Port-Hamiltonian Realizations of Linear Time Invariant Systems

Christopher Beattie* and Volker Mehrmann† and Hongguo Xu‡

January 7, 2016

Abstract

The question when a general linear time invariant control system is equivalent to a port-Hamiltonian systems is answered. Several equivalent characterizations are derived which extend the characterizations of [38] to the general non-minimal case. An explicit construction of the transformation matrices is presented. The methods are applied in the stability analysis of disc brake squeal.

Keywords: port-Hamiltonian system, passivity, stability, system transformation, linear matrix inequality, Lyapunov inequality, even pencil, quadratic eigenvalue problem.
AMS subject classification. 93A30, 93B17, 93B11.

1 Introduction

The synthesis of system models that describe complex physical phenomena often involves the coupling of independently developed subsystems originating within different disciplines. Systematic approaches to coupling such diversely generated subsystems prudently follows a system-theoretic network paradigm that focusses on the transfer of energy, mass, and other conserved quantities among the subsystems. When the subsystem models themselves arise from variational principles, then the aggregate system typically has structural features that reflects underlying conservation laws and often it may be characterized as a *port-Hamiltonian (PH) system*, see [11, 15, 27, 29, 30, 33, 32, 34, 35, 36] for some major references. Although PH systems may be formulated in a more general framework, we will restrict ourselves to *input-state-output PH systems*, which have the form

$$\begin{aligned}\dot{\mathbf{x}} &= (\mathbf{J}-\mathbf{R})\,\nabla_{\mathbf{x}}\mathcal{H}(\mathbf{x}) + (\mathbf{F}-\mathbf{P})\mathbf{u}(t), \\ \mathbf{y}(t) &= (\mathbf{F}+\mathbf{P})^T\nabla_{\mathbf{x}}\mathcal{H}(\mathbf{x}) + (\mathbf{S}+\mathbf{N})\mathbf{u}(t),\end{aligned} \tag{1}$$

*Department of Mathematics, Virginia Tech, Blacksburg, VA 24061, USA. beattie@vt.edu. Supported by *Einstein Foundation Berlin*, through an Einstein Visiting Fellowship.

[2]Institut für Mathematik MA 4-5, TU Berlin, Str. des 17. Juni 136, D-10623 Berlin, FRG. mehrmann@math.tu-berlin.de. The authors gratefully acknowledge the support by the Deutsche Forschungsgemeinschaft (DFG) as part of the collaborative research center SFB 1029 Sub- stantial efficiency increase in gas turbines through direct use of coupled unsteady combustion and flow dynamics, project A02 *Development of a reduced order model of pulsed detonation combuster* and by Einstein Center ECMath in Berlin.

[3]Department of Mathematics, University of Kansas, Lawrence, KS 66045, USA. xu@math.ku.edu. Partially supported by *Alexander von Humboldt Foundation* and by *Deutsche Forschungsgemeinschaft*, through the DFG Research Center MATHEON *Mathematics for Key Technologies*

where $\mathbf{x} \in \mathbb{R}^n$ is the n-dimensional *state vector*; $\mathcal{H} : \mathbb{R}^n \to [0, \infty)$ is the *Hamiltonian*, a continuously differentiable scalar-valued vector function describing the distribution of internal energy among the energy storage elements of the system; $\mathbf{J} = -\mathbf{J}^T \in \mathbb{R}^{n \times n}$ is the *structure matrix* describing the energy flux among energy storage elements within the system; $\mathbf{R} = \mathbf{R}^T \in \mathbb{R}^{n \times n}$ is the *dissipation matrix* describing energy dissipation/loss in the system; $\mathbf{F} \pm \mathbf{P} \in \mathbb{R}^{n \times m}$ are the *port* matrices, describing the manner in which energy enters and exits the system, and $\mathbf{S} + \mathbf{N}$, with $\mathbf{S} = \mathbf{S}^T \in \mathbb{R}^{m,m}$ and $\mathbf{N} = -\mathbf{N}^T \in \mathbb{R}^{m,m}$, describing the direct feed-through of input to output. The matrices, $\mathbf{R}$, $\mathbf{P}$, and $\mathbf{S}$ must satisfy

$$\mathbf{K} = \begin{bmatrix} \mathbf{R} & \mathbf{P} \\ \mathbf{P}^T & \mathbf{S} \end{bmatrix} \geq 0; \tag{2}$$

that is, $\mathbf{K}$ is symmetric positive-semidefinite. This implies, in particular, that $\mathbf{R}$ and $\mathbf{S}$ are also positive semidefinite, $\mathbf{R} \geq 0$ and $\mathbf{S} \geq 0$.

Port-Hamiltonian systems generalize the classical notion of *Hamiltonian systems* expressed in our notation as $\dot{\mathbf{x}} = \mathbf{J} \nabla_{\mathbf{x}} \mathcal{H}(\mathbf{x})$. The analog of the *conservation of energy* for Hamiltonian systems is for PH systems (1) the *dissipation inequality*:

$$\mathcal{H}(\mathbf{x}(t_1)) - \mathcal{H}(\mathbf{x}(t_0)) \leq \int_{t_0}^{t_1} \mathbf{y}(t)^T \mathbf{u}(t)\ dt, \tag{3}$$

which has a natural interpretation as an assertion that the increase in internal energy of the system, as measured by $\mathcal{H}$, cannot exceed the *total work* done on the system. $\mathcal{H}(\mathbf{x})$ is a *storage function* associated with the *supply rate*, $\mathbf{y}(t)^T \mathbf{u}(t)$. In the language of system theory, (3) constitutes the property that (1) is a *passive* system [10].

One may verify with elementary manipulations that the inequality in (3) is an immediate consequence of the inequality in (2), and holds even when the coefficient matrices $\mathbf{J}$, $\mathbf{R}$, $\mathbf{F}$, $\mathbf{P}$, $\mathbf{S}$, and $\mathbf{N}$ depend on $\mathbf{x}$ or explicitly on time t (see, [27]) or, indeed (with care taken to define suitable operator domains), when they are defined as linear operators acting on infinite dimensional spaces [21, 35]. Notice that with a null input, $\mathbf{u}(t) = 0$, the the dissipation inequality asserts that $\mathcal{H}(\mathbf{x})$ is non-increasing along any unforced system trajectory. Thus, $\mathcal{H}(\mathbf{x})$ defines a Lyapunov function for the unforced system, so PH systems are implicitly Lyapunov stable [19]. Similarly, $\mathcal{H}(\mathbf{x})$ is non-increasing along any system trajectory that produces a null output, $\mathbf{y}(t) = 0$, so PH systems also have Lyapunov stable *zero dynamics* [9].

PH systems constitute a class of systems that is closed under *power-conserving interconnection*. This means that port-connected PH systems produce an aggregate system that must also be port-Hamiltonian. This aggregate system hence, will be guaranteed to be both stable and passive. Modeling with PH systems, thus, represents physical properties in such a way as to facilitate automated modeling [23] and at same time encoding physical properties in the structure of equations. This framework also provides compelling motivation to identify and preserve PH structure when producing reduced order surrogate models, see [1, 18, 31].

Now consider a general linear time-invariant (LTI) system:

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}\,\mathbf{x} + \mathbf{B}\,\mathbf{u} \\ \mathbf{y} &= \mathbf{C}\,\mathbf{x} + \mathbf{D}\,\mathbf{u}, \end{aligned} \tag{4}$$

with $\mathbf{A} \in \mathbb{R}^{n \times n}$, $\mathbf{B} \in \mathbb{R}^{n \times m}$, $\mathbf{C} \in \mathbb{R}^{m \times n}$, and $\mathbf{D} \in \mathbb{R}^{m \times m}$. The system (4) is *passive* if there exists a continuously differentiable *storage function* $\mathcal{H} : \mathbb{R}^n \to [0, \infty)$ such that (3) holds for all admissible inputs $\mathbf{u}$.

So the natural question arises: *When is (4) equivalent to a port-Hamiltonian system?*

Here specifically, we consider *equivalence* to LTI port-Hamiltonian systems taking the form

$$
\begin{aligned}
\dot{\boldsymbol{\xi}} &= (\mathbf{J}-\mathbf{R})\mathbf{Q}\,\boldsymbol{\xi} \;+\; (\mathbf{F}-\mathbf{P})\,\boldsymbol{\varphi}, \\
\boldsymbol{\eta} &= (\mathbf{F}+\mathbf{P})^T\mathbf{Q}\,\boldsymbol{\xi} \;+\; (\mathbf{S}+\mathbf{N})\,\boldsymbol{\varphi},
\end{aligned}
\tag{5}
$$

with $\mathbf{J}=-\mathbf{J}^T$, $\mathbf{R}=\mathbf{R}^T\geq 0$, $\mathbf{Q}=\mathbf{Q}^T>0$, $\mathbf{S}=\mathbf{S}^T\geq 0$, $\mathbf{N}=-\mathbf{N}^T$, where $\mathbf{J}$, $\mathbf{R}$, $\mathbf{Q}\in\mathbb{R}^{n\times n}$, $\mathbf{F},\mathbf{P}\in\mathbb{R}^{n\times m}$, $\mathbf{S},\mathbf{N}\in\mathbb{R}^{m\times m}$, and $\mathbf{K}$ as defined in (2) is positive semidefinite.

The notion of *system equivalence* that we consider here engages three invertible transformations connecting (4) and (5), one on each of the input, output, and state space:

$$
\mathbf{u}(t)=\widetilde{\mathbf{V}}\boldsymbol{\varphi}(t), \quad \boldsymbol{\eta}(t)=\mathbf{V}^T\mathbf{y}(t), \quad \text{and} \quad \mathbf{x}(t)=\mathbf{T}\boldsymbol{\xi}(t) \quad (\text{with } \widetilde{\mathbf{V}},\ \mathbf{V},\ \mathbf{T} \text{ invertible}).
$$

Within this context, the *supply rate* associated with (4) is transformed as

$$
\mathbf{y}(t)^T\mathbf{u}(t)=\boldsymbol{\eta}(t)^T\mathbf{V}^{-1}\widetilde{\mathbf{V}}\boldsymbol{\varphi}(t).
$$

We wish to constrain the permissible transformations characterizing *system equivalence* so as to be *power conserving*; that is, so that supply rates remain invariant, $\mathbf{y}(t)^T\mathbf{u}(t)=\boldsymbol{\eta}(t)^T\boldsymbol{\varphi}(t)$. Thus, we assume that $\widetilde{\mathbf{V}}=\mathbf{V}$ and we say that (4) is *equivalent* to (5) if there exist invertible matrices, $\mathbf{V}$ and $\mathbf{T}$, such that

$$
\mathbf{u}(t)=\mathbf{V}\,\boldsymbol{\varphi}(t), \quad \boldsymbol{\eta}(t)=\mathbf{V}^T\,\mathbf{y}(t), \quad \text{and} \quad \mathbf{x}(t)=\mathbf{T}\,\boldsymbol{\xi}(t). \tag{6}
$$

Since PH systems are structurally passive, our starting point is the following characterization of passivity introduced in [39] for minimal linear time invariant systems. The system (4) is *minimal* if it is both controllable and observable. The system (4) (and more specifically, the pair of matrices $(\mathbf{A},\mathbf{B})$ with $\mathbf{A}\in\mathbb{R}^{n\times n}$, $\mathbf{B}\in\mathbb{R}^{n\times m}$) is *controllable* if rank $\begin{bmatrix} s\mathbf{I}-\mathbf{A} & \mathbf{B}\end{bmatrix}=n$ for all $s\in\mathbb{C}$. Similarly, the system (4) (and the pair $(\mathbf{A},\mathbf{C})$ with $\mathbf{A}\in\mathbb{R}^{n\times n}$, $\mathbf{C}\in\mathbb{R}^{m\times n}$ is *observable* if rank $\begin{bmatrix} s\mathbf{I}-\mathbf{A} \\ \mathbf{C}\end{bmatrix}=n$ for all $s\in\mathbb{C}$.

Theorem 1 ([39]) *Assume that the LTI system (4) is minimal. The matrix inequality*

$$
\begin{bmatrix} \mathbf{A}^T\mathbf{Q}+\mathbf{Q}\mathbf{A} & \mathbf{Q}\mathbf{B}-\mathbf{C}^T \\ \mathbf{B}^T\mathbf{Q}-\mathbf{C} & -(\mathbf{D}+\mathbf{D}^T) \end{bmatrix} \leq 0 \tag{7}
$$

has a solution $\mathbf{Q}=\mathbf{Q}^T>0$ *if and only if (4) is a* passive *system, in which case:*

i) $\mathcal{H}(\mathbf{x})=\frac{1}{2}\mathbf{x}^T\mathbf{Q}\mathbf{x}$ *defines a storage function for (4) associated with the supply rate* $\mathbf{y}^T\mathbf{u}$*, satisfying (3).*

ii) There exist maximum and minimum symmetric solutions to (7): $0<\mathbf{Q}_-\leq\mathbf{Q}_+$ *such that for all symmetric solutions* $\mathbf{Q}$ *to (7),* $\mathbf{Q}_-\leq\mathbf{Q}\leq\mathbf{Q}_+$.

This result yields an immediate consequence for PH realizations.

Corollary 2 *Assume that the LTI system (4) is minimal. Then (4) has a PH realization if and only if it is passive. Moreover, if (4) is passive then* every *equivalent system to (4) (as generated by transformations as in (6)) may also be directly expressed as a PH system.*

Proof. If (4) has a PH realization then *a fortiori* it is passive. Conversely, if (4) is passive then (7) has a positive definite solution $\widehat{\mathbf{Q}} = \widehat{\mathbf{Q}}^T = \mathbf{T}^T\mathbf{T}$, written in terms of a Cholesky or square root factor $\mathbf{T}$. Then we can define directly

$$\begin{array}{lll} \mathbf{Q} = \mathbf{I}, & \mathbf{J} = \frac{1}{2}(\mathbf{TAT}^{-1} - (\mathbf{TAT}^{-1})^T), & \mathbf{R} = -\frac{1}{2}(\mathbf{TAT}^{-1} + (\mathbf{TAT}^{-1})^T) \\ & \mathbf{F} = \frac{1}{2}\left(\mathbf{TB} + (\mathbf{CT}^{-1})^T\right), & \mathbf{P} = \frac{1}{2}\left(\mathbf{TB} + (\mathbf{CT}^{-1})^T\right), \\ & \mathbf{S} = \frac{1}{2}(\mathbf{D} + \mathbf{D}^T) & \mathbf{N} = \frac{1}{2}(\mathbf{D} - \mathbf{D}^T) \end{array} \tag{8}$$

and (7) can be written in terms of these defined quantities as

$$-2\begin{bmatrix} \mathbf{T}^T & 0 \\ 0 & \mathbf{I} \end{bmatrix}\begin{bmatrix} \mathbf{R} & \mathbf{P} \\ \mathbf{P}^T & \mathbf{S} \end{bmatrix}\begin{bmatrix} \mathbf{T} & 0 \\ 0 & \mathbf{I} \end{bmatrix} \le 0,$$

which verifies (2). Thus, $\mathbf{J}$, $\mathbf{R}$, $\mathbf{Q}$, $\mathbf{F}$, $\mathbf{P}$, $\mathbf{S}$, and $\mathbf{N}$ as defined in (8) indeed determine a PH system. $\square$

The matrix inequality (7) actually includes two properties, the *Lyapunov inequality* in the upper left block,

$$\mathbf{A}^T\mathbf{Q} + \mathbf{QA} \le 0, \tag{9}$$

via Lyapunov's theorem [26], guarantees that the uncontrolled system $\dot{x} = \mathbf{A}x$ is *stable*, i.e., $\mathbf{A}$ has all eigenvalues in the closed left half plane and those on the imaginary axis are semisimple, while passivity is encoded in the solvability of the full matrix inequality (7). If the inequality (9) is strict, then the system is *asymptotically stable*, i.e., all eigenvalues of $\mathbf{A}$ are in the open left half plane, and *strictly passive*, i.e., also the dissipation inequality (3) is strict, see [24] for a detailed analysis.

We will discuss the solution of these two types matrix inequalities in the more general situation of non-minimal systems in Section 2. Based on these results we will then construct explicit transformations from of a general LTI system (minimal or not) to port-Hamiltonian form (5) in Section 3.

2 Lyapunov and Riccati inequalities

The solution of Lyapunov and Riccati inequalities as they arise in the characterizations (9) and (7) is usually addressed via the solution of semidefinite programming, see [4]. We discuss the more explicit characterization via invariant subspaces.

2.1 Solution of Lyapunov inequalities.

The stability of $\mathbf{A}$ is a necessary condition for (9) and (7) which require that $\mathbf{TAT}^{-1} + \mathbf{T}^{-T}\mathbf{A}^T\mathbf{T}^T \le 0$, or equivalently, that the Lyapunov inequality (9) has a positive definite solution $\mathbf{Q} = \mathbf{T}^T\mathbf{T}$.

It is well known that the equality case in (9) always has a positive definite solution if $\mathbf{A}$ is stable, see [26]. In the following we recall, see e.g. [4], a characterization of the complete set of solutions of the inequality case.

If $\mathbf{A}$ is stable, but not asymptotically stable, then due to the fact that the eigenvalues on the imaginary axis are all semi-simple, using the real Jordan form of $\mathbf{A}$, see e.g. [20], there exist a nonsingular matrix $\mathbf{M} \in \mathbb{R}^{n\times n}$ such that

$$\mathbf{MAM}^{-1} = \operatorname{diag}\left(\mathbf{A}_1, \alpha_2\mathbf{J}_2, \ldots, \alpha_r\mathbf{J}_r\right), \tag{10}$$

where $\mathbf{A}_1 \in \mathbb{R}^{n_1 \times n_1}$ is asymptotically stable, $\alpha_2, \ldots, \alpha_r \geq 0$ are real and distinct, and $\mathbf{J}_j = \begin{bmatrix} 0 & \mathbf{I}_{n_j} \\ -\mathbf{I}_{n_j} & 0 \end{bmatrix}$, $j = 2, \ldots, r$. To characterize the solution set of (9), we make the ansatz that

$$\mathbf{Q} = \mathbf{M}^T \operatorname{diag}\left(\mathbf{Q}_1, \hat{\mathbf{Q}}_2, \ldots, \hat{\mathbf{Q}}_r\right) \mathbf{M}, \tag{11}$$

and separately consider the determination of the block $\mathbf{Q}_1$ and the other blocks. Let $\mathcal{W}(n_1)$ be the set of symmetric positive semidefinite matrices $\Theta_1 \in \mathbb{R}^{n_1 \times n_1}$ with the property that $\Theta_1 \mathbf{x} \neq 0$ for any eigenvector $\mathbf{x}$ of $\mathbf{A}_1$. Then for any $\Theta_1 \in \mathcal{W}(n_1)$ we define $\mathbf{Q}_1$ to be the unique symmetric positive definite solution of the Lyapunov equation $\mathbf{A}_1^T \mathbf{Q}_1 + \mathbf{Q}_1 \mathbf{A}_1 = -\Theta_1$, see [26]. The other matrices $\hat{\mathbf{Q}}_j$, $j = 2, \ldots, r$ are chosen of the form

$$\hat{\mathbf{Q}}_j = \begin{bmatrix} \mathbf{Y}_j & \mathbf{Z}_j \\ -\mathbf{Z}_j & \mathbf{Y}_j \end{bmatrix} > 0,$$

with $\mathbf{Z}_j = -\mathbf{Z}_j^T$, when $\alpha_j > 0$ or an arbitrary $\hat{\mathbf{Q}}_j > 0$ when $\alpha_j = 0$.

We have the following characterization of the solution set of (9).

Lemma 3 *Let $\mathbf{A} \in \mathbb{R}^{n \times n}$. Then the Lyapunov inequality (9) has a symmetric positive definite solution $\mathbf{Q} \in \mathbb{R}^{n \times n}$ if and only if $\mathbf{A}$ is stable.*

If $\mathbf{A}$ is asymptotically stable, then the solution set is given by the set of all symmetric positive definite solutions of the Lyapunov equation $\mathbf{A}^T \mathbf{Q} + \mathbf{Q}\mathbf{A} = -\Theta$, where Θ is any symmetric positive semidefinite matrix with the property that $\Theta \mathbf{x} \neq 0$ for any eigenvector $\mathbf{x}$ of $\mathbf{A}$, or in other words (A, Θ) is observable.

If $\mathbf{A}$ is stable, but not asymptotically stable, then with the transformation (10), any solution of (9) has the form (11) solving the Lyapunov equation

$$\mathbf{A}^T \mathbf{Q} + \mathbf{Q}\mathbf{A} = -\mathbf{M}^T \operatorname{diag}(\Theta_1, 0, \ldots, 0) \mathbf{M} \tag{12}$$

with $\Theta_1 \in \mathcal{W}(n_1)$.

Proof. Consider a transformation to the form (10) and set, for a symmetric matrix $\mathbf{Q}$, $\mathbf{M}^{-T}\mathbf{Q}\mathbf{M}^{-1} = [\mathbf{Q}_{ij}]$ partitioned accordingly. Form

$$\begin{aligned} &\mathbf{M}^{-T}(\mathbf{A}^T\mathbf{Q} + \mathbf{Q}\mathbf{A})\mathbf{M}^{-1} \\ &= \begin{bmatrix} \mathbf{A}_1^T \mathbf{Q}_{11} + \mathbf{Q}_{11}\mathbf{A}_1 & \mathbf{A}_1^T \mathbf{Q}_{12} + \alpha_2 \mathbf{Q}_{12}\mathbf{J}_2 & \ldots & \mathbf{A}_1^T \mathbf{Q}_{1,r} + \alpha_r \mathbf{Q}_{1,r}\mathbf{J}_r \\ \alpha_2 \mathbf{J}_2^T \mathbf{Q}_{12}^T + \mathbf{Q}_{12}^T \mathbf{A}_1 & \alpha_2(\mathbf{J}_2^T \mathbf{Q}_{22} + \mathbf{Q}_{22}\mathbf{J}_2) & \ldots & \alpha_2 \mathbf{J}_2^T \mathbf{Q}_{2,r} + \alpha_r \mathbf{Q}_{2,r}\mathbf{J}_r \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_r \mathbf{J}_r^T \mathbf{Q}_{1,r}^T + \mathbf{Q}_{1,r}^T \mathbf{A}_1 & \alpha_r \mathbf{J}_r^T \mathbf{Q}_{2,r}^T + \alpha_2 \mathbf{Q}_{2,r}^T \mathbf{J}_2 & \ldots & \alpha_r(\mathbf{J}_r^T \mathbf{Q}_{r,r} + \mathbf{Q}_{r,r}\mathbf{J}_r) \end{bmatrix}, \end{aligned}$$

again partitioned accordingly. For any $j = 2, \ldots, r$ with $\alpha_r \neq 0$ partition

$$\mathbf{Q}_{jj} = \begin{bmatrix} \mathbf{Q}_{j,11} & \mathbf{Q}_{j,12} \\ \mathbf{Q}_{j,12}^T & \mathbf{Q}_{j,22} \end{bmatrix}$$

according to the block structure in $\mathbf{J}_j$, so that

$$\mathbf{J}_j^T \mathbf{Q}_{jj} + \mathbf{Q}_{jj}\mathbf{J}_j = \begin{bmatrix} -\mathbf{Q}_{j,12} - \mathbf{Q}_{j,12}^T & \mathbf{Q}_{j,11} - \mathbf{Q}_{j,22} \\ \mathbf{Q}_{j,11} - \mathbf{Q}_{j,22} & \mathbf{Q}_{j,12} + \mathbf{Q}_{j,12}^T \end{bmatrix}.$$

Then, $\alpha_j(\mathbf{J}_j^T\mathbf{Q}_{jj}+\mathbf{Q}_{jj}\mathbf{J}_j) \leq 0$ if and only if $\mathbf{Q}_{j,12}+\mathbf{Q}_{j,12}^T = 0$ and it follows that $\mathbf{Q}_{j,11}-\mathbf{Q}_{j,22} = 0$. Thus any matrix of the form $\mathbf{Q}_{jj} = \begin{bmatrix} \mathbf{Y}_j & \mathbf{Z}_j \\ -\mathbf{Z}_j & \mathbf{Y}_j \end{bmatrix} > 0$ with $\mathbf{Z}_j = -\mathbf{Z}_j^T$, is a positive definite solution. If $\alpha_j = 0$, then clearly any $\mathbf{Q}_{jj} > 0$ is a solution.

In both cases the resulting diagonal blocks $\mathbf{Q}_{jj} > 0$ satisfy

$$\alpha_j(\mathbf{J}_j^T\mathbf{Q}_{jj} + \mathbf{Q}_{jj}\mathbf{J}_j) = 0.$$

Thus, for $i \neq j$, the blocks $\mathbf{Q}_{ij}$ have to satisfy one of the two matrix equations

$$\mathbf{A}_1^T\mathbf{Q}_{1j} + \alpha_j\mathbf{Q}_{1j}\mathbf{J}_j = 0,\ \alpha_i\mathbf{J}_i^T\mathbf{Q}_{ij} + \alpha_j\mathbf{Q}_{ij}\mathbf{J}_j = 0,$$

which implies that $\mathbf{Q}_{ij} = 0$ for all $i \neq j$, since the spectra of the respective pairs $\mathbf{A}_1, \alpha_j\mathbf{J}_j$ and $\alpha_i\mathbf{J}_i, \alpha_j\mathbf{J}_j$ are different and thus the corresponding Sylvester equations only have 0 as solution [26].

For any positive semidefinite matrix Θ_1 satisfying $\Theta_1\mathbf{x} \neq 0$ for any eigenvector $\mathbf{x}$ of $\mathbf{A}_1$, by Lyapunov's Theorem [26], there exists a symmetric positive definite matrix $\mathbf{Q}_1$ satisfying $\mathbf{A}_1^T\mathbf{Q}_1 + \mathbf{Q}_1\mathbf{A}_1 = -\Theta_1$. Thus, the Lyapunov inequality (9) has a solution $\mathbf{Q} > 0$ and it has the form (11) satisfying (12). The results for an asymptotically stable $\mathbf{A}$ can be easily derived as a special case with all $\hat{\mathbf{Q}}_2, \ldots, \hat{\mathbf{Q}}_r$ void in (11).

Conversely, if the Lyapunov inequality has a solution $\mathbf{Q} > 0$, and if $\mathbf{x} \neq 0$ is an eigenvector of $\mathbf{A}$, i.e., $\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$, then

$$(\bar{\lambda} + \lambda)\mathbf{x}^*\mathbf{Q}\mathbf{x} \leq 0.$$

Since $\mathbf{x}^*\mathbf{Q}\mathbf{x} > 0$, it follows that $\bar{\lambda} + \lambda \leq 0$, i.e., the real part of λ is non-positive. Suppose that $\mathbf{A}$ has a purely imaginary eigenvalue $i\alpha$. Since a similarity transformation of $\mathbf{A}$ does not change the Lyapunov inequality (9), we may assume that $\mathbf{A}$ is in (complex) Jordan canonical form, and we may consider each Jordan block separately. If $\mathbf{A}_k$ is a single real Jordan block of size $k \times k$, $k > 1$ with eigenvalue $i\alpha$, then for any Hermitian matrix $\hat{\mathbf{Q}}_k = [q_{ij}]$ we have that

$$\mathbf{A}_k^*\mathbf{Q}_k + \mathbf{Q}_k\mathbf{A}_k = \begin{bmatrix} 0 & q_{11} & \cdots & q_{1,k-1} \\ q_{11} & q_{12} + \bar{q}_{12} & \cdots & q_{1k} + q_{2,k-1} \\ \vdots & \vdots & \ddots & \vdots \\ \bar{q}_{1,k-1} & \bar{q}_{2,k-1} + \bar{q}_{1,k} & \cdots & q_{k-1,k} + \bar{q}_{k-1,k} \end{bmatrix}.$$

This matrix cannot be negative semidefinite, since otherwise $q_{11} = 0$, which would contradict that $\hat{\mathbf{Q}}_k$ is positive definite.

Therefore, $\mathbf{A}$ has to be stable, i.e., if it has purely imaginary eigenvalues, these eigenvalues must be semi-simple. □

Remark 1 The construction in Lemma 3 relies on the computation of the real Jordan form of $\mathbf{A}$, which usually cannot be computed in finite precision arithmetic. For the numerical computation of the solution it is better to use the real Schur form, see [16].

2.2 Solution of Riccati inequalities

By Corollary 2 we can characterize (at least in the minimal case) the existence of a transformation to PH form via the existence of a symmetric positive definite matrix $\mathbf{Q}$ solving the linear matrix inequality

$$\begin{bmatrix} -\mathbf{A}^T\mathbf{Q}-\mathbf{Q}\mathbf{A} & \mathbf{C}^T-\mathbf{Q}\mathbf{B} \\ \mathbf{C}-\mathbf{B}^T\mathbf{Q} & \mathbf{D}+\mathbf{D}^T \end{bmatrix} \geq 0, \tag{13}$$

In this subsection we will discuss the explicit solution of this matrix inequality in the general (non-minimal) case. It is clear that for the inequality to hold, $\mathbf{S} := \mathbf{D}+\mathbf{D}^T$ has to positive semidefinite. However, we will see below that we can restrict ourselves to the case that $\mathbf{S} := \mathbf{D}+\mathbf{D}^T$ is positive definite, so we only consider this case here and, using Schur complements, (13) is equivalent to the *Riccati inequality*

$$(\mathbf{A}-\mathbf{B}\mathbf{S}^{-1}\mathbf{C})^T\mathbf{Q}+\mathbf{Q}(\mathbf{A}-\mathbf{B}\mathbf{S}^{-1}\mathbf{C})+\mathbf{Q}\mathbf{B}\mathbf{S}^{-1}\mathbf{B}^T\mathbf{Q}+\mathbf{C}^T\mathbf{S}^{-1}\mathbf{C} \leq 0. \tag{14}$$

We first investigate the influence of the purely imaginary eigenvalues of $\mathbf{A}$. Clearly, a necessary condition for (13) to be solvable is that $\mathbf{Q}$ satisfies $\mathbf{A}^T\mathbf{Q}+\mathbf{Q}\mathbf{A} \leq 0$. Following Lemma 3, if $\mathbf{A}$ has the form (10) then $\mathbf{Q}$ must have the form (11), and $\mathbf{A}^T\mathbf{Q}+\mathbf{Q}\mathbf{A}$ has the form (12). Written in compact form, we get

$$\mathbf{M}\mathbf{A}\mathbf{M}^{-1} = \begin{bmatrix} \mathbf{A}_1 & 0 \\ 0 & \mathbf{A}_2 \end{bmatrix}, \quad \mathbf{M}^{-T}\mathbf{Q}\mathbf{M}^{-1} = \begin{bmatrix} \mathbf{Q}_1 & 0 \\ 0 & \mathbf{Q}_2 \end{bmatrix},$$

where

$$\mathbf{A}_2 = \mathrm{diag}(\alpha_2\mathbf{J}_2,\ldots,\alpha_r\mathbf{J}_r) = -\mathbf{A}_2^T, \quad \mathbf{Q}_2 = \mathrm{diag}(\hat{\mathbf{Q}}_2,\ldots,\hat{\mathbf{Q}}_r),$$

satisfying $\mathbf{A}_2^T\mathbf{Q}_2+\mathbf{Q}_2\mathbf{A}_2 = 0$, and

$$\mathbf{M}^{-T}(\mathbf{A}^T\mathbf{Q}+\mathbf{Q}\mathbf{A})\mathbf{M}^{-1} = \begin{bmatrix} \mathbf{A}_1^T\mathbf{Q}_1+\mathbf{Q}_1\mathbf{A}_1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Setting

$$\mathbf{M}\mathbf{B} = \begin{bmatrix} \mathbf{B}_1 \\ \mathbf{B}_2 \end{bmatrix}, \quad \mathbf{C}\mathbf{M}^{-1} = \begin{bmatrix} \mathbf{C}_1 & \mathbf{C}_2 \end{bmatrix}.$$

and premultiplying $\mathbf{M}^{-T}$ and post-multiplying $\mathbf{M}^{-1}$ to the first block row and column of (13), respectively, one has

$$\begin{bmatrix} -\mathbf{A}_1^T\mathbf{Q}_1-\mathbf{Q}_1\mathbf{A}_1 & 0 & \mathbf{C}_1^T-\mathbf{Q}_1\mathbf{B}_1 \\ 0 & 0 & \mathbf{C}_2^T-\mathbf{Q}_2\mathbf{B}_2 \\ \mathbf{C}_1-\mathbf{B}_1^T\mathbf{Q}_1 & \mathbf{C}_2-\mathbf{B}_2^T\mathbf{Q}_2 & \mathbf{S} \end{bmatrix} \geq 0.$$

Therefore, $\mathbf{Q}_2$ must be positive definite satisfying

$$\mathbf{B}_2^T\mathbf{Q}_2 = \mathbf{C}_2, \quad \mathbf{A}_2^T\mathbf{Q}_2+\mathbf{Q}_2\mathbf{A}_2 = 0, \tag{15}$$

and

$$\begin{bmatrix} -\mathbf{A}_1^T\mathbf{Q}_1-\mathbf{Q}_1\mathbf{A}_1 & \mathbf{C}_1^T-\mathbf{Q}_1\mathbf{B}_1 \\ \mathbf{C}_1-\mathbf{B}_1^T\mathbf{Q}_1 & \mathbf{S} \end{bmatrix} \geq 0, \tag{16}$$

or equivalently it has to satisfy the *reduced Riccati inequality*

$$\Psi(\mathbf{Q}_1) := (\mathbf{A}_1-\mathbf{B}_1\mathbf{S}^{-1}\mathbf{C}_1)^T\mathbf{Q}_1+\mathbf{Q}_1(\mathbf{A}_1-\mathbf{B}_1\mathbf{S}^{-1}\mathbf{C}_1)+\mathbf{Q}_1\mathbf{B}_1\mathbf{S}^{-1}\mathbf{B}_1^T\mathbf{Q}_1+\mathbf{C}_1^T\mathbf{S}^{-1}\mathbf{C}_1 \leq 0. \tag{17}$$

To study the solvability of (17), we note that $\mathbf{A}_1$ is asymptotically stable. We claim that $\mathbf{A}_1-\mathbf{B}_1\mathbf{S}^{-1}\mathbf{C}_1$ is necessarily asymptotically stable as well. To show this, suppose that the inequality (17) has a solution $\mathbf{Q}_1 > 0$. Since $\mathbf{Q}_1\mathbf{B}_1\mathbf{S}^{-1}\mathbf{B}_1^T\mathbf{Q}_1 + \mathbf{C}_1^T\mathbf{S}^{-1}\mathbf{C}_1 \geq 0$, it follows from Lemma 3 that $\mathbf{A}_1 - \mathbf{B}_1\mathbf{S}^{-1}\mathbf{C}_1$ is stable, and there exits an invertible matrix $\mathbf{M}_1$ such that $\mathbf{M}_1(\mathbf{A}_1 - \mathbf{B}_1\mathbf{S}^{-1}\mathbf{C}_1)\mathbf{M}_1^{-1}$ has real Jordan form (10), $\mathbf{M}_1^{-T}\mathbf{Q}_1\mathbf{M}_1^{-1}$ has the form (11) and following (12), we have

$$\mathbf{M}_1^{-T}(\mathbf{Q}_1\mathbf{B}_1\mathbf{S}^{-1}\mathbf{B}_1^T\mathbf{Q}_1 + \mathbf{C}_1^T\mathbf{S}^{-1}\mathbf{C}_1)\mathbf{M}_1^{-1} = \mathrm{diag}(\Theta, 0, \ldots, 0) \geq 0.$$

Then, due to the positive definiteness of $\mathbf{S}$ it follows that $\mathbf{C}_1\mathbf{M}_1^{-1} = \begin{bmatrix} \mathbf{C}_{11} & 0 & \ldots & 0 \end{bmatrix}$, and by making use of the block diagonal structure of $\mathbf{M}_1^{-T}\mathbf{Q}_1\mathbf{M}_1^{-1}$, we also have $\mathbf{M}_1\mathbf{B}_1 = \begin{bmatrix} \mathbf{B}_{11}^T & 0 & \ldots & 0 \end{bmatrix}^T$. Thus it follows that

$$\mathbf{M}_1\mathbf{A}_1\mathbf{M}_1^{-1} = \mathrm{diag}(\mathbf{A}_{11}, \alpha_2\mathbf{J}_2, \ldots, \alpha_r\mathbf{J}_r).$$

Since $\mathbf{A}_1$ is asymptotically stable, all $\alpha_j\mathbf{J}_j$ must be void, which implies that $\mathbf{A}_1 - \mathbf{B}_1\mathbf{S}^{-1}\mathbf{C}_1$ must be asymptotically stable as well.

In order to characterize the solution of the reduced Riccati inequality (17), we use the following lemmas. The first is the orthogonal version of the Kalman decomposition, [22, 37].

Lemma 4 *Consider a general LTI system of the form (4). Then there exists a real orthogonal matrix* $\mathbf{U}$ *such that*

$$\begin{aligned}
\mathbf{U}^T\mathbf{A}\mathbf{U} &= \left[\begin{array}{cc|c} \mathbf{A}_{11} & 0 & \mathbf{A}_{13} \\ \mathbf{A}_{21} & \mathbf{A}_{22} & \mathbf{A}_{23} \\ \hline 0 & 0 & \mathbf{A}_{33} \end{array}\right] =: \begin{bmatrix} \tilde{\mathbf{A}}_{11} & \tilde{\mathbf{A}}_{12} \\ 0 & \tilde{\mathbf{A}}_{22} \end{bmatrix} \\
\mathbf{U}^T\mathbf{B} &= \left[\begin{array}{c} \mathbf{B}_1 \\ \mathbf{B}_2 \\ \hline 0 \end{array}\right] =: \begin{bmatrix} \tilde{\mathbf{B}}_1 \\ 0 \end{bmatrix}, \quad \mathbf{C}\mathbf{U} = \left[\begin{array}{cc|c} \mathbf{C}_1 & 0 & \mathbf{C}_3 \end{array}\right] =: \begin{bmatrix} \tilde{\mathbf{C}}_1 & \tilde{\mathbf{C}}_2 \end{bmatrix},
\end{aligned} \tag{18}$$

where the pair $(\tilde{\mathbf{A}}_{11}, \tilde{\mathbf{B}}_1)$ *is controllable and the pair* $(\mathbf{A}_{11}, \mathbf{C}_1)$ *is observable.*

Proof. Applying first the controllability and then the observability staircase forms of [37] to the system, there exists a real orthogonal matrix $\mathbf{U}$ such that $\mathbf{U}^T\mathbf{A}\mathbf{U}$ is in the form (18) with $(\tilde{\mathbf{A}}_{11}, \tilde{\mathbf{B}}_1)$ controllable and $(\mathbf{A}_{11}, \mathbf{C}_1)$ observable. □

Note that by using the block structure in Lemma 4, we have that $(\mathbf{A}_{11}, \mathbf{B}_1)$ is controllable as well.

The next lemma considers the Riccati inequality in the form (refstcf).

Lemma 5 *Consider the reduced Riccati inequality (17) and suppose that both* $\mathbf{A}_1$ *and* $\mathbf{A}_1 - \mathbf{B}_1\mathbf{S}^{-1}\mathbf{C}_1$ *are asymptotically stable, with* $\mathbf{S} > 0$. *Then there exists a transformation to the condensed form (18) of* $\mathbf{A}_1, \mathbf{B}_1, \mathbf{C}_1$, *such that*

$$\begin{aligned}
\mathbf{U}^T(\mathbf{A}_1 - \mathbf{B}_1\mathbf{S}^{-1}\mathbf{C}_1)\mathbf{U} &= \begin{bmatrix} \tilde{\mathbf{A}}_{11} - \tilde{\mathbf{B}}_1\mathbf{S}^{-1}\tilde{\mathbf{C}}_1 & \tilde{\mathbf{A}}_{12} - \tilde{\mathbf{B}}_1\mathbf{S}^{-1}\tilde{\mathbf{C}}_2 \\ 0 & \tilde{\mathbf{A}}_{22} \end{bmatrix} \\
&= \left[\begin{array}{cc|c} \mathbf{A}_{11} - \mathbf{B}_1\mathbf{S}^{-1}\mathbf{C}_1 & 0 & \mathbf{A}_{13} - \mathbf{B}_1\mathbf{S}^{-1}\mathbf{C}_3 \\ \mathbf{A}_{21} - \mathbf{B}_2\mathbf{S}^{-1}\mathbf{C}_1 & \mathbf{A}_{22} & \mathbf{A}_{23} - \mathbf{B}_2\mathbf{S}^{-1}\mathbf{C}_3 \\ \hline 0 & 0 & \mathbf{A}_{33} \end{array}\right],
\end{aligned} \tag{19}$$

where $\mathbf{A}_{11} - \mathbf{B}_1\mathbf{S}^{-1}\mathbf{C}_1$, $\mathbf{A}_{22}$, *and* $\mathbf{A}_{33}$ *are all asymptotically stable. In addition, also* $\mathbf{A}_{11}$ *is asymptotically stable,* $(\tilde{\mathbf{A}}_{11}, \tilde{\mathbf{B}}_1)$ *and* $(\mathbf{A}_{11}, \mathbf{B}_1)$ *are controllable, and* $(\mathbf{A}_{11}, \mathbf{C}_1)$ *is observable.*

The proof is straightforward. $\square$

The following Lemma is also well-known, see e.g. [25, 28], but we present a proof in our notation.

Lemma 6 *Suppose that* $(\mathbf{A}, \mathbf{B})$ *is controllable,* $(\mathbf{A}, \mathbf{C})$ *is observable,* $\mathbf{A}$ *is asymptotically stable, and* $\mathbf{S} > 0$*. Then the Riccati equation*

$$(\mathbf{A} - \mathbf{B}\mathbf{S}^{-1}\mathbf{C})^T\mathbf{Q} + \mathbf{Q}(\mathbf{A} - \mathbf{S}^{-1}\mathbf{C}) + \mathbf{Q}\mathbf{B}\mathbf{S}^{-1}\mathbf{B}^T\mathbf{Q} + \mathbf{C}^T\mathbf{S}^{-1}\mathbf{C} = 0 \tag{20}$$

has a solution $\mathbf{Q} > 0$ *if and only if the* Hamiltonian matrix

$$\mathbf{H} := \begin{bmatrix} \mathbf{A} - \mathbf{B}\mathbf{S}^{-1}\mathbf{C} & \mathbf{B}\mathbf{S}^{-1}\mathbf{B}^T \\ -\mathbf{C}^T\mathbf{S}^{-1}\mathbf{C} & -(\mathbf{A} - \mathbf{B}\mathbf{S}^{-1}\mathbf{C})^T \end{bmatrix} \tag{21}$$

has a Lagrangian invariant subspace, i.e., there exist square matrices $\mathbf{W}_1$*,* $\mathbf{W}_2$*,* $\mathbf{E}$ *such that* $\mathbf{W}_1^T\mathbf{W}_2 = \mathbf{W}_2^T\mathbf{W}_1$*,* $\begin{bmatrix} \mathbf{W}_1^T & \mathbf{W}_2^T \end{bmatrix}^T$ *has full column rank, and*

$$\mathbf{H}\begin{bmatrix} \mathbf{W}_1 \\ \mathbf{W}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{W}_1 \\ \mathbf{W}_2 \end{bmatrix}\mathbf{E}. \tag{22}$$

Moreover, if (22) holds, then $\mathbf{Q} := \mathbf{W}_2\mathbf{W}_1^{-1} > 0$ *solves (20).*

Proof. Suppose that $\mathbf{Q} > 0$ solves (20). It is straightforward to prove that $\begin{bmatrix} \mathbf{W}_1^T & \mathbf{W}_2^T \end{bmatrix}^T = \begin{bmatrix} \mathbf{I} & \mathbf{Q} \end{bmatrix}^T$ satisfies the required conditions and (22) with $\mathbf{E} = \mathbf{A} + \mathbf{B}\mathbf{S}^{-1}\mathbf{B}^T\mathbf{Q}$.

Conversely, suppose there exist $\mathbf{W}_1$ and $\mathbf{W}_2$ satisfying all the described conditions. Without loss of generality we may assume $\begin{bmatrix} \mathbf{W}_1^T & \mathbf{W}_2^T \end{bmatrix}^T$ has orthogonal columns. Then it is well known, [6, 7] that there exists a *CS decomposition*, i.e., there exist orthogonal matrices $\mathbf{U}, \mathbf{V}$ and diagonal matrices Δ, Γ such that

$$\mathbf{U}^T\mathbf{W}_1\mathbf{V} = \begin{bmatrix} \Delta & 0 \\ 0 & 0 \end{bmatrix}, \quad \mathbf{U}^T\mathbf{W}_2\mathbf{V} = \begin{bmatrix} \Gamma & 0 \\ 0 & \mathbf{I} \end{bmatrix},$$

where Δ is invertible, and $\Delta^2 + \Gamma^2 = I$. (Note that the zero block in $\mathbf{U}^T\mathbf{W}_1\mathbf{V}$ could be void. Partition

$$\begin{aligned} \mathbf{U}^T\mathbf{A}\mathbf{U} &= \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix}, \quad \mathbf{U}^T\mathbf{B}\mathbf{S}^{-1}\mathbf{B}^T\mathbf{U} = \begin{bmatrix} \mathbf{G}_{11} & \mathbf{G}_{12} \\ \mathbf{G}_{12}^T & \mathbf{G}_{22} \end{bmatrix}, \\ \mathbf{U}^T\mathbf{C}^T\mathbf{S}^{-1}\mathbf{C}\mathbf{U} &= \begin{bmatrix} \mathbf{H}_{11} & \mathbf{H}_{12} \\ \mathbf{H}_{12}^T & \mathbf{H}_{22} \end{bmatrix}, \quad \mathbf{V}^T\mathbf{E}\mathbf{V} = \begin{bmatrix} \mathbf{E}_{11} & \mathbf{E}_{12} \\ \mathbf{E}_{21} & \mathbf{E}_{22} \end{bmatrix}. \end{aligned}$$

Then (22) takes the form

$$\begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} & \mathbf{G}_{11} & \mathbf{G}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} & \mathbf{G}_{12}^T & \mathbf{G}_{22} \\ -\mathbf{Y}_{11} & -\mathbf{Y}_{12} & -\mathbf{A}_{11}^T & -\mathbf{A}_{21}^T \\ -\mathbf{Y}_{12}^T & -\mathbf{Y}_{22} & -\mathbf{A}_{12}^T & -\mathbf{A}_{22} \end{bmatrix}\begin{bmatrix} \Delta & 0 \\ 0 & 0 \\ \Gamma & 0 \\ 0 & \mathbf{I} \end{bmatrix} = \begin{bmatrix} \Delta & 0 \\ 0 & 0 \\ \Gamma & 0 \\ 0 & \mathbf{I} \end{bmatrix}\begin{bmatrix} \mathbf{E}_{11} & \mathbf{E}_{12} \\ \mathbf{E}_{21} & \mathbf{E}_{22} \end{bmatrix}.$$

By comparing the (2,2) block on both sides one has $\mathbf{G}_{22} = 0$. Due to the positive semi-definiteness of $\mathbf{S}$, then also $\mathbf{G}_{12} = 0$. By comparing the (1,2) blocks on both sides of the above equation and using the nonsingularity of Δ, it follows that $\mathbf{E}_{12} = 0$. Then by comparing the (3,2) block on both sides one has $\mathbf{A}_{21} = 0$. Hence,

$$\mathbf{U}^T\mathbf{A}\mathbf{U} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ 0 & \mathbf{A}_{22} \end{bmatrix}, \quad \mathbf{U}^T\mathbf{B}\mathbf{S}^{-1}\mathbf{B}^T\mathbf{U} = \begin{bmatrix} \mathbf{G}_{11} & 0 \\ 0 & 0 \end{bmatrix},$$

implying that $(\mathbf{A}, \mathbf{B})$ is not controllable if $\mathbf{W}_1$ is not invertible. Therefore, $\mathbf{W}_1$ must be invertible. Similarly, the observability of $(\mathbf{A}, \mathbf{C})$ implies that $\mathbf{W}_2$ is invertible. Define $\mathbf{Q} = \mathbf{W}_2\mathbf{W}_1^{-1}$, which is then invertible and also symmetric because of the relation $\mathbf{W}_1^T\mathbf{W}_2 = \mathbf{W}_2^T\mathbf{W}_1$. From (22), we then obtain

$$\mathbf{H}\begin{bmatrix} \mathbf{I} \\ \mathbf{Q} \end{bmatrix} = \begin{bmatrix} \mathbf{I} \\ \mathbf{Q} \end{bmatrix}(\mathbf{W}_1\mathbf{R}\mathbf{W}_1^{-1}),$$

which implies that $\mathbf{Q}$ solves (20). The positive definiteness of $\mathbf{Q}$ follows from the fact that $\mathbf{Q}$ is invertible, $\mathbf{A}$ is asymptotically stable, and $\mathbf{Q}\mathbf{B}\mathbf{S}^{-1}\mathbf{B}^T\mathbf{Q} + \mathbf{C}^T\mathbf{S}^{-1}\mathbf{C} \geq 0$. $\square$

Remark 2 The previous lemmas still hold if $\mathbf{A}$ is replaced by $\mathbf{A} - \mathbf{B}\mathbf{S}^{-1}\mathbf{C}$, provided that this matrix is also asymptotically stable. Note that $(\mathbf{A}, \mathbf{B})$ is controllable if and only if $(\mathbf{A} - \mathbf{B}\mathbf{S}^{-1}\mathbf{C}, \mathbf{B})$ is controllable and $(\mathbf{A}, \mathbf{C})$ is observable if and only if $(\mathbf{A} - \mathbf{B}\mathbf{S}^{-1}\mathbf{C}, \mathbf{C})$ is observable.

Lemma 6 shows that the Riccati equation (20) has a solution $\mathbf{Q} > 0$ whenever the Hamiltonian matrix $\mathbf{H}$ has a Lagrangian invariant subspace. The existence of such an invariant subspace depends only on the purely imaginary eigenvalues of $\mathbf{H}$, e.g., [14]. When such an invariant subspace exists, then there are many such invariant subspaces. Note that the eigenvalues of $\mathbf{H}$ are symmetric with respect to the imaginary axis in the complex plane. If (22) holds, then the union of the eigenvalues of $\mathbf{E}$ and $-\mathbf{E}^T$ form the spectrum of $\mathbf{H}$. One particular choice is that the spectrum of $\mathbf{E}$ is in the closed left half complex plane, another choice is that it is in the closed right half complex plane. The two corresponding solutions of the Riccati equation (20) are the *minimal* solution $\mathbf{Q}_-$ and the *maximal* solution $\mathbf{Q}_+$ and all other solution of the Riccati equation lie (in the Loewner ordering of symmetric matrices) between these extremal solutions.

Example 1 Consider the example $\mathbf{B} = \mathbf{S} = 1$, $\mathbf{C} = -1$, $\mathbf{A} = -1 - \alpha$, where $\alpha > 0$. So $\mathbf{A}$ and $\mathbf{A} - \mathbf{B}\mathbf{S}^{-1}\mathbf{C} = -\alpha$ are both asymptotically stable. The Riccati equation (20) is $q^2 - 2\alpha q + 1 = 0$, which does not have a real positive semidefinite solution when $\alpha \in (0, 1)$. If $\alpha = 1$, then it has a unique solution $q = 1$ associated with $\mathbf{E} = 0$. If $\alpha > 1$, then it has two solutions $\alpha \pm \sqrt{\alpha^2 - 1} > 0$ with $\mathbf{E} = \pm\sqrt{\alpha^2 - 1}$.

Suppose that $\mathbf{A}_1, \mathbf{B}_1, \mathbf{C}_1$ from (17) have the condensed form (18) with an orthogonal matrix $\mathbf{U}$. Partition

$$\mathbf{U}^T\mathbf{Q}_1\mathbf{U} = \begin{bmatrix} \tilde{\mathbf{Q}}_{11} & \tilde{\mathbf{Q}}_{12} \\ \tilde{\mathbf{Q}}_{12}^T & \tilde{\mathbf{Q}}_{22} \end{bmatrix}.$$

The reduced Riccati inequality (17) then is equivalent to

$$\mathbf{U}^T\Psi(\mathbf{Q}_1)\mathbf{U} = \begin{bmatrix} \tilde{\Psi}_{11}(\mathbf{Q}_1) & \tilde{\Psi}_{12}(\mathbf{Q}_1) \\ \tilde{\Psi}_{12}(\mathbf{Q}_1)^T & \tilde{\Psi}_{22}(\mathbf{Q}_1) \end{bmatrix} \leq 0, \tag{23}$$

where (leaving off arguments)

$$\begin{array}{rcl}\tilde{\Psi}_{11} & = & (\tilde{\mathbf{A}}_{11}-\tilde{\mathbf{B}}_1\mathbf{S}^{-1}\tilde{\mathbf{C}}_1)^T\tilde{\mathbf{Q}}_{11}+\tilde{\mathbf{Q}}_{11}(\tilde{\mathbf{A}}_{11}-\tilde{\mathbf{B}}_1\mathbf{S}^{-1}\tilde{\mathbf{C}}_1)+\tilde{\mathbf{Q}}_{11}\tilde{\mathbf{B}}_1\mathbf{S}^{-1}\tilde{\mathbf{B}}_1^T\tilde{\mathbf{Q}}_{11}+\tilde{\mathbf{C}}_1^T\mathbf{S}^{-1}\tilde{\mathbf{C}}_1,\\ \tilde{\Psi}_{12} & = & (\tilde{\mathbf{A}}_{11}-\tilde{\mathbf{B}}_1\mathbf{S}^{-1}(\tilde{\mathbf{C}}_1-\tilde{\mathbf{B}}_1^T\tilde{\mathbf{Q}}_{11}))^T\tilde{\mathbf{Q}}_{12}+\tilde{\mathbf{Q}}_{12}\tilde{\mathbf{A}}_{22}+\tilde{\mathbf{Q}}_{11}(\tilde{\mathbf{A}}_{12}-\tilde{\mathbf{B}}_1\mathbf{S}^{-1}\tilde{\mathbf{C}}_2)+\tilde{\mathbf{C}}_1^T\mathbf{S}^{-1}\tilde{\mathbf{C}}_2\\ \tilde{\Psi}_{22} & = & \tilde{\mathbf{A}}_{22}^T\tilde{\mathbf{Q}}_{22}+\tilde{\mathbf{Q}}_{22}\tilde{\mathbf{A}}_{22}+(\tilde{\mathbf{A}}_{12}-\tilde{\mathbf{B}}_1\mathbf{S}^{-1}\tilde{\mathbf{C}}_2)^T\tilde{\mathbf{Q}}_{12}+\tilde{\mathbf{Q}}_{12}^T(\tilde{\mathbf{A}}_{12}-\tilde{\mathbf{B}}_1\mathbf{S}^{-1}\tilde{\mathbf{C}}_2),\\ & & \quad +\tilde{\mathbf{Q}}_{12}^T\tilde{\mathbf{B}}_1\mathbf{S}^{-1}\tilde{\mathbf{B}}_1^T\tilde{\mathbf{Q}}_{12}+\tilde{\mathbf{C}}_2^T\mathbf{S}^{-1}\tilde{\mathbf{C}}_2.\end{array}$$

For (17) to have a solution $\mathbf{Q}_1 > 0$, it is necessary that

$$\tilde{\Psi}_{11}(\mathbf{Q}_1) = \tilde{\Psi}_{11}(\tilde{\mathbf{Q}}_{11}) \leq 0$$

has a positive definite solution $\tilde{\mathbf{Q}}_{11}$ or equivalently, the dual Riccati inequality

$$(\tilde{\mathbf{A}}_{11}-\tilde{\mathbf{B}}_1\mathbf{S}^{-1}\tilde{\mathbf{C}}_1)\tilde{\mathbf{Y}}+\tilde{\mathbf{Y}}(\tilde{\mathbf{A}}_{11}-\tilde{\mathbf{B}}_1\mathbf{S}^{-1}\tilde{\mathbf{C}}_1)^T+\tilde{\mathbf{Y}}\tilde{\mathbf{C}}_1^T\mathbf{S}^{-1}\tilde{\mathbf{C}}_1\tilde{\mathbf{Y}}+\tilde{\mathbf{B}}_1\mathbf{S}^{-1}\tilde{\mathbf{B}}_1^T \leq 0$$

has a solution $\tilde{\mathbf{Y}} = \tilde{\mathbf{Q}}_{11}^{-1} > 0$. Using the partitioning in (18) for $\tilde{\mathbf{Y}} = \begin{bmatrix} \mathbf{Y}_{11} & \mathbf{Y}_{12} \\ \mathbf{Y}_{12}^T & \mathbf{Y}_{22} \end{bmatrix}$, then one can write this as

$$\begin{bmatrix} \Phi_{11}(\tilde{\mathbf{Y}}) & \Phi_{12}(\tilde{\mathbf{Y}}) \\ \Phi_{12}(\tilde{\mathbf{Y}})^T & \Phi_{22}(\tilde{\mathbf{Y}}) \end{bmatrix} \leq 0,$$

where

$$\Phi_{11}(\tilde{\mathbf{Y}}) = (\mathbf{A}_{11}-\mathbf{B}_1\mathbf{S}^{-1}\mathbf{C}_1)\mathbf{Y}_{11}+\mathbf{Y}_{11}(\mathbf{A}_{11}-\mathbf{B}_1\mathbf{S}^{-1}\mathbf{C}_1)^T+\mathbf{Y}_{11}\mathbf{C}_1^T\mathbf{S}^{-1}\mathbf{C}_1\mathbf{Y}_{11}+\mathbf{B}_1\mathbf{S}^{-1}\mathbf{B}_1^T.$$

Again, it is necessary that $\Phi_{11}(\tilde{\mathbf{Y}}) = \Phi_{11}(\mathbf{Y}_{11}) \leq 0$ has a solution $\mathbf{Y}_{11} > 0$, or equivalently that the dual inequality

$$(\mathbf{A}_{11}-\mathbf{B}_1\mathbf{S}^{-1}\mathbf{C}_1)^T\mathbf{Q}_{11}+\mathbf{Q}_{11}(\mathbf{A}_{11}-\mathbf{B}_1\mathbf{S}^{-1}\mathbf{C}_1)+\mathbf{Q}_{11}\mathbf{B}_1\mathbf{S}^{-1}\mathbf{B}_1^T\mathbf{Q}_{11}+\mathbf{C}_1^T\mathbf{S}^{-1}\mathbf{C}_1 \leq 0 \quad (24)$$

has a positive definite solution. Since $(\mathbf{A}_{11}, \mathbf{B}_1)$ is controllable and $(\mathbf{A}_{11}, \mathbf{C}_1)$ is observable, the inequality (24) is solvable if and only if the corresponding Riccati equation is solvable, see [4, 25]. From Lemma 6 we have as necessary condition for the solvability of (13) or (14) that $\mathbf{A}_{11} - \mathbf{B}_1\mathbf{S}^{-1}\mathbf{C}_1$ is asymptotically stable and that the Hamiltonian matrix

$$\mathbf{H} = \begin{bmatrix} \mathbf{A}_{11}-\mathbf{B}_1\mathbf{S}^{-1}\mathbf{C}_1 & \mathbf{B}_1\mathbf{S}^{-1}\mathbf{B}_1^T \\ -\mathbf{C}_1^T\mathbf{S}^{-1}\mathbf{C}_1 & -(\mathbf{A}_{11}-\mathbf{B}_1\mathbf{S}^{-1}\mathbf{C}_1)^T \end{bmatrix}. \quad (25)$$

has a Lagrangian invariant subspace.

We now show these two conditions as well as (15) are also sufficient for the existence of a positive definite solution by explicit constructions. Clearly, we only need to consider (16) or (17).

We first show that $\tilde{\Psi}_{11} \leq 0$ has a solution $\tilde{\mathbf{Q}}_{11} > 0$, where $\tilde{\Psi}_{11}$ is defined in (23). Suppose the Hamiltonian matrix $\mathbf{H}$ has a Lagrangian invariant subspace. By Lemma 6, we can determine a matrix $\mathbf{Q}_{11}^0 > 0$ that solves the equation (24) with the eigenvalues of $\mathbf{A}_{11}-\mathbf{B}_1\mathbf{S}^{-1}(\mathbf{C}_1-\mathbf{B}_1^T\mathbf{Q}_{11}^0)$ in the closed left half complex plane. Then $\tilde{\mathbf{Q}}_{11}^0 =: \begin{bmatrix} \mathbf{Q}_{11}^0 & 0 \\ 0 & 0 \end{bmatrix}$ is a solution of $\tilde{\Psi}_{11} = 0$, but $\tilde{\mathbf{Q}}_{11}^0$ is only positive semidefinite if $\mathbf{A}_{22}$ is present. If $\mathbf{A}_{11}-\mathbf{B}_1\mathbf{S}^{-1}(\mathbf{C}_1-\mathbf{B}_1^T\mathbf{Q}_{11}^0)$

has no purely imaginary eigenvalues, or equivalently $\mathbf{H}$ does not have purely imaginary eigenvalues, then the Hamiltonian matrix

$$\tilde{\mathbf{H}} = \begin{bmatrix} \tilde{\mathbf{A}}_{11} - \tilde{\mathbf{B}}_1\mathbf{S}^{-1}\tilde{\mathbf{C}}_1 & \tilde{\mathbf{B}}_1\mathbf{S}^{-1}\tilde{\mathbf{B}}_1^T \\ -\tilde{\mathbf{C}}_1^T\mathbf{S}^{-1}\tilde{\mathbf{C}}_1 & -(\tilde{\mathbf{A}}_{11} - \tilde{\mathbf{B}}_1\mathbf{S}^{-1}\tilde{\mathbf{C}}_1)^T \end{bmatrix}$$

does not have purely imaginary eigenvalues either, since the spectrum of $\tilde{\mathbf{H}}$ is the union of the spectra of $\mathbf{H}$, $\mathbf{A}_{22}$ and $-\mathbf{A}_{22}$, and $\mathbf{A}_{22}$ is asymptotically stable. Consider the Riccati equation

$$\tilde{\Psi}_{11} + \Xi = 0, \tag{26}$$

with $\Xi \geq 0$ being chosen such that $(\tilde{\mathbf{A}}_{11} - \tilde{\mathbf{B}}_1\mathbf{S}^{-1}\tilde{\mathbf{C}}_1, \tilde{\mathbf{C}}_1^T\mathbf{S}^{-1}\tilde{\mathbf{C}}_1 + \Xi)$ is observable; such Ξ always exists. Recall that $(\tilde{\mathbf{A}}_{11} - \tilde{\mathbf{B}}_1\mathbf{S}^{-1}\tilde{\mathbf{C}}_1, \tilde{\mathbf{B}}_1)$ is controllable. The Riccati equation (26) corresponds to the Hamiltonian matrix

$$\tilde{\mathbf{H}} - \begin{bmatrix} 0 & 0 \\ \Xi & 0 \end{bmatrix}.$$

For a sufficiently small (in norm) Ξ, by continuity, this new Hamiltonian matrix has a Lagrangian invariant subspace (when its eigenvalues are away from the imaginary axis), and by Lemma 6, $\tilde{\Psi} = -\Xi \leq 0$ has a positive definite solution $\tilde{\mathbf{Q}}_{11}$ with all the eigenvalues of $\tilde{\mathbf{A}}_{11} - \tilde{\mathbf{B}}_1\mathbf{S}^{-1}(\tilde{\mathbf{C}}_1 - \tilde{\mathbf{B}}_1^T\tilde{\mathbf{Q}}_{11})$ in the closed left half complex plane.

If $\mathbf{A}_{11} - \mathbf{B}_1\mathbf{S}^{-1}(\mathbf{C}_1 - \mathbf{B}_1^T\mathbf{Q}_{11}^0)$ has purely imaginary eigenvalues (including 0), then there exists an invertible matrix $\mathbf{L}$ such that

$$\mathbf{L}(\mathbf{A}_{11} - \mathbf{B}_1\mathbf{S}^{-1}(\mathbf{C}_1 - \mathbf{B}_1^T\mathbf{Q}_{11}^0)\mathbf{L}^{-1} = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix},$$

where Σ_1 has only purely imaginary eigenvalues and Σ_2 is asymptotically stable. So there exists an invertible matrix $\tilde{\mathbf{L}}$ such that

$$\tilde{\mathbf{L}}\tilde{\mathbf{A}}_{11}^0\tilde{\mathbf{L}}^{-1} = \tilde{\mathbf{L}} \begin{bmatrix} \mathbf{A}_{11} - \mathbf{B}_1\mathbf{S}^{-1}(\mathbf{C}_1 - \mathbf{B}_1^T\mathbf{Q}_{11}^0) & 0 \\ \mathbf{A}_{21} - \mathbf{B}_2\mathbf{S}^{-1}(\mathbf{C}_1 - \mathbf{B}_1^T\mathbf{Q}_{11}^0) & \mathbf{A}_{22} \end{bmatrix} \tilde{\mathbf{L}}^{-1} = \left[\begin{array}{c|cc} \Sigma_1 & 0 & 0 \\ \hline 0 & \Sigma_2 & 0 \\ 0 & \Sigma_{32} & \mathbf{A}_{22} \end{array}\right] =: \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2^0 \end{bmatrix},$$

where

$$\tilde{\mathbf{A}}_{11}^0 = \tilde{\mathbf{A}}_{11} - \tilde{\mathbf{B}}_1\mathbf{S}^{-1}(\tilde{\mathbf{C}}_1 - \tilde{\mathbf{B}}_1^T\tilde{\mathbf{Q}}_{11}^0)$$

and Σ_2^0 is asymptotically stable. Since similarity transformations will not affect our analysis, without loss of generality we may assume that both $\mathbf{L}$ and $\tilde{\mathbf{L}}$ are identity matrices.

Subtracting

$$(\tilde{\mathbf{A}}_{11} - \tilde{\mathbf{B}}_1\mathbf{S}^{-1}\tilde{\mathbf{C}}_1)^T\tilde{\mathbf{Q}}_{11}^0 + \tilde{\mathbf{Q}}_{11}^0(\tilde{\mathbf{A}}_{11} - \tilde{\mathbf{B}}_1\mathbf{S}^{-1}\tilde{\mathbf{C}}_1) + \tilde{\mathbf{Q}}_{11}^0\tilde{\mathbf{B}}_1\mathbf{S}^{-1}\tilde{\mathbf{B}}_1^T\tilde{\mathbf{Q}}_{11}^0 + \tilde{\mathbf{C}}_1^T\mathbf{S}^{-1}\tilde{\mathbf{C}}_1 = 0$$

from (26) yields the Riccati equation

$$(\tilde{\mathbf{A}}_{11}^0)^T\tilde{\mathbf{Y}} + \tilde{\mathbf{Y}}\tilde{\mathbf{A}}_{11}^0 + \tilde{\mathbf{Y}}\tilde{\mathbf{B}}_1\mathbf{S}^{-1}\tilde{\mathbf{B}}_1^T\tilde{\mathbf{Y}} + \Xi = 0, \tag{27}$$

where $\tilde{\mathbf{Y}} = \tilde{\mathbf{Q}}_{11} - \tilde{\mathbf{Q}}_{11}^0$. Repartition

$$\tilde{\mathbf{B}}_1 = \begin{bmatrix} \mathbf{B}_1^0 \\ \mathbf{B}_2^0 \end{bmatrix}$$

according to $\tilde{\mathbf{A}}_{11}^0$, and set

$$\Xi = \begin{bmatrix} 0 & 0 \\ 0 & \Xi_{22} \end{bmatrix}, \quad \tilde{\mathbf{Y}} = \begin{bmatrix} 0 & 0 \\ 0 & \mathbf{Y}_2 \end{bmatrix}.$$

Then the Riccati equation (27) is reduced to

$$(\Sigma_2^0)^T \mathbf{Y}_2 + \mathbf{Y}_2 \Sigma_2^0 + \mathbf{Y}_2 \mathbf{B}_2^0 \mathbf{S}^{-1} (\mathbf{B}_2^0)^T \mathbf{Y}_2 + \Xi_{22} = 0.$$

Since $(\tilde{\mathbf{A}}_{11}^0, \tilde{\mathbf{B}}_1)$ is controllable, so is $(\Sigma_2^0, \mathbf{B}_2^0)$. Recall also that Σ_2^0 is asymptotically stable. Analogous to the previous case one can choose (sufficiently small) $\Xi_{22} \geq 0$ with (Σ_2^0, Ξ_{22}) observable and then the reduced Riccati equation has a positive definite solution $\mathbf{Y}_2$ with the eigenvalues of $\Sigma_2^0 + \mathbf{B}_2^0 \mathbf{S}^{-1} (\mathbf{B}_2^0)^T \mathbf{Y}_2$ in the closed left half complex plane. Then

$$\tilde{\mathbf{Q}}_{11} = \tilde{\mathbf{Q}}_{11}^0 + \tilde{\mathbf{Y}} = \begin{bmatrix} \mathbf{Q}_{11}^0 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \mathbf{Y}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{Q}_{11}^0 & \mathbf{Q}_{12}^0 & 0 \\ (\mathbf{Q}_{12}^0)^T & \mathbf{Q}_{22}^0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & \mathbf{Y}_{11} & \mathbf{Y}_{12} \\ 0 & \mathbf{Y}_{12}^T & \mathbf{Y}_{22} \end{bmatrix} > 0,$$

where $\mathbf{Q}_{11}^0$ has the same size of Σ_1 and $\mathbf{Y}_{22}$ has the same size of $\mathbf{A}_{22}$, and is a solution to

$$\tilde{\Psi}_{11} = -\begin{bmatrix} 0 & 0 \\ 0 & \Xi_{22} \end{bmatrix} \leq 0,$$

since

$$\tilde{\mathbf{A}}_{11} - \tilde{\mathbf{B}}_1 \mathbf{S}^{-1} (\tilde{\mathbf{C}}_1 - \tilde{\mathbf{B}}_1^T \tilde{\mathbf{Q}}_{11}) = \tilde{\mathbf{A}}_{11}^0 + \tilde{\mathbf{B}}_1 \mathbf{S}^{-1} \tilde{\mathbf{B}}_1^T \tilde{\mathbf{Y}} = \begin{bmatrix} \Sigma_1 & \mathbf{B}_1^0 \mathbf{S}^{-1} (\mathbf{B}_2^0)^T \mathbf{Y}_2 \\ 0 & \Sigma_2^0 + \mathbf{B}_2^0 \mathbf{S}^{-1} (\mathbf{B}_2^0)^T \mathbf{Y}_2 \end{bmatrix},$$

has its eigenvalues is in the closed left half complex plane.

Once we have such a solution $\tilde{\mathbf{Q}}_{11}$, we can solve $\tilde{\Psi}_{12} = 0$ for $\tilde{\mathbf{Q}}_{12}$. This Sylvester equation has a unique solution because $\tilde{\mathbf{A}}_{22} = \mathbf{A}_{33}$ is asymptotically stable and the eigenvalues of $\tilde{\mathbf{A}}_{11} - \tilde{\mathbf{B}}_1 \mathbf{S}^{-1} (\tilde{\mathbf{C}}_1 - \tilde{\mathbf{B}}_1^T \tilde{\mathbf{Q}}_{11})$ are in the closed left half complex plane.

Having (leaving off arguments) solved the equality $\tilde{\Psi}_{12} = 0$, we finally need to solve the inequality $\tilde{\Psi}_{22} \leq 0$. Let

$$\Theta = \begin{bmatrix} \mathbf{I} & \tilde{\mathbf{Q}}_{11}^{-1} \tilde{\mathbf{Q}}_{12} \\ 0 & \mathbf{I} \end{bmatrix}.$$

From the equations $\tilde{\Psi}_{11} = -\Xi$ and $\tilde{\Psi}_{12} = 0$, we obtain

$$\Theta^{-T} (\mathbf{U}^T \Psi(\mathbf{Q}_1) \mathbf{U}) \Theta^{-1} = \Theta^{-T} \begin{bmatrix} -\Xi & 0 \\ 0 & \tilde{\Psi}_{22} \end{bmatrix} \Theta^{-1} = \begin{bmatrix} -\Xi & \Xi \tilde{\mathbf{Q}}_{11}^{-1} \tilde{\mathbf{Q}}_{12} \\ \tilde{\mathbf{Q}}_{12}^T \tilde{\mathbf{Q}}_{11}^{-1} \Xi & \tilde{\Psi}_{22} - \tilde{\mathbf{Q}}_{12}^T \tilde{\mathbf{Q}}_{11}^{-1} \Xi \tilde{\mathbf{Q}}_{11}^{-1} \tilde{\mathbf{Q}}_{12} \end{bmatrix}.$$

On the other hand

$$\Theta^{-T} (\mathbf{U}^T \mathbf{Q}_1 \mathbf{U}) \Theta^{-1} = \begin{bmatrix} \tilde{\mathbf{Q}}_{11} & 0 \\ 0 & \Pi \end{bmatrix}, \quad \Pi = \tilde{\mathbf{Q}}_{22} - \tilde{\mathbf{Q}}_{12}^T \tilde{\mathbf{Q}}_{11}^{-1} \tilde{\mathbf{Q}}_{12},$$

and

$$\begin{aligned} \Theta (\mathbf{U}^T \mathbf{A}_1 \mathbf{U}) \Theta^{-1} &= \begin{bmatrix} \tilde{\mathbf{A}}_{11} & \tilde{\mathbf{A}}_{12} - \tilde{\mathbf{A}}_{11} \tilde{\mathbf{Q}}_{11}^{-1} \tilde{\mathbf{Q}}_{12} + \tilde{\mathbf{Q}}_{11}^{-1} \tilde{\mathbf{Q}}_{12} \tilde{\mathbf{A}}_{22} \\ 0 & \tilde{\mathbf{A}}_{22} \end{bmatrix} \\ \Theta (\mathbf{U}^T \mathbf{B}_1) &= \mathbf{U}^T \mathbf{B}_1 = \begin{bmatrix} \tilde{\mathbf{B}}_1 \\ 0 \end{bmatrix}, \quad (\mathbf{C}_1 \mathbf{U}) \Theta^{-1} = \begin{bmatrix} \tilde{\mathbf{C}}_1 & \tilde{\mathbf{C}}_2 - \tilde{\mathbf{C}}_1 \tilde{\mathbf{Q}}_{11}^{-1} \tilde{\mathbf{Q}}_{12} \end{bmatrix}. \end{aligned}$$

The (2,2) block of $\Theta^{-T}(\mathbf{U}^T\Psi(\mathbf{Q}_1)\mathbf{U})\Theta^{-1}$ can be rewritten as

$$\tilde{\mathbf{A}}_{22}^T\Pi + \Pi\tilde{\mathbf{A}}_{22} + (\tilde{\mathbf{C}}_2 - \tilde{\mathbf{C}}_1\tilde{\mathbf{Q}}_{11}^{-1}\tilde{\mathbf{Q}}_{12})^T\mathbf{S}^{-1}(\tilde{\mathbf{C}}_2 - \tilde{\mathbf{C}}_1\tilde{\mathbf{Q}}_{11}^{-1}\tilde{\mathbf{Q}}_{12}),$$

so that

$$\tilde{\Psi}_{22} = \tilde{\mathbf{A}}_{22}^T\Pi + \Pi\tilde{\mathbf{A}}_{22} + (\tilde{\mathbf{C}}_2 - \tilde{\mathbf{C}}_1\tilde{\mathbf{Q}}_{11}^{-1}\tilde{\mathbf{Q}}_{12})^T\mathbf{S}^{-1}(\tilde{\mathbf{C}}_2 - \tilde{\mathbf{C}}_1\tilde{\mathbf{Q}}_{11}^{-1}\tilde{\mathbf{Q}}_{12}) + \tilde{\mathbf{Q}}_{12}^T\tilde{\mathbf{Q}}_{11}^{-1}\mathbf{F}\tilde{\mathbf{Q}}_{11}^{-1}\tilde{\mathbf{Q}}_{12}.$$

Since $\tilde{\mathbf{A}}_{22}$ is asymptotically stable and

$$(\tilde{\mathbf{C}}_2 - \tilde{\mathbf{C}}_1\tilde{\mathbf{Q}}_{11}^{-1}\tilde{\mathbf{Q}}_{12})^T\mathbf{S}^{-1}(\tilde{\mathbf{C}}_2 - \tilde{\mathbf{C}}_1\tilde{\mathbf{Q}}_{11}^{-1}\tilde{\mathbf{Q}}_{12}) + \tilde{\mathbf{Q}}_{12}^T\tilde{\mathbf{Q}}_{11}^{-1}\mathbf{F}\tilde{\mathbf{Q}}_{11}^{-1}\tilde{\mathbf{Q}}_{12} \geq 0,$$

one can always select $\tilde{\Xi} \geq 0$ such that the Lyapunov equation

$$\tilde{\Psi}_{22} = -\tilde{\Xi}$$

has a unique solution $\Pi > 0$. Clearly, for $\tilde{\mathbf{Q}}_{22} = \Pi + \tilde{\mathbf{Q}}_{12}^T\tilde{\mathbf{Q}}_{11}^{-1}\tilde{\mathbf{Q}}_{12}$, the matrix

$$\mathbf{Q}_1 = \mathbf{U}\begin{bmatrix} \tilde{\mathbf{Q}}_{11} & \tilde{\mathbf{Q}}_{12} \\ \tilde{\mathbf{Q}}_{12}^T & \tilde{\mathbf{Q}}_{22} \end{bmatrix}\mathbf{U}^T > 0$$

satisfies

$$\Psi(\mathbf{Q}_1) = -\mathbf{U}\begin{bmatrix} \xi & 0 \\ 0 & \tilde{\Xi} \end{bmatrix}\mathbf{U}^T \leq 0,$$

i.e., $\mathbf{Q}_1 > 0$ solves (16) and (17).

We summarize the conditions for the existence of a positive definite solution of the matrix inequality (13) in the following theorem.

Theorem 7 *Consider a general LTI system of the form (4) with $\mathbf{A}$ stable and $\mathbf{S} = \mathbf{D}+\mathbf{D}^T > 0$. Let $\mathbf{M}$ be invertible such that*

$$\mathbf{M}\mathbf{A}\mathbf{M}^{-1} = \begin{bmatrix} \mathbf{A}_1 & 0 \\ 0 & \mathbf{A}_2 \end{bmatrix}, \quad \mathbf{M}\mathbf{B} = \begin{bmatrix} \mathbf{B}_1 \\ \mathbf{B}_2 \end{bmatrix}, \quad \mathbf{C}\mathbf{M}^{-1} = \begin{bmatrix} \mathbf{C}_1 & \mathbf{C}_2 \end{bmatrix},$$

where $\mathbf{A}_2$ is diagonalizable and contains all the purely imaginary eigenvalues of $\mathbf{A}$. Let $\mathbf{A}_1, \mathbf{B}_1, \mathbf{C}_1$ have the condensed form (19) with an orthogonal matrix $\mathbf{U}$. Then the LMI (13) has a positive definite solution $\mathbf{Q}$ if and only if the following conditions hold.

(a) There exists a positive definite matrix $\mathbf{Q}_2$ satisfying

$$\mathbf{B}_2^T\mathbf{Q}_2 = \mathbf{C}_2, \quad \mathbf{A}_2\mathbf{Q}_2 = \mathbf{Q}_2\mathbf{A}_2.$$

(b) The block $\mathbf{A}_{11} - \mathbf{B}_1\mathbf{S}^{-1}\mathbf{C}_1$ is asymptotically stable.

(c) The Hamiltonian matrix $\mathbf{H}$ defined in (25) has a Lagrangian invariant subspace.

If these conditions hold, then the LMI (13) has a positive definite solution of the form $\mathbf{Q} = \mathbf{M}^T\begin{bmatrix} \mathbf{Q}_1 & 0 \\ 0 & \mathbf{Q}_2 \end{bmatrix}\mathbf{M} > 0$, where $\mathbf{Q}_1 > 0$ solves (17) and $\mathbf{Q}_2$ is determined from condition (a), and

$$\begin{aligned}&(\mathbf{A} - \mathbf{B}\mathbf{S}^{-1}\mathbf{C})^T\mathbf{Q} + \mathbf{Q}(\mathbf{A} - \mathbf{B}\mathbf{S}^{-1}\mathbf{C}) + \mathbf{Q}\mathbf{B}\mathbf{S}^{-1}\mathbf{B}^T\mathbf{Q} + \mathbf{C}^T\mathbf{S}^{-1}\mathbf{C} \\ &= -\mathbf{M}^T\begin{bmatrix} \mathbf{U}\begin{bmatrix} \Xi & 0 \\ 0 & \tilde{\Xi} \end{bmatrix}\mathbf{U}^T & 0 \\ 0 & 0 \end{bmatrix}\mathbf{M} \leq 0,\end{aligned}$$

where $\Xi, \tilde{\Xi}$ as in above construction.

Proof. The proof follows from the given construction. $\square$

Remark 3 Relation (22) is equivalent to the property that the *even matrix pencil*

$$\lambda \begin{bmatrix} 0 & \mathbf{I} & 0 \\ -\mathbf{I} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & \mathbf{A} & \mathbf{B} \\ \mathbf{A}^T & 0 & \mathbf{C}^T \\ \mathbf{B}^T & \mathbf{C} & \mathbf{S} \end{bmatrix} \tag{28}$$

is regular and has *index at most one*, i.e., the eigenvalues at ∞ are all semi-simple, (this follows because $\mathbf{S} > 0$) and that it has a deflating subspace spanned by the columns of $\begin{bmatrix} \mathbf{Q} & -\mathbf{I} & \mathbf{Y}^T \end{bmatrix}^T$ with $\mathbf{Y} = \mathbf{S}^{-1}(\mathbf{C} - \mathbf{B}^T\mathbf{Q})$, i.e.,

$$\begin{bmatrix} 0 & \mathbf{I} & 0 \\ -\mathbf{I} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{Q} \\ -\mathbf{I} \\ \mathbf{Y} \end{bmatrix} (\mathbf{A} - \mathbf{B}\mathbf{Y}) = \begin{bmatrix} 0 & \mathbf{A} & \mathbf{B} \\ \mathbf{A}^T & 0 & \mathbf{C}^T \\ \mathbf{B}^T & \mathbf{C} & \mathbf{S} \end{bmatrix} \begin{bmatrix} \mathbf{Q} \\ -\mathbf{I} \\ \mathbf{Y} \end{bmatrix}. \tag{29}$$

A pencil $\lambda\mathcal{N} - \mathcal{M}$ is called *even* if $\mathcal{N} = -\mathcal{N}^T$ and $\mathcal{M} = \mathcal{M}^T$. Since even pencils have the Hamiltonian spectral symmetry in the finite eigenvalues, this means that there are equally many eigenvalues in the open left and in the open right half plane.

Numerically, to compute $\mathbf{Q}$ it is preferable to work with the even pencil (29) over the Hamiltonian matrix (21), since explicit inversion of $\mathbf{S}$ can be avoided. Numerically stable structure preserving methods for this task are available, see [2, 3].

Note that this procedure can be directly applied to the original data in (4) if $\mathbf{S} = \mathbf{D} + \mathbf{D}^T$ is positive definite and we attempt to solve the Riccati equation of (14). The situation is more complicated if $\mathbf{S}$ is singular, because then it is well-known that the pencil (28) has Jordan blocks of size larger than 1 (the index of (28) is larger than 1) associated with the eigenvalue ∞, which means that the number of finite eigenvalues is less than $2n$. In this case one has to first deflate the high index and possibly singular part of the system. This can be done with the help of the staircase form [8] which has been implemented in a production code [5]. The situation is also more difficult if $\mathbf{S}$ is indefinite, since in this case both $\mathbf{B}\mathbf{S}^{-1}\mathbf{B}^T$ and $\mathbf{C}^T\mathbf{S}^{-1}\mathbf{C}$ cannot be guaranteed to be positive semidefinite and Lemma 6 may no longer be valid.

Remark 4 To show that the system (4) is passive it is sufficient that the LMI (13) has a positive semidefinite solution $\mathbf{Q}$. In this case the conditions can be relaxed. First of all, the condition $\mathbf{B}_2^T\mathbf{Q}_2 = \mathbf{C}_2$ can be relaxed to $\mathsf{Ker}\,\mathbf{B}_2 \subseteq \mathsf{Ker}\,\mathbf{C}_2^T$, $\operatorname{rank}\mathbf{C}_2\mathbf{B}_2 = \operatorname{rank}\mathbf{C}_2$, and $\mathbf{C}_2\mathbf{B}_2 \geq 0$. Also, $\mathbf{A}_2$ may have purely imaginary eigenvalues with Jordan blocks. For example in the extreme case when $\mathbf{C}_2 = 0$, $\mathbf{Q}_2 = 0$ always satisfies the conditions for any $\mathbf{A}_2$.

Secondly, for (16) or (17), we still require that $\mathbf{A}_{11} - \mathbf{B}_1\hat{\mathbf{S}}^{-1}\mathbf{C}_1$ is asymptotically stable and that the Hamiltonian matrix $\hat{\mathbf{H}}$ has a Lagrangian invariant subspace. Since this only requires that $\mathbf{Q}_1 \geq 0$, a solution can be determined in a simpler way. We may simply set $\tilde{\mathbf{Q}}_{11} = \tilde{\mathbf{Q}}_{11}^0$ and $\tilde{\Psi}_{11} = 0$. Then with the block structure the solution of $\tilde{\Psi}_{12} = 0$ has a form $\tilde{\mathbf{Q}}_{12} = \begin{bmatrix} \mathbf{Q}_{12} \\ 0 \end{bmatrix}$. To solve $\tilde{\Psi}_{22} = 0$ for $\tilde{\mathbf{Q}}_{22}$, one can show $\mathbf{Q}_1 = \mathbf{U} \begin{bmatrix} \tilde{\mathbf{Q}}_{11} & \tilde{\mathbf{Q}}_{12} \\ \tilde{\mathbf{Q}}_{12}^T & \tilde{\mathbf{Q}}_{22} \end{bmatrix} \mathbf{U}^T \geq 0$ and solves the Riccati equation $\Psi(\mathbf{Q}_1) = 0$. Also, under certain conditions $\tilde{\mathbf{A}}_{22} = \mathbf{A}_{33}$ does not have to be asymptotically stable.

This shows that for non-minimal systems passivity is more general than the PH property.

3 Construction of port-Hamiltonian realizations

In the last section we have seen that the existence of a port-Hamiltonian realization for (4) reduces to the existence of a nonsingular matrix $\mathbf{T} \in \mathbb{R}^{n\times n}$ or a positive definite matrix $\mathbf{Q} = \mathbf{T}^T\mathbf{T}$ such that the matrix inequalities (9) or (7) hold. In this section we develop a constructive procedure to check these conditions. In our previous considerations the matrix $\mathbf{V}$ that is used for basis change in the input space suffices to be invertible, but to implement the transformations in a numerical stable algorithms, we require $\mathbf{V}$ to be real orthogonal.

Consider $\mathbf{V}_0 = [\mathbf{V}_{0,1}, \mathbf{V}_{0,2}]$, where $\mathbf{V}_{0,1}$ is chosen so that the columns of $\mathbf{V}_{0,1}$ form an orthonormal basis of the kernel of $\mathbf{D}+\mathbf{D}^T$. To construct such a $\mathbf{V}_0$ we can use a singular value or rank-revealing QR decomposition, [16]. Then for the symmetric part $\mathbf{S}$ of the feedthrough matrix we have

$$\mathbf{S} = \frac{1}{2}(\mathbf{V}_0^T\mathbf{D}\mathbf{V}_0 + \mathbf{V}_0^T\mathbf{D}\mathbf{V}_0) = \frac{1}{2}\mathbf{V}_0^T\left(\mathbf{D}+\mathbf{D}^T\right)\mathbf{V}_0 = \begin{bmatrix} 0 & 0 \\ 0 & \mathbf{S}_2 \end{bmatrix}, \tag{30}$$

where $\mathbf{S}_2$ is symmetric and invertible. Set

$$\begin{bmatrix} \mathbf{B}_1 & \mathbf{B}_2 \end{bmatrix} = \mathbf{B}\mathbf{V}_0, \ \begin{bmatrix} \mathbf{C}_1^T & \mathbf{C}_2^T \end{bmatrix} := \mathbf{C}^T\mathbf{V}_0, \tag{31}$$

each partitioned compatibly with $\mathbf{V}_0$ as in (30).

Scaling the second block row and column of the matrix inequality (7) with $\mathbf{V}_0^T$ and $\mathbf{V}_0$ respectively, the matrix inequality

$$\begin{bmatrix} \mathbf{A}^T\mathbf{Q}+\mathbf{Q}\mathbf{A} & \mathbf{Q}\mathbf{B}_1 - \mathbf{C}_1^T & \mathbf{Q}\mathbf{B}_2 - \mathbf{C}_2^T \\ \mathbf{B}_1^T\mathbf{Q} - \mathbf{C}_1 & 0 & 0 \\ \mathbf{B}_2^T\mathbf{Q} - \mathbf{C}_2 & 0 & -\mathbf{S}_2 \end{bmatrix} \leq 0 \tag{32}$$

has a solution $\mathbf{Q} = \mathbf{Q}^T > 0$ if and only if the matrix inequality

$$\begin{bmatrix} \mathbf{A}^T\mathbf{Q}+\mathbf{Q}\mathbf{A} & \mathbf{Q}\mathbf{B}_2 - \mathbf{C}_2^T \\ \mathbf{B}_2^T\mathbf{Q} - \mathbf{C}_2 & -\mathbf{S}_2 \end{bmatrix} \leq 0 \tag{33}$$

has a positive definite solution Q satisfying the constraint $\mathbf{B}_1^T\mathbf{Q} - \mathbf{C}_1$. We will characterize conditions when this constraint is satisfied in the following subsections. For the solution of the matrix inequality, then we can restrict the input and output space to the invertible part of $\mathbf{D}+\mathbf{D}^T$, and once the transformation matrices have been constructed, this transformation can be extended to the full system, making sure that the constraint is satisfied.

3.1 Construction of the transformation in the case $\mathbf{D} = -\mathbf{D}^T$

To explicitly construct the transformation to port Hamiltonian form let us first discuss the special case that $\mathbf{D} = -\mathbf{D}^T$, i.e., $\mathbf{S} = 0$. Considering the matrix $\mathbf{K}$ in (2), to have $\mathbf{K} \geq 0$, we must have $\mathbf{P} = 0$, and the block $\mathbf{V}_{0,2}$ in the transformation of the feedthrough term is void, while $\mathbf{V}_0 = \mathbf{V}_{0,1}$ is any orthogonal matrix. Then we can express the conditions (32) and (33) in terms of the factor T in a factorization $\mathbf{Q} = \mathbf{T}\mathbf{T}^T$.

Corollary 8 *For a state-space system of the form* (4) *with* $\mathbf{D}+\mathbf{D}^T = 0$ *the following two statements are equivalent:*

1. There exists a change of basis $\mathbf{x} = \mathbf{T}\mathbf{z}$ *with an invertible matrix* $\mathbf{T} \in \mathbb{R}^{n\times n}$ *such that the resulting realization in the new basis has PH structure as in (5).*

2. There exists an invertible matrix $\mathbf{T}$ *such that*

$$a)\ (\mathbf{T}\mathbf{B})^T = \mathbf{C}\mathbf{T}^{-1} \qquad \text{and} \qquad b)\ (\mathbf{T}\mathbf{A}\mathbf{T}^{-1})^T + \mathbf{T}\mathbf{A}\mathbf{T}^{-1} \le 0. \tag{34}$$

Note that without the constraint (34a), if $\mathbf{A}$ is stable, then by Lemma 3 the second condition (34b) can always be satisfied. Adding the constraint (34a), however, makes the question nontrivial.

We have the following characterization of the transformation matrices $\mathbf{T}$ that satisfy (34a).

Lemma 9 *Consider* $\mathbf{B}, \mathbf{C}^T \in \mathbb{R}^{n\times m}$*, and assume that* $\operatorname{rank}\mathbf{B} = r$*.*

a) There exists an invertible transformation $\mathbf{T}$ *satisfying condition (34a) if and only if* $\mathsf{Ker}\,\mathbf{C}^T = \mathsf{Ker}\,\mathbf{B}$*,* $\operatorname{rank}\mathbf{C}\mathbf{B} = r$ *and* $\mathbf{C}\mathbf{B} \ge 0$*, or equivalently, there exists an invertible (orthogonal) matrix* $\mathbf{V}$ *such that*

$$\mathbf{B}\mathbf{V} = \begin{bmatrix} \mathbf{B}_1 & 0 \end{bmatrix}, \quad \mathbf{C}^T\mathbf{V} = \begin{bmatrix} \mathbf{C}_1^T & 0 \end{bmatrix}, \quad \mathbf{C}_1\mathbf{B}_1 = \mathbf{Y}\mathbf{Y}^T > 0,$$

where $\mathbf{B}_1, \mathbf{C}_1^T \in \mathbb{R}^{n\times r}$ *have full column rank and* $\mathbf{Y} \in \mathbb{R}^{r\times r}$ *is invertible.*

b) Let $\mathbf{N}_\mathbf{B} \in \mathbb{R}^{n\times(n-r)}$ *have columns that form a basis of* $\mathsf{Ker}\,\mathbf{B}^T$*. If Condition a) is satisfied, then any* $\mathbf{T}$ *satisfying condition (34a) has the form* $\mathbf{T} = \mathbf{U}\mathbf{T}_\mathbf{Z}\mathbf{T}_0$ *with*

$$\mathbf{T}_0 = \begin{bmatrix} \mathbf{N}_\mathbf{B}^T \\ \mathbf{Y}^{-1}\mathbf{C}_1 \end{bmatrix}, \ \mathbf{T}_\mathbf{Z} = \begin{bmatrix} \mathbf{Z} & 0 \\ 0 & \mathbf{I} \end{bmatrix}, \tag{35}$$

where $\mathbf{U} \in \mathbb{R}^{n\times n}$ *is an arbitrary orthogonal matrix and* $\mathbf{Z} \in \mathbb{R}^{(n-r)\times(n-r)}$ *is arbitrary nonsingular.*

Proof. Condition (34a) is equivalent to $\mathbf{C} = \mathbf{B}^T\mathbf{T}^T\mathbf{T}$. Necessary for the existence of an invertible $\mathbf{T}$ with this property are the conditions

$$\begin{aligned} \mathsf{Ker}\,\mathbf{C}^T &= \mathsf{Ker}\,\mathbf{T}^T\mathbf{T}\mathbf{B} = \mathsf{Ker}\,\mathbf{B}, \\ 0 &\le \mathbf{C}\mathbf{B} = \mathbf{B}^T\mathbf{T}^T\mathbf{T}\mathbf{B}, \\ \operatorname{rank}\mathbf{C}\mathbf{B} &= \operatorname{rank}\mathbf{B}^T\mathbf{T}^T\mathbf{T}\mathbf{B} = \operatorname{rank}\mathbf{B} = r. \end{aligned} \tag{36}$$

Conversely, $\mathsf{Ker}\,\mathbf{C}^T = \mathsf{Ker}\,\mathbf{B}$ is equivalent to the existence of an orthogonal matrix $\mathbf{V} \in \mathbb{R}^{m\times m}$ such that

$$\mathbf{B}\mathbf{V} = \begin{bmatrix} \mathbf{B}_1 & 0 \end{bmatrix}, \quad \mathbf{C}^T\mathbf{V} = \begin{bmatrix} \mathbf{C}_1^T & 0 \end{bmatrix}, \tag{37}$$

with $\mathbf{B}_1, \mathbf{C}_1^T \in \mathbb{R}^{n\times r}$ of full column rank. The conditions $\operatorname{rank}\mathbf{C}\mathbf{B} = r$ and $\mathbf{C}\mathbf{B} \ge 0$ together are equivalent to $\tilde{\mathbf{C}}_1\tilde{\mathbf{B}}_1 > 0$. So there exists an invertible matrix $\mathbf{Y}$ (e.g. a Cholesky factor or a positive square root) such that $\mathbf{C}_1\mathbf{B}_1 = \mathbf{Y}\mathbf{Y}^T$.

The matrix $\mathbf{T}_0$ as in (35) is then well defined. Furthermore, $\mathbf{T}_0$ is invertible, since if $\mathbf{T}_0\mathbf{Q} = 0$, then $\mathbf{C}_1\mathbf{Q} = 0$ and $\mathbf{N}_\mathbf{B}^T\mathbf{Q} = 0$. The latter statement implies that $\mathbf{Q} \in \mathsf{Ran}(\mathbf{B})$, so $\mathbf{Q} = \mathbf{B}_1\mathbf{z}$ for some $\mathbf{z}$ and, furthermore, $\mathbf{C}_1\mathbf{B}_1\mathbf{z} = 0$. This in turn implies that $\mathbf{z} = 0$ and $\mathbf{Q} = 0$; so $\mathbf{T}_0$ is injective, and hence invertible.

The invertibility of $\mathbf{T}_0$ implies that

$$\begin{aligned}\mathbf{B}^T\mathbf{T}_0^T\mathbf{T}_0 &= \mathbf{V}^{-T}\begin{bmatrix}\mathbf{B}_1^T\\ 0\end{bmatrix}[\mathbf{N}_\mathbf{B} \quad \mathbf{C}_1^T\mathbf{Y}^{-T}]\begin{bmatrix}\mathbf{N}_\mathbf{B}^T\\ \mathbf{Y}^{-1}\mathbf{C}_1\end{bmatrix}\\ &= \mathbf{V}^{-T}\begin{bmatrix}0 & (\mathbf{Y}^{-1}\mathbf{C}_1\mathbf{B}_1)^T\\ 0 & 0\end{bmatrix}\begin{bmatrix}\mathbf{N}_\mathbf{B}^T\\ \mathbf{Y}^{-1}\mathbf{C}_1\end{bmatrix}\\ &= \mathbf{V}^{-T}\begin{bmatrix}(\mathbf{C}_1\mathbf{B}_1)(\mathbf{C}_1\mathbf{B}_1)^{-1}\mathbf{C}_1\\ 0\end{bmatrix} = \mathbf{V}^{-T}\begin{bmatrix}\mathbf{C}_1\\ 0\end{bmatrix} = \mathbf{C}.\end{aligned}$$

Hence, (34a) holds with $\mathbf{T} = \mathbf{T}_0$.

Now suppose that $\mathbf{T}$ is any invertible transformation satisfying (34a). Then,

$$\mathbf{B}^T\mathbf{T}^T(\mathbf{T}\mathbf{T}_0^{-1}) = \mathbf{C}\mathbf{T}_0^{-1} = \mathbf{B}^T\mathbf{T}_0^T,$$

which is equivalent to

$$\mathbf{B}_1^T\mathbf{T}_0^T\left((\mathbf{T}\mathbf{T}_0^{-1})^T(\mathbf{T}\mathbf{T}_0^{-1}) - \mathbf{I}\right) = [0 \quad \mathbf{Y}]\left((\mathbf{T}\mathbf{T}_0^{-1})^T(\mathbf{T}\mathbf{T}_0^{-1}) - \mathbf{I}\right) = 0.$$

From this, it follows that

$$(\mathbf{T}\mathbf{T}_0^{-1})^T(\mathbf{T}\mathbf{T}_0^{-1}) = \begin{bmatrix}\mathbf{Z}^T\mathbf{Z} & 0\\ 0 & \mathbf{I}\end{bmatrix}$$

for some invertible matrix $\mathbf{Z}$, and that $\mathbf{T}\mathbf{T}_0^{-1}\begin{bmatrix}\mathbf{Z}^{-1} & 0\\ 0 & \mathbf{I}\end{bmatrix} = \mathbf{U}$ must be real orthogonal. $\square$

In order to explicitly construct a transformation matrix $\mathbf{T}_0$ as in (35), it will be useful to construct bi-orthogonal bases for the two subspaces $\mathsf{Ker}\,\mathbf{B}^T$ and $\mathsf{Ker}\,\mathbf{C}$. Toward this end, let $\mathbf{N}_\mathbf{C}$ contain columns that form a basis of $\mathsf{Ker}\,\mathbf{C}$, so that $\mathsf{Ran}\,\mathbf{N}_\mathbf{C} = \mathsf{Ker}\,\mathbf{C}$. Such a matrix is easily constructed in a numerically stable way via the singular value decomposition or a rank-revealing QR decomposition of $\mathbf{C}$, see [16]. Since $\mathbf{B}$ and $\mathbf{C}^T$ are assumed to satisfy $\mathsf{Ker}\,\mathbf{B} = \mathsf{Ker}\,\mathbf{C}^T$, we have singular value decompositions

$$\mathbf{B} = [\mathbf{U}_{\mathbf{B},1} \quad \mathbf{U}_{\mathbf{B},2}]\begin{bmatrix}\Sigma_\mathbf{B} & 0\\ 0 & 0\end{bmatrix}\mathbf{V}_\mathbf{B}{}^T, \quad \mathbf{C} = \mathbf{U}_\mathbf{C}\begin{bmatrix}\Sigma_\mathbf{C} & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}\mathbf{V}_{\mathbf{C},1}^T\\ \mathbf{V}_{\mathbf{C},2}^T\end{bmatrix} \tag{38}$$

with $\Sigma_\mathbf{B}, \Sigma_\mathbf{C} \in \mathbb{R}^{r\times r}$ both invertible. We then obtain

$$\mathbf{N}_\mathbf{C} = \mathbf{V}_{\mathbf{C},2}, \quad \mathbf{N}_\mathbf{B} = \mathbf{U}_{\mathbf{B},2}. \tag{39}$$

Observe that $\mathbf{N}_\mathbf{B}^T\mathbf{N}_\mathbf{C}$ is nonsingular, since if $\mathbf{N}_\mathbf{B}^T\mathbf{N}_\mathbf{C}\mathbf{Q} = 0$ and $\mathbf{z} = \mathbf{N}_\mathbf{C}\mathbf{Q}$, then $\mathbf{N}_\mathbf{B}^T\mathbf{z} = 0$ implies that $\mathbf{z} \in \mathsf{Ran}\,\mathbf{B}$. But then, $\mathbf{z} = \mathbf{B}_1\mathbf{Y} = \mathbf{N}_\mathbf{C}\mathbf{Q}$ implies $\mathbf{C}_1\mathbf{B}_1\mathbf{Y} = 0$, where $\mathbf{B}_1$ and $\mathbf{C}_1$ are defined in (37), and so, $\mathbf{Y} = 0$ and hence $\mathbf{z} = 0$. Then, since $\mathbf{N}_\mathbf{C}$ has full column rank, we have $\mathbf{Q} = 0$. Thus, $\mathbf{N}_\mathbf{B}^T\mathbf{N}_\mathbf{C}$ is injective, hence invertible.

Performing another singular value decomposition, $\mathbf{N}_\mathbf{B}^T\mathbf{N}_\mathbf{C} = \widetilde{\mathbf{U}}\widetilde{\Delta}\widetilde{\mathbf{V}}^T$, with $\tilde{\Delta}$ positive diagonal, and $\tilde{\mathbf{U}}, \tilde{\mathbf{V}}$ real orthogonal, we can perform a change of basis $\tilde{\mathbf{N}}_\mathbf{B} = \mathbf{N}_\mathbf{B}\widetilde{\mathbf{U}}\widetilde{\Delta}^{-1/2}$ and $\tilde{\mathbf{N}}_\mathbf{C} = \mathbf{N}_\mathbf{C}\widetilde{\mathbf{V}}\widetilde{\Delta}^{-1/2}$ and obtain that the columns of $\tilde{\mathbf{N}}_\mathbf{B}$ form a basis for $\mathsf{Ker}\,\mathbf{B}^T$, the columns of $\tilde{\mathbf{N}}_\mathbf{C}$ form a basis for $\mathsf{Ker}\,\mathbf{C}$ and these two bases are bi-orthogonal, i.e., $\tilde{\mathbf{N}}_\mathbf{B}^T\tilde{\mathbf{N}}_\mathbf{C} = \mathbf{I}$, and we have

$$\mathbf{T}_0 = \begin{bmatrix}\tilde{\mathbf{N}}_\mathbf{B}^T\\ \mathbf{Y}^{-1}\mathbf{C}_1\end{bmatrix}, \quad \mathbf{T}_0^{-1} = \begin{bmatrix}\tilde{\mathbf{N}}_\mathbf{C} & \mathbf{B}_1\mathbf{Y}^{-T}\end{bmatrix}. \tag{40}$$

Using the formula (35), we can express the conditions for a transformation to PH form that we have obtained so far in a more concrete way.

Corollary 10 *Consider system (4) with* $\mathbf{D} = -\mathbf{D}^T$ *and* $\operatorname{rank}\mathbf{B} = r$. *Let the columns of* $\tilde{\mathbf{N}}_{\mathbf{B}}$ *and* $\tilde{\mathbf{N}}_{\mathbf{C}}$ *span the kernels of* $\mathbf{B}^T$, $\mathbf{C}$ *and satisfy* $\tilde{\mathbf{N}}_{\mathbf{B}}^T\tilde{\mathbf{N}}_{\mathbf{C}} = \mathbf{I}$. *Then system (4) is equivalent to a PH system if and only if*

1. $\mathsf{Ker}\,\mathbf{C}^T = \mathsf{Ker}\,\mathbf{B}$, $\operatorname{rank}\mathbf{CB} = r$, $\mathbf{CB} \geq 0$, *and*

2. *there exists an invertible matrix* $\mathbf{Z}$ *such that*

$$\begin{bmatrix} \mathbf{Z} & 0 \\ 0 & \mathbf{I} \end{bmatrix} \mathbf{T}_0\mathbf{A}\mathbf{T}_0^{-1} \begin{bmatrix} \mathbf{Z}^{-1} & 0 \\ 0 & \mathbf{I} \end{bmatrix} + \left(\begin{bmatrix} \mathbf{Z} & 0 \\ 0 & \mathbf{I} \end{bmatrix} \mathbf{T}_0\mathbf{A}\mathbf{T}_0^{-1} \begin{bmatrix} \mathbf{Z}^{-1} & 0 \\ 0 & \mathbf{I} \end{bmatrix} \right)^T \leq 0, \tag{41}$$

and $\mathbf{T}_0$, $\mathbf{T}_0^{-1}$ *are defined in (40).*

Proof. The condition follows from Corollary 8 and the representation (35) by setting $\mathbf{U} = \mathbf{I}$ and $\mathbf{T}_0$ as in (40). $\square$

3.2 Construction of the transformation in the case of general D

For the case that $\mathbf{D}$ is general we will present a recursive procedure. The first step is to perform the transformations (30), (31), and to obtain the following characterization when a transformation to PH form exists.

Lemma 11 *Consider system (4) transformed as in (30) and (31). Let the columns of* $\tilde{\mathbf{N}}_{\mathbf{B}_1}$ *and* $\tilde{\mathbf{N}}_{\mathbf{C}_1}$ *form the kernels of* $\mathbf{B}_1^T$, $\mathbf{C}_1$ *respectively, and satisfy* $\tilde{\mathbf{N}}_{\mathbf{B}_1}^T\tilde{\mathbf{N}}_{\mathbf{C}_1} = \mathbf{I}$.

Then the system is equivalent to a PH system if and only if

1. $\mathsf{Ker}\,\mathbf{C}_1^T = \mathsf{Ker}\,\mathbf{B}_1$, $\operatorname{rank}\mathbf{C}_1\mathbf{B}_1 = \operatorname{rank}\mathbf{B}_1$, *and* $\mathbf{C}_1\mathbf{B}_1 \geq 0$, *and*

2. *there exists an invertible matrix* $\mathbf{Z}$ *such that*

$$\tilde{\mathbf{Y}} + \tilde{\mathbf{Y}}^T \geq 0, \tag{42}$$

with

$$\tilde{\mathbf{Y}} := \begin{bmatrix} \mathbf{Z} & 0 \\ 0 & \mathbf{I} \end{bmatrix} \begin{bmatrix} -\mathbf{T}_0\mathbf{A}\mathbf{T}_0^{-1} & -\mathbf{T}_0\mathbf{B}_2 \\ \mathbf{C}_2\mathbf{T}_0^{-1} & \mathbf{V}_{0,2}^T\mathbf{D}\mathbf{V}_{0,2} \end{bmatrix} \begin{bmatrix} \mathbf{Z}^{-1} & 0 \\ 0 & \mathbf{I} \end{bmatrix} \tag{43}$$

and

$$\mathbf{T}_0 = \begin{bmatrix} \tilde{\mathbf{N}}_{\mathbf{B}_1}^T \\ \mathbf{Y}^{-1}\mathbf{C}_1 \end{bmatrix}, \quad \mathbf{T}_0^{-1} = \begin{bmatrix} \tilde{\mathbf{N}}_{\mathbf{C}_1} & \mathbf{B}_1\mathbf{Y}^{-T} \end{bmatrix}, \quad \mathbf{C}_1\mathbf{B}_1 = \mathbf{Y}\mathbf{Y}^T, \tag{44}$$

and $\mathbf{B}_1, \mathbf{C}_1$ *are defined similar as in (37) just with* $\mathbf{B}, \mathbf{C}$ *being replaced by* $\mathbf{B}_1, \mathbf{C}_1$.

Proof. Condition (34a) in this case has the form $(\mathbf{T}\mathbf{B}_1)^T = \mathbf{C}_1\mathbf{T}^{-1}$. If this condition holds, then applying the result of Lemma 9 to $\mathbf{B}_1$ and $\mathbf{C}_1$ one has the form (35). The result is proved by applying this formula to (34b). $\square$

Note that $\tilde{\mathbf{Y}}$ is obtained from the one given in (34b) via a congruence (and similarity) transformation with $\operatorname{diag}(\mathbf{U}, \mathbf{I})$, where $\mathbf{U}$ is orthogonal.

We can repartition the middle factor of $\tilde{\mathbf{Y}}$ in (43) as

$$
\begin{aligned}
\begin{bmatrix} -\mathbf{T}_0\mathbf{A}\mathbf{T}_0^{-1} & -\mathbf{T}_0\mathbf{B}_2 \\ \mathbf{C}_2\mathbf{T}_0^{-1} & \mathbf{V}_2^T\mathbf{D}\mathbf{V}_2 \end{bmatrix} &= \left[\begin{array}{c|cc} -\tilde{\mathbf{N}}_{\mathbf{B}_1}^T\mathbf{A}\tilde{\mathbf{N}}_{\mathbf{C}_1} & -\tilde{\mathbf{N}}_{\mathbf{B}_1}^T\mathbf{A}\mathbf{B}_1\mathbf{Y}^{-T} & -\tilde{\mathbf{N}}_{\mathbf{B}_1}^T\mathbf{B}_2 \\ \hline -\mathbf{Y}^{-1}\mathbf{C}_1\mathbf{A}\tilde{\mathbf{N}}_{\mathbf{C}_1} & -\mathbf{Y}^{-1}\mathbf{C}_1\mathbf{A}\mathbf{B}_1\mathbf{Y}^{-T} & -\mathbf{Y}^{-1}\mathbf{C}_1\mathbf{B}_2 \\ \mathbf{C}_2\tilde{\mathbf{N}}_{\mathbf{C}_1} & \mathbf{C}_2\mathbf{B}_1\mathbf{Y}^{-T} & \mathbf{V}_{0,2}^T\mathbf{D}\mathbf{V}_{0,2} \end{array}\right] \\
&=: \begin{bmatrix} -\tilde{\mathbf{A}}_1 & -\tilde{\mathbf{B}}_1 \\ \tilde{\mathbf{C}}_1 & \tilde{\mathbf{D}}_1 \end{bmatrix}. \qquad (45)
\end{aligned}
$$

Thus, we have that

$$
\tilde{\mathbf{Y}} = \begin{bmatrix} \mathbf{Z} & 0 \\ 0 & \mathbf{I} \end{bmatrix} \begin{bmatrix} -\tilde{\mathbf{A}}_1 & -\tilde{\mathbf{B}}_1 \\ \tilde{\mathbf{C}}_1 & \tilde{\mathbf{D}}_1 \end{bmatrix} \begin{bmatrix} \mathbf{Z}^{-1} & 0 \\ 0 & \mathbf{I} \end{bmatrix},
$$

and condition (42) is analogous to condition (9), just replacing $\mathbf{T}$, $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$ with $\mathbf{Z}$, $\tilde{\mathbf{A}}_1$, $\tilde{\mathbf{B}}_1$, $\tilde{\mathbf{C}}_1$, $\tilde{\mathbf{D}}_1$. Hence, the existence of $\mathbf{Z}$ can be checked again by using Lemma 9.

This implies that the procedure of checking the existence of a transformation from (4) to a PH system can be performed in a recursive way. One first performs the transformation (30) and checks whether a condition as in Part 1 of Lemma 11 does not hold, in which case a transformation does not exist, or one has that $\tilde{\mathbf{D}} + \tilde{\mathbf{D}}^T$ in (45) is invertible. In the latter case, if the associated matrix inequality does not have a positive definite solution, then a transformation does not exist. Otherwise, a transformation exists and a corresponding transformation matrix $\mathbf{T}$ can be constructed by computing $\mathbf{Z}$ satisfying $\tilde{\mathbf{Y}}(\mathbf{Z}) + \tilde{\mathbf{Y}}^T(\mathbf{Z}) \geq 0$ and the matrix $\mathbf{T}_0$ is formed accordingly. After this the process is repeated in a recursive manner.

To formalize the recursive procedure, let

$$
\mathbf{G}_0 = \begin{bmatrix} -\mathbf{A} & -\mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix},
$$

and suppose that (34a) is satisfied. Then form

$$
\mathbf{G}_1 := \tilde{\mathbf{V}}_0^T\tilde{\mathbf{T}}_0\mathbf{G}_0\tilde{\mathbf{T}}_0^{-1}\tilde{\mathbf{V}}_0,
$$

where

$$
\tilde{\mathbf{T}}_0 = \begin{bmatrix} \mathbf{T}_0 & 0 \\ 0 & \mathbf{I} \end{bmatrix}, \quad \tilde{\mathbf{V}}_0 = \begin{bmatrix} \mathbf{I} & 0 \\ 0 & \mathbf{V}_0 \end{bmatrix},
$$

$\mathbf{V}_0$ is the matrix in the decomposition (30) times a permutation that interchanges the last block columns, and $\mathbf{T}_0$ is obtained from (44). In this way we obtain

$$
\mathbf{G}_1 = \left[\begin{array}{c|cc|c} -\tilde{\mathbf{N}}_{\mathbf{B}_1}^T\mathbf{A}\tilde{\mathbf{N}}_{\mathbf{C}_1} & -\tilde{\mathbf{N}}_{\mathbf{B}_1}^T\mathbf{A}\mathbf{B}_1\mathbf{Y}^{-T} & -\tilde{\mathbf{N}}_{\mathbf{B}_1}^T\mathbf{B}_2 & 0 \\ \hline -\mathbf{Y}^{-1}\mathbf{C}_1\mathbf{A}\tilde{\mathbf{N}}_{\mathbf{C}_1} & -\mathbf{Y}^{-1}\mathbf{C}_1\mathbf{A}\mathbf{B}_1\mathbf{Y}^{-T} & -\mathbf{Y}^{-1}\mathbf{C}_1\mathbf{B}_2 & -\mathbf{Y}^T \\ \mathbf{C}_2\tilde{\mathbf{N}}_{\mathbf{C}_1} & \mathbf{C}_2\mathbf{B}_1\mathbf{Y}^{-T} & \mathbf{V}_{0,2}^T\mathbf{D}\mathbf{V}_{0,2} & \mathbf{V}_{0,2}^T\mathbf{D}\mathbf{V}_{0,1} \\ \hline 0 & \mathbf{Y} & \mathbf{V}_{0,1}^T\mathbf{D}\mathbf{V}_{0,2} & \mathbf{V}_{0,1}^T\mathbf{D}\mathbf{V}_{0,1} \end{array}\right]
$$

by using the fact that

$$
\mathbf{C}_1\mathbf{T}_0^{-1} = (\mathbf{T}_0\mathbf{B}_1)^T = \mathbf{V}\left[\begin{array}{c|c} 0 & \mathbf{Y} \\ 0 & 0 \end{array}\right],
$$

where $\mathbf{V}$ is defined in (37) for $\mathbf{B}_1, \mathbf{C}_1$. By (30), we have that $\mathbf{V}_{0,2}^T \mathbf{D} \mathbf{V}_{0,1} = -\mathbf{V}_{0,1}^T \mathbf{D} \mathbf{V}_{0,2}$, $\mathbf{V}_{0,1}^T \mathbf{D} \mathbf{V}_{0,1} = -\mathbf{V}_{0,1}^T \mathbf{D}^T \mathbf{V}_{0,1}$, so that we can express $\mathbf{G}_1$ as

$$\mathbf{G}_1 = \left[\begin{array}{cc|c} -\tilde{\mathbf{A}}_1 & -\tilde{\mathbf{B}}_1 & 0 \\ \tilde{\mathbf{C}}_1 & \tilde{\mathbf{D}}_1 & -\Gamma_1 \\ \hline 0 & \Gamma_1^T & \Phi_1 \end{array}\right].$$

If $\begin{bmatrix} -\tilde{\mathbf{A}}_1 & -\tilde{\mathbf{B}}_1 \\ \tilde{\mathbf{C}}_1 & \tilde{\mathbf{D}}_1 \end{bmatrix}$ satisfies condition (34a), then in an analogous way we construct

$$\tilde{\mathbf{T}}_1 = \mathrm{diag}(\mathbf{T}_1, \mathbf{I}, \mathbf{I}), \quad \tilde{\mathbf{V}}_1 = \mathrm{diag}(\mathbf{I}, \mathbf{V}_1, \mathbf{I})$$

such that

$$\mathbf{G}_2 = \tilde{\mathbf{V}}_1^T \tilde{\mathbf{T}}_1 \mathbf{G}_1 \tilde{\mathbf{T}}_1^{-1} \tilde{\mathbf{V}}_1 = \left[\begin{array}{cc|cc} -\tilde{\mathbf{A}}_2 & -\tilde{\mathbf{B}}_2 & 0 & 0 \\ \tilde{\mathbf{C}}_2 & \tilde{\mathbf{D}}_2 & -\Gamma_2 & -\Gamma_{11} \\ \hline 0 & \Gamma_2^T & \Phi_2 & -\Gamma_{12} \\ 0 & \Gamma_{11}^T & \Gamma_{12}^T & \Phi_1 \end{array}\right]$$

and we continue. Since the size of the matrices $\mathbf{T}_j$ is monotonically decreasing, this procedure will terminate after a finite number of k steps with

$$\mathbf{G}_k = \tilde{\mathbf{V}}_{k-1}^T \tilde{\mathbf{T}}_{k-1} \ldots \tilde{\mathbf{V}}_0^T \tilde{\mathbf{T}}_0 \mathbf{G}_0 \tilde{\mathbf{T}}_0^{-1} \tilde{\mathbf{V}}_0 \ldots \tilde{\mathbf{T}}_{k-1}^{-1} \tilde{\mathbf{V}}_{k-1} = \left[\begin{array}{cc|c} -\tilde{\mathbf{A}}_k & -\tilde{\mathbf{B}}_k & 0 \\ \tilde{\mathbf{C}}_k & \tilde{\mathbf{D}}_k & -\tilde{\Gamma}_k \\ \hline 0 & \tilde{\Gamma}_k^T & \tilde{\Phi}_k \end{array}\right],$$

where $\tilde{\mathbf{D}}_k + \tilde{\mathbf{D}}_k^T$ is positive definite, $\tilde{\Phi}_k = -\tilde{\Phi}_k^T$,

$$\tilde{\mathbf{T}}_j = \mathrm{diag}(\mathbf{T}_j, \mathbf{I}, \mathbf{I}), \quad \tilde{\mathbf{V}}_j = \mathrm{diag}(\mathbf{I}_{\ell_j}, \mathbf{V}_j, \mathbf{I}),$$

and ℓ_j is the size of $\mathbf{T}_j$ for $j = 0, \ldots, k-1$.

If there exists an invertible matrix $\mathbf{T}_k$ such that

$$\begin{bmatrix} \mathbf{T}_k & 0 \\ 0 & \mathbf{I} \end{bmatrix} \begin{bmatrix} -\tilde{\mathbf{A}}_k & -\tilde{\mathbf{B}}_k \\ \tilde{\mathbf{C}}_k & \tilde{\mathbf{D}}_k \end{bmatrix} \begin{bmatrix} \mathbf{T}_k^{-1} & 0 \\ 0 & \mathbf{I} \end{bmatrix} + \left(\begin{bmatrix} \mathbf{T}_k & 0 \\ 0 & \mathbf{I} \end{bmatrix} \begin{bmatrix} -\tilde{\mathbf{A}}_k & -\tilde{\mathbf{B}}_k \\ \tilde{\mathbf{C}}_k & \tilde{\mathbf{D}}_k \end{bmatrix} \begin{bmatrix} \mathbf{T}_k^{-1} & 0 \\ 0 & \mathbf{I} \end{bmatrix} \right)^T \geq 0,$$

see the solvability conditions in the previous section, then with $\tilde{\mathbf{T}}_k = \mathrm{diag}(\mathbf{T}_k, \mathbf{I}, \mathbf{I})$, we have

$$\tilde{\mathbf{T}}_k \mathbf{G}_k \tilde{\mathbf{T}}_k^{-1} + (\tilde{\mathbf{T}}_k \mathbf{G}_k \tilde{\mathbf{T}}_k^{-1})^T \geq 0. \tag{46}$$

Observe that for each i, j with $j > i$ the matrices $\tilde{\mathbf{V}}_i$ and $\tilde{\mathbf{V}}_i^T$ commute with $\tilde{\mathbf{T}}_j$ and $\tilde{\mathbf{T}}_j^{-1}$, and thus setting

$$\tilde{\mathbf{T}} = \tilde{\mathbf{T}}_k \ldots \tilde{\mathbf{T}}_0, \quad \tilde{\mathbf{V}} = \tilde{\mathbf{V}}_0 \ldots \tilde{\mathbf{V}}_{k-1},$$

then

$$\tilde{\mathbf{T}}_k \mathbf{G}_k \tilde{\mathbf{T}}_k^{-1} = \tilde{\mathbf{V}}^T \tilde{\mathbf{T}} \mathbf{G}_0 \tilde{\mathbf{T}}^{-1} \tilde{\mathbf{V}},$$

and inequality (46) implies that

$$\tilde{\mathbf{T}} \mathbf{G}_0 \tilde{\mathbf{T}}^{-1} + (\tilde{\mathbf{T}} \mathbf{G}_0 \tilde{\mathbf{T}}^{-1}) \geq 0.$$

Then the desired transformation matrix $\mathbf{T}$ is positioned in the top diagonal block of $\tilde{\mathbf{T}}$, and the matrix $\mathbf{V}$ is positioned in the bottom diagonal block of $\tilde{\mathbf{V}}$.

Remark 5 The recursive procedure described above requires at each step the computation of three singular value decompositions, to check the ranks of the matrices $\mathbf{B}_j$, $\mathbf{C}_j$ and to construct the bi-orthogonal bases so that (44) holds.

While each step of this procedure can be implemented in a numerically stable way, the consecutive rank decisions and the consequence of performing these subtle decisions are hard to analyze, similar to the case of staircase algorithms [8, 12, 13]. In general the strategy should be adapted to the goal of the computation, which in our case is to obtain a representation in PH form that is robust to small perturbations.

3.3 Explicit solution of linear matrix inequalities via even pencils

We have seen that to check the existence of the transformation to PH form and to explicitly construct the transformation matrices $\mathbf{T}, \mathbf{V}$ we can consider the solution of the linear matrix inequality (7). As we have discussed before the best way to do this is via the transformation of the even pencil (28). In this subsection we combine the recursive procedure with the construction of a staircase like form for this even pencil..

Let for a given real symmetric matrix $\mathbf{Q}$ denote by

$$\Psi_0(\mathbf{Q}) := \begin{bmatrix} -\mathbf{A}^T\mathbf{Q} - \mathbf{Q}\mathbf{A} & \mathbf{C}^T - \mathbf{Q}\mathbf{B} \\ \mathbf{C} - \mathbf{B}^T\mathbf{Q} & \mathbf{D} + \mathbf{D}^T \end{bmatrix}$$

the corresponding block matrix which is supposed to be positive semidefinite.

Let $\mathbf{V}_{0,1}, \mathbf{B}_1, \mathbf{C}_1$ be defined as in (30), (31). If $\mathbf{B}_1$, $\mathbf{C}_1$ satisfy Part 1. of Lemma 11 then, since $\mathsf{Ker}\,\mathbf{C}_1^T = \mathsf{Ker}\,\mathbf{B}_1$, there exist real orthogonal matrices $\tilde{\mathbf{U}}_1$, $\tilde{\mathbf{V}}_1$ (which can be obtained by performing a permuted singular value decomposition of $\mathbf{B}_1$) such that

$$\tilde{\mathbf{U}}_1^T\mathbf{B}_1\tilde{\mathbf{V}}_{0,1} = \begin{bmatrix} 0 & 0 \\ \Sigma_{\mathbf{B}} & 0 \end{bmatrix}, \quad \tilde{\mathbf{V}}_{0,1}^T\mathbf{C}_1\tilde{\mathbf{U}}_1 = \begin{bmatrix} \mathbf{C}_{1,1} & \mathbf{C}_{1,2} \\ 0 & 0 \end{bmatrix} \tag{47}$$

where $\Sigma_{\mathbf{B}}$ is invertible and $\mathbf{C}_{1,2}\Sigma_{\mathbf{B}}$ is real symmetric and positive definite. Transforming the desired $\mathbf{Q}$ correspondingly as

$$\tilde{\mathbf{Q}} := \tilde{\mathbf{U}}_1^T\mathbf{Q}\tilde{\mathbf{U}}_1 = \begin{bmatrix} \mathbf{Q}_{11} & \mathbf{Q}_{12} \\ \mathbf{Q}_{12}^T & \mathbf{Q}_{22} \end{bmatrix},$$

then, since the linear matrix inequality (7) implies $\tilde{\mathbf{Q}}\mathbf{B}_1 = \mathbf{C}_1^T$, it follows in the transformed variables that

$$\mathbf{Q}_{22} := \mathbf{C}_{1,2}^T\Sigma_{\mathbf{B}}^{-1} > 0, \quad \mathbf{Q}_{12} := \mathbf{C}_{1,1}^T\Sigma_{\mathbf{B}}^{-1}, \tag{48}$$

and it remains to determine $\mathbf{Q}_{11}$ so that $\tilde{\mathbf{Q}} > 0$. To achieve this, we set

$$\mathbf{Q}_0 := \begin{bmatrix} \mathbf{Q}_{12}\mathbf{Q}_{22}^{-1}\mathbf{Q}_{12}^T & \mathbf{Q}_{12} \\ \mathbf{Q}_{12}^T & \mathbf{Q}_{22} \end{bmatrix} = \mathbf{T}_0^{-T}\begin{bmatrix} 0 & 0 \\ 0 & \mathbf{Q}_{22} \end{bmatrix}\mathbf{T}_0^{-1}, \quad \mathbf{T}_0 = \begin{bmatrix} \mathbf{I} & 0 \\ -\mathbf{Q}_{22}^{-1}\mathbf{Q}_{12}^T & \mathbf{I} \end{bmatrix}$$

and we clearly have that $\mathbf{Q}_0 \geq 0$. Then we can rewrite $\tilde{\mathbf{Q}}$ as

$$\tilde{\mathbf{Q}} = \begin{bmatrix} \mathbf{Q}_1 & 0 \\ 0 & 0 \end{bmatrix} + \mathbf{Q}_0 = \mathbf{T}_0^{-T}\left(\begin{bmatrix} \mathbf{Q}_1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \mathbf{Q}_{22} \end{bmatrix}\right)\mathbf{T}_0^{-1} = \mathbf{T}_0^{-T}\begin{bmatrix} \mathbf{Q}_1 & 0 \\ 0 & \mathbf{Q}_{22} \end{bmatrix}\mathbf{T}_0^{-1},$$

partitioned analogously and we obtain that $\tilde{\mathbf{Q}} > 0$ if and only if $\mathbf{Q}_1 > 0$. By performing a congruence transformation on $\Psi_0(\mathbf{Q})$ with

$$\mathbf{Z}_0 := \begin{bmatrix} \mathbf{T}_0 & 0 \\ 0 & \mathbf{V}_0 \end{bmatrix}, \quad \mathbf{T}_0 = \tilde{\mathbf{U}}_1\mathbf{T}_0, \quad \mathbf{V}_0 = \mathbf{V}_0 \begin{bmatrix} \tilde{\mathbf{V}}_{0,1} & 0 \\ 0 & \mathbf{I} \end{bmatrix}$$

and using the fact that $\tilde{\mathbf{Q}}(\tilde{\mathbf{U}}_1^T\mathbf{B}_1\tilde{\mathbf{V}}_{0,1}) = (\tilde{\mathbf{V}}_{0,1}^T\mathbf{C}_1\tilde{\mathbf{U}}_1)^T$ for any real symmetric $\tilde{\mathbf{Q}}_1$, it follows that

$$\mathbf{Z}_0^T\Psi_0(\mathbf{Q})\mathbf{Z}_0 = \begin{bmatrix} -(\mathbf{T}_0^{-1}\mathbf{A}\mathbf{T}_0)^T\mathbf{T}_0^T\mathbf{Q}\mathbf{T}_0 - \mathbf{T}_0^T\mathbf{Q}\mathbf{T}_0(\mathbf{T}_0^{-1}\mathbf{A}\mathbf{T}_0) & 0 & \mathbf{T}_0^T\mathbf{C}_2^T - \mathbf{T}_0^T\mathbf{Q}\mathbf{T}_0(\mathbf{T}_0^{-1}\mathbf{B}_2) \\ 0 & 0 & 0 \\ \mathbf{C}_2\mathbf{T}_0 - (\mathbf{T}_0^{-1}\mathbf{B}_2)^T\mathbf{T}_0^T\mathbf{Q}\mathbf{T}_0 & 0 & 2\mathbf{S}_2 \end{bmatrix}.$$

Partitioning

$$\mathbf{T}_0^{-1}\mathbf{A}\mathbf{T}_0 =: \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix}, \quad \mathbf{T}_0^{-1}\mathbf{B}_2 = \begin{bmatrix} \mathbf{B}_{13} \\ \mathbf{B}_{23} \end{bmatrix}, \quad \mathbf{C}_2\mathbf{T}_0 = \begin{bmatrix} \mathbf{C}_{31} & \mathbf{C}_{32} \end{bmatrix},$$

and using that

$$\mathbf{T}_0^T\mathbf{Q}\mathbf{T}_0 = \begin{bmatrix} \mathbf{Q}_1 & 0 \\ 0 & \mathbf{Q}_{22} \end{bmatrix},$$

we obtain that

$$\mathbf{Z}_0^T\Psi_0(\mathbf{Q})\mathbf{Z}_0 = \begin{bmatrix} -\mathbf{A}_{11}^T\mathbf{Q}_1 - \mathbf{Q}_1\mathbf{A}_{11} & -\mathbf{Q}_1\mathbf{A}_{12} - \mathbf{A}_{21}^T\mathbf{Q}_{22} & 0 & \mathbf{C}_{31}^T - \mathbf{Q}_1\mathbf{B}_{13} \\ -\mathbf{A}_{12}^T\mathbf{Q}_1 - \mathbf{Q}_{22}\mathbf{A}_{21} & -\mathbf{A}_{22}^T\mathbf{Q}_{22} - \mathbf{Q}_{22}\mathbf{A}_{22} & 0 & \mathbf{C}_{32}^T - \mathbf{Q}_{22}\mathbf{B}_{23} \\ 0 & 0 & 0 & 0 \\ \mathbf{C}_{31} - \mathbf{B}_{13}^T\mathbf{Q}_1 & \mathbf{C}_{32} - \mathbf{B}_{23}^T\mathbf{Q}_{22} & 0 & 2\mathbf{S}_2 \end{bmatrix}.$$

In this way, we have that (7) holds for some $\mathbf{Q} > 0$ if and only if

$$\begin{aligned} \Psi_1(\mathbf{Q}_1) &:= \left[\begin{array}{c|cc} -\mathbf{A}_{11}^T\mathbf{Q}_1 - \mathbf{Q}_1\mathbf{A}_{11} & -\mathbf{Q}_1\mathbf{A}_{12} - \mathbf{A}_{21}^T\mathbf{Q}_{22} & \mathbf{C}_{31}^T - \mathbf{Q}_1\mathbf{B}_{13} \\ \hline -\mathbf{A}_{12}^T\mathbf{Q}_1 - \mathbf{Q}_{22}\mathbf{A}_{21} & -\mathbf{A}_{22}^T\mathbf{Q}_{22} - \mathbf{Q}_{22}\mathbf{A}_{22} & \mathbf{C}_{32}^T - \mathbf{Q}_{22}\mathbf{B}_{23} \\ \mathbf{C}_{31} - \mathbf{B}_{13}^T\mathbf{Q}_1 & \mathbf{C}_{32} - \mathbf{B}_{23}^T\mathbf{Q}_{22} & 2\mathbf{S}_2 \end{array}\right] \\ &=: \begin{bmatrix} -\mathbf{A}_1^T\mathbf{Q}_1 - \mathbf{Q}_1\mathbf{A}_1 & \mathbf{C}_1^T - \mathbf{Q}_1\mathbf{B}_1 \\ \mathbf{C}_1 - \mathbf{B}_1^T\mathbf{Q}_1 & \mathbf{D}_1 + \mathbf{D}_1^T \end{bmatrix} \geq 0 \end{aligned}$$

holds for some real symmetric positive definite $\mathbf{Q}_1$, where

$$\mathbf{A}_1 = \mathbf{A}_{11}, \quad \mathbf{C}_1 = \begin{bmatrix} -\mathbf{Q}_{22}\mathbf{A}_{21} \\ \mathbf{C}_{3,1} \end{bmatrix}, \quad \mathbf{B}_1 = \begin{bmatrix} \mathbf{A}_{12} & \mathbf{B}_{1,3} \end{bmatrix},$$

and

$$\mathbf{D}_1 + \mathbf{D}_1^T = \begin{bmatrix} -\mathbf{A}_{22}^T\mathbf{Q}_{22} - \mathbf{Q}_{22}\mathbf{A}_{22} & \mathbf{C}_{32}^T - \mathbf{Q}_{22}\mathbf{B}_{23} \\ \mathbf{C}_{32} - \mathbf{B}_{23}^T\mathbf{Q}_{22} & 2\mathbf{S}_2 \end{bmatrix}.$$

This construction has reduced the solution of the linear matrix inequality (7) to the solution of a smaller linear matrix inequality of the same form. Thus, we can again proceed in a recursive manner with the same reduction process until either the condition in Part 1. of Lemma 9 no longer holds (in which case no solution exists) or $\mathbf{D}_k + \mathbf{D}_k^T$ is positive definite for some k.

This reduction process can be considered as the construction of a structured staircase form for the even pencil (28). By applying a congruence transformation to the pencil (28) with the matrix

$$\mathbf{Y}_0 = \begin{bmatrix} \mathbf{T}_0^{-T} & 0 & 0 \\ 0 & \mathbf{T}_0 & 0 \\ 0 & 0 & \mathbf{V}_0 \end{bmatrix},$$

it follows that

$$\lambda \left[\begin{array}{cc|cc|ccc} 0 & 0 & \mathbf{I} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{I} & 0 & 0 & 0 \\ \hline -\mathbf{I} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\mathbf{I} & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] - \left[\begin{array}{cc|cc|ccc} 0 & 0 & \mathbf{A}_{11} & \mathbf{A}_{12} & 0 & 0 & \mathbf{B}_{13} \\ 0 & 0 & \mathbf{A}_{21} & \mathbf{A}_{22} & \Sigma_B & 0 & \mathbf{B}_{23} \\ \hline \mathbf{A}_{11}^T & \mathbf{A}_{21}^T & 0 & 0 & 0 & 0 & \mathbf{C}_{31}^T \\ \mathbf{A}_{12}^T & \mathbf{A}_{22}^T & 0 & 0 & \mathbf{C}_{12}^T & 0 & \mathbf{C}_{32}^T \\ \hline 0 & \Sigma_B^T & 0 & \mathbf{C}_{12} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \mathbf{B}_{13}^T & \mathbf{B}_{23}^T & \mathbf{C}_{31} & \mathbf{C}_{32} & 0 & 0 & \tilde{\mathbf{S}}_1 \end{array}\right],$$

where $\tilde{\mathbf{S}}_1 := 2\mathbf{S}_2$. By performing another congruence transformation with the matrix

$$\tilde{\mathbf{Y}}_0 = \left[\begin{array}{cc|cc|ccc} \mathbf{I} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{I} & 0 & -\mathbf{Q}_{22} & 0 & 0 & 0 \\ \hline 0 & 0 & \mathbf{I} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{I} & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & \mathbf{I} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathbf{I} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{I} \end{array}\right],$$

the pencil becomes

$$\lambda \left[\begin{array}{cc|cc|ccc} 0 & 0 & \mathbf{I} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{I} & 0 & 0 & 0 \\ \hline -\mathbf{I} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\mathbf{I} & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$
$$- \left[\begin{array}{cc|cc|ccc} 0 & 0 & \mathbf{A}_{11} & \mathbf{A}_{12} & 0 & 0 & \mathbf{B}_{13} \\ 0 & 0 & \mathbf{A}_{21} & \mathbf{A}_{22} & \Sigma_B & 0 & \mathbf{B}_{23} \\ \hline \mathbf{A}_{11}^T & \mathbf{A}_{21}^T & 0 & -\mathbf{A}_{21}^T\mathbf{Q}_{22} & 0 & 0 & \mathbf{C}_{31}^T \\ \mathbf{A}_{12}^T & \mathbf{A}_{22}^T & -\mathbf{Q}_{22}\mathbf{A}_{21} & -\mathbf{A}_{22}^T\mathbf{Q}_{22} - \mathbf{Q}_{22}\mathbf{A}_{22} & 0 & 0 & \mathbf{C}_{32}^T - \mathbf{Q}_{22}\mathbf{B}_{23} \\ \hline 0 & \Sigma_B^T & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \mathbf{B}_{13}^T & \mathbf{B}_{23}^T & \mathbf{C}_{31} & \mathbf{C}_{32} - \mathbf{B}_{23}^T\mathbf{Q}_{22} & 0 & 0 & \tilde{\mathbf{S}}_1 \end{array}\right].$$

By further moving the last block row and column to the fifth position and then the 2nd block row and column to the fifth position, i.e., by performing a congruence permutation with

$$\left[\begin{array}{cc|cc|ccc} \mathbf{I} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathbf{I} & 0 & 0 \\ \hline 0 & \mathbf{I} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{I} & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & \mathbf{I} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{I} \\ 0 & 0 & 0 & \mathbf{I} & 0 & 0 & 0 \end{array}\right],$$

one obtains

$$\lambda \left[\begin{array}{ccc|ccc} 0 & \mathbf{I} & 0 & 0 & 0 & 0 \\ -\mathbf{I} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\Gamma_1 & 0 & 0 \\ \hline 0 & 0 & \Gamma_1^T & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] - \left[\begin{array}{ccc|ccc} 0 & \mathbf{A}_1 & \mathbf{B}_1 & 0 & 0 & 0 \\ \mathbf{A}_1^T & 0 & \mathbf{C}_1^T & \mathbf{A}_{21}^T & 0 & 0 \\ \mathbf{B}_1^T & \mathbf{C}_1 & \mathbf{D}_1 + \mathbf{D}_1^T & \Delta_1 & 0 & 0 \\ \hline 0 & \mathbf{A}_{21} & \Delta_1^T & 0 & \Sigma_1 & 0 \\ 0 & 0 & 0 & \Sigma_1^T & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right],$$

where

$$\Gamma_1 = \begin{bmatrix} \mathbf{I} \\ 0 \end{bmatrix}, \quad \Delta_1 = \begin{bmatrix} \mathbf{A}_{22}^T \\ \mathbf{B}_{23}^T \end{bmatrix}, \quad \Sigma_1 := \Sigma_B$$

and $\mathbf{A}_1, \mathbf{B}_1, \mathbf{C}_1$, $\mathbf{D}_1 + \mathbf{D}_1^T$ are as defined before. In this way, we may repeat the reduction process on the (1,1) block, which corresponds to Ψ_1. In order to exploit the block structures of the pencil we use a slightly different compression technique for $\mathbf{D}_1 + \mathbf{D}_1^T$. Note that we may write

$$\mathbf{D}_1 + \mathbf{D}_1^T = \begin{bmatrix} \mathbf{D}_{11} & \mathbf{D}_{12} \\ \mathbf{D}_{12}^T & \mathbf{S}_1 \end{bmatrix},$$

with $\mathbf{S}_1$ symmetric positive definite. Then we have

$$\mathbf{D}_1 + \mathbf{D}_1^T = \begin{bmatrix} \mathbf{I} & 0 \\ \mathbf{S}_1^{-1}\mathbf{D}_{12}^T & \mathbf{I} \end{bmatrix}^T \begin{bmatrix} \mathbf{D}_{11} - \mathbf{D}_{12}\mathbf{S}_1^{-1}\mathbf{D}_{12}^T & 0 \\ 0 & \mathbf{S}_1 \end{bmatrix} \begin{bmatrix} \mathbf{I} & 0 \\ \mathbf{S}_1^{-1}\mathbf{D}_{12}^T & \mathbf{I} \end{bmatrix}.$$

Let

$$\mathbf{D}_{11} - \mathbf{D}_{12}\mathbf{S}_1^{-1}\mathbf{D}_{12}^T = \mathbf{Z}_1 \begin{bmatrix} 0 & 0 \\ 0 & \tilde{\mathbf{S}}_2 \end{bmatrix} \mathbf{Z}_1^T,$$

where $\tilde{\mathbf{S}}_2$ is invertible and $\mathbf{Z}_1$ is orthogonal. Then

$$\begin{bmatrix} \mathbf{Z}_1 & 0 \\ -\mathbf{S}_1^{-1}\mathbf{D}_{12}^T\mathbf{Z}_1 & \mathbf{I} \end{bmatrix}^T (\mathbf{D}_1 + \mathbf{D}_1^T) \begin{bmatrix} \mathbf{Z}_1 & 0 \\ -\mathbf{S}_1^{-1}\mathbf{D}_{12}^T\mathbf{Z}_1 & \mathbf{I} \end{bmatrix} = \left[\begin{array}{c|cc} 0 & 0 & 0 \\ \hline 0 & \tilde{\mathbf{S}}_2 & 0 \\ 0 & 0 & \mathbf{S}_1 \end{array}\right] =: \begin{bmatrix} 0 & 0 \\ 0 & \mathbf{S}_2 \end{bmatrix}.$$

A necessary condition for the existence of a transformation to PH form is that $\mathbf{S}_2 > 0$ or equivalently $\tilde{\mathbf{S}}_2 > 0$. If this holds, then using the fact

$$\begin{bmatrix} \mathbf{Z}_1 & 0 \\ -\mathbf{S}_1^{-1}\mathbf{D}_{12}^T\mathbf{Z}_1 & \mathbf{I} \end{bmatrix}^T \Gamma_1 = \begin{bmatrix} \mathbf{Z}_1^T \\ 0 \end{bmatrix},$$

by performing a congruence transformation on the 3rd block rows and columns with

$$\begin{bmatrix} \mathbf{Z}_1 & 0 \\ -\mathbf{S}_1^{-1}\mathbf{D}_{12}^T\mathbf{Z}_1 & \mathbf{I} \end{bmatrix}$$

and another congruence transformation on the fourth block row and column with $\mathbf{Z}_1$ we obtain the pencil

$$\lambda \left[\begin{array}{cccc|ccc} 0 & \mathbf{I} & 0 & 0 & 0 & 0 & 0 \\ -\mathbf{I} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\Gamma_{11} & 0 & 0 \\ 0 & 0 & 0 & 0 & -\Gamma_{12} & 0 & 0 \\ \hline 0 & 0 & \Gamma_{11}^T & \Gamma_{12}^T & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] - \left[\begin{array}{cccc|ccc} 0 & \mathbf{A}_1 & \mathbf{B}_{11} & \mathbf{B}_{12} & 0 & 0 & 0 \\ \mathbf{A}_1^T & 0 & \mathbf{C}_{11}^T & \mathbf{C}_{21}^T & \Delta_{11} & 0 & 0 \\ \mathbf{B}_{11}^T & \mathbf{C}_{11} & 0 & 0 & \Delta_{21} & 0 & 0 \\ \mathbf{B}_{12}^T & \mathbf{C}_{21} & 0 & \mathbf{S}_2 & \Delta_{31} & 0 & 0 \\ \hline 0 & \Delta_{11}^T & \Delta_{21}^T & \Delta_{31}^T & 0 & \Sigma_1 & 0 \\ 0 & 0 & 0 & 0 & \Sigma_1^T & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right],$$

where

$$\begin{bmatrix} \Gamma_{11} \\ \Gamma_{12} \end{bmatrix} = \Gamma = \left[\begin{array}{cc} \mathbf{I} & 0 \\ \hline 0 & \mathbf{I} \\ 0 & 0 \end{array}\right].$$

In order to proceed, $\mathbf{B}_{11}, \mathbf{C}_{11}^T$ must satisfy the same conditions as $\mathbf{B}_1, \mathbf{C}_1^T$. If these conditions hold, then we can perform a second set of congruence transformation and transform the pencil to

$$\lambda \left[\begin{array}{ccc|ccc|ccc} 0 & \mathbf{I} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\mathbf{I} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\Gamma_2 & 0 & 0 & -\Gamma_{11} & 0 & 0 \\ \hline 0 & 0 & \Gamma_2^T & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\Gamma_{12} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\Gamma_{13} & 0 & 0 \\ \hline 0 & 0 & \Gamma_{11}^T & 0 & \Gamma_{12}^T & \Gamma_{13}^T & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$- \left[\begin{array}{ccc|ccc|ccc} 0 & \mathbf{A}_2 & \mathbf{B}_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ \mathbf{A}_2^T & 0 & \mathbf{C}_2^T & \Theta_2^T & 0 & 0 & \Delta_{11} & 0 & 0 \\ \mathbf{B}_2^T & \mathbf{C}_2 & \mathbf{D}_2 + \mathbf{D}_2^T & \Delta_2 & 0 & 0 & \Delta_{21} & 0 & 0 \\ \hline 0 & \Theta_2 & \Delta_2^T & 0 & \Sigma_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \Sigma_2^T & 0 & 0 & \Delta_{41} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \Delta_{51} & 0 & 0 \\ \hline 0 & \Delta_{11}^T & \Delta_{21}^T & 0 & \Delta_{41}^T & \Delta_{51}^T & 0 & \Sigma_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \Sigma_1^T & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right],$$

where

$$\begin{bmatrix} \Gamma_{11} \\ \Gamma_{12} \\ \Gamma_{13} \end{bmatrix} = \begin{bmatrix} 0 \\ \Gamma_{12} \\ \Gamma_{11} \end{bmatrix}, \quad \Gamma_2 = \begin{bmatrix} \mathbf{I} \\ 0 \end{bmatrix}.$$

This reduction process continues as long as all the required conditions hold, until for some k, $\mathbf{D}_k + \mathbf{D}_k^T > 0$. If this is the case, then the pencil (28) is reduced to even pencil that has the eigenvalue ∞ with equal algebraic and geometric multiplicity (it is of index one)

$$\lambda \begin{bmatrix} 0 & \mathbf{I} & 0 \\ -\mathbf{I} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & \mathbf{A}_k & \mathbf{B}_k \\ \mathbf{A}_l^T & 0 & \mathbf{C}_k^T \\ \mathbf{B}_k^T & \mathbf{C}_k & \mathbf{D}_k + \mathbf{D}_k^T \end{bmatrix},$$

and if it has a deflating subspace spanned by a matrix as in (29), then the passive system (4) can be transformed to a PH system.

Note the above process is actually a special staircase form reduction process that deflates the singular part and higher index of the eigenvalue infinity of the even pencil (28), [5, 8].

Remark 6 Note that to check the passivity of (4) it is only necessary to have a positive semidefinite solution to (7). Thus, if one only wants to check passivity, then Part 1. in Lemma 9,1 can be relaxed to $\mathsf{Ker}\,\mathbf{B}_1 \subseteq \mathsf{Ker}\,\mathbf{C}_1^T$. In this case the transformation to the form (47) can still be made, but $\mathbf{Q}_{22}$ in (48) is only positive semidefinite, and $\mathsf{Ker}\,\mathbf{Q}_{22} \subseteq \mathsf{Ker}\,\mathbf{Q}_{12}$. Then $\mathbf{Q}_0$ can still be defined but instead of $\mathbf{Q}_{11}^{-1}$ one needs to use the Moore-Penrose pseudoinverse, see [16], of $\mathbf{Q}_{11}$. However, in this case $\mathbf{Q}_0$ and the resulting solution $\mathbf{Q}$ cannot be positive definite. Thus, in this situation, (4) may be a passive system that cannot be transformed to a PH system. A simple example is the scalar system

$$\dot{\mathbf{z}} = -\mathbf{z} + 2\mathbf{u}, \quad \mathbf{y} = 0\mathbf{z} + 0\mathbf{u}.$$

It is passive (but not strictly passive) with $\mathbf{S} = \mathbf{z}^T[0]\mathbf{z}$, since $\mathbf{y}^T\mathbf{u} = 0$. But it cannot be transformed to a PH system, since for any $\mathbf{Q} = \alpha\mathbf{z}$ and $\mathbf{y} = \beta\mathbf{y}$, $\mathbf{u} = \beta^{-1}\mathbf{u}$ with $\alpha, \beta \neq 0$, one has

$$\dot{\mathbf{Q}} = -\mathbf{Q} + 2\alpha\beta\mathbf{u}, \quad \mathbf{y} = 0\mathbf{Q} + 0\mathbf{u}.$$

So we would obtain $\mathbf{J} = 0, \mathbf{R} = 1, \mathbf{B} = -\mathbf{P} = \alpha\beta$ and $\mathbf{S} = 0$. However,

$$\mathbf{K} = \begin{bmatrix} 1 & -\alpha\beta \\ -\alpha\beta & 0 \end{bmatrix}$$

is indefinite for any $\alpha\beta \neq 0$.

What goes wrong in this case in the transformation to PH form is that the resulting transformation matrix $\mathbf{T}$ that is needed to satisfy the condition on the input and output matrices becomes singular.

To illustrate the analysis procedures, consider the following example.

Example 2 In the finite element analysis of disc brake squeal [17] large scale (approx. 1 million degrees of freedom) second order differential equations arise which depend on parameters, e.g., the disc speed ω. If no further constraints are incorporated, then in the stationary case they take the form

$$\mathbf{M}\ddot{\mathbf{q}} + (\mathbf{C}_1 + \frac{\omega_r}{\omega}\mathbf{C}_R + \frac{\omega}{\omega_r}\mathbf{C}_G)\dot{\mathbf{q}} + (\mathbf{K}_1 + \mathbf{K}_R + (\frac{\omega}{\omega_r})^2\mathbf{K}_G)\mathbf{q} = 0,$$

where $\mathbf{M} = \mathbf{M}^T > 0$ is the mass matrix, $\mathbf{C}_1 = \mathbf{C}_1^T \geq 0$ models material damping, $\mathbf{C}_G = -\mathbf{C}_G^T$ models gyroscopic effects, $\mathbf{C}_R = \mathbf{C}_R^T \geq 0$ models friction induced damping, $\mathbf{K}_1 = \mathbf{K}_1^T > 0$

is the stiffness matrix, $\mathbf{K}_R = \mathbf{K}_2 + \mathbf{N}$ with $\mathbf{K}_2 = \mathbf{K}_2^T$ and $\mathbf{N} = -\mathbf{N}^T$, is a nonsymmetric matrix modeling circulatory effects, $\mathbf{K}_G = \mathbf{K}_G^T \geq 0$ is the geometric stiffness matrix, and ω is the rotational speed of the disc with reference velocity ω_r. In industrial brake models, the matrices $\mathbf{D} := \mathbf{C}_1 + \frac{\omega_r}{\omega}\mathbf{C}_R$, and $\mathbf{N}$ are sparse and have very low rank (approx 2000) corresponding to finite element nodes associated with the brake pad. Setting $\mathbf{G} := \frac{\omega}{\omega_r}\mathbf{C}_G$), $\mathbf{K} = \mathbf{K}_1 + \mathbf{K}_2 + (\frac{\omega}{\omega_r})^2\mathbf{K}_G$, we may assume that $\mathbf{K} > 0$

Thus, introducing $\mathbf{p} = \mathbf{M}\dot{\mathbf{q}}$, we can write the system in first order form

$$\begin{aligned}\begin{bmatrix} \dot{\mathbf{p}} \\ \dot{\mathbf{q}} \end{bmatrix} &= \left(\begin{bmatrix} -\mathbf{G} & -(\mathbf{I} + \frac{1}{2}\mathbf{N}\mathbf{K}^{-1}) \\ (\mathbf{I} + \frac{1}{2}\mathbf{K}^{-1}\mathbf{N}) & 0 \end{bmatrix} - \begin{bmatrix} \mathbf{D} & \frac{1}{2}\mathbf{N}\mathbf{K}^{-1} \\ \frac{1}{2}\mathbf{K}^{-1}\mathbf{N} & 0 \end{bmatrix} \right) \begin{bmatrix} \mathbf{M}^{-1} & 0 \\ 0 & \mathbf{K} \end{bmatrix} \begin{bmatrix} \mathbf{p} \\ \mathbf{q} \end{bmatrix} \\ &:= (\mathbf{J} - \mathbf{R})\mathbf{Q} \begin{bmatrix} \mathbf{p} \\ \mathbf{q} \end{bmatrix}.\end{aligned}$$

Since

$$-\mathbf{R} := \begin{bmatrix} \mathbf{D} & \frac{1}{2}\mathbf{N}\mathbf{K}^{-1} \\ \frac{1}{2}\mathbf{N}\mathbf{K}^{-1} & 0 \end{bmatrix}$$

is indefinite, as soon as $\mathbf{N} \neq 0$ it is clear that this system is not automatically stable and it is definitely unstable if $\mathbf{x}^*\mathbf{R}\mathbf{x} < 0$ for any eigenvector $\mathbf{x}$ of

$$\mathbf{J} := \begin{bmatrix} -\mathbf{G} & -(\mathbf{I} + \frac{1}{2}\mathbf{N}\mathbf{K}^{-1}) \\ (\mathbf{I} + \frac{1}{2}\mathbf{N}\mathbf{K}^{-1}) & 0 \end{bmatrix}.$$

4 Conclusions

In this paper we have extended the characterization of [38] when a system is equivalent to a port Hamiltonian system to the case of general non-minimal systems. We have presented an explicit construction procedure for the construction of the transformation matrices, which can be implemented as a numerical method. This implementation is currently under construction.

References

[1] C. Beattie and S. Gugercin, *Structure-preserving model reduction for nonlinear port-Hamiltonian systems*, 50th IEEE Conference on Decision and Control and European Control Conference (CDC-ECC), 2011, IEEE, 2011, pp. 6564–6569.

[2] P. Benner, R. Byers, V. Mehrmann, and H. Xu, *Numerical methods for linear-quadratic and $\mathcal{H}_\infty$ control problems*, Dynamical Systems, Control, Coding, Computer Vision: New Trends, Interfaces, and Interplay (G. Picci and D.S. Gilliam, eds.), Progress in Systems and Control Theory, vol. 25, Birkhäuser, Basel, 1999, pp. 203–222.

[3] P. Benner, P. Losse, V. Mehrmann, and M. Voigt, *Numerical linear algebra methods for linear differential-algebraic equations: A survey*, DAE Forum, Springer, 2015, p. To appear.

[4] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan, *Linear matrix inequalities in systems and control theory*, SIAM, Philadelphia, 1994.

[5] T. Brüll and V. Mehrmann, *STCSSP: A FORTRAN 77 routine to compute a structured staircase form for a (skew-)symmetric/(skew-)symmetric matrix pencil*, Preprint 31-2007, Institut für Mathematik, TU Berlin, 2007.

[6] R. Byers, *Hamiltonian and symplectic algorithms for the algebraic Riccati equation*, Ph.D. thesis, Cornell University, Dept. Comp. Sci., Ithaca, NY, 1983.

[7] ______, *A Hamiltonian QR-algorithm*, SIAM J. Sci. Statist. Comput. **7** (1986), 212–229.

[8] R. Byers, V. Mehrmann, and H. Xu, *A structured staircase algorithm for skew-symmetric/symmetric pencils*, Electron. Trans. Numer. Anal. **26** (2007), 1–13.

[9] C.I. Byrnes and A. Isidori, *Limit sets, zero dynamics, and internal models in the problem of nonlinear output regulation*, IEEE Transactions on Automatic Control **48** (2003), no. 10, 1712–1723.

[10] C.I. Byrnes, A. Isidori, and J.C. Willems, *Passivity, feedback equivalence, and the global stabilization of minimum phase nonlinear systems*, IEEE Trans. Autom. Control **36** (1991), 1228–1240.

[11] M. Dalsmo and A.J. van der Schaft, *On representations and integrability of mathematical structures in energy-conserving physical systems*, SIAM J. Control Optim. **37** (1999), 54–91.

[12] K. Demmel and B. Kågström, *The generalized Schur decomposition of an arbitrary pencil $A - zB$: Robust software with error bounds and applications. Part I: Theory and algorithms*, ACM Trans. Math. Softw. **19** (1993), no. 2, 160–174.

[13] ______, *The generalized Schur decomposition of an arbitrary pencil $A - zB$: Robust software with error bounds and applications. Part II: Software and applications*, ACM Trans. Math. Softw. **19** (1993), no. 2, 175–201.

[14] G. Freiling, V. Mehrmann, and H. Xu, *Existence, uniqueness and parametrization of Lagrangian invariant subspaces*, SIAM J. Matrix Anal. Appl. **23** (2002), 1045–1069.

[15] G. Golo, A.J. van der Schaft, P.C. Breedveld, and B.M. Maschke, *Hamiltonian formulation of bond graphs*, Nonlinear and Hybrid Systems in Automotive Control (A. Rantzer R. Johansson, ed.), Springer, Heidelberg, 2003, pp. 351–372.

[16] G. H. Golub and C. F. Van Loan, *Matrix computations*, 3rd ed., Johns Hopkins Univ. Press, Baltimore, 1996.

[17] N. Gräbner, V. Mehrmann, S. Quraishi, and C. Schrödera nd U. von Wagner, *Numerical methods for parametric model reduction in the simulation of disc brake squeal*, Tech. Report Preprint 16/2015, Institut für Mathematik, TU Berlin, 2015, Submitted for publication.

[18] S. Gugercin, R.V. Polyuga, C. Beattie, and A.J. van der Schaft, *Structure-preserving tangential interpolation for model reduction of port-Hamiltonian systems*, Automatica **48** (2012), 1963–1974.

[19] D. Hinrichsen and A. J. Pritchard, *Mathematical systems theory I. modelling, state space analysis, stability and robustness*, Springer-Verlag, New York, NY, 2005.

[20] R. A. Horn and C. R. Johnson, *Matrix analysis*, Cambridge University Press, Cambridge, 1985.

[21] B. Jacob and Zwart H, *Linear port-Hamiltonian systems on infinite-dimensional spaces*, Operator Theory: Advances and Applications, 223, Birkhäuser/Springer Basel AG, Basel CH, 2012.

[22] T. Kailath, *Linear systems*, vol. 1, Prentice-Hall Englewood Cliffs, NJ, 1980.

[23] C. Kleijn, *20-sim 4c 2.1 reference manual*, Controlab Products B.V., 2013.

[24] N. Kottensette and P.J. Antsaklis, *Relationships between positive real, passive, dissipative and positive systems*, American Control Conference (Baltimore, MD, USA), 2010.

[25] P. Lancaster and L. Rodman, *The algebraic Riccati equation*, Oxford University Press, Oxford, 1995.

[26] P. Lancaster and M. Tismenetsky, *The theory of matrices*, 2nd ed., Academic Press, Orlando, 1985.

[27] B.M. Maschke, A.J. van der Schaft, and P.C. Breedveld, *An intrinsic Hamiltonian formulation of network dynamics: non-standard poisson structures and gyrators*, J. Franklin Inst. **329** (1992), 923–966.

[28] V. Mehrmann, *The autonomous linear quadratic control problem, theory and numerical solution*, Lecture Notes in Control and Inform. Sci., vol. 163, Springer-Verlag, Heidelberg, July 1991.

[29] R. Ortega, A.J. van der Schaft, Y. Mareels, and B.M. Maschke, *Putting energy back in control*, Control Syst. Mag. **21** (2001), 18–33.

[30] R. Ortega, A.J. van der Schaft, B.M. Maschke, and G. Escobar, *Interconnection and damping assignment passivity-based control of port-controlled Hamiltonian systems*, Automatica **38** (2002), 585–596.

[31] R. V. Polyuga and A.J. van der Schaft, *Structure preserving model reduction of port-Hamiltonian systems by moment matching at infinity*, Automatica **46** (2010), 665–672.

[32] A.J. van der Schaft, *Port-Hamiltonian systems: an introductory survey*, Proc. of the International Congress of Mathematicians, vol. III, Invited Lectures (Madrid, Spain) (J.L. Verona M. Sanz-Sole and J. Verdura, eds.), pp. 1339–1365.

[33] ______, *Port-Hamiltonian systems: network modeling and control of nonlinear physical systems*, Advanced Dynamics and Control of Structures and Machines, CISM Courses and Lectures, Vol. 444, Springer Verlag, New York, N.Y., 2004.

[34] A.J. van der Schaft and B.M. Maschke, *The Hamiltonian formulation of energy conserving physical systems with external ports*, Arch. Elektron. Übertragungstech. **45** (1995), 362–371.

[35] ______, *Hamiltonian formulation of distributed-parameter systems with boundary energy flow*, J. Geom. Phys. **42** (2002), 166–194.

[36] ______, *Port-Hamiltonian systems on graphs*, SIAM J. Control Optim. (2013).

[37] P. Van Dooren, *The computation of Kronecker's canonical form of a singular pencil*, Linear Algebra Appl. **27** (1979), 103–121.

[38] J. C. Willems, *Dissipative dynamical systems – Part I: General theory*, Arch. Ration. Mech. Anal. **45** (1972), 321–351.

[39] ______, *Dissipative dynamical systems – Part II: Linear systems with quadratic supply rates*, Arch. Ration. Mech. Anal. **45** (1972), 352–393.