Methods for the Temporal Approximation of Nonlinear, Nonautonomous Evolution Equations

vorgelegt von
M. Sc.
Monika Eisenmann
ORCID: 0000-0001-7428-9546

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Vorsitzender: Prof. Dr. Martin Skutella
Gutachter: Prof. Dr. Etienne Emmrich
Gutachter: Dr. Raphael Kruse
Gutachterin: Prof. Dr. Mechthild Thalhammer

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Abstract

Differential equations are an important building block for modeling processes in physics, biology, and social sciences. Usually, their exact solution is not known explicitly though. Therefore, numerical schemes to approximate the solution are of great importance. In this thesis, we consider the temporal approximation of nonlinear, nonautonomous evolution equations on a finite time horizon. We present two independent approaches that can be used to find a temporal approximation of the solution.

As the solution of a nonlinear equation typically lacks global higher-order regularity, it cannot be expected to obtain higher-order convergence rates. Thus, we only concentrate on schemes that are formally of first order.

In the first part of the thesis, we consider the question of how nonsmooth temporal data can be handled. A common method for the approximation of the integral of an irregular function is a Monte Carlo type quadrature rule. We take on this idea and use a similar approach to approximate the solution to a nonautonomous evolution equation. If the data is evaluated at the points of a randomly shifted grid, we can prove the convergence of the backward Euler scheme. Moreover, we prove explicit error estimates. Here, we introduce a second set of randomized points, where the data is evaluated, and make additional assumptions on the data and the solution.

Secondly, we approximate the solution via an operator splitting based scheme. We work with both an implicit-explicit splitting and a product type splitting. First, we decompose the operator into a monotone and a bounded part. The implicit-explicit splitting is used to obtain one implicit equation that contains the monotone part. The bounded part is solved in an explicit fashion. This way, we only solve as many implicit equations as necessary. Further, we use a product type splitting on the monotone part. Even though this leads to more problems, they are potentially easier to solve individually. For this splitting scheme, we follow a similar approach as in the first part of the thesis. After proving the convergence of the scheme, we provide error bounds under additional assumptions on both the data and the solution.

In order to provide an interesting field of application, we show that the schemes can be applied for the temporal approximation of certain nonlinear, parabolic problems.

Zusammenfassung

Differentialgleichungen bilden einen wichtigen Bestandteil für die Modellierung von Prozessen in der Physik, Biologie und Sozialwissenschaft. Allerdings lässt sich ihre Lösung nur in seltenen Fällen analytisch bestimmen. Aus diesem Grund ist eine numerische Näherung der Lösung von großer Wichtigkeit. In dieser Arbeit betrachten wir die Zeitdiskretisierung von nichtlinearen, nichtautonomen Evolutionsgleichungen auf einem endlichen Zeitintervall. Wir präsentieren zwei unabhängig voneinander anwendbare Lösungsverfahren für die zeitliche Approximation der Lösung.

Da die Lösung einer nichtlinearen Gleichung häufig irregulär ist, können keine besonders hohen Konvergenzraten erwartet werden. Aus diesem Grund konzentrieren wir uns ausschließlich auf Verfahren, die formal eine Konvergenzordnung von eins aufweisen.

Im ersten Teil der Arbeit beschäftigen wir uns mit der Frage, wie zeitlich irreguläre Daten behandelt werden können. Für die Approximation des Integrals einer nichtglatten Funktion ist ein Monte-Carlo-Algorithmus häufig eine gute Wahl. Wir verfolgen hier einen ähnlichen Ansatz, um die Lösung einer nichtautonomen Evolutionsgleichung zu approximieren. Wir zeigen die Konvergenz des impliziten Euler Verfahrens unter der Verwendung eines zufällig verschobenes Zeitgitters. Weiterhin können unter zusätzlichen Voraussetzungen an die Daten und die Lösung explizite Fehlerschranken angegeben werden. Um diese zu zeigen, wenden wir eine weitere Randomisierung an.

Der zweite Teil der Arbeit enthält ein Approximationsverfahren, das ein Operatorsplitting nutzt. Wir verwenden sowohl ein implizit-explizites Splitting als auch ein Produktsplitting. Hierbei zerlegen wir den Operator zunächst in einen monotonen und einen beschränkten Anteil. Das implizit-explizit Splitting wird genutzt, um eine implizite Gleichung zu erhalten, die den monotonen Anteil enthält. Der beschränkte Anteil kann in einer expliziten Gleichung gelöst werden. Auf diese Weise entstehen nur so viele implizite Gleichungen, wie tatsächlich notwendig sind. Weiterhin verwenden wir das Produktsplitting, um die implizite Gleichung weiter aufzuteilen. Hierbei erhalten wir zwar mehr Gleichungen, diese sind aber möglicherweise leichter zu lösen. Für das Splittingverfahren gehen wir ähnlich vor wie im ersten Teil der Arbeit. Nachdem die Konvergenz des Verfahrens gezeigt ist, wenden wir uns auch hier expliziten Fehlerabschätzungen zu, die unter zusätzlichen Voraussetzungen an die Daten und die Lösung möglich sind.

Schlußendlich präsentieren wir für beide Verfahren ein Anwendungsbeispiel aus dem Bereich der nichtlinearen, parabolischen Differentialgleichungen.

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Contents

Introduction

This work intends to present results on the approximation of nonlinear, nonautonomous evolution equations on a finite time horizon. This type of equation appears when modeling complex processes in physics, biology, and social sciences. Yet the solution can rarely be written down explicitly. Therefore, it is important to find ways to approximate a solution efficiently. In this thesis, we will present two independent approaches that can be helpful for the temporal approximation of such equations. For the presented schemes, our aim is always twofold: We begin to prove the convergence of a proposed scheme, while we do not make any additional regularity assumptions on the solution. This verifies that our approaches work in general settings. For practical uses, a certain classification of the size of the error becomes important to rule out an arbitrarily slow convergence of the method. Thus, our second aim is to show certain error bounds if the exact solution u is more regular. This quantifies the error at least under additional assumptions.

We only concentrate on methods with a convergence order of at most one. Higher-order schemes usually only lead to a better convergence rate if the solution is sufficiently smooth. For general nonlinear problems, the solution usually lacks global higher-order spatial and temporal regularity. Thus, we only concentrate on simpler schemes.

In the first part of this work, we consider a randomized scheme for the approximation of the solution to an evolution equation. Precisely, let $T \in (0,\infty)$ as well as a Banach space V and a Hilbert space H be given such that $V \overset{d}{\hookrightarrow} H \cong H^* \overset{d}{\hookrightarrow} V^*$. We consider the problem

$$\begin{cases} u'(t) + A(t)u(t) = f(t) & \text{in } V^*, \quad \text{for almost all } t \in (0,T), \\ u(0) = u_0 & \text{in } H. \end{cases}$$

Here, $\{A(t)\}_{t\in[0,T]}$ is a family of operators such that $A(t)\colon V \to V^*$ for every $t \in [0,T]$. Further, $A(t)$, $t \in [0,T]$, is a monotone, coercive, radially continuous operator, of at most polynomial growth. The function $f\colon [0,T] \to V^*$ is integrable and $u_0 \in H$. Our starting point for the approximation is rather simple: We approximate the solution of the evolution equation with the well-known backward Euler scheme. To this end, for $N \in \mathbb{N}$, let the equidistant temporal grid $0 = t_0 < t_1 < \cdots < t_N = T$ be given with $t_n = nk$, $n \in \{0,\dots,N\}$, and the step size $k = \frac{T}{N}$. In order to find an approximation $\mathbf{U}^n \approx u(t_n)$, a recursion of the type

$$\frac{\mathbf{U}^n - \mathbf{U}^{n-1}}{k} + \mathbf{A}^n\mathbf{U}^n = \mathbf{f}^n, \quad n \in \{1,\dots,N\}, \tag{1}$$

with $\mathbf{U}^0 = u_0$ can be solved. Here, $(\mathbf{A}^n)_{n\in\{1,\dots,N\}}$ and $(\mathbf{f}^n)_{n\in\{1,\dots,N\}}$ are approximations for the operator and the right-hand side.

The question we want to address is what kind of approximations $(\mathbf{A}^n)_{n\in\{1,\dots,N\}}$ and $(\mathbf{f}^n)_{n\in\{1,\dots,N\}}$ should be used. While standard point evaluations for merely integrable data are not well-defined, a suitable choice is

$$\mathbf{A}^n = \frac{1}{k}\int_{t_{n-1}}^{t_n} A(t)\,\mathrm{d}t, \quad \mathbf{f}^n = \frac{1}{k}\int_{t_{n-1}}^{t_n} f(t)\,\mathrm{d}t, \quad n \in \{1,\dots,N\}.$$

In order to obtain such values in practice, quadrature rules are applied. As these rules mostly depend on point evaluations, irregular data remains problematic. For functions with very low regularity, a useful approach to approximate their integral is given by Monte Carlo integration techniques. Instead of using this approach to obtain such integrals, we include it directly to our scheme. This can be done via randomized point evaluations and measuring only the expectation of the error.

Here, we propose two different ways to randomize the evaluation points. First, we consider a temporal grid, which is randomly shifted. We evaluate the data at these points. Under general assumptions on the data and no additional regularity condition imposed on the solution u, we can show that piecewise polynomial prolongations of the values $(\mathbf{U}^n)_{n\in\{1,\dots,N\}}$ converge to the solution in a certain probabilistic sense.

Secondly, we always choose a randomized point between two randomly shifted grid points. When the data is evaluated at these points, we can provide certain error bounds if the solution is more regular and the data fulfills some additional assumptions. More precisely, we assume that the solution u is an element of a fractional Sobolev space and we suppose that the operator $A(t)$, $t \in [0,T]$, fulfills a stronger monotonicity condition and is Lipschitz continuous on bounded sets. Then we can provide error estimates, where we prove that the expectation of the error is sufficiently small.

A second, independent method to improve the computation of a solution, is to decompose the operator and consider an operator splitting. Here, we allow a monotone part $A(t)\colon V \to V^*$, $t \in [0,T]$, as before and a Lipschitz continuous part $B(t)\colon H \to H$, $t \in [0,T]$. Then we consider

$$\begin{cases} u'(t) + A(t)u(t) + B(t)u(t) = f(t) & \text{in } V^*, \quad \text{for almost all } t \in (0,T), \\ u(0) = u_0 & \text{in } H. \end{cases}$$

We again have the same starting point and want to solve the recursion

$$\frac{\mathbf{U}^n - \mathbf{U}^{n-1}}{k} + \mathbf{A}^n\mathbf{U}^n + \mathbf{B}^n\mathbf{U}^n = \mathbf{f}^n, \quad n \in \{1,\dots,N\}, \tag{2}$$

with $\mathbf{U}^0 = u_0^k$. This time our aim is different and we assume that suitable approximations u_0^k, $(\mathbf{A}^n)_{n\in\{1,\dots,N\}}$, $(\mathbf{B}^n)_{n\in\{1,\dots,N\}}$, and $(\mathbf{f}^n)_{n\in\{1,\dots,N\}}$ are known. We concentrate on finding modifications for one single backward Euler step

$$(I + k\mathbf{A}^n + k\mathbf{B}^n)\mathbf{U}^n = k\mathbf{f}^n + \mathbf{U}^{n-1}, \quad n \in \{1,\dots,N\},$$

that make this step potentially easier to solve. When approximating the solution to an operator equation that is governed by a monotone operator, it is convenient to use a backward Euler scheme. Compared to the forward Euler scheme, this has much better stability properties. The downside of the backward Euler method is that it becomes necessary to solve an implicit equation in each step. In our setting, we assume that the operator $B(t)$, $t \in [0,T]$, is bounded on the pivot space H. Here, the better stability properties of the implicit scheme are not present. Thus, we exchange $\mathbf{B}^n\mathbf{U}^n$ in (2) by $\mathbf{B}^n\mathbf{U}^{n-1}$. We then work with the implicit-explicit structure given by

$$(I + k\mathbf{A}^n)\mathbf{U}^n = k\mathbf{f}^n + \mathbf{U}_0^n \quad \text{with} \quad \mathbf{U}_0^n = (I - k\mathbf{B}^n)\mathbf{U}^{n-1}$$

for $n \in \{1,\dots,N\}$. This has the advantage that not more implicit equations appear than really necessary. As the implicit equation containing $\mathbf{A}^n$ can be complex to solve, we further introduce $M \in \mathbb{N}$ values $\mathbf{A}_m^n$ and $\mathbf{f}_m^n$, $m \in \{1,\dots,M\}$, such that $\mathbf{A}^n = \sum_{m=1}^M \mathbf{A}_m^n$ and $\mathbf{f}^n = \sum_{m=1}^M \mathbf{f}_m^n$. Then the backward Euler step containing $\mathbf{A}^n$ is equivalent to

$$\Big(I + k\sum_{m=1}^M \mathbf{A}_m^n\Big)\mathbf{U}^n = k\sum_{m=1}^M \mathbf{f}_m^n + \mathbf{U}_0^n.$$

An application of the well-known product splitting leads to the system of equations given by

$$\big(I + k\mathbf{A}_m^n\big)\mathbf{U}_m^n = k\mathbf{f}_m^n + \mathbf{U}_{m-1}^n, \quad m \in \{1,\dots,M\}.$$

Finally, we obtain a system of the type

$$\mathbf{U}_0^n = (I - k\mathbf{B}^n)\mathbf{U}^{n-1}$$

and

$$(I + k\mathbf{A}_m^n)\mathbf{U}_m^n = k\mathbf{f}_m^n + \mathbf{U}_{m-1}^n, \quad m \in \{1,\dots,M\},$$

with $\mathbf{U}^n = \mathbf{U}_M^n$ for $n \in \{1,\dots,N\}$ and $\mathbf{U}^0 = u_0^k$. This method is not intended to lead to an increased convergence rate. But we will see that the additional error caused by the splitting scheme does not affect the magnitude of the error. Compared to the standard backward Euler scheme, this approach leads to more subproblems. These can potentially be easier to solve such that the total computational time may decrease. A suitable choice of the decomposition for $A(t)$ and $f(t)$, $t \in [0,T]$, can even lead to a problem that is easier to parallelize. In modern hardware structures, parallelization can be a powerful tool to accelerate the algorithm.

We follow a similar intention as for the randomized scheme and prove the convergence of piecewise polynomial prolongations of the values $(\mathbf{U}^n)_{n\in\{1,\dots,N\}}$. Under the additional regularity condition that u is Hölder continuous and that the operator $A(t)$, $t \in [0,T]$, fulfills a stronger monotonicity condition and a bounded Lipschitz condition, we can prove explicit error bounds.

This thesis consists of extensions of the following works, which were developed over the last years.

(i) M. Eisenmann and E. Hansen. [34]

Convergence analysis of domain decomposition based time integrators for degenerate parabolic equations. *Numer. Math.*, 140(4):913–938, 2018.

(ii) M. Eisenmann and E. Hansen. [35]

A variational approach to splitting schemes, with applications to domain decomposition integrators. *ArXiv Preprint, arXiv:1902.10023*, 2019.

(iii) M. Eisenmann, M. Kovács, R. Kruse, and S. Larsson. [37]

On a randomized backward Euler method for nonlinear evolution equations with time-irregular coefficients. *Found. Comput. Math.*, Jan 2019 (Online First).

(iv) M. Eisenmann and R. Kruse. [38]

Two quadrature rules for stochastic Itô-integrals with fractional Sobolev regularity. *Commun. Math. Sci.*, 16(8):2125–2146, 2018.

In the separate chapters, we explain in more detail how the content of this thesis is related to the works mentioned above. Furthermore, two additional papers appeared within the last years. Their content is not included directly in this thesis.

(v) M. Eisenmann, E. Emmrich, and V. Mehrmann. [33]

Convergence of the backward Euler scheme for the operator-valued Riccati differential equation with semi-definite data. *Evol. Equ. Control Theory*, 8(2):315–342, 2019.

(vi) M. Eisenmann, M. Kovács, R. Kruse, and S. Larsson. [36]

Error estimates of the backward Euler-Maruyama method for multi-valued stochastic differential equations. *ArXiv Preprint, arXiv:1906.11538*, 2019.

This monograph mainly consists of variations for the well-known Rothe method, which is used in [99]. We prove convergence results of a semidiscrete scheme in a variational setting. Similar approaches can be found in [35, 37, 40, 41, 42, 44].

When it comes to discretizing evolution equations, different strategies can be used. It is possible to use a semi-discretization either via a temporal discretization or a spatial discretization. In order to obtain a full discretization, these two strategies can be combined. This leads, in particular, to an implementable method. A more basic introduction to the numerical approximation of linear evolution equations can be found in [80, 111]. There, the solutions of linear parabolic equations are approximated using a fully discretized scheme involving a Galerkin approximation.

We only concentrate on a temporal discretization in this work and use the concept of variational solutions. A full discretization can be obtained in a somewhat natural way. One of the main aspects of the concept of variational solutions is to look at the problem in a certain tested way. A Galerkin scheme can easily be integrated through a finite-dimensional space of test functions. This also includes the finite element method.

Two important classes of methods to approximate the solution of a differential equation are Runge–Kutta and multistep methods. For a more basic introduction, we refer the reader to [105]. Applications of Runge–Kutta methods to evolution equations can be found in [44, 53, 60, 87, 95] and multistep methods in [40, 41, 58, 81]. The backward Euler method is one of the most simple prototypes of these classes of algorithms. Randomized point evaluations for both the backward and the forward Euler scheme have been considered in [25, 37, 71, 75] for different types of problem classes.

An introduction to operator splittings can be found in [70]. There exist many different schemes that are based on operator splittings. Similar ones to the product splitting can be found in [34, 35, 106]. Approaches with an implicit-explicit splitting are discussed in [2, 5, 17, 24, 64]. The main difference of an operator splitting based scheme compared to Runge–Kutta or multistep methods is that instead of evaluating the data at different points, we decompose the data in several parts but evaluate them at the same point. Various types of useful decompositions can be used. For many differential equations, there exists an intuitive choice for a splitting given by different structures within one equation. Often, these different structures can be handled easier by themselves. For example, a linear main part and nonlinear perturbation can be split, see [65, 107, 108]. It is also possible to split different partial derivatives or to decompose the domain, see [34, 35, 61, 62].

In our work, we only assume that the solution of the problem fulfills an additional regularity condition that involves a fractional derivative when proving error bounds. Similar error bounds for linear problems can be found in [15, 69]. The analysis becomes more involved as soon as a nonlinearity is part of the equation. A first generalization is to consider a linear main part and allow for some nonlinear perturbation. A numerical analysis for such semilinear problems can be found in [3, 63, 79, 88, 92]. When the equation is only linear with respect to the highest appearing derivative it is called quasilinear. Such problems have been considered in [4, 53, 87]. The numerical treatment of a fully nonlinear equation has been studied in [52, 95]. Evolution equations containing a maximal monotone main part are treated in [58, 59, 60, 64, 66, 101].

This monograph is build up as follows. At the end of the introduction, a collection of the used notation can be found. In Chapter 1, we begin with a short recollection of both the solvability of evolution equations and appropriate regularity results. We give some examples of equations, where the solution is more regular. These examples contain settings that fit the conditions imposed for our explicit error bounds. The two approaches discussed above are explained in detail in Chapter 2 and Chapter 3. In Chapter 2, we consider randomized schemes to approximate the solution of an evolution equation. Here, we begin to prove the convergence of the scheme when the temporal grid is randomly shifted. In order to obtain more information about the magnitude of the error, we prove error bounds for the scheme. The chapter ends with an example of a parabolic problem of p-Laplacian type. The following Chapter 3 is build up similarly. We begin to prove the convergence of a method that is based on an operator splitting scheme. We also prove that under some additional assumptions, certain error bounds can be provided. Again, we show that the abstract theory has applications for nonlinear parabolic problems. At the end of the monograph, we collect some auxiliary results in the appendix. Here, useful inequalities can be found. This is followed by a brief introduction to spaces of Bochner integrable functions on a general measure space and some results from stochastic analysis that are needed for the randomized schemes.

Notation

Let $\mathcal{D} \subset \mathbb{R}^d$, $d \geq 1$, be a bounded domain. We denote the boundary of $\mathcal{D}$ by $\partial\mathcal{D}$ and its closure by $\overline{\mathcal{D}}$. The space of uniformly continuous functions $v\colon \overline{\mathcal{D}} \to \mathbb{R}$ is denoted by $C(\overline{\mathcal{D}})$, while the space of continuously differentiable functions on $\mathcal{D}$ is denoted by $C^1(\mathcal{D})$. On $\overline{\mathcal{D}}$ we also consider the space

$$C^1(\overline{\mathcal{D}}) = \Big\{ v \in C(\overline{\mathcal{D}}) : \partial_i v \text{ exists and } \partial_i v \in C(\overline{\mathcal{D}}), i \in \{1, \dots, d\} \Big\},$$

where $\partial_i v := \frac{\partial v}{\partial x_i}$ for $i \in \{1, \dots, d\}$. The space $C_c^\infty(\mathcal{D})$ contains all functions $v\colon \mathcal{D} \to \mathbb{R}$ that are infinitely many times differentiable and have a compact support in $\mathcal{D}$. Furthermore, for a function $v \in C^1(\mathcal{D})$, we write $\nabla v = \big(\partial_1 v, \dots, \partial_d v\big)^T$ for its gradient while the divergence of a function $v = (v_1, \dots, v_d)^T \in C^1(\mathcal{D})^d$ is denoted by $\nabla \cdot v = \sum_{i=1}^d \partial_i v_i$. For $T \in (0, \infty)$ and a function $v\colon (0,T) \times \mathcal{D} \to \mathbb{R}$, we write $\partial_t v$ for the partial derivative with respect to the temporal parameter.

For $p \in [1, \infty]$ and $\ell \in \mathbb{N}$, we write $L^p(\mathcal{D})^\ell$ for the space of Lebesgue measurable functions $v\colon \mathcal{D} \to \mathbb{R}^\ell$ such that

$$\|v\|_{L^p(\mathcal{D})^\ell} = \begin{cases} \Big(\displaystyle\int_{\mathcal{D}} |v|^p \,\mathrm{d}x \Big)^{\frac{1}{p}}, & p \in [1, \infty), \\ \operatorname*{ess\,sup}_{\mathcal{D}} |v|, & p = \infty \end{cases}$$

is finite. If $\ell = 1$, we just write $L^p(\mathcal{D})$. Here, $\operatorname{ess\,sup}_{\mathcal{D}}$ is the essential supremum over $\mathcal{D}$. For more details and properties, see [1, Chapter 2]. The Sobolev space $W^{1,p}(\mathcal{D})$ consists of all functions $v \in L^p(\mathcal{D})$ such that every weak partial derivative $\partial_i v$, $i \in \{1, \dots, d\}$, exists and is an element of $L^p(\mathcal{D})$. This space is equipped with the norm

$$\|v\|_{W^{1,p}(\mathcal{D})} = \begin{cases} \left(\|v\|_{L^p(\mathcal{D})}^p + \|\nabla v\|_{L^p(\mathcal{D})^d}^p\right)^{\frac{1}{p}}, & p \in [1,\infty), \\ \|v\|_{L^\infty(\mathcal{D})} + \|\nabla v\|_{L^\infty(\mathcal{D})^d}, & p = \infty \end{cases}$$

for $v \in W^{1,p}(\mathcal{D})$. If all the mixed partial derivatives up to order $j \in \mathbb{N}$ exist and are elements of $L^p(\mathcal{D})$, we denote the space of such functions by $W^{j,p}(\mathcal{D})$. Additionally, for $p \in [1,\infty)$ we consider the space $W_0^{1,p}(\mathcal{D})$, which is the closure of $C_c^\infty(\mathcal{D})$ with respect to the norm of $W^{1,p}(\mathcal{D})$. Due to Poincaré's inequality (cf. [19, Corollary 9.19]) the seminorm $\|v\|_{W_0^{1,p}(\mathcal{D})} = \|\nabla v\|_{L^p(\mathcal{D})^d}$, $v \in W_0^{1,p}(\mathcal{D})$, is a full norm on this space. In the case $p = 2$, we also write $H^j(\mathcal{D}) = W^{j,2}(\mathcal{D})$, $j \in \mathbb{N}$, and $H_0^1(\mathcal{D}) = W_0^{1,2}(\mathcal{D})$. More details and properties for Sobolev spaces can be found in [1, 19, 82]. For fractional differentiability exponents, we use the same notation and refer the reader to [26, Chapter 4] for the precise definition.

In the following, let $(X, \|\cdot\|_X)$ be a real Banach space. We write X^* for its dual space that is equipped with the induced norm given by

$$\|f\|_{X^*} = \sup_{\substack{x \in X, \\ \|x\|_X \le 1}} \langle f, x\rangle_{X^* \times X}, \quad f \in X^*,$$

where $\langle f, x\rangle_{X^* \times X} = f(x)$ stands for the duality pairing. For a finite value $T \in (0,\infty)$, the space of uniformly continuous functions $v\colon [0,T] \to X$ is denoted by $C([0,T];X)$. A norm on this space is given by

$$\|v\|_{C([0,T];X)} = \max_{t \in [0,T]} \|v(t)\|_X$$

for $v \in C([0,T];X)$. We denote the Hölder seminorm to the exponent $\alpha \in (0,1]$ of a function $v\colon [0,T] \to X$ by

$$|v|_{C^{0,\alpha}([0,T];X)} = \sup_{\substack{s,t \in [0,T], \\ s \neq t}} \frac{\|v(s) - v(t)\|_X}{|s-t|^\alpha}.$$

We then call the space

$$C^{0,\alpha}([0,T];X) = \{v \in C([0,T];X) : |v|_{C^{0,\alpha}([0,T];X)} < \infty\}$$

the Hölder space with exponent α and use the norm

$$\|v\|_{C^{0,\alpha}([0,T];X)} = \|v\|_{C([0,T];X)} + |v|_{C^{0,\alpha}([0,T];X)}$$

for $v \in C^{0,\alpha}([0,T];X)$. For $\alpha = 1$ this is the space of Lipschitz continuous functions with values in X. For more details on continuous X-valued functions, see [39, Abschnitt 7.1] and for Hölder continuous functions, see [6, Section 3.7].

For $p \in [1,\infty]$ the space of Bochner integrable functions on $[0,T]$ with values in X is denoted by $L^p(0,T;X)$. This space is equipped with the norm

$$\|v\|_{L^p(0,T;X)} = \begin{cases} \left(\int_0^T \|v(t)\|_X^p \,\mathrm{d}t\right)^{\frac{1}{p}}, & p \in [1,\infty), \\ \operatorname*{ess\,sup}_{t \in [0,T]} \|v(t)\|_X, & p = \infty \end{cases}$$

for $v \in L^p(0,T;X)$. For more details, see [39, Abschnitt 7.1] or [96, Section 4.2]. In Appendix A.2 there is a short explanation and collection of results for Bochner integrable functions that are defined on a general measure space.

For $p \in [1,\infty)$, the space $W^{1,p}(0,T;X)$ contains the functions in $L^p(0,T;X)$ that possess a weak derivative in $L^p(0,T;X)$. A norm for this space is given by

$$\|v\|_{W^{1,p}(0,T;X)} = \big(\|v\|^p_{L^p(0,T;X)} + \|v'\|^p_{L^p(0,T;X)}\big)^{\frac{1}{p}},$$

compare [99, Section 7.1]. For $\alpha \in (0,1)$, $p \in [1,\infty)$, and $v\colon [0,T] \to X$ we consider the Sobolev–Slobodeckiĭ seminorm

$$|v|_{W^{\alpha,p}(0,T;X)} = \Big(\int_0^T \int_0^T \frac{\|v(s)-v(t)\|^p_X}{|s-t|^{\alpha p+1}} \,\mathrm{d}s\,\mathrm{d}t\Big)^{\frac{1}{p}}.$$

Then the Sobolev–Slobodeckiĭ space is given by

$$W^{\alpha,p}(0,T;X) = \{v \in L^p(0,T;X) : |v|_{W^{\alpha,p}(0,T;X)} < \infty\},$$

which is endowed with the norm

$$\|v\|_{W^{\alpha,p}(0,T;X)} = \big(\|v\|^p_{L^p(0,T;X)} + |v|^p_{W^{\alpha,p}(0,T;X)}\big)^{\frac{1}{p}}$$

for $v \in W^{\alpha,p}(0,T;X)$. A full introduction and properties can be found in [26, Chapter 4] or [27]. In [102], there is a wide range of embedding theorems for such function spaces.

The spaces V, H, and V^* are called a Gelfand triple if $(V, \|\cdot\|_V)$ is a separable, reflexive Banach space that is continuously and densely embedded into the separable Hilbert space $(H, (\cdot,\cdot)_H, \|\cdot\|_H)$. The space H is identified with its dual and we consider

$$V \overset{d}{\hookrightarrow} H \cong H^* \overset{d}{\hookrightarrow} V^*.$$

For $f \in V^*$ and $v \in V$ we write $\langle f, v\rangle_{V^* \times V} = f(v)$, which is the continuous extension of the inner product of H. On such spaces, we introduce

$$\mathcal{W}^p(0,T) = \{v \in L^p(0,T;V) : v' \text{ exists and } v' \in L^q(0,T;V^*)\},$$

where $p \in (1,\infty)$, $q = \frac{p}{p-1}$, and v' denotes the weak derivative of v. In the case $p = 2$, we write $\mathcal{W}(0,T) = \mathcal{W}^2(0,T)$. The space $\mathcal{W}^p(0,T)$ is equipped with the norm

$$\|v\|_{\mathcal{W}^p(0,T)} = \|v\|_{L^p(0,T;V)} + \|v'\|_{L^q(0,T;V^*)}$$

for $v \in \mathcal{W}^p(0,T)$ and is continuously embedded into $C([0,T];H)$. For more details, see [39, Abschnitt 8.1 and 8.4], [99, Section 7.2], and [96, Section 4.2].

Chapter 1

Solvability and Properties of the Solutions to Evolution Equations

Before we come to the numerical analysis of evolution equations, we state a general setting with known existence results, where the solution is also unique. This is in mind, we explain some settings where the solution fulfills additional regularity conditions.

1.1 Existence and Uniqueness

In this thesis, we work with variational solutions of evolution equations. We only give a brief introduction to this concept. For more details, we refer the reader to the following monographs [39, 46, 49, 99, 117, 118]. Another widely spread solution concept is the theory of mild solutions. Sometimes, we refer to results in the literature, where this notion of solution is used. For more details, see [16, 89, 118].

In the following, let $(H, (\cdot,\cdot)_H, \|\cdot\|_H)$ be a real, separable Hilbert space and let $(V, \|\cdot\|_V)$ be a real, separable, reflexive Banach space, which is continuously and densely embedded into H. Identifying H with its dual, we obtain the Gelfand triple

$$V \overset{d}{\hookrightarrow} H \cong H^* \overset{d}{\hookrightarrow} V^*.$$

For a finite end time $T \in (0,\infty)$, we consider a family $\{A(t)\}_{t\in[0,T]}$ of operators such that $A(t)\colon V \to V^*$ for every $t \in [0,T]$. We assume that the mapping $Av\colon [0,T] \to V^*$ given by $t \mapsto A(t)v$ is Bochner measurable for every $v \in V$. Further, we suppose that $A(t)$ is radially continuous for every $t \in [0,T]$, i.e., the mapping $s \mapsto \langle A(t)(v+sw), w\rangle_{V^*\times V}$ is continuous on $[0,1]$ for every $v, w \in V$. For $\kappa \in [0,\infty)$, we assume that $A(t) + \kappa I$ is monotone, i.e., the inequality

$$\langle A(t)v - A(t)w, v-w\rangle_{V^*\times V} + \kappa\|v-w\|_H^2 \geq 0$$

is fulfilled for all $v, w \in V$. For a fixed $p \in (1,\infty)$, we assume that the operator $A(t)$ fulfills a growth condition and $A(t)+\kappa I$ fulfills a semi-coercivity condition such that there exist $\beta, \lambda \in [0,\infty)$ and $\mu \in (0,\infty)$ with

$$\|A(t)v\|_{V^*} \leq \beta\big(1 + \|v\|_V^{p-1}\big), \quad \langle A(t)v, v\rangle_{V^*\times V} + \kappa\|v\|_H^2 + \lambda \geq \mu|v|_V^p$$

for every $t \in [0,T]$ and $v \in V$. Here, $|\cdot|_V$ is a seminorm on V such that there exists $c_V \in (0,\infty)$ with $\|\cdot\|_V \leq c_V\big(\|\cdot\|_H + |\cdot|_V\big)$. Note that since V is continuously embedded into H, it is always possible to choose $|\cdot|_V = \|\cdot\|_V$. In this case, the operator $A(t)+\kappa I$, $t \in [0,T]$, is coercive. As pointed out in [41], it is possible to rewrite the semi-coercivity condition as a coercivity condition of the type

$$\begin{aligned}\langle (A(t)+(\kappa+\mu)I)v, v\rangle_{V^*\times V} &\geq \mu\big(|v|_V^p + \|v\|_H^2\big) - \lambda \geq \mu\big(|v|_V^{\tilde{p}} + \|v\|_H^{\tilde{p}} - 1\big) - \lambda \\ &\geq 2^{1-\tilde{p}}\mu\big(|v|_V + \|v\|_H\big)^{\tilde{p}} - \mu - \lambda \geq 2^{1-\tilde{p}}\mu c_V^{-\tilde{p}}\|v\|_V^{\tilde{p}} - \mu - \lambda\end{aligned} \tag{1.1}$$

for $v \in V$ with $\tilde{p} = \min\{2, p\}$. In Chapter 3, we will use the semi-coercivity condition as this slightly improves some convergence results compared to using (1.1).

For a source term $f \in L^q(0,T;V^*) + L^1(0,T;H)$, $q = \frac{p}{p-1}$, and $u_0 \in H$, we consider the initial value problem

$$\begin{cases} u' + Au = f & \text{in } L^q(0,T;V^*) + L^1(0,T;H), \\ u(0) = u_0 & \text{in } H. \end{cases}$$

We are looking for a solution u that is an element of the space

$$\mathcal{W}_1^p(0,T) = \{v \in L^p(0,T;V) : v' \text{ exists and } v' \in L^q(0,T;V^*) + L^1(0,T;H)\},$$

where v' denotes the weak derivative of v. This space is continuously embedded into $C([0,T];H)$, compare [109, Chapter III, Section 1.5]. Thus, the initial condition is well-defined.

The existence of a solution to the initial value problem has been proved in [99, Theorem 8.9], [39, Satz 8.4.2], and [85, Section 2.7] if V is compactly embedded into H. A further existence results for $\kappa = 0$ can be found in [118, Chapter 30]. In Chapter 2, we also only consider this particular case. It is not a real restriction for $p \in [2,\infty)$ as it is possible to use a transformation trick, compare [117, Remark 23.25] or [49, Folgerung on page 211]. For $t \in [0,T]$, we transform $A(t)$ and $f(t)$ to $\tilde{A}(t) = e^{-\kappa t}A(t)e^{\kappa t} + \kappa I$ and $\tilde{f}(t) = e^{-\kappa t}f(t)$ and find a solution to the problem

$$\begin{cases} \tilde{u}' + \tilde{A}\tilde{u} = \tilde{f} & \text{in } L^q(0,T;V^*) + L^1(0,T;H), \\ \tilde{u}(0) = u_0 & \text{in } H. \end{cases} \tag{1.2}$$

The family $\{\tilde{A}(t)\}_{t\in[0,T]}$ of operators $\tilde{A}(t)\colon V \to V^*$, $t \in [0,T]$, fulfills all the condition imposed above for $\kappa = 0$. The solution of the original problem is given by $u(t) = e^{\kappa t}\tilde{u}(t)$ in H for $t \in [0,T]$. Usually, it is a requirement to assume that V is compactly embedded into H when allowing for $\kappa \in (0,\infty)$. With this transformation, we do not have to make this assumption.

Furthermore, the solutions of both the original and the transformed problem are unique, compare [99, Theorem 8.31]. Thus, from an analytical point of view, it makes sense to use the transformed problem. In applications, it might be preferable to work with the original problem without any transformation. In Chapter 3, we allow for arbitrary $\kappa \in [0,\infty)$ in terms of an additionally appearing family $\{B(t)\}_{t\in[0,T]}$ of operators such that $B(t)\colon H \to H$. This kind of operator cannot easily be included in Chapter 2 due to a missing compactness result, which we will point out later.

1.2 Regularity of the Solution

For the results in Sections 2.2 and 3.2, we need some additional regularity of the solution. Precisely, we need that the solution is in a certain Sobolev–Slobodeckiĭ space or in a Hölder space with values in V depending on the exact statement. These particular conditions are not available under general assumptions on the data. We will focus briefly on some settings where we can obtain this higher-order regularity. For a general overview, we refer the reader to [89] and [7, Chapter III]. Some further results for a fractional derivative can be found in [78, Chapter IV, Section 5] and [84, Chapter 4, Section 5].

For general nonlinear problems, it is difficult to state suitable regularity results that fit our assumptions. Here, it becomes necessary to look at the specific problem more closely to obtain any appropriate results if possible. In the following, we concentrate on known regularity results for linear problems. For certain semilinear problems, bootstrap arguments can be applied to recover the regularity of the linear problem, compare [110, Section 3] or Example 1.2.4 below. Some further regularity results for more specific nonlinear problems can be found in [28, 51].

In the following, we assume that V is a real, separable Hilbert space, which is continuously and densely embedded into the real, separable Hilbert space H such that we again obtain a Gelfand triple $V \overset{d}{\hookrightarrow} H \cong H^* \overset{d}{\hookrightarrow} V^*$. We assume that the family $\{A(t)\}_{t\in[0,T]}$ of operators $A(t)\colon V \to V^*$, $t \in [0,T]$, fulfills the conditions stated in the previous section for $p = 2$ and that $A(t)\colon V \to V^*$ is linear for every $t \in [0,T]$. For $f \in L^2(0,T;V^*)$, we consider the linear equation

$$\begin{cases} u' + Au = f & \text{in } L^2(0,T;V^*), \\ u(0) = u_0 & \text{in } H. \end{cases} \tag{1.3}$$

For such a linear problem, compatibility conditions lead to a more regular solution, see [39, Abschnitt 8.5], [46, Chapter 7, Theorem 6], [48, Chapter 10, Section 6–7], and [114, §27].

To this end, we assume that the temporal derivative of f exists and it fulfills $f' \in L^2(0,T;V^*)$. We also need an additional assumption for the family of operators $\{A(t)\}_{t\in[0,T]}$. Here, we assume that the classical derivative of $t \mapsto \langle A(t)v, w\rangle_{V^*\times V}$ exists on $[0,T]$, is measurable, and there exists $\beta' \in [0,\infty)$ such that $|\frac{\mathrm{d}}{\mathrm{d}t}\langle A(t)v, w\rangle_{V^*\times V}| \leq \beta'\|v\|_V\|w\|_V$ for every $v, w \in V$ and $t \in [0,T]$. This derivative can be used to define the operator $A'(t)\colon V \to V^*$, $t \in [0,T]$, by $\langle A'(t)v, w\rangle_{V^*\times V} = \frac{\mathrm{d}}{\mathrm{d}t}\langle A(t)v, w\rangle_{V^*\times V}$ for $v, w \in V$. The operator $A'(t)$, $t \in [0,T]$, is linear and bounded independently of t. Further, if the initial conditions $u(0) = u_0$ in V and $u_0' := f(0) - A(0)u_0$ in H are fulfilled, it follows that $u, u' \in L^2(0,T;V)$ and $u'' \in L^2(0,T;V^*)$. This shows, in particular, that $u \in W^{1,2}(0,T;V)$ and $u' \in W^{1,2}(0,T;V^*)$. Using embedding theorems, we obtain that

$$\begin{aligned} u &\in W^{1,2}(0,T;V) \hookrightarrow W^{\alpha,2}(0,T;V) \hookrightarrow C^{0,\alpha-\frac{1}{2}}([0,T];V), \\ u' &\in W^{1,2}(0,T;V^*) \hookrightarrow W^{\alpha,2}(0,T;V^*) \hookrightarrow C^{0,\alpha-\frac{1}{2}}([0,T];V^*), \end{aligned} \tag{1.4}$$

for every $\alpha \in (0,1)$, cf. [102, Corollary 26]. This includes the regularity conditions stated in Theorem 2.2.7 and 3.2.3 in the case $p = 2$. For a nonlinear, yet autonomous, operator, a similar idea has been considered in [99, Theorem 8.18].

A further approach to prove higher-order regularity of the solution to the linear problem (1.3) is the concept of maximal L^p-regularity. The initial value problem (1.3) is said to have maximal L^p-regularity in H for some $p \in [2,\infty)$ if for every $f \in L^p(0,T;H)$ the unique solution $u \in \mathcal{W}(0,T)$ fulfills that $u' \in L^p(0,T;H)$ and $Au \in L^p(0,T;H)$. Suitable regularity results can be obtained by embedding theorems as we will see in one of the examples below. A survey about this concept can be found in [93]. Sufficient conditions to obtain maximal L^p-regularity are stated in [9, 10, 11, 32, 47, 55].

Another approach is to only look for local regularity. Due to the parabolic smoothing property, it is possible to prove certain regularity results away from the initial data. If we assume that $f \in L^2(0,T;V^*)$ fulfills $t \mapsto tf'(t) \in L^2(0,T;V^*)$ and $A'(t)\colon V \to V^*$, $t \in [0,T]$, is linear and bounded independently of t, then the solution of (1.3) fulfills that $t \mapsto tu'(t) \in \mathcal{W}(0,T)$. See [39, Satz 8.5.3] and [111, Chapter 3] for more details. For a fully nonlinear problem, a regularity result of this type can be found in [95, Lemma 3]. These local results are not enough for our analysis and are more useful in a non-smooth data analysis as has been done in [79, 88, 111]. Still, they indicate that after a certain time the methods should work well if the error from the beginning has not become too large. Moreover, under the assumption that the solution is bounded, it is possible to prove local Hölder regularity for nonlinear problems, compare [28, Chapter III].

In the following, we present a few different examples and discuss the regularity of the solution.

Example 1.2.1. For a finite end time $T \in (0,\infty)$ and a bounded Lipschitz domain $\mathcal{D} \subset \mathbb{R}^d$, $d \in \mathbb{N}$, we look at the problem

$$\begin{cases} \partial_t u(t,x) - \nabla \cdot \big(\mathbf{a}(t,x)\nabla u(t,x)\big) + \mathbf{b}(t,x,u(t,x)) = f(t,x), & (t,x) \in (0,T)\times\mathcal{D},\\ u(t,x) = 0, & (t,x) \in (0,T)\times\partial\mathcal{D},\\ u(0,x) = u_0(x), & x \in \mathcal{D}. \end{cases}$$

We assume that $\mathbf{a} = (\mathbf{a}_{ij})_{i,j\in\{1,\dots,d\}}\colon [0,T]\times\overline{\mathcal{D}} \to \mathbb{R}^{d,d}$ is an element of $L^\infty((0,T)\times\mathcal{D};\mathbb{R}^{d,d})$ and there exists $\mu \in (0,\infty)$ with

$$\mathbf{a}(t,x)z\cdot z = \sum_{i,j=1}^{d} \mathbf{a}_{ij}(t,x)z_i z_j \geq \mu|z|^2 \tag{1.5}$$

for all $t \in [0,T]$, almost all $x \in \mathcal{D}$, and all $z \in \mathbb{R}^d$. We assume that $\mathbf{b}\colon [0,T]\times\overline{\mathcal{D}}\times\mathbb{R} \to \mathbb{R}$ fulfills $\mathbf{b}(\cdot,\cdot,z) \in L^\infty((0,T)\times\mathcal{D})$ for every $z \in \mathbb{R}$ and there exist $\kappa,\rho \in [0,\infty)$ such that

$$|\mathbf{b}(t,x,z) - \mathbf{b}(t,x,\tilde{z})| \leq \kappa|z-\tilde{z}|, \quad |\mathbf{b}(t,x,0)| \leq \rho \tag{1.6}$$

for all $t \in [0,T]$, almost all $x \in \mathcal{D}$, and all $z,\tilde{z} \in \mathbb{R}$.

We choose the Hilbert spaces $V = H_0^1(\mathcal{D})$ and $H = L^2(\mathcal{D})$ equipped with the norms given in the introduction. We obtain the Gelfand triple $V \overset{d}{\hookrightarrow} H \cong H^* \overset{d}{\hookrightarrow} V^*$, where we identify H with its dual space. We introduce the operators $A(t)\colon V \to V^*$ and $B(t)\colon H \to H$, $t \in [0,T]$, which are given by

$$\langle A(t)v,w\rangle_{V^*\times V} = \int_{\mathcal{D}} \mathbf{a}(t,\cdot)\nabla v\cdot\nabla w\,\mathrm{d}x, \quad v,w \in V, \tag{1.7}$$

$$(B(t)v,w)_H = \int_{\mathcal{D}} \mathbf{b}(t,\cdot,v)w\,\mathrm{d}x, \quad v,w \in H. \tag{1.8}$$

We assume that for $f\colon [0,T]\times\overline{\mathcal{D}} \to \mathbb{R}$ the abstract function $[f(t)](x) = f(t,x)$, $(t,x) \in (0,T)\times\mathcal{D}$, is an element of $L^2(0,T;V^*)$. For $u_0 \in H$, we obtain the variational formulation of the problem

$$\begin{cases} u' + Au + Bu = f & \text{in } L^2(0,T;V^*),\\ u(0) = u_0 & \text{in } H. \end{cases} \tag{1.9}$$

In the following, we verify that this equation fits into the setting introduced in the previous section. The proof for this is quite basic. As all the examples mentioned in this section have the same underlying structure, we add it for the sake of completeness.

First, we prove that $t \mapsto B(t)v$ is measurable for every $v \in H$. The measurability of $t \mapsto A(t)v$ for $v \in V$ can be argued analogously. By assumption, $t \mapsto \mathbf{b}(t,x,z)$ is measurable for almost every $x \in \mathcal{D}$ and every $z \in \mathbb{R}$. Thus, there exists a sequence $(\mathbf{b}_i)_{i\in\mathbb{N}}$ of functions $\mathbf{b}_i\colon [0,T] \times \overline{\mathcal{D}} \times \mathbb{R} \to \mathbb{R}$, $i \in \mathbb{N}$, that are simple with respect to the first argument such that $\mathbf{b}_i(t,x,z) \to \mathbf{b}(t,x,z)$ as $i \to \infty$ and $|\mathbf{b}_i(t,x,z)| \leq |\mathbf{b}(t,x,z)|$, $i \in \mathbb{N}$, for almost every $(t,x) \in (0,T) \times \mathcal{D}$ and every $z \in \mathbb{R}^d$. For $t \in [0,T]$, we defined the simple operator $B_i(t)\colon H \to H$ given by

$$(B_i(t)v, w)_H = \int_{\mathcal{D}} \mathbf{b}_i(t,\cdot,v)w \,\mathrm{d}x, \quad v,w \in H.$$

Applying the conditions from (1.6), it follows that

$$\begin{aligned}
\big|\big(\mathbf{b}(t,\cdot,v) - \mathbf{b}_i(t,\cdot,v)\big)w\big| &\leq 2|\mathbf{b}(t,\cdot,v)||w| \\
&\leq 2|\mathbf{b}(t,\cdot,v) - \mathbf{b}(t,\cdot,0)||w| + 2|\mathbf{b}(t,\cdot,0)||w| \\
&\leq 2\kappa|v||w| + 2\rho|w| \qquad (1.10)
\end{aligned}$$

for every $v,w \in H$, for almost every $t \in (0,T)$, and almost everywhere in $\mathcal{D}$. As (1.10) is an integrable function on $\mathcal{D}$, we can apply Lebesgue's dominated convergence theorem to obtain that

$$\lim_{i\to\infty} (B(t)v - B_i(t)v, w)_H = \int_{\mathcal{D}} \lim_{i\to\infty} \big(\mathbf{b}(t,\cdot,v) - \mathbf{b}_i(t,\cdot,v)\big)w \,\mathrm{d}x = 0$$

for every $v,w \in H$ and almost every $t \in (0,T)$. This implies that $t \mapsto B(t)v$, $v \in H$, is weakly measurable. As H is also separable, the mapping is Bochner measurable.

It is easy to see that $v \mapsto A(t)v$ is linear and inserting the definition of $A(t)$ implies

$$\langle A(t)v, w\rangle_{V^*\times V} = \int_{\mathcal{D}} \mathbf{a}(t,\cdot)\nabla v \cdot \nabla w \,\mathrm{d}x \leq \|\mathbf{a}\|_{L^\infty((0,T)\times\mathcal{D};\mathbb{R}^{d,d})}\|v\|_V\|w\|_V \qquad (1.11)$$

for every $v,w \in V$ and $t \in [0,T]$. Hence, this shows that $v \mapsto A(t)v$ is continuous because $\|A(t)\|_{\mathcal{L}(V,V^*)} \leq \|\mathbf{a}\|_{L^\infty((0,T)\times\mathcal{D};\mathbb{R}^{d,d})}$ holds true for every $t \in [0,T]$, where $\|\cdot\|_{\mathcal{L}(V,V^*)}$ denotes the induced operator norm. Further, the Lipschitz continuity of $\mathbf{b}$ in the third argument shows

$$(B(t)v_1 - B(t)v_2, w)_H = \int_{\mathcal{D}} \big(\mathbf{b}(t,\cdot,v_1) - \mathbf{b}(t,\cdot,v_2)\big)w \,\mathrm{d}x \leq \kappa\|v_1 - v_2\|_H\|w\|_H$$

for every $v_1, v_2, w \in H$ and $t \in [0,T]$. Therefore, we see that $B(t)\colon H \to H$ is Lipschitz continuous, as $\|B(t)v_1 - B(t)v_2\|_H \leq \kappa\|v_1 - v_2\|_H$ is fulfilled for every $v_1, v_2 \in H$ and $t \in [0,T]$. This shows that $v \mapsto A(t)v + B(t)v$ is continuous and $A(t) + B(t)$ radially continuous for every $t \in [0,T]$.

Next, we prove that $A(t) + B(t) + \kappa I$, $t \in [0,T]$, is monotone. To this end, we first notice that

$$\langle A(t)v, v\rangle_{V^*\times V} = \int_{\mathcal{D}} \mathbf{a}(t,\cdot)\nabla v \cdot \nabla v \,\mathrm{d}x \geq \mu \int_{\mathcal{D}} |\nabla v|^2 \,\mathrm{d}x = \mu\|v\|_V^2$$

is fulfilled for every $v \in V$ and $t \in [0,T]$ due to (1.5). Thus, we see that

$$\begin{aligned}
&\langle A(t)v_1 - A(t)v_2 + B(t)v_1 - B(t)v_2, v_1 - v_2\rangle_{V^*\times V} + \kappa\|v_1 - v_2\|_H^2 \\
&= \int_{\mathcal{D}} \mathbf{a}(t,\cdot)\big(\nabla v_1 - \nabla v_2\big) \cdot \big(\nabla v_1 - \nabla v_2\big) \,\mathrm{d}x \\
&\quad + \int_{\mathcal{D}} \big(\mathbf{b}(t,\cdot,v_1) - \mathbf{b}(t,\cdot,v_2)\big)(v_1 - v_2) \,\mathrm{d}x + \kappa\|v_1 - v_2\|_H^2 \\
&\geq \mu\|v_1 - v_2\|_V^2
\end{aligned}$$

holds for every $v_1, v_2 \in V$ and $t \in [0,T]$.

Using (1.6), it follows that

$$(B(t)0, w)_H = \int_{\mathcal{D}} \mathbf{b}(t,\cdot,0) w \,\mathrm{d}x \le \rho |\mathcal{D}|^{\frac{1}{2}} \|w\|_H \tag{1.12}$$

and together with the Lipschitz continuity of $B(t)$

$$\|B(t)v\|_H \le \|B(t)v - B(t)0\|_H + \|B(t)0\|_H \le \kappa \|v\|_H + \rho |\mathcal{D}|^{\frac{1}{2}} \tag{1.13}$$

for every $v, w \in H$ and $t \in [0,T]$, where $|\mathcal{D}|$ denotes the Lebesgue measure of $\mathcal{D}$. Thus, we see that $A(t) + B(t) + (\kappa+1)I$ is coercive since

$$\begin{aligned}
\langle A(t)v + B(t)v, v\rangle_{V^*\times V} + (\kappa+1)\|v\|_H^2 &\ge \mu \|v\|_V^2 - \|B(t)v\|_H \|v\|_H + (\kappa+1)\|v\|_H^2 \\
&\ge \mu \|v\|_V^2 - \kappa \|v\|_H^2 - \rho |\mathcal{D}|^{\frac{1}{2}} \|v\|_H + (\kappa+1)\|v\|_H^2 \\
&\ge \mu \|v\|_V^2 - \frac{\rho^2}{4} |\mathcal{D}|
\end{aligned}$$

is fulfilled for every $v \in V$ and $t \in [0,T]$, where we applied the weighted Young inequality.

Combining (1.11) and (1.13), the operator $A(t) + B(t)$, $t \in [0,T]$, is bounded in the sense that

$$\|A(t)v + B(t)v\|_{V^*} \le \|\mathbf{a}\|_{L^\infty((0,T)\times\mathcal{D};\mathbb{R}^{d,d})} \|v\|_V + c_1 \big(\kappa \|v\|_H + \rho |\mathcal{D}|^{\frac{1}{2}}\big),$$

holds. Here, $c_1 \in (0,\infty)$ denotes the embedding constant of H into V^*.

In [39, Satz 8.3.5], it is shown that there exists a unique $u \in \mathcal{W}(0,T)$ that solves (1.9).

Example 1.2.2. We consider the same setting as in Example 1.2.1 with $\mathbf{b} \equiv 0$. Additionally, we assume that for every $i, j \in \{1, \dots, d\}$ the function $t \mapsto \mathbf{a}_{ij}(t,x)$ is absolutely continuous for almost every $x \in \mathcal{D}$, $\mathbf{a}_{ij}(0,\cdot) \in W^{1,\infty}(\mathcal{D})$, and $\partial_t \mathbf{a}_{ij} \in L^\infty((0,T)\times\mathcal{D})$. The operator $A'(t)\colon V \to V^*$ fulfills

$$\langle A'(t)v, w\rangle_{V^*\times V} = \int_{\mathcal{D}} \partial_t \mathbf{a}(t,\cdot) \nabla v \cdot \nabla w \,\mathrm{d}x \le \|\partial_t \mathbf{a}\|_{L^\infty((0,T)\times\mathcal{D};\mathbb{R}^{d,d})} \|v\|_V \|w\|_V$$

for every $v, w \in V$ and $t \in [0,T]$. This implies that $A'(t)\colon V \to V^*$, $t \in [0,T]$, is linear and bounded independently of t. Moreover, we choose $f \in W^{1,2}(0,T;V^*)$ with $f(0) \in H$ and $u_0 \in H^2(\mathcal{D}) \cap V$. Then it follows that $u_0' := f(0) + \nabla \cdot \big(\mathbf{a}(0,\cdot)\nabla u_0\big) \in H$. We can apply [39, Satz 8.5.1], where regularity is obtained through compatibility conditions of the data. Then we obtain that $u \in W^{1,2}(0,T;V)$ and $u' \in W^{1,2}(0,T;V^*)$. Using the embedding theorem from [102, Corollary 26], it follows that

$$\begin{aligned}
u &\in W^{1,2}(0,T;V) \hookrightarrow W^{\alpha,2}(0,T;V) \hookrightarrow C^{0,\alpha-\frac{1}{2}}([0,T];V), \\
u' &\in W^{1,2}(0,T;V^*) \hookrightarrow W^{\alpha,2}(0,T;V^*) \hookrightarrow C^{0,\alpha-\frac{1}{2}}([0,T];V^*).
\end{aligned}$$

for every $\alpha \in (0,1)$.

Example 1.2.3. We consider the same setting as in Example 1.2.1 with $\mathbf{b} \equiv 0$. We apply [29, Corollary 7.1] for $\alpha \in (0,\frac{1}{2})$, $f \in L^2(0,T;H^{2\alpha-1}(\mathcal{D}))$, $u_0 \in H_0^{2\alpha}(\mathcal{D})$. Further, we assume that for $\varepsilon \in \big(0,\frac{1}{2}\big)$ the coefficients $\mathbf{a}_{ij}$, $i,j \in \{1,\dots,d\}$, are in $W^{\alpha+\varepsilon,\frac{1}{\alpha}}(0,T;L^\infty(\mathcal{D}))$ and therefore continuous. Then it follows that $u \in W^{\alpha,2}(0,T;V) \cap W^{1,2}(0,T;H^{2\alpha-1}(\mathcal{D}))$.

Additionally, we can find a corresponding result for the derivative u'. To this end, we use that in [29, Theorem 6.2, Corollary 7.1] it is noted that $A \in W^{\alpha+\varepsilon,\frac{1}{\alpha}}(0,T;\mathcal{L}(V,V^*))$. Together with a suitable result for the Nemytskiĭ operator of A from [29, Lemma 5.3] and an extension result from [27, Theorem 5.4], it follows that $Au \in W^{\alpha,2}(0,T;V^*)$. For a function $f \in W^{\alpha,2}(0,T;V^*)$, this also shows that $u' \in W^{\alpha,2}(0,T;V^*)$.

Example 1.2.4. Again, we consider the same setting as in Example 1.2.1. At first, we set $\mathbf{b} \equiv 0$ but show later that we can recover the regularity for more general $\mathbf{b}$. In the following, we denote the Friedrichs extension of an operator $A(t)$ by $A_F(t)\colon \mathrm{dom}(A_F(t)) \to H$, $A_F(t)u = A(t)u$, where $\mathrm{dom}(A_F(t)) = H^2(\mathcal{D}) \cap H_0^1(\mathcal{D})$ for every $t \in [0,T]$. We assume that there exists $p \in (2,\infty)$ such that $\mathbf{a}$ fulfills the additional condition

$$|\mathbf{a}_{ij}(t,x) - \mathbf{a}_{ij}(s,x)| \le \omega(|t-s|)$$

for every $i,j \in \{1,\dots,d\}$, $s,t \in [0,T]$, as well as almost every $x \in \mathcal{D}$. Here, $\omega\colon [0,T] \to [0,\infty)$ is a non-decreasing function that fulfills

$$\int_0^T \frac{\omega(t)}{t^{\frac{3}{2}}}\,\mathrm{d}t < \infty \quad \text{and} \quad \int_0^T \Big(\frac{\omega(t)}{t}\Big)^p\,\mathrm{d}t < \infty.$$

Then we can apply [55, Theorem 2] and find that (1.9) has maximal L^p-regularity for all u_0 within the interpolation space $(H, \mathrm{dom}(A_F(0)))_{1-\frac{1}{p},p}$. As a proper explanation of the concept of interpolation spaces is out of place in this section, we only refer the reader to [90, Chapter 1] and [112, Chapter 1] for further details.

This means that for every $f \in L^p(0,T;H)$ the solution u is an element of $W^{1,p}(0,T;H)$ and $Au \in L^p(0,T;H)$. Analogously to [37, Theorem 7.2], it follows that $u \in C^{0,\alpha}([0,T];V)$ for every $\alpha \in [0,\frac{1}{2}-\frac{1}{p}-\varepsilon)$ and an arbitrary $\varepsilon \in (0,\frac{1}{2}-\frac{1}{p})$. Note that using the first embedding result from [8, Theorem 5.2] in the proof of [37, Theorem 7.2] instead, it also follows that $u \in W^{\alpha+\frac{1}{p},2}(0,T;V)$. Regularity results for u' can be obtained analogously as in the previous example after possibly asking for some additional regularity for the data.

Using a similar idea as in [37, Theorem 7.8] and [92, Theorem 2.10], we can allow for a nontrivial function $\mathbf{b}$ and can still recover the same regularity for the solution. To this end, let $u \in C([0,T];H)$ be the unique solution of (1.9). The function $g = f - Bu$ fulfills

$$\begin{aligned}\|g\|_{L^p(0,T;H)} &\le \|f\|_{L^p(0,T;H)} + \|Bu - B0\|_{L^p(0,T;H)} + \|B0\|_{L^p(0,T;H)}\\ &\le \|f\|_{L^p(0,T;H)} + \kappa\|u\|_{L^p(0,T;H)} + \Big(\int_0^T \|B(t)0\|_H^p\,\mathrm{d}t\Big)^{\frac{1}{p}}\\ &\le \|f\|_{L^p(0,T;H)} + \kappa\|u\|_{L^p(0,T;H)} + T^{\frac{1}{p}}\rho|\mathcal{D}|^{\frac{1}{2}},\end{aligned}$$

where we use the Lipschitz continuity of $B(t)$, $t \in [0,T]$, and (1.12). Thus, we have $g \in L^p(0,T;H)$ and it follows that the solution v of

$$\begin{cases} v' + Av = g & \text{in } L^2(0,T;V^*),\\ v(0) = u_0 & \text{in } H \end{cases} \tag{1.14}$$

is an element of $C^{0,\alpha}([0,T];V)$ and $W^{\alpha+\frac{1}{p},2}(0,T;V)$ as we have seen above.

Now, we can use a bootstrap argument to show that the solution u of (1.9) has the same regularity. Both (1.9) and (1.14) have a unique solution. Inserting u in (1.14), we see that u also solves this problem. Thus, u and v coincide and fulfill the same regularity condition.

Chapter 2

Randomized Schemes for Nonlinear, Nonautonomous Evolution Equations

In this chapter, we introduce randomized schemes that can be used to approximate the solution of a nonlinear, nonautonomous evolution equation on a finite time interval. Precisely, for $T \in (0,\infty)$, we consider

$$\begin{cases} u'(t) + A(t)u(t) = f(t) & \text{in } V^*, \quad \text{for almost all } t \in (0,T), \\ u(0) = u_0 & \text{in } H \end{cases} \tag{2.1}$$

for a Gelfand triple $V \overset{d}{\hookrightarrow} H \cong H^* \overset{d}{\hookrightarrow} V^*$ as well as a family $\{A(t)\}_{t\in[0,T]}$ of monotone operators $A(t)\colon V \to V^*$, $t \in [0,T]$, a source term $f\colon [0,T] \to V^*$, and an initial value $u_0 \in H$. The approach presented here mainly offers advantages for a temporal approximation. Hence, we only consider a discretization in time. This leads to a semidiscrete problem. We begin to follow a standard approach given by the backward Euler scheme. For $N \in \mathbb{N}$, we consider an equidistant partition $0 = t_0 < \cdots < t_N = T$ with $k = \frac{T}{N}$ and $t_n = nk$, $n \in \{0,\dots,N\}$, of the interval $[0,T]$ to find an approximation $\mathbf{U}^n$ of $u(t_n)$, $n \in \{1,\dots,N\}$. To this end, we solve the recursion

$$\frac{\mathbf{U}^n - \mathbf{U}^{n-1}}{k} + \mathbf{A}^n\mathbf{U}^n = \mathbf{f}^n \quad \text{in } V^*, \quad n \in \{1,\dots,N\},$$

for $\mathbf{U}^0 = u_0$, where $(\mathbf{A}^n)_{n\in\{1,\dots,N\}}$ and $(\mathbf{f}^n)_{n\in\{1,\dots,N\}}$ are approximations of the data. A common choice of such values is of the form

$$\mathbf{A}^n = \frac{1}{k}\int_{t_{n-1}}^{t_n} A(t)\,\mathrm{d}t, \quad \mathbf{f}^n = \frac{1}{k}\int_{t_{n-1}}^{t_n} f(t)\,\mathrm{d}t, \quad n \in \{1,\dots,N\}.$$

These values prove themselves to be very suitable if they are known. In practice though, they are not necessarily available and additional approximation techniques are needed. An easy to get, yet not always well-defined, alternative of merely integrable data would be

$$\mathbf{A}^n = A(t_n), \quad \mathbf{f}^n = f(t_n), \quad n \in \{1,\dots,N\}.$$

Due to the low regularity, point evaluations will in general not offer a suitable approximation. As a pre-designed grid and even the amount of points in a sequence of such grids is only a null set, it is possible to find functions that "fool" the scheme. This can be done by redefining the data on the grid points. In order to bypass this problem, we will work with two different types of randomization. The first is ideal for proving convergence in a setting without any further regularity assumptions on the solution. For a complete probability space $(\Omega_\theta, \mathcal{F}^\theta, \mathbf{P}_\theta)$ and a uniformly distributed random variable $\theta\colon \Omega_\theta \to [0,1]$, we consider the randomly shifted grid

$$0 = t_0^\theta < t_1^\theta < \cdots < t_N^\theta = T - k(1-\theta) \quad \text{with } t_n^\theta = t_{n-1} + k\theta, \quad n \in \{1, \dots, N\}.$$

Note that we write θ as an index to the probability space. This is not supposed to show any dependence on θ but only means that this is the probability space that θ is defined on. Later, we also introduce a further family of random variables on a different probability space. This second probability space is also indexed so there will be no mix up between the two spaces. For

$$\mathbf{A}^n = A(t_n^\theta), \quad \mathbf{f}^n = f(t_n^\theta), \quad n \in \{1, \dots, N\},$$

we prove in Section 2.1 that the piecewise polynomial prolongations of $(\mathbf{U}^n)_{n\in\{1,\dots,N\}}$ converge to u pointwise strongly in $L^2(\Omega_\theta; H)$. Furthermore, depending on the monotonicity condition imposed on $A(t)$, $t \in [0,T]$, we obtain weak or strong convergence in the space $L^p(0,T; L^p(\Omega_\theta; V))$, where the value p depends on $A(t)$, $t \in [0,T]$. Measuring the expectation of the error on a randomly shifted grid, offers a way to handle data that is non-smooth with respect to the temporal input. This convergence result can be obtained with fairly general assumptions on the data and no additional regularity requirements on the solution. If higher regularity of the solution might be available though, we prove in Section 2.2 that a simple modification of the scheme leads to explicit error bounds with an order that depends on the Sobolev–Slobodeckiĭ regularity of the solution. Here, a second randomization can be used to exploit the additional regularity. To this end, for a second complete probability space $(\Omega_\tau, \mathcal{F}^\tau, \mathbf{P}_\tau)$, let $(\tau_n)_{n\in\{1,\dots,N\}}$ be a family of independent, uniformly distributed random variables with $\tau_n\colon \Omega_\tau \to [0,1]$. For

$$\xi_n = t_{n-1}^\theta + (t_n^\theta - t_{n-1}^\theta)\tau_n, \quad n \in \{1, \dots, N\}, \tag{2.2}$$

we use

$$\mathbf{A}^n = A(\xi_n), \quad \mathbf{f}^n = f(\xi_n), \quad n \in \{1, \dots, N\},$$

to prove that there exists $C \in (0,\infty)$, which depends on u and u', such that

$$\max_{n\in\{1,\dots,N\}} \mathbf{E}\big[\|u(t_n^\theta) - \mathbf{U}^n\|_H^2\big] + \sum_{n=1}^N \mathbf{E}\big[(t_n^\theta - t_{n-1}^\theta)\|u(t_n^\theta) - \mathbf{U}^n\|_V^p\big] \le Ck^{\alpha\frac{p}{p-1}} \tag{2.3}$$

is fulfilled for every $N \in \mathbb{N}$. Here, p depends on $A(t)$, $t \in [0,T]$, and $\alpha \in (0,1)$ denotes the differentiability exponent of the Sobolev–Slobodeckiĭ spaces that contain u and u'. We write $\mathbf{E}$ for the expectation on the product probability space $(\Omega_\theta \times \Omega_\tau, \mathcal{F}^\theta \otimes \mathcal{F}^\tau, \mathbf{P}_\theta \otimes \mathbf{P}_\tau)$. Even though, we have to make additional regularity assumptions on the solution, we will see that they are rather low. In order to demonstrate the relevance of these theoretical convergence results and error bounds, we show how they can be applied to a nonlinear parabolic problem. This includes, in particular, the well-known parabolic p-Laplacian equation. Note that the porous media equation in a very weak formulation as considered in [34, 45] also fits in the abstract framework.

The approach of using randomized schemes has already been studied for certain classes of problems in the literature. Within the context of Monte Carlo algorithms, these methods are well-known for the approximation of integrals. See [94] for an introduction. The concept of stratified sampling was first introduced in [56, 57]. There, a subdivision of the domain of integration is made and one random value within each subset is chosen. This approach is useful to make sure that the random variables are distributed more evenly compared to a standard Monte Carlo algorithm. In [103, 104], a randomization was used to approximate the solution of an ordinary differential equation for the first time. There are many works thereafter that continue this approach. The use of explicit randomized schemes was further studied in [25, 71, 75]. When compared to deterministic schemes, randomized schemes can often handle low regularity assumptions on the data better while their complexity does not increase. In the theory of information-based complexity, upper and lower bounds of classes of randomized schemes are studied, see [68, 72]. Similar to ours, yet explicit, schemes for stochastic ordinary differential equations have been considered in [77, 97, 98] or for stochastic partial differential equations in [76]. In [18], a randomized grid was used for this type of problem. Note that a randomization of the grid is only helpful if the regularity is measured in an appropriate way. As pointed out in [50], there are problem classes where an additional randomization does not yield any advantages compared to a deterministic time grid. Data from a Sobolev–Slobodeckiĭ space seems to be well suited for this kind of scheme.

The central idea of the work in the following chapter is based on both [37] and [38]. In [38], a randomized grid was used to prove the convergence of a quadrature rule to approximate a stochastic Itô-integral. It was shown that the rate of convergence depends on the Sobolev–Slobodeckiĭ regularity of the integrand instead of the Hölder regularity. In [67], it is even pointed out that the obtained rate is optimal. Therefore, better error estimates can be proved if the differentiability exponent of the Sobolev–Slobodeckiĭ space is higher than of the Hölder space. In [37], a randomization of the type $\tilde{\xi}_n = t_{n-1} + k\tau_n$, $n \in \{1, \dots, N\}$, with t_{n-1} and τ_n as introduced above, was used to approximate the solution of a problem like (2.1) with a Lipschitz continuous operator $A(t)$, $t \in [0, T]$. There, the solution is assumed to be Hölder continuous to prove a result like (2.3) with an error bound whose order depends on the Hölder regularity of the solution. The results of this chapter are a combination of these works. More precisely, in Theorem 2.1.11 and Theorem 2.1.12, we prove the convergence of the piecewise polynomial prolongations of the solution to the semidiscrete problem to the exact solution of the evolution equation (2.1). Here, the data is evaluated on a randomly shifted grid. This expands the theory from [37] to a more general problem class and shows that even without any additional regularity assumptions made on the exact solution a randomized scheme leads to a useful numerical approximation. In Theorem 2.2.6 and 2.2.7, we evaluate the data at the points $(\xi_n)_{n\in\{1,\dots,N\}}$ explained in (2.2) and obtain error bounds that depend on the Sobolev–Slobodeckiĭ regularity of the exact solution. This lowers the regularity assumptions from [37] and allows for operators $A(t)$, $t \in [0, T]$, which fulfill a bounded Lipschitz condition instead of a global Lipschitz condition.

This chapter is organized as follows. In Section 2.1, we begin to state the precise assumptions made. Then we prove the convergence of the scheme with evaluations on a randomly shifted grid. This is followed by a setting where explicit error estimates are proved in Section 2.2. For these bounds, we need to make additional assumptions on the solution and the data. The chapter is concluded with an example of a p-Laplacian type problem. Here, we show that the abstract theory can be applied to such a problem.

2.1 Convergence on a Randomly Shifted Grid

In this section, it is our overall goal to prove the convergence of the backward Euler scheme on a randomly shifted grid. We allow for a fairly general setting without any further regularity assumption on the solution. First, we introduce a nonlinear, nonautonomous operator $A(t)$, $t \in [0,T]$, and a source term f.

Assumption 2.1.1. *Let $T \in (0,\infty)$, $p \in [2,\infty)$ be given. Let $(H,(\cdot,\cdot)_H,\|\cdot\|_H)$ be a real, separable Hilbert space and $(V,\|\cdot\|_V)$ be a real, separable, reflexive Banach space, which is continuously and densely embedded into H. Let $\{A(t)\}_{t\in[0,T]}$ be a family of operators $A(t)\colon V \to V^*$ such that the following conditions are fulfilled:*

(1) The mapping $Av\colon [0,T] \to V^*$ given by $t \mapsto A(t)v$ is measurable for every $v \in V$.

(2) The operator $A(t)\colon V \to V^*$, $t \in [0,T]$, is radially continuous, i.e., the mapping $s \mapsto \langle A(t)(v+sw), w\rangle_{V^*\times V}$ is continuous on $[0,1]$ for every $v,w \in V$.

(3) The operator $A(t)\colon V \to V^*$, $t \in [0,T]$, is monotone, i.e.,

$$\langle A(t)v - A(t)w, v-w\rangle_{V^*\times V} \geq 0$$

is fulfilled for every $v,w \in V$.

(4) The operator $A(t)\colon V \to V^*$, $t \in [0,T]$, is uniformly bounded such that there exists $\beta \in [0,\infty)$, which does not depend on t, with

$$\|A(t)v\|_{V^*} \leq \beta\big(1+\|v\|_V^{p-1}\big)$$

for every $v \in V$.

(5) The operator $A(t)\colon V \to V^*$, $t \in [0,T]$, fulfills a coercivity condition in the sense that there exist $\mu \in (0,\infty)$ and $\lambda \in [0,\infty)$, which do not depend on t, such that

$$\langle A(t)v, v\rangle_{V^*\times V} \geq \mu\|v\|_V^p - \lambda$$

for every $v \in V$.

In the following, we always identify H with its dual space and consider the Gelfand triple

$$V \overset{d}{\hookrightarrow} H \cong H^* \overset{d}{\hookrightarrow} V^*.$$

In applications, it can be of advantage to generalize the conditions (3) and (5) of the previous assumption in such a way that there exists $\kappa \in [0,\infty)$ such that $A(t)+\kappa I$, $t \in [0,T]$, fulfills these conditions. Here, $I\colon V \to V^*$ denotes the identity mapping. Then it is necessary to suppose that V is compactly embedded into H. Due to the randomization, we cannot use this compact embedding in a straight forward way. If the underlying problem contains strictly positive κ, we can still use the transformation from (1.2) to obtain data that fulfills our requirements.

Furthermore, it is also possible to consider the case $p \in (1,2)$ in Assumption 2.1.1, compare also [40, 41, 42]. Here, it will become necessary to work with slightly different function spaces. In the second section of this chapter, we impose a stronger monotonicity condition on the operator $A(t)$, $t \in [0,T]$. It is possible to show, that there exists no operator that fulfills this condition for $p \in (1,2)$. Thus, we concentrate on $p \in [2,\infty)$ for simplicity and to be more consistent throughout this chapter.

We consider a source term $f \in L^q(0,T;V^*)$ with $q = \frac{p}{p-1}$, where p is the same as in Assumption 2.1.1. Note that it is not trivial to allow for a more general source term $f \in L^q(0,T;V^*) + L^1(0,T;H)$. We will explain this in more detail at a later point. In

Section 1.1, we have seen that the evolution equation (2.1) is uniquely solvable under the imposed conditions on $A(t)$, $t \in [0,T]$, and f for an initial value $u_0 \in H$.

It will also be important to consider the operator $A(t)$, $t \in [0,T]$, as a mapping on the space of Bochner integrable functions $L^p(0,T;V)$ into its dual space. To this end, we collect some properties of the Nemytskiĭ operator in the following lemma. We omit the proof as it can be found in [39, Lemma 8.4.4] or [118, Section 30].

Lemma 2.1.2. *Let Assumption 2.1.1 be fulfilled. Then $v \mapsto Av$ with $(Av)(t) = A(t)v(t)$ maps $L^p(0,T;V)$ into $L^q(0,T;V^*)$, where $q = \frac{p}{p-1}$. Then the operator is radially continuous, i.e., the mapping $s \mapsto \langle A(v+sw), w\rangle_{L^q(0,T;V^*)\times L^p(0,T;V)}$ is continuous on $[0,1]$ for all $v, w \in L^p(0,T;V)$. Further, A fulfills a monotonicity, a boundedness, and a coercivity condition such that*

$$\begin{aligned}
&\langle Av - Aw, v - w\rangle_{L^q(0,T;V^*)\times L^p(0,T;V)} \geq 0,\\
&\|Av\|_{L^q(0,T;V^*)} \leq \beta\big(T^{\frac{1}{q}} + \|v\|^{p-1}_{L^p(0,T;V)}\big),\\
&\langle Av, v\rangle_{L^q(0,T;V^*)\times L^p(0,T;V)} + \lambda T \geq \mu \|v\|^p_{L^p(0,T;V)}
\end{aligned}$$

hold true for all $v, w \in L^p(0,T;V)$.

For the temporal discretization of (2.1), we introduce a randomly shifted grid to bypass classical point evaluations at the points of a predetermined temporal grid.

Assumption 2.1.3. *Let $T \in (0,\infty)$ and $N \in \mathbb{N}$ be given. Consider the equidistant partition $0 = t_0 < \dots < t_N = T$ with $k = \frac{T}{N}$ and $t_n = nk$, $n \in \{0,\dots,N\}$. Further, let $(\Omega_\theta, \mathcal{F}^\theta, \mathbf{P}_\theta)$ be a complete probability space such that $L^1(\Omega_\theta)$ is separable. Let $\theta\colon \Omega_\theta \to [0,1]$ be a uniformly distributed random variable. The randomly shifted grid is denoted by $0 = t_0^\theta < t_1^\theta < \dots < t_N^\theta = T - k(1-\theta)$ with $t_n^\theta = t_{n-1} + k\theta$ for $n \in \{1,\dots,N\}$.*

The expectation on the probability space $(\Omega_\theta, \mathcal{F}^\theta, \mathbf{P}_\theta)$ is denoted by $\mathbf{E}_\theta$. It is necessary to assume that $L^1(\Omega_\theta)$ is a separable space to argue that some of the Bochner spaces, which will appear further below, are separable. For applications, this assumption is unproblematic. We only need one uniformly distributed random variable $\theta\colon \Omega_\theta \to [0,1]$ in this section. Thus, we could simply take $\Omega_\theta = [0,1]$ equipped with the Lebesgue σ-algebra $\mathcal{F}^\theta$ and the Lebesgue-measure $\mathbf{P}_\theta$ and choose $\theta(\omega) = \omega$ for $\omega \in [0,1]$.

Under Assumptions 2.1.1 and 2.1.3 as well as $f \in L^q(0,T;V^*)$, we can now consider the recursion

$$\begin{cases}\mathbf{U}^n + kA(t_n^\theta)\mathbf{U}^n = kf(t_n^\theta) + \mathbf{U}^{n-1} & \text{almost surely in } V^*, \quad n \in \{1,\dots,N\},\\ \mathbf{U}^0 = u_0 & \text{in } H,\end{cases} \tag{2.4}$$

which is the classical backward Euler scheme, but on a randomly shifted grid. In the following, we will prove that (2.4) admits a unique solution $(\mathbf{U}^n)_{n\in\{1,\dots,N\}}$. For $n \in \{1,\dots,N\}$ the mapping $\mathbf{U}^n\colon \Omega_\theta \to V$ is $\mathcal{F}^\theta$-measurable and its expectation fulfills an a priori bound. We begin by proving a general auxiliary result to show the $\mathcal{F}^\theta$-measurability of a solution to such an implicit equation. A similar result can be found in [54, Lemma 3.8]. The structure of the proof is comparable to [31, Proposition 1] and [37, Lemma 4.3]. We adapt it to fit our setting in an infinite-dimensional space.

Lemma 2.1.4. *Let $(\overline{\Omega}, \overline{\mathcal{F}}, \overline{\mathbf{P}})$ be a complete probability space and let $(V, \|\cdot\|_V)$ be a real, separable Banach space. Further, let $h\colon \overline{\Omega}\times V \to V^*$ fulfill the following conditions for some $\mathcal{N} \subset \overline{\Omega}$ with $\overline{\mathbf{P}}(\mathcal{N}) = 0$:*

(1) The mapping $u \mapsto \langle h(\omega, u), v\rangle_{V^*\times V}$ is continuous for every $v \in V$ and $\omega \in \overline{\Omega} \setminus \mathcal{N}$.

(2) The mapping $\omega \mapsto h(\omega, u)$ is $\overline{\mathcal{F}}$-measurable for every $u \in V$.

(3) For every $\omega \in \overline{\Omega} \setminus \mathcal{N}$, there exists a unique element $\mathbf{U}(\omega) \in V$ such that $h(\omega, \mathbf{U}(\omega)) = 0$.

Consider the mapping $\mathbf{U}\colon \overline{\Omega} \to V$, $\omega \mapsto \mathbf{U}(\omega)$, where $\mathbf{U}(\omega)$ is the unique element in V described in (3) for $\omega \in \overline{\Omega} \setminus \mathcal{N}$ and $\mathbf{U}(\omega) = 0$ for $\omega \in \mathcal{N}$. Then $\mathbf{U}$ is $\overline{\mathcal{F}}$-measurable.

Proof. For $\varepsilon > 0$ and an arbitrary $v \in V$, we introduce the multivalued function $\mathbf{U}_\varepsilon^v\colon \overline{\Omega} \to \mathcal{P}(V)$ by

$$\mathbf{U}_\varepsilon^v(\omega) = \big\{u \in V : \langle h(\omega, u), v\rangle_{V^*\times V} \in I_\varepsilon\big\},$$

where $\mathcal{P}(V)$ denotes the power set of V and $I_\varepsilon = (-\varepsilon, \varepsilon)$. Note that when considering measurability conditions for set-valued mappings, it is usually necessary to work with an image that is a closed set, compare [14, Section 8.1]. Since we do not use any specific results imposed on such mappings, it is unproblematic to define it like this.

Let C be an open set within the Borel σ-algebra $\mathcal{B}(V)$ of V. It is our first intention to prove that for C the set

$$\begin{aligned}(\mathbf{U}_\varepsilon^v)^{-1}(C) &= \{\omega \in \overline{\Omega} : \text{there exists } u \in C \text{ such that } u \in \mathbf{U}_\varepsilon^v(\omega)\}\\ &= \{\omega \in \overline{\Omega} : \text{there exists } u \in C \text{ such that } \langle h(\omega, u), v\rangle_{V^*\times V} \in I_\varepsilon\}\end{aligned}$$

is an element of $\overline{\mathcal{F}}$. Note that if C only consists of a single element, it follows that

$$(\mathbf{U}_\varepsilon^v)^{-1}(\{u\}) = \{\omega \in \overline{\Omega} : u \in \mathbf{U}_\varepsilon^v(\omega)\} = \big(\langle h(\cdot, u), v\rangle_{V^*\times V}\big)^{-1}\big(I_\varepsilon\big)$$

is an element of $\overline{\mathcal{F}}$ since $\omega \mapsto \langle h(\omega, u), v\rangle_{V^*\times V}$ is measurable. If C contains more than one element, we can still write

$$(\mathbf{U}_\varepsilon^v)^{-1}(C) = \bigcup_{u\in C} \big(\langle h(\cdot, u), v\rangle_{V^*\times V}\big)^{-1}\big(I_\varepsilon\big).$$

For a countable set C, it is easy to see that $(\mathbf{U}_\varepsilon^v)^{-1}(C) \in \overline{\mathcal{F}}$ as it is a countable union of sets in $\overline{\mathcal{F}}$. By assumption, the space V is separable. Thus, there exists a countable, dense subset Q of V such that

$$(\mathbf{U}_\varepsilon^v)^{-1}(C \cap Q) = \bigcup_{u\in C\cap Q} \big(\langle h(\cdot, u), v\rangle_{V^*\times V}\big)^{-1}\big(I_\varepsilon\big)$$

is an element of $\overline{\mathcal{F}}$. Using the continuity of $u \mapsto \langle h(\omega, u), v\rangle_{V^*\times V}$ for every $\omega \in \overline{\Omega} \setminus \mathcal{N}$, we will justify that $(\mathbf{U}_\varepsilon^v)^{-1}(C \cap Q) = (\mathbf{U}_\varepsilon^v)^{-1}(C)$. As $C \cap Q \subset C$ holds true, $(\mathbf{U}_\varepsilon^v)^{-1}(C \cap Q) \subseteq (\mathbf{U}_\varepsilon^v)^{-1}(C)$ follows directly. Thus, it only remains to prove $(\mathbf{U}_\varepsilon^v)^{-1}(C) \subseteq (\mathbf{U}_\varepsilon^v)^{-1}(C \cap Q)$. Here, we consider two cases. In the first case, we assume that $(\mathbf{U}_\varepsilon^v)^{-1}(C) \subseteq \mathcal{N}$. Then the completeness of the probability space yields $(\mathbf{U}_\varepsilon^v)^{-1}(C) \in \overline{\mathcal{F}}$. Else, for $\omega \in (\mathbf{U}_\varepsilon^v)^{-1}(C) \setminus \mathcal{N}$, there exists $u_1 \in C$ such that $u_1 \in \mathbf{U}_\varepsilon^v(\omega)$. As $u \mapsto \langle h(\omega, u), v\rangle_{V^*\times V}$ is continuous,

$$D = \mathbf{U}_\varepsilon^v(\omega) = \big(\langle h(\omega, \cdot), v\rangle_{V^*\times V}\big)^{-1}\big(I_\varepsilon\big)$$

is an open set in V and $u_1 \in C \cap D$. Since both C and D are open, their intersection is open and, as we have just seen, it is nonempty. Thus, there exists $u_2 \in C \cap D \cap Q$ such that $\langle h(\omega,u_2),v\rangle_{V^*\times V}\in I_\varepsilon$. Altogether, this implies $\omega\in(\mathbf{U}_\varepsilon^v)^{-1}(C\cap Q)$ and therefore, in particular, $(\mathbf{U}_\varepsilon^v)^{-1}(C)=(\mathbf{U}_\varepsilon^v)^{-1}(C\cap Q)$.

For every $\omega\in\overline{\Omega}\setminus\mathcal{N}$ there exists a unique element $\mathbf{U}(\omega)\in V$ such that $h(\omega,\mathbf{U}(\omega))=0$ in V^*. Hence, there exists at least one element $u\in V$ such that $u\in\mathbf{U}_\varepsilon^v(\omega)$ for every $v\in Q$ with $\|v\|_V\le 1$. For such an element u, it follows that

$$\|h(\omega,u)\|_{V^*}=\sup_{\substack{v\in V,\\ \|v\|_V\le 1}}|\langle h(\omega,u),v\rangle_{V^*\times V}|=\sup_{\substack{v\in Q,\\ \|v\|_V\le 1}}|\langle h(\omega,u),v\rangle_{V^*\times V}|\le\varepsilon,$$

as every element of V^* is continuous. This in mind, we obtain that the following intersection fulfills

$$\bigcap_{\substack{v\in Q,\\ \|v\|_V\le 1}}\mathbf{U}_\varepsilon^v(\omega)=\bigcap_{\substack{v\in Q,\\ \|v\|_V\le 1}}\big\{u\in V:\langle h(\omega,u),v\rangle_{V^*\times V}\in I_\varepsilon\big\}=\big\{u\in V: h(\omega,u)\in B_{\varepsilon,V^*}(0)\big\},$$

where $B_{\varepsilon,V^*}(0)$ denotes the open ball in V^* with radius ε and center $0\in V^*$. Therefore, we can write for the unique element $\mathbf{U}(\omega)\in V$ such that $h(\omega,\mathbf{U}(\omega))=0$ in V^* that

$$\mathbf{U}(\omega)\in\bigcap_{i\in\mathbb{N}}\bigcap_{\substack{v\in Q,\\ \|v\|_V\le 1}}\mathbf{U}_{i^{-1}}^v(\omega)\subseteq\bigcap_{i\in\mathbb{N}}\big\{u\in V: h(\omega,u)\in B_{i^{-1},V^*}(0)\big\}=\{\mathbf{U}(\omega)\}.$$

For an arbitrary open set $C\in\mathcal{B}(V)$, we then obtain

$$\begin{aligned}\mathbf{U}^{-1}(C)&=\{\omega\in\overline{\Omega}:\text{ there exists } u\in C \text{ such that } h(\omega,u)=0\}\\ &=\bigcap_{i\in\mathbb{N}}\bigcap_{\substack{v\in Q,\\ \|v\|_V\le 1}}\{\omega\in\overline{\Omega}:\text{ there exists } u\in C\text{ such that }\langle h(\omega,u),v\rangle_{V^*\times V}\in I_{i^{-1}}\}\\ &=\bigcap_{i\in\mathbb{N}}\bigcap_{\substack{v\in Q,\\ \|v\|_V\le 1}}(\mathbf{U}_{i^{-1}}^v)^{-1}(C)=\bigcap_{i\in\mathbb{N}}\bigcap_{\substack{v\in Q,\\ \|v\|_V\le 1}}(\mathbf{U}_{i^{-1}}^v)^{-1}(C\cap Q)\in\overline{\mathcal{F}},\end{aligned}$$

since we have a countable intersection of measurable sets. This proves the measurability of $\omega\mapsto\mathbf{U}(\omega)$. □

Lemma 2.1.5. *Let Assumptions 2.1.1 and 2.1.3 be fulfilled and for $q=\frac{p}{p-1}$, let $f\in L^q(0,T;V^*)$ be given. Then there exists a unique solution $(\mathbf{U}^n)_{n\in\{1,\dots,N\}}$ to the recursion (2.4). The mapping $\mathbf{U}^n\colon\Omega_\theta\to V$, $\omega\mapsto\mathbf{U}^n(\omega)$ is $\mathcal{F}^\theta$-measurable for every $n\in\{1,\dots,N\}$.*

Proof. The set

$$\mathcal{N}=\{\omega\in\Omega_\theta:\text{there exists } n\in\{1,\dots,N\}\text{ such that }\|f(t_n^\theta(\omega))\|_{V^*}=\infty\}\tag{2.5}$$

is a null set in Ω_θ due to the integrability of f. In the following, let $n\in\{1,\dots,N\}$ and $\omega\in\Omega_\theta\setminus\mathcal{N}$ be arbitrary but fixed. Assuming that $\mathbf{U}^{n-1}(\omega)$ exists, we apply the Browder–Minty Theorem, see [99, Theorem 2.14]. To this end, we consider the equation

$$\big(I+kA(t_n^\theta(\omega))\big)\mathbf{U}=kf(t_n^\theta(\omega))+\mathbf{U}^{n-1}(\omega)\quad\text{in } V^*.\tag{2.6}$$

The operator $I+kA(t_n^\theta(\omega))$ is radially continuous, as both I and $A(t_n^\theta(\omega))$ are radially continuous (cf. Assumption 2.1.1 (2)). Moreover, I is strictly monotone and $A(t_n^\theta(\omega))$ is monotone (cf. Assumption 2.1.1 (3)), so the operator $I+kA(t_n^\theta(\omega))$ is also strictly monotone. For arbitrary $v \in V$, we obtain

$$\langle\big(I + kA(t_n^\theta(\omega))\big)v, v\rangle_{V^*\times V} \geq \|v\|_H^2 + k(\mu\|v\|_V^p - \lambda) \to \infty \quad \text{as } \|v\|_V \to \infty,$$

due to Assumption 2.1.1 (5). Thus, $I + kA(t_n^\theta(\omega))$ is radially continuous, strictly monotone and coercive. So there exists a unique element $\mathbf{U} = \mathbf{U}^n(\omega) \in V$ such that (2.6) is fulfilled.

It remains to prove the $\mathcal{F}^\theta$-measurability of the mapping $\omega \mapsto \mathbf{U}^n(\omega)$. This can be done by applying Lemma 2.1.4 to the function

$$h_n\colon \Omega_\theta \times V \to V^*, \quad h_n(\omega, \mathbf{U}) = \big(I + kA(t_n^\theta(\omega))\big)\mathbf{U} - kf(t_n^\theta(\omega)) - \mathbf{U}^{n-1}(\omega)$$

for $n \in \{1, \dots, N\}$. By Assumption 2.1.1 (2) and (3), the operator $A(t)$, $t \in [0, T]$, is radially continuous and monotone. Thus, it is also hemicontinuous (cf. [49, Kapitel III, Lemma 1.3]), i.e., $\mathbf{U} \mapsto \langle h_n(\omega, \mathbf{U}), v\rangle_{V^*\times V}$ is continuous for every $\omega \in \Omega_\theta \setminus \mathcal{N}$ and $v \in V$. Moreover, the mapping $\omega \mapsto h_n(\omega, v)$ is $\mathcal{F}^\theta$-measurable for every $v \in V$ due to Assumption 2.1.1 (1) and the measurability of f and t_n^θ. By the argumentation above, there exists a unique element $\mathbf{U}^n(\omega) \in V$ that is the root of $h_n(\omega, \cdot)$. Thus, it follows that $\mathbf{U}^n\colon \Omega_\theta \to V$, $\omega \mapsto \mathbf{U}^n(\omega)$ is $\mathcal{F}^\theta$-measurable. □

Now that the existence of a unique $\mathcal{F}^\theta$-measurable family $(\mathbf{U}^n)_{n\in\{1,\dots,N\}}$ is proven, we need a priori bounds for the solution. First, we state a result that shows that the appearing terms containing f are bounded.

Lemma 2.1.6. *Let Assumption 2.1.3 be fulfilled and let $(X, \|\cdot\|_X)$ be a real Banach space. For $f \in L^q(0, T; X)$, $q \in [1, \infty)$,*

$$k\sum_{n=1}^N \mathbf{E}_\theta\big[\|f(t_n^\theta)\|_X^q\big] = \|f\|_{L^q(0,T;X)}^q$$

is fulfilled.

Proof. For $n \in \{1, \dots, N\}$, we use a substitution as in (A.2) and can write

$$\mathbf{E}_\theta\big[\|f(t_n^\theta)\|_X^q\big] = \int_0^1 \|f(t_{n-1} + ks)\|_X^q \,\mathrm{d}s = \frac{1}{k}\int_{t_{n-1}}^{t_n} \|f(s)\|_X^q \,\mathrm{d}s.$$

Thus, it follows that

$$k\sum_{n=1}^N \mathbf{E}_\theta\big[\|f(t_n^\theta)\|_X^q\big] = \sum_{n=1}^N \int_{t_{n-1}}^{t_n} \|f(s)\|_X^q \,\mathrm{d}s = \int_0^T \|f(s)\|_X^q \,\mathrm{d}s.$$

□

The following lemma is a central part of our argumentation. These bounds in mind, we can prove the boundedness of sequences of prolongations of the values obtained in our semidiscrete scheme (2.4). The structure of the proof to these bounds consists of standard techniques. However, in our setting, it is not trivial to include a source term $f \in L^q(0, T; V^*) + L^1(0, T; H)$, as in [109, Chapter III, Section 1.5], or a semi-coercivity condition as in [99, Theorem 8.9]. The main difficulty consists of the additional expectation we have to include to the bounds. Whereas it is easy to see that $\|w\|_H^2 < \infty$ for $w \in H$ implies that $\|w\|_H^p < \infty$ for every $p \in [2, \infty)$, it is not possible to conclude that $\mathbf{E}_\theta\big[\|W\|_H^2\big] < \infty$ also implies $\mathbf{E}_\theta\big[\|W\|_H^p\big] < \infty$ for a random variable $W\colon \Omega_\theta \to H$. The techniques proposed in [21, Lemma 3.1] could offer a possibility to allow for these more general assumptions. This remains a question for future work.

Lemma 2.1.7. *Let Assumptions 2.1.1 and 2.1.3 be fulfilled. Further, let $f \in L^q(0,T;V^*)$, $q = \frac{p}{p-1}$, and $u_0 \in H$ be given. Then there exists $K \in (0,\infty)$ such that for all $k = \frac{T}{N}$, $N \in \mathbb{N}$, the unique solution $(\mathbf{U}^n)_{n\in\{1,\dots,N\}}$ of (2.4) fulfills*

$$\max_{n\in\{1,\dots,N\}} \mathbf{E}_\theta\big[\|\mathbf{U}^n\|_H^2\big] + \sum_{n=1}^N \mathbf{E}_\theta\big[\|\mathbf{U}^n - \mathbf{U}^{n-1}\|_H^2\big] + k\sum_{n=1}^N \mathbf{E}_\theta\big[\|\mathbf{U}^n\|_V^p\big] \le K \tag{2.7}$$

and

$$k^{1-q}\sum_{i=1}^N \mathbf{E}_\theta\big[\|\mathbf{U}^i - \mathbf{U}^{i-1}\|_{V^*}^q\big] = k\sum_{i=1}^N \mathbf{E}_\theta\Big[\Big\|\frac{\mathbf{U}^i - \mathbf{U}^{i-1}}{k}\Big\|_{V^*}^q\Big] \le K. \tag{2.8}$$

Proof. In the following, let $i \in \{1,\dots,N\}$ be fixed. Furthermore, for the set $\mathcal{N}$ defined in (2.5), we consider the following calculations for $\omega \in \Omega_\theta \setminus \mathcal{N}$ without explicitly stating ω in each step. For a single Euler step (2.4), it holds true that

$$\mathbf{U}^i - \mathbf{U}^{i-1} + kA(t_i^\theta)\mathbf{U}^i = kf(t_i^\theta) \quad \text{in } V^*.$$

Testing this equation with $\mathbf{U}^i$, we obtain

$$(\mathbf{U}^i - \mathbf{U}^{i-1}, \mathbf{U}^i)_H + k\langle A(t_i^\theta)\mathbf{U}^i, \mathbf{U}^i\rangle_{V^*\times V} = k\langle f(t_i^\theta), \mathbf{U}^i\rangle_{V^*\times V}. \tag{2.9}$$

Recalling the identity from Lemma A.1.4 and applying it to (2.9), it follows

$$\begin{aligned} &\frac{1}{2}\big(\|\mathbf{U}^i\|_H^2 - \|\mathbf{U}^{i-1}\|_H^2 + \|\mathbf{U}^i - \mathbf{U}^{i-1}\|_H^2\big) + k\langle A(t_i^\theta)\mathbf{U}^i, \mathbf{U}^i\rangle_{V^*\times V} \\ &= k\langle f(t_i^\theta), \mathbf{U}^i\rangle_{V^*\times V}. \end{aligned} \tag{2.10}$$

The weighted Young inequality applied to the right-hand side of (2.10) shows

$$k\langle f(t_i^\theta), \mathbf{U}^i\rangle_{V^*\times V} \le kc_1\|f(t_i^\theta)\|_{V^*}^q + k\frac{\mu}{2}\|\mathbf{U}^i\|_V^p,$$

where $c_1 = \frac{(p\mu)^{1-q}}{q2^{1-q}}$. Inserting this bound and the coercivity condition from Assumption 2.1.1 (5) in (2.10), it follows that

$$\frac{1}{2}\big(\|\mathbf{U}^i\|_H^2 - \|\mathbf{U}^{i-1}\|_H^2 + \|\mathbf{U}^i - \mathbf{U}^{i-1}\|_H^2\big) + k\frac{\mu}{2}\|\mathbf{U}^i\|_V^p \le k\lambda + kc_1\|f(t_i^\theta)\|_{V^*}^q.$$

Multiplying both sides with the factor two and summing up this inequality from $i = 1$ to $n \in \{1,\dots,N\}$, yields

$$\begin{aligned} &\|\mathbf{U}^n\|_H^2 + \sum_{i=1}^n \|\mathbf{U}^i - \mathbf{U}^{i-1}\|_H^2 + k\mu\sum_{i=1}^n \|\mathbf{U}^i\|_V^p \\ &\le \|\mathbf{U}^0\|_H^2 + 2t_n\lambda + 2kc_1\sum_{i=1}^n \|f(t_i^\theta)\|_{V^*}^q. \end{aligned} \tag{2.11}$$

Taking the expectation, it follows that

$$\begin{aligned} &\mathbf{E}_\theta\big[\|\mathbf{U}^n\|_H^2\big] + \sum_{i=1}^n \mathbf{E}_\theta\big[\|\mathbf{U}^i - \mathbf{U}^{i-1}\|_H^2\big] + k\mu\sum_{i=1}^n \mathbf{E}_\theta\big[\|\mathbf{U}^i\|_V^p\big] \\ &\le \|u_0\|_H^2 + 2T\lambda + 2kc_1\sum_{i=1}^N \mathbf{E}_\theta\big[\|f(t_i^\theta)\|_{V^*}^q\big] = \|u_0\|_H^2 + 2T\lambda + 2c_1\|f\|_{L^q(0,T;V^*)}^q, \end{aligned}$$

due to Lemma 2.1.6. It remains to prove the second a priori bound (2.8). To this end, let $v \in V$ be arbitrary but fixed. Then we test (2.4) with v to obtain

$$\Big(\frac{\mathbf{U}^n - \mathbf{U}^{n-1}}{k}, v\Big)_H = \langle f(t_n^\theta) - A(t_n^\theta)\mathbf{U}^n, v\rangle_{V^* \times V} \le \|f(t_n^\theta) - A(t_n^\theta)\mathbf{U}^n\|_{V^*}\|v\|_V,$$

$n \in \{1, \dots, N\}$. Together with the boundedness condition from Assumption 2.1.1 (4) this implies

$$\Big\|\frac{\mathbf{U}^n - \mathbf{U}^{n-1}}{k}\Big\|_{V^*} \le \|f(t_n^\theta)\|_{V^*} + \|A(t_n^\theta)\mathbf{U}^n\|_{V^*} \le \|f(t_n^\theta)\|_{V^*} + \beta\big(1 + \|\mathbf{U}^n\|_V^{p-1}\big).$$

Taking the q-th power and the expectation as well as summing up the inequality from $n = 1$ to N shows that

$$\sum_{n=1}^N \mathbf{E}_\theta\Big[\Big\|\frac{\mathbf{U}^n - \mathbf{U}^{n-1}}{k}\Big\|_{V^*}^q\Big] \le \sum_{n=1}^N \mathbf{E}_\theta\big[\big(\|f(t_n^\theta)\|_{V^*} + \beta\big(1 + \|\mathbf{U}^n\|_V^{p-1}\big)\big)^q\big].$$

We then multiply by k and take the $\frac{1}{q}$-th power again in order to use the triangle inequality and obtain the desired bound

$$\begin{aligned}
&\Big(k\sum_{n=1}^N \mathbf{E}_\theta\Big[\Big\|\frac{\mathbf{U}^n - \mathbf{U}^{n-1}}{k}\Big\|_{V^*}^q\Big]\Big)^{\frac{1}{q}} \\
&\le \Big(k\sum_{n=1}^N \mathbf{E}_\theta\big[\|f(t_n^\theta)\|_{V^*}^q\big]\Big)^{\frac{1}{q}} + \Big(k\sum_{n=1}^N \beta^q\Big)^{\frac{1}{q}} + \beta\Big(k\sum_{n=1}^N \mathbf{E}_\theta\big[\|\mathbf{U}^n\|_V^p\big]\Big)^{\frac{1}{q}} \\
&= \|f\|_{L^q(0,T;V^*)} + T^{\frac{1}{q}}\beta + \beta\Big(k\sum_{n=1}^N \mathbf{E}_\theta\big[\|\mathbf{U}^n\|_V^p\big]\Big)^{\frac{1}{q}}.
\end{aligned}$$

This is bounded independently of the step size k due to Lemma 2.1.6 and the first a priori bound (2.7). □

For the time discrete solution $(\mathbf{U}^n)_{n\in\{1,\dots,N\}}$ to (2.4) corresponding to the shifted grid stated in Assumption 2.1.3, we construct piecewise polynomial prolongations defined on the entire interval $[0, T]$. To this end, we introduce the piecewise constant prolongations for $t \in (t_{n-1}, t_n]$, $n \in \{1, \dots, N\}$,

$$\bar{U}^k(t) = \mathbf{U}^n, \quad A^k(t) = A(t_n^\theta), \quad f^k(t) = f(t_n^\theta) \tag{2.12}$$

as well as the piecewise affine-linear function

$$U^k(t) = \mathbf{U}^{n-1} + \frac{t - t_{n-1}}{k}(\mathbf{U}^n - \mathbf{U}^{n-1}) \tag{2.13}$$

on $\Omega_\theta \backslash \mathcal{N}$, where $\mathcal{N}$ is defined in (2.5). For $t = 0$, we set $\bar{U}^k(0) = U^k(0) = \mathbf{U}^0$, $A^k(0) = A(t_1^\theta)$, and $f^k(0) = f(t_1^\theta)$. Then we have

$$\begin{cases} (U^k)'(t) + A^k(t)\bar{U}^k(t) = f^k(t) & \text{in } L^q(\Omega_\theta; V^*), \quad t \in (0, T), \\ U^k(0) = u_0 & \text{in } H, \end{cases} \tag{2.14}$$

where $(U^k)'$ denotes the weak derivative of U^k. Here, the weak derivative coincides with the classical derivative, where the latter exists. Note that due to the a priori bounds of

Lemma 2.1.7, the boundedness condition for $A(t)$, $t \in [0,T]$, from Assumption 2.1.1 (4), and the fact that $f \in L^q(0,T;V^*)$ the equation is indeed fulfilled in $L^q(\Omega_\theta;V^*)$.

In the following, we always consider step sizes $k = \frac{T}{N_\ell}$, where $(N_\ell)_{\ell\in\mathbb{N}}$ is a sequence of natural numbers such that $N_\ell \to \infty$ as $\ell \to \infty$. We abbreviate the corresponding sequence $(\bar{U}^{\frac{T}{N_\ell}})_{\ell\in\mathbb{N}}$ by $(\bar{U}^k)_{k>0}$ and analogously for the other functions introduced above.

This in mind, we can prove the convergence of $\bar{U}^k$ and U^k to the exact solution pointwise strongly in $L^2(\Omega_\theta;H)$. Further, we can prove that $\bar{U}^k$ converges to u in $L^p(0,T;L^p(\Omega_\theta;V))$. The Bochner spaces $L^2(\Omega_\theta;H)$ and $L^p(\Omega_\theta;V)$ appear because of the additional dependence of the solution $(\mathbf{U}^n)_{n\in\{1,\dots,N\}}$ to values of the space Ω_θ. More information on a Bochner space defined on a general measure space can be found in Appendix A.2. There, a collection of the properties and important results can be found. This information available, a space $L^p(0,T;L^p(\Omega_\theta;V))$ or alike has a similar structure. Since $(0,T)$ equipped with the Lebesgue σ-algebra and the Lebesgue measure is a finite measure space, the properties of $L^p(\Omega_\theta;V)$ can be transferred to $L^p(0,T;L^p(\Omega_\theta;V))$.

Lemma 2.1.8. *Let Assumptions 2.1.1 and 2.1.3 be fulfilled and let $f \in L^q(0,T;V^*)$, $q = \frac{p}{p-1}$, and $u_0 \in H$ be given. Further, let $(N_\ell)_{\ell\in\mathbb{N}}$ be a sequence of natural numbers with $N_\ell \to \infty$ as $\ell \to \infty$, let $k = \frac{T}{N_\ell}$ be the corresponding step sizes, and let the sequences of piecewise constant and piecewise linear prolongations be given as in* (2.12) *and* (2.13)*. Then there exists a subsequence of step sizes, again denoted by k, such that*

$$\begin{aligned}
&\bar{U}^k \rightharpoonup U && \text{in } L^p(0,T;L^p(\Omega_\theta;V)),\\
&\bar{U}^k \overset{*}{\rightharpoonup} U, \quad U^k \overset{*}{\rightharpoonup} U && \text{in } L^\infty(0,T;L^2(\Omega_\theta;H)),\\
&(U^k)' \rightharpoonup U' && \text{in } L^q(0,T;L^q(\Omega_\theta;V^*))
\end{aligned}$$

as $k \to 0$. The function U is an element of $L^p(0,T;L^p(\Omega_\theta;V)) \cap L^\infty(0,T;L^2(\Omega_\theta;H))$ and U' is the weak temporal derivative of U, which is an element of $L^q(0,T;L^q(\Omega_\theta;V^*))$.

In particular, this shows that U is an element of the space

$$\begin{aligned}
\mathcal{W}^p_{\Omega_\theta}(0,T) = \{&W \in L^p(0,T;L^p(\Omega_\theta;V)) :\\
&W' \text{ exists and } W' \in L^q(0,T;L^q(\Omega_\theta;V^*))\},
\end{aligned} \tag{2.15}$$

which is continuously embedded into $C([0,T];L^2(\Omega_\theta;H))$.

Proof of Lemma 2.1.8. For simplicity, we do not denote every subsequence differently within this proof and we drop the index ℓ. Due to the a priori bound (2.7) from Lemma 2.1.7, the sequence $(\bar{U}^k)_{k>0}$ of piecewise constant prolongations is bounded in both $L^p(0,T;L^p(\Omega_\theta;V))$ and $L^\infty(0,T;L^2(\Omega_\theta;H))$. Further, the sequence $(U^k)_{k>0}$ of piecewise linear prolongations is bounded in the space $L^\infty(0,T;L^2(\Omega_\theta;H))$. Using the second a priori bound (2.8) from Lemma 2.1.7, it follows that the sequence $((U^k)')_{k>0}$ is bounded in $L^q(0,T;L^q(\Omega_\theta;V^*))$. As $L^p(0,T;L^p(\Omega_\theta;V))$ is a reflexive Banach space, there exists a subsequence of $(\bar{U}^k)_{k>0}$ and an element $U \in L^p(0,T;L^p(\Omega_\theta;V))$ such that $\bar{U}^k \rightharpoonup U$ in $L^p(0,T;L^p(\Omega_\theta;V))$ as $k \to 0$. An analogous argumentation yields that there exists a subsequence of $\big((U^k)'\big)_{k>0}$ and $W \in L^q(0,T;L^q(\Omega_\theta;V^*))$ such that $(U^k)' \rightharpoonup W$ in $L^q(0,T;L^q(\Omega_\theta;V^*))$ as $k \to 0$.

As stated in Assumption 2.1.1, the space $L^1(\Omega_\theta)$ is separable. Applying the fact that H is separable as well as [96, Proposition 2.3.24, Proposition 4.2.22], it follows that $L^2(\Omega_\theta;H)$ is separable. Thus, the space $L^\infty(0,T;L^2(\Omega_\theta;H))$ is the dual space of the separable Banach space $L^1(0,T;L^2(\Omega_\theta;H))$, compare Appendix A.2. Thus, we can extract weakly$*$ converging subsequences of $(\bar{U}^k)_{k>0}$ and $(U^k)_{k>0}$. Due to the uniqueness of the limit of weak and weak$*$ convergent sequences, it follows that $\bar{U}^k \overset{*}{\rightharpoonup} U$ in $L^\infty(0,T;L^2(\Omega_\theta;H))$ as $k \to 0$ and therefore $U \in L^p(0,T;L^p(\Omega_\theta;V)) \cap L^\infty(0,T;L^2(\Omega_\theta;H))$. Furthermore, there exists an element $\tilde{U} \in L^\infty(0,T;L^2(\Omega_\theta;H))$ such that $U^k \overset{*}{\rightharpoonup} \tilde{U}$ in $L^\infty(0,T;L^2(\Omega_\theta;H))$ as $k \to 0$.

In order to prove that the two limits U and $\tilde{U}$ coincide, we consider

$$\begin{aligned}\int_0^T \mathbf{E}_\theta\big[\|\bar{U}^k(t) - U^k(t)\|_H^2\big]\,\mathrm{d}t &= \sum_{n=1}^N \int_{t_{n-1}}^{t_n} \mathbf{E}_\theta\Big[\Big\|\mathbf{U}^n - \mathbf{U}^{n-1} - \frac{t-t_{n-1}}{k}\big(\mathbf{U}^n - \mathbf{U}^{n-1}\big)\Big\|_H^2\Big]\,\mathrm{d}t\\ &= \frac{1}{k^2}\sum_{n=1}^N \mathbf{E}_\theta\big[\|\mathbf{U}^n - \mathbf{U}^{n-1}\|_H^2\big] \int_{t_{n-1}}^{t_n} (t_n - t)^2\,\mathrm{d}t\\ &= \frac{k}{3}\sum_{n=1}^N \mathbf{E}_\theta\big[\|\mathbf{U}^n - \mathbf{U}^{n-1}\|_H^2\big] \to 0 \quad \text{as } k \to 0,\end{aligned}$$

where we also used the a priori bound (2.7) from Lemma 2.1.7. This shows that $U = \tilde{U}$ in $L^2(0,T;L^2(\Omega_\theta;H))$. The spaces $L^\infty(0,T;L^2(\Omega_\theta;H))$ and $L^p(0,T;L^p(\Omega_\theta;V))$ are continuously embedded into $L^2(0,T;L^2(\Omega_\theta;H))$ and U is an element of $L^p(0,T;L^p(\Omega_\theta;V)) \cap L^\infty(0,T;L^2(\Omega_\theta;H))$ and $\tilde{U}$ of $L^\infty(0,T;L^2(\Omega_\theta;H))$. This shows that $U = \tilde{U}$ in both $L^p(0,T;L^p(\Omega_\theta;V))$ and $L^\infty(0,T;L^2(\Omega_\theta;H))$ as the embedding is always injective.

It remains to prove that W is the weak derivative of U with respect to the temporal input. For arbitrary $v \in L^p(\Omega_\theta;V)$ and $\varphi \in C_c^\infty(0,T)$, it follows that

$$\mathbf{E}_\theta\Big[\int_0^T \big((U^k)'(t),v\big)_H \varphi(t)\,\mathrm{d}t\Big] = \mathbf{E}_\theta\Big[\int_0^T (U^k(t),v)_H \varphi'(t)\,\mathrm{d}t\Big]$$

since $(U^k)'$ is the weak derivative of U^k almost surely. Thus, an application of Fubini's theorem then yields

$$\begin{aligned}\int_0^T \mathbf{E}_\theta\big[\langle W(t),v\rangle_{V^*\times V}\big]\varphi(t)\,\mathrm{d}t &= \lim_{k\to 0}\int_0^T \mathbf{E}_\theta\big[\big((U^k)'(t),v\big)_H\big]\varphi(t)\,\mathrm{d}t\\ &= -\lim_{k\to 0}\int_0^T \mathbf{E}_\theta\big[(U^k(t),v)_H\big]\varphi'(t)\,\mathrm{d}t\\ &= -\int_0^T \mathbf{E}_\theta\big[(U(t),v)_H\big]\varphi'(t)\,\mathrm{d}t.\end{aligned}$$

Applying [49, Kapitel IV, Lemma 1.7], it follows that $W = U'$ in $L^q(0,T;L^q(\Omega_\theta;V^*))$ and, in particular, that $U \in \mathcal{W}^p_{\Omega_\theta}(0,T)$. ☐

Lemma 2.1.9. *Let Assumption 2.1.3 be fulfilled, let $(X,\|\cdot\|_X)$ be a real Banach space, $q \in [1,\infty)$, and $v \in L^q(0,T;X)$. Then the piecewise constant function $v^k\colon [0,T] \to L^q(\Omega_\theta;X)$ given by $v^k(0) = v(t_1^\theta)$ and $v^k(t) = v(t_n^\theta)$ in $L^q(\Omega_\theta;X)$ for $t \in (t_{n-1},t_n]$, $n \in \{1,\dots,N\}$, fulfills*

$$v^k \to v \quad \text{in } L^q(0,T;L^q(\Omega_\theta;X))$$

as $k \to 0$.

Proof. As the space $C([0,T];X)$ is dense in $L^q(0,T;X)$, for every $\varepsilon > 0$, there exists a function $v_\varepsilon \in C([0,T];X)$ such that

$$\|v_\varepsilon - v\|_{L^q(0,T;X)} < \frac{\varepsilon}{4}.$$

In particular, the function v_ε is uniformly continuous and there exists $k_0 > 0$ such that

$$\|v_\varepsilon(s) - v_\varepsilon(t)\|_X < \frac{\varepsilon}{2T^{\frac{1}{q}}}$$

is fulfilled for all $s, t \in [0, T]$ with $|s - t| \leq k$ and $k \leq k_0$. In order to prove the assertion, we notice that

$$\|v^k - v\|^q_{L^q(0,T;L^q(\Omega_\theta;X))} = \sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} \mathbf{E}_\theta\big[\|v(t_n^\theta) - v(t)\|^q_X\big] \,\mathrm{d}t.$$

Then for $n \in \{1, \dots, N\}$ and almost every $t \in (t_{n-1}, t_n]$, a substitution as in (A.2) yields

$$\mathbf{E}_\theta\big[\|v(t_n^\theta) - v(t)\|^q_X\big] = \int_0^1 \|v(t_{n-1} + ks) - v(t)\|^q_X \,\mathrm{d}s = \frac{1}{k}\int_{t_{n-1}}^{t_n} \|v(s) - v(t)\|^q_X \,\mathrm{d}s,$$

such that for $k \leq k_0$

$$\begin{aligned}
&\Big(\frac{1}{k}\sum_{n=1}^{N}\int_{t_{n-1}}^{t_n}\int_{t_{n-1}}^{t_n} \|v(s) - v(t)\|^q_X \,\mathrm{d}s\,\mathrm{d}t\Big)^{\frac{1}{q}}\\
&\leq \Big(\frac{1}{k}\sum_{n=1}^{N}\int_{t_{n-1}}^{t_n}\int_{t_{n-1}}^{t_n} \|v(s) - v_\varepsilon(s)\|^q_X \,\mathrm{d}t\,\mathrm{d}s\Big)^{\frac{1}{q}}\\
&\quad + \Big(\frac{1}{k}\sum_{n=1}^{N}\int_{t_{n-1}}^{t_n}\int_{t_{n-1}}^{t_n} \|v_\varepsilon(s) - v_\varepsilon(t)\|^q_X \,\mathrm{d}t\,\mathrm{d}s\Big)^{\frac{1}{q}}\\
&\quad + \Big(\frac{1}{k}\sum_{n=1}^{N}\int_{t_{n-1}}^{t_n}\int_{t_{n-1}}^{t_n} \|v_\varepsilon(t) - v(t)\|^q_X \,\mathrm{d}t\,\mathrm{d}s\Big)^{\frac{1}{q}}\\
&\leq 2\Big(\int_0^T \|v(s) - v_\varepsilon(s)\|^q_X \,\mathrm{d}s\Big)^{\frac{1}{q}} + \Big(\frac{1}{k}\sum_{n=1}^{N}\int_{t_{n-1}}^{t_n}\int_{t_{n-1}}^{t_n} \frac{\varepsilon^q}{2^q T} \,\mathrm{d}t\,\mathrm{d}s\Big)^{\frac{1}{q}} < \varepsilon.
\end{aligned}$$

Altogether, we have proved that

$$\|v^k - v\|_{L^q(0,T;L^q(\Omega_\theta;X))} < \varepsilon,$$

which verifies the statement of the lemma as $\varepsilon > 0$ can be chosen arbitrarily. □

The next lemma contains a comparable result for the operator A^k. Note that in contrast to deterministic methods that use values $\frac{1}{k}\int_{t_{n-1}}^{t_n} A(t)\,\mathrm{d}t$ instead of $A(t_n^\theta)$ for $n \in \{1, \dots, N\}$, we prove that $A^k v^k \to Av$ in $L^q(0, T; L^q(\Omega_\theta; V^*))$ rather than $A^k v \to Av$ in $L^q(0, T; L^q(\Omega_\theta; V^*))$ as $k \to 0$ for $v \in L^p(0, T; V)$, where v^k is a piecewise constant function similar to the one introduced in the previous lemma. In the proof, $A^k v^k$ leads to a function $t \mapsto A(t)v(t)$ whereas $A^k v$ leads to a function $(s, t) \mapsto A(s)v(t)$, which is harder to compare to the claimed limit.

Lemma 2.1.10. *Let Assumptions 2.1.1 and 2.1.3 be fulfilled and let an arbitrary $v \in L^p(0, T; V)$ be given. Then for A^k defined in* (2.12) *and the piecewise constant function $v^k\colon [0, T] \to L^p(\Omega_\theta; V)$ given by $v^k(0) = v(t_1^\theta)$ and $v^k(t) = v(t_n^\theta)$ in $L^p(\Omega_\theta; V)$ for $t \in (t_{n-1}, t_n]$, $n \in \{1, \dots, N\}$, it follows that*

$$A^k v^k \to Av \quad \text{in } L^q(0, T; L^q(\Omega_\theta; V^*))$$

as $k \to 0$.

Proof. In order to estimate $A^k v^k - Av$ in the norm of $L^q(0,T;L^q(\Omega_\theta;V^*))$, we use a substitution as in (A.2) to obtain

$$\begin{aligned}
&\|A^k v^k - Av\|^q_{L^q(0,T;L^q(\Omega_\theta;V^*))}\\
&= \sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} \mathbf{E}_\theta \big[\|A(t_n^\theta)v(t_n^\theta) - A(t)v(t)\|^q_{V^*}\big] \,\mathrm{d}t\\
&= \sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} \int_0^1 \|A(t_{n-1}+ks)v(t_{n-1}+ks) - A(t)v(t)\|^q_{V^*} \,\mathrm{d}s \,\mathrm{d}t\\
&= \frac{1}{k} \sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} \int_{t_{n-1}}^{t_n} \|A(s)v(s) - A(t)v(t)\|^q_{V^*} \,\mathrm{d}s \,\mathrm{d}t.
\end{aligned}$$

Using the function $h(s) := A(s)v(s)$, we can follow an analogous argumentation as in the proof of Lemma 2.1.9. □

The previous lemmas show that the single summands in (2.14) are converging as $k \to 0$. The remaining question is how these limits relate to equation (2.1). And indeed, combining the results, shows that the limit U from Lemma 2.1.8 is the solution of the initial value problem (2.1).

Theorem 2.1.11. *Let Assumptions 2.1.1 and 2.1.3 be fulfilled and let* $f \in L^q(0,T;V^*)$, $q = \frac{p}{p-1}$, *as well as* $u_0 \in H$ *be given. Further, let* $(N_\ell)_{\ell\in\mathbb{N}}$ *be a sequence of natural numbers with* $N_\ell \to \infty$ *as* $\ell \to \infty$ *and let* $k = \frac{T}{N_\ell}$ *be the corresponding step sizes. Then the sequences of piecewise constant and piecewise linear prolongations as given in* (2.12) *and* (2.13) *fulfill*

$$\begin{aligned}
&\bar{U}^k \rightharpoonup u && \text{in } L^p(0,T;L^p(\Omega_\theta;V)),\\
&\bar{U}^k \overset{*}{\rightharpoonup} u, \quad U^k \overset{*}{\rightharpoonup} u && \text{in } L^\infty(0,T;L^2(\Omega_\theta;H)),\\
&(U^k)' \rightharpoonup u' && \text{in } L^q(0,T;L^q(\Omega_\theta;V^*)),\\
&A^k\bar{U}^k \rightharpoonup Au && \text{in } L^q(0,T;L^q(\Omega_\theta;V^*))
\end{aligned}$$

as $k \to 0$, *where* u *is the solution to* (2.1) *and* u' *its weak derivative. Furthermore, it holds true that* $\bar{U}^k(t) \rightharpoonup u(t)$ *and* $U^k(t) \rightharpoonup u(t)$ *in* $L^2(\Omega_\theta;H)$ *as* $k \to 0$ *for all* $t \in [0,T]$.

Note that it is also possible to prove that $U^k \rightharpoonup u$ in $L^p(0,T;L^p(\Omega_\theta;V))$ as $k \to 0$ if we choose $U^k(0) = u_0^k$ in V, where $u_0^k \to u_0$ in H as $k \to 0$ and $\big(k^{\frac{1}{p}}\|u_0^k\|_V\big)_{k>0}$ is a bounded sequence. Such a sequence always exists because V is densely embedded into H. A short construction of such a sequence can also be found in Chapter 3. If in this case a scheme without randomization is used, the sequence of piecewise linear prolongations is bounded in the space $\mathcal{W}^p(0,T)$. Due to the Lions–Aubin lemma, this space is compactly embedded into $L^2(0,T;H)$ if V is compactly embedded into H. Such a compact embedding is useful when H-valued perturbations are considered. In our setting, it is not possible to apply the Lions–Aubin lemma because of the additional Ω_θ dependence. We could still obtain that the sequence $(U^k)_{k>0}$ is bounded in the space

$$\mathcal{W}^p_{\Omega_\theta}(0,T) = \{W \in L^p(0,T;L^p(\Omega_\theta;V)) : W' \text{ exists and } W' \in L^q(0,T;L^q(\Omega_\theta;V^*))\}$$

if the initial value is approximated with values in V. But now $L^p(\Omega_\theta;V)$ does not have to be compactly embedded into $L^2(\Omega_\theta;H)$ even if V is compactly embedded into H. Thus, if we want to consider equations with an H-valued perturbations, a transformation as in (1.2) has to be used. Even without the compact embedding from the Lions–Aubin lemma, we will see in Theorem 2.1.12 below that certain strong convergence results can be obtained.

Proof of Theorem 2.1.11. For simplicity, we again do not denote the subsequences differently within this proof and we drop the index ℓ. In the following, let $U \in \mathcal{W}^p_{\Omega_\theta}(0,T)$ be the limit of the sequences of piecewise constant and piecewise linear prolongations obtained in Lemma 2.1.8. An application of the a priori bound (2.7) from Lemma 2.1.7 and the boundedness condition from Assumption 2.1.1 (4) yields

$$\begin{aligned}
\|A^k\bar{U}^k\|_{L^q(0,T;L^q(\Omega_\theta;V^*))} &= \Big(k\sum_{n=1}^N \mathbf{E}_\theta\big[\|A(t_n^\theta)\mathbf{U}^n\|_{V^*}^q\big]\Big)^{\frac{1}{q}} \\
&\le \Big(k\sum_{n=1}^N \mathbf{E}_\theta\big[\beta^q\big(1+\|\mathbf{U}^n\|_V^{p-1}\big)^q\big]\Big)^{\frac{1}{q}} \le \beta T^{\frac{1}{q}} + \beta K^{\frac{1}{q}}.
\end{aligned}$$

As $L^q(0,T;L^q(\Omega_\theta;V^*))$ is a reflexive Banach space, we can extract a weakly converging subsequence such that

$$A^k\bar{U}^k \rightharpoonup b \quad \text{in } L^q(0,T;L^q(\Omega_\theta;V^*)) \quad \text{as } k \to 0$$

for $b \in L^q(0,T;L^q(\Omega_\theta;V^*))$. Next, we identify $f-b$ with the weak derivative of U. To this end, for arbitrary $v \in L^p(\Omega_\theta;V)$ and $\varphi \in C_c^\infty(0,T)$ we see that

$$\begin{aligned}
\int_0^T \mathbf{E}_\theta\big[\langle U'(t),v\rangle_{V^*\times V}\big]\varphi(t)\,\mathrm{d}t &= \lim_{k\to 0}\int_0^T \mathbf{E}_\theta\big[\langle (U^k)'(t),v\rangle_{V^*\times V}\big]\varphi(t)\,\mathrm{d}t \\
&= \lim_{k\to 0}\int_0^T \mathbf{E}_\theta\big[\langle f^k(t)-A^k(t)\bar{U}^k(t),v\rangle_{V^*\times V}\big]\varphi(t)\,\mathrm{d}t \\
&= \int_0^T \mathbf{E}_\theta\big[\langle f(t)-b(t),v\rangle_{V^*\times V}\big]\varphi(t)\,\mathrm{d}t,
\end{aligned}$$

where we also used $f^k \to f$ and $(U^k)' \rightharpoonup U'$ in $L^q(0,T;L^q(\Omega_\theta;V^*))$ as $k \to 0$, compare Lemma 2.1.8 and Lemma 2.1.9. Thus, $b = f - U'$ in $L^q(0,T;L^q(\Omega_\theta;V^*))$ is fulfilled.

Due to the a priori bound (2.7) from Lemma 2.1.7, for $t \in [0,T]$ we can again extract a weakly converging subsequence of $(U^k(t))_{k>0}$ such that

$$U^k(t) \rightharpoonup \tilde{U}(t) \quad \text{in } L^2(\Omega_\theta;H) \quad \text{as } k \to 0$$

with $\tilde{U}(t) \in L^2(\Omega_\theta;H)$. Assuming that $t \in [0,T]$, $\varphi \in C^1([0,T])$, and $v \in L^p(\Omega_\theta;V)$, we can see

$$\begin{aligned}
&\mathbf{E}_\theta\big[(U(t),v)_H\big]\varphi(t) - \mathbf{E}_\theta\big[(U(0),v)_H\big]\varphi(0) - \int_0^t \mathbf{E}_\theta\big[(U(s),v)_H\big]\varphi'(s)\,\mathrm{d}s \\
&= \int_0^t \mathbf{E}_\theta\big[\langle U'(s),v\rangle_{V^*\times V}\big]\varphi(s)\,\mathrm{d}s \\
&= \lim_{k\to 0}\int_0^t \mathbf{E}_\theta\big[\langle (U^k)'(s),v\rangle_{V^*\times V}\big]\varphi(s)\,\mathrm{d}s \\
&= \lim_{k\to 0}\Big(\mathbf{E}_\theta\big[(U^k(t),v)_H\big]\varphi(t) - \mathbf{E}_\theta\big[(U^k(0),v)_H\big]\varphi(0) - \int_0^t \mathbf{E}_\theta\big[(U^k(s),v)_H\big]\varphi'(s)\,\mathrm{d}s\Big) \\
&= \mathbf{E}_\theta\big[(\tilde{U}(t),v)_H\big]\varphi(t) - \mathbf{E}_\theta\big[(u_0,v)_H\big]\varphi(0) - \int_0^t \mathbf{E}_\theta\big[(U(s),v)_H\big]\varphi'(s)\,\mathrm{d}s.
\end{aligned}$$

In the single steps, we use $(U^k)' \rightharpoonup U'$ in $L^q(0,T;L^q(\Omega_\theta;V^*))$ as well as $U^k \overset{*}{\rightharpoonup} U$ in $L^\infty(0,T;L^2(\Omega_\theta;H))$, compare Lemma 2.1.8. As for every $k > 0$ the equality $U^k(0) = \mathbf{U}^0 =$

u_0 in H holds true, this implies $U(t) = \tilde{U}(t)$ in $L^2(\Omega_\theta; H)$. For the piecewise constant prolongation $\bar{U}^k$, we see that

$$\begin{aligned}
\mathbf{E}_\theta\big[\|\bar{U}^k(t) - U^k(t)\|_{V^*}^q\big] &= \mathbf{E}_\theta\Big[\Big\|\mathbf{U}^n - \mathbf{U}^{n-1} - \frac{t-t_{n-1}}{k}\big(\mathbf{U}^n - \mathbf{U}^{n-1}\big)\Big\|_{V^*}^q\Big] \\
&= \Big(\frac{t_n - t}{k}\Big)^q \mathbf{E}_\theta\big[\|\mathbf{U}^n - \mathbf{U}^{n-1}\|_{V^*}^q\big] \\
&\le \sum_{i=1}^N \mathbf{E}_\theta\big[\|\mathbf{U}^i - \mathbf{U}^{i-1}\|_{V^*}^q\big] \le k^{q-1}K
\end{aligned}$$

for all $t \in (t_{n-1}, t_n]$, $n \in \{1, \dots, N\}$, where we applied the a priori bound (2.8) from Lemma 2.1.7. Thus, it follows that $\mathbf{E}_\theta\big[\|\bar{U}^k(t) - U^k(t)\|_{V^*}^q\big] \to 0$ as $k \to 0$ for every $t \in [0,T]$. This implies that the limits of $(\bar{U}^k(t))_{k>0}$ and $(U^k(t))_{k>0}$ coincide in $L^q(\Omega_\theta; V^*)$. Since the sequence $(\bar{U}^k(t))_{k>0}$, $t \in [0,T]$, is bounded in $L^2(\Omega_\theta; H)$ due to the a priori bound (2.7) from Lemma 2.1.7, we can extract a weakly converging subsequence. The $L^2(\Omega_\theta; H)$-valued limit of $(\bar{U}^k(t))_{k>0}$ then coincides with the weak limit $U(t)$ of $(U^k(t))_{k>0}$ in $L^q(\Omega_\theta; V^*)$ for every $t \in [0,T]$. Since $L^2(\Omega_\theta; H)$ is continuously embedded into $L^q(\Omega_\theta; V^*)$, it follows that $(\bar{U}^k(t))_{k>0}$ converges weakly to $U(t)$ in $L^2(\Omega_\theta; H)$ for every $t \in [0,T]$.

It remains to verify that $b = AU$ in $L^q(0,T; L^q(\Omega_\theta; V^*))$. Testing the differential equation in (2.14) with $\bar{U}^k \in L^p(0,T; L^p(\Omega_\theta; V))$ and integrating from 0 to $t \in (t_{n-1}, t_n]$, $n \in \{1, \dots, N\}$, we can write that

$$\begin{aligned}
&\int_0^t \mathbf{E}_\theta\big[\langle A^k(s)\bar{U}^k(s), \bar{U}^k(s)\rangle_{V^*\times V}\big]\,\mathrm{d}s \\
&= \int_0^t \mathbf{E}_\theta\big[\langle f^k(s) - (U^k)'(s), \bar{U}^k(s)\rangle_{V^*\times V}\big]\,\mathrm{d}s \\
&= \int_0^{t_n} \mathbf{E}_\theta\big[\langle f^k(s) - (U^k)'(s), \bar{U}^k(s)\rangle_{V^*\times V}\big]\,\mathrm{d}s \\
&\quad - \int_t^{t_n} \mathbf{E}_\theta\big[\langle f^k(s) - (U^k)'(s), \bar{U}^k(s)\rangle_{V^*\times V}\big]\,\mathrm{d}s.
\end{aligned}$$

Next, we insert the structure of the prolongations as well as the identity from Lemma A.1.4 to obtain that

$$\begin{aligned}
&\int_0^{t_n} \mathbf{E}_\theta\big[\langle (U^k)'(s), \bar{U}^k(s)\rangle_{V^*\times V}\big]\,\mathrm{d}s \\
&= \frac{1}{k}\sum_{i=1}^n \int_{t_{i-1}}^{t_i} \mathbf{E}_\theta\big[(\mathbf{U}^i - \mathbf{U}^{i-1}, \mathbf{U}^i)_H\big]\,\mathrm{d}s \\
&= \sum_{i=1}^n \mathbf{E}_\theta\big[(\mathbf{U}^i - \mathbf{U}^{i-1}, \mathbf{U}^i)_H\big] \\
&= \frac{1}{2}\sum_{i=1}^n \big(\mathbf{E}_\theta\big[\|\mathbf{U}^i\|_H^2\big] - \mathbf{E}_\theta\big[\|\mathbf{U}^{i-1}\|_H^2\big] + \mathbf{E}_\theta\big[\|\mathbf{U}^i - \mathbf{U}^{i-1}\|_H^2\big]\big) \\
&\ge \frac{1}{2}\mathbf{E}_\theta\big[\|\mathbf{U}^n\|_H^2\big] - \frac{1}{2}\|\mathbf{U}^0\|_H^2 = \frac{1}{2}\mathbf{E}_\theta\big[\|\bar{U}^k(t)\|_H^2\big] - \frac{1}{2}\|u_0\|_H^2.
\end{aligned}$$

Recall that $f^k \to f$ in $L^q(0,T; L^q(\Omega_\theta; V^*))$ and $\bar{U}^k \rightharpoonup U$ in $L^p(0,T; L^p(\Omega_\theta; V))$ as $k \to 0$. As for every $t \in (t_{n-1}, t_n]$, $n \in \{1, \dots, N\}$, we have that $\bar{U}^k(t_n) = \bar{U}^k(t) \rightharpoonup U(t)$ in $L^2(\Omega_\theta; H)$

as $k \to 0$, the lower semi-continuity of the norm then implies that

$$
\begin{aligned}
&\limsup_{k\to 0} \int_0^t \mathbf{E}_\theta\big[\langle A^k(s)\bar{U}^k(s), \bar{U}^k(s)\rangle_{V^*\times V}\big]\,\mathrm{d}s\\
&= \limsup_{k\to 0} \Big(\int_0^{t_n} \mathbf{E}_\theta\big[\langle f^k(s) - (U^k)'(s), \bar{U}^k(s)\rangle_{V^*\times V}\big]\,\mathrm{d}s\\
&\qquad\qquad - \int_t^{t_n} \mathbf{E}_\theta\big[\langle f^k(s) - (U^k)'(s), \bar{U}^k(s)\rangle_{V^*\times V}\big]\,\mathrm{d}s\Big)\\
&= \int_0^t \mathbf{E}_\theta\big[\langle f(s), U(s)\rangle_{V^*\times V}\big]\,\mathrm{d}s - \liminf_{k\to 0} \int_0^{t_n} \mathbf{E}_\theta\big[\langle (U^k)'(s), \bar{U}^k(s)\rangle_{V^*\times V}\big]\,\mathrm{d}s\\
&\le \int_0^t \mathbf{E}_\theta\big[\langle f(s), U(s)\rangle_{V^*\times V}\big]\,\mathrm{d}s - \liminf_{k\to 0} \frac{1}{2}\big(\mathbf{E}_\theta\big[\|\bar{U}^k(t)\|_H^2\big] - \|u_0\|_H^2\big)\\
&\le \int_0^t \mathbf{E}_\theta\big[\langle f(s), U(s)\rangle_{V^*\times V}\big]\,\mathrm{d}s - \frac{1}{2}\big(\mathbf{E}_\theta\big[\|U(t)\|_H^2\big] - \|u_0\|_H^2\big).
\end{aligned}
$$

Here, we also used that $(\langle f^k - (U^k)', \bar{U}^k\rangle_{V^*\times V})_{k>0}$ is bounded uniformly with respect to k in $L^1((0,T)\times\Omega_\theta)$. As U is an element of the space $\mathcal{W}^p_{\Omega_\theta}(0,T)$, which is defined in (2.15), it follows that

$$
\frac{1}{2}\big(\mathbf{E}_\theta\big[\|U(t)\|_H^2\big] - \|u_0\|_H^2\big) = \frac{1}{2}\int_0^t \frac{\mathrm{d}}{\mathrm{d}s}\mathbf{E}_\theta\big[\|U(s)\|_H^2\big]\,\mathrm{d}s = \int_0^t \mathbf{E}_\theta\big[\langle U'(s), U(s)\rangle_{V^*\times V}\big]\,\mathrm{d}s,
$$

for every $t \in [0,T]$. This implies that

$$
\begin{aligned}
&\limsup_{k\to 0} \int_0^t \mathbf{E}_\theta\big[\langle A^k(s)\bar{U}^k(s), \bar{U}^k(s)\rangle_{V^*\times V}\big]\,\mathrm{d}s\\
&\le \int_0^t \mathbf{E}_\theta\big[\langle f(s) - U'(s), U(s)\rangle_{V^*\times V}\big]\,\mathrm{d}s = \int_0^t \mathbf{E}_\theta\big[\langle b(s), U(s)\rangle_{V^*\times V}\big]\,\mathrm{d}s.
\end{aligned}
\tag{2.16}
$$

In order to prove that $AU = b$ in $L^q(0,T;L^q(\Omega_\theta;V^*))$, it is problematic to apply the Minty monotonicity trick directly. Without further information, the limit U is a random variable that might have a dependence on Ω_θ. This makes it difficult to apply Lemma 2.1.10 as the function v in the statement has to be independent of Ω_θ. But we can use the fact that we already know that the initial value problem (2.1) has a unique solution $u \in \mathcal{W}^p(0,T)$, which is constant with respect to Ω_θ. We then define the piecewise constant function $u^k\colon [0,T] \to L^p(\Omega_\theta;V)$ given by $u^k(0) = u(t_1^\theta)$ and $u^k(t) = u(t_n^\theta)$ in $L^p(\Omega_\theta;V)$ for $t \in (t_{n-1}, t_n]$, $n \in \{1,\dots,N\}$. Because of the monotonicity condition from Assumption 2.1.1 (3), we see that

$$
\int_0^t \mathbf{E}_\theta\big[\langle A^k(s)\bar{U}^k(s) - A^k(s)u^k(s), \bar{U}^k(s) - u^k(s)\rangle_{V^*\times V}\big]\,\mathrm{d}s \ge 0.
$$

Rearranging the terms yields the inequality

$$
\begin{aligned}
&\int_0^t \mathbf{E}_\theta\big[\langle A^k(s)\bar{U}^k(s), \bar{U}^k(s)\rangle_{V^*\times V}\big]\,\mathrm{d}s\\
&\ge \int_0^t \mathbf{E}_\theta\big[\langle A^k(s)\bar{U}^k(s), u^k(s)\rangle_{V^*\times V}\big]\,\mathrm{d}s + \int_0^t \mathbf{E}_\theta\big[\langle A^k(s)u^k(s), \bar{U}^k(s) - u^k(s)\rangle_{V^*\times V}\big]\,\mathrm{d}s.
\end{aligned}
$$

Using the definition of b as well as (2.16), Lemma 2.1.9 and Lemma 2.1.10, we obtain that

$$\begin{aligned}
&\int_0^t \mathbf{E}_\theta\big[\langle b(s), U(s)\rangle_{V^*\times V}\big]\,\mathrm{d}s\\
&\geq \limsup_{k\to 0} \int_0^t \mathbf{E}_\theta\big[\langle A^k(s)\bar{U}^k(s), \bar{U}^k(s)\rangle_{V^*\times V}\big]\,\mathrm{d}s\\
&\geq \lim_{k\to 0} \int_0^t \mathbf{E}_\theta\big[\langle A^k(s)\bar{U}^k(s), u^k(s)\rangle_{V^*\times V}\big]\,\mathrm{d}s\\
&\quad + \lim_{k\to 0} \int_0^t \mathbf{E}_\theta\big[\langle A^k(s)u^k(s), \bar{U}^k(s) - u^k(s)\rangle_{V^*\times V}\big]\,\mathrm{d}s\\
&= \int_0^t \mathbf{E}_\theta\big[\langle b(s), u(s)\rangle_{V^*\times V}\big]\,\mathrm{d}s + \int_0^t \mathbf{E}_\theta\big[\langle A(s)u(s), U(s) - u(s)\rangle_{V^*\times V}\big]\,\mathrm{d}s.
\end{aligned}$$

This implies

$$\int_0^t \mathbf{E}_\theta\big[\langle b(s) - A(s)u(s), U(s) - u(s)\rangle_{V^*\times V}\big]\,\mathrm{d}s \geq 0. \tag{2.17}$$

This in mind and employing that U and u are elements of $\mathcal{W}^p_{\Omega_\theta}(0,T)$, we see that

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}s}\|U(s) - u(s)\|^2_{L^2(\Omega_\theta;H)} &= \mathbf{E}_\theta\big[\langle U'(s) - u'(s), U(s) - u(s)\rangle_{V^*\times V}\big]\\
&= \mathbf{E}_\theta\big[\langle f(s) - b(s) - f(s) + A(s)u(s), U(s) - u(s)\rangle_{V^*\times V}\big]\\
&= -\mathbf{E}_\theta\big[\langle b(s) - A(s)u(s), U(s) - u(s)\rangle_{V^*\times V}\big]
\end{aligned}$$

for almost every $s \in (0,T)$. Integrating this equality from 0 to $t \in [0,T]$ and applying (2.17), shows that

$$\begin{aligned}
&\frac{1}{2}\|U(t) - u(t)\|^2_{L^2(\Omega_\theta;H)} - \frac{1}{2}\|U(0) - u(0)\|^2_{L^2(\Omega_\theta;H)}\\
&= -\int_0^t \mathbf{E}_\theta\big[\langle b(s) - A(s)u(s), U(s) - u(s)\rangle_{V^*\times V}\big]\,\mathrm{d}s \leq 0.
\end{aligned}$$

Since we have already seen that $U(0)$ and $u(0)$ coincide in $L^2(\Omega_\theta; H)$, it follows that $U(t) = u(t)$ in $L^2(\Omega_\theta; H)$ for all $t \in [0,T]$. This shows, in particular, that U is constant in Ω_θ.

The last step is to prove that $Au = b$ in $L^q(0,T; L^q(\Omega_\theta; V^*))$. This also proves that b is constant on Ω_θ. As we have seen that $U = u$ in $L^\infty(0,T;H)$ and both U' and u' exist it follows that $U' = u'$ in $L^q(0,T; L^q(\Omega_\theta; V^*))$ and because u' is constant on Ω_θ the same is true for U'. Also we have seen that $b = U' - f$ in $L^q(0,T; L^q(\Omega_\theta; V^*))$. Since $U' - f = u' - f = Au$ in $L^q(0,T;V^*)$, it follows that $b = Au$ in $L^q(0,T;V^*)$.

So far, we have only proved that every converging subsequence of $(\bar{U}^k)_{k>0}$ converges to u weakly in $L^p(0,T; L^p(\Omega_\theta; V))$. An application of the subsequence principle, see [116, Proposition 10.13] or [49, Kapitel I, Lemma 5.4], yields that the original sequence converges weakly to u in $L^p(0,T; L^p(\Omega_\theta; V))$. Analogously, it is possible to prove that the other assertions of this theorem hold true for the original sequence. □

The previous theorem verifies that the sequences of prolongations converge to the solution of (2.1) in the weak sense. We can strengthen the result from this theorem and show that we obtain a strong pointwise convergence in $L^2(\Omega_\theta; H)$. If $A(t)$, $t \in [0,T]$, fulfills a stronger monotonicity assumption, we can even show a strong convergence result for the piecewise constant prolongation in $L^p(0,T; L^p(\Omega_\theta; V))$.

Theorem 2.1.12. *Let Assumptions 2.1.1 and 2.1.3 be fulfilled and let $f \in L^q(0,T;V^*)$, $q = \frac{p}{p-1}$, as well as $u_0 \in H$ be given. Further, let $(N_\ell)_{\ell\in\mathbb{N}}$ be a sequence of natural numbers with $N_\ell \to \infty$ as $\ell \to \infty$ and let $k = \frac{T}{N_\ell}$ be the corresponding step sizes. Then the sequences of piecewise constant and piecewise linear prolongations given in* (2.12) *and* (2.13) *fulfill that*

$$\bar{U}^k(t) \to u(t), \quad U^k(t) \to u(t) \quad \text{in } L^2(\Omega_\theta; H) \quad \text{as } k \to 0$$

for every $t \in [0,T]$, where u is the solution to the initial value problem (2.1)*. Furthermore, under the additional assumption that $A(t)$, $t \in [0,T]$, fulfills a p-monotonicity condition such that there exists $\eta \in (0,\infty)$, which does not depend on $t \in [0,T]$, with*

$$\langle A(t)v - A(t)w, v-w\rangle_{V^*\times V} \geq \eta \|v-w\|_V^p \tag{2.18}$$

for every $v, w \in V$, the sequence $(\bar{U}^k)_{k>0}$ satisfies that

$$\bar{U}^k \to u \quad \text{in } L^p(0,T;L^p(\Omega_\theta;V)) \quad \text{as } k \to 0.$$

Proof. For simplicity, we drop the index ℓ within this proof. The main idea of this proof is to use the weak convergence results proved in Theorem 2.1.11 to deduce the strong convergence in the same space. We combine the monotonicity conditions from Assumption 2.1.1 (3) and from (2.18). We notice that the case $\eta = 0$ in (2.18) is exactly the monotonicity condition from Assumption 3.1.2 (3). Thus, we include $\eta = 0$ to (2.18). We point out the additional result for $\eta \in (0,\infty)$ later in the proof. We consider the piecewise constant function $u^k \colon [0,T] \to L^p(\Omega_\theta;V)$ given by $u^k(0) = u(t_1^\theta)$ and $u^k(t) = u(t_n^\theta)$ in $L^p(\Omega_\theta;V)$ for $t \in (t_{n-1}, t_n]$, $n \in \{1,\dots,N\}$. Using the p-monotonicity condition from (2.18), yields

$$\begin{aligned}
&\mathbf{E}_\theta\big[\|u^k(t) - \bar{U}^k(t)\|_H^2\big] + 2\eta \int_0^t \mathbf{E}_\theta\big[\|u^k(s) - \bar{U}^k(s)\|_V^p\big] \,\mathrm{d}s\\
&\leq \mathbf{E}_\theta\big[\|u^k(t) - \bar{U}^k(t)\|_H^2\big] + 2\int_0^t \mathbf{E}_\theta\big[\langle A^k(s)u^k(s) - A^k(s)\bar{U}^k(s), u^k(s) - \bar{U}^k(s)\rangle_{V^*\times V}\big] \,\mathrm{d}s\\
&= \mathbf{E}_\theta\big[\|u^k(t)\|_H^2\big] + 2\int_0^t \mathbf{E}_\theta\big[\langle A^k(s)u^k(s), u^k(s)\rangle_{V^*\times V}\big] \,\mathrm{d}s\\
&\quad - 2\mathbf{E}_\theta\big[(u^k(t), \bar{U}^k(t))_H\big] - 2\int_0^t \mathbf{E}_\theta\big[\langle A^k(s)u^k(s), \bar{U}^k(s)\rangle_{V^*\times V}\big] \,\mathrm{d}s\\
&\quad - 2\int_0^t \mathbf{E}_\theta\big[\langle A^k(s)\bar{U}^k(s), u^k(s)\rangle_{V^*\times V}\big] \,\mathrm{d}s\\
&\quad + \mathbf{E}_\theta\big[\|\bar{U}^k(t)\|_H^2\big] + 2\int_0^t \mathbf{E}_\theta\big[\langle A^k(s)\bar{U}^k(s), \bar{U}^k(s)\rangle_{V^*\times V}\big] \,\mathrm{d}s\\
&=: \Gamma_1^k(t) + \Gamma_2^k(t) + \Gamma_3^k(t),
\end{aligned}$$

for every $t \in [0,T]$, where

$$\begin{aligned}
\Gamma_1^k(t) &= \mathbf{E}_\theta\big[\|u^k(t)\|_H^2\big] + 2\int_0^t \mathbf{E}_\theta\big[\langle A^k(s)u^k(s), u^k(s)\rangle_{V^*\times V}\big] \,\mathrm{d}s,\\
\Gamma_2^k(t) &= -2\mathbf{E}_\theta\big[(u^k(t), \bar{U}^k(t))_H\big] - 2\int_0^t \mathbf{E}_\theta\big[\langle A^k(s)u^k(s), \bar{U}^k(s)\rangle_{V^*\times V}\big] \,\mathrm{d}s\\
&\quad - 2\int_0^t \mathbf{E}_\theta\big[\langle A^k(s)\bar{U}^k(s), u^k(s)\rangle_{V^*\times V}\big] \,\mathrm{d}s,\\
\Gamma_3^k(t) &= \mathbf{E}_\theta\big[\|\bar{U}^k(t)\|_H^2\big] + 2\int_0^t \mathbf{E}_\theta\big[\langle A^k(s)\bar{U}^k(s), \bar{U}^k(s)\rangle_{V^*\times V}\big] \,\mathrm{d}s.
\end{aligned}$$

As u is an element of $L^p(0,T;V)$, we can apply Lemma 2.1.10 to obtain that $A^k u^k \to Au$ in $L^q(0,T;L^q(\Omega_\theta;V^*))$ as $k \to 0$. We can apply Lemma 2.1.9 to obtain that $u^k \to u$ in $L^p(0,T;L^p(\Omega_\theta;V))$ as $k \to 0$. The solution u to (2.1) is, in particular, an element of $C([0,T];H)$. For $t \in [0,T]$, we always choose $n \in \{1,\dots,N\}$ such $t \in (t_{n-1},t_n]$ or $n=1$ if $t=0$. Then it follows that

$$\|u(t) - u^k(t)\|_{L^2(\Omega_\theta;H)} = \|u(t) - u(t_n^\theta)\|_{L^2(\Omega_\theta;H)} \to 0 \quad k \to 0.$$

This means that $u^k(t) \to u(t)$ in $L^2(\Omega_\theta;H)$ as $k \to 0$ for every $t \in [0,T]$. Thus, it follows that

$$\begin{aligned}\lim_{k\to 0}\Gamma_1^k(t) &= \|u(t)\|^2_{L^2(\Omega_\theta;H)} + 2\int_0^t \langle A(s)u(s), u(s)\rangle_{V^*\times V}\,\mathrm{d}s\\ &= \|u(t)\|_H^2 + 2\int_0^t \langle A(s)u(s), u(s)\rangle_{V^*\times V}\,\mathrm{d}s.\end{aligned}$$

Recall that in Theorem 2.1.11 it was proved that $\bar{U}^k(t) \rightharpoonup u(t)$ in $L^2(\Omega_\theta;H)$ for every $t\in[0,T]$, $\bar{U}^k \rightharpoonup u$ in $L^p(0,T;L^p(\Omega_\theta;V))$ and $A^k\bar{U}^k \rightharpoonup Au$ in $L^q(0,T;L^q(\Omega_\theta;V^*))$ as $k \to 0$. Together with the convergence results mentioned for Γ_1^k, this yields

$$\begin{aligned}\lim_{k\to 0}\Gamma_2^k(t) &= -2(u(t),u(t))_H - 2\int_0^t \langle A(s)u(s), u(s)\rangle_{V^*\times V}\,\mathrm{d}s - 2\int_0^t \langle A(s)u(s), u(s)\rangle_{V^*\times V}\,\mathrm{d}s\\ &= -2\|u(t)\|_H^2 - 4\int_0^t \langle A(s)u(s), u(s)\rangle_{V^*\times V}\,\mathrm{d}s.\end{aligned}$$

Handling Γ_3^k needs a little more attention. For every $t \in (t_{n-1},t_n]$, $n\in\{1,\dots,N\}$, we can write

$$\begin{aligned}\Gamma_3^k(t) &= \mathbf{E}_\theta\big[\|\bar{U}^k(t)\|_H^2\big] + 2\int_0^t \mathbf{E}_\theta\big[\langle A^k(s)\bar{U}^k(s), \bar{U}^k(s)\rangle_{V^*\times V}\big]\,\mathrm{d}s\\ &= \mathbf{E}_\theta\big[\|\mathbf{U}^n\|_H^2\big] + 2\int_0^{t_n} \mathbf{E}_\theta\big[\langle A^k(s)\bar{U}^k(s) - f^k(s), \bar{U}^k(s)\rangle_{V^*\times V}\big]\,\mathrm{d}s\\ &\quad + 2\int_0^t \mathbf{E}_\theta\big[\langle f^k(s), \bar{U}^k(s)\rangle_{V^*\times V}\big]\,\mathrm{d}s\\ &\quad - 2\int_t^{t_n} \mathbf{E}_\theta\big[\langle A^k(s)\bar{U}^k(s) - f^k(s), \bar{U}^k(s)\rangle_{V^*\times V}\big]\,\mathrm{d}s.\end{aligned}$$

Inserting equation (2.14), it follows that

$$\int_0^{t_n} \mathbf{E}_\theta\big[\langle A^k(s)\bar{U}^k(s) - f^k(s), \bar{U}^k(s)\rangle_{V^*\times V}\big]\,\mathrm{d}s = -\int_0^{t_n} \mathbf{E}_\theta\big[\langle (U^k)'(s), \bar{U}^k(s)\rangle_{V^*\times V}\big]\,\mathrm{d}s.$$

Applying the specific structure of the piecewise constant and piecewise linear prolongation from (2.12) and (2.13), the integral containing the weak derivative of U^k can be estimated by

$$\begin{aligned}-2\int_0^{t_n} \mathbf{E}_\theta\big[\langle (U^k)'(s), \bar{U}^k(s)\rangle_{V^*\times V}\big]\,\mathrm{d}s &= -2\sum_{i=1}^n \mathbf{E}_\theta\big[(\mathbf{U}^i - \mathbf{U}^{i-1}, \mathbf{U}^i)_H\big]\\ &\le -\sum_{i=1}^n \big(\mathbf{E}_\theta\big[\|\mathbf{U}^i\|_H^2\big] - \mathbf{E}_\theta\big[\|\mathbf{U}^{i-1}\|_H^2\big]\big)\\ &= -\mathbf{E}_\theta\big[\|\mathbf{U}^n\|_H^2\big] + \mathbf{E}_\theta\big[\|\mathbf{U}^0\|_H^2\big].\end{aligned}$$

Here, we used the telescopic structure of the sum as well as the identity from Lemma A.1.4. A combination of the previous arguments then gives us the bound

$$\begin{aligned}\Gamma_3^k(t) \le \|u_0\|_H^2 &+ 2\int_0^t \mathbf{E}_\theta\big[\langle f^k(s), \bar{U}^k(s)\rangle_{V^*\times V}\big]\,\mathrm{d}s \\ &- 2\int_t^{t_n} \mathbf{E}_\theta\big[\langle A^k(s)\bar{U}^k(s) - f^k(s), \bar{U}^k(s)\rangle_{V^*\times V}\big]\,\mathrm{d}s\end{aligned} \tag{2.19}$$

for every $t \in (t_{n-1}, t_n]$, $n \in \{1, \dots, N\}$. In Lemma 2.1.9, we proved that $f^k \to f$ in $L^q(0,T; L^q(\Omega_\theta; V^*))$ as $k \to 0$. Further, in Theorem 2.1.11, we showed that $\bar{U}^k \rightharpoonup u$ in $L^p(0,T; L^p(\Omega_\theta; V))$ as $k \to 0$. Thus, it follows that the first integral on the right-hand side of (2.19) converges to $\int_0^t \langle f(s), u(s)\rangle_{V^*\times V}\,\mathrm{d}s$. For the second integral, we notice that

$$\begin{aligned}&\langle A^k(s)\bar{U}^k(s) - f^k(s), \bar{U}^k(s)\rangle_{V^*\times V} \\ &\le \|A^k(s)\bar{U}^k(s)\|_{V^*}\|\bar{U}^k(s)\|_V + \|f^k(s)\|_{V^*}\|\bar{U}^k(s)\|_V \\ &\le \beta\big(\|\bar{U}^k(s)\|_V + \|\bar{U}^k(s)\|_V^p\big) + \frac{1}{q}\|f^k(s)\|_{V^*}^q + \frac{1}{p}\|\bar{U}^k(s)\|_V^p =: g(s)\end{aligned}$$

for almost all $s \in (0,T)$. The function g is bounded by a function in $L^1((0,T)\times\Omega_\theta)$, compare Lemma 2.1.6 and the a priori bound (2.7) from Lemma 2.1.7. Thus, the second integral in (2.19) tends to zero as $|t_n - t| \to 0$. Thus, it follows that

$$\limsup_{k\to 0} \Gamma_3^k(t) \le \|u_0\|_H^2 + 2\int_0^t \langle f(s), u(s)\rangle_{V^*\times V}\,\mathrm{d}s.$$

Now, it remains to combine all these results to find that

$$\begin{aligned}&\limsup_{k\to 0}\Big(\mathbf{E}_\theta\big[\|u^k(t) - \bar{U}^k(t)\|_H^2\big] + 2\eta\int_0^t \mathbf{E}_\theta\big[\|u^k(s) - \bar{U}^k(s)\|_V^p\big]\,\mathrm{d}s\Big) \\ &\le \|u(t)\|_H^2 + 2\int_0^t \langle A(s)u(s), u(s)\rangle_{V^*\times V}\,\mathrm{d}s \\ &\quad - 2\|u(t)\|_H^2 - 4\int_0^t \langle A(s)u(s), u(s)\rangle_{V^*\times V}\,\mathrm{d}s \\ &\quad + \|u_0\|_H^2 + 2\int_0^t \langle f(s), u(s)\rangle_{V^*\times V}\,\mathrm{d}s \\ &= \|u_0\|_H^2 - \|u(t)\|_H^2 + 2\int_0^t \langle f(s) - A(s)u(s), u(s)\rangle_{V^*\times V}\,\mathrm{d}s \\ &= \|u_0\|_H^2 - \|u(t)\|_H^2 + 2\int_0^t \langle u'(s), u(s)\rangle_{V^*\times V}\,\mathrm{d}s.\end{aligned}$$

Since $u \in \mathcal{W}^p(0,T)$, we can apply a partial integration rule to obtain

$$\|u_0\|_H^2 - \|u(t)\|_H^2 + 2\int_0^t \langle u'(s), u(s)\rangle_{V^*\times V}\,\mathrm{d}s = \|u_0\|_H^2 - \|u(t)\|_H^2 + \int_0^t \frac{\mathrm{d}}{\mathrm{d}s}\|u(s)\|_H^2\,\mathrm{d}s = 0.$$

As $u^k(t) \to u(t)$ in $L^2(\Omega_\theta; H)$ as $k \to 0$ for every $t \in [0,T]$, an application of the triangular inequality yields that $\bar{U}^k(t) \to u(t)$ in $L^2(\Omega_\theta; H)$ as $k \to 0$ for every $t \in [0,T]$. In the case of $\eta \in (0,\infty)$, a similar argumentation yields that $\bar{U}^k \to u$ in $L^p(0,T; L^p(\Omega_\theta; V))$ as $k \to 0$ as the same thing is true for $(u^k)_{k>0}$.

It remains to prove that $U^k(t) \to u(t)$ in $L^2(\Omega_\theta; H)$ as $k \to 0$ for every $t \in [0,T]$. This is mainly due to the fact that the limit of both the sequence of the piecewise constant and piecewise linear prolongations coincide in a suitable sense, see also [43] for some further results. Recall the definition of $\bar{U}^k$ and U^k from (2.12) and (2.13), respectively. An application of the triangle inequality then yields that

$$
\begin{aligned}
&\|U^k(t) - u(t)\|_{L^2(\Omega_\theta;H)} \\
&\le \Big\| \frac{t_n - t}{k} \big(\bar{U}^k(t-k) - u(t)\big) \Big\|_{L^2(\Omega_\theta;H)} + \Big\| \frac{t - t_{n-1}}{k} \big(\bar{U}^k(t) - u(t)\big) \Big\|_{L^2(\Omega_\theta;H)} \\
&\le \|\bar{U}^k(t-k) - u(t)\|_{L^2(\Omega_\theta;H)} + \|\bar{U}^k(t) - u(t)\|_{L^2(\Omega_\theta;H)} \\
&\le \|\bar{U}^k(t-k) - u(t-k)\|_{L^2(\Omega_\theta;H)} + \|u(t-k) - u(t)\|_{L^2(\Omega_\theta;H)} + \|\bar{U}^k(t) - u(t)\|_{L^2(\Omega_\theta;H)}
\end{aligned}
$$

for every $t \in (t_{n-1}, t_n]$, $n \in \{1, \dots, N\}$. Using that $u \in \mathcal{W}^p(0,T) \hookrightarrow C([0,T];H)$ and $\bar{U}^k(t) \to u(t)$ in $L^2(\Omega_\theta; H)$ as $k \to 0$ for every $t \in [0,T]$, it also follows that $U^k(t) \to u(t)$ in $L^2(\Omega_\theta; H)$ as $k \to 0$. □

2.2 Explicit Error Estimates

Having proved convergence under no additional regularity assumptions, we now consider a second type of randomization. This is appropriate to prove explicit error estimates of the scheme. Here, the size of the error depends on the regularity of the exact solution. In the previous section, we used a randomized grid. Now, we still use a randomized grid but evaluate the data at a randomized point in between the randomly shifted grid points. Precisely, the random points are given in the following assumption.

Assumption 2.2.1. *Let $T \in (0,\infty)$ and $N \in \mathbb{N}$ be given. Consider the equidistant partition $0 = t_0 < \dots < t_N = T$ with $k = \frac{T}{N}$ and $t_n = nk$, $n \in \{0, \dots, N\}$. For a complete probability space $(\Omega_\theta, \mathcal{F}^\theta, \mathbf{P}_\theta)$ and a uniformly distributed random variable $\theta \colon \Omega_\theta \to [0,1]$, the randomly shifted grid is denoted by $0 = t_0^\theta < t_1^\theta < \dots < t_N^\theta = T - k(1-\theta)$ with $t_n^\theta = t_{n-1} + k\theta$ for $n \in \{1, \dots, N\}$. The step size is denoted by $k_n = t_n^\theta - t_{n-1}^\theta$ for $n \in \{1, \dots, N\}$.*

Let $(\Omega_\tau, \mathcal{F}^\tau, \mathbf{P}_\tau)$ be a second complete probability space and let $(\tau_n)_{n\in\{1,\dots,N\}}$ be a family of independent, uniformly distributed random variables such that $\tau_n \colon \Omega_\tau \to [0,1]$. On the product probability space $(\Omega, \mathcal{F}, \mathbf{P}) = (\Omega_\theta \times \Omega_\tau, \mathcal{F}^\theta \otimes \mathcal{F}^\tau, \mathbf{P}_\theta \otimes \mathbf{P}_\tau)$, let $\xi_n \colon \Omega \to [0,1]$ be given by $\xi_n = t_{n-1}^\theta + k_n \tau_n$ for $n \in \{1, \dots, N\}$.

The expectations on the probability spaces $(\Omega_\tau, \mathcal{F}^\tau, \mathbf{P}_\tau)$ and $(\Omega, \mathcal{F}, \mathbf{P})$ are denoted by $\mathbf{E}_\tau$ and $\mathbf{E}$, respectively. Note that the grid is not equidistant since $k_1 = \theta k$ but $k_n = k$ for $n \in \{2, \dots, N\}$. We still have $k_n \le k$ for every $n \in \{1, \dots, N\}$. This specific structure of randomization will be used to show our desired error bounds. We evaluate the data at the randomized points ξ_n, $n \in \{1, \dots, N\}$, and compare $(\mathbf{U}^n)_{n\in\{1,\dots,N\}}$ with $u(t_n^\theta)_{n\in\{1,\dots,N\}}$.

This randomization is a mixture of the one considered in [37] for the approximation of nonautonomous evolution equations and the one in [38] for the quadrature of stochastic Itô-integrals. Similar to [38], we can now weaken the regularity assumption on the solution. Instead of asking for a Hölder continuous solution as has been done in [37], we now assume that it is an element of a Sobolev–Slobodeckiĭ space. Since the exponent for the fractional derivative in a Sobolev–Slobodeckiĭ space can be larger than the exponent in a Hölder space, our setting can fill a gap between rates of convergence that are seen in numerical examples and rates that are theoretically derived.

In order to prove error estimates of schemes for nonlinear problems, it is possible to use a linear approximation of the operator given by its derivative. In [86] or [95], the fully nonlinear equation $u'(t) = F(t, u(t))$ for $t \in (0, T)$ with an initial condition is linearized along the exact solution u. If the partial derivative $A(t) = \partial_u F(t, u)$ exists, we can work with it instead of the fully nonlinear problem. In the following, we want to work with an approach that does not rely on such an approximation via a partial derivative. But we follow an approach that uses the fact that $A(t)$, $t \in [0, T]$, is Lipschitz continuous on a bounded set. Our starting point is to generalize the approach from [37], where error estimates are proved for a globally Lipschitz continuous operator. This is extended to a generalized Lipschitz condition, see (2.23) below. Now, we can also consider operators $A(t)$, $t \in [0, T]$, that do not have to be of linear growth. An example, which fits in our framework, is the classical p-Laplacian in a variational formulation. Note that for $p = 2$, the framework is nearly the same as in [37]. In [41, 42, 44] error estimates are provided without the additional bounded Lipschitz condition for the operator $A(t)$, $t \in [0, T]$. One advantage of our approach is that we do not have to impose any temporal regularity on f and A. This could be of advantage if $u' = f - Au$ fulfills a stronger regularity condition than the functions f and Au separately. The case where a function u' is more regular than f and Au separately probably does not contain many relevant problems. Thus, it would also be interesting to consider the techniques from [41, 42, 44] in the context of randomized schemes.

With the random point $(\xi_n)_{n\in\{1,\dots,N\}}$ from Assumption 2.2.1, we consider the scheme

$$\begin{cases} \mathbf{U}^n + k_n A(\xi_n)\mathbf{U}^n = k_n f(\xi_n) + \mathbf{U}^{n-1} & \text{almost surely in } V^*, \quad n \in \{1, \dots, N\}, \\ \mathbf{U}^0 = u_0 & \text{in } H, \end{cases} \tag{2.20}$$

for an initial value $u_0 \in H$ and a source term $f \in L^q(0, T; V^*)$.

In Assumption 2.2.1, we introduced $N+1$ independent random variables. In this section, it becomes necessary to consider two filtrations $(\mathcal{F}_n^\tau)_{n\in\{0,\dots,N\}} \subset \mathcal{F}^\tau$ and $(\mathcal{F}_n)_{n\in\{0,\dots,N\}} \subset \mathcal{F} = \mathcal{F}^\theta \otimes \mathcal{F}^\tau$. The first is given by

$$\begin{aligned} \mathcal{F}_0^\tau &:= \sigma\big(\mathcal{N} \in \mathcal{F}^\tau : \mathbf{P}_\tau(\mathcal{N}) = 0\big) \\ \mathcal{F}_n^\tau &:= \sigma\big(\sigma\big(\tau_i : i \in \{1, \dots, n\}\big) \cup \mathcal{F}_0^\tau\big), \quad n \in \{1, \dots, N\}, \end{aligned} \tag{2.21}$$

where σ denotes the generated σ-algebra, compare (A.4). Further, we consider

$$\mathcal{F}_n := \mathcal{F}^\theta \otimes \mathcal{F}_n^\tau, \quad n \in \{0, \dots, N\}. \tag{2.22}$$

In particular, it is clear that $\mathcal{F}_n^\tau \subset \mathcal{F}_m^\tau$ and $\mathcal{F}_n \subset \mathcal{F}_m$ for $n \le m$. Note that, for every $n \in \{1, \dots, N\}$, the mapping $\xi_n \colon \Omega \to [0, 1]$ is $\mathcal{F}_n$-measurable as a composition of measurable functions. Also, $\xi_n(\omega_\theta, \cdot) \colon \Omega_\tau \to [t_{n-1}^\theta(\omega_\theta), t_n^\theta(\omega_\theta)]$ is a uniformly distributed random variable, which is $\mathcal{F}_n^\tau$-measurable for every $n \in \{1, \dots, N\}$ and $\omega_\theta \in \Omega_\theta$. We begin by proving that (2.20) is uniquely solvable and its solution is adapted to the filtrations introduced above.

Lemma 2.2.2. *Let Assumptions 2.1.1 and 2.2.1 be fulfilled and let $f \in L^q(0, T; V^*)$ as well as $u_0 \in H$ be given. For a step size $k = \frac{T}{N}$, there exists a unique solution $(\mathbf{U}^n)_{n\in\{1,\dots,N\}}$ to the recursion (2.20). For every $n \in \{1, \dots, N\}$, the mapping $\mathbf{U}^n \colon \Omega \to V$, $\omega \mapsto \mathbf{U}^n(\omega)$ is $\mathcal{F}_n$-measurable, while $\mathbf{U}^n(\omega_\theta, \cdot) \colon \Omega_\tau \to V$, $\omega_\tau \mapsto \mathbf{U}^n(\omega_\theta, \omega_\tau)$ is $\mathcal{F}_n^\tau$-measurable for almost every $\omega_\theta \in \Omega_\theta$.*

Proof. The existence of $(\mathbf{U}^n)_{n\in\{1\dots,N\}}$ can be proved analogously as in Lemma 2.1.5.

In order to prove the measurability conditions, we again use Lemma 2.1.4. For $n \in \{1, \dots, N\}$, we consider the mappings

$$h_n \colon \Omega \times V \to V^*, \quad h_n(\omega, \mathbf{U}) = \big(I + k_n(\omega) A(\xi_n(\omega))\big)\mathbf{U} - k_n(\omega) f(\xi_n(\omega)) - \mathbf{U}^{n-1}(\omega)$$

as well as

$$h_n^\tau \colon \Omega_\tau \times V \to V^*, \quad h_n(\omega_\tau, \mathbf{U}) = \big(I + k_n(\omega_\theta)A(\xi_n(\omega_\theta,\omega_\tau))\big)\mathbf{U} \\ - k_n(\omega_\theta) f(\xi_n(\omega_\theta,\omega_\tau)) - \mathbf{U}^{n-1}(\omega_\theta,\omega_\tau)$$

for almost every $\omega_\theta \in \Omega_\theta$. Note that for h_n, we consider $k_n \colon \Omega \to [0,1]$ given by $k_n(\omega_\theta,\omega_\tau) = k_n(\omega_\theta)$. This mapping is also measurable with respect to $\mathcal{F}_n$. As in the proof of Lemma 2.1.5, we obtain that $\omega \mapsto \mathbf{U}^n(\omega)$ is $\mathcal{F}_n$-measurable and $\omega_\tau \mapsto \mathbf{U}^n(\omega_\theta,\omega_\tau)$ is $\mathcal{F}_n^\tau$-measurable for almost every $\omega_\theta \in \Omega_\theta$. □

Now that the existence of a solution to (2.20) is covered, let it be mentioned that the a priori bound from Lemma 2.1.7 holds true for this scheme if $\mathbf{E}_\theta$ is replaced by the expectation $\mathbf{E}$ on the probability space $(\Omega, \mathcal{F}, \mathbf{P})$. It remains to make sure that we have a bound for the terms containing f. To this end, we also mention a counterpart to Lemma 2.1.6 for the second randomization.

Lemma 2.2.3. *Let Assumption 2.2.1 be fulfilled and let $(X, \|\cdot\|_X)$ be a real Banach space. Then for $f \in L^q(0,T;X)$, $q \in [1,\infty)$, the bound*

$$\sum_{n=1}^N \mathbf{E}\big[k_n \|f(\xi_n)\|_X^q\big] \le 2\|f\|_{L^q(0,T;X)}^q$$

is fulfilled.

Proof. For the first summand, we use a substitution as in (A.2) and $k_1 = t_1^\theta = k\theta$ to obtain that

$$\begin{aligned}\mathbf{E}_\theta\big[k_1 \mathbf{E}_\tau\big[\|f(\xi_1)\|_X^q\big]\big] &= \int_0^1 ks \int_0^1 \|f(kst)\|_X^q \,\mathrm{d}t\,\mathrm{d}s \\ &= \frac{1}{k}\int_0^{t_1} s \int_0^1 \|f(st)\|_X^q \,\mathrm{d}t\,\mathrm{d}s \\ &= \frac{1}{k}\int_0^{t_1} \int_0^s \|f(t)\|_X^q \,\mathrm{d}t\,\mathrm{d}s \le \int_0^{t_1} \|f(t)\|_X^q \,\mathrm{d}t.\end{aligned}$$

For $n \in \{2,\dots,N\}$, we can argue similarly but notice that the step size k_n is equal to the maximal step size k. Then we see that

$$\begin{aligned}\mathbf{E}_\theta\big[k_n \mathbf{E}_\tau\big[\|f(\xi_n)\|_X^q\big]\big] &= k\int_0^1 \int_0^1 \|f(t_{n-2} + sk + kt)\|_X^q \,\mathrm{d}t\,\mathrm{d}s \\ &= \int_{t_{n-2}}^{t_{n-1}} \int_0^1 \|f(s+kt)\|_X^q \,\mathrm{d}t\,\mathrm{d}s \\ &= \frac{1}{k}\int_{t_{n-2}}^{t_{n-1}} \int_s^{s+k} \|f(t)\|_X^q \,\mathrm{d}t\,\mathrm{d}s \le \int_{t_{n-2}}^{t_n} \|f(t)\|_X^q \,\mathrm{d}t.\end{aligned}$$

Thus, altogether this proves the bound

$$\sum_{n=1}^N \mathbf{E}\big[k_n\|f(\xi_n)\|_X^q\big] \le \int_0^{t_1} \|f(t)\|_X^q \,\mathrm{d}t + \sum_{n=2}^N \int_{t_{n-2}}^{t_n} \|f(t)\|_X^q \,\mathrm{d}t \le 2\int_0^T \|f(t)\|_X^q \,\mathrm{d}t.$$

□

In order to use the higher-order regularity of a function in the error estimates, we consider the following two lemmas.

Lemma 2.2.4. *Let Assumption 2.2.1 be fulfilled and let* $(X, \|\cdot\|_X)$ *be a real Banach space. For* $\alpha \in (0,1)$ *and* $q \in [1,\infty)$*, let* $v \in W^{\alpha,q}(0,T;X)$ *be given. Then the bound*

$$\sum_{n=1}^{N} \mathbf{E}\big[k_n \|v(\xi_n) - v(t_n^\theta)\|_X^q\big] \le k^{q\alpha} |v|^q_{W^{\alpha,q}(0,T;X)}$$

is fulfilled.

Proof. Since we do not work on an equidistant grid, there are two cases that need to be considered separately. The distance between 0 and t_1^θ is given by the random value $k_1 = \theta k$. After a substitution as in (A.2), we see for the first summand using $\xi_1 = t_0^\theta + k_1\tau_1 = k\theta\tau_1$

$$\begin{aligned}
\mathbf{E}_\theta\big[k_1 \mathbf{E}_\tau\big[\|v(\xi_1) - v(t_1^\theta)\|_X^q\big]\big] &= \int_0^1 ks \int_0^1 \|v(kst) - v(ks)\|_X^q \,\mathrm{d}t\,\mathrm{d}s \\
&= \frac{1}{k}\int_0^{t_1} s \int_0^1 \|v(st) - v(s)\|_X^q \,\mathrm{d}t\,\mathrm{d}s \\
&= \frac{1}{k}\int_0^{t_1} \int_0^s \|v(t) - v(s)\|_X^q \,\mathrm{d}t\,\mathrm{d}s \\
&\le k^{q\alpha} \int_0^{t_1} \int_0^s \frac{\|v(t) - v(s)\|_X^q}{|t-s|^{q\alpha+1}} \,\mathrm{d}t\,\mathrm{d}s.
\end{aligned}$$

For $n \in \{2, \dots, N\}$, we use that the distance between t_n^θ and t_{n-1}^θ is always given by k. Thus, all further terms can be estimated by

$$\begin{aligned}
&\mathbf{E}_\theta\big[k_n \mathbf{E}_\tau\big[\|v(\xi_n) - v(t_n^\theta)\|_X^q\big]\big] \\
&= k\int_0^1 \int_0^1 \|v(t_{n-2} + ks + kt) - v(t_{n-1} + ks)\|_X^q \,\mathrm{d}t\,\mathrm{d}s \\
&= \int_{t_{n-1}}^{t_n} \int_0^1 \|v(s + kt - k) - v(s)\|_X^q \,\mathrm{d}t\,\mathrm{d}s \\
&= \frac{1}{k}\int_{t_{n-1}}^{t_n} \int_{s-k}^s \|v(t) - v(s)\|_X^q \,\mathrm{d}t\,\mathrm{d}s \\
&\le k^{q\alpha} \int_{t_{n-1}}^{t_n} \int_{s-k}^s \frac{\|v(t) - v(s)\|_X^q}{|t-s|^{q\alpha+1}} \,\mathrm{d}t\,\mathrm{d}s.
\end{aligned}$$

Combining these estimates, yields

$$\begin{aligned}
&\sum_{n=1}^{N} \mathbf{E}\big[k_n \|v(\xi_n) - v(t_n^\theta)\|_X^q\big] \\
&\le k^{q\alpha} \int_0^{t_1} \int_0^s \frac{\|v(t) - v(s)\|_X^q}{|t-s|^{q\alpha+1}} \,\mathrm{d}t\,\mathrm{d}s + k^{q\alpha} \int_{t_1}^{T} \int_{s-k}^s \frac{\|v(t) - v(s)\|_X^q}{|t-s|^{q\alpha+1}} \,\mathrm{d}t\,\mathrm{d}s \\
&\le k^{q\alpha} \int_0^{T} \int_0^T \frac{\|v(t) - v(s)\|_X^q}{|t-s|^{q\alpha+1}} \,\mathrm{d}t\,\mathrm{d}s = k^{q\alpha} |v|^q_{W^{\alpha,q}(0,T;X)}.
\end{aligned}$$

□

Lemma 2.2.5. *Let Assumption 2.2.1 be fulfilled and let $(X, \|\cdot\|_X)$ be a real Banach space. For $\alpha \in (0,1)$ and $q \in [1,\infty)$, let $v \in W^{\alpha,q}(0,T;X)$ be given. Then the bound*

$$\sum_{n=1}^{N} \mathbf{E}\Big[\int_{t_{n-1}^{\theta}}^{t_n^{\theta}} \|v(t)-v(\xi_n)\|_X^q \,\mathrm{d}t\Big] \leq 2k^{q\alpha} |v|^q_{W^{\alpha,q}(0,T;X)}$$

is fulfilled.

Proof. The proof is very similar to the proof of Lemma 2.2.4 but an additional integral appears in the estimates. Again, we consider the first summand separately. A substitution as in (A.2) and using the equality $\xi_1 = t_0^{\theta} + k_1\tau_1 = k\theta\tau_1$ yields

$$\begin{aligned}
\mathbf{E}_\theta\Big[\mathbf{E}_\tau\Big[\int_0^{t_1^\theta} \|v(t)-v(\xi_1)\|_X^q\,\mathrm{d}t\Big]\Big] &= \int_0^1\int_0^1\int_0^{ks} \|v(t)-v(ksr)\|_X^q \,\mathrm{d}t\,\mathrm{d}r\,\mathrm{d}s\\
&= \frac{1}{k}\int_0^{t_1}\int_0^1\int_0^s \|v(t)-v(sr)\|_X^q\,\mathrm{d}t\,\mathrm{d}r\,\mathrm{d}s\\
&= \frac{1}{k}\int_0^{t_1}\frac{1}{s}\int_0^s\int_0^s \|v(t)-v(r)\|_X^q\,\mathrm{d}t\,\mathrm{d}r\,\mathrm{d}s\\
&\leq \frac{1}{k}\int_0^{t_1} s^{q\alpha}\int_0^s\int_0^s \frac{\|v(t)-v(r)\|_X^q}{|t-r|^{q\alpha+1}}\,\mathrm{d}t\,\mathrm{d}r\,\mathrm{d}s\\
&\leq k^{q\alpha}\int_0^{t_1}\int_0^{t_1} \frac{\|v(t)-v(r)\|_X^q}{|t-r|^{q\alpha+1}}\,\mathrm{d}t\,\mathrm{d}r.
\end{aligned}$$

Similarly, for $n \in \{2,\dots,N\}$, we see that

$$\begin{aligned}
\mathbf{E}_\theta\Big[\mathbf{E}_\tau\Big[\int_{t_{n-1}^\theta}^{t_n^\theta} \|v(t)-v(\xi_n)\|_X^q\,\mathrm{d}t\Big]\Big] &= \int_0^1\int_0^1\int_{t_{n-2}+ks}^{t_{n-1}+ks} \|v(t)-v(t_{n-2}+ks+kr)\|_X^q\,\mathrm{d}t\,\mathrm{d}r\,\mathrm{d}s\\
&= \frac{1}{k}\int_{t_{n-2}}^{t_{n-1}}\int_0^1\int_s^{s+k} \|v(t)-v(s+kr)\|_X^q\,\mathrm{d}t\,\mathrm{d}r\,\mathrm{d}s\\
&= \frac{1}{k^2}\int_{t_{n-2}}^{t_{n-1}}\int_s^{s+k}\int_s^{s+k} \|v(t)-v(r)\|_X^q\,\mathrm{d}t\,\mathrm{d}r\,\mathrm{d}s\\
&\leq k^{q\alpha-1}\int_{t_{n-2}}^{t_{n-1}}\int_s^{s+k}\int_s^{s+k} \frac{\|v(t)-v(r)\|_X^q}{|t-r|^{q\alpha+1}}\,\mathrm{d}t\,\mathrm{d}r\,\mathrm{d}s\\
&\leq k^{q\alpha}\int_{t_{n-2}}^{t_n}\int_{t_{n-2}}^{t_n} \frac{\|v(t)-v(r)\|_X^q}{|t-r|^{q\alpha+1}}\,\mathrm{d}t\,\mathrm{d}r.
\end{aligned}$$

A combination of the estimates then shows

$$\begin{aligned}
&\sum_{n=1}^{N}\mathbf{E}\Big[\int_{t_{n-1}^\theta}^{t_n^\theta} \|v(t)-v(\xi_n)\|_X^q\,\mathrm{d}t\Big]\\
&\leq k^{q\alpha}\int_0^{t_1}\int_0^{t_1}\frac{\|v(t)-v(r)\|_X^q}{|t-r|^{q\alpha+1}}\,\mathrm{d}t\,\mathrm{d}r + k^{q\alpha}\sum_{n=2}^{N}\int_{t_{n-2}}^{t_n}\int_{t_{n-2}}^{t_n}\frac{\|v(t)-v(r)\|_X^q}{|t-r|^{q\alpha+1}}\,\mathrm{d}t\,\mathrm{d}r\\
&\leq 2k^{q\alpha}\int_0^T\int_0^T \frac{\|v(t)-v(r)\|_X^q}{|t-r|^{q\alpha+1}}\,\mathrm{d}t\,\mathrm{d}r = 2k^{q\alpha}|v|^q_{W^{\alpha,q}(0,T;X)}.
\end{aligned}$$

□

The previous lemma can also be proved with the help of Lemma 2.2.4. Using the triangular inequality yields

$$\begin{aligned}
&\Big(\sum_{n=1}^{N}\mathbf{E}\Big[\int_{t_{n-1}^{\theta}}^{t_n^{\theta}}\|v(t)-v(\xi_n)\|_X^q\,\mathrm{d}t\Big]\Big)^{\frac{1}{q}}\\
&\leq\Big(\sum_{n=1}^{N}\mathbf{E}\Big[\int_{t_{n-1}^{\theta}}^{t_n^{\theta}}\|v(t)-v(t_n^{\theta})\|_X^q\,\mathrm{d}t\Big]\Big)^{\frac{1}{q}}+\Big(\sum_{n=1}^{N}\mathbf{E}\big[k_n\|v(t_n^{\theta})-v(\xi_n)\|_X^q\big]\Big)^{\frac{1}{q}}\\
&=2\Big(\sum_{n=1}^{N}\mathbf{E}\big[k_n\|v(t_n^{\theta})-v(\xi_n)\|_X^q\big]\Big)^{\frac{1}{q}}\leq 2k^{\alpha}|v|_{W^{\alpha,q}(0,T;X)}.
\end{aligned}$$

For large q, this leads to a worse constant though.

Now, we are well prepared to prove the two main statements of this section. Both show error bounds for the expectation of the distance between the exact solution at a shifted grid point and the numerical approximation. The magnitude of the error depends on the regularity of the exact solution u. Here, we measure the temporal regularity of u and u' within a space of Sobolev–Slobodeckiĭ functions. In the first theorem, we assume that the temporal derivative of the exact solution is H valued. In the second theorem, we show that this can be weakened to values in V^*. To obtain the same error bound, we require more temporal regularity for the V^*-valued result though.

Further, we state two different bounded Lipschitz conditions in each theorem. In the first one, the operator is Lipschitz continuous on bounded sets in H. Alternatively, it is also possible to ask for a Lipschitz condition on a bounded set in V. As every bounded set in V is also bounded in H, but not necessarily the other way around, the second condition is more general. For the second Lipschitz condition, we additionally need that the solution u is an element of $L^{\infty}(0,T;V)$. Note that this assumption is fulfilled directly if the differentiability exponent of the Sobolev–Slobodeckiĭ space is large enough (cf. [102, Corollary 32]). In this section, we only make the assumption that the specific regularity is fulfilled without any further explanation. Some information for additional regularity and more concrete examples that fit this setting can be found in Section 1.2.

Theorem 2.2.6. *Let Assumptions 2.1.1 and 2.2.1 be fulfilled and let* $f\in L^q(0,T;V^*)$, $q=\frac{p}{p-1}$, *as well as the initial value* $u_0\in H$ *be given. Let the operator* $A(t)$, $t\in[0,T]$, *fulfill a bounded Lipschitz condition in the sense that for every* $R\in(0,\infty)$ *there exists* $L(R)\in[0,\infty)$ *such that*

$$\|A(t)v-A(t)w\|_{V^*}\leq L(R)\|v-w\|_V \tag{2.23}$$

is fulfilled for all $t\in[0,T]$ *and* $v,w\in V$ *with* $\|v\|_H,\|w\|_H\leq R$. *If the exact solution* u *is an element of* $L^{\infty}(0,T;V)$, *then it is sufficient that* (2.23) *is fulfilled for all* $v,w\in V$ *with* $\|v\|_V,\|w\|_V\leq R$. *Furthermore, let* $A(t)$, $t\in[0,T]$, *satisfy a* p*-monotonicity condition such that there exists* $\eta\in(0,\infty)$ *with*

$$\langle A(t)v-A(t)w,v-w\rangle_{V^*\times V}\geq\eta\|v-w\|_V^p \tag{2.24}$$

for all $v,w\in V$ *and* $t\in[0,T]$.

Let the exact solution u *be an element of* $W^{\alpha,q}(0,T;V)$ *for* $\alpha\in(0,1)$. *Further, let the temporal derivative* u' *of the exact solution be an element of* $W^{\gamma,2}(0,T;H)$ *for* $\gamma\in\big(0,\frac{1}{2}\big)$. *Then there exists* $C\in(0,\infty)$ *such that for every maximal step size* $k=\frac{T}{N}$, $N\in\mathbb{N}$, *the error* *estimate*

$$\max_{n\in\{1,\dots,N\}} \mathbf{E}\big[\|u(t_n^\theta) - \mathbf{U}^n\|_H^2\big] + \sum_{n=1}^{N} \mathbf{E}\big[k_n\|u(t_i^\theta) - \mathbf{U}^n\|_V^p\big] \le C(k^{2\gamma+1} + k^{q\alpha})$$

is fulfilled for the solution $(\mathbf{U}^n)_{n\in\{1,\dots,N\}}$ *to* (2.20).

Due to the boundedness condition from Assumption 2.1.1 (4), for the relevant examples that fulfill (2.23), we get $L(R) = L_0(\max\{\|v\|_H, \|w\|_H\})(1 + \|v\|_V^{p-2} + \|w\|_V^{p-2})$ with $L_0 \colon \mathbb{R} \to [0,\infty)$. It is also possible to include the case $\gamma = 0$ to the assumptions of the theorem in the sense that we assume $u' \in L^2(0,T;H)$. The proof remains similar. Instead of applying Lemma 2.2.5 to the terms containing the derivative u', it then becomes necessary to apply Lemma 2.2.3.

Proof of Theorem 2.2.6. In the following, let $i \in \{1,\dots,N\}$ be fixed. As the exact solution is an element of $\mathcal{W}^p(0,T)$ its derivative is in $L^q(0,T;V^*)$. Further, $u(t_i^\theta) - \mathbf{U}^i$ is an element of $L^p(\Omega;V)$ and we can write

$$\mathbf{E}\big[(u(t_i^\theta) - u(t_{i-1}^\theta), u(t_i^\theta) - \mathbf{U}^i)_H\big] = \mathbf{E}\Big[\int_{t_{i-1}^\theta}^{t_i^\theta} \langle u'(t), u(t_i^\theta) - \mathbf{U}^i\rangle_{V^*\times V}\,\mathrm{d}t\Big]. \tag{2.25}$$

Here, we technically do not have any point evaluation of u as we consider the expectation of the equality. The scheme (2.20) tested with $u(t_i^\theta) - \mathbf{U}^i$ yields the equality

$$\mathbf{E}\big[(\mathbf{U}^i - \mathbf{U}^{i-1}, u(t_i^\theta) - \mathbf{U}^i)_H\big] = \mathbf{E}\big[k_i\langle f(\xi_i) - A(\xi_i)\mathbf{U}^i, u(t_i^\theta) - \mathbf{U}^i\rangle_{V^*\times V}\big]. \tag{2.26}$$

Using the identity from Lemma A.1.4, the difference of the left-hand sides of (2.25) and (2.26) can be written as

$$\begin{aligned}
&\mathbf{E}\big[(u(t_i^\theta) - \mathbf{U}^i - u(t_{i-1}^\theta) + \mathbf{U}^{i-1}, u(t_i^\theta) - \mathbf{U}^i)_H\big]\\
&= \frac{1}{2}\big(\mathbf{E}\big[\|u(t_i^\theta) - \mathbf{U}^i\|_H^2\big] - \mathbf{E}\big[\|u(t_{i-1}^\theta) - \mathbf{U}^{i-1}\|_H^2\big] + \mathbf{E}\big[\|u(t_i^\theta) - \mathbf{U}^i - u(t_{i-1}^\theta) + \mathbf{U}^{i-1}\|_H^2\big]\big).
\end{aligned}$$

The difference of the right-hand sides of (2.25) and (2.26) can be rewritten by adding and subtracting terms containing $A(\xi_i)$. This allows us to use the structure of the operator $A(\xi_i)$ more efficiently. We then obtain

$$\begin{aligned}
&\mathbf{E}\Big[\int_{t_{i-1}^\theta}^{t_i^\theta} \langle u'(t), u(t_i^\theta) - \mathbf{U}^i\rangle_{V^*\times V}\,\mathrm{d}t\Big]\\
&\quad - \mathbf{E}\big[k_i\langle f(\xi_i) - A(\xi_i)\mathbf{U}^i, u(t_i^\theta) - \mathbf{U}^i\rangle_{V^*\times V}\big]\\
&= \mathbf{E}\Big[\int_{t_{i-1}^\theta}^{t_i^\theta} \langle u'(t), u(t_i^\theta) - \mathbf{U}^i\rangle_{V^*\times V}\,\mathrm{d}t\Big]\\
&\quad - \mathbf{E}\big[k_i\langle f(\xi_i) - A(\xi_i)u(\xi_i), u(t_i^\theta) - \mathbf{U}^i\rangle_{V^*\times V}\big]\\
&\quad - \mathbf{E}\big[k_i\langle A(\xi_i)u(\xi_i) - A(\xi_i)u(t_i^\theta), u(t_i^\theta) - \mathbf{U}^i\rangle_{V^*\times V}\big]\\
&\quad - \mathbf{E}\big[k_i\langle A(\xi_i)u(t_i^\theta) - A(\xi_i)\mathbf{U}^i, u(t_i^\theta) - \mathbf{U}^i\rangle_{V^*\times V}\big]\\
&= \mathbf{E}\Big[\int_{t_{i-1}^\theta}^{t_i^\theta} \langle u'(t) - u'(\xi_i), u(t_i^\theta) - \mathbf{U}^i\rangle_{V^*\times V}\,\mathrm{d}t\Big]\\
&\quad - \mathbf{E}\big[k_i\langle A(\xi_i)u(\xi_i) - A(\xi_i)u(t_i^\theta), u(t_i^\theta) - \mathbf{U}^i\rangle_{V^*\times V}\big]\\
&\quad - \mathbf{E}\big[k_i\langle A(\xi_i)u(t_i^\theta) - A(\xi_i)\mathbf{U}^i, u(t_i^\theta) - \mathbf{U}^i\rangle_{V^*\times V}\big]\\
&=: \Gamma_1 + \Gamma_2 + \Gamma_3,
\end{aligned}$$

where

$$\begin{aligned}\Gamma_1 &= \mathbf{E}\Big[\int_{t_{i-1}^\theta}^{t_i^\theta} \langle u'(t) - u'(\xi_i), u(t_i^\theta) - \mathbf{U}^i\rangle_{V^*\times V}\,\mathrm{d}t\Big],\\ \Gamma_2 &= -\mathbf{E}\big[k_i\langle A(\xi_i)u(\xi_i) - A(\xi_i)u(t_i^\theta), u(t_i^\theta) - \mathbf{U}^i\rangle_{V^*\times V}\big],\\ \Gamma_3 &= -\mathbf{E}\big[k_i\langle A(\xi_i)u(t_i^\theta) - A(\xi_i)\mathbf{U}^i, u(t_i^\theta) - \mathbf{U}^i\rangle_{V^*\times V}\big].\end{aligned} \tag{2.27}$$

Thus, we obtain that

$$\begin{aligned}&\mathbf{E}\big[\|u(t_i^\theta) - \mathbf{U}^i\|_H^2\big] - \mathbf{E}\big[\|u(t_{i-1}^\theta) - \mathbf{U}^{i-1}\|_H^2\big] + \mathbf{E}\big[\|u(t_i^\theta) - \mathbf{U}^i - u(t_{i-1}^\theta) + \mathbf{U}^{i-1}\|_H^2\big]\\ &= 2\Gamma_1 + 2\Gamma_2 + 2\Gamma_3.\end{aligned} \tag{2.28}$$

Since we added and subtracted the terms containing $A(\xi_i)u(\xi_i)$ and $A(\xi_i)u(t_i^\theta)$ above, we can now estimate Γ_1, Γ_2, and Γ_3 more easily. For Γ_1, we use the regularity of u'. In order to estimate Γ_2, we use the bounded Lipschitz condition and the regularity of u, while for Γ_3 we use the monotonicity of $A(\xi_i)$. Precisely, for Γ_1, we obtain that

$$\begin{aligned}\Gamma_1 &= \mathbf{E}\Big[\int_{t_{i-1}^\theta}^{t_i^\theta} \langle u'(t) - u'(\xi_i), u(t_i^\theta) - \mathbf{U}^i\rangle_{V^*\times V}\,\mathrm{d}t\Big]\\ &= \mathbf{E}\Big[\int_{t_{i-1}^\theta}^{t_i^\theta} \langle u'(t) - u'(\xi_i), u(t_i^\theta) - \mathbf{U}^i - u(t_{i-1}^\theta) + \mathbf{U}^{i-1}\rangle_{V^*\times V}\,\mathrm{d}t\Big] &(2.29)\\ &\quad + \mathbf{E}\Big[\int_{t_{i-1}^\theta}^{t_i^\theta} \langle u'(t) - u'(\xi_i), u(t_{i-1}^\theta) - \mathbf{U}^{i-1}\rangle_{V^*\times V}\,\mathrm{d}t\Big]. &(2.30)\end{aligned}$$

Using the Cauchy–Schwarz inequality as well as the weighted Young inequality, we find the estimate for (2.29)

$$\begin{aligned}&\mathbf{E}\Big[\int_{t_{i-1}^\theta}^{t_i^\theta} \langle u'(t) - u'(\xi_i), u(t_i^\theta) - \mathbf{U}^i - u(t_{i-1}^\theta) + \mathbf{U}^{i-1}\rangle_{V^*\times V}\,\mathrm{d}t\Big]\\ &\le \mathbf{E}\Big[\Big(\int_{t_{i-1}^\theta}^{t_i^\theta} \|u'(t) - u'(\xi_i)\|_H^2\,\mathrm{d}t\Big)^{\frac12}\big(k_i\|u(t_i^\theta) - \mathbf{U}^i - u(t_{i-1}^\theta) + \mathbf{U}^{i-1}\|_H^2\big)^{\frac12}\Big]\\ &\le \Big(\mathbf{E}\Big[k_i\int_{t_{i-1}^\theta}^{t_i^\theta} \|u'(t) - u'(\xi_i)\|_H^2\,\mathrm{d}t\Big]\Big)^{\frac12}\big(\mathbf{E}\big[\|u(t_i^\theta) - \mathbf{U}^i - u(t_{i-1}^\theta) + \mathbf{U}^{i-1}\|_H^2\big]\big)^{\frac12}\\ &\le \frac12\mathbf{E}\Big[k_i\int_{t_{i-1}^\theta}^{t_i^\theta} \|u'(t) - u'(\xi_i)\|_H^2\,\mathrm{d}t\Big] + \frac12\mathbf{E}\big[\|u(t_i^\theta) - \mathbf{U}^i - u(t_{i-1}^\theta) + \mathbf{U}^{i-1}\|_H^2\big].\end{aligned}$$

This structure is useful, as we can absorb the second summand in the last row using one of the terms on the left-hand side of (2.28). Further, we can write for (2.30)

$$\begin{aligned}&\mathbf{E}\Big[\int_{t_{i-1}^\theta}^{t_i^\theta} \langle u'(t) - u'(\xi_i), u(t_{i-1}^\theta) - \mathbf{U}^{i-1}\rangle_{V^*\times V}\,\mathrm{d}t\Big]\\ &= \mathbf{E}_\theta\Big[\mathbf{E}_\tau\Big[\int_{t_{i-1}^\theta}^{t_i^\theta} \langle u'(t), u(t_{i-1}^\theta) - \mathbf{U}^{i-1}\rangle_{V^*\times V}\,\mathrm{d}t\Big]\Big]\\ &\quad - \mathbf{E}_\theta\big[k_i\mathbf{E}_\tau\big[\langle u'(\xi_i), u(t_{i-1}^\theta) - \mathbf{U}^{i-1}\rangle_{V^*\times V}\big]\big].\end{aligned}$$

In the following, we denote the conditional expectation with respect to $\mathcal{F}_{i-1}^{\tau}$ by $\mathbf{E}_{\tau}\big[\,\cdot\,|\mathcal{F}_{i-1}^{\tau}\big]$, compare Appendix A.3. We notice that $\Omega_{\tau} \ni \omega_{\tau} \mapsto u(t_{i-1}^{\theta}(\omega_{\theta})) - \mathbf{U}^{i-1}(\omega_{\theta},\omega_{\tau})$ is measurable with respect to the σ-algebra $\mathcal{F}_{i-1}^{\tau}$ for almost every $\omega_{\theta} \in \Omega_{\theta}$, compare Lemma 2.2.2. Thus, we use the tower property for the conditional expectation to obtain

$$\begin{aligned}\mathbf{E}_{\tau}\big[\langle u'(\xi_i), u(t_{i-1}^{\theta}) - \mathbf{U}^{i-1}\rangle_{V^*\times V}\big] &= \mathbf{E}_{\tau}\big[\mathbf{E}_{\tau}\big[\langle u'(\xi_i), u(t_{i-1}^{\theta}) - \mathbf{U}^{i-1}\rangle_{V^*\times V}|\mathcal{F}_{i-1}^{\tau}\big]\big]\\ &= \mathbf{E}_{\tau}\big[\langle \mathbf{E}_{\tau}\big[u'(\xi_i)|\mathcal{F}_{i-1}^{\tau}\big], u(t_{i-1}^{\theta}) - \mathbf{U}^{i-1}\rangle_{V^*\times V}\big]\end{aligned}$$

almost surely in Ω_{θ}. As the generated σ-algebra $\sigma(\tau_i)$ is independent of $\mathcal{F}_{i-1}^{\tau}$, it follows that $\mathbf{E}_{\tau}\big[u'(\xi_i)|\mathcal{F}_{i-1}^{\tau}\big] = \mathbf{E}_{\tau}\big[u'(\xi_i)\big]$ almost surely in Ω_{θ}. Then we find that

$$\begin{aligned}\langle \mathbf{E}_{\tau}\big[u'(\xi_i)|\mathcal{F}_{i-1}^{\tau}\big], u(t_{i-1}^{\theta}) - \mathbf{U}^{i-1}\rangle_{V^*\times V} &= \langle \mathbf{E}_{\tau}\big[u'(\xi_i)\big], u(t_{i-1}^{\theta}) - \mathbf{U}^{i-1}\rangle_{V^*\times V}\\ &= \frac{1}{k_i}\int_{t_{i-1}^{\theta}}^{t_i^{\theta}} \langle u'(t), u(t_{i-1}^{\theta}) - \mathbf{U}^{i-1}\rangle_{V^*\times V}\,\mathrm{d}t\end{aligned}$$

almost surely in Ω and therefore, in particular,

$$\mathbf{E}\Big[\int_{t_{i-1}^{\theta}}^{t_i^{\theta}} \langle u'(t) - u'(\xi_i), u(t_{i-1}^{\theta}) - \mathbf{U}^{i-1}\rangle_{V^*\times V}\,\mathrm{d}t\Big] = 0.$$

Altogether, this proves a bound for Γ_1 that is given by

$$\Gamma_1 \le \frac{1}{2}\mathbf{E}\Big[k_i\int_{t_{i-1}^{\theta}}^{t_i^{\theta}} \|u'(t) - u'(\xi_i)\|_H^2\,\mathrm{d}t\Big] + \frac{1}{2}\mathbf{E}\big[\|u(t_i^{\theta}) - \mathbf{U}^i - u(t_{i-1}^{\theta}) + \mathbf{U}^{i-1}\|_H^2\big].$$

In order to estimate Γ_2, we apply the bounded Lipschitz condition of the operator $A(t)$, $t \in [0,T]$, from (2.23). Thus, we see that

$$\begin{aligned}\Gamma_2 &= -\mathbf{E}\big[k_i\langle A(\xi_i)u(\xi_i) - A(\xi_i)u(t_i^{\theta}), u(t_i^{\theta}) - \mathbf{U}^i\rangle_{V^*\times V}\big]\\ &\le L(R)\mathbf{E}\big[k_i\|u(\xi_i) - u(t_i^{\theta})\|_V\|u(t_i^{\theta}) - \mathbf{U}^i\|_V\big]\\ &\le c_1\mathbf{E}\big[k_i\|u(\xi_i) - u(t_i^{\theta})\|_V^q\big] + \frac{\eta}{2}\mathbf{E}\big[k_i\|u(t_i^{\theta}) - \mathbf{U}^i\|_V^p\big],\end{aligned}$$

for $c_1 = \frac{(p\eta)^{1-q}L(R)^q}{2^{1-q}q}$. Here, we can choose the parameter R for (2.23) as $R = \|u\|_{L^{\infty}(0,T;H)}$. Recall that a weak solution u of (2.1) is an element of $\mathcal{W}^p(0,T) \hookrightarrow C([0,T];H)$. Thus, this particular R is finite. If the solution fulfills $u \in L^{\infty}(0,T;V)$, we can choose $R = \|u\|_{L^{\infty}(0,T;V)}$ and it is sufficient that (2.23) is fulfilled for $v, w \in V$ with $\|v\|_V, \|w\|_V \le R$. Last, observe that

$$\Gamma_3 = -\mathbf{E}\big[k_i\langle A(\xi_i)u(t_i^{\theta}) - A(\xi_i)\mathbf{U}^i, u(t_i^{\theta}) - \mathbf{U}^i\rangle_{V^*\times V}\big] \le -\eta\mathbf{E}\big[k_i\|u(t_i^{\theta}) - \mathbf{U}^i\|_V^p\big]$$

is fulfilled due to the monotonicity condition from (2.24). After an insertion of these bounds into (2.28), we see that

$$\begin{aligned}&\mathbf{E}\big[\|u(t_i^{\theta}) - \mathbf{U}^i\|_H^2\big] - \mathbf{E}\big[\|u(t_{i-1}^{\theta}) - \mathbf{U}^{i-1}\|_H^2\big] + \mathbf{E}\big[\|u(t_i^{\theta}) - \mathbf{U}^i - u(t_{i-1}^{\theta}) + \mathbf{U}^{i-1}\|_H^2\big]\\ &= 2\Gamma_1 + 2\Gamma_2 + 2\Gamma_3\\ &\le \mathbf{E}\Big[k_i\int_{t_{i-1}^{\theta}}^{t_i^{\theta}} \|u'(t) - u'(\xi_i)\|_H^2\,\mathrm{d}t\Big] + \mathbf{E}\big[\|u(t_i^{\theta}) - \mathbf{U}^i - u(t_{i-1}^{\theta}) + \mathbf{U}^{i-1}\|_H^2\big]\\ &\quad + 2c_1\mathbf{E}\big[k_i\|u(\xi_i) - u(t_i^{\theta})\|_V^q\big] - \eta\mathbf{E}\big[k_i\|u(t_i^{\theta}) - \mathbf{U}^i\|_V^p\big].\end{aligned}$$

This implies, in particular,

$$\begin{aligned}
&\mathbf{E}\big[\|u(t_i^\theta)-\mathbf{U}^i\|_H^2\big]-\mathbf{E}\big[\|u(t_{i-1}^\theta)-\mathbf{U}^{i-1}\|_H^2\big]+\eta\mathbf{E}\big[k_i\|u(t_i^\theta)-\mathbf{U}^i\|_V^p\big]\\
&\le \mathbf{E}\Big[k_i\int_{t_{i-1}^\theta}^{t_i^\theta}\|u'(t)-u'(\xi_i)\|_H^2\,\mathrm{d}t\Big]+2c_1\mathbf{E}\big[k_i\|u(\xi_i)-u(t_i^\theta)\|_V^q\big].
\end{aligned}$$

Summing up the inequality from $i=1$ to $n\in\{1,\dots,N\}$, we can make use of the telescopic sum structure, the fact that $u(0)-\mathbf{U}^0=u(0)-u_0=0$, as well as Lemma 2.2.4 and Lemma 2.2.5 and obtain

$$\begin{aligned}
&\mathbf{E}\big[\|u(t_n^\theta)-\mathbf{U}^n\|_H^2\big]+\eta\sum_{i=1}^{n}\mathbf{E}\big[k_i\|u(t_i^\theta)-\mathbf{U}^i\|_V^p\big]\\
&\le k\sum_{i=1}^{N}\mathbf{E}\Big[\int_{t_{i-1}^\theta}^{t_i^\theta}\|u'(t)-u'(\xi_i)\|_H^2\,\mathrm{d}t\Big]+2c_1\sum_{i=1}^{N}\mathbf{E}\big[k_i\|u(\xi_i)-u(t_i^\theta)\|_V^q\big]\\
&\le 2k^{2\gamma+1}|u'|^2_{W^{\gamma,2}(0,T;H)}+2c_1k^{q\alpha}|u|^q_{W^{\alpha,q}(0,T;V)}.
\end{aligned}\tag{2.31}$$

□

The next theorem contains a comparable result, where the temporal regularity condition of u' changes while the spatial regularity condition can be relaxed to V^*.

Theorem 2.2.7. *Let Assumptions 2.1.1 and 2.2.1 be fulfilled and let $f\in L^q(0,T;V^*)$, $q=\frac{p}{p-1}$, as well as the initial value $u_0\in H$ be given. Furthermore, let the operator $A(t)$, $t\in[0,T]$, fulfill the bounded Lipschitz condition (2.23) and the p-monotonicity condition (2.24) as in Theorem 2.2.6.*

Let the exact solution u be an element of $W^{\alpha,q}(0,T;V)$ for $\alpha\in(0,1)$. Further, let the temporal derivative u' of the exact solution be an element of $W^{\gamma,q}(0,T;V^*)$ for $\gamma\in(0,1)$. Then there exists $C\in(0,\infty)$ such that for every maximal step size $k=\frac{T}{N}$, $N\in\mathbb{N}$, the error estimate

$$\max_{n\in\{1,\dots,N\}}\mathbf{E}\big[\|u(t_n^\theta)-\mathbf{U}^n\|_H^2\big]+\sum_{n=1}^{N}\mathbf{E}\big[k_n\|u(t_n^\theta)-\mathbf{U}^n\|_V^p\big]\le C(k^{q\gamma}+k^{q\alpha})$$

is fulfilled for the solution $(\mathbf{U}^n)_{n\in\{1,\dots,N\}}$ to (2.20).

Proof. Analogously to the proof of Theorem 2.2.6, we can write

$$\begin{aligned}
&\mathbf{E}\big[\|u(t_i^\theta)-\mathbf{U}^i\|_H^2\big]-\mathbf{E}\big[\|u(t_{i-1}^\theta)-\mathbf{U}^{i-1}\|_H^2\big]\\
&\quad+\mathbf{E}\big[\|u(t_i^\theta)-\mathbf{U}^i-u(t_{i-1}^\theta)+\mathbf{U}^{i-1}\|_H^2\big]\\
&=2\Gamma_1+2\Gamma_2+2\Gamma_3,
\end{aligned}\tag{2.32}$$

where Γ_1, Γ_2, and Γ_3 are given in (2.27). Again, we consider Γ_1, Γ_2, and Γ_3 separately, where we can use analogous bounds as in the proof of Theorem 2.2.6. First, we obtain

$$\begin{aligned}
\Gamma_2&=-\mathbf{E}\big[k_i\langle A(\xi_i)u(\xi_i)-A(\xi_i)u(t_i^\theta),u(t_i^\theta)-\mathbf{U}^i\rangle_{V^*\times V}\big]\\
&\le c_1\mathbf{E}\big[k_i\|u(\xi_i)-u(t_i^\theta)\|_V^q\big]+\frac{\eta}{4}\mathbf{E}\big[k_i\|u(t_i^\theta)-\mathbf{U}^i\|_V^p\big],
\end{aligned}$$

for $c_1=\frac{(p\eta)^{1-q}L(R)^q}{4^{1-q}q}$. Again, we choose the parameter R for (2.23) as $R=\|u\|_{L^\infty(0,T;H)}$. If the solution fulfills $u\in L^\infty(0,T;V)$, we choose $R=\|u\|_{L^\infty(0,T;V)}$ and only require that

(2.23) is fulfilled for $v, w \in V$ with $\|v\|_V, \|w\|_V \leq R$. For Γ_3, we apply the monotonicity condition (2.24) and find

$$\Gamma_3 \leq -\eta \mathbf{E}\big[k_i \|u(t_i^\theta) - \mathbf{U}^i\|_V^p\big].$$

When estimating Γ_1, we now have to use the different regularity assumption on u'. Here, we apply the weighted Young inequality to see that

$$\begin{aligned}\Gamma_1 &= \mathbf{E}\Big[\int_{t_{i-1}^\theta}^{t_i^\theta} \langle u'(t) - u'(\xi_i), u(t_i^\theta) - \mathbf{U}^i\rangle_{V^*\times V}\,\mathrm{d}t\Big]\\ &\leq \mathbf{E}\Big[\int_{t_{i-1}^\theta}^{t_i^\theta} \|u'(t) - u'(\xi_i)\|_{V^*}\|u(t_i^\theta) - \mathbf{U}^i\|_V\,\mathrm{d}t\Big]\\ &\leq c_2\mathbf{E}\Big[\int_{t_{i-1}^\theta}^{t_i^\theta} \|u'(t) - u'(\xi_i)\|_{V^*}^q\,\mathrm{d}t\Big] + \frac{\eta}{4}\mathbf{E}\big[k_i\|u(t_i^\theta) - \mathbf{U}^i\|_V^p\big],\end{aligned}$$

with the constant $c_2 = \frac{(p\eta)^{1-q}}{4^{1-q}q}$. Inserting the bounds in (2.32), shows that

$$\begin{aligned}&\mathbf{E}\big[\|u(t_i^\theta) - \mathbf{U}^i\|_H^2\big] - \mathbf{E}\big[\|u(t_{i-1}^\theta) - \mathbf{U}^{i-1}\|_H^2\big]\\ &\leq 2\Gamma_1 + 2\Gamma_2 + 2\Gamma_3\\ &\leq 2c_2\mathbf{E}\Big[\int_{t_{i-1}^\theta}^{t_i^\theta} \|u'(t) - u'(\xi_i)\|_{V^*}^q\,\mathrm{d}t\Big] + 2c_1\mathbf{E}\big[k_i\|u(\xi_i) - u(t_i^\theta)\|_V^q\big] - \eta\mathbf{E}\big[k_i\|u(t_i^\theta) - \mathbf{U}^i\|_V^p\big].\end{aligned}$$

The remainder of the proof can be done analogously to the end of the proof of Theorem 2.2.6. □

Note that in the proof of Theorem 2.2.7 we do not use the independence of $\{\tau_n\}_{n\in\{1,\dots,N\}}$. Thus, here it would also be possible to choose the same random variable $\tau_n = \tau$ for every $n \in \{1,\dots,N\}$, which is uniformly distributed in $[0,1]$.

When we compare the two results from the previous theorems, different error rates can be seen. If u' is an element of $W^{\gamma,2}(0,T;H)$ then the error can be smaller than for $u' \in W^{\gamma,q}(0,T;V^*)$ if u is smooth enough. Still, the first result is not necessarily stronger. In [90, Proposition 6.6], it is demonstrated how the temporal regularity decreases when the spatial regularity becomes higher. Thus, in practice, the two results should lead to comparable error estimates.

2.3 Example: A Problem of p-Laplacian Type

For a finite end time $T \in (0,\infty)$ and a bounded Lipschitz domain $\mathcal{D} \subset \mathbb{R}^d$, $d \in \mathbb{N}$, we regard

$$\begin{cases}\partial_t u(t,x) - \nabla\cdot\mathbf{a}(t,x,\nabla u(t,x)) = f(t,x), & (t,x) \in (0,T)\times\mathcal{D},\\ u(t,x) = 0, & (t,x) \in (0,T)\times\partial\mathcal{D},\\ u(0,x) = u_0(x), & x \in \mathcal{D}.\end{cases} \tag{2.33}$$

Here, the mapping $\mathbf{a}\colon [0,T]\times\overline{\mathcal{D}}\times\mathbb{R}^d \to \mathbb{R}^d$ fulfills the assumption below and $f\colon [0,T]\times\overline{\mathcal{D}} \to \mathbb{R}$ as well as $u_0\colon \overline{\mathcal{D}} \to \mathbb{R}$ will be specified later.

Assumption 2.3.1. *Let $p \in [2,\infty)$ be given and $q = \frac{p}{p-1}$. Let $\mathbf{a}\colon [0,T]\times\overline{\mathcal{D}}\times\mathbb{R}^d \to \mathbb{R}^d$ fulfill the following conditions:*

(1) The map $(t,x) \mapsto \mathbf{a}(t,x,z)$ *is measurable for every* $z \in \mathbb{R}^d$*, while* $z \mapsto \mathbf{a}(t,x,z)$ *is continuous for every* $t \in [0,T]$ *and almost every* $x \in \mathcal{D}$*.*

(2) The map $\mathbf{a}$ *fulfills a monotonicity condition in the sense that the inequality* $(\mathbf{a}(t,x,z) - \mathbf{a}(t,x,\tilde{z})) \cdot (z - \tilde{z}) \geq 0$ *is satisfied for every* $t \in [0,T]$*, almost every* $x \in \mathcal{D}$*, and every* $z, \tilde{z} \in \mathbb{R}^d$*.*

(3) The map $\mathbf{a}$ *fulfills a growth condition in the sense that there exist* $d_1 \in [0,\infty)$ *and a nonnegative function* $d_2 \in L^q(\mathcal{D})$ *such that for every* $t \in [0,T]$*, almost every* $x \in \mathcal{D}$*, and every* $z \in \mathbb{R}^d$ *the inequality* $|\mathbf{a}(t,x,z)| \leq d_1|z|^{p-1} + d_2(x)$ *is satisfied.*

(4) The map $\mathbf{a}$ *fulfills a coercivity condition in the sense that there exist* $d_3 \in (0,\infty)$ *and a nonnegative* $d_4 \in L^1(\mathcal{D})$ *such that for every* $t \in [0,T]$*, almost every* $x \in \mathcal{D}$*, as well as every* $z \in \mathbb{R}^d$ *the condition* $\mathbf{a}(t,x,z) \cdot z \geq d_3|z|^p - d_4(x)$ *is satisfied.*

Assumption 2.3.2. *Let Assumption 2.3.1 be fulfilled. Additionally, there exists* $d_5 \in (0,\infty)$ *such that*

$$(\mathbf{a}(t,x,z) - \mathbf{a}(t,x,\tilde{z})) \cdot (z - \tilde{z}) \geq d_5|z - \tilde{z}|^p$$

is satisfied for every $t \in [0,T]$*, almost every* $x \in \mathcal{D}$*, and every* $z, \tilde{z} \in \mathbb{R}^d$*.*

Assumption 2.3.3. *Let Assumption 2.3.1 be fulfilled. Additionally, there exists* $d_6 \in [0,\infty)$ *such that*

$$|\mathbf{a}(t,x,z) - \mathbf{a}(t,x,\tilde{z})| \leq d_6\big(1 + \max\{|z|^{p-2}, |\tilde{z}|^{p-2}\}\big)|z - \tilde{z}| \tag{2.34}$$

is satisfied for every $t \in [0,T]$*, almost every* $x \in \mathcal{D}$*, and every* $z, \tilde{z} \in \mathbb{R}^d$*.*

A prototype example for the function $\mathbf{a}$ is given by $\mathbf{a}(t,x,z) = \mathbf{a}(z) = |z|^{p-2}z$. Then (2.33) is the p-Laplace equation. It is easy to see that $\mathbf{a}$ fulfills Assumption 2.3.1. In [28, Chapter I, Lemma 4.4] it is proved that this $\mathbf{a}$ fulfills Assumption 2.3.2. In order to see that the function fulfills Assumption 2.3.3, we notice that for $z \in \mathbb{R}^d$ with $z \neq 0$

$$\partial_i \mathbf{a}(z) = \frac{p-2}{2}\Big(\sum_{j=1}^{d} z_j^2\Big)^{\frac{p-4}{2}} 2z_i z + |z|^{p-2}e_i = (p-2)|z|^{p-4}z_i z + |z|^{p-2}e_i$$

is fulfilled for $i \in \{1,\dots,d\}$ and the i-th unit vector e_i in $\mathbb{R}^d$. Moreover, we have $\partial_i \mathbf{a}(0) = 0$ for every $i \in \{1,\dots,d\}$. Then the Jacobian matrix fulfills

$$|\nabla \mathbf{a}(z)| = \big|(p-2)|z|^{p-4}zz^T + |z|^{p-2}I\big| \leq (p-1)|z|^{p-2}$$

for $z \in \mathbb{R}^d$. An application of the mean value theorem then shows that (2.34) is fulfilled.

In order to formulate the problem (2.33) in a weak formulation, we consider the spaces $V = W_0^{1,p}(\mathcal{D})$ and $H = L^2(\mathcal{D})$, where $p \in [2,\infty)$ is chosen as in Assumption 2.3.1. We equip the spaces with the norms introduced in the notation section in the introduction. Then we assume that for $f\colon [0,T]\times\overline{\mathcal{D}} \to \mathbb{R}$ the abstract function $[f(t)](x) = f(t,x)$, $(t,x) \in (0,T)\times\mathcal{D}$, is an element of $L^q(0,T;V^*)$ and $u_0 \in H$. Further, the operator $A(t)\colon V \to V^*$ is given by

$$\langle A(t)v, w\rangle = \int_{\mathcal{D}} \mathbf{a}(t,\cdot,\nabla v) \cdot \nabla w \,\mathrm{d}x$$

for $t \in [0,T]$ and $v, w \in V$. Then we consider the variational formulation of (2.33) given by

$$\begin{cases} u' + Au = f & \text{in } L^q(0,T;V^*), \\ u(0) = u_0 & \text{in } H. \end{cases}$$

Theorem 2.3.4. *Let Assumption 2.3.1 be fulfilled. Let $f \in L^q(0,T;V^*)$ and $u_0 \in H$ be given. Further, let $(N_\ell)_{\ell\in\mathbb{N}}$ be a sequence of natural numbers with $N_\ell \to \infty$ as $\ell \to \infty$, $k = \frac{T}{N_\ell}$, $t_n = nk$, $n \in \{0,\dots,N_\ell\}$, and consider the corresponding randomly shifted grid introduced in Assumption 2.1.3. Then the backward Euler scheme*

$$\begin{cases} \frac{1}{k}\big(\mathbf{U}^n - \mathbf{U}^{n-1}\big) + A(t_n^\theta)\mathbf{U}^n = f(t_n^\theta) & \text{in } L^q(\Omega_\theta;V^*), \quad n \in \{1,\dots,N_\ell\},\\ \mathbf{U}^0 = u_0 & \text{in } H \end{cases}$$

admits a unique solution $(\mathbf{U}^n)_{n\in\{1,\dots,N_\ell\}}$ in $L^p(\Omega_\theta;V)$. All the convergence results from Theorem 2.1.11 and Theorem 2.1.12 hold true. In particular, the sequences of piecewise constant and piecewise linear prolongations of $(\mathbf{U}^n)_{n\in\{1,\dots,N_\ell\}}$ converge to the weak solution u of (2.33) pointwise strongly in $L^2(\Omega_\theta;H)$ as $k \to 0$.

If Assumption 2.3.2 is satisfied additionally, then the sequence of piecewise constant prolongations converges to u strongly in $L^p(0,T;L^p(\Omega_\theta;V))$ as $k \to 0$.

Proof. In order to apply Theorem 2.1.11 and Theorem 2.1.12 from Section 2.1, it only remains to verify that $A(t)$, $t \in [0,T]$, fulfills Assumption 2.1.1. To this end, let $v, w \in V$ be given. Then we see that

$$\begin{aligned} \langle A(t)v, w\rangle_{V^*\times V} &= \int_{\mathcal{D}} \mathbf{a}(t,\cdot,\nabla v)\cdot\nabla w \,\mathrm{d}x \\ &\le \int_{\mathcal{D}} \big(d_1|\nabla v|^{p-1} + d_2\big)|\nabla w| \,\mathrm{d}x \\ &\le \max\big\{d_1, \|d_2\|_{L^q(\mathcal{D})}\big\}\big(1 + \|v\|_V^{p-1}\big)\|w\|_V, \end{aligned} \tag{2.35}$$

which proves both that $A(t)$, $t \in [0,T]$, is well-defined and that the boundedness condition from Assumption 2.1.1 (4) is fulfilled.

Since $t \mapsto \mathbf{a}(t,x,z)$ is measurable for almost every $x \in \mathcal{D}$ and every $z \in \mathbb{R}^d$, there exists a sequence $(\mathbf{a}_i)_{i\in\mathbb{N}}$ of functions $\mathbf{a}_i\colon [0,T]\times\overline{\mathcal{D}}\times\mathbb{R}^d \to \mathbb{R}^d$, $i \in \mathbb{N}$, that are simple with respect to the first argument such that $\mathbf{a}_i(t,x,z) \to \mathbf{a}(t,x,z)$ in $\mathbb{R}^d$ as $i \to \infty$ and $|\mathbf{a}_i(t,x,z)| \le |\mathbf{a}(t,x,z)|$, $i \in \mathbb{N}$, for almost every $(t,x) \in (0,T)\times\mathcal{D}$ and every $z \in \mathbb{R}^d$. Then $A_i(t)\colon V \to V^*$ given by

$$\langle A_i(t)v, w\rangle_{V^*\times V} = \int_{\mathcal{D}} \mathbf{a}_i(t,\cdot,\nabla v)\cdot\nabla w \,\mathrm{d}x, \quad v, w \in V$$

is a simple function with respect to $t \in (0,T)$. Using a similar bound as in (2.35), it follows that $\big(\mathbf{a}(t,\cdot,\nabla v) - \mathbf{a}_i(t,\cdot,\nabla v)\big)\cdot\nabla w$ is bounded by a function that is integrable on $\mathcal{D}$. We can apply Lebesgue's dominated convergence theorem to obtain that

$$\lim_{i\to\infty} \langle A(t)v - A_i(t)v, w\rangle_{V^*\times V} = \int_{\mathcal{D}} \lim_{i\to\infty} \big(\mathbf{a}(t,\cdot,\nabla v) - \mathbf{a}_i(t,\cdot,\nabla v)\big)\cdot\nabla w \,\mathrm{d}x = 0$$

for every $v, w \in V$ and almost every $t \in (0,T)$. This implies that $t \mapsto A(t)v$ is weakly measurable since V^* is reflexive. As V^* is also separable, the mapping is Bochner measurable.

In order to prove that $A(t)\colon V \to V^*$, $t \in [0,T]$, is radially continuous, let $(s_i)_{i\in\mathbb{N}}$ be a convergent sequence in $[0,1]$ with the limit $s \in [0,1]$. Using the fact that (2.35) is finite, it follows that $\mathbf{a}(t,\cdot,\nabla v + s_i\nabla w)\cdot\nabla w$ is bounded by an integrable function on $\mathcal{D}$ for every $v, w \in V$. Then we can apply Lebesgue's dominated convergence theorem and it follows that

$$\begin{aligned}\lim_{i\to\infty}\langle A(t)(v+s_iw),w\rangle_{V^*\times V} &= \lim_{i\to\infty}\int_{\mathcal{D}}\mathbf{a}(t,\cdot,\nabla v+s_i\nabla w)\cdot\nabla w\,\mathrm{d}x\\ &= \int_{\mathcal{D}}\lim_{i\to\infty}\mathbf{a}(t,\cdot,\nabla v+s_i\nabla w)\cdot\nabla w\,\mathrm{d}x\\ &= \int_{\mathcal{D}}\mathbf{a}(t,\cdot,\nabla v+s\nabla w)\cdot\nabla w\,\mathrm{d}x\end{aligned}$$

for every $v,w\in V$ and $t\in[0,T]$ due to Assumption 2.3.1 (1).

The monotonicity condition for $A(t)$, $t\in[0,T]$, is a direct consequence of Assumption 2.3.1 (2). This can be seen as

$$\langle A(t)v-A(t)w,v-w\rangle_{V^*\times V}=\int_{\mathcal{D}}(\mathbf{a}(t,\cdot,\nabla v)-\mathbf{a}(t,\cdot,\nabla w))\cdot(\nabla v-\nabla w)\,\mathrm{d}x\geq 0$$

is fulfilled for every $v,w\in V$ and $t\in[0,T]$. Analogously, the condition from Assumption 2.3.2 implies that

$$\langle A(t)v-A(t)w,v-w\rangle_{V^*\times V}\geq d_5\int_{\mathcal{D}}|\nabla v-\nabla w|^p\,\mathrm{d}x=d_5\|v-w\|_V^p$$

for every $v,w\in V$ and $t\in[0,T]$. So we see that (2.18) is fulfilled.

It remains to verify the coercivity condition from Assumption 2.1.1 (5). Here, we apply Assumption 2.3.1 (4) to see that

$$\langle A(t)v,v\rangle_{V^*\times V}\geq\int_{\mathcal{D}}\big(d_3|\nabla v|^p-d_4\big)\,\mathrm{d}x=d_3\|v\|_V^p-\|d_4\|_{L^1(\mathcal{D})}$$

for every $v\in V$ and $t\in[0,T]$. Therefore, as the operator $A(t)$, $t\in[0,T]$, fulfills all the necessary conditions, we can apply Theorem 2.1.11 and Theorem 2.1.12 to finish the proof. □

To prove explicit error bounds, we make an additional regularity assumption on the solution of (2.33). We do not discuss this condition here. In Section 1.2, more details and suitable examples can be found.

Theorem 2.3.5. *Let Assumptions 2.3.1, 2.3.2, and 2.3.3 be fulfilled and let $f\in L^q(0,T;V^*)$ and $u_0\in H$ be given. For $\alpha\in(0,1)$ and $\gamma\in(0,1)$, assume that the exact solution u is an element of $W^{\alpha,2}(0,T;V)$ for $p=2$ and of $L^\infty(0,T;V)\cap W^{\alpha,q}(0,T;V)$ for $p\in(2,\infty)$ while its temporal derivative u' belongs to $W^{\gamma,2}(0,T;H)$. For every $N\in\mathbb{N}$, $k=\frac{T}{N}$, and the corresponding random values $(\xi_n)_{n\in\{1,\dots,N\}}$ introduced in Assumption 2.2.1, the scheme*

$$\begin{cases}\frac{1}{k_n}\big(\mathbf{U}^n-\mathbf{U}^{n-1}\big)+A(\xi_n)\mathbf{U}^n=f(\xi_n) & \text{in } L^q(\Omega;V^*),\quad n\in\{1,\dots,N\},\\ \mathbf{U}^0=u_0 & \text{in } H\end{cases}$$

admits a unique solution $(\mathbf{U}^n)_{n\in\{1,\dots,N\}}$ in $L^p(\Omega;V)$. Then there exists $C\in(0,\infty)$ such that

$$\max_{n\in\{1,\dots,N\}}\mathbf{E}\big[\|u(t_n^\theta)-\mathbf{U}^n\|_H^2\big]+\sum_{n=1}^N\mathbf{E}\big[k_n\|u(t_n^\theta)-\mathbf{U}^n\|_V^p\big]\leq C(k^{q\gamma+1}+k^{q\alpha})$$

is fulfilled.

Proof. The proof can be carried out by applying Theorem 2.2.6. To this end, it only remains to verify that $A(t)$, $t \in [0,T]$, fulfills (2.23). The proof of the other conditions for $A(t)$, $t \in [0,T]$, has already been done for Theorem 2.3.4. To this end, we use the bounded Lipschitz condition from Assumption 2.3.3. For $p = 2$, (2.34) is a global Lipschitz condition. We obtain that

$$\begin{aligned}\langle A(t)v_1 - A(t)v_2, w\rangle_{V^*\times V} &= \int_{\mathcal{D}} \big(\mathbf{a}(t,\cdot,\nabla v_1) - \mathbf{a}(t,\cdot,\nabla v_2)\big)\cdot \nabla w \,\mathrm{d}x\\ &\le 2d_6 \int_{\mathcal{D}} |\nabla v_1 - \nabla v_2||\nabla w| \,\mathrm{d}x\\ &\le 2d_6 \|\nabla v_1 - \nabla v_2\|_{L^2(\mathcal{D})^d}\|\nabla w\|_{L^2(\mathcal{D})^d} = 2d_6\|v_1 - v_2\|_V\|w\|_V\end{aligned}$$

for every $v_1, v_2, w \in V$ and $t \in [0,T]$. This proves the global Lipschitz condition

$$\|A(t)v_1 - A(t)v_2\|_{V^*} \le 2d_6\|v_1 - v_2\|_V$$

for every $v_1, v_2 \in V$ and $t \in [0,T]$. In the case $p \in (2,\infty)$, we can argue in a similar fashion but have to handle the Lipschitz constant that depends on the input $v_1, v_2 \in V$. Here, we use Lemma A.1.3 to obtain that

$$\begin{aligned}&\langle A(t)v_1 - A(t)v_2, w\rangle_{V^*\times V}\\ &= \int_{\mathcal{D}} \big(\mathbf{a}(t,\cdot,\nabla v_1) - \mathbf{a}(t,\cdot,\nabla v_2)\big)\cdot \nabla w \,\mathrm{d}x\\ &\le d_6 \int_{\mathcal{D}} |\nabla v_1 - \nabla v_2||\nabla w| \,\mathrm{d}x + d_6 \int_{\mathcal{D}} \max\big\{|\nabla v_1|^{p-2}, |\nabla v_2|^{p-2}\big\}|\nabla v_1 - \nabla v_2||\nabla w| \,\mathrm{d}x\\ &\le d_6 \|1\|_{L^{\frac{p}{p-2}}(\mathcal{D})}\|\nabla v_1 - \nabla v_2\|_{L^p(\mathcal{D})^d}\|\nabla w\|_{L^p(\mathcal{D})^d}\\ &\quad + d_6\Big(\int_{\mathcal{D}} \max\big\{|\nabla v_1|^p, |\nabla v_2|^p\big\} \,\mathrm{d}x\Big)^{\frac{p-2}{p}}\|\nabla v_1 - \nabla v_2\|_{L^p(\mathcal{D})^d}\|\nabla w\|_{L^p(\mathcal{D})^d}\end{aligned}$$

for every $v_1, v_2, w \in V$ and $t \in [0,T]$. Since $\frac{p-2}{p} \in (0,1)$ for $p \in (2,\infty)$ we get that

$$\begin{aligned}\Big(\int_{\mathcal{D}} \max\big\{|\nabla v_1|^p, |\nabla v_2|^p\big\} \,\mathrm{d}x\Big)^{\frac{p-2}{p}} &\le \Big(\int_{\mathcal{D}} |\nabla v_1|^p \,\mathrm{d}x + \int_{\mathcal{D}} |\nabla v_2|^p \,\mathrm{d}x\Big)^{\frac{p-2}{p}}\\ &\le \Big(\int_{\mathcal{D}} |\nabla v_1|^p \,\mathrm{d}x\Big)^{\frac{p-2}{p}} + \Big(\int_{\mathcal{D}} |\nabla v_2|^p \,\mathrm{d}x\Big)^{\frac{p-2}{p}}\\ &= \|\nabla v_1\|_{L^p(\mathcal{D})^d}^{p-2} + \|\nabla v_2\|_{L^p(\mathcal{D})^d}^{p-2}\\ &\le 2\max\big\{\|\nabla v_1\|_{L^p(\mathcal{D})^d}^{p-2}, \|\nabla v_2\|_{L^p(\mathcal{D})^d}^{p-2}\big\}.\end{aligned}$$

Thus, for $R \in (0,\infty)$ and all $v_1, v_2 \in V$ with $\|v_1\|_V, \|v_2\|_V \le R$, we obtain the bound

$$\begin{aligned}\|A(t)v_1 - A(t)v_2\|_{V^*} &\le d_6\Big(\|1\|_{L^{\frac{p}{p-2}}(\mathcal{D})} + 2\max\big\{\|v_1\|_V^{p-2}, \|v_2\|_V^{p-2}\big\}\Big)\|v_1 - v_2\|_V\\ &=: L(R)\|v_1 - v_2\|_V,\end{aligned}$$

which proves the weaker form of (2.23) that is needed if $u \in L^\infty(0,T;V)$. ☐

Theorem 2.3.6. *Let Assumptions 2.3.1, 2.3.2, and 2.3.3 be fulfilled and let* $f \in L^q(0,T;V^*)$ *and* $u_0 \in H$ *be given. For* $\alpha \in (0,1)$ *and* $\gamma \in (0,1)$*, assume that the exact solution* u *is an element of* $W^{\alpha,2}(0,T;V)$ *for* $p = 2$ *and of* $L^\infty(0,T;V) \cap W^{\alpha,q}(0,T;V)$ *for* $p \in (2,\infty)$

while its temporal derivative u' belongs to $W^{\gamma,q}(0,T;V^*)$. For every $N \in \mathbb{N}$, $k = \frac{T}{N}$, and the corresponding random values $(\xi_n)_{n\in\{1,\dots,N\}}$ introduced in Assumption 2.2.1, the scheme

$$\begin{cases} \frac{1}{k_n}\big(\mathbf{U}^n - \mathbf{U}^{n-1}\big) + A(\xi_n)\mathbf{U}^n = f(\xi_n) & \text{in } L^q(\Omega;V^*), \quad n \in \{1,\dots,N\}, \\ \mathbf{U}^0 = u_0 & \text{in } H \end{cases}$$

admits a unique solution $(\mathbf{U}^n)_{n\in\{1,\dots,N\}}$ in $L^p(\Omega;V)$. Then there exists $C \in (0,\infty)$ such that

$$\max_{n\in\{1,\dots,N\}} \mathbf{E}\big[\|u(t_n^\theta) - \mathbf{U}^n\|_H^2\big] + \sum_{n=1}^{N} \mathbf{E}\big[k_n\|u(t_n^\theta) - \mathbf{U}^n\|_V^p\big] \le C(k^{q\gamma} + k^{q\alpha})$$

is fulfilled.

Proof. The proof can be done analogously to the proof of Theorem 2.3.6, where we now use Theorem 2.2.7. □

Chapter 3

An Operator Splitting Based Scheme for Nonlinear, Nonautonomous Evolution Equations

An operator splitting offers the opportunity to obtain easier solvable subproblems, which in some settings can even be solved in parallel. Due to modern hardware structures, methods that are based on parallelization become more and more useful for a faster computation. To this end, we will present a numerical scheme based on an operator splitting in order to discretize a nonlinear, nonautonomous evolution equation on a finite time horizon. Precisely, for $T \in (0,\infty)$, we consider

$$\begin{cases} u'(t) + A(t)u(t) + B(t)u(t) = f(t) & \text{in } V^*, \quad \text{for almost all } t \in (0,T), \\ u(0) = u_0 & \text{in } H \end{cases} \tag{3.1}$$

for a Gelfand triple $V \overset{d}{\hookrightarrow} H \cong H^* \overset{d}{\hookrightarrow} V^*$ as well as families $\{A(t)\}_{t\in[0,T]}$ and $\{B(t)\}_{t\in[0,T]}$ of operators $A(t)\colon V \to V^*$ and $B(t)\colon H \to H$. Here, $A(t)$, $t \in [0,T]$, is an operator of monotone type and $B(t)$, $t \in [0,T]$, is Lipschitz continuous. Further, we allow for an integrable source term $f\colon [0,T] \to V^*$ and an initial value $u_0 \in H$. Standard examples for our problem class contain p-Laplacian type and porous media type problems with lower order perturbations. As the solutions of such nonlinear equations usually lack global higher-order temporal and spatial regularity, we concentrate on a lower-order scheme.

As in the previous chapter, our starting point is to look at the well-known backward Euler scheme. For $N \in \mathbb{N}$, we consider an equidistant grid $0 = t_0 < t_1 < \cdots < t_N = T$ for points $t_n = nk$, $n \in \{0,\dots,N\}$, as well as a step size $k = \frac{T}{N}$. Then the solution to the recursion

$$\frac{\mathbf{U}^n - \mathbf{U}^{n-1}}{k} + \mathbf{A}^n\mathbf{U}^n + \mathbf{B}^n\mathbf{U}^n = \mathbf{f}^n \quad \text{in } V^*, \quad n \in \{1,\dots,N\},$$

with $\mathbf{U}^0 = u_0^k$ in H can be used to obtain an approximation $\mathbf{U}^n \approx u(t_n)$, $n \in \{1,\dots,N\}$. Here, u_0^k, $(\mathbf{f}^n)_{n\in\{1,\dots,N\}}$, $(\mathbf{A}^n)_{n\in\{1,\dots,N\}}$, as well as $(\mathbf{B}^n)_{n\in\{1,\dots,N\}}$ are approximations of the data that we assume to be known. This scheme is formally of first order and we want to present modifications that preserve the order but lead to several subproblems, which can potentially be solved more efficiently. For the first modification, we notice that from a numerical point of view, on first sight, it could seem like a good option to exchange $\mathbf{A}^n\mathbf{U}^n$ and $\mathbf{B}^n\mathbf{U}^n$ by $\mathbf{A}^n\mathbf{U}^{n-1}$ and $\mathbf{B}^n\mathbf{U}^{n-1}$, respectively. This leads to a similar scheme as the forward Euler method. In terms of an implementable full discretization, it has the advantage that no nonlinear, implicit equation has to be solved. But note that if an operator $\mathbf{A}^n \colon V \to V^*$ is interpreted as an H valued operator in terms of the restriction $\mathbf{A}_F^n$ given by $\mathbf{A}_F^n \colon \operatorname{dom}(\mathbf{A}_F^n) \subset H \to H$, $\mathbf{A}_F^n v = \mathbf{A}^n v$ with $\operatorname{dom}(\mathbf{A}_F^n) = \{v \in V : \mathbf{A}_F^n v \in H\}$, it can be unbounded. Thus, in a fully discretized scheme, it can become necessary to choose a temporal discretization parameter that depends on the spatial discretization parameter. This coupling is highly undesirable in applications. Due to the monotonicity of the operator $A(t)$, $t \in [0,T]$, the backward Euler method has better stability properties as the underlying physical system is dissipative. Here, no coupling of the discretization parameters appears. Therefore, we do not exchange $\mathbf{A}^n\mathbf{U}^n$ by $\mathbf{A}^n\mathbf{U}^{n-1}$. As the operator $\mathbf{B}^n$ is bounded in H, these problems do not appear. Hence, we use $\mathbf{B}^n\mathbf{U}^{n-1}$ and propose the implicit-explicit scheme

$$\frac{\mathbf{U}^n - \mathbf{U}^{n-1}}{k} + \mathbf{A}^n\mathbf{U}^n + \mathbf{B}^n\mathbf{U}^{n-1} = \mathbf{f}^n \quad \text{in } V^*, \quad n \in \{1, \dots, N\},$$

with $\mathbf{U}^0 = u_0^k$ in H. Altogether, this yields the equations

$$(I + k\mathbf{A}^n)\mathbf{U}^n = k\mathbf{f}^n + \mathbf{U}_0^n \quad \text{in } V^* \quad \text{with} \quad \mathbf{U}_0^n = (I - k\mathbf{B}^n)\mathbf{U}^{n-1} \quad \text{in } H$$

for $n \in \{1, \dots, N\}$ and $\mathbf{U}^0 = u_0^k$ in H. Next, we decompose the operator $A(t)$, $t \in [0,T]$, and the source term f. To this end, we assume that there exist $M \in \mathbb{N}$ families $\{A_m(t)\}_{t\in[0,T]}$ of operators $A_m(t) \colon V_m \to V_m^*$ and functions $f_m \colon [0,T] \to V_m^*$ for $m \in \{1, \dots, M\}$, where $V_m \subset H$ and $\bigcap_{m=1}^M V_m = V$. These operators and functions have to fulfill the sum property

$$\sum_{m=1}^M A_m(t)v = A(t)v, \quad \sum_{m=1}^M f_m(t) = f(t) \quad \text{in } V^*$$

for every $v \in V$ and almost every $t \in (0,T)$. For $m \in \{1, \dots, M\}$ and approximations $(\mathbf{f}_m^n)_{n\in\{1,\dots,N\}}$, $(\mathbf{A}_m^n)_{n\in\{1,\dots,N\}}$, which also fulfill a corresponding sum property, we use a product splitting scheme to approximate a backward Euler step containing $\mathbf{A}^n$ and $\mathbf{f}^n$. The idea of such a scheme is that for real numbers a_m, $m \in \{1, \dots, M\}$, some basic calculations show that

$$\begin{aligned}
&\Bigl(1 + k\sum_{m=1}^M a_m\Bigr)^{-1} - \prod_{m=1}^M \bigl(1 + ka_m\bigr)^{-1} \\
&= k^2\Bigl(1 + k\sum_{m=1}^M a_m\Bigr)^{-1}\Bigl(\sum_{\substack{j,m=1,\\ j<m}}^M a_j a_m\Bigr)\prod_{m=1}^M \bigl(1 + ka_m\bigr)^{-1}
\end{aligned}$$

is fulfilled. This suggests that the splitting error, i.e., the difference of one Euler step $\bigl(I + k\mathbf{A}^n\bigr)^{-1} = \Bigl(I + k\sum_{m=1}^M \mathbf{A}_m^n\Bigr)^{-1}$ and the product $\prod_{m=1}^M \bigl(I + k\mathbf{A}_m^n\bigr)^{-1}$, is sufficiently small. Altogether, this gives rise to consider the following system of equations

$$\mathbf{U}_0^n = (I - k\mathbf{B}^n)\mathbf{U}^{n-1} \quad \text{in } H$$

and

$$(I + k\mathbf{A}_m^n)\mathbf{U}_m^n = k\mathbf{f}_m^n + \mathbf{U}_{m-1}^n \quad \text{in } V_m^*, \quad m \in \{1, \dots, M\},$$

for $n \in \{1, \dots, N\}$ with

$$\mathbf{U}^n = \mathbf{U}_M^n \quad \text{in } H, \quad n \in \{1, \dots, N\}, \quad \text{and} \quad \mathbf{U}^0 = u_0^k \quad \text{in } H.$$

This structure has some similarities to a Runge–Kutta method. In such a scheme, for one temporal step, the data can be evaluated at different points to receive several explicit or implicit equations. A linear combination of the solutions of them is used to obtain $\mathbf{U}^n$. In contrast to this, we decompose the data into different parts. The decomposed data is used to receive several equations that are used to obtain $\mathbf{U}^n$.

Under no additional regularity assumptions on the solution, we can prove that sequences of piecewise polynomial prolongations of the values $(\mathbf{U}^n)_{n \in \{1,\dots,N\}}$ converge pointwise strongly to the solution in H. Depending on the monotonicity condition, we can also show that the sequence of piecewise constant prolongations converges weakly or even strongly to the solution in $L^p(0,T;V_M)$, where p depends on the operator $A(t)$, $t \in [0,T]$.

Under the assumption that the exact solution is more regular and the operators $A_m(t)$, $t \in [0,T]$ fulfill a bounded Lipschitz condition and a stronger monotonicity condition for $m \in \{1, \dots, M\}$, we even obtain explicit error estimates. Precisely, there exists $C \in (0, \infty)$, which depends on u, such that

$$\max_{n \in \{1,\dots,N\}} \|u(t_n) - \mathbf{U}^n\|_H^2 + k \sum_{n=1}^{N} \|u(t_n) - \mathbf{U}^n\|_{V_M}^p \leq C k^{\frac{p}{p-1}\alpha}$$

for all $N \in \mathbb{N}$ and $k = \frac{T}{N}$, where $\alpha \in (0,1]$ is the exponent of the Hölder space that contains the solution u and p depends on $A(t)$, $t \in [0,T]$. In particular, we see that the order of the error bounds can be the same as the convergence rate of the classical backward Euler scheme for suitable data.

Splitting schemes offer a useful tool in decreasing the computational costs of algorithms. A general introduction can be found in [70]. A well-known field of applications to operator splittings is given by evolution equations with different structures. In order to name a few examples, reaction-diffusion equations have been studied in terms of an operator splitting in [12, 64, 74], the Riccati differential equation in [65, 108], and the Navier–Stokes problem in [107]. Another useful way of splitting operators is a dimension splitting, where each $A_m(t)$, $t \in [0,T]$ and $m \in \{1, \dots, M\}$, contains different partial derivatives, see [62, 106]. A modern alternative to dimension splitting is given by domain decomposition schemes, see [13, 34, 61, 91, 113]. This approach is even more suitable for a parallel implementation as the communication between subproblems is smaller. Moreover, in contrast to a dimension splitting, non-Cartesian spatial domain can be considered in a domain decomposition based scheme.

The analysis of splitting schemes for evolution equations has mostly been done in a semigroup framework. General results as presented in [20, 23] can be used to prove the convergence of several schemes. In [34, 61, 64, 66], a convergence analysis with these techniques for the product splitting, sum splitting, Douglas–Rachford scheme, Peaceman–Rachford scheme, Crank–Nicolson scheme, and implicit-explicit splitting can be found. Thus, this approach can be used for many examples. When it comes to a setting, where we want to allow for a temporal dependence of the operator or a time-dependent source term, the results from [20, 23] cannot be applied directly anymore. In the spirit of the work of [35, 106], we want to prove the convergence of a splitting scheme in a variational framework. This enables us to also look at a nonautonomous problem.

The structure of this work is comparable to [35], where a sum splitting scheme was analyzed. The product splitting scheme in a variational approach has been considered in

[106]. Both the sum and the product splitting in a mild framework have been analyzed [34]. The main advantage of this work is the additional, possibly non-monotone, operator $B(t)$, $t \in [0,T]$, which is handled in terms of a forward Euler step. Similar implicit-explicit splittings can be found in [2, 5, 17, 24, 64]. In a similar fashion to [35], we prove in Theorem 3.1.18 and Theorem 3.1.19 that sequences of piecewise polynomial prolongations of $(\mathbf{U}^n)_{n\in\{1,\dots,N\}}$ converge to the solution of the evolution equation (3.1).

In Theorem 3.2.3, we provide error estimates for the method. While there are known results for this in the semigroup setting, see [61, 64], this is still new within a variational approach to splitting schemes.

These results are well applicable to dimension splitting and domain decomposition methods if $B \equiv 0$. For a nontrivial operator $B(t)$, $t \in [0,T]$, a compact embedding result remains to be proved within this context. In the last section of this chapter, we provide an example where the necessary compact embedding can easily be obtained.

In the first section of this chapter, we begin by introducing the exact assumptions needed on the data and prove the convergence of the piecewise polynomial prolongations without any further regularity assumptions made on the solution. In the second section, we show that under an additional bounded Lipschitz condition and a stronger monotonicity assumption on the operators $A_m(t)$, $t \in [0,T]$ and $m \in \{1,\dots,M\}$, we can even prove explicit error bounds. The size of the error depends on an additional regularity assumption on the solution. At the end of the chapter, we show that the theoretical results from the first two sections can be applied to a nonlinear parabolic problem.

3.1 Convergence of the Splitting Scheme

In this first section, we focus on proving the convergence of the implicit-explicit product splitting in a general framework. To this end, we begin to state the exact assumptions that have to be made on the data. This in mind, we can introduce our scheme and prove that it is has a unique solution. The solution also fulfills a priori bounds. Using these bounds, we can argue that the piecewise constant and piecewise linear prolongations of the solution to the semidiscrete problem are bounded in suitable spaces. Therefore, we can extract weakly or weakly$*$ converging subsequences. It remains to identify the limit with the equation, where among other things the Minty monotonicity trick will be used. The following setting is similar to both [35] and [106]. Let us begin by introducing the structure of the spaces which will be used in the following.

Assumption 3.1.1. *Let $(H, (\cdot,\cdot)_H, \|\cdot\|_H)$ be a real, separable Hilbert space and $(V, \|\cdot\|_V)$ be a real, separable, reflexive Banach space, which is continuously and densely embedded into H. Further, there exist a seminorm $|\cdot|_V$ on V and $c_V \in (0,\infty)$ such that $\|\cdot\|_V \leq c_V\big(\|\cdot\|_H + |\cdot|_V\big)$ is fulfilled.*

For $M \in \mathbb{N}$ and $m \in \{1,\dots,M\}$, let $(V_m, \|\cdot\|_{V_m})$ be real reflexive Banach spaces, which are continuously embedded into H, such that $\bigcap_{m=1}^M V_m = V$ and $\sum_{m=1}^M \|\cdot\|_{V_m}$ is equivalent to $\|\cdot\|_V$. For every $m \in \{1,\dots,M\}$, there exist a seminorm $|\cdot|_{V_m}$ on V_m and $c_{V_m} \in (0,\infty)$ such that $\|\cdot\|_{V_m} \leq c_{V_m}\big(\|\cdot\|_H + |\cdot|_{V_m}\big)$ and $\sum_{m=1}^M |\cdot|_{V_m}$ is equivalent to $|\cdot|_V$.

Note that asking for the existence of the seminorms in the previous assumption is no additional restriction on the spaces. As V and V_m, $m \in \{1,\dots,M\}$, are continuously embedded into H it is possible to use the full norm as the seminorm. If we consider, for example, $H = L^2(\mathcal{D})$ and $V = W^{1,p}(\mathcal{D})$ for $p \in [1,\infty)$ on a bounded domain $\mathcal{D} \subset \mathbb{R}^d$, $d \in \mathbb{N}$, it is possible to use the seminorm $|v|_V = \big(\int_{\mathcal{D}} |\nabla v|^p \,\mathrm{d}x\big)^{\frac{1}{p}}$. In this case, the seminorm is not a full norm. This setting is closely related to [99, Chapter 8] and allows for a different coercivity condition for the operators defined below.

These spaces in mind, we can identify H with its dual space H^* and obtain the Gelfand triples

$$V \overset{d}{\hookrightarrow} H \cong H^* \overset{d}{\hookrightarrow} V^* \quad \text{and} \quad V_m \overset{d}{\hookrightarrow} H \cong H^* \overset{d}{\hookrightarrow} V_m^*, \quad m \in \{1,\dots,M\}.$$

Note that for every $m \in \{1,\dots,M\}$, V_m is densely embedded into H because $V_m \supseteq V$ and V is densely embedded into H. The operator $A(t)$, $t \in [0,T]$, acts on the spaces defined above and fulfills the next assumption.

Assumption 3.1.2. *Let the spaces H and V be given as stated in Assumption 3.1.1. Furthermore, for $T \in (0,\infty)$ as well as $p \in [2,\infty)$, let $\{A(t)\}_{t\in[0,T]}$ be a family of operators $A(t)\colon V \to V^*$ that satisfy the following conditions:*

(1) The mapping $Av\colon [0,T] \to V^*$ given by $t \mapsto A(t)v$ is continuous almost everywhere in $(0,T)$ for all $v \in V$.

(2) The operator $A(t)\colon V \to V^*$, $t \in [0,T]$, is radially continuous, i.e., the mapping $s \mapsto \langle A(t)(v+sw), w\rangle_{V^*\times V}$ is continuous on $[0,1]$ for all $v,w \in V$.

(3) The operator $A(t)\colon V \to V^*$, $t \in [0,T]$, is monotone, i.e.,

$$\langle A(t)v - A(t)w, v-w\rangle_{V^*\times V} \geq 0$$

is fulfilled for all $v,w \in V$.

(4) The operator $A(t)\colon V \to V^*$, $t \in [0,T]$, is uniformly bounded such that there exists $\beta \in [0,\infty)$, which does not depend on t, with

$$\|A(t)v\|_{V^*} \leq \beta\big(1 + \|v\|_V^{p-1}\big)$$

for all $v \in V$.

(5) The operator $A(t)\colon V \to V^*$, $t \in [0,T]$, fulfills a uniform semi-coercivity condition such that there exist $\mu \in (0,\infty)$ and $\lambda \in [0,\infty)$, which do not depend on t, with

$$\langle A(t)v, v\rangle_{V^*\times V} + \lambda \geq \mu|v|_V^p$$

for all $v \in V$.

The assumption $p \in [2,\infty)$ can be weakened to $p \in (1,\infty)$ for this section, compare [40, 41, 42] for more details. It will then be necessary to choose some of the appearing function spaces differently to ensure that the spaces are embedded into each other. In Section 3.2, we cannot directly consider the case $p \in (1,2)$. Here, we make a stronger monotonicity condition, compare (3.48) below. There exists no operator that fulfills this condition for $p \in (1,2)$. Altogether, for simplicity and to keep the assumptions consistent, we concentrate on the case $p \in [2,\infty)$ throughout the entire chapter.

For the existence of a solution, Assumption 3.1.2 (1) can be generalized to assuming that the mapping $Av\colon [0,T] \to V^*$ given by $t \mapsto A(t)v$ is Bochner measurable for every $v \in V$, compare [118, Chapter 30]. We use a stronger condition to keep the proof of Lemma 3.1.15 below more simple. Furthermore, we use a semi-coercivity condition instead of a standard coercivity assumption, which contains the full norm. A similar condition was imposed in [99, Chapter 8]. As pointed out in [41] such a condition implies, in particular, that

$$\langle A(t)v, v\rangle_{V^*\times V} + \tilde{\nu}\|v\|_H^2 + \tilde{\lambda} \geq \tilde{\mu}\|v\|_V^2$$

is fulfilled. This implies that $A(t)$, $t \in [0,T]$, fulfills an ordinary coercivity condition but with different exponents. See (1.1) for the exact definition of the coefficients. We will use the semi-coercivity condition as this enables us to prove certain bounds in $L^p(0,T;V)$ while the coercivity condition above only allows for bounds in $L^2(0,T;V)$.

In the following, we want to decompose the operator $A(t)$, $t \in [0,T]$, in M operators that all have the same structure and act on the spaces V_m, $m \in \{1,\dots,M\}$, from Assumption 3.1.1.

Assumption 3.1.3. *For $M \in \mathbb{N}$, let H, V and V_m, $m \in \{1,\dots,M\}$, be as stated in Assumption 3.1.1. For $T \in (0,\infty)$, let $\{A(t)\}_{t\in[0,T]}$ be a family of operators $A(t)\colon V \to V^*$ as given in Assumption 3.1.2. For $m \in \{1,\dots,M\}$, let $A_m(t)\colon V_m \to V_m^*$, $t \in [0,T]$, also fulfill Assumption 3.1.2, with V replaced by V_m such that the sum property*

$$\sum_{m=1}^{M} A_m(t)v = A(t)v \quad \text{in } V^*$$

is satisfied for all $t \in [0,T]$ and $v \in V$.

Remark 3.1.4. Note that the optimal coefficients β, λ, μ for the operators $A(t)$ and $A_m(t)$, $t \in [0,T]$ and $m \in \{1,\dots,M\}$, do not necessarily have to coincide. For the sake of simplicity, we assume that the set of coefficients is the same for all appearing operators.

A comparable setting can be found in [83, Chapitre 2, Section 1.7]. Here, the operators $A_m(t)$, $t \in [0,T]$ and $m \in \{1,\dots,M\}$, fulfill a similar assumption as Assumption 3.1.2. But p can be a different value p_m for each operator. This could be an interesting generalization for our proposed operator splitting.

Additionally, we introduce a Lipschitz continuous operator $B(t)$, $t \in [0,T]$, stated below.

Assumption 3.1.5. *Let H be given as stated in Assumption 3.1.1. Furthermore, for $T \in (0,\infty)$, let $\{B(t)\}_{t\in[0,T]}$ be a family of operators $B(t)\colon H \to H$ that satisfy the following conditions:*

(1) The mapping $Bv\colon [0,T] \to H$ given by $t \mapsto B(t)v$ is continuous almost everywhere in $(0,T)$ for every $v \in H$.

(2) The operator $B(t)\colon H \to H$, $t \in [0,T]$, fulfills a uniform Lipschitz condition such that there exists $\kappa \in [0,\infty)$, which does not depend on t, with

$$\|B(t)v - B(t)w\|_H \leq \kappa\|v - w\|_H$$

for all $v, w \in H$.

(3) The operator $B(t)\colon H \to H$, $t \in [0,T]$, is uniformly bounded in $0 \in H$ such that there exists $\rho \in [0,\infty)$ with $\|B(t)0\|_H \leq \rho$ for all $t \in [0,T]$.

Moreover, if κ is strictly larger than zero, then let the space V_M from Assumption 3.1.1 be compactly embedded into H.

Remark 3.1.6. Observe that every operator $B(t)$, $t \in [0,T]$, which fulfills Assumption 3.1.5, also fulfills

$$\|B(t)v\|_H \leq \kappa(1 + \|v\|_H), \quad \big|(B(t)v, v)_H\big| \leq \kappa(1 + \|v\|_H^2)$$

for all $v \in H$ and $t \in [0,T]$ after possibly enlarging κ.

We also want to consider the operators of Assumptions 3.1.2, 3.1.3, and 3.1.5 as operators acting on Bochner spaces. To this end, we consider their Nemytskiĭ operators and state some useful properties in the lemma below. A proof can be found in [39, Lemma 8.4.4] or [118, Section 30].

Lemma 3.1.7. *Let the spaces V and H be given as in Assumption 3.1.1. For $T \in (0,\infty)$ and $p \in [2,\infty)$, let $A(t)\colon V \to V^*$ be an operator as stated in Assumption 3.1.2 and $B(t)\colon H \to H$ as in Assumption 3.1.5 for $t \in [0,T]$. Then for $q = \frac{p}{p-1}$ the operators $(Av)(t) = A(t)v(t)$ and $(Bv)(t) = B(t)v(t)$ map $L^p(0,T;V)$ into $L^q(0,T;V^*)$ and $L^2(0,T;H)$ into $L^2(0,T;H)$, respectively.*

The operator $A\colon L^p(0,T;V) \to L^q(0,T;V^*)$ is radially continuous, i.e., the mapping $s \mapsto \langle A(v+sw), w\rangle_{L^q(0,T;V^*)\times L^p(0,T;V)}$ is continuous on $[0,1]$ for all $v, w \in L^p(0,T;V)$. Furthermore, A fulfills a monotonicity, a boundedness, and a coercivity condition such that it holds true that

$$\begin{aligned}
&\langle Av - Aw, v - w\rangle_{L^q(0,T;V^*)\times L^p(0,T;V)} \geq 0,\\
&\|Av\|_{L^q(0,T;V^*)} \leq \beta\big(T^{\frac{1}{q}} + \|v\|_{L^p(0,T;V)}^{p-1}\big),\\
&\langle Av, v\rangle_{L^q(0,T;V^*)\times L^p(0,T;V)} + \mu\|v\|_{L^p(0,T;H)}^p + \lambda T \geq 2^{1-p}\mu c_V^{-p}\|v\|_{L^p(0,T;V)}^p
\end{aligned}$$

for all $v, w \in L^p(0,T;V)$. The operator $B\colon L^2(0,T;H) \to L^2(0,T;H)$ is Lipschitz continuous and bounded at $0 \in L^2(0,T;H)$ such that

$$\begin{aligned}
&\|Bv - Bw\|_{L^2(0,T;H)} \leq \kappa\|v - w\|_{L^2(0,T;H)}\\
&\|B0\|_{L^2(0,T;H)} \leq T^{\frac{1}{2}}\rho
\end{aligned}$$

is fulfilled for all $v, w \in L^2(0,T;H)$.

Note that for every $m \in \{1,\dots,M\}$, the Nemytskiĭ operator of $A_m(t)$, $t \in [0,T]$, introduced in Assumption 3.1.3 maps $L^p(0,T;V_m)$ into $L^q(0,T;V_m^*)$ and fulfills the same bounds stated in the previous lemma with V replaced by V_m. It remains to state the assumptions on the source term f and its decomposition.

Assumption 3.1.8. *Let V and V_m, $m \in \{1,\dots,M\}$, be given as in Assumption 3.1.1 and $q = \frac{p}{p-1}$, where $p \in [2,\infty)$ is the same as in Assumption 3.1.2. Let $f \in L^q(0,T;V^*)$ be given and assume that there exist functions $f_m \in L^q(0,T;V_m^*)$, $m \in \{1,\dots,M\}$, such that*

$$\sum_{m=1}^{M} f_m(t) = f(t) \text{ in } V^*, \quad \|f_m(t)\|_{V_m^*} \leq \|f(t)\|_{V^*}, \quad \text{for almost all } t \in (0,T).$$

It is also possible to allow for a more general source term $f \in L^q(0,T;V^*) + L^1(0,T;H)$, compare [106] and [109, Chapter III, Section 1.5]. For simplicity, we only concentrate on functions from $L^q(0,T;V^*)$. As discussed in Section 1.1, the evolution equation (3.1) is uniquely solvable if Assumptions 3.1.2 and 3.1.5 are fulfilled, f is an element of $L^q(0,T;V^*)$, and $u_0 \in H$.

In order to discretize the equation, we consider an equidistant grid on $[0,T]$, where $N \in \mathbb{N}$, $k = \frac{T}{N}$, and $t_n = nk$, $n \in \{0,\dots,N\}$. For $m \in \{1,\dots,M\}$ and $n \in \{1,\dots,N\}$, we introduce

$$\mathbf{A}_m^n v = \frac{1}{k}\int_{t_{n-1}}^{t_n} A_m(t)v \,\mathrm{d}t \quad \text{in } V_m^*, \quad \mathbf{B}^n w = \frac{1}{k}\int_{t_{n-1}}^{t_n} B(t)w \,\mathrm{d}t \quad \text{in } H \tag{3.2}$$

for $v \in V_m$ and $w \in H$ as well as

$$\mathbf{f}_m^n = \frac{1}{k}\int_{t_{n-1}}^{t_n} f_m(t) \,\mathrm{d}t \quad \text{in } V_m^*. \tag{3.3}$$

We use these values to construct an approximation $\mathbf{U}^n \approx u(t_n)$ of the solution u of the evolution equation (3.1) at the grid points. To this end, we examine the semidiscrete problem

$$\frac{\mathbf{U}_0^n - \mathbf{U}^{n-1}}{k} + \mathbf{B}^n \mathbf{U}^{n-1} = 0 \quad \text{in } H, \tag{3.4}$$

and

$$\frac{\mathbf{U}_m^n - \mathbf{U}_{m-1}^n}{k} + \mathbf{A}_m^n \mathbf{U}_m^n = \mathbf{f}_m^n \quad \text{in } V_m^*, \quad m \in \{1,\dots,M\}, \tag{3.5}$$

for $n \in \{1,\dots,N\}$ with

$$\mathbf{U}^n = \mathbf{U}_M^n \quad \text{in } H, \quad n \in \{1,\dots,N\}, \quad \text{and} \quad \mathbf{U}^0 = u_0^k \quad \text{in } H. \tag{3.6}$$

Depending on the statement, u_0^k has to be in V_M or H. In order to prove that the approximation converges to the solution, we require that $u_0^k \to u_0$ in H as $k \to 0$. For some results, we further need that $(k^{\frac{1}{p}}\|u_0^k\|_{V_M})_{k>0}$ is uniformly bounded with respect to k. In order to see that such a sequence exists, let $(u_0^i)_{i\in\mathbb{N}}$ be a sequence in V_M such that $u_0^i \to u_0$ in H as $i \to \infty$. This sequence exists because V_M is densely embedded into H. For the construction of a sequence that fulfills this boundedness condition, we use a sequence $(k_j)_{j\in\mathbb{N}}$ such that $k_j \to 0$ as $j \to \infty$. Then we set $u_0^{k_1} = u_0^1$ in V_M. As $k_j^{-\frac{1}{p}} \to \infty$ as $j \to \infty$ there exists $j_1 \in \mathbb{N}$ such that $\|u_0^2\|_{V_M} \le k_{j_1}^{-\frac{1}{p}}$. This in mind, we write $u_0^{k_1} = \dots = u_0^{k_{j_1-1}}$ and $u_0^{k_{j_1}} = u_0^2$ in V_M. Analogously, there exists $j_2 \in \mathbb{N}$ such that $\|u_0^3\|_{V_M} \le k_{j_2}^{-\frac{1}{p}}$ and we write $u_0^{k_{j_1}} = \dots = u_0^{k_{j_2-1}}$ and $u_0^{k_{j_2}} = u_0^3$ in V_M. Repeating this argument, we obtain an appropriate sequence to approximate the initial value.

The following two lemmas show that the discrete values $(\mathbf{A}_m^n)_{n\in\{1,\dots,N\}}$, $m \in \{1,\dots,M\}$, and $(\mathbf{B}^n)_{n\in\{1,\dots,N\}}$ fulfill the same properties as their underlying operators $A_m(t)$, $m \in \{1,\dots,M\}$, and $B(t)$ do for every $t \in [0,T]$.

Lemma 3.1.9. *Let Assumptions 3.1.1, 3.1.2, and 3.1.3 be fulfilled. For $n \in \{1,\dots,N\}$ and $m \in \{1,\dots,M\}$, the operator $\mathbf{A}_m^n \colon V_m \to V_m^*$ defined in (3.2) is radially continuous, i.e., the mapping $s \mapsto \langle \mathbf{A}_m^n(v+sw), w\rangle_{V_m^*\times V_m}$ is continuous on $[0,1]$ for all $v, w \in V_m$. Furthermore, it fulfills a monotonicity, a boundedness, and a coercivity condition such that*

$$\langle \mathbf{A}_m^n v - \mathbf{A}_m^n w, v - w\rangle_{V_m^*\times V_m} \ge 0, \tag{3.7}$$

$$\|\mathbf{A}_m^n v\|_{V_m^*} \le \beta\big(1 + \|v\|_{V_m}^{p-1}\big), \tag{3.8}$$

$$\langle \mathbf{A}_m^n v, v\rangle_{V_m^*\times V_m} + \lambda \ge \mu |v|_{V_m}^p \tag{3.9}$$

are fulfilled for all $v, w \in V_m$.

Proof. Let $(s_i)_{i\in\mathbb{N}}$ be a sequence in $[0,1]$ that converges to $s \in [0,1]$. Then we see that

$$\begin{aligned}\langle A_m(t)(v+s_iw),w\rangle_{V_m^*\times V_m} &\leq \|A_m(t)(v+s_iw)\|_{V_m^*}\|w\|_{V_m}\\ &\leq \beta\big(1+\|v+s_iw\|_{V_m}^{p-1}\big)\|w\|_{V_m}\\ &\leq \beta\big(1+2^{p-2}\|v\|_{V_m}^{p-1}+2^{p-2}\|w\|_{V_m}^{p-1}\big)\|w\|_{V_m}.\end{aligned}$$

for $v, w \in V_m$ due to the boundedness condition of $A(t)$, $t \in [0,T]$, from Assumption 3.1.2 (4). We can then apply the radial continuity of $A(t)$, $t \in [0,T]$, from Assumption 3.1.2 (2) and Lebesgue's dominated convergence theorem to obtain

$$\begin{aligned}\lim_{i\to\infty}\langle \mathbf{A}_m^n(v+s_iw),w\rangle_{V_m^*\times V_m} &= \lim_{i\to\infty}\frac{1}{k}\int_{t_{n-1}}^{t_n}\langle A_m(t)(v+s_iw),w\rangle_{V_m^*\times V_m}\,\mathrm{d}t\\ &= \frac{1}{k}\int_{t_{n-1}}^{t_n}\lim_{i\to\infty}\langle A_m(t)(v+s_iw),w\rangle_{V_m^*\times V_m}\,\mathrm{d}t\\ &= \frac{1}{k}\int_{t_{n-1}}^{t_n}\langle A_m(t)(v+sw),w\rangle_{V_m^*\times V_m}\,\mathrm{d}t\\ &= \langle \mathbf{A}_m^n(v+sw),w\rangle_{V_m^*\times V_m}\end{aligned}$$

for $v, w \in V_m$. Therefore, $\mathbf{A}_m^n$ is radially continuous. In order to prove (3.7)–(3.9), we apply Assumption 3.1.2 (3)–(5) and obtain the monotonicity condition

$$\langle \mathbf{A}_m^n v-\mathbf{A}_m^n w, v-w\rangle_{V_m^*\times V_m} = \frac{1}{k}\int_{t_{n-1}}^{t_n}\langle A_m(t)v-A_m(t)w,v-w\rangle_{V_m^*\times V_m}\,\mathrm{d}t \geq 0,$$

the boundedness condition

$$\|\mathbf{A}_m^n v\|_{V_m^*} \leq \frac{1}{k}\int_{t_{n-1}}^{t_n}\|A_m(t)v\|_{V_m^*}\,\mathrm{d}t \leq \beta\big(1+\|v\|_{V_m}^{p-1}\big),$$

and the coercivity condition

$$\langle \mathbf{A}_m^n v, v\rangle_{V_m^*\times V_m} = \frac{1}{k}\int_{t_{n-1}}^{t_n}\langle A_m(t)v,v\rangle_{V_m^*\times V_m}\,\mathrm{d}t \geq \frac{1}{k}\int_{t_{n-1}}^{t_n}\big(\mu|v|_{V_m}^p-\lambda\big)\,\mathrm{d}t = \mu|v|_{V_m}^p-\lambda$$

for all $v, w \in V_m$. □

Lemma 3.1.10. *Let Assumptions 3.1.1 and 3.1.5 be fulfilled. The operator* $\mathbf{B}^n\colon H \to H$ *defined in* (3.2) *fulfills*

$$\|\mathbf{B}^n v-\mathbf{B}^n w\|_H \leq \kappa\|v-w\|_H, \quad \|\mathbf{B}^n 0\|_H \leq \rho, \tag{3.10}$$

as well as

$$\|\mathbf{B}^n v\|_H \leq \kappa\big(1+\|v\|_H\big), \quad \big|(\mathbf{B}^n v,v)_H\big| \leq \kappa\big(1+\|v\|_H^2\big) \tag{3.11}$$

for all $v, w \in H$ *and* $n \in \{1,\dots,N\}$.

We omit the proof of this lemma. It can be done analogously to the proof of Lemma 3.1.9, where we also use the bounds proposed in Remark 3.1.6. Now, we are well prepared to prove that the operator splitting scheme (3.4)–(3.6) is uniquely solvable and its solution fulfills a priori bounds.

Lemma 3.1.11. *Let Assumptions 3.1.1, 3.1.2, 3.1.3, 3.1.5, and 3.1.8 be fulfilled. For a step size $k = \frac{T}{N}$, $N \in \mathbb{N}$, and $u_0^k \in H$, the semidiscrete problem* (3.4)–(3.6) *is uniquely solvable.*

Proof. Let $n \in \{1, \dots, N\}$ and $m \in \{1, \dots, M\}$ be fixed in the following. Assuming that $\mathbf{U}^{n-1} \in H$ exists, then $\mathbf{U}_0^n \in H$ is given by the explicit equation

$$\mathbf{U}_0^n = \big(I - k\mathbf{B}^n\big)\mathbf{U}^{n-1} \quad \text{in } H.$$

If now $\mathbf{U}_{m-1}^n \in H$ exists, we want to show that there exists a unique element $\mathbf{U}_m^n \in V_m$ that solves

$$\big(I + k\mathbf{A}_m^n\big)\mathbf{U}_m^n = k\mathbf{f}_m^n + \mathbf{U}_{m-1}^n \quad \text{in } V_m^*. \tag{3.12}$$

As proven in Lemma 3.1.9 the operator $\mathbf{A}_m^n$ is radially continuous. This implies that $I + k\mathbf{A}_m^n$ is radially continuous. Applying the monotonicity condition (3.7) from Lemma 3.1.9, it can be seen that $I + k\mathbf{A}_m^n$ is strictly monotone. Using the inequality $\|v\|_{V_m} \leq c_{V_m}\big(\|v\|_H + |v|_{V_m}\big)$ for the V_m-norm stated in Assumption 3.1.1 and the coercivity condition (3.9) from Lemma 3.1.9, we obtain

$$\begin{aligned}
\frac{\langle (I + k\mathbf{A}_m^n)v, v\rangle_{V_m^* \times V_m}}{\|v\|_{V_m}} &\geq \frac{\|v\|_H^2 + \mu|v|_{V_m}^p}{c_{V_m}\big(\|v\|_H + |v|_{V_m}\big)} - \frac{\lambda}{\|v\|_{V_m}} \\
&\geq \frac{\min\{1, \mu\}}{c_{V_m}} \cdot \frac{\|v\|_H^2 + |v|_{V_m}^p}{\|v\|_H + |v|_{V_m}} - \frac{\lambda}{\|v\|_{V_m}} \to \infty \quad \text{as } \|v\|_{V_m} \to \infty.
\end{aligned}$$

Hence, there exists a unique element $\mathbf{U}_m^n \in V_m$ that solves (3.12) due to the Browder–Minty theorem, see [99, Theorem 2.14]. ☐

Lemma 3.1.12. *Let Assumptions 3.1.1, 3.1.2, 3.1.3, 3.1.5, and 3.1.8 be fulfilled and let $u_0^k \in H$ be given. Then there exists $K \in (0, \infty)$ such that for every step size $k = \frac{T}{N}$, $N \in \mathbb{N}$, the unique solution of* (3.4)–(3.6) *fulfills the a priori estimates*

$$\max_{n \in \{1,\dots,N\}} \|\mathbf{U}^n\|_H^2 + \sum_{n=1}^N \sum_{m=1}^M \|\mathbf{U}_m^n - \mathbf{U}_{m-1}^n\|_H^2 \leq K, \tag{3.13}$$

$$\max_{n \in \{1,\dots,N\}} \|\mathbf{U}_0^n\|_H^2 + \sum_{n=1}^N \|\mathbf{U}_0^n - \mathbf{U}^{n-1}\|_H^2 \leq K, \tag{3.14}$$

$$\max_{n \in \{1,\dots,N\}} \|\mathbf{U}_m^n\|_H^2 + k\sum_{n=1}^N \|\mathbf{U}_m^n\|_{V_m}^p \leq K, \quad m \in \{1, \dots, M\}, \tag{3.15}$$

as well as

$$k^{1-q} \sum_{n=1}^N \|\mathbf{U}^n - \mathbf{U}^{n-1}\|_{V^*}^q = k\sum_{n=1}^N \Big\|\frac{\mathbf{U}^n - \mathbf{U}^{n-1}}{k}\Big\|_{V^*}^q \leq K. \tag{3.16}$$

In order to prove the a priori bounds, we follow a similar structure as in [99, 106]. Since we only assumed that $A_m(t)$, $t \in [0, T]$, fulfills a semi-coercivity condition for every $m \in \{1, \dots, M\}$ a Gronwall-like-argument becomes necessary. In [99, Lemma 8.6], the classical Gronwall lemma leads to a step size restriction, which is f depended. For some appearing terms, we avoid the classical Gronwall argument and use Lemma A.1.2 instead. The main advantage of this argumentation is that we do not have a restriction for the step size k.

Proof of Lemma 3.1.12. Let $i \in \{1, \dots, N\}$ and $m \in \{1, \dots, M\}$ be fixed in the following. We test (3.4) with $\mathbf{U}_0^i \in H$ and use the identity from Lemma A.1.4 to obtain that

$$\frac{1}{2}\big(\|\mathbf{U}_0^i\|_H^2 - \|\mathbf{U}^{i-1}\|_H^2 + \|\mathbf{U}_0^i - \mathbf{U}^{i-1}\|_H^2\big) = -k(\mathbf{B}^i\mathbf{U}^{i-1}, \mathbf{U}_0^i)_H. \tag{3.17}$$

An application of the conditions in (3.11) yields

$$\begin{aligned}
-(\mathbf{B}^i\mathbf{U}^{i-1}, \mathbf{U}_0^i)_H &= -(\mathbf{B}^i\mathbf{U}^{i-1}, \mathbf{U}^{i-1})_H - (\mathbf{B}^i\mathbf{U}^{i-1}, \mathbf{U}_0^i - \mathbf{U}^{i-1})_H \\
&\leq \kappa\big(1 + \|\mathbf{U}^{i-1}\|_H^2\big) + k\|\mathbf{B}^i\mathbf{U}^{i-1}\|_H^2 + \frac{1}{4k}\|\mathbf{U}_0^i - \mathbf{U}^{i-1}\|_H^2 \\
&\leq \kappa\big(1 + \|\mathbf{U}^{i-1}\|_H^2\big) + k\kappa^2\big(1 + \|\mathbf{U}^{i-1}\|_H\big)^2 + \frac{1}{4k}\|\mathbf{U}_0^i - \mathbf{U}^{i-1}\|_H^2 \\
&\leq c_1\big(1 + \|\mathbf{U}^{i-1}\|_H^2\big) + \frac{1}{4k}\|\mathbf{U}_0^i - \mathbf{U}^{i-1}\|_H^2
\end{aligned}$$

for $c_1 = \kappa + 2\kappa^2 T$. We insert this bound into (3.17) to obtain

$$\|\mathbf{U}_0^i\|_H^2 - \|\mathbf{U}^{i-1}\|_H^2 + \frac{1}{2}\|\mathbf{U}_0^i - \mathbf{U}^{i-1}\|_H^2 \leq 2kc_1\big(1 + \|\mathbf{U}^{i-1}\|_H^2\big). \tag{3.18}$$

Similarly, we test (3.5) with $\mathbf{U}_m^i \in V_m$ and again use the identity from Lemma A.1.4 to find that

$$\begin{aligned}
&\frac{1}{2}\big(\|\mathbf{U}_m^i\|_H^2 - \|\mathbf{U}_{m-1}^i\|_H^2 + \|\mathbf{U}_m^i - \mathbf{U}_{m-1}^i\|_H^2\big) + k\langle \mathbf{A}_m^i\mathbf{U}_m^i, \mathbf{U}_m^i\rangle_{V_m^*\times V_m} \\
&= k\langle \mathbf{f}_m^i, \mathbf{U}_m^i\rangle_{V_m^*\times V_m} \leq k\|\mathbf{f}_m^i\|_{V_m^*}\|\mathbf{U}_m^i\|_{V_m}.
\end{aligned} \tag{3.19}$$

We then multiply the inequality by two, insert the coercivity condition (3.9) stated in Lemma 3.1.9, and the inequality $\|v\|_{V_m} \leq c_{V_m}\big(\|v\|_H + |v|_{V_m}\big)$ for the V_m-norm from Assumption 3.1.1 as well as Young's inequality to obtain that

$$\begin{aligned}
&\|\mathbf{U}_m^i\|_H^2 - \|\mathbf{U}_{m-1}^i\|_H^2 + \|\mathbf{U}_m^i - \mathbf{U}_{m-1}^i\|_H^2 + 2k\mu|\mathbf{U}_m^i|_{V_m}^p \\
&\leq 2kc_{V_m}\|\mathbf{f}_m^i\|_{V_m^*}\big(\|\mathbf{U}_m^i\|_H + |\mathbf{U}_m^i|_{V_m}\big) + 2k\lambda \\
&\leq 2kc_{V_m}\|\mathbf{f}_m^i\|_{V_m^*}\|\mathbf{U}_m^i\|_H + kc_2\|\mathbf{f}_m^i\|_{V_m^*}^q + k\mu|\mathbf{U}_m^i|_{V_m}^p + 2k\lambda,
\end{aligned}$$

with $c_2 = \frac{(2c_{V_m})^q(p\mu)^{1-q}}{q}$. After absorbing the summand containing the V_m-seminorm, it follows that

$$\begin{aligned}
&\|\mathbf{U}_m^i\|_H^2 - \|\mathbf{U}_{m-1}^i\|_H^2 + \|\mathbf{U}_m^i - \mathbf{U}_{m-1}^i\|_H^2 + k\mu|\mathbf{U}_m^i|_{V_m}^p \\
&\leq 2kc_{V_m}\|\mathbf{f}_m^i\|_{V_m^*}\|\mathbf{U}_m^i\|_H + kc_2\|\mathbf{f}_m^i\|_{V_m^*}^q + 2k\lambda.
\end{aligned}$$

We sum up the inequality from $m = 1$ to M, add (3.18), and insert $\mathbf{U}_M^i = \mathbf{U}^i$ in H to see that

$$\begin{aligned}
&\|\mathbf{U}^i\|_H^2 - \|\mathbf{U}^{i-1}\|_H^2 + \sum_{m=1}^M \|\mathbf{U}_m^i - \mathbf{U}_{m-1}^i\|_H^2 + \frac{1}{2}\|\mathbf{U}_0^i - \mathbf{U}^{i-1}\|_H^2 + k\mu\sum_{m=1}^M |\mathbf{U}_m^i|_{V_m}^p \\
&\leq k\sum_{m=1}^M \Big(2c_{V_m}\|\mathbf{f}_m^i\|_{V_m^*}\|\mathbf{U}_m^i\|_H + c_2\|\mathbf{f}_m^i\|_{V_m^*}^q\Big) + 2k\lambda M + 2kc_1\big(1 + \|\mathbf{U}^{i-1}\|_H^2\big).
\end{aligned}$$

Summing up the previous inequality from $i = 1$ to $n \in \{1, \dots, N\}$, shows that

$$\begin{aligned}
&\|\mathbf{U}^n\|_H^2 - \|\mathbf{U}^0\|_H^2 + \sum_{i=1}^{n}\sum_{m=1}^{M} \|\mathbf{U}_m^i - \mathbf{U}_{m-1}^i\|_H^2 + \frac{1}{2}\sum_{i=1}^{n} \|\mathbf{U}_0^i - \mathbf{U}^{i-1}\|_H^2 + k\mu \sum_{i=1}^{n}\sum_{m=1}^{M} |\mathbf{U}_m^i|_{V_m}^p \\
&\le k \sum_{i=1}^{n}\sum_{m=1}^{M} \Big(2c_{V_m} \|\mathbf{f}_m^i\|_{V_m^*} \|\mathbf{U}_m^i\|_H + c_2 \|\mathbf{f}_m^i\|_{V_m^*}^q \Big) + 2T\lambda M + 2kc_1 \sum_{i=0}^{n-1} \big(1 + \|\mathbf{U}^i\|_H^2\big).
\end{aligned} \tag{3.20}$$

The sums containing $\mathbf{f}_m^i$ can be bounded using Assumption 3.1.8 as well as Hölder's inequality. Then we see that

$$\begin{aligned}
k \sum_{i=1}^{n}\sum_{m=1}^{M} \|\mathbf{f}_m^i\|_{V_m^*}^q &= k \sum_{i=1}^{n}\sum_{m=1}^{M} \Big\| \frac{1}{k} \int_{t_{i-1}}^{t_i} f_m(t) \,\mathrm{d}t \Big\|_{V_m^*}^q \\
&\le \sum_{i=1}^{n}\sum_{m=1}^{M} \int_{t_{i-1}}^{t_i} \|f_m(t)\|_{V_m^*}^q \,\mathrm{d}t \le M \|f\|_{L^q(0,T;V^*)}^q
\end{aligned} \tag{3.21}$$

and

$$k \|\mathbf{f}_m^i\|_{V_m^*} \le k \Big\| \frac{1}{k} \int_{t_{i-1}}^{t_i} f_m(t) \,\mathrm{d}t \Big\|_{V_m^*} \le \int_{t_{i-1}}^{t_i} \|f(t)\|_{V^*} \,\mathrm{d}t. \tag{3.22}$$

Inserting these inequalities and $\mathbf{U}^0 = u_0^k$ in H to (3.20), we obtain that

$$\begin{aligned}
&\|\mathbf{U}^n\|_H^2 + \sum_{i=1}^{n}\sum_{m=1}^{M} \|\mathbf{U}_m^i - \mathbf{U}_{m-1}^i\|_H^2 + \frac{1}{2}\sum_{i=1}^{n} \|\mathbf{U}_0^i - \mathbf{U}^{i-1}\|_H^2 + k\mu \sum_{i=1}^{n}\sum_{m=1}^{M} |\mathbf{U}_m^i|_{V_m}^p \\
&\le \|u_0^k\|_H^2 + k \sum_{i=1}^{n}\sum_{m=1}^{M} \Big(2c_{V_m} \|\mathbf{f}_m^i\|_{V_m^*} \|\mathbf{U}_m^i\|_H + c_2 \|\mathbf{f}_m^i\|_{V_m^*}^q \Big) + 2T\lambda M + 2kc_1 \sum_{i=0}^{n-1} \big(1 + \|\mathbf{U}^i\|_H^2\big) \\
&\le \big(1 + 2Tc_1\big) \|u_0^k\|_H^2 + 2c_3 \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \|f(t)\|_{V^*} \,\mathrm{d}t \sum_{m=1}^{M} \|\mathbf{U}_m^i\|_H + c_2 M \|f\|_{L^q(0,T;V^*)}^q \\
&\quad + 2T(\lambda M + c_1) + 2kc_1 \sum_{i=1}^{n-1} \|\mathbf{U}^i\|_H^2 \\
&\le \big(1 + 2Tc_1\big) \|u_0^k\|_H^2 + 2c_3 \|f\|_{L^1(0,T;V^*)} \max_{i \in \{1,\dots,N\}} \sum_{m=1}^{M} \|\mathbf{U}_m^i\|_H + c_2 M \|f\|_{L^q(0,T;V^*)}^q \\
&\quad + 2T(\lambda M + c_1) + 2kc_1 \sum_{i=1}^{n-1} \|\mathbf{U}^i\|_H^2,
\end{aligned}$$

where $c_3 = \max_{m \in \{1,\dots,M\}} c_{V_m}$. This can now be estimated using Lemma A.1.1 such that we arrive at

$$\begin{aligned}
&\|\mathbf{U}^n\|_H^2 + \sum_{i=1}^{n}\sum_{m=1}^{M} \|\mathbf{U}_m^i - \mathbf{U}_{m-1}^i\|_H^2 + \frac{1}{2}\sum_{i=1}^{n} \|\mathbf{U}_0^i - \mathbf{U}^{i-1}\|_H^2 + k\mu \sum_{i=1}^{n}\sum_{m=1}^{M} |\mathbf{U}_m^i|_{V_m}^p \\
&\le \Big(\big(1 + 2Tc_1\big) \|u_0^k\|_H^2 + 2c_3 \|f\|_{L^1(0,T;V^*)} \max_{i \in \{1,\dots,N\}} \sum_{m=1}^{M} \|\mathbf{U}_m^i\|_H + c_2 M \|f\|_{L^q(0,T;V^*)}^q \\
&\qquad + 2T(\lambda M + c_1) \Big) \exp(2Tc_1).
\end{aligned}$$

As the right-hand side is independent of n, the inequality is also fulfilled if we take the maximum over all $n \in \{1, \dots, N\}$ on the left-hand side. A telescopic sum argument and $\mathbf{U}^i = \mathbf{U}^i_M$ in H imply that

$$\begin{aligned} \|\mathbf{U}^i_m\|_H^2 &= \Big\| \mathbf{U}^i - \sum_{j=m+1}^{M} \big(\mathbf{U}^i_j - \mathbf{U}^i_{j-1}\big) \Big\|_H^2 \le M\Big(\|\mathbf{U}^i\|_H^2 + \sum_{j=m+1}^{M} \|\mathbf{U}^i_j - \mathbf{U}^i_{j-1}\|_H^2 \Big) \\ &\le M\Big(\|\mathbf{U}^i\|_H^2 + \sum_{j=1}^{M} \|\mathbf{U}^i_j - \mathbf{U}^i_{j-1}\|_H^2 \Big) \end{aligned} \tag{3.23}$$

and therefore

$$\sum_{m=1}^{M} \|\mathbf{U}^i_m\|_H \le M^{\frac{3}{2}} \Big(\|\mathbf{U}^i\|_H^2 + \sum_{j=1}^{M} \|\mathbf{U}^i_j - \mathbf{U}^i_{j-1}\|_H^2 \Big)^{\frac{1}{2}}.$$

We then abbreviate the terms

$$\begin{aligned} x^2 &= \max_{n \in \{1,\dots,N\}} \Big(\|\mathbf{U}^n\|_H^2 + \sum_{i=1}^{n} \sum_{m=1}^{M} \|\mathbf{U}^i_m - \mathbf{U}^i_{m-1}\|_H^2 + \frac{1}{2} \sum_{i=1}^{n} \|\mathbf{U}^i_0 - \mathbf{U}^{i-1}\|_H^2 \\ &\qquad + k\mu \sum_{i=1}^{n} \sum_{m=1}^{M} |\mathbf{U}^i_m|_{V_m}^p \Big), \\ a &= c_3 M^{\frac{3}{2}} \|f\|_{L^1(0,T;V^*)} \exp(2Tc_1), \\ b^2 &= \big((1 + 2Tc_1) \|u_0^k\|_H^2 + c_2 M \|f\|^q_{L^q(0,T;V^*)} + 2T(\lambda M + c_1) \big) \exp(2Tc_1) \end{aligned}$$

to obtain $x^2 \le 2ax + b^2$. An application of Lemma A.1.2 then yields the bound $x \le 2a + b$. This means that there exists $K_1 \in (0, \infty)$, which does not depend on the step size, such that

$$\|\mathbf{U}^n\|_H^2 + \sum_{i=1}^{n} \sum_{m=1}^{M} \|\mathbf{U}^i_m - \mathbf{U}^i_{m-1}\|_H^2 + \frac{1}{2} \sum_{i=1}^{n} \|\mathbf{U}^i_0 - \mathbf{U}^{i-1}\|_H^2 + k\mu \sum_{i=1}^{n} \sum_{m=1}^{M} |\mathbf{U}^i_m|_{V_m}^p \le K_1 \tag{3.24}$$

for every $n \in \{1, \dots, N\}$. Using (3.18) and (3.23), it follows that

$$\|\mathbf{U}^n_0\|_H^2 \le 2Tc_1 + \big(1 + 2Tc_1\big) K_1, \quad \|\mathbf{U}^n_m\|_H^2 \le M K_1 \tag{3.25}$$

for every $m \in \{1, \dots, M\}$.

In order to prove a bound for $(\mathbf{U}^n_m)_{n \in \{1,\dots,N\}}$ in V_m for every $m \in \{1, \dots, M\}$, we use the previous estimate as well as the inequality $\|v\|_{V_m} \le c_{V_m} \big(\|v\|_H + |v|_{V_m} \big)$ for the V_m-norm from Assumption 3.1.1. Then we obtain that

$$\begin{aligned} \Big(k \sum_{i=1}^{N} \sum_{m=1}^{M} \|\mathbf{U}^i_m\|_{V_m}^p \Big)^{\frac{1}{p}} &\le \Big(k \sum_{i=1}^{N} \sum_{m=1}^{M} c_{V_m}^p \big(\|\mathbf{U}^i_m\|_H + |\mathbf{U}^i_m|_{V_m} \big)^p \Big)^{\frac{1}{p}} \\ &\le c_3 \Big(k \sum_{i=1}^{N} \sum_{m=1}^{M} \|\mathbf{U}^i_m\|_H^p \Big)^{\frac{1}{p}} + c_3 \Big(k \sum_{i=1}^{N} \sum_{m=1}^{M} |\mathbf{U}^i_m|_{V_m}^p \Big)^{\frac{1}{p}} \\ &\le c_3 T^{\frac{1}{p}} M^{\frac{1}{2} + \frac{1}{p}} K_1^{\frac{1}{2}} + c_3 \Big(\frac{K_1}{\mu} \Big)^{\frac{1}{p}} =: K_2. \end{aligned} \tag{3.26}$$

In order to prove the a priori bound (3.16), we rewrite the difference $\mathbf{U}^n - \mathbf{U}^{n-1}$ using (3.5) as well as (3.4) and insert $\mathbf{U}_M^n = \mathbf{U}^n$ in H to obtain

$$\mathbf{U}^n - \mathbf{U}^{n-1} = \sum_{m=1}^{M} \big(\mathbf{U}_m^n - \mathbf{U}_{m-1}^n\big) + \mathbf{U}_0^n - \mathbf{U}^{n-1} = k \sum_{m=1}^{M} \big(\mathbf{f}_m^n - \mathbf{A}_m^n \mathbf{U}_m^n\big) - k\mathbf{B}^n \mathbf{U}^{n-1}$$

in V^* for $n \in \{1, \dots, N\}$. Testing the equation with $v \in V$, shows that

$$\begin{aligned}
&(\mathbf{U}^n - \mathbf{U}^{n-1}, v)_H \\
&= k \sum_{m=1}^{M} \langle \mathbf{f}_m^n - \mathbf{A}_m^n \mathbf{U}_m^n, v\rangle_{V_m^* \times V_m} - k(\mathbf{B}^n \mathbf{U}^{n-1}, v)_H \\
&\le k c_4 \|v\|_V \sum_{m=1}^{M} \big(\|\mathbf{f}_m^n\|_{V_m^*} + \|\mathbf{A}_m^n \mathbf{U}_m^n\|_{V_m^*}\big) + k c_4 \|v\|_V \|\mathbf{B}^n \mathbf{U}^{n-1}\|_H \\
&\le k c_4 \|v\|_V \sum_{m=1}^{M} \big(\|\mathbf{f}_m^n\|_{V_m^*} + \beta\big(1 + \|\mathbf{U}_m^n\|_{V_m}^{p-1}\big)\big) + k c_4 \kappa \|v\|_V \big(1 + \|\mathbf{U}^{n-1}\|_H\big),
\end{aligned}$$

where $c_4 \in (0, \infty)$ is the maximal embedding constant from the embeddings of V into V_m^*, $m \in \{1, \dots, M\}$, and H into V^*. This implies

$$\Big\|\frac{\mathbf{U}^n - \mathbf{U}^{n-1}}{k}\Big\|_{V^*} \le c_4 \sum_{m=1}^{M} \big(\|\mathbf{f}_m^n\|_{V_m^*} + \beta\big(1 + \|\mathbf{U}_m^n\|_{V_m}^{p-1}\big)\big) + c_4 \kappa \big(1 + \|\mathbf{U}^{n-1}\|_H\big).$$

Taking the q-power, summing up from $n = 1$ to N, multiplying by the step size k and again taking the $\frac{1}{q}$-power, it follows that

$$\begin{aligned}
&\Big(k \sum_{n=1}^{N} \Big\|\frac{\mathbf{U}^n - \mathbf{U}^{n-1}}{k}\Big\|_{V^*}^q\Big)^{\frac{1}{q}} \\
&\le c_4 \Big(k \sum_{n=1}^{N} \Big(\sum_{m=1}^{M} \big(\|\mathbf{f}_m^n\|_{V_m^*} + \beta\big(1 + \|\mathbf{U}_m^n\|_{V_m}^{p-1}\big)\big) + c_4 \kappa \big(1 + \|\mathbf{U}^{n-1}\|_H\big)\Big)^q\Big)^{\frac{1}{q}} \\
&\le c_4 \Big(k \sum_{n=1}^{N} \sum_{m=1}^{M} \|\mathbf{f}_m^n\|_{V_m^*}^q\Big)^{\frac{1}{q}} + c_4 \Big(k \sum_{n=1}^{N} \sum_{m=1}^{M} \beta^q\Big)^{\frac{1}{q}} + c_4 \Big(k \sum_{n=1}^{N} \sum_{m=1}^{M} \beta^q \|\mathbf{U}_m^n\|_{V_m}^p\Big)^{\frac{1}{q}} \\
&\quad + c_4 \Big(k \sum_{n=1}^{N} \sum_{m=1}^{M} \kappa^q\Big)^{\frac{1}{q}} + c_4 \Big(k \sum_{n=1}^{N} \sum_{m=1}^{M} \kappa^q \|\mathbf{U}^{n-1}\|_H^q\Big)^{\frac{1}{q}} \\
&\le c_4 M^{\frac{1}{q}} \|f\|_{L^q(0,T;V^*)} + c_4(\beta + \kappa)(TM)^{\frac{1}{q}} + c_4 \beta K_2^{\frac{1}{q}} + c_4 \kappa K_1^{\frac{1}{2}} (TM)^{\frac{1}{q}},
\end{aligned}$$

where we used (3.21), (3.24), and (3.26). A combination of (3.24), (3.25), (3.26), and the previous inequality shows the desired bounds. □

For the time discrete solutions $(\mathbf{U}^n)_{n\in\{1,\dots,N\}}$ and $(\mathbf{U}_m^n)_{n\in\{1,\dots,N\}}$, $m \in \{1, \dots, M\}$, to (3.4)–(3.6) corresponding to the grid $0 = t_0 < t_1 < \dots < t_N = T$ with $k = \frac{T}{N}$ and $t_n = nk$, $n \in \{0, \dots, N\}$, we construct piecewise polynomial prolongations defined on the entire interval $[0, T]$. To this end, we introduce the piecewise constant prolongations for $t \in (t_{n-1}, t_n]$, $n \in \{1, \dots, N\}$,

$$A_m^k(t) = \mathbf{A}_m^n, \quad B^k(t) = \mathbf{B}^n, \quad f_m^k(t) = \mathbf{f}_m^n \tag{3.27}$$

with $A_m^k(0) = \mathbf{A}_m^1$, $B^k(0) = \mathbf{B}^1$, and $f_m^k(0) = \mathbf{f}_m^1$ for $m \in \{1, \dots, M\}$ and

$$\bar{U}_m^k(t) = \mathbf{U}_m^n, \quad \bar{U}^k(t) = \mathbf{U}^n, \quad \underline{U}^k(t) = \mathbf{U}^{n-1} \tag{3.28}$$

for $m \in \{0, \dots, M\}$ as well as the piecewise affine-linear function

$$U^k(t) = \mathbf{U}^{n-1} + \frac{t - t_{n-1}}{k}(\mathbf{U}^n - \mathbf{U}^{n-1}), \tag{3.29}$$

with

$$\bar{U}_m^k(0) = \bar{U}^k(0) = \underline{U}^k(0) = U^k(0) = u_0^k \quad \text{in } H, \quad m \in \{0, \dots, M\}. \tag{3.30}$$

In the following, we always consider step sizes $k = \frac{T}{N_\ell}$, where $(N_\ell)_{\ell \in \mathbb{N}}$ is a sequence of natural numbers with $N_\ell \to \infty$ as $\ell \to \infty$. For simplicity, a sequence $(\bar{U}^{\frac{T}{N_\ell}})_{\ell \in \mathbb{N}}$ is abbreviated by $(\bar{U}^k)_{k>0}$ and analogously for the other functions introduced above.

The function U^k is weakly differentiable. Note that its weak derivative coincides with the classical derivative at the points where the latter exists. Using (3.4) and (3.5) as well as $\mathbf{U}^n = \mathbf{U}_M^n$ in H, its weak derivative can be rewritten as

$$\begin{aligned}(U^k)'(t) = \frac{1}{k}\big(\mathbf{U}^n - \mathbf{U}^{n-1}\big) &= \frac{1}{k}\sum_{m=1}^{M}\big(\mathbf{U}_m^n - \mathbf{U}_{m-1}^n\big) + \frac{1}{k}\big(\mathbf{U}_0^n - \mathbf{U}^{n-1}\big) \\ &= \sum_{m=1}^{M}\big(\mathbf{f}_m^n - \mathbf{A}_m^n\mathbf{U}_m^n\big) - \mathbf{B}^n\mathbf{U}^{n-1} \quad \text{in } V^*\end{aligned}$$

for $t \in (t_{n-1}, t_n)$, $n \in \{1, \dots, N\}$. Therefore, we see that

$$\begin{cases} (U^k)' + \sum_{m=1}^M A_m^k \bar{U}_m^k + B^k \underline{U}^k = \sum_{m=1}^M f_m^k & \text{in } L^q(0,T;V^*), \\ U^k(0) = u_0^k & \text{in } H. \end{cases} \tag{3.31}$$

For every $m \in \{1, \dots, M\}$, the operator A_m^k maps $L^p(0,T;V_m)$ into $L^q(0,T;V_m^*)$, compare Lemma 3.1.7. Together with the a priori bounds from Lemma 3.1.12 this shows that (3.31) is well-defined. For the following calculations, it will be helpful to have an integrated version of (3.31).

Lemma 3.1.13. *Let Assumptions 3.1.1, 3.1.2, 3.1.3, 3.1.5, and 3.1.8 be fulfilled. For $N \in \mathbb{N}$, $k = \frac{T}{N}$, and grid points $t_n = nk$, $n \in \{0, \dots, N\}$, as well as $u_0^k \in H$, let the piecewise constant and piecewise linear prolongations be given as in* (3.27)–(3.29). *Then*

$$\begin{aligned}&\frac{1}{2}\|\bar{U}^k(t_n)\|_H^2 - \frac{1}{2}\|\bar{U}^k(0)\|_H^2 + \sum_{m=1}^{M}\int_0^{t_n}\langle A_m^k(t)\bar{U}_m^k(t), \bar{U}_m^k(t)\rangle_{V_m^* \times V_m}\,\mathrm{d}t \\ &+ \int_0^{t_n}(B^k(t)\underline{U}^k(t), \bar{U}_0^k(t))_H\,\mathrm{d}t \le \sum_{m=1}^{M}\int_0^{t_n}\langle f_m^k(t), \bar{U}_m^k(t)\rangle_{V_m^* \times V_m}\,\mathrm{d}t\end{aligned}$$

is fulfilled for every $n \in \{1, \dots, N\}$.

Proof. In order to prove the inequality, we test (3.4) with $\mathbf{U}_0^i \in H$ to get

$$\frac{1}{k}(\mathbf{U}_0^i - \mathbf{U}^{i-1}, \mathbf{U}_0^i)_H + (\mathbf{B}^i\mathbf{U}^{i-1}, \mathbf{U}_0^i)_H = 0, \tag{3.32}$$

and (3.5) with $\mathbf{U}_m^i \in V_m$ to obtain

$$\frac{1}{k}(\mathbf{U}_m^i - \mathbf{U}_{m-1}^i, \mathbf{U}_m^i)_H + \langle \mathbf{A}_m^i \mathbf{U}_m^i, \mathbf{U}_m^i \rangle_{V_m^* \times V_m} = \langle \mathbf{f}_m^i, \mathbf{U}_m^i \rangle_{V_m^* \times V_m} \tag{3.33}$$

for $i \in \{1, \dots, N\}$ and $m \in \{1, \dots, M\}$. Summing up (3.33) from $m = 1$ to M and adding (3.32), yields

$$\begin{aligned} &\frac{1}{k}\sum_{m=1}^{M}(\mathbf{U}_m^i - \mathbf{U}_{m-1}^i, \mathbf{U}_m^i)_H + \frac{1}{k}(\mathbf{U}_0^i - \mathbf{U}^{i-1}, \mathbf{U}_0^i)_H \\ &\quad + \sum_{m=1}^{M}\langle \mathbf{A}_m^i \mathbf{U}_m^i, \mathbf{U}_m^i \rangle_{V_m^* \times V_m} + (\mathbf{B}^i \mathbf{U}^{i-1}, \mathbf{U}_0^i)_H = \sum_{m=1}^{M}\langle \mathbf{f}_m^i, \mathbf{U}_m^i \rangle_{V_m^* \times V_m} \end{aligned}$$

for $i \in \{1, \dots, N\}$. Another summation of this equality from $i = 1$ to $n \in \{1, \dots, N\}$ and a multiplication with k shows that

$$\sum_{i=1}^{n}\sum_{m=1}^{M}(\mathbf{U}_m^i - \mathbf{U}_{m-1}^i, \mathbf{U}_m^i)_H + \sum_{i=1}^{n}(\mathbf{U}_0^i - \mathbf{U}^{i-1}, \mathbf{U}_0^i)_H \tag{3.34}$$

$$+ k\sum_{i=1}^{n}\sum_{m=1}^{M}\langle \mathbf{A}_m^i \mathbf{U}_m^i, \mathbf{U}_m^i \rangle_{V_m^* \times V_m} + k\sum_{i=1}^{n}(\mathbf{B}^i \mathbf{U}^{i-1}, \mathbf{U}_0^i)_H \tag{3.35}$$

$$= k\sum_{i=1}^{n}\sum_{m=1}^{M}\langle \mathbf{f}_m^i, \mathbf{U}_m^i \rangle_{V_m^* \times V_m}. \tag{3.36}$$

We can write for (3.34)

$$\begin{aligned} &\sum_{i=1}^{n}\sum_{m=1}^{M}(\mathbf{U}_m^i - \mathbf{U}_{m-1}^i, \mathbf{U}_m^i)_H + \sum_{i=1}^{n}(\mathbf{U}_0^i - \mathbf{U}^{i-1}, \mathbf{U}_0^i)_H \\ &\geq \frac{1}{2}\sum_{i=1}^{n}\sum_{m=1}^{M}\left(\|\mathbf{U}_m^i\|_H^2 - \|\mathbf{U}_{m-1}^i\|_H^2\right) + \frac{1}{2}\sum_{i=1}^{n}\left(\|\mathbf{U}_0^i\|_H^2 - \|\mathbf{U}^{i-1}\|_H^2\right) \\ &= \frac{1}{2}\sum_{i=1}^{n}\left(\|\mathbf{U}_M^i\|_H^2 - \|\mathbf{U}_0^i\|_H^2\right) + \frac{1}{2}\sum_{i=1}^{n}\left(\|\mathbf{U}_0^i\|_H^2 - \|\mathbf{U}^{i-1}\|_H^2\right) \\ &= \frac{1}{2}\left(\|\mathbf{U}^n\|_H^2 - \|\mathbf{U}^0\|_H^2\right) = \frac{1}{2}\left(\|\bar{U}^k(t_n)\|_H^2 - \|\bar{U}^k(0)\|_H^2\right), \end{aligned}$$

due to the identity from Lemma A.1.4, the telescopic structure, and $\mathbf{U}_M^i = \mathbf{U}^i$ in H. Inserting the definition of the piecewise constant prolongations from (3.27) and (3.28) in (3.35) and (3.36), yields

$$\begin{aligned} &\frac{1}{2}\|\bar{U}^k(t_n)\|_H^2 - \frac{1}{2}\|\bar{U}^k(0)\|_H^2 + \sum_{m=1}^{M}\int_0^{t_n}\langle A_m^k(t)\bar{U}_m^k(t), \bar{U}_m^k(t)\rangle_{V_m^* \times V_m}\,\mathrm{d}t \\ &+ \int_0^{t_n}(B^k(t)\underline{U}^k(t), \bar{U}_0^k(t))_H\,\mathrm{d}t \leq \sum_{m=1}^{M}\int_0^{t_n}\langle f_m^k(t), \bar{U}_m^k(t)\rangle_{V_m^* \times V_m}\,\mathrm{d}t. \end{aligned}$$

□

It remains to look closer at the behavior of the prolongations from (3.27), (3.28), and (3.29) as the step size k tends to zero. The following lemma shows that the sequences of such prolongations converge in a suitable sense.

Lemma 3.1.14. *Let Assumptions 3.1.1, 3.1.2, 3.1.3, 3.1.5, and 3.1.8 be fulfilled. Further, let $(N_\ell)_{\ell\in\mathbb{N}}$ be a sequence of natural numbers with $N_\ell \to \infty$ as $\ell \to \infty$, let the step sizes be given by $k = \frac{T}{N_\ell}$, and let $(u_0^k)_{k>0}$ be a bounded sequence in H. Then for the sequences of piecewise constant and piecewise linear prolongations as given in* (3.28) *and* (3.29)*, there exists a subsequence of step sizes, again denoted by k, such that*

$$\begin{aligned}
&\bar{U}_m^k \rightharpoonup U && \text{in } L^p(0,T;V_m), && m \in \{1,\dots,M\},\\
&\bar{U}_m^k \overset{*}{\rightharpoonup} U, \quad \underline{U}^k \overset{*}{\rightharpoonup} U, \quad U^k \overset{*}{\rightharpoonup} U && \text{in } L^\infty(0,T;H), && m \in \{0,\dots,M\},\\
&(U^k)' \rightharpoonup U' && \text{in } L^q(0,T;V^*)
\end{aligned}$$

as $k \to 0$. The limit U is an element of $L^p(0,T;V) \cap L^\infty(0,T;H)$ and its weak derivative fulfills $U' \in L^q(0,T;V^*)$. This implies, in particular, that U is an element of $\mathcal{W}^p(0,T)$.

Furthermore, let V_M be compactly embedded into H, let u_0^k be in V_M for every $k > 0$, and let $(k^{\frac{1}{p}}\|u_0^k\|_{V_M})_{k>0}$ be uniformly bounded. Then it additionally follows that

$$\begin{aligned}
&U^k \rightharpoonup U && \text{in } L^p(0,T;V_M),\\
&\bar{U}^k \to U, \quad \underline{U}^k \to U, \quad U^k \to U && \text{in } L^2(0,T;H)
\end{aligned}$$

as $k \to 0$.

Note that the compact embedding of V_M into H is only necessary to prove that the sequences of piecewise constant and piecewise linear prolongations converge strongly in $L^2(0,T;H)$. If Assumption 3.1.2 is generalized to $p \in (1,\infty)$, it is still possible to prove $U^k \to U$ in $L^2(0,T;H)$. We can use the compact embedding argument from the Lions–Aubin lemma (cf. Lemma A.2.5) to obtain strong convergence in $L^p(0,T;H)$. Together with the a priori bound (3.13) from Lemma 3.1.12 and Lemma A.2.3, it follows that the sequence converges strongly in $L^2(0,T;H)$.

Proof of Lemma 3.1.14. For simplicity, we do not denote the subsequences differently within this proof and we drop the index ℓ. Using the a priori bound (3.15) from Lemma 3.1.12, it follows that the sequence $(\bar{U}_m^k)_{k>0}$ of piecewise constant prolongations is bounded in $L^p(0,T;V_m)$ and $L^\infty(0,T;H)$ for every $m \in \{1,\dots,M\}$. Since $L^p(0,T;V_m)$ is a reflexive Banach space and $L^\infty(0,T;H)$ is the dual of the separable Banach space $L^1(0,T;H)$, there exists an element $U_m \in L^p(0,T;V_m) \cap L^\infty(0,T;H)$ such that

$$\bar{U}_m^k \rightharpoonup U_m \quad \text{in } L^p(0,T;V_m), \quad \bar{U}_m^k \overset{*}{\rightharpoonup} U_m \quad \text{in } L^\infty(0,T;H)$$

as $k \to 0$ for every $m \in \{1,\dots,M\}$. Analogously, there exists $U_0 \in L^\infty(0,T;H)$ such that

$$\bar{U}_0^k \overset{*}{\rightharpoonup} U_0 \quad \text{in } L^\infty(0,T;H) \quad \text{as } k \to 0,$$

where we use the a priori bound (3.14) from Lemma 3.1.12. In the following, we will prove that $U_0 = U_1 = \ldots = U_M =: U$ in $L^p(0,T;V)$ and $L^\infty(0,T;H)$ is fulfilled. To this end, it is sufficient to show that U_0 and U_1 coincide in $L^p(0,T;V_1)$ and $L^\infty(0,T;H)$. The other equalities follow analogously. For the difference of the two functions, we see that

$$\bar{U}_1^k(t) - \bar{U}_0^k(t) = \mathbf{U}_1^n - \mathbf{U}_0^n = k\big(\mathbf{f}_1^n - \mathbf{A}_1^n\mathbf{U}_1^n\big) = \int_{t_{n-1}}^{t_n} \big(f_1(s) - A_1^k(s)\bar{U}_1^k(s)\big)\,\mathrm{d}s$$

in V^* for $t \in (t_{n-1}, t_n]$, $n \in \{1, \dots, N\}$, due to the definition of the scheme (3.5). Therefore, we obtain

$$\begin{aligned}\|\bar{U}_1^k(t) - \bar{U}_0^k(t)\|_{V_1^*} &= \Big\| \int_{t_{n-1}}^{t_n} \big(f_1(s) - A_1^k(s)\bar{U}_1^k(s)\big)\,\mathrm{d}s \Big\|_{V_1^*} \\ &\le \int_{t_{n-1}}^{t_n} \|f_1(s) - A_1^k(s)\bar{U}_1^k(s)\|_{V_1^*}\,\mathrm{d}s \\ &\le k^{\frac{1}{p}} \Big(\int_{t_{n-1}}^{t_n} \|f_1(s) - A_1^k(s)\bar{U}_1^k(s)\|_{V_1^*}^q\,\mathrm{d}s \Big)^{\frac{1}{q}},\end{aligned}$$

where we can bound the integral by

$$\begin{aligned}\Big(\int_{t_{n-1}}^{t_n} \|f_1(s) - A_1^k(s)\bar{U}_1^k(s)\|_{V_1^*}^q\,\mathrm{d}s \Big)^{\frac{1}{q}} &\le \Big(\int_{t_{n-1}}^{t_n} \|f_1(s)\|_{V_1^*}^q\,\mathrm{d}s \Big)^{\frac{1}{q}} + k^{\frac{1}{q}} \|\mathbf{A}_1^n \mathbf{U}_1^n\|_{V_1^*} \\ &\le \|f_1\|_{L^q(0,T;V_1^*)} + k^{\frac{1}{q}} \beta \big(1 + \|\mathbf{U}_1^n\|_{V_1}^{p-1}\big).\end{aligned}$$

This is bounded independently of the step size k and $n \in \{1, \dots, N\}$ due the a priori bound (3.15) from Lemma 3.1.12. Thus, we have proved that $\|\bar{U}_1^k(t) - \bar{U}_0^k(t)\|_{V_1^*} \to 0$ as $k \to 0$ for every $t \in [0,T]$. Further, this also shows that $\big(\|\bar{U}_1^k(t) - \bar{U}_0^k(t)\|_{V_1^*}\big)_{k>0}$ is uniformly bounded independently of $t \in [0,T]$. Thus, we can apply Lebesgue's dominated convergence theorem to see

$$\|\bar{U}_1^k - \bar{U}_0^k\|_{L^q(0,T;V_1^*)} \to 0 \quad \text{as } k \to 0.$$

Hence, U_0 and U_1 coincide in $L^q(0,T;V_1^*)$. Both spaces $L^p(0,T;V_1)$ and $L^\infty(0,T;H)$ are embedded into $L^q(0,T;V_1^*)$. This implies that $U_0 = U_1$ is fulfilled in all three spaces as the embedding is always injective and $U_0 \in L^\infty(0,T;H)$ and $U_1 \in L^p(0,T;V_1) \cap L^\infty(0,T;H)$. The fact that $U := U_0 = U_1 = \cdots = U_M$ in $\bigcap_{m=1}^M L^p(0,T;V_m)$ and $L^\infty(0,T;H)$ can be proved analogously. Due to Assumption 3.1.1, we know that $\bigcap_{m=1}^M V_m = V$ and the norm $\sum_{m=1}^M \|\cdot\|_{V_m}$ is equivalent to $\|\cdot\|_V$. This shows, in particular, that $U \in \bigcap_{m=1}^M L^p(0,T;V_m) = L^p(0,T;V)$. Note that the functions $\bar{U}_M^k$ and $\bar{U}^k$ coincide by definition. Therefore, it follows that $\bar{U}^k \overset{*}{\rightharpoonup} U$ in $L^\infty(0,T;H)$ as $k \to 0$.

Another application of the a priori bound (3.13) from Lemma 3.1.12 shows that $(U^k)_{k>0}$ is bounded in $L^\infty(0,T;H)$. Again, we find a subsequence and $\tilde{U} \in L^\infty(0,T;H)$ such that

$$U^k \overset{*}{\rightharpoonup} \tilde{U} \quad \text{in } L^\infty(0,T;H) \quad \text{as } k \to 0.$$

Furthermore, the difference of $\bar{U}^k$ and U^k converges to zero in $L^q(0,T;V^*)$ since

$$\begin{aligned}\int_0^T \|\bar{U}^k(t) - U^k(t)\|_{V^*}^q\,\mathrm{d}t &= \sum_{n=1}^N \int_{t_{n-1}}^{t_n} \Big\| \mathbf{U}^n - \mathbf{U}^{n-1} - \frac{t - t_{n-1}}{k}(\mathbf{U}^n - \mathbf{U}^{n-1}) \Big\|_{V^*}^q \,\mathrm{d}t \\ &= \frac{1}{k^q} \sum_{n=1}^N \|\mathbf{U}^n - \mathbf{U}^{n-1}\|_{V^*}^q \int_{t_{n-1}}^{t_n} (t_n - t)^q\,\mathrm{d}t \\ &= \frac{k}{q+1} \sum_{n=1}^N \|\mathbf{U}^n - \mathbf{U}^{n-1}\|_{V^*}^q \le \frac{k^q}{q+1} K \to 0 \qquad \text{as } k \to 0,\end{aligned}$$

where we used the a priori bound (3.16) from Lemma 3.1.12. Therefore, the limits of $(\bar{U}^k)_{k>0}$ and $(U^k)_{k>0}$ coincide in $L^q(0,T;V^*)$. The limits U and $\tilde{U}$ are elements of the space $L^\infty(0,T;H)$. This space is continuously embedded into $L^q(0,T;V^*)$. It then follows that $\tilde{U} = U$ in $L^\infty(0,T;H)$ because the embedding is injective.

The sequence $((U^k)')_{k>0}$ is bounded in $L^q(0,T;V^*)$ due to the a priori bound (3.16) from Lemma 3.1.12. As this space is a reflexive Banach space, we can extract a subsequence and find $W \in L^q(0,T;V^*)$ such that

$$(U^k)' \rightharpoonup W \quad \text{in } L^q(0,T;V^*) \quad \text{ as } k \to 0.$$

In order to prove that the sequence $(U^k)'$ converges to the weak derivative of U weakly in $L^q(0,T;V^*)$, we use that $U^k \overset{*}{\rightharpoonup} U$ in $L^\infty(0,T;H)$ as $k \to 0$ and see

$$\begin{aligned}
-\int_0^T \langle W(t), v\rangle_{V^*\times V}\varphi(t)\,\mathrm{d}t &= -\lim_{k\to 0}\int_0^T \langle (U^k)'(t), v\rangle_{V^*\times V}\varphi(t)\,\mathrm{d}t \\
&= \lim_{k\to 0}\int_0^T (U^k(t), v)_H\varphi'(t)\,\mathrm{d}t = \int_0^T (U(t),v)_H\varphi'(t)\,\mathrm{d}t
\end{aligned}$$

for $v \in V$ and $\varphi \in C_c^\infty(0,T)$. Applying [49, Kapitel IV, Lemma 1.7], it follows that $W = U'$ in $L^q(0,T;V^*)$ and therefore, in particular, $U \in \mathcal{W}^p(0,T)$.

In the following, we require that V_M is compactly embedded into H. We use the a priori bound (3.15) from Lemma 3.1.12 and $\mathbf{U}^n = \mathbf{U}^n_M$ in H for every $n \in \{1,\dots,N\}$ to see

$$\begin{aligned}
&\|U^k\|_{L^p(0,T;V_M)} \\
&= \Big(\sum_{n=1}^N \int_{t_{n-1}}^{t_n} \Big\|\mathbf{U}^{n-1} + \frac{t-t_{n-1}}{k}\big(\mathbf{U}^n - \mathbf{U}^{n-1}\big)\Big\|_{V_M}^p \,\mathrm{d}t\Big)^{\frac{1}{p}} \\
&= \Big(\sum_{n=1}^N \int_{t_{n-1}}^{t_n} \Big\|\frac{t_n - t}{k}\mathbf{U}^{n-1} + \frac{t-t_{n-1}}{k}\mathbf{U}^n\Big\|_{V_M}^p \,\mathrm{d}t\Big)^{\frac{1}{p}} \\
&\le \Big(\frac{1}{k^p}\sum_{n=1}^N \|\mathbf{U}^{n-1}\|_{V_M}^p \int_{t_{n-1}}^{t_n} (t_n - t)^p\,\mathrm{d}t\Big)^{\frac{1}{p}} + \Big(\frac{1}{k^p}\sum_{n=1}^N \|\mathbf{U}^{n}\|_{V_M}^p \int_{t_{n-1}}^{t_n} (t - t_{n-1})^p\,\mathrm{d}t\Big)^{\frac{1}{p}} \\
&\le \Big(\frac{k}{p+1}\sum_{n=1}^N \|\mathbf{U}^{n-1}\|_{V_M}^p\Big)^{\frac{1}{p}} + \Big(\frac{k}{p+1}\sum_{n=1}^N \|\mathbf{U}^{n}\|_{V_M}^p\Big)^{\frac{1}{p}} \\
&\le \Big(\frac{k}{p+1}\Big)^{\frac{1}{p}}\|u_0^k\|_{V_M} + 2\Big(\frac{K}{p+1}\Big)^{\frac{1}{p}}.
\end{aligned}$$

This is bounded as $\big(k^{\frac{1}{p}}\|u_0^k\|_{V_M}\big)_{k>0}$ is uniformly bounded with respect to k. As the weak limit of a sequence is unique, this implies that $U^k \rightharpoonup U$ in $L^p(0,T;V_M)$ as $k \to 0$, where we again choose a suitable subsequence if necessary. Since $(U^k)' \rightharpoonup U'$ in $L^q(0,T;V^*)$ as $k \to 0$, it follows that $U^k \rightharpoonup U$ in the space

$$\mathcal{W}^p_M(0,T) = \{v \in L^p(0,T;V_M) : v' \text{ exists and } v' \in L^q(0,T;V^*)\}$$

as $k \to 0$. By assumption, the space V_M is separable, reflexive, and compactly embedded into H. Furthermore, H^* is embedded into the reflexive space V^* and we see that

$$V_M \overset{d}{\hookrightarrow} H \cong H^* \overset{d}{\hookrightarrow} V^*$$

is fulfilled. Thus, we can apply Lemma A.2.5 and obtain that $\mathcal{W}^p_M(0,T)$ is compactly embedded into $L^2(0,T;H)$. As the embedding is compact, it follows that $U^k \to U$ in

$L^2(0,T;H)$ as $k \to 0$. This in mind, we can also prove that $\bar{U}^k \to U$ and $\underline{U}^k \to U$ in $L^2(0,T;H)$ as $k \to 0$. We use the a priori bounds (3.13) and (3.14) from Lemma 3.1.12 and $\mathbf{U}^n = \mathbf{U}^n_M$ in H to see that

$$\begin{aligned}
\Big(\sum_{n=1}^{N} \|\mathbf{U}^n - \mathbf{U}^{n-1}\|_H^2\Big)^{\frac{1}{2}} &= \Big(\sum_{n=1}^{N} \Big\| \sum_{m=1}^{M} (\mathbf{U}^n_m - \mathbf{U}^n_{m-1}) + \mathbf{U}^n_0 - \mathbf{U}^{n-1}\Big\|_H^2\Big)^{\frac{1}{2}} \\
&\le \Big(\sum_{n=1}^{N} \Big(\sum_{m=1}^{M} \|\mathbf{U}^n_m - \mathbf{U}^n_{m-1}\|_H + \|\mathbf{U}^n_0 - \mathbf{U}^{n-1}\|_H\Big)^2\Big)^{\frac{1}{2}} \\
&\le \Big(\sum_{n=1}^{N}\sum_{m=1}^{M} \|\mathbf{U}^n_m - \mathbf{U}^n_{m-1}\|_H^2 + \|\mathbf{U}^n_0 - \mathbf{U}^{n-1}\|_H^2\Big)^{\frac{1}{2}} \\
&\le (2K)^{\frac{1}{2}}.
\end{aligned} \tag{3.37}$$

Considering the difference of $\bar{U}^k$ and U^k in the $L^2(0,T;H)$-norm squared, it follows that

$$\begin{aligned}
\|\bar{U}^k - U^k\|^2_{L^2(0,T;H)} &= \sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} \Big\| \mathbf{U}^n - \mathbf{U}^{n-1} - \frac{t - t_{n-1}}{k} (\mathbf{U}^n - \mathbf{U}^{n-1})\Big\|_H^2 \,\mathrm{d}t \\
&= \frac{1}{k^2} \sum_{n=1}^{N} \|\mathbf{U}^n - \mathbf{U}^{n-1}\|_H^2 \int_{t_{n-1}}^{t_n} (t_n - t)^2 \,\mathrm{d}t \\
&= \frac{k}{3} \sum_{n=1}^{N} \|\mathbf{U}^n - \mathbf{U}^{n-1}\|_H^2 \le \frac{2k}{3} K \to 0 \quad \text{ as } k \to 0.
\end{aligned}$$

Therefore, we have shown that

$$\|\bar{U}^k - U\|_{L^2(0,T;H)} \le \|\bar{U}^k - U^k\|_{L^2(0,T;H)} + \|U^k - U\|_{L^2(0,T;H)} \to 0 \quad \text{ as } k \to 0.$$

Similarly, we consider the difference of $\underline{U}^k$ and $\bar{U}^k$ in $L^2(0,T;H)$ to find

$$\begin{aligned}
\|\bar{U}^k - \underline{U}^k\|^2_{L^2(0,T;H)} &= \sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} \|\mathbf{U}^n - \mathbf{U}^{n-1}\|_H^2 \,\mathrm{d}t \\
&= k \sum_{n=1}^{N} \|\mathbf{U}^n - \mathbf{U}^{n-1}\|_H^2 \le 2kK \to 0 \quad \text{ as } k \to 0,
\end{aligned}$$

where we again use the bound from (3.37). The last desired convergence result then is fulfilled as we have shown

$$\|\underline{U}^k - U\|_{L^2(0,T;H)} \le \|\underline{U}^k - \bar{U}^k\|_{L^2(0,T;H)} + \|\bar{U}^k - U\|_{L^2(0,T;H)} \to 0 \quad \text{ as } k \to 0.$$

□

Lemma 3.1.15. *Let Assumptions 3.1.1, 3.1.2, and 3.1.3 be fulfilled. Then for every $m \in \{1, \dots, M\}$ the operator A^k_m defined in* (3.27) *fulfills that*

$$A^k_m v \to A_m v \quad \text{in } L^q(0,T;V^*_m)$$

as $k \to 0$ for every $v \in L^p(0,T;V_m)$.

Proof. Let $m \in \{1, \dots, M\}$ be arbitrary but fixed in the following. We want to estimate $A_m^k v - A_m v$ within the $L^q(0,T;V_m^*)$-norm. To this end, we notice that

$$\begin{aligned}
\|A_m^k v - A_m v\|_{L^q(0,T;V_m^*)}^q &= \int_0^T \|A_m^k(t)v(t) - A_m(t)v(t)\|_{V_m^*}^q \,\mathrm{d}t \\
&= \sum_{n=1}^N \int_{t_{n-1}}^{t_n} \Big\| \frac{1}{k} \int_{t_{n-1}}^{t_n} \big(A_m(s)v(t) - A_m(t)v(t)\big) \,\mathrm{d}s \Big\|_{V_m^*}^q \,\mathrm{d}t.
\end{aligned}$$

For $t \in [0,T]$ such that $s \mapsto A_m(s)w$ is continuous for all $w \in V_m$, we always choose $n \in \{1, \dots, N\}$ such that $t \in (t_{n-1}, t_n]$. Then it follows that

$$\Big\| \frac{1}{k} \int_{t_{n-1}}^{t_n} \big(A_m(s)v(t) - A_m(t)v(t)\big) \,\mathrm{d}s \Big\|_{V_m^*} \leq \frac{1}{k} \int_{t_{n-1}}^{t_n} \|A_m(s)v(t) - A_m(t)v(t)\|_{V_m^*} \,\mathrm{d}s \to 0$$

as $k \to 0$. Furthermore, an application of the boundedness condition for $A_m(t)$, $t \in [0,T]$, from Assumption 3.1.2 (4), shows that

$$\begin{aligned}
&\Big\| \frac{1}{k} \int_{t_{n-1}}^{t_n} \big(A_m(s)v(t) - A_m(t)v(t)\big) \,\mathrm{d}s \Big\|_{V_m^*} \\
&\leq \frac{1}{k} \int_{t_{n-1}}^{t_n} \|A_m(s)v(t) - A_m(t)v(t)\|_{V_m^*} \,\mathrm{d}s \\
&\leq \frac{1}{k} \int_{t_{n-1}}^{t_n} \|A_m(s)v(t)\|_{V_m^*} \,\mathrm{d}s + \frac{1}{k} \int_{t_{n-1}}^{t_n} \|A_m(t)v(t)\|_{V_m^*} \,\mathrm{d}s \\
&\leq 2\beta\big(1 + \|v(t)\|_{V_m}^{p-1}\big) =: g(t)
\end{aligned}$$

for almost all $t \in (t_{n-1}, t_n)$, $n \in \{1, \dots, N\}$. Since $\|v(t)\|_{V_m}^{(p-1)q} = \|v(t)\|_{V_m}^p$ and $v \in L^p(0,T;V_m)$, it follows that $g \in L^q(0,T)$. Now, we can apply Lebesgue's dominated convergence theorem and see

$$\lim_{k \to 0} \|A_m^k v - A_m v\|_{L^q(0,T;V_m^*)}^q = \int_0^T \lim_{k \to 0} \|A_m^k(t)v(t) - A_m(t)v(t)\|_{V_m^*}^q \,\mathrm{d}t = 0.$$

□

Lemma 3.1.16. *Let Assumptions 3.1.1 and 3.1.5 be fulfilled. Then the operator B^k defined in (3.27) fulfills that*

$$B^k v \to Bv \quad \text{in } L^2(0,T;H)$$

as $k \to 0$ for every $v \in L^2(0,T;H)$.

We omit the proof, as it can be done analogously to the proof of Lemma 3.1.15.

Lemma 3.1.17. *Let Assumptions 3.1.1 and 3.1.8 be fulfilled. For every $m \in \{1, \dots, M\}$, it follows that f_m^k defined in (3.27) fulfills $f_m^k \to f_m$ in $L^q(0,T;V_m^*)$ as $k \to 0$.*

Again, we omit the proof as it is essentially the same as the proof of Lemma 2.1.9. The previous lemmas in mind, we are well prepared to prove that the limit $U \in \mathcal{W}^p(0,T)$ from Lemma 3.1.14 is the solution to the initial value problem (3.1). At first, we show in Theorem 3.1.18 that the sequences of piecewise constant and piecewise linear prolongations of the solution from (3.4)–(3.6) converge to the solution u of the evolution equation (3.1) in a weak sense. Also, a strong convergence result can be proved. If V_M is compactly embedded into H, we can argue directly that certain strong convergence results are fulfilled. This compact embedding is only necessary if the Lipschitz continuous, H-valued operator $B(t)$, $t \in [0,T]$, is not constantly zero. If this operator is constantly zero and the embedding from V_M into H is not compact, we can still show a pointwise strong convergence result in Theorem 3.1.19 below.

Theorem 3.1.18. *Let Assumptions 3.1.1, 3.1.2, 3.1.3, 3.1.5, and 3.1.8 be fulfilled and let $u_0 \in H$ be given. For a sequence $(N_\ell)_{\ell \in \mathbb{N}}$ of natural numbers with $N_\ell \to \infty$ as $\ell \to \infty$, step sizes $k = \frac{T}{N_\ell}$, and a bounded sequence $(u_0^k)_{k>0}$ in H such that $u_0^k \to u_0$ in H as $k \to 0$, let the sequences of piecewise constant and piecewise linear prolongations from* (3.28) *and* (3.29) *be given. If κ in Assumption 3.1.5 (2) is zero, then*

$$\begin{aligned}
&\bar{U}_m^k \rightharpoonup u && \text{in } L^p(0,T;V_m), \quad m \in \{1,\dots,M\},\\
&\bar{U}_m^k \overset{*}{\rightharpoonup} u, \quad \bar{U}^k \overset{*}{\rightharpoonup} u, \quad U^k \overset{*}{\rightharpoonup} u && \text{in } L^\infty(0,T;H), \quad m \in \{0,\dots,M\},\\
&(U^k)' \rightharpoonup u' && \text{in } L^q(0,T;V^*),\\
&\sum_{m=1}^{M} A_m^k \bar{U}_m^k \rightharpoonup Au && \text{in } L^q(0,T;V^*),\\
&B^k \underline{U}^k \to Bu && \text{in } L^2(0,T;H)
\end{aligned}$$

as $k \to 0$. Here, u is the solution of (3.1) *and u' its weak derivative. Also, it holds true that both $\bar{U}^k(t) \rightharpoonup u(t)$ and $U^k(t) \rightharpoonup u(t)$ in H as $k \to 0$ for every $t \in [0,T]$.*

Furthermore, let $(u_0^k)_{k>0}$ be in V_M, let $\big(k^{\frac{1}{p}} \|u_0^k\|_{V_M}\big)_{k>0}$ be uniformly bounded with respect to k, and let the space V_M be compactly embedded into H. Then it follows for an arbitrary value $\kappa \in [0,\infty)$ from Assumption 3.1.5 (2) that the results above are fulfilled and additionally it follows that

$$\begin{aligned}
&U^k \rightharpoonup u && \text{in } L^p(0,T;V_M),\\
&\bar{U}^k \to u, \quad \underline{U}^k \to u, \quad U^k \to u && \text{in } L^2(0,T;H)
\end{aligned}$$

as $k \to 0$.

It would also be possible to prove $\underline{U}^k(t) \rightharpoonup u(t)$ and $\bar{U}_m^k(t) \rightharpoonup u(t)$, $m \in \{0,\dots,M\}$, in H as $k \to 0$ for every $t \in [0,T]$. For simplicity, we concentrate on the sequences $(\bar{U}^k(t))_{k>0}$ and $(U^k(t))_{k>0}$.

Proof of Theorem 3.1.18. For simplicity, we do not denote the subsequences differently in this proof and we drop the index ℓ. In the cases that κ from Assumption 3.1.5 (2) is zero, it is easy to see that $B^k \underline{U}^k = B^k U \to BU$ in $L^2(0,T;H)$ as $k \to 0$ due to Lemma 3.1.16. If κ is strictly larger than zero and V_M is compactly embedded into H, we can apply Lemma 3.1.14 to obtain that $\underline{U}^k \to U$ in $L^2(0,T;H)$ as $k \to 0$. Since the inequality

$$\|BU - B^k \underline{U}^k\|_{L^2(0,T;H)} \le \|BU - B^k U\|_{L^2(0,T;H)} + \|B^k U - B^k \underline{U}^k\|_{L^2(0,T;H)} \tag{3.38}$$

is fulfilled for every $k > 0$, we can consider the two summands separately. For the first summand, we can apply Lemma 3.1.16 to obtain

$$\|BU - B^k U\|_{L^2(0,T;H)} \to 0 \quad \text{as } k \to 0.$$

For the square of the second summand on the right-hand side of (3.38), we use the Lipschitz continuity of the operator $\mathbf{B}^n$, $n \in \{1, \dots, N\}$, compare (3.10) of Lemma 3.1.10, and the fact that $\underline{U}^k \to U$ in $L^2(0,T;H)$ as $k \to 0$. Then we see that

$$\begin{aligned} \|B^k U - B^k \underline{U}^k\|_{L^2(0,T;H)}^2 &= \sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} \|\mathbf{B}^n U(t) - \mathbf{B}^n \mathbf{U}^{n-1}\|_H^2 \,\mathrm{d}t \\ &\le \kappa^2 \sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} \|U(t) - \mathbf{U}^{n-1}\|_H^2 \,\mathrm{d}t \\ &= \kappa^2 \|U - \underline{U}^k\|_{L^2(0,T;H)}^2 \to 0 \end{aligned}$$

as $k \to 0$. Altogether, this proves that

$$B^k \underline{U}^k \to BU \quad \text{in } L^2(0,T;H) \quad \text{as } k \to 0. \tag{3.39}$$

Due to the a priori bound (3.15) from Lemma 3.1.12 and the boundedness condition (3.8) from Lemma 3.1.9, we find that

$$\begin{aligned} \|A_m^k \bar{U}_m^k\|_{L^q(0,T;V_m^*)} &= \Big(k \sum_{n=1}^{N} \|\mathbf{A}_m^n \mathbf{U}_m^n\|_{V_m^*}^q\Big)^{\frac{1}{q}} \\ &\le \Big(k \sum_{n=1}^{N} \beta^q \big(1 + \|\mathbf{U}_m^n\|_{V_m}^{p-1}\big)^q\Big)^{\frac{1}{q}} \le \beta\big(T^{\frac{1}{q}} + K^{\frac{1}{q}}\big). \end{aligned}$$

As $L^q(0,T;V_m^*)$ is a reflexive Banach space, we can extract a weakly converging subsequence such that

$$A_m^k \bar{U}_m^k \rightharpoonup b_m \quad \text{in } L^q(0,T;V_m^*) \quad \text{as } k \to 0$$

for $b_m \in L^q(0,T;V_m^*)$. Next, we identify the derivative of U with the equation. Using Lemma 3.1.14, Lemma 3.1.17, (3.39), and $b := \sum_{m=1}^{M} b_m$ in $L^q(0,T;V^*)$, we obtain the following equality

$$U' = \operatorname*{w-lim}_{k\to 0} (U^k)' = \operatorname*{w-lim}_{k\to 0} \Big(\sum_{m=1}^{M} \big(f_m^k - A_m^k \bar{U}_m^k\big) - B^k \underline{U}^k \Big) = f - b - BU$$

in $L^q(0,T;V^*)$. By w-lim we denote the limiting process with respect to the weak topology in $L^q(0,T;V^*)$. Since $U \in \mathcal{W}^p(0,T)$ and $\mathcal{W}^p(0,T)$ is continuously embedded into $C([0,T];H)$, we can work with the continuous representative of U in the following.

Another application of the a priori bound (3.13) from Lemma 3.1.12 shows that the sequence $(U^k(t))_{k>0}$, $t \in [0,T]$, is bounded in H. As H is reflexive, for every $t \in [0,T]$ there exist a subsequence and an element $\tilde{U}(t) \in H$ with

$$U^k(t) \rightharpoonup \tilde{U}(t) \quad \text{in } H \tag{3.40}$$

as $k \to 0$. This in mind, we prove $U(t) = \tilde{U}(t)$ and $U(0) = u_0$ for every $t \in [0,T]$. First, we recall that $(U^k)' \rightharpoonup U'$ in $L^q(0,T;V^*)$ and $U^k \overset{*}{\rightharpoonup} U$ in $L^\infty(0,T;H)$ as $k \to 0$, compare

Lemma 3.1.14. For arbitrary but fixed $x \in V$ and $\varphi \in C^1([0,T])$, we then find that

$$\begin{aligned}
&(U(t),x)_H\varphi(t) - (U(0),x)_H\varphi(0) - \int_0^t (U(s),x)_H\varphi'(s)\,\mathrm{d}s \\
&= \int_0^t \langle U'(s),x\rangle_{V^*\times V}\varphi(s)\,\mathrm{d}s = \lim_{k\to 0}\int_0^t ((U^k)'(s),x)_H\varphi(s)\,\mathrm{d}s \\
&= \lim_{k\to 0}\Big((U^k(t),x)_H\varphi(t) - (u_0^k,x)_H\varphi(0) - \int_0^t (U^k(s),x)_H\varphi'(s)\,\mathrm{d}s\Big) \\
&= (\tilde{U}(t),x)_H\varphi(t) - (u_0,x)_H\varphi(0) - \int_0^t (U(s),x)_H\varphi'(s)\,\mathrm{d}s
\end{aligned}$$

for $t \in [0,T]$. This implies $U(t) = \tilde{U}(t)$ and $U(0) = u_0$ in H. For the piecewise constant prolongation $\bar{U}^k$, we see that

$$\begin{aligned}
\|\bar{U}^k(t) - U^k(t)\|_{V^*}^q &= \Big\|\mathbf{U}^n - \mathbf{U}^{n-1} - \frac{t-t_{n-1}}{k}\big(\mathbf{U}^n - \mathbf{U}^{n-1}\big)\Big\|_{V^*}^q \\
&\le \Big(\frac{t_n - t}{k}\Big)^q \|\mathbf{U}^n - \mathbf{U}^{n-1}\|_{V^*}^q \le \sum_{i=1}^N \|\mathbf{U}^i - \mathbf{U}^{i-1}\|_{V^*}^q \le k^{q-1}K
\end{aligned}$$

for every $t \in (t_{n-1},t_n]$, $n \in \{1,\dots,N\}$. For the last inequality, we use the a priori bound (3.16) from Lemma 3.1.12. Thus, it follows that $\|\bar{U}^k(t) - U^k(t)\|_{V^*} \to 0$ as $k \to 0$ for every $t \in [0,T]$. This means that the limits of $(\bar{U}^k(t))_{k>0}$ and $(U^k(t))_{k>0}$ coincide in V^*. Due to the a priori bound (3.13) from Lemma 3.1.12, $(\bar{U}^k(t))_{k>0}$ is bounded in H for every $t \in [0,T]$. Thus, we can extract a subsequence that converges weakly to an element of H. As H is continuously embedded into V^*, the limit has to coincide with $U(t)$ in H as the embedding is injective. This implies that $\bar{U}^k(t) \rightharpoonup U(t)$ in H as $k \to 0$ for every $t \in [0,T]$.

It remains to prove that $b = AU$ in $L^q(0,T;V^*)$. Recall that for every $m \in \{1,\dots,M\}$, the sequence $(\bar{U}_m^k)_{k>0}$ converges weakly to U in $L^p(0,T;V_m)$, $(\bar{U}_0^k)_{k>0}$ converges weakly$*$ to U in $L^\infty(0,T;H)$, and U is an element of $\mathcal{W}^p(0,T)$, compare Lemma 3.1.14. Using (3.39), the statements of Lemma 3.1.13 and Lemma 3.1.17 as well as the lower semi-continuity of the norm, it follows that

$$\begin{aligned}
&\limsup_{k\to 0}\Big(\sum_{m=1}^M \int_0^T \langle A_m^k(t)\bar{U}_m^k(t),\bar{U}_m^k(t)\rangle_{V_m^*\times V_m}\,\mathrm{d}t\Big) \\
&\le \limsup_{k\to 0}\Big(\sum_{m=1}^M \int_0^T \langle f_m^k(t),\bar{U}_m^k(t)\rangle_{V_m^*\times V_m}\,\mathrm{d}t - \int_0^T (B^k(t)\underline{U}^k(t),\bar{U}_0^k(t))_H\,\mathrm{d}t \\
&\qquad\qquad + \frac{1}{2}\|\bar{U}^k(0)\|_H^2 - \frac{1}{2}\|\bar{U}^k(T)\|_H^2\Big) \\
&\le \sum_{m=1}^M \int_0^T \langle f_m(t),U(t)\rangle_{V_m^*\times V_m}\,\mathrm{d}t - \int_0^T (B(t)U(t),U(t))_H\,\mathrm{d}t + \frac{1}{2}\|U(0)\|_H^2 - \frac{1}{2}\|U(T)\|_H^2 \\
&= \int_0^T \langle f(t) - B(t)U(t),U(t)\rangle_{V^*\times V}\,\mathrm{d}t - \int_0^T \langle U'(t),U(t)\rangle_{V^*\times V}\,\mathrm{d}t = \int_0^T \langle b(t),U(t)\rangle_{V^*\times V}\,\mathrm{d}t,
\end{aligned}$$

which implies

$$\limsup_{k\to 0}\sum_{m=1}^M \int_0^T \langle A_m^k(t)\bar{U}_m^k(t),\bar{U}_m^k(t)\rangle_{V_m^*\times V_m}\,\mathrm{d}t \le \int_0^T \langle b(t),U(t)\rangle_{V^*\times V}\,\mathrm{d}t. \tag{3.41}$$

Due to the monotonicity condition for $A_m(t)$, $t \in [0,T]$ and $m \in \{1,\dots,M\}$, from Assumption 3.1.2 (3) we can write

$$\sum_{m=1}^{M} \int_0^T \langle A_m^k(t)\bar{U}_m^k(t) - A_m^k(t)v(t), \bar{U}_m^k(t) - v(t)\rangle_{V_m^* \times V_m} \,\mathrm{d}t \geq 0$$

for every $v \in L^p(0,T;V)$. Therefore, an application of Lemma 3.1.14 and Lemma 3.1.15 shows

$$\begin{aligned}
&\sum_{m=1}^{M} \int_0^T \langle A_m^k(t)\bar{U}_m^k(t), \bar{U}_m^k(t)\rangle_{V_m^* \times V_m} \,\mathrm{d}t \\
&\geq \sum_{m=1}^{M} \int_0^T \big(\langle A_m^k(t)\bar{U}_m^k(t), v(t)\rangle_{V_m^* \times V_m} + \langle A_m^k(t)v(t), \bar{U}_m^k(t) - v(t)\rangle_{V_m^* \times V_m}\big) \,\mathrm{d}t \\
&\xrightarrow{k\to 0} \sum_{m=1}^{M} \int_0^T \big(\langle b_m(t), v(t)\rangle_{V_m^* \times V_m} + \langle A_m(t)v(t), U(t) - v(t)\rangle_{V_m^* \times V_m}\big) \,\mathrm{d}t \\
&= \int_0^T \big(\langle b(t), v(t)\rangle_{V^* \times V} + \langle A(t)v(t), U(t) - v(t)\rangle_{V^* \times V}\big) \,\mathrm{d}t,
\end{aligned}$$

which implies

$$\begin{aligned}
&\liminf_{k\to 0} \sum_{m=1}^{M} \int_0^T \langle A_m^k(t)\bar{U}_m^k(t), \bar{U}_m^k(t)\rangle_{V_m^* \times V_m} \,\mathrm{d}t \\
&\geq \int_0^T \big(\langle b(t), v(t)\rangle_{V^* \times V} + \langle A(t)v(t), U(t) - v(t)\rangle_{V^* \times V}\big) \,\mathrm{d}t.
\end{aligned}$$

Applying (3.41), this yields

$$\int_0^T \langle b(t), U(t) - v(t)\rangle_{V^* \times V} \,\mathrm{d}t \geq \int_0^T \langle A(t)v(t), U(t) - v(t)\rangle_{V^* \times V} \,\mathrm{d}t.$$

We then choose $v = U - sw$ for $s \in (0,1)$ and $w \in L^p(0,T;V)$ and apply the Minty monotonicity trick, see [99, Lemma 2.13], to prove that $AU = b$ in $L^q(0,T;V^*)$. To this end, we notice that

$$\int_0^T \langle b(t), sw(t)\rangle_{V^* \times V} \,\mathrm{d}t \geq \int_0^T \langle A(t)\big(U(t) - sw(t)\big), sw(t)\rangle_{V^* \times V} \,\mathrm{d}t,$$

implies

$$\int_0^T \langle b(t), w(t)\rangle_{V^* \times V} \,\mathrm{d}t \geq \int_0^T \langle A(t)U(t), w(t)\rangle_{V^* \times V} \,\mathrm{d}t.$$

In the previous step, we divided by $s > 0$, considered $s \to 0$, and used the radial continuity of $A(t)$, $t \in [0,T]$, from Assumption 3.1.2 (2). Together with the same argumentation for $s \in (-1,0)$, this proves that

$$\int_0^T \langle b(t), w(t)\rangle_{V^* \times V} \,\mathrm{d}t = \int_0^T \langle A(t)U(t), w(t)\rangle_{V^* \times V} \,\mathrm{d}t$$

for every $w \in L^p(0,T;V)$. This shows that $AU = b$ in $L^q(0,T;V^*)$. Therefore, $U = u$ is the unique solution of the evolution problem (3.1).

Since every converging subsequence of $(\bar{U}^k_m)_{k>0}$ converges to the unique solution u of (3.1), we can apply the subsequence principle, see [116, Proposition 10.13] or [49, Kapitel I, Lemma 5.4], to prove that the original sequence $(\bar{U}^k_m)_{k>0}$ converges to the solution u of (3.1). Analogously, we see that every other convergence result claimed in the theorem is fulfilled for the original sequence. $\square$

Theorem 3.1.19. *Let Assumptions 3.1.1, 3.1.2, 3.1.3, 3.1.5, and 3.1.8 be fulfilled and let $u_0 \in H$ be given. For a sequence $(N_\ell)_{\ell\in\mathbb{N}}$ of natural numbers with $N_\ell \to \infty$ as $\ell \to \infty$, step sizes $k = \frac{T}{N_\ell}$, and a bounded sequence $(u^k_0)_{k>0}$ in H such that $u^k_0 \to u_0$ in H as $k \to 0$, let the sequences of piecewise constant and piecewise linear prolongations from* (3.28) *and* (3.29) *be given. If κ from Assumption 3.1.5 (2) is zero, then it follows that*

$$\bar{U}^k(t) \to u(t), \quad U^k(t) \to u(t) \quad \text{in } H \quad \text{as } k \to 0,$$

for every $t \in [0,T]$, where u is the solution of (3.1)*. Under the additional assumption that for $m \in \{1,\dots,M\}$ there exists $\eta \in (0,\infty)$, which does not depend on t, with*

$$\langle A_m(t)v - A_m(t)w, v-w\rangle_{V_m^*\times V_m} \ge \eta|v-w|^p_{V_m} \tag{3.42}$$

for all $v,w \in V_m$ the sequence $(\bar{U}^k_m)_{k>0}$ converges strongly to the solution u of (3.1) *in $L^p(0,T;V_m)$.*

Furthermore, let u^k_0 be in V_M for every $k>0$, let $\big(k^{\frac{1}{p}}\|u^k_0\|_{V_M}\big)_{k>0}$ be uniformly bounded with respect to k, and let the space V_M be compactly embedded into H. Then the statement above is fulfilled for an arbitrary operator $B(t)$, $t\in[0,T]$, that fulfills Assumption 3.1.5.

Proof. For simplicity, we drop the index ℓ within this proof. In order to estimate the error, we split it up in separate parts, which can be handled more easily. We combine the monotonicity conditions from Assumption 3.1.2 (3) and from (3.42). This can be done by including $\eta = 0$ to (3.42). The case $\eta = 0$ is exactly the monotonicity condition from Assumption 3.1.2 (3). We point out the additional result for $\eta \in (0,\infty)$ at the end of the proof. Using the condition (3.42), we then obtain that

$$\begin{aligned}
&\|u(t)-\bar{U}^k(t)\|^2_H + 2\eta\sum_{m=1}^M\int_0^t |u(s)-\bar{U}^k_m(s)|^p_{V_m}\,\mathrm{d}s\\
&\le \|u(t)-\bar{U}^k(t)\|^2_H + 2\sum_{m=1}^M\int_0^t \langle A^k_m(s)u(s)-A^k_m(s)\bar{U}^k_m(s), u(s)-\bar{U}^k_m(s)\rangle_{V_m^*\times V_m}\,\mathrm{d}s\\
&= \|u(t)\|^2_H + 2\sum_{m=1}^M\int_0^t \langle A^k_m(s)u(s),u(s)\rangle_{V_m^*\times V_m}\,\mathrm{d}s\\
&\quad - 2(u(t),\bar{U}^k(t))_H - 2\sum_{m=1}^M\int_0^t \langle A^k_m(s)u(s),\bar{U}^k_m(s)\rangle_{V_m^*\times V_m}\,\mathrm{d}s\\
&\quad - 2\sum_{m=1}^M\int_0^t \langle A^k_m(s)\bar{U}^k_m(s),u(s)\rangle_{V_m^*\times V_m}\,\mathrm{d}s\\
&\quad + \|\bar{U}^k(t)\|^2_H + 2\sum_{m=1}^M\int_0^t \langle A^k_m(s)\bar{U}^k_m(s),\bar{U}^k_m(s)\rangle_{V_m^*\times V_m}\,\mathrm{d}s\\
&=: \Gamma^k_1(t)+\Gamma^k_2(t)+\Gamma^k_3(t)
\end{aligned}$$

with

$$\Gamma_1^k(t) = \|u(t)\|_H^2 + 2\sum_{m=1}^M \int_0^t \langle A_m^k(s)u(s), u(s)\rangle_{V_m^*\times V_m}\,\mathrm{d}s,$$

$$\begin{aligned}\Gamma_2^k(t) &= -2(u(t), \bar{U}^k(t))_H - 2\sum_{m=1}^M \int_0^t \langle A_m^k(s)u(s), \bar{U}_m^k(s)\rangle_{V_m^*\times V_m}\,\mathrm{d}s\\ &\quad - 2\sum_{m=1}^M \int_0^t \langle A_m^k(s)\bar{U}_m^k(s), u(s)\rangle_{V_m^*\times V_m}\,\mathrm{d}s,\end{aligned}$$

$$\Gamma_3^k(t) = \|\bar{U}^k(t)\|_H^2 + 2\sum_{m=1}^M \int_0^t \langle A_m^k(s)\bar{U}_m^k(s), \bar{U}_m^k(s)\rangle_{V_m^*\times V_m}\,\mathrm{d}s$$

for every $t \in [0,T]$. Recall that in Theorem 3.1.18, we proved that $\bar{U}_m^k \rightharpoonup u$ in $L^p(0,T;V_m)$ and $\sum_{m=1}^M A_m^k\bar{U}_m^k \rightharpoonup Au$ in $L^q(0,T;V^*)$ as $k \to 0$. Applying the result from Lemma 3.1.15, it follows that

$$\begin{aligned}\lim_{k\to 0}\Gamma_1^k(t) &= \|u(t)\|_H^2 + 2\int_0^t \langle A(s)u(s), u(s)\rangle_{V^*\times V}\,\mathrm{d}s,\\ \lim_{k\to 0}\Gamma_2^k(t) &= -2\|u(t)\|_H^2 - 4\int_0^t \langle A(s)u(s), u(s)\rangle_{V^*\times V}\,\mathrm{d}s\end{aligned}$$

for every $t \in [0,T]$. In order to estimate Γ_3^k, we need a few additional arguments. Here, we assume that $t \in (t_{n-1}, t_n]$, $n \in \{1,\dots,N\}$, and obtain

$$\begin{aligned}\Gamma_3^k(t) &= \|\bar{U}^k(t)\|_H^2 + 2\sum_{m=1}^M \int_0^{t_n} \langle A_m^k(s)\bar{U}_m^k(s), \bar{U}_m^k(s)\rangle_{V_m^*\times V_m}\,\mathrm{d}s\\ &\quad - 2\sum_{m=1}^M \int_t^{t_n} \langle A_m^k(s)\bar{U}_m^k(s), \bar{U}_m^k(s)\rangle_{V_m^*\times V_m}\,\mathrm{d}s.\end{aligned} \tag{3.43}$$

For the first appearing sum of integrals, we can apply Lemma 3.1.13 to see that

$$\begin{aligned}&2\sum_{m=1}^M \int_0^{t_n} \langle A_m^k(s)\bar{U}_m^k(s), \bar{U}_m^k(s)\rangle_{V_m^*\times V_m}\,\mathrm{d}s\\ &\le 2\sum_{m=1}^M \int_0^{t_n} \langle f_m^k(s), \bar{U}_m^k(s)\rangle_{V_m^*\times V_m}\,\mathrm{d}s - 2\int_0^{t_n} (B^k(t)\underline{U}^k(s), \bar{U}_0^k(s))_H\,\mathrm{d}s\\ &\quad - \big(\|\bar{U}^k(t_n)\|_H^2 - \|\bar{U}^k(0)\|_H^2\big),\end{aligned}$$

which we can reinsert in (3.43). Together with the fact that $\bar{U}^k(0) = u_0^k$ and $\bar{U}^k(t_n) = \bar{U}^k(t)$ in H, we obtain a bound for $\Gamma_3^k(t)$ given by

$$\begin{aligned}\Gamma_3^k(t) &\le \|u_0^k\|_H^2 + 2\sum_{m=1}^M \int_0^t \langle f_m^k(s), \bar{U}_m^k(s)\rangle_{V_m^*\times V_m}\,\mathrm{d}s\\ &\quad - 2\int_0^t (B^k(s)\underline{U}^k(s), \bar{U}_0^k(s))_H\,\mathrm{d}s - 2\int_t^{t_n} (B^k(s)\underline{U}^k(s), \bar{U}_0^k(s))_H\,\mathrm{d}s\\ &\quad + 2\sum_{m=1}^M \int_t^{t_n} \langle f_m^k(s) - A_m^k(s)\bar{U}_m^k(s), \bar{U}_m^k(s)\rangle_{V_m^*\times V_m}\,\mathrm{d}s\end{aligned} \tag{3.44}$$

for every $t \in (t_{n-1}, t_n]$, $n \in \{1, \dots, N\}$. As proven in Lemma 3.1.17, the sequence $(f_m^k)_{k>0}$ converges strongly to f_m in $L^q(0,T;V_m^*)$ as $k \to 0$. Thus, together with Theorem 3.1.18 it follows that

$$\begin{aligned}\sum_{m=1}^M \int_0^t \langle f_m^k(s), \bar{U}_m^k(s)\rangle_{V_m^* \times V_m} \,\mathrm{d}s &\to \sum_{m=1}^M \int_0^t \langle f_m(s), u(s)\rangle_{V_m^* \times V_m} \,\mathrm{d}s \\ &= \int_0^t \langle f(s), u(s)\rangle_{V^* \times V} \,\mathrm{d}s \quad \text{as } k \to 0.\end{aligned}$$

Similarly, we see that

$$\int_0^t (B^k(s)\underline{U}^k(s), \bar{U}_0^k(s))_H \,\mathrm{d}s \to \int_0^t (B(s)u(s), u(s))_H \,\mathrm{d}s \quad \text{as } k \to 0,$$

since $B^k\underline{U}^k \to Bu$ in $L^2(0,T;H)$ and $\bar{U}_0^k \overset{*}{\rightharpoonup} u$ in $L^\infty(0,T;H)$ as $k \to \infty$, compare Theorem 3.1.18. For the remaining integrals in (3.44), we notice that the functions $g_m^k \colon [0,T] \to \mathbb{R}$, $m \in \{0, \dots, M\}$, given by

$$(B^k(s)\underline{U}^k(s), \bar{U}_0^k(s))_H \le \kappa\big(1 + \|\underline{U}^k(s)\|_H\big)\|\bar{U}_0^k(s)\|_H =: g_0^k(s)$$

and

$$\begin{aligned}&\langle f_m^k(s) - A_m^k(s)\bar{U}_m^k(s), \bar{U}_m^k(s)\rangle_{V_m^* \times V_m} \\ &\le \|f_m^k(s)\|_{V_m^*}\|\bar{U}_m^k(s)\|_{V_m} + \|A_m^k(s)\bar{U}_m^k(s)\|_{V_m^*}\|\bar{U}_m^k(s)\|_{V_m} \\ &\le \frac{1}{q}\|f_m^k(s)\|_{V_m^*}^q + \frac{1}{p}\|\bar{U}_m^k(s)\|_{V_m}^p + \beta\big(\|\bar{U}_m^k(s)\|_{V_m} + \|\bar{U}_m^k(s)\|_{V_m}^p\big) =: g_m^k(s)\end{aligned}$$

for almost every $s \in (0,T)$ are bounded by an $L^1(0,T)$-function uniformly in k due to the a priori bounds (3.13), (3.14), and (3.15) from Lemma 3.1.12. Thus, the remaining integrals in (3.44) tend to zero as $0 \le t_n - t \le k \to 0$ and it follows that

$$\limsup_{k\to 0} \Gamma_3^k(t) \le \|u_0\|_H^2 + 2\int_0^t \langle f(s), u(s)\rangle_{V^* \times V} \,\mathrm{d}s - 2\int_0^t (B(s)u(s), u(s))_H \,\mathrm{d}s.$$

The previous arguments have shown

$$\begin{aligned}&\limsup_{k\to 0} \big(\Gamma_1^k(t) + \Gamma_2^k(t) + \Gamma_3^k(t)\big) \\ &\le \|u(t)\|_H^2 + 2\int_0^t \langle A(s)u(s), u(s)\rangle_{V^* \times V} \,\mathrm{d}s \\ &\quad - 2\|u(t)\|_H^2 - 4\int_0^t \langle A(s)u(s), u(s)\rangle_{V^* \times V} \,\mathrm{d}s \\ &\quad + \|u_0\|_H^2 + 2\int_0^t \langle f(s) - B(s)u(s), u(s)\rangle_{V^* \times V} \,\mathrm{d}s \\ &= -\|u(t)\|_H^2 + \|u_0\|_H^2 + 2\int_0^t \langle f(s) - A(s)u(s) - B(s)u(s), u(s)\rangle_{V^* \times V} \,\mathrm{d}s \\ &= -\|u(t)\|_H^2 + \|u_0\|_H^2 + 2\int_0^t \langle u'(s), u(s)\rangle_{V^* \times V} \,\mathrm{d}s \\ &= -\|u(t)\|_H^2 + \|u_0\|_H^2 + \int_0^t \frac{\mathrm{d}}{\mathrm{d}t}\|u(s)\|_H^2 \,\mathrm{d}s = 0\end{aligned}$$

for every $t \in [0,T]$. We also use the fact that $u \in \mathcal{W}^p(0,T)$ and therefore a partial integration rule can be applied. Altogether, this implies the strong convergence of $(\bar{U}^k(t))_{k>0}$ in H for every $t \in [0,T]$ as we have proved

$$\lim_{k\to 0}\Big(\|u(t)-\bar{U}^k(t)\|_H^2 + 2\eta\sum_{m=1}^{M}\int_0^t |u(s)-\bar{U}_m^k(s)|_{V_m}^p\,\mathrm{d}s\Big) = 0. \tag{3.45}$$

Recall the definition of $\bar{U}^k$ and U^k from (3.28) and (3.29), respectively. Then we obtain that

$$\begin{aligned}\|U^k(t)-u(t)\|_H &\le \Big\|\frac{t_n-t}{k}\big(\bar{U}^k(t-k)-u(t)\big)\Big\|_H + \Big\|\frac{t-t_{n-1}}{k}\big(\bar{U}^k(t)-u(t)\big)\Big\|_H\\ &\le \|\bar{U}^k(t-k)-u(t)\|_H + \|\bar{U}^k(t)-u(t)\|_H\\ &\le \|\bar{U}^k(t-k)-u(t-k)\|_H + \|u(t-k)-u(t)\|_H + \|\bar{U}^k(t)-u(t)\|_H\end{aligned}$$

for every $t \in [0,T]$. Using that $u \in \mathcal{W}^p(0,T) \hookrightarrow C([0,T];H)$ and $\bar{U}^k(t) \to u(t)$ in H as $k \to 0$ for every $t \in [0,T]$, it also follows that $U^k(t) \to u(t)$ in H as $k \to 0$.

Now, we consider the case $\eta \in (0,\infty)$ for $m \in \{1,\dots,M\}$ and prove $\bar{U}_m^k \to u$ in $L^p(0,T;V_m)$ as $k \to 0$. The difference of the piecewise constant prolongations $\bar{U}_m^k$ and $\bar{U}^k$ converges to zero in $L^2(0,T;H)$. This is true as

$$\|\bar{U}^k-\bar{U}_m^k\|_{L^2(0,T;H)}^2 = k\sum_{n=1}^{N}\|\mathbf{U}^n-\mathbf{U}_m^n\|_H^2 \le kM\sum_{n=1}^{N}\sum_{j=1}^{M}\|\mathbf{U}_j^n-\mathbf{U}_{j-1}^n\|_H^2 \to 0 \quad \text{as } k\to 0,$$

where we used the a priori bound (3.13) from Lemma 3.1.12. Since $(\bar{U}^k)_{k>0}$ is bounded in $L^2(0,T;H)$ and converges to u pointwise strongly in H, this shows that $\bar{U}^k \to u$ in $L^2(0,T;H)$ as $k \to 0$. We see, in particular, that $\bar{U}_m^k \to u$ in $L^2(0,T;H)$ as $k \to 0$ due to the previous estimate. The a priori bound (3.15) from Lemma 3.1.12 even shows that the sequence $(\bar{U}_m^k)_{k>0}$ is bounded in $L^\infty(0,T;H)$. Therefore, it converges to u also in the space $L^p(0,T;H)$ as $k \to 0$, compare Lemma A.2.3. Using (3.45) and the inequality $\|v\|_{V_m} \le c_{V_m}\big(\|v\|_H + |v|_{V_m}\big)$ for the V_m-norm stated in Assumption 3.1.1, it follows that

$$\begin{aligned}\|u-\bar{U}_m^k\|_{L^p(0,T;V_m)} &= \Big(\int_0^T \|u(t)-\bar{U}_m^k(t)\|_{V_m}^p\,\mathrm{d}t\Big)^{\frac{1}{p}}\\ &\le c_{V_m}\Big(\int_0^T \big(\|u(t)-\bar{U}_m^k(t)\|_H + |u(t)-\bar{U}_m^k(t)|_{V_m}\big)^p\,\mathrm{d}t\Big)^{\frac{1}{p}}\\ &\le c_{V_m}\|u-\bar{U}_m^k\|_{L^p(0,T;H)} + c_{V_m}\Big(\int_0^T |u(t)-\bar{U}_m^k(t)|_{V_m}^p\,\mathrm{d}t\Big)^{\frac{1}{p}} \to 0\end{aligned}$$

as $k \to 0$. This proves that $\bar{U}_m^k \to u$ in $L^p(0,T;V_m)$ as $k \to 0$ if (3.42) is fulfilled for η that is strictly larger than zero for $A_m(t)$, $t \in [0,T]$.

□

3.2 An Explicit Error Estimate

After a convergence analysis of the implicit-explicit product splitting scheme under no additional regularity assumptions on the solution, we now regard the question whether a more regular solution u of the evolution equation (3.1) will lead to explicit error bounds. In the following, we assume that for $\alpha \in (0,1]$ the function u is an element of the space of Hölder continuous functions $C^{0,\alpha}([0,T];V)$. Here, we will not go into detail to explain when this condition is fulfilled. More information about additional regularity of the solution and some examples that fit this setting can be found in Section 1.2. In the following, we will need a similar condition for a counterpart to the approximations $\bar{U}_m^k$, $m \in \{0,\dots,M\}$. To this end, we introduce the functions below.

Assumption 3.2.1. *Let Assumptions 3.1.1, 3.1.2, 3.1.3, 3.1.5, and 3.1.8 be fulfilled and let $u_0 \in V$ be given. For $\alpha \in (0,1]$, let the solution u of the evolution equation (3.1) be an element of $C^{0,\alpha}([0,T];V)$. For $N \in \mathbb{N}$, consider $t_n = kn$ for $n \in \{0,\dots,N\}$, where $k = \frac{T}{N}$. Moreover, for every $m \in \{0,\dots,M\}$, let the function $U_m\colon [0,T] \to V^*$ be given by*

$$U_m(t) = u(t_{n-1}) + \sum_{j=1}^{m} \int_{t_{n-1}}^{t} \big(f_j(s) - A_j(s)u(s)\big)\,\mathrm{d}s - \int_{t_{n-1}}^{t} B(s)u(s)\,\mathrm{d}s \quad \text{in } V^*,$$

for every $t \in (t_{n-1},t_n]$, $n \in \{1,\dots,N\}$ with $U_m(0) = u_0$. Suppose that for $m \in \{1,\dots,M\}$, the function fulfills

$$\|U_m(t) - u(t_{n-1})\|_{V_m} \le L_{\alpha,m}|t - t_{n-1}|^{\alpha} \tag{3.46}$$

for every $t \in (t_{n-1},t_n]$, $n \in \{1,\dots,N\}$.

Note that we use the convention $\sum_{j=1}^{0} = 0$. It does not make sense to assume that a function U_m, $m \in \{1,\dots,M\}$, fulfills a Hölder condition like u. The functions U_m do not even have to be continuous for $m \in \{1\dots,M-1\}$ as $\lim_{t\searrow t_n} U_m(t)$ does not have to coincide with $U_m(t_n)$. Thus, we only ask for a corresponding condition on subintervals. This assumption is not necessary for U_0.

When it comes to verifying such a condition in applications, first note that $U_M = u$ in $C^{0,\alpha}([0,T];V)$. Thus, the condition (3.46) does not impose any additional regularity on the function U_M. For $m \in \{1,\dots,M-1\}$, the condition means that

$$\Big\| \sum_{j=1}^{m} \int_{t_{n-1}}^{t} \big(f_j(s) - A_j(s)u(s)\big)\,\mathrm{d}s - \int_{t_{n-1}}^{t} B(s)u(s)\,\mathrm{d}s \Big\|_{V_m} \le L_{\alpha,m}|t - t_{n-1}|^{\alpha}$$

has to be fulfilled. In our example from Section 3.3, this follows with the help of an embedding argument. Also, a suitable order of appearance of A_m and f_m can be helpful when proving (3.46). These regularity conditions in mind, we can get the following bounds.

Lemma 3.2.2. *Let Assumptions 3.1.1, 3.1.2, 3.1.3, 3.1.5, 3.1.8, and 3.2.1 be fulfilled. Then for every $r \in [1,\infty)$ there exists a constant $C \in (0,\infty)$ such that*

$$\int_0^T \|u(t) - U_m(t)\|_{V_m}^r \,\mathrm{d}t \le Ck^{\alpha r}$$

and

$$\sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} \|U_m(t) - U_m(t_n)\|_{V_m}^r \,\mathrm{d}t \le Ck^{\alpha r}$$

are fulfilled.

Proof. For the first estimate, we use the Hölder continuity of u and the regularity condition (3.46) from Assumption 3.2.1 to obtain that

$$\begin{aligned}\|u(t) - U_m(t)\|_{V_m} &\leq \|u(t) - u(t_{n-1})\|_{V_m} + \|u(t_{n-1}) - U_m(t)\|_{V_m} \\ &\leq c_1\|u(t) - u(t_{n-1})\|_V + \|u(t_{n-1}) - U_m(t)\|_{V_m} \\ &\leq c_1 L_\alpha |t - t_{n-1}|^\alpha + L_{\alpha,m}|t - t_{n-1}|^\alpha\end{aligned}$$

for every $t \in (t_{n-1}, t_n]$. The constant $c_1 \in (0, \infty)$ is the embedding constant from V into V_m and $L_\alpha \in [0, \infty)$ is the Hölder seminorm of u. Thus, it follows

$$\int_0^T \|u(t) - U_m(t)\|_{V_m}^r \,\mathrm{d}t \leq \sum_{i=1}^{N} \int_{t_{n-1}}^{t_n} k^{\alpha r}\big(c_1 L_\alpha + L_{\alpha,m}\big)^r \,\mathrm{d}t = k^{\alpha r} T\big(c_1 L_\alpha + L_{\alpha,m}\big)^r.$$

For the second estimate, we again use (3.46) to see that

$$\begin{aligned}&\Big(\sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} \|U_m(t_n) - U_m(t)\|_{V_m}^r \,\mathrm{d}t\Big)^{\frac{1}{r}} \\ &\leq \Big(\sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} \|U_m(t_n) - u(t_{n-1})\|_{V_m}^r \,\mathrm{d}t\Big)^{\frac{1}{r}} + \Big(\sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} \|u(t_{n-1}) - U_m(t)\|_{V_m}^r \,\mathrm{d}t\Big)^{\frac{1}{r}} \\ &\leq 2\Big(\sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} k^{\alpha r} L_{\alpha,m}^r \,\mathrm{d}t\Big)^{\frac{1}{r}} \leq 2k^\alpha L_{\alpha,m} T^{\frac{1}{r}}.\end{aligned}$$

□

This auxiliary statement in mind, we can now turn to the main statement of this section. We prove explicit error bounds for the approximation of a nonlinear evolution equation. In [86] or [95], a fully nonlinear problem $u'(t) = F(t, u(t))$ for $t \in (0, T)$ with an initial condition is linearized along the exact solution u in order to find an approximation of u. Then it is possible to use the partial derivative $A(t) = \partial_u F(t, u)$ if F is smooth enough. In the following, we do not rely on a linear approximation of our nonlinear equation but use a similar approach as in Section 2.2. Again, we use the analysis from [37] for a globally Lipschitz continuous operator $A(t)$, $t \in [0, T]$, as a starting point. We suppose that u and U_m, $m \in \{1, \dots, M\}$, fulfill the additional regularity condition from Assumption 3.2.1. Further, we assume that for every $m \in \{1, \dots, M\}$ the operator $A_m(t)$, $t \in [0, T]$, fulfills a bounded Lipschitz condition and a p-monotonicity condition. Then we can obtain an explicit error bound. The size of the error depends both on $q = \frac{p}{p-1}$ and the Hölder exponent α of the exact solution u. The assumption that an operator $A(t)$, $t \in [0, T]$, fulfills a bounded Lipschitz condition is fulfilled in standard examples as the p-Laplacian. Note that in contrast to Section 2.2, we do not ask for any regularity conditions for u'. This is technically not necessary since we assume that the integrals from (3.2) and (3.3) are known. To obtain such values in practice, usually requires a certain temporal regularity condition on the data. We exchanged this by a regularity condition for u' in Section 2.2.

Note that for $p = 2$ and $\alpha = 1$, we have an error bound of order one. This is also the formal rate of convergence of the standard backward Euler scheme. Thus, the additional splitting error of our proposed scheme only affects the error in terms of constants, which do not depend on the step size.

Theorem 3.2.3. *Let Assumptions 3.1.1, 3.1.2, 3.1.3, 3.1.5, 3.1.8, and 3.2.1 be fulfilled and let the initial value $u_0 \in V$ be given.*

For every $m \in \{1,\dots,M\}$, let the operator $A_m(t)$, $t \in [0,T]$, fulfill a bounded Lipschitz condition in the sense that for every $R \in (0,\infty)$ there exists $L(R) \in [0,\infty)$ such that

$$\|A_m(t)v - A_m(t)w\|_{V_m^*} \le L(R)\|v-w\|_{V_m} \tag{3.47}$$

is fulfilled for all $t \in [0,T]$ and $v,w \in V_m$ with $\|v\|_{V_m}, \|w\|_{V_m} \le R$. Furthermore, let every $A_m(t)$, $t \in [0,T]$, satisfy a p-monotonicity condition such that there exists $\eta \in (0,\infty)$ with

$$\langle A_m(t)v - A_m(t)w, v-w\rangle_{V_m^* \times V_m} \ge \eta\|v-w\|_{V_m}^p \tag{3.48}$$

for all $v,w \in V_m$ and $t \in [0,T]$. Then there exists $C \in (0,\infty)$ such that for every step size $k = \frac{T}{N}$, $N \in \mathbb{N}$, with $2\kappa k \in [0,1)$ and $u_0^k = u_0$ in V the solution of (3.4)–(3.6) fulfills that

$$\max_{n\in\{1,\dots,N\}} \|u(t_n) - \mathbf{U}^n\|_H^2 + k\sum_{n=1}^{N}\sum_{m=1}^{M}\|u(t_n)-\mathbf{U}_m^n\|_{V_m}^p \le Ck^{\alpha q} \tag{3.49}$$

for $q = \frac{p}{p-1}$, $t_n = nk$, $n \in \{0,\dots,N\}$.

The bounded Lipschitz condition we require is more general than a Lipschitz condition on bounded sets in H. Since V_m is continuously embedded into H, there exists a constant $c_1 \in (0,\infty)$ such that $\|v\|_H \le c_1\|v\|_{V_m}$ for every $v \in V_m$. Thus, if (3.47) is fulfilled for every $v,w \in V_m$ with $\|v\|_H, \|w\|_H \le c_1 R$, it is also fulfilled for $v,w \in V_m$ with $\|v\|_{V_m}, \|w\|_{V_m} \le R$ since $\|v\|_H \le c_1\|v\|_{V_m} \le c_1 R$.

Proof of Theorem 3.2.3. In the following, let $i \in \{1,\dots,N\}$ be fixed. Recalling the definition of the function U_m from Assumption 3.2.1, we first notice that $U_m \in L^\infty(0,T;V_m)$ for every $m \in \{1,\dots,M\}$ since

$$\begin{aligned}
\|U_m\|_{L^\infty(0,T;V_m)} &= \operatorname*{ess\,sup}_{t\in[0,T]} \|U_m(t)\|_{V_m} \\
&\le \max_{n\in\{1,\dots,N\}} \Big(\operatorname*{ess\,sup}_{t\in(t_{n-1},t_n]} \|U_m(t) - u(t_{n-1})\|_{V_m} + \|u(t_{n-1})\|_{V_m}\Big) \\
&\le k^\alpha L_{\alpha,m} + \|u\|_{L^\infty(0,T;V_m)} < \infty
\end{aligned}$$

due to (3.46) and the fact that u is also bounded in V_m. Further, we can write that

$$U_m(t_i) - U_{m-1}(t_i) = \int_{t_{i-1}}^{t_i} \big(f_m(t) - A_m(t)u(t)\big)\,\mathrm{d}t \quad \text{in } V_m^*$$

for $m \in \{1,\dots,M\}$ and

$$U_0(t_i) - u(t_{i-1}) = -\int_{t_{i-1}}^{t_i} B(t)u(t)\,\mathrm{d}t \quad \text{in } H.$$

These equations can now be tested with the corresponding element $U_m(t_i) - \mathbf{U}_m^i \in V_m$, $m \in \{1,\dots,M\}$, and $U_0(t_i) - \mathbf{U}_0^i \in H$, respectively. Then we obtain

$$\begin{aligned}
&(U_m(t_i) - U_{m-1}(t_i), U_m(t_i) - \mathbf{U}_m^i)_H \\
&= \int_{t_{i-1}}^{t_i} \langle f_m(t) - A_m(t)u(t), U_m(t_i) - \mathbf{U}_m^i\rangle_{V_m^*\times V_m}\,\mathrm{d}t
\end{aligned} \tag{3.50}$$

for $m \in \{1, \dots, M\}$ and

$$(U_0(t_i) - u(t_{i-1}), U_0(t_i) - \mathbf{U}_0^i)_H = -\int_{t_{i-1}}^{t_i} (B(t)u(t), U_0(t_i) - \mathbf{U}_0^i)_H \,\mathrm{d}t. \tag{3.51}$$

We now sum up (3.50) from $m = 1$ to M and add (3.51). This yields

$$\begin{aligned}
&\sum_{m=1}^{M} (U_m(t_i) - U_{m-1}(t_i), U_m(t_i) - \mathbf{U}_m^i)_H + (U_0(t_i) - u(t_{i-1}), U_0(t_i) - \mathbf{U}_0^i)_H \\
&= \sum_{m=1}^{M} \int_{t_{i-1}}^{t_i} \langle f_m(t) - A_m(t)u(t), U_m(t_i) - \mathbf{U}_m^i \rangle_{V_m^* \times V_m} \,\mathrm{d}t \\
&\quad - \int_{t_{i-1}}^{t_i} (B(t)u(t), U_0(t_i) - \mathbf{U}_0^i)_H \,\mathrm{d}t.
\end{aligned} \tag{3.52}$$

We show a corresponding equality for the numerical scheme. Here, we test the equation (3.5) with $U_m(t_i) - \mathbf{U}_m^i \in V_m$ to obtain

$$(\mathbf{U}_m^i - \mathbf{U}_{m-1}^i, U_m(t_i) - \mathbf{U}_m^i)_H = k\langle \mathbf{f}_m^i - \mathbf{A}_m^i \mathbf{U}_m^i, U_m(t_i) - \mathbf{U}_m^i \rangle_{V_m^* \times V_m} \tag{3.53}$$

for $m \in \{1, \dots, M\}$. Further, we test (3.4) with $U_0(t_i) - \mathbf{U}_0^i \in H$ and get

$$(\mathbf{U}_0^i - \mathbf{U}^{i-1}, U_0(t_i) - \mathbf{U}_0^i)_H = -k(\mathbf{B}^i \mathbf{U}^{i-1}, U_0(t_i) - \mathbf{U}_0^i)_H. \tag{3.54}$$

Then we sum up (3.53) from $m = 1$ to M and add (3.54) to see

$$\begin{aligned}
&\sum_{m=1}^{M} (\mathbf{U}_m^i - \mathbf{U}_{m-1}^i, U_m(t_i) - \mathbf{U}_m^i)_H + (\mathbf{U}_0^i - \mathbf{U}^{i-1}, U_0(t_i) - \mathbf{U}_0^i)_H \\
&= k \sum_{m=1}^{M} \langle \mathbf{f}_m^i - \mathbf{A}_m^i \mathbf{U}_m^i, U_m(t_i) - \mathbf{U}_m^i \rangle_{V_m^* \times V_m} - k(\mathbf{B}^i \mathbf{U}^{i-1}, U_0(t_i) - \mathbf{U}_0^i)_H \\
&= \sum_{m=1}^{M} \int_{t_{i-1}}^{t_i} \langle f_m(t) - A_m(t)\mathbf{U}_m^i, U_m(t_i) - \mathbf{U}_m^i \rangle_{V_m^* \times V_m} \,\mathrm{d}t \\
&\quad - \int_{t_{i-1}}^{t_i} (B(t)\mathbf{U}^{i-1}, U_0(t_i) - \mathbf{U}_0^i)_H \,\mathrm{d}t.
\end{aligned} \tag{3.55}$$

In the last step, we inserted the definition of $\mathbf{A}_m^i$, $\mathbf{f}_m^i$, $m \in \{1, \dots, M\}$, and $\mathbf{B}^i$ from (3.2). In order to estimate the error, we consider the difference of (3.52) and (3.55). We can write for the difference of the left-hand sides

$$\begin{aligned}
&\sum_{m=1}^{M} (U_m(t_i) - U_{m-1}(t_i), U_m(t_i) - \mathbf{U}_m^i)_H + (U_0(t_i) - u(t_{i-1}), U_0(t_i) - \mathbf{U}_0^i)_H \\
&\quad - \sum_{m=1}^{M} (\mathbf{U}_m^i - \mathbf{U}_{m-1}^i, U_m(t_i) - \mathbf{U}_m^i)_H + (\mathbf{U}_0^i - \mathbf{U}^{i-1}, U_0(t_i) - \mathbf{U}_0^i)_H \\
&= \sum_{m=1}^{M} \big((U_m(t_i) - \mathbf{U}_m^i) - (U_{m-1}(t_i) - \mathbf{U}_{m-1}^i), U_m(t_i) - \mathbf{U}_m^i \big)_H \\
&\quad + \big((U_0(t_i) - \mathbf{U}_0^i) - (u(t_{i-1}) - \mathbf{U}^{i-1}), U_0(t_i) - \mathbf{U}_0^i \big)_H.
\end{aligned}$$

After inserting the identity from Lemma A.1.4, we see that this is equal to

$$
\begin{aligned}
&\frac{1}{2}\sum_{m=1}^{M}\big(\|U_m(t_i)-\mathbf{U}_m^i\|_H^2-\|U_{m-1}(t_i)-\mathbf{U}_{m-1}^i\|_H^2\big)\\
&\quad+\frac{1}{2}\sum_{m=1}^{M}\|U_m(t_i)-\mathbf{U}_m^i-U_{m-1}(t_i)+U_{m-1}^i\|_H^2\\
&\quad+\frac{1}{2}\big(\|U_0(t_i)-\mathbf{U}_0^i\|_H^2-\|u(t_{i-1})-\mathbf{U}^{i-1}\|_H^2+\|U_0(t_i)-\mathbf{U}_0^i-u(t_{i-1})+\mathbf{U}^{i-1}\|_H^2\big)\\
&=\frac{1}{2}\big(\|u(t_i)-\mathbf{U}^i\|_H^2-\|u(t_{i-1})-\mathbf{U}^{i-1}\|_H^2\big)\\
&\quad+\frac{1}{2}\sum_{m=1}^{M}\|U_m(t_i)-\mathbf{U}_m^i-U_{m-1}(t_i)+\mathbf{U}_{m-1}^i\|_H^2 \qquad (3.56)\\
&\quad+\frac{1}{2}\|U_0(t_i)-\mathbf{U}_0^i-u(t_{i-1})+\mathbf{U}^{i-1}\|_H^2,
\end{aligned}
$$

where we insert $U_M(t_i)=u(t_i)$ and $\mathbf{U}_M^i=\mathbf{U}^i$ in H. Next, we rewrite the right-hand side of the difference of (3.52) and (3.55). Then we see that

$$
\begin{aligned}
&\sum_{m=1}^{M}\int_{t_{i-1}}^{t_i}\langle f_m(t)-A_m(t)u(t),U_m(t_i)-\mathbf{U}_m^i\rangle_{V_m^*\times V_m}\,\mathrm{d}t\\
&\quad-\sum_{m=1}^{M}\int_{t_{i-1}}^{t_i}\langle f_m(t)-A_m(t)\mathbf{U}_m^i,U_m(t_i)-\mathbf{U}_m^i\rangle_{V_m^*\times V_m}\,\mathrm{d}t\\
&\quad-\int_{t_{i-1}}^{t_i}(B(t)u(t)-B(t)\mathbf{U}^{i-1},U_0(t_i)-\mathbf{U}_0^i)_H\,\mathrm{d}t\\
&=-\sum_{m=1}^{M}\int_{t_{i-1}}^{t_i}\langle A_m(t)u(t)-A_m(t)\mathbf{U}_m^i,U_m(t_i)-\mathbf{U}_m^i\rangle_{V_m^*\times V_m}\,\mathrm{d}t\\
&\quad-\int_{t_{i-1}}^{t_i}(B(t)u(t)-B(t)\mathbf{U}^{i-1},U_0(t_i)-\mathbf{U}_0^i)_H\,\mathrm{d}t\\
&=:\Gamma_1+\Gamma_2+\Gamma_3+\Gamma_4,
\end{aligned}
$$

where

$$
\begin{aligned}
\Gamma_1&=-\sum_{m=1}^{M}\int_{t_{i-1}}^{t_i}\langle A_m(t)u(t)-A_m(t)U_m(t),U_m(t_i)-\mathbf{U}_m^i\rangle_{V_m^*\times V_m}\,\mathrm{d}t,\\
\Gamma_2&=-\sum_{m=1}^{M}\int_{t_{i-1}}^{t_i}\langle A_m(t)U_m(t)-A_m(t)U_m(t_i),U_m(t_i)-\mathbf{U}_m^i\rangle_{V_m^*\times V_m}\,\mathrm{d}t,\\
\Gamma_3&=-\sum_{m=1}^{M}\int_{t_{i-1}}^{t_i}\langle A_m(t)U_m(t_i)-A_m(t)\mathbf{U}_m^i,U_m(t_i)-\mathbf{U}_m^i\rangle_{V_m^*\times V_m}\,\mathrm{d}t,\\
\Gamma_4&=-\int_{t_{i-1}}^{t_i}(B(t)u(t)-B(t)\mathbf{U}^{i-1},U_0(t_i)-\mathbf{U}_0^i)_H\,\mathrm{d}t.
\end{aligned}
$$

We added and subtracted the terms containing $A_m(t)U_m(t)$ and $A_m(t)U_m(t_i)$ in order to estimate Γ_1, Γ_2, Γ_3, and Γ_4 more easily. For Γ_1 and Γ_2, we can use the bounded Lipschitz condition and the results from Lemma 3.2.2. In order to estimate Γ_3, we use the monotonicity of $A_m(t)$, $t \in [0,T]$, while for Γ_4 we use the Lipschitz continuity of $B(t)$, $t \in [0,T]$.

Precisely, for Γ_1, we obtain that

$$\begin{aligned}
\Gamma_1 &\leq \sum_{m=1}^{M} \int_{t_{i-1}}^{t_i} \|A_m(t)u(t) - A_m(t)U_m(t)\|_{V_m^*} \|U_m(t_i) - \mathbf{U}_m^i\|_{V_m} \,\mathrm{d}t \\
&\leq L_A(R) \sum_{m=1}^{M} \int_{t_{i-1}}^{t_i} \|u(t) - U_m(t)\|_{V_m} \|U_m(t_i) - \mathbf{U}_m^i\|_{V_m} \,\mathrm{d}t \\
&\leq c_1 \sum_{m=1}^{M} \int_{t_{i-1}}^{t_i} \|u(t) - U_m(t)\|_{V_m}^q \,\mathrm{d}t + k\frac{\eta}{4} \sum_{m=1}^{M} \|U_m(t_i) - \mathbf{U}_m^i\|_{V_m}^p,
\end{aligned}$$

where $c_1 = L_A(R)^q \frac{(p\eta)^{1-q}}{4^{1-q}q}$ and we choose R for the condition (3.47) to be

$$R = \max\big\{\|u\|_{L^\infty(0,T;V_1)}, \ldots, \|u\|_{L^\infty(0,T;V_M)}, \|U_1\|_{L^\infty(0,T;V_1)}, \ldots, \|U_M\|_{L^\infty(0,T;V_M)}\big\}. \tag{3.57}$$

For Γ_2, we can argue similarly to obtain that

$$\begin{aligned}
\Gamma_2 &\leq \sum_{m=1}^{M} \int_{t_{i-1}}^{t_i} \|A_m(t)U_m(t) - A_m(t)U_m(t_i)\|_{V_m^*} \|U_m(t_i) - \mathbf{U}_m^i\|_{V_m} \,\mathrm{d}t \\
&\leq L_A(R) \sum_{m=1}^{M} \int_{t_{i-1}}^{t_i} \|U_m(t) - U_m(t_i)\|_{V_m} \|U_m(t_i) - \mathbf{U}_m^i\|_{V_m} \,\mathrm{d}t \\
&\leq c_2 \sum_{m=1}^{M} \int_{t_{i-1}}^{t_i} \|U_m(t) - U_m(t_i)\|_{V_m}^q \,\mathrm{d}t + k\frac{\eta}{4} \sum_{m=1}^{M} \|U_m(t_i) - \mathbf{U}_m^i\|_{V_m}^p,
\end{aligned}$$

where $c_2 = L_A(R)^q \frac{(p\eta)^{1-q}}{4^{1-q}q}$ and R can again be chosen as in (3.57). For Γ_3, we use the monotonicity condition from (3.48) to see that

$$\Gamma_3 \leq -k\eta \sum_{m=1}^{M} \|U_m(t_i) - \mathbf{U}_m^i\|_{V_m}^p.$$

Furthermore, the Lipschitz continuity of $B(t)$, $t \in [0,T]$, shows that

$$\begin{aligned}
\Gamma_4 &\leq \kappa \int_{t_{i-1}}^{t_i} \|u(t) - \mathbf{U}^{i-1}\|_H \|U_0(t_i) - \mathbf{U}_0^i\|_H \,\mathrm{d}t \\
&\leq \frac{\kappa}{2} \int_{t_{i-1}}^{t_i} \|u(t) - \mathbf{U}^{i-1}\|_H^2 \,\mathrm{d}t + k\frac{\kappa}{2} \|U_0(t_i) - \mathbf{U}_0^i\|_H^2 \\
&\leq \kappa \int_{t_{i-1}}^{t_i} \|u(t) - u(t_{i-1})\|_H^2 \,\mathrm{d}t + k\kappa \|u(t_{i-1}) - \mathbf{U}^{i-1}\|_H^2 \\
&\quad + k\kappa \|U_0(t_i) - \mathbf{U}_0^i - u(t_{i-1}) + \mathbf{U}^{i-1}\|_H^2 + k\kappa \|u(t_{i-1}) - \mathbf{U}^{i-1}\|_H^2 \\
&\leq \kappa \int_{t_{i-1}}^{t_i} \|u(t) - u(t_{i-1})\|_H^2 \,\mathrm{d}t + 2k\kappa \|u(t_{i-1}) - \mathbf{U}^{i-1}\|_H^2 \\
&\quad + \frac{1}{2} \|U_0(t_i) - \mathbf{U}_0^i - u(t_{i-1}) + \mathbf{U}^{i-1}\|_H^2
\end{aligned}$$

for $2\kappa k \in [0, 1)$. We can now combine the calculations for both the left-hand side and the right-hand side of the difference of (3.52) and (3.55). The difference of left-hand sides can be found in (3.56), while the difference of the right-hand sides is given by $\Gamma_1 + \Gamma_2 + \Gamma_3 + \Gamma_4$.

$$\begin{aligned}
&\frac{1}{2}\big(\|u(t_i) - \mathbf{U}^i\|_H^2 - \|u(t_{i-1}) - \mathbf{U}^{i-1}\|_H^2\big) + \frac{1}{2}\|U_0(t_i) - \mathbf{U}_0^i - u(t_{i-1}) + \mathbf{U}^{i-1}\|_H^2 \\
&\quad + \frac{1}{2}\sum_{m=1}^{M} \|U_m(t_i) - \mathbf{U}_m^i - U_{m-1}(t_i) + \mathbf{U}_{m-1}^i\|_H^2 \\
&= \Gamma_1 + \Gamma_2 + \Gamma_3 + \Gamma_4 \\
&\le c_1 \sum_{m=1}^{M} \int_{t_{i-1}}^{t_i} \|u(t) - U_m(t)\|_{V_m}^q \,\mathrm{d}t + k\frac{\eta}{4}\sum_{m=1}^{M} \|U_m(t_i) - \mathbf{U}_m^i\|_{V_m}^p \\
&\quad + c_2 \sum_{m=1}^{M} \int_{t_{i-1}}^{t_i} \|U_m(t) - U_m(t_i)\|_{V_m}^q \,\mathrm{d}t + k\frac{\eta}{4}\sum_{m=1}^{M} \|U_m(t_i) - \mathbf{U}_m^i\|_{V_m}^p \\
&\quad - k\eta \sum_{m=1}^{M} \|U_m(t_i) - \mathbf{U}_m^i\|_{V_m}^p \\
&\quad + \kappa \int_{t_{i-1}}^{t_i} \|u(t) - u(t_{i-1})\|_H^2 \,\mathrm{d}t + 2k\kappa \|u(t_{i-1}) - \mathbf{U}^{i-1}\|_H^2 \\
&\quad + \frac{1}{2}\|U_0(t_i) - \mathbf{U}_0^i - u(t_{i-1}) + \mathbf{U}^{i-1}\|_H^2.
\end{aligned}$$

This inequality can be rearranged to a suitable bound. Further, we multiply by two to obtain

$$\begin{aligned}
&\|u(t_i) - \mathbf{U}^i\|_H^2 - \|u(t_{i-1}) - \mathbf{U}^{i-1}\|_H^2 + k\eta \sum_{m=1}^{M} \|U_m(t_i) - \mathbf{U}_m^i\|_{V_m}^p \\
&\le 2c_1 \sum_{m=1}^{M} \int_{t_{i-1}}^{t_i} \|u(t) - U_m(t)\|_{V_m}^q \,\mathrm{d}t + 2c_2 \sum_{m=1}^{M} \int_{t_{i-1}}^{t_i} \|U_m(t) - U_m(t_i)\|_{V_m}^q \,\mathrm{d}t \\
&\quad + 2\kappa \int_{t_{i-1}}^{t_i} \|u(t) - u(t_{i-1})\|_H^2 \,\mathrm{d}t + 4k\kappa \|u(t_{i-1}) - \mathbf{U}^{i-1}\|_H^2.
\end{aligned}$$

A summation from $i = 1$ to $n \in \{1, \dots, N\}$ and the fat that $u(0) = u_0 = \mathbf{U}^0$ in V then implies that

$$\begin{aligned}
&\|u(t_n) - \mathbf{U}^n\|_H^2 + k\eta \sum_{i=1}^{n} \sum_{m=1}^{M} \|U_m(t_i) - \mathbf{U}_m^i\|_{V_m}^p \\
&\le 2c_1 \sum_{m=1}^{M} \int_0^T \|u(t) - U_m(t)\|_{V_m}^q \,\mathrm{d}t + 2c_2 \sum_{i=1}^{N} \sum_{m=1}^{M} \int_{t_{i-1}}^{t_i} \|U_m(t) - U_m(t_i)\|_{V_m}^q \,\mathrm{d}t \\
&\quad + 2\kappa \sum_{i=1}^{N} \int_{t_{i-1}}^{t_i} \|u(t) - u(t_{i-1})\|_H^2 \,\mathrm{d}t + 4k\kappa \sum_{i=1}^{n-1} \|u(t_i) - \mathbf{U}^i\|_H^2.
\end{aligned}$$

As the function u is Hölder continuous with values in V, it is also an element of the space

$C^{0,\alpha}([0,T];H)$. Together with the results from Lemma 3.2.2, it then follows that

$$\|u(t_n) - \mathbf{U}^n\|_H^2 + k\eta \sum_{i=1}^{n} \sum_{m=1}^{M} \|U_m(t_i) - \mathbf{U}_m^i\|_{V_m}^p \le c_3\big(k^{\alpha q} + k^{2\alpha}\big) + 4k\kappa \sum_{i=1}^{n-1} \|u(t_i) - \mathbf{U}^i\|_H^2$$

for a constant $c_3 \in (0,\infty)$, which does not depend on the step size k. An application of Lemma A.1.1 shows that

$$\|u(t_n) - \mathbf{U}^n\|_H^2 + k\eta \sum_{i=1}^{n} \sum_{m=1}^{M} \|U_m(t_i) - \mathbf{U}_m^i\|_{V_m}^p \le c_3\big(k^{\alpha q} + k^{2\alpha}\big) \exp(4\kappa T). \tag{3.58}$$

Moreover, applying (3.46) from Assumption 3.2.1 and the Hölder continuity of u, it follows that

$$\begin{aligned} \|u(t_i) - \mathbf{U}_m^i\|_{V_m} &\le \|u(t_i) - u(t_{i-1})\|_{V_m} + \|u(t_{i-1}) - U_m(t_i)\|_{V_m} + \|U_m(t_i) - \mathbf{U}_m^i\|_{V_m} \\ &\le c_m |u|_{C^{0,\alpha}([0,T];V)} k^\alpha + L_{\alpha,m} k^\alpha + \|U_m(t_i) - \mathbf{U}_m^i\|_{V_m} \\ &\le c_5 k^\alpha + \|U_m(t_i) - \mathbf{U}_m^i\|_{V_m}, \end{aligned}$$

where $c_m \in (0,\infty)$ is the embedding constant from V into V_m and

$$c_5 = \max_{m \in \{1,\dots,M\}} \big(c_m |u|_{C^{0,\alpha}([0,T];V)} + L_{\alpha,m}\big).$$

Altogether, this implies

$$\begin{aligned} k \sum_{i=1}^{n} \sum_{m=1}^{M} \|u(t_i) - \mathbf{U}_m^i\|_{V_m}^p &\le k \sum_{i=1}^{n} \sum_{m=1}^{M} \big(c_5 k^\alpha + \|U_m(t_i) - \mathbf{U}_m^i\|_{V_m}\big)^p \\ &\le 2^{p-1} k c_5 \sum_{i=1}^{n} \sum_{m=1}^{M} k^{\alpha p} + 2^{p-1} k \sum_{i=1}^{n} \sum_{m=1}^{M} \|U_m(t_i) - \mathbf{U}_m^i\|_{V_m}^p \\ &\le 2^{p-1} c_5 k^{\alpha p} T M + 2^{p-1} c_3 \big(k^{\alpha q} + k^{2\alpha}\big) \frac{\exp(4\kappa T)}{\eta}. \end{aligned}$$

Together with (3.58) this finishes the proof as $p \in [2,\infty)$. □

3.3 Example: A Nonlinear Parabolic Problem

In order to demonstrate that our abstract theory applies to more concrete problems, we consider a nonlinear parabolic problem and split the equation into the appearing terms. This enables us to look at different problems that can be solved more efficiently individually. A similar example was presented in [35]. The abstract theory in the previous two sections now offers a possibility to allow for a somewhat more general setting. Here, we can also permit non-monotone lower-order terms due to the additional Lipschitz continuous operator $B(t)$, $t \in [0,T]$, from the theory above. Furthermore, we obtain explicit error bounds under some additional assumptions.

For a finite end time $T \in (0,\infty)$ and a bounded Lipschitz domain $\mathcal{D} \subset \mathbb{R}^d$, $d \in \mathbb{N}$, we consider the problem

$$\begin{cases} \partial_t u(t,x) + \mathbf{a}_1(t,x,u(t,x)) - \nabla \cdot \mathbf{a}_2(t,x,\nabla u(t,x)) + \mathbf{b}(t,x,u(t,x)) = f(t,x), \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (t,x) \in (0,T) \times \mathcal{D}, \\ u(t,x) = 0, \qquad\qquad\qquad\qquad\qquad\quad (t,x) \in (0,T) \times \partial\mathcal{D}, \\ u(0,x) = u_0(x), \qquad\qquad\qquad\qquad\quad x \in \mathcal{D}. \end{cases} \tag{3.59}$$

Further, $\mathbf{a}_1\colon [0,T]\times\overline{\mathcal{D}}\times\mathbb{R}\to\mathbb{R}$ and $\mathbf{a}_2\colon [0,T]\times\overline{\mathcal{D}}\times\mathbb{R}^d\to\mathbb{R}^d$ fulfill Assumption 3.3.1 below and $\mathbf{b}\colon [0,T]\times\overline{\mathcal{D}}\times\mathbb{R}\to\mathbb{R}$ fulfills Assumption 3.3.4 below. Moreover, $f\colon [0,T]\times\overline{\mathcal{D}}\to\mathbb{R}$ and $u_0\colon \overline{\mathcal{D}}\to\mathbb{R}$ are functions we will be specify later.

Assumption 3.3.1. *Let $p\in[2,\infty)$ and $\ell\in\{1,d\}$ be given and $q=\frac{p}{p-1}$. Let $\mathbf{a}\colon [0,T]\times\overline{\mathcal{D}}\times\mathbb{R}^\ell\to\mathbb{R}^\ell$ fulfill the following conditions:*

(1) The mapping $t\mapsto\mathbf{a}(t,x,z)$ is continuous almost everywhere in $(0,T)$ for almost every $x\in\mathcal{D}$ and every $z\in\mathbb{R}^\ell$, $x\mapsto\mathbf{a}(t,x,z)$ is measurable for every $t\in[0,T]$ and $z\in\mathbb{R}^\ell$, while $z\mapsto\mathbf{a}(t,x,z)$ is continuous for every $t\in[0,T]$ and almost every $x\in\mathcal{D}$.

(2) The mapping $\mathbf{a}$ fulfills a monotonicity condition such that for every $t\in[0,T]$, almost every $x\in\mathcal{D}$, as well as every $z,\tilde z\in\mathbb{R}^\ell$ the inequality $(\mathbf{a}(t,x,z)-\mathbf{a}(t,x,\tilde z))\cdot(z-\tilde z)\geq 0$ is satisfied.

(3) The mapping $\mathbf{a}$ fulfills a growth condition in the sense that there exist $d_1\in[0,\infty)$ and a nonnegative function $d_2\in L^q(\mathcal{D})$ such that for every $t\in[0,T]$, almost every $x\in\mathcal{D}$, as well as every $z\in\mathbb{R}^\ell$ the inequality $|\mathbf{a}(t,x,z)|\leq d_1|z|^{p-1}+d_2(x)$ is satisfied.

(4) The mapping $\mathbf{a}$ fulfills a coercivity condition in the sense that there exist $d_3\in(0,\infty)$ and a nonnegative $d_4\in L^1(\mathcal{D})$ such that for every $t\in[0,T]$, almost every $x\in\mathcal{D}$, as well as every $z\in\mathbb{R}^\ell$ the condition $\mathbf{a}(t,x,z)\cdot z\geq d_3|z|^p-d_4(x)$ is satisfied.

Assumption 3.3.2. *Let Assumption 2.3.1 be fulfilled. Additionally, there exists $d_5\in(0,\infty)$ such that*

$$(\mathbf{a}(t,x,z)-\mathbf{a}(t,x,\tilde z))\cdot(z-\tilde z)\geq d_5|z-\tilde z|^p$$

is satisfied for every $t\in[0,T]$, almost every $x\in\mathcal{D}$, and every $z,\tilde z\in\mathbb{R}^\ell$.

Assumption 3.3.3. *Let Assumption 2.3.1 be fulfilled. Additionally, there exists $d_6\in[0,\infty)$ such that*

$$|\mathbf{a}(t,x,z)-\mathbf{a}(t,x,\tilde z)|\leq d_6\big(1+\max\{|z|^{p-2},|\tilde z|^{p-2}\}\big)|z-\tilde z|$$

is satisfied for every $t\in[0,T]$, almost every $x\in\mathcal{D}$, and every $z,\tilde z\in\mathbb{R}^\ell$.

As explained in Section 2.3, a standard example that fulfills all the assumptions above is $\mathbf{a}(t,x,z)=\mathbf{a}(z)=|z|^{p-2}z$.

Assumption 3.3.4. *Let $\mathbf{b}\colon [0,T]\times\overline{\mathcal{D}}\times\mathbb{R}\to\mathbb{R}$ fulfill the following conditions:*

(1) The mapping $t\mapsto\mathbf{b}(t,x,z)$ is continuous almost everywhere in $(0,T)$ for almost every $x\in\mathcal{D}$ and every $z\in\mathbb{R}$ and $x\mapsto\mathbf{b}(t,x,z)$ is measurable for every $t\in[0,T]$ and $z\in\mathbb{R}$.

(2) The mapping $\mathbf{b}$ fulfills a Lipschitz condition in the sense that there exists $e_1\in[0,\infty)$ such that for every $t\in[0,T]$, almost every $x\in\mathcal{D}$, as well as every $z,\tilde z\in\mathbb{R}$ the inequality $|\mathbf{b}(t,x,z)-\mathbf{b}(t,x,\tilde z)|\leq e_1|z-\tilde z|$ is satisfied.

(3) The mapping $\mathbf{b}$ fulfills a growth condition at the point zero in the sense that there exists a nonnegative function $e_2\in L^2(\mathcal{D})$ such that for every $t\in[0,T]$ and almost every $x\in\mathcal{D}$ the inequality $|\mathbf{b}(t,x,0)|\leq e_2(x)$ is satisfied.

Next, let $H = L^2(\mathcal{D})$ and $V = W_0^{1,p}(\mathcal{D})$ be equipped with the norms introduced in the notation section in the introduction. The value p is the same as in Assumption 3.3.1. Further, we consider $V_1 = L^p(\mathcal{D})$ equipped with the standard norm and $V_2 = V$ with the same norm as for V. The seminorms are all chosen as the full norm in the corresponding space.

For $t \in [0,T]$, the operators $A_m(t)\colon V_m \to V_m^*$, $m \in \{1,2\}$, and $B(t)\colon H \to H$ are given by

$$\langle A_1(t)v, w\rangle_{V_1^* \times V_1} = \int_{\mathcal{D}} \mathbf{a}_1(t,\cdot,v)w \,\mathrm{d}x, \quad v,w \in V_1, \tag{3.60}$$

$$\langle A_2(t)v, w\rangle_{V_2^* \times V_2} = \int_{\mathcal{D}} \mathbf{a}_2(t,\cdot,\nabla v)\cdot\nabla w \,\mathrm{d}x, \quad w,v \in V_2 \tag{3.61}$$

$$(B(t)v, w)_H = \int_{\mathcal{D}} \mathbf{b}(t,\cdot,v)w \,\mathrm{d}x, \quad v,w \in H \tag{3.62}$$

and $A(t)\colon V \to V^*$ is given by $A(t) = A_1(t) + A_2(t)$. We assume that for $f\colon [0,T]\times\overline{\mathcal{D}} \to \mathbb{R}$ the abstract function $[f(t)](x) = f(t,x)$, $(t,x) \in (0,T)\times\mathcal{D}$, is an element of $L^q(0,T;V^*)$. This function is decomposed into $f_1 = 0$ and $f_2 = f$ in $L^q(0,T;V^*)$. For $u_0 \in H$, we can now state (3.59) in a variational formulation given by

$$\begin{cases} u' + Au + Bu = f & \text{in } L^q(0,T;V^*),\\ u(0) = u_0 & \text{in } H. \end{cases} \tag{3.63}$$

This evolution equation in mind, we obtain convergence results for the product splitting scheme.

Theorem 3.3.5. *Let $\mathbf{a}_1\colon [0,T]\times\overline{\mathcal{D}}\times\mathbb{R} \to \mathbb{R}$ and $\mathbf{a}_2\colon [0,T]\times\overline{\mathcal{D}}\times\mathbb{R}^d \to \mathbb{R}^d$ fulfill Assumption 3.3.1 and let $\mathbf{b}\colon [0,T]\times\overline{\mathcal{D}}\times\mathbb{R}\to\mathbb{R}$ fulfill Assumption 3.3.4. Let $f \in L^q(0,T;V^*)$ and $u_0 \in H$ be given.*

Furthermore, let $(N_\ell)_{\ell\in\mathbb{N}}$ be a sequence of natural numbers with $N_\ell \to \infty$ as $\ell \to \infty$, $k = \frac{T}{N_\ell}$, $t_n = nk$, $n \in \{0,\dots,N_\ell\}$, and $(u_0^k)_{k>0}$ in V such that $u_0^k \to u_0$ in H as $k \to 0$ and $(k^{\frac{1}{p}}\|u_0^k\|_V)_{k>0}$ is uniformly bounded with respect to k. Then the scheme,

$$\begin{aligned} \frac{\mathbf{U}_0^n - \mathbf{U}^{n-1}}{k} + \mathbf{B}^n\mathbf{U}^{n-1} &= 0 \quad \text{in } H,\\ \frac{\mathbf{U}_m^n - \mathbf{U}_{m-1}^n}{k} + \mathbf{A}_m^n\mathbf{U}_m^n &= \mathbf{f}_m^n \quad \text{in } V_m^*, \quad m \in \{1,2\}, \end{aligned}$$

for $n \in \{1,\dots,N_\ell\}$ with $\mathbf{U}^n = \mathbf{U}_2^n$ and $\mathbf{U}^0 = u_0^k$ admits a unique solution $(\mathbf{U}^n)_{n\in\{1,\dots,N_\ell\}}$ in H. Here, the discretizations of the data are given by

$$\mathbf{A}_m^n = \frac{1}{k}\int_{t_{n-1}}^{t_n} A_m(t)\,\mathrm{d}t, \quad m \in \{1,2\}, \quad \mathbf{B}^n = \frac{1}{k}\int_{t_{n-1}}^{t_n} B(t)\,\mathrm{d}t,$$

$\mathbf{f}_1^n = 0$, and $\mathbf{f}_2^n = \frac{1}{k}\int_{t_{n-1}}^{t_n} f(t)\,\mathrm{d}t$. Then all the convergence results from Theorem 3.1.18 and 3.1.19 hold true. In particular, the sequences of the piecewise constant and piecewise linear prolongations of $(\mathbf{U}^n)_{n\in\{1,\dots,N_\ell\}}$ converge to the solution u of (3.63) pointwise strongly in H as $k \to 0$.

If $\mathbf{a}_2$ also fulfills Assumption 3.3.2, then the sequence of piecewise constant prolongations of the values $(\mathbf{U}^n)_{n\in\{1,\dots,N_\ell\}}$ converges to u strongly in $L^p(0,T;V)$ as $k \to 0$.

In our scheme, the order of the appearing operators $A_1(t)$ and $A_2(t)$, $t \in [0,T]$, is important. As we need the embedding of V_2 into H to be compact, it is not possible to change their order. Moreover, as the solution $(\mathbf{U}^n)_{n\in\{1,\dots,N_\ell\}}$ is in the space V_2, it makes sense to choose the smallest space as the last. Then we obtain the best regularity result for our numerical approximation. The choice of $(\mathbf{f}_1^n)_{n\in\{1,\dots,N_\ell\}}$ and $(\mathbf{f}_2^n)_{n\in\{1,\dots,N_\ell\}}$ is not unique. Choosing one function as zero, seems like a good choice when it comes to computing a solution. If only a function $f \in L^q(0,T;V_1^*)$ is given, it is also possible to set $\mathbf{f}_2^n = 0$ in V_2^* and $\mathbf{f}_1^n = \frac{1}{k}\int_{t_{n-1}}^{t_n} f(t)\,\mathrm{d}t$ in V_1^* for $n \in \{1,\dots,N_\ell\}$.

Proof of Theorem 3.3.5. In order to apply Theorem 3.1.18 and Theorem 3.1.19, it only remains to verify that $A_m(t)$, $t \in [0,T]$ and $m \in \{1,2\}$, fulfill Assumption 3.1.2 and 3.1.3 and $B(t)$, $t \in [0,T]$, fulfills Assumption 3.1.5. For the decomposition of f, it is easy to see that Assumption 3.1.8 is fulfilled.

In order to prove that the operator $A_1(t)$, $t \in [0,T]$, is well-defined, we apply Assumption 3.3.1 (3), which yields that

$$\begin{aligned}\langle A_1(t)v, w\rangle_{V_1^*\times V_1} &= \int_{\mathcal{D}} \mathbf{a}_1(t,\cdot,v)w\,\mathrm{d}x \le \int_{\mathcal{D}} \big(d_1|v|^{p-1} + d_2\big)|w|\,\mathrm{d}x\\ &\le \max\big\{d_1, \|d_2\|_{L^q(\mathcal{D})}\big\}\big(1 + \|v\|_{V_1}^{p-1}\big)\|w\|_{V_1} \end{aligned}\tag{3.64}$$

for every $v, w \in V_1$ and $t \in [0,T]$. This also proves that $A_1(t)$, $t \in [0,T]$, fulfills the boundedness condition from Assumption 3.1.2 (4).

Next, we prove the continuity of $t \mapsto A_1(t)v$ almost everywhere in $(0,T)$ for every $v \in V_1$. To this end, let $t \in [0,T]$ and $(t_i)_{i\in\mathbb{N}}$ with $t_i \to t$ be such that $\mathbf{a}_1(t_i,x,z) \to \mathbf{a}_1(t,x,z)$ as $i \to \infty$ for almost every $x \in \mathcal{D}$ and every $z \in \mathbb{R}$. An application of Hölder's inequality yields

$$\begin{aligned}\langle A_1(t_i)v - A_1(t)v, w\rangle_{V_1^*\times V_1} &= \int_{\mathcal{D}} \big(\mathbf{a}_1(t_i,\cdot,v) - \mathbf{a}_1(t,\cdot,v)\big)w\,\mathrm{d}x\\ &\le \Big(\int_{\mathcal{D}} |\mathbf{a}_1(t_i,\cdot,v) - \mathbf{a}_1(t,\cdot,v)|^q\,\mathrm{d}x\Big)^{\frac{1}{q}}\|w\|_{V_1}\end{aligned}$$

for every $v, w \in V_1$ and $t \in [0,T]$. Similarly to (3.64), we can obtain $|\mathbf{a}_1(t_i,\cdot,v) - \mathbf{a}_1(t,\cdot,v)|^q$ is bounded by a function that is integrable on $\mathcal{D}$. Then we can apply Lebesgue's dominated convergence theorem and it follows that

$$\lim_{i\to\infty} \|A_1(t_i)v - A_1(t)v\|_{V_1^*} = \lim_{i\to\infty}\Big(\int_{\mathcal{D}} |\mathbf{a}_1(t_i,\cdot,v) - \mathbf{a}_1(t,\cdot,v)|^q\,\mathrm{d}x\Big)^{\frac{1}{q}} = 0$$

for every $v \in V_1$ and $t \in [0,T]$.

In order to prove that $A_1(t)\colon V_1 \to V_1^*$, $t \in [0,T]$, is radially continuous, let $(s_i)_{i\in\mathbb{N}}$ be a convergent sequence in $[0,1]$ with the limit $s \in [0,1]$. As (3.64) is finite, it follows that $\mathbf{a}_1(t,\cdot,v+s_iw)w$ is bounded by an integrable function on $\mathcal{D}$. Thus, we can apply Lebesgue's dominated convergence theorem and it follows that

$$\begin{aligned}\lim_{i\to\infty}\langle A_1(t)(v+s_iw), w\rangle_{V_1^*\times V_1} &= \lim_{i\to\infty}\int_{\mathcal{D}} \mathbf{a}_1(t,\cdot,v+s_iw)w\,\mathrm{d}x\\ &= \int_{\mathcal{D}} \lim_{i\to\infty}\mathbf{a}_1(t,\cdot,v+s_iw)w\,\mathrm{d}x = \int_{\mathcal{D}} \mathbf{a}_1(t,\cdot,v+sw)w\,\mathrm{d}x\end{aligned}$$

for every $v, w \in V_1$ and $t \in [0,T]$ due to Assumption 3.3.1 (1). The monotonicity condition for $A_1(t)$, $t \in [0,T]$, from Assumption 3.1.2 (3) is a consequence of Assumption 3.3.1 (2).

Here, we see that

$$\langle A_1(t)v - A_1(t)w, v - w\rangle_{V_1^*\times V_1} = \int_{\mathcal{D}} (\mathbf{a}_1(t,\cdot,v) - \mathbf{a}_1(t,\cdot,w))(v-w)\,\mathrm{d}x \geq 0$$

for $v, w \in V_1$ and $t \in [0,T]$. In order to verify the coercivity condition from Assumption 3.1.2 (5), we apply Assumption 3.3.1 (3) to see that

$$\langle A_1(t)v, v\rangle_{V_1^*\times V_1} \geq \int_{\mathcal{D}} \big(d_3|v|^p - d_4\big)\,\mathrm{d}x = d_3|v|_{V_1}^p - \|d_4\|_{L^1(\mathcal{D})}.$$

The proof that $A_2(t)\colon V_2 \to V_2^*$ is well-defined and fulfills Assumption 3.1.2 (1)–(5) can be done analogously to $A_1(t)$, $t \in [0,T]$. The functions $v, w \in V_1$ just have to be replaced by $\nabla v, \nabla w$ for $v, w \in V_2$.

In order to see that $B(t)$, $t \in [0,T]$, fulfills Assumption 3.1.5, we notice that the continuity of $t \mapsto B(t)v$ almost everywhere in $(0,T)$ for every $v \in H$ is a consequence of Assumption 3.3.4 (1) and can be proved analogously to the corresponding condition for $t \mapsto A_1(t)v$, $v \in V_1$. The Lipschitz condition from Assumption 3.1.5 (2) is fulfilled as

$$\begin{aligned}(B(t)v_1 - B(t)v_2, w)_H &= \int_{\mathcal{D}} \big(\mathbf{b}(t,\cdot,v_1) - \mathbf{b}(t,\cdot,v_2)\big)w\,\mathrm{d}x\\ &\leq \int_{\mathcal{D}} e_1|v_1 - v_2||w|\,\mathrm{d}x \leq e_1\|v_1 - v_2\|_H\|w\|_H\end{aligned}$$

holds true for all $v_1, v_2, w \in H$ and $t \in [0,T]$. Therefore,

$$\|B(t)v_1 - B(t)v_2\|_H \leq e_1\|v_1 - v_2\|_H$$

is fulfilled for all $v_1, v_2 \in H$ and $t \in [0,T]$. In a similar fashion, we can show Assumption 3.1.5 (3) since

$$(B(t)0, w)_H = \int_{\mathcal{D}} \mathbf{b}(t,\cdot,0)w\,\mathrm{d}x \leq \int_{\mathcal{D}} |e_2||w|\,\mathrm{d}x \leq \|e_2\|_{L^2(\mathcal{D})}\|w\|_H$$

is fulfilled for every $w \in H$ and $t \in [0,T]$ because of Assumption 3.3.4 (3). This implies $\|B(t)0\| \leq \|e_2\|_{L^2(\mathcal{D})}$ and $t \in [0,T]$. A combination of the two inequalities also shows that the operator is indeed well-defined. Also the space $V_2 = W_0^{1,p}(\Omega)$ is compactly embedded into $H = L^2(\Omega)$, compare [1, Theorem 6.3].

Therefore, for $t \in [0,T]$ the operators $A_1(t)$, $A_2(t)$, $A(t) = A_1(t) + A_2(t)$, and $B(t)$ fulfill all the necessary conditions and we can apply Theorem 3.1.18 and Theorem 3.1.19.

Moreover, if the stronger monotonicity condition from Assumption 3.3.2 is fulfilled, we see that (3.42) is satisfied for $A_2(t)$, $t \in [0,T]$, since

$$\begin{aligned}\langle A_2(t)v - A_2(t)w, v - w\rangle_{V_2^*\times V_2} &= \int_{\mathcal{D}} (\mathbf{a}_2(t,\cdot,\nabla v) - \mathbf{a}_2(t,\cdot,\nabla w))\cdot(\nabla v - \nabla w)\,\mathrm{d}x\\ &\geq d_5\int_{\mathcal{D}} |\nabla v - \nabla w|^p\,\mathrm{d}x = d_5\|v - w\|_{V_2}^p\end{aligned}$$

for $v, w \in V_2$ and $t \in [0,T]$. Thus, the sequence of piecewise constant prolongations $(\bar{U}^k)_{k>0} = (\bar{U}_2^k)_{k>0}$ converges strongly to the exact solution u in $L^p(0,T;V_2) = L^p(0,T;V)$ as $k \to 0$ due to Theorem 3.1.19. □

It remains to verify that the results from Section 3.2 are also applicable to the nonlinear parabolic problem. In order to obtain explicit error bounds, we need to make additional assumptions on the mappings $\mathbf{a}_m$, $m \in \{1,2\}$, and on the exact solution u of (3.63). At this point, we do not explain how the additional regularity of u can be obtained. For more information about additional regularity and some examples, see Section 1.2.

Theorem 3.3.6. *Let all the assumptions from Theorem 3.3.5 be fulfilled and consider the same scheme. In addition, let p be either an element of $[2,\infty) \cap [d,\infty)$ or $\big[2, \frac{2d-p}{d-p}\big) \cap [2,d)$, where d is the dimension of the underlying space $\mathbb{R}^d \supset \mathcal{D}$. Assume that $\mathbf{a}_1$ and $\mathbf{a}_2$ fulfill Assumptions 3.3.2 and 3.3.3 and let the nonnegative functions d_2 and e_2 from the boundedness condition for $\mathbf{a}_1$ and $\mathbf{b}$ in Assumption 3.3.1 (3) and Assumption 3.3.4 (3) be elements of $L^p(\mathcal{D})$. For $\alpha \in (0,1]$, let the solution u of (3.63) be an element of $C^{0,\alpha}([0,T];V)$.*

Then the scheme obtains a unique solution $(\mathbf{U}^n)_{n\in\{1,\dots,N\}}$ in H and there exists a constant $C \in (0,\infty)$ such that

$$\max_{n\in\{1,\dots,N\}} \|u(t_n) - \mathbf{U}^n\|_H^2 + k\sum_{n=1}^N \|u(t_n) - \mathbf{U}^n\|_V^p \le Ck^{\alpha q}$$

is fulfilled for every step size k, which is small enough.

Proof. In order to prove this theorem, we will apply Theorem 3.2.3. In the proof of Theorem 3.3.5 we have already seen that $A_1(t)$, $A_2(t)$, and $B(t)$ fulfill Assumptions 3.1.2, 3.1.3, and 3.1.5 for $t \in [0,T]$. The stronger monotonicity condition from (3.48) is fulfilled as an application of Assumption 3.3.2 yields

$$\langle A_m(t)v - A_m(t)w, v-w\rangle_{V_m^*\times V_m} \ge d_5\|v-w\|_{V_m}^p$$

for $m \in \{1,2\}$.

Next, we use Assumption 3.3.3 to prove the bounded Lipschitz condition for $A_1(t)$, $t \in [0,T]$, from (3.47). Here, we consider two cases, at first we prove the condition for $p=2$ and after that for $p \in (2,\infty)$. In the case $p=2$, we have a global Lipschitz condition. Inserting the definition of $A_1(t)$, $t\in[0,T]$, we obtain that

$$\begin{aligned}\langle A_1(t)v_1 - A_1(t)v_2, w\rangle_{V_1^*\times V_1} &= \int_{\mathcal{D}} \big(\mathbf{a}_1(t,\cdot,v_1) - \mathbf{a}_1(t,\cdot,v_2)\big) w \,\mathrm{d}x\\ &\le 2d_6 \int_{\mathcal{D}} |v_1 - v_2||w|\,\mathrm{d}x \le 2d_6\|v_1-v_2\|_{V_1}\|w\|_{V_1}\end{aligned}$$

and therefore

$$\|A_1(t)v_1 - A_1(t)v_2\|_{V_1^*} \le 2d_6\|v_1 - v_2\|_{V_1}$$

for every $v_1, v_2, w \in V_1$ and $t \in [0,T]$. For $p \in (2,\infty)$, we obtain a Lipschitz constant, which depends on the inserted functions. Thus, an additional application of Lemma A.1.3 becomes necessary to obtain

$$\begin{aligned}&\langle A_1(t)v_1 - A_1(t)v_2, w\rangle_{V_1^*\times V_1}\\ &= \int_{\mathcal{D}} \big(\mathbf{a}_1(t,\cdot,v_1) - \mathbf{a}_1(t,\cdot,v_2)\big) w\,\mathrm{d}x\\ &\le d_6 \int_{\mathcal{D}} \big(1 + \max\big\{|v_1|^{p-2}, |v_2|^{p-2}\big\}\big)|v_1 - v_2||w|\,\mathrm{d}x\\ &\le d_6\Big(\|1\|_{L^{\frac{p}{p-2}}(\mathcal{D})} + \Big(\int_{\mathcal{D}} \max\big\{|v_1|^p, |v_2|^p\big\}\,\mathrm{d}x\Big)^{\frac{p-2}{p}}\Big)\|v_1 - v_2\|_{L^p(\mathcal{D})}\|w\|_{L^p(\mathcal{D})}\end{aligned}$$

for every $v_1, v_2, w \in V_1$ and $t \in [0,T]$. Since $\frac{p-2}{p} \in (0,1)$ for $p \in (2,\infty)$, it follows that

$$\begin{aligned}\Big(\int_{\mathcal{D}} \max\big\{|v_1|^p, |v_2|^p\big\}\,\mathrm{d}x\Big)^{\frac{p-2}{p}} &\le \Big(\int_{\mathcal{D}} |v_1|^p\,\mathrm{d}x + \int_{\mathcal{D}} |v_2|^p\,\mathrm{d}x\Big)^{\frac{p-2}{p}}\\ &\le \Big(\int_{\mathcal{D}} |v_1|^p\,\mathrm{d}x\Big)^{\frac{p-2}{p}} + \Big(\int_{\mathcal{D}} |v_2|^p\,\mathrm{d}x\Big)^{\frac{p-2}{p}}\\ &= \|v_1\|_{L^p(\mathcal{D})}^{p-2} + \|v_2\|_{L^p(\mathcal{D})}^{p-2} \le 2\max\big\{\|v_1\|_{V_1}^{p-2}, \|v_2\|_{V_1}^{p-2}\big\}.\end{aligned}$$

Thus, for $R \in (0,\infty)$ and all $v_1, v_2 \in V_1$ with $\|v_1\|_{V_1}, \|v_2\|_{V_1} \le R$, we obtain the bound

$$\begin{aligned}\|A_1(t)v_1 - A_1(t)v_2\|_{V_1^*} &\le d_6\Big(\|1\|_{L^{\frac{p}{p-2}}(\mathcal{D})} + 2\max\big\{\|v_1\|_{V_1}^{p-2}, \|v_2\|_{V_1}^{p-2}\big\}\Big)\|v_1 - v_2\|_{V_1}\\ &=: L(R)\|v_1 - v_2\|_{V_1},\end{aligned}$$

which proves (3.47). An analogous bound for $A_2(t)$, $t \in [0,T]$, can be proved by replacing v_1, v_2, w with their gradient ∇v_1, ∇v_2, $\nabla w \in L^p(\mathcal{D})^d$.

In order to prove the required regularity conditions from Theorem 3.2.3, recall that by assumption u is an element of $C^{0,\alpha}([0,T];V)$. The function U_2 defined in Assumption 3.2.1 coincides with u in $C^{0,\alpha}([0,T];V)$. Thus, it follows by the regularity assumption $u \in C^{0,\alpha}([0,T];V)$

$$\|U_2(t) - u(t_{n-1})\|_{V_2} = \|u(t) - u(t_{n-1})\|_V \le |u|_{C^{0,\alpha}([0,T];V)}|t - t_{n-1}|^\alpha$$

for every $t \in (t_{n-1}, t_n]$, $n \in \{1,\dots,N\}$. The function U_1 defined in Assumption 3.2.1 is given by

$$U_1(t) = u(t_{n-1}) - \int_{t_{n-1}}^t \big(A_1(s)u(s) + B(s)u(s)\big)\,\mathrm{d}s$$

for every $t \in (t_{n-1}, t_n]$, $n \in \{1,\dots,N\}$. We then obtain that

$$\|U_1(t) - u(t_{n-1})\|_{V_1}^p \le |t - t_{n-1}|^{p-1}\int_{t_{n-1}}^t \|A_1(s)u(s) + B(s)u(s)\|_{L^p(\mathcal{D})}^p\,\mathrm{d}s,$$

where the integrand can be bounded by

$$\begin{aligned}&\|A_1(s)u(s) + B(s)u(s)\|_{L^p(\mathcal{D})}\\ &\le \|\mathbf{a}_1(s,\cdot,u(s))\|_{L^p(\mathcal{D})} + \|\mathbf{b}(s,\cdot,u(s))\|_{L^p(\mathcal{D})}\\ &\le d_1\||u(s)|^{p-1}\|_{L^p(\mathcal{D})} + \|d_2\|_{L^p(\mathcal{D})} + e_1\|u(s)\|_{L^p(\mathcal{D})} + \|e_2\|_{L^p(\mathcal{D})}\end{aligned} \tag{3.65}$$

for every $s \in [0,T]$. In order to prove that the last row is finite, we use the fact that $d_2, e_2 \in L^p(\mathcal{D})$ and the Sobolev embedding theorem, compare [1, Theorem 4.12]. If $p \in [d,\infty)$, then $V = W_0^{1,p}(\mathcal{D})$ is continuously embedded into $L^r(\mathcal{D})$ for every $r \in [p,\infty)$. Thus, for $u(s) \in V$ it follows that $\||u(s)|^{p-1}\|_{L^p(\mathcal{D})}$ and $\|u(s)\|_{L^p(\mathcal{D})}$ are finite. If $p \in [2,d)$, then the space V is continuously embedded into $L^r(\mathcal{D})$ for $r \in \big[p, \frac{dp}{d-p}\big]$. As we assume that $p \in \big[2, \frac{2d-p}{d-p}\big]$ in this case, it follows that

$$p(p-1) \le p\Big(\frac{2d-p}{d-p} - 1\Big) = p\Big(\frac{2d-p}{d-p} - \frac{d-p}{d-p}\Big) = \frac{dp}{d-p}$$

and therefore the terms containing $u(s)$ in (3.65) are finite. This shows that

$$\|U_1(t) - u(t_{n-1})\|_{V_1} \le c_2|t - t_{n-1}| \le c_2 T^{1-\alpha}|t - t_{n-1}|^\alpha$$

for $c_2 \in (0,\infty)$, which does not depend on the step size k.

□

Appendix A

Appendix

A.1 Useful Inequalities

In the following, we collect a few inequalities and identities that appear throughout the analysis in the chapters above. We begin by a discrete Gronwall lemma.

Lemma A.1.1. *Let $(u_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ be two nonnegative sequences that satisfy, for given $a \in [0,\infty)$ and $N \in \mathbb{N}$, that*

$$u_n \leq a + \sum_{i=1}^{n-1} b_i u_i, \quad n \in \{1,\dots,N\}.$$

Then it follows that

$$u_n \leq a \exp\Big(\sum_{i=1}^{n-1} b_i\Big), \quad n \in \{1,\dots,N\}.$$

In the case that a sum $\sum_{i=1}^{0}$ appears, we use the convention that it is zero. A proof of this lemma can be found in [22]. In some settings, it is possible to use the following inequality as an alternative.

Lemma A.1.2. *Let $a, b, x \in [0,\infty)$ be given such that $x^2 \leq 2ax + b^2$ is fulfilled. Then it also follows that $x \leq 2a + b$.*

Proof. Since $x^2 \leq 2ax + b^2$ is fulfilled, it follows that

$$(x-a)^2 = x^2 - 2ax + a^2 \leq a^2 + b^2.$$

Taking the square root on both sides, this yields

$$|x-a| \leq \sqrt{a^2+b^2} \leq a + b.$$

As $x - a \leq |x-a|$ is fulfilled, we obtain the desired bound after adding a to both sides of the inequality. □

The following lemma is a Hölder type inequality. We omit the proof, it can be done by an inductive application of the ordinary Hölder inequality.

Lemma A.1.3. *Let $\mathcal{D} \subseteq \mathbb{R}^d$, $d \geq 1$, and $N \in \mathbb{N}$ be given. Then for $p_n \in [1,\infty]$, $n \in \{1,\ldots,N\}$, such that $\sum_{n=1}^N \frac{1}{p_n} = 1$ it follows that*

$$\int_{\mathcal{D}} \prod_{n=1}^N |u_n| \,\mathrm{d}x \leq \prod_{n=1}^N \|u_n\|_{L^{p_n}(\mathcal{D})}$$

for $u_n \in L^{p_n}(\mathcal{D})$.

Last, we state an identity, which appears several times throughout the analysis. Again, we omit the proof. It can be done by rewriting the right-hand side of the equality with the definition of the norm via the inner product.

Lemma A.1.4. *Let $(H,(\cdot,\cdot)_H,\|\cdot\|_H)$ be a real Hilbert space. The identity*

$$(v-w,v)_H = \frac{1}{2}\big(\|v\|_H^2 - \|w\|_H^2 + \|v-w\|_H^2\big)$$

is fulfilled for every $v,w \in H$.

A.2 Bochner Integral

We shortly recall the main statements for Bochner integrable functions on a general measure space. For a complete introduction, we refer the reader to [115, Chapter V, Section 4–5], [30, Chapter II.2], [96, Section 4.2], and [100, Kapitel 2].

In the following, we assume that $(X,\|\cdot\|_X)$ is a real Banach space and $(\Omega,\mathcal{F},\mu)$ is a finite measure space. Then we consider an abstract function $v\colon \Omega \to X$ and call it simple if for a finite number $N \in \mathbb{N}$ there exist $x_1,\ldots,x_N \in X$ and mutually disjoint sets $C_1,\ldots,C_N \in \mathcal{F}$ such that $v = \sum_{n=1}^N x_n \chi_{C_n}$, where χ_{C_n} is the characteristic function with respect to the set C_n, $n \in \{1,\ldots N\}$. We call a function v Bochner measurable if there exists a sequence $(v_n)_{n\in\mathbb{N}}$ of simple functions such that $v_n(\omega) \to v(\omega)$ in X as $n \to \infty$ for almost every $\omega \in \Omega$. A function $v\colon \Omega \to X$ is called weakly measurable if $\omega \mapsto \langle f, v(\omega)\rangle_{X^*\times X}$ is Lebesgue measurable for every $f \in X^*$. Under suitable assumptions, these two concepts are equivalent as we see in the next theorem.

Theorem A.2.1. *Let $(\Omega,\mathcal{F},\mu)$ be a finite measure space and let $(X,\|\cdot\|_X)$ be a real, separable Banach space. For a function $v\colon \Omega \to X$, the following statements are equivalent:*

(a) The function v is Bochner measurable.

(b) The function v fulfills that $v^{-1}(C) \in \mathcal{F}$ for all open sets $C \subseteq X$.

(c) The function v is weakly measurable.

This theorem is a consequence of Pettis theorem. A proof can be found in [96, Theorem 4.2.4]. We are foremost interested in functions that are also Bochner integrable. A Bochner measurable function $v\colon \Omega \to X$ is called Bochner integrable if there exists a sequence $(v_n)_{n\in\mathbb{N}}$ of simple functions such that $v_n(\omega) \to v(\omega)$ in X as $n \to \infty$ for almost every $\omega \in \Omega$ and for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $n,m \geq N$ the inequality $\int_\Omega \|v_n - v_m\|_X \,\mathrm{d}\mu < \varepsilon$ is fulfilled. The integral of v is then given by

$\int_\Omega v \,\mathrm{d}\mu = \lim_{n\to\infty} \int_\Omega v_n \,\mathrm{d}\mu$. Moreover, for $p \in [1,\infty]$ we introduce the space $L^p(\Omega; X)$ that consists of all Bochner measurable functions $v\colon \Omega \to X$ such that

$$\|v\|_{L^p(\Omega;X)} = \begin{cases} \Big(\int_\Omega \|v\|_X^p \,\mathrm{d}\mu\Big)^{\frac{1}{p}}, & p \in [1,\infty), \\ \operatorname*{ess\,sup}_\Omega \|v\|_X, & p = \infty \end{cases} \tag{A.1}$$

is finite. This function space fulfills the following properties.

Lemma A.2.2. *Let $(\Omega, \mathcal{F}, \mu)$ be a finite measure space and let $(X, \|\cdot\|_X)$ be a real Banach space. Then the following properties are fulfilled:*

(a) The space $L^p(\Omega; X)$ equipped with the norm given in (A.1) *is a Banach space for every $p \in [1,\infty]$.*

(b) The set of simple X-valued functions is dense in $L^p(\Omega; X)$ for $p \in [1,\infty)$.

(c) If both $L^1(\Omega)$ and X are separable, it follows that $L^p(\Omega; X)$ is separable for $p \in [1,\infty)$.

(d) If X is reflexive, then the space $L^p(\Omega; X)$ is reflexive for every $p \in (1,\infty)$.

(e) If X is continuously embedded into $(Y, \|\cdot\|_Y)$, then $L^q(\Omega; X)$ is continuously embedded into $L^p(\Omega; Y)$ for $p, q \in [1,\infty]$ and $p \le q$.

These statements can be found in [30] and [96, Proposition 2.3.24, Proposition 4.2.22]. Furthermore, if X is continuously and densely embedded into a space Y, then the simple X-valued functions are also dense in $L^q(\Omega; Y)$ for every $q \in [1,\infty)$. Thus, the space $L^p(\Omega; X)$ is continuously and densely embedded into $L^q(\Omega; Y)$ for $q \in [1,\infty)$ and $p \in [q,\infty]$. Moreover, if X has the Radon–Nikodym property, it follows that $\big(L^p(\Omega; X)\big)^* = L^q(\Omega; X^*)$ for $p \in [1,\infty)$ and $q = \frac{p}{p-1}$. Note that the Radon–Nikodym property is fulfilled if, for example, X is reflexive or separable, compare [96, Theorem 4.2.25 and 4.2.26].

Lemma A.2.3. *Let $(\Omega, \mathcal{F}, \mu)$ be a finite measure space, let $(X, \|\cdot\|_X)$ be a real Banach space, and let $p, r \in [1,\infty]$ be such that $p < r$. For a sequence $(v_n)_{n\in\mathbb{N}}$ that is bounded in $L^r(\Omega; X)$ and $v \in L^p(\Omega; X)$ such that $v_n \to v$ in $L^p(\Omega; X)$ as $n \to \infty$, it follows that $v_n \to v$ in $L^q(\Omega; X)$ as $n \to \infty$ for every $q \in [p, r)$.*

Proof. Since $\frac{1}{q} \in (\frac{1}{r}, \frac{1}{p}]$ there exists $\theta \in [0,1)$ such that $\frac{1}{q} = \frac{\theta}{r} + \frac{1-\theta}{p}$. Choosing $\alpha = \frac{r}{\theta q}$ and $\tilde{\alpha} = \frac{p}{(1-\theta)q}$, it follows that $\frac{1}{\alpha} + \frac{1}{\tilde{\alpha}} = 1$ and we can apply Hölder's inequality to

$$\begin{aligned} \|w\|_{L^q(\Omega;X)}^q &= \int_\Omega \|w\|_X^{\theta q} \|w\|_X^{(1-\theta)q} \,\mathrm{d}\mu \\ &\le \Big(\int_\Omega \|w\|_X^r \,\mathrm{d}\mu\Big)^{\frac{1}{\alpha}} \Big(\int_\Omega \|w\|_X^p \,\mathrm{d}\mu\Big)^{\frac{1}{\tilde{\alpha}}} = \|w\|_{L^r(\Omega;X)}^{\theta q} \|w\|_{L^p(\Omega;X)}^{(1-\theta)q} \end{aligned}$$

for $w \in L^r(\Omega; X)$. Then we obtain

$$\|v_n - v_m\|_{L^q(\Omega;X)} \le \|v_n - v_m\|_{L^r(\Omega;X)}^\theta \|v_n - v_m\|_{L^p(\Omega;X)}^{1-\theta} \to 0 \quad \text{as } m, n \to \infty,$$

since the sequence $(v_n)_{n\in\mathbb{N}}$ is bounded in $L^r(\Omega; X)$ and convergent in $L^p(\Omega; X)$. This shows that $(v_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^q(\Omega; X)$ and therefore convergent to v due to the uniqueness of the limit. $\square$

A proof of the next theorem can be found in [96, Proposition 4.2.12].

Theorem A.2.4. *Let $(\Omega, \mathcal{F}, \mu)$ be a finite measure space and let $(X, \|\cdot\|_X)$ be a real Banach space. Further, let $v\colon \Omega \to X$ be a Bochner measurable function. Then v is Bochner integrable if and only if $\|v\|_X : \Omega \to \mathbb{R}$ is integrable.*

In the special case that $\Omega = [0,T]$ equipped with the Lebesgue sets and the Lebesgue measure, we can state the lemma of Lions–Aubin. This gives us a compact embedding argument for Bochner integrable functions. We refer the reader to [99, Lemma 7.7] for a proof of this statement.

Lemma A.2.5. *Let X_{-1}, X_0, and X_1 be real Banach spaces such that $X_1 \hookrightarrow X_0 \hookrightarrow X_{-1}$, X_1 is separable, reflexive, and compactly embedded into X_0, and X_{-1} is reflexive. For $p \in (1,\infty)$ and $q \in [1,\infty]$, the space*

$$\mathcal{W} = \{v \in L^p(0,T;X_1) : v' \text{ exists and } v' \in L^q(0,T;X_{-1})\}$$

is compactly embedded into $L^p(0,T;X_0)$.

A.3 Stochastic Background

As we are dealing with a randomized scheme in Chapter 2, we will give a short overview of the probabilistic results needed. For more details, we refer the reader to [73].

In the following, let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space. For a real, separable Banach space $(X, \|\cdot\|_X)$, we call a mapping $U\colon \Omega \to X$ a random variable if it is measurable with respect to the σ-algebra $\mathcal{F}$ and the Borel σ-algebra $\mathcal{B}(X)$ in X. Precisely, this means that for every $C \in \mathcal{B}(X)$ the set

$$U^{-1}(C) = \{\omega \in \Omega \,: U(\omega) \in C\} \subseteq \Omega$$

is an element of $\mathcal{F}$. The expectation, i.e., the integral of a random variable U with respect to the measure $\mathbf{P}$ is denoted by

$$\mathbf{E}[U] = \int_\Omega U(\omega) \,\mathrm{d}\mathbf{P}(\omega).$$

In our theory, we often work with random variables $\xi\colon \Omega \to [a,b]$, $a,b \in \mathbb{R}$, $a < b$, that are uniformly distributed. For such a mapping, the density is given by $\frac{1}{b-a}$ and a substitution yields that

$$\mathbf{E}[v(\xi)] = \int_\Omega v(\xi(\omega)) \,\mathrm{d}\mathbf{P}(\omega) = \frac{1}{b-a} \int_a^b v(t) \,\mathrm{d}t, \tag{A.2}$$

where $v\colon [a,b] \to X$ is a Bochner integrable function. Note that the theory from the previous section applies as every probability space is, in particular, a measurable space.

Within the theory of Monte Carlo algorithms, it will be important to consider independent random variables. To this end, let us recall the concept of independence. We call a family $(C_n)_{n\in\mathbb{N}}$ of elements in $\mathcal{F}$ independent if for every finite subset $I \subset \mathbb{N}$

$$\mathbf{P}\Big(\bigcap_{n\in I} C_n\Big) = \prod_{n\in I} \mathbf{P}(C_n) \tag{A.3}$$

is fulfilled. Similarly, we call a family $(\mathcal{F}_n)_{n\in\mathbb{N}}$ of σ-algebras independent if again for every finite subset of $I \subset \mathbb{N}$ and $(C_n)_{n\in I}$ such that $C_n \in \mathcal{F}_n$, $n \in I$, it follows that the family

$(C_n)_{n\in I}$ is independent in the sense of (A.3). This at hand, we can now transfer the concept of independence to a family $(U_n)_{n\in I}$, $I \subset \mathbb{N}$ and finite, of random variables. If the generated σ-algebras

$$\sigma(U_n) = \{U_n^{-1}(C) : C \in \mathcal{B}(X)\}, \quad n \in I, \tag{A.4}$$

are independent, we also call $(U_n)_{n\in I}$ independent.

In some settings, it can be important to consider a certain family $(\mathcal{F}_n)_{n\in\mathbb{N}}$ of σ-algebras. It is called a filtration if $\mathcal{F}_n$ is a subset of $\mathcal{F}$ and $\mathcal{F}_m$ for every $n, m \in N$ with $n \leq m$.

A random variable $U\colon \Omega \to X$ can be measurable with respect to $\mathcal{F}_m$ but not with respect to $\mathcal{F}_n$ for $n < m$, $n, m \in \mathbb{N}$. It can then be helpful to consider the conditional expectation $\mathbf{E}[U|\mathcal{F}_n]\colon \Omega \to X$ of U with respect to $\mathcal{F}_n$. More precisely, the $\mathcal{F}_n$-measurable mapping $\mathbf{E}[U|\mathcal{F}_n]$ is uniquely determined by the relation

$$\mathbf{E}[U\chi_C] = \mathbf{E}[\mathbf{E}[U|\mathcal{F}_n]\chi_C]$$

for every $C \in \mathcal{F}_n$. A useful property of the conditional expectation is the tower property. This states that for two σ-algebras $\mathcal{F}_n$ and $\mathcal{F}_m$ of the filtration $(\mathcal{F}_n)_{n\in\mathbb{N}}$ with $n \leq m$ we obtain that

$$\mathbf{E}[\mathbf{E}[U|\mathcal{F}_n]|\mathcal{F}_m] = \mathbf{E}[\mathbf{E}[U|\mathcal{F}_m]|\mathcal{F}_n] = \mathbf{E}[U|\mathcal{F}_n].$$

If the random variable U is measurable with respect to $\mathcal{F}_n$, then the conditional expectation is the function itself, i.e., $\mathbf{E}[U|\mathcal{F}_n] = U$. Furthermore, if $\sigma(U)$ is independent of $\mathcal{F}_n$, we obtain that $\mathbf{E}[U|\mathcal{F}_n] = \mathbf{E}[U]$.

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