Bilinear Discretization of Integrable Quadratic Vector Fields: Algebraic Structure and Algebro-Geometric Solutions

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Abstract

This thesis discusses the integrability properties of a class of bilinear discretizations of integrable quadratic vector fields, the so called Hirota-Kimura type discretizations. This method tends to produce integrable birational mappings. The integrability properties of these mappings are discussed in detail and - where possible - solved exactly in terms of elliptic functions or their relatives. Integrability of the mappings under consideration is typically characterized by conserved quantitites, invariant volume forms and particular invariance relations, formulated in the language of so called HK bases.

After a short introduction into the theory of finite dimensional integrable systems in the continuous and discrete setting, a general methodology for discovery and proof of integrability of birational mappings is developed. This methodology is based on the concept of HK bases. Having recalled the basics of the theory of elliptic functions, the relations between HK bases and elliptic solutions of integrable birational mappings is explored. This makes it possible to formulate a general approach to the explicit integration of integrable birational mappings, provided they are solvable in terms of elliptic functions. The appealing feature of this approach is that it does not require knowledge of additional structures typically characterizing integrability (e.g. Lax pairs).

Having discussed the general properties of the HK type discretizations, several examples are discussed with the help of the previously introduced methods. In particular, discretizations of the following systems are considered: Euler top, Zhukovsky-Volterra system, three and four dimensional periodic Volterra systems, Clebsch system, Kirchhoff System, and Lagrange top. HK bases, conserved quantities and invariant volume forms are found for all examples. Furthermore, explicit solutions in terms of elliptic functions or their relatives are obtained for the Volterra systems and the Kirchhoff system.

Methodologically this work is based on the concept of experimental mathematics. This means that discovery and proof of most of the presented results are based on computer experiments and the usage of specialized symbolic computations.

Contents

1

Introduction

The theory of integrable systems is a rich and old field of mathematics. In a sense it is as old as the the subject of differential equations itself. Since Newton's solution of the Kepler problem, which might be considered as the first integrable system in the history of mathematics, mathematicians and physicists have been trying to find differential equations which could be "integrated", that is solved exactly in terms of previously known functions.

After Newton, Euler and Lagrange discovered two new integrable systems, which are now known as the Euler Top and the Lagrange Top. The study of the functions which characterized their solutions fueled the subsequent development of analysis, leading to the systematic study of elliptic functions and their higher genus analogs by Gauss, Abel, Jacobi and their contemporaries.

At this time there was, however, no precise notion of the term integrability. Back then, integrability of a system of differential equation, would usually mean, that the equations of motion could be reduced to simpler equations whose solutions were then found by inversion of elliptic or hyperelliptic integrals. A precise notion of integrability was first formulated by Liouville. He showed that Hamilton's equations could be transformed into a simple linear set of differential equations if the system of equations possessed enough independent conserved quantities.

Soon, new integrable systems were discovered; among them were the so called Kirchhoff case of rigid body motion in an ideal fluid, the related Clebsch system and the celebrated Kovalevskaia top. While there was still the faint hope that all differential equations describing physical phenomena could be integrated, Poincare eventually proved that this was not possible in the case of the three-body problem. From this point on interest in integrable systems slowly faded, as they were more and more being regarded as very remarkable yet isolated curiosities.

The big revival of integrable systems then began in the 1960's with the discovery of soliton solutions of the Korteweg-de Vries equation by Gardner, Kruskal, Green and Miura. Quickly, the connection to the Lax formalism and the related inverse scattering transform were established. Soon after, an enormous amount of new integrable systems were discovered, among them for instance the famous Toda lattice. Moreover, it became evident, that almost all classical integrable systems could be cast into Lax form.

With the advent of computers research has naturally shifted focus from the study of differential equations to the study of difference equations. In the discrete realm one now faces similar problems as Newton, Euler, Lagrange, Hamilton and their colleagues did. There are numerous examples of discrete equations that admit conserved quantities, Lax formulations and explicit solutions in terms of elliptic functions or their higher genus analogs. A general framework into which one could cast these discrete systems has however not been found. Some of these new integrable equations can be understood as discrete analogs to equations integrable in the sense of Liouville, yet there are classes of equations which fit into other recently developed frameworks of discrete integrability. In the near future there is hence the possibility for the appearance of a lot of intruiging and fascinating results in the theory of discrete integrable systems.

A modern subfield of the theory of discrete integrability is the field integrable discretizations. An integrable discretization of a continuous time integrable system is a system of difference equations obtained via discretization (in the sense of numerical analysis) which shares the original integrable structures of the continuous time system. In this thesis we will study a particular class of integrable discretizations, the so called Hirota-Kimura-type (HK type) discretizations.

The objective of this thesis is two-fold:

1. It will be shown that the HK type discretization scheme tends to produce integrable mappings. Furthermore, we will study in detail the integrability properties of the HK type discretizations in the case of several examples. The integrability properties being studied are conserved quantities, invariant volume forms and special invariance relations characterizing explicit solutions.

2. To accomplish the first goal, one is in need of suitable theoretical and algorithmic tools to study possibly integrable birational maps. Hence, this thesis contains a detailed exposition of these tools, which have only recently been developed by the author of this thesis together with Yu. B. Suris and M. Petrera.

1.1 Methodological Remarks

Methodologically this thesis is based on the concept of experimental mathematics. Bailey, et al. [11] define this particular brach of mathematics in the following way:

> *Experimental Mathematics is that branch of mathematics that concerns itself ultimately with the codification and transmission of insights within the mathematical community through the use of experimental (in either the Galilean, Baconian, Aristotelian or Kantian sense) exploration of conjectures and more informal beliefs and a careful analysis of the data acquired in this pursuit.*

Practitioners of experimental mathematics heavily rely on computer experiments in order to identify interesting mathematical structures and previously hidden patterns with the aim of formulating conjectures and finding ideas about how to prove these conjectures. The need for computer experiments in mathematical research mainly originates from the immense complexity of modern mathematical problems. This is even more true in the case of discrete integrable systems. Experimental methods have therefore played a central role in this work. The discovery of most results and their proofs originates from results of suitable computer experiments. Hence, this thesis will contain an exposition of the mathematics behind the relevant computer experiments.

Moreover, the inherently large complexity of many problems surrounding the integrability of the HK type discretizations prevents one from doing most computations by hand, especially in the case of higher dimensional systems (in our case $N > 3$). Therefore, a large number of results in this thesis depend on computer-aided proofs, performed using the software packages MAPLE, SINGULAR [24] and FORM [55]. It will be mentioned at the relevant points when this was the case exactly. It would certainly be desirable to find smarter methods for proving a large number of results contained in this thesis. Yet, for the HK type discretizations the absence of the "usual" integrability structures has so far prevented the discovery of such methods.

In the spirit of experimental mathematics this thesis also presents results which are based on numerical computer experiments, but which have not been rigorously proven. These results will be marked as such at the relevant points. Instead of "Proposition" or "Theorem" we will designate them with **"Experimental Result"**. Although one might criticize the lack of a formal proof, one should note that the evidence supporting these results is strong enough to clear any doubts one might have. Also, one should note that these results will usually be used as intermediate steps towards the *formulation* of a mathematical statement, which will then be *proven* rigorously.

This dissertation includes a CD-ROM which contains the MAPLE worksheets and SINGULAR programs used for the computer assisted proof and discovery of those results in this thesis which are not directly verifiable by hand.

1.2 Outline of the Thesis

This thesis is organized as follows: First, in Chapter 2 the neccessary theoretical foundations behind the theory of integrable systems (in finite dimensions) in the continuous and discrete setting will be established. Furthermore, several tools needed during the study of possibly integrable birational mappings will be introduced and discussed. Chapter 3 continues by recalling the basic facts of the theory of elliptic functions. They will later appear in the explicit solutions of some of the integrable HK type discretizations. Also, we will see the implications that the existence of explicit solutions in terms of elliptic functions has on the existence of HK-bases. We also discuss related computer experiments which aid during the formulation of ansätze for explicit solutions. In Chapter 4 we will then be introduced to the the Hirota-Kimura-type discretizations and get to know them better by considering simple examples demonstrating their basic features and relations to the methods outlined in Chapter 2. Finally, the remaining two chapters present detailed expositions of more complicated cases of integrable Hirota-Kimura type discretizations. There we will prove not only integrability of the systems under consideration, but also derive explicit solutions. The central results of this thesis are then summarized in the final chapter.

Except for Section 2.4 Chapters 1 to 3 consist mainly of a review of existing literature. The basic results and recipes pertaining to the theory of HK bases are based on joint research with the author's advisor Prof. Dr. Yuri Suris and also Dr. Matteo Petrera, with some original extensions added by the author. These results have partially been published in [45]. Most statements in Chapter 3 are well known facts from the theory of elliptic functions and are found in any standard textbook about this subject. The research presented in Chapter 5 has been carried out together with Prof. Dr. Yuri Suris, the results presented in Sections (4.1.1) and (4.2) are based on joint research with Prof. Dr. Yuri Suris and Dr. Matteo Petrera and have been published in [43]. The results relevant to the HK type discretization of the Kirchhoff System represent the author's own research. The results for the HK type discretization of the Clebsch System are also based on joint research with Prof. Dr. Yuri Suris and Dr. Matteo Petrera and have been first published in [45].

2

Integrability in the Continuous and Discrete Realm

This chapter is meant to give a short overview of the theory of finite dimensional integrable systems in both the continuous and the discrete setting. In the continuous setting our focus will be on the theory of completely integrable Hamiltonian systems. In the discrete setting we will consider discretizations of completely integrable Hamiltonian systems and integrable birational maps. In the following sections we will briefly introduce the following key notions:

1. Complete integrability in the sense of Liouville-Arnold.

2. Algebraic complete integrability.

3. Integrable discretizations.

4. Algebraic entropy, singularity confinement and Diophantine integrability.

Each of the above notions can be taken as one definition of the term integrability, yet we will not adopt one single notion of them as the basis for this work, but rather understand them as basic points of orientation. The most important one will be the concept of integrable discretizations. This notion is well-defined and established in the literature. It will serve as the basis of our discussions, yet we will leave aside the aspect of Poissonicity and shift focus to particular invariance relations and invariant volume forms. Both will typically characterize integrability of our examples. In practice we will use the language of HK bases to describe the specific integrability aspects of our examples.

In this thesis we will call a discrete dynamical system integrable, once we have found enough integrals of motion and related invariance relations, such that this would in principle enable us to derive explicit solutions in terms of known special functions. In a sense, this approach should be seen analogous to the one taken by mathematicians before the first historical formalization of integrability by Liouville and his contemporaries.

2.1 Hamiltonian Systems

We will now provide a brief overview of Hamiltonian systems on Poisson manifolds. This provides the most direct way to one of the key notions underlying this work: the concept of *complete integrability in the sense of Liouville-Arnold*. This presentation will closely follow the expositions by Suris [52] and Perelomov [42].

Definition 2.1. **[52]** *Let M be a smooth manifold and let $\mathcal{F}(M)$ be the space of smooth functions on M. A bilinear operation $\{\cdot,\cdot\} : \mathcal{F}(M)\times\mathcal{F}(M) \to \mathcal{F}(M)$ is called* **Poisson bracket** *(or* **Poisson structure***) if it satisfies the following conditions:*

1. *skew-symmetry:*
$$\{F, G\} = -\{G, F\} \tag{2.1}$$

2. *Jacobi identity:*
$$\{F, \{G, H\}\} + \{G, \{H, F\}\} + \{H, \{F, G\}\} = 0 \tag{2.2}$$

3. *Leibniz rule:*
$$\{F, GH\} = \{F, G\} H + \{F, H\} G \tag{2.3}$$

The pair $(M, \{\cdot,\cdot\})$ is called Poisson manifold.

Definition 2.2. **[52]** *Let $(M, \{\cdot,\cdot\})$ be a Poisson manifold and $H \in \mathcal{F}(M)$. The unique vector field $X_H : M \to TM$ satisfying*

$$X_H \cdot F = \{H, F\} \tag{2.4}$$

for all $F \in \mathcal{F}(M)$ is called **Hamiltonian vector field** *of the Hamilton function H. The flow $\phi^t : M \to M$ of X_H, that is the solution of the differential equation*

$$\dot{x}(t) = X_H(x(t)) \quad x(t) \in M \tag{2.5}$$

is called Hamiltonian flow of the Hamilton function H. The expression $X_H \cdot F$ denotes the Lie derivative of F along the vector field X_H. If M is n-dimensional and x_i are local coordinates on M and X_H^i denotes the i-th component of X_H^i, then

$$X_H \cdot F = \sum_{i=1}^{n} X_H^i \frac{\partial F}{\partial x_i}.$$

We may therefore write the differential equation governing the flow ϕ^t as

$$\dot{x} = \{H, x\}. \tag{2.6}$$

Definition 2.3. **[52]** *Let ϕ^t be Hamiltonian flow on a Poisson manifold $(M, \{\cdot,\cdot\})$. A function $F \in \mathcal{F}(M)$ is called* **integral of motion** *(first integral, conserved quantity) for the flow ϕ^t, if*

$$F \circ \phi^t = F. \tag{2.7}$$

Definition 2.4. *Let $(M, \{\cdot,\cdot\})$ be a Poisson manifold. Two functions H,$F \in \mathcal{F}(M)$ are said to be* **in involution** *if*

$$\{F, H\} = 0 \tag{2.8}$$

A function C, which is involution with every other function in $\mathcal{F}(M)$ is called Casimir function.

Proposition 2.1. [52] *Let ϕ^t be a Hamiltonian flow on a Poisson manifold $(M, \{\cdot,\cdot\})$ with Hamilton function H. Then H is an integral of motion for ϕ^t. Furthermore, a function $F \in \mathcal{F}(M)$ is an integral of motion for ϕ^t if and only if $\{F, H\} = 0$. In particular:*

$$\frac{d}{dt}(F \circ \phi^t) = \left\{H, F \circ \phi^t\right\}. \tag{2.9}$$

Proposition 2.2. [52] *Let $(M, \{\cdot,\cdot\})$ be a a Poisson manifold and $H, F \in \mathcal{F}(M)$. Also, let ϕ^t be the Hamiltonian flow of X_H and ψ^s the Hamiltonian flow of X_F. If $\{F, H\} = 0$, then the flows ϕ^t and ψ^s commute:*

$$\phi^t \circ \psi^s = \psi^s \circ \phi^t \quad \forall s, t \in \mathbb{R}. \tag{2.10}$$

Definition 2.5. *Let $(M, \{\cdot,\cdot\}_M)$ and $(N, \{\cdot,\cdot\}_N)$ be two Poisson manifolds and $f : M \to N$ be a mapping between them. f is called* **Poisson mapping** *if it preserves the Poisson brackets:*

$$\{F, G\}_N \circ f = \{F \circ f, G \circ f\}_M \quad \forall F, G \in \mathcal{F}(N). \tag{2.11}$$

Definition 2.6. [52] [42] *Let M be a manifold. A nondegenerate closed two-form ω is called* **symplectic structure**. *The pair (M, ω) is called symplectic manifold.*

Symplectic manifolds form an important subclass of Poisson manifolds, since there exists a canonical way of defining a Poisson structure from a given symplectic one. Hence every symplectic manifold is also a Poisson manifold [52] [42]. (However, the converse statement is in general not true). We also accept the fact that the dimension of a symplectic manifold always is an even number. A Hamiltonian system can also be defined on a symplectic Manifold. However, this definition of a Hamiltonian system then turns out to be compatible with definition 2.2: The definition of a Hamiltonian system using a symplectic structure is equivalent to the definition a Hamiltonian system on a symplectic manifold using the canonically obtained Poisson bracket on the symplectic manifold [52] [42]. Hamiltonian flows on symplectic manifolds have the important property that they are symplectic maps, that is they preserve the symplectic form. This fact [52] [42] corresponds to the preservation of some phase space volume (See the next example).

Example Consider the canonical phase space $\mathbb{R}^{2n} = \mathbb{R}^{2n}(p, q)$ with the Poisson bracket

$$\{F, G\} = \sum_{i=1}^{n} \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q_i} - \frac{\partial G}{\partial p_i}\frac{\partial F}{\partial q_i}. \tag{2.12}$$

One easily verifies that (2.12) defines a Poisson-bracket turning $\mathbb{R}^{2n}(p, q)$ into a Poisson manifold. We take a function $H \in \mathcal{F}(\mathbb{R}^{2n})$ and find the corresponding Hamiltonian system to read

$$\dot{x} = (\dot{p}, \dot{q}) = \{H, x\} = \left(-\frac{\partial H}{\partial q}, \frac{\partial H}{\partial p}\right). \tag{2.13}$$

Hence we see that in this case we obtain Hamilton' s classical equations:

$$\dot{p} = -\frac{\partial H}{\partial q}, \qquad \dot{q} = \frac{\partial H}{\partial p}. \tag{2.14}$$

$\mathbb{R}^{2n}$ also is a symplectic manifold with the 2-form ω being

$$\omega = \sum_{k=1}^{n} dp_k \wedge dq_k, \tag{2.15}$$

which is preserved by the flow of (2.13).

The bracket (2.12) is the so called *canonical* bracket of $\mathbb{R}^{2n}$. In general, a Poisson bracket on $\mathbb{R}^n = \mathbb{R}^n(x_1, x_2, ..., x_n)$ may be defined by its values on pairs of coordinate functions:

$$\{F, G\} = \sum_{i=1}^{n}\sum_{j=1}^{n} \{x_i, x_j\} \frac{\partial F}{\partial x_i}\frac{\partial G}{\partial x_j}. \tag{2.16}$$

Thus, we may write

$$\{F, G\}(x) = \nabla F(x)^T B(x) \nabla G(x),$$

where the entries of the matrix B are defined by $B_{ij} = \{x_i, x_j\}$. B is called *Poisson matrix*. Given a Hamilton function H, the corresponding Hamiltonian system takes the form

$$\dot{x} = B(x)\nabla H(x).$$

A map $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ is then Poisson with respect to the Bracket 2.16, iff

$$D\varphi(x)^T B(x) D\varphi(x) = D\varphi(x).$$

Example **[42]** **[40]** The dynamics of a three dimensional rigid body may also be described by a Hamiltonian system of equations. Let $m = (m_1, m_2, m_3) \in \mathbb{R}^3$ denote total angular momentum of the rigid body and $p = (p_1, p_2, p_3) \in \mathbb{R}^3$ denote its total linear momentum. The equations of motion then read

$$\dot{m}_i = \{H, m_i\}, \quad \dot{p}_i = \{H, p_i\}, \tag{2.17}$$

where H is the Hamilton function of the system and

$$\{m_i, m_j\} = \epsilon_{ijk} m_k, \qquad \{m_i, p_j\} = \epsilon_{ijk} p_k, \qquad \{p_i, p_j\} = 0. \tag{2.18}$$

ϵ_{ijk} denotes the Levi-Civita symbol:

$$\epsilon_{ijk} = \begin{cases} 1 & \text{if } (i,j,k) \text{ is an even permutation of } (1,2,3), \\ -1 & \text{if } (i,j,k) \text{ is an odd permutation of } (1,2,3), \\ 0 & \text{if } i = j \text{ or } j = k \text{ or } i = k. \end{cases}$$

The bracket (6.2) is actually the Lie-Poisson bracket on the dual of the Lie-algebra $e(3)$ of the Lie-group $E(3)$ of euclidean motions (see below). We will return to this example when we will discuss the Kirchhoff-type systems.

Lie-Poisson Brackets [42] [52] Important examples of Poisson brackets are the Lie-Poisson brackets. They are defined on the dual space $\mathfrak{g}^*$ of a Lie algebra $(\mathfrak{g}, [\cdot,\cdot])$. For an element $X \in \mathfrak{g}$ one associates a linear functional $X^* \in \mathfrak{g}^*$ via

$$X^* : \mathfrak{g}^* \to \mathbb{R}, L \mapsto L(X) =: \langle L, X\rangle.$$

The Lie-Poisson bracket on $\mathfrak{g}^*$ is then defined by

$$\{F, G\}(L) = \langle L, [\nabla F(L), \nabla G(L)]\rangle, \quad \forall F, G \in \mathcal{F}(\mathfrak{g}^*).$$

2.2 Complete Integrability

Definition 2.7. **[52] [42]** *A Hamiltonian system on a 2N-dimensional symplectic manifold* $(M, \{\cdot,\cdot\})$ *with Hamilton function* $H \in \mathcal{F}(M)$ *is called* **completely integrable** *(in the sense of Liouville-Arnold), if it possesses N first integrals* $F_1,\ldots,F_N \in \mathcal{F}(M)$ *with* $H = G(F_1, ..., F_N)$ *such that*

1. $F_1,\ldots,F_N$ *are functionally independent, i.e. their gradients are linearly independent;*

2. $F_1,\ldots,F_N$ *are in involution with each other:*

$$\{F_i, F_j\} = 0 \quad 1 \le k, j \le N. \tag{2.19}$$

Theorem 2.1. **[42] [52]** *The solution of a completely integrable Hamiltonian system on a 2N-dimensional symplectic manifold* $(M, \{\cdot,\cdot\})$ *is obtained by "quadrature". More specifically, the following holds:*

1. *Let* $F_1,\ldots,F_N$ *be the integrals of motion of a Hamiltonian system on the manifold* M *and let* $\mathcal{T}$ *be a connected component of a common level set*

$$\{Q \in M \mid F_k(Q) = c_k, k = 1..N\}. \tag{2.20}$$

   Then $\mathcal{T}$ *is diffeomorphic to* $\mathbb{T}^d \times \mathbb{R}^{N-d}$ *with some* $0 \le d \le N$. *If* $\mathcal{T}$ *is compact, then it is diffeomorphic to* $\mathbb{T}^N$. *Here* $\mathbb{T}^d$ *denotes the N dimensional Torus.*

2. *If* $\mathcal{T}$ *is compact, the in some neighborhood* $\mathcal{T} \times \Omega$ *of* $\mathcal{T}$, *where* $\Omega \in \mathbb{R}^N$ *is an open ball, there exist coordinates (so called action-angle coordinates)* $(I, \theta) = (I_k, \theta_k)_{k=1}^N$, *where* $I \in \Omega$ *and* $\theta \in \mathbb{T}^N$ *with the following properties:*

   - *The "actions"* I_k *depend only on* F_j*'s:*

$$I_k = I_k(F_1, ..., F_N) \quad k = 1, .., N. \tag{2.21}$$

   - *The Poisson brackets between the coordinate functions are canonical:*

$$\{I_k, I_j\} = \{\theta_k, \theta_j\} = 0, \quad \{I_k, \theta_j\} = \delta_{kj} \quad 1 \le k, j \le N. \tag{2.22}$$

Therefore, for an arbitrary Hamilton function $H = H(F_1, ..., F_N)$ depending only on F_j's the Hamiltonian equations of motion have the form

$$\dot{I}_k = 0, \quad \dot{\theta}_k = \omega_k(I_1, ..., I_N), \quad k = 1, ..., N. \tag{2.23}$$

Hence, in the action-angle coordinates the evolution of the Hamiltonian equations is actually a linear motion on a torus.

Moreover, for an arbitrary symplectic map $\Phi : M \to M$ admitting $F_1, ..., F_N$ as integrals of motion, the equations of motion in the coordinates (I, θ) take the form

$$\widetilde{I_k} = I_k, \quad \widetilde{\theta_k} = \theta_k + \Omega_k(I_1, ..., I_N), \quad k = 1, ..., N. \tag{2.24}$$

If a Poisson bracket possesses Casimir functions, the conditions for complete integrability slightly change. Suppose for instance that the Poisson structure a Hamiltonian system with N degrees of freedom ($2N$-dimensional phase space) has M Casimir functions and P conserved quantities which are not Casimir functions. Then, if all Casimir functions and other conserved quantities are in involution and functionally independent, we define the Hamiltonian system to be integrable if the following formula holds:

$$2N - 2P = M. \tag{2.25}$$

The reason for this is the fact that Casimir functions generate trivial Hamiltonian equations creating what is called a *Poisson submanifold.* For further details the reader is referred to [52].

Lax pairs [42] [62] [9] The above notions have in some form already been known in the 19th century. The theory of integrable systems did, however, not develop any further, until in the year 1967 Gardner, Green, Kruskal and Miura invented the inverse scattering transform for the Korteveg-de Vries Equation leading to the discovery of soliton solutions of several nonlinear PDE's and lattice equations [22]. Nowadays, the theory of integrable systems is therefore also called soliton theory. The main tool of soliton theory is the notion of *Lax pairs.*

Suppose that a system of ordinary differential equations can equivalently be formulated as

$$\dot{L}(t) = [L(t), M(t)] = L(t)M(t) - M(t)L(t) \tag{2.26}$$

where L and M are matrices of the same dimension depending on the time variable t through phase variables. L and M are then called *Lax pair.* Eq. (2.26) admits a solution of the form

$$L(t) = U(t)^{-1}L(0)U(t), \quad M = \dot{U}(t)U(t)^{-1},$$

It follows that the eigenvalues of L remain constant as L evolves through time. It is said that L has an *isospectral* evolution. An important consequence then is that the original equations of motion equivalent to the Lax formulation (2.26) possess a number of conserved quantities given by the eigenvalues of L. Hence trace and determinant of L remain constant as well. In some cases, the Lax matrices L and M also depend analytically on a complex parameter λ, which is called spectral parameter. It is also important to mention that a Lax pair is not unique. Two different Lax pairs may also consist of matrices of *different* dimension.

Remarkably, almost all known integrable systems have a Lax formulation. The Lax formulation gives rise to *algebro-geometric integration* which is an elegant method for the explicit integration of an integrable system [10].

Algebraically Completely Integrable Systems In a lot of cases the integrals of motion are rational functions in the phase variables and the torus in Theorem 2.1 on which the motion takes place turns out to be the real part of a complex torus. This complex torus is an abelian variety. The solutions of the original system of equations can then be expressed by abelian functions which in turn can be expressed in terms of multi-dimensional theta functions. We will call systems with this behavior *algebraically completely integrable* or a.c.i. for short. All the discretizations that we will study in this thesis are discretizations of a.c.i. systems.

2.3 Integrable Discretizations

The central mathematical tool used in all areas of science are differential equations. Most of the fundamental systems of equations appearing in (mathematical) physics constitute either integrable systems or possess special qualitative features which are often analytically expressed by conserved quantities. The study of the behavior of such systems is often only manageable using numerical computations. Therefore one needs a way of discretizing differential equations such that they can approximately be solved by a computer. Of course, there exist numerous approaches of discretizing a system of differential equations, yet most of them fail to reproduce a discrete counterpart of the qualitative features of the original system, i.e. the discretization does not preserve some (or all) conserved quantities (or the corresponding symmetries). This usually leads to a loss of qualitative features. These qualitative features are however crucial for the study of the long term dynamics of a system of differential equations (for instance in astrophysics). This motivation has lead to the development of the new field of geometric integration [25].

A special problem related to this approach is the problem of integrable discretization. It is easily stated: how to discretize one or several independent variables of a given system of integrable differential equations while at the same time preserving the integrability property of the original continuous system? Note that we are now also requiring that the integrability property is preserved under discretization. Hence we are interested in finding a discretization which can in some sense be solved explicitly, i.e. we can express the n-th iterate explicitly as a function of the initial data and time.

As one might expect, there is no general answer to this question, yet there are different frameworks in which to embed this problem. One possible way is to adopt a "Hamiltonian point of view": one views the Poisson structure of an integrable system and its conserved quantities as the fundamental objects, for which one tries to find discrete counterparts.

An in-depth introduction to the problem of integrable discretization is found in [52]. There, one may also find some general remarks on the history of this subject and references to already established approaches to the problem of integrable discretization. Let us now continue and formally define what we mean by an integrable discretization.

Definition 2.8. *A function $h : \mathbb{R}^n \to \mathbb{R}$ is called an* **integral***, or a* **conserved quantity***, of the map $f : \mathbb{R}^n \to \mathbb{R}^n$, if for every $x_0 \in \mathbb{R}^n$ there holds*

$$h(f(x_0)) = h(x_0), \tag{2.27}$$

so that

$$h \circ f^i(x_0) = h(x_0) \quad \forall i \in \mathbb{Z}. \tag{2.28}$$

Thus, each orbit of the map f lies on a certain level set of its integral h. As a consequence, if one knows d functionally independent integrals $h_1, \dots, h_d$ of f, one can claim that each orbit of f is confined to an $(n-d)$-dimensional invariant set, which is a common level set of the functions $h_1, \dots, h_d$.

Suppose now that we are given a completely integrable Hamiltonian flow on a Poisson manifold $(M, \{\cdot,\cdot\})$

$$\dot{x} = f(x) = \{H, x\}, \tag{2.29}$$

possessing a number of independent conserved quantities I_k. We now formally define what we mean by an integrable discretization:

Definition 2.9. [52] *An* **integrable discretization** *of the flow (2.29) is a one parameter family of diffeomorphisms $\Psi_\epsilon : M \to M$ depending on the (small) parameter ϵ which satisfies the following conditions:*

1. *The continuous flow is approximated in the following sense:*

$$\Psi_\epsilon(x) = x + \epsilon f(x) + O(\epsilon^2). \tag{2.30}$$

2. *The map Ψ_ϵ is Poisson with respect to the bracket $\{\cdot,\cdot\}$ or a different bracket $\{\cdot,\cdot\}_\epsilon = \{\cdot,\cdot\}) + O(\epsilon)$.*

3. *The map Ψ_ϵ is an integrable map, i.e. possesses a sufficient number of discrete integrals of motion $I_k(x;\epsilon)$ in involution which approximate the integrals of motion of the continuous system: $I_k(x;\epsilon) = I_k(x) + O(h)$.*

In order to simplify the notation we write

$$\widetilde{x} = \Psi_\epsilon(x). \tag{2.31}$$

Thus, for a fixed value of ϵ, we obtain a map $x \mapsto \widetilde{x}$.

This definition is, of course, justified by the last statement in Theorem 2.1.

Remarks In the above definition one could of course require that the continuous flow is approximated up to a higher order. As we are, however, mainly interested in preserving the integrable structure during the discretization, we will be content even with lower orders of approximation. The specific maps which we will later deal with are described by implicit equations of motion which are of the type

$$\widetilde{x} - x = \Psi(x, \widetilde{x}, \epsilon). \tag{2.32}$$

In all cases which we will be discuss

$$\Psi(x, x, 0) = f(x) \tag{2.33}$$

holds. The implicit function theorem then guarantees the local solvability of equations of the type (2.32).

2.4 Detecting and Proving Integrability of Birational Maps

We now study the problem of integrability detection and the eventual proving of integrability in the case of birational maps. One may define birational maps in affine space and also in projective space. In affine space one defines a birational map in the following way:

Definition 2.10. *Let p_i, $q_i \in \mathbb{R}[x_1, \ldots, x_n]$, so that each pair p_i and q_i are coprime polynomials. The rational map $x \mapsto f(x)$, where*

$$f(x) = (p_1(x)/q_1(x), \ldots, p_n(x)/q_n(x)),$$

is called **birational**, *if f^{-1} exists everywhere except on some closed set $U \subset \mathbb{R}^n$ and is also given by a rational map. Although f is not defined at zeros of the denominators q_i, we will still write $f : \mathbb{R}^n \to \mathbb{R}^n$. If $i \in \{1, \ldots, n\}$ and f is of the above form, then we further define*

$$den_i f = q_i, \quad num_i f = p_i.$$

Definition 2.11. *For a birational map $f : \mathbb{R}^n \to \mathbb{R}^n$, we define the following two sets:*

$$S_I = \{x \in \mathbb{R}^n \mid den_i f(x) = 0 \text{ for some } i\}.$$

$$S_{II} = \{x \in \mathbb{R}^n \mid \det Df(x) = 0\}.$$

S_I and S_{II} are called **singular sets** *of f. Elements of these sets are called singularities.*

In this thesis, we will usually work in the affine setting when considering concrete examples. Yet, in order to explain the concept of algebraic entropy in the next subsection, we will have to work in projective space.

Definition 2.12. *A map $f : \mathbb{RP}^n \to \mathbb{RP}^n$, defined by*

$$z = [z_0 : z_1 : \ldots, z_n] \mapsto [p_0(z) : \ldots : p_n(z)],$$

with homogeneous polynomials p_i of the same degree is called birational if it is bijective everywhere except on a Zariski closed set $\Sigma \subset \mathbb{RP}^n$. We define the singular set S of f by

$$S = \{z \in \mathbb{RP}^n \mid p_i(z) = 0 \ \forall i = 0, .., n\}.$$

Hence, S contains all points whose image under f would not be defined in $\mathbb{RP}^n$.

Given a birational map in affine space, one can obtain a projective version of f. Assume that we are given a birational map in affine space by

$$f : \mathbb{R}^n \to \mathbb{R}^n, \quad x \mapsto (p_1(x)/q_1(x), \ldots, p_n(x)/q_n(x)).$$

Without loss of generality we may assume that all denominators q_i are equal to one and the same polynomial q. By setting $x_i = z_i/z_0$ we introduce projective coordinates and thus obtain a projective version of f:

$$f : \mathbb{RP}^n \to \mathbb{RP}^n, \quad [z_0 : z_1 : \ldots, z_n] \mapsto [z_0^N q(x) : z_0^N p_1(x) : \ldots : z_0^N p_1(x)]$$

Here, N is the maximal degree of the polynomials p_i and q_i.

Studying the integrability of birational maps one faces the problem that there exists no commonly accepted notion of the integrability of a birational map. For instance, a birational map may be called integrable if it is a symplectic (Poisson) map with a suitable number of conserved quantities. In light of the Liouville-Arnold theorem we will hence call maps of this type Liouville-Arnold integrable. Alternatively, one might also call the map integrable, if

1. its algebraic entropy is zero, that is degrees of the numerators and denominators of the iterates of f grow polynomially,
2. its singularities are confined,
3. or if the heights of numerators and denominators grow polynomially (diophantine integrability).

In this thesis we will study the integrability of birational maps with the notion of Liouville-Arnold integrability as our main theoretical basis. Hence, to prove their integrability, we will have to find a sufficiently large number of integrals of motion for the maps in question. Yet, as mentioned in the introduction of this chapter, we will leave aside the aspect of Poissonicity and shift focus to particular invariance relations and invariant volume forms, both of which will characterize integrability of our examples.

The remaining three concepts mentioned above will also be useful for us. This is due to the fact that they arise as necessary conditions of Liouville-Arnold integrability. Together with them being relatively easy to detect using computer experiments they may be used as integrability detectors of birational maps. In the following two sections we will now show how to detect the integrability of birational maps using the algebraic entropy and the Diophantine integrability approach. Then, we will introduce the concept of Hirota-Kimura bases, which may be used for the detection, as well as the eventual proving of integrability.

We conclude this part by mentioning a prototypical family of integrable birational mappings, the so called QRT maps (see for instance [46, 47] or the recent monograph [19]). They were discovered by Quispel, Roberts and Thompson in 1988. The QRT maps are an 18 parameter family of maps $\mathbb{R}^2 \to \mathbb{R}^2$, $(x, y) \mapsto (\widetilde{x}, \widetilde{y})$ defined by

$$\begin{aligned} \widetilde{x} &= \frac{f_1(y) - x f_2(y)}{f_2(y) - x f_3(y)}, \\ \widetilde{y} &= \frac{g_1(\widetilde{x}) - y g_2(\widetilde{x})}{g_2(\widetilde{x}) - y g_3(\widetilde{x})}, \end{aligned}$$

where

$$\begin{aligned} f(x) &= (A_1 X) \times (A_2 X), \\ g(y) &= (A_1^T X) \times (A_1^T X), \end{aligned}$$

with

$$X = (x^2, x, 1)^T, \quad A_1, A_2 \in Mat_{3\times 3}.$$

These mappings have a conserved quantity K defined by

$$K(x, y) = \frac{\langle X, A_1 Y\rangle}{\langle X, A_2 Y\rangle},$$

where $\langle \cdot, \cdot \rangle$ is the standard scalar product and again $Y = (y^2, y, 1)^T$. Moreoever, one can show that each member of the QRT family has a one parameter family of invariant curves

$$P(x, y) = q_0 x^2 y^2 + q_1 x^2 y + q_2 x y^2 + q_3 x^2 + q_4 y^2 + q_5 xy + q_6 x + q_7 y + q_8 = 0, \quad (2.34)$$

with the coefficients q_i depending on the 18 parameters of the map and the conserved quantity K. These curves can be parametrized by elliptic functions (see Chapter 3) leading to explicit solutions of the QRT maps.

2.4.1 Algebraic Entropy and Diophantine Integrability

In this section we present three methods of integrability detection for birational maps and explore their relations. The common idea behind these approaches is the study of the so called *complexity*[1] of a birational map and to use this complexity as a measure of integrability, where low complexity would usually mean integrability of the mapping. The basic ideas behind this approach go back to the work of Arnold. In [3] certain growth properties of mappings were studied and related to their integrability properties. Veselov later applied this idea to polynomial mappings and was able to demonstrate a relation between the growths of degrees of the iterations of polynomial mappings and their integrability properties [57]. In particular, it was first shown, that polynomial growth of degrees would usually mean that a mapping is integrable,

[1] In this general formulation this should not be understood as a well-defined notion.

indeed. Eventually, this lead to the development of integrability detection methods by Viallet, Bellon, Hietarinta and many others [8, 29, 58]. We will now discuss the essential concepts and methods. As was mentioned before, the concepts on which the following integrability detectors are based on can themselves be taken as definitions of integrability in the discrete setting.

Algebraic Entropy. The algebraic entropy approach was pioneered by Viallet and Hietarinta. It is based on the observation that the degrees of the numerators and denominators of the iterates of f grow polynomially. We now define the notion of algebraic entropy following [8]. For this aim we need to clarify what we mean by the degree of the iterates of a birational map f set in projective space. When we calculate the composition of f with itself, common factors in all components of $f^2 = f \circ f$ might appear. So, we define the reduced second iterate $f^{[2]}$ of f by taking f^2 and cancelling all common factors. The reduced iterates $f^{[k]}$ are then defined inductively. Now we may define the notion of algebraic entropy.

Definition 2.13. *Let $f : \mathbb{RP}^n \to \mathbb{RP}^n$ be birational. The algebraic entropy of f is defined as*

$$ent(f) = \lim_{k\to\infty} \frac{1}{k} \log(d_k),$$

where $d_k = \max_i \deg f_i^{[k]}$.

The above limit always exists [8]. In general, the sequence d_k grows exponentially, so that $\mathrm{ent}(f) \neq 0$. If d_k grows polynomially, i.e. $d_k = \mathcal{O}(k^d)$, for some fixed d, then $\mathrm{ent}(f) = 0$. A remarkable result due to Bellon [7] is the following:

Fact 2.1. *If f is a birational map, integrable in the sense of Liouville-Arnold, then $ent(f) = 0$.*

This remarkable behaviour of integrable maps now provides us with a simple method of detecting integrability.

(AE) For a given birational map $f : \mathbb{RP}^n \to \mathbb{RP}^n$ consider the images of a line $r(\lambda)$ under successive iterations of f: choose

$$r(\lambda) = [1 : \lambda r_1 : \ldots : \lambda r_n],$$

with some fixed rational values r_i and consider for $k \in \mathbb{N}$

$$d_k = \max_i \deg_\lambda f_i^{[k]}(r(\lambda)).$$

Randomly choose rational values r_i and compute the first elements of the sequence d_k. Repeat this procedure several times. If the sequence appears to grow polynomially in all cases, then f is most likely integrable.

Polynomial growth of d_k is, of course, related to the appearance of common factors in the components of $f_i^k(r(\lambda))$. Usually, in the integrable case, one can observe that d_k first grows exponentially until some index k_0, after which there appear common factors in $f_i^k(r(\lambda))$ for $k \geq k_0$. We note that the algebraic entropy method of integrability detection has one drawback: when computing $f_i^k(r(\lambda))$ one has to use a symbolic manipulator like MAPLE. Clearly, in the process of iterating the map f with the *symbolic* initial data $r(\lambda)$, the expressions for $f_i^k(r(\lambda))$ might swell up to considerable lengths. Hence, in some situations it might happen that one is not able to compute as many $f_i^k(r(\lambda))$ as one would need in order to identify the critical index k_0, where a degree-drop occurs. In such situations it can prove useful to use other methods, such as the Diophantine integrability test or the HK bases approach.

Before discussing the other concepts let us briefly explain the origin of the degree-drop phenomenon. It is closely is related to the nature of the singularities of f. For a birational map $f : \mathbb{RP}^n \to \mathbb{RP}^n$ we have

$$f \circ f^{-1}(x) = \sigma_1(x) \cdot id, \quad f^{-1}(x) \circ f = \sigma_2(x) \cdot id,$$

so that

$$\Sigma = \{x \in \mathbb{RP}^n \mid \sigma_1(x) = 0\} \cup \{x \in \mathbb{RP}^n \mid \sigma_2(x) = 0\}.$$

It may now happen for some index k that the image of a point p under f^k lies in S, so that $f^{k+1}(p)$ is not defined. This means, that for arbitary x

$$f^{k+1}(x) = \kappa(x) \cdot f^{[k+1]}(x),$$

with a polynomial κ, such that $\kappa(p) = 0$. We see that, in this situation, common factors appear in all components of f^{k+1} and also that κ must consist of factors of σ_1 and σ_2.

At this point it seems worthwhile to consider an example illustrating the method (AE). We investigate the birational map defined by

$$f : \mathbb{R}^3 \to \mathbb{R}^3, \quad (x_1, x_2, x_3) \mapsto \left(x_2, x_3, \frac{1 + x_2 + x_3}{x_1}\right), \tag{2.35}$$

which is a member of the Lyness family of mappings. The inverse of f is easily found:

$$f^{-1} : \mathbb{R}^3 \to \mathbb{R}^3, \quad (x_1, x_2, x_3) \mapsto \left(\frac{1 + x_1 + x_2}{x_3}, x_1, x_2\right). \tag{2.36}$$

We apply the method (AE) and obtain the following sequence for d_k:

$$1, 2, 3, 3, 3, 3, 3, 2, 1, 2, 3, 3, 3, 3, 3, 2, 1, 2, \ldots$$

This clearly suggests that $\mathrm{ent}(f) = 0$. We investigate this situation more closely. The projectivization of f is given by

$$[x_0 : x_1 : x_2 : x_3] \mapsto [x_0x_1 : x_1x_2 : x_1x_3, (x_0 + x_2 + x_3)x_0],$$

while the projective version of f^{-1} reads as

$$[x_0 : x_1 : x_2 : x_3] \mapsto [x_0x_3 : (x_0 + x_1 + x_2)x_0 : x_1x_3, x_2x_3].$$

By considering their composition we find that

$$\sigma_1(x) = x_0x_1(x_0 + x_2 + x_3), \quad \sigma_2(x) = x_0x_1(x_0 + x_1 + x_2),$$

so that the singular set Σ is given by

$$\Sigma = \{x_0 = 0\} \cup \{x_1 = 0\} \cup \{x_0 + x_1 + x_2 = 0\} \cup \{x_0 + x_2 + x_3 = 0\}.$$

The sequence of degrees found using (AE) suggests that a factorization will occur after four iterations. Computing these first four iterates of f symbolically using MAPLE, we find that each of the factors x_0, x_1, $x_0 + x_1 + x_2$, and $x_0 + x_2 + x_3$ appears in every component of $f^4(x)$ thus explaining the degree-drop.

Singularity Confinement For the sake of completeness we briefly mention another method of integrability detection. It is based on the notion of singularity confinement which can be seen as a discrete analog of the Painlevé property. Following [39] we define it in the affine setting in the following way:

Definition 2.14. *Let $f : \mathbb{C}^n \to \mathbb{C}^n$ be birational and $x_0 \in S_I \cup S_{II}$. If there exists a number $k \in \mathbb{N}$, such that the two limits*

$$\lim_{x \to x_0} f^k(x), \quad \lim_{x \to x_0} \det D(f^k)(x),$$

exist and $\det D(f^k)(x_0) \neq 0$*, then the singularity x_0 is said to be confined.*

In [39] it is shown that a birational map $f : \mathbb{R}^n \to \mathbb{R}^n$ with $n-1$ independent rational conserved quantities must possess a sufficiently large set of confined singularities. Testing for singularity confinement thus constitutes another method of integrability detection. One should note that this concept may be extended to the projective setting and is closely related to the algebraic entropy approach [8, 29, 58].

Diophantine Integrability. The concept of Diophantine integrability, introduced by Halburd [26], is very similar in spirit to the algebraic entropy approach. Here, instead of looking at the sequence of degrees of iterates of the map f one iterates fixed rational initial data $p \in \mathbb{Q}^n$ and observes the so called heights of the iterates $f^k(p)$.

Definition 2.15. *The height of the rational number $r = p/q$, such that p and q are coprime integers, is defined as*

$$h(r) = \max\{|p|, |q|\}.$$

For $R = (p_1/q_1, \ldots, p_n/q_n) \in \mathbb{Q}^n$, we define

$$h(R) = \max_i h(R_i), \quad H(R) = \log h(R).$$

$h(r)$ is called th Archimedan height of r, $H(R)$ logarithmic height of R.

Definition 2.16. *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be birational. For $x_0 \in \mathbb{Q}^n$, we define*

$$H_k(x_0) = H(f^k(x_0)).$$

If

$$\lim_{k\to\infty} \frac{1}{k} \log H_k(x_0) = 0,$$

for all $x_0 \in \mathbb{Q}^n$, such that $f^k(x_0)$ is well-defined for all k, then f is said to possess the Diophantine integrability property.

Fact 2.2. *If the birational map $f : \mathbb{R}^n \to \mathbb{R}^n$, where $n = 2, 3$, is Liouville-Arnold integrable then it has the Diophantine integrability property [26].*

Hence, we are now in the posssession of another integrability detector:

(DI) For a given map f, compute the first elements of the sequence H_k. If the points $(\log(k), \log(H_k))$ asymptotically tend to a straight line, then f is most likely integrable. If $(\log(k), \log(H_k))$ form an exponential shape, then f is most likely not integrable.

At the time of writing there were no results relating the Diophantine integrability property to Liouville-Arnold integrability, if $n > 3$. Hence, this is an obvious drawback of the Diophantine integrability approach. Yet, preliminary numerical results indicate that it is suitable as an integrability detector, even if $n > 3$. Further research in this direction could hence prove useful.

We now conclude this section with an example application of the Diophantine integrability test. As an example we consider the map

$$f : \mathbb{R}^2 \to \mathbb{R}^2, \quad (x_1, x_2) \mapsto \left(-x_1 - x_2 + 1 + \frac{1}{x_1}, x_1\right), \tag{2.37}$$

which can be found in [39]. We apply the method (DI) for several randomly chosen initial data and create plots of the points $(\log(k), \log(H_k))$. In every case we obtain a picture similar to Figure 2.1. This suggests that the map is in fact integrable. Indeed, one easily verifies that it has a polynomial integral given by

$$H(x_1, x_2) = x_1x_2(x_1 + x_2) - x_1x_2 - x_1 - x_2.$$

It is an interesting and nontrivial problem to find integrals of motion for a possibly integrable birational map. There is, however, an experimental approach which allows for their discovery. This approach uses so called HK bases. They will be descibed in the next section.

2.4.2 Hirota-Kimura Bases

Definition 2.17. *A set of functions $\Phi = (\varphi_1, \ldots, \varphi_l)$, linearly independent over $\mathbb{R}$, is called a* **Hirota-Kimura basis (HK basis)***, if for every $x \in \mathbb{R}^n$ there exists a vector $c = (c_1, \ldots, c_l) \neq 0$ such that*

$$c_1\varphi_1(f^i(x)) + \ldots + c_l\varphi_l(f^i(x)) = 0 \tag{2.38}$$

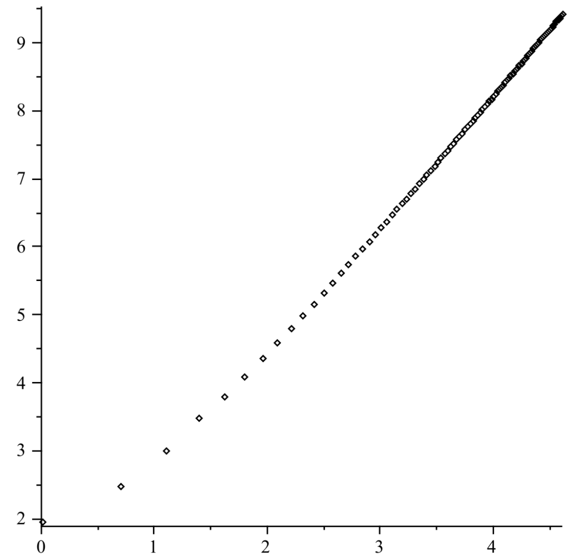


Figure 2.1: Plot of $\log(H_k)$ versus $\log(k)$ for the first 100 iterates of the map 2.37 with initial data $x_1 = 13/4$ and $x_2 = 3/11$.

holds true for all $i \in \mathbb{Z}$. For a given $x \in \mathbb{R}^n$, the vector space consisting of all $c \in \mathbb{R}^l$ with this property will be denoted by $K_\Phi(x)$ and called the null-space of the basis Φ (at the point x).

Thus, for a HK basis Φ and for $c \in K_\Phi(x)$ the function $h = c_1\varphi_1 + ... + c_l\varphi_l$ vanishes along the f-orbit of x. Let us stress that we cannot claim that $h = c_1\varphi_1 + ... + c_l\varphi_l$ is an integral of motion, since vectors $c \in K_\Phi(x)$ do not have to belong to $K_\Phi(y)$ for initial points y not lying on the orbit of x. However, for any x the orbit $\{f^i(x)\}$ is confined to the common zero level set of d functions

$$h_j = c_1^{(j)}\varphi_1 + \ldots + c_l^{(j)}\varphi_l = 0, \quad j = 1, \ldots, d,$$

where the vectors $c^{(j)} = \left(c_1^{(j)}, \ldots, c_l^{(j)}\right) \in \mathbb{R}^l$ form a basis of $K_\Phi(x)$. We will say that the HK basis Φ is **regular**, if the differentials $dh_1, \ldots, dh_d$ are lineraly independent along the the common zero level set of the functions $h_1, \ldots, h_d$. Thus, knowledge of a regular HK basis with a d-dimensional null-space leads to a similar conclusion as knowledge of d independent integrals of f, namely to the conclusion that the orbits lie on $(n-d)$-dimensional invariant sets. Note, however, that a HK basis gives no immediate information on how these invariant sets foliate the phase space $\mathbb{R}^n$, since the vectors $c^{(j)}$, and therefore the functions h_j, change from one initial point x to another.

Although the notions of integrals and of HK bases cannot be immediately translated into one another, they turn out to be closely related.

The simplest situation for a HK basis corresponds to $l = 2$, $\dim K_\Phi(x) = d = 1$. In this case we immediately see that $h = \varphi_1/\varphi_2$ is an integral of motion of the map

f. Conversely, for any rational integral of motion $h = \varphi_1/\varphi_2$ its numerator and denominator φ_1, φ_2 satisfy

$$c_1\varphi_1(f^i(x)) + c_2\varphi_2(f^i(x)) = 0, \quad i \in \mathbb{Z},$$

with $c_1 = 1$, $c_2 = -h(x)$, and thus build a HK basis with $l = 2$. Thus, the notion of a HK basis *generalizes* (for $l \geq 3$) the notion of integrals of motion. Another example can for instance be found in the theory of QRT maps. Here, the invariant curves (2.34) can also be interpreted as HK bases of the form

$$\Phi = (x^2y^2, x^2y, y^2x, x^2, y^2, xy, x, y, 1),$$

with the one dimensional nullspace $K_{\Phi(x_0)} = [q_0 : \ldots : q_8]$.

Knowing a HK basis Φ with $\dim K_\Phi(x) = d \geq 1$ allows one to find integrals of motion for the map f. Indeed, from Definition 2.17 there follows immediately:

Proposition 2.1. *If Φ is a HK basis for a map f, then*

$$K_\Phi(f(x)) = K_\Phi(x).$$

Thus, the d-dimensional null-space $K_\Phi(x) \in Gr(d, l)$, regarded as a function of the initial point $x \in \mathbb{R}^n$, is constant along trajectories of the map f, i.e., it is a $Gr(d, l)$-valued integral. Its Plücker coordinates are then scalar integrals:

Corollary 2.1. *Let Φ be a HK basis for f with $\dim K_\Phi(x) = d$ for all $x \in \mathbb{R}^n$. Take a basis of $K_\Phi(x)$ consisting of d vectors $c^{(i)} \in \mathbb{R}^l$ and put them into the columns of a $l \times d$ matrix $C(x)$. For any d-index $\alpha = (\alpha_1, \ldots, \alpha_d) \subset \{1, 2, \ldots, n\}$ let $C_\alpha = C_{\alpha_1 \ldots \alpha_d}$ denote the $d \times d$ minor of the matrix C built from the rows $\alpha_1, \ldots, \alpha_d$. Then for any two d-indices α, β the function C_α/C_β is an integral of f.*

Especially simple is the situation when the null-space of a HK basis has dimension $d = 1$.

Corollary 2.2. *Let Φ be a HK basis for f with $\dim K_\Phi(x) = 1$ for all $x \in \mathbb{R}^n$. Let $K_\Phi(x) = [c_1(x) : \ldots : c_l(x)] \in \mathbb{RP}^{l-1}$. Then the functions c_j/c_k are integrals of motion for f.*

An interesting (and difficult) question is about the number of functionally independent integrals obtained from a given HK basis according to Corollaries 2.1 and 2.2. It is possible for a HK basis with a one-dimensional null-space to produce more than one independent integral. The first examples of this mechanism (with $d = 1$) were found in [35] and (somewhat implicitly) in [30].

It should also be mentioned that HK bases appeared in a disguised form in the continuous time theory long ago. We consider here two relevant examples. Classically, integration of a given system of ODEs in terms of elliptic functions started with the derivation of an equation of the type $\dot{y}^2 = P_4(y)$, where y is one of the components of the solution, and $P_4(y)$ is a polynomial of degree 4 with constant coefficients (depending on parameters of the system and on its integrals of motion), see examples in later chapters. This can be interpreted as the claim about $\Phi = (\dot{y}^2,\, y^4,\, y^3,\, y^2,\, y,\, 1)$ being a HK basis with a one-dimensional null-space.

Moreover, according to [1, Sect. 7.6.6], for any algebraically integrable system, one can choose projective coordinates $y_0, y_1, \ldots, y_n$ so that *quadratic Wronskian equations* are satisfied:

$$\dot{y}_i y_j - y_i \dot{y}_j = \sum_{k,l=0}^{n} \alpha_{ij}^{kl} y_k y_l,$$

with coefficients α_{ij}^{kl} depending on integrals of motion of the original system. Again, this admits an immediate interpretation in terms of HK bases consisting of the Wronskians and the quadratic monomials of the coordinate functions: $\Phi_{ij} = \big(\dot{y}_i y_j - y_i \dot{y}_j,\, \{y_k y_l\}_{k,l=0}^{n}\big)$. Thus, these HK bases consist not only of simple monomials, but include also more complicated functions composed of the vector field of the system at hand. We will encounter discrete counterparts of these HK bases, as well.

2.4.3 Algorithmic Detection of HK Bases

At the moment there exist no general theretical conditions implying the existence of a HK basis. Hence, the only way to find them remains the experimental way. We therefore present two experimental methods of finding candidates for HK-bases of a birational map f. One will be called (N), the other one (V). Later on in this thesis, we present statements supported purely by numerical evidence. These results are those designated by **"Experimental Result"** and have been obtained either by (N) or (V).

Before we formulate the first method, we need to fix some notation. In particular, for a given set of functions $\Phi = (\varphi_1, \ldots, \varphi_l)$ and for any interval $[j,k] \subset \mathbb{Z}$ we denote

$$X_{[j,k]}(x) = \begin{pmatrix} \varphi_1(f^j(x)) & .. & \varphi_l(f^j(x)) \\ \varphi_1(f^{j+1}(x)) & .. & \varphi_l(f^{j+1}(x)) \\ \ldots & & \ldots \\ \varphi_1(f^k(x)) & .. & \varphi_l(f^k(x)) \end{pmatrix}. \tag{2.39}$$

In particular, $X_{(-\infty,\infty)}(x)$ will denote the double infinite matrix of the type (2.39). Obviously,

$$\ker X_{(-\infty,\infty)}(x) = K_\Phi(x).$$

Theorem 2.2. *Let*

$$\dim\ker X_{[0,s-1]}(x) = \begin{cases} l-s & \text{for } \ 1 \le s \le l-1, \\ 1 & \text{for } \ s = l, \end{cases} \tag{2.40}$$

hold for all $x \in \mathbb{R}^n$. Then for any $x \in \mathbb{R}^n$ there holds:

$$\ker X_{(-\infty,\infty)}(x) = \ker X_{[0,l-2]}(x),$$

and, in particular,

$$\dim\ker X_{(-\infty,\infty)}(x) = 1.$$

Hence, $\Phi = (\varphi_1, \ldots, \varphi_l)$ is a HK-basis with $\dim K_\Phi(x) = 1$.

These results lead us to formulate the following *numerical algorithm for the estimation of* $\dim K_\Phi(x)$ *for a hypothetic HK-basis* $\Phi = (\varphi_1, \dots, \varphi_l)$.

(N) For several randomly chosen initial points $x \in \mathbb{R}^n$, compute $\dim \ker X_{[0,s-1]}(x)$ for $1 \le s \le l$. If for every x the condition of the previous theorem is satisfied, then Φ is likely to be a HK-basis for f, with $\dim K_\Phi(x) = 1$

Finding a suitable candidate for a HK-basis could in some instances take up a considerable amount of time. To counter this problem, we will now present an algorithm which simplifies the search for potential polynomial HK-bases. Its main ideas are based on the paper [32], where similar methods have been applied to the computation of invariants of group actions of algebraic groups. In what is to follow now, we will use the concept of Gröbner-bases (for a simple introduction see for instance [18]) and related notions from commutative algebra and algebraic geometry. Gröbner-bases, invented by Buchberger in his Ph.D. thesis [12], can be thought of as canonical sets of generators for a polynomial ideal, which may in particular be used to solve the ideal membership problem. The following definition is just one of many ways to define Gröbner bases:

Definition 2.18. *Let R be a polynomial ring together with a monomial order M and $I \subset R$ be an ideal. A set of polynomials $G = \{g_1, \dots, g_k\}$, such that $I = \langle g_1, \dots, g_k \rangle$ is called a* **Gröbner basis** *relative to M, if multivariate polynomial division of any polynomial $p \in I$ by G with respect to M gives zero. A Gröbner-basis is called reduced, if the leading monomial of any g_i is equal to one and no monomial in any element of the basis is in the ideal generated by the leading terms of the other elements of the basis.*

Assume now that we are given an integrable birational map $f : \mathbb{C}^n \to \mathbb{C}^n$ with a polynomial HK-basis $\Phi = (\varphi_1, \dots, \varphi_l)$, such that $\dim K_\Phi(x_0) = d$. We choose a basis of for all $\Phi(x_0)$ given by the d vectors $c_1(x_0)$,...,$c_d(x_0)$. Then, for a fixed x_0 we consider the polynomial ideal

$$J(x_0) = \langle c_1^T(x_0)\Phi(X), \dots, c_d^T(x_0)\Phi(X) \rangle \subset \mathbb{C}[X] = \mathbb{C}[X_1, \dots, X_N].$$

Clearly, since Φ is a HK-basis, any polynomial $p(X) \in J(x_0)$ vanishes if $X \in \mathcal{O}(x_0) = \{f^k(x_0) \mid k \in \mathbb{Z}\}$. Consider now, for a fixed x_0, the ideal

$$I(\mathcal{O}(x_0)) = \{p \in \mathbb{C}[X] \mid p(X) = 0 \ \ \forall X \in \mathcal{O}(x_0)\}.$$

Since $I(\mathcal{O}(x_0))$ is radical, it follows from Hilbert's Nullstellensatz that $I(\overline{\mathcal{O}(x_0)})$ is the ideal of functions vanishing on the variety $\overline{\mathcal{O}(x_0)}$[2]. Hence, $J(x_0) \subset I(\overline{\mathcal{O}(x_0)})$. To find a HK-basis, we can thus consider the ideal $I(\overline{\mathcal{O}(x_0)})$ and try to find a set of generators for it.

In principle, $I(\overline{\mathcal{O}(x_0)})$ is completely defined by a finite subset of points in the orbit of f going through x_0. So, it is reasonable to try to find generators by taking a finite

[2]Here we mean, of course, the closure in the Zariski topology.

subset S of the orbit of f through x_0 and compute the set $I(S)$ of polynomials vanishing on this set. Naively, one could construct generators of $I(S)$ by simply assigning zeros given by the elements of S. This would, however, lead to very large polynomials of high degrees. A different approach to the construction of these polynomials is through the computation of a canonical set of generators which are in a sense "small" with respect to their size and degrees. This can be achieved using the so called Buchberger-Möller algorithm or its variants [20,41]. Given a finite set of points $S \subset \mathbb{R}^n$, algorithms of this type compute a reduced Gröbner basis G of $I(S)$ with respect to a chosen monomial order.

A crucial observation is the following: *If we have computed a (Gröbner-)basis G of $I(S)$ with $S \subset \mathcal{O}(x_0)$ and there exists an element $g \in G$, such that the number of terms of g is less than $|S|$, then the monomials of p are a suitable candidate for a HK basis.* Indeed, if $g \in I(S)$, such that

$$g = c_1(x_0)\varphi_1(X) + \dots c_l(x_0)\varphi_l(X),$$

where φ_i are monomials, then

$$c_1(x_0)\varphi_1(X) + \dots c_l(x_0)\varphi_l(X) = 0,$$

for $X \in S$. Hence, if $l < |S|$, then one can conjecture that $\dim K_\Phi(x_0) > 1$, where $\Phi = (\varphi_1, \dots, \varphi_l)$. This observation is the basis of the following algorithm:

(V)
1. Choose $x_0 \in \mathbb{Q}^n$, and a number m in $\mathbb{N}$.
2. Using exact rational arithmetic, compute the first m iterates of x_0: $x_k = f^{(k)}(x_0)$.
3. Let $S = \{x_0, .., x_m\}$. Choose a monomial order[3] and compute a Gröbner basis G of $I(S)$ using a variant of the Buchberger-Möller Algorithm [20,41].
4. Output the set $V(x_0)$ consisting of all $g \in G$, such that the number of terms of g is less than $m + 1 = |S|$.

If one repeats the algorithm (V) several times for different, randomly chosen initial data and observes that all elements of all $V(x_0)$ are spanned by one and the same set of monomials Φ, then one can for all practical considerations be sure that Φ is a HK basis. The number of elements in $V(x_0)$ will be a first estimate for $\dim K_\Phi(x_0)$. An example of the concrete usage and typical output of (V) is given in Appendix A.

In general, the algorithm (V) will provide good insights into the structure of HK bases for a given map f and should be the preferred tool when looking for HK bases. Sometimes, however, when HK bases consist of a large number of monomials it can prove useful to solely rely on the algorithm (N). In this case the computation of Gröbner bases for the vanishing ideals can become very demanding.

To further justify the usage of the algorithm (V) we discuss properties of reduced Gröbner-bases of $I(\overline{\mathcal{O}(x_0)})$. In particular, we can show the following: If we have

[3] In practice a good choice has proven to be degree reverse lexicographic ordering.

obtained a reduced Gröbner-basis $G = \{g_1, \ldots, g_d\}$ of $I(\overline{\mathcal{O}(x_0)})$, the coefficients of each g_i will be integrals of motion for f. Hence, each g_i gives a one-dimensional HK-basis. To understand this, we consider an element element $p_{x_0} \in I(\mathcal{O}(x_0))$. It can be written as

$$p_{x_0}(X) = \sum_{\alpha \in A} c_\alpha(x_0) X^\alpha,$$

where A is a set of multindices of the form $\alpha = (\alpha_1, \ldots, \alpha_n)$, $X^\alpha = x_1^{\alpha_1} \cdot \ldots \cdot x_n^{\alpha_n}$ and $c_\alpha(x_0)$ are rational functions of x_0. It is easy to see that

$$p_{f(x_0)}(X) = 0, \quad \text{for } X \in \mathcal{O}(x_0),$$

so that $p_{f(x_0)} \in I(\mathcal{O}(x_0)) = I(\overline{\mathcal{O}(x_0)})$. Let $G(x_0) = \{g_1, \ldots, g_d\}$ be a reduced Gröbner-basis of $I(\overline{\mathcal{O}(x_0)})$ with respect to some monomial order. Because G is reduced, the coefficient at the leading monomial of any $g_i(x_0)$ is equal to one:

$$g_i(x_0) = X^\alpha + \sum_{\beta \in B} c_\beta(x_0) X^\beta.$$

Moreover, from the previous considerations, it is clear that $g_i(f(x_0)) \in I(\overline{\mathcal{O}(x_0)})$, so that

$$g_i(x_0) - g_i(f(x_0)) = \sum_{\beta \in B} (c_\beta(x_0) - c_\beta(f(x_0)) X^\beta \in I(\overline{\mathcal{O}(x_0)}).$$

Since $g_i(x_0)$ and $g_i(f(x_0))$ have the same leading monomial, their difference is in normal form with respect to G. Since this difference belongs to $I(\overline{\mathcal{O}(x_0)})$, it must hence be zero. This implies that $c_\beta(x_0) = c_\beta(f(x_0)$. It should be mentioned that the above considerations are essentially the ideas behind the proofs of Lemma 2.13 and Theorem 2.14 in [32].

2.4.4 HK Bases and Symbolic Computation

When we will rely on experimentally obtained results in this thesis, we will usually be in a situation where these results will be used as intermediate steps towards a final mathematical statement, which will be proven rigorously (see for instance Chapter 5). In this way the intermediate statements, which were a priori numerically supported results, do not require additional proof. In some cases, however, we will be interested in rigorous mathematical proofs. More concretely, we will be faced with the problem of how to prove rigorously an experimental result stating the existence of a HK basis. Because of the growing complexity of the iterates $f^i(x)$ this can be a highly nontrivial task.

The typical situation is the following: Having found a a candidate for a HK-basis Φ with $\dim K_\Phi(x_0) = d$ numerically using (N) or (V), *prove* that Φ is a HK basis, indeed. Recall that this means to prove that the system of equations (2.38) with $i = i_0, i_0 + 1, \ldots, i_0 + l - d$ admits (for some, and then for all $i_0 \in \mathbb{Z}$) a d-dimensional space of solutions. For the sake of clarity, we restrict our following discussion to the most important case $d = 1$. Thus, one has to prove that the homogeneous system

$$(c_1\varphi_1 + c_2\varphi_2 + ... + c_l\varphi_l) \circ f^i(x) = 0, \qquad i = i_0, i_0 + 1, \dots, i_0 + l - 1 \tag{2.41}$$

admits for every $x \in \mathbb{R}^n$ a one-dimensional vector space of nontrivial solutions. The main obstruction for a symbolic solution of the system (2.41) is the growing complexity of the iterates $f^i(x)$. While the expression for $f(x)$ is typically of a moderate size, already the second iterate $f^2(x)$ becomes typically prohibitively big. In such a situation a symbolic solution of the linear system (2.41) should be considered as impossible, as soon as $f^2(x)$ is involved, for instance, if $l \geq 3$ and one considers the linear system with $i = 0, 1, \dots, l - 1$.

Therefore it becomes crucial to reduce the number of iterates involved in (2.41) as far as possible. A reduction of this number by 1 becomes in many cases crucial. One can imagine several ways to accomplish this.

(A) Take into account that, because of the reversibility $f^{-1}(x, \epsilon) = f(x, -\epsilon)$, the negative iterates f^{-i} are of the same complexity as f^i. Therefore, one can reduce the complexity of the functions involved in (2.41) by choosing $i_0 = -[l/2]$ instead of the naive choice $i_0 = 0$.

For instance, in the case $l = 3$ one should consider the system (2.41) with $i = -1, 0, 1$, and not with $i = 0, 1, 2$. However, already in the case $l = 4$ this simple recipe does not allow us to avoid considering f^2. In this case, the following way of dealing with the system (2.41) becomes useful.

(B) Set $c_l = -1$ and consider instead of the homogeneous system (2.41) of l equations the non-homogeneous system

$$(c_1\varphi_1 + c_2\varphi_2 + ... + c_{l-1}\varphi_{l-1}) \circ f^i(x) = \varphi_l \circ f^i(x), \quad i = i_0, i_0 + 1, \dots, i_0 + l - 2, \tag{2.42}$$

of $l - 1$ equations. Having found the (unique) solution $\big(c_1(x), \dots, c_{l-1}(x)\big)$, prove that these functions are integrals of motion, that is,

$$c_1(f(x)) = c_1(x), \quad \dots, \quad c_{l-1}(f(x)) = c_{l-1}(x). \tag{2.43}$$

Thus, for instance, in the case $l = 4$ one has to deal with the non-homogeneous system of equations (2.42) with $i = -1, 0, 1$. Unfortunately, even if one is able to solve this system symbolically, the task of a symbolic verification of eq. (2.43) might become very hard due to complexity of the solutions $\big(c_1(x), \dots, c_{l-1}(x)\big)$. When the map f is given implicitly by a polynomial system of the type

$$g_i(\widetilde{x}, x) = 0, \quad i = 1, \dots, n, \tag{2.44}$$

which we may solve explicitly for $\widetilde{x}$, then we may efficiently handle the above problem using the following method:

(G) In order to verify that a rational function $c(x) = p(x)/q(x)$ is an integral of motion of the map $\widetilde{x} = f(x)$ coming from a system (2.44):

i) find a Gröbner basis G of the ideal I generated by the components of eq. (2.44) considered as polynomials of $2n$ variables $x, \widetilde{x}$.

ii) check, via polynomial division through elements of G, whether the polynomial $\delta(x, \widetilde{x}) = p(\widetilde{x})q(x) - p(x)q(\widetilde{x})$ belongs to the ideal I.

An advantage of this method is that neither of its two steps needs the complicated explicit expressions for the map f. Nevertheless, both steps might be very demanding, especially the second step in case of a complicated integral $c(x)$. This method has been used, for instance, in [35], where the task of verifying the equations of the type (2.43) has been accomplished using the above method.

In some situations, a symbolic verification of eq. (2.43) can, however, be avoided by means of the following tricks.

(C) Solve system (2.42) for two different but overlapping ranges $i \in [i_0, i_0 + l - 2]$ and $i \in [i_1, i_1 + l - 2]$. If the solutions coincide, then eq. (2.43) holds automatically.

Indeed, in this situation the functions $\big(c_1(x), \ldots, c_{l-1}(x)\big)$ solve the system with $i \in [i_0, i_0 + l - 2] \cup [i_1, i_1 + l - 2]$ consisting of more than $l - 1$ equations.

A clever modification of this idea, which allows one to avoid solving the second system, is as follows.

(D) Suppose that the index range $i \in [i_0, i_0 + l - 2]$ in eq. (2.42) contains 0 but is non-symmetric. If the solution of this system $\big(c_1(x, \epsilon), \ldots, c_{l-1}(x, \epsilon)\big)$ is even with respect to ϵ, then eqs. (2.43) hold automatically.

Indeed, the reversibility of the map $f^{-1}(x, \epsilon) = f(x, -\epsilon)$ yields in this case that equations of the system (2.42) are satisfied for $i \in [-(i_0 + l - 2), -i_0]$, as well, and the intervals $[i_0, i_0 + l - 2]$ and $[-(i_0 + l - 2), -i_0]$ overlap but do not coincide, by condition.

The most powerful method of reducing the number of iterations to be considered is as follows.

(E) Often, the solutions $\big(c_1(x), \ldots, c_{l-1}(x)\big)$ satisfy some linear relations with constant coefficients. Find (observe) such relations *numerically.* Each such (still hypothetic) relation can be used to replace one equation in the system (2.42). Solve the resulting system *symbolically*, and proceed as in recipes (C) or (D) in order to verify eqs. (2.43).

The detection and identification of linear relations among the solutions $\big(c_1(x), \ldots, c_{l-1}(x)\big)$ can in most instances be simplified using the PLSQ algorithm. This will for instance be done in Chaper 6, where the above methods will be applied to the HK type discretization of the Clebsch System. A concrete example of how to identify a linear relation using the PSLQ algorithm is found in the appendix. Whenever we have explicit symbolic expressions for $c_i(x)$ at our disposal, we will usually use different methods for the identification of linear relations. Examples for these kinds of situations are found in Chapter 6, when we discuss the HK type discretizations of the Kirchhoff system and the Lagrange top.

2.4.5 Invariant Volume Forms for Integrable Birational Maps

Having found a suitably large number of independent integrals for a birational map, one can usually find an invariant volume form.

Definition 2.19. *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be birational. The differential $n-$form*

$$\omega = \frac{1}{\phi(x)} dx_1 \wedge \ldots \wedge dx_n$$

with some polynomial ϕ is called an invariant volume form for f, if ω is invariant under the pullback of f, i.e.

$$f^* \omega = \omega.$$

In other words:

$$\det Df(x) = \frac{\phi(f(x))}{\phi(x)}.$$

If a map $f : \mathbb{R}^n \to \mathbb{R}^n$ has an invariant volume form and $n-2$ integrals of motion, one may construct a Poisson structure for the map and thus prove its Poissonicity [13].

Theorem 2.3. [13] *Let $f : M \to M$ be a smooth mapping on the n-dimensional manifold M and let ω be an invariant volume form for f. Let $I_1, \ldots, I_{n-2}$ be independent integrals of motion for f, so that $dI_1 \wedge \ldots \wedge dI_{n-2} \neq 0$. Define τ as the dual n-vectorfield to ω, such that $\tau \rfloor \omega = 1$. Then, the bi-vectorfield $\sigma = \tau \rfloor dI_1 \rfloor \ldots \rfloor dI_{n-2}$ is an invariant Poisson structure for f.*

Let us consider the simplest case of this theorem when $n = 3$. If $\omega = 1/\phi(x) dx_1 \wedge dx_2 \wedge dx_3$ is a three-form, so that $f^*\omega = \omega$, then its dual tri-vectorfield τ is given by

$$\tau = \phi(x) \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial x_3}.$$

Contracting τ with the exterior derivative of the integral $I = I_1$ we obtain the invariant Poisson structure

$$\sigma = \tau \rfloor dI = \phi(x) \left(\frac{\partial I}{\partial x_3} \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2} + \frac{\partial I}{\partial x_1} \frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial x_3} + \frac{\partial I}{\partial x_2} \frac{\partial}{\partial x_3} \wedge \frac{\partial}{\partial x_1} \right).$$

In terms of coordinates, this means, that the map f is Poisson with respect to the bracket

$$\{x_1, x_2\} = \phi(x) \frac{\partial I}{\partial x_3}, \quad \{x_2, x_3\} = \phi(x) \frac{\partial I}{\partial x_1}, \quad \{x_3, x_1\} = \phi(x) \frac{\partial I}{\partial x_2}.$$

In most examples that we will encounter in this thesis, an invariant volume form for a map f with integrals $h_1 = p_q/q_1$,..., $h_d = p_d/q_d$ can be constructed by taking ϕ

as a power of one the numerators or denominators of the integrals h_i (or powers of factors of numeratos or denominators). This remarkable fact can in many examples be explained by considering the singular set S_{II} of f.

Let for instance $f : \mathbb{C}^n \to \mathbb{C}^n$ be birational with independent integrals $h_1 = p_1/q_1$,..., $h_{n-1} = p_{n-1}/q_{n-1}$. Following [39] the image of S_{II} under f must in general be contained in

$$\bigcap_{c \in \mathbb{C}^{n-1}} \overline{\{x \in \mathbb{C}^n \mid p_1(x) - c_1 q_1(x) = 0, \ldots, p_{n-1}(x) - c_{n-1} q_{n-1}(x) = 0\}} \tag{2.45}$$

$$= \{x \in \mathbb{C}^n \mid p_1(x) = q_1(x) = 0, \ldots, p_{n-1}(x) = q_{n-1}(x) = 0\}. \tag{2.46}$$

Because h_i are integrals, we have

$$p_i(f(x)) = p_i(x) R_i(x), \quad q_i(f(x)) = q_i(x) R_i(x),$$

for some rational functions R_i. Hence, if $x_0 \in S_{II}$, then $R_i(x_0) = 0$. Therefore, in the numerator of $R_i(x)$ there must appear a factor of the numerator of $\det Df(x)$. To construct an invariant volume form given by ϕ, it needs to have the property

$$\phi(f(x)) = \det Df(x) \phi(x).$$

Hence, it is reasonable to assume that one can obtain an invariant volume form for f by taking a suitable rational combination of p_i and q_i. For instance, if we have found d integrals for a map f where a polynomial q appears as a factor in all denominators of these d integrals, then q or q^k for some $k > 1$ typically is a suitable first ansatz for the density ϕ of a possible invariant volume form. We will encounter such examples in Chapters 4 and 6.

2.4.6 Summary

Concluding this section, we present a short summary of our findings regarding the detection and eventual proving of integrability of birational maps. This summary will be given in the form of a simple recipe. Let us hence assume that we are given some birational map f and that we would like to

1. obtain a conjecture whether f is integrable or not and
2. find an appropriate ansatz to compute its integrals of motion and an invariant Poisson structure.

These tasks may be accomplished by following this recipe:

1. Get a first estimate of the complexity of f. This can be accomplished by simply computing a reasonably large number of exact (rational) iterates. If the complexity of f is high, then computation times of higher iterates will most likely increase exponentially in time (See Section 4.2.3 for an example).
2. Depending on the size of the symbolic expressions, apply either (AE) or (DI) in order to confirm the first estimate.

3. If f passed either (AE) or (DI), apply the algorithm (V) in order to get a candidate for a HK-Basis.

4. Compute the integrals of f symbolically using the ansatz obtained in the previous step. This task may computation-wise be the most demanding part of this recipe. Be aware of the recipes discussed in the previous section.

5. Having found a number integrals, try to find an invariant measure. This can under certain circumstances be accomplished using the singular set approach outlined in the previous section.

6. If possible, use the invariant volume form to construct a Poisson structure using the contraction procedure from [13] outlined in the previous section.

3

Elements of the Theory of Elliptic Functions

As was mentioned earlier in Chapter 1, algebraically completely integrable systems may be solved exactly in terms of abelian functions. In the simplest case, when the genus of the spectral curve equals one, this means that solutions are given in terms of elliptic functions (or their relatives). Hence, we will now recall some of the basic facts from the theory of elliptic functions. Our focus will be on the Weierstrassian theory, the reason for this being its formal simplicity and also some computational aspects.

As we will see later, elliptic functions also appear as solutions of the HK type discretizations. Yet, the problem of how to determine elliptic solutions of discrete integrable systems is in general highly nontrivial. A possible way of how to approach this problem will be presented in the last section of this chapter. There we will encounter a general approach to the integration of birational maps having elliptic solutions. This approach uses HK bases in conjunction with experimental methods. The mechanism behind this approach is due to addition theorems and other relations satisfied by pairs of two elliptic functions of the same periods. These relations will be studied in the second section of this chapter.

3.1 Basic Theory

Definition 3.1. *Let $f:\mathbb{C}\to\mathbb{C}$ be meromorphic and v_1,$v_2\in\mathbb{C}$, such that their ratio is not a real number. f is called* **elliptic** *if for all $u\in\mathbb{C}$ it satisfies $f(u+v_1)=f(u)$ and $f(u+v_2)=f(u)$. An elliptic function hence is a doubly periodic meromorphic function.*

From this definition there follows immediately that for any elliptic function f there holds

$$f(u+mv_1+nv_2)=f(u),$$

for all $u\in\mathbb{C}$ and all integers m and n.

Any number $w\in\mathbb{C}$, such that $f(u+w)=f(w)$ for all $u\in\mathbb{C}$, is called a period of f. If there exist v_1, $v_2\in\mathbb{C}$ such that any other period w can be written as $w=mv_1+nv_2$ with two integers m and n, then v_1 and v_2 are called a fundamental pair of periods. Any elliptic function has a a fundamental pair of periods. This pair is, however, not unique. Given two fundamental periods v_1 and v_2, they form a parallelogram in the complex plane. The complex plane can thus be tesselated by translating this parallelogram over integer multiples of the two periods.

The fundamental properties of elliptic functions may be summarized in the following theorem.

Theorem 3.1. *1. Inside one fundamental parallelogram the number of zeros of an elliptic function is always equal to the number of its poles (counting multiplicities).*

2. An elliptic function may be characterized up to a multiplicative constant by giving its zeros, poles, and periods.

3. The sum of the residues with respect to all poles inside one fundamental parallelogram is zero.

4. The sum of the all poles inside one fundamental parallelogram is equal to the sum of all zeros inside the same parallelogram.

5. Every nonconstant elliptic function has at least two poles inside one fundamental parallelogram.

Definition 3.2. *The number of poles of an elliptic function (counting multiplicities) is called the order of f and is denoted by $\mathrm{ord} f$.*

There are in principle two approaches with which one could construct concrete examples of elliptic functions. The first (older) approach is due to Jacobi and makes use of theta functions. The second approach goes back to Weierstrass. For our purposes it will be useful to follow the Weierstrass approach.

For the remainder of this section we assume that we are given two complex numbers ω_1 and ω_2 which are independent, that is their ratio has a non zero imaginary part.

Definition 3.3. *The function*

$$\wp(z,\omega_1,\omega_2) = \frac{1}{z^2} + \sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ (m,n)\neq 0}} \left\{ \frac{1}{(z-2m\omega_1-2n\omega_2)^2} - \frac{1}{(2m\omega_1+2n\omega_2)^2} \right\}$$

is called the Weierstrass $\wp$ function with half-periods ω_1, ω_2. We will usually supress the dependence on the half-periods and simply write $\wp(z)$.

Proposition 3.1. *The function $\wp$ has the following properties:*

1. It has a double pole at $2m\omega_1 + 2n\omega_2$ with zero residues.

2. It is a second order elliptic function with fundamental periods $2\omega_1$ and $2\omega_2$.

3. It is an even function.

4. Its derivate $\wp'$ is an odd elliptic function of third order.

5. It satisfies the differential equation

$$\wp'(z)^2 = 4\wp(z) - g_2\wp(z) - g_3,$$

where

$$g_2 = 60 \sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ (m,n)\neq 0}} \frac{1}{(2m\omega_1 + 2n\omega_2)^4}, \quad g_3 = 140 \sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ (m,n)\neq 0}} \frac{1}{(2m\omega_1 + 2n\omega_2)^6}.$$

g_2, g_3 are called (Weierstrass) invariants.

6. *The periods of $\wp$ may be obtained from g_2 and g_3, so that $\wp$ may be fully characterized by giving g_2 and g_3.*
7. *The field of elliptic functions is generated by $\wp$ and $\wp'$. This means that any elliptic function may be written as a rational function of $\wp$ and $\wp'$.*

We define two more functions which may be used to construct elliptic functions, given either their poles and residues, or their poles and zeros.

Definition 3.4. *The function*

$$\sigma(z,\omega_1,\omega_2) = \prod_{\substack{(m,n)\in\mathbb{Z}^2 \\ (m,n)\neq 0}} \left(1 - \frac{z}{(2m\omega_1 + 2n\omega_2)}\right) \exp\left(\frac{z}{(2m\omega_1 + 2n\omega_2)} + \frac{z^2}{2(2m\omega_1 + 2n\omega_2)^2}\right)$$

is called the Weierstrass σ-function with half-periods ω_1, ω_2. We will usually supress the dependence on the half-periods and simply write $\sigma(z)$.

Definition 3.5. *The function*

$$\zeta(z,\omega_1,\omega_2) = \frac{1}{z} + \sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ (m,n)\neq 0}} \left\{\frac{1}{(z - 2m\omega_1 - 2n\omega_2)} + \frac{1}{(2m\omega_1 + 2n\omega_2)} + \frac{z}{(2m\omega_1 + 2n\omega_2)^2}\right\}$$

is called the Weierstrass ζ-function with half-periods ω_1, ω_2.

The fundamental properties of σ and ζ are summarized in the following two theorems.

Theorem 3.2. *For the function ζ the following statements hold:*

1. *It has simple poles at $2m\omega_1 + 2n\omega_2$ with residues equal to one.*
2. *It is an odd function.*
3. *$\zeta'(z) = -\wp(z)$.*
4. *$\zeta(z + 2\omega_i) = \zeta(z) + 2\zeta(\omega_i)$.*

Theorem 3.3. *For the function σ the following statements hold:*

1. *It has simple zeros at $2m\omega_1 + 2n\omega_2$.*

2. *It is an odd function.*
3. $\zeta(z) = \sigma'(z)/\sigma(z)$.
4. $\sigma(z + 2\omega_i) = -\sigma(z)\exp(2\eta_i(z + \omega_i))$.

With the help of the σ-function we may construct an elliptic functions starting from its zeros and poles.

Theorem 3.4. *Suppose that f is an elliptic function with periods $2\omega_1$, $2\omega_2$. Let $z_1,\ldots,z_n$ denote its zeros and $p_1,\ldots,p_n$ its poles (counting multiplicities), chosen such that*

$$z_1 + \ldots + z_n = p_1 + \ldots + p_n.$$

Up to a multiplicative constant $C \in \mathbb{C}$, f is then given by

$$f(z) = \frac{\sigma(z - z_1) \cdot \ldots \cdot \sigma(z - z_n)}{\sigma(z - p_1) \cdot \ldots \cdot \sigma(z - p_n)}.$$

Similarly, one may use to the ζ-function to construct an elliptic function:

Theorem 3.5. *Suppose that f is an elliptic function with periods $2\omega_1$, $2\omega_2$. Let $p_1,\ldots,p_n$ denote its poles and assume that all p_i are disctinct, so that f has simple poles only. Up to an additive constant $C \in \mathbb{C}$, f is then given by*

$$f(z) = r_1\zeta(t - p_1) + r_1\zeta(t - p_1)\ldots + r_n\zeta(t - p_n),$$

where $r_i = res_{p_i}(f)$.

Like all elliptic functions and their relatives the Weierstrass family of functions satisfies an enormous amount of functional identities. The most fundamental identity for the σ-function is the celebrated three-term identity:

$$\begin{aligned}\sigma(z + a)\sigma(z - a)\sigma(b + c)\sigma(b - c) + \sigma(z + b)\sigma(z - b)\sigma(c + a)\sigma(c - a)\\ +\sigma(z + c)\sigma(z - c)\sigma(a + b)\sigma(a - b) = 0.\end{aligned} \tag{3.1}$$

Differentiating this identity we obtain the following formula for the ζ-function:

$$\zeta(a) + \zeta(b) + \zeta(c) - \zeta(a + b + c) = \frac{\sigma(a + b)\sigma(b + c)\sigma(c + a)}{\sigma(a)\sigma(b)\sigma(c)\sigma(a + b + c)}. \tag{3.2}$$

The ζ-function can be related to $\wp$ via

$$\frac{1}{2}\frac{\wp'(u) - \wp'(v)}{\wp(u) - \wp(v)} = \zeta(u + v) - \zeta(u) - \zeta(v). \tag{3.3}$$

Furthermore, one may derive

$$\wp(u + \nu) + \wp(u) + \wp(v) = \frac{1}{4}\left(\frac{\wp'(u) - \wp'(u)}{\wp(u) - \wp(v)}\right)^2, \tag{3.4}$$

which is the well known addition-formula for the $\wp$-function. Taking together the last two identities, we obtain the so called Frobenius-Stickelberger formula,

$$(\zeta(x) + \zeta(y) + \zeta(z))^2 = \wp(x) + \wp(y) + \wp(z), \tag{3.5}$$

which holds if $x + y + z = 0$. Eventually, we mention the remarkable formula

$$\wp(u) - \wp(v) = \frac{\sigma(v-u)\sigma(v+u)}{\sigma^2(v)\sigma(u)}. \tag{3.6}$$

The reason for the existence of the multitude of functional relations encountered in the theory of elliptic functions stems from the fact that any elliptic function satisfies an addition theorem. This behaviour and its implications for our work will discussed in the following sections.

3.2 Relations Between Elliptic Functions And Addition Theorems

A well known classical result (see for instance section 20.54 in [61]) in the theory of elliptic functions is the fact that any two elliptic functions with the same periods satisfy an algebraic relation. Specifically, we have the following theorem.

Theorem 3.6. *Let f and g be two elliptic functions with the same periods, such that* $\operatorname{ord} f = n$ *and* $\operatorname{ord} g = m$. *Then there exists an algebraic relation of the form*

$$P(f, g) = 0,$$

with an irreducible bivariate polynomial $P(X, Y)$ satisfying

$$\deg_X P \leq m, \quad \deg_Y P \leq n, \quad \deg P \leq n + m.$$

The coefficients of P are unique up to multiplication with a scalar.

Proof. First, write f and g in terms of $\wp$ and $\wp'$, so that one obtains the three equations

$$f = R_1(\wp, \wp'), \quad g = R_2(\wp, \wp'), \quad \wp'^2 = 4\wp^3 - g_2\wp - g_3,$$

with some rational functions R_1, R_2. Eliminating $\wp$, $\wp'$ from these three equations, leaves one polyomial equation for f and g. The first part of the theorem is thus proven. For the second part, consider the following: For any value $z = f(u)$ there correspond n values u_i of u and thus n values $g(u_i) = w_i$. Also, for any value $w = g(u)$ there correspond m values u_i of u and thus m values $g(u_i) = z_i$. Hence, if we fix some value z, then there exist n values w_i, such that

$$P(z, w_i) = 0, \quad i = 1..n,$$

and if we fix some value w, then there exist m values z_i such that

$$P(z_i, w) = 0, \quad i = 1..m.$$

Hence, P, considered as a univariate polynomial in w, has n roots and if we consider it as a polynomial in z, it possesses m roots. Hence, $\deg_X P \leq m$ and $\deg_Y P \leq n$. $\square$

Of course, the above relation may be of lower degree, as is for instance the case for $f = \wp$ and $g = \wp'$ or $f = \wp$ and $g = \wp^2$. This behavior is a common phenomenon related to shared poles of f and g. If one imposes additional conditions on the poles of f and g, we may obtain lower degree bounds for P, indeed. In particular, we have the following theorem.

Theorem 3.7. *Let f and g be two elliptic functions with the same periods, each of them of of order n and having pairwise distinct simple poles. If f and g have k common poles, then* $\deg P \le 2n - k$.

Proof. We count the number of independent conditions on the coefficients of P which are required to "kill" all poles of the nonconstant part of P (so that the Liouville theorem applies). These conditions always form a linear homogeneous system of equations satisfied by coefficients of P. We start by investigating the simplest case and proceed inductively. Let $k = 1$ and denote the common pole of f and g by p_1. The expression $P(f(u), g(u))$ has exactly one term which has a singularity at $u = p_1$ of order $2n$. The coefficient of P at this term must hence be zero. Hence, we have $\deg P \le 2n-1$. Now, let $k = 2$ and denote the common pole of f and g by p_1 and p_2. Since f and g have at least one common pole, we have that $\deg P \le 2n-1$. The expression $P(f(u), g(u))$ has exactly two terms which have a singularity at $u = p_1$ and $u = p_2$ of order $2n-1$. Hence we get two independent homogeneous linear equations satisfied by the two coefficients. Hence, they must be zero. Thus, $\deg P \le 2n - 2$. Since for arbitrary k, the maximal number of monomials in P of total degree $2n - k$ is equal to k, we may repeat this argument until $k = n$. □

Since any two elliptic functions with the same periods satisfy some algebraic relation, it follows that any elliptic function satisfies an algebraic differential equation, i.e. a polynomial relation between an elliptic function x and its derivative $\dot{x}$. A particular example for this is the following theorem by Halphen [27].

Theorem 3.8. *The general solution of the differential equation*

$$\dot{y}^2 = \alpha_0 y^4 + 4\alpha_1 y^3 + 6\alpha_2 y^2 + 4\alpha_3 y + \alpha_4 \tag{3.7}$$

is given by the (time shifts of) the second order elliptic function

$$y(t) = -\frac{\alpha_1}{\alpha_0} + \zeta(u+v) - \zeta(u) - \zeta(v) = -\frac{\alpha_1}{\alpha_0} + \frac{1}{2} \cdot \frac{\wp'(u) - \wp'(v)}{\wp(u) - \wp(v)}, \quad u = \sqrt{\alpha_0} t. \tag{3.8}$$

Here the invariants of the Weierstrass $\wp$-function are given by

$$g_2 = \frac{\alpha_0\alpha_4 - 4\alpha_1\alpha_3 + 3\alpha_2^2}{\alpha_0^2}, \quad g_3 = \frac{\alpha_0\alpha_2\alpha_4 + 2\alpha_1\alpha_2\alpha 3 - \alpha_2^3 - \alpha_0\alpha_3^2 - \alpha_1^2\alpha_4}{\alpha_0^3}, \tag{3.9}$$

while the point v of the corresponding elliptic curve is determined by the relations

$$\wp(v) = \frac{\alpha_1^2 - \alpha_0\alpha_2}{\alpha_0}, \quad \wp'(v) = \frac{\alpha_3\alpha_0^2 - 3\alpha_0\alpha_1\alpha_2 + 2\alpha_1^3}{\alpha_0^3}. \tag{3.10}$$

If f is an elliptic function, then $\tilde{f}(t) = f(t+h)$ for an arbitrary $h \in \mathbb{C}$ is also elliptic with the same periods. Hence, f and $\tilde{f}$ are also connected by a polynomial relation. We call such a relation an addition theorem. The situation, where $\operatorname{ord} f = 2$, is of particular interest.

Theorem 3.9. *Let f be an elliptic function of order two. Then, for arbitrary $h \in \mathbb{C}$, $f(t)$ and $\tilde{f}(t) = f(t+h)$ satisfy an an algebraic relation of the form*

$$P(f, \tilde{f}) = 0,$$

where P is a symmetric, biquadratic polynomial.

A concrete example can be obtained when considering the $\wp$-function. If we take Eq. (3.4) and eliminate all derivatives via $\wp'^2 = 4\wp^3 - g_2\wp - g_3$, we obtain the equation

$$\left(XY + YZ + ZX + \frac{g_2}{4}\right)^2 - 4(XYZ - g_3)(X + Y + Z) = 0, \tag{3.11}$$

where

$$X = \wp(x), \quad Y = \wp(y), \quad Z = \wp(z),$$

such that $x + y + z = 0$. Setting $z = h$ we obtain the symmetric biquadratic relation for $\wp$. Similar results exist for the Jacobian elliptic function sn [6]. The following theorem is the converse statement of the previous theorem.

Theorem 3.10. *Let $P(X, Y)$ be a symmetric, biquadratic polynomial. Then, the curve*

$$\mathcal{C} = \big\{(X, Y) \in \mathbb{C}^2 \mid P(X, Y) = 0\big\}$$

has genus one and may be parametrized by

$$X = f(t), \quad Y = f(t + h),$$

with some shift $h \in \mathbb{C}$ and a second order elliptic function f.

This result may be traced back to the work of Leonhard Euler. One of the first known applications of these results to the theory of discrete integrable systems is due to R. Baxter [6].

Suppose that we would like to determine invariants characterizing the elliptic curve corresponding to the relation

$$P(y, \widetilde{y}) = \alpha_0 y^2 \widetilde{y}^2 + \alpha_1 y\widetilde{y}(y + \widetilde{y}) + \alpha_2(y^2 + \widetilde{y}^2) + \alpha_3 y\widetilde{y} + \alpha_4(y + \widetilde{y}) + \alpha_5 = 0,$$

where $\widetilde{y}(t) = y(t + 2\epsilon)$. We already know, that y and $\widetilde{y}$ are up to time shifts given by a second order elliptic function f. Now, we would like to determine the corresponding Weierstrass invariants g_2 and g_3 in analogy to Theorem 3.8. For this aim we consider the system of differential equations

$$\begin{aligned} \dot{y} &= \frac{\partial P(y, \widetilde{y})}{\partial \widetilde{y}}, \\ \dot{\widetilde{y}} &= -\frac{\partial P(y, \widetilde{y})}{\partial y}, \end{aligned}$$

which is a Hamiltonian system with Hamilton function

$$H(y,\widetilde{y}) = \alpha_0 y^2\widetilde{y}^2 + \alpha_1 y\widetilde{y}(y+\widetilde{y}) + \alpha_2(y^2+\widetilde{y}^2) + \alpha_3 y\widetilde{y} + \alpha_4(y+\widetilde{y}).$$

Using H we may eliminate either y or $\widetilde{y}$ and obtain

$$\dot{y}^2 = P_4(y), \quad \dot{\widetilde{y}}^2 = P_4(\widetilde{y}), \tag{3.12}$$

where

$$\begin{aligned} P_4(y) &= (\alpha_1^2 - 4\alpha_0\alpha_2)y^4 + (2\alpha_1\alpha_3 - 4\alpha_0\alpha_4 - 4\alpha_1\alpha_2)y^3 \\ &\quad + (4\alpha_0 H - 4\alpha_2^2 + \alpha_3^2 - 2\alpha_1\alpha_4)u^2 + (2\alpha_3\alpha_4 + 4\alpha_1 H - 4\alpha_2\alpha_4) + 4\alpha_2 H + \alpha_4^2. \end{aligned}$$

At this point we may apply Theorem 3.8 to (3.12) and determine g_2 and g_3 in terms of α_i, once one has fixed the value of $H = -\alpha_5$. For a more extensive treatment of the uniformization problem for biquadratic curves we refer to the monograph [19].

The genus of a a curve of higher degree, i.e. $\mathcal{C} = \big\{(X,Y) \in \mathbb{C}^2 \mid P(X,Y) = 0\big\}$, where $\deg P > 3$, is in general not equal to one, so that it may not be parametrized by elliptic functions. Of course, there exist curves of higher degree which may be parametrized by elliptic functions. We will in fact encounter such curves in later chapters. For our purposes it will, however, not be necessary to know how one could exactly parametrize these curves in terms of elliptic functions. Usually we will be content with knowing that a particular curve has genus one. From the knowledge of the curve's degree we will then be able to deduce further information on the elliptic functions which parametrize the particular curve in question. Regarding the general problem of the parametrization of genus one curves the reader is referred to Clebsch's classical treatments [16] and [15].

3.3 Elliptic Functions, Experimental Mathematics And Discrete Integrability

This section is meant to be a synthesis of the concepts introduced in this chapter and the previous one. We will demonstrate how one can use the HK bases approach in order to systematically obtain explicit solutions for discrete integrable systems given by birational maps, provided solutions are given in terms of elliptic functions. The appealing features of this approach are the following:

1. It is systematic: guessing ansätze for integrals of motion or explicit solutions can be avoided to a large extent.

2. We do not look for or try to construct additional integrable structures (for instance Lax pairs), a process which would usually also require large amounts of guesswork and/or research experience.

We sketch the essentials of this method. Concrete examples will be discussed in Chapters 5 and 6. To better understand what is to follow, we recall the classical way of integrating a system of ordinary differential equations

$$\dot{x} = g(x), \tag{3.13}$$

which has a number of conserved quantities and whose solutions are elliptic functions. In this situation we know that any component x_i satisfies an algebraic differential equation of the type

$$P_i(x_i, \dot{x}_i) = 0, \tag{3.14}$$

with some polynomial P whose degree depends on the order of x. Moreover, if all components x_i are elliptic with respect to the same period lattice, any two functions x_i and x_j will satisfy a polynomial relation

$$Q_{ij}(x_i, x_j) = 0. \tag{3.15}$$

The relations (3.15) can be obtained by considering the integrals of motion of (3.13) followed by algebraic manipulations. With the help of these relations one would then try to eliminate as many variables and their derivatives as possible to try to find the relations (3.14). Eventually, one can then find explicit expressions by inversion of the elliptic integrals appearing in (3.14) when solving for $\dot{x}_i$.

We try to adopt the classical approach to the case of integrable birational maps and assume that we are given a birational map f on the phase space $\mathbb{R}^n$ with coordinates x_i. We want to test whether it is integrable and solvable in terms of elliptic functions. Furthermore, we want to obtain an Ansatz for explicit solutions. This will be made possible by trying to find invariance relations similar to (3.14) and (3.15). In principle, this would be possible by direct algebraic manipulation of the equations defining f, yet typically these expressions are much more complex (in terms of the size of the involved expressions) when compared to the continuous setting. Hence, except for some very simple examples, one is usually not able to perform all the neccessary computations by hand or even by using a symbolic manipulator like MAPLE. Therefore, the only feasible way to continue remains in most cases the experimental way. One should note here that this is a typical situation in experimental mathematics.

As a first step we run the algorithm (V). Assume that this enables us to find a HK-basis describing the invariant manifolds of f given by $\Phi = (\phi_1, \dots, \phi_l)$, where ϕ_i are polynomials in x and $\dim K_\Phi(x) = d$. Assuming that d is the "correct" dimension of the invariant varieties of f, we now like to get an idea whether f can be solved in terms of elliptic functions. From the previous section, we know that, if f can indeed be solved in terms of elliptic functions, then $(x_i, \widetilde{x}_i) = (x_i(t), x_i(t+h))$ considered as functions of the discrete time $t \in h\mathbb{Z}$ will satisfy a polynomial relation. This relation can be detected numerically by the algorithm (V). Hence, we now run the algorithm (V), but this time apply it to the "map" $(x_i, \widetilde{x}_i) \mapsto (\widetilde{x}_i, \widetilde{\widetilde{x}}_i)$. If the map f is solvable in terms of elliptic functions, then this will give us for each coordinate x_i a polynomial relation of the type

$$P(x_i, \widetilde{x}_i) = 0.$$

The degree of P in x_i and $\widetilde{x}_i$ must be the same and is equal to the order of the elliptic function x_i. At this point one should find out whether all functions x_i are elliptic with the same periods. For this aim one should compute the invariants g_2, g_3 of all the curves given by the above relations. If the absolute invariants for all curves coincide, then all x_i are elliptic functions with the same periods. The computation of the invariants g_2, g_3 may easily be accomplished using algorithms by van Hoeij[1] [54].

Now we may assume that all x_i are given by elliptic functions with the same periods. Furthermore, for the sake of simplicity we suppose that their order is the same for all x_i. In order to characterize the elliptic functions x_i further, we have to gather information about their poles and zeros. This can be achieved by investigating the relations among the elliptic functions x_i. Let us assume that ord $x_i = k$ for all i. From the HK-basis Φ we may derive the relations among the x_i, in particular we can obtain relations of the form

$$Q(x_i, x_j) = 0.$$

The degree of Q now tells us whether x_i and x_j have a number of common poles. In particular, if $\deg Q = 2k - m$, then x_i and x_j must have m common poles. Similarly, we may find information about possible common zeros of x_i and x_j. For this aim we investigate the relations among $1/x_i$ and $1/x_j$ and proceed similarly. After successive application of this method, that is by finding as many invariance relations as possible and analyzing them in the light of information about zeros and poles of x_i, we will eventually obtain enough information about the solutions in order to fully characterize poles and zeros of x_i. As we have seen, x_i are then fully characterized up to their multiplicative constants. We will see later that a complete solution of f in terms of elliptic functons is then relatively easy to find and verify using rigorous mathematical analysis.

[1]Implementations of these algorithms are included in MAPLE's *algcurves* package

4

The Hirota-Kimura Type Discretizations

The discretization method studied in this thesis seems to be introduced in the geometric integration literature by W. Kahan in the unpublished notes [34]. It is applicable to any system of ordinary differential equations for $x : \mathbb{R} \to \mathbb{R}^n$ with a quadratic vector field:

$$\dot{x} = g(x) = Q(x) + Bx + c, \tag{4.1}$$

where each component of $Q : \mathbb{R}^n \to \mathbb{R}^n$ is a quadratic form, while $B \in \mathrm{Mat}_{n\times n}$ and $c \in \mathbb{R}^n$. Kahan's discretization reads as

$$\frac{\widetilde{x} - x}{\epsilon} = Q(x, \widetilde{x}) + \frac{1}{2}B(x + \widetilde{x}) + c, \tag{4.2}$$

where

$$Q(x, \widetilde{x}) = \frac{1}{2}\left[Q(x + \widetilde{x}) - Q(x) - Q(\widetilde{x})\right]$$

is the symmetric bilinear form corresponding to the quadratic form Q. Here and below we use the following notational convention which will allow us to omit a lot of indices: for a sequence $x : \mathbb{Z} \to \mathbb{R}$ we write x for x_k and $\widetilde{x}$ for x_{k+1}. Eq. (4.2) is *linear* with respect to $\widetilde{x}$ and therefore defines a *rational* map $\widetilde{x} = f(x, \epsilon)$. Clearly, this map approximates the time-ϵ-shift along the solutions of the original differential system, so that $x_k \approx x(k\epsilon)$. (Sometimes it will be more convenient to use 2ϵ for the time step, in order to avoid appearance of various powers of 2 in numerous formulas.) Since eq. (4.2) remains invariant under the interchange $x \leftrightarrow \widetilde{x}$ with the simultaneous sign inversion $\epsilon \mapsto -\epsilon$, one has the *reversibility* property

$$f^{-1}(x, \epsilon) = f(x, -\epsilon). \tag{4.3}$$

In particular, the map f is *birational.* Probably unaware of the work by Kahan, this scheme was first applied to integrable systems, namely the Euler top and the Lagrange top, by Hirota and Kimura [30, 35]. Since we will be studying Kahan's scheme in an "integrable" context , we hence adopt the name *Hirota-Kimura type discretizations.*

When Hirota and Kimura applied Kahan's scheme to the Euler Top [30] and the Lagrange Top [35] they obtained in both cases integrable maps. The derivation of their results was, however, rather cryptic and almost incomprehensible. Hence, a lot of researchers ignored these results. At the 2006 Oberwolfach Meeting "Geometric Numerical Integration" T. Ratiu [48] then presented the two claims that the Kahan discretization of both the Clebsch System and the Kovalevskaia system were also integrable. While the second claim turned out to be wrong, the first claim turned out to be correct. This lead to a greater interest in the Kahan discretizations of integrable systems. In particular, it turned out that in most cases Kahan's scheme produced new integrable mappings, when it was applied to algebraically integrable systems with a quadratic vector field. At this point, however, it still remains a mystery, as to what underlying structures are responsible for this behavior.

Before discussing the "integrable" aspects of the HK type scheme, we mention some of its general properties, which follow directly from the definitions.

Proposition 4.1. *1. The scheme (4.2) is of order 2, i.e.*

$$\widetilde{x} = x + \epsilon g(x) + \frac{1}{2}\epsilon^2 Dg(x)g(x) + \mathcal{O}(\epsilon^3),$$

so that one time step of Kahan's scheme coincides with the flow of (4.1) up to the second order [1].

2. *For linear systems, i.e. if $Q = 0$, the scheme (4.2) coincides with the implicit midpoint rule applied to (4.1). Thus, if (4.1) is a canonical, linear Hamiltonian system, then the map $f : x \mapsto \widetilde{x}$ obtained from (4.2) is symplectic. Moreover, if a linear system of the form (4.1) has a quadratic conserved quantity, then this quantity is preserved*[2] *by f.*

3. *One time step of the scheme (4.2) can be interpreted as one Newton iteration applied to both the implicit midpoint rule or the implicit trapezoidal rule [60].*

Proposition 4.1. *Any map $x \mapsto \widetilde{x}$ obtained from (4.2) with time step 2ϵ can be put in matrix form as*

$$\widetilde{x} = A^{-1}(x,\epsilon)\left(x + \epsilon B x + \epsilon c\right), \quad A(x,\epsilon) = \left(I - \epsilon Dg(x)\right).$$

For the Jacobian of $x \mapsto \widetilde{x}$ there holds the formula

$$\det \frac{\partial \widetilde{x}}{\partial x} = \frac{\det A(\widetilde{x}, -\epsilon)}{\det A(x,\epsilon)}. \tag{4.4}$$

Proof. The first statement follows directly from (4.2) because of

$$Q(x,\widetilde{x}) = \frac{1}{2}\big(DQ(x)\widetilde{x} + DQ(\widetilde{x})x\big) = DQ(x)\widetilde{x}.$$

Differentiating (4.2) and considering that

$$\frac{\partial}{\partial x}Q(x,\widetilde{x}) = \frac{\partial}{\partial x}DQ(x)\widetilde{x} = DQ(\widetilde{x}) + DQ(x)\frac{\partial \widetilde{x}}{\partial x},$$

one obtains

$$\frac{\partial \widetilde{x}}{\partial x} - I = \epsilon DQ(\widetilde{x}) + \epsilon DQ(x)\frac{\partial \widetilde{x}}{\partial x} + \epsilon B + \epsilon B\frac{\partial \widetilde{x}}{\partial x}.$$

Solving for $\partial\widetilde{x}/\partial x$ then gives $\partial\widetilde{x}/\partial x = A(x,\epsilon)^{-1}A(\widetilde{x},-\epsilon)$, which implies (4.4). □

[1] This is, of course, a direct consequence of the reversibility property (4.3).

[2] One may refer to [25] for more information about the structure preservation of the implicit midpoint rule and related numerical integrators.

Another interesting aspect is the following.

Proposition 4.2. *Let $B = 0$ and $c = 0$. The map $f : x \mapsto \widetilde{x}$ obtained from (4.2) can then be written as a product of two involutions: Let $I_1 : x \mapsto x^+$ be defined by*

$$x^+ + x = \epsilon Q(x, x^+) \tag{4.5}$$

and $I_2(x) = -x$. Then $I_1(I_1(x)) = x$ and $\widetilde{x} = I_2(I_1(x))$.

Proof. The equation

$$x^+ + x = \epsilon Q(x, x^+)$$

is symmetric w.r.t the interchange $x^+ \leftrightarrow x$ and for fixed x uniquely solvable for x^+, thus I_1 is an involution. The remaining statements follow directly from the definition of $x \mapsto \widetilde{x}$. □

This result is already implicitly present in the work of Jonas. In [33] an involution of the type (4.5) is studied, which admits for a nice geometrical interpretation. In particular, Jonas studied the involution

$$x^+ + x + yz^+ + zy^+ = 0, \quad y^+ + y + zx^+ + xz^+ = 0, \quad z^+ + z + xy^+ + yx^+ = 0,$$

where (x, y, z) and (x^+, y^+, z^+) are the cosines of the side lengths of two spherical triangles with complementary angles. Moreover, Jonas showed that this involution could be "integrated" in terms of elliptic functions. One could hence consider this involution as one of the first examples of an integrable mapping.

One of the first applications [35] of Kahan's scheme was to the famous Lotka-Volterra system modelling the interacton of two species, one being predators and the other one their prey. The equations of motion in this case read

$$\dot{x} = x(1 - y), \qquad \dot{y} = y(x - 1).$$

This system is Poisson with respect to the Poisson structure

$$\{x, y\} = xy, \tag{4.6}$$

and possesses the conserved quantity

$$H(x, y) = x + y - \log(xy).$$

The scheme (4.2) applied to this system then gives

$$(\widetilde{x} - x)/\epsilon = (\widetilde{x} + x) - (\widetilde{x}y + x\widetilde{y}), \qquad (\widetilde{y} - y)/\epsilon = (\widetilde{x}y + x\widetilde{y}) - (\widetilde{y} + y), \tag{4.7}$$

and defines an explicit birational map $(x, y) \mapsto (\tilde{x}, \tilde{y})$. Plotting orbits of this map suggests, that it has favorable numerical properties when being compared with more standard methods. One should note here the nonspiralling solutions of Kahan's scheme which correspond to the existence of the conserved quantity H of the continuous equations of motion. This behavior can somewhat be explained by the fact that the Kahan's

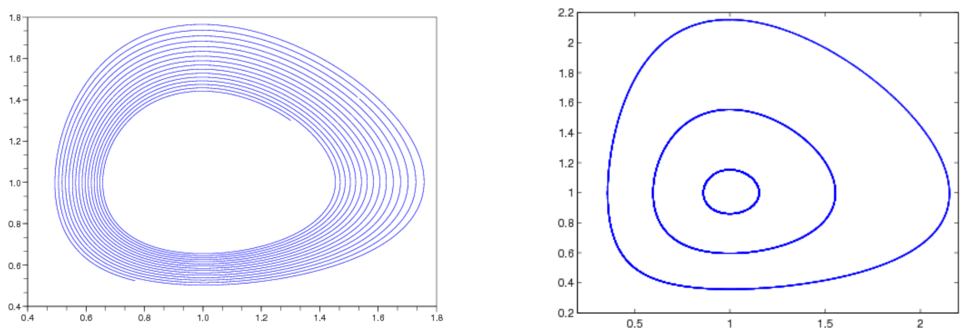


Figure 4.1: Left: explicit Euler method, $\epsilon = 0.01$ applied to the Lotka-Volterra equations. Right: Kahan's discretization, $\epsilon = 0.1$.

map is Poisson w.r.t. the Poisson structure (4.6) [50]. This can equivalently be formulated by saying that the map 4.7 preserves the invariant measure form

$$\omega = \frac{1}{xy} dx \wedge dy.$$

This statement can be generalized to two bigger classes of equations. The first class reads:

$$\dot{x}_i = \sum_{j=1}^{N} a_{ij} x_j^2 + c_i, \quad 1 \le i \le N, \tag{4.8}$$

with a skew-symmetric matrix $A = (a_{ij})_{i,j=1}^N = -A^{\mathrm{T}}$. Kahan's discretization reads:

$$\widetilde{x}_i - x_i = \epsilon \sum_{j=1}^{N} a_{ij} x_j \widetilde{x}_j + \epsilon c_i, \quad 1 \le i \le N. \tag{4.9}$$

Proposition 4.2. *The map $\widetilde{x} = f(x, \epsilon)$ defined by equations (4.9) has an invariant volume form:*

$$\det \frac{\partial \widetilde{x}}{\partial x} = \frac{\phi(\widetilde{x}, \epsilon)}{\phi(x, \epsilon)} \quad \Leftrightarrow \quad f^* \omega = \omega, \quad \omega = \frac{dx_1 \wedge \ldots \wedge dx_N}{\phi(x, \epsilon)}, \tag{4.10}$$

where $\phi(x, \epsilon) = \det(I - \epsilon A X)$ with $X = \operatorname{diag}(x_1, \ldots, x_N)$ is an even polynomial in ϵ.

Proof. Equations (4.9) can be put as

$$\widetilde{x} = A^{-1}(x, \epsilon)(x + \epsilon c), \qquad A(x, \epsilon) = I - \epsilon A X. \tag{4.11}$$

Due to formula (4.4) it remains to notice that $\det A(x, \epsilon) = \det A(x, -\epsilon)$. Indeed, due to the skew-symmetry of A, we have: $\det(I - \epsilon A X) = \det(I - \epsilon X^{\mathrm{T}} A^{\mathrm{T}}) = \det(I + \epsilon X A) = \det(I + \epsilon A X)$. $\square$

The second class consists of equations of the Lotka-Volterra type:

$$\dot{x}_i = x_i\left(b_i + \sum_{j=1}^{N} a_{ij}x_j\right), \quad 1 \le i \le N, \tag{4.12}$$

with a skew-symmetric matrix $A = (a_{ij})_{i,j=1}^{N} = -A^{\mathrm{T}}$. The Kahan's discretization (with the stepsize 2ϵ) reads:

$$\widetilde{x}_i - x_i = \epsilon b_i(x_i + \widetilde{x}_i) + \epsilon\sum_{j=1}^{N} a_{ij}(x_i\widetilde{x}_j + \widetilde{x}_i x_j), \quad 1 \le i \le N, \tag{4.13}$$

Proposition 4.3. *The map $\widetilde{x} = f(x,\epsilon)$ defined by equations (4.13) has an invariant volume form:*

$$\det\frac{\partial\widetilde{x}}{\partial x} = \frac{\widetilde{x}_1\widetilde{x}_2\cdots\widetilde{x}_N}{x_1x_2\cdots x_N} \quad \Leftrightarrow \quad f^*\omega = \omega, \quad \omega = \frac{dx_1\wedge\ldots\wedge dx_N}{x_1x_2\cdots x_N}. \tag{4.14}$$

Proof. Equations (4.13) are equivalent to

$$\frac{\widetilde{x}_i}{1+\epsilon b_i + \epsilon\sum_{j=1}^{N} a_{ij}\widetilde{x}_j} = \frac{x_i}{1-\epsilon b_i - \epsilon\sum_{j=1}^{N} a_{ij}x_j} =: y_i. \tag{4.15}$$

We denote $d_i(x,\epsilon) = 1 - \epsilon b_i - \epsilon\sum_{j=1}^{N} a_{ij}x_j$. In the matrix form equation (4.13) can be put as

$$\widetilde{x} = A^{-1}(x,\epsilon)(I+\epsilon B)x, \tag{4.16}$$

where the i-th diagonal entry of $A(x,\epsilon)$ equals $d_i(x,\epsilon)$, while the ij-th off-diagonal entry equals $-\epsilon x_i a_{ij}$. In other words, $A(x,\epsilon) = D(I-\epsilon YA)$, where $D = D(x,\epsilon) = \mathrm{diag}(d_1,\ldots,d_N)$ and $Y = \mathrm{diag}(y_1,\ldots y_N)$. Formula (4.4) holds true also in the present case, and it implies:

$$\det\frac{\partial\widetilde{x}}{\partial x} = \frac{\det D(\widetilde{x},-\epsilon)}{\det D(x,\epsilon)}\cdot\frac{\det(I+\epsilon YA)}{\det(I-\epsilon YA)}.$$

The second factor equals 1 due to the skew-symmetry of A, while the first factor equals

$$\frac{d_1(\widetilde{x},-\epsilon)\cdots d_N(\widetilde{x},-\epsilon)}{d_1(x,\epsilon)\cdots d_N(x,\epsilon)} = \frac{\widetilde{x}_1\cdots\widetilde{x}_N}{x_1\cdots x_N},$$

by virtue of (4.15). □

4.1 First Integrable Examples

We will now discuss some first examples of integrable HK type discretizations, namely the discrete Weierstrass system, the discrete Euler top and the discrete Zhukovsky Volterra system. During this discussion we will encounter first examples of the integrability properties of the Hirota-Kimura type discretizations. In particular, we will see first examples of HK-bases. When discussing the discretization of the Zhukovsky Volterra system we will also see, that not all of the HK type discretizations produce integrable mappings.

4.1.1 Weierstrass Differential Equation

Consider the second-order differential equation

$$\ddot{x} = 6x^2 - \alpha. \tag{4.17}$$

Its general solution is given by the Weierstrass elliptic function $\wp(t) = \wp(t, g_2, g_3)$ with the invariants $g_2 = 2\alpha$, g_3 arbitrary, and by its time shifts. Actually, the parameter g_3 can be interpreted as the value of an integral of motion (conserved quantity) of system (4.17):

$$\dot{x}^2 - 4x^3 + 2\alpha x = -g_3.$$

Being re-written as a system of first-order equations with a quadratic vector field,

$$\begin{cases} \dot{x} = y, \\ \dot{y} = 6x^2 - \alpha, \end{cases} \tag{4.18}$$

equation (4.17) becomes suitable for an application of the HK type discretization scheme:

$$\begin{cases} \widetilde{x} - x = \dfrac{\epsilon}{2}\left(\widetilde{y} + y\right), \\ \widetilde{y} - y = \epsilon\left(6x\widetilde{x} - \alpha\right). \end{cases} \tag{4.19}$$

Eqs. (4.19), put as a linear system for $(\widetilde{x}, \widetilde{y})$, reads:

$$\begin{pmatrix} 1 & -\epsilon/2 \\ -6\epsilon x & 1 \end{pmatrix} \begin{pmatrix} \widetilde{x} \\ \widetilde{y} \end{pmatrix} = \begin{pmatrix} x + \epsilon y/2 \\ y - \epsilon\alpha \end{pmatrix}.$$

This can be immediately solved, yielding an explicit birational map $(\widetilde{x}, \widetilde{y}) = f(x, y, \epsilon)$:

$$\begin{cases} \widetilde{x} = \dfrac{x + \epsilon y - \epsilon^2\alpha/2}{1 - 3\epsilon^2 x}, \\ \widetilde{y} = \dfrac{y + \epsilon(6x^2 - \alpha) + 3\epsilon^2 xy}{1 - 3\epsilon^2 x}. \end{cases} \tag{4.20}$$

This map turns out to be integrable: it possesses an invariant two-form

$$\omega = \frac{dx \wedge dy}{1 - 3\epsilon^2 x}, \tag{4.21}$$

and an integral of motion (conserved quantity):

$$I(x, y, \epsilon) = \frac{y^2 - 4x^3 + 2\alpha x + \epsilon^2 x(y^2 - 2\alpha x) - \epsilon^4\alpha^2 x}{1 - 3\epsilon^2 x}. \tag{4.22}$$

Both these objects are $O(\epsilon^2)$-perturbations of the corresponding objects for the continuous time system (4.18). The statement about the invariant 2-form (4.21) is not difficult to prove. In particular, using formula (4.4) we obtain

$$\det \frac{\partial(\widetilde{x}, \widetilde{y})}{\partial(x, y)} = \frac{1 - 3\epsilon^2\widetilde{x}}{1 - 3\epsilon^2 x},$$

which is equivalent to the preservation of (4.21). The statement about the conserved quantity is most simply verified with any computer system for symbolic manipulations.

System (4.19) is known in the literature on integrable maps, although in a somewhat different form. Indeed, it is equivalent to the second order difference equation

$$\widetilde{x} - 2x + \underset{\sim}{x} = \epsilon^2 \left[3x(\widetilde{x} + \underset{\sim}{x}) - \alpha\right] \quad \Leftrightarrow \quad \widetilde{x} - 2x + \underset{\sim}{x} = \frac{\epsilon^2(6x^2 - \alpha)}{1 - 3\epsilon^2 x}.$$

This equation belongs to the class of integrable *QRT systems* [46, 51]; in order to see this, one should re-write it as

$$\widetilde{x} - 2x + \underset{\sim}{x} = \frac{\epsilon^2(6x^2 - \alpha)(1 + \epsilon^2 x)}{1 - 2\epsilon^2 x - 3\epsilon^4 x^2}.$$

This difference equation generates a map $(x, \underset{\sim}{x}) \mapsto (\widetilde{x}, x)$ which is symplectic, that is, preserves the two-form $\omega = dx \wedge d\widetilde{x}$, and possesses a biquadratic integral of motion

$$I(x, \widetilde{x}, \epsilon) = (\widetilde{x} - x)^2 - 2\epsilon^2 x\widetilde{x}(x + \widetilde{x}) + \epsilon^2\alpha(x + \widetilde{x}) - \epsilon^4(3x^2\widetilde{x}^2 - \alpha x\widetilde{x}).$$

Under the change of variables $(x, \widetilde{x}) \mapsto (x, y)$ given by the first equation in (4.20), these integrability attributes turn into the two-form (4.21) and the conserved quantity (4.22) (up to an additive constant).

4.1.2 Euler Top

The differential equations of motion of the Euler top read

$$\left\{\begin{array}{l} \dot{x}_1 = \alpha_1 x_2 x_3, \\ \dot{x}_2 = \alpha_2 x_3 x_1, \\ \dot{x}_3 = \alpha_3 x_1 x_2, \end{array}\right. \tag{4.23}$$

with real parameters α_i. This is one of the most famous integrable systems of the classical mechanics, with a big literature devoted to it. It can be explicitly integrated in terms of elliptic functions, and admits two functionally independent integrals of motion. Actually, a quadratic function $H(x) = \gamma_1 x_1^2 + \gamma_2 x_2^2 + \gamma_3 x_3^2$ is an integral for eqs. (4.23) as soon as $\gamma_1\alpha_1 + \gamma_2\alpha_2 + \gamma_2\alpha_2 = 0$. In particular, the following three functions are integrals of motion:

$$H_1 = \alpha_2 x_3^2 - \alpha_3 x_2^2, \qquad H_2 = \alpha_3 x_1^2 - \alpha_1 x_3^2, \qquad H_3 = \alpha_1 x_2^2 - \alpha_2 x_1^2.$$

Clearly, only two of them are functionally independent because of $\alpha_1 H_1 + \alpha_2 H_2 + \alpha_3 H_3 = 0$. These integrals appear also on the right-hand sides of the quadratic (in this case even linear) expressions for the Wronskians of the coordinates x_j:

$$\left\{\begin{array}{l} \dot{x}_2 x_3 - x_2 \dot{x}_3 = H_1 x_1, \\ \dot{x}_3 x_1 - x_3 \dot{x}_1 = H_2 x_2, \\ \dot{x}_1 x_2 - x_1 \dot{x}_2 = H_3 x_3. \end{array}\right. \tag{4.24}$$

Moreover, one easily sees that the coordinates x_j satisfy the following differential equations with the coefficients depending on the integrals of motion:

$$\begin{cases} \dot{x}_1^2 = (H_3 + \alpha_2 x_1^2)(\alpha_3 x_1^2 - H_2), \\ \dot{x}_2^2 = (H_1 + \alpha_3 x_2^2)(\alpha_1 x_2^2 - H_3), \\ \dot{x}_3^2 = (H_2 + \alpha_1 x_3^2)(\alpha_2 x_3^2 - H_1). \end{cases}$$

The fact that the polynomials on the right-hand sides of these equations are of degree four implies that the solutions are given by elliptic functions.

The HK discretization of the Euler top [30] is:

$$\begin{cases} \widetilde{x}_1 - x_1 = \epsilon\alpha_1(\widetilde{x}_2 x_3 + x_2\widetilde{x}_3), \\ \widetilde{x}_2 - x_2 = \epsilon\alpha_2(\widetilde{x}_3 x_1 + x_3\widetilde{x}_1), \\ \widetilde{x}_3 - x_3 = \epsilon\alpha_3(\widetilde{x}_1 x_2 + x_1\widetilde{x}_2). \end{cases} \tag{4.25}$$

The map $f : x \mapsto \widetilde{x}$ obtained by solving (4.25) for $\widetilde{x}$ is given by:

$$\widetilde{x} = f(x,\epsilon) = A^{-1}(x,\epsilon)x, \quad A(x,\epsilon) = \begin{pmatrix} 1 & -\epsilon\alpha_1 x_3 & -\epsilon\alpha_1 x_2 \\ -\epsilon\alpha_2 x_3 & 1 & -\epsilon\alpha_2 x_1 \\ -\epsilon\alpha_3 x_2 & -\epsilon\alpha_3 x_1 & 1 \end{pmatrix}. \tag{4.26}$$

It might be instructive to have a look at the explicit formulas for this map:

$$\begin{cases} \widetilde{x}_1 = \dfrac{x_1 + 2\epsilon\alpha_1 x_2 x_3 + \epsilon^2 x_1(-\alpha_2\alpha_3 x_1^2 + \alpha_3\alpha_1 x_2^2 + \alpha_1\alpha_2 x_3^2)}{\Delta(x,\epsilon)}, \\[2ex] \widetilde{x}_2 = \dfrac{x_2 + 2\epsilon\alpha_2 x_3 x_1 + \epsilon^2 x_2(\alpha_2\alpha_3 x_1^2 - \alpha_3\alpha_1 x_2^2 + \alpha_1\alpha_2 x_3^2)}{\Delta(x,\epsilon)}, \\[2ex] \widetilde{x}_3 = \dfrac{x_3 + 2\epsilon\alpha_3 x_1 x_2 + \epsilon^2 x_3(\alpha_2\alpha_3 x_1^2 + \alpha_3\alpha_1 x_2^2 - \alpha_1\alpha_2 x_3^2)}{\Delta(x,\epsilon)}, \end{cases} \tag{4.27}$$

where

$$\Delta(x,\epsilon) = \det A(x,\epsilon) = 1 - \epsilon^2(\alpha_2\alpha_3 x_1^2 + \alpha_3\alpha_1 x_2^2 + \alpha_1\alpha_2 x_3^2) - 2\epsilon^3\alpha_1\alpha_2\alpha_3 x_1 x_2 x_3. \tag{4.28}$$

We will use the abbreviation dET for this map. As always the case for a HK discretization, dET is birational, with the reversibility property expressed as $f^{-1}(x,\epsilon) = f(x,-\epsilon)$. We now summarize the known results regarding the integrability of dET.

Proposition 4.3. [30, 44] *The quantities*

$$F_1 = \frac{1 - \epsilon^2\alpha_3\alpha_1 x_2^2}{1 - \epsilon^2\alpha_1\alpha_2 x_3^2}, \quad F_2 = \frac{1 - \epsilon^2\alpha_1\alpha_2 x_3^2}{1 - \epsilon^2\alpha_2\alpha_3 x_1^2}, \quad F_3 = \frac{1 - \epsilon^2\alpha_2\alpha_3 x_1^2}{1 - \epsilon^2\alpha_3\alpha_1 x_2^2}$$

are conserved quantities of dET. Of course, there are only two independent integrals since $F_1F_2F_3 = 1$.

The relation between F_i and the integrals H_i of the continuous time Euler top is straightforward: $F_i = 1 + \epsilon^2\alpha_i H_i + O(\epsilon^4)$. As a corollary of Proposition 4.3, we find that, for any conserved quantity H of the Euler top which is a linear combination of the integrals H_1, H_2, H_3, the three functions $H/(1-\epsilon^2\alpha_j\alpha_k x_i^2)$ are conserved quantities of dET. Hereafter (i,j,k) are cyclic permutations of $(1,2,3)$. In particular, the functions

$$H_i(\epsilon) = \frac{\alpha_j x_k^2 - \alpha_k x_j^2}{1-\epsilon^2\alpha_j\alpha_k x_i^2} \tag{4.29}$$

are conserved quantities of dET. Again, only two of them are independent, since

$$\alpha_1 H_1(\epsilon) + \alpha_2 H_2(\epsilon) + \alpha_3 H_3(\epsilon) + \epsilon^4\alpha_1\alpha_2\alpha_3 H_1(\epsilon)H_2(\epsilon)H_3(\epsilon) = 0.$$

Proposition 4.4. [44] *The map dET possesses an invariant volume form:*

$$\det\frac{\partial\widetilde{x}}{\partial x} = \frac{\phi(\widetilde{x})}{\phi(x)} \quad\Leftrightarrow\quad f^*\omega = \omega, \quad \omega = \frac{dx_1\wedge dx_2\wedge dx_3}{\phi(x)},$$

where $\phi(x)$ is any of the functions

$$\phi(x) = (1-\epsilon^2\alpha_i\alpha_j x_k^2)(1-\epsilon^2\alpha_j\alpha_k x_i^2) \quad \text{or} \quad (1-\epsilon^2\alpha_i\alpha_j x_k^2)^2.$$

(The ratio of any two functions $\phi(x)$ is an integral of motion, due to Proposition 4.3).

Proof. Direct computation. □

A proper discretization of the Wronskian differential equations (4.24) is given by the following statement.

Proposition 4.5. *The following relations hold true for dET:*

$$\left\{\begin{array}{l} \widetilde{x}_2 x_3 - x_2\widetilde{x}_3 = \epsilon H_1(\epsilon)(\widetilde{x}_1 + x_1), \\ \widetilde{x}_3 x_1 - x_3\widetilde{x}_1 = \epsilon H_2(\epsilon)(\widetilde{x}_3 + x_3), \\ \widetilde{x}_1 x_2 - x_1\widetilde{x}_2 = \epsilon H_3(\epsilon)(\widetilde{x}_3 + x_3), \end{array}\right. \tag{4.30}$$

with the functions $H_i(\epsilon)$ from (4.29).

The proof is based on relations

$$\widetilde{x}_i + x_i = \frac{2(1-\epsilon^2\alpha_j\alpha_k x_i^2)(x_i + \epsilon\alpha_i x_j x_k)}{\Delta(x,\epsilon)}, \tag{4.31}$$

$$\widetilde{x}_j x_k - x_j\widetilde{x}_k = \frac{2\epsilon(\alpha_j x_k^2 - \alpha_k x_j^2)(x_i + \epsilon\alpha_i x_j x_k)}{\Delta(x,\epsilon)}, \tag{4.32}$$

which follow easily from the explicit formulas (4.27). They should be compared with

$$\widetilde{x}_i - x_i = \epsilon\alpha_i(\widetilde{x}_j x_k + x_j\widetilde{x}_k) = \frac{2\epsilon\alpha_i(x_j + \epsilon\alpha_j x_k x_i)(x_k + \epsilon\alpha_k x_i x_j)}{\Delta(x,\epsilon)}. \tag{4.33}$$

A probable way to the discovery of the conserved quantities of dET in [30] was through finding the HK bases for this map. In this respect, one has the following results. All HK bases can easily detected with the algorithm (V) (see Appendix A).

Proposition 4.4. [45]

(a) *The set* $\Phi = (x_1^2, x_2^2, x_3^2, 1)$ *is a HK basis for dET with* $\dim K_\Phi(x) = 2$. *Therefore, any orbit of dET lies on the intersection of two quadrics in* $\mathbb{R}^3$.

(b) *The set* $\Phi_0 = (x_1^2, x_2^2, x_3^2)$ *is a HK basis for dET with* $\dim K_{\Phi_0}(x) = 1$. *At each point* $x \in \mathbb{R}^3$ *we have:*

$$K_{\Phi_0}(x) = [c_1 : c_2 : c_3] = [\,\alpha_2 x_3^2 - \alpha_3 x_2^2 : \alpha_3 x_1^2 - \alpha_1 x_3^2 : \alpha_1 x_2^2 - \alpha_2 x_1^2\,].$$

Setting $c_3 = -1$, *the following functions are integrals of motion of dET:*

$$c_1(x) = \frac{\alpha_3 x_2^2 - \alpha_2 x_3^2}{\alpha_1 x_2^2 - \alpha_2 x_1^2}, \qquad c_2(x) = \frac{\alpha_1 x_3^2 - \alpha_3 x_1^2}{\alpha_1 x_2^2 - \alpha_2 x_1^2}. \tag{4.34}$$

(c) *The set* $\Phi_{12} = (x_1^2, x_2^2, 1)$ *is a further HK basis for dET with* $\dim K_{\Phi_{12}}(x) = 1$. *At each point* $x \in \mathbb{R}^3$ *we have:* $K_{\Phi_{12}}(x) = [d_1 : d_2 : -1]$, *where*

$$d_1(x) = -\frac{\alpha_2(1 - \epsilon^2\alpha_3\alpha_1 x_2^2)}{\alpha_1 x_2^2 - \alpha_2 x_1^2}, \qquad d_2(x) = \frac{\alpha_1(1 - \epsilon^2\alpha_2\alpha_3 x_1^2)}{\alpha_1 x_2^2 - \alpha_2 x_1^2}. \tag{4.35}$$

These functions are integrals of motion of dET independent on the integrals (4.34). We have: $K_\Phi(x) = K_{\Phi_0} \oplus K_{\Phi_{12}}$.

Proof. To prove statement b), we solve the system

$$\begin{cases} c_1 x_1^2 + c_2 x_2^2 &= x_3^2, \\ c_1 \widetilde{x}_1^2 + c_2 \widetilde{x}_2^2 &= \widetilde{x}_3^2. \end{cases}$$

The solution is given, according to the Cramer's rule, by ratios of determinants of the type

$$\begin{vmatrix} x_i^2 & x_j^2 \\ \widetilde{x}_i^2 & \widetilde{x}_j^2 \end{vmatrix} = \frac{4\epsilon(\alpha_j x_i^2 - \alpha_i x_j^2)(x_1 + \epsilon\alpha_1 x_2 x_3)(x_2 + \epsilon\alpha_2 x_3 x_1)(x_3 + \epsilon\alpha_3 x_1 x_2)}{\Delta^2(x,\epsilon)} \tag{4.36}$$

(here we used (4.32), (4.33)). In the ratios of such determinants everything cancels out, except for the factors $\alpha_j x_i^2 - \alpha_i x_j^2$, so we end up with (4.34). The cancelation of the denominators $\Delta^2(x,\epsilon)$ is, of course, no wonder, but the cancelation of all the non-even factors in the numerators is rather *remarkable and miraculous* and is not granted by any well-understood mechanism. Since the components of the solution do not depend on ϵ, we conclude that functions (4.34) are integrals of motion of dET.

To prove statement c), we solve the system

$$\begin{cases} d_1 x_1^2 + d_2 x_2^2 &= 1, \\ d_1 \widetilde{x}_1^2 + d_2 \widetilde{x}_2^2 &= 1. \end{cases}$$

The solution is given by eq. (4.35), due to eq. (4.36) and the similar formula

$$\begin{vmatrix} 1 & x_i^2 \\ 1 & \widetilde{x}_i^2 \end{vmatrix} = \frac{4\epsilon\alpha_i(1 - \epsilon^2\alpha_j\alpha_k x_i^2)(x_1 + \epsilon\alpha_1 x_2 x_3)(x_2 + \epsilon\alpha_2 x_3 x_1)(x_3 + \epsilon\alpha_3 x_1 x_2)}{\Delta^2(x,\epsilon)}$$

(which, in turn, follows from (4.31) and (4.32)). This time the solution does depend on ϵ, but consists of manifestly even functions of ϵ. Everything non-even luckily cancels, again. Therefore, functions (4.35) are integrals of motion of dET (recipe (D)). $\square$

Although each one of the HK bases Φ_0, Φ_1 delivers apparently two integrals of motion (4.34), each pair turns out to be *functionally dependent*, as

$$\alpha_1 c_1(x) + \alpha_2 c_2(x) = \alpha_3, \quad \alpha_1 d_1(x) + \alpha_2 d_2(x) = \epsilon^2 \alpha_1 \alpha_2 \alpha_3.$$

However, functions c_1, c_2 are independent on d_1, d_2, since the former depend on x_3, while the latter do not.

Of course, permutational symmetry yields that each of the sets of monomials $\Phi_{23} = (x_2^2,\, x_3^2,\, 1)$ and $\Phi_{13} = (x_1^2,\, x_3^2,\, 1)$ is a HK basis, as well, with $\dim K_{\Phi_{23}}(x) = \dim K_{\Phi_{13}}(x) = 1$. But we do not obtain additional linearly independent null-spaces, as any two of the four found one-dimensional null-spaces span the full null-space $K_\Phi(x)$.

Summarizing, we have found a HK basis with a two-dimensional null-space, as well as two functionally independent conserved quantities for the HK discretization of the Euler top. Both results yield integrability of this discretization, in the sense that its orbits are confined to closed curves in $\mathbb{R}^3$. Moreover, each such curve is an intersection of two quadrics, which in the general position case is an elliptic curve.

Proposition 4.6. *Each component x_i of any solution of dET satisfies a relation of the type $P_i(x_i, \widetilde{x}_i) = 0$, where P_i is a biquadratic polynomial whose coefficients are integrals of motion of dET:*

$$P_i(x_i, \widetilde{x}_i) = p_i^{(3)} x_i^2 \widetilde{x}_i^2 + p_i^{(2)} (x_i^2 + \widetilde{x}_i^2) + p_i^{(1)} x_i \widetilde{x}_i + p_i^{(0)} = 0,$$

with

$$p_i^{(3)} = -4\epsilon^2 \alpha_j \alpha_k, \quad p_i^{(2)} = [1 + \epsilon^2 \alpha_j H_j(\epsilon)][1 - \epsilon^2 \alpha_k H_k(\epsilon)],$$
$$p_i^{(1)} = -2[1 - \epsilon^2 \alpha_j H_j(\epsilon)][1 + \epsilon^2 \alpha_k H_k(\epsilon)], \quad p_i^{(0)} = 4\epsilon^2 H_j(\epsilon) H_k(\epsilon).$$

Proof. From eqs. (4.25) and (4.30) there follows:

$$(\widetilde{x}_i - x_i)^2/(\epsilon\alpha_i)^2 + [\epsilon H_i(\epsilon)]^2 (\widetilde{x}_i + x_i)^2 = 2(\widetilde{x}_j^2 x_k^2 + x_j^2 \widetilde{x}_k^2).$$

It remains to express x_j^2 and x_k^2 through x_i^2 and integrals $H_j(\epsilon)$, $H_k(\epsilon)$ given in eq. (4.29). $\square$

It follows from Proposition 4.6 that solutions $x_i(t)$ as functions of the discrete time $t \in 2\epsilon\mathbb{Z}$ are given by elliptic functions of order 2.

Note that that Propositions 4.5, 4.6 can be interpreted as existence of further HK bases. For instance, according to Proposition 4.5, each pair $(\widetilde{x}_j x_k - x_j \widetilde{x}_k, \widetilde{x}_i + x_i)$ is a HK basis with a 1-dimensional null-space. Similarly, Proposition 4.6 says that for each $i = 1, 2, 3$, the set $x_i^p \widetilde{x}_i^q$ $(0 \le p, q \le 2)$ is a HK basis with a 1-dimensional null-space. Of course, due to the dependence on the shifted variables $\widetilde{x}$, these HK bases consist of complicated functions of x rather than of monomials. A further instance of HK bases of this sort is given in the following statement. Compared with Proposition 4.4, it says that for dET, *for each HK basis consisting of monomials quadratic in x, the corresponding set of monomials bilinear in $x, \widetilde{x}$ is a HK basis, as well.* This seems to be a quite general phenomenon, further issues of which will appear later several times.

Proposition 4.5.

(a) *The set* $\Psi = (\widetilde{x}_1 x_1,\, \widetilde{x}_2 x_2,\, \widetilde{x}_3 x_3,\, 1)$ *is a HK basis for dET with* $\dim K_\Psi(x) = 2$.

(b) *The set* $\Psi_0 = (\widetilde{x}_1 x_1,\, \widetilde{x}_2 x_2,\, \widetilde{x}_3 x_3)$ *is a HK basis for dET with* $\dim K_{\Psi_0}(x) = 1$. *At each point* $x \in \mathbb{R}^3$*, the homogeneous coordinates* $\bar{c}_i$ *of the null-space* $K_{\Psi_0}(x) = [\bar{c}_1 : \bar{c}_2 : \bar{c}_3]$ *are given by*

$$\bar{c}_i = (\alpha_j x_k^2 - \alpha_k x_j^2)\left[1 - \epsilon^2(\alpha_i\alpha_j x_k^2 + \alpha_k\alpha_i x_j^2 - \alpha_j\alpha_k x_i^2)\right].$$

The quotients $\bar{c}_i/\bar{c}_j$ *are integrals of motion of dET.*

(c) *The set* $\Psi_{12} = (\widetilde{x}_1 x_1,\, \widetilde{x}_2 x_2,\, 1)$ *is a further HK basis for dET with* $\dim K_{\Psi_{12}}(x) = 1$. *At each point* $x \in \mathbb{R}^3$*, there holds:* $K_{\Psi_{12}}(x) = [\bar{d}_1 : \bar{d}_2 : -1]$*, where*

$$\begin{aligned}
\bar{d}_1(x) &= -\frac{\alpha_2(1-\epsilon^2\alpha_3\alpha_1 x_2^2)}{\alpha_1 x_2^2 - \alpha_2 x_1^2}\,\frac{1-\epsilon^2(\alpha_2\alpha_3 x_1^2 - \alpha_3\alpha_1 x_2^2 + \alpha_1\alpha_2 x_3^2)}{1-\epsilon^2(\alpha_2\alpha_3 x_1^2 + \alpha_3\alpha_1 x_2^2 - \alpha_1\alpha_2 x_3^2)},\\
\bar{d}_2(x) &= \frac{\alpha_1(1-\epsilon^2\alpha_2\alpha_3 x_1^2)}{\alpha_1 x_2^2 - \alpha_2 x_1^2}\,\frac{1-\epsilon^2(\alpha_3\alpha_1 x_2^2 - \alpha_2\alpha_3 x_1^2 + \alpha_1\alpha_2 x_3^2)}{1-\epsilon^2(\alpha_3\alpha_1 x_2^2 + \alpha_2\alpha_3 x_1^2 - \alpha_1\alpha_2 x_3^2)},
\end{aligned}$$

are integrals of dET. We have: $K_\Psi(x) = K_{\Psi_0}(x) \oplus K_{\Psi_{12}}(x)$.

Proof. This is easily checked with a symbolic manipulator like MAPLE. □

Concluding this section we mention that a Poisson structure for dET may be found using the contraction procedure outlined in Chapter 2, Section 2.4.5 [44].

4.2 A More Complicated Example: The Zhukovski-Volterra System

The gyroscopic Zhukovski-Volterra (ZV) system is a generalization of the Euler top. It describes the free motion of a rigid body carrying an asymmetric rotor (gyrostat) [59]. Equations of motion of the ZV system read

$$\left\{\begin{aligned}
\dot{x}_1 &= \alpha_1 x_2 x_3 + \beta_3 x_2 - \beta_2 x_3,\\
\dot{x}_2 &= \alpha_2 x_3 x_1 + \beta_1 x_3 - \beta_3 x_1,\\
\dot{x}_3 &= \alpha_3 x_1 x_2 + \beta_2 x_1 - \beta_1 x_2,
\end{aligned}\right. \tag{4.37}$$

with α_i, β_i being real parameters of the system. For $(\beta_1, \beta_2, \beta_3) = (0, 0, 0)$, the flow (4.37) reduces to the Euler top (4.23). The ZV system is (Liouville and algebraically) integrable under the condition

$$\alpha_1 + \alpha_2 + \alpha_3 = 0. \tag{4.38}$$

It can be explicitly integrated in terms of elliptic functions, see [59] and also [5] for a more recent exposition. The following quantities are integrals of motion of the ZV system:

$$\begin{aligned} H_1 &= \alpha_2 x_3^2 - \alpha_3 x_2^2 - 2(\beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3), \\ H_2 &= \alpha_3 x_1^2 - \alpha_1 x_3^2 - 2(\beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3), \\ H_3 &= \alpha_1 x_2^2 - \alpha_2 x_1^2 - 2(\beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3). \end{aligned} \tag{4.39}$$

Clearly, only two of them are functionally independent because of $\alpha_1 H_1 + \alpha_2 H_2 + \alpha_3 H_3 = 0$. Note that

$$H_2 - H_1 = \alpha_3 C, \quad H_3 - H_2 = \alpha_1 C, \quad H_1 - H_3 = \alpha_2 C, \tag{4.40}$$

with $C = x_1^2 + x_2^2 + x_3^2$.

As in the Euler case, the Wronskians of the coordinates x_j admit quadratic expressions with coefficients dependent on the integrals of motion:

$$\left\{ \begin{aligned} \dot{x}_2 x_3 - x_2 \dot{x}_3 &= H_1 x_1 + x_1(\beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3) + \beta_1 C, \\ \dot{x}_3 x_1 - x_3 \dot{x}_1 &= H_2 x_2 + x_2(\beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3) + \beta_2 C, \\ \dot{x}_1 x_2 - x_1 \dot{x}_2 &= H_3 x_3 + x_3(\beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3) + \beta_3 C. \end{aligned} \right. \tag{4.41}$$

The HK discretization of the ZV system is:

$$\left\{ \begin{aligned} \widetilde{x}_1 - x_1 &= \epsilon \left[\alpha_1(\widetilde{x}_2 x_3 + x_2 \widetilde{x}_3) + \beta_3(\widetilde{x}_2 + x_2) - \beta_2(\widetilde{x}_3 + x_3) \right], \\ \widetilde{x}_2 - x_2 &= \epsilon \left[\alpha_2(\widetilde{x}_3 x_1 + x_3 \widetilde{x}_1) + \beta_1(\widetilde{x}_3 + x_3) - \beta_3(\widetilde{x}_1 + x_1) \right], \\ \widetilde{x}_3 - x_3 &= \epsilon \left[\alpha_3(\widetilde{x}_1 x_2 + x_1 \widetilde{x}_2) + \beta_2(\widetilde{x}_1 + x_1) - \beta_1(\widetilde{x}_2 + x_2) \right]. \end{aligned} \right. \tag{4.42}$$

The map $f : x \mapsto \widetilde{x}$ obtained by solving (4.42) for $\widetilde{x}$ is given by:

$$\widetilde{x} = f(x, \epsilon) = A^{-1}(x, \epsilon)(\mathbb{1} + \epsilon B)x,$$

with

$$A(x, \epsilon) = \begin{pmatrix} 1 & -\epsilon\alpha_1 x_3 & -\epsilon\alpha_1 x_2 \\ -\epsilon\alpha_2 x_3 & 1 & -\epsilon\alpha_2 x_1 \\ -\epsilon\alpha_3 x_2 & -\epsilon\alpha_3 x_1 & 1 \end{pmatrix} - \epsilon B, \quad B = \begin{pmatrix} 0 & \beta_3 & -\beta_2 \\ -\beta_3 & 0 & \beta_1 \\ \beta_2 & -\beta_1 & 0 \end{pmatrix}.$$

We will call this map dZV.

4.2.1 ZV System with Two Vanishing β_k's

In the case when two out of three β_k's vanish, say $\beta_2 = \beta_3 = 0$, the condition (4.38) is not necessary for integrability of the ZV system. The functions H_2 and H_3 as given in (4.39) (with $\beta_2 = \beta_3 = 0$) are in this case conserved quantities without any condition on α_k's, while their linear combinations H_1 and C are given by

$$\begin{aligned} H_1 &= -\frac{1}{\alpha_1}(\alpha_2 H_2 + \alpha_3 H_3) = \alpha_2 x_3^2 - \alpha_3 x_2^2 + 2\beta_1 \frac{\alpha_2 + \alpha_3}{\alpha_1} x_1, \\ C &= \frac{1}{\alpha_1}(H_3 - H_2) = x_2^2 + x_3^2 - \frac{\alpha_2 + \alpha_3}{\alpha_1} x_1^2. \end{aligned}$$

Wronskian relations (4.41) are replaced by

$$\begin{cases} \dot{x}_2 x_3 - x_2 \dot{x}_3 = H_1 x_1 - \beta_1 \dfrac{\alpha_2+\alpha_3}{\alpha_1}\, x_1^2 + \beta_1 C, \\ \dot{x}_3 x_1 - x_3 \dot{x}_1 = H_2 x_2 + \beta_1 x_1 x_2, \\ \dot{x}_1 x_2 - x_1 \dot{x}_2 = H_3 x_3 + \beta_1 x_1 x_3. \end{cases} \tag{4.43}$$

With the help of the algorithm (V) it is easy to find HK bases for this case of the map dZV:

Proposition 4.6.

(a) *The set* $\Phi = (x_1^2, x_2^2, x_3^2, x_1, 1)$ *is a HK basis for dZV with* $\beta_2 = \beta_3 = 0$*, with* $\dim K_\Phi(x) = 2$*. Any orbit of dZV with* $\beta_2 = \beta_3 = 0$ *is thus confined to the intersection of two quadrics in* $\mathbb{R}^3$*.*

(b) *The set* $\Phi_0 = (x_1^2, x_2^2, x_3^2, 1)$ *is a HK basis for dZV with* $\beta_2 = \beta_3 = 0$*, with* $\dim K_{\Phi_0}(x) = 1$*. At each point* $x \in \mathbb{R}^3$ *we have:* $K_{\Phi_0}(x) = [-1 : d_2 : d_3 : d_4]$*, where*

$$d_2 = \frac{\alpha_1}{\alpha_2+\alpha_3}[1 - \epsilon^2\beta_1^2 - \epsilon^2\alpha_3 H_2(\epsilon)], \quad d_3 = \frac{\alpha_1}{\alpha_2+\alpha_3}[1 - \epsilon^2\beta_1^2 + \epsilon^2\alpha_2 H_3(\epsilon)],$$

$$d_4 = \frac{1}{\alpha_2+\alpha_3}[H_2(\epsilon) - H_3(\epsilon)].$$

(c) *The set* $\Phi_{23} = (x_2^2, x_3^2, x_1, 1)$ *is a HK basis for dZV with* $\beta_2 = \beta_3 = 0$*, with* $\dim K_{\Phi_{23}}(x) = 1$*. At each point* $x \in \mathbb{R}^3$ *we have:* $K_{\Phi_{23}}(x) = [c_1 : c_2 : c_3 : c_4]$*, where*

$$c_1 = \alpha_1[\alpha_3 + \epsilon^2\beta_1^2\alpha_2 + \epsilon^2\alpha_2\alpha_3 H_2(\epsilon)], \quad c_2 = -\alpha_1[\alpha_2 + \epsilon^2\beta_1^2\alpha_3 - \epsilon^2\alpha_2\alpha_3 H_3(\epsilon)],$$

$$c_3 = -2\beta_1(\alpha_2+\alpha_3), \quad c_4 = -[\alpha_2 H_2(\epsilon) + \alpha_3 H_3(\epsilon)].$$

Here the functions

$$H_2(\epsilon) = \frac{\alpha_3 x_1^2 - \alpha_1 x_3^2 - 2\beta_1 x_1 + \epsilon^2\beta_1^2\alpha_1 x_2^2}{1 - \epsilon^2\alpha_3\alpha_1 x_2^2},$$
$$H_3(\epsilon) = \frac{\alpha_1 x_2^2 - \alpha_2 x_1^2 - 2\beta_1 x_1 - \epsilon^2\beta_1^2\alpha_1 x_3^2}{1 - \epsilon^2\alpha_1\alpha_2 x_3^2},$$

are conserved quantities for the map dZV with $\beta_2 = \beta_3 = 0$ *.*

Proof. Direct computation. □

Unlike the case of dET, we see that here a HK basis with a one dimensional null-space already provides more than one independent integral of motion.

"Bilinear" versions of the above HK bases also exist:

Proposition 4.7. *The set* $\Psi = (x_1\widetilde{x}_1, x_2\widetilde{x}_2, x_3\widetilde{x}_3, x_1 + \widetilde{x}_1, 1)$ *is a HK basis for dZV with* $\beta_2 = \beta_3 = 0$*, with* $\dim K_\Psi(x) = 2$*. The sets*

$$\Psi_0 = (x_1\widetilde{x}_1, x_2\widetilde{x}_2, x_3\widetilde{x}_3, 1) \quad \textit{and} \quad \Psi_{23} = (x_2\widetilde{x}_2, x_3\widetilde{x}_3, x_1 + \widetilde{x}_1, 1)$$

are HK bases with one-dimensional null-spaces.

Proof. This is also easy to verify using a symbolic manipulator. □

The following statement is a starting point towards an explicit integration of the map dZV with $\beta_2 = \beta_3 = 0$ in terms of elliptic functions.

Proposition 4.7. *The component x_1 of the solution of the difference equations (4.42) satisfies a relation of the type*

$$P(x_1, \widetilde{x}_1) = p_0 x_1^2 \widetilde{x}_1^2 + p_1 x_1 \widetilde{x}_1 (x_1 + \widetilde{x}_1) + p_2 (x_1^2 + \widetilde{x}_1^2) + p_3 x_1 \widetilde{x}_1 + p_4 (x_1 + \widetilde{x}_1) + p_5 = 0,$$

coefficients of the biquadratic polynomial P being conserved quantities of dZV with $\beta_2 = \beta_3 = 0$.

Proof. The proof is parallel to that of Proposition 4.6. □

The conserved quantities of Proposition 4.6 appear on the right-hand sides of the following relations which are the discrete versions of the Wronskian relations (4.43):

Proposition 4.8. *The following relations hold true for dZV with $\beta_2 = \beta_3 = 0$:*

$$\begin{cases} \widetilde{x}_2 x_3 - x_2 \widetilde{x}_3 = \epsilon \left[c_1 (\widetilde{x}_1 + x_1) + 2c_2 \widetilde{x}_1 x_1 + 2c_3 \right], \\ \widetilde{x}_3 x_1 - x_3 \widetilde{x}_1 = \epsilon \left[H_2(\epsilon)(\widetilde{x}_2 + x_2) + \beta_1 (\widetilde{x}_1 x_2 + x_1 \widetilde{x}_2) \right], \\ \widetilde{x}_1 x_2 - x_1 \widetilde{x}_2 = \epsilon \left[H_3(\epsilon)(\widetilde{x}_3 + x_3) + \beta_1 (\widetilde{x}_1 x_3 + x_1 \widetilde{x}_3) \right], \end{cases}$$

with

$$c_1 = -\frac{\alpha_2 H_2(\epsilon) + \alpha_3 H_3(\epsilon)}{\alpha_1 \Delta}, \quad c_2 = -\frac{\beta_1(\alpha_2 + \alpha_3)}{\alpha_1 \Delta}, \quad c_3 = \frac{\beta_1 (H_3(\epsilon) - H_2(\epsilon))}{\alpha_1 \Delta},$$

$$\Delta = 1 + \epsilon^4 [\alpha_2 H_3(\epsilon) - \beta_1^2][\alpha_3 H_2(\epsilon) + \beta_1^2].$$

Proof. Direct verification using a symbolic manipulator. □

Finally, the HK discretization of the ZV system with $\beta_2 = \beta_3 = 0$ turns out to possess an invariant measure.

Proposition 4.9. *The map dZV with $\beta_2 = \beta_3 = 0$ possesses an invariant volume form:*

$$\det \frac{\partial \widetilde{x}}{\partial x} = \frac{\phi(\widetilde{x})}{\phi(x)} \quad \Leftrightarrow \quad f^* \omega = \omega, \quad \omega = \frac{dx_1 \wedge dx_2 \wedge dx_3}{\phi(x)},$$

with $\phi(x) = (1 - \epsilon^2 \alpha_3 \alpha_1 x_2^2)(1 - \epsilon^2 \alpha_1 \alpha_2 x_3^2)$.

Again, at this point one could continue and construct an invariant Poisson structure for the map dZV with $\beta_2 = \beta_3 = 0$.

4.2.2 ZV System with One Vanishing β_k

In the case $\beta_3 = 0$ (say) and generic values of other parameters, the ZV system has only one integral H_3 and is therefore non-integrable. One of the Wronskian relations holds true in this general situation:

$$\dot{x}_1 x_2 - x_1 \dot{x}_2 = H_3 x_3 + \beta_1 x_1 x_3 + \beta_2 x_2 x_3, \tag{4.44}$$

Under condition (4.38), the ZV system becomes integrable, with all the results formulated in the general case.

Similarly, the map dZV with $\beta_3 = 0$ and generic values of other parameters possesses one conserved quantity:

$$H_3(\epsilon) = \frac{\alpha_1 x_2^2 - \alpha_2 x_1^2 - 2(\beta_1 x_1 + \beta_2 x_2) - \epsilon^2(\beta_1^2 \alpha_1 + \beta_2^2 \alpha_2) x_3^2}{1 - \epsilon^2 \alpha_1 \alpha_2 x_3^2}.$$

Clearly, this fact can be re-formulated as the existence of a HK basis $\Phi = (x_1^2, x_2^2, x_3^2, x_1, x_2, 1)$ with $\dim K_\Phi = 1$. The Wronskian relation (4.44) possesses a decent discretization:

$$(\widetilde{x}_1 x_2 - x_1 \widetilde{x}_2)/\epsilon = H_3(\epsilon)(x_3 + \widetilde{x}_3) + \beta_1(\widetilde{x}_1 x_3 + x_1 \widetilde{x}_3) + \beta_2(\widetilde{x}_2 x_3 + x_2 \widetilde{x}_3). \tag{4.45}$$

However, it seems that the map dZV with $\beta_3 = 0$ does not acquire an additional integral of motion under condition (4.38). It might be conjectured that in order to assure the integrability of the dZV map with $\beta_3 = 0$, its other parameters have to satisfy some relation which is an $O(\epsilon)$-deformation of (4.38).

4.2.3 ZV System with All β_k's Non-Vanishing

Here we encounter again the phenomenon that not all HK type discretizations of integrable systems are integrable. In particular, numerical experiments using (DI) indicate non-integrability for the map (4.42) with non-vanishing β_k's (see Figure 4.3). Furthermore, this claim is supported by the exponential growth of the computation times for higher iterates (see Figure 4.2). It remains an open problem, as to how one could rigorously prove non-integrability for this map.

Nevertheless, some other relation between the parameters might yield integrability. In this connection we notice that the map dZV with $(\alpha_1, \alpha_2, \alpha_3) = (\alpha, -\alpha, 0)$ admits a polynomial conserved quantity

$$H = -\alpha x_3^2 - 2(\beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3) + \epsilon^2 \alpha(\beta_2 x_1 - \beta_1 x_2)^2.$$

4.3 Integrability of the HK type Discretizations

As mentioned earlier, the HK type scheme tends to produce integrable discretizations. In fact, the list of integrable HK type discretizations presented in [43] is rather impressive. It includes the discretizations of the following systems:

1. Weierstrass differential equation,

Figure 4.2: Left: computation time in seconds of k-th iterate vs. k for an orbit of the map dZV with $\alpha = (1, 2, 3)$, $\beta = (1, 2, 0)$ (integrable). Right: computation time of k-th iterate vs. k for an orbit of the map dZV with $\alpha = (1, 2, 3)$, $\beta = (1, 3, 0)$ (nonintegrable). (See Sect. 4.2.3 for the definiton of dZV)

(a) (b)

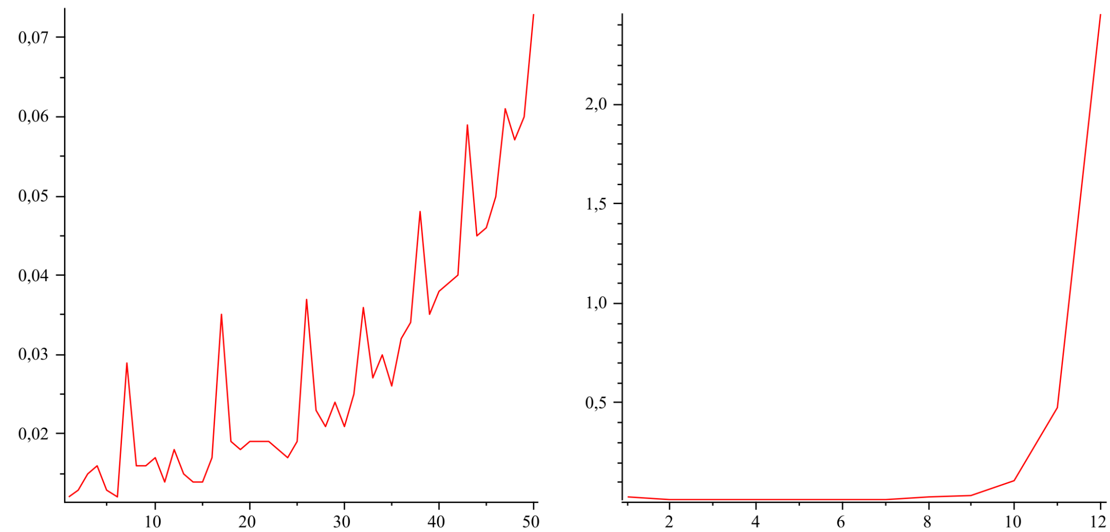
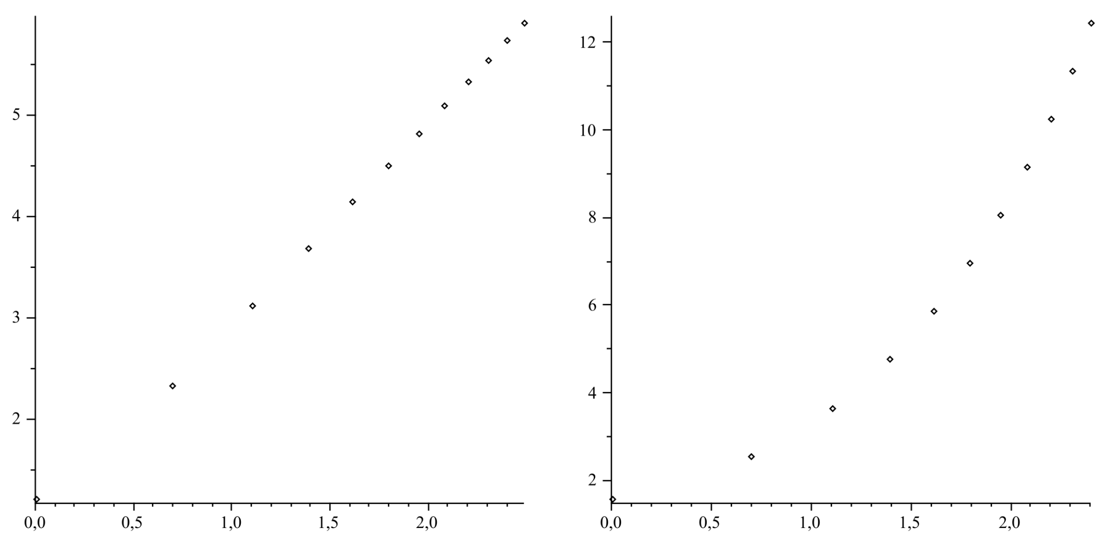


Figure 4.3: Left: Plot of $\log h_k$ versus $\log k$ for the first 11 iterations of the map (4.42) with parameters $\alpha = (1, 2, 3)$, $\beta = (2, 0, 0)$, $\epsilon = 1$ and initial data $x_0 = (1, 2, 3)$. Right: Plot of $\log h_k$ versus $\log k$ for the first 11 iterations of the map (4.42) with parameters $\alpha = (1, 2, 3)$, $\beta = (2, 1, 3)$, $\epsilon = 1$ and initial data $x_0 = (1, 2, 3)$

(a) (b)



2. Three-dimensional Suslov system [53],
3. Reduced Nahm equations [31],
4. Euler top,
5. Subcases of the Zhukosky-Volterra system [59],
6. The periodic Volterra chains with $N = 3$ and $N = 4$ particles,
7. The Dressing chain with $N = 3$ particles,
8. A system of coupled Euler tops,
9. Three wave system [2],
10. Lagrange top,
11. Kirchhoff case of the rigid body motion in an ideal fluid [34],
12. Clebsch case of the rigid body motion in an ideal fluid [17],
13. $\mathfrak{su}(2)$ rational Gaudin system with $N = 2$ spins [23].

The large number of these examples originally lead to the conjecture that the HK type discretization of any algebraically integrable system with a quadratic right hand side is integrable. Yet, numerical experiments using the integrability detectors from Chapter 2 do, however, indicate the non-integrability of the HK type discretizations of the following systems:

1. General case of the Zhukosky-Volterra system,
2. Kovalevskaia top,
3. Periodic Volterra chain with $N > 4$ particles,
4. Dressing chain with $N > 4$ particles [56].

There is currently no explanation as to where this behavior of the HK type discretizations might originate from. This fact, together with the enormous number of positive (i.e. integrable) examples makes the study of the HK type discretizations even more intruiging and calls for further research.

5

3D and 4D Volterra Lattices

Having been introduced to the basic features of some of the integrable HK type discretizations, we now discuss two examples where we will apply the methodology outlined in Chapter 3 in order to obtain explicit solutions in terms elliptic functions. The two examples being studied are the three and four dimensional periodic Volterra chains. In both cases we will first solve the continuous equations of motion and then find explicit solutions for the discrete systems. Relevant computer experiments can be found in the form of MAPLE worksheets on the attached CD-ROM.

5.1 Elliptic Solutions of the Infinite Volterra Chain

We consider the infinite Volterra chain (VC). Its equations of motion read

$$\dot{x}_n = x_n(x_{n+1} - x_{n-1}), \quad n \in \mathbb{Z}. \tag{5.1}$$

This system has two families of elliptic solutions. The first family of elliptic solutions is given by

$$\begin{aligned} x_n(t) &= \zeta\big(t+nv\big) - \zeta\big(t+(n-1)v\big) + \zeta(v) - \zeta(2v) & (5.2)\\ &= \frac{\sigma\big(t+(n+1)v\big)\sigma\big(t+(n-2)v\big)}{\sigma\big(t+nv\big)\sigma\big(t+(n-1)v\big)\sigma(2v)}. & (5.3)\end{aligned}$$

The equivalence of these two representations is either easily checked by looking at poles and zeroes of the both elliptic functions, or just by using the well known fundamental formula (3.2).

The check that (5.2), (5.3) is indeed a solution of VC is now elementary: take the logarithmic derivative of (5.3) and then use (5.2) with shifted indices:

$$\begin{aligned} \frac{\dot{x}_n}{x_n} &= \zeta\big(t+(n+1)v\big) + \zeta\big(t+(n-2)v\big) - \zeta\big(t+nv\big) - \zeta\big(t+(n-1)v\big)\\ &= x_{n+1} - x_{n-1}. \end{aligned}$$

The second family of elliptic solutions (reduces to the first one if $v_1 = v_2 = v$) is given by

$$\begin{aligned} x_{2n-1}(t) &= \zeta\big(t+nv_1+(n-1)v_2\big) - \zeta\big(t+(n-1)(v_1+v_2)\big) + \zeta(v_2) - \zeta(v_1+v_2)\\ &= \frac{\sigma\big(t+n(v_1+v_2)\big)\sigma\big(t+(n-1)v_1+(n-2)v_2\big)\sigma(v_1)}{\sigma\big(t+nv_1+(n-1)v_2\big)\sigma\big(t+(n-1)(v_1+v_2)\big)\sigma(v_2)\sigma(v_1+v_2)}, \end{aligned} \tag{5.4}$$

$$\begin{aligned} x_{2n}(t) &= \zeta\big(t+n(v_1+v_2)\big) - \zeta\big(t+nv_1+(n-1)v_2\big) + \zeta(v_1) - \zeta(v_1+v_2)\\ &= \frac{\sigma\big(t+(n+1)v_1+nv_2\big)\sigma\big(t+(n-1)(v_1+v_2)\big)\sigma(v_2)}{\sigma\big(t+n(v_1+v_2)\big)\sigma\big(t+nv_1+(n-1)v_2\big)\sigma(v_1)\sigma(v_1+v_2)}. \end{aligned} \tag{5.5}$$

Again, the verification of equations of motion is straightforward:

$$\begin{aligned}
\frac{\dot{x}_{2n-1}}{x_{2n-1}} &= \zeta\big(t+n(v_1+v_2)\big)+\zeta\big(t+(n-1)v_1+(n-2)v_2\big)\\
&\quad -\zeta\big(t+nv_1+(n-1)v_2\big)-\zeta\big(t+(n-1)(v_1+v_2)\big)\\
&= x_{2n}-x_{2n-2},\\
\frac{\dot{x}_{2n}}{x_{2n}} &= \zeta\big(t+(n+1)v_1+nv_2)\big)+\zeta\big(t+(n-1)(v_1+v_2)\big)\\
&\quad -\zeta\big(t+n(v_1+v_2)\big)-\zeta\big(t+nv_1+(n-1)v_2\big)\\
&= x_{2n+1}-x_{2n-1}.
\end{aligned}$$

The first family admits an N-periodic reduction ($n \in \mathbb{Z}/N\mathbb{Z}$), if $Nv \equiv 0$ modulo the period lattice. The second family admits a ($2N$)-periodic reduction, if $N(v_1+v_2) \equiv 0$ modulo the period lattice. We will show that for the 3-periodic and the 4-periodic VC, these elliptic solutions are general solutions.

5.2 Three-periodic Volterra chain: Equations of Motion and Explicit Solution

The 3-periodic reduction of the Volterra chain (VC_3) reads:

$$\begin{cases}
\dot{x}_1 = x_1(x_2-x_3),\\
\dot{x}_2 = x_2(x_3-x_1),\\
\dot{x}_3 = x_3(x_1-x_2).
\end{cases} \tag{5.6}$$

This system is completely integrable, with the following two independent integrals of motion:

$$H_1 = x_1+x_2+x_3, \quad H_2 = x_1x_2x_3. \tag{5.7}$$

Theorem 5.1. *The general solution of (5.6) is given by formulas (5.2) or (5.3) with v being a one third of a period, i.e., $3v \equiv 0$ modulo the period lattice: in terms of ζ-functions,*

$$\begin{aligned}
x_1 &= \zeta(t+v)-\zeta(t)+\zeta(v)-\zeta(2v),\\
x_2 &= \zeta(t+2v)-\zeta(t+v)+\zeta(v)-\zeta(2v),\\
x_3 &= \zeta(t+3v)-\zeta(t+2v)+\zeta(v)-\zeta(2v),
\end{aligned} \tag{5.8}$$

or, in terms of σ-functions,

$$\begin{aligned}
x_1 &= \frac{\sigma(t-v)\sigma(t+2v)}{\sigma(t)\sigma(t+v)\sigma(2v)},\\
x_2 &= \frac{\sigma(t)\sigma(t+3v)}{\sigma(t+v)\sigma(t+2v)\sigma(2v)},\\
x_3 &= \frac{\sigma(t+v)\sigma(t+4v)}{\sigma(t+2v)\sigma(t+3v)\sigma(2v)}.
\end{aligned} \tag{5.9}$$

Proof. Eliminating x_j, x_k from equation of motion for x_i with the help of integrals of motion, one arrives at

$$\dot{x}_i^2 = x_i^2(x_i - H_1)^2 - 4H_2 x_i. \tag{5.10}$$

This shows that the general solution is given by elliptic functions. To make this more precise, we use Halphen's method from Chapter 3 in order to integrate this equation. We get

$$\alpha_0 = 1, \quad \alpha_1 = -\frac{1}{2}H_1, \quad \alpha_2 = -\frac{1}{12}H_1^2, \quad \alpha_3 = -H_2, \quad \alpha_4 = 0.$$

Thus, we find:

$$\wp(v) = \frac{1}{12}H_1^2, \quad \wp'(v) = -H_2, \quad 3\wp^2(v) - g_2 = -\frac{1}{16}H_1^4 + 2H_1H_2.$$

The last formula can be brought with the help of the previous two ones into the form

$$H_1H_2 = 6\wp^2(v) - \frac{1}{2}g_2 = \wp''(v) \quad \Rightarrow \quad 12\wp(v)(\wp'(v))^2 = (\wp''(v))^2.$$

This has to be compared with the duplication formula for the Weierstrass function,

$$\wp(2v) = \frac{1}{4}\left(\frac{\wp''(v)}{\wp'(v)}\right)^2 - 2\wp(v).$$

As a result, we find $\wp(2v) = \wp(v)$, so that $2v \equiv -v$, or $3v \equiv 0$. From the above formulas there follows:

$$H_1 = -\frac{\wp''(v)}{\wp'(v)} = 4\zeta(v) - 2\zeta(2v).$$

(Note that setting $u_1 = u_2 = v$, $u_3 = -2v \equiv v$ in the Frobenius-Stickelberger formula,

$$\wp(u_1) + \wp(u_2) + \wp(u_3) = \big(\zeta(u_1) + \zeta(u_2) + \zeta(u_3)\big)^2, \quad u_1 + u_2 + u_3 = 0,$$

leads to $3\wp(v) = \big(2\zeta(v) - \zeta(2v)\big)^2$ for $3v \equiv 0$.) Finally, each of the coordinates x_i is a time shift of

$$x(t) = \zeta(t+v) - \zeta(t) - \zeta(v) + \frac{1}{2}H_1 = \zeta(t+v) - \zeta(t) + \zeta(v) - \zeta(2v).$$

We may eliminate any x_k between equations (5.7), getting $x_ix_j(x_i+x_j) - H_1x_ix_j + H_2 = 0$. Hence, any pair of functions (x_i, x_j) satisfies a polynomial relation of degree 3, which implies that any two functions x_i and x_j must have one common pole. Therefore, the solutions of eqs. (5.6) must be as in (5.8). □

5.3 HK type Discretization of $\mathbf{VC}_3$

The HK discretization of system (5.6) (with the time step 2ϵ) is:

$$\begin{cases} \widetilde{x}_1 - x_1 = \epsilon x_1(\widetilde{x}_2 - \widetilde{x}_3) + \epsilon \widetilde{x}_1(x_2 - x_3), \\ \widetilde{x}_2 - x_2 = \epsilon x_2(\widetilde{x}_3 - \widetilde{x}_1) + \epsilon \widetilde{x}_2(x_3 - x_1), \\ \widetilde{x}_3 - x_3 = \epsilon x_3(\widetilde{x}_1 - \widetilde{x}_2) + \epsilon \widetilde{x}_3(x_1 - x_2). \end{cases} \tag{5.11}$$

The map $f : x \mapsto \widetilde{x}$ obtained by solving (5.11) for $\widetilde{x}$ is given by:

$$\widetilde{x} = f(x, \epsilon) = A^{-1}(x, \epsilon)x,$$

with

$$A(x, \epsilon) = \begin{pmatrix} 1 + \epsilon(x_3 - x_2) & -\epsilon x_1 & \epsilon x_1 \\ \epsilon x_2 & 1 + \epsilon(x_1 - x_3) & -\epsilon x_2 \\ -\epsilon x_3 & \epsilon x_3 & 1 + \epsilon(x_2 - x_1) \end{pmatrix}.$$

Explicitly:

$$\widetilde{x}_i = x_i \, \frac{1 + 2\epsilon(x_j - x_k) + \epsilon^2\big((x_j + x_k)^2 - x_i^2\big)}{1 - \epsilon^2(x_1^2 + x_2^2 + x_3^2 - 2x_1x_2 - 2x_2x_3 - 2x_3x_1)}. \tag{5.12}$$

This map will be called dVC_3. The following form of equations of motion will be useful, as well:

$$\begin{aligned} \frac{\widetilde{x}_1}{1 + \epsilon(\widetilde{x}_2 - \widetilde{x}_3)} &= \frac{x_1}{1 - \epsilon(x_2 - x_3)}, \\ \frac{\widetilde{x}_2}{1 + \epsilon(\widetilde{x}_3 - \widetilde{x}_1)} &= \frac{x_2}{1 - \epsilon(x_3 - x_1)}, \\ \frac{\widetilde{x}_3}{1 + \epsilon(\widetilde{x}_1 - \widetilde{x}_2)} &= \frac{x_3}{1 - \epsilon(x_1 - x_2)}. \end{aligned} \tag{5.13}$$

Proposition 5.1. *The map dVC_3 possesses an invariant volume form:*

$$\det \frac{\partial \widetilde{x}}{\partial x} = \frac{\phi(\widetilde{x})}{\phi(x)} \quad \Leftrightarrow \quad f^*\omega = \omega, \quad \omega = \frac{dx_1 \wedge dx_2 \wedge dx_3}{\phi(x)},$$

with $\phi(x) = x_1x_2x_3$.

Proof. This follows directly from Proposition 4.3. □

Concerning integrability of dVC_3, we note first of all that H_1 is an obvious conserved quantity. The second one is most easily obtained from the following discrete Wronskian relations.

Proposition 5.2. *For the map dVC_3, the following relations hold:*

$$(\widetilde{x}_i x_j - x_i \widetilde{x}_j)/\epsilon = H_1(\widetilde{x}_i x_j + x_i \widetilde{x}_j) - 6H_2(\epsilon)(1 - \tfrac{1}{3}\epsilon^2 H_1^2), \tag{5.14}$$

where $H_2(\epsilon)$ is a conserved quantity, given by

$$H_2(\epsilon) = \frac{x_1x_2x_3}{1 - \epsilon^2(x_1^2 + x_2^2 + x_3^2 - 2x_1x_2 - 2x_2x_3 - 2x_3x_1)}. \tag{5.15}$$

Proof. Define $H_2(\epsilon)$ by equation (5.14). It is easily computed with explicit formulas (5.12). The result given by (5.15) is a manifestly even function of ϵ and therefore an integral of motion. □

5.4 Solution of the Discrete Equations of Motion

We show how one can approach the problem of solving the discrete equations of motion. What follows is an application of the method outlined at the end of Chapter 3.

Experimental Result 5.1. *The pairs $(x_i, \widetilde{x}_i)$ lie on a symmetric biquadratic curve*

$$P(x_i, \widetilde{x}_i) = p_1 x_i^2 \widetilde{x}_i^2 + p_2 x_i \widetilde{x}_i (x_i + \widetilde{x}_i) + p_3 (x_i^2 + \widetilde{x}_i^2) + p_4 x_i \widetilde{x}_i + p_5 (x_i + \widetilde{x}_i) + p_6$$

with constant coefficients $[p_1 : \ldots : p_6]$ (which can be expressed through integrals of motion).

Proposition 5.1. *The pairs (x_i, x_j) lie on a biquadratic curve of degree 3,*

$$Q(x_i, x_j) = q_1 x_i x_j (x_i + x_j) + q_2 (x_i^2 + x_j^2) + q_3 x_i x_j + q_4 (x_i + x_j) + q_5,$$

with constant coefficients $[q_1 : \ldots : q_5]$ (which can be expressed through integrals of motion).

Proof. Elimination of x_k from (5.15) via $x_k = H_1 - x_i - x_j$. □

The first statement yields that each variable x_i as a function t is an elliptic function of order two (i.e., with two poles within one parallelogram of periods). The second statement yields that every pair of functions x_i and x_j has one common pole.

One can refine this information further. For instance, the biquadratic curves from the first result coincide for all three components x_1, x_2, x_3, therefore all three components are time shifts of one and the same functions. For symmetry reasons, we may assume that $x_2(t) = x_1(t + v)$ and $x_3(v) = x_2(t + v)$, where $3v \equiv 0$. We may therefore assume that the denominators of the functions $x_1(t)$, $x_2(t)$, $x_3(t)$ are

$$\sigma(t)\sigma(t + v), \quad \sigma(t + v)\sigma(t + 2v), \quad \sigma(t + 2v)\sigma(t + 3v),$$

respectively, just as in the solution of VC$_3$. The following observation is crucial.

Experimental Result 5.2. *For any indices i, j, the pairs $(x_i, 1/\widetilde{x}_j)$ lie on a biquadratic curve of degree 3, so that the functions x_i, $1/\widetilde{x}_j$ have a common pole.*

This yields that zeros of $x_i(t)$ are the (2ϵ)-shift and the (-2ϵ)-shift of the common pole of $x_j(t)$ and $x_k(t)$. We arrive at the conclusion that

$$\begin{aligned} x_1 &= \rho\,\frac{\sigma(t - v - 2\epsilon)\sigma(t + 2v + 2\epsilon)}{\sigma(t)\sigma(t + v)}, \\ x_2 &= \rho\,\frac{\sigma(t - 2\epsilon)\sigma(t + 3v + 2\epsilon)}{\sigma(t + v)\sigma(t + 2v)}, \\ x_3 &= \rho\,\frac{\sigma(t + v - 2\epsilon)\sigma(t + 4v + 2\epsilon)}{\sigma(t + 2v)\sigma(t + 3v)}. \end{aligned} \tag{5.16}$$

(The other choice of the signs of the time shifts leads to the same functions, up to a constant factor.) The constant factor ρ can be determined with the help of the following considerations. The functions participating in the representation (5.13) of equation of motion of dVC$_3$ have the following remarkable property.

Experimental Result 5.3. *For any cyclic permutation* (i, j, k) *of* $(1, 2, 3)$*, the elliptic functions*

$$\frac{x_i}{1 \pm \epsilon(x_j - x_k)}$$

are of order 2.

As a consequence of this proposition combined with (5.13), one easily sees that the two zeros of $x_1/\big(1-\epsilon(x_2-x_3)\big)$ must be $v-2\epsilon, v$, while the two zeros of $x_1/\big(1+\epsilon(x_2-x_3)\big)$ must be $v, v+2\epsilon$. In other words, the following relations must hold true:

$$1 - \epsilon(x_2 - x_3)\big|_{t=v+2\epsilon} = 0, \quad 1 + \epsilon(x_2 - x_3)\big|_{t=v-2\epsilon} = 0. \tag{5.17}$$

Upon using formulas (5.16) and taking into account that $3v \equiv 0$, both requirements in (5.17) result in one and the same formula for the factor ρ, namely,

$$\frac{1}{\epsilon\rho} = \frac{\sigma(2v+4\epsilon)\sigma(v)}{\sigma(2\epsilon)\sigma(v+2\epsilon)} + \frac{\sigma(2v)\sigma(v+4\epsilon)}{\sigma(v-2\epsilon)\sigma(2\epsilon)}. \tag{5.18}$$

To simplify this expression, we observe that

$$\sigma(2v+4\epsilon)\sigma(v)\sigma(v-2\epsilon)\sigma(2\epsilon)+\sigma(2v)\sigma(v+4\epsilon)\sigma(v+2\epsilon)\sigma(2\epsilon) = \sigma(2v+2\epsilon)\sigma(v+2\epsilon)\sigma(v)\sigma(4\epsilon).$$

This follows from the famous three-term functional equation for the σ-function (3.1) with the choice

$$z = \frac{3v}{2} + 2\epsilon, \quad a = \frac{v}{2} + 2\epsilon, \quad b = \frac{v}{2} - 2\epsilon, \quad c = -\frac{v}{2}.$$

Thus, we get

$$\frac{1}{\epsilon\rho} = \frac{\sigma(2v+2\epsilon)\sigma(v)\sigma(4\epsilon)}{\sigma(v-2\epsilon)\sigma^2(2\epsilon)}. \tag{5.19}$$

We arrive at the following statement.

Theorem 5.2. *The general solution of* dVC_3 *is given by formulas (5.16) and (5.19) with* $3v \equiv 0$*. In terms of* ζ*-functions,*

$$\begin{aligned} x_1 &= \rho_1(\zeta(t+v) - \zeta(t) + \zeta(v+2\epsilon) - \zeta(2v+2\epsilon)), \\ x_2 &= \rho_1(\zeta(t+2v) - \zeta(t+v) + \zeta(v+2\epsilon) - \zeta(2v+2\epsilon)), \\ x_3 &= \rho_1(\zeta(t+3v) - \zeta(t+2v) + \zeta(v+2\epsilon) - \zeta(2v+2\epsilon)), \end{aligned} \tag{5.20}$$

with

$$\frac{1}{\epsilon\rho_1} = \frac{\sigma^2(v)\sigma(4\epsilon)}{\sigma(v+2\epsilon)\sigma(v-2\epsilon)\sigma^2(2\epsilon)} = 2\zeta(2\epsilon) - \zeta(v+2\epsilon) + \zeta(v-2\epsilon). \tag{5.21}$$

Proof. We will verify that formulas (5.16) and (5.19) with $3v \equiv 0$ give solutions of equations of motion (5.13), indeed.

First verification. We have:

$$\begin{aligned}
\frac{1}{\epsilon\rho}-\frac{1}{\rho}(x_2-x_3) \quad &= \\
=\frac{\sigma(2v)\sigma(v+4\epsilon)}{\sigma(v-2\epsilon)\sigma(2\epsilon)}&+\frac{\sigma(2v+4\epsilon)\sigma(v)}{\sigma(v+2\epsilon)\sigma(2\epsilon)}-\frac{\sigma(t-2\epsilon)\sigma(t+3v+2\epsilon)}{\sigma(t+v)\sigma(t+2v)}+\frac{\sigma(t+v-2\epsilon)\sigma(t+4v+2\epsilon)}{\sigma(t+2v)\sigma(t+3v)} \\
=\frac{\sigma(t+v)\sigma(t+2v)\sigma(2v)\sigma(v+4\epsilon)-\sigma(t-2\epsilon)\sigma(t+3v+2\epsilon)\sigma(v-2\epsilon)\sigma(2\epsilon)}{\sigma(t+v)\sigma(t+2v)\sigma(v-2\epsilon)\sigma(2\epsilon)}& \\
+\frac{\sigma(t+2v)\sigma(t+3v)\sigma(2v+4\epsilon)\sigma(v)+\sigma(t+v-2\epsilon)\sigma(t+4v+2\epsilon)\sigma(v+2\epsilon)\sigma(2\epsilon)}{\sigma(t+2v)\sigma(t+3v)\sigma(v+2\epsilon)\sigma(2\epsilon)}&
\end{aligned}$$

Applying formula (3.1) twice, first with

$$z=t+\frac{3v}{2},\quad a=\frac{v}{2},\quad b=\frac{3v}{2}+2\epsilon,\quad c=\frac{v}{2}-2\epsilon,$$

and then with

$$z=t+\frac{5v}{2},\quad a=\frac{v}{2},\quad b=\frac{3v}{2}+2\epsilon,\quad c=\frac{v}{2}+2\epsilon,$$

we obtain:

$$\begin{aligned}
\frac{1}{\epsilon\rho}-\frac{1}{\rho}(x_2-x_3) \quad &= \\
=\frac{\sigma(t+2v-2\epsilon)\sigma(t+v+2\epsilon)\sigma(2v+2\epsilon)\sigma(v+2\epsilon)}{\sigma(t+v)\sigma(t+2v)\sigma(v-2\epsilon)\sigma(2\epsilon)}& \\
+\frac{\sigma(t+3v+2\epsilon)\sigma(t+2v-2\epsilon)\sigma(2v+2\epsilon)\sigma(v+2\epsilon)}{\sigma(t+2v)\sigma(t+3v)\sigma(v+2\epsilon)\sigma(2\epsilon)}& \\
=\frac{\sigma(t+2v-2\epsilon)\sigma(2v+2\epsilon)\sigma(v+2\epsilon)}{\sigma(t+2v)\sigma(2\epsilon)}&\left(\frac{\sigma(t+v+2\epsilon)}{\sigma(v-2\epsilon)\sigma(t+v)}+\frac{\sigma(t+3v+2\epsilon)}{\sigma(v+2\epsilon)\sigma(t+3v)}\right). \qquad (5.22)
\end{aligned}$$

A similar computation:

$$\begin{aligned}
\frac{1}{\epsilon\rho}+\frac{1}{\rho}(x_2-x_3) \quad &= \\
=\frac{\sigma(2v)\sigma(v+4\epsilon)}{\sigma(v-2\epsilon)\sigma(2\epsilon)}&+\frac{\sigma(2v+4\epsilon)\sigma(v)}{\sigma(v+2\epsilon)\sigma(2\epsilon)}+\frac{\sigma(t-2\epsilon)\sigma(t+3v+2\epsilon)}{\sigma(t+v)\sigma(t+2v)}-\frac{\sigma(t+v-2\epsilon)\sigma(t+4v+2\epsilon)}{\sigma(t+2v)\sigma(t+3v)} \\
=\frac{\sigma(t+v)\sigma(t+2v)\sigma(2v+4\epsilon)\sigma(v)+\sigma(t-2\epsilon)\sigma(t+3v+2\epsilon)\sigma(v+2\epsilon)\sigma(2\epsilon)}{\sigma(t+v)\sigma(t+2v)\sigma(v+2\epsilon)\sigma(2\epsilon)}& \\
+\frac{\sigma(t+2v)\sigma(t+3v)\sigma(2v)\sigma(v+4\epsilon)-\sigma(t+v-2\epsilon)\sigma(t+4v+2\epsilon)\sigma(v-2\epsilon)\sigma(2\epsilon)}{\sigma(t+2v)\sigma(t+3v)\sigma(v-2\epsilon)\sigma(2\epsilon)}&
\end{aligned}$$

Applying formula (3.1) twice, first with

$$z=t+\frac{3v}{2},\quad a=\frac{v}{2},\quad b=\frac{3v}{2}+2\epsilon,\quad c=\frac{v}{2}+2\epsilon,$$

and then with

$$z = t + \frac{5v}{2}, \quad a = \frac{v}{2}, \quad b = \frac{3v}{2} + 2\epsilon, \quad c = \frac{v}{2} - 2\epsilon,$$

we obtain:

$$\begin{aligned}
&\frac{1}{\epsilon\rho} + \frac{1}{\rho}(x_2 - x_3) \quad = \\
&= \frac{\sigma(t+2v+2\epsilon)\sigma(t+v-2\epsilon)\sigma(2v+2\epsilon)\sigma(v+2\epsilon)}{\sigma(t+v)\sigma(t+2v)\sigma(v+2\epsilon)\sigma(2\epsilon)} \\
&\quad + \frac{\sigma(t+3v-2\epsilon)\sigma(t+2v+2\epsilon)\sigma(2v+2\epsilon)\sigma(v+2\epsilon)}{\sigma(t+2v)\sigma(t+3v)\sigma(v-2\epsilon)\sigma(2\epsilon)} \\
&= \frac{\sigma(t+2v+2\epsilon)\sigma(2v+2\epsilon)\sigma(v+2\epsilon)}{\sigma(t+2v)\sigma(2\epsilon)} \left(\frac{\sigma(t+v-2\epsilon)}{\sigma(v+2\epsilon)\sigma(t+v)} + \frac{\sigma(t+3v-2\epsilon)}{\sigma(v-2\epsilon)\sigma(t+3v)} \right).
\end{aligned} \tag{5.23}$$

Now a simple computation shows that, up to a constant factor,

$$\begin{aligned}
\frac{1-\epsilon(x_2-x_3)}{x_1} &= \frac{1+\epsilon(\widetilde{x}_2-\widetilde{x}_3)}{\widetilde{x}_1} \\
&\simeq \frac{\sigma(t+v+2\epsilon)\sigma(t+3v)\sigma(v+2\epsilon)+\sigma(t+v)\sigma(t+3v+2\epsilon)\sigma(v-2\epsilon)}{\sigma(t+2v)\sigma(t+2v+2\epsilon)}.
\end{aligned}$$

Indeed, the quantities in the first line are proportional to the quantity in the second line with the factors

$$\frac{\sigma(t)\sigma(t+2v-2\epsilon)}{\sigma(t-v-2\epsilon)\sigma(t+3v)}, \quad \text{resp.} \quad \frac{\sigma(t+2\epsilon)\sigma(t+2v)}{\sigma(t-v)\sigma(t+3v+2\epsilon)},$$

which are both constant (and equal), since they are elliptic functions without zeros and poles, due to $3v \equiv 0$.

Second verification. Applying formula (3.2) twice, one obtains

$$\begin{aligned}
\frac{1}{\epsilon\rho_1} - \frac{1}{\rho_1}(x_2-x_3) \quad &= \quad 2\zeta(2\epsilon) - \zeta(v+2\epsilon) + \zeta(v-2\epsilon) - 2\zeta(t+2v) + \zeta(t+v) + \zeta(t+3v) \\
&= \Big(\zeta(2\epsilon) - \zeta(v+2\epsilon) - \zeta(t+2v) + \zeta(t+3v)\Big) + \Big(\zeta(2\epsilon) + \zeta(v-2\epsilon) + \zeta(t+v) - \zeta(t+2v)\Big) \\
&= \frac{\sigma(v)\sigma(t+2v-2\epsilon)\sigma(t+3v+2\epsilon)}{\sigma(2\epsilon)\sigma(v+2\epsilon)\sigma(t+2v)\sigma(t+3v)} + \frac{\sigma(v)\sigma(t+v+2\epsilon)\sigma(t+2v-2\epsilon)}{\sigma(2\epsilon)\sigma(v-2\epsilon)\sigma(t+v)\sigma(t+2v)} \\
&= \frac{\sigma(v)\sigma(t+2v-2\epsilon)}{\sigma(2\epsilon)\sigma(t+2v)} \left(\frac{\sigma(t+3v+2\epsilon)}{\sigma(v+2\epsilon)\sigma(t+3v)} + \frac{\sigma(t+v+2\epsilon)}{\sigma(v-2\epsilon)\sigma(t+v)} \right).
\end{aligned} \tag{5.24}$$

Similarly:

$$\begin{aligned}
\frac{1}{\epsilon\rho_1} + \frac{1}{\rho_1}(\widetilde{x}_2 - \widetilde{x}_3) \quad &= \\
&= 2\zeta(2\epsilon) - \zeta(v+2\epsilon) + \zeta(v-2\epsilon) + 2\zeta(t+2v+2\epsilon) - \zeta(t+v+2\epsilon) - \zeta(t+3v+2\epsilon) \\
&= \Big(\zeta(2\epsilon) - \zeta(v+2\epsilon) - \zeta(t+v+2\epsilon) + \zeta(t+2v+2\epsilon)\Big) \\
&\quad + \Big(\zeta(2\epsilon) + \zeta(v-2\epsilon) + \zeta(t+2v+2\epsilon) - \zeta(t+3v+2\epsilon)\Big) \\
&= \frac{\sigma(v)\sigma(t+v)\sigma(t+2v+4\epsilon)}{\sigma(2\epsilon)\sigma(v+2\epsilon)\sigma(t+v+2\epsilon)\sigma(t+2v+2\epsilon)} + \frac{\sigma(v)\sigma(t+3v)\sigma(t+2v+4\epsilon)}{\sigma(2\epsilon)\sigma(v-2\epsilon)\sigma(t+2v+2\epsilon)\sigma(t+3v+2\epsilon)} \\
&= \frac{\sigma(v)\sigma(t+2v+4\epsilon)}{\sigma(2\epsilon)\sigma(t+2v+2\epsilon)}\left(\frac{\sigma(t+v)}{\sigma(v+2\epsilon)\sigma(t+v+2\epsilon)} + \frac{\sigma(t+3v)}{\sigma(v-2\epsilon)\sigma(t+3v+2\epsilon)}\right).
\end{aligned} \tag{5.25}$$

From this point, the second verification proceeds literally as the first one. □

It remains to express v and the invariants g_2 and g_3 in terms of the integrals of motion. This can be achieved using the method described at the end of Chapter 3, Section 3.2.

5.5 Periodic Volterra Chain with $N = 4$ Particles

Equations of motion of VC_4 are:

$$\begin{cases}
\dot{x}_1 = x_1(x_2 - x_4), \\
\dot{x}_2 = x_2(x_3 - x_1), \\
\dot{x}_3 = x_3(x_4 - x_2), \\
\dot{x}_4 = x_4(x_1 - x_3).
\end{cases} \tag{5.26}$$

This system possesses three obvious integrals of motion: $H_1 = x_1 + x_2 + x_3 + x_4$, $H_2 = x_1x_3$, and $H_3 = x_2x_4$.

Theorem 5.3. *The general solution of VC_4 is given by*

$$\begin{aligned}
x_1 &= \zeta(t+v_1) - \zeta(t) + \zeta(v_2) - \zeta(v_1+v_2), \\
x_2 &= \zeta(t+v_1+v_2) - \zeta(t+v_1) + \zeta(v_1) - \zeta(v_1+v_2), \\
x_3 &= \zeta(t+2v_1+v_2) - \zeta(t+v_1+v_2) + \zeta(v_2) - \zeta(v_1+v_2), \\
x_4 &= \zeta(t+2v_1+2v_2) - \zeta(t+2v_1+v_2) + \zeta(v_1) - \zeta(v_1+v_2),
\end{aligned}$$

where $2(v_1 + v_2) \equiv 0$. *In terms of* σ*-functions:*

$$x_1 = \rho_1 \frac{\sigma(t - v_2)\sigma(t + v_1 + v_2)}{\sigma(t)\sigma(t + v_1)}, \tag{5.27}$$

$$x_2 = \rho_2 \frac{\sigma(t)\sigma(t + 2v_1 + v_2)}{\sigma(t + v_1)\sigma(t + v_1 + v_2)}, \tag{5.28}$$

$$x_3 = \rho_1 \frac{\sigma(t + v_1)\sigma(t + 2v_1 + 2v_2)}{\sigma(t + v_1 + v_2)\sigma(t + 2v_1 + v_2)}, \tag{5.29}$$

$$x_4 = \rho_2 \frac{\sigma(t + v_1 + v_2)\sigma(t + 3v_1 + 2v_2)}{\sigma(t + 2v_1 + v_2)\sigma(t + 2v_1 + 2v_2)}, \tag{5.30}$$

where

$$\rho_1 = \frac{\sigma(v_1)}{\sigma(v_2)\sigma(v_1 + v_2)}, \quad \rho_2 = \frac{\sigma(v_2)}{\sigma(v_1)\sigma(v_1 + v_2)}. \tag{5.31}$$

Proof. One easily finds that x_1, x_3 satisfy the differential equation

$$\dot{x}_1^2 = (x_1^2 - H_1 x_1 + H_2)^2 - 4H_3 x_1^2,$$

while x_2, x_4 satisfy a similar equation with $H_2 \leftrightarrow H_3$. This immediately leads to solution in terms of elliptic functions. We apply Halphen's method to the above equation for x_1 and obtain

$$\alpha_0 = 1, \quad \alpha_1 = -\frac{1}{2}H_1, \quad \alpha_2 = \frac{1}{6}H_1^2 + \frac{1}{3}H_2 - \frac{2}{3}H_3, \quad \alpha_3 = \frac{1}{2}H_1 H_2, \quad \alpha_4 = H_2^2,$$

so that

$$\begin{aligned} g_2 &= \frac{1}{12}{H_1}^4 - \frac{2}{3}{H_1}^2 H_2 - \frac{2}{3}{H_1}^2 H_3 + \frac{4}{3}{H_2}^2 - \frac{4}{3}H_2 H_3 + \frac{4}{3}{H_3}^2 \\ g_3 &= -\frac{1}{216}{H_1}^6 + \frac{1}{18}{H_1}^4 H_2 - \frac{2}{9}{H_1}^2 {H_2}^2 + \frac{1}{18}{H_1}^4 H_3 \\ &\quad + \frac{8}{27}{H_2}^3 - \frac{4}{9}H_3 {H_2}^2 - \frac{2}{9}{H_3}^2 {H_1}^2 - \frac{1}{9}H_2 {H_1}^2 H_3 - \frac{4}{9}H_2 {H_3}^2 + \frac{8}{27}{H_3}^3. \end{aligned}$$

g_2 and g_3 are symmetric with respect to the interchange $H_2 \leftrightarrow H_3$ (as they should be), which implies that all functions x_i are elliptic functions with respect to the same period lattice.

We conclude that x_1 and x_3 are given by time shifts of the function

$$x_{1,3}(t) = \zeta(t + v_1) - \zeta(t) - \zeta(v_1) + \frac{1}{2}H_1,$$

where

$$\wp(v_1) = \frac{1}{12}H_1^2 - \frac{1}{3}H_2 + \frac{2}{3}H_3, \quad \wp'(v_1) = -H_3 H_1.$$

Similarly, x_2 and x_4 are time shifts of the function

$$x_{2,4}(t) = \zeta(t + v_2) - \zeta(t) - \zeta(v_2) + \frac{1}{2}H_1,$$

with
$$\wp(v_2) = \frac{1}{12}H_1^2 - \frac{1}{3}H_3 + \frac{2}{3}H_2, \quad \wp'(v_2) = -H_2H_1.$$
With the help of the addition formula
$$\wp(v_1) + \wp(v_2) + \wp(v_1 + v_2) = \frac{1}{4}\left(\frac{\wp'(v_1) - \wp'(v_2)}{\wp(v_1) - \wp(v_2)}\right)^2$$
we find
$$\wp(v_1 + v_2) = \frac{1}{12}H_1^2 - \frac{1}{3}H_2 - \frac{1}{3}H_3,$$
which gives
$$\wp'(v_1 + v_2)^2 = 4\wp(v_1 + v_2)^3 - g_2\wp(v_1 + v_2) - g_3 = 0.$$
This implies that $\wp(v_1 + v_2)$ is one of the roots of the Weierstrass cubic $4z^3 - g_2z - g_3$, which means that $\wp(v_1 + v_2) = \wp(\omega_i)$, where ω_i is one of the half periods of the period lattice corresponding to g_2 and g_3. Hence, $v_1 + v_2$ must be equal to a half period modulo the period lattice. Therefore, $2(v_1 + v_2) \equiv 0$.

From the above formulas there also follows that
$$\zeta(v_1 + v_2) - \zeta(v_1) - \zeta(v_2) = \frac{1}{2}\frac{\wp'(v_1) - \wp'(v_2)}{\wp(v_1) - \wp(v_2)} = -\frac{1}{2}H_1.$$
Finally, since H_1, H_2 and H_3 must be conserved quantities, it is easy to convince oneself that x_i must be of the form stated in the theorem. □

5.6 HK type Discretization of VC$_4$

The HK discretization (denoted by dVC$_4$) of VC$_4$ reads:
$$\begin{cases} \widetilde{x}_1 - x_1 = \epsilon x_1(\widetilde{x}_2 - \widetilde{x}_4) + \epsilon\widetilde{x}_1(x_2 - x_4), \\ \widetilde{x}_2 - x_2 = \epsilon x_2(\widetilde{x}_3 - \widetilde{x}_1) + \epsilon\widetilde{x}_2(x_3 - x_1), \\ \widetilde{x}_3 - x_3 = \epsilon x_3(\widetilde{x}_4 - \widetilde{x}_2) + \epsilon\widetilde{x}_3(x_4 - x_2), \\ \widetilde{x}_4 - x_4 = \epsilon x_4(\widetilde{x}_1 - \widetilde{x}_3) + \epsilon\widetilde{x}_4(x_1 - x_4). \end{cases} \tag{5.32}$$
It possesses an obvious integral $H_1 = x_1 + x_2 + x_3 + x_4$. Equations of motion can be equivalently re-written as
$$\frac{\widetilde{x}_1}{1 + \epsilon(\widetilde{x}_2 - \widetilde{x}_4)} = \frac{x_1}{1 - \epsilon(x_2 - x_4)}, \tag{5.33}$$
$$\frac{\widetilde{x}_2}{1 + \epsilon(\widetilde{x}_3 - \widetilde{x}_1)} = \frac{x_2}{1 - \epsilon(x_3 - x_1)}, \tag{5.34}$$
$$\frac{\widetilde{x}_3}{1 + \epsilon(\widetilde{x}_4 - \widetilde{x}_2)} = \frac{x_3}{1 - \epsilon(x_4 - x_2)}, \tag{5.35}$$
$$\frac{\widetilde{x}_4}{1 + \epsilon(\widetilde{x}_1 - \widetilde{x}_3)} = \frac{x_2}{1 - \epsilon(x_1 - x_3)}, \tag{5.36}$$
which immediately leads to two further integrals of motion,
$$H_2(\epsilon) = \frac{x_1x_3}{1 - \epsilon^2(x_2 - x_4)^2}, \quad H_3(\epsilon) = \frac{x_2x_4}{1 - \epsilon^2(x_1 - x_3)^2}. \tag{5.37}$$

5.7 Solution of the Discrete Equations of Motion

We now proceed and show how one can obtain elliptic solutions for dVC_4.

Experimental Result 5.4. *For the iterates of the map dVC_4 the pairs $(x_i, \widetilde{x}_i)$ lie on a biquartic curve of genus 1 with constant coefficients (which can be expressed through integrals of motion). The biquartic curves coincide for x_1 and x_3, as well as for x_2 and x_4*

This yields that x_i as functions of t are elliptic functions of degree 4 (i.e., with four poles within one parallelogram of periods). Moreover, x_1 and x_3 are time shifts of one and the same function, and the same for x_2 and x_4.

Proposition 5.2. *For the iterates of map dVC_4:*

a) *The pairs (x_i, x_j) with i, j of different parity lie on a quartic curve whose coefficients are constant (expressed through integrals of motion);*

b) *The pairs (x_i, x_j) with i, j of the same parity lie on a curve of degree 2 with constant coefficients.*

Proof. Statements a) and b) both follow by eliminating x_k, x_ℓ from integrals $H_1, H_2(\epsilon), H_3(\epsilon)$. □

Hence, all x_i have the same poles.

Experimental Result 5.5. *The pairs $(x_1 + x_3, \widetilde{x}_1 + \widetilde{x}_3)$ lie on a biquadratic curve, and the same holds true for the pairs $(x_2 + x_4, \widetilde{x}_2 + \widetilde{x}_4)$.*

Thus, functions $x_1 + x_3$ and $x_2 + x_4$ are of degree 2, which has the following explanation: for any $(2T)$-periodic function $f(t)$, the function $g(t) = f(t) + f(t + T)$ is T-periodic. Thus, the time shift relating x_1 and x_3 should be a half-period, and the same for x_2 and x_4. Therefore, we always assume

$$x_3(t) = x_1(t + v_1 + v_2), \quad x_4(t) = x_2(t + v_1 + v_2), \quad 2(v_1 + v_2) \equiv 0. \tag{5.38}$$

We denote the common poles of x_i by 0, $-v_1$, $-(v_1 + v_2)$, $-(2v_1 + v_2)$.

The next piece of information:

Experimental Result 5.6. *The pairs $(1/x_i, 1/\widetilde{x}_i)$ lie on a biquartic curve of degree 6.*

This means that $x_i(t)$ and $\widetilde{x}_i(t)$ have two common zeros. We denote the zeros of x_1 by $-a, -(a - 2\epsilon), -b, -(b - 2\epsilon)$, and the zeros of x_2 by $-c, -(c - 2\epsilon), -d, -(d - 2\epsilon)$. Thus, we can finally write down the factorized expressions for x_1, x_2:

$$\begin{aligned}
x_1 &= \rho_1 \frac{\sigma(t+a)\sigma(t+a+2v_1+2v_2-2\epsilon)\sigma(t+b)\sigma(t+b-2\epsilon)}{\sigma(t)\sigma(t+v_1)\sigma(t+v_1+v_2)\sigma(t+2v_1+v_2)}, \\
x_2 &= \rho_2 \frac{\sigma(t+c)\sigma(t+c+2v_1+2v_2-2\epsilon)\sigma(t+d)\sigma(t+d-2\epsilon)}{\sigma(t+v_1)\sigma(t+v_1+v_2)\sigma(t+2v_1+v_2)\sigma(t+2v_1+2v_2)}.
\end{aligned}$$

This choice of the factorization is justified by the continuous limit $\epsilon \to 0$ to the expressions (5.27), (5.28) tells us that

$$a \approx -v_2, \quad b \approx v_1 + v_2, \quad b + a - 2\epsilon = v_1, \tag{5.39}$$

$$c \approx 0, \quad d \approx 2v_1 + v_2, \quad c + d - 2\epsilon = 2v_1 + v_2. \tag{5.40}$$

Eliminating b and d from the above expressions, we find:

$$\begin{aligned} x_1 &= \rho_1 \frac{\sigma(t+a)\sigma(t-a+v_1+2\epsilon)\sigma(t-a+v_1)\sigma(t+a+2v_1+2v_2-2\epsilon)}{\sigma(t)\sigma(t+v_1)\sigma(t+v_1+v_2)\sigma(t+2v_1+v_2)}, \\ x_2 &= \rho_2 \frac{\sigma(t+c)\sigma(t-c+2v_1+v_2)\sigma(t-c+2v_1+v_2+2\epsilon)\sigma(t+c+2v_1+2v_2-2\epsilon)}{\sigma(t+v_1)\sigma(t+v_1+v_2)\sigma(t+2v_1+v_2)\sigma(t+2v_1+2v_2)}. \end{aligned}$$

Factorized expressions for x_3, x_4 follow by (5.38). However, it turns out to be convenient to have expressions for these variables with the same denominators as for x_1, x_2, respectively. This is achieved by using the quasi-periodicity of the σ-function with respect to the period $2(v_1+v_2) \equiv 0$:

$$\begin{aligned} x_3 &= \rho_1 \frac{\sigma(t-a-v_2+2\epsilon)\sigma(t+a+v_1+v_2)\sigma(t+a+v_1+v_2-2\epsilon)\sigma(t-a+2v_1+v_2)}{\sigma(t)\sigma(t+v_1)\sigma(t+v_1+v_2)\sigma(t+2v_1+v_2)}, \\ x_4 &= \rho_2 \frac{\sigma(t-c+v_1+2\epsilon)\sigma(t+c+v_1+v_2)\sigma(t+c+v_1+v_2-2\epsilon)\sigma(t-c+3v_1+2v_2)}{\sigma(t+v_1)\sigma(t+v_1+v_2)\sigma(t+2v_1+v_2)\sigma(t+2v_1+2v_2)}. \end{aligned}$$

Next, we have to find the remaining unknowns a and c, as well as ρ_1 and ρ_2. This can be achieved with the help of the further piece of information about the solutions which is obtained in the computer-assisted manner.

Experimental Result 5.7. *For each* $i = 1,2,3,4$, *the pairs*

$$\left(\frac{x_i}{1 \pm \epsilon(x_j - x_k)}, \frac{\widetilde{x}_i}{1 \pm \epsilon(\widetilde{x}_j - \widetilde{x}_k)} \right),$$

where $j = i+1 \pmod 4$, $k = i-1 \pmod 4$, *lie on a symmetric biquadratic curve. Thus, the elliptic functions*

$$\frac{x_i}{1 \pm \epsilon(x_j - x_k)}$$

are of order 2.

From (5.33) there follows that the two zeros of $x_1/(1-\epsilon(x_2-x_4))$ are $-a, -b$, while the two zeros of $x_1/(1+\epsilon(x_2-x_4))$ are $-(a-2\epsilon), -(b-2\epsilon)$. Therefore,

$$1 - \epsilon(x_2 - x_4)|_{t=-a+2\epsilon} = 0, \quad 1 - \epsilon(x_2 - x_4)|_{t=-b+2\epsilon} = 0, \tag{5.41}$$

$$1 + \epsilon(x_2 - x_4)|_{t=-a} = 0, \quad 1 + \epsilon(x_2 - x_4)|_{t=-b} = 0. \tag{5.42}$$

Similarly, from (5.34) there follows that the two zeros of $x_2/(1-\epsilon(x_3-x_1))$ are $-c, -d$, while the two zeros of $x_2/(1+\epsilon(x_3-x_1))$ are $-(c-2\epsilon), -(d-2\epsilon)$, so that

$$1 - \epsilon(x_3 - x_1)|_{t=-c+2\epsilon} = 0, \quad 1 - \epsilon(x_3 - x_1)|_{t=-d+2\epsilon} = 0, \tag{5.43}$$

$$1+\epsilon(x_3-x_1)|_{t=-c}=0, \quad 1+\epsilon(x_3-x_1)|_{t=-d}=0. \tag{5.44}$$

From (5.35) we deduce that the two zeros of $x_3/(1-\epsilon(x_4-x_2))$ are $-a+v_1+v_2, -b+v_1+v_2$, while the two zeros of $x_3/(1+\epsilon(x_4-x_2))$ are $-(a-2\epsilon)+v_1+v_2, -(b-2\epsilon)+v_1+v_2$, so that

$$1-\epsilon(x_4-x_2)|_{t=-a+2\epsilon+v_1+v_2}=0, \quad 1-\epsilon(x_4-x_2)|_{t=-b+2\epsilon+v_1+v_2}=0, \tag{5.45}$$

$$1+\epsilon(x_4-x_2)|_{t=-a+v_1+v_2}=0, \quad 1+\epsilon(x_4-x_2)|_{t=-b+v_1+v_2}=0. \tag{5.46}$$

Finally, from (5.36) we conclude that the two zeros of $x_4/(1-\epsilon(x_1-x_3))$ are $-c+v_1+v_2, -d+v_1+v_2$, while the two zeros of $x_4/(1+\epsilon(x_1-x_3))$ are $-(c-2\epsilon)+v_1+v_2, -(d-2\epsilon)+v_1+v_2$, so that

$$1-\epsilon(x_1-x_3)|_{t=-c+2\epsilon+v_1+v_2}=0, \quad 1-\epsilon(x_1-x_3)|_{t=-d+2\epsilon+v_1+v_2}=0, \tag{5.47}$$

$$1+\epsilon(x_1-x_3)|_{t=-c+v_1+v_2}=0, \quad 1+\epsilon(x_1-x_3)|_{t=-d+v_1+v_2}=0. \tag{5.48}$$

Let us first concentrate on equations (5.43), (5.44), (5.47), (5.48). They result in eight conditions for a, c and ρ_1. We show that actually almost all these conditions are equivalent, so that we are actually left with one condition for c and one expression for ρ_1 through c and a. For this aim, we first apply the tree-term formula (3.1) to obtain

$$\begin{aligned}
\frac{1}{\rho_1}(x_1-x_3) &= \\
&= \frac{\sigma(t+a)\sigma(t-a+v_1)\sigma(t-a+v_1+2\epsilon)\sigma(t+a+2v_1+2v_2-2\epsilon)}{\sigma(t)\sigma(t+v_1)\sigma(t+v_1+v_2)\sigma(t+2v_1+v_2)} \\
&\quad -\frac{\sigma(t-a-v_2+2\epsilon)\sigma(t+a+v_1+v_2-2\epsilon)\sigma(t+a+v_1+v_2)\sigma(t-a+2v_1+v_2)}{\sigma(t)\sigma(t+v_1)\sigma(t+v_1+v_2)\sigma(t+2v_1+v_2)} \\
&= -\frac{\sigma(2t+2v_1+v_2)\sigma(v_1+v_2)\sigma(v_1+v_2-2\epsilon)\sigma(2a+v_2-2\epsilon)}{\sigma(t)\sigma(t+v_1)\sigma(t+v_1+v_2)\sigma(t+2v_1+v_2)},
\end{aligned}$$

where we have used the following values of the variables:

$$z=t+\frac{v_1}{2}, \quad a=a-\frac{v_1}{2}, \quad b=\frac{v_1}{2}+v_2+a-2\epsilon, \quad c=t+\frac{3v_1}{2}+v_2.$$

This function changes its sign by the shift $t\mapsto t+v_1+v_2$, therefore conditions (5.47), (5.48) are equivalent to (5.43), (5.44). Similarly, this function changes it sign by $t\mapsto -t-2v_1-v_2$, therefore conditions (5.43) and (5.44) are equivalent. Thus, we can consider the first conditions in each of (5.43), (5.44) only. They result in two values for ρ_1:

$$\begin{aligned}
\epsilon\rho_1 &= \frac{\sigma(-c)\sigma(-c+v_1)\sigma(-c+v_1+v_2)\sigma(-c+2v_1+v_2)}{\sigma(-2c+2v_1+v_2)\sigma(v_1+v_2)\sigma(v_1+v_2-2\epsilon)\sigma(2a+v_2-2\epsilon)} \\
&= -\frac{\sigma(-c+2\epsilon)\sigma(-c+v_1+2\epsilon)\sigma(-c+v_1+v_2+2\epsilon)\sigma(-c+2v_1+v_2+2\epsilon)}{\sigma(-2c+2v_1+v_2+4\epsilon)\sigma(v_1+v_2)\sigma(v_1+v_2-2\epsilon)\sigma(2a+v_2-2\epsilon)}.
\end{aligned} \tag{5.49}$$

The condition that these two expressions coincide reads:

$$\frac{\sigma(c)\sigma(v_1-c)\sigma(v_1+v_2-c)\sigma(2v_1+v_2-c)\sigma(2v_1+v_2+4\epsilon-2c)}{\sigma(2\epsilon-c)\sigma(v_1+2\epsilon-c)\sigma(v_1+v_2+2\epsilon-c)\sigma(2v_1+v_2+2\epsilon-c)\sigma(2v_1+v_2-2c)}=1. \tag{5.50}$$

The second computation is absolutely similar:

$$\begin{aligned}
\frac{1}{\rho_2}(x_2-x_4) \quad &= \\
&= \frac{\sigma(t+c)\sigma(t-c+2v_1+v_2)\sigma(t-c+2v_1+v_2+2\epsilon)\sigma(t+c+2v_1+2v_2-2\epsilon)}{\sigma(t+v_1)\sigma(t+v_1+v_2)\sigma(t+2v_1+v_2)\sigma(t+2v-1+2v_2)} \\
&\quad -\frac{\sigma(t-c+v_1+2\epsilon)\sigma(t+c+v_1+v_2-2\epsilon)\sigma(t+c+v_1+v_2)\sigma(t-c+3v_1+2v_2)}{\sigma(t+v_1)\sigma(t+v_1+v_2)\sigma(t+2v_1+v_2)\sigma(t+2v_1+2v_2)} \\
&= \frac{\sigma(2t+3v_1+2v_2)\sigma(v_1+v_2)\sigma(v_1+v_2-2\epsilon)\sigma(2c-v_1-2\epsilon)}{\sigma(t+v_1)\sigma(t+v_1+v_2)\sigma(t+2v_1+v_2)\sigma(t+2v_1+2v_2)},
\end{aligned}$$

where we have used the three-term relation (3.1) with the following values of the variables:

$$z=t+v_1+\frac{v_2}{2}, \quad a=c-v_1-\frac{v_2}{2}, \quad b=-c-\frac{v_2}{2}+2\epsilon, \quad c=t+2v_1+\frac{3v_2}{2}.$$

Also this function changes its sign by the shift $t\mapsto t+v_1+v_2$, therefore conditions (5.45), (5.46) are equivalent to (5.41), (5.42). This function changes it sign also by $t\mapsto -t-3v_1-2v_2$, which, combined with the first shift performed twice, gives changing the sign under $t\mapsto -t-v_1$. Therefore conditions (5.41) and (5.42) are equivalent. We can consider the conditions corresponding to $t=-a-v_1-v_2$ and to $t=-a-v_1-v_2+2\epsilon$ only. They result in two values for ρ_2:

$$\begin{aligned}
\epsilon\rho_2 \quad &= \quad \frac{\sigma(-a-v_2)\sigma(-a)\sigma(-a+v_1)\sigma(-a+v_1+v_2)}{\sigma(-2a+v_1)\sigma(v_1+v_2)\sigma(v_1+v_2-2\epsilon)\sigma(2c-v_1-2\epsilon)} \qquad (5.51) \\
&= \quad -\frac{\sigma(-a-v_2+2\epsilon)\sigma(-a+2\epsilon)\sigma(-a+v_1+2\epsilon)\sigma(-a+v_1+v_2+2\epsilon)}{\sigma(-2a+v_1+4\epsilon)\sigma(v_1+v_2)\sigma(v_1+v_2-2\epsilon)\sigma(2c-v_1-2\epsilon)}.
\end{aligned}$$

Requiring that these two answers for ρ_2 coincide, we obtain one condition for a:

$$\frac{\sigma(a+v_2)\sigma(-a)\sigma(-a+v_1)\sigma(-a+v_1+v_2)\sigma(-2a+v_1+4\epsilon)}{\sigma(2\epsilon-a-v_2)\sigma(-a+2\epsilon)\sigma(-a+v_1+2\epsilon)\sigma(-a+v_1+v_2+2\epsilon)\sigma(-2a+v_1)}=1. \tag{5.52}$$

It is easy to see that equation (5.50) for c and equation (5.52) for $\bar{a}=a+v_2$ are obtained from one another by the flip $v_1\leftrightarrow v_2$ (as they should). We are now ready to prove the final result of this section.

Theorem 5.4. *The general solution of dVC_4 is given by*

$$\begin{aligned}
x_1 &= \rho_1 \frac{\sigma(t+a)\sigma(t-a+v_1+2\epsilon)\sigma(t-a+v_1)\sigma(t+a+2v_1+2v_2-2\epsilon)}{\sigma(t)\sigma(t+v_1)\sigma(t+v_1+v_2)\sigma(t+2v_1+v_2)},\\
x_2 &= \rho_2 \frac{\sigma(t+c)\sigma(t-c+2v_1+v_2)\sigma(t-c+2v_1+v_2+2\epsilon)\sigma(t+c+2v_1+2v_2-2\epsilon)}{\sigma(t+v_1)\sigma(t+v_1+v_2)\sigma(t+2v_1+v_2)\sigma(t+2v_1+2v_2)},\\
x_3 &= \rho_1 \frac{\sigma(t-a-v_2+2\epsilon)\sigma(t+a+v_1+v_2)\sigma(t+a+v_1+v_2-2\epsilon)\sigma(t-a+2v_1+v_2)}{\sigma(t)\sigma(t+v_1)\sigma(t+v_1+v_2)\sigma(t+2v_1+v_2)}\\
x_4 &= \rho_2 \frac{\sigma(t-c+v_1+2\epsilon)\sigma(t+c+v_1+v_2)\sigma(t+c+v_1+v_2-2\epsilon)\sigma(t-c+3v_1+2v_2)}{\sigma(t+v_1)\sigma(t+v_1+v_2)\sigma(t+2v_1+v_2)\sigma(t+2v_1+2v_2)},
\end{aligned}$$

where $2(v_1+v_2) \equiv 0$,

$$\begin{aligned}
\rho_1 &= \frac{\sigma(-c)\sigma(-c+v_1)\sigma(-c+v_1+v_2)\sigma(-c+2v_1+v_2)}{\epsilon\sigma(-2c+2v_1+v_2)\sigma(v_1+v_2)\sigma(v_1+v_2-2\epsilon)\sigma(2a+v_2-2\epsilon)},\\
\rho_2 &= \frac{\sigma(-a-v_2)\sigma(-a)\sigma(-a+v_1)\sigma(-a+v_1+v_2)}{\epsilon\sigma(-2a+v_1)\sigma(v_1+v_2)\sigma(v_1+v_2-2\epsilon)\sigma(2c-v_1-2\epsilon)},
\end{aligned}$$

and the constants a, c are defined by Eqs. (5.52), (5.50).

Proof. We show how to verify Eq. (5.34). The remaining three equations may be dealt with in completely the same way. Under conditions (5.52) and (5.50) $1+\epsilon(x_3-x_1)$ has the zeros $-c$, $-d$, $-c-v_1-v_2+2\epsilon$, $-d-v_1-v_2+2\epsilon$, while $1-\epsilon(x_3-x_1)$ has the zeros $-c+2\epsilon$, $-d+2\epsilon$,$-c-v_1-v_2$, $-d-v_1-v_2$. Hence, with the help of the periodicity condition $2(v_1+v_2) \equiv 0$, it is easy to see that there holds

$$\begin{aligned}
&1+\epsilon(x_3-x_1) = \qquad (5.53)\\
&\quad = C_1 \frac{\sigma(t+c)\sigma(t-c+2\epsilon+2v_1+v_2)\sigma(t+c+v_1+v_2-2\epsilon)\sigma(t-c+3v_1+2v_2)}{\sigma(t+v_1)\sigma(t+v_1+v_2)\sigma(t+2v_1+v_2)\sigma(t+2v_1+2v_2)},
\end{aligned}$$

as well as

$$\begin{aligned}
&1-\epsilon(x_3-x_1) = \qquad (5.54)\\
&\quad = C_2 \frac{\sigma(t+c+v_1+v_2)\sigma(t+2\epsilon-c+3v_1+2v_2)\sigma(t+c-2\epsilon)\sigma(t-c+2v_1+v_2)}{\sigma(t+v_1)\sigma(t+v_1+v_2)\sigma(t+2v_1+v_2)\sigma(t+2v_1+2v_2)},
\end{aligned}$$

with some constants C_1, C_2. With the help of the three-term identity for the σ-function we see that the difference x_3-x_1 has a zero at $t=-v_1-v_2/2$. We therefore determine C_1 and C_2 by setting $t=-v_1-v_2/2$ in (5.53) and (5.54), giving

$$\begin{aligned}
C_1 &= \frac{\sigma(-1/2v_2)\sigma(1/2v_2)\sigma(v_1+1/2v_2)\sigma(v_1+3/2v_2)}{\sigma(-v_1-1/2v_2+c)\sigma(v_1+1/2v_2-c+2\epsilon)\sigma(1/2v_2+c-2\epsilon)\sigma(2v_1+3/2v_2-c)},\\
C_2 &= \frac{\sigma(-1/2v_2)\sigma(1/2v_2)\sigma(v_1+1/2v_2)\sigma(v_1+3/2v_2)}{\sigma(1/2v_2+c)\sigma(2v_1+3/2v_2+2\epsilon-c)\sigma(-v_1-1/2v_2+c-2\epsilon)\sigma(v_1+1/2v_2-c)}.
\end{aligned}$$

Hence,

$$\frac{C_1}{C_2} = \frac{\sigma(1/2v_2+c)\sigma(-2v_1-3/2v_2-2\epsilon+c)}{\sigma(1/2v_2+c-2\epsilon)\sigma(-2v_1-3/2v_2+c)}. \qquad (5.55)$$

Now, by virtue of (5.53) and (5.54), the equation of motion (5.34) becomes

$$\begin{aligned}&\frac{\rho_2\sigma(t+c+2v_1+2v_2)\sigma(t-c+2\epsilon+2v_1+v_2)}{C_1\sigma(t+c+v_1+v_2)\sigma(t+2\epsilon-c+3v_1+2v_2)}\\&\quad=\frac{\rho_2\sigma(t+c)\sigma(t-c+2\epsilon+2v_1+v_2)\sigma(t+c+2v_1+2v_2-2\epsilon)}{C_2\sigma(t+c+v_1+v_2)\sigma(t+2\epsilon-c+3v_1+2v_2)\sigma(t+c-2\epsilon)},\end{aligned}$$

which reduces to

$$\frac{C_2}{C_1}=\frac{\sigma(t+c)\sigma(t+c+2v_1+2v_2-2\epsilon)}{\sigma(t+c+2v_1+2v_2)\sigma(t+c-2\epsilon)},$$

but this identity is easily verified using (5.55) and the quasi-periodicity of the σ-function. □

6

Integrable Cases of the Euler Equations on $e(3)$

In this chapter we will study the HK type discretizations of integrable cases of the Euler equations on $e(3)$. In particular, we we will consider the Kirchoff system, the Lagrange top and the Clebsch system. For the discrete versions of the latter two systems we will describe their HK-bases, conserved quantities and invariant volume forms. In the case of the Kirchhoff system we will also derive explicit solutions in terms of elliptic and double-Bloch functions. As a first example we will, however, study the HK type discretization of the Clebsch System. With this example we will see how one can apply the various recipes described in Chapter 2. Moreover, it should become evident how one may approach the study of more complicated birational maps.

The Euler equations on $e(3)$ read:

$$\begin{cases} \dot{m} = m \times \dfrac{\partial H}{\partial m} + p \times \dfrac{\partial H}{\partial p}, \\ \dot{p} = p \times \dfrac{\partial H}{\partial m}, \end{cases} \tag{6.1}$$

with $m = (m_1, m_2, m_3)^{\mathrm{T}} \in \mathbb{R}^3$ and $p = (p_1, p_2, p_3)^{\mathrm{T}} \in \mathbb{R}^3$. The physical meaning of m is the total angular momentum, whereas p represents the total linear momentum of the system. A detailed introduction to the general context of rigid body dynamics and its mathematical foundations can be found in [40]. When H is a quadratic form in m and p, eqs. (6.1) are called *Kirchhoff equations*. In this case they can be used to model the motion of a rigid body submerged in an ideal fluid. Any system of the type (6.1) is Hamiltonian with the Hamilton function $H = H(m, p)$ with respect to the Poisson bracket

$$\{m_i, m_j\} = \epsilon_{ijk} m_k, \qquad \{m_i, p_j\} = \epsilon_{ijk} p_k, \qquad \{p_i, p_j\} = 0 \tag{6.2}$$

(the Lie-Poisson bracket on e(3)*), and admits the Hamilton function H and the Casimir functions

$$C_1 = p_1^2 + p_2^2 + p_3^2, \quad C_2 = m_1 p_1 + m_2 p_2 + m_3 p_3 \tag{6.3}$$

as integrals of motion. For complete integrability of a system of the type (6.1), it should admit a fourth independent integral of motion. The following subcases of eqs. (6.1) are known to be integrable [42]:

1. Lagrange top.

2. Motion of a rigid body in an ideal fluid - Kirchhoff's case.

3. Motion of a rigid body in an ideal fluid - Clebsch's case.

4. Kovalevskaia top.

Each of these systems corresponds to a particular choice of the Hamiltonian H. Except for the Kovalevskaia top more details will be presented later.

6.1 Clebsch System

A famous integrable case of the Kirchhoff equations was discovered by Clebsch [17] and is characterized by the Hamilton function $H = \frac{1}{2}H_1$, where

$$H_1 = \langle m, Am\rangle + \langle p, Bp\rangle = \frac{1}{2}\sum_{k=1}^{3}(a_k m_k^2 + b_k p_k^2). \tag{6.4}$$

The vectors $A = \mathrm{diag}(a_1, a_2, a_3)$ and $B = \mathrm{diag}(b_1, b_2, b_3)$ satisfy the condition

$$\frac{b_1 - b_2}{a_3} + \frac{b_2 - b_3}{a_1} + \frac{b_3 - b_1}{a_2} = 0. \tag{6.5}$$

This condition is also equivalent to saying that the quantity

$$\theta = \frac{b_j - b_k}{a_i(a_j - a_k)} \tag{6.6}$$

takes one and the same value for all permutations (i, j, k) of the indices (1,2,3).

For an embedding of this system into the modern theory of integrable systems see [42, 49]. Equations of motion of the Clebsch case read:

$$\begin{cases} \dot m = m \times Am + p \times Bp\,, \\ \dot p = p \times Am. \end{cases} \tag{6.7}$$

In components:

$$\begin{aligned}
\dot m_1 &= (a_3 - a_2)m_2 m_3 + (b_3 - b_2)p_2 p_3, \\
\dot m_2 &= (a_1 - a_3)m_3 m_1 + (b_1 - b_3)p_3 p_1, \\
\dot m_3 &= (a_2 - a_1)m_1 m_2 + (b_2 - b_1)p_1 p_2. \\
\dot p_1 &= a_3 m_3 p_2 - a_2 m_2 p_3, \\
\dot p_2 &= a_1 m_1 p_3 - a_3 m_3 p_1, \\
\dot p_3 &= a_2 m_2 p_1 - a_1 m_1 p_2.
\end{aligned} \tag{6.8}$$

Condition (6.5) can be resolved for a_i as

$$a_1 = \frac{b_2 - b_3}{\omega_2 - \omega_3}\,, \quad a_2 = \frac{b_3 - b_1}{\omega_3 - \omega_1}\,, \quad a_3 = \frac{b_1 - b_2}{\omega_1 - \omega_2}\,. \tag{6.9}$$

For fixed values of ω_i and varying values of b_i, equations of motion of the Clebsch case share the integrals of motion: the Casimirs C_1, C_2, cf. eq. (6.3), and the Hamiltonians

$$I_i = p_i^2 + \frac{m_j^2}{\omega_i - \omega_k} + \frac{m_k^2}{\omega_i - \omega_j}\,, \quad (i, j, k) = \text{c.p.}(1, 2, 3). \tag{6.10}$$

There are four independent functions among C_i, I_i, because of $C_1 = I_1 + I_2 + I_3$. Note that $H_1 = b_1 I_1 + b_2 I_2 + b_3 I_3$. One can denote all models with the same ω_i as a hierarchy, single flows of which are characterized by the parameters b_i. Usually, one denotes as "the first flow" of this hierarchy the one corresponding to the choice $b_i = \omega_i$, so that $a_i = 1$. Thus, the first flow is characterized by the value $\theta = \infty$ of the constant (6.6).

We will now study the first flow of the Clebsch hierarchy and its HK type discretization. This flow is generated by the Hamilton function $H = \frac{1}{2} H_1$, where

$$H_1 = m_1^2 + m_2^2 + m_3^2 + \omega_1 p_1^2 + \omega_2 p_2^2 + \omega_3 p_3^2. \tag{6.11}$$

The corresponding equations of motion read:

$$\begin{cases} \dot{m} = p \times \Omega p, \\ \dot{p} = p \times m, \end{cases}$$

where $\Omega = \mathrm{diag}(\omega_1, \omega_2, \omega_3)$ is the matrix of parameters. In components:

$$\begin{aligned} \dot{m}_1 &= (\omega_3 - \omega_2) p_2 p_3, \\ \dot{m}_2 &= (\omega_1 - \omega_3) p_3 p_1, \\ \dot{m}_3 &= (\omega_2 - \omega_1) p_1 p_2, \\ \dot{p}_1 &= m_3 p_2 - m_2 p_3, \\ \dot{p}_2 &= m_1 p_3 - m_3 p_1, \\ \dot{p}_3 &= m_2 p_1 - m_1 p_2. \end{aligned} \tag{6.12}$$

The fourth independent quadratic integral can be chosen as

$$H_2 = \omega_1 m_1^2 + \omega_2 m_2^2 + \omega_3 m_3^2 - \omega_2 \omega_3 p_1^2 - \omega_3 \omega_1 p_2^2 - \omega_1 \omega_2 p_3^2. \tag{6.13}$$

Note that $H_1 = \omega_1 I_1 + \omega_2 I_2 + \omega_3 I_3$, $H_1 = -\omega_2 \omega_3 I_1 - \omega_3 \omega_1 I_2 - \omega_1 \omega_2 I_3$.

We mention the following Wronskian relation:

$$(\dot{m}_1 p_1 - m_1 \dot{p}_1) + (\dot{m}_2 p_2 - m_2 \dot{p}_2) + (\dot{m}_3 p_3 - m_3 \dot{p}_3) = 0, \tag{6.14}$$

which holds true for the first Clebsch flow.

The Hirota-Kimura discretization of the first Clebsch flow (proposed in [48]) is:

$$\begin{aligned} \widetilde{m}_1 - m_1 &= \epsilon(\omega_3 - \omega_2)(\widetilde{p}_2 p_3 + p_2 \widetilde{p}_3), \\ \widetilde{m}_2 - m_2 &= \epsilon(\omega_1 - \omega_3)(\widetilde{p}_3 p_1 + p_3 \widetilde{p}_1), \\ \widetilde{m}_3 - m_3 &= \epsilon(\omega_2 - \omega_1)(\widetilde{p}_1 p_2 + p_1 \widetilde{p}_2), \\ \widetilde{p}_1 - p_1 &= \epsilon(\widetilde{m}_3 p_2 + m_3 \widetilde{p}_2) - \epsilon(\widetilde{m}_2 p_3 + m_2 \widetilde{p}_3), \\ \widetilde{p}_2 - p_2 &= \epsilon(\widetilde{m}_1 p_3 + m_1 \widetilde{p}_3) - \epsilon(\widetilde{m}_3 p_1 + m_3 \widetilde{p}_1), \\ \widetilde{p}_3 - p_3 &= \epsilon(\widetilde{m}_2 p_1 + m_2 \widetilde{p}_1) - \epsilon(\widetilde{m}_1 p_2 + m_1 \widetilde{p}_2). \end{aligned}$$

As usual, it leads to a reversible birational map $\widetilde{x} = f(x,\epsilon)$, $x = (m,p)^{\mathrm{T}}$, given by $f(x,\epsilon) = A^{-1}(x,\epsilon)x$ with

$$A(m,p,\epsilon) = \begin{pmatrix} 1 & 0 & 0 & 0 & \epsilon\omega_{23}p_3 & \epsilon\omega_{23}p_2 \\ 0 & 1 & 0 & \epsilon\omega_{31}p_3 & 0 & \epsilon\omega_{31}p_1 \\ 0 & 0 & 1 & \epsilon\omega_{12}p_2 & \epsilon\omega_{12}p_1 & 0 \\ 0 & \epsilon p_3 & -\epsilon p_2 & 1 & -\epsilon m_3 & \epsilon m_2 \\ -\epsilon p_3 & 0 & \epsilon p_1 & \epsilon m_3 & 1 & -\epsilon m_1 \\ \epsilon p_2 & -\epsilon p_1 & 0 & -\epsilon m_2 & \epsilon m_1 & 1 \end{pmatrix},$$

where the abbreviation $\omega_{ij} = \omega_i - \omega_j$ is used. This map will be referred to as dC.

A remark on the complexity of the iterates of f is in order here. Each component of $(\widetilde{m},\widetilde{p}) = f(m,p)$ is a rational function with the numerator and the denominator being polynomials on m_k, p_k of total degree 6. The numerators of $\widetilde{p}_k$ consist of 31 monomials, the numerators of $\widetilde{m}_k$ consist of 41 monomials, the common denominator consists of 28 monomials. It should be taken into account that the coefficients of all these polynomials depend, in turn, polynomially on ϵ and ω_k, which additionally increases their complexity for a symbolic manipulator. Expressions for the second iterate swell to considerable length, thus prohibiting naive attempts to compute them symbolically. Using the software FORM [55] together with MAPLE's LargeExpressions package [14] and an appropriate veiling strategy it is, however, possible to obtain $f^2(m,p)$ with a reasonable amount of memory. Some impression on the complexity can be obtained from Table 6.1. The resulting expressions are too big to be used in further symbolic computations. Consider, for instance, the numerator of the p_1-component of $f^2(m,p)$. As a polynomial of m_k, p_k, it contains 64 056 monomials; their coefficients are, in turn, polynomials of ϵ and ω_k, and, considered as a polynomial of the phase variables and the parameters, this expression contains 1 647 595 terms.

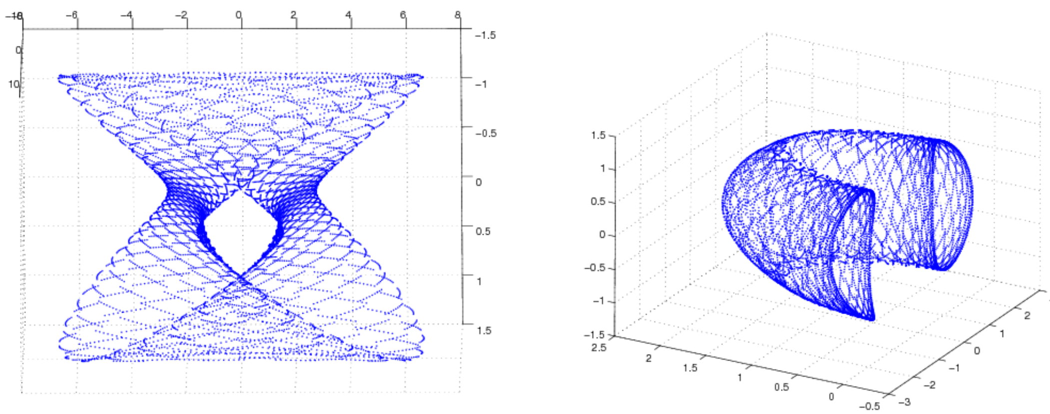
	deg	$\deg_{p_1}$	$\deg_{p_2}$	$\deg_{p_3}$	$\deg_{m_1}$	$\deg_{m_2}$	$\deg_{m_3}$
Common denominator of f^2	27	24	24	24	12	12	12
Numerator of p_1-comp. of f^2	27	25	24	24	12	12	12
Numerator of p_2-comp. of f^2	27	24	25	24	12	12	12
Numerator of p_3-comp. of f^2	27	24	24	25	12	12	12
Numerator of m_1-comp. of f^2	33	28	28	28	15	14	14
Numerator of m_2-comp. of f^2	33	28	28	28	14	15	14
Numerator of m_3-comp. of f^2	33	28	28	28	14	14	15

Table 6.1: Degrees of the numerators and the denominator of the second iterate $f^2(m,p)$

With the help of the algorithms (V) and (N) we come to the following result:

Theorem 6.1. *The set of functions*

$$\Phi = (p_1^2, p_2^2, p_3^2, m_1^2, m_2^2, m_3^2, m_1p_1, m_2p_2, m_3p_3, 1)$$



(a) m_1, m_2, m_3 (b) p_1, p_2, p_3

Figure 6.1: An orbit of the map dC with $\omega_1 = 1$, $\omega_2 = 0.2$, $\omega_3 = 30$ and $\epsilon = 1$; initial point $(m_0, p_0) = (1, 1, 1, 1, 1, 1)$.

is a HK basis for the map dC, with $\dim K_\Phi(m,p) = 4$. *Thus, any orbit of the map dC lies on an intersection of four quadrics in* $\mathbb{R}^6$.

At this point Theorem 6.1 remains a numerical result, based on the algorithms (N) and (V). A direct symbolical proof of this statement is impossible, since it requires dealing with f^i, $i \in [-4, 4]$, and the fourth iterate f^4 is a forbiddingly large expression. In order to *prove* Theorem 6.1 and to extract from it four independent integrals of motion, it is desirable to find HK-(sub)bases with a smaller number of monomials, corresponding to some (preferably one-dimensional) subspaces of $K_\Phi(m,p)$. A much more detailed information on the HK-bases is provided by the following statement.

Theorem 6.2. *The following four sets of functions are HK-bases for the map dC with one-dimensional null-spaces:*

$$\Phi_0 = (p_1^2, p_2^2, p_3^2, 1), \tag{6.15}$$

$$\Phi_1 = (p_1^2, p_2^2, p_3^2, m_1^2, m_2^2, m_3^2, m_1 p_1), \tag{6.16}$$

$$\Phi_2 = (p_1^2, p_2^2, p_3^2, m_1^2, m_2^2, m_3^2, m_2 p_2), \tag{6.17}$$

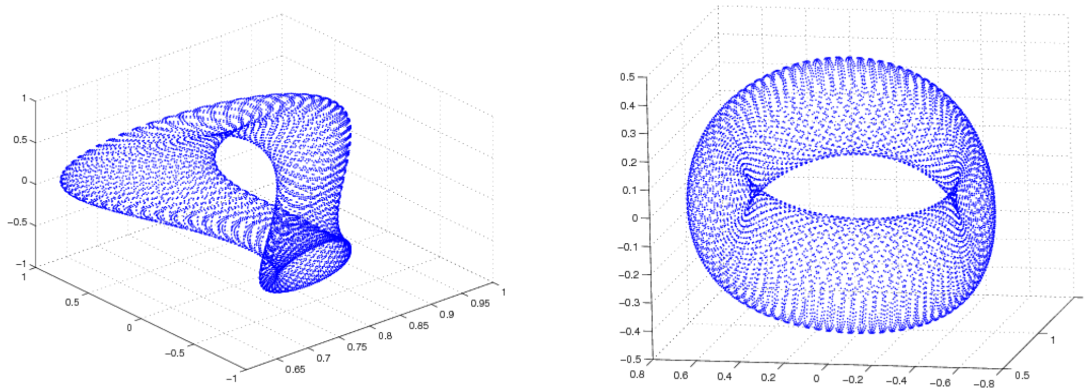
$$\Phi_3 = (p_1^2, p_2^2, p_3^2, m_1^2, m_2^2, m_3^2, m_3 p_3). \tag{6.18}$$

If all the null-spaces are considered as subspaces of $\mathbb{R}^{10}$, *so that*

$$\begin{array}{rcl}
K_{\Phi_0} & = & [c_1 : \; c_2 : \; c_3 : \; 0 : \; 0 : \; 0 : \; 0 : \; 0 : \; 0 : \; c_{10}], \\
K_{\Phi_1} & = & [\alpha_1 : \; \alpha_2 : \; \alpha_3 : \; \alpha_4 : \; \alpha_5 : \; \alpha_6 : \; \alpha_7 : \; 0 : \; 0 : \; 0], \\
K_{\Phi_2} & = & [\beta_1 : \; \beta_2 : \; \beta_3 : \; \beta_4 : \; \beta_5 : \; \beta_6 : \; 0 : \; \beta_8 : \; 0 : \; 0], \\
K_{\Phi_3} & = & [\gamma_1 : \; \gamma_2 : \; \gamma_3 : \; \gamma_4 : \; \gamma_5 : \; \gamma_6 : \; 0 : \; 0 : \; \gamma_9 : \; 0],
\end{array}$$

then there holds:

$$K_\Phi = K_{\Phi_0} \oplus K_{\Phi_1} \oplus K_{\Phi_2} \oplus K_{\Phi_3}.$$



(a) m_1, m_2, m_3 (b) p_1, p_2, p_3

Figure 6.2: An orbit of the map dC with $\omega_1 = 0.1$, $\omega_2 = 0.2$, $\omega_3 = 0.3$ and $\epsilon = 1$; initial point $(m_0, p_0) = (1, 1, 1, 1, 1, 1)$.

Also this statement was first found with the help of numerical experiments based on the algorithms (V) and (N). In what follows, we will discuss how these claims can be given a rigorous (computer assisted) proof, and how much additional information (for instance, about conserved quantities for the map dC) can be extracted from such a proof. MAPLE worksheets used for the computer assisted proofs in the following subsections are found on the attached CD-ROM.

6.1.1 First HK Basis

Theorem 6.3. *The set (6.15) is a HK basis for the map dC with* $\dim K_{\Phi_0}(m,p) = 1$. *At each point* $(m,p) \in \mathbb{R}^6$ *there holds:*

$$\begin{aligned} K_{\Phi_0}(m,p) &= [c_1 : c_2 : c_3 : c_{10}] \\ &= \left[\frac{1 + \epsilon^2(\omega_1 - \omega_2)p_2^2 + \epsilon^2(\omega_1 - \omega_3)p_3^2}{p_1^2 + p_2^2 + p_3^2} : \frac{1 + \epsilon^2(\omega_2 - \omega_1)p_1^2 + \epsilon^2(\omega_2 - \omega_3)p_3^2}{p_1^2 + p_2^2 + p_3^2} : \right. \\ &\qquad \left. \frac{1 + \epsilon^2(\omega_3 - \omega_1)p_1^2 + \epsilon^2(\omega_3 - \omega_2)p_2^2}{p_1^2 + p_2^2 + p_3^2} : -1 \right] \\ &= \left[\frac{1}{J} + \epsilon^2\omega_1 : \frac{1}{J} + \epsilon^2\omega_2 : \frac{1}{J} + \epsilon^2\omega_3 : -1 \right], \end{aligned} \tag{6.19}$$

where

$$J(m,p,\epsilon) = \frac{p_1^2 + p_2^2 + p_3^2}{1 - \epsilon^2(\omega_1 p_1^2 + \omega_2 p_2^2 + \omega_3 p_3^2)}. \tag{6.20}$$

The function (6.20) is an integral of motion of the map dC.

Proof. The statement of the theorem means that for every $(m,p) \in \mathbb{R}^6$ the space of solutions of the homogeneous system

$$(c_1 p_1^2 + c_2 p_2^2 + c_3 p_3^2 + c_{10}) \circ f^i(m,p) = 0, \qquad i = 0, \ldots, 3,$$

is one-dimensional. This system involves the third iterate of f, therefore its symbolical treatment is impossible. According to the strategy (B), we set $c_{10} = -1$ and consider the non-homogeneous system

$$(c_1 p_1^2 + c_2 p_2^2 + c_3 p_3^2) \circ f^i(m,p) = 1, \qquad i = 0, 1, 2. \tag{6.21}$$

This system involves the second iterate of f, which still precludes its symbolical treatment. There are now several possibilities to proceed.

- First, we could follow the recipe (E) and find further information about the solutions c_i. For this aim, we plot the points $(c_1(m,p), c_2(m,p), c_3(m,p))$ for different initial data $(m,p) \in \mathbb{R}^6$. Figure 6.3 shows such a plot, with 300 initial data (m,p) randomly chosen from the set $[0,1]^6$. The points $(c_1(m,p), c_2(m,p), c_3(m,p))$

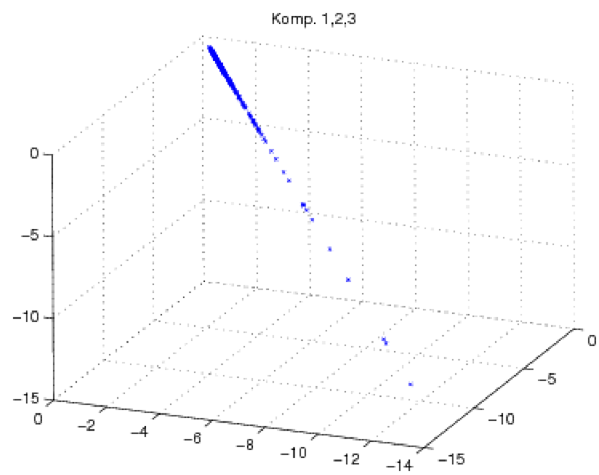


Figure 6.3: Plot of the coefficients c_1, c_2, c_3

seem to lie on a line in $\mathbb{R}^3$, which means that there should be two linear dependencies between the functions c_1, c_2 and c_3. In order to identify these linear dependencies, we run the PSLQ algorithm (See [4, 21] and the appendix) with the vectors $(c_1, c_2, 1)$ as input. On this way we obtain the conjecture

$$c_1 - c_2 = \epsilon^2(\omega_1 - \omega_2).$$

Similarly, running the PSLQ algorithm with the vectors $(c_2, c_3, 1)$ as input leads to the conjecture

$$c_2 - c_3 = \epsilon^2(\omega_2 - \omega_3).$$

Having identified (numerically!) these two linear relations, we use them instead of two equations in the system (6.21), say the equations for $i = 1, 2$. The resulting system becomes extremely simple:

$$\begin{cases} c_1 p_1^2 + c_2 p_2^2 + c_3 p_3^2 = 1, \\ c_1 - c_2 = \epsilon^2(\omega_1 - \omega_2), \\ c_2 - c_3 = \epsilon^2(\omega_2 - \omega_3). \end{cases}$$

It contains no iterates of f at all and can be solved immediately by hands, with the result (6.19). It should be stressed that this result still remains conjectural, and one has to prove *a posteriori* that the functions c_1, c_2, c_3 are integrals of motion.

- Alternatively, we can combine the above approach based on the prescription (E) with the recipe (D). For this, we use just one of the linear dependencies found above to replace the equation in (6.21) with $i = 2$, and then let MAPLE solve the remaining system. The output is still as in (6.19), but arguing this way one does not need to verify *a posteriori* that c_1, c_2, c_3 are integrals of motion, because they are manifestly even functions of ϵ, while the symmetry of the linear system with respect to ϵ has been broken.

To finish the proof along the lines of the first of the possible arguments above, we show how to verify the statement that the function J in (6.20) is an integral of motion, i.e., that

$$\frac{p_1^2 + p_2^2 + p_3^2}{1 - \epsilon^2(\omega_1 p_1^2 + \omega_2 p_2^2 + \omega_3 p_3^2)} = \frac{\widetilde{p}_1^2 + \widetilde{p}_2^2 + \widetilde{p}_3^2}{1 - \epsilon^2(\omega_1 \widetilde{p}_1^2 + \omega_2 \widetilde{p}_2^2 + \omega_3 \widetilde{p}_3^2)}.$$

This is equivalent to

$$\begin{aligned} &\widetilde{p}_1^2 - p_1^2 + \widetilde{p}_2^2 - p_2^2 + \widetilde{p}_3^2 - p_3^2 \\ &\quad = \epsilon^2 \left[(\omega_2 - \omega_1)(\widetilde{p}_1^2 p_2^2 - \widetilde{p}_2^2 p_1^2) + (\omega_3 - \omega_2)(\widetilde{p}_2^2 p_3^2 - \widetilde{p}_3^2 p_2^2) + (\omega_1 - \omega_3)(\widetilde{p}_3^2 p_1^2 - \widetilde{p}_1^2 p_3^2) \right]. \end{aligned}$$

On the left-hand side of this equation we replace $\widetilde{p}_i - p_i$ through the expressions from the last three equations of motion (6.15. On the right-hand side we replace $\epsilon(\omega_k - \omega_j)(\widetilde{p}_j p_k + p_j \widetilde{p}_k)$ by $\widetilde{m}_i - m_i$, according to the first three equations of motion (6.15). This brings the equation we want to prove into the form

$$\begin{aligned} &(\widetilde{p}_1 + p_1)(\widetilde{m}_3 p_2 + m_3 \widetilde{p}_2 - \widetilde{m}_2 p_3 - m_2 \widetilde{p}_3) + \\ &(\widetilde{p}_2 + p_2)(\widetilde{m}_1 p_3 + m_1 \widetilde{p}_3 - \widetilde{m}_3 p_1 - m_3 \widetilde{p}_1) + \\ &(\widetilde{p}_3 + p_3)(\widetilde{m}_2 p_1 + m_2 \widetilde{p}_1 - \widetilde{m}_1 p_2 - m_1 \widetilde{p}_2) = \\ &\quad = (\widetilde{p}_1 p_2 - p_1 \widetilde{p}_2)(\widetilde{m}_3 - m_3) + (\widetilde{p}_2 p_3 - p_2 \widetilde{p}_3)(\widetilde{m}_1 - m_1) + (\widetilde{p}_3 p_1 - p_3 \widetilde{p}_1)(\widetilde{m}_2 - m_2). \end{aligned}$$

The latter equation is an algebraic identity in twelve variables $m_k, p_k, \widetilde{m}_k, \widetilde{p}_k$. This finishes the proof. $\square$

Remarkably, the "simple" conserved J quantity can also be found from the following natural discretization of the Wronskian relation (6.14).

Proposition 6.1. *The set* $\Gamma = (\widetilde{m}_1 p_1 - m_1 \widetilde{p}_1,\, \widetilde{m}_2 p_2 - m_2 \widetilde{p}_2,\, \widetilde{m}_3 p_3 - m_3 \widetilde{p}_3)$ *is a HK basis for the map dC with* $\dim K_\Gamma(x) = 1$. *At each point* $x \in \mathbb{R}^6$ *we have:* $K_\Gamma(x) = [e_1 : e_2 : e_3]$, *where*

$$e_i = 1 + \epsilon^2(\omega_i - \omega_j)p_j^2 + \epsilon^2(\omega_i - \omega_k)p_k^2, \quad (i,j,k) = \text{c.p.}(1,2,3). \tag{6.22}$$

The conserved quantities e_i/e_j can be put as

$$e_i/e_j = (1 + \epsilon^2\omega_i J)/(1 + \epsilon^2\omega_j J). \tag{6.23}$$

6.1.2 Remaining HK Bases

We now consider the remaining HK-bases Φ_1,Φ_2 and Φ_3. Here we are dealing with the three linear systems

$$\begin{aligned}
(\alpha_1 p_1^2 + \alpha_2 p_2^2 + \alpha_3 p_3^2 + \alpha_4 m_1^2 + \alpha_5 m_2^2 + \alpha_6 m_3^2) \circ f^i(m,p) &= m_1 p_1 \circ f^i(m,p), &(6.24)\\
(\beta_1 p_1^2 + \beta_2 p_2^2 + \beta_3 p_3^2 + \beta_4 m_1^2 + \beta_5 m_2^2 + \beta_6 m_3^2) \circ f^i(m,p) &= m_2 p_2 \circ f^i(m,p), &(6.25)\\
(\gamma_1 p_1^2 + \gamma_2 p_2^2 + \gamma_3 p_3^2 + \gamma_4 m_1^2 + \gamma_5 m_2^2 + \gamma_6 m_3^2) \circ f^i(m,p) &= m_3 p_3 \circ f^i(m,p), &(6.26)
\end{aligned}$$

already made non-homogeneous by normalizing the last coefficient in each system, as in recipe (B), with $l = 7$. The claim about each of the systems is that it admits a unique solution for $i \in \mathbb{Z}$. It is enough to solve each system for two different but intersecting ranges of $l - 1 = 6$ consecutive indices i, such as $i \in [-2, 3]$ and $i \in [-3, 2]$, and to show that solutions coincide for both ranges (recipe (C)). Actually, since the index range $i \in [-2, 3]$ is non-symmetric, it would be enough to consider the system for this one range and to show that the solutions $\alpha_j, \beta_j, \gamma_j$ are even functions with respect to ϵ (recipe (D)). However, symbolic manipulations with the iterates f^i for $i = \pm 2, \pm 3$ are impossible. In what follows, we will gradually extend the available information about the coefficients $\alpha_j, \beta_j, \gamma_j$, which at the end will allow us to get the analytic expressions for all of them and to prove that they are integrals, indeed.

6.1.3 First Additional HK Basis

Theorem 6.2 shows that, after finding the HK basis Φ_0 with $\dim K_{\Phi_0}(x) = 1$ it is enough to concentrate on (sub)-bases not containing the constant function $\varphi_{10}(m,p) = 1$. It turns out to be possible to find a HK basis without φ_{10} and with a one-dimensional null-space, which is more amenable to a symbolic treatment than Φ_1, Φ_2, Φ_3. Numerical algorithm (N) suggests that the following set of functions is a HK basis with $d = 1$:

$$\Psi = (p_1^2, p_2^2, p_3^2, m_1 p_1, m_2 p_2, m_3 p_3). \tag{6.27}$$

Theorem 6.4. *The set (6.27) is a HK basis for the map dC with* $\dim K_\Psi(m,p) = 1$. *At every point* $(m,p) \in \mathbb{R}^6$ *there holds:*

$$K_\Psi(m,p) = [-1 : -1 : -1 : d_7 : d_8 : d_9],$$

with

$$d_k = \frac{(p_1^2 + p_2^2 + p_3^2)(1 + \epsilon^2 d_k^{(2)} + \epsilon^4 d_k^{(4)} + \epsilon^6 d_k^{(6)})}{\Delta}, \quad k = 7, 8, 9, \tag{6.28}$$

$$\Delta = m_1p_1 + m_2p_2 + m_3p_3 + \epsilon^2\Delta^{(4)} + \epsilon^4\Delta^{(6)} + \epsilon^6\Delta^{(8)}, \tag{6.29}$$

where $d_k^{(2q)}$ and $\Delta^{(2q)}$ are homogeneous polynomials of degree $2q$ in phase variables. In particular,

$$\begin{aligned}
d_7^{(2)} &= m_1^2 + m_2^2 + m_3^2 + (\omega_2 + \omega_3 - 2\omega_1)p_1^2 + (\omega_3 - \omega_2)p_2^2 + (\omega_2 - \omega_3)p_3^2,\\
d_8^{(2)} &= m_1^2 + m_2^2 + m_3^2 + (\omega_3 - \omega_1)p_1^2 + (\omega_3 + \omega_1 - 2\omega_2)p_2^2 + (\omega_1 - \omega_3)p_3^2,\\
d_9^{(2)} &= m_1^2 + m_2^2 + m_3^2 + (\omega_2 - \omega_1)p_1^2 + (\omega_1 - \omega_2)p_2^2 + (\omega_1 + \omega_2 - 2\omega_3)p_3^2,
\end{aligned}$$

and

$$\Delta^{(4)} = m_1p_1d_7^{(2)} + m_2p_2d_8^{(2)} + m_3p_3d_9^{(2)}.$$

(All other polynomials are too messy to be given here.) The functions d_7, d_8, d_9 are integrals of the map dC. They are dependent due to the linear relation

$$(\omega_2 - \omega_3)d_7 + (\omega_3 - \omega_1)d_8 + (\omega_1 - \omega_2)d_9 = 0. \tag{6.30}$$

Any two of them are functionally independent. Moreover, any two of them together with J are still functionally independent.

Proof. As already mentioned, numerical experiments suggest that for any $(m, p) \in \mathbb{R}^6$ there exists a one-dimensional space of vectors $(d_1, d_2, d_3, d_7, d_8, d_9)$ satisfying

$$(d_1p_1^2 + d_2p_2^2 + d_3p_3^2 + d_7m_1p_1 + d_8m_2p_2 + d_9m_3p_3) \circ f^i(m, p) = 0$$

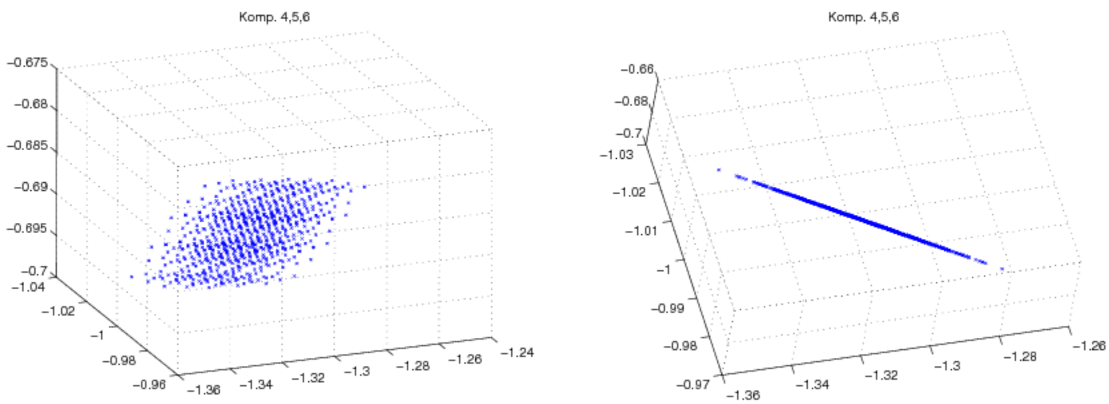
for $i = 0, 1, \ldots, 5$. According to recipe (A), one can equally well consider this system for $i = -2, -1, \ldots, 3$, which however still contains the third iterate of f and is therefore not manageable. Therefore, we apply recipe (E) and look for linear relations between the (numerical) solutions. Two such relations can be observed immediately, namely

$$d_1 = d_2 = d_3. \tag{6.31}$$

Accepting these (still hypothetical) relations and applying recipe (B), i.e., setting the common value of (6.31) equal to -1, we arrive at the non-homogeneous system of only 3 linear relations

$$(d_7m_1p_1 + d_8m_2p_2 + d_9m_3p_3) \circ f^i(m, p) = (p_1^2 + p_2^2 + p_3^2) \circ f^i(m, p) \tag{6.32}$$

for $i = -1, 0, 1$. Fortunately, it is possible to find one more linear relation between d_7, d_8, d_9. This was discovered numerically: we produced a three-dimensional plot of the points $(d_7(m, p), d_8(m, p), d_9(m, p))$ which can be seen in Fig. 6.4 in two different projections. This figure suggests that all these points lie on a plane in $\mathbb{R}^3$, the second picture bsubseing a "side view" along a direction parallel to this plane. Thus, it is plausible that one more linear relation exists. With the help of the PSLQ algorithm this hypothetic relation can then be identified as eq. (6.30). Now the ansatz (6.32) is



(a) (b)

Figure 6.4: Plot of the points (d_7, d_8, d_9) for 729 values of (m, p) from a six-dimensional grid around the point $(1, 1, 1, 1, 1, 1)$ with a grid size of 0.01 and the parameters $\epsilon = 0.1$, $\omega_1 = 0.1$, $\omega_2 = 0.2$, $\omega_3 = 0.3$.

reduced to the following system of three equations for (d_7, d_8, d_9), which involves only one iterate of the map f:

$$\begin{cases} (d_7 m_1 p_1 + d_8 m_2 p_2 + d_9 m_3 p_3) \circ f^i(m, p) = (p_1^2 + p_2^2 + p_3^2) \circ f^i(m, p), \quad i = 0, 1, \\ (\omega_2 - \omega_3) d_7 + (\omega_3 - \omega_1) d_8 + (\omega_2 - \omega_2) d_9 = 0. \end{cases} \tag{6.33}$$

This system can be solved by MAPLE, resulting in functions given in eqs. (6.28), (6.29). They are manifestly even functions of ϵ, while the system has no symmetry with respect to $\epsilon \mapsto -\epsilon$. This proves that they are integrals of motion for the map f. This argument slightly generalizes the recipes (D) and (E), and, since it is used not only here but also on several further occasions in this chapter, we give here its formalization.

Proposition 6.2. *Consider a map $f : \mathbb{R}^6 \to \mathbb{R}^6$ depending on a parameter ϵ, reversible in the sense of eq. (4.3). Let $I(m, p, \epsilon)$ be an integral of f, even in ϵ, and let $A_1, A_2, A_3 \in \mathbb{R}$. Suppose that the set of functions $\Phi = (\varphi_1, \ldots, \varphi_4)$ is such that the system of three linear equations for (a_1, a_2, a_3),*

$$\begin{cases} (a_1 \varphi_1 + a_2 \varphi_2 + a_3 \varphi_3) \circ f^i(m, p, \epsilon) = \varphi_4 \circ f^i(m, p, \epsilon), \quad i = 0, 1, \\ A_1 a_1 + A_2 a_2 + A_3 a_3 = I(m, p, \epsilon), \end{cases} \tag{6.34}$$

admits a unique solution which is even with respect to ϵ. Then this solution (a_1, a_2, a_3) consists of integrals of the map f, and Φ is a HK basis with $\dim K_\Phi(m, p) = 1$.

Proof. Since (a_1, a_2, a_3) are even functions of ϵ, they satisfy also the system (6.34) with

$\epsilon \mapsto -\epsilon$, which, due to the reversibility, can be represented as

$$\begin{cases} (a_1\varphi_1 + a_2\varphi_2 + a_3\varphi_3) \circ f^i(m,p,\epsilon) = \varphi_4 \circ f^i(m,p,\epsilon), \quad i = 0, -1, \\ A_1 a_1 + A_2 a_2 + A_3 a_3 = I(m,p,\epsilon). \end{cases} \tag{6.35}$$

Since the functions (a_1, a_2, a_3) are *uniquely determined* by any of the systems (6.34) or (6.35), we conclude that they remain invariant under the change $(m,p) \mapsto f(m,p,\epsilon)$, or, in other words, that they are integrals of motion. Finally, we can conclude that these functions satisfy equation $(a_1\varphi_1 + a_2\varphi_2 + a_3\varphi_3) \circ f^i = \varphi_4 \circ f^i$ for all $i \in \mathbb{Z}$ (and can be uniquely determined by this property), and that linear relation $A_1a_1 + A_2a_2 + A_3a_3 = I$ is satisfied, as well. □

Application of Proposition 6.2 to system (6.33) shows that d_7, d_8, d_9 are integrals of motion, since they are even in ϵ. Note that here, as always in similar context, the evenness of solutions is due to "miraculous cancellation" of the equal non-even polynomials which factor out both in the numerators and denominators of the solutions. In the present computation, these common non-even factors are of degree 2 in ϵ.

It remains to prove that any two of the integrals d_7, d_8, d_9 together with the previously found integral J are functionally independent. For this aim, we show that from such a triple of integrals one can construct another triple of integrals which yields in the limit $\epsilon \to 0$ three independent conserved quantities H_3, H_4, H_1 of the continuous Clebsch system. Indeed:

$$\begin{aligned} J &= p_1^2 + p_2^2 + p_3^2 + O(\epsilon^2) = H_3 + O(\epsilon^2), \\ \frac{J}{d_{k+6}} &= m_1p_1 + m_2p_2 + m_3p_3 + O(\epsilon^2) = H_4 + O(\epsilon^2). \end{aligned}$$

On the other hand, it is easy to derive:

$$\frac{d_7}{d_8} = 1 + \epsilon^2(d_7^{(2)} - d_8^{(2)}) + O(\epsilon^4) = 1 + \epsilon^2(\omega_2 - \omega_1)(p_1^2 + p_2^2 + p_3^2) + O(\epsilon^4),$$

and, taking this into account and computing the terms of order ϵ^4, one finds:

$$\frac{d_7}{d_8} - 1 - \epsilon^2(\omega_2 - \omega_1)J = \epsilon^4(\omega_2 - \omega_1)(2H_4^2 + \omega_2 H_3^2 - 2H_3H_1) + O(\epsilon^6),$$

from which one easily extracts H_1. This proves our claim. □

Concluding this section, we mention that - with the help of the integrals d_i - we may also find an invariant volume form for the map dC:

Experimental Result 6.1. *The map dC possesses an invariant volume form:*

$$\det \frac{\partial \widetilde{x}}{\partial x} = \frac{\phi(\widetilde{x})}{\phi(x)} \quad \Leftrightarrow \quad f^*\omega = \omega, \quad \omega = \frac{dm_1 \wedge dm_2 \wedge dm_3 \wedge dp_1 \wedge dp_2 \wedge dp_3}{\phi(x)}$$

with $\phi(x) = \frac{\Delta(x,\epsilon)}{p_1^2+p_2^2+p_3^2}$, *where* Δ *is defined in Theorem 6.4.*

A possible proof of this statement would follow that of Proposition 6.9.

6.1.4 Second Additional HK Basis

From the (still hypothetic) properties (6.24)–(6.26) of the bases Φ_1, Φ_2, Φ_3 there follows that for any $(m,p) \in \mathbb{R}^6$ the system of linear equations

$$(g_1p_1^2+g_2p_2^2+g_3p_3^2+g_4m_1^2+g_5m_2^2+g_6m_3^2)\circ f^i(m,p) = (m_1p_1+m_2p_2+m_3p_3)\circ f^i(m,p) \tag{6.36}$$

has a unique solution $(g_1, g_2, g_3, g_4, g_5, g_6)$. Indeed, the solution should be given by

$$g_j = \alpha_j + \beta_j + \gamma_j, \quad j = 1, \dots, 6. \tag{6.37}$$

As for the bases Φ_1, Φ_2, Φ_3, the solution of (6.36) can be determined by solving these equations for two different but intersecting ranges of 6 consecutive values of i, say for $i \in [-3, 2]$ and $i \in [-2, 3]$. However, it turns out that, due to the existence of several linear relations between the solutions g_j, system (6.36) is much easier to deal with than systems (6.24)–(6.26), so that the functions g_j can be determined and studied independently of $\alpha_j, \beta_j, \gamma_j$.

Theorem 6.5. *The set of functions*

$$\Theta = (p_1^2, p_2^2, p_3^2, m_1^2, m_2^2, m_3^2, m_1p_1 + m_2p_2 + m_3p_3)$$

is a HK basis for the map dCwith $\dim K_\Theta(m,p) = 1$. *At every point* $(m,p) \in \mathbb{R}^6$ *there holds:*

$$K_\Theta(m,p) = [g_1 : g_2 : g_3 : g_4 : g_5 : g_6 : -1].$$

Here g_1, g_2, g_3 *are integrals of the map dC given by*

$$g_k = \frac{g_k^{(4)} + \epsilon^2 g_k^{(6)} + \epsilon^4 g_k^{(8)} + \epsilon^6 g_k^{(10)}}{2(p_1^2+p_2^2+p_3^2)\Delta}, \qquad k = 1, 2, 3,$$

where $g_k^{(2q)}$ *are homogeneous polynomials of degree* $2q$ *in phase variables, and* Δ *is given in eq. (6.29). For instance,*

$$g_k^{(4)} = 2H_4^2 - H_3H_1 + \omega_k H_3^2.$$

Integrals g_4, g_5, g_6 *are given by*

$$g_4 = \frac{g_2 - g_3}{\omega_2 - \omega_3}, \quad g_5 = \frac{g_3 - g_1}{\omega_3 - \omega_1}, \quad g_6 = \frac{g_1 - g_2}{\omega_1 - \omega_2}.$$

Proof. Since system (6.36) involves too many iterates of f for a symbolical treatment, we look for linear relations between the (numerical) solutions of this system. Application of the PSLQ algorithm allows us to identify three such relations, as given in eq. (6.38). This reduces system (6.36) to the following one:

$$\begin{aligned}\Big[g_1\Big(p_1^2 + \frac{m_2^2}{\omega_1-\omega_3} + \frac{m_3^2}{\omega_1-\omega_2}\Big) + g_2\Big(p_2^2 + \frac{m_1^2}{\omega_2-\omega_3} + \frac{m_3^2}{\omega_2-\omega_1}\Big) \\ + g_3\Big(p_3^2 + \frac{m_1^2}{\omega_3-\omega_2} + \frac{m_2^2}{\omega_3-\omega_1}\Big)\Big] \circ f^i(m,p) = (m_1p_1+m_2p_2+m_3p_3)\circ f^i(m,p).\end{aligned} \tag{6.38}$$

Thus, one can say that we are dealing with a reduced Hirota-Kimura basis consisting of $l = 4$ functions

$$\widetilde{\Theta} = (I_1, I_2, I_3, C_2),$$

see (6.10). Interestingly, this is a basis of integrals for the continuous-time Clebsch system. System (6.38) has to be solved for two different but intersecting ranges of $l - 1 = 3$ consecutive indices i. It would be enough to show that the solution for one non-symmetric range, e.g., for $i \in [0, 2]$, consists of even functions of ϵ. However, this non-symmetric system involves with necessity the second iterate f^2. To avoid dealing with f^2, one more linear relation for g_1, g_2, g_3 would be needed. Such a relation has been found with the help of PSLQ algorithm, it does not have constant coefficients anymore but involves the previously found integrals d_7, d_8, d_9:

$$(\omega_2 - \omega_3)g_1 + (\omega_3 - \omega_1)g_2 + (\omega_1 - \omega_2)g_3 = \frac{1}{2}(\omega_2 - \omega_3)(\omega_3 - \omega_1)(d_8 - d_7). \quad (6.39)$$

Of course, due to eq. (6.30), the right-hand side of eq. (6.39) can be equivalently put as

$$\frac{1}{2}(\omega_3 - \omega_1)(\omega_1 - \omega_2)(d_9 - d_8) = \frac{1}{2}(\omega_1 - \omega_2)(\omega_2 - \omega_3)(d_7 - d_9).$$

The linear system consisting of eq. (6.38) for $i = 0, 1$ and eq. (6.39) can be solved by MAPLE with the result given in theorem. Since (d_7, d_8, d_9) are already proven to be integrals of motion, and since the solutions (g_1, g_2, g_3) are manifestly even in ϵ, Proposition 6.2 yields that (g_1, g_2, g_3) are integrals of the map f. $\square$

Theorem 6.5 gives us the third HK basis with a one-dimensional null-space for the discrete Clebsch system. Thus, it shows that every orbit lies in the intersection of three quadrics in $\mathbb{R}^6$. What concerns the integrals of motion, it turns out that the basis Θ does not provide us with additional ones: a numerical check with gradients shows that integrals g_1, g_2, g_3 are functionally dependent from the previously found ones. At this point we are lacking one more HK basis with a one-dimensional null-space, linearly independent from K_{Φ_0}, K_Ψ, K_Θ, and one more integral of motion, functionally independent from J and d_7, d_8.

6.1.5 Proof for the Bases Φ_1, Φ_2, Φ_3

Now we return to the bases Φ_1, Φ_2, Φ_3 discussed in Sect. 6.1.2. In order to be able to solve systems (6.24)–(6.26) symbolically and to prove that the solutions $\alpha_j, \beta_j, \gamma_j$ are indeed integrals, we have to find additional linear relations for these quantities (recipe (E)). Within each set of coefficients we were able to identify just one relation:

$$(\omega_1 - \omega_3)\alpha_5 = (\omega_1 - \omega_2)\alpha_6, \quad (6.40)$$

$$(\omega_2 - \omega_3)\beta_4 = (\omega_2 - \omega_1)\beta_6, \quad (6.41)$$

$$(\omega_3 - \omega_2)\gamma_4 = (\omega_3 - \omega_1)\gamma_5. \quad (6.42)$$

This reduces the number of equations in each system by one, which however does not resolve our problems. A way out consists in looking for linear relations among all the coefficients $\alpha_j, \beta_j, \gamma_j$. Remarkably, six more independent linear relations of this kind can be identified:

$$\alpha_4 = \beta_5 = \gamma_6, \tag{6.43}$$

$$\frac{\alpha_2 - \alpha_3 - (\omega_2 - \omega_3)\alpha_4}{\omega_2 - \omega_3} = \frac{\beta_2 - \beta_3 - (\omega_2 - \omega_3)\beta_4}{\omega_3 - \omega_1} = \frac{\gamma_2 - \gamma_3 - (\omega_2 - \omega_3)\gamma_4}{\omega_1 - \omega_2}, \tag{6.44}$$

$$\frac{\alpha_3 - \alpha_1 - (\omega_3 - \omega_1)\alpha_5}{\omega_2 - \omega_3} = \frac{\beta_3 - \beta_1 - (\omega_3 - \omega_1)\beta_5}{\omega_3 - \omega_1} = \frac{\gamma_3 - \gamma_1 - (\omega_3 - \omega_1)\gamma_5}{\omega_1 - \omega_2}. \tag{6.45}$$

There are two more similar relations:

$$\frac{\alpha_1 - \alpha_2 - (\omega_1 - \omega_2)\alpha_6}{\omega_2 - \omega_3} = \frac{\beta_1 - \beta_2 - (\omega_1 - \omega_2)\beta_6}{\omega_3 - \omega_1} = \frac{\gamma_1 - \gamma_2 - (\omega_1 - \omega_2)\gamma_6}{\omega_1 - \omega_2},$$

but they follow from the already listed ones (6.40)–(6.45). We stress that all these linear relations were identified numerically, with the help of the PSLQ algorithm, and remain at this stage hypothetic.

With nine linear relations (6.40)–(6.45), we have to solve systems (6.24)–(6.26) *simultaneously* for a range of 3 consecutive indices i. Taking this range as $i = -1, 0, 1$ we can avoid dealing with f^2, which however would leave us with the problem of a proof that the solutions are integrals. Alternatively, we can choose the range $i = 0, 1, 2$, and then the solutions are automatically integrals, as soon as it is established that they are even functions of ϵ.

A symbolic solution of the system consisting of 18 linear equations, namely eqs. (6.24)–(6.26) with $i = 0, 1, 2$ along with nine simple equations (6.40)–(6.45), would require astronomical amounts of memory, because of the complexity of f^2. However, this task becomes manageable and even simple for fixed (numerical) values of the phase variables (m, p) and of the parameters ω_i, while leaving ϵ a symbolic variable. For rational values of m_k, p_k, ω_k all computations can be done precisely (in rational arithmetic). This means that α_j, β_j, and γ_j can be evaluated, as functions of ϵ, at arbitrary points in $\mathbb{Q}^9(m, p, \omega_1, \omega_2, \omega_3)$. A big number of such evaluations provides us with a convincing evidence in favor of the claim that these functions are even in ϵ.

In order to obtain a rigorous proof without dealing with f^2, further linear relations would be necessary. Before introducing these, we present some preliminary considerations. Assuming that Φ_1, Φ_2, Φ_3 are HK-bases with one-dimensional null-spaces, results of Theorem 6.4 on the HK basis Ψ tell us that the row vector (d_7, d_8, d_9) is the unique left null-vector for the matrix

$$M_2 = \begin{pmatrix} \alpha_4 & \alpha_5 & \alpha_6 \\ \beta_4 & \beta_5 & \beta_6 \\ \gamma_4 & \gamma_5 & \gamma_6 \end{pmatrix},$$

normalized so that

$$(d_7, d_8, d_9)M_1 = (1, 1, 1), \quad \text{where} \quad M_1 = \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{pmatrix}.$$

Note that due to eqs. (6.40)–(6.43) the matrix M_2 has at most four (linearly) independent entries. Denoting the common values in these equations by A, B, C, D, respectively, we find:

$$M_2 = \begin{pmatrix} \alpha_4 & \alpha_5 & \alpha_6 \\ \beta_4 & \beta_5 & \beta_6 \\ \gamma_4 & \gamma_5 & \gamma_6 \end{pmatrix} = \begin{pmatrix} D & A/(\omega_1-\omega_3) & A/(\omega_1-\omega_2) \\ B/(\omega_2-\omega_3) & D & B/(\omega_2-\omega_1) \\ C/(\omega_3-\omega_2) & C/(\omega_3-\omega_1) & D \end{pmatrix}. \tag{6.46}$$

The existence of the left null-vector (d_7, d_8, d_9) shows that $\det(M_2) = 0$, or, equivalently,

$$D^2 - \frac{AB}{(\omega_1-\omega_3)(\omega_2-\omega_3)} - \frac{BC}{(\omega_2-\omega_1)(\omega_3-\omega_1)} - \frac{CA}{(\omega_3-\omega_2)(\omega_1-\omega_2)} = 0. \tag{6.47}$$

From eqs. (6.46) and (6.47) one easily derives that the row

$$\left(D - \frac{B}{\omega_2-\omega_3} - \frac{C}{\omega_3-\omega_2}\,,\, D - \frac{A}{\omega_1-\omega_3} - \frac{C}{\omega_3-\omega_1}\,,\, D - \frac{A}{\omega_1-\omega_2} - \frac{B}{\omega_2-\omega_1}\right)$$
$$= (\alpha_4 - \beta_4 - \gamma_4,\ -\alpha_5 + \beta_5 - \gamma_5,\ -\alpha_6 - \beta_6 + \gamma_6)$$

is a left null-vector of the matrix M_2, and therefore (d_7, d_8, d_9) is proportional to this vector. The proportionality coefficient can be now determined with the help of the PSLQ algorithm and turns out to be extremely simple. Namely, the following relations hold:

$$\alpha_4 - \beta_4 - \gamma_4 = D - \frac{B-C}{\omega_2-\omega_3} = \frac{1}{2}d_7, \tag{6.48}$$

$$-\alpha_5 + \beta_5 - \gamma_5 = D - \frac{C-A}{\omega_3-\omega_1} = \frac{1}{2}d_8, \tag{6.49}$$

$$-\alpha_6 - \beta_6 + \gamma_6 = D - \frac{A-B}{\omega_1-\omega_2} = \frac{1}{2}d_9. \tag{6.50}$$

Only two of them are independent, because of eq. (6.30). We note also that, according to eq. (6.37), one has

$$\alpha_4 + \beta_4 + \gamma_4 = D + \frac{B-C}{\omega_2-\omega_3} = g_4, \tag{6.51}$$

$$\alpha_5 + \beta_5 + \gamma_5 = D + \frac{C-A}{\omega_3-\omega_1} = g_5, \tag{6.52}$$

$$\alpha_6 + \beta_6 + \gamma_6 = D + \frac{A-B}{\omega_1-\omega_2} = g_6. \tag{6.53}$$

Equations (6.48)–(6.53) and (6.47) are already enough to determine all four integrals A, B, C, D, that is, all $\alpha_j, \beta_j, \gamma_j$ with $j = 4, 5, 6$, *provided* it is proven that they are indeed integrals. These (conditional) results read:

$$A = \frac{1+\epsilon^2 A^{(2)}+\epsilon^4 A^{(4)}+\epsilon^6 A^{(6)}+\epsilon^8 A^{(8)}}{2\epsilon^2\Delta}, \tag{6.54}$$

$$B = \frac{1+\epsilon^2 B^{(2)}+\epsilon^4 B^{(4)}+\epsilon^6 B^{(6)}+\epsilon^8 B^{(8)}}{2\epsilon^2\Delta}, \tag{6.55}$$

$$C = \frac{1+\epsilon^2 C^{(2)}+\epsilon^4 C^{(4)}+\epsilon^6 C^{(6)}+\epsilon^8 C^{(8)}}{2\epsilon^2\Delta}, \tag{6.56}$$

$$D = \frac{p_1^2+p_2^2+p_3^2+\epsilon^2 D^{(4)}+\epsilon^4 D^{(6)}+\epsilon^6 D^{(8)}}{2\Delta}, \tag{6.57}$$

where $A^{(2q)}$, $B^{(2q)}$, $C^{(2q)}$, $D^{(2q)}$ are homogeneous polynomials of degree $2q$ in phase variables, for instance,

$$\begin{aligned}
A^{(2)} &= B^{(2)} = C^{(2)} \\
&= m_1^2+m_2^2+m_3^2+(\omega_2+\omega_3-2\omega_1)p_1^2+(\omega_3+\omega_1-2\omega_2)p_2^2+(\omega_1+\omega_2-2\omega_3)p_3^2, \\
D^{(4)} &= (m_1p_1+m_2p_2+m_3p_3)^2 \\
&\quad +(p_1^2+p_2^2+p_3^2)\Big((\omega_2+\omega_3-2\omega_1)p_1^2+(\omega_3+\omega_1-2\omega_2)p_2^2+(\omega_1+\omega_2-2\omega_3)p_3^2\Big).
\end{aligned}$$

We remark that eq. (6.47) tells us that no more than three of the functions A, B, C, D are actually functionally independent. Computation with gradients shows that A, B, C are functionally independent, indeed. Moreover, all other previously found integrals J, d_7, d_8, d_9, and g_1, g_2, g_3 are functionally dependent on these ones.

Theorem 6.6. *The sets (6.16)–(6.18) are HK-bases for the map dC with* $\dim K_{\Phi_1}(m,p) = \dim K_{\Phi_2}(m,p) = \dim K_{\Phi_3}(m,p) = 1$. *At each point* $(m,p) \in \mathbb{R}^6$ *there holds:*

$$\begin{aligned}
K_{\Phi_1}(m,p) &= [\alpha_1 : \alpha_2 : \alpha_3 : \alpha_4 : \alpha_5 : \alpha_6 : -1], \\
K_{\Phi_2}(m,p) &= [\beta_1 : \beta_2 : \beta_3 : \beta_4 : \beta_5 : \beta_6 : -1], \\
K_{\Phi_3}(m,p) &= [\gamma_1 : \gamma_2 : \gamma_3 : \gamma_4 : \gamma_5 : \gamma_6 : -1],
\end{aligned}$$

where α_j,β_j, *and* γ_j *are rational functions of* (m,p), *even with respect to* ϵ. *They are integrals of motion for the map dC and satisfy linear relations (6.40)–(6.45). For* $j = 4,5,6$ *they are given by eqs. (6.46), (6.56), (6.57). For* $j = 1,2,3$ *they are of the form*

$$h = \frac{h^{(2)}+\epsilon^2 h^{(4)}+\epsilon^4 h^{(6)}+\epsilon^6 h^{(8)}+\epsilon^8 h^{(10)}+\epsilon^{10} h^{(12)}}{2\epsilon^2(p_1^2+p_2^2+p_3^2)\Delta}, \tag{6.58}$$

where h *stands for any of the functions* $\alpha_j, \beta_j, \gamma_j$, $j = 1,2,3$, *and the corresponding* $h^{(2q)}$ *are homogeneous polynomials in phase variables of degree* $2q$. *For instance,*

$$\begin{array}{lll}
\alpha_1^{(2)} = C_1 - I_1, & \alpha_2^{(2)} = -I_1, & \alpha_3^{(2)} = -I_1, \\
\beta_1^{(2)} = -I_2, & \beta_2^{(2)} = C_1 - I_2, & \beta_3^{(2)} = -I_2, \\
\gamma_1^{(2)} = -I_3, & \gamma_2^{(2)} = -I_3, & \gamma_3^{(2)} = C_1 - I_3.
\end{array} \tag{6.59}$$

The four functions J, α_1, β_1 *and* γ_1 *are functionally independent.*

Proof. The proof consists of several steps.

Step 1. Consider the system for 18 unknowns $\alpha_j, \beta_j, \gamma_j$, $j = 1, \ldots, 6$, consisting of 17 linear equations: eqs. (6.24)–(6.26) with $i = 0, 1$, eqs. (6.40)–(6.45), and eqs. (6.48), (6.49). This system is underdetermined, so that in principle it admits a one-parameter family of solutions. Remarkably, the symbolic MAPLE solution shows that all variables $\alpha_j, \beta_j, \gamma_j$ with $j = 4, 5, 6$ are determined by this system uniquely, the results coinciding with eqs. (6.46), (6.56)–(6.57). (Actually, the MAPLE answers are much more complicated, and their simplification has been performed with SINGULAR, which was used to cancel out common factors from the huge expressions in numerators and denominators of these rational functions.) Since these uniquely determined $\alpha_j, \beta_j, \gamma_j$ with $j = 4, 5, 6$ are even functions of ϵ, this proves that they (i.e., A, B, C, D) are integrals of motion.

Step 2. Having determined $\alpha_j, \beta_j, \gamma_j$ with $j = 4, 5, 6$, we are in a position to *compute* $\alpha_j, \beta_j, \gamma_j$ with $j = 1, 2, 3$. For instance, to obtain the values of α_j with $j = 1, 2, 3$, we consider the symmetric linear system (6.24) with $i = -1, 0, 1$ (and with already found $\alpha_4, \alpha_5, \alpha_6$). This system has been solved by MAPLE. The solutions are huge rational functions which however turn out to admit massive cancellations. These cancellations have been performed with the help of SINGULAR. The resulting expressions for $\alpha_1, \alpha_2, \alpha_3$ turn out to satisfy the ansatz (6.58) with the leading terms given in the first line of eq. (6.59). However, this computation does not prove that the functions so obtained are indeed integrals of motion. To prove this, one could, in principle, either check directly the identities $\alpha_j \circ f = \alpha_j$, $j = 1, 2, 3$, or verify equation (6.24) with $i = 2$. Both ways are prohibitively expensive, so that we have to look for an alternative one.

Step 3. The results of Step 2 yield an explicit expression for the function

$$F = (\omega_2 - \omega_3)\alpha_1 + (\omega_3 - \omega_1)\alpha_2 + (\omega_1 - \omega_2)\alpha_3, \tag{6.60}$$

which is of the form

$$F = \frac{(\omega_2 - \omega_3)(1 + \epsilon^2 F^{(2)} + \epsilon^4 F^{(4)} + \epsilon^6 F^{(6)} + \epsilon^8 F^{(8)})}{2\epsilon^2 \Delta}.$$

It is of a crucial importance for our purposes that it can be proven directly that F is an integral of motion. We have proved this with the method (G) based on the Gröbner basis for the ideal generated by discrete equations of motion. The application of this method to F is more feasible that to any single of α_j, $j = 1, 2, 3$, because of the cancellation of the huge polynomial coefficient of ϵ^{10} in the numerator of F.

Step 4. The result of Step 3 allows us to proceed as follows. Consider the system of three linear equations for $\alpha_1, \alpha_2, \alpha_3$, consisting of (6.24) with $i = 0, 1$, and of

$$(\omega_2 - \omega_3)\alpha_1 + (\omega_3 - \omega_1)\alpha_2 + (\omega_1 - \omega_2)\alpha_3 = F,$$

where F is the explicit expression obtained and proven to be an integral on Step 3. This system can now be solved by MAPLE; the results, again simplified with SINGULAR, are even functions of ϵ (actually, the same ones obtained on Step 1 from the symmetric system). Non-even polynomials in ϵ of degree 7 cancel in a miraculous way from the numerators and the denominator. Now Proposition 6.2 assures that these solutions are integrals of motion.

Step 5. Finally, in order to find $\beta_1, \beta_2, \beta_3$ and $\gamma_1, \gamma_2, \gamma_3$, we solve the two systems consisting of (6.25), resp. (6.26) with $i = 0, 1$, and the first, resp. the second linear relation in eq. (6.44). The results are even functions of ϵ, satisfying the ansatz (6.58) with the leading terms given in eq. (6.59). Proposition 6.2 yields that also these functions are integrals of motion. □

6.2 General Flow of the Clebsch System

We conclude the discussion of the Clebsch System with some findings regarding the general flow. Not all of the following results have been proven rigorously in the sense of the previous section. This will be pointed out at the relevant points.

The HK discretization of the flow (6.7) reads as

$$\begin{cases} \widetilde{m} - m &= \epsilon(\widetilde{m} \times Am + m \times A\widetilde{m} + \widetilde{p} \times Bp + p \times B\widetilde{p}), \\ \widetilde{p} - p &= \epsilon\left(\widetilde{p} \times Am + p \times A\widetilde{m}\right), \end{cases}$$

or in components:

$$\begin{aligned}
\widetilde{m}_1 - m_1 &= \epsilon(a_3 - a_2)(\widetilde{m}_2 m_3 + m_2 \widetilde{m}_3) + \epsilon(b_3 - b_2)(\widetilde{p}_2 p_3 + p_2 \widetilde{p}_3), \\
\widetilde{m}_2 - m_2 &= \epsilon(a_1 - a_3)(\widetilde{m}_3 m_1 + m_3 \widetilde{m}_1) + \epsilon(b_1 - b_3)(\widetilde{p}_3 p_1 + p_3 \widetilde{p}_1), \\
\widetilde{m}_3 - m_3 &= \epsilon(a_2 - a_1)(\widetilde{m}_1 m_2 + m_1 \widetilde{m}_2) + \epsilon(b_2 - b_1)(\widetilde{p}_1 p_2 + p_1 \widetilde{p}_2), \\
\widetilde{p}_1 - p_1 &= \epsilon a_3(\widetilde{m}_3 p_2 + m_3 \widetilde{p}_2) - \epsilon a_2(\widetilde{m}_2 p_3 + m_2 \widetilde{p}_3), \\
\widetilde{p}_2 - p_2 &= \epsilon a_1(\widetilde{m}_1 p_3 + m_1 \widetilde{p}_3) - \epsilon a_3(\widetilde{m}_3 p_1 + m_3 \widetilde{p}_1), \\
\widetilde{p}_3 - p_3 &= \epsilon a_2(\widetilde{m}_2 p_1 + m_2 \widetilde{p}_1) - \epsilon a_1(\widetilde{m}_1 p_2 + m_1 \widetilde{p}_2). \qquad (6.61)
\end{aligned}$$

In what follows, we will use the abbreviations $b_{ij} = b_i - b_j$ and $a_{ij} = a_i - a_j$. The linear system (6.61) defines an explicit, birational map $f : \mathbb{R}^6 \to \mathbb{R}^6$,

$$\begin{pmatrix} \widetilde{m} \\ \widetilde{p} \end{pmatrix} = f(m, p, \epsilon) = M^{-1}(m, p, \epsilon) \begin{pmatrix} m \\ p \end{pmatrix}, \qquad (6.62)$$

where

$$M(m, p, \epsilon) = \begin{pmatrix}
1 & \epsilon a_{23} m_3 & \epsilon a_{23} m_2 & 0 & \epsilon b_{23} p_3 & \epsilon b_{23} p_2 \\
\epsilon a_{31} m_3 & 1 & \epsilon a_{31} m_1 & \epsilon b_{31} p_3 & 0 & \epsilon b_{31} p_1 \\
\epsilon a_{12} m_2 & \epsilon a_{12} m_1 & 1 & \epsilon b_{12} p_2 & \epsilon b_{12} p_1 & 0 \\
0 & \epsilon a_2 p_3 & -\epsilon a_3 p_2 & 1 & -\epsilon a_3 m_3 & \epsilon a_2 m_2 \\
-\epsilon a_1 p_3 & 0 & \epsilon a_3 p_1 & \epsilon a_3 m_3 & 1 & -\epsilon a_1 m_1 \\
\epsilon a_1 p_2 & -\epsilon a_2 p_1 & 0 & -\epsilon a_2 m_2 & \epsilon a_1 m_1 & 1
\end{pmatrix}.$$

This map will be denoted dGC in what follows.

A "simple" integral of the map dGC can be obtained by discretizing the following Wronskian relation with constant coefficients, which holds for the general flow of the Clebsch system (6.8):

$$A_1(\dot{m}_1 p_1 - m_1\dot{p}_1) + A_2(\dot{m}_2 p_2 - m_2\dot{p}_2) + A_3(\dot{m}_3 p_3 - m_3\dot{p}_3) = 0, \tag{6.63}$$

with

$$A_i = a_i a_j + a_i a_k - a_j a_k, \quad (i,j,k) = \text{c.p.}(1,2,3). \tag{6.64}$$

Proposition 6.3. *The set* $\Gamma = (\widetilde{m}_1 p_1 - m_1\widetilde{p}_1,\ \widetilde{m}_2 p_2 - m_2\widetilde{p}_2,\ \widetilde{m}_3 p_3 - m_3\widetilde{p}_3)$ *is a HK basis for the map dGC with* $\dim K_\Gamma(x) = 1$. *At each point* $x \in \mathbb{R}^6$ *we have:* $K_\Gamma(x) = [e_1 : e_2 : e_3]$, *where, for* $(i,j,k) = \text{c.p.}(1,2,3)$,

$$e_i = A_i + \epsilon^2 a_i(b_i - b_j)A_k\Theta_j + \epsilon^2 a_i(b_i - b_k)A_j\Theta_k, \tag{6.65}$$

with

$$\Theta_i = p_i^2 + \frac{a_i}{\theta a_j a_k} m_i^2 \tag{6.66}$$

(recall that θ is defined by equation (6.6); we assume here that $\theta \neq \infty$).

Proof. Direct verification using MAPLE. □

As in the case of the first flow, the integrals e_i/e_j can be expressed through one symmetric integral: $e_i/e_j = (A_i - \theta a_i L)/(A_j - \theta a_j L)$, where

$$L(m,p,\epsilon) = \frac{a_2a_3A_1\Theta_1 + a_3a_1A_2\Theta_2 + a_1a_2A_3\Theta_3}{1 + \epsilon^2\theta a_1a_2a_3(\Theta_1 + \Theta_2 + \Theta_3)}.$$

The quantities e_i and the integral L can be also obtained from a different (monomial) HK basis, given in the following proposition.

Proposition 6.4. *The set of functions* $\Phi_0 = (p_1^2,\ p_2^2,\ p_3^2,\ m_1^2,\ m_2^2,\ m_3^2,\ 1)$ *is a HK basis for the map dGC with* $\dim K_{\Phi_0}(m,p) = 1$. *At each point* $(m,p) \in \mathbb{R}^6$ *there holds:*

$$K_{\Phi_0}(m,p) = [a_2a_3e_1 : a_3a_1e_2 : a_1a_2e_3 : (a_1/\theta)e_1 : (a_2/\theta)e_2 : (a_3/\theta)e_3 : -e_0],$$

where

$$e_0 = a_2a_3A_1\Theta_1 + a_3a_1A_2\Theta_2 + a_1a_2A_3\Theta_3 \tag{6.67}$$

is an integral of motion of the continuous time flow (6.8).

Proof. Direct verification using MAPLE. □

Experimental Result 6.2. a) *The set* $\Phi = (p_1^2,\ p_2^2,\ p_3^2,\ m_1^2,\ m_2^2,\ m_3^2,\ m_1p_1,\ m_2p_2,\ m_3p_3,\ 1)$ *is a HK basis for the map dGC with* $\dim K_\Phi(m,p) = 4$. *Thus, any orbit of the map dGC lies on an intersection of four quadrics in* $\mathbb{R}^6$.

b) *Each of the sets of functions* $\Psi_0 = (\widetilde{p}_1p_1,\ \widetilde{p}_2p_2,\ \widetilde{p}_3p_3,\ \widetilde{m}_1m_1,\ \widetilde{m}_2m_2,\ \widetilde{m}_3m_3,\ 1)$ *and (6.16)–(6.18) are HK-bases for the maps dC and dGC with a one-dimensional null-space.*

6.3 Kirchhoff System

The integrable case of this system found in the original paper by Kirchhoff [36] and carrying his name is characterized by the Hamilton function $H = \frac{1}{2}H_1$, where

$$H_1 = a_1(m_1^2 + m_2^2) + a_3 m_3^2 + b_1(p_1^2 + p_2^2) + b_3 p_3^2. \tag{6.68}$$

The differential equations of the Kirchhoff case are:

$$\begin{aligned}
\dot{m}_1 &= (a_3 - a_1)m_2 m_3 + (b_3 - b_1)p_2 p_3, \\
\dot{m}_2 &= (a_1 - a_3)m_1 m_3 + (b_1 - b_3)p_1 p_3, \\
\dot{m}_3 &= 0, \\
\dot{p}_1 &= a_3 p_2 m_3 - a_1 p_3 m_2, \\
\dot{p}_2 &= a_1 p_3 m_1 - a_3 p_1 m_3, \\
\dot{p}_3 &= a_1(p_1 m_2 - p_2 m_1).
\end{aligned} \tag{6.69}$$

Along with the Hamilton function H and the Casimir functions (6.3), it possesses the obvious fourth integral, due to the rotational symmetry of the system:

$$H_2 = m_3. \tag{6.70}$$

Note that the Kirchhoff case ($a_1 = a_2$ and $b_1 = b_2$) can be considered as a particular case of the Clebsch case, but is special in many respects (the symmetry resulting in the existence of the Noether integral m_3, solvability in elliptic functions, in contrast to the general Clebsch system being solvable in terms of theta-functions of genus $g = 2$, etc.).

We mention also the following Wronskian relation which follows easily from equations of motion:

$$a_1(\dot{m}_1 p_1 - m_1 \dot{p}_1) + a_1(\dot{m}_2 p_2 - m_2 \dot{p}_2) + (2a_3 - a_1)(\dot{m}_3 p_3 - m_3 \dot{p}_3) = 0. \tag{6.71}$$

Before discussing the HK type discretization of the Kirchhoff system we show how one can integrate the Kirchhoff system. Again, we perform this task the "classical" way. Our focus will be more on the *general form of solutions* and the steps necessary to deduce them.

First, we introduce the following notation:

$$M_1 = m_1 + i m_2, \quad M_2 = m_1 - i m_2, \quad P_1 = p_1 + i p_2, \quad P_2 = p_1 - i p_2.$$

System (6.69) then takes the following form:

$$\begin{array}{rcrl}
\dot{P}_1 &=& -i & (a_1 p_3 M_1 - a_3 m_3 P_1) \\
\dot{P}_2 &=& i & (a_1 p_3 M_2 - a_3 m_3 P_2), \\
\dot{M}_1 &=& -i & ((a_3 - a_1)m_3 M_1 + (b_3 - b_1)p_3 P_1), \\
\dot{M}_2 &=& i & ((a_3 - a_1)m_3 M_2 + (b_3 - b_1)p_3 P_2), \\
\dot{p}_3 &=& -\frac{1}{2}i a_1 & (P_1 M_2 - P_2 M_1), \\
\dot{m}_3 &=& & 0.
\end{array} \tag{6.72}$$

In this notation, the conserved quantities become

$$P_1P_2+p_3^2 = I_1, \quad \frac{1}{2}(P_1M_2+P_2M_1)+m_3p_3 = I_2, \quad a_1M_1M_2+b_1P_1P_2+a_3m_3^2+b_3p_3^2 = I_3.$$

We will now show how to solve system (6.72). Essentially, we reproduce the classical work of Halphen [28].

Proposition 6.5. *The component p_3 of system (6.72) satisfies the differential equation*

$$\dot{p_3}^2 = a_1A_1(A_2-p_3^2)(I_1-p_3^2) - a_1^2(I_2-m_3p_3)^2, \tag{6.73}$$

where A_1,a_3 depend on the constants a_i, b_i and the conserved quantities:

$$A_1 = b_3-b_1, \quad A_2 = \frac{I_3-b_1I_1-a_3m_3^2}{b_3-b_1}.$$

Proof. From the equations of motion and the expressions of the integral I_2 it follows that

$$(M_2P_1+P_2M_1) = 2(I_2-m_3p_3), \quad (M_2P_1-P_2M_1) = -\frac{2i}{a_1}\dot{p_3}. \tag{6.74}$$

There holds:

$$(M_2P_1+P_2M_1)^2-(M_2P_1-P_2M_1)^2 = 4M_1M_2P_1P_2.$$

Substituting expressions (6.74) into the left hand side and using integrals I_1 and I_3, it follows that

$$\dot{p_3}^2 = a_1A_1(A_2-p_3^2)(I_1-p_3^2) - a_1^2(I_2-m_3p_3)^2, \tag{6.75}$$

with A_1 and A_2 as stated above. □

Following the procedure by Halphen we thus find that p_3 is given by time-shifts of

$$p_3(t) = \zeta(u+\nu)-\zeta(u)-\zeta(\nu) = \frac{1}{2}\frac{\wp'(u)-\wp'(\nu)}{\wp(u)-\wp(\nu)}, \quad u = \sqrt{a_1A_1}t,$$

where the invariants g_2, g_3 and ν are defined according to Theorem 3.8. In what follows we consider p_3 as a function of u and denote differentiation w.r.t. u by $'$.

Theorem 6.7. *Any solution of (6.72) has the form*

$$\begin{aligned}
P_1 &= C_1\frac{\sigma(u+\alpha+\nu)\sigma(u+\beta+\nu)}{\sigma(u)\sigma(u+\nu)}\exp(Lu), && (6.76)\\
P_2 &= C_2\frac{\sigma(u-\alpha)\sigma(u-\beta)}{\sigma(u)\sigma(u+\nu)}\exp(-Lu), && (6.77)\\
M_1 &= C_3\frac{\sigma(u+\alpha_1+\nu)\sigma(u+\beta_1+\nu)}{\sigma(u)\sigma(u+\nu)}\exp(Lu), && (6.78)\\
M_2 &= C_4\frac{\sigma(u-\alpha_1)\sigma(u-\beta_1)}{\sigma(u)\sigma(u+\nu)}\exp(-Lu), && (6.79)\\
p_3 &= \zeta(u+\nu)-\zeta(u)-\zeta(\nu), && (6.80)
\end{aligned}$$

where α, β, $-\alpha-\nu$, $-\beta-\nu$ are defined as the zeros of $p_3^2 - I_1$ and α_1, β_1, $-\alpha_1-\nu$, $-\beta_1-\nu$ as those of $p_3^2 - A_2$. They satisfy the following two relations:

$$\alpha+\beta = \alpha_1+\beta_1,$$

$$\frac{\sigma(\alpha_1+\nu)\sigma(\beta_1+\nu)\sigma(\alpha)\sigma(\beta)}{\sigma(\alpha+\nu)\sigma(\beta+\nu)\sigma(\alpha_1)\sigma(\beta_1)} = -1.$$

Furthermore:

$$L = \frac{1}{\sqrt{a_1A_1}}\left(ia_3m_3 - \frac{1}{2}D\right), \qquad D = \frac{2ia_1m_3}{\sqrt{a_1A_1}} - \zeta(\alpha) - \zeta(\beta) + \zeta(\beta+\nu) + \zeta(\alpha+\nu).$$

The constants C_1 and C_2 satisfy the relation

$$C_1C_2 = \frac{\sigma(\nu)^2}{\sigma(\alpha)\sigma(\alpha+\nu)\sigma(\beta)\sigma(\beta+\nu)},$$

while C_3 and C_4 depend on C_1 and C_2:

$$\begin{aligned} C_3 &= -\frac{\sqrt{a_1A_1}}{ia_1C_2}\frac{\sigma(\nu)^2}{\sigma(\alpha)\sigma(\beta)\sigma(\alpha_1+\nu)\sigma(\beta_1+\nu)}, \\ C_4 &= \frac{\sqrt{a_1A_1}}{ia_1C_1}\frac{\sigma(\nu)^2}{\sigma(\alpha_1)\sigma(\beta_1)\sigma(\alpha+\nu)\sigma(\beta+\nu)}. \end{aligned}$$

Proof. Considering the first equation in (6.72) and dividing both sides by P_1, we obtain

$$\frac{d}{dt}\log P_1 = \frac{\dot{P}_1}{P_1} = -ia_1p_3\frac{M_1}{P_1} + ia_3m_3. \tag{6.81}$$

Utilizing integrals I_1 and I_2, we have

$$\frac{2(I_2 - m_3p_3)}{I_1 - p_3^2} = \frac{P_1M_2 + P_2M_1}{P_1P_2} = \frac{M_2}{P_2} + \frac{M_1}{P_1}. \tag{6.82}$$

Taking the equation for p_3 in eqs. of motion (6.72), it follows that

$$-\frac{2i}{a_1}\frac{\dot{p_3}}{I_1 - p_3^2} = \frac{P_1M_2 - P_2M_1}{P_1P_2} = \frac{M_2}{P_2} - \frac{M_1}{P_1}, \tag{6.83}$$

leading to:

$$ia_1p_3\frac{M_1}{P_1} = ia_1\frac{p_3(m_3p_3 - I_2)}{p_3^2 - I_1} + \frac{p_3\dot{p_3}}{p_3^2 - I_1}.$$

We will now express the right hand side in terms of ζ-functions. To simplify the first part of the sum, we consider the function

$$\phi(u) = \frac{p_3(m_3p_3 - I_2)}{p_3^2 - I_1}.$$

The residue of ϕ at a singularity u is given by

$$R(u) := \mathrm{res}_u \phi = \frac{m_3 p_3 - I_2}{2p_3'}.$$

ϕ is a fourth order elliptic function of u and its four poles are the zeros of $p_3^2 - I_1$. One observes that, if p_3 has a zero at α, then there must be another zero at $-\alpha - \nu$. Hence, we may assume these four zeros inside one parallelogram of periods to be

$$\alpha, \beta, -\alpha-\nu, -\beta-\nu.$$

We determine the value of R at these zeros. For this aim we substitute each of them into (6.73), which gives

$$\dot{p}_3(u_0)^2 + a_1^2(I_2 - m_3 p_3(u_0)) = 0,$$

where u_0 stands for one of the four zeros. Factoring the right hand side we obtain

$$(\dot{p}_3(u_0) + ia_1(I_2 - m_3p_3))\,(\dot{p}_3(u_0) - ia_1(I_2 - m_3p_3)) = 0.$$

If u_0 is a zero of one of these two factors, $-u_0 - \nu$ must be a zero of the other factor, since p_3 remains invariant and $\dot{p}_3$ changes sign under $u \to -u - \nu$. Thus, we find

$$R(\alpha) = \frac{\sqrt{a_1 A_1}}{2ia_1}, \quad R(-\alpha-\nu) = -\frac{\sqrt{a_1 A_1}}{2ia_1}, \quad R(\beta) = \frac{\sqrt{a_1 A_1}}{2ia_1}, \quad R(-\beta-\nu) = -\frac{\sqrt{a_1 A_1}}{2ia_1}.$$

Hence:

$$\phi(u) = \frac{\sqrt{a_1 A_1}}{2ia_1}(\zeta(u-\alpha) + \zeta(u+\beta) - \zeta(u+\alpha+\nu) - \zeta(u+\beta+\nu) + D),$$

with a constant D, which can be determined as

$$D = \frac{2ia_1 m_3}{\sqrt{a_1 A_1}} - \zeta(\alpha) - \zeta(\beta) + \zeta(\beta+\nu) + \zeta(\alpha+\nu),$$

since $\phi(0) = m_3$.

We shift attention to the second part of the sum in (6.81). There we have

$$p_3^2 - I_1 = (p_3 - p_3(\alpha))\,(p_3 - p_3(\beta))\,,$$

which may written in terms of σ-functions using

$$\zeta(u+\nu) - \zeta(u) - \zeta(a+\nu) + \zeta(a) = \frac{\sigma(\nu)\sigma(u-a)\sigma(u+a+\nu)}{\sigma(u)\sigma(u+\nu)\sigma(a)\sigma(a+\nu)}, \tag{6.84}$$

eventually leading to

$$\frac{p_3 p_3'}{p_3^2 - I_1} = \frac{1}{2}\left(\zeta(u-\alpha) + \zeta(u+\alpha+\nu) + \zeta(u-\beta) + \zeta(u+\beta+\nu) - 2\zeta(u) - 2\zeta(u+\nu)\right). \tag{6.85}$$

Taking everything together and substituting into (6.81) we obtain

$$\begin{aligned}\frac{d}{du}\log P_1 &= \frac{1}{\sqrt{a_1 A1}}\frac{\dot{P}_1}{P_1} = \\ &= \zeta(u+\alpha+\nu)+\zeta(u+\beta+\nu)-\zeta(u)-\zeta(u+\nu)+\frac{1}{\sqrt{a_1A_1}}\left(ia_3m_3-\frac{1}{2}D\right),\end{aligned}$$

which may now be integrated to

$$P_1 = C_1\frac{\sigma(u+\alpha+\nu)\sigma(u+\beta+\nu)}{\sigma(u)\sigma(u+\nu)}\exp\left(\frac{1}{\sqrt{a_1A_1}}\left(ia_3m_3-\frac{1}{2}D\right)u\right). \tag{6.86}$$

Repeating this procedure for P_2 it is now easy to see that

$$P_2 = C_2\frac{\sigma(u-\alpha)\sigma(u-\beta)}{\sigma(u)\sigma(u+\nu)}\exp\left(-\frac{1}{\sqrt{a_1A_1}}\left(ia_3m_3-\frac{1}{2}D\right)u\right). \tag{6.87}$$

Since $P_1P_2 = p_3^2 - I_1 = (p_3 - p_3(\alpha))\,(p_3 - p_3(\beta))$, one obtains with the help of formula (6.84):

$$C_1C_2 = \frac{\sigma(\nu)^2}{\sigma(\alpha)\sigma(\alpha+\nu)\sigma(\beta)\sigma(\beta+\nu)}.$$

Solutions for M_1 and M_2 are obtained in a different fashion. First, we consider the zeros of $A_2 - p_3^2$. They can be set as

$$\alpha_1, \beta_1, -\alpha_1-\nu, -\beta_1-\nu,$$

and can be assumed to lie inside the same parallelogram of periods as α, β. From (6.82), (6.83) there follows:

$$ia_1P_1M_2 = \dot{p}_3 + ia_1(I_2 - m_3p_3) =: \Phi_1, \quad -ia_1P_2M_1 = \dot{p}_3 - ia_1(I_2 - m_3p_3) =: \Phi_2.$$

Two zeros of Φ_1 must be $-\alpha-\nu$ and $-\beta-\nu$. Without loss of generality the remaining two can, due to (6.75), be set as α_1 and β_1, because p_3 remains invariant and $\dot{p}_3$ changes sign under $u \to -u-\nu$. We use this information to write Φ_1 in terms of σ-functions. This yields

$$\frac{1}{\sqrt{a_1A_1}}\Phi_1 = \frac{\sigma(\nu)^2\sigma(u-\alpha_1)\sigma(u-\beta_1)\sigma(u+\alpha+\nu)\sigma(u+\beta+\nu)}{\sigma(\alpha_1)\sigma(\beta_1)\sigma(\alpha+\nu)\sigma(\beta+\nu)\sigma(u)^2\sigma(u+\nu)^2}, \tag{6.88}$$

because $\lim_{u\to 0}\left(u^2\Phi_1\right) = 1/\sqrt{a_1A_1}$. Similarly, we obtain:

$$\frac{1}{\sqrt{a_1A_1}}\Phi_2 = \frac{\sigma(\nu)^2\sigma(u-\alpha)\sigma(u-\beta)\sigma(u+\alpha_1+\nu)\sigma(u+\beta_1+\nu)}{\sigma(\alpha)\sigma(\beta)\sigma(\alpha_1+\nu)\sigma(\beta_1+\nu)\sigma(u)^2\sigma(u+\nu)^2}. \tag{6.89}$$

Since Φ_1 and Φ_2 are elliptic functions, there must hold

$$\alpha+\beta = \alpha_1+\beta_1.$$

As we have $\lim_{u\to-\nu}\left[(u+\nu)^2\Phi_{1/2}\right]=-\sqrt{a_1A_1}$, we obtain one more condition:

$$\frac{\sigma(\alpha_1+\nu)\sigma(\beta_1+\nu)\sigma(\alpha)\sigma(\beta)}{\sigma(\alpha+\nu)\sigma(\beta+\nu)\sigma(\alpha_1)\sigma(\beta_1)}=-1.$$

Finally, using (6.89) and dividing by (6.87), we obtain

$$M_1=-\frac{\sqrt{a_1A_1}}{ia_1C_2}\frac{\sigma(\nu)^2\sigma(u+\alpha_1+\nu)\sigma(u+\beta_1+\nu)}{\sigma(\alpha)\sigma(\beta)\sigma(\alpha_1+\nu)\sigma(\beta_1+\nu)\sigma(u)\sigma(u+\nu)}\exp\left(\frac{1}{\sqrt{a_1A_1}}\left(ia_3m_3-\frac{1}{2}D\right)u\right),$$

and similarly:

$$M_2=\frac{\sqrt{a_1A_1}}{ia_1C_1}\frac{\sigma(\nu)^2\sigma(u-\alpha_1)\sigma(u-\beta_1)}{\sigma(\alpha_1)\sigma(\beta_1)\sigma(\alpha+\nu)\sigma(\beta+\nu)\sigma(u)\sigma(u+\nu)}\exp\left(-\frac{1}{\sqrt{a_1A_1}}\left(ia_3m_3-\frac{1}{2}D\right)u\right).$$

This concludes the proof. □

6.4 HK type Discretization of the Kirchhoff System

6.4.1 Equations of Motion

Applying the Hirota-Kimura approach to (6.69), we obtain the following system of equations:

$$\begin{aligned}
\widetilde{m}_1-m_1 &= \epsilon(a_3-a_1)(\widetilde{m}_2m_3+m_2\widetilde{m}_3)+\epsilon(b_3-b_1)(\widetilde{p}_2p_3+p_2\widetilde{p}_3),\\
\widetilde{m}_2-m_2 &= \epsilon(a_1-a_3)(\widetilde{m}_1m_3+m_1\widetilde{m}_3)+\epsilon(b_1-b_3)(\widetilde{p}_1p_3+p_1\widetilde{p}_3),\\
\widetilde{m}_3-m_3 &= 0,\\
\widetilde{p}_1-p_1 &= \epsilon a_3(\widetilde{p}_2m_3+p_2\widetilde{m}_3)-\epsilon a_1(\widetilde{p}_3m_2+p_3\widetilde{m}_2),\\
\widetilde{p}_2-p_2 &= \epsilon a_1(\widetilde{p}_3m_1+p_3\widetilde{m}_1)-\epsilon a_3(\widetilde{p}_1m_3+p_1\widetilde{m}_3),\\
\widetilde{p}_3-p_3 &= \epsilon a_1(\widetilde{p}_1m_2+p_1\widetilde{m}_2)-\epsilon a_1(\widetilde{p}_2m_1+p_2\widetilde{m}_1).
\end{aligned}\tag{6.90}$$

As usual, these equations define a birational map $\widetilde{x}=f(x,\epsilon)$, $x=(m,p)^{\mathrm{T}}$, reversible the usual sense:

$$f^{-1}(x,\epsilon)=f(x,-\epsilon).$$

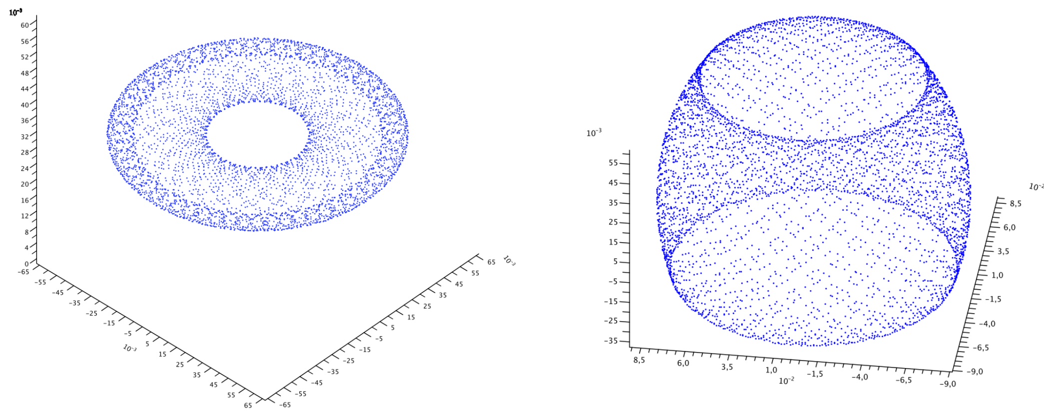
We will refer to this map as dK. Obviously, m_3 is a conserved quantity of dK.

6.4.2 HK Bases and Conserved Quantities

All HK bases presented in this section can easily be detected using (V). We start their investigation by considering the following natural discretization of the Wronskian relation (6.71) providing a "simple" conserved quantity.

Proposition 6.6. *The set* $\Gamma=(\widetilde{m}_1p_1-m_1\widetilde{p}_1,\,\widetilde{m}_2p_2-m_2\widetilde{p}_2,\,\widetilde{m}_3p_3-m_3\widetilde{p}_3)$ *is a HK basis for the map dK with* $\dim K_\Gamma(x)=1$*. At each point* $x\in\mathbb{R}^6$ *we have:* $K_\Gamma(x)=[1:1:-\gamma_3]$*, where* γ_3 *is a conserved quantity of dK given by*

$$\gamma_3=\frac{\Delta_0}{a_1\Delta_1},\tag{6.91}$$



(a) m_1, m_2, m_3 (b) p_1, p_2, p_3

Figure 6.5: An orbit of the map dK with $a_1 = 1$,$a_3 = 2$,$b_1 = 2$,$b_3 = 3$ and $\epsilon = 1$; initial point $(m_0, p_0) = (0.01, 0.02, 0.03, 0.04, 0.05, 0.06)$.

where

$$\Delta_0 = a_1 - 2a_3 + \epsilon^2 a_1^2(a_1 - a_3)(m_1^2 + m_2^2) + \epsilon^2 a_1 a_3(b_1 - b_3)(p_1^2 + p_2^2), \quad (6.92)$$

$$\Delta_1 = 1 + \epsilon^2 a_3(a_1 - a_3)m_3^2 + \epsilon^2 a_1(b_1 - b_3)p_3^2. \quad (6.93)$$

Proof. We let MAPLE compute the quantity

$$\gamma_3 := \frac{(\widetilde{m}_1 p_1 - m_1 \widetilde{p}_1) + (\widetilde{m}_2 p_2 - m_2 \widetilde{p}_2)}{(\widetilde{m}_3 p_3 - m_3 \widetilde{p}_3)},$$

which results in (6.91) – an even function of ϵ and therefore a conserved quantity. □

Interestingly enough, this same integral may also be obtained from another HK basis:

Proposition 6.7. *The set* $\Phi_0 = (m_1^2 + m_2^2, p_1^2 + p_2^2, p_3^2, 1)$ *is a HK Basis for the map* dK *with* $\dim K_{\Phi_0}(x) = 1$*. The linear combination of these functions vanishing along the orbits can be put as* $\Delta_0 - \gamma_3 a_1 \Delta_1 = 0$.

Proof. The statement of the proposition deals with the solution of a linear system of equations consisting of

$$(c_1(m_1^2 + m_2^2) + c_2(p_1^2 + p_2^2) + c_3 p_3^2) \circ f^i(m, p, \epsilon) = 1 \quad (6.94)$$

for all $i \in \mathbb{Z}$. We solve this system with $i = -1, 0, 1$ (numerically or symbolically), and observe that the solutions satisfy $a_3(b_1 - b_3)c_1 = a_1(a_1 - a_3)c_2$. Then, we consider the system of three equations for c_1, c_2, c_3 consisting of the latter linear relation between c_1, c_2, and of equations (6.94) for $i = 0, 1$. This system is easily solved symbolically (by MAPLE), its unique solution can be put as in the proposition. Its components are manifestly even functions of ϵ, thus conserved quantities. □

Proposition 6.8. a) *The set* $\Phi = (m_1^2+m_2^2,\, p_1m_1+p_2m_2,\, p_1^2+p_2^2,\, p_3^2,\, p_3,\, 1)$ *is a HK basis for the map dK with* $\dim K_\Phi(x)=3$.

b) *The set* $\Phi_1 = (1,\, p_3,\, p_3^2,\, m_1^2+m_2^2)$ *is a HK basis for the map dK with a one-dimensional null-space. At each point* $x\in\mathbb{R}^6$ *we have:* $K_{\Phi_1}(x) = [c_0 : c_1 : c_2 : -1]$. *The functions* c_0, c_1, c_2 *are conserved quantities of the map dK, given by*

$$\begin{aligned}
c_0 &= \frac{a_1(m_1^2+m_2^2)-(b_1-b_3)p_3^2+\epsilon^2c_0^{(4)}+\epsilon^4c_0^{(6)}+\epsilon^6c_0^{(8)}+\epsilon^8c_0^{(10)}}{a_1\Delta_1\Delta_2},\\
c_1 &= -\frac{2\epsilon^2a_3(b_1-b_3)m_3\Big(C_2+\epsilon^2c_1^{(4)}+\epsilon^4c_1^{(6)}+\epsilon^6c_1^{(8)}\Big)}{\Delta_1\Delta_2},\\
c_2 &= \frac{(b_1-b_3)\Big(1+\epsilon^2c_2^{(2)}+\epsilon^4c_2^{(4)}+\epsilon^6c_2^{(6)}+\epsilon^8c_2^{(8)}\Big)}{a_1\Delta_1\Delta_2},
\end{aligned}$$

where Δ_1 *is given in (6.93), and* $\Delta_2 = 1+\epsilon^2\Delta_2^{(2)}+\epsilon^4\Delta_2^{(4)}+\epsilon^6\Delta_2^{(6)}$; *coefficients* $c_k^{(q)}$ *and* $\Delta_2^{(q)}$ *are homogeneous polynomials of degree q in the phase variables. In particular:*

$$\begin{aligned}
c_2^{(2)} &= -2a_1^2(m_1^2+m_2^2)-(a_1^2-2a_1a_3+3a_3^2)m_3^2+a_1(b_1-b_3)(p_1^2+p_2^2)-a_1(b_1-b_3)p_3^2,\\
\Delta_2^{(2)} &= a_1^2(m_1^2+m_2^2)+(a_1^2-3a_1a_3+3a_3^2)m_3^2-a_1(b_1-b_3)(p_1^2+p_2^2)+a_1(b_1-b_3)p_3^2.
\end{aligned}$$

c) *The set* $\Phi_2 = (1, p_3, p_3^2, m_1p_1+m_2p_2)$ *is a HK basis for the map dK with a one-dimensional null-space. At each point* $x\in\mathbb{R}^6$ *we have:* $K_{\Phi_2}(x) = [d_0 : d_1 : d_2 : -1]$. *The functions* d_0, d_1, d_2 *are conserved quantities of the map dK, given by*

$$\begin{aligned}
d_0 &= \frac{C_2+\epsilon^2d_0^{(4)}+\epsilon^4d_0^{(6)}+\epsilon^6d_0^{(8)}+\epsilon^8d_0^{(10)}}{\Delta_1\Delta_2},\\
d_1 &= \frac{m_3\Big(-1+\epsilon^2d_1^{(2)}+\epsilon^4d_1^{(4)}+\epsilon^6d_1^{(6)}+\epsilon^8d_1^{(8)}\Big)}{\Delta_1\Delta_2},\\
d_2 &= \frac{a_1(b_3-b_1)\epsilon^2\Big(C_2+\epsilon^2c_1^{(4)}+\epsilon^4c_1^{(6)}+\epsilon^6c_1^{(8)}\Big)}{\Delta_1\Delta_2},
\end{aligned}$$

where $d_k^{(q)}$ *are homogeneous polynomials of degree q in the phase variables. In particular,*

$$d_1^{(2)} = -a_1a_3(m_1^2+m_2^2)-(a_1^2-3a_1a_3+3a_3^2)m_3^2+(a_1-a_3)(b_1-b_3)(p_1^2+p_2^2)-3a_1(b_1-b_3)p_3^2.$$

d) *The set* $\Phi_3 = (1,\, p_3,\, p_3^2,\, p_1^2+p_2^2)$ *is a HK basis for the map dK with a one-dimensional null-space. At each point* $x\in\mathbb{R}^6$ *we have:* $K_{\Phi_3}(x) = [e_0 : e_1 : e_2 : -1]$. *The functions*

e_0, e_1, e_2 are conserved quantities of the map dK, given by

$$\begin{aligned} e_0 &= \frac{C_1 + \epsilon^2 e_0^{(4)} + \epsilon^4 e_0^{(6)} + \epsilon^6 e_0^{(8)} + \epsilon^8 e_0^{(10)}}{\Delta_1 \Delta_2}, \\ e_1 &= \frac{2\epsilon^2 a_1(a_3 - a_1) m_3 \Big(C_2 + \epsilon^2 c_1^{(4)} + \epsilon^4 c_1^{(6)} + \epsilon^6 c_1^{(8)} \Big)}{\Delta_1 \Delta_2}, \\ e_2 &= \frac{-1 + \epsilon^2 e_2^{(2)} + \epsilon^4 e_2^{(4)} + \epsilon^6 e_6^{(6)} + \epsilon^8 e_8^{(8)}}{\Delta_1 \Delta_2}, \end{aligned}$$

where $e_k^{(q)}$ are polynomials of degree q in the phase variables. In particular,

$$e_2^{(2)} = -a_1^2(m_1^2 + m_2^2) - (2a_1^2 - 4a_1 a_3 + 3a_3^2) m_3^2 + 2a_1(b_1 - b_3)(p_1^2 + p_2^2) - a_1(b_1 - b_3)p_3^2.$$

Proof. b) The statement deals with the solution of the linear system

$$(c_0 + c_1 p_3 + c_2 p_3^2) \circ f^i(m, p, \epsilon) = (m_1^2 + m_2^2) \circ f^i(m, p, \epsilon), \tag{6.95}$$

for $i \in \mathbb{Z}$. To prove the statement, we consider (6.95) for $i = -1, 0, 1$. This system is solved symbolically using MAPLE giving explicit expressions for c_0, c_1, c_2. With the help of recipe (G) and SINGULAR we verify that c_2 is a conserved quantity. Statement b.) thus follows along the lines of Proposition 6.2.

c.) Again, we consider the linear system of equations

$$(d_0 + d_1 p_3 + d_2 p_3^2) \circ f^i(m, p, \epsilon) = (m_1 p_2 + m_2 p_1) \circ f^i(m, p, \epsilon) \tag{6.96}$$

for $i = -1, 0, 1$. This system is solved symbolically using MAPLE for the unknowns d_0, d_1, d_2. One then observes that c_1 and d_2 satisfy the linear relation

$$2(a_3 - a_1) m_3 d_2 = (b_3 - b_1) c_1,$$

so that d_2 must be a conserved quantity. Statement c.) thus follows from Proposition 6.2. Statement d.) is proven completely analogously with the help of the relation

$$a_1(a_3 - a_3) c_1 = a_3(b_3 - b_1) e_1.$$

For details regarding this computation the reader is referred to the MAPLE worksheets on the attached CD-ROM. □

Proposition 6.9. *The map dK possesses an invariant volume form:*

$$\det \frac{\partial \widetilde{x}}{\partial x} = \frac{\phi(\widetilde{x})}{\phi(x)} \quad \Leftrightarrow \quad f^* \omega = \omega, \quad \omega = \frac{dm_1 \wedge dm_2 \wedge dm_3 \wedge dp_1 \wedge dp_2 \wedge dp_3}{\phi(x)}$$

with $\phi(x) = \Delta_2(x, \epsilon)$.

Proof. We write the map dK in matrix form:

$$f(m,p) = A^{-1}(m,p,\epsilon)(m,p)^T.$$

Due to formula (4.4) we therefore have to show that

$$\det A(\tilde{m},\tilde{p},-\epsilon)\phi(m,p) = \det A(m,p,\epsilon)\phi(\tilde{m},\tilde{p})$$

holds. The size of the involved expressions excludes a simple minded direct computation. We therefore apply a Gröbner basis technique similar to the recipe (G). In particular, we reduce the polynomial $P = \det A(\tilde{m},\tilde{p},-\epsilon)\phi(m,p) - \det A(m,p,\epsilon)\phi(\tilde{m},\tilde{p})$, where $\tilde{p}$, $\tilde{m}$ have to be considered as independent variables, with respect to a Gröbner basis generated by the equations defining the map dK. The result is that P indeed reduces to zero. For details regarding this computation the reader is referred to the MAPLE worksheets and SINGULAR programs on the attached CD-ROM. □

As the map dK has an invariant volume form and $n-2$ independent first integrals, it should hence be possible to construct a suitable Poisson structure using the known contraction procedure outlined in Chapter 2, Section 2.4.5.

6.5 Solution of the Discrete Kirchhoff System

We now show how one can approach the explicit integration of a map of type dK. Again, our focus will be more on the *general layout of solutions* and the methods necessary for their discovery.

For this purpose we first apply the transformation

$$M_1 = m_1 + im_2, \quad M_2 = m_1 - im_2, \quad P_1 = p_1 + ip_2, \quad P_2 = p_1 - ip_2,$$

to (6.90) and obtain the following system of difference equations:

$$\begin{array}{rcrl}
\tilde{P_1} - P_1 & = & -i & \epsilon\big(a_1(\tilde{p_3}M_1 + p_3\tilde{M_1}) - a_3m_3(P_1 + \tilde{P_1})\big), \\
\tilde{P_2} - P_2 & = & i & \epsilon\big(a_1(\tilde{p_3}M_2 + p_3\tilde{M_2}) - a_3m_3(P_2 + \tilde{P_2})\big), \\
\tilde{M_1} - M_1 & = & -i & \epsilon\big((a_3 - a_1)m_3(M_1 + \tilde{M_1}) + (b_3 - b_1)(\tilde{p_3}P_1 + p_3\tilde{P_1})\big), \\
\tilde{M_2} - M_2 & = & i & \epsilon\big((a_3 - a_1)m_3(M_2 + \tilde{M_2}) + (b_3 - b_1)(\tilde{p_3}P_2 + p_3\tilde{P_2})\big), \\
\tilde{p_3} - p_3 & = & -\frac{1}{2}ia_1\epsilon & \big(\tilde{P_1}M_2 + P_1\tilde{M_2} - \tilde{P_2}M_1 - P_2\tilde{M_1}\big), \\
\tilde{m_3} - m_3 & = & 0. &
\end{array} \tag{6.97}$$

The above system may be considered as an instance of the slightly more general system

$$\begin{array}{rcl}
\tilde{P_1} - P_1 & = & \alpha_1\epsilon(\tilde{p_3}M_1 + p_3\tilde{M_1}) + \beta_1\epsilon(P_1 + \tilde{P_1}), \\
\tilde{P_2} - P_2 & = & -\alpha_1\epsilon(\tilde{p_3}M_2 + p_3\tilde{M_2}) - \beta_1\epsilon(P_2 + \tilde{P_2}), \\
\tilde{M_1} - M_1 & = & \alpha_2\epsilon(\tilde{p_3}P_1 + p_3\tilde{P_1}) + \beta_2\epsilon(M_1 + \tilde{M_1}), \\
\tilde{M_2} - M_2 & = & -\alpha_2\epsilon(\tilde{p_3}P_2 + p_3\tilde{P_2}) - \beta_2\epsilon(M_2 + \tilde{M_2}), \\
\tilde{p_3} - p_3 & = & \alpha\epsilon(\tilde{P_1}M_2 + P_1\tilde{M_2} - \tilde{P_2}M_1 - P_2\tilde{M_1}),
\end{array} \tag{6.98}$$

by setting

$$\alpha_1 = -ia_1, \quad \alpha_2 = -i(b_3 - b_1), \quad \beta_1 = ia_3 m_3, \quad \beta_2 = -i(a_3 - a_1)m_3, \quad 2\alpha = -ia_1.$$

For the sake of simplicity we restrict the following discussions to the treatment of system (6.98) with $\alpha = 1$. The birational map $(P_1, M_1, P_2, M_2, p_3) \mapsto (\tilde{P}_1, \tilde{M}_1, \tilde{P}_2, \tilde{M}_2, \tilde{p}_3)$ obtained from solving eqs. (6.98) will be called *dKC*, where the letter C stands for "complex".

The known results regarding the integrability of the map dK carry over directly to dKC. We summarize these facts in the following proposition:

Proposition 6.10. a) *The set* $\Phi = (M_1M_2, P_1P_2, M_1P_2 + P_2M_1, p_3^2, p_3, 1)$ *is a HK basis for the map dK with* $\dim K_\Phi(x) = 3$.

b) *The set* $\Phi_1 = (1, p_3, p_3^2, P_1P_2)$ *is a HK basis for the map dK with a one-dimensional null-space. At each point* $x \in \mathbb{C}^5$ *we have:* $K_{\Phi_1}(x) = [c_0 : c_1 : c_2 : -1]$. *The functions* c_0, c_1, c_2 *are independent conserved quantities of the map dKC.*

c) *The set* $\Phi_2 = (1, p_3, p_3^2, M_1M_2)$ *is a HK basis for the map dK with a one-dimensional null-space. At each point* $x \in \mathbb{C}^5$ *we have:* $K_{\Phi_2}(x) = [e_0 : e_1 : e_2 : -1]$. *The functions* e_0, e_1, e_2 *are conserved quantities of the map dKC.*

d) *The set* $\Phi_3 = (1, p_3, p_3^2, M_1P_2 + P_2M_1)$ *is a HK basis for the map dK with a one-dimensional null-space. At each point* $x \in \mathbb{C}^5$ *we have:* $K_{\Phi_3}(x) = [d_0 : d_1 : d_2 : -1]$. *The functions* d_0, d_1, d_2 *are conserved quantities of the map dKC.*

e) *The conserved quantities* c_i, e_i *and* d_i *satisfy the following relations:*

$$\beta_1\alpha_2c_1 - \beta_2\alpha_1e_1 = 0, \tag{6.99}$$
$$\beta_1d_2 - \alpha_1e_1 = 0, \tag{6.100}$$
$$(1 - \epsilon^2\beta_2\beta_1)d_2 + \epsilon^2\alpha_1\alpha_2d_0 = 0, \tag{6.101}$$
$$-\frac{1}{2}\alpha_1\alpha_2 + \alpha_2c_2 + (-\alpha_1\alpha_2^2c_0 + \alpha_1\beta_2^2e_2)\epsilon^2 = 0 \tag{6.102}$$
$$\frac{1}{2}\alpha_1 + c_2 + \frac{1}{2}(2\alpha_1\alpha_2c_0 - \alpha_1\beta_2^2)\epsilon^2 + (\beta_2^2\alpha_1^2e_0 - \beta_2^2\beta_1^2c_2)\epsilon^4 = 0, \tag{6.103}$$

$$\begin{aligned} -2{\beta_1}^3\alpha_2{c_2}^2 - 2{\beta_1}^3\alpha_2\alpha_1c_2 + {\beta_1}^2\alpha_1\alpha_2c_2d_1 + {\alpha_1}^3\beta_1{e_1}^2 & \\ +\alpha_2{\beta_1}^2\alpha_1\beta_2c_2 - {\alpha_1}^3\beta_2{e_1}^2 + r_1^{(2)}\epsilon^2 + r_2^{(4)}\epsilon^4 &= 0. \end{aligned} \tag{6.104}$$

The polynomials $r_i^{(2q)}$ *read*

$$\begin{aligned} r_1^{(2)} &= \alpha_2{\beta_1}^4\alpha_1\beta_2c_2 - {\alpha_1}^3\alpha_2{\beta_1}^2\beta_2e_0 + 2{\alpha_1}^2\alpha_2{\beta_1}^3c_2e_0 + {\beta_1}^2{\alpha_1}^3\alpha_2e_0d_1 - 2{\beta_1}^5\alpha_2{c_2}^2 \\ &\quad -2{\alpha_1}^2\alpha_2{\beta_1}^2\beta_2c_2e_0 - {\beta_1}^4\alpha_1\alpha_2c_2d_1 + 2{\beta_1}^4\alpha_2\beta_2{c_2}^2, \\ r_2^{(4)} &= -4{\alpha_1}^2\alpha_2{\beta_1}^4\beta_2c_2e_0 + 2{\beta_1}^6\alpha_2\beta_2{c_2}^2 + 2{\alpha_1}^4\beta_2\alpha_2{\beta_1}^2{e_0}^2. \end{aligned}$$

Proof. We consider the system of linear equations

$$(d_0 + d_1p_3 + d_2p_3^2) \circ f^i(m, p, \epsilon) = (M_1P_2 + P_2M_1) \circ f^i(m, p, \epsilon),$$

with $i = -1, 0, 1$. It is easily solved using MAPLE giving the desired expressions for d_0, d_1, d_2. We observe that relation (6.101) holds. This proves statement d.) along the lines of Proposition 6.2. Similarly, we let MAPLE compute the remaining conserved quantities, considering the linear systems

$$(c_0 + c_1 p_3 + c_2 p_3^2) \circ f^i(m, p, \epsilon) = (P_1 P_2) \circ f^i(m, p, \epsilon),$$

as well as

$$(e_0 + e_1 p_3 + e_2 p_3^2) \circ f^i(m, p, \epsilon) = (M_1 M_2) \circ f^i(m, p, \epsilon),$$

for $i = -1, 0, 1$. We observe that relations (6.99) and (6.100) hold and verify using exact evaluation of gradients that c_0, c_1, c_2 are independent. Thus, statements b.) and c.) follow again by Proposition 6.2. It remains to show that relations (6.102)-(6.104) hold, but this is easily verified symbolically using MAPLE. The corresponding worksheets relevant for this computation are found on the attached CD-ROM. □

We briefly sketch how relations (6.102)-(6.104) have been discovered. Since we have all explicit expressions at our disposal, we do not apply the PSLQ algorithm, but rather rely on a two different methods. While relations (6.99)-(6.101) are easily observed from their explicit expressions, identification of the remaining three relations is more difficult. Relations (6.102) and (6.103) have been found by considering the system of the five equations

$$A_1 c_0(x_0) + A_2 c_2(x_0) + A_3 e_0(x_0) + A_4 e_2(x_0) + A_5 = 0, \quad x_0 \in S = \{q_1, q_2, q_3, q_4, q_5\},$$

where q_i are five randomly chosen points in $\mathbb{Q}^6$, which all lie on different orbits. Solving this linear system for A_i symbolically using MAPLE, one finds that it admits a two dimensional space of solutions. From this space one extracts two independent elements giving (6.102) and (6.103).

Relation (6.104) is found using a different approach. One considers the integral $d_1 = d_1(M_1, M_2, P_1, P_2, p_3)$ and observes that one can write it in terms of $P_1 P_2$, $M_1 M_2$ and $M_1 P_2 + P_2 M_1$, so that

$$d_1 = R(P_1 P_2, M_1 M_2, M_1 P_2 + P_2 M_1),$$

where R is a rational function R. The arguments $P_1 P_2$, $M_1 M_2$, and $M_1 P_2 + P_2 M_1$ are in turn expressed using integrals and p_3:

$$\begin{aligned} P_1 P_2 &= c_0 + c_1 p_3 + c_2 p_3^2, && (6.105)\\ M_1 M_2 &= e_0 + e_1 p_3 + e_2 p_3^2, && (6.106)\\ M_1 P_2 + P_1 M_2 &= d_0 + d_1 p_3 + d_2 p_3^2. && (6.107) \end{aligned}$$

This gives

$$d_1 = R(c_0, c_1, c_2, e_0, e_1, e_2, d_0, d_1, d_2, p_3). \tag{6.108}$$

Using (6.99)-(6.103) one can then eliminate five of the nine variables c_i, e_i, d_i from (6.108). Solving (6.108) system for d_1, all terms containing p_3 cancel:

$$d_1 = Q(c_0, c_1, c_2, d_1).$$

This equation gives (6.104). Of course, all of the above computations should (and have been) performed with MAPLE. For more details the reader is referred to the MAPLE worksheets on the attached CD-ROM.

Recall that we were able to reduce the continuous Kirchhoff equations to a quadrature of the form

$$\dot{p}_3^2 = P_4(p_3), \tag{6.109}$$

with a quartic polynomial P_4 whose coefficients are expressed through integrals of motion. In the discrete setting there holds an analogous statement.

Proposition 6.11. *The component p_3 of the solution of difference equations (6.98) satisfies a relation of the type*

$$P(p_3, \widetilde{p}_3) = q_0 p_3^2 \widetilde{p}_3^2 + q_1 p_3 \widetilde{p}_3 (p_3 + \widetilde{p}_3) + q_2 (p_3^2 + \widetilde{p}_3^2) + q_3 p_3 \widetilde{p}_3 + q_4 (p_3 + \widetilde{p}_3) + q_5 = 0, \tag{6.110}$$

coefficients of the biquadratic polynomial P being conserved quantities of dKC. In particular, there holds

$$q_1 = 0.$$

Proof. Using eqs. of motion (6.98) we express the difference $\tilde{p_3} - p_3$ explicitly in terms of the phase variables. One observes that the resulting expressions can in turn be written in terms of P_1P_2, M_1M_2, $M_1P_2 + P_2M_1$, and $M_1P_2 - P_2M_1$:

$$\tilde{p_3} - p_3 = R(p_3, P_1P_2, M_1M_2, M_1P_2 + P_1M_2, M_1P_2 - P_2M_1),$$

where R is a rational function with numerator and denominator being linear in $M_1P_2 - P_2M_1$ (explicit expressions are too messy to be given here). The arguments P_1P_2, M_1M_2, and $M_1P_2 + P_2M_1$ of R are in turn expressed using integrals and p_3 by virtue of (6.105)-(6.107), so that

$$\tilde{p_3} - p_3 = R(c_0, c_1, c_2, d_0, d_1, d_2, e_0, e_1, e_2, p_3, M_1P_2 - P_2M_1).$$

We solve for $M_1P_2 - P_2M_1$ and obtain

$$M_1P_2 - P_2M_1 = Q(c_0, c_1, c_2, d_0, d_1, d_2, e_0, e_1, e_2, p_3, \tilde{p_3}), \tag{6.111}$$

with a suitable rational function Q. Since

$$Q^2 = (M_1P_2 - P_2M_1)^2 = (M_1P_2 + P_2M_1)^2 - 4P_1P_2M_1M_2,$$

we obtain

$$Q^2 - (M_1P_2 + P_2M_1)^2 + 4P_1P_2M_1M_2 = 0, \tag{6.112}$$

which only depends on integrals of motion and p_3, $\tilde{p_3}$ due to (6.111) and (6.105)-(6.107). Using (6.99)-(6.104) one can then eliminate the six variables d_i, and e_i from (6.112). Remarkably, the remaining expression on the right hand side factors into two terms, one depending on p_3 only and the other one being the sought after symmetric biquadratic relation among p_3 and $\tilde{p_3}$. These computations should be performed with MAPLE. For details regarding this computation the reader is referred to the MAPLE worksheets on the attached CD-ROM. □

From this theorem we can thus claim that p_3 is a second order elliptic function with two poles, which we may assume to be 0 and $-\nu$. Hence,

$$p_3(u) = \rho(\zeta(u) - \zeta(u+\nu) + A_1), \tag{6.113}$$

with constants ρ and A_1. The corresponding invariants may be determined according to the method outlined at the end of Chapter 3, Section (3.2). We continue with the integration of the map dKC.

Theorem 6.8. *Any solution of the map dKC has the form*

$$\begin{aligned}
P_1(u) &= C_1 \frac{\sigma(u+\alpha+\nu)\sigma(u+\beta+\nu)}{\sigma(u)\sigma(u+\nu)} K^{\frac{1}{2\epsilon}u}, && (6.114)\\
P_2(u) &= C_2 \frac{\sigma(u-\alpha)\sigma(u-\beta)}{\sigma(u)\sigma(u+\nu)} K^{-\frac{1}{2\epsilon}u}, && (6.115)\\
M_1(u) &= C_3 \frac{\sigma(u+\alpha_1+\nu)\sigma(u+\beta_1+\nu)}{\sigma(u)\sigma(u+\nu)} K^{\frac{1}{2\epsilon}u}, && (6.116)\\
M_2(u) &= C_4 \frac{\sigma(u-\alpha_1)\sigma(u-\beta_1)}{\sigma(u)\sigma(u+\nu)} K^{-\frac{1}{2\epsilon}u}, && (6.117)\\
p_3(u) &= \rho\left(\zeta(u) - \zeta(u+\nu) + \frac{1}{2}\zeta(\nu-2\epsilon) + \frac{1}{2}\zeta(\nu+2\epsilon)\right), && (6.118)
\end{aligned}$$

where ρ and K are constants and α, β, $-\alpha-\nu$, $-\beta-\nu$ are defined as the zeros of $c_0 + c_1 p_3 + c_2 p_3^2$, and α_1, β_1, $-\alpha_1-\nu$, $-\beta_1-\nu$ as those of $d_0 + d_1 p_3 + d_2 p_3^2$. The constants C_i satisfy the relations (6.128), (6.129), (6.130), and (6.131). Furthermore, there holds:

$$\alpha + \beta = \alpha_1 + \beta_1.$$

The rest of this section is devoted to the proof of this theorem. Recall that the crucial step during the integration of the continuous equations was to express the logarithmic derivatives of P_i, M_i in terms of p_3. For the discrete equations this corresponds to expressing ratios of the type $\tilde{P}_i/P_i$ in terms of p_3 and $\tilde{p_3}$. Hence, we now investigate the existence of bilinear HK-Bases. They are easily detected with the help of the algorithms (N) and (V).

Proposition 6.12. a) *The set $\Phi_1 = (1, p_3 + \tilde{p_3}, p_3\tilde{p_3}, P_1\tilde{P}_2 + \tilde{P}_1 P_2)$ is a HK basis for the map dK with a one-dimensional null-space. At each point $x \in \mathbb{C}^5$ we have: $K_{\Phi_1}(x) = [\gamma_0 : \gamma_1 : \gamma_2 : -1]$. The functions $\gamma_0, \gamma_1, \gamma_2$ are conserved quantities of the map dKC.*

b) *The set $\Phi_2 = (1, p_3 + \tilde{p_3}, p_3\tilde{p_3}, P_1\tilde{P}_2 - \tilde{P}_1 P_2)$ is a HK basis for the map dK with a one-dimensional null-space. At each point $x \in \mathbb{C}^5$ we have: $K_{\Phi_2}(x) = [\delta_0 : \delta_1 : \delta_2 : -1]$. The functions $\delta_0, \delta_1, \delta_2$ are conserved quantities of the map dKC.*

c) *The set $\Phi_3 = (1, p_3 + \tilde{p_3}, p_3\tilde{p_3}, M_1\tilde{M}_2 + \tilde{M}_1 M_2)$ is a HK basis for the map dK with a one-dimensional null-space. At each point $x \in \mathbb{C}^5$ we have: $K_{\Phi_3}(x) = [\kappa_0 : \kappa_1 : \kappa_2 : -1]$. The functions $\kappa_0, \kappa_1, \kappa_2$ are conserved quantities of the map dKC.*

d) *The set* $\Phi_3 = (1, p_3 + \tilde{p_3}, p_3\tilde{p_3}, M_1\tilde{M}_2 - \tilde{M}_1 M_2)$ *is a HK basis for the map dK with a one-dimensional null-space. At each point* $x \in \mathbb{C}^5$ *we have:* $K_{\Phi_3}(x) = [\lambda_0 : \lambda_1 : \lambda_2 : -1]$. *The functions* $\lambda_0, \lambda_1, \lambda_2$ *are conserved quantities of the map dKC.*

Proof. To prove statement a.) we employ the technique used in the proof of Proposition 6.11. We consider $P_1\tilde{P}_2 + \tilde{P}_1 P_2$ and write it explicitly in terms of the phase variables using eqs. of motion (6.98). One observes that the resulting expression can be expressed in terms of P_1P_2, M_1M_2, $M_1P_2 + P_2M_1$, and $M_1P_2 - P_2M_1$. We write the first three in terms of the conserved quantities and substitute $M_1P_2 - P_2M_1$ with (6.111) as obtained in Proposition 6.11. After expressing d_i and e_i through c_i all terms that are non-symmetric w.r.t. to the interchange $p_3 \leftrightarrow \tilde{p_3}$ cancel, so that we are left with the eq.

$$\gamma_0 + \gamma_1(p_3 + \tilde{p_3}) + \gamma_2 p_3\tilde{p_3} = P_1\tilde{P}_2 + \tilde{P}_1 P_2,$$

where the coefficients γ_i are conserved quantities which are expressed in terms of c_i. Statements b.) through d.) are proven completely analogously. Again, we note that these computations should be performed with MAPLE. Relevant worksheets can be found on the attached CD-ROM. □

Since there holds

$$\gamma_0 + \gamma_1(p_3 + \tilde{p_3}) + \gamma_2 p_3\tilde{p_3} = P_1\tilde{P}_2 + \tilde{P}_1 P_2,$$

as well as

$$\delta_0 + \delta_1(p_3 + \tilde{p_3}) + \delta_2 p_3\tilde{p_3} = P_1\tilde{P}_2 - \tilde{P}_1 P_2,$$

we obtain by subtraction of these two equations and subsequent division by P_1P_2 the following relation:

$$\frac{\tilde{P}_1}{P_1} = \frac{V_0 + V_1(p_3 + \tilde{p_3}) + V_2 p_3\tilde{p_3}}{c_0 + c_1 p_3 + c_2 p_3^2}, \tag{6.119}$$

Similarly, we obtain

$$\frac{\tilde{P}_2}{P_2} = \frac{\overline{V}_0 + \overline{V}_1(p_3 + \tilde{p_3}) + \overline{V}_2 p_3\tilde{p_3}}{c_0 + c_1 p_3 + c_2 p_3^2}, \tag{6.120}$$

with constants V_i,$\overline{V}_i$ depending on integrals of motion. From (6.119) we see that

$$\frac{\tilde{P}_1}{\underset{\sim}{P_1}} = \frac{V_0 + V_1(p_3 + \tilde{p_3}) + V_2 p_3\tilde{p_3}}{V_0 + V_1(p_3 + \underset{\sim}{p_3}) + V_2 p_3 \underset{\sim}{p_3}}. \tag{6.121}$$

We have thus shown that $\tilde{P}_1/\underset{\sim}{P_1}$ is an elliptic function. We now deduce the order of this function.

Proposition 6.13. *The functions* $\tilde{P}_i/\underset{\sim}{P_i}$ *and* $\tilde{M}_i/\underset{\sim}{M_i}$ *are elliptic functions of order 4.*

Proof. We show using MAPLE that, for some generically chosen rational initial data, the pairs $(\tilde{\tilde{P}}_i/P_i, \tilde{P}_i/\underset{\sim}{P_i})$ do not lie on a curve of bidegree $(2,2)$ or $(3,3)$. This proves that the order of $\tilde{P}_1/\underset{\sim}{P_1}$ is at least four. The same holds for $\tilde{M}_i/\underset{\sim}{M_i}$.

Furthermore, it follows from (6.121) that the maximal order of $\tilde{\tilde{P}}_i/P_i$ is 6. To show that the only possible order of $\tilde{\tilde{P}}_i/P_i$ is 4, we write

$$\frac{\tilde{P}_1}{\underset{\sim}{P_1}} = \frac{\tilde{P}_1\tilde{P}_2}{\underset{\sim}{P_1}\,\underset{\sim}{P_2}}\frac{\underset{\sim}{P_2}}{\tilde{P}_2} = F(p_3,\tilde{p_3})\frac{\underset{\sim}{P_2}}{\tilde{P}_2},$$

where F is an elliptic function of order 8, since $P_1P_2 = c_0 + c_1p_3 + c_2p_3^2$ is an elliptic function of order 4, as p_3 is an elliptic function of order 2. If the order of $\tilde{P}_1/\underset{\sim}{P_1}$ was 6, then $\underset{\sim}{P_2}/\tilde{P}_2$ would be an elliptic function of order two. This possibility can be excluded, as we have previously shown that $\underset{\sim}{P_2}/\tilde{P}_2$ must have at least order 4. Analogously, one excludes the possibility that $\tilde{P}_1/\underset{\sim}{P_1}$ has order 5. Hence, $\tilde{P}_1/\underset{\sim}{P_1}$ is an elliptic function of order 4. The same arguments hold for the remaining variables. $\square$

Before proceeding to the last step of the integration we observe that under the transformation $\epsilon \to -\epsilon$, we have $\tilde{P}_1/\underset{\sim}{P_1} = W \to 1/W$, so that poles and zeros of W are exchanged under the change of sign of ϵ. This implies

$$\frac{\tilde{P}_1}{\underset{\sim}{P_1}} = K_1\frac{\sigma(u-v_1-2\epsilon)\sigma(u-v_2-2\epsilon)\sigma(u-2\epsilon)\sigma(u+\nu-2\epsilon)}{\sigma(u-v_1+2\epsilon)\sigma(u-v_2+2\epsilon)\sigma(u+2\epsilon)\sigma(u+\nu+2\epsilon)},$$

with some complex numbers v_1, v_2, K_1 depending on integrals of motion. One may assume that the solutions of dKC are meromorphic, quasiperiodic functions. Under these analyticity assumptions on P_1, the above equation functional equation has the solution

$$P_1 = C_1\frac{\sigma(u-v_1)\sigma(u-v_2)}{\sigma(u)\sigma(u+\nu)}\exp(Lu),$$

where C_1 is an arbitrary constant and L is determined as $L = \frac{1}{2\epsilon}\log K_1$ (see [37,38]). This solution is unique up to a multiplication by an entire periodic function ϕ which we may simply assume to be constant, as this would merely effect a rescaling of the parameters in the equations of motion. Our solution for P_1 is hence of the form

$$P_1 = C_1\frac{\sigma(u-v_1)\sigma(u-v_2)}{\sigma(u)\sigma(u+\nu)}K_1^{\frac{1}{2\epsilon}u}.$$

Functions of this form are called double Bloch functions in [37]. Similarly, one obtains explicit solutions for the remaining variables. In total, this gives

$$P_1(u) = C_1\frac{\sigma(u-v_1)\sigma(u-v_2)}{\sigma(u)\sigma(u+\nu)}K_1^{\frac{1}{2\epsilon}u}, \tag{6.122}$$

$$P_2(u) = C_2\frac{\sigma(u-v_3)\sigma(u-v_4)}{\sigma(u)\sigma(u+\nu)}K_2^{\frac{1}{2\epsilon}u}, \tag{6.123}$$

$$M_1(u) = C_3\frac{\sigma(u-w_1)\sigma(u-w_2)}{\sigma(u)\sigma(u+\nu)}K_3^{\frac{1}{2\epsilon}u}, \tag{6.124}$$

$$M_2(u) = C_4\frac{\sigma(u-w_3)\sigma(u-w_4)}{\sigma(u)\sigma(u+\nu)}K_4^{\frac{1}{2\epsilon}u}, \tag{6.125}$$

$$p_3(u) = \rho(\zeta(u)-\zeta(u+\nu)+A_1), \tag{6.126}$$

with suitable zeros v_i, w_i and constants C_i, K_i. In particular, due to the relations (6.119) and (6.120), one can define v_i as the four zeros of the function elliptic function $c_0 + c_1 p_3 + c_2 p_3^2$, such that v_1 and v_2 are not zeros of $V_0 + V_1(p_3 + \tilde{p_3}) + V_2 p_3 \tilde{p_3}$. Completely analogously, w_i are defined as the four zeros of $d_0 + d_1 p_3 + d_2 p_3^2$.

We now characterize the remaining constants appearing in these solutions and also deduce further relations satisfied by v_i and w_i. From relations (6.105)-(6.107) it follows that $P_1 P_2$, $M_1 M_2$, and $M_1 P_2 + M_2 P_1$ must be elliptic functions. Hence, there must hold

$$K_1 = 1/K_2, \quad K_3 = 1/K_4.$$

From (6.111) it follows together with (6.107) that $M_1 P_2$ and $M_2 P_1$ must be elliptic functions as well. This implies that $K_2 = 1/K_3$, so that

$$K_1 = K, \quad K_2 = 1/K, \quad K_3 = K, \quad K_4 = 1/K.$$

To find A_1, we consider the symmetric biquadratic relation satisfied by p_3:

$$p_3^2 \widetilde{p}_3^2 + \frac{q_2}{q_0}(p_3^2 + \widetilde{p}_3^2) + \frac{q_3}{q_0} p_3 \widetilde{p}_3 + \frac{q_4}{q_0}(p_3 + \widetilde{p}_3) + \frac{q_5}{q_0} = 0.$$

We compute the Laurent expansions of this relation around the poles 0, -2ϵ, $-\nu$, and $-\nu - 2\epsilon$. Here, we get we get

$$\begin{aligned}
\frac{\left(\rho^4 \left(\zeta(\nu + 2\epsilon) - A_1 - \zeta(2\epsilon)\right)^2 + \frac{q_2}{q_0}\rho^2\right)}{u^2} + \ldots &= 0, \\
\frac{\left(\rho^4 \left(\zeta(\nu - 2\epsilon) - A_1 + \zeta(2\epsilon)\right)^2 + \frac{q_2}{q_0}\rho^2\right)}{(u + 2\epsilon)^2} + \ldots &= 0, \\
\frac{\left(\rho^4 \left(\zeta(\nu - 2\epsilon) - A_1 + \zeta(2\epsilon)\right)^2 - \frac{q_2}{q_0}\rho^2\right)}{(u + \nu)^2} + \ldots &= 0, \\
\frac{\left(\rho^4 \left(\zeta(\nu + 2\epsilon) - A_1 - \zeta(2\epsilon)\right)^2 + \frac{q_2}{q_0}\rho^2\right)}{(u + \nu + 2\epsilon)^2} + \ldots &= 0,
\end{aligned}$$

so that

$$A_1 = \frac{1}{2}\zeta(\nu - 2\epsilon) + \frac{1}{2}\zeta(\nu + 2\epsilon). \tag{6.127}$$

Zeros v_i of $P_1 P_2 = c_0 + c_1 p_3 + c_2 p_3^2$ may again[1] be taken as

$$v_1 = -\alpha - \nu, \quad v_2 = -\beta - \nu, \quad v_3 = \alpha, \quad v_4 = \beta.$$

Similarly, zeros w_i of $M_1 M_2 = d_0 + d_1 p_3 + d_2 p_3^2$ can be assumed as

$$w_1 = -\alpha_1 - \nu, \quad w_2 = -\beta_1 - \nu, \quad w_3 = \alpha_1, \quad v_4 = \beta_1,$$

[1] This should be compared to the solution of the continuous equations.

such that they lie in the same parallelogram of periods as the previous ones. The ellipticity M_1P_2 and M_2P_1 then implies that

$$\alpha + \beta = \alpha_1 + \beta_1.$$

As we have

$$P_1P_2 = c_0 + c_1p_3 + c_2p_3^2 = c_2(p_3 - p_3(\alpha))(p_3 - p_3(\beta)),$$

as well as

$$M_1M_2 = d_0 + d_1p_3 + d_2p_3^2 = d_2(p_3 - p_3(\alpha_1))(p_3 - p_3(\beta_1)),$$

we obtain with the help of formula (6.84) the following two conditions:

$$c_2\rho^2 = \frac{C_1C_2}{\sigma(\nu)^2}\sigma(\alpha)\sigma(\beta)\sigma(\alpha+\nu)\sigma(\beta+\nu), \tag{6.128}$$

$$d_2\rho^2 = \frac{C_3C_4}{\sigma(\nu)^2}\sigma(\alpha_1)\sigma(\beta_1)\sigma(\alpha_1+\nu)\sigma(\beta_1+\nu),. \tag{6.129}$$

Finally, we consider the principal parts of $P_1M_2+P_2M_1 = e_0+e_1p_3+e_2p_3^2$. Computing the Laurent expansion around the poles $u=0$ and $u=-\nu$ and comparing the terms at $1/u^2$ and $1/(u+\nu)^2$, we get two more conditions:

$$\begin{aligned} e_2\rho^2 = {} & \frac{C_1C_4}{\sigma(\nu)^2}\sigma(\alpha)\sigma(\beta)\sigma(\alpha_1+\nu)\sigma(\beta_1+\nu) \\ & +\frac{C_2C_3}{\sigma(\nu)^2}\sigma(\alpha+\nu)\sigma(\beta+\nu)\sigma(\alpha_1)\sigma(\beta_1), \end{aligned} \tag{6.130}$$

$$\begin{aligned} e_2\rho^2 = {} & \frac{C_1C_4}{\sigma(\nu)^2}\sigma(\alpha+\nu)\sigma(\beta+\nu)\sigma(\alpha_1)\sigma(\beta_1) \\ & +\frac{C_2C_3}{\sigma(\nu)^2}\sigma(\alpha)\sigma(\beta)\sigma(\alpha_1+\nu)\sigma(\beta_1+\nu). \end{aligned} \tag{6.131}$$

This concludes the proof of Theorem 6.8.

6.6 Lagrange Top

The Lagrange top was the second integrable system, after Euler top, to which the Hirota-Kimura discretization was successfully applied [35]. To complete the discussion of the HK type discretizations of the Kirchhoff Equations, we now reproduce and re-derive here the results of that paper, and also add some new results.

The equations of motion of the Lagrange top are also of Kirchhoff type. The Hamilton function of the Lagrange top is given by $H = \frac{1}{2}H_1$, where

$$H_1 = m_1^2 + m_2^2 + \alpha m_3^2 + 2\gamma p_3. \tag{6.132}$$

Thus, equations of motion of LT read

$$\begin{aligned}
\dot{m}_1 &= (\alpha-1)m_2m_3+\gamma p_2,\\
\dot{m}_2 &= (1-\alpha)m_1m_3-\gamma p_1,\\
\dot{m}_3 &= 0,\\
\dot{p}_1 &= \alpha p_2m_3-p_3m_2,\\
\dot{p}_2 &= p_3m_1-\alpha p_1m_3,\\
\dot{p}_3 &= p_1m_2-p_2m_1.
\end{aligned}\tag{6.133}$$

It follows immediately that the fourth integral of motion is simply

$$H_2=m_3.\tag{6.134}$$

Traditionally, the explicit integration of the LT in terms of elliptic functions starts with the following observation: the component p_3 of the solution satisfies the differential equation

$$\dot{p}_3^2=P_3(p_3)\tag{6.135}$$

with a cubic polynomial P_3 whose coefficients are expressed through integrals of motion:

$$P_3(p_3)=(H_1-\alpha m_3^2-2\gamma p_3)(C_1-p_3^2)-(C_2-m_3p_3)^2.$$

We mention also the following Wronskian relation which follows easily from equations of motion:

$$(\dot{m}_1p_1-m_1\dot{p}_1)+(\dot{m}_2p_2-m_2\dot{p}_2)+(2\alpha-1)(\dot{m}_3p_3-m_3\dot{p}_3)=0.\tag{6.136}$$

Applying the Hirota-Kimura discretization scheme to equations (6.133), we obtain the following discrete system:

$$\begin{aligned}
\widetilde{m}_1-m_1 &= \epsilon(\alpha-1)(\widetilde{m}_2m_3+m_2\widetilde{m}_3)+\epsilon\gamma(p_2+\widetilde{p}_2)\\
\widetilde{m}_2-m_2 &= \epsilon(1-\alpha)(\widetilde{m}_1m_3+m_1\widetilde{m}_3)-\epsilon\gamma(p_1+\widetilde{p}_1)\\
\widetilde{m}_3-m_3 &= 0\\
\widetilde{p}_1-p_1 &= \epsilon\alpha(p_2\widetilde{m}_3+\widetilde{p}_2m_3)-\epsilon(p_3\widetilde{m}_2+\widetilde{p}_3m_2)\\
\widetilde{p}_2-p_2 &= \epsilon(p_3\widetilde{m}_1+\widetilde{p}_3m_1)-\epsilon\alpha(p_1\widetilde{m}_3+\widetilde{p}_1m_3)\\
\widetilde{p}_3-p_3 &= \epsilon(p_1\widetilde{m}_2+\widetilde{p}_1m_2-p_2\widetilde{m}_1-\widetilde{p}_2m_1)
\end{aligned}\tag{6.137}$$

As usual, this can be solved for $(\widetilde{m},\widetilde{p})$, thus yielding the reversible and birational map $x\mapsto\widetilde{x}=f(x,\epsilon)=A^{-1}(x,\epsilon)(I+\epsilon B)x$, where $x=(m_1,m_2,m_3,p_1,p_2,p_3)^{\mathrm{T}}$, and

$$A(x,\epsilon)=\begin{pmatrix}
1 & \epsilon(1-\alpha)m_3 & \epsilon(1-\alpha)m_2 & 0 & 0 & 0\\
-\epsilon(1-\alpha)m_3 & 1 & -\epsilon(1-\alpha)m_1 & 0 & 0 & 0\\
0 & 0 & 1 & 0 & 0 & 0\\
0 & \epsilon p_3 & -\epsilon\alpha p_2 & 1 & -\epsilon\alpha m_3 & \epsilon m_2\\
-\epsilon p_3 & 0 & \epsilon\alpha p_1 & \epsilon\alpha m_3 & 1 & -\epsilon m_1\\
\epsilon p_2 & -\epsilon p_1 & 0 & -\epsilon m_2 & \epsilon m_1 & 1
\end{pmatrix}-\epsilon B,$$

$$B = \begin{pmatrix} 0 & 0 & 0 & 0 & \gamma & 0 \\ 0 & 0 & 0 & -\gamma & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

This map will be called dLT in the sequel. Obviously, m_3 serves as a conserved quantity for dLT. The remaining three conserved quantities can be found with the help of the HK bases approach. A simple conserved quantity can be found from the following statement which serves as a natural discretization of the Wronskian relation (6.136).

Proposition 6.14. *The set* $\Gamma = (\widetilde{m}_1 p_1 - m_1 \widetilde{p}_1,\, \widetilde{m}_2 p_2 - m_2 \widetilde{p}_2,\, \widetilde{m}_3 p_3 - m_3 \widetilde{p}_3)$ *is a HK basis for the map dLT with* $\dim K_\Gamma(x) = 1$. *At each point* $x \in \mathbb{R}^6$ *we have:* $K_\Gamma(x) = [1 : 1 : b_3]$, *where* b_3 *is a conserved quantity of dLT given by*

$$b_3 = \frac{(2\alpha - 1)m_3 + \epsilon^2(\alpha - 1)m_3(m_1^2 + m_2^2) + \epsilon^2\gamma(m_1 p_1 + m_2 p_2)}{m_3 \Delta_1}, \tag{6.138}$$

where

$$\Delta_1 = 1 + \epsilon^2\alpha(1-\alpha)m_3^2 - \epsilon^2\gamma p_3. \tag{6.139}$$

Proof. A straightforward computation with MAPLE of the quantity

$$b_3 := -\frac{(\widetilde{m}_1 p_1 - m_1 \widetilde{p}_1) + (\widetilde{m}_2 p_2 - m_2 \widetilde{p}_2)}{(\widetilde{m}_3 p_3 - m_3 \widetilde{p}_3)}$$

leads to the value (6.138). It is an even function of ϵ and therefore a conserved quantity. $\square$

Further integrals of motion were found by Hirota and Kimura. We reproduce here their results with new simplified proofs.

Proposition 6.15. [35]

a) *The set* $\Phi = (m_1^2 + m_2^2,\, p_1 m_1 + p_2 m_2,\, p_1^2 + p_2^2,\, p_3^2,\, p_3,\, 1)$ *is a HK basis for the map dLT with* $\dim K_\Phi(x) = 3$.

b) *The set* $\Phi_1 = (1,\, p_3,\, p_3^2,\, m_1^2 + m_2^2)$ *is a HK basis for the map dLT with a one-dimensional null-space. At each point* $x \in \mathbb{R}^6$ *we have:* $K_{\Phi_1}(x) = [c_0 : c_1 : c_2 : -1]$. *The functions* c_0, c_1, c_2 *are conserved quantities of the map dLT, given by*

$$\begin{aligned}
c_0 &= \frac{m_1^2 + m_2^2 + 2\gamma p_3 + \epsilon^2 c_0^{(4)} + \epsilon^4 c_0^{(6)} + \epsilon^6 c_0^{(8)} + \epsilon^8 c_0^{(10)}}{\Delta_1 \Delta_2}, \\
c_1 &= -\frac{2\gamma\Big(1 - \epsilon^2\alpha(1-\alpha)m_3^2\Big)\Big(1 + \epsilon^2 c_2^{(2)} + \epsilon^4 c_2^{(4)} + \epsilon^6 c_2^{(6)}\Big)}{\Delta_1 \Delta_2}, \\
c_2 &= -\frac{\epsilon^2\gamma^2\Big(1 + \epsilon^2 c_2^{(2)} + \epsilon^4 c_2^{(4)} + \epsilon^6 c_2^{(6)}\Big)}{\Delta_1 \Delta_2}.
\end{aligned}$$

Here Δ_1 is given in (6.139), and $\Delta_2 = 1 + \epsilon^2\Delta_2^{(2)} + \epsilon^4\Delta_2^{(4)} + \epsilon^6\Delta_2^{(6)}$; coefficients $\Delta^{(q)}$ and $c_k^{(q)}$ are polynomials of degree q in the phase variables. In particular:

$$\begin{aligned} c_2^{(2)} &= m_1^2 + m_2^2 + (1 - 2\alpha + 2\alpha^2)m_3^2 - 2\gamma p_3, \\ \Delta_2^{(2)} &= m_1^2 + m_2^2 + (1 - 3\alpha + 3\alpha^2)m_3^2 - \gamma p_3. \end{aligned}$$

c) *The set $\Phi_2 = (1,\, p_3,\, p_3^2,\, m_1p_1 + m_2p_2)$ is a HK basis for the map dLT with a one-dimensional null-space. At each point $x \in \mathbb{R}^6$ we have: $K_{\Phi_2}(x) = [d_0 : d_1 : d_2 : -1]$. The functions d_0, d_1, d_2 are conserved quantities of the map dLT, given by*

$$\begin{aligned} d_0 &= \frac{m_1p_1 + m_2p_2 + m_3p_3 + \epsilon^2 d_0^{(4)} + \epsilon^4 d_0^{(6)} + \epsilon^6 d_0^{(8)} + \epsilon^8 d_0^{(10)}}{\Delta_1\Delta_2}, \\ d_1 &= -\frac{m_3 + \epsilon^2 d_1^{(3)} + \epsilon^4 d_1^{(5)} + \epsilon^6 d_1^{(7)} + \epsilon^8 d_1^{(9)}}{\Delta_1\Delta_2}, \\ d_2 &= -\frac{\epsilon^2\gamma(1-\alpha)m_3\Big(1 + \epsilon^2 c_2^{(2)} + \epsilon^4 c_2^{(4)} + \epsilon^6 c_2^{(6)}\Big)}{\Delta_1\Delta_2}, \end{aligned}$$

where $d_k^{(q)}$ are polynomials of degree q in the phase variables. In particular,

$$d_1^{(3)} = \gamma(m_1p_1 + m_2p_2) - \gamma(3 - 2\alpha)m_3p_3 + \alpha m_3(m_1^2 + m_2^2) + (1 - 3\alpha + 3\alpha^2)m_3^3.$$

d) *The set $\Phi_3 = (1,\, p_3,\, p_3^2,\, p_1^2 + p_2^2)$ is a HK basis for the map dLT with a one-dimensional null-space. At each point $x \in \mathbb{R}^6$ we have: $K_{\Phi_3}(x) = [e_0 : e_1 : e_2 : -1]$. The functions e_0, e_1, e_2 are conserved quantities of the map dLT, given by*

$$\begin{aligned} e_0 &= \frac{p_1^2 + p_2^2 + p_3^2 + \epsilon^2 e_0^{(4)} + \epsilon^4 e_0^{(6)} + \epsilon^6 e_0^{(8)} + \epsilon^8 e_0^{(10)}}{\Delta_1\Delta_2}, \\ e_1 &= -\frac{2\epsilon^2\Big(e_1^{(3)} + \epsilon^2 e_1^{(5)} + \epsilon^4 e_1^{(7)} + \epsilon^6 e_1^{(9)}\Big)}{\Delta_1\Delta_2}, \\ e_2 &= -\frac{\Big(1 + \epsilon^2(1-\alpha)^2 m_3^2\Big)\Big(1 + \epsilon^2 c_2^{(2)} + \epsilon^4 c_2^{(4)} + \epsilon^6 c_2^{(6)}\Big)}{\Delta_1\Delta_2}, \end{aligned}$$

where $e_k^{(q)}$ are polynomials of degree q in the phase variables. In particular,

$$e_1^{(3)} = \gamma(p_1^2 + p_2^2 + p_3^2) - (1-\alpha)m_3(m_1p_1 + m_2p_2 + m_3p_3).$$

Proof. The proof is parallel to that of Proposition 6.8. All statements follow by Proposition 6.2 with the help of the three linear relations

$$\frac{1}{2}\gamma\epsilon^2 c_1 = \Big(1 - \epsilon^2\alpha(1-\alpha)m_3^2\Big)c_2, \tag{6.140}$$

$$\gamma d_2 = (1-\alpha)m_3 c_2, \tag{6.141}$$

$$\epsilon^2\gamma^2 e_2 = \Big(1 + \epsilon^2(1-\alpha)^2 m_3^2\Big)c_2. \tag{6.142}$$

□

We note that for $\alpha = 1$ the integrals d_0, d_1, d_2 simplify to

$$d_0 = \frac{m_1p_1 + m_2p_2 + m_3p_3}{1 - \epsilon^2\gamma p_3}, \quad d_1 = -\frac{m_3 + \epsilon^2\gamma(m_1p_1 + m_2p_2)}{1 - \epsilon^2\gamma p_3}, \quad d_2 = 0. \tag{6.143}$$

It is possible to find a further simple, in fact polynomial, integral for the map dLT.

Proposition 6.16. [35] *The function*

$$F = m_1^2 + m_2^2 + 2\gamma p_3 - \epsilon^2\big((1-\alpha)m_3m_1 + \gamma p_1\big)^2 - \epsilon^2\big((1-\alpha)m_3m_2 + \gamma p_2\big)^2.$$

is a conserved quantity for the map dLT.

Proof. Setting

$$C = 1 - \epsilon^2(1-\alpha)^2m_3^2, \quad D = -2\epsilon^2\gamma(1-\alpha)m_3, \quad E = -\epsilon^2\gamma^2,$$

one can check that $Cc_1+Dd_1+Ee_1 = 0$ and $Cc_2+Dd_2+Ee_2 = -2\gamma$. This yields for the conserved quantity $F = Cc_0 + Dd_0 + Ee_0$ the expression given in the proposition. □

Considering the leading terms of the power expansions in ϵ, one sees immediately that the integrals c_0, d_0, e_0, and m_3 are functionally independent. Using exact evaluation of gradients we can also verify independence of other sets of integrals. It turns out that for $\alpha \neq 1$ each one of the quadruples $\{d_0, d_1, d_2, m_3\}$ and $\{e_0, e_1, e_2, m_3\}$ consists of independent integrals.

A direct "bilinearization" of the HK bases of Proposition 6.15 provides us with an alternative source of integrals of motion:

Experimental Result 6.3. *The set*

$$\Psi = (m_1\widetilde{m}_1 + m_2\widetilde{m}_2,\, p_1\widetilde{m}_1 + \widetilde{p}_1m_1 + p_2\widetilde{m}_2 + \widetilde{p}_2m_2,\, p_1\widetilde{p}_1 + p_2\widetilde{p}_2,\, p_3\widetilde{p}_3,\, p_3 + \widetilde{p}_3,\, 1)$$

is a HK basis for the map dLT with $\dim K_\Psi(x) = 3$. *Each of the following subsets of* Ψ*,*

$$\begin{aligned}
\Psi_1 &= (1,\, p_3 + \widetilde{p}_3,\, p_3\widetilde{p}_3,\, m_1\widetilde{m}_1 + m_2\widetilde{m}_2),\\
\Psi_2 &= (1,\, p_3 + \widetilde{p}_3,\, p_3\widetilde{p}_3,\, m_1\widetilde{p}_1 + \widetilde{m}_1p_1 + m_2\widetilde{p}_2 + \widetilde{m}_2p_2),\\
\Psi_3 &= (1,\, p_3 + \widetilde{p}_3,\, p_3\widetilde{p}_3,\, p_1\widetilde{p}_1 + p_2\widetilde{p}_2),
\end{aligned}$$

is a HK basis with a one-dimensional null-space.

Concerning solutions of dLT as functions of the (discrete) time t, the crucial result is given in the following statement which should be considered as the proper discretization of the differential equation (6.135).

Proposition 6.17. [35] *The component* p_3 *of the solution of difference equations (6.137) satisfies a relation of the type*

$$Q(p_3, \widetilde{p}_3) = q_0p_3^2\widetilde{p}_3^2 + q_1p_3\widetilde{p}_3(p_3 + \widetilde{p}_3) + q_2(p_3^2 + \widetilde{p}_3^2) + q_3p_3\widetilde{p}_3 + q_4(p_3 + \widetilde{p}_3) + q_5 = 0,$$

coefficients of the biquadratic polynomial Q *being conserved quantities of dLT. Hence,* $p_3(t)$ *is an elliptic function of order 2.*

Proof. Completely analogous to the proof of Proposition 6.11. □

Also, the map dLT admits an invariant Poisson structure, we have the following statement.

Proposition 6.18. *The map dLT possesses an invariant volume form:*

$$\det \frac{\partial \widetilde{x}}{\partial x} = \frac{\phi(\widetilde{x})}{\phi(x)} \quad \Leftrightarrow \quad f^*\omega = \omega, \quad \omega = \frac{dm_1 \wedge dm_2 \wedge dm_3 \wedge dp_1 \wedge dp_2 \wedge dp_3}{\phi(x)}$$

with $\phi(x) = \Delta_2(x, \epsilon)$.

Proof. The proof is parallel to that of Proposition 6.9. □

Again, we note that one could use this result in order to construct an invariant Poisson structure for dLT.

7

Conclusion and Future Perspectives

We now conclude this thesis by summarizing its main results. First, we describe the findings of this thesis from a more general point of view:

1. The HK type discretization scheme produces an *impressive number of new integrable birational maps*. Integrability for these maps is characterized by the existence of a sufficient number of conserved quantities, invariant volume forms, and HK bases which serve as discrete counterparts to the known invariance relations of the continuous time systems.

2. We have developed a set of *experimental tools* together with a systematic approach which allows for *efficient integrability detection* of birational maps, as well as discovery of their conserved quantities and more general invariance relations formulated in the general framework of HK bases. The most important tools are the algorithms (N) and (V). Usage of these algorithms simplifies the discovery of HK bases and conserved quantities and also the derivation of explicit solutions.

3. It has been developed a *novel methodology for finding explicit solutions* for birational maps, provided solutions are expressed in terms of elliptic functions. This approach does not require the knowledge of additional features (attributes) of integrable systems like Lax pairs, bi-Hamiltonian structures or similar.This can be seen as modern analog to the classical approach of solving integrable systems in terms of elliptic functions.

4. Furthermore, in the form of the recipes described in Chapter 2, there now exists a methodology based on specialized symbolic computational techniques which allow for *rigorous proofs* of integrability, originally found via (N) and (V). It is thus possible to tackle the inherent complexity of discrete integrable systems using clever application of symbolic computation.

The more concrete results of this thesis are the following:

1. The HK type discretizations of the three and four-dimensional periodic Volterra chains are integrable in the sense that they admit $N-1$ independent conserved quantities, possess invariant volume forms and may be integrated exactly in terms of elliptic functions.

2. The HK type discretizations of the Lagrange Top, the Kirchhoff System and the Clebsch System are integrable in the sense that they admit 4 independent

conserved quantities and possess invariant volume forms. In the case of the Clebsch system, the rigorous proof of this result required heavy usage of the techniques outlined in Chapter 2.

3. Moreover, for the HK type discretization of the Kirchhoff System, it has been possible to derive explicit solutions in terms of double-Bloch functions.

4. The original results by Hirota and Kimura regarding the Lagrange Top [35] have been rederived and proven rigorously.

5. Using numerical computation we have obtained evidence that several of the HK type discretizations are most likely not integrable.

Since it has become evident, that not all of the HK type discretizations are integrable, one wonders as to where this behavior originates from. Currently there exists no satisfying solution to this problem. Of course, one might argue that the form of the difference equations obtained via the HK bilinear approach simply resembles addition theorems of elliptic or hyperelliptic functions. Yet, the sheer number of integrable example of the HK type discretizations shows that this answer is unsatisfactory and suggests that there exist undiscovered structures which could help explaining this behavior. This work should hence be seen as a first step towards a demystification of this situation.

Although we have encountered a number of new and interesting results in this work, there is the need for further study of the integrability properties of the HK type discretizations. Future research could for instance follow some of the following paths:

1. One possible path could be to adapt the methods from Kowalewski-Painlevè Analysis. By studying suitable series expansions of the solutions of the equations of the type
$$\widetilde{x} - x = \epsilon Q(\widetilde{x}, x) + \epsilon B(\widetilde{x} + x) + \epsilon C$$
it might be possible to deduce neccessary conditions for the integrability of the HK type discretizations.

2. The study of the singular sets of birational mappings obtained via the HK type discretization scheme has already given insight into the existence of invariant volume forms. A more general study of the geometry of these sets could also prove useful during the uncovering of other integrable structures.

3. Also, it appears worthwhile to study the question of how the method of obtaining explicit solutions in terms of elliptic functions for some HK type discretizations could be adopted to the case where solutions are most likely given by hyperelliptic functions. The continuous time Clebsch System is, for instance, explicitly solvable in terms of genus 2 theta functions. The findings of this thesis suggest that this is also true for the HK type discretization of the Clebsch System.

4. More generally, the success of the application of the algorithm (V) shows that there might be the opportunity to further refine the methods of computing invariants of discrete dynamical systems given by birational maps by considering more links to commutative algebra and invariant theory.

In conclusion, it seems apt to claim that the HK type discretizations are interesting mathematical objects which deserve further study and attention by experts acquainted with the many mathematical subjects involved in the study of integrable systems.

Appendix A

MAPLE Session Illustrating the Application of the Algorithm (V)

The following MAPLE session demonstrates the usage of the algorithm (V) in the case of the HK-type discretization of the Euler top. The computation of the Gröbner bases is done using algorithms described in [41] and [20]. MAPLE already includes all necessary implementations starting from version 11.

First, we load the required packages.

```
> restart:
> with(LinearAlgebra):
> with(PolynomialIdeals):
```

Set the dimension of phase space:

```
> N:=3;
```

Define the continuous equations:

```
> f_q := (x,alpha) -> Vector([ (alpha[3]-alpha[2])*x[2]*x[3],
> (alpha[1]-alpha[3])*x[3]*x[1],
> (alpha[2]-alpha[1])*x[1]*x[2] ]):
> f_b := (x,beta) -> Vector([ 0,0,0]):
```

Set up the corresponding Hirota-Kimura map:

```
> F_HK:=proc(x,alpha,beta,epsilon) local eq,xx,X,f_,i,sol;global N;
> eq:={};
> xx:='xx';
> X:=Vector(N,symbol=xx);
> f_:=f_q(X+x,alpha)-f_q(x,alpha)-f_q(X,alpha)+f_b(X+x,beta);
> for i from 1 to N do
> eq := eq union {X[i]-x[i] = epsilon*(f_[i])};
> end;
> sol:=solve(eq,{seq(xx[i],i=1..N)});
> assign(sol);
> return Vector([seq(xx[i],i=1..N)]);
> end proc:
```

$$N := 3$$

Use $m = 10$ iterates.

```
> m:=10:
```

Set the initial data and parameters.

```
> alpha:=[1,7,6]:
> beta:=[]:
> f:=[]:
> x:=Vector([1,2,3]):
```

Iterate the map.

```
> for i from 1 to m do
> f := [op(f), [seq(x[i],i=1..3)]];
> x:=F_HK(x,alpha,beta,1);;
> end:
```

Compute the vanishing ideal and inspect the number of terms of its generators. The resulting list is a candidate for a basis of $I(\mathcal{O}(x_0))$, where $x_0 = (1, 2, 3)$.

```
> vars:=[X1,X2,X3]:
> V:=VanishingIdeal(f[1..m)],vars,tdeg(seq(vars[i],i=1..nops(vars))),5,0):
> g:=Generators(V):
> hk:=[]:
> for i from 1 to nops(g) do
> if ( nops(g[i]) < m ) then hk:=[op(hk),g[i]];  end;
> end:
> hk;
```

$$[-3 + 372\,X1^2 - 41\,X3^2, -39 + 372\,X2^2 - 161\,X3^2]$$

Similarly, we can look for other invariance relations. In particular, we now look for the symmetric biquadratic curves for the pairs $(x_1, \widetilde{x}_1)$.

```
> m:=10:
> alpha:=[1,7,6]:
> beta:=[]:
> f:=[]:
> x:=Vector([1,2,3]):
> for i from 1 to m do
> F:=F_HK(x,alpha,beta,1);
> f := [op(f), [x[1],F[1]]];
> x:=F;
> end:
> vars:=[X1,F1]:
> V:=VanishingIdeal(f[1..m)],vars,tdeg(seq(vars[i],i=1..nops(vars))),5,0):
> g:=Generators(V):
> hk:=[]:
> for i from 1 to nops(g) do
> if ( nops(g[i]) < m ) then hk:=[op(hk),g[i]];  end;
> end:
> hk;
```

$$[-9 + 59892\,X1^2 F1^2 + 80\,X1\,F1 - 484\,X1^2 - 484\,F1^2]$$

Appendix B

The PSLQ Algorithm

Another tool which we have used in this thesis and which belongs to the standard "tools of the trade" in experimental mathematics is the PSLQ algorithm [4, 21]. It was invented by Bailey and Ferguson and has successfully been used for the discovery of a lot of beautiful results. One of these results for instance is the BPP formula

$$\pi = \sum_{n=0}^{\infty} \frac{1}{16^n} \left(\frac{4}{8n+1} - \frac{2}{8n+4} - \frac{1}{8n+5} - \frac{1}{8n+6} \right)$$

which spawns an algorithm for the computation of the n-th hexadecimal digit of π *without* knowing the $n-1$ digits before [11].

Let us shortly sketch the main features of the PSLQ algorithm. Given a set of n real numbers x_i we are interested in the question whether there exists an integer relation among the x_i, that is we want to know whether there exist n numbers $a_i \in \mathbb{Z}$ such that

$$\sum_{i=0}^{n} a_i x_i = 0$$

holds. The PSLQ Algorithm can be used to find such numbers a_i. It requires as input a vector of n high precision (usually around several hundred digits) floating point numbers x_i. The algorithm always terminates after a number of steps bounded by a polynomial in n. Its output is a vector of n integer numbers a_i which either are a candidate for an integer relation or an "upper bound" for the existence of a linear relation, i.e. the meaning of the output a_i is that there are no integer relations with numbers whose absolute value is less than those in the output. Since the algorithm is based on computations which are not exact, the user has to confirm using a rigorous proof that the output is an integer relation for the numbers x_i indeed.

As mentioned, the above formula for π was discovered with the help of the PSLQ algorithm. Here, the PSLQ algorithm was run with the input being the vector

$$[\pi, x_1, x_2, x_3, x_4, x_5, x_6, x_7],$$

where

$$x_i = \sum_{n=0}^{\infty} \frac{1}{16^n(8n+j)},$$

which, together with π, were evaluated numerically up to several hundred digits. The output of the PSLQ algorithm then read

$$[1, -4, 0, 0, 2, 1, 1, 0],$$

which suggested the above formula. This result was later proven rigorously by Bailey, Borwein and Plouffe [11].

In the following we now show, how one can use MAPLE and the PSLQ algorithm in order to identify a linear relation between two functions. As a concrete example we take the integrals of motion c_1 and c_2 from Theorem 6.3.

First, we load the required package.

```
> restart;
> with(IntegerRelations):
```

We set up the two functions c_1 and c_2:

```
> c1:=(1+epsilon^2*(omega_1-omega_2)*p_2^2+
> epsilon^2*(omega_1-omega_3)*p_3^2)/(p_1^2+p_2^2+p_3^2);
> c2:=(1+epsilon^2*(omega_2-omega_1)*p_1^2+
> epsilon^2*(omega_2-omega_3)*p_3^2)/(p_1^2+p_2^2+p_3^2);
```

$$c1 := \frac{1+\epsilon^2\,(omega1-omega2)\,p\_2^2+\epsilon^2\,(omega1-omega3)\,p\_3^2}{p\_1^2+p\_2^2+p\_3^2}$$

$$c2 := \frac{1+\epsilon^2\,(omega2-omega1)\,p\_1^2+\epsilon^2\,(omega2-omega3)\,p\_3^2}{p\_1^2+p\_2^2+p\_3^2}$$

Now, we choose some numerical values for the parameters ω_i and ϵ.

```
> omega_1:=1;omega_2:=20;omega_3:=13;
> epsilon:=7;
> roll:=rand(1..10)/100000:
```

$$omega1 := 1$$
$$omega2 := 20$$
$$omega3 := 13$$
$$\epsilon := 7$$

We now run the PSLQ algorithm five times using different, randomly chosen values for p_i.

```
> for i from 1 to 5 do
> p_1:=roll();p_2:=roll();p_3:=roll();
> print(PSLQ([c1,c2,1]));
> end:
```

$$[1,-1,931]$$
$$[1,-1,931]$$
$$[1,-1,931]$$
$$[1,-1,931]$$
$$[1,-1,931]$$

This output suggests, that $c_1(x_0) - c_2(x_0) = -931 = 7^2(1 - 20)$. Hence, we can conjecture, that there holds $c_1(x_0) - c_2(x_0) = \epsilon^2(\omega_1 - \omega_2)$.

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