)X_{11}+A_{\mathcal{C}}X_{21}+A_{\mathcal{V}}X_{31} &= A_{\mathcal{I}}, &\text{(C.2)}\\
-A_{\mathcal{C}}^TX_{11} &= 0, &\text{(C.3)}\\
-A_{\mathcal{V}}^TX_{11}+X_{31} &= 0. &\text{(C.4)}
\end{aligned}$$

Substituting X_{31} from (C.4) in (C.2), we obtain that

$$(A_{\mathcal{R}}\mathcal{R}^{-1}A_{\mathcal{R}}^T+A_{\mathcal{I}}A_{\mathcal{I}}^T+A_{\mathcal{V}}A_{\mathcal{V}}^T)X_{11}+A_{\mathcal{C}}X_{21}=A_{\mathcal{I}}. \tag{C.5}$$

It follows from (C.3) that $X_{11}=Q_{\mathcal{C}}X_{11}$. A multiplication of equation (C.5) from the left by $Q_{\mathcal{C}}^T$ leads to

$$Q_{\mathcal{C}}^T(A_{\mathcal{R}}\mathcal{R}^{-1}A_{\mathcal{R}}^T+A_{\mathcal{I}}A_{\mathcal{I}}^T+A_{\mathcal{V}}A_{\mathcal{V}}^T)Q_{\mathcal{C}}X_{11}=Q_{\mathcal{C}}^TA_{\mathcal{I}},$$

and, hence,

$$X_{11} = H_6^{-1} Q_{\mathcal{C}}^T A_{\mathcal{I}} + Q_{\mathcal{R}\mathcal{I}\mathcal{V}-\mathcal{C}} X_{11}.$$

Therefore, we get

$$X_{11} = Q_{\mathcal{C}} X_{11} = Q_{\mathcal{C}} H_6^{-1} Q_{\mathcal{C}}^T A_{\mathcal{I}} + Q_{\mathcal{C}\mathcal{R}\mathcal{I}\mathcal{V}} X_{11}.$$

Since $Q_{\mathcal{C}\mathcal{R}\mathcal{I}\mathcal{V}}$ is a projector onto $\ker[A_{\mathcal{C}},\, A_{\mathcal{R}},\, A_{\mathcal{I}},\, A_{\mathcal{V}}]^T$, we have

$$\begin{aligned} \mathcal{M}_{11} &= I - 2A_{\mathcal{I}}^T X_{11} = I - 2A_{\mathcal{I}}^T (Q_{\mathcal{C}} H_6^{-1} Q_{\mathcal{C}}^T A_{\mathcal{I}} + Q_{\mathcal{C}\mathcal{R}\mathcal{I}\mathcal{V}} X_{11}) \\ &= I - 2A_{\mathcal{I}}^T Q_{\mathcal{C}} H_6^{-1} Q_{\mathcal{C}}^T A_{\mathcal{I}} \end{aligned}$$

and

$$\begin{aligned} \mathcal{M}_{21} &= -2X_{31} = -2A_{\mathcal{V}}^T X_{11} = -2A_{\mathcal{V}}^T (Q_{\mathcal{C}} H_6^{-1} Q_{\mathcal{C}}^T A_{\mathcal{I}} + Q_{\mathcal{C}\mathcal{R}\mathcal{I}\mathcal{V}} X_{11}) \\ &= -2A_{\mathcal{V}}^T Q_{\mathcal{C}} H_6^{-1} Q_{\mathcal{C}}^T A_{\mathcal{I}}. \end{aligned}$$

We now compute $\mathcal{M}_{22}$. Substituting $X_2 = [\, X_{12}^T,\, X_{22}^T,\, X_{32}^T\,]^T$ in the second equation in (C.1), we have

$$(A_{\mathcal{R}} \mathcal{R}^{-1} A_{\mathcal{R}}^T + A_{\mathcal{I}} A_{\mathcal{I}}^T) X_{12} + A_{\mathcal{C}} X_{22} + A_{\mathcal{V}} X_{32} = 0, \tag{C.6}$$

$$-A_{\mathcal{C}}^T X_{12} = 0, \tag{C.7}$$

$$-A_{\mathcal{V}}^T X_{12} + X_{32} = I. \tag{C.8}$$

Then we obtain from (C.6) and (C.8) that

$$(A_{\mathcal{R}} \mathcal{R}^{-1} A_{\mathcal{R}}^T + A_{\mathcal{I}} A_{\mathcal{I}}^T + A_{\mathcal{V}} A_{\mathcal{V}}^T) X_{12} + A_{\mathcal{C}} X_{22} = -A_{\mathcal{V}}.$$

Furthermore, equation (C.7) yields $X_{12} = Q_{\mathcal{C}} X_{12}$ and, hence,

$$Q_{\mathcal{C}}^T (A_{\mathcal{R}} \mathcal{R}^{-1} A_{\mathcal{R}}^T + A_{\mathcal{I}} A_{\mathcal{I}}^T + A_{\mathcal{V}} A_{\mathcal{V}}^T) Q_{\mathcal{C}} X_{12} = -Q_{\mathcal{C}}^T A_{\mathcal{V}}.$$

By the same argumentation as for X_{11}, we obtain

$$X_{12} = -Q_{\mathcal{C}} H_6^{-1} Q_{\mathcal{C}}^T A_{\mathcal{V}} + Q_{\mathcal{C}\mathcal{R}\mathcal{I}\mathcal{V}} X_{12}.$$

Thus, $\mathcal{M}_{22} = I - 2X_{32} = -I + 2A_{\mathcal{V}}^T Q_{\mathcal{C}} H_6^{-1} Q_{\mathcal{C}}^T A_{\mathcal{V}}$. ◻

Appendix D. Topological conditions for the invertibility of $I - \mathcal{M}_0 \mathcal{M}_0^T$. In order to prove Theorem 4.7 we need the following result.

LEMMA D.1. *Let $P \in \mathbb{R}^{n,n}$ be an orthogonal projector, i.e., $P^2 = P$ and $P = P^T$. Assume that P is partitioned as*

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix}$$

with $P_{11} \in \mathbb{R}^{k,k}$ for some $k \le n$. Then all eigenvalues of P_{11} satisfy $\lambda_j(P_{11}) < 1$ if and only if

$$\operatorname{im} \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix} \cap \operatorname{im} \begin{bmatrix} I_k \\ 0_{n-k,k} \end{bmatrix} = \{0\}. \tag{D.1}$$

Proof. First assume that the converse of (D.1) holds true. Let $v_1 \in \mathbb{R}^k \backslash \{0\}$ such that $v = [\, v_1^T \,,\, 0\,]^T \in \operatorname{im} P$. Then $P_{11}v_1 = v_1$. This is a contradiction to the fact that $\lambda_j(P_{11}) < 1$ for $j = 1, \ldots, k$.

Assume now that $v_1 \in \mathbb{R}^k \backslash \{0\}$ with $P_{11}v_1 = \lambda v_1$ for some $\lambda \geq 1$. Since $\|P\|_2 \leq 1$, we have

$$\left\| \begin{bmatrix} \lambda v_1 \\ P_{12}^T v_1 \end{bmatrix} \right\|^2 = \left\| \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix} \begin{bmatrix} v_1 \\ 0 \end{bmatrix} \right\|^2 \leq \left\| \begin{bmatrix} v_1 \\ 0 \end{bmatrix} \right\|^2.$$

Then $\lambda = 1$ and $P_{12}^T v_1 = 0$. Hence,

$$\begin{bmatrix} v_1 \\ 0 \end{bmatrix} \in \operatorname{im} \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix} \cap \operatorname{im} \begin{bmatrix} I_k \\ 0_{n-k,k} \end{bmatrix},$$

which is a contradiction to (D.1). ∎

Proof of Theorem 4.7. The invertibility of $I - \mathcal{M}_0^T \mathcal{M}_0$ is equivalent to the fact that $\sigma = 1$ is not a singular value of $\mathcal{M}_0$ and also of the symmetric matrix

$$\begin{aligned} \mathcal{M}_0 S_{\text{ext}} &= \begin{bmatrix} I - 2A_{\mathcal{I}}^T Q_{\mathcal{C}} H_6^{-1} Q_{\mathcal{C}}^T A_{\mathcal{I}} & -2A_{\mathcal{I}}^T Q_{\mathcal{C}} H_6^{-1} Q_{\mathcal{C}}^T A_{\mathcal{V}} \\ -2A_{\mathcal{V}}^T Q_{\mathcal{C}} H_6^{-1} Q_{\mathcal{C}}^T A_{\mathcal{I}} & I - 2A_{\mathcal{V}}^T Q_{\mathcal{C}} H_6^{-1} Q_{\mathcal{C}}^T A_{\mathcal{V}} \end{bmatrix} \\ &= \begin{bmatrix} I_{n_{\mathcal{I}}} & 0 \\ 0 & I_{n_{\mathcal{V}}} \end{bmatrix} - 2 \begin{bmatrix} A_{\mathcal{I}}^T \\ A_{\mathcal{V}}^T \end{bmatrix} Q_{\mathcal{C}} H_6^{-1} Q_{\mathcal{C}}^T \begin{bmatrix} A_{\mathcal{I}}, & A_{\mathcal{V}} \end{bmatrix}, \end{aligned}$$

with S_{ext} as in (2.3). The symmetry of this matrix then implies that $I - \mathcal{M}_0^T \mathcal{M}_0$ is invertible if and only if the spectrum of

$$\mathcal{K} = \begin{bmatrix} A_{\mathcal{I}}^T \\ A_{\mathcal{V}}^T \end{bmatrix} Q_{\mathcal{C}} H_6^{-1} Q_{\mathcal{C}}^T \begin{bmatrix} A_{\mathcal{I}}, & A_{\mathcal{V}} \end{bmatrix}$$

contains neither 0 nor 1.

Since H_6 is symmetric and positive definite, we have that $\lambda = 0$ is not an eigenvalue of $\mathcal{K}$ if and only if

$$\ker Q_{\mathcal{C}}^T \begin{bmatrix} A_{\mathcal{I}}, & A_{\mathcal{V}} \end{bmatrix} = \{0\}.$$

We now analyze whether $\mathcal{K}$ has the eigenvalue $\lambda = 1$. Let $\mathcal{R}^{-1/2}$ be the matrix square root of $\mathcal{R}^{-1}$ and consider the matrix

$$\mathcal{P} = \begin{bmatrix} A_{\mathcal{I}}^T \\ A_{\mathcal{V}}^T \\ \mathcal{R}^{-1/2} A_{\mathcal{R}}^T \end{bmatrix} Q_{\mathcal{C}} H_6^{-1} Q_{\mathcal{C}}^T \begin{bmatrix} A_{\mathcal{I}}, & A_{\mathcal{V}}, & A_{\mathcal{R}} \mathcal{R}^{-1/2} \end{bmatrix}.$$

Then we have $\mathcal{P}^T = \mathcal{P}$ and

$$\begin{aligned} I - Q_{\mathcal{RIV}-\mathcal{C}} &= H_6^{-1} Q_{\mathcal{C}}^T (A_{\mathcal{R}} \mathcal{R}^{-1} A_{\mathcal{R}}^T + A_{\mathcal{I}} A_{\mathcal{I}}^T + A_{\mathcal{V}} A_{\mathcal{V}}^T) Q_{\mathcal{C}}, \\ \begin{bmatrix} A_{\mathcal{I}}^T \\ A_{\mathcal{V}}^T \end{bmatrix} Q_{\mathcal{C}} (I - Q_{\mathcal{RIV}-\mathcal{C}}) &= \begin{bmatrix} A_{\mathcal{I}}^T \\ A_{\mathcal{V}}^T \end{bmatrix} Q_{\mathcal{C}}. \end{aligned}$$

Thus, $\mathcal{P}$ is an orthogonal projector. Since $\mathcal{K}$ is an upper left block of $\mathcal{P}$, Lemma D.1 leads to the fact that $\lambda = 1$ is not an eigenvalue of $\mathcal{K}$ if and only if

$$\operatorname{im} \mathcal{P} \cap \operatorname{im} \begin{bmatrix} I_{n_\mathcal{I}} & 0 \\ 0 & I_{n_\mathcal{V}} \\ 0 & 0 \end{bmatrix} = \{0\}.$$

In this case, $v_1 \in \mathbb{R}^{n_\mathcal{I}}$, $v_2 \in \mathbb{R}^{n_\mathcal{V}}$ with $v_1 = 0$ and $v_2 = 0$ are the only vectors with

$$\begin{bmatrix} v_1 \\ v_2 \\ 0 \end{bmatrix} \in \operatorname{im} \begin{bmatrix} A_\mathcal{I}^T \\ A_\mathcal{V}^T \\ \mathcal{R}^{-1/2} A_\mathcal{R}^T \end{bmatrix} Q_\mathcal{C}.$$

Hence, for all $v \in \ker A_\mathcal{R}^T Q_\mathcal{C}$ holds $A_\mathcal{V}^T Q_\mathcal{C} v = 0$ and $A_\mathcal{I}^T Q_\mathcal{C} v = 0$. Then for the projector $Q_{\mathcal{R}-\mathcal{C}}$ onto $\ker A_\mathcal{R}^T Q_\mathcal{C}$ and the projector $Q_{\mathcal{RC}} = Q_\mathcal{C} Q_{\mathcal{R}-\mathcal{C}}$ onto $\ker[\, A_\mathcal{R}, A_\mathcal{C}\,]^T$, we obtain

$$\begin{bmatrix} A_\mathcal{I}^T \\ A_\mathcal{V}^T \end{bmatrix} Q_\mathcal{C} Q_{\mathcal{R}-\mathcal{C}} = \begin{bmatrix} A_\mathcal{I}^T \\ A_\mathcal{V}^T \end{bmatrix} Q_{\mathcal{RC}} = 0. \qquad \square$$

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