Dissipativity of linear quadratic systems

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Chapter 1

Introduction

The starting point of this dissertation was the following problem, which emerged from a cooperation with *CST AG, Darmstadt* (`http://www.cst.com`). Assume that a *standard state-space system* of the form

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t) + Du(t), \end{aligned} \tag{1.1}$$

with $A \in \mathbb{C}^{n,n}$, $B \in \mathbb{C}^{n,m}$, $C \in \mathbb{C}^{l,n}$, and $D \in \mathbb{C}^{l,m}$ is given, that describes the electromagnetic behavior of a passive electronic device, e.g., a network cable connector or an antenna which does not generate energy. Assume further that in spite of the underlying physical problem our model (1.1) is one that generates energy (in some sense which, of course, has to be closer specified, see Definition 3.1). Then, it is natural to ask if one can determine a nearby system

$$\begin{aligned} \dot{x}(t) &= \tilde{A}x(t) + \tilde{B}u(t), \\ y(t) &= \tilde{C}x(t) + \tilde{D}u(t), \end{aligned} \tag{1.2}$$

with $\tilde{A} \in \mathbb{C}^{n,n}$, $\tilde{B} \in \mathbb{C}^{n,m}$, $\tilde{C} \in \mathbb{C}^{l,n}$, and $\tilde{D} \in \mathbb{C}^{l,m}$ which is passive. With "nearby" we mean that the difference of the block matrices

$$\left\| \begin{bmatrix} A & B \\ C & D \end{bmatrix} - \begin{bmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & \tilde{D} \end{bmatrix} \right\|_F$$

is small.
This problem is well-known in the literature. Solutions have been obtained via semi-definite programming methods in [11, 15, 16, 17] and via the perturbation of Hamiltonian matrices in [21, 22, 31, 32, 33]. Unfortunately, the semi-definite programming methods are very expensive computationally and the methods that employ the perturbation of a Hamiltonian matrix sometimes fail. Furthermore, none of these methods extends to *descriptor systems*

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t) + Du(t), \end{aligned}$$

with $E \in \mathbb{C}^{\rho,n}$, $A \in \mathbb{C}^{\rho,n}$, $B \in \mathbb{C}^{\rho,m}$, $C \in \mathbb{C}^{l,n}$, and $D \in \mathbb{C}^{l,m}$ or *behavioral systems*

$$F\dot{z}(t) + Gz(t) = 0,$$

with $F, G \in \mathbb{C}^{p,q}$. However, such systems are the appropriate model class in most electrical applications. In this dissertation we will propose such a passivation algorithm for descriptor systems, see Algorithm 4.9, which is a generalization of the methods which employ the perturbation of a Hamiltonian matrix. Although a generalization of the results to behavioral systems is also possible it will not be conducted here, since all our test examples (provided by CST AG, Darmstadt) take the form of standard state-space systems (1.1).
For the theoretical considerations in Chapter 2 and Chapter 3, however, we use behavioral systems without exception. This is mainly done for two reasons. First, the results become more general, although this increased generality may not play a prominent role in practice and, second, the results become simpler. This increased simplicity makes the theorem statements and proofs shorter and more readable, since a lower number of letters is needed (e.g., F, G, z instead of E, A, B, C, D, u, x, y). Thus, it fosters understanding, since the mind can concentrate on the things that are of real importance and not get impeded by excessive elementary matrix manipulations.
However, one must not forget that in the end one wants to apply the results to systems which are most likely descriptor systems. This is the reason why we will try to avoid statements which involve image representations, as far as possible. We mainly think of an image representations as a parameterization of the controllable part of a system.
Behavioral systems have thoroughly been studied [27, 43, 44] via the Smith canonical form. While the Smith canonical form (see Theorem 2.3) will also be used in this thesis, we will further make use of the Kronecker canonical form (see Theorem 2.14). Since the Kronecker canonical form refers to first-order matrix polynomials, the corresponding results cannot directly (only via linearization) be applied to higher-order system, as it is possible with the Smith canonical form. In return, the Kronecker canonical form grants deeper insight into the first-order system.
The research and results which are summarized in this thesis and that lead to the proposition of the algorithms in Chapter 4 started from the following observations. First, it was necessary to understand why the problem of passivation is so intimately linked with a certain Hamiltonian eigenvalue problem (Why can we perturb a Hamiltonian matrix to passivate a system?). A good reference to understand this relationship is [5]. There it is shown that the singular values of the transfer function can be determined via the computation of the purely imaginary eigenvalues of a Hamiltonian matrix. However, the zero singular values of the transfer function can also be interpreted as the zeros of a so-called Popov function, compare Definition 3.4. Another important observation for the progression of this thesis was that Hamiltonian eigenvalue problems are closely related to generalized para-Hermitian eigenvalues problems (see Definition 2.22; sometimes also called even eigenvalue problems) as described in [8]. Indeed, every Hamiltonian eigenvalue problem can be formulated as a generalized para-Hermitian eigenvalue problem and the other way round [4]. Combining these observations led to the first main result of this thesis, namely Theorem 3.7, which states

that the zeros of any Popov function are essentially given by the zeros of a para-Hermitian matrix N, which can easily be obtained from the original system data.
When looking at this matrix N in the special case of descriptor systems (which is done in Section 4.1), one immediately notices that its coefficients are the same that occur in the boundary value problem which stands behind the standard linear quadratic optimal control problem, compare [24]. That the linear system given by N is also intimately connected to a linear quadratic control problem in a more general behavioral setting is then shown in Section 3.2. Furthermore, the similarity of the optimal control problem to the so-called available storage and required supply (see Definition 3.14) is notable. We follow mainly the ideas of [41, 42], to introduce storage functions in general (see Definition 3.1) and especially the available storage and required supply. In Section 3.3 we show that for dissipative systems the available storage and the required supply are the extremal storage functions. Although this result is well-known (e.g., [35, Theorem 5.7] and [41, Theorem 2]), we present a simple and self-contained exposition here.
When thinking of linear quadratic optimal control, the algebraic Riccati equation is one of the things that comes to mind almost immediately [40]. It is well-known that one method to solve the algebraic Riccati equation is via the solution of a Hamiltonian eigenvalue problem, also for descriptor systems [24]. For descriptor systems, the algebraic Riccati equation can also be generalized and is then called Lur'e equation, compare [30]. The Lur'e equation can also be interpreted as a linear matrix inequality with a rank minimizing condition. The role of such linear matrix inequalities (without rank minimizing condition) in systems theory is well understood [6]. For behavioral systems, linear matrix inequalities have also been studied [36]. Here we will introduce another type of linear matrix inequality which allows to make weaker assumptions than in [36]. Also from our results it is possible to derive linear matrix inequalities which have recently been proposed for descriptor systems [9] and which had a major influence on the new formulation given in this thesis.
This thesis is structured in such a way that the material, which grants the deepest insight into the system theoretic principles is gathered in the main part, i.e., Chapters 2 and 3. The technical part is deferred into Appendix A. The Notation used is quite standard and summed up in Table 1.1 and Table 1.2. Some Definitions which are introduced later are summed up in Table 1.3.

Table 1.1: Notation - 1/2

$\mathbb{C}^+$	denotes $\{z \in \mathbb{C} : \mathrm{Re}\,(z) > 0\}$
$\mathbb{C}^-$	denotes $\{z \in \mathbb{C} : \mathrm{Re}\,(z) < 0\}$
$\mathcal{C}^q_\infty$	$\{z : \mathbb{R} \to \mathbb{C}^q \,\vert\, z \text{ is infinitely often differentiable}\}$
$\mathcal{C}^q_+$	the set of all functions $z \in \mathcal{C}^q_\infty$ for which z and all its derivatives are exponentially decaying for $t \to \infty$, i.e., all $z \in \mathcal{C}^q_\infty$ such that for every $i \in \mathbb{N}_0$ there exist $a_i, b_i > 0$ with $\left\Vert z^{(i)}(t)\right\Vert_2 \leq a_i e^{-b_i t}$ for all $t \geq 0$
$\mathcal{C}^q_-$	the set of all functions $z \in \mathcal{C}^q_\infty$ for which z and all its derivatives are exponentially decaying for $t \to -\infty$, i.e., all $z \in \mathcal{C}^q_\infty$ such that for every $i \in \mathbb{N}_0$ there exist $a_i, b_i > 0$ with $\left\Vert z^{(i)}(t)\right\Vert_2 \leq a_i e^{b_i t}$ for all $t \leq 0$
$\mathcal{C}^q_c$	$\{z \in \mathcal{C}^q_\infty \,\vert\, z \text{ has compact support}\}$
$\mathcal{C}_\infty$	short for $\mathcal{C}^1_\infty$
$\mathcal{C}_+$	short for $\mathcal{C}^1_+$
$\mathcal{C}_-$	short for $\mathcal{C}^1_-$
$\mathcal{C}_c$	short for $\mathcal{C}^1_c$
$\mathbb{C}[\lambda]$	the ring of polynomials with coefficients in $\mathbb{C}$
$\mathbb{C}[\lambda]_K$	the set of polynomials with coefficients in $\mathbb{C}$ and degree less than or equal to $K \in \mathbb{N}$
$\mathbb{C}(\lambda)$	the field of rational functions with coefficients in $\mathbb{C}$
$\mathbb{C}[\lambda]^{p,q}$	a p-by-q matrix with polynomial entries
$\mathbb{C}[\lambda]^{p,q}_K$	a p-by-q matrix with polynomial entries of degree less than or equal to $K \in \mathbb{N}$
$\mathbb{C}(\lambda)^{p,q}$	a p-by-q matrix with rational entries
$\mathrm{rank}_{\mathbb{C}(\lambda)}\,(R)$	where $R \in \mathbb{C}(\lambda)^{p,q}$; denotes the rank of R over the field $\mathbb{C}(\lambda)$; also called generic rank
$\mathrm{kernel}_{\mathbb{C}(\lambda)}\,(R)$	where $R \in \mathbb{C}(\lambda)^{p,q}$; denotes the kernel of R over the field $\mathbb{C}(\lambda)$ which is a subset of $\mathbb{C}(\lambda)^q$
$\mathrm{image}_{\mathbb{C}(\lambda)}\,(R)$	where $R \in \mathbb{C}(\lambda)^{p,q}$; denotes the range of R over the field $\mathbb{C}(\lambda)$ which is a subset of $\mathbb{C}(\lambda)^p$
$\mathrm{rank}\,(P(\lambda))$	where $P \in \mathbb{C}[\lambda]^{p,q}$ and $\lambda \in \mathbb{C}$; denotes the rank of $P(\lambda) \in \mathbb{C}^{p,q}$ in the usual way
$\mathrm{kernel}\,(P(\lambda))$	where $P \in \mathbb{C}[\lambda]^{p,q}$ and $\lambda \in \mathbb{C}$; denotes the kernel of $P(\lambda) \in \mathbb{C}^{p,q}$ in the usual way
$\mathrm{image}\,(P(\lambda))$	where $P \in \mathbb{C}[\lambda]^{p,q}$ and $\lambda \in \mathbb{C}$; denotes the range of $P(\lambda) \in \mathbb{C}^{p,q}$ in the usual way
$\mathrm{diag}\,(A_1, \ldots, A_r)$	where $A_1, \ldots, A_r$ are matrices; denotes the block diagonal matrix which has the (not necessarily square) matrices $A_1, \ldots, A_r$ on the block diagonal and zeros everywhere else
$z^{(i)}$	the i-th derivative of the function z

Table 1.2: Notation - 2/2

$P\left(\frac{d}{dt}\right)z$	where $P \in \mathbb{C}[\lambda]^{p,q}$ has the form $P(\lambda) = \sum_{i=0}^{K} \lambda^i P_i$ and $z \in \mathcal{C}_\infty^q$; denotes the function $\sum_{i=0}^{K} P_i z^{(i)}$
$y^* P\left(\frac{d}{dt}\right)z$	where $P \in \mathbb{C}[\lambda]^{p,q}$, $y \in \mathcal{C}_\infty^p$, and $z \in \mathcal{C}_\infty^q$ means the function given by the inner product $y^*\left(P\left(\frac{d}{dt}\right)z\right)$, i.e., the differential operator $P\left(\frac{d}{dt}\right)$ is always assumed to obtain its input from the right side
Δ_K^q	where $K, q \in \mathbb{N}$; denotes the polynomial given by $$\Delta_K^q(\lambda) := \begin{bmatrix} I_q \\ \lambda I_q \\ \vdots \\ \lambda^{K-1} I_q \end{bmatrix} \in \mathbb{C}[\lambda]^{qK,q}$$
$\Delta_K z$	where $K \in \mathbb{N}$ and $z \in \mathcal{C}_\infty^q$; denotes the function $$\Delta_K z := \begin{bmatrix} z \\ \vdots \\ z^{(K-1)} \end{bmatrix} \in \mathcal{C}_\infty^{qK},$$ and thus we have $\Delta_K z = \Delta_K^q\left(\frac{d}{dt}\right)z$
$\langle f, g\rangle_+$	where $f, g \in \mathcal{C}_+^n$; denotes the L_2 inner product on the positive half axis given by $$\langle f, g\rangle_+ := \int_0^\infty g^*(t) f(t) dt$$
$\langle f, g\rangle_-$	where $f, g \in \mathcal{C}_-^n$; denotes the L_2 inner product on the negative half axis given by $$\langle f, g\rangle_- := \int_{-\infty}^0 g^*(t) f(t) dt$$
$\Vert f\Vert_+$	where $f \in \mathcal{C}_+^n$; denotes the L_2 measure on the positive half axis given by $$\Vert f\Vert_+ := \sqrt{\langle f, f\rangle_+}$$
$\Vert f\Vert_-$	where $f \in \mathcal{C}_-^n$; denotes the L_2 measure on the negative half axis given by $$\Vert f\Vert_- := \sqrt{\langle f, f\rangle_-}$$

Table 1.3: Some Definitions

$\mathfrak{Z}(R)$, $\mathfrak{P}(R)$, and $\mathfrak{D}(R)$	where $R \in \mathbb{C}(\lambda)^{p,q}$ is a rational matrix; denotes the set of zeros, poles, and domain of definition of R, compare Definition 2.5
$P^{\langle k \rangle}$	where $P \in \mathbb{C}[\lambda]_K^{p,q}$; denotes the k-times shifted polynomial $P^{\langle k \rangle} \in \mathbb{C}[\lambda]_{K-k}^{p,q}$, compare Definition 2.25
$R^{\sim}$	where $R \in \mathbb{C}(\lambda)^{p,q}$; denotes the para-Hermitian of R, i.e., the matrix $R^{\sim} \in \mathbb{C}(\lambda)^{q,p}$ with $R^{\sim}(\lambda) := R^*(-\overline{\lambda})$, see Definition 2.22
$\mathfrak{B}(P)$, $\mathfrak{B}_+(P)$, $\mathfrak{B}_-(P)$, and $\mathfrak{B}_c(P)$	the behavior, the positive decaying behavior, the negative decaying behavior, and the compact behavior of P; see Definition 2.15
$R(P)$, $R_+(P)$, $R_-(P)$, and $R_c(P)$	the reachable set, the positive decaying reachable set, the negative decaying reachable set, and the compact reachable set of P; see Definition 2.16
U and V	kernel and co-kernel matrix; see Definition 2.8
Π	Popov function; see Definition 3.4.
Z	the variable in the linear matrix inequality (3.14); or an actual solution of it; see Definition 3.23
Θ	a storage function; see Definition 3.1
Θ_+ and Θ_-	the available storage and required supply; compare Definition 3.14
$\eta(A)$	the signsum function, i.e., the number of non-negative eigenvalues minus the number of negative eigenvalues of the Hermitian matrix $A = A^*$, compare Definition 3.9
dissipativity	a property of the complete system $\mathfrak{B}(P)$; see Definition 3.1
cyclo-dissipativity	a property of the controllable part of a system $\mathfrak{B}_c(P)$; see Definition 3.2
signsum plot	see Figure 4.3 and surrounding text
completely controllable	type of controllability only defined for descriptor systems; see Definition 4.4

Chapter 2

Preliminaries

In this chapter we will repeat some well known facts concerning polynomial and rational matrices, systems theory from a behavioral point of view (as described in [27, 44, 43]), and the Kronecker canonical form.
The notation used here differs from the standard notation of behavioral systems theory in that we will not formally introduce the term *image representation*. Instead, we use what is called *kernel matrix* in this thesis (see Definition 2.8). A polynomial kernel matrix without zeros can be thought of as an image representation of the controllable part, see Lemma 2.18. In contrast to an image representation, a kernel matrix $U \in \mathbb{C}(\lambda)^{q,m}$ is allowed to be rational. This approach resembles the one used in [39] and the increased generality is necessary to derive some of the results in Section 4.1, where explicit representations of kernel matrices are given, see (4.4) and (4.6), which are rational matrices

2.1 Rational and polynomial matrices

In this section we introduce the Smith and McMillan canonical form, so that we can specify what zeros and poles of rational matrices are. Also the kernel matrix is defined and we state some of its properties.

Definition 2.1. A square polynomial matrix $Q \in \mathbb{C}[\lambda]^{p,p}$ is called *unimodular* if it has a polynomial inverse, i.e., if there exists a $\tilde{Q} \in \mathbb{C}[\lambda]^{p,p}$ such that $\tilde{Q}Q = Q\tilde{Q} = I$.

Lemma 2.2. *A polynomial matrix $Q \in \mathbb{C}[\lambda]^{p,p}$ is unimodular if and only if its determinant is a non-zero constant.*

Proof. [27, Exercise 2.6] Since $1 = \det I = \det(QQ^{-1}) = \det(Q)\det(Q^{-1})$, we see that $\det(Q) = \det(Q^{-1})^{-1}$. If Q is unimodular this implies that both $\det(Q)$ and $\det(Q^{-1})$ are polynomials and thus non-zero constants. If, on the other hand, $\det(Q)$ is a non-zero constant we obtain that the inverse of Q is polynomial by using Laplace expansion. □

Theorem 2.3 (Smith canonical form)**.** *Let $P \in \mathbb{C}[\lambda]^{p,q}$. Then there exist $r \in \mathbb{N}$ and unimodular matrices $S \in \mathbb{C}[\lambda]^{p,p}$ and $T \in \mathbb{C}[\lambda]^{q,q}$ such that*

$$P = S \begin{bmatrix} \operatorname{diag}(d_1, \dots, d_r) & 0 \\ 0 & 0 \end{bmatrix} T, \tag{2.1}$$

where $d_1, \dots, d_r \in \mathbb{C}[\lambda]$ with $d_i \neq 0$ for $i = 1, \dots, r$ and d_{i+1} divides d_i for $i = 1, \dots, r-1$. The rank of P is given by $r = \operatorname{rank}_{\mathbb{C}(\lambda)}(P)$.

Proof. A completely self contained proof can be found in [20, Chapter S1.1] and another in [18, p.141, Theorem 3]. □

The problem with the Smith canonical form is that the degree of the d_i's cannot be bounded by the degree of P. For example, P can have degree 1 while the associated d_1 has arbitrary high degree. For first order systems, this problem can be avoided by using the Kronecker canonical form (see Theorem 2.14).

Theorem 2.4 (McMillan canonical form)**.** *Let $R \in \mathbb{C}(\lambda)^{p,q}$. Then there exist $r \in \mathbb{N}$ and unimodular matrices $S \in \mathbb{C}[\lambda]^{p,p}$ and $T \in \mathbb{C}[\lambda]^{q,q}$ such that*

$$R = S \begin{bmatrix} \operatorname{diag}\left(\frac{\alpha_1}{\beta_1}, \dots, \frac{\alpha_r}{\beta_r}\right) & 0 \\ 0 & 0 \end{bmatrix} T, \tag{2.2}$$

where $\alpha_1, \dots, \alpha_r, \beta_1, \dots, \beta_r \in \mathbb{C}[\lambda]$ with $\alpha_i, \beta_i \neq 0$ for $i = 1, \dots, r$, α_{i+1} divides α_i for $i = 1, \dots, r-1$, and β_i divides β_{i+1} for $i = 1, \dots, r-1$. The rank of R is given by $r = \operatorname{rank}_{\mathbb{C}(\lambda)}(R)$.

Proof. [47, Lemma 3.26] Write $R = \frac{1}{d}P$ with $d \in \mathbb{C}[\lambda]$, $P \in \mathbb{C}[\lambda]^{p,q}$ and use the Smith canonical form. □

Definition 2.5. Let $R \in \mathbb{C}(\lambda)^{p,q}$ and its McMillan form be given by (2.2). Then we call

$$\mathfrak{Z}(R) := \{\lambda \in \mathbb{C} \,|\, \alpha_1(\lambda) \cdot \ldots \cdot \alpha_r(\lambda) = 0\},$$

the *zeros of* R and

$$\mathfrak{P}(R) := \{\lambda \in \mathbb{C} \,|\, \beta_1(\lambda) \cdot \ldots \cdot \beta_r(\lambda) = 0\},$$

the *poles of* R. Also, we call $\mathfrak{D}(R) := \mathbb{C} \setminus \mathfrak{P}(R)$ the *domain of definition of* R and assume w.l.o.g. that for every $\lambda_0 \in \mathfrak{D}(R)$ the matrix $R(\lambda_0) \in \mathbb{C}^{p,q}$ is well defined.

We see that for polynomial matrices $P \in \mathbb{C}[\lambda]^{p,q}$ we have $\mathfrak{P}(P) = \emptyset$ and thus $\mathfrak{D}(P) = \mathbb{C}$.

Lemma 2.6. *Let $R \in \mathbb{C}(\lambda)^{p,q}$. Then we have $\operatorname{rank}_{\mathbb{C}(\lambda)}(R) = \max_{\lambda \in \mathfrak{D}(R)} \operatorname{rank}(R(\lambda))$. Defining $\Lambda := \{\lambda \in \mathfrak{D}(R) \,|\, \operatorname{rank}(R(\lambda)) < \operatorname{rank}_{\mathbb{C}(\lambda)}(R)\}$ we have $\Lambda \subset \mathfrak{Z}(R) \subset \Lambda \cup \mathfrak{P}(R)$.*

Proof. [47, Lemma 3.29] Using the McMillan form of R we obtain the results. □

Lemma 2.6 especially shows that for polynomial matrices $P \in \mathbb{C}[\lambda]^{p,q}$ we have

$$\mathfrak{Z}(P) = \{\lambda \in \mathbb{C} \big| \operatorname{rank}(P(\lambda)) < \operatorname{rank}_{\mathbb{C}(\lambda)}(P)\},$$

i.e., that the zeros correspond to those points in the complex plane, where the rank drops below the generic rank. Note, that a unimodular matrix can have no zeros, since at such a zero the determinant would vanish which contradicts Lemma 2.2.

Theorem 2.7. *Let $P \in \mathbb{C}[\lambda]^{p,q}$ and set $r := \operatorname{rank}_{\mathbb{C}(\lambda)}(P)$. Then there exist polynomial matrices $U \in \mathbb{C}[\lambda]^{q,q-r}$ and $V \in \mathbb{C}[\lambda]^{q,r}$ with the properties*

1. $PU = 0$,
2. $\operatorname{rank}_{\mathbb{C}(\lambda)}(PV) = r$ *and* $\mathfrak{Z}(P) = \mathfrak{Z}(PV)$,
3. $\begin{bmatrix} U & V \end{bmatrix}$ *is unimodular.*

Proof. Let a Smith form of P be given by (2.1). Partition the inverse of T according to the block structure of the diagonal matrix in the Smith form as $T^{-1} =: \begin{bmatrix} V & U \end{bmatrix}$, i.e., such that V has r columns and U has $q-r$ columns. Then clearly also $\begin{bmatrix} U & V \end{bmatrix}$ is unimodular, since it can be obtained from T^{-1} through a block column permutation and 3. is proved. We see that 1. holds, since

$$PU = S \begin{bmatrix} \operatorname{diag}(d_1, \dots, d_r) & 0 \\ 0 & 0 \end{bmatrix} TU = S \begin{bmatrix} \operatorname{diag}(d_1, \dots, d_r) & 0 \\ 0 & 0 \end{bmatrix} TT^{-1} \begin{bmatrix} 0 \\ I_{q-r} \end{bmatrix} = 0.$$

Property 2. holds, since a Smith form of PV can be obtained via

$$PV = S \begin{bmatrix} \operatorname{diag}(d_1, \dots, d_r) & 0 \\ 0 & 0 \end{bmatrix} TT^{-1} \begin{bmatrix} I_r \\ 0 \end{bmatrix} = S \begin{bmatrix} \operatorname{diag}(d_1, \dots, d_r) \\ 0 \end{bmatrix},$$

and all $d_i \neq 0$ for $i = 1, \dots, r$. □

Theorem 2.7 motivates the following definition.

Definition 2.8. Let $P \in \mathbb{C}[\lambda]^{p,q}$ and $r := \operatorname{rank}_{\mathbb{C}(\lambda)}(P)$ be its rank. Then $U \in \mathbb{C}(\lambda)^{q,q-r}$ and $V \in \mathbb{C}(\lambda)^{q,r}$ are called *kernel matrix and co-kernel matrix* of P, resp., if they fulfill the following properties

1. $PU = 0$,
2. $\operatorname{rank}_{\mathbb{C}(\lambda)}(PV) = r$,
3. $\begin{bmatrix} U & V \end{bmatrix}$ is invertible over $\mathbb{C}(\lambda)$.

Theorem 2.7 shows that for every matrix polynomial there exist *polynomial* kernel and co-kernel matrices which have no zeros (since $\begin{bmatrix} U & V \end{bmatrix}$ is unimodular). We allow the kernel matrix to be a rational function, because for regular first order state-space systems we will later present kernel matrices in explicit form, see (4.4) and (4.6), which happen to be rational matrices.

Lemma 2.9. *Let $P \in \mathbb{C}[\lambda]^{p,q}$ and set $r := \operatorname{rank}_{\mathbb{C}(\lambda)}(P)$. Let $U \in \mathbb{C}(\lambda)^{q,q-r}$ and $V \in \mathbb{C}(\lambda)^{q,r}$ be a kernel and co-kernel matrix of P. Let a Smith form of P be given by (2.1), and partition the inverse of T as $T^{-1} = \begin{bmatrix} T_1 & T_2 \end{bmatrix}$ with T_1 having r columns and T_2 having $q-r$ columns, partitioned analogously to the Smith form.*
Then, there exists an invertible $U_2 \in \mathbb{C}(\lambda)^{q-r,q-r}$ with $\mathfrak{P}(U) = \mathfrak{P}(U_2)$ and $\mathfrak{Z}(U) = \mathfrak{Z}(U_2)$, an invertible matrix $V_1 \in \mathbb{C}(\lambda)^{r,r}$, and $V_2 \in \mathbb{C}(\lambda)^{q-r,r}$ such that

$$U = T_2 U_2, \text{ and } V = T_1 V_1 + T_2 V_2.$$

If U and V are polynomial, then also U_2, V_1, and V_2 are polynomial.

Proof. Set $\tilde{U} := TU$ and $\tilde{V} := TV$ and partition these matrices via

$$\tilde{U} =: \begin{array}{c} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} \end{array} \begin{array}{c} r \\ q-r \end{array} \quad \text{and} \quad \tilde{V} =: \begin{array}{c} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \end{array} \begin{array}{c} r \\ q-r \end{array},$$

i.e., such that V_1 and U_2 are square. Taking the Smith form into consideration, we find that

$$0 = S^{-1}PU = \begin{bmatrix} \operatorname{diag}(d_1, \dots, d_r) & 0 \\ 0 & 0 \end{bmatrix} \tilde{U} = \begin{bmatrix} \operatorname{diag}(d_1, \dots, d_r) U_1 \\ 0 \end{bmatrix},$$

and, thus, that $U_1 = 0$, since $\operatorname{diag}(d_1, \dots, d_r)$ is invertible. Since U has full column rank (property 3. of Definition 2.8), so does $\tilde{U}$, which implies that U_2 is invertible. From the relation $TU = \begin{bmatrix} 0 & U_2^T \end{bmatrix}^T$ we find that U and U_2 have in principle the same McMillan form and thus the same zeros and poles. For $\tilde{V}$, on the other hand, we find that

$$r = \operatorname{rank}_{\mathbb{C}(\lambda)}(PV) = \operatorname{rank}_{\mathbb{C}(\lambda)}\left(S^{-1}PV\right) = \operatorname{rank}_{\mathbb{C}(\lambda)}\left(\begin{bmatrix} \operatorname{diag}(d_1, \dots, d_r) V_1 \\ 0 \end{bmatrix}\right),$$

and, thus, that V_1 is invertible, since, again, $\operatorname{diag}(d_1, \dots, d_r)$ is invertible. □

Remark 2.10. Consider a polynomial kernel matrix without zeros, e.g., the one from Theorem 2.7. Then we conclude with Lemma 2.9 that in this case U_2 is an invertible polynomial matrix without zeros. Using the Smith form of U_2 this implies that in this case U_2 is unimodular.

Lemma 2.11. *Let $R \in \mathbb{C}(\lambda)^{p,r}$ be a rational matrix with full column rank, i.e., with* $\operatorname{rank}_{\mathbb{C}(\lambda)}(R) = r$. *Then there exists a left inverse $X \in \mathbb{C}(\lambda)^{r,p}$ of R (i.e., a rational matrix X such that $XR = I_r$) such that $\mathfrak{P}(X) = \mathfrak{Z}(R)$ and $\mathfrak{P}(R) = \mathfrak{Z}(X)$. Furthermore, if R has no zeros then X is a polynomial.*

Proof. [46, p. 173] Let the McMillan form of R be given by (2.2) and define

$$X := T^{-1} \begin{bmatrix} \operatorname{diag}\left(\frac{\beta_1}{\alpha_1}, \dots, \frac{\beta_r}{\alpha_r}\right) & 0 \end{bmatrix} S^{-1}.$$

For this X one can obtain a McMillan form by permuting the first r rows and columns and thus the claim is shown. □

2.1.1 Linearization and the Kronecker canonical form

It is common practice in systems theory to transform higher order systems into systems of first order, for example to run algorithms on them which only work for first order systems. This process is called *linearization* or *first-order reduction*. For behavioral systems such a linearization is also possible.

Definition 2.12. For $P \in \mathbb{C}[\lambda]_K^{p,q}$ in the form $P(\lambda) = \sum_{i=0}^{K} \lambda^i P_i$ with $P_i \in \mathbb{C}^{p,q}$ we call

$$\lambda F + G := \lambda \begin{bmatrix} I_q & & & \\ & \ddots & & \\ & & I_q & \\ & & & P_K \end{bmatrix} + \begin{bmatrix} 0 & -I_q & & \\ & \ddots & \ddots & \\ & & 0 & -I_q \\ P_0 & \dots & P_{K-2} & P_{K-1} \end{bmatrix} \in \mathbb{C}[\lambda]_1^{p+q(K-1),qK} \tag{2.3}$$

the *canonical linearization of* P.

Lemma 2.13. *Let* $\lambda F + G \in \mathbb{C}[\lambda]_1^{p+q(K-1),qK}$ *be the canonical linearization of* $P \in \mathbb{C}[\lambda]_K^{p,q}$. *Then we have* $\operatorname{rank}_{\mathbb{C}(\lambda)}(\lambda F + G) = q(K-1) + \operatorname{rank}_{\mathbb{C}(\lambda)}(P)$ *and* $\mathfrak{Z}(P) = \mathfrak{Z}(\lambda F + G)$.

Proof. Through pre- and post-multiplication with unimodular matrices we obtain that

$$\begin{aligned} \lambda F + G &= \begin{bmatrix} \lambda I & -I_q & & \\ & \ddots & \ddots & \\ & & \lambda I & -I_q \\ P_0 & \dots & P_{K-2} & \lambda P_K + P_{K-1} \end{bmatrix} \sim \begin{bmatrix} \lambda I & -I_q & & \\ & \ddots & \ddots & \\ & & \lambda I & -I_q \\ P_0 & \dots & \lambda^2 P_K + \lambda P_{K-1} + P_{K-2} & 0 \end{bmatrix} \\ &\sim \begin{bmatrix} \lambda I & -I_q & & \\ & \ddots & \ddots & \\ & & 0 & -I_q \\ P_0 & \dots & \lambda^2 P_K + \lambda P_{K-1} + P_{K-2} & 0 \end{bmatrix} \sim \dots \sim \begin{bmatrix} 0 & -I_q & & \\ & \ddots & \ddots & \\ & & 0 & -I_q \\ P(\lambda) & \dots & 0 & 0 \end{bmatrix}, \end{aligned}$$

and thus with Lemma 2.6 the assertion. □

In this thesis we need the canonical linearization of a matrix polynomial $P \in \mathbb{C}[\lambda]^{p,q}$ so that we can apply the following theorem.

Theorem 2.14 (Kronecker canonical form)**.** *Let* $\lambda F + G \in \mathbb{C}[\lambda]_1^{p,q}$. *Then there exist nonsingular matrices* $S \in \mathbb{C}^{p,p}$ *and* $T \in \mathbb{C}^{q,q}$ *and* $\epsilon, \rho, \sigma, \eta, s, u, v, w \in \mathbb{N}_0$ *such that*

$$\lambda F + G = S \cdot \operatorname{diag}(\mathcal{L}, \mathcal{J}, \mathcal{N}, \mathcal{M}) \cdot T, \tag{2.4}$$

where $\mathcal{L} \in \mathbb{C}[\lambda]_1^{\epsilon,\epsilon+s}$, $\mathcal{J} \in \mathbb{C}[\lambda]_1^{\rho,\rho}$, $\mathcal{N} \in \mathbb{C}[\lambda]_1^{\sigma,\sigma}$, *and* $\mathcal{M} \in \mathbb{C}[\lambda]_1^{\eta+w,\eta}$ *can be further partitioned into*

$$\begin{aligned} \mathcal{L} &=: \operatorname{diag}(\mathcal{L}_{\epsilon_1}, \dots, \mathcal{L}_{\epsilon_s}), & \mathcal{J} &=: \operatorname{diag}(\mathcal{J}_{\rho_1}, \dots, \mathcal{J}_{\rho_u}), \\ \mathcal{N} &=: \operatorname{diag}(\mathcal{N}_{\sigma_1}, \dots, \mathcal{N}_{\sigma_v}), & \mathcal{M} &=: \operatorname{diag}(\mathcal{M}_{\eta_1}, \dots, \mathcal{M}_{\eta_w}), \end{aligned}$$

with $\epsilon = \epsilon_1 + \ldots + \epsilon_s$, $\rho = \rho_1 + \ldots + \rho_u$, $\sigma = \sigma_1 + \ldots + \sigma_v$, *and* $\eta = \eta_1 + \ldots + \eta_w$ *and the blocks* $\mathcal{L}_{\epsilon_j}$, $\mathcal{J}_{\rho_j}$, $\mathcal{N}_{\sigma_j}$, *and* $\mathcal{M}_{\eta_j}$ *have the following form:*

1. *Every entry $\mathcal{L}_{\epsilon_j}$ has the size $\epsilon_j \times (\epsilon_j + 1)$, $\epsilon_j \in \mathbb{N}_0$ and the form*

$$\mathcal{L}_{\epsilon_j}(\lambda) := \lambda \begin{bmatrix} 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \end{bmatrix}. \tag{2.5}$$

2. *Every entry $\mathcal{J}_{\rho_j}$ has the size $\rho_j \times \rho_j$, $\rho_j \in \mathbb{N}$ and the form*

$$\mathcal{J}_{\rho_j}(\lambda) := \lambda \begin{bmatrix} 1 & & & \\ & \ddots & & \\ & & \ddots & \\ & & & 1 \end{bmatrix} - \begin{bmatrix} \lambda_j & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_j \end{bmatrix}, \tag{2.6}$$

where $\lambda_j \in \mathbb{C}$ is a zero of $\lambda F + G$.

3. *Every entry $\mathcal{N}_{\sigma_j}$ has the size $\sigma_j \times \sigma_j$, $\sigma_j \in \mathbb{N}$ and the form*

$$\mathcal{N}_{\sigma_j}(\lambda) := \lambda \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{bmatrix} + \begin{bmatrix} 1 & & & \\ & \ddots & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}. \tag{2.7}$$

4. *Every entry $\mathcal{M}_{\eta_j}$ has the size $(\eta_j + 1) \times \eta_j$, $\eta_j \in \mathbb{N}_0$ and the form*

$$\mathcal{M}_{\eta_j}(\lambda) := \lambda \begin{bmatrix} 1 & & \\ 0 & \ddots & \\ & \ddots & 1 \\ & & 0 \end{bmatrix} + \begin{bmatrix} 0 & & \\ 1 & \ddots & \\ & \ddots & 0 \\ & & 1 \end{bmatrix}. \tag{2.8}$$

Proof. A proof can be found in [19, p. 37]. □

2.2 Linear systems

With the preliminaries from Section 2.1 we are ready to introduce some well known results concerning systems theory from a behavioral viewpoint. A more throughout discussion can be found in [27, 44, 43]. Note that the notations used in this thesis differs slightly from the standard notation, for example, we will introduce the so-called reachable sets (Definition 2.16) which were not introduced in [27, 44, 43]. Also, we then introduce controllability, stabilizability, and autonomous systems via these reachable sets.

Definition 2.15. Let $P \in \mathbb{C}[\lambda]^{p,q}$ be a polynomial matrix. Then we call

$$\mathfrak{B}(P) := \left\{ z \in \mathcal{C}_\infty^q \middle| P\left(\tfrac{d}{dt}\right) z = 0 \right\} = \text{kernel}_{\mathcal{C}_\infty}\left(P\left(\tfrac{d}{dt}\right)\right)$$

the *behavior of* P,

$$\mathfrak{B}_+(P) := \left\{ z \in \mathcal{C}_+^q \middle| P\left(\tfrac{d}{dt}\right) z = 0 \right\} = \text{kernel}_{\mathcal{C}_+}\left(P\left(\tfrac{d}{dt}\right)\right)$$

the *positive decaying behavior of* P,

$$\mathfrak{B}_-(P) := \left\{ z \in \mathcal{C}_-^q \middle| P\left(\tfrac{d}{dt}\right) z = 0 \right\} = \text{kernel}_{\mathcal{C}_-}\left(P\left(\tfrac{d}{dt}\right)\right)$$

the *negative decaying behavior of* P, and

$$\mathfrak{B}_c(P) := \left\{ z \in \mathcal{C}_c^q \middle| P\left(\tfrac{d}{dt}\right) z = 0 \right\} = \text{kernel}_{\mathcal{C}_c}\left(P\left(\tfrac{d}{dt}\right)\right)$$

the *compact behavior of* P. The elements of $\mathfrak{B}(P)$, $\mathfrak{B}_+(P)$, $\mathfrak{B}_-(P)$, and $\mathfrak{B}_c(P)$ are called *trajectories of* P.

We have $\mathfrak{B}_c(P) \subset \mathfrak{B}_\pm(P) \subset \mathfrak{B}(P)$. For obvious reasons, P is also called a *kernel representation*. In the following we will also call P a *system* or a *plant*.

Definition 2.16. Let $P \in \mathbb{C}[\lambda]_K^{p,q}$. Then we call

$$R(P) := \left\{ \hat{z} \in \mathbb{C}^{qK} \middle| \exists z \in \mathfrak{B}(P) \text{ such that } \hat{z} = \Delta_K z(0) \right\}$$

the *reachable set of* P,

$$R_+(P) := \left\{ \hat{z} \in \mathbb{C}^{qK} \middle| \exists z \in \mathfrak{B}_+(P) \text{ such that } \hat{z} = \Delta_K z(0) \right\}$$

the *positive decaying reachable set of* P,

$$R_-(P) := \left\{ \hat{z} \in \mathbb{C}^{qK} \middle| \exists z \in \mathfrak{B}_-(P) \text{ such that } \hat{z} = \Delta_K z(0) \right\}$$

the *negative decaying reachable set of* P, and

$$R_c(P) := \left\{ \hat{z} \in \mathbb{C}^{qK} \middle| \exists z \in \mathfrak{B}_c(P) \text{ such that } \hat{z} = \Delta_K z(0) \right\}$$

the *compact reachable set of* P.

We have $R_c(P) \subset R_\pm(P) \subset R(P)$. To understand the difference between $R_c(P)$ and $R(P)$ let $\hat{z} \in R_c(P)$ and let $z \in \mathfrak{B}_c(P)$ be some trajectory with compact support such that $\hat{z} = \Delta_K z(0)$. This means that there exists an $T > 0$ such that z vanishes outside of $[-T, T]$. Thus, there exists a trajectory which takes the state $\hat{z}$ at time 0 to the state 0 at time T. In other words, $\hat{z}$ can be controlled to 0. This is the reason why one can think of $\mathfrak{B}_c(P)$ as the *controllable part* of the system $\mathfrak{B}(P)$. In a similar fashion, we see that $\mathfrak{B}_+(P)$ represents the *stabilizable part* and $\mathfrak{B}_-(P)$ represents the *anti-stabilizable part* of $\mathfrak{B}(P)$.

Definition 2.17. We call $P \in \mathbb{C}[\lambda]^{p,q}$ *controllable* if $R_c(P) = R(P)$. We call P *stabilizable* if $R_+(P) = R(P)$ and we call P *anti-stabilizable* if $R_-(P) = R(P)$. We say that P is *autonomous* if $R_c(P) = \{0\}$.

Note that our definitions of controllability, stabilizability, and autonomous system differ from the standard definitions given in [27, 44, 43]. However, in Lemma 2.21 alternative characterizations of controllability, stabilizability, and autonomous system will be given, which show that our definitions are equivalent to the ones used in [27, 44, 43].

Lemma 2.18. *Let $P \in \mathbb{C}[\lambda]^{p,q}$ with $r := \operatorname{rank}_{\mathbb{C}(\lambda)}(P)$ and let $U \in \mathbb{C}[\lambda]^{q,q-r}$ be a polynomial kernel matrix of P without zeros. Then*

$$\mathfrak{B}_c(P) = \operatorname{image}_{\mathcal{C}_c}\left(U\left(\tfrac{d}{dt}\right)\right).$$

Proof. [27, Theorem 6.6.1] The inclusion "$\supset$" follows directly by equating coefficients in $PU = 0$. For the inclusion "$\subset$" let $z \in \mathfrak{B}_c(P)$ be arbitrary and let a Smith form of P be given by (2.1). Partition the inverse of T as $T^{-1} = \begin{bmatrix} T_1 & T_2 \end{bmatrix}$ with T_1 having r columns and T_2 having $q - r$ columns. Using Lemma 2.9 and Remark 2.10, we obtain the existence of a unimodular U_2 such that $U = T_2U_2$. Since $z \in \mathfrak{B}_c(P) = \mathfrak{B}_c(S^{-1}P)$ and by defining the functions $\tilde{z}_1 \in \mathcal{C}_c^r$, $\tilde{z}_2 \in \mathcal{C}_c^{q-r}$, and $\tilde{z} \in \mathcal{C}_c^q$, through

$$\begin{bmatrix} \tilde{z}_1 \\ \tilde{z}_2 \end{bmatrix} := \tilde{z} := T\left(\tfrac{d}{dt}\right) z,$$

we also see that

$$0 = S^{-1}\left(\tfrac{d}{dt}\right) P\left(\tfrac{d}{dt}\right) z \quad = \quad \begin{bmatrix} \operatorname{diag}\left(d_1\left(\tfrac{d}{dt}\right), \ldots, d_r\left(\tfrac{d}{dt}\right)\right) \tilde{z}_1 \\ 0 \end{bmatrix}.$$

Further defining the functions $z_i \in \mathcal{C}_c$ for $i = 1, \ldots, r$ by partitioning

$$\begin{bmatrix} z_1 & \cdots & z_r \end{bmatrix}^T := \tilde{z}_1,$$

we obtain that $0 = d_i\left(\frac{d}{dt}\right) z_i$ for all $t \in \mathbb{R}$ and all $i = 1, \ldots, r$. If d_i is a non-zero constant we immediately see that this implies that $z_i = 0$. If, however, d_i is another non-zero polynomial this means that the scalar-valued function z_i satisfies a scalar homogeneous linear ordinary differential equation. The fact that z_i has compact support gives us the initial condition $z(R) = 0$ (where $R \in \mathbb{R}$ is small enough or large enough). From the theory of linear ordinary differential equations (reducing the system to first order and writing down the explicit solution formula) we again see that $z_i = 0$. Thus, it follows that $\tilde{z}_1 = 0$ and we deduce that

$$z \quad = \quad T^{-1}\left(\tfrac{d}{dt}\right) \tilde{z} = \begin{bmatrix} T_1\left(\tfrac{d}{dt}\right) & T_2\left(\tfrac{d}{dt}\right) \end{bmatrix} \begin{bmatrix} 0 \\ \tilde{z}_2 \end{bmatrix} = T_2\left(\tfrac{d}{dt}\right) \tilde{z}_2.$$

Since $U = T_2U_2$, with U_2 being a unimodular polynomial, also U_2^{-1} is a unimodular polynomial and we have $UU_2^{-1} = T_2$. Setting $\alpha := U_2^{-1}\left(\frac{d}{dt}\right) \tilde{z}_2 \in \mathcal{C}_c^{q-r}$ and we finally get

$$z = T_2\left(\tfrac{d}{dt}\right) \tilde{z}_2 = U\left(\tfrac{d}{dt}\right) U_2^{-1}\left(\tfrac{d}{dt}\right) \tilde{z}_2 = U\left(\tfrac{d}{dt}\right) \alpha,$$

which finishes the proof. $\square$

Lemma 2.18 is the reason why we say that a polynomial kernel matrix without zeros is an *image representation* of the controllable part of a system.
The following lemma describes the reachable and compact reachable set of a first order system with the help of the Kronecker canonical form.

Lemma 2.19. *Let $P(\lambda) = \lambda F + G \in \mathbb{C}[\lambda]_1^{p,q}$ be a first order matrix polynomial and let the Kronecker canonical form of $\lambda F + G$ be given by (2.4). Then the reachable set of P is given by*

$$R(P) = \left\{ T^{-1} \begin{bmatrix} \hat{z} \\ 0_{\sigma+\eta} \end{bmatrix} \middle| \hat{z} \in \mathbb{C}^{\epsilon+s+\rho} \right\},$$

and the compact reachable set of P is given by

$$R_c(P) = \left\{ T^{-1} \begin{bmatrix} \hat{z} \\ 0_{\rho+\sigma+\eta} \end{bmatrix} \middle| \hat{z} \in \mathbb{C}^{\epsilon+s} \right\},$$

where for $n \in \mathbb{N}$ the vector $0_n \in \mathbb{C}^n$ is the vector consisting of only zeros.

Proof. We first show that the behavior of P is given by

$$\mathfrak{B}(P) := \left\{ T^{-1} \begin{bmatrix} \Delta_{\epsilon_1+1} z_1 \\ \vdots \\ \Delta_{\epsilon_s+1} z_s \\ e^{\mathcal{J}(0)t}\hat{x} \\ 0_{\sigma+\eta} \end{bmatrix} \middle| z_1, \dots, z_s \in \mathcal{C}_\infty, \hat{x} \in \mathbb{C}^\rho \right\}. \tag{2.9}$$

To show this representations it is sufficient to examine the behavior of each of the blocks in the Kronecker canonical from. We start with the blocks of type (2.5). Thus, for $j \in \{1, \dots, s\}$ let $x \in \mathcal{C}_\infty^{\epsilon_j+1}$ be a solution of the system

$$\begin{bmatrix} 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \vdots \\ \dot{x}_{\epsilon_j+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_{\epsilon_j+1} \end{bmatrix}.$$

This is equivalent to the system of scalar equations $\dot{x}_i = x_{i+1}$ for $i = 1, \dots, \epsilon_j$. This shows that $x_i = x_1^{(i-1)}$ for $i = 1, \dots, \epsilon_j + 1$ and thus one can write x in the form $\Delta_{\epsilon_j+1} z_j$ by setting $z_j := x_1 \in \mathcal{C}_\infty$. For blocks of the type (2.6) we obtain that $\mathfrak{B}(\mathcal{J}_{\rho_j}) = \left\{ e^{\mathcal{J}_{\rho_j}(0)t}\hat{x} \middle| \hat{x} \in \mathbb{C}^{\rho_j} \right\}$ from the standard theory of ordinary differential equations. For the blocks of type (2.7) and (2.8) we see that $\mathfrak{B}(\mathcal{N}_{\sigma_j}) = \mathfrak{B}(\mathcal{M}_{\sigma_j}) = \{0\}$. Using Hermite interpolation and the fact that $e^0 = I$ we deduce the statement about the reachable set $R(P)$ from (2.9). The proof for the compact reachable set works analogously. □

Remark 2.20. By splitting up the blocks of type (2.6) into blocks with $\mathrm{Re}\,(\lambda_j) < 0$, blocks with $\mathrm{Re}\,(\lambda_j) > 0$, and blocks with $\mathrm{Re}\,(\lambda_j) = 0$ one can also give an explicit representation (as in Lemma 2.19) of $R_+(P)$ and $R_-(P)$ which we will not do here for the sake of simplicity.

Lemma 2.21. *Let $P \in \mathbb{C}[\lambda]^{p,q}$. Then we have the following:*

1. *P is controllable if and only if $\mathfrak{Z}(P) = \emptyset$.*
2. *P is stabilizable if and only if $\mathfrak{Z}(P) \subset \mathbb{C}^-$.*
3. *P is anti-stabilizable if and only if $\mathfrak{Z}(P) \subset \mathbb{C}^+$.*
4. *P is autonomous if $q = \operatorname{rank}_{\mathbb{C}(\lambda)}(P)$.*

Proof. [27, Theorem 5.2.10 and Theorem 5.2.30] Let $\lambda F + G$ be the canonical linearization of P from (2.3). Then we see that $R(P) = R(\lambda F + G)$, $R_+(P) = R_+(\lambda F + G)$, $R_-(P) = R_-(\lambda F + G)$, and $R_c(P) = R_c(\lambda F + G)$. Using Lemma 2.13 we derive point 4. and conclude that it is sufficient to show points 1. to 3. for first order polynomials. Thus, the assertions follow with Lemma 2.19 and the subsequent Remark 2.20. $\square$

2.3 Para-Hermitian matrices and the shift operator

In this section we define the para-Hermitian of a rational matrix and the shift operator for polynomial matrices. We then show (in Lemma 2.26) how both are connected with respect to systems theory.

Definition 2.22. [46, Def. 2] For $R \in \mathbb{C}(\lambda)^{p,q}$ we define $R^{\sim} \in \mathbb{C}(\lambda)^{q,p}$ through

$$R^{\sim}(\lambda) := R^*(-\overline{\lambda}),$$

the *para-Hermitian of R*. Further, R is called *para-Hermitian* if $R = R^{\sim}$.

Some properties of the para-Hermitian are summed up in the following lemma.

Lemma 2.23. *We have*

1. *$(A^{-1})^{\sim} = (A^{\sim})^{-1}$ for a square rational matrix A which is invertible over $\mathbb{C}(\lambda)$.*
2. *$(BC)^{\sim} = C^{\sim}B^{\sim}$ for arbitrary rational matrices B and C of proper dimension.*
3. *$(B^{\sim})^{\sim} = B$ for every rational matrix B.*
4. *If D is a para-Hermitian rational matrix so is $U^{\sim}DU$ for every rational matrix U of appropriate dimension.*

Proof. [46, p.173] The proof can be conducted in the same way as the proof for the Hermitian of matrices with complex entries. $\square$

Lemma 2.23 especially justifies the notation $A^{-\sim} := (A^{\sim})^{-1}$ to denote the para-Hermitian of the inverse of a square invertible rational matrix. Also, we note that the para-Hermitian of a unimodular matrix is again unimodular.

Lemma 2.24. *For every $R \in \mathbb{C}(\lambda)^{p,q}$ we have $\mathfrak{Z}\left(R^{\sim}\right) = -\overline{\mathfrak{Z}\left(R\right)}$ and $\mathfrak{P}\left(R^{\sim}\right) = -\overline{\mathfrak{P}\left(R\right)}$.*

Proof. [46] Let the McMillan form of R be given by (2.2). Then, we have that

$$R^{\sim} = T^{\sim} \begin{bmatrix} \operatorname{diag}\left(\frac{\alpha_1}{\beta_1}, \ldots, \frac{\alpha_r}{\beta_r}\right)^{\sim} & 0 \\ 0 & 0 \end{bmatrix} S^{\sim} = T^{\sim} \begin{bmatrix} \operatorname{diag}\left(\frac{\alpha_1^{\sim}}{\beta_1^{\sim}}, \ldots, \frac{\alpha_r^{\sim}}{\beta_r^{\sim}}\right) & 0 \\ 0 & 0 \end{bmatrix} S^{\sim}.$$

Since every α_i and β_i can be factored into a product of linear polynomials, it is sufficient to show that for $p(\lambda) := \lambda - a$ with $a \in \mathbb{C}$ we have $\mathfrak{Z}\left(p^{\sim}\right) = -\overline{\mathfrak{Z}\left(p\right)}$. However, since $(\lambda - a)^{\sim} = (-\overline{\lambda} - a)^{*} = -\lambda - \overline{a}$, this is clearly the case. □

Lemma 2.24 especially shows that for para-Hermitian matrices the zeros and poles are symmetric with respect to the imaginary axis.

Definition 2.25. Let $P \in \mathbb{C}[\lambda]_K^{p,q}$ be a matrix polynomial of the form $P(\lambda) = \sum_{i=0}^K \lambda^i P_i$ and let $k \in \mathbb{N}$. Then we define the *k-times shifted polynomial* $P^{\langle k \rangle} \in \mathbb{C}[\lambda]_{K-k}^{p,q}$ through

$$P^{\langle k \rangle}(\lambda) := \sum_{i=k}^{K} \lambda^{i-k} P_i = \sum_{j=0}^{K-k} \lambda^j P_{j+k}.$$

Lemma 2.26. *Let $P \in \mathbb{C}[\lambda]^{p,q}$, let $y \in \mathcal{C}_\infty^p$, and $z \in \mathcal{C}_\infty^q$. Then for all $t_0, t_1 \in \mathbb{R}$ with $t_0 \leq t_1$ we have that*

$$\begin{aligned} & \int_{t_0}^{t_1} z^*(t) P^{\sim}\left(\tfrac{d}{dt}\right) y(t) dt \\ = & \int_{t_0}^{t_1} \left(P\left(\tfrac{d}{dt}\right) z(t)\right)^* y(t) dt + \sum_{k=1}^{\infty} (-1)^k \left(P^{\langle k \rangle}\left(\tfrac{d}{dt}\right) z(t)\right)^* y^{(k-1)}(t) \Big|_{t_0}^{t_1}, \end{aligned}$$

where the infinite series is indeed only a finite sum, since for k big enough $P^{\langle k \rangle}$ vanishes.

Proof. [29, Proof of Proposition 6.1] Assume that P has the form $P(\lambda) = \sum_{i=0}^K \lambda^i P_i$. Using repeated partial integration we see that for $i = 0, \ldots, K$ we have

$$\begin{aligned} & \int_{t_0}^{t_1} z^*(t) P_i^* y^{(i)}(t) dt = z^*(t) P_i^* y^{(i-1)}(t) \Big|_{t_0}^{t_1} - \int_{t_0}^{t_1} \dot{z}^*(t) P_i^* y^{(i-1)}(t) dt \\ = & (-1)^0 z^*(t) P_i^* y^{(i-1)}(t) \Big|_{t_0}^{t_1} + (-1)^1 \dot{z}^*(t) P_i^* y^{(i-2)}(t) \Big|_{t_0}^{t_1} + (-1)^2 \int_{t_0}^{t_1} \ddot{z}^*(t) P_i^* y^{(i-2)}(t) dt \\ = & \ldots = \sum_{j=0}^{i-1} (-1)^j \left(z^{(j)}(t)\right)^* P_i^* y^{(i-1-j)}(t) \Big|_{t_0}^{t_1} + (-1)^i \int_{t_0}^{t_1} \left(z^{(i)}(t)\right)^* P_i^* y(t) dt. \end{aligned}$$

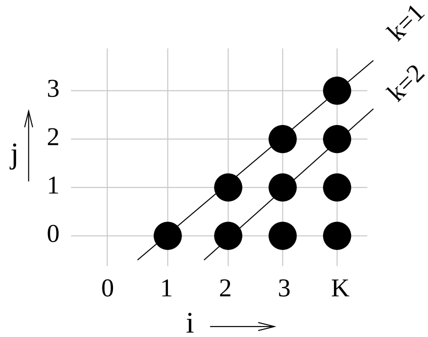
Using the formula

$$P^{\sim}(\lambda) = \sum_{i=0}^{K} (-1)^i P_i^* \lambda^i$$

this implies

$$\begin{aligned}
\int_{t_0}^{t_1} z^*(t) P^{\sim}\left(\tfrac{d}{dt}\right) y(t) &= \sum_{i=0}^{K} (-1)^i \int_{t_0}^{t_1} z^*(t) P_i^* y^{(i)}(t) dt \\
= \sum_{i=0}^{K} (-1)^i & \left[\sum_{j=0}^{i-1} (-1)^j \left(z^{(j)}(t)\right)^* P_i^* y^{(i-1-j)}(t) \Big|_{t_0}^{t_1} + (-1)^i \int_{t_0}^{t_1} \left(z^{(i)}(t)\right)^* P_i^* y(t) dt \right] \\
= \sum_{i=0}^{K} \sum_{j=0}^{i-1} & (-1)^{i+j} \left(z^{(j)}(t)\right)^* P_i^* y^{(i-1-j)}(t) \Big|_{t_0}^{t_1} + \int_{t_0}^{t_1} \left(P\left(\tfrac{d}{dt}\right) z(t)\right)^* y(t) dt.
\end{aligned}$$

Finally, using reindexing as in



we see that for all $f : \mathbb{N}^2 \to \mathbb{C}$ we have $\sum_{i=0}^{K} \sum_{j=0}^{i-1} f(i,j) = \sum_{k=1}^{K} \sum_{l=0}^{K-k} f(k+l, l)$ and thus

$$\begin{aligned}
& \sum_{i=0}^{K} \sum_{j=0}^{i-1} (-1)^{i+j} \left(z^{(j)}(t)\right)^* P_i^* y^{(i-1-j)}(t) \Big|_{t_0}^{t_1} \\
= & \sum_{k=1}^{K} \sum_{l=0}^{K-k} (-1)^{k+l+l} \left(z^{(l)}(t)\right)^* P_{k+l}^* y^{(k+l-1-l)}(t) \Big|_{t_0}^{t_1} \\
= & \sum_{k=1}^{K} \sum_{l=0}^{K-k} (-1)^{k} \left(z^{(l)}(t)\right)^* P_{k+l}^* y^{(k-1)}(t) \Big|_{t_0}^{t_1} \\
= & \sum_{k=1}^{K} (-1)^{k} \left(\sum_{l=0}^{K-k} P_{k+l} z^{(l)}(t) \right)^* y^{(k-1)}(t) \Big|_{t_0}^{t_1} \\
= & \sum_{k=1}^{K} (-1)^{k} \left(P^{\langle k \rangle}\left(\tfrac{d}{dt}\right) z(t) \right)^* y^{(k-1)}(t) \Big|_{t_0}^{t_1},
\end{aligned}$$

which proves the assertion. □

Lemma 2.26 implies that $P\left(\frac{d}{dt}\right)$ and $P^{\sim}\left(\frac{d}{dt}\right)$ are adjoint operators when considered with respect to the scalar product $\langle f, g \rangle_{L_2} := \int_{\mathbb{R}} f^*(t) g(t) dt$ over the smooth functions with compact support.

Chapter 3

Dissipativity

In this chapter we present the main results, which all revolve around the notion of dissipativity and cyclo-dissipativity. First, we introduce dissipativity and cyclo-dissipativity in a formal way. Then, in the following sections characterizations of dissipativity and cyclo-dissipativity are presented.

Definition 3.1. Let $P(\lambda) \in \mathbb{C}[\lambda]_K^{p,q}$ and $H = H^* \in \mathbb{C}^{qK,qK}$. Then we call (P, H) *dissipative* if there exists a so-called *storage function* $\Theta : \mathbb{C}^{qK} \to \mathbb{R}$, i.e., a continuous function Θ with $\Theta(0) = 0$ such that the *dissipation inequality*

$$\Theta(\Delta_K z(t_1)) - \Theta(\Delta_K z(t_0)) \leq \int_{t_0}^{t_1} (\Delta_K z(t))^* H (\Delta_K z(t)) \, dt \tag{3.1}$$

is fulfilled for all $t_0 \leq t_1$ and all $z \in \mathfrak{B}(P)$.

The term on the right hand side of the dissipation inequality (3.1) can be viewed as a measure of the amount of energy which is supplied to the system given by P along the trajectory z in the time frame t_0 to t_1. The function $(\Delta_K z(t))^* H (\Delta_K z(t))$ thus measures the amount of energy supplied to the system given by P along the trajectory z at the time point t. The left hand side of the dissipation inequality (3.1) can be viewed as a measure of the gain in energy which is stored in the system internally, along the trajectory z, and the function $\Theta(\Delta_K z(t))$ measures the internal energy at the time point t. In other words, the dissipation inequality (3.1) states that the system P cannot generate energy (with energy supply measured by means of H), i.e., it only dissipates energy. The existence of a storage function guarantees that one can measure the internally stored energy in such a way that never more energy is stored than the amount of energy supplied to the system. The matrix H encapsulates the notion of energy which we want to impose onto the system and we will also call H the *supply*.

Definition 3.2. Let $P \in \mathbb{C}[\lambda]_K^{p,q}$ and $H = H^* \in \mathbb{C}^{qK,qK}$. Then we call (P, H) *cyclo-dissipative* if

$$0 \leq \int_{-\infty}^{\infty} (\Delta_K z(t))^* H (\Delta_K z(t)) \, dt, \tag{3.2}$$

is fulfilled for all $z \in \mathfrak{B}_c(P)$.

Cyclo-dissipativity thus only demands that every trajectory of the system which starts at zero and comes back to zero at some time later does not generate energy. We will later see that dissipativity implies cyclo-dissipativity, see Theorem 3.20. However, cyclo-dissipativity is only a property of the controllable part $\mathfrak{B}_c(P)$ whereas dissipativity is a property involving the complete system $\mathfrak{B}(P)$. We will later see that for controllable systems dissipativity and cyclo-dissipativity are equivalent, see Corollary 3.21. The following example shows that cyclo-dissipativity in general does not imply dissipativity.

Example 3.3. Consider the system

$$P(\lambda) := \begin{bmatrix} 0 & \lambda + 1 \end{bmatrix},$$

together with the supply

$$H = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Since for this combination of system and supply we have

$$\int_{-\infty}^{\infty} z^*(t) H z(t) dt = 0,$$

for all $z \in \mathfrak{B}_c(P)$, we conclude that (P, H) is cyclo-dissipative. To show that (P, H) is not dissipative assume to the contrary that there exists a storage function $\Theta : \mathbb{C}^2 \to \mathbb{R}$. Let $\tilde{b} : \mathbb{R} \to \mathbb{R}$ be an infinitely often differentiable function which fulfills $\tilde{b}(t) = 0$ for all $t \in (-\infty, 0] \cup [1, \infty)$ and $b(t) > 0$ for all $t \in (0, 1)$. For $\alpha \in \mathbb{R}$ define the family of functions $z_\alpha \in \mathcal{C}_+^2$ through

$$z_\alpha(t) := \begin{bmatrix} \alpha \tilde{b}(t) \\ e^{-t} \end{bmatrix}.$$

Then for all $\alpha \in \mathbb{R}$ we would have

$$\begin{aligned} c \;&:=\; \Theta\left(\begin{bmatrix} 0 \\ e^{-1} \end{bmatrix}\right) - \Theta\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \Theta(z_\alpha(1)) - \Theta(z_\alpha(0)) \\ &\leq\; \int_0^1 z_\alpha^*(t) H z_\alpha(t) dt = 2\alpha \underbrace{\int_0^1 \tilde{b}(t) e^{-t} dt}_{=:d} = 2\alpha d \end{aligned}$$

where $c \in \mathbb{R}$ is a constant which does not depend on α and $d \neq 0$. This is a contradiction, an thus (P, H) cannot be dissipative.

3.1 Popov functions

In this section we present a frequency domain characterization of cyclo-dissipativity via a certain para-Hermitian matrix polynomial (3.3) which can directly be formed from the system P and the supply H.

Definition 3.4. Let $P \in \mathbb{C}[\lambda]_K^{p,q}$ and $H = H^* \in \mathbb{C}^{qK,qK}$. Let $r := \operatorname{rank}_{\mathbb{C}(\lambda)}(P)$ and let $U \in \mathbb{C}(\lambda)^{q,q-r}$ be a kernel matrix of P. Then we call the para-Hermitian matrix $\Pi = \Pi^\sim \in \mathbb{C}(\lambda)^{q-r,q-r}$ defined through

$$\Pi := U^\sim \Delta_K^{q\ \sim} H \Delta_K^q U$$

a *Popov function of* (P, H) or **the** *Popov function of* (P, H) *associated with* U.

Theorem 2.7 implies that for every matrix polynomial P and Hermitian matrix H there exists a Popov function which is polynomial. Lemma 2.9 implies that for any two Popov functions Π_1, Π_2 there exists an invertible matrix U_2 such that $\Pi_1 = U_2^\sim \Pi_2 U_2$. In the following well-known Theorem 3.5 cyclo-dissipativity is characterized via a frequency-domain condition.

Theorem 3.5. *Let $P \in \mathbb{C}[\lambda]_K^{p,q}$ and $H = H^* \in \mathbb{C}^{qK,qK}$. Let Π be a Popov function of (P, H). Then (P, H) is cyclo-dissipative if and only if Π is positive semi-definite along the imaginary axis, i.e., we have*

$$\Pi(i\omega) \geq 0$$

for all $\omega \in \mathbb{R}$ such that $i\omega \in \mathfrak{D}(\Pi)$.

Proof. [45, Proposition 5.2] A self contained proof can be found in Appendix A.2. □

Theorem 3.5 can be summarized in the following words. A system is cyclo-dissipative if and only if one of its Popov functions is positive semi-definite along the imaginary axis. In this case every Popov function is positive semi-definite along the imaginary axis.

Lemma 3.6. *Let $P \in \mathbb{C}[\lambda]_K^{p,q}$ and $H = H^* \in \mathbb{C}^{qK,qK}$. Let $r := \operatorname{rank}_{\mathbb{C}(\lambda)}(P)$ and let $U \in \mathbb{C}(\lambda)^{q,q-r}$ and $V \in \mathbb{C}(\lambda)^{q,r}$ be kernel and co-kernel matrices of P. Let $\Pi := U^\sim \Delta_K^{q\ \sim} H \Delta_K^q U$ be the Popov function associated with U. Then there exists an invertible $W \in \mathbb{C}(\lambda)^{p+q,p+q}$ such that*

$$W^\sim \begin{bmatrix} 0 & P \\ P^\sim & \Delta_K^{q\ \sim} H \Delta_K^q \end{bmatrix} W = \begin{bmatrix} 0 & 0 & PV \\ 0 & \Pi & 0 \\ V^\sim P^\sim & 0 & 0 \end{bmatrix},$$

with $\mathfrak{Z}(W) \subset \mathfrak{Z}(U) \cup \mathfrak{Z}(V)$ and $\mathfrak{P}(W) \subset \mathfrak{P}(U) \cup \mathfrak{P}(V) \cup \mathfrak{P}(V^\sim) \cup \mathfrak{Z}((PV)^\sim)$.

Proof. Using the abbreviation $\tilde{H} := \Delta_K^{q\ \sim} H \Delta_K^q \in \mathbb{C}[\lambda]^{q,q}$ and defining the block transformation matrix

$$S := \begin{bmatrix} I_p & 0 \\ 0 & \begin{bmatrix} U & V \end{bmatrix} \end{bmatrix} \in \mathbb{C}(\lambda)^{p+q,p+q},$$

we have

$$S^\sim N S = \begin{bmatrix} 0 & PU & PV \\ U^\sim P^\sim & U^\sim \tilde{H} U & U^\sim \tilde{H} V \\ V^\sim P^\sim & V^\sim \tilde{H} U & V^\sim \tilde{H} V \end{bmatrix} = \begin{bmatrix} 0 & 0 & PV \\ 0 & \Pi & U^\sim \tilde{H} V \\ V^\sim P^\sim & V^\sim \tilde{H} U & V^\sim \tilde{H} V \end{bmatrix}.$$

To eliminate the blocks below PV and to the right of $(PV)^\sim$, let $X \in \mathbb{C}(\lambda)^{r,p}$ be a left inverse of PV as in Lemma 2.11. Then, set

$$Y := -X^\sim V^\sim \tilde{H} U \in \mathbb{C}(\lambda)^{p,q-r} \quad \text{and} \quad Z := -\tfrac{1}{2} X^\sim V^\sim \tilde{H} V \in \mathbb{C}(\lambda)^{p,p}$$

and with this define the block transformation matrix

$$T := \begin{bmatrix} I_p & Y & Z \\ 0 & I_{q-r} & 0 \\ 0 & 0 & I_r \end{bmatrix} \in \mathbb{C}(\lambda)^{p+q,p+q}.$$

Since we have that

$$\begin{aligned} V^{\sim}P^{\sim}Y + V^{\sim}\tilde{H}U &= -(XPV)^{\sim}V^{\sim}\tilde{H}U + V^{\sim}\tilde{H}U = 0, \\ Z^{\sim}PV + V^{\sim}P^{\sim}Z + V^{\sim}\tilde{H}V &= -\tfrac{1}{2}V^{\sim}\tilde{H}V - \tfrac{1}{2}V^{\sim}\tilde{H}V + V^{\sim}\tilde{H}V = 0, \end{aligned}$$

we find that

$$\begin{aligned} T^{\sim}S^{\sim}NST &= \begin{bmatrix} 0 & 0 & PV \\ 0 & \Pi & Y^{\sim}PV + U^{\sim}\tilde{H}V \\ V^{\sim}P^{\sim} & V^{\sim}P^{\sim}Y + V^{\sim}\tilde{H}U & Z^{\sim}PV + V^{\sim}P^{\sim}Z + V^{\sim}\tilde{H}V \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & PV \\ 0 & \Pi & 0 \\ V^{\sim}P^{\sim} & 0 & 0 \end{bmatrix}. \end{aligned}$$

We define $W := ST$. Using $\mathfrak{P}(X^{\sim}) = \mathfrak{Z}((PV)^{\sim})$ we conclude that $\mathfrak{Z}(W) \subset \mathfrak{Z}(S) \cup \mathfrak{Z}(T) \subset \mathfrak{Z}(U) \cup \mathfrak{Z}(V)$ and $\mathfrak{P}(W) \subset \mathfrak{P}(S) \cup \mathfrak{P}(T) \subset \mathfrak{P}(U) \cup \mathfrak{P}(V) \cup \mathfrak{P}(V^{\sim}) \cup \mathfrak{P}(X^{\sim}) = \mathfrak{P}(U) \cup \mathfrak{P}(V) \cup \mathfrak{P}(V^{\sim}) \cup \mathfrak{Z}((PV)^{\sim})$ and thus we obtain the assertion. □

The following is a generalization of [5, Theorem 1].

Theorem 3.7. *Let $P \in \mathbb{C}[\lambda]_K^{p,q}$ and $H = H^* \in \mathbb{C}^{qK,qK}$. Let $r := \operatorname{rank}_{\mathbb{C}(\lambda)}(P)$ and let $U \in \mathbb{C}(\lambda)^{q,q-r}$ and $V \in \mathbb{C}(\lambda)^{q,r}$ be kernel and co-kernel matrices of P. Let $\Pi := U^{\sim}\Delta_K^{q\,\sim} H \Delta_K^q U$ be the Popov function associated with U. Define the* exceptional set *as*

$$\begin{aligned} \mathcal{E} \;:=\; & \mathfrak{Z}(U) \cup \mathfrak{Z}(V) \cup \mathfrak{Z}(U^{\sim}) \cup \mathfrak{Z}(V^{\sim}) \cup \\ & \mathfrak{P}(U) \cup \mathfrak{P}(V) \cup \mathfrak{P}(U^{\sim}) \cup \mathfrak{P}(V^{\sim}) \cup \mathfrak{Z}((PV)^{\sim}) \cup \mathfrak{Z}(PV). \end{aligned}$$

Then we have

$$\mathfrak{Z}\left(\begin{bmatrix} 0 & P \\ P^{\sim} & \Delta_K^{q\,\sim} H \Delta_K^q \end{bmatrix}\right) \setminus \mathcal{E} = \mathfrak{Z}(\Pi) \setminus \mathcal{E}.$$

Proof. Using Lemma 3.6 and Lemma 2.6 we verify the assertion. □

Remark 3.8. Looking back at Theorem 2.7 we see that we can choose the kernel and co-kernel matrix such that the exceptional set in the previous Theorem 3.7 becomes $\mathcal{E} = \mathfrak{Z}(P) \cup \mathfrak{Z}(P^{\sim})$. With Lemma 2.21 this shows that for controllable systems we can assume w.l.o.g. that the $\mathcal{E} = \emptyset$.

Definition 3.9. We define the *sign-sum function* η which maps from the Hermitian matrices to $\mathbb{Z}$ in the following way. Assume that the Hermitian matrix $A = A^* \in \mathbb{C}^{m,m}$ has π positive eigenvalues, ν negative eigenvalues, and μ zero eigenvalues, i.e., let the inertia index of A be given by (π, ν, μ). Then we set

$$\eta(A) := \pi + \mu - \nu.$$

We conclude that a Hermitian matrix $A = A^* \in \mathbb{C}^{m,m}$ is positive semi-definite if and only if $\eta(A) = m$.

Lemma 3.10. *Let $A = A^* \in \mathbb{C}^{m,m}$ and $B \in \mathbb{C}^{p,r}$. Then we have*

$$\eta\left(\begin{bmatrix} 0 & 0 & B \\ 0 & A & 0 \\ B^* & 0 & 0 \end{bmatrix}\right) = \eta(A) + p + r - 2 \cdot \operatorname{rank}(B)$$

Proof. [10, Section 4] Since under congruence transformations the inertia does not change, also the sign-sum function does not change under congruence transformations. Denoting the rank of B by $\gamma := \operatorname{rank}(B)$ we verify the congruence transformations

$$\begin{aligned}
\begin{bmatrix} 0 & 0 & B \\ 0 & A & 0 \\ B^* & 0 & 0 \end{bmatrix} &\sim \begin{bmatrix} 0_\gamma & 0 & 0 & I_\gamma & 0 \\ 0 & 0_{p-\gamma} & 0 & 0 & 0 \\ 0 & 0 & A & 0 & 0 \\ I_\gamma & 0 & 0 & 0_\gamma & 0 \\ 0 & 0 & 0 & 0 & 0_{r-\gamma} \end{bmatrix} \sim \begin{bmatrix} A & 0 & 0 & 0 \\ 0 & 0 & I_\gamma & 0 \\ 0 & I_\gamma & 0 & 0 \\ 0 & 0 & 0 & 0_{p+r-2\gamma} \end{bmatrix} \\
&\sim \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} I_\gamma & -\frac{1}{\sqrt{2}} I_\gamma & 0 \\ 0 & \frac{1}{\sqrt{2}} I_\gamma & \frac{1}{\sqrt{2}} I_\gamma & 0 \\ 0 & 0 & 0 & I \end{bmatrix} \begin{bmatrix} A & 0 & 0 & 0 \\ 0 & 0 & I_\gamma & 0 \\ 0 & I_\gamma & 0 & 0 \\ 0 & 0 & 0 & 0_{p+r-2\gamma} \end{bmatrix} \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} I_\gamma & \frac{1}{\sqrt{2}} I_\gamma & 0 \\ 0 & -\frac{1}{\sqrt{2}} I_\gamma & \frac{1}{\sqrt{2}} I_\gamma & 0 \\ 0 & 0 & 0 & I \end{bmatrix} \\
&= \begin{bmatrix} A & 0 & 0 & 0 \\ 0 & I_\gamma & 0 & 0 \\ 0 & 0 & -I_\gamma & 0 \\ 0 & 0 & 0 & 0_{p+r-2\gamma} \end{bmatrix},
\end{aligned}$$

and thus the assertion. □

Theorem 3.11. *Let $P \in \mathbb{C}[\lambda]_K^{p,q}$ and $H = H^* \in \mathbb{C}^{qK,qK}$. Let $r := \operatorname{rank}_{\mathbb{C}(\lambda)}(P)$. Define the para-Hermitian matrix polynomial N through*

$$N := \begin{bmatrix} 0 & P \\ P^\sim & \Delta_K^{q\,\sim} H \Delta_K^q \end{bmatrix} \in \mathbb{C}[\lambda]^{p+q,p+q}. \tag{3.3}$$

Then (P, H) is cyclo-dissipative if and only if we have

$$\eta(N(i\omega)) = p + q - 2r, \tag{3.4}$$

for all $\omega \in \mathbb{R}$ such that $i\omega \notin \mathfrak{Z}(P)$.

Proof. Let $U \in \mathbb{C}[\lambda]^{q,q-r}$ and $V \in \mathbb{C}[\lambda]^{q,r}$ be the polynomial kernel and co-kernel matrices from Theorem 2.7. Define the Popov function $\Pi := U^\sim \Delta_K^{q\,\sim} H \Delta_K^q U \in \mathbb{C}[\lambda]^{q-r,q-r}$ and observe that with Theorem 3.5 cyclo-dissipativity of (P, H) is equivalent to the condition that $\eta(\Pi(i\omega)) = q - r$ all $\omega \in \mathbb{R}$. Since Π is polynomial and as such also continuous, we may conclude that dissipativity is also equivalent to the condition that $\eta\left(\Pi(i\omega)\right) = q - r$ for all $\omega \in \mathbb{R}$ such that $i\omega \notin \mathfrak{Z}\left(P\right)$. Finally, we find that with the described choice of V we have that for all $i\omega \notin \mathfrak{Z}\left(P\right)$ the rank of $(PV)(i\omega)$ is r and thus using Lemma 3.6 and Lemma 3.10 we conclude that

$$\eta\left(N(i\omega)\right) = \eta\left(\begin{bmatrix} 0 & 0 & (PV)(i\omega) \\ 0 & \Pi(i\omega) & 0 \\ (PV)^*(i\omega) & 0 & 0 \end{bmatrix}\right) = \eta\left(\Pi(i\omega)\right) + p + r - 2r,$$

which proves the claim. $\square$

As shown in Algorithm 4.7 one can use Theorem 3.11 to design a simple check for cyclo-dissipativity.

3.2 Linear quadratic optimal control

In the previous section we saw that the para-Hermitian matrix polynomial N as defined in (3.3) plays a prominent role in linear quadratic systems theory. It is thus natural to ask if the behavior of N has some meaning. The following two theorems show that this is indeed the case.

Theorem 3.12. *Let $P \in \mathbb{C}[\lambda]_K^{p,q}$ and $H = H^* \in \mathbb{C}^{qK,qK}$, and define the para-Hermitian matrix polynomial N through*

$$N := \begin{bmatrix} 0 & P \\ P^{\sim} & {\Delta_K^q}^{\sim} H \Delta_K^q \end{bmatrix} \in \mathbb{C}[\lambda]^{p+q,p+q}. \tag{3.5}$$

Let (P, H) be cyclo-dissipative. Then the following statements hold:

1. *If $(\mu_+, z_+) \in \mathfrak{B}_+(N)$ then for all $t_0 \in \mathbb{R}$ we have*

$$\int_{t_0}^{\infty} \left(\Delta_K z_+(t)\right)^* H \left(\Delta_K z_+(t)\right) dt = \inf_{\substack{z \in \mathfrak{B}_+(P) \\ z(t) = z_+(t),\ t \le t_0}} \int_{t_0}^{\infty} \left(\Delta_K z(t)\right)^* H \left(\Delta_K z(t)\right) dt. \tag{3.6}$$

2. *If $(\mu_-, z_-) \in \mathfrak{B}_-(N)$ then for all $t_0 \in \mathbb{R}$ we have*

$$\int_{-\infty}^{t_0} \left(\Delta_K z_-(t)\right)^* H \left(\Delta_K z_-(t)\right) dt = \inf_{\substack{z \in \mathfrak{B}_-(P) \\ z(t) = z_-(t),\ t \ge t_0}} \int_{-\infty}^{t_0} \left(\Delta_K z(t)\right)^* H \left(\Delta_K z(t)\right) dt. \tag{3.7}$$

Proof. To prove the first assertion let $t_0 \in \mathbb{R}$ be arbitrary and let $v \in \mathfrak{B}_+(P)$ be arbitrary such that $v(t) = z_+(t)$ for $t \le t_0$. Then we see that for every $s \in \mathbb{R}$ the function defined as $z_s := s z_+ + (1-s)v \in \mathfrak{B}_+(P)$ is a trajectory of the system. Also we have that $z_s(t) = z_+(t)$ for all $t \le t_0$ and all $s \in \mathbb{R}$. Define the function $\Phi_v : \mathbb{R} \to \mathbb{R}$ through

$$\Phi_v(s) \;:=\; \int_{t_0}^{\infty} \left(\Delta_K z_s(t)\right)^* H \left(\Delta_K z_s(t)\right)$$

$$
\begin{aligned}
&= \int_{t_0}^{\infty} \left(s\Delta_K z_+(t) + (1-s)\Delta_K v(t)\right)^* H \left(s\Delta_K z_+(t) + (1-s)\Delta_K v(t)\right) dt \\
&= \int_{t_0}^{\infty} s^2 \left(\Delta_K z_+(t)\right)^* H \left(\Delta_K z_+(t)\right) + 2s(1-s)\mathrm{Re}\left(\left(\Delta_K v(t)\right)^* H \left(\Delta_K z_+(t)\right)\right) \\
&\quad +(1-s)^2 \left(\Delta_K v(t)\right)^* H \left(\Delta_K v(t)\right) dt.
\end{aligned}
$$

In the following we are going to show that Φ_v has a minimum in $s = 1$. Since v is assumed to be arbitrary, this then implies the optimality of z_+. To show that Φ_v has a minimum in $s = 1$ we consider its first and second derivative. Differentiation of Φ_v yields

$$
\begin{aligned}
\frac{d}{ds}\Phi_v(s) &= \int_{t_0}^{\infty} 2s \left(\Delta_K z_+(t)\right)^* H \left(\Delta_K z_+(t)\right) + 2(1-2s)\mathrm{Re}\left(\left(\Delta_K v(t)\right)^* H \left(\Delta_K z_+(t)\right)\right) \\
&\quad -2(1-s) \left(\Delta_K v(t)\right)^* H \left(\Delta_K v(t)\right) dt,
\end{aligned}
$$

and evaluation at the point $s = 1$ implies

$$
\begin{aligned}
\frac{d}{ds}\Phi_v(1) &= \int_{t_0}^{\infty} 2 \left(\Delta_K z_+(t)\right)^* H \left(\Delta_K z_+(t)\right) - 2\mathrm{Re}\left(\left(\Delta_K v(t)\right)^* H \left(\Delta_K z_+(t)\right)\right) dt \\
&= 2\int_{t_0}^{\infty} \mathrm{Re}\left(\left(\Delta_K z_+(t)\right)^* H \left(\Delta_K z_+(t)\right) - \left(\Delta_K v(t)\right)^* H \left(\Delta_K z_+(t)\right)\right) dt \\
&= 2\mathrm{Re}\left(\int_{t_0}^{\infty} \left(\Delta_K z_+(t) - \Delta_K v(t)\right)^* H \left(\Delta_K z_+(t)\right) dt\right).
\end{aligned}
$$

Defining $y := z_+ - v$ we see that $y \in \mathfrak{B}_+(P)$ with

$$
y(t) = 0 \tag{3.8}
$$

for all $t \leq t_0$ and we have that

$$
\frac{d}{ds}\Phi_v(1) = 2\mathrm{Re}\left(\int_{t_0}^{\infty} \left(\Delta_K y(t)\right)^* H \left(\Delta_K z_+(t)\right) dt\right).
$$

Using Lemma 2.26 we find that

$$
\begin{aligned}
&\int_{t_0}^{\infty} \left(\Delta_K y(t)\right)^* H \left(\Delta_K z_+(t)\right) dt \\
&= \int_{t_0}^{\infty} \left(\Delta_K^q \left(\tfrac{d}{dt}\right) y(t)\right)^* H \Delta_K z_+(t) dt \\
&= \int_{t_0}^{\infty} y(t) \Delta_K^{q\,\sim} \left(\tfrac{d}{dt}\right) H \Delta_K z_+(t) dt \\
&\quad - \sum_{k=1}^{\infty} (-1)^k \left(\Delta_K^{q\,\langle k\rangle} \left(\tfrac{d}{dt}\right) y(t)\right)^* H \Delta_K z_+^{(k-1)}(t) \Big|_{t_0}^{\infty} \\
&= \int_{t_0}^{\infty} y(t) \Delta_K^{q\,\sim} \left(\tfrac{d}{dt}\right) H \Delta_K^q \left(\tfrac{d}{dt}\right) z_+(t) dt,
\end{aligned}
$$

since $y^{(j)}(t_0) = 0$ and $\lim_{t\to\infty} y^{(j)}(t) = 0$ for $j = 0, 1, \ldots$ due to (3.8). Using Lemma 2.26 again we find

$$\begin{aligned}
\frac{d}{ds}\Phi_v(1) &= 2\mathrm{Re}\left(\int_{t_0}^{\infty} y(t)\Delta_K^{q\,\sim}\left(\tfrac{d}{dt}\right) H\Delta_K^q\left(\tfrac{d}{dt}\right) z_+(t)dt\right) \\
&= 2\mathrm{Re}\left(-\int_{t_0}^{\infty} y^*(t)P^{\sim}\left(\tfrac{d}{dt}\right)\mu_+(t)dt\right) \\
&= 2\mathrm{Re}\left(-\int_{t_0}^{\infty}\left(\underbrace{P\left(\tfrac{d}{dt}\right)y(t)}_{=0}\right)^*\mu_+(t)dt\right) \\
&\quad -2\mathrm{Re}\left(\sum_{k=1}^{\infty}(-1)^k\left(P^{\langle k\rangle}\left(\tfrac{d}{dt}\right)y(t)\right)^*\mu_+^{(k-1)}(t)\Big|_{t_0}^{\infty}\right) = 0,
\end{aligned}$$

where for the last identity we again used (3.8). The second derivative of Φ_v is given by

$$\begin{aligned}
\left(\tfrac{d}{dt}\right)^2\Phi_v(s) &= 2\int_{t_0}^{\infty}(\Delta_K z_+(t))^* H(\Delta_K z_+(t)) - 2\mathrm{Re}\left((\Delta_K v(t))^* H(\Delta_K z_+(t))\right) \\
&\quad + (\Delta_K v(t))^* H(\Delta_K v(t))\,dt \\
&= 2\int_{t_0}^{\infty}(\Delta_K z_+(t) - \Delta_K v(t))^* H(\Delta_K z_+(t) - \Delta_K v(t))\,dt \\
&= 2\int_{t_0}^{\infty}(\Delta_K y(t))^* H(\Delta_K y(t))\,dt \\
&= 2\int_{-\infty}^{\infty}(\Delta_K y(t))^* H(\Delta_K y(t))\,dt \geq 0,
\end{aligned}$$

where the last inequality follows from the dissipativity of the system together with Lemma A.6 and the last identity follows from (3.8). □

The following theorem is the converse of the previous theorem.

Theorem 3.13. *Let $P \in \mathbb{C}[\lambda]_K^{p,q}$ and $H = H^* \in \mathbb{C}^{qK,qK}$ and define the para-Hermitian matrix polynomial N through (3.5). Let $z_+ \in \mathfrak{B}_+(P)$ solve (3.6) (or let $z_- \in \mathfrak{B}_-(P)$ solve (3.7)). Then, (P, H) is cyclo-dissipative and there exists a co-state function $\mu_+ \in \mathcal{C}_+^p$ (or $\mu_- \in \mathcal{C}_-^p$) such that we have $(\mu_+, z_+) \in \mathfrak{B}_+(N)$ (or $(\mu_-, z_-) \in \mathfrak{B}_-(N)$).*

Proof. Let $r := \mathrm{rank}_{\mathbb{C}(\lambda)}(P)$ and let $U \in \mathbb{C}[\lambda]^{q,q-r}$ and $V \in \mathbb{C}[\lambda]^{q,r}$ be polynomial kernel and co-kernel matrices without zeros, according to Theorem 2.7. Let $t_0 \in \mathbb{R}$ and let $z_0 \in \mathfrak{B}_+(P)$ be an arbitrary function with $z_0(t) = 0$ for all $t \leq t_0$. Set $z_\epsilon := z_+ + \epsilon z_0$, for $\epsilon \in \mathbb{R}$. Then we have

$$\begin{aligned}
&\int_{t_0}^{\infty}(\Delta_K z_\epsilon(t))^* H(\Delta_K z_\epsilon(t))\,dt \\
=\; &\int_{t_0}^{\infty}(\Delta_K z_+(t) + \epsilon\Delta_K z_0(t))^* H(\Delta_K z_+(t) + \epsilon\Delta_K z_0(t))\,dt
\end{aligned}$$

$$\begin{aligned} &= \int_{t_0}^{\infty} (\Delta_K z_+(t))^* H (\Delta_K z_+(t))\, dt + 2\epsilon \int_{t_0}^{\infty} \mathrm{Re}\left((\Delta_K z_0(t))^* H (\Delta_K z_+(t))\right) dt \\ &\quad + \epsilon^2 \int_{t_0}^{\infty} (\Delta_K z_0(t))^* H (\Delta_K z_0(t))\, dt, \end{aligned}$$

and thus, since z_+ is an optimal trajectory in the sense of the assumption, we find that

$$\begin{aligned} 0 &\leq \int_{t_0}^{\infty} (\Delta_K z_\epsilon(t))^* H (\Delta_K z_\epsilon(t))\, dt - \int_{t_0}^{\infty} (\Delta_K z_+(t))^* H (\Delta_K z_+(t))\, dt \qquad (3.9) \\ &= 2\epsilon \int_{t_0}^{\infty} \mathrm{Re}\left((\Delta_K z_0(t))^* H (\Delta_K z_+(t))\right) dt + \epsilon^2 \int_{t_0}^{\infty} (\Delta_K z_0(t))^* H (\Delta_K z_0(t))\, dt, \end{aligned}$$

for all $\epsilon \in \mathbb{R}$. This implies that

$$0 \leq \int_{t_0}^{\infty} (\Delta_K z_0(t))^* H (\Delta_K z_0(t))\, dt = \int_{-\infty}^{\infty} (\Delta_K z_0(t))^* H (\Delta_K z_0(t))\, dt,$$

since $z_0(t) = 0$ for all $t \leq t_0$. Because $\mathfrak{B}_c(P)$ is shift invariant and a subset of the functions which are in $\mathfrak{B}_+(P)$ and fulfill the additional property that they vanish for $t \leq t_0$, this implies cyclo-dissipativity of (P, H), since z_0 is allowed to be arbitrary. Also we see that (3.9) implies

$$0 = \int_{t_0}^{\infty} \mathrm{Re}\left((\Delta_K z_0(t))^* H (\Delta_K z_+(t))\right) dt.$$

If $z_0 \in \mathfrak{B}_+(P)$ is a trajectory of P with $z_0(t) = 0$ for $t \leq t_0$, so is iz_0 with i being the imaginary unit. Thus we obtain

$$\begin{aligned} 0 &= \int_{t_0}^{\infty} \mathrm{Re}\left((\Delta_K i z_0(t))^* H (\Delta_K z_+(t))\right) dt = \int_{t_0}^{\infty} \mathrm{Re}\left(i\,(\Delta_K z_0(t))^* H (\Delta_K z_+(t))\right) dt \\ &= -\int_{t_0}^{\infty} \mathrm{Im}\left((\Delta_K z_0(t))^* H (\Delta_K z_+(t))\right) dt. \end{aligned}$$

Using Lemma 2.26 this implies

$$\begin{aligned} 0 &= \int_{t_0}^{\infty} (\Delta_K z_0(t))^* H (\Delta_K z_+(t))\, dt \\ &= \int_{t_0}^{\infty} \left(\Delta_K^q\left(\tfrac{d}{dt}\right) z_0(t)\right)^* H \left(\Delta_K^q\left(\tfrac{d}{dt}\right) z_+(t)\right) dt \\ &= \int_{t_0}^{\infty} z_0^*(t) \Delta_K^{q\sim}\left(\tfrac{d}{dt}\right) H \Delta_K^q\left(\tfrac{d}{dt}\right) z_+(t) dt \end{aligned}$$

for all $z_0 \in \mathfrak{B}_+(P)$ with $z_0(t) = 0$ for $t \leq t_0$. Using Lemma 2.18 we see that for every $\alpha \in \mathcal{E}_{q-r}^+$ with $\alpha(t) = 0$ for all $t \leq t_0$, we have that the specific $z_0 := U\left(\frac{d}{dt}\right)\alpha \in \mathfrak{B}_+(P)$ is a trajectory of the system. In this case also $z_0(t) = U\left(\frac{d}{dt}\right)\alpha(t) = 0$ for all $t \leq t_0$. Thus, by setting $\tilde{H} := \Delta_K^{q\sim} H \Delta_K^q$ and using Lemma 2.26 we have

$$0 = \int_{t_0}^{\infty} \left(U\left(\tfrac{d}{dt}\right)\alpha(t)\right)^* \tilde{H}\left(\tfrac{d}{dt}\right) z_+(t) dt$$

$$= \int_{t_0}^{\infty} \alpha^*(t) U^\sim\left(\tfrac{d}{dt}\right) \tilde{H}\left(\tfrac{d}{dt}\right) z_+(t) dt,$$

for all $\alpha \in \mathcal{E}^+_{q-r}$ with $\alpha(t) = 0$ for $t \le t_0$. Since $t_0 \in \mathbb{R}$ is arbitrary, we deduce that

$$U^\sim\left(\tfrac{d}{dt}\right) \tilde{H}\left(\tfrac{d}{dt}\right) z_+(t) = 0,$$

for all $t \in \mathbb{R}$. Let $\mu_+ \in \mathcal{E}^+_p$ be a solution of the problem

$$(PV)^\sim\left(\tfrac{d}{dt}\right) \mu_+(t) = -V^\sim\left(\tfrac{d}{dt}\right) \tilde{H}\left(\tfrac{d}{dt}\right) z_+(t),$$

which exists due to Lemma A.15 and since we know from Theorem 2.7 that PV has full column rank. Using that U is a kernel matrix we find

$$\begin{aligned}
P^\sim\left(\tfrac{d}{dt}\right)\mu_+ &= \begin{bmatrix} U\left(\tfrac{d}{dt}\right) & V\left(\tfrac{d}{dt}\right)\end{bmatrix}^{-\sim} \begin{bmatrix} U\left(\tfrac{d}{dt}\right) & V\left(\tfrac{d}{dt}\right)\end{bmatrix}^{\sim} P^\sim\left(\tfrac{d}{dt}\right)\mu_+ \\
&= \begin{bmatrix} U\left(\tfrac{d}{dt}\right) & V\left(\tfrac{d}{dt}\right)\end{bmatrix}^{-\sim} \begin{bmatrix} (PU)^\sim\left(\tfrac{d}{dt}\right)\mu_+ \\ (PV)^\sim\left(\tfrac{d}{dt}\right)\mu_+ \end{bmatrix} \\
&= \begin{bmatrix} U\left(\tfrac{d}{dt}\right) & V\left(\tfrac{d}{dt}\right)\end{bmatrix}^{-\sim} \begin{bmatrix} 0 \\ (PV)^\sim\left(\tfrac{d}{dt}\right)\mu_+ \end{bmatrix} \\
&= \begin{bmatrix} U\left(\tfrac{d}{dt}\right) & V\left(\tfrac{d}{dt}\right)\end{bmatrix}^{-\sim} \begin{bmatrix} -U^\sim\left(\tfrac{d}{dt}\right)\tilde{H}\left(\tfrac{d}{dt}\right) z_+ \\ -V^\sim\left(\tfrac{d}{dt}\right)\tilde{H}\left(\tfrac{d}{dt}\right) z_+ \end{bmatrix} \\
&= \begin{bmatrix} U\left(\tfrac{d}{dt}\right) & V\left(\tfrac{d}{dt}\right)\end{bmatrix}^{-\sim} \begin{bmatrix} -U^\sim\left(\tfrac{d}{dt}\right) \\ -V^\sim\left(\tfrac{d}{dt}\right) \end{bmatrix} \tilde{H}\left(\tfrac{d}{dt}\right) z_+ = -\tilde{H}\left(\tfrac{d}{dt}\right) z_+,
\end{aligned}$$

which finishes the proof. □

Another way to discuss the importance of cyclo-dissipativity for the linear quadratic optimal control problem is by considering a system which is not cyclo-dissipative, i.e., assume that there would exist a trajectory $\tilde{z} \in \mathfrak{B}_c(P)$ with compact support such that

$$0 > \int_{-\infty}^{\infty} \left(\Delta_K \tilde{z}(t)\right)^* H \left(\Delta_K \tilde{z}(t)\right) dt.$$

Then one can concatenate the non-trivial part of $\tilde{z}$ over and over again to obtain a trajectory of arbitrary low cost which still has compact support. Thus, in this case non of the optimal control problems (3.6) or (3.7) is solvable.

Theorems 3.12 and 3.13 can be summarized in the following words. Cyclo-dissipativity is equivalent to the solvability of one of the optimal control problems (3.6) and (3.7). Looking back at the definition of cyclo-dissipativity (Definition 3.2) this shows that cyclo-dissipativity is some kind of positive semi-definiteness of the cost functional H on the linear subspace given by $\mathfrak{B}_c(P)$. Also, if the optimal control problem is solvable (or the system is cyclo-dissipative) then its solutions can be obtained from the behavior of N as defined in (3.5).

When looking at the optimal control problems (3.6) and (3.7) we notice that those are not the optimal control problems which are usually considered. In the literature boundary value problems associated with systems of the form (3.5) are frequently connected to optimal control problems where not the whole past is fixed (i.e., $z(t) = z_+(t)$ for all $t \le t_0$), but only one point is fixed (i.e., something like $z(t_0) = \hat{z}$), compare [24]. In the following section we will see that both problems are equivalent, compare Theorem 3.15.

3.3 Available storage and required supply

In this section we introduce the available storage and the required supply and then show how they are connected to cyclo-dissipativity. Especially, we will see that for controllable systems cyclo-dissipativity and dissipativity are equivalent, by showing that for controllable systems the available storage and required supply are storage functions.

Definition 3.14. Let $P \in \mathbb{C}[\lambda]_K^{p,q}$ and $H = H^* \in \mathbb{C}^{qK,qK}$. Then we call the function $\Theta_+ : \mathbb{C}^{qK} \to \mathbb{R} \cup \{\pm\infty\}$ defined through

$$\begin{aligned} \Theta_+(\hat{z}) \;&:=\; \sup_{\substack{z\in\mathfrak{B}_+(P)\\ \Delta_K z(0)=\hat{z}}} -\int_0^\infty (\Delta_K z(t))^* H (\Delta_K z(t))\, dt \\ &=\; -\inf_{\substack{z\in\mathfrak{B}_+(P)\\ \Delta_K z(0)=\hat{z}}} \int_0^\infty (\Delta_K z(t))^* H (\Delta_K z(t))\, dt, \end{aligned} \tag{3.10}$$

the *available storage* of (P, H) and the function $\Theta_- : \mathbb{C}^{qK} \to \mathbb{R} \cup \{\pm\infty\}$ defined through

$$\Theta_-(\hat{z}) := \inf_{\substack{z\in\mathfrak{B}_-(P)\\ \Delta_K z(0)=\hat{z}}} \int_{-\infty}^0 (\Delta_K z(t))^* H (\Delta_K z(t))\, dt, \tag{3.11}$$

the *required supply* of (P, H).

Obviously, the available storage looks much like (3.6) and the required supply looks much like (3.7). The following Theorem 3.15 shows that both are indeed the same.

Theorem 3.15. *Let $P \in \mathbb{C}[\lambda]_K^{p,q}$ and $H = H^* \in \mathbb{C}^{qK,qK}$. Let $z_+ \in \mathfrak{B}_+(P)$ and $z_- \in \mathfrak{B}_-(P)$. Then we have*

$$-\Theta_+(\Delta_K z_+(0)) \;=\; \inf_{\substack{z\in\mathfrak{B}_+(P)\\ z(t)=z_+(t),t\le 0}} \int_0^\infty (\Delta_K z(t))^* H (\Delta_K z(t))\, dt$$

and

$$\Theta_-(\Delta_K z_-(0)) \;=\; \inf_{\substack{z\in\mathfrak{B}_-(P)\\ z(t)=z_-(t),t\ge 0}} \int_{-\infty}^0 (\Delta_K z(t))^* H (\Delta_K z(t))\, dt.$$

If $K = 1$ we also have that

$$-\Theta_+(\Delta_K z_+(0)) \;=\; \inf_{\substack{z\in\mathfrak{B}_+(P)\\ P^{\langle k\rangle}\left(\frac{d}{dt}\right)z(0)=P^{\langle k\rangle}\left(\frac{d}{dt}\right)z_+(0),\ k\ge 1}} \int_0^\infty (\Delta_K z(t))^* H (\Delta_K z(t))\, dt,$$

and

$$\Theta_-(\Delta_K z_-(0)) \;=\; \inf_{\substack{z\in\mathfrak{B}_-(P)\\ P^{\langle k\rangle}\left(\frac{d}{dt}\right)z(0)=P^{\langle k\rangle}\left(\frac{d}{dt}\right)z_-(0),\ k\ge 1}} \int_{-\infty}^0 (\Delta_K z(t))^* H (\Delta_K z(t))\, dt.$$

Proof. See Appendix A.1. □

We conjecture that the assumption $K = 1$ in Theorem 3.15 can be dropped because of the following reason. In [29] it has been shown that a so-called *state-map* for a system of the form $P \in \mathbb{C}[\lambda]_K^{p,q}$ is given by

$$\begin{bmatrix} P^{\langle 1 \rangle} \\ \vdots \\ P^{\langle K \rangle} \end{bmatrix} \in \mathbb{C}[\lambda]^{pK,q}.$$

Further, in [35] it was shown that every storage function is a function of the state. However, in Theorem 3.20 we will see that for cyclo-dissipative systems, both the available storage and the required supply are storage functions.

Remark 3.16. In the following we will frequently state inequalities in which one or both sides are allowed to be ∞ or $-\infty$. Therefore we introduce the convention that the inequalities $\infty \leq \infty$, $-\infty \leq -\infty$, and $-\infty < \infty$ are considered to be true but not the inequalities $\infty < \infty$, $-\infty < -\infty$, and $\infty \leq -\infty$. Of course, the inequality $-\infty < a < \infty$ is considered to be true for all $a \in \mathbb{R}$. Also, $a \cdot \infty = \infty$ and $a \cdot (-\infty) = (-\infty)$ for all $a \in \mathbb{C} \setminus \{0\}$. The expressions $0 \cdot \infty$ and $0 \cdot (-\infty)$ will not be used in this thesis and are considered to be undefined. Similar, $a + \infty = \infty$ and $a - \infty = -\infty$ for all $a \in \mathbb{C}$ and the expression $\infty - \infty$ will not be used.

Remark 3.17. If $z_+ \in \mathfrak{B}_+(P)$ then the integral

$$\int_0^\infty (\Delta_K z_+(t))^* H (\Delta_K z_+(t))\, dt$$

exists. This implies $\Theta_+(\Delta_K z_+(0)) \in \mathbb{R} \cup \{\infty\}$, since then in (3.10) the supremum of a non-empty set is taken. We conclude that $\Theta_+(\hat{z}) \in \mathbb{R} \cup \{\infty\}$ for all $\hat{z} \in R_+(P)$.
If $z_- \in \mathfrak{B}_-(P)$ then the integral

$$\int_{-\infty}^0 (\Delta_K z_-(t))^* H (\Delta_K z_-(t))\, dt$$

exists. This implies $\Theta_-(\Delta_K z_-(0)) \in \mathbb{R} \cup \{-\infty\}$, since then in (3.11) the infimum of a non-empty set is taken. We conclude that $\Theta_-(\hat{z}) \in \mathbb{R} \cup \{-\infty\}$ for all $\hat{z} \in R_-(P)$.

We can characterize cyclo-dissipativity via the available storage and required supply in the following way.

Lemma 3.18. *Let $P \in \mathbb{C}[\lambda]_K^{p,q}$ and $H = H^* \in \mathbb{C}^{qK,qK}$. Then (P, H) is cyclo-dissipative if and only if we have*

$$\Theta_+(\hat{z}) \leq \Theta_-(\hat{z}), \tag{3.12}$$

for all $\hat{z} \in R_c(P)$.

Proof. First, assume that condition (3.12) is fulfilled and let $\tilde{z} \in \mathfrak{B}_c(P)$ be arbitrary. Then due to Remark 3.17 we see that both the available storage $\Theta_+(\Delta_K \tilde{z}(0)) \in \mathbb{R}$ and the required supply $\Theta_-(\Delta_K \tilde{z}(0)) \in \mathbb{R}$ are real numbers. Thus we obtain

$$
\begin{aligned}
0 &\leq \Theta_-(\Delta_K \tilde{z}(0)) - \Theta_+(\Delta_K \tilde{z}(0)) \\
&= \inf_{\substack{z \in \mathfrak{B}_-(P) \\ \Delta_K z(0) = \Delta_K \tilde{z}(0)}} \int_{-\infty}^{0} (\Delta_K z(t))^* H (\Delta_K z(t))\, dt + \\
&\quad \inf_{\substack{z \in \mathfrak{B}_+(P) \\ \Delta_K z(0) = \Delta_K \tilde{z}(0)}} \int_{0}^{\infty} (\Delta_K z(t))^* H (\Delta_K z(t))\, dt \\
&\leq \int_{-\infty}^{0} (\Delta_K \tilde{z}(t))^* H (\Delta_K \tilde{z}(t))\, dt + \int_{0}^{\infty} (\Delta_K \tilde{z}(t))^* H (\Delta_K \tilde{z}(t))\, dt \\
&= \int_{-\infty}^{\infty} (\Delta_K \tilde{z}(t))^* H (\Delta_K \tilde{z}(t))\, dt,
\end{aligned}
$$

and with this cyclo-dissipativity of (P, H).
For the converse assume that (P, H) is cyclo-dissipative. Assume to the contrary that there would be a $\hat{z} \in R_c(P)$ such that $\Theta_+(\hat{z}) > \Theta_-(\hat{z})$. This shows that there exist $z_+ \in \mathfrak{B}_+(P)$ and $z_- \in \mathfrak{B}_-(P)$ such that $\Delta_K z_+(0) = \hat{z} = \Delta_K z_-(0)$ and

$$
-\int_{0}^{\infty} (\Delta_K z_+(t))^* H (\Delta_K z_+(t))\, dt > \int_{-\infty}^{0} (\Delta_K z_-(t))^* H (\Delta_K z_-(t))\, dt.
$$

Define η through

$$
\eta := \int_{-\infty}^{0} (\Delta_K z_-(t))^* H (\Delta_K z_-(t))\, dt + \int_{0}^{\infty} (\Delta_K z_+(t))^* H (\Delta_K z_+(t))\, dt < 0.
$$

Then $-\frac{\eta}{2} > 0$ and thus there exists an $\epsilon > 0$ such that

$$
3\epsilon^2 \|H\| + 2\epsilon \|H\| \|\Delta_K z_+\|_+ < -\frac{\eta}{2}.
$$

With this ϵ, construct a function $\tilde{z} \in \mathfrak{B}_+(P) \cup \mathfrak{B}_-(P)$ from z_+ and z_- such that we have $\|\Delta_K(\tilde{z} - z_+)\|_+ + \|\Delta_K(\tilde{z} - z_-)\|_- < \epsilon$. Such a $\tilde{z}$ exists due to Lemma A.5. Since then

$$
\begin{aligned}
&\langle H\Delta_K(z_+ - \tilde{z}), \Delta_K(z_+ - \tilde{z})\rangle_+ \\
= \ &\langle H\Delta_K z_+, \Delta_K z_+\rangle_+ - 2\mathrm{Re}\left(\langle H\Delta_K z_+, \Delta_K \tilde{z}\rangle_+\right) + \langle H\Delta_K \tilde{z}, \Delta_K \tilde{z}\rangle_+,
\end{aligned}
$$

we also obtain that

$$
\begin{aligned}
&\left|\langle H\Delta_K z_+, \Delta_K z_+\rangle_+ - \langle H\Delta_K \tilde{z}, \Delta_K \tilde{z}\rangle_+\right| \\
= \ &\left|\langle H\Delta_K(z_+ - \tilde{z}), \Delta_K(z_+ - \tilde{z})\rangle_+ + 2\mathrm{Re}\left(\langle H\Delta_K z_+, \Delta_K \tilde{z}\rangle_+\right)\right. \\
&\left. -2\langle H\Delta_K \tilde{z}, \Delta_K \tilde{z}\rangle_+\right|
\end{aligned}
$$

$$
\begin{aligned}
&\leq \left|\langle H\Delta_K(z_+-\tilde z), \Delta_K(z_+-\tilde z)\rangle_+\right| + 2\left|\langle H\Delta_K z_+, \Delta_K\tilde z\rangle_+ - \langle H\Delta_K\tilde z, \Delta_K\tilde z\rangle_+\right| \\
&\leq \|H\|\|\Delta_K(z_+-\tilde z)\|_+^2 + 2\|H\|\|\Delta_K(z_+-\tilde z)\|_+\|\Delta_K\tilde z\|_+ \\
&< \epsilon^2\|H\| + 2\epsilon\|H\|\|\Delta_K(\tilde z - z_+) + \Delta_K z_+\|_+ \\
&\leq \epsilon^2\|H\| + 2\epsilon\|H\|\left(\|\Delta_K(\tilde z - z_+)\|_+ + \|\Delta_K z_+\|_+\right) \\
&< 3\epsilon^2\|H\| + 2\epsilon\|H\|\|\Delta_K z_+\|_+ < -\frac{\eta}{2}.
\end{aligned}
$$

Analogously we deduce that

$$
\left|\langle H\Delta_K z_-, \Delta_K z_-\rangle_- - \langle H\Delta_K\tilde z, \Delta_K\tilde z\rangle_-\right| < -\frac{\eta}{2}.
$$

Finally, using the assumption of cyclo-dissipativity together with Lemma A.6 we conclude that

$$
\begin{aligned}
0 &\leq \int_{-\infty}^{\infty} (\Delta_K\tilde z(t))^* H\,(\Delta_K\tilde z(t))\,dt \\
&= \langle H\Delta_K\tilde z, \Delta_K\tilde z\rangle_+ + \langle H\Delta_K\tilde z, \Delta_K\tilde z\rangle_- \\
&\leq \left|\langle H\Delta_K\tilde z, \Delta_K\tilde z\rangle_+ - \langle H\Delta_K z_+, \Delta_K z_+\rangle_+\right| + \\
&\quad\ \left|\langle H\Delta_K\tilde z, \Delta_K\tilde z\rangle_- - \langle H\Delta_K z_-, \Delta_K z_-\rangle_-\right| + \eta \\
&< -\frac{\eta}{2} - \frac{\eta}{2} + \eta = 0,
\end{aligned}
$$

which is a contradiction. $\square$

The following proof is an adaption of the proof of [41, Theorem 1].

Lemma 3.19. *Let $P \in \mathbb{C}[\lambda]_K^{p,q}$, $H = H^* \in \mathbb{C}^{qK,qK}$, $z_+ \in \mathfrak{B}_+(P)$, $z_- \in \mathfrak{B}_-(P)$, and $t_0, t_1 \in \mathbb{R}$ with $t_0 \leq t_1$. Then we have*

$$
\begin{aligned}
\Theta_+(\Delta_K z_+(t_1)) \;&-\; \int_{t_0}^{t_1} (\Delta_K z_+(t))^* H\,(\Delta_K z_+(t))\,dt \;\leq\; \Theta_+(\Delta_K z_+(t_0)), \\
\Theta_-(\Delta_K z_-(t_1)) \;&\leq\; \int_{t_0}^{t_1} (\Delta_K z_-(t))^* H\,(\Delta_K z_-(t))\,dt \;+\; \Theta_-(\Delta_K z_-(t_0)).
\end{aligned}
$$

Proof. Using Theorem 3.15 we find

$$
\begin{aligned}
&\Theta_+(\Delta_K z_+(t_0)) \\
= &-\inf_{\substack{z\in\mathfrak{B}_+(P)\\ \Delta_K z(0)=\Delta_K z_+(t_0)}} \int_0^{\infty} (\Delta_K z(t))^* H\,(\Delta_K z(t))\,dt \\
= &-\inf_{\substack{z\in\mathfrak{B}_+(P)\\ \Delta_K z(t_0)=\Delta_K z_+(t_0)}} \int_{t_0}^{\infty} (\Delta_K z(t))^* H\,(\Delta_K z(t))\,dt \\
\geq &-\inf_{\substack{z\in\mathfrak{B}_+(P)\\ z(t)=z_+(t), t\leq t_1}} \int_{t_0}^{\infty} (\Delta_K z(t))^* H\,(\Delta_K z(t))\,dt
\end{aligned}
$$

$$
\begin{aligned}
&= -\inf_{\substack{z\in\mathfrak{B}_+(P)\\ z(t)=z_+(t),t\le t_1}} \int_{t_0}^{t_1} (\Delta_K z_+(t))^* H (\Delta_K z_+(t))\, dt + \int_{t_1}^{\infty} (\Delta_K z(t))^* H (\Delta_K z(t))\, dt \\
&= -\int_{t_0}^{t_1} (\Delta_K z_+(t))^* H (\Delta_K z_+(t))\, dt - \inf_{\substack{z\in\mathfrak{B}_+(P)\\ z(t)=z_+(t),t\le t_1}} \int_{t_1}^{\infty} (\Delta_K z(t))^* H (\Delta_K z(t))\, dt \\
&= -\int_{t_0}^{t_1} (\Delta_K z_+(t))^* H (\Delta_K z_+(t))\, dt + \Theta_+(\Delta_K z_+(t_1))
\end{aligned}
$$

and also

$$
\begin{aligned}
&\Theta_-(\Delta_K z_-(t_1)) \\
&= \inf_{\substack{z\in\mathfrak{B}_-(P)\\ \Delta_K z(0)=\Delta_K z_-(t_1)}} \int_{-\infty}^{0} (\Delta_K z(t))^* H (\Delta_K z(t))\, dt \\
&= \inf_{\substack{z\in\mathfrak{B}_-(P)\\ \Delta_K z(t_1)=\Delta_K z_-(t_1)}} \int_{-\infty}^{t_1} (\Delta_K z(t))^* H (\Delta_K z(t))\, dt \\
&\le \inf_{\substack{z\in\mathfrak{B}_-(P)\\ z(t)=z_-(t),t\ge t_0}} \int_{-\infty}^{t_1} (\Delta_K z(t))^* H (\Delta_K z(t))\, dt \\
&= \inf_{\substack{z\in\mathfrak{B}_-(P)\\ z(t)=z_-(t),t\ge t_0}} \int_{-\infty}^{t_0} (\Delta_K z(t))^* H (\Delta_K z(t))\, dt + \int_{t_0}^{t_1} (\Delta_K z_-(t))^* H (\Delta_K z_-(t))\, dt \\
&= \int_{t_0}^{t_1} (\Delta_K z_-(t))^* H (\Delta_K z_-(t))\, dt + \inf_{\substack{z\in\mathfrak{B}_-(P)\\ z(t)=z_-(t),t\ge t_0}} \int_{-\infty}^{t_0} (\Delta_K z(t))^* H (\Delta_K z(t))\, dt \\
&= \int_{t_0}^{t_1} (\Delta_K z_-(t))^* H (\Delta_K z_-(t))\, dt + \Theta_-(\Delta_K z_-(t_0)),
\end{aligned}
$$

which is the assertion. □

The following proof is an adaption of the proof of [41, Theorem 2].

Theorem 3.20. *Let $P \in \mathbb{C}[\lambda]_K^{p,q}$ and $H = H^* \in \mathbb{C}^{qK,qK}$. Then the following statements hold:*

1. *If (P, H) is dissipative then (P, H) is cyclo-dissipative.*
2. *For every storage function Θ of (P, H), $\hat{z}_+ \in R_+(P)$, and $\hat{z}_- \in R_-(P)$ we have*

$$\Theta_+(\hat{z}_+) \le \Theta(\hat{z}_+) \text{ and } \Theta(\hat{z}_-) \le \Theta_-(\hat{z}_-).$$

3. *If P is stabilizable and (P, H) is dissipative then the available storage Θ_+ is a storage function.*

4. *If P is anti-stabilizable and (P,H) is dissipative then the required supply Θ_- is a storage function.*

Proof. First assume that Θ is a storage function and let $\hat{z} \in \mathfrak{B}_c(P)$ be arbitrary. Choose $R \in \mathbb{R}^+$ such that $\hat{z}(t) = 0$ for all $|t| \geq R$. Then we have

$$\begin{aligned} 0 &= \Theta(0) - \Theta(0) = \Theta(\Delta_K \hat{z}(R)) - \Theta(\Delta_K \hat{z}(-R)) \\ &\leq \int_{-R}^{R} (\Delta_K \hat{z}(t))^* H (\Delta_K \hat{z}(t))\, dt = \int_{-\infty}^{\infty} (\Delta_K \hat{z}(t))^* H (\Delta_K \hat{z}(t))\, dt, \end{aligned}$$

which means cyclo-dissipativity and 1. is shown. To show 2. let $z_+ \in \mathfrak{B}_+(P)$ be fixed. Then for every $z \in \mathfrak{B}_+(P)$ with $\Delta_K z_+(0) = \Delta_K z(0)$ we have from the definition of the storage function that

$$\begin{aligned} -\Theta(\Delta_K z_+(0)) &= \Theta(\lim_{s\to\infty} \Delta_K z(s)) - \Theta(\Delta_K z(0)) = \lim_{s\to\infty} \Theta(\Delta_K z(s)) - \Theta(\Delta_K z(0)) \\ &\leq \lim_{s\to\infty} \int_0^s (\Delta_K z(t))^* H (\Delta_K z(t))\, dt = \int_0^\infty (\Delta_K z(t))^* H (\Delta_K z(t))\, dt. \end{aligned}$$

Since $z \in \mathfrak{B}_+(P)$ was allowed to be arbitrary, this implies

$$-\Theta(\Delta_K z_+(0)) \leq \inf_{\substack{z\in\mathfrak{B}_+(P) \\ \Delta_K z(0) = \Delta_K z_+(0)}} \int_0^\infty (\Delta_K z(t))^* H (\Delta_K z(t))\, dt = -\Theta_+(\Delta_K z_+(t)).$$

Analogously, we obtain that for all $z_- \in \mathfrak{B}_-(P)$ we have $\Theta(\Delta_K z_-(t)) \leq \Theta_-(\Delta_K z_-(t))$ and 2. is shown.
For 3. assume that Θ is a storage function of (P,H). Then using the previously shown point 2. together with Remark 3.17 we see that we have $\Theta_+(\hat{z}_+) \in \mathbb{R}$ for all $\hat{z}_+ \in R_+(P)$. Using the assumption that P is stabilizable we conclude that $\Theta_+(\hat{z}) \in \mathbb{R}$ for all $\hat{z} \in R(P)$. Thus the inequalities from Lemma 3.19 can be transformed to match the dissipation inequality from Definition 3.1. The proof of 4. works analogously. □

The following corollary is similar to [45, Theorem 5.7].

Corollary 3.21. *Let $P \in \mathbb{C}[\lambda]_K^{p,q}$ be and $H = H^* \in \mathbb{C}^{qK,qK}$. Then the following statements hold:*

1. *For controllable P we have that (P,H) is dissipative if and only if (P,H) is cyclo-dissipative.*

2. *For every storage function Θ we have*

$$\Theta_+(\hat{z}) \leq \Theta(\hat{z}) \leq \Theta_-(\hat{z}),$$

for all $\hat{z} \in R_c(P)$, i.e., the available storage and the required supply constitute the extremal storage functions on the controllable part.

Proof. From Theorem 3.20 we conclude that dissipativity implies cyclo-dissipativity and also part 2. This only leaves to show that for controllable P cyclo-dissipativity implies dissipativity. Thus assume that P is controllable and (P, H) is cyclo-dissipative. Using Theorem 3.18 we deduce that $\Theta_+(\hat{z}) \leq \Theta_-(\hat{z})$ for all $\hat{z} \in R_c(P) = R(P)$. Together with Remark 3.17 this implies that both $\Theta_+(\hat{z})$ and $\Theta_-(\hat{z})$ are real numbers for all $\hat{z} \in R(P)$ which is why we deduce from Lemma 3.19 that both the available storage and the required supply constitute a storage function. $\square$

3.4 Linear matrix inequalities

In this section we will show that (under some additional assumptions) for first-order systems $P(\lambda) = \lambda F + G \in \mathbb{C}[\lambda]_1^{p,q}$ the solution of a certain linear matrix inequality (3.14) is equivalent to dissipativity. Using the canonical linearization (2.3) one can in principle generalize the results to higher order systems.

Theorem 3.22. *Consider $P(\lambda) = \lambda F + G \in \mathbb{C}[\lambda]_1^{p,q}$ together with $H = H^* \in \mathbb{C}^{q,q}$. Let $\lambda F + G$ be stabilizable (or anti-stabilizable). Then (P, H) is dissipative if and only if there exists a $Z \in \mathbb{C}^{p,q}$ such that $Z^*F = F^*Z$ and*

$$0 \leq \hat{z}^* \left[H + Z^*G + G^*Z\right] \hat{z}, \tag{3.13}$$

for all $\hat{z} \in R(P)$.

Proof. To show that (3.13) implies dissipativity, define the function $\Theta(\hat{z}) := \hat{z}^*F^*Z\hat{z}$. Let $z \in \mathfrak{B}(P)$. The assumption gives

$$\begin{aligned} \frac{d}{dt}\Theta(z(t)) &= \dot{z}^*(t)F^*Zz(t) + z^*(t)Z^*F\dot{z}(t) \\ &= z^*(t)\left[-G^*Z - Z^*G\right] z(t) \leq z^*(t)Hz(t) \end{aligned}$$

for all $t \in \mathbb{R}$, from which we conclude that

$$\Theta(z(t_1)) - \Theta(z(t_0)) = \int_{t_0}^{t_1} \frac{d}{dt}\Theta(z(t))dt \leq \int_{t_0}^{t_1} z^*(t)Hz(t)dt,$$

and, thus, that Θ is really a storage function. To show that dissipativity implies the existence of a $Z \in \mathbb{C}^{p,q}$ such that (3.13) holds, assume that the system is stabilizable. Observe that from point 3. of Theorem 3.20 we obtain that the available storage is a storage function. Using Theorem A.24 we deduce the existence of a $Z \in \mathbb{C}^{p,q}$ such that $Z^*F = F^*Z$ and such that the available storage fulfills $\Theta_+(\hat{z}) := \hat{z}F^*Z\hat{z}$ for all $\hat{z} \in R_+(P) = R(P)$. Dividing the dissipation inequality for Θ_+ by $t_1 - t_0$, letting $t_1 \to t_0$, and using the mean value theorem we find

$$z^*(t)\left[-G^*Z - Z^*G\right] z(t) = \frac{d}{dt}\Theta_+(z(t)) \leq z^*(t)Hz(t)$$

for all $z \in \mathfrak{B}(P)$ and, thus, the assertion holds. For anti-stabilizable systems one uses the required supply instead of the available storage. $\square$

In this subsection we show that under certain conditions the inequality (3.13) is equivalent to a linear matrix inequality which not only holds on the subspace $R(P)$.

Definition 3.23. Consider $P(\lambda) = \lambda F + G \in \mathbb{C}[\lambda]_1^{p,q}$ together with $H = H^* \in \mathbb{C}^{q,q}$. Then we call the system of equations

$$\begin{array}{rcl} F^* Z & = & Z^* F \\ 0 & \leq & H + Z^* G + G^* Z \end{array} \tag{3.14}$$

the *linear matrix inequality derived from* (P, H) (where $Z \in \mathbb{C}^{p,q}$ is the unknown).

For a notion of linear matrix inequalities, which is more general than Definition 3.23, see [6]. From Theorem 3.22 it is clear that if the linear matrix inequality (3.14) has a solution the system is dissipative. The reverse is not always true as one can see from the following example.

Example 3.24. With $z(t) = \begin{bmatrix} z_1(t) & z_2(t) \end{bmatrix}^T$ consider the system

$$\underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{=:F} \dot{z}(t) + \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{=:G} z(t) = 0.$$

Define $P(\lambda) := \lambda F + G$. Clearly, $\mathfrak{B}(P) = \mathfrak{B}_c(P) = \{0\}$ and thus the system is dissipative with respect to any $H = H^* \in \mathbb{C}^{2,2}$ by definition. However, we cannot find $Z \in \mathbb{C}^{2,2}$ such that $F^* Z = Z^* F$ and $0 \leq G^* Z + Z^* G + H$ for any given $H = H^* \in \mathbb{C}^{2,2}$ as we will see in the following.
Using the notation $Z = [z_{ij}]$ we find that the identity $F^* Z = Z^* F$ is equivalent to

$$\begin{bmatrix} 0 & 0 \\ z_{11} & z_{12} \end{bmatrix} = \begin{bmatrix} 0 & \overline{z_{11}} \\ 0 & \overline{z_{12}} \end{bmatrix}.$$

Thus, Z has to take the form

$$Z = \begin{bmatrix} 0 & r \\ z_{21} & z_{22} \end{bmatrix},$$

where $r \in \mathbb{R}$ and $z_{21}, z_{22} \in \mathbb{C}$ are allowed to be arbitrary. Using the notation $H = [h_{ij}]$ we can rewrite the inequality $0 \leq G^* Z + Z^* G + H$ through

$$0 \leq \begin{bmatrix} 0 & r + \overline{z_{21}} \\ r + z_{21} & z_{22} + \overline{z_{22}} \end{bmatrix} + \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix},$$

which will never be possible if $h_{11} < 0$.

The problem with Example 3.24 is that the pencil $\lambda F + G$ corresponds to a block of type (2.7) and size $\sigma_j = 2$ in the Kronecker canonical form. Thus it seems acceptable to only consider pencils for which the Kronecker canonical form only has blocks of type (2.7) with size $\sigma_j < 2$. Also, we will assume that all blocks of type (2.8) have size $\eta_j \leq 2$.

Theorem 3.25. *Let the system $P(\lambda) = \lambda F + G \in \mathbb{C}[\lambda]_1^{p,q}$ be stabilizable (or anti-stabilizable). Let the Kronecker canonical form of $\lambda F + G$ be given by (2.4) and assume that all blocks of type (2.7) have sizes $\sigma_j < 2$ and all blocks of type (2.8) have sizes $\eta_j \leq 2$. Let $H = H^* \in \mathbb{C}^{q,q}$ and let (P, H) be dissipative. Then there exists a solution $Z \in \mathbb{C}^{p,q}$ of the linear matrix inequality (3.14) associated with (P, H).*

Proof. Using Theorem 3.22 and setting $\tilde{z} := T\hat{z}$ we see that the assumptions imply the existence of a $\hat{Z} \in \mathbb{C}^{p,q}$ such that $F^*\hat{Z} = \hat{Z}^*F$ and

$$\begin{aligned}
0 &\leq \hat{z}^*(H + G^*\hat{Z} + \hat{Z}^*G)\hat{z} \\
&= \hat{z}^*T^*T^{-*}(H + G^*S^{-*}S^*\hat{Z} + \hat{Z}^*SS^{-1}G)T^{-1}T\hat{z} \\
&= \tilde{z}^*(\tilde{H} + \tilde{G}^*\tilde{Z} + \tilde{Z}^*\tilde{G})\tilde{z},
\end{aligned}$$

for all $\hat{z} \in R(P)$, where we introduced the symbols $\tilde{H} := T^{-*}HT^{-1}$, $\tilde{Z} = S^*\hat{Z}T^{-1}$, and $\tilde{G} := S^{-1}GT^{-1}$. Also we introduce $\tilde{F} = S^{-1}FT^{-1}$ and observe that the Kronecker canonical from (2.4) implies that $\tilde{F} = \operatorname{diag}(F_{\mathcal{L}}, I, 0, F_{\mathcal{M}})$ and $\tilde{G} = \operatorname{diag}(G_{\mathcal{L}}, G_{\mathcal{J}}, I, G_{\mathcal{M}})$, where $\lambda F_{\mathcal{L}} + G_{\mathcal{L}}$ contains all blocks of type (2.5), $\lambda F_{\mathcal{M}} + G_{\mathcal{M}}$ contains all blocks of type (2.8), and $\lambda I + G_{\mathcal{J}}$ contains all blocks of type (2.6). Partition $\tilde{Z} = [Z_{i,j}]_{i,j=1,\dots,4}$, $H = [H_{i,j}]_{i,j=1,\dots,4}$, and $\tilde{z} = [\tilde{z}_i]_{i=1,\dots,4}$ with $\tilde{z}_1 \in \mathbb{C}^{\epsilon+s}$, $\tilde{z}_2 \in \mathbb{C}^{\rho}$, $\tilde{z}_3 \in \mathbb{C}^{\sigma}$, and $\tilde{z}_4 \in \mathbb{C}^{\eta}$ according to the block diagonal structure of $S^{-1}(\lambda F + G)T^{-1} = \operatorname{diag}(\mathcal{L}, \mathcal{J}, \mathcal{N}, \mathcal{M})$. Then we deduce that

$$\begin{bmatrix} F_{\mathcal{L}}^*Z_{11} & F_{\mathcal{L}}^*Z_{12} & F_{\mathcal{L}}^*Z_{13} & F_{\mathcal{L}}^*Z_{14} \\ Z_{21} & Z_{22} & Z_{23} & Z_{24} \\ 0 & 0 & 0 & 0 \\ F_{\mathcal{M}}^*Z_{41} & F_{\mathcal{M}}^*Z_{42} & F_{\mathcal{M}}^*Z_{43} & F_{\mathcal{M}}^*Z_{44} \end{bmatrix} = \tilde{F}^*\tilde{Z} \;=\; \tilde{Z}^*\tilde{F} = \begin{bmatrix} Z_{11}^*F_{\mathcal{L}} & Z_{21}^* & 0 & Z_{41}^*F_{\mathcal{M}} \\ Z_{12}^*F_{\mathcal{L}} & Z_{22}^* & 0 & Z_{42}^*F_{\mathcal{M}} \\ Z_{13}^*F_{\mathcal{L}} & Z_{23}^* & 0 & Z_{43}^*F_{\mathcal{M}} \\ Z_{14}^*F_{\mathcal{L}} & Z_{24}^* & 0 & Z_{44}^*F_{\mathcal{M}} \end{bmatrix}.$$

and that

$$\begin{aligned}
0 \leq \begin{bmatrix} \tilde{z}_1 \\ \tilde{z}_2 \\ \tilde{z}_3 \\ \tilde{z}_4 \end{bmatrix}^* &\left(\begin{bmatrix} H_{11} & H_{12} & H_{13} & H_{14} \\ H_{12}^* & H_{22} & H_{23} & H_{24} \\ H_{13}^* & H_{23}^* & H_{33} & H_{34} \\ H_{14}^* & H_{24}^* & H_{34}^* & H_{44} \end{bmatrix} + \right. \\
+ &\left. \begin{bmatrix} G_{\mathcal{L}}^*Z_{11} & G_{\mathcal{L}}^*Z_{12} & G_{\mathcal{L}}^*Z_{13} & G_{\mathcal{L}}^*Z_{14} \\ G_{\mathcal{J}}^*Z_{21} & G_{\mathcal{J}}^*Z_{22} & G_{\mathcal{J}}^*Z_{23} & G_{\mathcal{J}}^*Z_{24} \\ Z_{31} & Z_{32} & Z_{33} & Z_{34} \\ G_{\mathcal{M}}^*Z_{41} & G_{\mathcal{M}}^*Z_{42} & G_{\mathcal{M}}^*Z_{43} & G_{\mathcal{M}}^*Z_{44} \end{bmatrix} + \begin{bmatrix} Z_{11}^*G_{\mathcal{L}} & Z_{21}^*G_{\mathcal{J}} & Z_{31}^* & Z_{41}^*G_{\mathcal{M}} \\ Z_{12}^*G_{\mathcal{L}} & Z_{22}^*G_{\mathcal{J}} & Z_{32}^* & Z_{42}^*G_{\mathcal{M}} \\ Z_{13}^*G_{\mathcal{L}} & Z_{23}^*G_{\mathcal{J}} & Z_{33}^* & Z_{43}^*G_{\mathcal{M}} \\ Z_{14}^*G_{\mathcal{L}} & Z_{24}^*G_{\mathcal{J}} & Z_{34}^* & Z_{44}^*G_{\mathcal{M}} \end{bmatrix} \right) \begin{bmatrix} \tilde{z}_1 \\ \tilde{z}_2 \\ \tilde{z}_3 \\ \tilde{z}_4 \end{bmatrix},
\end{aligned}$$

for all $\tilde{z} = T\hat{z}$ with $\hat{z} \in R(P)$. Due to Lemma 2.19 we know that

$$R(P) = \left\{ T^{-1} \begin{bmatrix} \tilde{z}_1 \\ \tilde{z}_2 \\ 0_{\sigma+\eta} \end{bmatrix} \middle| \tilde{z}_1 \in \mathbb{C}^{\epsilon+s}, \tilde{z}_2 \in \mathbb{C}^{\rho} \right\}$$

and thus we conclude that

$$0 \;\leq\; \tilde{z}^* \left(\tilde{H} + \tilde{G}^*\tilde{Z} + \tilde{Z}^*\tilde{G} \right) \tilde{z}$$

$$= \begin{bmatrix} \tilde{z}_1 \\ \tilde{z}_2 \end{bmatrix}^* \begin{bmatrix} H_{11} + G_{\mathcal{L}}^* Z_{11} + Z_{11}^* G_{\mathcal{L}} & H_{12} + G_{\mathcal{L}}^* Z_{12} + Z_{21}^* G_{\mathcal{J}} \\ H_{12}^* + G_{\mathcal{J}}^* Z_{21} + Z_{12}^* G_{\mathcal{L}} & H_{22} + G_{\mathcal{J}}^* Z_{22} + Z_{22}^* G_{\mathcal{J}} \end{bmatrix} \begin{bmatrix} \tilde{z}_1 \\ \tilde{z}_2 \end{bmatrix}$$

for arbitrary $\tilde{z}_1$, $\tilde{z}_2$. This means that there exist Z_{11}, Z_{12}, Z_{21}, and Z_{22} such that

$$\begin{bmatrix} F_{\mathcal{L}}^* Z_{11} & F_{\mathcal{L}}^* Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} = \begin{bmatrix} Z_{11}^* F_{\mathcal{L}} & Z_{21}^* \\ Z_{12}^* F_{\mathcal{L}} & Z_{22}^* \end{bmatrix},$$

and

$$0 \leq \begin{bmatrix} H_{11} + G_{\mathcal{L}}^* Z_{11} + Z_{11}^* G_{\mathcal{L}} & H_{12} + G_{\mathcal{L}}^* Z_{12} + Z_{21}^* G_{\mathcal{J}} \\ H_{12}^* + G_{\mathcal{J}}^* Z_{21} + Z_{12}^* G_{\mathcal{L}} & H_{22} + G_{\mathcal{J}}^* Z_{22} + Z_{22}^* G_{\mathcal{J}} \end{bmatrix}.$$

We set $Z_{33} := -\frac{1}{2} H_{33}$, $Z_{43} := 0$, $Z_{34} := -H_{34}$, $Z_{23} := 0$, $Z_{32} := -H_{23}^*$, $Z_{13} := 0$, and $Z_{31} := -H_{13}^*$. We use Lemma A.27 to construct a matrix Z_{44} such that $Z_{44}^* F_{\mathcal{M}} = F_{\mathcal{M}}^* Z_{44}$ and $0 = H_{44} + Z_{44}^* G_{\mathcal{M}} + G_{\mathcal{M}}^* Z_{44}$. We use Lemma A.28 to construct matrices Z_{14} and Z_{41} such that $F_{\mathcal{L}}^* Z_{14} = Z_{41}^* F_{\mathcal{M}}$ and $0 = H_{14} + G_{\mathcal{L}}^* Z_{14} + Z_{41}^* G_{\mathcal{M}}$. We use Lemma A.29 to construct matrices Z_{24} and Z_{42} such that $Z_{24} = Z_{42}^* F_{\mathcal{M}}$ and $0 = H_{24} + G_{\mathcal{J}}^* Z_{24} + Z_{42}^* G_{\mathcal{M}}$. Thus, all in all we constructed a matrix $\tilde{Z}$ such that $\tilde{Z}^* F = F^* \tilde{Z}$ and

$$\begin{aligned}
&\tilde{H} + \tilde{G}^* \tilde{Z} + \tilde{Z}^* \tilde{G} \\
= &\left(\begin{bmatrix} H_{11} & H_{12} & H_{13} & H_{14} \\ H_{12}^* & H_{22} & H_{23} & H_{24} \\ H_{13}^* & H_{23}^* & H_{33} & H_{34} \\ H_{14}^* & H_{24}^* & H_{34}^* & H_{44} \end{bmatrix} + \begin{bmatrix} G_{\mathcal{L}}^* Z_{11} & G_{\mathcal{L}}^* Z_{12} & G_{\mathcal{L}}^* Z_{13} & G_{\mathcal{L}}^* Z_{14} \\ G_{\mathcal{J}}^* Z_{21} & G_{\mathcal{J}}^* Z_{22} & G_{\mathcal{J}}^* Z_{23} & G_{\mathcal{J}}^* Z_{24} \\ Z_{31} & Z_{32} & Z_{33} & Z_{34} \\ G_{\mathcal{M}}^* Z_{41} & G_{\mathcal{M}}^* Z_{42} & G_{\mathcal{M}}^* Z_{43} & G_{\mathcal{M}}^* Z_{44} \end{bmatrix} \right. \\
+ &\left. \begin{bmatrix} Z_{11}^* G_{\mathcal{L}} & Z_{21}^* G_{\mathcal{J}} & Z_{31}^* & Z_{41}^* G_{\mathcal{M}} \\ Z_{12}^* G_{\mathcal{L}} & Z_{22}^* G_{\mathcal{J}} & Z_{32}^* & Z_{42}^* G_{\mathcal{M}} \\ Z_{13}^* G_{\mathcal{L}} & Z_{23}^* G_{\mathcal{J}} & Z_{33}^* & Z_{43}^* G_{\mathcal{M}} \\ Z_{14}^* G_{\mathcal{L}} & Z_{24}^* G_{\mathcal{J}} & Z_{34}^* & Z_{44}^* G_{\mathcal{M}} \end{bmatrix} \right) \\
= &\begin{bmatrix} H_{11} + G_{\mathcal{L}}^* Z_{11} + Z_{11}^* G_{\mathcal{L}} & H_{12} + G_{\mathcal{L}}^* Z_{12} + Z_{21}^* G_{\mathcal{J}} & 0 & 0 \\ H_{21} + G_{\mathcal{J}}^* Z_{21} + Z_{12}^* G_{\mathcal{L}} & H_{22} + G_{\mathcal{J}}^* Z_{22} + Z_{22}^* G_{\mathcal{J}} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \geq 0,
\end{aligned}$$

which is why we set $Z := S^{-*} \tilde{Z} T$ and the claim is shown. □

Corollary 3.26 (Kalman-Yakubovich-Popov Lemma). *Let $P(\lambda) = \lambda F + G \in \mathbb{C}[\lambda]_1^{p,q}$ be stabilizable (or anti-stabilizable) and let $H = H^* \in \mathbb{C}^{q,q}$. Let the Kronecker canonical form of $\lambda F + G$ be given by (2.4) and assume that all blocks of type (2.7) have sizes $\sigma_j < 2$ and all blocks of type (2.8) have sizes $\eta_j \leq 2$.*
Then (P, H) is dissipative if and only if there exists a solution of the linear matrix inequality (3.14) associated with (P, H).

Proof. That dissipativity implies solvability of the linear matrix inequality is the statement of Theorem 3.25. For the other direction assume that Z solves (3.14). Then also condition (3.13) is fulfilled and dissipativity follows from Theorem 3.22. □

3.4.1 Spectral factorization of Popov functions

In this subsection we use the results obtained in the previous section to explicitly give spectral factorizations of Popov functions of dissipative systems.

Definition 3.27. Let $\Pi = \Pi^{\sim} \in \mathbb{C}(\lambda)^{n,n}$ be a para-Hermitian matrix. If there exists a $m \in \mathbb{N}$ and a $K \in \mathbb{C}(\lambda)^{m,n}$ such that

$$\Pi = K^{\sim}K,$$

we say that K is a *spectral factor of* Π. Also we refer to the product $K^{\sim}K$ as a *spectral factorization of* Π.

The following theorem and corollary are restatements of [36, Theorem 5.3] in our notation.

Theorem 3.28. *Let $P(\lambda) = \lambda F + G \in \mathbb{C}[\lambda]_1^{p,q}$ and $H = H^* \in \mathbb{C}^{q,q}$. Set $r := \operatorname{rank}_{\mathbb{C}(\lambda)}(P)$ and let $U \in \mathbb{C}(\lambda)^{q,q-r}$ be a kernel matrix of P. Consider the associated Popov function $\Pi := U^{\sim}HU$. Then we have the following:*

1. *If there exists a solution $Z \in \mathbb{C}^{p,q}$ of the linear matrix inequality (3.14) associated with (P, H) and the matrix L is a Cholesky factor of*

$$0 \leq G^*Z + Z^*G + H = L^*L,$$

   then $K(\lambda) := LU(\lambda)$ is a spectral factor of $\Pi(\lambda)$.

2. *If on the other hand P is stabilizable (or anti-stabilizable), in the Kronecker canonical form (2.4) all blocks of type (2.7) have sizes $\sigma_j < 2$ and all blocks of type (2.8) have sizes $\eta_j \leq 2$, and there exists a spectral factor of Π then the linear matrix inequality (3.14) associated with (P, H) has a solution $Z \in \mathbb{C}^{p,q}$.*

Proof. [36, Theorem 5.3] For 1. note that we have

$$\begin{aligned} & K^{\sim}(\lambda)K(\lambda) = U^{\sim}(\lambda)L^*LU(\lambda) = U^{\sim}(\lambda)\left[G^*Z + Z^*G + H\right]U(\lambda) \\ = \;& \Pi(\lambda) + U^*(-\overline{\lambda})G^*ZU(\lambda) + U^{\sim}(\lambda)Z^*GU(\lambda) \\ = \;& \Pi(\lambda) + \left(GU(-\overline{\lambda})\right)^* ZU(\lambda) + U^{\sim}(\lambda)Z^*\left(GU(\lambda)\right) \\ = \;& \Pi(\lambda) + \left(\overline{\lambda}FU(-\overline{\lambda})\right)^* ZU(\lambda) + U^{\sim}(\lambda)Z^*\left(-\lambda FU(\lambda)\right) \\ = \;& \Pi(\lambda) + \lambda U^{\sim}(\lambda)F^*ZU(\lambda) - \lambda U^{\sim}(\lambda)Z^*FU(\lambda) = \Pi(\lambda), \end{aligned}$$

since $(\lambda F + G)U(\lambda) = 0$ implies $GU(\lambda) = -\lambda FU(\lambda)$ and also $GU(-\overline{\lambda}) = \overline{\lambda}FU(-\overline{\lambda})$. For part 2. note that for all $\omega \in \mathbb{R}$ with $i\omega \in \mathfrak{D}(K) \cap \mathfrak{D}(U)$ we have

$$\Pi(i\omega) = K^{\sim}(i\omega)K(i\omega) = K^*(-\overline{i\omega})K(i\omega) = K^*(i\omega)K(i\omega) \geq 0.$$

Using the continuity of Π this implies that also $\Pi(i\omega) \geq 0$ for all $\omega \in \mathbb{R}$ such that $i\omega \in \mathfrak{D}(U)$. Using Theorem 3.5 this proves that (P, H) is cyclo-dissipative. Using Corollary 3.21 we find that (P, H) is dissipative. Using Theorem 3.25 and the additional assumptions we deduce the existence of a solution of the linear matrix inequality. $\square$

Corollary 3.29. *Let the system $P(\lambda) = \lambda F + G \in \mathbb{C}[\lambda]_1^{p,q}$ be stabilizable (or anti-stabilizable). Let the Kronecker canonical form of $\lambda F + G$ be given by (2.4) and assume that all blocks of type (2.7) have sizes $\sigma_j < 2$ and all blocks of type (2.8) have sizes $\eta_j \leq 2$. Let $H = H^* \in \mathbb{C}^{q,q}$. Set $r := \operatorname{rank}_{\mathbb{C}(\lambda)}(P)$ and let $U \in \mathbb{C}(\lambda)^{q,q-r}$ be a kernel matrix of P.*
Then there exists a solution of the linear matrix inequality (3.14) associated with (P, H) if and only if there exists a spectral factor of the Popov function $\Pi := U^{\sim} H U$.

Proof. [36, Theorem 5.3] The claim follows directly by combining points 1. and 2. of the previous Theorem 3.28. □

Corollary 3.29 is related to the Youla factorization. One can show that for every para-Hermitian rational function which is positive semi-definite along the imaginary axis there exists a spectral factorization [46, Theorem 2]. Also, for every para-Hermitian polynomial function there exists a polynomial spectral factor [46, Corollary 2].

Chapter 4

Applications

We use the results from the previous chapter for several applications. We start by applying the results to state-space systems in Section 4.1 to obtain some well known results. After that an algorithm to check cyclo-dissipativity (in Section 4.2) and an algorithm to enforce cyclo-dissipativity of systems which are close to cyclo-dissipative (in Section 4.3) will be proposed. Since the enforcement algorithm is only a heuristic method, we will apply it to several systems to see how it performs.

4.1 Application to descriptor systems

In this section we interpret the results from Chapter 3 in terms of state-space descriptor systems with an output equation to obtain a number of well known results. State-space descriptor systems with an output equation take the form

$$\begin{array}{rcl} E\dot{x}(t) & = & Ax(t) + Bu(t), \\ y(t) & = & Cx(t) + Du(t), \end{array} \tag{4.1}$$

where $E, A \in \mathbb{C}^{\rho,n}$, $B \in \mathbb{C}^{\rho,m}$, $C \in \mathbb{C}^{l,n}$, $D \in \mathbb{C}^{l,m}$, $x \in \mathcal{C}_\infty^n$ is called the *state*, $u \in \mathcal{C}_\infty^m$ is called the *input*, and $y \in \mathcal{C}_\infty^l$ is called the *output*. In the literature, see e.g. [3, Section 5.9], for such systems the supply is frequently measured in the form

$$\begin{bmatrix} y \\ u \end{bmatrix}^* \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \begin{bmatrix} y \\ u \end{bmatrix}, \tag{4.2}$$

where $Q = Q^* \in \mathbb{C}^{l,l}$, $S \in \mathbb{C}^{l,m}$, and $R = R^* \in \mathbb{C}^{m,m}$. Using the equation for y in (4.1) we can rewrite the supply (4.2) to depend on the state variables (instead of the output variables) by

$$\begin{aligned} \begin{bmatrix} y \\ u \end{bmatrix}^* \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \begin{bmatrix} y \\ u \end{bmatrix} &= \begin{bmatrix} x \\ u \end{bmatrix}^* \begin{bmatrix} C^* & 0 \\ D^* & I \end{bmatrix} \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \begin{bmatrix} C & D \\ 0 & I \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} \\ = \quad \begin{bmatrix} x \\ u \end{bmatrix}^* &\begin{bmatrix} C^*QC & C^*QD + C^*S \\ D^*QC + S^*C & D^*QD + D^*S + S^*D + R \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} =: \begin{bmatrix} x \\ u \end{bmatrix}^* \begin{bmatrix} \tilde{Q} & \tilde{S} \\ \tilde{S}^* & \tilde{R} \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} \end{aligned}$$

where $\tilde{Q} = \tilde{Q}^* \in \mathbb{C}^{n,n}$, $\tilde{S} \in \mathbb{C}^{n,m}$, $\tilde{R} = \tilde{R}^* \in \mathbb{C}^{m,m}$. Introducing the notation $q_1 := n + m$, $p_1 := \rho$

$$F_1 := \begin{bmatrix} E & 0 \end{bmatrix}, \quad G_1 := \begin{bmatrix} -A & -B \end{bmatrix}, \quad P_1(\lambda) := \lambda F_1 + G_1, \quad H_1 := \begin{bmatrix} \tilde{Q} & \tilde{S} \\ \tilde{S}^* & \tilde{R} \end{bmatrix}, \tag{4.3}$$

and $z := \begin{bmatrix} x^T & u^T \end{bmatrix}^T$ we see that $F_1, G_1 \in \mathbb{C}^{p_1,q_1}$, $P_1 \in \mathbb{C}[\lambda]^{p_1,q_1}$, $H_1 = H_1^* \in \mathbb{C}^{q_1,q_1}$, and $z \in \mathcal{C}_\infty^{q_1}$ and we can rewrite the first equation of (4.1) as the behavioral system $P_1\left(\frac{d}{dt}\right) z = 0$ and the supply as $z^* H_1 z$.
Assuming that the pencil $\lambda E - A$ is invertible over $\mathbb{C}(\lambda)$, we can give an explicit representation of a kernel and a co-kernel matrix.

Lemma 4.1. *Let P_1 be defined through (4.3) and assume that the pencil $\lambda E - A$ is invertible over $\mathbb{C}(\lambda)$. Then we have $\operatorname{rank}_{\mathbb{C}(\lambda)}(P_1) = n$ and the matrices $U_1 \in \mathbb{C}(\lambda)^{q,m}$ and $V_1 \in \mathbb{C}[\lambda]^{q,n}$ given by*

$$U_1(\lambda) := \begin{bmatrix} (\lambda E - A)^{-1} B \\ I_m \end{bmatrix} \text{ and } V_1(\lambda) := \begin{bmatrix} I_n \\ 0 \end{bmatrix}, \tag{4.4}$$

constitute a kernel and a co-kernel matrix of P_1 with $\mathfrak{D}(U_1) \subset \mathbb{C} \setminus \mathfrak{Z}(\lambda E - A)$.

Proof. Invertibility of $\lambda E - A$ implies that $\rho = n$. Since $P_1(\lambda) = \lambda F_1 + G_1 = \begin{bmatrix} \lambda E - A & -B \end{bmatrix}$ and $\lambda E - A$ is assumed to be invertible, we have $\operatorname{rank}_{\mathbb{C}(\lambda)}(P_1) = n$ and thus U_1 and V_1 are matrices of proper dimension according to Definition 2.8. We still have to show the three properties from Definition 2.8. To see property 1. observe that

$$P_1 U = \begin{bmatrix} \lambda E - A & -B \end{bmatrix} \begin{bmatrix} (\lambda E - A)^{-1} B \\ I_m \end{bmatrix} = B - B = 0.$$

For 2. notice that

$$\begin{aligned} \operatorname{rank}_{\mathbb{C}(\lambda)}(P_1 V) &= \operatorname{rank}_{\mathbb{C}(\lambda)}\left(\begin{bmatrix} \lambda E - A & -B \end{bmatrix} \begin{bmatrix} I_n \\ 0 \end{bmatrix} \right) = \operatorname{rank}_{\mathbb{C}(\lambda)}(\lambda E - A) \\ &= n = \operatorname{rank}_{\mathbb{C}(\lambda)}(P_1). \end{aligned}$$

Finally, since we have

$$\det \begin{bmatrix} U_1 & V_1 \end{bmatrix} = \det \begin{bmatrix} (\lambda E - A)^{-1} B & I_n \\ I_m & 0 \end{bmatrix} = \pm 1,$$

we conclude that $\begin{bmatrix} U_1 & V_1 \end{bmatrix}$ is invertible. □

Another way to write system (4.1) with supply (4.2) in behavioral form is by introducing the notation $q_2 := l + n + m$, $p_2 := \rho + l$

$$F_2 := \begin{bmatrix} 0 & E & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad G_2 := \begin{bmatrix} 0 & -A & -B \\ I & -C & -D \end{bmatrix}, \quad P_2 := \lambda F_2 + G_2, \quad H_2 := \begin{bmatrix} Q & 0 & S \\ 0 & 0 & 0 \\ S^* & 0 & R \end{bmatrix}, \tag{4.5}$$

and $z := \begin{bmatrix} y^T & x^T & u^T \end{bmatrix}^T$ to see that $F_2, G_2 \in \mathbb{C}^{p_2,q_2}$, $P_2 \in \mathbb{C}[\lambda]^{p_2,q_2}$, $H_2 = H_2^* \in \mathbb{C}^{q_2,q_2}$, and $z \in \mathcal{C}_\infty^{q_2}$. We again find that we can rewrite (4.1) as the behavioral system $P_2\left(\frac{d}{dt}\right) z = 0$ and the supply as $z^* H_2 z$. For this kind of system (with invertible $\lambda E - A$) a kernel and a co-kernel matrix are given by

$$U_2(\lambda) := \begin{bmatrix} C(\lambda E - A)^{-1}B + D \\ (\lambda E - A)^{-1}B \\ I_m \end{bmatrix} \text{ and } V_2(\lambda) := \begin{bmatrix} I_l & 0 \\ 0 & I_n \\ 0 & 0 \end{bmatrix}, \tag{4.6}$$

as one can show with almost the same proof which was used in Lemma 4.1.
With these particular kernel matrices (4.4) or (4.6) we can also give the more well known explicit representation of a particular Popov function as

$$\begin{aligned} \Pi(\lambda) &= U_1^\sim H_1 U_1 = U_2^\sim H_2 U_2 \\ &= \begin{bmatrix} (-\overline{\lambda}E - A)^{-1}B \\ I_m \end{bmatrix}^* \begin{bmatrix} \tilde{Q} & \tilde{S} \\ \tilde{S}^* & \tilde{R} \end{bmatrix} \begin{bmatrix} (\lambda E - A)^{-1}B \\ I_m \end{bmatrix} \\ &= \begin{bmatrix} C(-\overline{\lambda}E - A)^{-1}B + D \\ I_m \end{bmatrix}^* \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \begin{bmatrix} C(\lambda E - A)^{-1}B + D \\ I_m \end{bmatrix} \\ &= B^*(-\lambda E^* - A^*)^{-1}\tilde{Q}(\lambda E - A)^{-1}B + B^*(-\lambda E^* - A^*)^{-1}\tilde{S} \\ &\quad + \tilde{S}^*(\lambda E - A)^{-1}B + \tilde{R} \end{aligned} \tag{4.7}$$

depending on which representation one prefers. The following corollary sums up the results of Section 3.1 when considered with respect to regular state-space systems.

Corollary 4.2. *Consider the system (4.1) together with the supply (4.2). Let $\lambda E - A$ be invertible. Introduce the notation from (4.3) and (4.5). Then the following are equivalent:*

1. *(P_1, H_1) is cyclo-dissipative.*

2. *(P_2, H_2) is cyclo-dissipative.*

3. *The Popov function (4.7) fulfills $\Pi(i\omega) \geq 0$ for all $\omega \in \mathbb{R}$ such that $i\omega \notin \mathfrak{Z}(\lambda E - A)$.*

4. *For all $\omega \in \mathbb{R}$ such that $i\omega \notin \mathfrak{Z}(P_1)$ the sign-sum function fulfills*

$$\eta\left(i\omega \begin{bmatrix} 0 & E & 0 \\ -E^* & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & -A & -B \\ -A^* & \tilde{Q} & \tilde{S} \\ -B^* & \tilde{S}^* & \tilde{R} \end{bmatrix}\right) = m.$$

5. *For all $\omega \in \mathbb{R}$ such that $i\omega \notin \mathfrak{Z}(P_2)$ the sign-sum function fulfills*

$$\eta\left(i\omega \begin{bmatrix} 0 & 0 & 0 & E & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -E^* & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & -A & -B \\ 0 & 0 & I & -C & -D \\ 0 & I & Q & 0 & S \\ -A^* & -C^* & 0 & 0 & 0 \\ -B^* & -D^* & S^* & 0 & R \end{bmatrix}\right) = m.$$

Proof. Equivalence of 1. and 2. follows form elementary considerations. From Theorem 3.5 we deduce the equivalence of 1. and 3. Finally, using

$$p_1 + q_1 - 2 \cdot \operatorname{rank}_{\mathbb{C}(\lambda)}(P_1) = n + (n+m) - 2n = m,$$

we deduce the equivalence of 1. and 4. and using

$$p_2 + q_2 - 2 \cdot \operatorname{rank}_{\mathbb{C}(\lambda)}(P_2) = (n+l) + (l+n+m) - 2(n+l) = m,$$

we deduce the equivalence of 2. and 5. from Theorem 3.11. □

In Corollary 4.2 we used two different kinds of para-Hermitian matrix pencils, namely the one from point 4. and the one from point 5. The advantage of the pencil from 5. is that it can be constructed from the original system (4.1) and supply (4.2) without any matrix multiplications or matrix additions. However, to construct the pencil from point 4. multiple matrix multiplications and matrix additions have to be performed in general. On the other hand, an advantage of the pencil from point 4. is that it is smaller in size than the pencil from point 5., although this difference in the dimension in most cases will not be significant, since the number of output equations p is usually small.
The following corollary sums up the results from Section 3.2 when considered with respect to state-space systems. We only state the results for the positive time-axis problem although a similar statement can be written down for the negative time-axis problem.

Corollary 4.3. *Consider the system (4.1) together with the supply (4.2). Introduce the notation from (4.3) and from (4.5). Let $\hat{z} = (\hat{x}, \hat{u}) \in \mathfrak{B}_+(P_1)$ and define $\hat{y} := C\hat{x} + D\hat{u}$ so that $(\hat{y}, \hat{z}) \in \mathfrak{B}_+(P_2)$. Then the following are equivalent:*

1. *(P_1, H_1) is cyclo-dissipative and there exists a $\hat{\mu} \in \mathcal{C}_+^n$ such that*

$$\begin{bmatrix} 0 & E & 0 \\ -E^* & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\hat{\mu}}(t) \\ \dot{\hat{x}}(t) \\ \dot{\hat{u}}(t) \end{bmatrix} = \begin{bmatrix} 0 & A & B \\ A^* & -\tilde{Q} & -\tilde{S} \\ B^* & -\tilde{S}^* & -\tilde{R} \end{bmatrix} \begin{bmatrix} \hat{\mu}(t) \\ \hat{x}(t) \\ \hat{u}(t) \end{bmatrix}.$$

2. *(P_2, H_2) is cyclo-dissipative and there exist $\hat{\mu} \in \mathcal{C}_+^n$ and $\hat{\nu} \in \mathcal{C}_+^l$ such that*

$$\begin{bmatrix} 0 & 0 & 0 & E & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -E^* & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\hat{\mu}}(t) \\ \dot{\hat{\nu}}(t) \\ \dot{\hat{y}}(t) \\ \dot{\hat{x}}(t) \\ \dot{\hat{u}}(t) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & A & B \\ 0 & 0 & -I & C & D \\ 0 & -I & -Q & 0 & -S \\ A^* & C^* & 0 & 0 & 0 \\ B^* & D^* & -S^* & 0 & -R \end{bmatrix} \begin{bmatrix} \hat{\mu}(t) \\ \hat{\nu}(t) \\ \hat{y}(t) \\ \hat{x}(t) \\ \hat{u}(t) \end{bmatrix}.$$

3. *$\hat{z}$ solves the optimal control problem on the positive time-axis, i.e.,*

$$\int_{t_0}^{\infty} \begin{bmatrix} \hat{y}(t) \\ \hat{u}(t) \end{bmatrix}^* \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \begin{bmatrix} \hat{y}(t) \\ \hat{u}(t) \end{bmatrix} dt = \inf_{\substack{E\dot{x}=Ax+Bu \\ y=Cx+Bu \\ Ex(t_0)=E\hat{x}(t_0)}} \int_{t_0}^{\infty} \begin{bmatrix} y(t) \\ u(t) \end{bmatrix}^* \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \begin{bmatrix} y(t) \\ u(t) \end{bmatrix} dt,$$

for all $t_0 \in \mathbb{R}$.

Proof. The equivalence of 1. and 2. and equivalence of 1. and 3. can be obtained from Theorems 3.12, 3.13, and 3.15.

□

To sum up the results of Section 3.4 for state-space systems we introduce a new type of controllability.

Definition 4.4. [9, Definition 2] Let $E, A \in \mathbb{R}^{n,n}$, $\lambda E - A$ be invertible, and $B \in \mathbb{R}^{n,m}$. Then the triplet (E, A, B) is called *completely controllable* if

$$\operatorname{rank}\left(\begin{bmatrix}\alpha E - \beta A & B\end{bmatrix}\right) = n \quad \text{for all } (\alpha, \beta) \in \mathbb{C}^2 \setminus \{0, 0\}.$$

Lemma 4.5. *Let $E, A \in \mathbb{C}^{n,n}$, $\lambda E - A$ be invertible, $B \in \mathbb{C}^{n,m}$, and let the triplet (E, A, B) be completely controllable. Introduce the notation from (4.3). Then P_1 is controllable and in the Kronecker canonical form (2.4) of the pencil $\lambda F_1 + G_1$ all blocks of type (2.7) have sizes $\sigma_j < 2$ and all blocks of type (2.8) have sizes $\eta_j \leq 0$.*

Proof. By Lemma 2.21 we know that controllability of P_1 is equivalent to P_1 having no zeros. To see that P_1 has no zeros note that we have $n = \operatorname{rank}_{\mathbb{C}(\lambda)}(P_1)$, since $\lambda E - A$ is invertible. On the other hand we have for all $\lambda_0 \in \mathbb{C}$ that

$$\begin{aligned}\operatorname{rank}(P_1(\lambda_0)) &= \operatorname{rank}\left(\begin{bmatrix}\lambda_0 E - A & -B\end{bmatrix}\begin{bmatrix}I & \\ & -I\end{bmatrix}\right)\\ &= \operatorname{rank}\left(\begin{bmatrix}\lambda_0 E - A & B\end{bmatrix}\right) = n,\end{aligned}$$

as one can see by setting $\alpha = \lambda_0$ and $\beta = 1$ in Definition 4.4. This shows that $\operatorname{rank}(P_1(\lambda_0)) = n = \operatorname{rank}_{\mathbb{C}(\lambda)}(P_1)$ for all $\lambda_0 \in \mathbb{C}$ which by Lemma 2.6 proves that P_1 has no zeros.
To show the assertion about the blocks in the Kronecker canonical form assume to the contrary that the Kronecker canonical form (2.4) of $\lambda F_1 + G_1$ has a block of type (2.7) with size $\sigma_j \geq 2$ or a block of type (2.8) with size $\eta_j > 0$. In any case we have

$$\begin{aligned}\lambda F_1 + G_1 &= \lambda\begin{bmatrix}E & 0\end{bmatrix} + \begin{bmatrix}-A & -B\end{bmatrix}\\ &= S\left(\lambda\begin{bmatrix}K_1 & \\ & R_1\end{bmatrix} + \begin{bmatrix}K_0 & \\ & R_0\end{bmatrix}\right)T,\end{aligned}$$

where the pencil $\lambda K_1 + K_0$ is either a block of type (2.7) with size $\sigma_j \geq 2$ or a block of type (2.8) with size $\eta_j > 0$. In both cases we can define the row vectors

$$v_1 := \begin{bmatrix}0 & \dots & 0 & 1\end{bmatrix} \text{ and } v_2 := \begin{bmatrix}0 & \dots & 0 & 1 & 0\end{bmatrix}$$

to obtain that $v_1 K_1 = 0$ and $v_2 K_1 = v_1 K_0$. Define the row vectors $w_1 := \begin{bmatrix}v_1 & 0\end{bmatrix} S^{-1}$ and $w_2 := \begin{bmatrix}v_2 & 0\end{bmatrix} S^{-1}$ to obtain that

$$w_1\begin{bmatrix}E & 0\end{bmatrix} = \begin{bmatrix}v_1 & 0\end{bmatrix} S^{-1}\begin{bmatrix}E & 0\end{bmatrix} = \begin{bmatrix}v_1 & 0\end{bmatrix}\begin{bmatrix}K_1 & \\ & R_1\end{bmatrix} T = 0$$

and also

$$\begin{aligned} w_2 \begin{bmatrix} E & 0 \end{bmatrix} &= \begin{bmatrix} v_2 & 0 \end{bmatrix} S^{-1} \begin{bmatrix} E & 0 \end{bmatrix} = \begin{bmatrix} v_2 & 0 \end{bmatrix} \begin{bmatrix} K_1 & \\ & R_1 \end{bmatrix} T \\ &= \begin{bmatrix} v_2 K_1 & 0 \end{bmatrix} T = \begin{bmatrix} v_1 K_0 & 0 \end{bmatrix} T = \begin{bmatrix} v_1 & 0 \end{bmatrix} \begin{bmatrix} K_0 & \\ & R_0 \end{bmatrix} T \\ &= \begin{bmatrix} v_1 & 0 \end{bmatrix} S^{-1} \begin{bmatrix} -A & -B \end{bmatrix} = w_1 \begin{bmatrix} -A & -B \end{bmatrix}. \end{aligned}$$

All in all we obtained $w_1, w_2 \neq 0$ with $w_1 E = 0$, $w_2 E = -w_1 A$, and $w_1 B = 0$. This implies that $w_1 \begin{bmatrix} E & B \end{bmatrix} = 0$, from which we deduce that $\operatorname{rank}\left(\begin{bmatrix} E & B \end{bmatrix}\right) < n$, since $w_1 \neq 0$. This, however, is a contradiction to the assumption of complete controllability as one can see by choosing $\alpha = 1$ and $\beta = 0$ in Definition 4.4. □

Corollary 4.6. *Consider the system (4.1) with $\lambda E - A$ invertible and supply (4.2). Assume that (E, A, B) is completely controllable. Introduce the notation from (4.3). Then the following statements are equivalent:*

1. *(P_1, H_1) is dissipative.*
2. *There exists a spectral factorization of the Popov function (4.7).*
3. *There exist $X \in \mathbb{R}^{n,n}$ and $Y \in \mathbb{R}^{n,m}$ such that*

$$\begin{aligned} \begin{bmatrix} A^*X + X^*A - \tilde{Q} & A^*Y + X^*B - \tilde{S} \\ B^*X + Y^*A - \tilde{S}^* & B^*Y + Y^*B - \tilde{R} \end{bmatrix} &\leq 0, \\ E^*X = X^*E \qquad\qquad\qquad E^*Y &= 0. \end{aligned} \tag{4.8}$$

Proof. Using Lemma 4.5 we find that under the assumptions made in this corollary also the assumptions of Corollary 3.29 are fulfilled. Thus, to show the equivalence of 2. and 3. we only have to show that the linear matrix inequality (3.14) associated with (P_1, H_1) with the notation from (4.3) is the same as (4.8). Therefore, we partition $Z \in \mathbb{C}^{n,n+m}$ from (3.14) into

$$Z =: \begin{bmatrix} X & Y \end{bmatrix},$$

with $X \in \mathbb{C}^{n,n}$ and $Y \in \mathbb{C}^{n,m}$, analogously to the partitioning of F_1 and G_1 given by (4.3). Then, we see that the linear matrix inequality (3.14) reads

$$\begin{bmatrix} E^*X & E^*Y \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} E^* \\ 0 \end{bmatrix} \begin{bmatrix} X & Y \end{bmatrix} = F^*Z = Z^*F = \begin{bmatrix} X^* \\ Y^* \end{bmatrix} \begin{bmatrix} E & 0 \end{bmatrix} = \begin{bmatrix} X^*E & 0 \\ Y^*E & 0 \end{bmatrix}$$

and

$$\begin{aligned} 0 &\leq G^*Z + Z^*G + H = \begin{bmatrix} -A^* \\ -B^* \end{bmatrix} \begin{bmatrix} X & Y \end{bmatrix} + \begin{bmatrix} X^* \\ Y^* \end{bmatrix} \begin{bmatrix} -A & -B \end{bmatrix} + \begin{bmatrix} \tilde{Q} & \tilde{S} \\ \tilde{S}^* & \tilde{R} \end{bmatrix} \\ &= \begin{bmatrix} -A^*X - X^*A + \tilde{Q} & -A^*Y - X^*B + \tilde{S} \\ -B^*X - Y^*A + \tilde{S}^* & -B^*Y - Y^*B + \tilde{R} \end{bmatrix}, \end{aligned}$$

which proves the claim. □

We remark the strong similarity of (4.8) to the linear matrix inequality in [9, Theorem 13].

4.2 Checking dissipativity

In Theorem 3.11 we saw that cyclo-dissipativity can be checked via the sign-sum function of a certain para-Hermitian matrix along the imaginary axis. In this section we will first develop a graphical representation of polynomial para-Hermitian matrices which we then use to check the condition in Theorem 3.11. This graphical representation also yields further insight into the problem and will also turn out to be helpful in developing the dissipativity enforcement algorithm.

Consider an arbitrary para-Hermitian matrix polynomial $N = N^{\sim} \in \mathbb{C}[\lambda]^{p,p}$. Since such a polynomial is Hermitian along the imaginary axis, i.e., $N(i\omega) = N^*(-\overline{i\omega}) = N^*(i\omega)$ for all $\omega \in \mathbb{R}$, we can compute the p-many eigenvalues of $N(i\omega)$ for each $\omega \in \mathbb{R}$. Ordering these eigenvalues by their value we see that for every para-Hermitian matrix polynomial N there exists a unique continuous function $f_N : \mathbb{R} \to \mathbb{R}^p$ such that $f_N(\omega)$ contains all the eigenvalues of $N(i\omega)$ in ordered sequence.

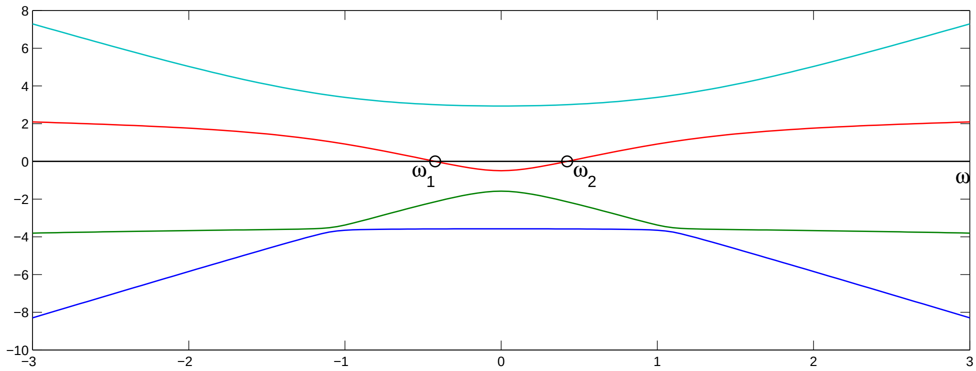


Figure 4.1: f_{N_1} with $N_1 = N_1^{\sim} \in \mathbb{C}[\lambda]^{4,4}$

Figure 4.1 depicts the function f_{N_1} for a para-Hermitian matrix polynomial N_1 of size 4-by-4. We do not suggest that this order is the most natural for all possible para-Hermitian matrices. From Figure 4.1 we see that there exist (innumerable many) points $\omega \in \mathbb{R}$ such that $N(i\omega)$ has 4 non-zero eigenvalues which implies that $\operatorname{rank}_{\mathbb{C}(\lambda)}(N_1) = 4$. However, at the two points ω_1 and ω_2 which are marked by a circle in Figure 4.1 the rank of N_1 drops, i.e., we have $\operatorname{rank}(N_1(i\omega_k)) = 3$, since at these points we only have three non-zero eigenvalues and one zero eigenvalue. Using Lemma 2.6 we conclude that $i\omega_1$ and $i\omega_2$ are zeros of N_1.

Via the above construction we see that the purely imaginary zeros of a para-Hermitian matrix polynomial segment the imaginary axis into intervals where the sign-sum function $\eta(N(i\omega))$ is constant. In the above example from Figure 4.1, e.g., the purely imaginary zeros of N_1 are given by $i\omega_1$ and $i\omega_2$ and we observe that

$$\eta(N_1(i\omega)) = \begin{cases} 0, & \omega \in (-\infty, \omega_1] \cup [\omega_2, \infty) \\ -2, & \omega \in (\omega_1, \omega_2) \end{cases}.$$

With respect to dissipativity we can build the para-Hermitian polynomial N as in (3.3) and draw the corresponding figure as in Figure 4.1. From this picture we can then theoretically check condition (3.4). However, for systems of larger dimension this becomes quite unhandy. For example, Figure 4.2 corresponds to a system $P \in \mathbb{C}[\lambda]^{20,20}$ of moderate size and is already completely unreadable.

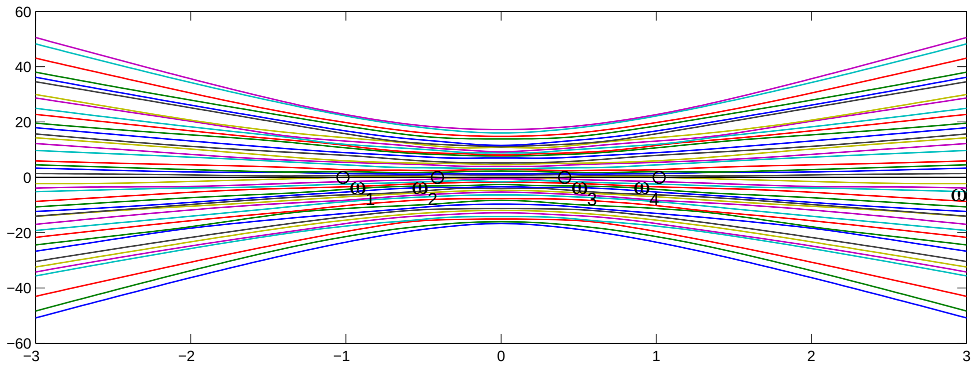


Figure 4.2: f_{N_2} with $N_2 = N_2^{\sim} \in \mathbb{C}[\lambda]^{40,40}$

Fortunately, we do not need all the information depicted in Figure 4.2! Since we are only interested in the value of the sign**sum** function, we can cancel non-negative curves against negative ones and only depict the curves which are left in the middle. Assuming that condition (3.4) for the example in Figure 4.2 reads $\eta\left(N_2(i\omega)\right) = 2$ we can strip the 19 largest and the 19 smallest lines from Figure 4.2 to end up with Figure 4.3.

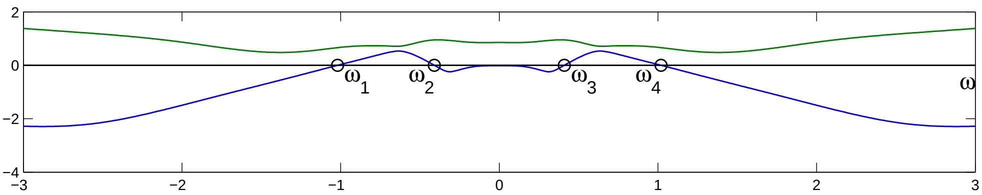


Figure 4.3: Sign-sum plot of a system

Consider a system $P \in \mathbb{C}[\lambda]_K^{p,q}$ with a supply $H = H^* \in \mathbb{C}^{qK,qK}$ and build the para-Hermitian polynomial N as in (3.3). Then we will refer to the figure which corresponds to Figure 4.3 as the *sign-sum plot of* (P, H) in the following. From Theorem 3.11 we conclude that a system is cyclo-dissipative if and only if every line in the sign-sum plot is greater or equal to zero. Clearly, the system depicted in Figure 4.3 is not dissipative.
The above considerations lead to the following Algorithm 4.7 to check cyclo-dissipativity. From Corollary 3.21 part 1. we see that for controllable systems the algorithm is also a dissipativity check. For uncontrollable systems the algorithm only checks a property of the controllable part. For autonomous systems the algorithm will always return `true`.

Algorithm 4.7. (Cyclo-dissipativity check)

Input: A system $P \in \mathbb{C}[\lambda]_K^{p,q}$ and a supply $H = H^* \in \mathbb{C}^{qK,qK}$.

Output: Whether (P, H) is cyclo-dissipative.

Step 1: Determine the normal rank of P and save as $r := \operatorname{rank}_{\mathbb{C}(\lambda)}(P)$.

Step 2: Form the para-Hermitian matrix polynomial

$$N := \begin{bmatrix} 0 & P \\ P^{\sim} & \Delta_K^{q\ \sim} H \Delta_K^q \end{bmatrix} \in \mathbb{C}[\lambda]^{p+q,p+q}.$$

Step 3: Compute the M-many distinct purely imaginary zeros of N as $i\omega_1, \ldots, i\omega_M$ with $\omega_1 < \ldots < \omega_M$. Set $\omega_0 := -\infty$ and $\omega_{M+1} := \infty$.

Step 4: For $k = 0, \ldots, M$ fix points $i\alpha_k \in i\mathbb{R}$ such that α_k is in the interior of (ω_k, ω_{k+1}).

Step 5: If for $k = 0, \ldots, M$ we have $\eta(N(i\alpha_k)) = p + q - 2r$ return `true`, otherwise, return `false`.

Some implementational remarks follow. In step 1 it is the best to know r beforehand, e.g., from theoretic considerations. If one does not, we suggest using linearization and the `GUPTRI` algorithm (see [12]) to compute r.
Step 3 is the crux of Algorithm 4.7. If P is of order bigger than one, we suggest to linearize the system to first order. For first order systems we suggest the para-Hermitian `STCSSP` method in [8, 7] or the unstructured `GUPTRI` algorithm (see [12]) to discover the regular part. On the regular part (or if N is regular in the first place) we suggest to use LAPACK's QZ-algorithm `ZGGES` to compute the purely imaginary zeros. For large and sparse matrices we suggest to employ a shift-and-invert'ed Arnoldi method with shifts along the imaginary axis, compare [25].
For step 5 we suggest to use LAPACK's Hermitian eigenvalue solver `ZHEEV` to compute the eigenvalues of $N(i\alpha_k)$ and then count the number of negative and non-negative eigenvalues. For higher speed one could also use LAPACK's LDL^T factorization `ZHETRF`. The LDL^T factorization approach also works for large sparse matrices.

4.3 Enforcing dissipativity

In this section we propose a heuristic method to enforce dissipativity of systems of the form (4.1) with respect to the supply (4.2). Consider the following example, which has been provided by *CST AG, Darmstadt*. It is a state-space system describing the electromagnetic behavior of a coaxial cable and takes the form

$$\begin{aligned} \dot{x} &= Ax + Bu, \\ y &= Cx + Du, \end{aligned} \tag{4.9}$$

where $A \in \mathbb{R}^{n,n}$, $B \in \mathbb{R}^{n,m}$, $C \in \mathbb{R}^{l,n}$, and $D \in \mathbb{R}^{l,m}$ are all real matrices with $n = 35$, $l = 2$, and $m = 2$ and the supply is measured by

$$\begin{bmatrix} y \\ u \end{bmatrix}^* \begin{bmatrix} -I_l & 0 \\ 0 & I_m \end{bmatrix} \begin{bmatrix} y \\ u \end{bmatrix} = \|u\|_2^2 - \|y\|_2^2. \tag{4.10}$$

Since the system matrices are real, we only show the sign-sum plot for $\omega \geq 0$. The negative part is the mirror image of the positive part.

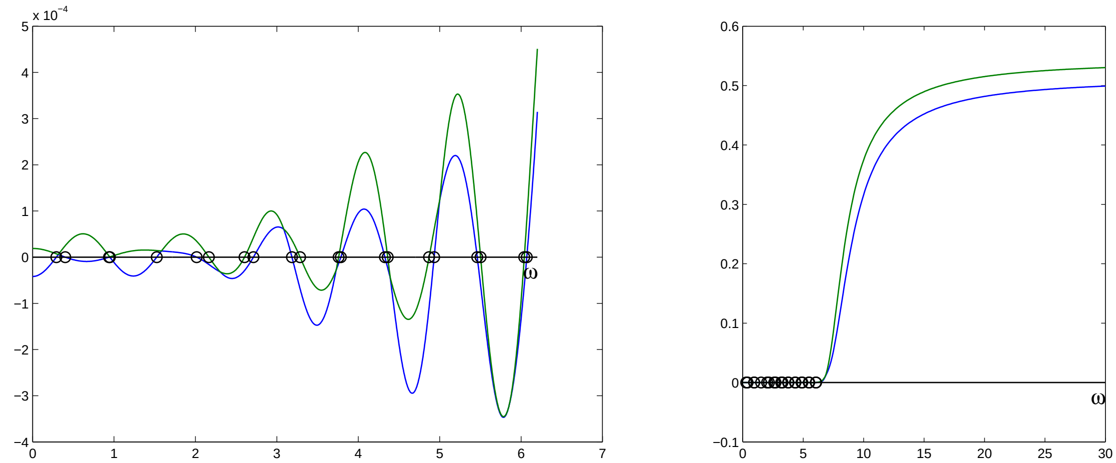


Figure 4.4: Sign-sum plot for the coaxial cable (4.9) from *CST AG* at different scales

Clearly, the coaxial cable is not dissipative. However, everybody will agree that it is close to dissipative, since the lines in the sign-sum plot do not go below $-4.0 \cdot 10^{-4}$ which is not too much when considering that for $\omega = 30$ the lines have a magnitude of approximately 0.5. We therefore ask the following question. *What is the minimum perturbation of A, B, C, D that makes the system (4.9) dissipative with respect to the given supply (4.10)?* To be more specific, we ask if we can compute ΔA, ΔB, ΔC, and ΔD such that

$$\sqrt{\|\Delta A\|_F^2 + \|\Delta B\|_F^2 + \|\Delta C\|_F^2 + \|\Delta D\|_F^2}, \tag{4.11}$$

is minimal among all matrices that satisfy the constraint that the perturbed system

$$\begin{aligned} \dot{x} &= (A + \Delta A)x + (B + \Delta B)u, \\ y &= (C + \Delta C)x + (D + \Delta D)u, \end{aligned}$$

is dissipative with respect to (4.10)? Since the answer to this question will most likely be quite involved, we refrain to the following simpler question. *What is a reasonably small perturbation of A, B, C, D that makes the system (4.9) dissipative with respect to (4.10)?* In our example we may suggest from Figure 4.4 that a reasonably small perturbation is on the order of 10^{-4}. We will use the following idea, which is taken from [1], to try to obtain such a perturbation. In [1] Hamiltonian matrices are considered. It is well-known that Hamiltonian matrices have a spectrum which is symmetric with respect to the imaginary axis, just like the zeros of para-Hermitian matrices. Indeed, the problem of finding the zeros of a para-Hermitian matrix of first order can be transformed into a Hamiltonian eigenvalue problem, cf. [4]. Especially, in [1] formulas are developed to move purely imaginary eigenvalues away from the imaginary axis via minimal Hamiltonian perturbations. To adopt this idea to our case, our first goal is to move all the imaginary zeros in Figure 4.4 away from the imaginary axis so that both lines never assume values below or equal to zero.

Lemma 4.8. *Let $N = N^{\sim} \in \mathbb{C}[\lambda]^{p,p}$ be a para-Hermitian matrix and let $\hat{\omega} \in \mathbb{R}$ be such that $i\hat{\omega} \notin \mathfrak{Z}(N)$. Then the smallest constant (para-)Hermitian perturbation $\Delta N = \Delta N^* \in \mathbb{C}^{p,p}$ which satisfies*

$$\operatorname{rank}\left(N(i\hat{\omega}) + \Delta N\right) < \operatorname{rank}_{\mathbb{C}(\lambda)}(N),$$

is given by

$$\Delta N = -\mu v v^*, \tag{4.12}$$

where $(\mu, v) \in \mathbb{R} \times \mathbb{C}^p$ is an eigenpair of the Hermitian matrix $N(i\hat{\omega})$ with the additional property that μ is a non-zero eigenvalue with minimum absolute value.

Proof. [M. Karow; personal communication] Set $r := \operatorname{rank}_{\mathbb{C}(\lambda)}(N)$. Since $i\hat{\omega}$ is not a zero of N, we know that $\operatorname{rank}(N(i\hat{\omega})) = r$ and thus we can compute its eigenvalue decomposition as

$$N(i\hat{\omega}) = \begin{bmatrix} V_1 & v & V_2 \end{bmatrix} \begin{bmatrix} D_1 & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} V_1^* \\ v^* \\ V_2^* \end{bmatrix} = \begin{bmatrix} V_1 D_1 V_1^* + \mu v^* v \end{bmatrix},$$

where $V_1 \in \mathbb{C}^{p,r-1}$, $v \in \mathbb{C}^p$, $V_2 \in \mathbb{C}^{p-r}$, $D_1 \in \mathbb{R}^{r-1,r-1}$, and $\mu \in \mathbb{R}$ such that D_1 and μ are invertible, the composed matrix $\begin{bmatrix} V_1 & v & V_2 \end{bmatrix}$ is unitary, and μ is smaller than any value on the diagonal of D_1 by absolute value. Thus, the smallest perturbation to make the rank of $N(i\hat{\omega})$ drop below r is given by (4.12). □

The zeros of a matrix polynomial are given by the points in the complex plane where the rank drops below the normal rank, see Lemma 2.6. Thus, if we have $\operatorname{rank}_{\mathbb{C}(\lambda)}(N) = \operatorname{rank}_{\mathbb{C}(\lambda)}(N + \Delta N)$, we see that by the perturbation (4.12), we have created the purely imaginary zero $i\hat{\omega}$.

Remember that for $\hat{\omega} \in \mathbb{R}$ the (in absolute value) small eigenvalues of $N(i\hat{\omega})$ are given by the lines in the sign-sum plot. Thus, what we will do is to select one point on a line of the sign-sum plot to fix an $\hat{\omega}$ and a μ as in Lemma 4.8. Then we compute the associated eigenvector v and perturb the constant coefficient in N by the perturbation given in (4.12). For the coaxial cable discussed in this section we obtain results which are depicted in Figure 4.5.

We see that the dashed lines (corresponding to the perturbed matrix) overlap with the solid lines (corresponding to the unperturbed matrix) largely. Only in the area where we selected the eigenvalue a noticeable change in the sign-sum plot occurred. We note that somehow only the line moves that has been selected, creating a zero as predicted by Lemma 4.8.

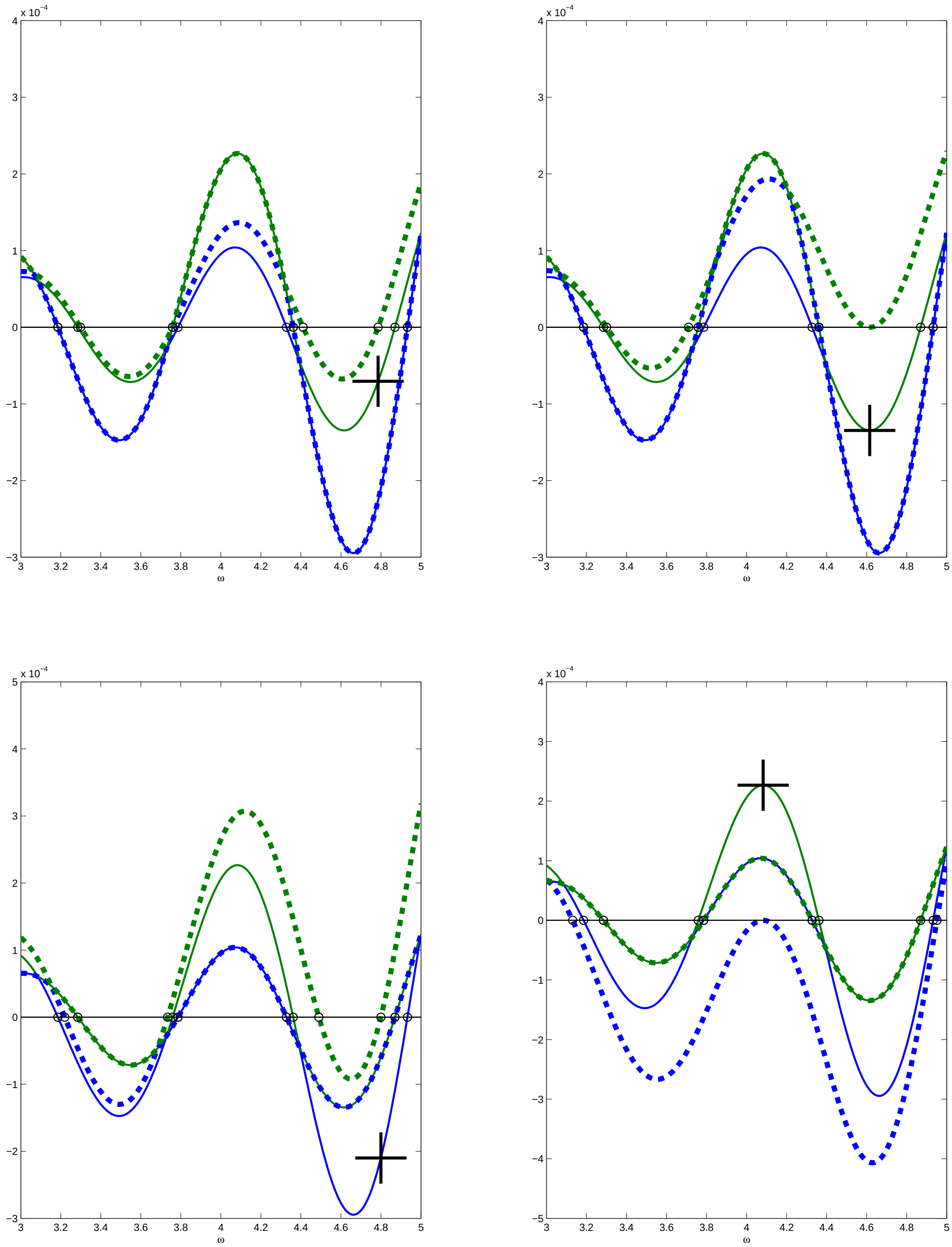


Figure 4.5: Sign-sum plot of (4.9) for different choices of $\hat{\omega}$ and μ. The dashed lines correspond to the perturbed N.

Furthermore, the assumption from Lemma 4.8 that μ has to have minimum absolute value seems not to be essential. Also we note that whenever we select the local extremum of one of the lines of the sign-sum plot the resulting perturbed sign-sum plot only touches zero. Theoretical proof that this is always the case is given in [1] for the Hamiltonian case.
However, having computed a perturbation of N does not yet mean that we have computed a perturbation of the original system matrices A, B, C, D from (4.9). To do so, we remember that in Section 4.1 we saw that we can build a para-Hermitian matrix pencil from (4.9) and (4.10) for the coaxial cable by

$$N(\lambda) := \lambda \begin{bmatrix} 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -I^* & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & -A & -B \\ 0 & 0 & I & -C & -D \\ 0 & I & -I & 0 & 0 \\ -A^* & -C^* & 0 & 0 & 0 \\ -B^* & -D^* & 0 & 0 & I \end{bmatrix}. \tag{4.13}$$

Computing a perturbation via Lemma 4.8, we immediately see that from ΔN we can obtain a perturbations $\Delta A, \Delta B, \Delta C, \Delta D$ by simply setting the entries which we do not want to perturb to zero. This changes the sign-sum plot again as depicted in the upper part of Figure 4.6.
In the upper part of Figure 4.6 we see that the perturbation which only works on A, B, C, D is (figuratively speaking) not strong enough to move the line above 0. To compensate for this we apply the perturbations with a factor $\alpha \in [1, \infty)$, i.e., we perform the updates

$$A \to A + \alpha\Delta A, \qquad B \to B + \alpha\Delta B, \qquad C \to C + \alpha\Delta C, \qquad D \to D + \alpha\Delta D.$$

Thus, in the top right plot of Figure 4.6 we see the effect of $\alpha = 1$. In the lower plots of Figure 4.6 the effect for higher α's is depicted.
Applying this process multiple times we finally end up with a dissipative system, compare Figure 4.7 (Right) for which the norm (4.11) had size 0.0016 while

$$\sqrt{\|A\|_F^2 + \|B\|_F^2 + \|C\|_F^2 + \|D\|_F^2} = 46.42.$$

Since drawing the sign-sum plot is computationally expensive (for every tic on the abscissa one has to compute the eigenvalues of a Hermitian matrix) and since the manual selection of $\hat{\omega}$ and μ to compute a perturbation is cumbersome, we would like to automate this process. To do so, we make the observation that for para-Hermitian matrices of first order $N(\lambda) = \lambda N_1 + N_0 \in \mathbb{C}[\lambda]_1^{p,p}$, the slope of a line at a point $i\omega_0$, where it hits the zero axis, can be computed via

$$-iv^* N_1 v,$$

where v is the generalized eigenvector associated with the generalized eigenvalue $i\omega_0$, compare [34]. We conclude that in the automated algorithm all $\hat{\omega}$ should be chosen between a purely imaginary zero with negative slope and a purely imaginary zero with positive slope. We give the following pseudo-code which is only meant exemplarily to demonstrate how the discussed ideas can be integrated.

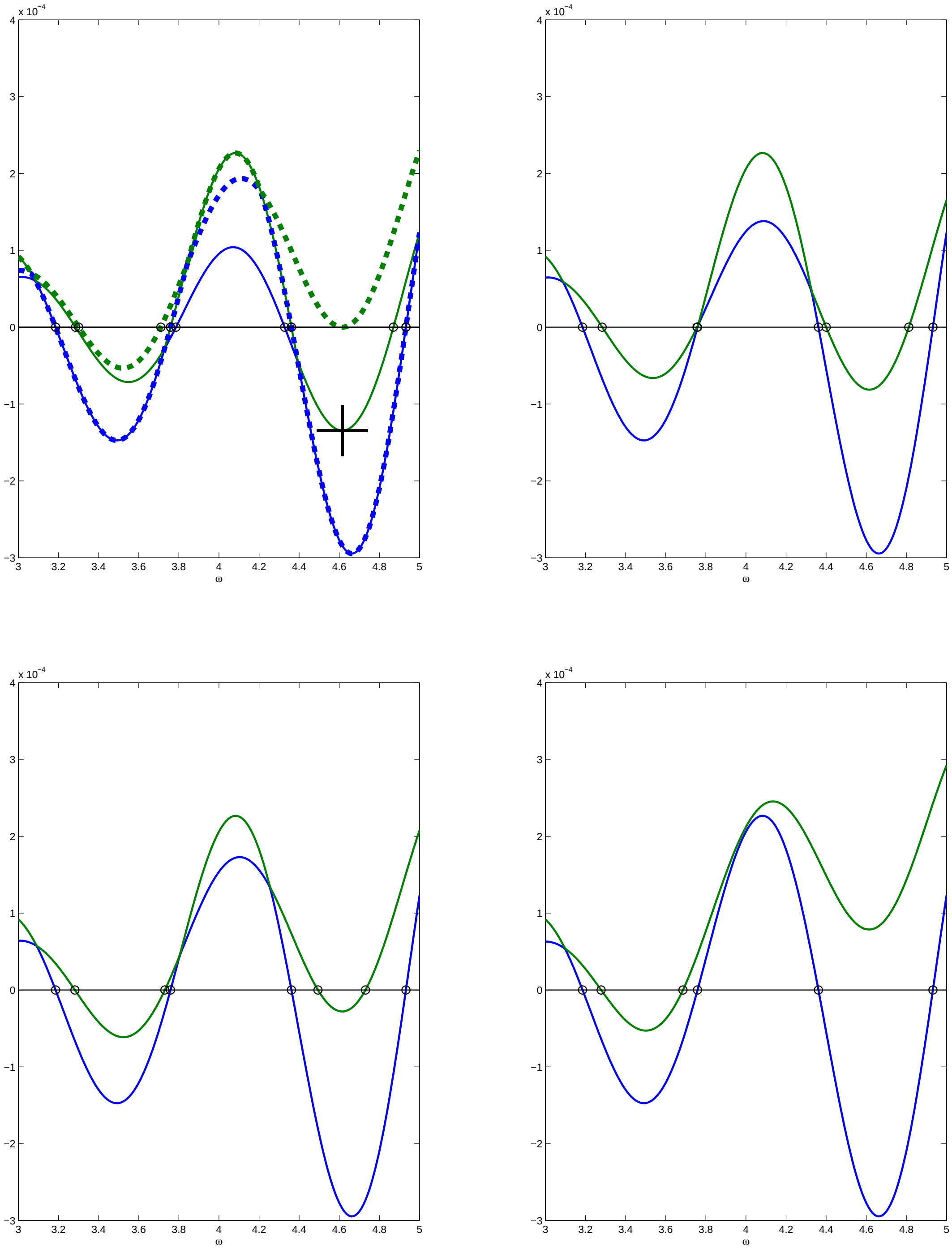


Figure 4.6: Upper Left: sign-sum plot of perturbed para-Hermitian matrix; Upper Right: sign-sum plot of para-Hermitian matrix with perturbations only in A, B, C, D. Lower Left: with $\alpha = 2$; Lower Right: with $\alpha = 4$.

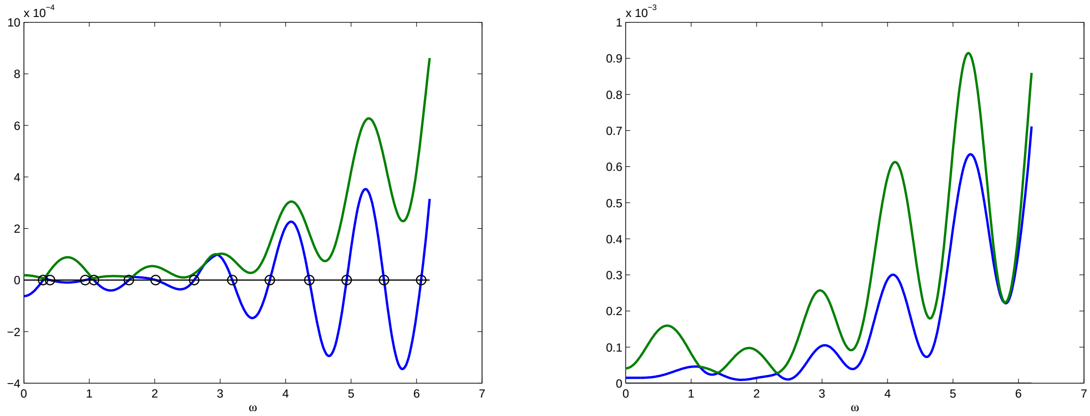


Figure 4.7: Left: after 3 perturbations; Right: after 8 perturbations. Each with $\alpha = 4$.

Algorithm 4.9. (Cyclo-dissipativity enforcement)

Input: The system matrices E, A, B, C, D from (4.1) describing the system, the matrices Q, S, R from (4.2) describing the supply, and an $\alpha \in [1, \infty)$.

Output: Matrices $\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}$ such that the system

$$\begin{aligned} E\dot{x}(t) &= \tilde{A}x(t) + \tilde{B}u(t), \\ y(t) &= \tilde{C}x(t) + \tilde{D}u(t), \end{aligned}$$

is dissipative with respect to (4.2) and

$$\sqrt{\left\|A - \tilde{A}\right\|_F^2 + \left\|B - \tilde{B}\right\|_F^2 + \left\|C - \tilde{C}\right\|_F^2 + \left\|D - \tilde{D}\right\|_F^2}$$

is not too big or *an error message*.

Step 1: Set $\tilde{\eta} := \rho + n + m - 2 \cdot \operatorname{rank}_{\mathbb{C}(\lambda)} \left(\begin{bmatrix} \lambda E - A & B \end{bmatrix}\right)$.

Step 2: Form the para-Hermitian matrix pencil

$$N(\lambda) := \lambda N_1 + N_0 := \lambda \begin{bmatrix} 0 & 0 & 0 & E & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -E^* & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & -A & -B \\ 0 & 0 & I & -C & -D \\ 0 & I & Q & 0 & S \\ -A^* & -C^* & 0 & 0 & 0 \\ -B^* & -D^* & S^* & 0 & R \end{bmatrix}.$$

Step 3: Compute the M-many distinct purely imaginary generalized eigenvalues of N as $i\omega_1, \ldots, i\omega_M$ with $\omega_1 < \ldots < \omega_M$ and the associated generalized eigenvectors $v_1, \ldots, v_M$.

Step 4: Compute the slopes as $\sigma_1, \ldots, \sigma_M$ as $\sigma_j := \operatorname{Im}\left(v_j^* N_1 v_j\right)$.

Step 5:

$onstack \leftarrow 0$
$processed \leftarrow 0$
$\Delta N \leftarrow 0$

for $k = 1, \ldots, M$ **do**
  if $\sigma_j < 0$ **then**
    $onstack \leftarrow onstack + 1$
  else if $\sigma_j > 0$ **then**
    $onstack \leftarrow onstack - 1$
  end if
  if $onstack = 0$ **then**
    $\hat{\omega} \leftarrow \left(\sum_{j=processed+1}^{k} \omega_j\right) \div (k - processed)$
    Compute the eigenvalues and eigenvectors of $N(i\hat{\omega})$ and cancel non-negative against negative eigenvalues to end up with $\tilde{\eta}$ eigenvalues. Of these discard all non-negative eigenvalues and keep R-many negative eigenvalues $\beta_1, \ldots, \beta_R$ with their associated eigenvectors $w_1, \ldots, w_R$
    $\Delta N \leftarrow \Delta N - \beta_1 w_1 w_1^* - \ldots - \beta_R w_R w_R^*$
    $processed \leftarrow k$
  else if $onstack < 0$ **then**
    Error! *Dissipativation impossible!*
  end if
end for
if $onstack > 0$ **then**
  Error! *Dissipativation impossible!*
end if

Step 6: If $\|\Delta N\| = 0$ set $\tilde{A} \leftarrow A, \tilde{B} \leftarrow B, \tilde{C} \leftarrow C, \tilde{D} \leftarrow D$ and return.

Step 7: From ΔN obtain $\Delta A, \Delta B, \Delta C, \Delta D$ by setting the other entries to zero according to the structure of (4.13).

Step 8: Update $A \leftarrow A + \alpha\Delta A, B \leftarrow B + \alpha\Delta B, C \leftarrow C + \alpha\Delta C, D \leftarrow D + \alpha\Delta D$, and goto Step 2.

Remark 4.10. A problem with the pseudocode given in Algorithm 4.9 becomes obvious when one has multiple purely imaginary eigenvalues. In this case Step 4 does not give the right slopes. Instead, for every multiple eigenvalue $\omega_j = \ldots = \omega_k$ one has to build the matrix of all associated eigenvectors $V = [v_j, \ldots, v_k]$, form the product $V^* N_1 V$, compute the (purely imaginary) eigenvalues of this matrix, and take the imaginary parts of these eigenvalues [M. Karow; personal communication]. Then, of course, the problem arises in which order the slopes are injected into the for-loop of step 5. This problem has been addressed in the MATLAB code in Appendix B. Also, Algorithm 4.9 can become an infinite loop which is why in the code in Appendix B a parameter which specifies the maximum number of iterations has been introduced. Another point in which the MATLAB code in Appendix B and Algorithm 4.9 differ is that the error messages are simply ignored. This is done, because in the implementation used we cannot safely distinguish between eigenvalues which are purely imaginary and those which are not. In practice this aborted the algorithm although one could simply continue and obtain reasonable results. Note, that this problem can be avoided by exploiting the para-Hermitian structure of the problem, compare [8, 28].

In the following we will use Algorithm 4.9 on various inputs and see how it behaves. All systems that will be tested are supplied by *CST AG, Darmstadt*. All tests were performed using a Intel(R) Core(TM)2 Duo CPU E6850 at 3.00GHz with 4GB memory and a Linux operating system.
We start by varying the parameter α for the now familiar coaxial cable (4.9). The results are shown in Figure 4.8. Note that for values of α greater then 5 nothing spectacular happens, i.e., the CPU time and the number of iterations remain constant and the relative residual continues it's smooth increase.

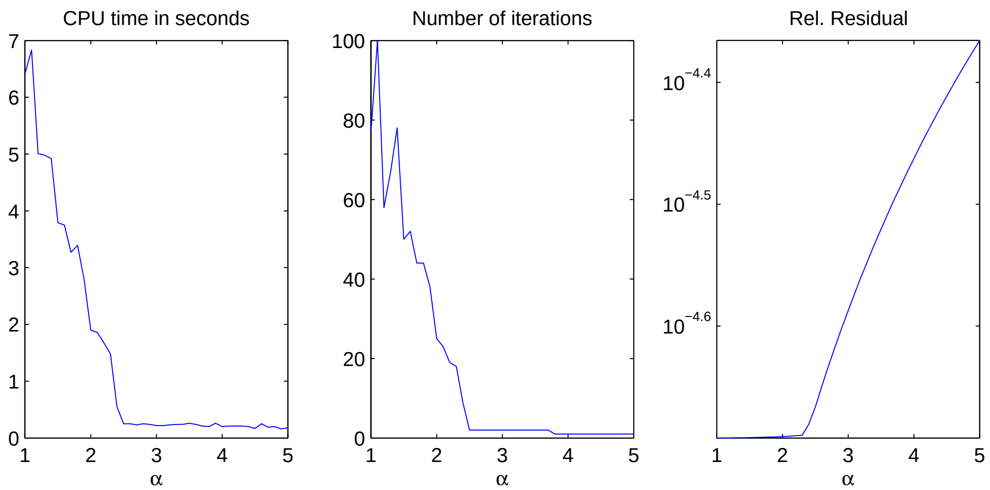


Figure 4.8: Algorithm 4.9 on the coaxial cable with α varying from 1 to 5 in equidistant steps of 0.1.

The following example (called *Branch 4*) also is an electromagnetic model of the form (4.9) with supply (4.10) but in this case $n = 100$, $l = 4$, and $m = 4$. The corresponding sign-sum plot can be found in Figure 4.9. Again, we measured several performance indices for varying α in Figure 4.10. Note that in Figure 4.10 for several choices of α the maximum number of iterations has been reached. This means that the algorithm failed to do the dissipativation (in 150 iterations). This probably comes from the problem that we failed to separate purely imaginary zeros from zeros with very small real part. We see that the choice $\alpha = 5$ seems to be a good one. Computing the perturbation with $\alpha = 5$ and drawing the sign-sum plot for the perturbed system has been done in Figure 4.11.

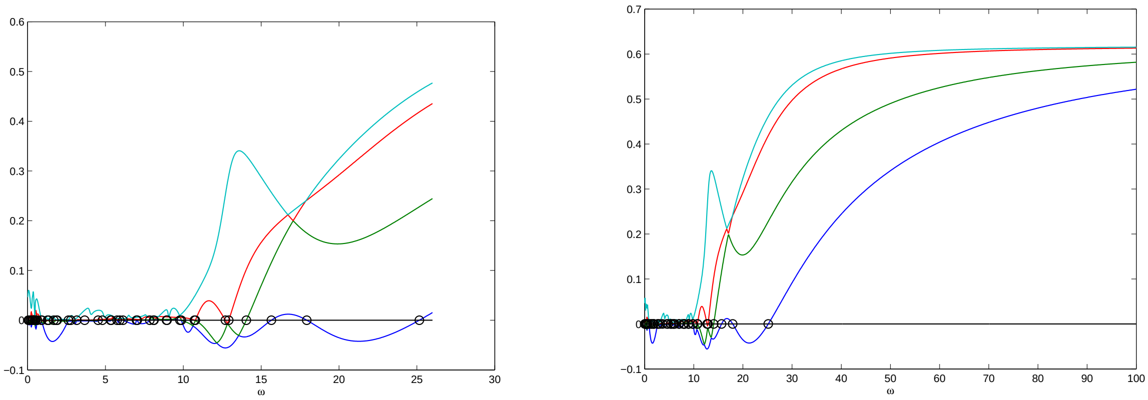


Figure 4.9: Sign-sum plot of the Branch 4 example from CST AG at different scales

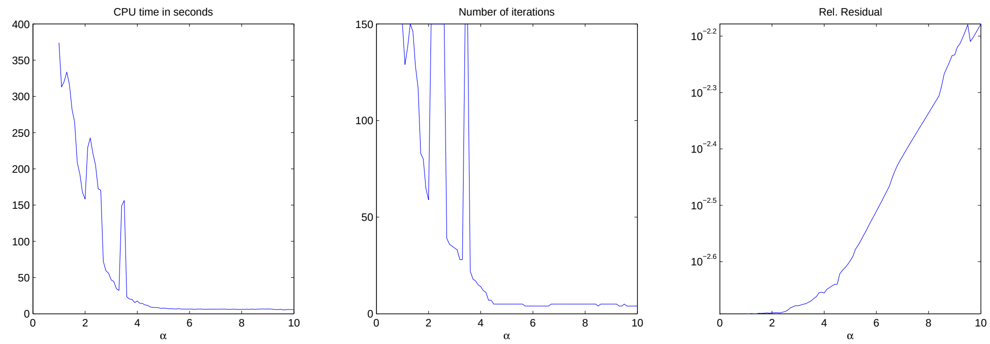


Figure 4.10: Algorithm 4.9 on Branch 4 with α varying from 1 to 10 in equidistant steps of 0.1 with maximum number of iterations restricted to 150.

The following example (called *RJ 45*) describes the electromagnetic behavior of an RJ 45 network connector and also has the form (4.9) with supply (4.10) with $n = 160$, $l = 8$, and $m = 8$. In this example the matrix A and B are sparse matrices. Especially the matrix A is almost diagonal. It thus would make sense to adopt this sparsity pattern of A and B in Algorithm 4.9, which we will not do here.

The dissipativation Algorithm 4.9 did not work on the RJ 45 connector at all. To understand why consider Figure 4.12. In the top left we see that three $\hat{\omega}$'s have been selected (the blue crosses, one invisible at approximately -1.5). Applying the computed perturbation shows that in the resulting sign-sum plot (bottom left) the lowest line is always below zero, which is why only one $\hat{\omega} = 0$ is selected in the second iteration. Thus the second perturbation only works in the vicinity of $\hat{\omega} = 0$ and thus the application of the new perturbation (bottom right) has not changed the sign-sum plot in a noticeable way. This behavior thus continues over and over again and the algorithm stagnates. Thus, for this problem another logic of choosing the $\hat{\omega}$'s would be appropriate which then might not work with other problems, e.g., the coaxial cable. Different choices of α also were not able to better the situation.

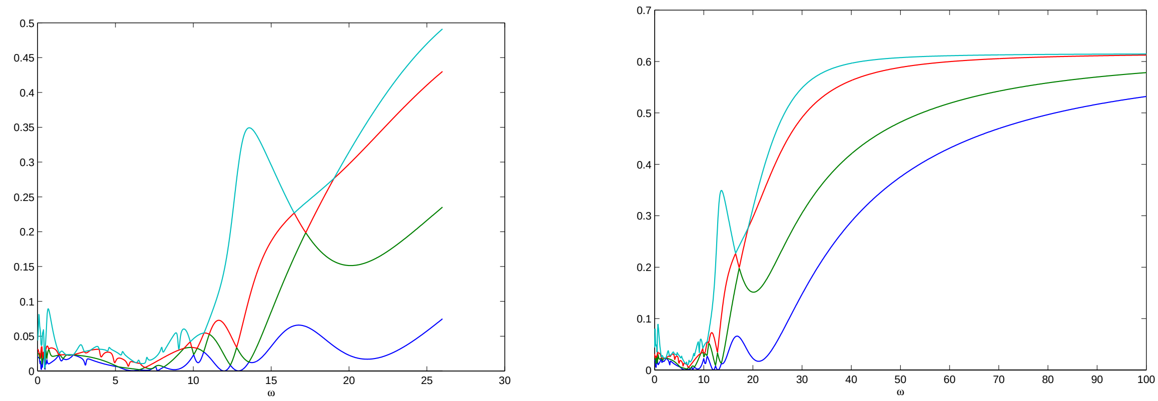


Figure 4.11: Sign-sum plot of the dissipativated Branch 4 example with $\alpha = 5$ at different scales

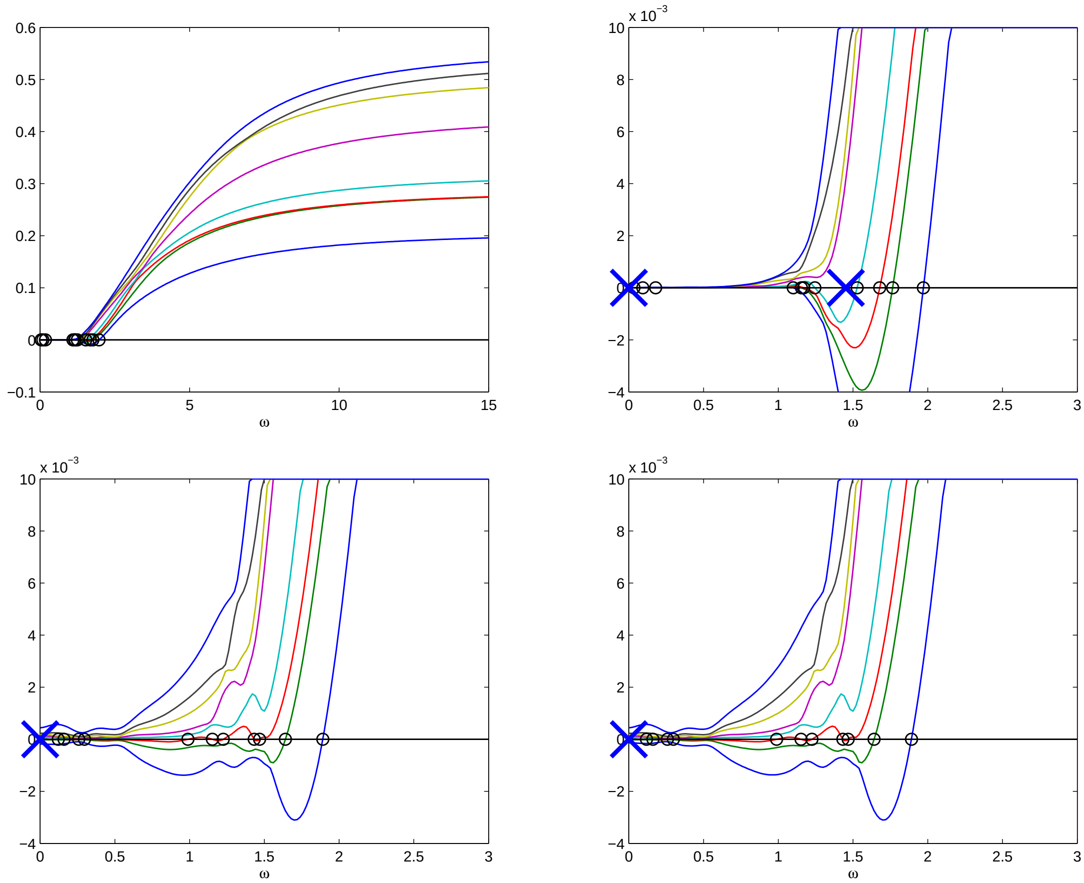


Figure 4.12: Sign-sum plot of the RJ 45 from CST AG: unperturbed at two scales (top), after one iteration (bottom left), after two iterations (bottom right) with $\alpha = 5$. Blue crosses mark the selected $\hat{\omega}$ as in Algorithm 4.9.

Chapter 5

Conclusion and Outlook

In this thesis three new theoretical results have been obtained. From these three results two algorithms have been derived.
The first important result Lemma 3.6 showed that any Popov function of a system with a supply $(P, H) \in \mathbb{C}[\lambda]_K^{p,q} \times \mathbb{C}^{qK,qK}$ is hidden in the matrix

$$N := \begin{bmatrix} 0 & P \\ P^{\sim} & {\Delta_K^q}^{\sim} H \Delta_K^q \end{bmatrix}.$$

This result was obtained by elementary matrix computations. The advantage of N against Popov functions is that Popov functions are not easily computed from P, whereas it is trivial to form the para-Hermitian matrix N from P. We then showed in Theorem 3.11 that one can check cyclo-dissipativity (which for controllable systems is equivalent to dissipativity) of (P, H) via the inertia information of N along the imaginary axis. To prove this we used the well-known result, that cyclo-dissipativity can be checked via the inertia information of a Popov function along the imaginary axis.
The next result was formulated in Section 3.2. There we met the para-Hermitian matrix N again and we stated that the solution of the infinite horizon linear quadratic control problem (which stands behind the available storage and required supply) is contained in $\mathfrak{B}(N)$, once the system is cyclo-dissipative. Since it is known that the algebraic Riccati equation plays an important role in linear quadratic optimal control [40] and the extremal solutions of the algebraic Riccati equation can be obtained from the eigenvalues in the left or right open half plane of a para-Hermitian eigenvalue problem [24, 14], we are especially interested in the autonomous part of $\mathfrak{B}(N)$. Furthermore, the results in Section 3.2 showed that solvability of the linear quadratic control problem is equivalent to cyclo-dissipativity, which was known before [37, 38, 26]. The proofs for the results basically consist of calculus of variation with some adaptions to behavioral systems.
The final theoretical result is given by the equivalence of dissipativity and solvability of the linear matrix inequality (3.14) under some controllability assumptions. This result is also called Kalman-Yakubovich-Popov lemma. In contrast to [36] we made use of the Kronecker canonical form to study this problem and thus were able to make weaker assumptions. Also, we used a different kind of linear matrix inequality than the one used in [36].

Having obtained the theoretical results for behavioral systems in Chapter 3 we then applied them to descriptor systems in Section 4.1. It is evident that the statements in Section 4.1 are much more complicated than the statements in Chapter 3 and one does not want to think what the proofs would look like, if one were to conduct the proofs directly for descriptor systems. Also, in Section 4.1 we saw that there are two ways to formulate a descriptor system with an output equation of the form (4.1) as a behavioral system, namely (4.3) and (4.5). The theory from Chapter 3 can then be applied to both formulations to obtain different results. However, the obtained results only differ slightly.

From the theoretical results we proposed two algorithms, one the check cyclo-dissipativity and one to enforce cyclo-dissipativity. While we can be sure from the theoretical considerations that the dissipativity check works properly, the dissipativity enforcement algorithm is a heuristic method that can fail. We saw such an example for which the dissipativity enforcement algorithm failed (see Figure 4.12). In the Introduction (Chapter 1) we already noted that it is known that eigenvalue perturbation methods for system passivation sometimes fail and Algorithm 4.9 is not an exception to this. We also proposed that it might be possible to modify the algorithm (by choosing the $\hat{\omega}$'s as in Algorithm 4.9 in another way) so that it works properly with the example from Figure 4.12, although the modified version might then no longer be working for the previous examples where the original enforcement Algorithm 4.9 has worked. In any case, there will be no selection of the $\hat{\omega}$'s which is optimal for every problem and it is doubtful that the approach used in Algorithm 4.9 is appropriate. A promising alternative is to construct an algorithm which computes perturbations of the para-Hermitian pencil (4.13) through the solution of a least squares problem, similar to [33]. This approach could also be much more efficient computationally since one would not have to compute the eigenvalues of $N(i\hat{\omega})$ for every selected $\hat{\omega}$ as in Step 5 of Algorithm 4.9. Also, Step 7 of Algorithm 4.9 could readily be integrated into the solution of the least squares problem. Furthermore, it would be helpful to develop a deeper understanding of eigenvalue perturbation theory for para-Hermitian matrix pencils, similar to [1].

An interesting open problem is the computation of an Riccati-like optimal feedback controller for behavioral systems via eigenvalue methods. To understand what that means consider the first-order system $P(\lambda) := \lambda F + G \in \mathbb{C}[\lambda]_1^{p,q}$ together with some supply $H = H^* \in \mathbb{C}^{q,q}$. Assume that $Z \in \mathbb{C}^{p,q}$ is the solution of the linear matrix inequality (3.14) which belongs to the available storage. This means that $G^*Z + Z^*G + H$ is positive semi-definite. Thus, there exists a low-rank Cholesky factor $L \in \mathbb{C}^{r,q}$ such that $G^*Z + Z^*G + H = L^*L$. In this case we have

$$\underbrace{\left(\lambda \begin{bmatrix} 0 & F \\ -F^* & 0 \end{bmatrix} + \begin{bmatrix} 0 & G \\ G^* & H \end{bmatrix}\right)}_{=N(\lambda)} \begin{bmatrix} Z \\ I \end{bmatrix} = \begin{bmatrix} I & 0 \\ -Z^* & L \end{bmatrix} \begin{bmatrix} \lambda F + G \\ L \end{bmatrix},$$

which shows that the optimal behavior can be extracted from the para-Hermitian matrix pencil N by imposing further (purely algebraic) equations on the original system $\lambda F + G$, which are specified by L. The matrix L is called a behavioral controller. Following the ideas in [30] it should be possible to compute such a controller efficiently.

Finally, another intriguing question is the following. Consider the pencil from (4.13) which arose by forming the para-Hermitian matrix N with the notations from (4.5) and $Q = -I$, $R = I$, and $S = 0$. We see that this pencil is equivalent to

$$\lambda \begin{bmatrix} 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -I^* & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & -A & -B \\ 0 & I & 0 & -C & -D \\ 0 & 0 & -I & 0 & 0 \\ -A^* & -C^* & 0 & 0 & 0 \\ -B^* & -D^* & 0 & 0 & I \end{bmatrix}.$$

When we consider the behavior of this system we see that the block in the middle will always be zero. Thus, omitting the block in the middle we obtain the para-Hermitian matrix pencil

$$\lambda \begin{bmatrix} 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 \\ -I^* & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & A & B \\ 0 & -I & C & D \\ A^* & C^* & 0 & 0 \\ B^* & D^* & 0 & -I \end{bmatrix}.$$

Setting

$$\hat{P}(\lambda) := \begin{bmatrix} \lambda I - A & -B \\ -C & -D \end{bmatrix}, \text{ and } \hat{H} := \begin{bmatrix} 0 & 0 \\ 0 & -I \end{bmatrix} \tag{5.1}$$

we can rewrite this pencil as

$$\begin{bmatrix} \hat{H} & \hat{P} \\ \hat{P}^{\sim} & \hat{H} \end{bmatrix}. \tag{5.2}$$

It is well known that algebraic properties of P as defined in (5.1) are very important in H_∞-control and it would be interesting to see if the results from Section 3.2 about linear quadratic optimal control can be generalized to make assertions about the pencil in (5.2). In other words, does the behavior of (5.2) describe something meaningful?

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Appendix A

Involved proofs

A.1 A property of the available storage and the required supply

In this section we show that the available storage and the required supply are equivalent to the optimal control problems which are studied in Section 3.2. For first-order systems we will also see another equivalent formulation of the available storage and the required supply which resembles the observations from [29] (where state-maps for linear systems were given) and [35] (where it was shown that every storage function is a function of the state).

Lemma A.1. *Let $\epsilon \in \mathbb{N}_0$ and $\alpha \in \mathcal{C}_\infty$ be such that $\Delta_\epsilon \alpha(0) = 0$. Let $\tilde{b} : \mathbb{R} \to \mathbb{R}$ be a smooth transition from 1 to 0, i.e., an infinitely often differentiable function with*

$$\tilde{b}(t) = \begin{cases} 1 & t < 0 \\ 0 & t > 1 \end{cases}.$$

Then, there exists a constant $C \in \mathbb{R}^+$ such that we have

$$\left\| \left(\tfrac{d}{dt}\right)^i \left[\tilde{b}(k\cdot)\alpha(\cdot)\right] \right\|_+ \leq \frac{C}{k^{\epsilon-i}} \frac{1}{\sqrt{k}} \leq C \frac{1}{\sqrt{k}},$$

for all $i = 0, \ldots, \epsilon$ and all $k = 1, 2, \ldots$.

Proof. Define the constant $B \in \mathbb{R}^+$ through

$$B := \max_{j=0,\ldots,\epsilon} \max_{t \in \mathbb{R}} \left| \tilde{b}^{(j)}(t) \right|,$$

and the constant $A \in \mathbb{R}^+$ through

$$A := \max_{t \in [0,1]} \left| \alpha^{(\epsilon)}(t) \right|.$$

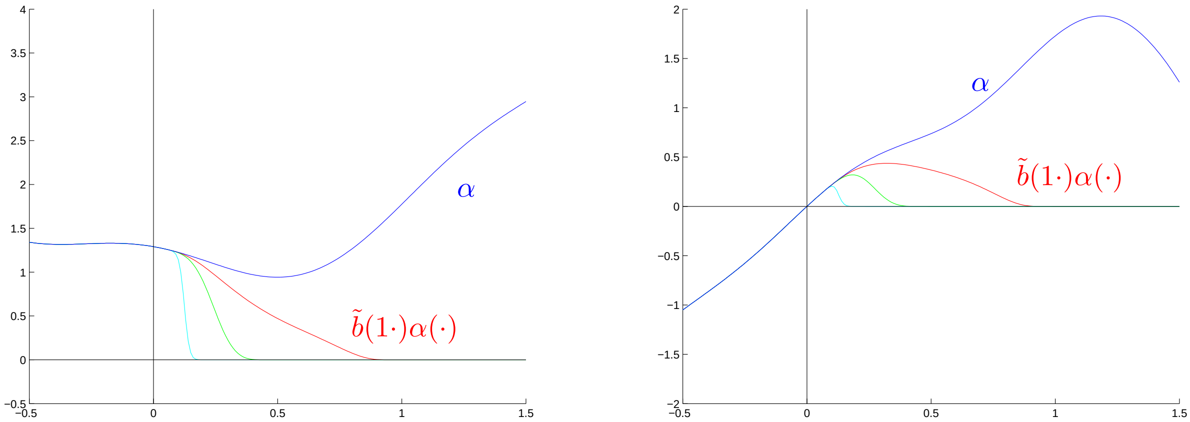

Figure A.1: Left: $\epsilon = 0$; Right: $\epsilon = 1$

Using Taylor expansion we find that for $\epsilon \geq i \geq j \geq 0$ and for $t \in [0,1]$ there exists a $\xi \in [0,1]$ such that we have

$$\begin{aligned} \left|\alpha^{(i-j)}(t)\right| &= \left|\left(\sum_{k=i-j}^{\epsilon-1} \frac{t^{k-i+j}}{(k-i+j)!}\alpha^{(k)}(0)\right) + \frac{t^{\epsilon-i+j}}{(\epsilon-i+j)!}\alpha^{(\epsilon)}(\xi)\right| \\ &= \left|\frac{t^{\epsilon-i+j}}{(\epsilon-i+j)!}\alpha^{(\epsilon)}(\xi)\right| \leq \frac{t^{\epsilon-i+j}}{(\epsilon-i+j)!}A. \end{aligned}$$

We observe that we have $b(kt) = 1$ for $t < 0$ as well as $b(kt) = 0$ for $t > \frac{1}{k}$. With the above observations and the Leibniz rule for differentiation we see that for all $i = 0, \ldots, \epsilon$ and all $k = 1, 2, \ldots$ we have

$$\begin{aligned} & \left\| \left(\tfrac{d}{dt}\right)^i \left[\tilde{b}(k\cdot)\alpha(\cdot)\right] \right\|_+ \\ = & \left\| \sum_{j=0}^{i} \binom{i}{j} k^j \tilde{b}^{(j)}(k\cdot)\alpha^{(i-j)}(\cdot) \right\|_+ \\ \leq & \sum_{j=0}^{i} \binom{i}{j}^2 k^j \left\| \tilde{b}^{(j)}(k\cdot)\alpha^{(i-j)}(\cdot) \right\|_+ \\ = & \sum_{j=0}^{i} \binom{i}{j}^2 k^j \sqrt{\int_0^\infty \left| \tilde{b}^{(j)}(kt)\alpha^{(i-j)}(t) \right|^2 dt} \\ = & \sum_{j=0}^{i} \binom{i}{j}^2 k^j \sqrt{\int_0^{\frac{1}{k}} \left| \tilde{b}^{(j)}(kt)\alpha^{(i-j)}(t) \right|^2 dt} \end{aligned}$$

$$
\begin{aligned}
&\leq \sum_{j=0}^{i} \binom{i}{j}^2 k^j B \sqrt{\int_0^{\frac{1}{k}} \left|\alpha^{(i-j)}(t)\right|^2 dt} \\
&\leq \sum_{j=0}^{i} \binom{i}{j}^2 k^j B \sqrt{\int_0^{\frac{1}{k}} \frac{t^{2\epsilon-2i+2j}}{(\epsilon-i+j)!^2} A^2 dt} \\
&= \sum_{j=0}^{i} \binom{i}{j}^2 k^j \frac{BA}{(\epsilon-i+j)!} \sqrt{\int_0^{\frac{1}{k}} t^{2\epsilon-2i+2j} dt} \\
&= \sum_{j=0}^{i} \binom{i}{j}^2 k^j \frac{BA}{(\epsilon-i+j)!} \sqrt{\frac{1}{2\epsilon-2i+2j+1} t^{2\epsilon-2i+2j+1} \Big|_0^{\frac{1}{k}}} \\
&= \sum_{j=0}^{i} \binom{i}{j}^2 \frac{k^j}{k^{\epsilon-i+j}\sqrt{k}} \frac{BA}{(\epsilon-i+j)!} \frac{1}{\sqrt{2\epsilon-2i+2j+1}} \\
&= \sum_{j=0}^{i} \binom{i}{j}^2 \frac{1}{k^{\epsilon-i}\sqrt{k}} \frac{BA}{(\epsilon-i+j)!} \frac{1}{\sqrt{2\epsilon-2i+2j+1}} \\
&= \frac{1}{k^{\epsilon-i}\sqrt{k}} \sum_{j=0}^{i} \binom{i}{j}^2 \frac{BA}{(\epsilon-i+j)!} \frac{1}{\sqrt{2\epsilon-2i+2j+1}} \\
&\leq \frac{C}{k^{\epsilon-i}\sqrt{k}}
\end{aligned}
$$

by setting

$$
C := \max_{i=0,\ldots,\epsilon} \left(\sum_{j=0}^{i} \binom{i}{j}^2 \frac{BA}{(\epsilon-i+j)!} \frac{1}{\sqrt{2\epsilon-2i+2j+1}} \right),
$$

which yields the assertion. $\square$

Lemma A.2. *Let $P(\lambda) = \lambda F + G \in \mathbb{C}[\lambda]_1^{p,q}$ with $r := \operatorname{rank}_{\mathbb{C}(\lambda)}(P)$. Let $\hat{z} \in \mathfrak{B}(P)$ be such that $F\hat{z}(0) = 0$. Then for any $\theta > 0$ we have the following:*

1. *There exists a trajectory $z_+ \in \mathfrak{B}(P)$ such that $z_+(t) = \hat{z}(t)$ for all $t \leq 0$, $z_+(t) = 0$ for all $t \geq \theta$, and $\|z_+\|_+ < \theta$.*

2. *There exists a trajectory $z_- \in \mathfrak{B}(P)$ such that $z_-(t) = \hat{z}(t)$ for all $t \geq 0$, $z_-(t) = 0$ for all $t \leq -\theta$, and $\|z_-\|_- < \theta$.*

Proof. Let the Kronecker canonical form of $\lambda F+G$ be given by (2.4) and use (2.9) to conclude that there exist $z_1,\ldots,z_s \in \mathcal{C}_\infty$ and $\hat{x} \in \mathbb{C}^\rho$ such that $\hat{z}$ takes the form

$$\hat{z}(t) = T^{-1} \begin{bmatrix} \Delta_{\epsilon_1+1} z_1(t) \\ \vdots \\ \Delta_{\epsilon_s+1} z_s(t) \\ e^{\mathcal{J}(0)t}\hat{x} \\ 0_{\sigma+\eta} \end{bmatrix},$$

for all $t \in \mathbb{R}$, where $0_{\sigma+\eta}$ denotes the zero vector of size $(\sigma+\eta)$-by-1. Further, introduce the notation $\mathcal{L}_{\epsilon_i}(\lambda) =: \lambda\mathcal{L}^1_{\epsilon_i} + \mathcal{L}^0_{\epsilon_i}$, with $\mathcal{L}^1_{\epsilon_i}, \mathcal{L}^0_{\epsilon_i} \in \mathbb{C}^{\epsilon_i,\epsilon_i+1}$ to denote the left and right matrices in (2.5). Then, from the assumption $F\hat{z}(0) = 0$ we obtain

$$0 = S^{-1}FT^{-1}T\hat{z}(0) = \begin{bmatrix} \mathcal{L}^1_{\epsilon_1}\Delta_{\epsilon_1+1} z_1(0) \\ \vdots \\ \mathcal{L}^1_{\epsilon_s}\Delta_{\epsilon_s+1} z_s(0) \\ \hat{x} \\ 0_{\sigma+\eta} \end{bmatrix},$$

which means that $\hat{z}$ takes the form

$$\hat{z} = T^{-1} \begin{bmatrix} \Delta_{\epsilon_1+1} z_1 \\ \vdots \\ \Delta_{\epsilon_s+1} z_s \\ 0_{\rho+\sigma+\eta} \end{bmatrix},$$

and we have $\mathcal{L}^1_{\epsilon_j}\Delta_{\epsilon_j+1} z_j(0) = 0$, for $j = 1,\ldots,s$. By using the definition of $\mathcal{L}^1_{\epsilon_j}$ from (2.5) we see that this implies

$$\Delta_{\epsilon_j} z_j(0) = 0, \tag{A.1}$$

for $j = 1,\ldots,s$. Let $\tilde{b} : \mathbb{R} \to \mathbb{R}$ be a smooth transition from 1 to 0, i.e., an infinitely often differentiable function such that

$$\tilde{b}(t) = \begin{cases} 1 & t < 0 \\ 0 & t > 1 \end{cases}.$$

Define the sequence of functions $z_k \in \mathcal{C}^q_\infty$ through

$$z_k(t) = T^{-1} \begin{bmatrix} \Delta_{\epsilon_1+1}\left[b(kt)z_1(t)\right] \\ \vdots \\ \Delta_{\epsilon_s+1}\left[b(kt)z_s(t)\right] \\ 0_{\rho+\sigma+\eta} \end{bmatrix},$$

and observe that (2.9) implies that all $z_k \in \mathfrak{B}(P)$ are trajectories of the system. With this, (A.1), and Lemma A.1 we deduce that

$$\|z_k\|^2_+ \;\leq\; \|T\|^2 \sum_{j=1}^{s} \left\|\Delta_{\epsilon_j+1}\left[b(kt)z_j(t)\right]\right\|^2_+ = \|T\|^2 \sum_{j=1}^{s}\sum_{i=0}^{\epsilon_j} \left\|\left(\tfrac{d}{dt}\right)^i \left[b(kt)z_j(t)\right]\right\|^2_+$$

$$\leq \quad \|T\|^2 \sum_{j=1}^{s} \sum_{i=0}^{\epsilon_j} C_j \frac{1}{k} = \frac{D}{k},$$

by setting

$$D := \|T\|^2 \sum_{j=1}^{s} \sum_{i=0}^{\epsilon_j} C_j = \|T\|^2 \sum_{j=1}^{s} C_j(\epsilon_j + 1).$$

Since the construction of the z_k implies that for all $k = 1, 2, \ldots$ we have that $\hat{z}(t) = z_k(t)$ for $t \leq 0$ and $\hat{z}(t) = 0$ for $t \geq \frac{1}{k}$, we only have to choose k big enough and the claim is proved. The proof of point 2. works analogously by choosing a smooth transition from 0 to 1. $\square$

Theorem A.3. *Let $P \in \mathbb{C}[\lambda]_K^{p,q}$ and $H = H^* \in \mathbb{C}^{qK,qK}$. Let $z_+ \in \mathfrak{B}_+(P)$ and $z_- \in \mathfrak{B}_-(P)$. Then we have*

$$-\Theta_+(\Delta_K z_+(0)) \quad = \quad \inf_{\substack{z \in \mathfrak{B}_+(P) \\ z(t) = z_+(t), t \leq 0}} \int_0^\infty (\Delta_K z(t))^* H (\Delta_K z(t))\, dt$$

and

$$\Theta_-(\Delta_K z_-(0)) \quad = \quad \inf_{\substack{z \in \mathfrak{B}_-(P) \\ z(t) = z_-(t), t \geq 0}} \int_{-\infty}^0 (\Delta_K z(t))^* H (\Delta_K z(t))\, dt.$$

If $K = 1$ we also have that

$$-\Theta_+(\Delta_K z_+(0)) \quad = \quad \inf_{\substack{z \in \mathfrak{B}_+(P) \\ P^{\langle k \rangle}\left(\frac{d}{dt}\right) z(0) = P^{\langle k \rangle}\left(\frac{d}{dt}\right) z_+(0),\ k \geq 1}} \int_0^\infty (\Delta_K z(t))^* H (\Delta_K z(t))\, dt,$$

and

$$\Theta_-(\Delta_K z_-(0)) \quad = \quad \inf_{\substack{z \in \mathfrak{B}_-(P) \\ P^{\langle k \rangle}\left(\frac{d}{dt}\right) z(0) = P^{\langle k \rangle}\left(\frac{d}{dt}\right) z_-(0),\ k \geq 1}} \int_{-\infty}^0 (\Delta_K z(t))^* H (\Delta_K z(t))\, dt.$$

Proof. We only show the result for the available storage. The proof for the required supply works analogously. Using the canonical linearization (2.3) we see that it is sufficient to proof that for first order systems $\lambda F + G \in \mathbb{C}[\lambda]_1^{p,q}$ and for $H = H^* \in \mathbb{C}^{q,q}$ we have

$$\begin{aligned}
\inf_{\substack{z \in \mathfrak{B}_+(\lambda F + G) \\ z(t) = z_+(t), t \leq 0}} \int_0^\infty z^*(t) H z(t) dt \quad &= \quad \inf_{\substack{z \in \mathfrak{B}_+(\lambda F + G) \\ z(0) = z_+(0)}} \int_0^\infty z^*(t) H z(t) dt \\
&= \quad \inf_{\substack{z \in \mathfrak{B}_+(\lambda F + G) \\ F z(0) = F z_+(0)}} \int_0^\infty z^*(t) H z(t) dt.
\end{aligned}$$

From the basic theory of sets we immediately see that

$$\inf_{\substack{z \in \mathfrak{B}_+(\lambda F + G) \\ z(t) = z_+(t), t \leq 0}} \int_0^\infty z^*(t) H z(t) dt \quad \geq \quad \inf_{\substack{z \in \mathfrak{B}_+(\lambda F + G) \\ z(0) = z_+(0)}} \int_0^\infty z^*(t) H z(t) dt$$

$$\geq \inf_{\substack{z \in \mathfrak{B}_+(\lambda F+G) \\ Fz(0)=Fz_+(0)}} \int_0^\infty z^*(t)Hz(t)dt.$$

Let $z \in \mathfrak{B}_+(P)$ be arbitrary with $Fz(0) = Fz_+(0)$. We will show in the following that for every such trajectory and every $\theta > 0$, there exists a trajectory $z_\theta \in \mathfrak{B}_+(P)$ such that

$$\int_0^\infty z^*(t)Hz(t)dt + \theta \geq \int_0^\infty z_\theta^*(t)Hz_\theta(t)dt,$$

while at the same time $z_\theta(t) = z_+(t)$ for all $t \leq 0$. From this one obtains that

$$\inf_{\substack{z \in \mathfrak{B}_+(\lambda F+G) \\ Fz(0)=Fz_+(0)}} \int_0^\infty z^*(t)Hz(t)dt \geq \inf_{\substack{z \in \mathfrak{B}_+(\lambda F+G) \\ z(t)=z_+(t), t\leq 0}} \int_0^\infty z^*(t)Hz(t)dt$$

and thus the claim follows. Thus, let $z \in \mathfrak{B}_+(P)$ be arbitrary with $Fz(0) = Fz_+(0)$ and let $\theta > 0$. Define $y := z - z_+ \in \mathfrak{B}_+(\lambda F + G)$ and notice that this implies that $Fy(0) = 0$. Using Lemma A.2 we find that there exists a trajectory $y_\theta \in \mathfrak{B}_+(P)$ such that $y_\theta(t) = y(t)$ for $t \leq 0$ and

$$\|y_\theta\|_+ < \sqrt{\|z\|_+^2 + \frac{\theta}{\|H\|}} - \|z\|_+ . \tag{A.2}$$

We define $z_\theta := z - y_\theta \in \mathfrak{B}_+(P)$, and see that with this for $t \leq 0$ we have

$$z_\theta(t) = z(t) - y_\theta(t) = z(t) - y(t) = z(t) - (z(t) - z_+(t)) = z_+(t).$$

Also, we obtain the inequality

$$\begin{aligned} & \int_0^\infty z_\theta^*(t)Hz_\theta(t)dt = \langle Hz_\theta, z_\theta\rangle_+ = \langle Hz - Hy_\theta, z - y_\theta\rangle_+ \\ = \ & \langle Hz, z\rangle_+ - 2\mathrm{Re}\left(\langle Hz, y_\theta\rangle_+\right) + \langle Hy_\theta, y_\theta\rangle_+ \\ \leq \ & \langle Hz, z\rangle_+ + 2\left|\langle Hz, y_\theta\rangle_+\right| + \left|\langle Hy_\theta, y_\theta\rangle_+\right| \\ \leq \ & \langle Hz, z\rangle_+ + 2\|H\|\,\|z\|_+\,\|y_\theta\|_+ + \|H\|\,\|y_\theta\|_+^2 . \end{aligned} \tag{A.3}$$

From (A.2) we obtain that

$$\|z\|_+^2 + \frac{\theta}{\|H\|} > \left(\|y_\theta\|_+ + \|z\|_+\right)^2 = \|y_\theta\|_+^2 + 2\|y_\theta\|_+\|z\|_+ + \|z\|_+^2 ,$$

and thus

$$\theta > \|H\|\,\|y_\theta\|_+^2 + 2\|H\|\,\|y_\theta\|_+\|z\|_+ .$$

Together with (A.3) this yields

$$\int_0^\infty z_\theta^*(t)Hz_\theta(t)dt \;<\; \langle Hz, z\rangle_+ + \theta = \int_0^\infty z^*(t)Hz(t)dt + \theta,$$

and thus the assertion follows. □

A.2 Characterizations of cyclo-dissipativity

Lemma A.4. *Let $P \in \mathbb{C}[\lambda]^{p,q}$ with $r = \operatorname{rank}_{\mathbb{C}(\lambda)}(P)$ and let $U \in \mathbb{C}[\lambda]^{q,q-r}$ be a polynomial kernel matrix of P without zeros. Then we have the following:*

1. *If $z \in \mathfrak{B}(P)$ is such that for some fixed $t_0 \in \mathbb{R}$ we have $z^{(k)}(t_0) = 0$, for all $k \in \mathbb{N}$ then there exists an $\alpha \in \mathcal{C}_\infty^{q-r}$ such that $z = U\left(\frac{d}{dt}\right)\alpha$.*

2. *If $z \in \mathfrak{B}_+(P) \cap \mathfrak{B}_-(P)$ then there exists an $\alpha \in \mathcal{C}_+^{q-r} \cap \mathcal{C}_-^{q-r}$ such that $z = U\left(\frac{d}{dt}\right)\alpha$.*

Proof. We show point 1. and 2. simultaneously by allowing t_0 from point 1. to assume the value $\pm\infty$. Let the Smith form of P be given by (2.1). Define $y := T\left(\frac{d}{dt}\right)z$. Since T has the representation $T(\lambda) = \sum_{i=0}^{\tau} \lambda^i T_i$, for some $\tau \in \mathbb{N}$ and $T_i \in \mathbb{C}^{q,q}$, we see that for all $k \in \mathbb{N}_0$ we have $y^{(k)}(t_0) = \sum_{i=0}^{\tau} T_i z^{(i+k)}(t_0) = 0$, due to the assumption. Denoting the elements of y by y_i and using that $S^{-1}\left(\frac{d}{dt}\right)P\left(\frac{d}{dt}\right)z = 0$, we find that $d_i\left(\frac{d}{dt}\right)y_i(t) = 0$ for $i = 1, \ldots, r$. For each $i = 1, \ldots, r$ we distinguish two cases. The first case is when $d_i(\lambda) \equiv \tilde{d}_i$ is a constant non-zero polynomial. Then $\tilde{d}_i y_i(t) = 0$ implies that $y_i(t) = 0$ for all $t \in \mathbb{R}$. The second case is when $d_i(\lambda)$ is a polynomial of degree higher than or equal to 1. In this case $d_i\left(\frac{d}{dt}\right)y_i(t) = 0$ constitutes a differential equation. Since we have already derived the initial/boundary conditions $y_i^{(k)}(t_0) = 0$ for all $k \in \mathbb{N}$, we see from the basic theory of homogeneous linear differential equations that in this case also $y_i(t) = 0$ for all $t \in \mathbb{R}$.
Thus, we have shown that y takes the form $y = \begin{bmatrix} 0 & \tilde{y}^T \end{bmatrix}^T$, with $\tilde{y} \in \mathcal{C}_{(\pm)}^{q-r}$. Partition the inverse of T as $T^{-1} = \begin{bmatrix} T_1 & T_2 \end{bmatrix}$ with T_1 having r columns and T_2 having $q-r$ columns. Using Lemma 2.9 and Remark 2.10 we obtain the existence of a unimodular $U_2 \in \mathbb{C}[\lambda]^{q-r,q-r}$ such that $U = T_2 U_2$. Setting $\alpha := U_2^{-1}\left(\frac{d}{dt}\right)\tilde{y}$ we find that

$$z = T^{-1}\left(\tfrac{d}{dt}\right)y = T_2\left(\tfrac{d}{dt}\right)\tilde{y} = T_2\left(\tfrac{d}{dt}\right)U_2\left(\tfrac{d}{dt}\right)U_2^{-1}\left(\tfrac{d}{dt}\right)\tilde{y} = U\left(\tfrac{d}{dt}\right)\alpha,$$

which proves the claim. □

Lemma A.5. *Consider the system $P \in \mathbb{C}[\lambda]_K^{p,q}$ and let $z_+ \in \mathfrak{B}_+(P)$ and $z_- \in \mathfrak{B}_-(P)$ be such that $\Delta_K z_+(0) = \Delta_K z_-(0)$. Then for every $\epsilon > 0$ there exists a trajectory $\tilde{z} \in \mathfrak{B}_+(P) \cap \mathfrak{B}_-(P)$ such that $\|\Delta_K(\tilde{z} - z_+)\|_+ + \|\Delta_K(\tilde{z} - z_-)\|_- < \epsilon$.*

Proof. [27, Corollary 2.4.12] Using the canonical linearization (2.3) we see that it is sufficient to prove the result for first order systems $P(\lambda) = \lambda F + G$. For first order systems, observe that $\hat{z} := z_+ - z_- \in \mathfrak{B}(P)$ is a trajectory of the system which satisfies $\hat{z}(0) = z_+(0) - z_-(0) = 0$. Thus, with Lemma A.2 we can construct a $\hat{z}_1 \in \mathfrak{B}(P)$ such that $\hat{z}_1(t) = \hat{z}(t)$ for all $t \geq 0$, $\hat{z}_1(t) = 0$ for all $t \leq -\epsilon$, and $\|\hat{z}_1\|_- < \epsilon$. Set $\tilde{z} := \hat{z}_1 + z_-$. Then for all $t \geq 0$ we have $\tilde{z}(t) = \hat{z}(t) + z_-(t) = z_+(t)$, for all $t \leq -\epsilon$ we have $\tilde{z}(t) = z_-(t)$, and we have $\|\tilde{z} - z_-\|_- = \|\hat{z}_1\|_- < \epsilon$. Clearly, we also have $\|\tilde{z} - z_+\|_+ = 0$. This implies the assertion. □

Lemma A.6. *Let $P \in \mathbb{C}[\lambda]_K^{p,q}$ and $H = H^* \in \mathbb{C}^{qK,qK}$. Then (P, H) is cyclo-dissipative if and only if*

$$0 \leq \int_{-\infty}^{\infty} (\Delta_K z(t))^* H (\Delta_K z(t))\, dt,$$

for all $z \in \mathfrak{B}_+(P) \cup \mathfrak{B}_-(P)$.

Proof. The "if" part is trivial. For the "only if" part assume to the contrary that there was a trajectory $z \in \mathfrak{B}_+(P) \cap \mathfrak{B}_-(P)$ such that

$$\int_{-\infty}^{\infty} (\Delta_K z(t))^* H (\Delta_K z(t))\, dt < 0. \tag{A.4}$$

Let $U \in \mathbb{C}[\lambda]^{q,q-r}$ be a polynomial kernel matrix of P without zeros, where $r = \operatorname{rank}_{\mathbb{C}(\lambda)}(P)$. According to point 2. of Lemma A.4 this implies the existence of an $\alpha \in \mathcal{C}_+^{q-r} \cap \mathcal{C}_-^{q-r}$ such that $z = U\left(\frac{d}{dt}\right)\alpha$. Using Lemma 2.11 we obtain the existence of a polynomial matrix $X \in \mathbb{C}[\lambda]^{q-r,q}$ without zeros such that $XU = I_{q-r}$. Let U take the form $U(\lambda) = \sum_{i=0}^{M} U_i \lambda^i$, with $M \in \mathbb{N}$ and $U_i \in \mathbb{C}^{q,q-r}$ and let $\tilde{b}$ be a smooth transition from 1 to 0, i.e., let $\tilde{b} \in \mathcal{C}_\infty$ fulfill $\tilde{b}(t) = 1$ for $t \leq 0$ and $\tilde{b}(t) = 0$ for $t \geq 1$. Define a family of functions $z_T \in \mathfrak{B}_c(P)$ through

$$z_T(t) := U\left(\tfrac{d}{dt}\right)\left[\alpha(t)\tilde{b}(t-T)\tilde{b}(-t-T)\right],$$

for $T \in \mathbb{R}$. With this we obtain from the assumption of cyclo-dissipativity that

$$\begin{aligned} 0 \;\leq\; & \int_{-\infty}^{\infty} (\Delta_K z_T(t))^* H (\Delta_K z_T(t))\, dt = \int_{-T}^{T} (\Delta_K z(t))^* H (\Delta_K z(t))\, dt \\ + \; & \int_{-\infty}^{-T} (\Delta_K z_T(t))^* H (\Delta_K z_T(t))\, dt + \int_{T}^{\infty} (\Delta_K z_T(t))^* H (\Delta_K z_T(t))\, dt. \end{aligned}$$

Using (A.4) we see that there exists an $\epsilon > 0$ and a T_0 such that for all $T \geq T_0$ we have

$$\int_{-T}^{T} (\Delta_K z(t))^* H (\Delta_K z(t))\, dt < -\epsilon,$$

e.g., one can choose

$$\epsilon = -\frac{1}{2}\int_{-\infty}^{\infty} (\Delta_K z(t))^* H (\Delta_K z(t))\, dt.$$

We also observe that

$$\begin{aligned} & \left| \int_{T}^{\infty} (\Delta_K z_T(t))^* H (\Delta_K z_T(t))\, dt \right| \\ = \; & \left| \int_{0}^{\infty} (\Delta_K z_T(t+T))^* H (\Delta_K z_T(t+T))\, dt \right| \\ = \; & \left| \langle H\Delta_K z_T(\cdot + T), \Delta_K z_T(\cdot + T)\rangle_+ \right| \\ \leq \; & \|H\| \, \|\Delta_K z_T(\cdot + T)\|_+^2 \\ = \; & \|H\| \left\| \Delta_K U\left(\tfrac{d}{dt}\right)\left[\alpha(\cdot + T)\tilde{b}(\cdot)\right] \right\|_+^2 \\ = \; & \|H\| \sum_{i=0}^{K-1} \left\| \left(\tfrac{d}{dt}\right)^i U\left(\tfrac{d}{dt}\right)\left[\alpha(\cdot + T)\tilde{b}(\cdot)\right] \right\|_+^2 \end{aligned}$$

$$
\begin{aligned}
&= \|H\| \sum_{i=0}^{K-1} \left\| \left(\tfrac{d}{dt}\right)^i \sum_{j=0}^{M} \left(\tfrac{d}{dt}\right)^j U_j \left[\alpha(\cdot + T)\tilde{b}(\cdot)\right] \right\|_+^2 \\
&\leq \|H\| \sum_{i=0}^{K-1} \sum_{j=0}^{M} \left\| U_j \left(\tfrac{d}{dt}\right)^{i+j} \left[\alpha(\cdot + T)\tilde{b}(\cdot)\right] \right\|_+^2 \\
&\leq \|H\| \sum_{i=0}^{K-1} \sum_{j=0}^{M} \|U_j\|^2 \left\| \left(\tfrac{d}{dt}\right)^{i+j} \left[\alpha(\cdot + T)\tilde{b}(\cdot)\right] \right\|_+^2 \\
&= \|H\| \sum_{i=0}^{K-1} \sum_{j=0}^{M} \|U_j\|^2 \left\| \sum_{k=0}^{i+j} \binom{i+j}{k} \alpha^{(k)}(\cdot + T)\tilde{b}^{(i+j-k)}(\cdot) \right\|_+^2 \\
&\leq \|H\| \sum_{i=0}^{K-1} \sum_{j=0}^{M} \|U_j\|^2 \sum_{k=0}^{i+j} \binom{i+j}{k} \left\| \alpha^{(k)}(\cdot + T)\tilde{b}^{(i+j-k)}(\cdot) \right\|_+^2, \qquad \text{(A.5)}
\end{aligned}
$$

and analogously

$$
\begin{aligned}
&\left| \int_{-\infty}^{-T} \left(\Delta_K z_T(t)\right)^* H \left(\Delta_K z_T(t)\right) dt \right| \\
&= \|H\| \left\| \Delta_K U\left(\tfrac{d}{dt}\right) \left[\alpha(\cdot - T)\tilde{b}(-\cdot)\right] \right\|_-^2 \\
&\leq \|H\| \sum_{i=0}^{K-1} \sum_{j=0}^{M} \|U_j\|^2 \left\| \sum_{k=0}^{i+j} \binom{i+j}{k} \alpha^{(k)}(\cdot - T)\tilde{b}^{(i+j-k)}(-\cdot)(-1)^{i+j-k} \right\|_-^2 \\
&\leq \|H\| \sum_{i=0}^{K-1} \sum_{j=0}^{M} \|U_j\|^2 \sum_{k=0}^{i+j} \binom{i+j}{k} \left\| \alpha^{(k)}(\cdot - T)\tilde{b}^{(i+j-k)}(-\cdot) \right\|_-^2. \qquad \text{(A.6)}
\end{aligned}
$$

Define

$$
B := \max_{\substack{t\in\mathbb{R} \\ k=0,\dots,M+K-1}} \left|\tilde{b}^{(k)}(t)\right|_2 = \max_{\substack{t\in[0,1] \\ k=0,\dots,M+K-1}} \left|\tilde{b}^{(k)}(t)\right|_2
$$

and choose $C, D > 0$ such that

$$
\left\|\alpha^{(k)}(t)\right\| \leq Ce^{-D|t|},
$$

for all $t \in \mathbb{R}$ and all $k = 0, \dots, M+K-1$. Then we obtain

$$
\begin{aligned}
&\left\| \alpha^{(k)}(\cdot + T)\tilde{b}^{(i+j-k)}(\cdot) \right\|_+^2 = \int_0^\infty \left\| \alpha^{(k)}(t + T)\tilde{b}^{(i+j-k)}(t) \right\|_2^2 dt \\
&\leq B^2 \int_0^\infty \left\|\alpha^{(k)}(t+T)\right\|_2^2 dt \leq B^2 \int_0^\infty C^2 e^{-2D(t+T)} dt \qquad \text{(A.7)} \\
&= B^2C^2e^{-2DT} \int_0^\infty e^{-2Dt} dt = B^2C^2e^{-2DT} \frac{1}{-2D} e^{-2Dt}\Big|_0^\infty = \frac{B^2C^2}{2D} e^{-2DT},
\end{aligned}
$$

and analogously

$$
\begin{aligned}
&\left\|\alpha^{(k)}(\cdot - T)\tilde{b}^{(i+j-k)}(-\cdot)\right\|_{+}^{2} \leq B^2 \int_{-\infty}^{0} C^2 e^{-2D|t-T|} dt = B^2 \int_{-\infty}^{0} C^2 e^{-2D(T-t)} dt \\
= \;& B^2 C^2 e^{-2DT} \int_{-\infty}^{0} e^{2Dt} dt = B^2 C^2 e^{-2DT} \frac{1}{2D} e^{2Dt}\Big|_{-\infty}^{0} = \frac{B^2 C^2}{2D} e^{-2DT},
\end{aligned}
\tag{A.8}
$$

for all $i = 0, \ldots, K-1$, $j = 0, \ldots, M$, and all $k = 0, \ldots, i+j$. Combining (A.5) and (A.7) yields

$$
\begin{aligned}
&\left|\int_{T}^{\infty} (\Delta_K z_T(t))^* H (\Delta_K z_T(t))\, dt\right| \\
\leq \;& \|H\| \sum_{i=0}^{K-1} \sum_{j=0}^{M} \|U_j\|^2 \sum_{k=0}^{i+j} \binom{i+j}{k} \frac{B^2 C^2}{2D} e^{-2DT} \leq e^{-2DT} A,
\end{aligned}
$$

whereas combining (A.6) and (A.8) yields

$$
\begin{aligned}
&\left|\int_{-\infty}^{-T} (\Delta_K z_T(t))^* H (\Delta_K z_T(t))\, dt\right| \\
\leq \;& \|H\| \sum_{i=0}^{K-1} \sum_{j=0}^{M} \|U_j\|^2 \sum_{k=0}^{i+j} \binom{i+j}{k} \frac{B^2 C^2}{2D} e^{-2DT} \leq e^{-2DT} A,
\end{aligned}
$$

if one defines

$$
A := \|H\| \sum_{i=0}^{K-1} \sum_{j=0}^{M} \|U_j\|^2 \sum_{k=0}^{i+j} \binom{i+j}{k} \frac{B^2 C^2}{2D} \geq 0.
$$

Now, choosing $T_1 \geq T_0$ such that $2e^{-2DT_1} A \leq \epsilon$, we finally see that

$$
\begin{aligned}
0 \;&\leq \int_{-\infty}^{\infty} (\Delta_K z_{T_1}(t))^* H (\Delta_K z_{T_1}(t))\, dt \\
&= \int_{-T_1}^{T_1} (\Delta_K z(t))^* H (\Delta_K z(t))\, dt + \int_{T_1}^{\infty} (\Delta_K z_{T_1}(t))^* H (\Delta_K z_{T_1}(t))\, dt \\
&\quad + \int_{-\infty}^{-T_1} (\Delta_K z_{T_1}(t))^* H (\Delta_K z_{T_1}(t))\, dt \\
&< -\epsilon + \left|\int_{T_1}^{\infty} (\Delta_K z_{T_1}(t))^* H (\Delta_K z_{T_1}(t))\, dt\right| + \left|\int_{-\infty}^{-T_1} (\Delta_K z_{T_1}(t))^* H (\Delta_K z_{T_1}(t))\, dt\right| \\
&\leq -\epsilon + 2e^{-2DT_1} A \leq -\epsilon + \epsilon = 0,
\end{aligned}
$$

which is a contradiction and thus the claim is proved. □

Lemma A.7. *Let $P \in \mathbb{C}[\lambda]^{p,q}$ with $r := \operatorname{rank}_{\mathbb{C}(\lambda)}(P)$ and let $U \in \mathbb{C}(\lambda)^{q,q-r}$ be a kernel matrix of P. Let $Z : \mathbb{C} \to \mathbb{C}^q$ be a continuous function such that $0 = P(\lambda)Z(\lambda)$, for $\lambda \in \mathbb{C}$. Then, there exists a function $\alpha : \mathbb{C} \setminus \mathfrak{Z}(U) \to \mathbb{C}^{q-r}$ such that*

$$Z(\lambda) = U(\lambda)\alpha(\lambda),$$

for all $\lambda \in \mathfrak{D}(U) \setminus \mathfrak{Z}(U)$.

Proof. Let a Smith form of P be given by (2.1) and partition the inverse of T as $T^{-1} = \begin{bmatrix} T_1 & T_2 \end{bmatrix}$ with T_1 having r columns and T_2 having $q-r$ columns. Using Lemma 2.9 we obtain the existence of an invertible $U_2 \in \mathbb{C}(\lambda)^{q-r,q-r}$ such that $U = T_2U_2$ and $\mathfrak{Z}(U) = \mathfrak{Z}(U_2)$ and $\mathfrak{P}(U) = \mathfrak{P}(U_2)$. Define the continuous functions $Z_1 : \mathbb{C} \to \mathbb{C}^r$, $Z_2 : \mathbb{C} \to \mathbb{C}^{q-r}$, and $\tilde{Z} : \mathbb{C} \to \mathbb{C}^q$ through

$$\begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} := \tilde{Z} := TZ,$$

and observe that this implies

$$0 = S^{-1}PZ = \begin{bmatrix} \operatorname{diag}(d_1, \dots, d_r) & 0 \\ 0 & 0 \end{bmatrix} TZ = \begin{bmatrix} \operatorname{diag}(d_1, \dots, d_r) Z_1 \\ 0 \end{bmatrix},$$

which in turn implies that $Z_1 \equiv 0$, since $\operatorname{diag}(d_1, \dots, d_r)$ is invertible and continuous. From this we deduce that

$$Z = T^{-1}\tilde{Z} = \begin{bmatrix} T_1 & T_2 \end{bmatrix} \begin{bmatrix} 0 \\ Z_2 \end{bmatrix} = T_2Z_2.$$

With this notation at hand define $\alpha := U_2^{-1}Z_2$. With Lemma 2.11 we find that $\mathfrak{D}\left(U_2^{-1}\right) = \mathbb{C} \setminus \mathfrak{Z}(U)$. Since Z_2 is defined on all of $\mathbb{C}$, the function α can be considered to be function defined on $\mathbb{C} \setminus \mathfrak{Z}(U)$. Finally, from the equation $U = T_2U_2$ we find that $UU_2^{-1} = T_2$ and conclude that

$$Z = T_2Z_2 = UU_2^{-1}Z_2 = U\alpha,$$

which is the assertion. □

Definition A.8. For a function with compact support $z \in \mathcal{C}_c^q$ we define its *two sided Laplace-transform* $Z : \mathbb{C} \to \mathbb{C}^q$ via

$$Z(\lambda) := \int_{-\infty}^{\infty} e^{-\lambda t} z(t) dt.$$

The two sided Laplace-transform is also denoted by $Z = \mathfrak{L}\{z\}$.

Lemma A.9. *Let $z_1, z_2 \in \mathcal{C}_c^q$ be two functions with compact support. Then their two sided Laplace-transforms Z_1, Z_2 are continuous functions that are well defined for all $\lambda \in \mathbb{C}$, i.e., $Z_1(\lambda)$ and $Z_2(\lambda)$ are well defined in the complete complex plane. Furthermore, the Parseval/Plancherel identity*

$$\int_{-\infty}^{\infty} z_1^*(t) z_2(t) dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} Z_1^*(i\omega) Z_2(i\omega) d\omega$$

holds. For the derivative of a function z_1 we have the formula

$$\mathfrak{L}\left\{\dot{z}_1\right\}(\lambda) = \lambda Z_1(\lambda).$$

Proof. [23, pp. 8-13] We dismiss the subscript and simply write z when meaning either z_1 or z_2. Since z is assumed to have compact support, there has to be a $R \in \mathbb{R}^+$ such that $z(t) = 0$ for every $|t| \geq R$. Thus, for every $\lambda \in \mathbb{C}$ the integral

$$Z(\lambda) = \int_{-\infty}^{\infty} e^{-\lambda t} z(t) dt = \int_{-R}^{R} e^{-\lambda t} z(t) dt$$

is an integral of a continuous function over a compact interval and thus exists.
Partial integration shows that we have

$$\begin{aligned} \mathfrak{L}\left\{\dot{z}\right\} &= \int_{-\infty}^{\infty} e^{-\lambda t}\dot{z}(t)dt = \int_{-R}^{R} e^{-\lambda t}\dot{z}(t)dt = e^{-\lambda t}z(t)\Big|_{-R}^{R} - \int_{-R}^{R}(-\lambda)e^{-\lambda t}z(t)dt \\ &= \lambda \int_{-R}^{R} e^{-\lambda t}z(t)dt = \lambda \int_{-\infty}^{\infty} e^{-\lambda t}z(t)dt = \lambda Z(\lambda). \end{aligned}$$

To see that Z is a continuous function let $\{\lambda_k\}_{k\in\mathbb{N}} \subset \mathbb{C}$ be a sequence that converges to a $\lambda \in \mathbb{C}$. Then the functions

$$z_k(t) := e^{-\lambda_k t} z(t) \in \mathcal{C}_c^q$$

constitute a sequence of functions which converges uniformly to the function $e^{-\lambda t}z(t)$. Thus, from basic calculus we know that

$$\lim_{k\to\infty} Z(\lambda_k) = \lim_{k\to\infty}\int_{-\infty}^{\infty} e^{\lambda_k t}z(t)dt = \lim_{k\to\infty}\int_{-R}^{R} e^{\lambda_k t}z(t)dt = \int_{-R}^{R}\lim_{k\to\infty} e^{\lambda_k t}z(t)dt = Z(\lambda),$$

which proves the continuity. The Parseval/Plancherel identity is harder to prove, see [13, §12]. ☐

Lemma A.10. *Let $P \in \mathbb{C}[\lambda]^{p,q}$ with $r = \operatorname{rank}_{\mathbb{C}(\lambda)}(P)$ and let $U \in \mathbb{C}[\lambda]^{q,q-r}$ be a polynomial kernel matrix of P without zeros. Let $\omega_0 \in \mathbb{R}$ and $v \in \mathbb{C}^{q-r}$ be arbitrary but fixed. Define*

$$\tilde{\omega}_0 := \begin{cases} |\omega_0| & \text{if } \omega_0 \neq 0 \\ 1 & \text{if } \omega_0 = 0 \end{cases}.$$

Then we can construct a sequence of trajectories $\{z_k\}_{k\in\mathbb{N}} = \left\{U\left(\frac{d}{dt}\right)v_k\right\}_{k\in\mathbb{N}} \subset \mathfrak{B}_c(P)$ with $v_k \in \mathcal{C}_c^{q-r}$ such that the following properties are satisfied for all $k \in \mathbb{N}$.

1. $z_k(t) = U\left(\frac{d}{dt}\right)ve^{\omega_0 t} = U(\omega_0)ve^{\omega_0 t}$ *for all* $t \in \left[-\frac{2\pi k}{\tilde{\omega}_0}, \frac{2\pi k}{\tilde{\omega}_0}\right]$.

2. $z_0(t) = z_k\left(t + \frac{2\pi k}{\tilde{\omega}_0}\right)$ *for all* $t \in \left[0, \frac{2\pi}{\tilde{\omega}_0}\right]$.

3. $z_0(t) = z_k\left(t - \frac{2\pi k}{\tilde{\omega}_0}\right)$ *for all* $t \in \left[-\frac{2\pi}{\tilde{\omega}_0}, 0\right]$.

4. $z_k(t) = 0$ *for all* $t \in \left]-\infty, -\frac{2\pi(k+1)}{\tilde{\omega}_0}\right] \cup \left[\frac{2\pi(k+1)}{\tilde{\omega}_0}, \infty\right[$.

Proof. [45, Proof of Theorem 3.1] Let $\tilde{b} : \mathbb{R} \to \mathbb{R}$ be a smooth transition from 0 to 1, i.e., an infinitely often differentiable function such that

$$\tilde{b}(t) = \begin{cases} 0 & t < -1 \\ 1 & t > 0 \end{cases}.$$

Define the sequence of functions $b_k : \mathbb{R} \to \mathbb{R}$ through

$$b_k(t) = \tilde{b}\left(t\tfrac{\tilde{\omega}_0}{2\pi} + k\right)\tilde{b}\left(-t\tfrac{\tilde{\omega}_0}{2\pi} + k\right),$$

for $k \in \mathbb{N}$ and observe that all b_k are infinitely often differentiable and have the properties

$$b_k(t) = \begin{cases} 0 & \text{for } t \in \left]-\infty, -\frac{2\pi(k+1)}{\tilde{\omega}_0}\right] \cup \left[\frac{2\pi(k+1)}{\tilde{\omega}_0}, \infty\right[\\ 1 & \text{for } t \in \left[-\frac{2\pi k}{\tilde{\omega}_0}, \frac{2\pi k}{\tilde{\omega}_0}\right] \end{cases}. \tag{A.9}$$

Next, define the sequence of trajectories $z_k \in \mathfrak{B}_c(P)$ through

$$z_k(t) := U\left(\tfrac{d}{dt}\right)\left[ve^{\omega_0 t}b_k(t)\right].$$

The construction implies that for $t \in \left[-\frac{2\pi k}{\tilde{\omega}_0}, \frac{2\pi k}{\tilde{\omega}_0}\right]$ we have

$$z_k(t) = U\left(\tfrac{d}{dt}\right)\left[ve^{\omega_0 t}\right] = U(\omega_0)ve^{\omega_0 t},$$

and thus we have shown part 1. Part 4. follows from (A.9). To see part 2. we find that for all $k \in \mathbb{N}$ and $t \in \left[0, \frac{2\pi}{\tilde{\omega}_0}\right]$ we have

$$\begin{aligned} z_k\left(t + \tfrac{2\pi k}{\tilde{\omega}_0}\right) &= U\left(\tfrac{d}{dt}\right)\left[ve^{\omega_0\left(t+\frac{2\pi k}{\tilde{\omega}_0}\right)}b_k\left(t + \tfrac{2\pi k}{\tilde{\omega}_0}\right)\right] \\ &= U\left(\tfrac{d}{dt}\right)\left[ve^{\omega_0 t + \frac{\omega_0}{\tilde{\omega}_0}2\pi k}\tilde{b}\left(t\tfrac{\tilde{\omega}_0}{2\pi} + k + k\right)\tilde{b}\left(-t\tfrac{\tilde{\omega}_0}{2\pi} - k + k\right)\right] \\ &= U\left(\tfrac{d}{dt}\right)\left[ve^{\omega_0 t}\underbrace{e^{\frac{\omega_0}{\tilde{\omega}_0}2\pi k}}_{=1}\tilde{b}\left(t\tfrac{\tilde{\omega}_0}{2\pi}\right)\tilde{b}\left(-t\tfrac{\tilde{\omega}_0}{2\pi}\right)\right] \\ &= U\left(\tfrac{d}{dt}\right)\left[ve^{\omega_0 t}b_0(t)\right] = z_0(t). \end{aligned}$$

Part 3. can be shown analogously and thus the proof is finished. □

Theorem A.11. *Let* $P \in \mathbb{C}[\lambda]_K^{p,q}$ *and* $H = H^* \in \mathbb{C}^{qK,qK}$. *Let* $r := \operatorname{rank}_{\mathbb{C}(\lambda)}(P)$, *let* $U \in \mathbb{C}(\lambda)^{q,q-r}$ *be a kernel matrix, and let* Π *be the Popov function of* (P, H) *associated with* U. *Then* (P, H) *is cyclo-dissipative if and only if*

$$\Pi(i\omega) \geq 0,$$

for all $\omega \in \mathbb{R}$ *such that* $i\omega \in \mathfrak{D}(\Pi) = \mathfrak{D}(U) \cap \mathfrak{D}(U^\sim)$.

Proof. [45, Proposition 5.2] First, assume that $\Pi(i\omega) \geq 0$ for all $i\omega \in \mathfrak{D}(\Pi)$ and let $z \in \mathfrak{B}_c(P)$ be arbitrary. Using Lemma A.9 we find that

$$\begin{aligned}
&\int_{-\infty}^{\infty} (\Delta_K z(t))^* H (\Delta_K z(t))\, dt \\
=& \int_{-\infty}^{\infty} \left(\Delta_K^q \left(\tfrac{d}{dt}\right) z(t)\right)^* H \left(\Delta_K^q \left(\tfrac{d}{dt}\right) z(t)\right) dt \\
=& \frac{1}{2\pi}\int_{-\infty}^{\infty} \left(\mathfrak{L}\left\{\Delta_K^q \left(\tfrac{d}{dt}\right) z\right\}(i\omega)\right)^* H \left(\mathfrak{L}\left\{\Delta_K^q \left(\tfrac{d}{dt}\right) z\right\}(i\omega)\right) dt \\
=& \frac{1}{2\pi}\int_{-\infty}^{\infty} (\Delta_K^q(i\omega)Z(i\omega))^* H (\Delta_K^q(i\omega)Z(i\omega))\, dt \\
=& \frac{1}{2\pi}\int_{-\infty}^{\infty} Z^*(i\omega)\Delta_K^{q\,\sim}(i\omega)H\Delta_K^q(i\omega)Z(i\omega)dt.
\end{aligned}$$

By taking the two sided Laplace-transform of the identity $P\left(\frac{d}{dt}\right) z = 0$, using the linearity of the two sided Laplace-transform, and using Lemma A.9 we obtain

$$0 = \mathfrak{L}\left\{P\left(\tfrac{d}{dt}\right) z\right\}(\lambda) = \sum_{j=0}^{d} P_j \mathfrak{L}\left\{z^{(j)}\right\}(\lambda) = P(\lambda)Z(\lambda).$$

Thus, Lemma A.7 shows that there exists an $\alpha : \mathbb{C}\backslash\mathfrak{Z}(U) \to \mathbb{C}^{q-r}$ such that $Z(\lambda) = U(\lambda)\alpha(\lambda)$ for all $\lambda \in \mathfrak{D}(U) \setminus \mathfrak{Z}(U)$. Since we can divide any integral over $\mathbb{R}$ into a finite number of distinct intervals such that the interior of each interval does not contain any zero or pole of U, we can write (in a slightly symbolic fashion) that

$$\begin{aligned}
\int_{-\infty}^{\infty} (\Delta_K z(t))^* H (\Delta_K z(t))\, dt &= \frac{1}{2\pi}\int_{-\infty}^{\infty} Z^*(i\omega)\Delta_K^{q\,\sim}(i\omega)H\Delta_K^q(i\omega)Z(i\omega)dt \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} \alpha^*(i\omega)U^*(i\omega)\Delta_K^{q\,\sim}(i\omega)H\Delta_K^q(i\omega)U(i\omega)\alpha(i\omega)dt \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} \alpha^*(i\omega)\Pi(i\omega)\alpha(i\omega)dt \geq 0,
\end{aligned}$$

since Π is assumed to be positive semi-definite along the imaginary axis. This means that (P, H) is cyclo-dissipative.

For the reverse direction assume to the contrary that there exists an ω_0 and a $v \in \mathbb{C}^{q-r}$ such that $v^*\Pi(i\omega_0)v < 0$. Let a Smith form of P be given by (2.1). Partition the inverse of T as $T^{-1} = \begin{bmatrix} T_1 & T_2 \end{bmatrix}$ with T_1 having r columns and T_2 having $q-r$ columns. Using Lemma 2.9 we obtain the existence of an invertible U_2 such that $U = T_2U_2$ with $\mathfrak{P}(U_2) = \mathfrak{P}(U)$ and $\mathfrak{Z}(U_2) = \mathfrak{Z}(U)$. W.l.o.g we may assume that $i\omega_0$ is not a pole or zero of both U_2 and U because if it is then one can choose a new ω_0 in the neighborhood of the old ω_0 where $\Pi(i\omega_0$ is still not positive semi-definite. Let $\tilde{\Pi} = T_2^{\sim}\Delta_K^{q\,\sim}H\Delta_K^q T_2$ be the Popov function associated with T_2 and set $\tilde{v} = U_2(i\omega_0)v$. Then also

$$\tilde{v}^*\tilde{\Pi}(i\omega_0)\tilde{v} = v^*U_2^*(i\omega_0)T_2^*(i\omega_0)\Delta_K^{q\,*}(i\omega_0)H\Delta_K^q(i\omega_0)T_2(i\omega_0)U_2(i\omega_0)v$$

$$= \quad v^* U^*(i\omega_0) {\Delta_K^q}^*(i\omega_0) H \Delta_K^q(i\omega_0) U(i\omega_0) v = v^* \Pi(i\omega_0) v < 0.$$

Using Lemma A.10 we construct a sequence of trajectories $\{z_k\}_{k\in\mathbb{N}} \subset \mathfrak{B}_c(P)$ such that

$$\begin{aligned}
z_k(t) = &\quad T_2(i\omega_0)\tilde{v}e^{i\omega_0 t}, &&\text{for } t \in \left[-\frac{2\pi k}{\tilde{\omega}_0}, \frac{2\pi k}{\tilde{\omega}_0}\right], \\
z_0\,(t) = &\quad z_k\left(t + \tfrac{2\pi k}{\tilde{\omega}_0}\right), &&\text{for } t \in \left[0, \frac{2\pi}{\tilde{\omega}_0}\right], \\
z_0\,(t) = &\quad z_k\left(t - \tfrac{2\pi k}{\tilde{\omega}_0}\right), &&\text{for } t \in \left[-\frac{2\pi}{\tilde{\omega}_0}, 0\right], \\
z_k(t) = &\quad 0, &&\text{for } t \in \left]-\infty, -\frac{2\pi(k+1)}{\tilde{\omega}_0}\right] \cup \left[\frac{2\pi(k+1)}{\tilde{\omega}_0}, \infty\right[,
\end{aligned}$$

where $\tilde{\omega}_0$ is defined as in the statement of Lemma A.10. This implies that for $k \in \mathbb{N}$ and $t \in \left[-\frac{2\pi k}{\tilde{\omega}_0}, \frac{2\pi k}{\tilde{\omega}_0}\right]$ we have

$$\Delta_K z_k(t) = \Delta_K^q\left(\tfrac{d}{dt}\right) T_2(i\omega_0)\tilde{v}e^{i\omega_0 t} = \Delta_K^q(i\omega_0) T_2(i\omega_0)\tilde{v}e^{i\omega_0 t},$$

and thus for $k \in \mathbb{N}$ we see that using the transformation rules $\phi_k(t) = t + \frac{2\pi k}{\tilde{\omega}_0}$ and $\psi_k(t) = t - \frac{2\pi k}{\tilde{\omega}_0}$ we obtain

$$\begin{aligned}
&\int_{-\infty}^{\infty} (\Delta_K z_k(t))^* H (\Delta_K z_k(t))\, dt \\
= &\int_{-\frac{2\pi k}{\tilde{\omega}_0}}^{\frac{2\pi k}{\tilde{\omega}_0}} (\Delta_K z_k(t))^* H (\Delta_K z_k(t))\, dt \\
&+ \int_{-\frac{2\pi(k+1)}{\tilde{\omega}_0}}^{-\frac{2\pi k}{\tilde{\omega}_0}} (\Delta_K z_k(t))^* H (\Delta_K z_k(t))\, dt + \int_{\frac{2\pi k}{\tilde{\omega}_0}}^{\frac{2\pi(k+1)}{\tilde{\omega}_0}} (\Delta_K z_k(t))^* H (\Delta_K z_k(t))\, dt \\
= &\int_{-\frac{2\pi k}{\tilde{\omega}_0}}^{\frac{2\pi k}{\tilde{\omega}_0}} \left(\Delta_K^q(i\omega_0) T_2(i\omega_0)\tilde{v}e^{i\omega_0 t}\right)^* H \left(\Delta_K^q(i\omega_0) T_2(i\omega_0)\tilde{v}e^{i\omega_0 t}\right) dt \\
&+ \int_{\psi_k\left(-\frac{2\pi}{\tilde{\omega}_0}\right)}^{\psi_k(0)} (\Delta_K z_k(t))^* H (\Delta_K z_k(t))\, dt + \int_{\phi_k(0)}^{\phi_k\left(\frac{2\pi}{\tilde{\omega}_0}\right)} (\Delta_K z_k(t))^* H (\Delta_K z_k(t))\, dt \\
= &\int_{-\frac{2\pi k}{\tilde{\omega}_0}}^{\frac{2\pi k}{\tilde{\omega}_0}} e^{-i\omega_0 t}\tilde{v}^* T_2^*(i\omega_0) \left(\Delta_K^q(i\omega_0)\right)^* H \Delta_K^q(i\omega_0) T_2(i\omega_0)\tilde{v}e^{i\omega_0 t} dt \\
&+ \int_{-\frac{2\pi}{\tilde{\omega}_0}}^{0} \dot{\psi}_k(t) \left(\Delta_K z_k(\psi_k(t))\right)^* H \left(\Delta_K z_k(\psi_k(t))\right) dt \\
&+ \int_{0}^{\frac{2\pi}{\tilde{\omega}_0}} \dot{\phi}_k(t) \left(\Delta_K z_k(\phi_k(t))\right)^* H \left(\Delta_K z_k(\phi_k(t))\right) dt
\end{aligned}$$

$$
\begin{aligned}
&= \tilde{v}^*\tilde{\Pi}(i\omega_0)\tilde{v}\int_{-\frac{2\pi k}{\tilde{\omega}_0}}^{\frac{2\pi k}{\tilde{\omega}_0}} dt \\
&\quad + \int_{-\frac{2\pi}{\tilde{\omega}_0}}^{0} \left(\Delta_K z_0(t)\right)^* H\left(\Delta_K z_0(t)\right) dt + \int_0^{\frac{2\pi}{\tilde{\omega}_0}} \left(\Delta_K z_0(t)\right)^* H\left(\Delta_K z_0(t)\right) dt \\
&= \frac{4\pi k}{\tilde{\omega}_0}\tilde{v}^*\tilde{\Pi}(i\omega_0)\tilde{v} + c,
\end{aligned}
$$

by setting

$$
c := \int_{-\frac{2\pi}{\tilde{\omega}_0}}^{0} \left(\Delta_K z_0(t)\right)^* H\left(\Delta_K z_0(t)\right) dt + \int_0^{\frac{2\pi}{\tilde{\omega}_0}} \left(\Delta_K z_0(t)\right)^* H\left(\Delta_K z_0(t)\right) dt.
$$

Clearly, c is a constant which does not depend on k. Thus, we have shown that there exists a sequence of trajectories of the system with compact support $z_k \in \mathfrak{B}_c(P)$ and a $c \in \mathbb{R}$ such that

$$
\int_{-\infty}^{\infty} \left(\Delta_K z_k(t)\right)^* H\left(\Delta_K z_k(t)\right) dt = \frac{4\pi k}{\tilde{\omega}_0}\tilde{v}^*\tilde{\Pi}(i\omega_0)\tilde{v} + c,
$$

which yields a contradiction to the assumption of cyclo-dissipativity by choosing k large enough. □

A.3 Differential equations with exponentially decaying inhomogeneity

In this section we show that differential equations with an exponentially decaying inhomogeneity have at least one solution which itself is exponentially decaying. Only the final Lemma A.15 is needed in the main text.

Lemma A.12. *Let $f \in \mathcal{C}_+$ (or $f \in \mathcal{C}_-$) and $a \in \mathbb{C}$ be arbitrary. Then there exists a $y \in \mathcal{C}_+$ (or $y \in \mathcal{C}_-$, resp.) such that for all $t \in \mathbb{R}$ we have*

$$
\dot{y}(t) = ay(t) + f(t). \tag{A.10}
$$

Proof. For the case that $f \in \mathcal{C}_+$ we know that for $i \in \mathbb{N}$ there exist constants $d_i, \delta_i > 0$ such that

$$
\left|f^{(i)}(t)\right| \le d_i e^{-\delta_i t},
$$

for all $t \ge 0$. We distinguish two cases. First assume that $\mathrm{Re}\,(a) \ge 0$. Define

$$
y_0 := -\int_0^{\infty} e^{-as} f(s) ds.
$$

Note that y_0 is well defined, since in this case e^{-as} is bounded for all $s \geq 0$ by 1 and f is exponentially decaying and infinitely often differentiable. With the variation-of-constants formula and y_0 as an initial condition we obtain that

$$\begin{aligned}
y(t) &= e^{at} y_0 + e^{at} \int_0^t e^{-as} f(s) ds \\
&= e^{at} \left(- \int_0^\infty e^{-as} f(s) ds + \int_0^t e^{-as} f(s) ds \right) \\
&= -e^{at} \int_t^\infty e^{-as} f(s) ds
\end{aligned}$$

is a solution of (A.10). We have

$$\begin{aligned}
|y(t)| &= \left| e^{at} \right| \left| \int_t^\infty e^{-as} f(s) ds \right| \\
&\leq e^{\mathrm{Re}(a)t} \int_t^\infty \left| e^{-as} f(s) \right| ds \\
&\leq e^{\mathrm{Re}(a)t} \int_t^\infty e^{-\mathrm{Re}(a)s} d_0 e^{-\delta_0 s} ds \\
&= d_0 e^{\mathrm{Re}(a)t} \int_t^\infty e^{-(\mathrm{Re}(a)+\delta_0)s} ds \\
&= d_0 e^{\mathrm{Re}(a)t} \left(- \frac{1}{\mathrm{Re}\,(a) + \delta_0} e^{-(\mathrm{Re}(a)+\delta_0)s} \Big|_t^\infty \right) \\
&= \frac{d_0}{\mathrm{Re}\,(a) + \delta_0} e^{\mathrm{Re}(a)t} e^{-(\mathrm{Re}(a)+\delta_0)t} \\
&= \frac{d_0}{\mathrm{Re}\,(a) + \delta_0} e^{-\delta_0 t} =: c_0 e^{-\gamma_0 t},
\end{aligned}$$

by setting $c_0 := \frac{d_0}{\mathrm{Re}(a)+\delta_0}$ and $\gamma_0 := \delta_0$. Since y solves the differential equation (A.10), we see that

$$y^{(i)}(t) = a^i y(t) + \sum_{j=0}^{i-1} a^j f^{(i-1-j)}(t).$$

Since $\mathcal{C}_+$ is a vector space, this shows that all derivatives of y are also exponentially decaying, which implies $y \in \mathcal{C}_+$.
If $\mathrm{Re}\,(a) < 0$ then multiple solutions exist. We choose $y_0 := 0$ and observe that in this case

$$y(t) = \int_0^t e^{a(t-s)} f(s) ds,$$

is a solution of (A.10). W.l.o.g. we assume that $\delta_0 < -\mathrm{Re}\,(a)$ (otherwise chose δ_0 smaller, which is still appropriate). Then $\delta_0 + \mathrm{Re}\,(a) < 0$ and we have

$$|y(t)| \leq \int_0^t \left| e^{a(t-s)} f(s) \right| ds$$

$$
\begin{aligned}
&\leq \int_0^t e^{\mathrm{Re}(a)(t-s)} d_0 e^{-\delta_0 s} ds \\
&= d_0 e^{\mathrm{Re}(a)t} \int_0^t e^{-s(\mathrm{Re}(a)+\delta_0)} ds \\
&= -\frac{d_0}{\mathrm{Re}(a)+\delta_0} e^{\mathrm{Re}(a)t} \left(e^{-s(\mathrm{Re}(a)+\delta_0)} \Big|_0^t \right) \\
&= -\frac{d_0}{\mathrm{Re}(a)+\delta_0} e^{\mathrm{Re}(a)t} \left(e^{-t(\mathrm{Re}(a)+\delta_0)} - 1 \right) \\
&= -\frac{d_0}{\mathrm{Re}(a)+\delta_0} \left(e^{-\delta_0 t} - e^{\mathrm{Re}(a)t} \right) \\
&=: c_0 \left(e^{-\gamma_0 t} - e^{\mathrm{Re}(a)t} \right) \\
&\leq c_0 e^{-\gamma_0 t},
\end{aligned}
$$

for all $t \geq 0$, which shows that y is exponentially decaying. As above we deduce that then all derivatives of y are also exponentially decaying, since y solves the differential equation (A.10), and thus $y \in \mathcal{C}_+$. The proof for $f \in \mathcal{C}_-$ is analogously. □

Theorem A.13. *Let $f \in \mathcal{C}_+^n$ (of $f \in \mathcal{C}_-^n$) and $A \in \mathbb{C}^{n,n}$ be arbitrary. Then there exists $y \in \mathcal{C}_+^n$ (or $y \in \mathcal{C}_-^n$, resp.) such that for all $t \in \mathbb{R}$ we have*

$$\dot{y}(t) = Ay(t) + f(t).$$

Proof. Using the Jordan canonical form of A the problem decomposes into a finite number of subproblems of which each has the form

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \vdots \\ \dot{y}_{n_i} \end{bmatrix} = \begin{bmatrix} a & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & a \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{n_i} \end{bmatrix} + \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_{n_i} \end{bmatrix},$$

with $n_i \in \mathbb{N}$. Starting from the last variable and last equation one can use Lemma A.12 to show that there exists a $y_{n_i} \in \mathcal{C}_+$ such that the last equation is fulfilled. Using the fact that $\mathcal{C}_+$ is a vector space we see that $y_{n_i} + f_{n_i-1} \in \mathcal{C}_+$. Thus, using Lemma A.12 again, we find that there exists a $y_{n_i-1} \in \mathcal{C}_+$ such that also the second last equation is fulfilled. Proceeding this way we obtain the claim for the subproblem and thus also for an arbitrary matrix A. The proof for $f \in \mathcal{C}_-^n$ is analogously. □

Lemma A.14. *Let $d \in \mathbb{C}[\lambda] \setminus \{0\}$ be a non-zero polynomial and let $b \in \mathcal{C}_+$ (or $b \in \mathcal{C}_-$). Then there exists $x \in \mathcal{C}_+$ ($x \in \mathcal{C}_-$, resp.) such that*

$$d\left(\tfrac{d}{dt}\right) x = b.$$

Proof. We distinguish two cases. For the first case assume that $d \equiv c \in \mathbb{C} \setminus \{0\}$ is a non-zero constant. In this case we set $x := \frac{1}{c} b$ and obtain the assertion immediately. For the second case let $d(\lambda) = \sum_{i=0}^{K} d_i \lambda^i$ with $K \in \mathbb{N} \setminus \{0\}$ and $d_K \neq 0$. This means that we are looking for a solution $x \in \mathcal{C}_+$ of the differential equation $\sum_{i=0}^{K} d_i x^{(i)}(t) = b(t)$. Reducing this higher order, scalar differential equation to a first order, matrix differential equation with the help of the companion form of d, and applying Theorem A.13 we immediately obtain the result. The proof for $b \in \mathcal{C}_-$ is analogously. $\square$

Lemma A.15. *Let $P \in \mathbb{C}[\lambda]^{p,q}$ be a polynomial matrix with full row rank* $\operatorname{rank}_{\mathbb{C}(\lambda)}(P) = p$. *Let $b \in \mathcal{C}_+^p$ (or $b \in \mathcal{C}_-^p$) be arbitrary. Then there exists $x \in \mathcal{C}_+^q$ (or $x \in \mathcal{C}_-^q$, resp.) such that*

$$P\left(\tfrac{d}{dt}\right) x = b.$$

Proof. Let a Smith-form (2.1) of P be given by $P = SDT$ where $S \in \mathbb{C}[\lambda]^{p,p}$ and $T \in \mathbb{C}[\lambda]^{q,q}$ are unimodular and $D \in \mathbb{C}[\lambda]^{p,q}$ is diagonal of the form

$$D = \begin{bmatrix} d_1 & & & 0 & \cdots & 0 \\ & \ddots & & \vdots & & \vdots \\ & & d_p & 0 & \cdots & 0 \end{bmatrix},$$

with $d_1, \ldots, d_p \in \mathbb{C}[\lambda] \setminus \{0\}$. Set $\tilde{b} := S^{-1}\left(\tfrac{d}{dt}\right) b$. Then we see that $\tilde{b} \in \mathcal{C}_+^p$ is itself exponentially decaying. For $i = 1, \ldots, p$ denote the elements of $\tilde{b}$ by $\tilde{b}_i$ and construct $x_i \in \mathcal{C}_+$ with the help of Lemma A.14 as exponentially decaying solutions of the scalar equations $d_i\left(\tfrac{d}{dt}\right) x_i = \tilde{b}_i$. Define $\tilde{x} \in \mathcal{C}_+^q$ through

$$\tilde{x} := \begin{bmatrix} x_1 & \cdots & x_p & 0 \cdots & 0 \end{bmatrix}^T$$

and notice that this implies $D\left(\tfrac{d}{dt}\right) \tilde{x} = \tilde{b}$. Thus, setting $x := T^{-1}\left(\tfrac{d}{dt}\right) \tilde{x} \in \mathcal{C}_+^q$ proves the claim. The proof for $b \in \mathcal{C}_-^p$ is analogously. $\square$

A.4 Quadraticity of the available storage and the required supply

Looking at Definition 3.14 we see that the available storage Θ_+ and the required supply Θ_- both correspond to the solution of an optimal control problem with a quadratic cost functional subject to linear constraints. This suggests that Θ_+ and Θ_- themselves might be quadratic functions. In this section we show that this is indeed the case by following the ideas in [2, Chapter II].

Definition A.16. Let $W \subset \mathbb{C}^n$ be a linear subspace. Then the function

$$B : W \times W \to \mathbb{C}$$

is called a *sesquilinear form on W* if the conditions

$$B(x, y) = \overline{B(y, x)}$$

$$
\begin{aligned}
B(x, y_1 + y_2) &= B(x, y_1) + B(x, y_2) \\
B(x, \alpha y) &= \overline{\alpha} B(x, y)
\end{aligned}
$$

hold for all $x, y, y_1, y_2 \in W$ and all $\alpha \in \mathbb{C}$.

A special case of the following lemma was used in [2, Lemma II.2.2.] without giving proof.

Proposition A.17. Let $W \subset \mathbb{C}^n$ be a linear subspace and let $B : W \times W \to \mathbb{C}$ be a sesquilinear form on W. Then there exists a unique Hermitian matrix $\tilde{X} = \tilde{X}^* \in \mathbb{C}^{n,n}$ such that

$$
\begin{aligned}
y^* \tilde{X} x &= B(x, y) && \text{for all } x, y \in W, && \text{(A.11)} \\
\tilde{X} x &= 0 && \text{for all } x \perp W. && \text{(A.12)}
\end{aligned}
$$

In particular, the function $\Theta : W \to \mathbb{R}$ defined by $\Theta(x) := B(x, x)$ is quadratic.

Proof. To see the existence, let $v_1, \ldots, v_m$ be an orthonormal basis of W, set $V := [v_1, \ldots, v_m]$ and define the matrix $X = [x_{i,j}] \in \mathbb{C}^{m,m}$ through $x_{i,j} = B(v_j, v_i)$. Let $x, y \in W$ be arbitrary. Then there exist coordinate vectors $\alpha, \beta \in \mathbb{C}^m$ such that $x = V\alpha = \sum_{i=1}^m \alpha_i v_i$ and $y = V\beta = \sum_{i=1}^m \beta_i v_i$. This implies that

$$
\begin{aligned}
B(x, y) &= \sum_{i,j=1}^m B(\alpha_j v_j, \beta_i v_i) = \sum_{i,j=1}^m \alpha_j \overline{\beta_i} B(v_j, v_i) = \sum_{i,j=1}^m \alpha_j \overline{\beta_i} x_{i,j} \\
&= \sum_{i=1}^m \overline{\beta_i} \sum_{j=1}^m \alpha_j x_{i,j} = [\overline{\beta_1}, \ldots, \overline{\beta_m}] \begin{bmatrix} \sum_{j=1}^m \alpha_j x_{1,j} \\ \vdots \\ \sum_{j=1}^m \alpha_j x_{m,j} \end{bmatrix} = \beta^* X \alpha.
\end{aligned}
$$

Set $\tilde{X} := V X V^*$. Then, for the arbitrary $x, y \in W$ from above, we see that $y^* \tilde{X} x = \beta^* V^* V X V^* V \alpha = \beta^* X \alpha = B(x, y)$. Also, we see that for any $x \perp W$, i.e., any $x \in \mathbb{C}^n$ with $V^* x = 0$, we have that $\tilde{X} x = V X V^* x = 0$.
To see the uniqueness let $\tilde{X}_1$ and $\tilde{X}_2$ be two matrices satisfying the properties (A.11) and (A.12). Then, for $i = 1, \ldots, n$ the unit vector $e_i \in \mathbb{C}^n$ can be written as $e_i = \tilde{v}_i + \tilde{w}_i$, where $\tilde{v}_i \in W$ and $\tilde{w}_i \perp W$. Thus, for $i, j \in \{1, \ldots, n\}$ and $k = 1, 2$ we have

$$
e_i^* \tilde{X}_k e_j = \tilde{v}_i^* \tilde{X}_k \tilde{v}_j + \tilde{w}_i^* \tilde{X}_k \tilde{v}_j + \tilde{v}_i^* \tilde{X}_k \tilde{w}_j + \tilde{w}_i^* \tilde{X}_k \tilde{w}_j = \tilde{v}_i^* \tilde{X}_k \tilde{v}_j,
$$

due to (A.12). Because of (A.11) this implies $e_i^* \tilde{X}_1 e_j = v_i^* \tilde{X}_1 v_j = B(v_j, v_i) = v_i^* \tilde{X}_2 v_j = e_i^* \tilde{X}_2 e_j$, i.e., $\tilde{X}_1 = \tilde{X}_2$. $\square$

The following Lemma is an extension of [2, Lemma II.2.2.] to the complex Hermitian case.

Lemma A.18. *Let $W \subset \mathbb{C}^n$ be a linear subspace and consider a function $\Theta : W \to \mathbb{R}$. Then there exists a unique Hermitian matrix $\tilde{X} = \tilde{X}^* \in \mathbb{C}^{n,n}$ such that*

$$
x^* \tilde{X} x = \Theta(x) \qquad \textit{for all } x \in W,
$$

$$\tilde{X}x = 0 \qquad\qquad \textit{for all } x \perp W,$$

if and only if for all $\alpha \in \mathbb{C}$, $\mu \in \mathbb{R}^+ \setminus \{1\}$ *and all vectors* $x_1, x_2 \in W$ *we have*

$$\begin{aligned} \Theta(\alpha x_1) &= |\alpha|^2 \Theta(x_1), && \text{(A.13)} \\ \Theta(x_1 + x_2) + \Theta(x_1 - x_2) &= 2\Theta(x_1) + 2\Theta(x_2), && \text{(A.14)} \\ \Theta(x_1 + \mu x_2) - \Theta(x_1 - \mu x_2) &= \mu\Theta(x_1 + x_2) - \mu\Theta(x_1 - x_2). && \text{(A.15)} \end{aligned}$$

Proof. The "only if" part is trivial. For the "if" part let $y_1, y_2, y_3 \in W$ be arbitrary vectors. Then from (A.14) (with $x_1 = y_1 + y_2$, $x_2 = y_1 + y_3$) we obtain that

$$\Theta(y_1 + y_2) + \Theta(y_1 + y_3) = \tfrac{1}{2}\left[\Theta(2y_1 + y_2 + y_3) + \Theta(y_2 - y_3)\right]$$

and also from (A.14) (with $x_1 = y_1 - y_3$, $x_2 = y_1 - y_2$) we obtain that

$$\Theta(y_1 - y_3) + \Theta(y_1 - y_2) = \tfrac{1}{2}\left[\Theta(2y_1 - y_2 - y_3) + \Theta(y_2 - y_3)\right].$$

Subtracting these two equations yields

$$\begin{aligned} & \Theta(y_1 + y_2) - \Theta(y_1 - y_3) + \Theta(y_1 + y_3) - \Theta(y_1 - y_2) \\ = \; & \tfrac{1}{2}\left[\Theta(2y_1 + y_2 + y_3) - \Theta(2y_1 - y_2 - y_3)\right], \end{aligned}$$

which by (A.13) (with $\alpha = -1$) and (A.15) (with $\mu = 2$, $x_1 = y_2 + y_3$, $x_2 = y_1$) is equivalent to the equation

$$\begin{aligned} & \Theta(y_1 + y_2) - \Theta(y_1 - y_3) + \Theta(y_1 + y_3) - \Theta(y_1 - y_2) \\ & + \Theta(y_1 - y_2 - y_3) - \Theta(y_1 + y_2 + y_3) \\ = \; & \tfrac{1}{2}\left[\Theta(2y_1 + y_2 + y_3) - \Theta(2y_1 - y_2 - y_3) \right. \\ & \left. - 2\Theta(y_1 + y_2 + y_3) + 2\Theta(y_1 - y_2 - y_3)\right] \\ = \; & \tfrac{1}{2}\left[\Theta(y_2 + y_3 + 2y_1) - \Theta(y_2 + y_3 - 2y_1) \right. \\ & \left. - 2\Theta(y_2 + y_3 + y_1) + 2\Theta(y_2 + y_3 - y_1)\right] = 0. \end{aligned}$$

Thus, we have shown that

$$\begin{gathered} \Theta(y_1 + y_2) - \Theta(y_1 - y_3) + \Theta(y_1 + y_3) - \Theta(y_1 - y_2) = \\ \Theta(y_1 + y_2 + y_3) - \Theta(y_1 - y_2 - y_3), \end{gathered} \qquad \text{(A.16)}$$

for all $y_1, y_2, y_3 \in W$. Define the function $B : W \times W \to \mathbb{C}$ by

$$B(x, y) := \Theta(x + y) - \Theta(x - y) + i\left[\Theta(x + iy) - \Theta(x - iy)\right].$$

Then, using (A.16) we see that for all $x, y, y_1, y_2 \in W$ and all $\alpha \in \mathbb{C}$ we have

$$\begin{aligned} B(x, y_1 + y_2) &= \Theta(x + y_1 + y_2) - \Theta(x - y_1 - y_2) \\ &\quad + i\left[\Theta(x + iy_1 + iy_2) - \Theta(x - iy_1 - iy_2)\right] \end{aligned}$$

$$
\begin{aligned}
&= \Theta(x+y_1) - \Theta(x-y_2) + \Theta(x+y_2) - \Theta(x-y_1) \\
&\quad + i\left[\Theta(x+iy_1) - \Theta(x-iy_2) + \Theta(x+iy_2) - \Theta(x-iy_1)\right] \\
&= B(x,y_1) + B(x,y_2).
\end{aligned}
$$

Using (A.13) we see that

$$
\begin{aligned}
B(x,y) &= \Theta(x+y) - \Theta(x-y) + i\left[\Theta(x+iy) - \Theta(x-iy)\right] \\
&= \Theta(y+x) - |-1|^2\,\Theta(y-x) + i\left[|i|^2\,\Theta(y-ix) - |-i|^2\,\Theta(y+ix)\right] \\
&= \Theta(y+x) - \Theta(y-x) - i\left[\Theta(y+ix) - \Theta(y-ix)\right] \\
&= \overline{B(y,x)}.
\end{aligned}
$$

Obviously, (A.15) also holds for $\mu = 0$ and $\mu = 1$. To see that (A.15) together with (A.13) even implies that (A.15) holds for all $\mu \in \mathbb{R}$ note that for any real $\lambda < 0$ we have $\mu := -\lambda > 0$ and thus we obtain

$$
\begin{aligned}
&\Theta(x_1 + \lambda x_2) - \Theta(x_1 - \lambda x_2) \\
&= -\left[\Theta(x_1 + \mu x_2) - \Theta(x_1 - \mu x_2)\right] \\
&= -\left[\mu\Theta(x_1 + x_2) - \mu\Theta(x_1 - x_2)\right] \\
&= \lambda\Theta(x_1 + x_2) - \lambda\Theta(x_1 - x_2).
\end{aligned}
$$

Thus, using (A.16) again, we see that condition (A.15) for all $\mu \in \mathbb{R}$ leads to

$$
\begin{aligned}
&\Theta(x_1 + \alpha x_2) - \Theta(x_1 - \alpha x_2) \\
&= \Theta(x_1 + \mathrm{Re}\,(\alpha)\,x_2 + i\mathrm{Im}\,(\alpha)\,x_2) \\
&\quad -\Theta(x_1 - \mathrm{Re}\,(\alpha)\,x_2 - i\mathrm{Im}\,(\alpha)\,x_2) \\
&= \Theta(x_1 + \mathrm{Re}\,(\alpha)\,x_2) - \Theta(x_1 - i\mathrm{Im}\,(\alpha)\,x_2) \\
&\quad +\Theta(x_1 + i\mathrm{Im}\,(\alpha)\,x_2) - \Theta(x_1 - \mathrm{Re}\,(\alpha)\,x_2) \\
&= \Theta(x_1 + \mathrm{Re}\,(\alpha)\,x_2) - \Theta(x_1 - \mathrm{Im}\,(\alpha)\,ix_2) \\
&\quad +\Theta(x_1 + \mathrm{Im}\,(\alpha)\,ix_2) - \Theta(x_1 - \mathrm{Re}\,(\alpha)\,x_2) \\
&= \Theta(x_1 + \mathrm{Re}\,(\alpha)\,x_2) - \Theta(x_1 - \mathrm{Re}\,(\alpha)\,x_2) \\
&\quad +\Theta(x_1 + \mathrm{Im}\,(\alpha)\,ix_2) - \Theta(x_1 - \mathrm{Im}\,(\alpha)\,ix_2) \\
&= \mathrm{Re}\,(\alpha)\,\Theta(x_1 + x_2) - \mathrm{Re}\,(\alpha)\,\Theta(x_1 - x_2) \\
&\quad +\mathrm{Im}\,(\alpha)\,\Theta(x_1 + ix_2) - \mathrm{Im}\,(\alpha)\,\Theta(x_1 - ix_2) \\
&= \mathrm{Re}\,(\alpha)\,(\Theta(x_1 + x_2) - \Theta(x_1 - x_2)) \\
&\quad +\mathrm{Im}\,(\alpha)\,(\Theta(x_1 + ix_2) - \Theta(x_1 - ix_2),
\end{aligned}
$$

for all $\alpha \in \mathbb{C}$ and all $x_1, x_2 \in W$. Thus, we see that

$$
\begin{aligned}
&B(x, \alpha y) \\
&= \Theta(x + \alpha y) - \Theta(x - \alpha y) + i\left[\Theta(x + i\alpha y) - \Theta(x - i\alpha y)\right] \\
&= \mathrm{Re}\,(\alpha)\,(\Theta(x+y) - \Theta(x-y)) + \mathrm{Im}\,(\alpha)\,(\Theta(x+iy) - \Theta(x-iy))
\end{aligned}
$$

$$
\begin{array}{rl}
+ & i\mathrm{Re}\,(\alpha)\,(\Theta(x+iy)-\Theta(x-iy)) + i\mathrm{Im}\,(\alpha)\,(\Theta(x+iiy)-\Theta(x-iiy)) \\
= & \mathrm{Re}\,(\alpha)\,(\Theta(x+y)-\Theta(x-y)) - i\mathrm{Im}\,(\alpha)\,(\Theta(x+y)-\Theta(x-y)) \\
+ & i\mathrm{Re}\,(\alpha)\,(\Theta(x+iy)-\Theta(x-iy)) + \mathrm{Im}\,(\alpha)\,(\Theta(x+iy)-\Theta(x-iy)) \\
= & \overline{\alpha}(\Theta(x+y)-\Theta(x-y)) + i\overline{\alpha}(\Theta(x+iy)-\Theta(x-iy)) \\
= & \overline{\alpha}B(x,y),
\end{array}
$$

which shows that B is a sesquilinear form. With Proposition A.17 this proves the existence of a unique $\tilde{X} = \tilde{X}^* \in \mathbb{C}^{n,n}$ such that $B(x,y) = y^*\tilde{X}x$ for all $x, y \in W$ and $\tilde{X}x = 0$ for all $x \perp W$. Thus for all $x \in W$ we have

$$
\begin{array}{rl}
& x^*\left(\frac{1}{4}\tilde{X}\right)x \\
= & \frac{1}{4}B(x,x) = \frac{1}{4}\left(\Theta(2x) - \Theta(0) - i\left[\Theta(x+ix) - \Theta(x-ix)\right]\right) \\
= & \frac{1}{4}4\Theta(x) - 0 - i\left[\Theta(x+ix) - |i|^2\Theta(x-ix)\right] \\
= & \Theta(x) - i\left[\Theta(x+ix) - \Theta(i(x-ix))\right] \\
= & \Theta(x) - i\left[\Theta(x+ix) - \Theta(ix+x))\right] \\
= & \Theta(x),
\end{array}
$$

where we used (A.13) extensively. This proves the claim. □

Lemma A.19. *Let $P \in \mathbb{C}[\lambda]_K^{p,q}$ and $H = H^* \in \mathbb{C}^{qK,qK}$. Then the available storage and the required supply fulfill the following properties:*

1. *For all $\hat{z}_+ \in R_+(P)$ and $\alpha \in \mathbb{C} \setminus \{0\}$ we have*

$$\Theta_+(\alpha\hat{z}_+) = |\alpha|^2\Theta_+(\hat{z}_+).$$

2. *For all $\hat{z}_- \in R_-(P)$ and $\alpha \in \mathbb{C} \setminus \{0\}$ we have*

$$\Theta_-(\alpha\hat{z}_-) = |\alpha|^2\Theta_-(\hat{z}_-).$$

3. *If (P, H) is cyclo-dissipative then we have $\Theta_+(0) = 0 = \Theta_-(0)$.*

The case $\alpha = 0$ in points 1. and 2. has to be excluded, since $0 \cdot \infty$ is not well defined, c.f. Remark 3.16.

Proof. First note that with $\alpha \neq 0$ we also have that $\alpha z_+ \in \mathfrak{B}_+(P)$ and thus

$$
\begin{array}{rcl}
\Theta_+(\alpha z_+(t)) & = & -\inf\limits_{\substack{z\in\mathfrak{B}_+(P) \\ \Delta_K z(0)=\Delta_K\alpha z_+(t)}} \displaystyle\int_0^\infty \left(\Delta_K z(t)\right)^* H\left(\Delta_K z(t)\right) dt \\
& = & -\inf\limits_{\substack{\alpha y\in\mathfrak{B}_+(P) \\ \alpha\Delta_K y(0)=\alpha\Delta_K z_+(t)}} \displaystyle\int_0^\infty \left(\Delta_K \alpha y(t)\right)^* H\left(\Delta_K \alpha y(t)\right) dt
\end{array}
$$

$$= \quad -|\alpha|^2 \inf_{\substack{y\in\mathfrak{B}_+(P)\\ \Delta_K y(0)=\Delta_K z_+(t)}} \int_0^\infty (\Delta_K y(t))^* H (\Delta_K y(t))\, dt = |\alpha|^2 \Theta_+(z_+(t))$$

and analogously we find that $\Theta_-(\alpha z_-(t)) = |\alpha|^2\Theta_-(z_-(t))$.
To see the part 3. we first note that $\Theta_+(0) \geq 0$, since the trivial trajectory $0 \in \mathfrak{B}_+(P)$ is exponentially decaying on the positive time axis. Also we see that $\Theta_-(0) \leq 0$, since $0 \in \mathfrak{B}_-(P)$ is exponentially decaying on the negative time axis. Using Lemma 3.18 and the cyclo-dissipativity this leads to

$$0 \leq \Theta_+(0) \leq \Theta_-(0) \leq 0,$$

and thus the claim follows. □

The following Lemma A.20 is a modification of [2, Theorem II.2.1.].

Lemma A.20. *Let $P \in \mathbb{C}[\lambda]_K^{p,q}$ and $H = H^* \in \mathbb{C}^{qK,qK}$. Assume that (P, H) is dissipative. Then, there exists a matrix $\tilde{X}_+ = \tilde{X}_+^* \in \mathbb{C}^{qK,qK}$ such that*

$$\Theta_+(\hat{z}) = \hat{z}^* \tilde{X}_+ \hat{z},$$

for all $\hat{z} \in R_+(P)$ and $\tilde{X}_+\hat{z} = 0$ for all $\hat{z} \perp R_+(P)$ (and a matrix $\tilde{X}_- = \tilde{X}_-^* \in \mathbb{C}^{qK,qK}$ such that $\Theta_-(\hat{z}) = \hat{z}^*\tilde{X}_-\hat{z}$ for all $\hat{z} \in R_-(P)$ and $\tilde{X}_-\hat{z} = 0$ for all $\hat{z} \perp R_-(P)$).

Proof. Using Remark 3.17 and point 2. of Lemma 3.20 we see that the available storage is finite on $W := R_+(P)$. Thus, we want to show the three conditions from Lemma A.18. Condition (A.13) is fulfilled due to Lemma A.19. To see condition (A.15) let $\hat{z}_1, \hat{z}_2 \in W$ and $\mu \in \mathbb{R}^+$ with $\mu > 0$ be arbitrary. Then, we see that for all $z_1, z_2 \in \mathfrak{B}_+(P)$ we have

$$\begin{aligned} & \|z_1 + \mu z_2\|_+^2 + \mu \|z_1 - z_2\|_+^2 = \langle z_1 + \mu z_2, z_1 + \mu z_2\rangle_+ + \mu \langle z_1 - z_2, z_1 - z_2\rangle_+ \\ = \; & (1+\mu) \langle z_1, z_1\rangle_+ + (\mu^2+\mu) \langle z_2, z_2\rangle_+ + 2\mathrm{Re}\left(\langle \mu z_2, z_1\rangle_+\right) - \mu 2\mathrm{Re}\left(\langle \mu z_2, z_1\rangle_+\right) \\ = \; & (1+\mu) \|z_1\|_+^2 + (\mu^2+\mu) \|z_2\|_+^2, \end{aligned}$$

since μ is assumed to be real. Analogously we obtain that

$$\begin{aligned} & \mu \|z_1 + z_2\|_+^2 + \|z_1 - \mu z_2\|_+^2 = \mu \langle z_1 + z_2, z_1 + z_2\rangle_+ + \langle z_1 - \mu z_2, z_1 - \mu z_2\rangle_+ \\ = \; & (\mu+1) \|z_1\|_+^2 + (\mu+\mu^2) \|z_2\|_+^2 . \end{aligned}$$

This shows that we have

$$\|z_1 + \mu z_2\|_+^2 + \mu \|z_1 - z_2\|_+^2 = \mu \|z_1 + z_2\|_+^2 + \|z_1 - \mu z_2\|_+^2 , \tag{A.17}$$

for all $z_1, z_2 \in \mathfrak{B}_+(P)$. Let $\epsilon > 0$ be arbitrary and choose $z_3, z_4 \in \mathfrak{B}_+(P)$ such that $\Delta_K z_3(0) = \hat{z}_1 + \hat{z}_2$, $\Delta_K z_4(0) = \hat{z}_1 - \mu\hat{z}_2$, and

$$\begin{aligned} \|z_3\|_+^2 &\leq -\Theta_+(\hat{z}_1 + \hat{z}_2) + \epsilon, \\ \|z_4\|_+^2 &\leq -\Theta_+(\hat{z}_1 - \mu\hat{z}_2) + \epsilon. \end{aligned} \tag{A.18}$$

Set $\tilde{z}_1 := \frac{1}{1+\mu}(\mu z_3 + z_4)$ and $\tilde{z}_2 := \frac{1}{1+\mu}(z_3 - z_4)$. Then we have

$$\begin{aligned}
\tilde{z}_1 + \tilde{z}_2 &= \frac{1}{1+\mu}(\mu z_3 + z_4) + \frac{1}{1+\mu}(z_3 - z_4) = z_3, \\
\tilde{z}_1 - \mu\tilde{z}_2 &= \frac{1}{1+\mu}(\mu z_3 + z_4) - \frac{\mu}{1+\mu}(z_3 - z_4) = z_4,
\end{aligned}$$

and also

$$\begin{aligned}
\Delta_K \tilde{z}_1(0) &= \Delta_K \left(\frac{1}{1+\mu}(\mu z_3 + z_4) \right)(0) = \frac{\mu}{1+\mu}\Delta_K z_3(0) + \frac{1}{1+\mu}\Delta_K z_4(0) \\
&= \frac{\mu}{1+\mu}(\hat{z}_1 + \hat{z}_2) + \frac{1}{1+\mu}(\hat{z}_1 - \mu\hat{z}_2) = \hat{z}_1 \\
\Delta_K \tilde{z}_2(0) &= \ldots = \hat{z}_2.
\end{aligned}$$

This shows that using (A.18) and (A.17) we can obtain

$$\begin{aligned}
-\Theta_+(\hat{z}_1 + \mu\hat{z}_2) - \mu\Theta_+(\hat{z}_1 - \hat{z}_2) &\leq \|\tilde{z}_1 + \mu\tilde{z}_2\|_+^2 + \mu\,\|\tilde{z}_1 - \tilde{z}_2)\|_+^2 \\
&= \mu\,\|\tilde{z}_1 + \tilde{z}_2\|_+^2 + \|\tilde{z}_1 - \mu\tilde{z}_2\|_+^2 \\
&= \mu\,\|z_3\|_+^2 + \|z_4\|_+^2 \\
&\leq \mu(-\Theta_+(\hat{z}_1 + \hat{z}_2) + \epsilon) - \Theta_+(\hat{z}_1 - \mu\hat{z}_2) + \epsilon \\
&= -\mu\Theta_+(\hat{z}_1 + \hat{z}_2) - \Theta_+(\hat{z}_1 - \mu\hat{z}_2) + (1+\mu)\epsilon.
\end{aligned}$$

For $\epsilon \to 0$ this gives

$$-\Theta_+(\hat{z}_1 + \mu\hat{z}_2) - \mu\Theta_+(\hat{z}_1 - \hat{z}_2) \leq -\mu\Theta_+(\hat{z}_1 + \hat{z}_2) - \Theta_+(\hat{z}_1 - \mu\hat{z}_2). \tag{A.19}$$

Let $\epsilon > 0$ again be arbitrary and choose $z_5, z_6 \in \mathfrak{B}_+(P)$ such that $\Delta_K z_5(0) = \hat{z}_1 + \mu\hat{z}_2$, $\Delta_K z_6(0) = \hat{z}_1 - \hat{z}_2$, and

$$\begin{aligned}
\|z_5\|_+^2 &\leq -\Theta_+(\hat{z}_1 + \mu\hat{z}_2) + \epsilon, \\
\|z_6\|_+^2 &\leq -\Theta_+(\hat{z}_1 - \hat{z}_2) + \epsilon.
\end{aligned} \tag{A.20}$$

Reset $\tilde{z}_1 := \frac{1}{1+\mu}(z_5 + \mu z_6)$ and $\tilde{z}_2 := \frac{1}{1+\mu}(z_5 - z_6)$. Then we have

$$\begin{aligned}
\tilde{z}_1 + \mu\tilde{z}_2 &= \frac{1}{1+\mu}(z_5 + \mu z_6) + \frac{\mu}{1+\mu}(z_5 - z_6) = z_5, \\
\tilde{z}_1 - \tilde{z}_2 &= \frac{1}{1+\mu}(z_5 + \mu z_6) - \frac{1}{1+\mu}(z_5 - z_6) = z_6.
\end{aligned}$$

and also

$$\Delta_K \tilde{z}_1(0) = \Delta_K \left(\frac{1}{1+\mu}(z_5 + \mu z_6) \right)(0) = \frac{1}{1+\mu}\Delta_K z_5(0) + \frac{\mu}{1+\mu}\Delta_K z_6(0)$$

$$\begin{aligned} &= \frac{1}{1+\mu}(\hat{z}_1 + \mu\hat{z}_2) + \frac{\mu}{1+\mu}(\hat{z}_1 - \hat{z}_2) = \hat{z}_1 \\ \Delta_K \tilde{z}_2(0) &= \ldots = \hat{z}_2. \end{aligned}$$

This shows that using (A.20) and (A.17) we can obtain

$$\begin{aligned} -\Theta_+(\hat{z}_1 - \mu\hat{z}_2) - \mu\Theta_+(\hat{z}_1 + \hat{z}_2) &\leq \|\tilde{z}_1 - \mu\tilde{z}_2\|_+^2 + \mu\,\|\tilde{z}_1 + \tilde{z}_2)\|_+^2 \\ &= \mu\,\|\tilde{z}_1 - \tilde{z}_2\|_+^2 + \theta_+(\tilde{z}_1 + \mu\tilde{z}_2) \\ &= \mu\,\|z_6\|_+^2 + \|z_5\|_+^2 \\ &\leq \mu(-\Theta_+(\hat{z}_1 - \hat{z}_2) + \epsilon) - \Theta_+(\hat{z}_1 + \mu\hat{z}_2) + \epsilon \\ &= -\mu\Theta_+(\hat{z}_1 - \hat{z}_2) - \Theta_+(\hat{z}_1 + \mu\hat{z}_2) + (1+\mu)\epsilon. \end{aligned}$$

For $\epsilon \to 0$ this gives

$$-\Theta_+(\hat{z}_1 - \mu\hat{z}_2) - \mu\Theta_+(\hat{z}_1 + \hat{z}_2) \leq -\mu\Theta_+(\hat{z}_1 - \hat{z}_2) - \Theta_+(\hat{z}_1 + \mu\hat{z}_2). \tag{A.21}$$

Combining (A.19) and (A.21) proves that we have

$$-\Theta_+(\hat{z}_1 - \mu\hat{z}_2) - \mu\Theta_+(\hat{z}_1 + \hat{z}_2) = -\mu\Theta_+(\hat{z}_1 - \hat{z}_2) - \Theta_+(\hat{z}_1 + \mu\hat{z}_2),$$

and thus condition (A.15) is fulfilled. To see condition (A.14) let $\hat{z}_1, \hat{z}_2 \in W$ be arbitrary. Then we see that

$$\begin{aligned} \|z_1 + z_2\|_+^2 + \|z_1 - z_2\|_+^2 &= \langle z_1 + z_2, z_1 + z_2\rangle_+ + \langle z_1 - z_2, z_1 - z_2\rangle_+ \\ &= 2\langle z_1, z_1\rangle_+ + 2\langle z_2, z_2\rangle_+ + 2\mathrm{Re}\left(\langle z_1, z_2\rangle_+\right) - 2\mathrm{Re}\left(\langle z_1, z_2\rangle_+\right) \\ &= 2\,\|z_1\|_+^2 + 2\,\|z_2\|_+^2\,, \end{aligned} \tag{A.22}$$

for all $z_1, z_2 \in \mathfrak{B}_+(P)$. With this let $\epsilon > 0$ be arbitrary and let $z_3, z_4 \in \mathfrak{B}_+(P)$ be such that $\Delta_K z_3(0) = \hat{z}_1 + \hat{z}_2$, $\Delta_K z_4(0) = \hat{z}_1 - \hat{z}_2$, and

$$\begin{aligned} \|z_3\|_+^2 &\leq -\Theta_+(\hat{z}_1 + \hat{z}_2) + \epsilon, \\ \|z_4\|_+^2 &\leq -\Theta_+(\hat{z}_1 - \hat{z}_2) + \epsilon. \end{aligned} \tag{A.23}$$

Set $\tilde{z}_1 := \frac{z_3+z_4}{2}$ and $\tilde{z}_2 := \frac{z_3-z_4}{2}$. Then we have

$$\begin{aligned} \tilde{z}_1 + \tilde{z}_2 &= z_3, \\ \tilde{z}_1 - \tilde{z}_2 &= z_4, \end{aligned}$$

as well as

$$\begin{aligned} \Delta_K \tilde{z}_1(0) &= \tfrac{1}{2}\Delta_K z_3(0) + \tfrac{1}{2}\Delta_K z_4(0) = \tfrac{1}{2}\hat{z}_1 + \tfrac{1}{2}\hat{z}_2 + \tfrac{1}{2}\hat{z}_1 - \tfrac{1}{2}\hat{z}_2 = \hat{z}_1, \\ \Delta_K \tilde{z}_2(0) &= \tfrac{1}{2}\Delta_K z_3(0) - \tfrac{1}{2}\Delta_K z_4(0) = \tfrac{1}{2}\hat{z}_1 + \tfrac{1}{2}\hat{z}_2 - \tfrac{1}{2}\hat{z}_1 + \tfrac{1}{2}\hat{z}_2 = \hat{z}_2. \end{aligned}$$

This shows that using (A.23) and (A.22) we see that

$$-2\Theta_+(\hat{z}_1) - 2\Theta_+(\hat{z}_2) \leq 2\,\|\tilde{z}_1\|_+^2 + 2\,\|\tilde{z}_2\|_+^2$$

$$
\begin{aligned}
&= \|\tilde{z}_1 + \tilde{z}_2\|_+^2 + \|\tilde{z}_1 - \tilde{z}_2\|_+^2 \\
&= \|z_3\|_+^2 + \|z_4\|_+^2 \\
&\leq -\Theta_+(\hat{z}_1 + \hat{z}_2) + \epsilon - \Theta_+(\hat{z}_1 - \hat{z}_2) + \epsilon \\
&= -\Theta_+(\hat{z}_1 + \hat{z}_2) - \Theta_+(\hat{z}_1 - \hat{z}_2) + 2\epsilon.
\end{aligned}
$$

For $\epsilon \to 0$ this gives

$$
-2\Theta_+(\hat{z}_1) - 2\Theta_+(\hat{z}_2) \leq -\Theta_+(\hat{z}_1 + \hat{z}_2) - \Theta_+(\hat{z}_1 - \hat{z}_2). \tag{A.24}
$$

Let $\epsilon > 0$ again be arbitrary and let z_5, z_6 be such that $\Delta_K z_5(0) = \hat{z}_1$, $\Delta_K z_6(0) = \hat{z}_2$, and

$$
\begin{aligned}
\|z_5\|_+^2 &\leq -\Theta_+(\hat{z}_1) + \epsilon, \\
\|z_6\|_+^2 &\leq -\Theta_+(\hat{z}_2) + \epsilon.
\end{aligned} \tag{A.25}
$$

Using (A.25) and (A.17) we see that this gives us

$$
\begin{aligned}
-\Theta_+(\hat{z}_1 + \hat{z}_2) - \Theta_+(\hat{z}_1 - \hat{z}_2) &\leq \|z_5 + z_6\|_+^2 + \|z_5 - z_6\|_+^2 \\
&= 2\,\|z_5\|_+^2 + 2\,\|z_6\|_+^2 \\
&\leq -2\Theta_+(\hat{z}_1) + 2\epsilon - 2\Theta_+(\hat{z}_2) + 2\epsilon \\
&\leq -2\Theta_+(\hat{z}_1) - 2\Theta_+(\hat{z}_2) + 4\epsilon.
\end{aligned}
$$

For $\epsilon \to 0$ this gives

$$
-\Theta_+(\hat{z}_1 + \hat{z}_2) - \Theta_+(\hat{z}_1 - \hat{z}_2) \leq -2\Theta_+(\hat{z}_1) - 2\Theta_+(\hat{z}_2). \tag{A.26}
$$

Combining (A.24) and (A.26) proves that we have

$$
-\Theta_+(\hat{z}_1 + \hat{z}_2) - \Theta_+(\hat{z}_1 - \hat{z}_2) = -2\Theta_+(\hat{z}_1) - 2\Theta_+(\hat{z}_2),
$$

and thus condition (A.14) is fulfilled. The proof for the required supply can be conducted in the same way. □

The above results show that the available storage and the required supply are quadratic functions. In the rest of the section we will show that the quadratic matrix $\tilde{X}$ has a special form.

Definition A.21. Let $F \in \mathbb{C}^{m,n}$ and let $W \subset \mathbb{C}^n$ be a subspace. Then we say that a function $\Theta : W \to \mathbb{R}$ is F-neutral on W if it has the property

$$
\Theta(x + y) = \Theta(x), \tag{A.27}
$$

for all $x, y \in W$ with $y \in \text{kernel}\,(F)$.

Lemma A.22. *Let $\tilde{X} = \tilde{X}^* \in \mathbb{C}^{n,n}$, $F \in \mathbb{C}^{m,n}$, and let $W \subset \mathbb{C}^n$ be a linear subspace. Consider the quadratic function $\Theta(x) := x^*\tilde{X}x$. Assume that Θ is F-neutral on W and assume that $\tilde{X}x = 0$ for all $x \perp W$. Then there exists a matrix $Z \in \mathbb{C}^{m,n}$ such that*

$$
\tilde{X} = F^*Z = Z^*F.
$$

Proof. Let the columns of $V_1 \in \mathbb{C}^{n,r}$ form an orthonormal basis of the linear vector space $\mathcal{V}_1 := \operatorname{image}(F^*) \cap W$, let the columns of $V_2 \in \mathbb{C}^{n,s}$ form an orthonormal basis of the linear vector space $\mathcal{V}_2 := \operatorname{kernel}(F) \cap W$, and let the columns of $V_3 \in \mathbb{C}^{n,t}$ form an orthonormal basis of the linear vector space $\mathcal{V}_3 := W^\perp$. It is easy to see that $\mathcal{V}_1$, $\mathcal{V}_2$, and $\mathcal{V}_3$ are orthogonal to each other. Also one may verify that $\mathcal{V}_1 \oplus \mathcal{V}_2 \oplus \mathcal{V}_3 = \mathbb{C}^n$. Thus the matrix $V := \begin{bmatrix} V_1 & V_2 & V_3 \end{bmatrix} \in \mathbb{C}^{n,n}$ is unitary and $r + s + t = n$. Since Θ is F-neutral on W, we obtain that

$$\Theta(V_1\alpha_1 + V_2\alpha_2) = \Theta(V_1\alpha_1),$$

for all $\alpha_1 \in \mathbb{C}^r$ and $\alpha_2 \in \mathbb{C}^s$. Since $\tilde{X}V_3 = 0$, we also have

$$\begin{aligned} \Theta(v + V_3\alpha_3) &= (v + V_3\alpha_3)^* \tilde{X} (v + V_3\alpha_3) \\ &= v^*\tilde{X}v + v^*\tilde{X}V_3\alpha_3 + \alpha_3^* V_3^* \tilde{X} v + \alpha_3^* V_3^* \tilde{X} V_3 \alpha_3 \\ &= v^*\tilde{X}v = \Theta(v) \end{aligned}$$

for all $\alpha_3 \in \mathbb{C}^t$ and $v \in \mathbb{C}^n$. This especially implies that

$$\begin{aligned} \Theta\left(V \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix}\right) &= \Theta\left(V \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ 0 \end{bmatrix} + V_3\alpha_3\right) = \Theta\left(V \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ 0 \end{bmatrix}\right) \\ &= \Theta\left(V_1\alpha_1 + V_2\alpha_2\right) = \Theta\left(V_1\alpha_1\right) = \Theta\left(V \begin{bmatrix} \alpha_1 \\ 0 \\ 0 \end{bmatrix}\right). \end{aligned}$$

Introducing the notation

$$V^*\tilde{X}V =: \begin{bmatrix} X_{11} & X_{12} & X_{13} \\ X_{21} & X_{22} & X_{23} \\ X_{31} & X_{32} & X_{33} \end{bmatrix},$$

with partitioning analogous to the partitioning of V, we deduce that

$$\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix}^* \begin{bmatrix} X_{11} & X_{12} & X_{13} \\ X_{21} & X_{22} & X_{23} \\ X_{31} & X_{32} & X_{33} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \alpha_1^* X_{11} \alpha_1 = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix}^* \begin{bmatrix} X_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix},$$

which shows that

$$V^*\tilde{X}V = \begin{bmatrix} \tilde{X}_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Since the vectors of V_1 span part of $\operatorname{image}(F^*)$, there exists a matrix $G \in \mathbb{C}^{m,r}$ such that $V_1 = F^*G$. With this we have

$$\tilde{X} = V \begin{bmatrix} \tilde{X}_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} V^* = \begin{bmatrix} V_1X_{11} & 0 & 0 \end{bmatrix} \begin{bmatrix} V_1^* \\ V_2^* \\ V_3^* \end{bmatrix}$$

$$= \quad V_1 X_{11} V_1^* = F^* \underbrace{G X_{11} V_1^*}_{=:Z} = F^* Z,$$

and also $F^*Z = Z^*F$, since $\tilde{X} = \tilde{X}^*$. □

Statements which resemble the following can be found in [35, Section 6].

Lemma A.23. *Let $P(\lambda) = \lambda F + G \in \mathbb{C}[\lambda]_1^{p,q}$ be a first-order matrix polynomial and let $H = H^* \in \mathbb{C}^{q,q}$. Assume that (P, H) is dissipative. Then the available storage is F-neutral on $R_+(P)$ and the required supply is F-neutral on $R_-(P)$.*

Proof. Let $\hat{z}, \hat{y} \in R_+(P)$ with $\hat{y} \in \operatorname{kernel}(F)$ and let $\tilde{z}, \tilde{y} \in \mathfrak{B}_+(P)$ be such that $\tilde{z}(0) = \hat{z}$ and $\tilde{y}(0) = \hat{y}$. We see that $P^{\langle 1 \rangle}\left(\frac{d}{dt}\right) = F$ and $P^{\langle k \rangle}\left(\frac{d}{dt}\right) = 0$ for $k \geq 2$. Thus, using Theorem A.3 twice we find

$$\begin{aligned} \Theta_+(\hat{z} + \hat{y}) &= -\inf_{\substack{z \in \mathfrak{B}_+(P) \\ z(0) = (\tilde{z} + \tilde{y})(0)}} \int_0^\infty z^*(t) H z(t) dt = -\inf_{\substack{z \in \mathfrak{B}_+(P) \\ Fz(0) = F(\tilde{z} + \tilde{y})(0)}} \int_0^\infty z^*(t) H z(t) dt \\ &= -\inf_{\substack{z \in \mathfrak{B}_+(P) \\ Fz(0) = F\tilde{z}(0)}} \int_0^\infty z^*(t) H z(t) dt = \Theta_+(\hat{z}), \end{aligned}$$

which shows that Θ_+ is F-neutral on $R_+(P)$. The proof for Θ_- works analogously. □

A result which is similar to the following theorem, has already been used in [36, Theorem 3.1].

Theorem A.24. *Let $P(\lambda) = \lambda F + G \in \mathbb{C}[\lambda]^{p,q}$ and $H = H^* \in \mathbb{C}^{q,q}$. Let (P, H) be dissipative. Then there exists a matrix $Z_+ \in \mathbb{C}^{p,q}$ such that $F^* Z_+ = Z_+^* F$ and*

$$\Theta_+(\hat{z}_+) \quad = \quad \hat{z}_+^* F^* Z_+ \hat{z}_+,$$

for all $\hat{z}_+ \in R_+(P)$ and there exists a matrix $Z_- \in \mathbb{C}^{p,q}$ such that $F^* Z_- = Z_-^* F$ and

$$\Theta_-(\hat{z}_-) \quad = \quad \hat{z}_-^* F^* Z_- \hat{z}_-,$$

for all $\hat{z}_- \in R_-(P)$.

Proof. Using Lemma A.20 we conclude that there exist matrices $\tilde{X}_+, \tilde{X}_- \in \mathbb{C}^{q,q}$ such that $\Theta_+(\hat{z}_+) = \hat{z}_+^* \tilde{X}_+ \hat{z}_+$ for all $\hat{z}_+ \in R_+(P)$ and $\Theta_-(\hat{z}_-) = \hat{z}_-^* \tilde{X}_- \hat{z}_- \hat{z}_- \in R_-(P)$. With this we obtain the assertion via Lemma A.23 and Lemma A.22. □

A.5 Proofs associated with linear matrix inequalities

Lemma A.25. *Let two matrices $F, G \in \mathbb{C}^{\eta+1,\eta}$ of the form (2.8)*

$$F = \begin{bmatrix} 1 & & \\ 0 & \ddots & \\ & \ddots & 1 \\ & & 0 \end{bmatrix}, \qquad G = \begin{bmatrix} 0 & & \\ 1 & \ddots & \\ & \ddots & 0 \\ & & 1 \end{bmatrix}$$

and a Hermitian matrix $H = H^* \in \mathbb{C}^{\eta,\eta}$ be given. Then there exists a $X = X^* \in \mathbb{C}^{\eta+1,\eta+1}$ such that

$$0 = G^*XF \quad + \quad F^*XG + H.$$

Proof. Denote the entries of the Hermitian matrix X in the form

$$X = \begin{bmatrix} x_{1,1} & \dots & x_{1,\eta+1} \\ \vdots & & \vdots \\ x_{\eta+1,1} & \dots & x_{\eta+1,\eta+1} \end{bmatrix} = \left[x_{i,j}\right]_{i,j=1,\dots,\eta+1} = \left[\overline{x_{j,i}}\right]_{i,j=1,\dots,\eta+1}$$

and the Hermitian matrix H in the form

$$H = \begin{bmatrix} h_{1,1} & \dots & h_{1,\eta} \\ \vdots & & \vdots \\ h_{\eta,1} & \dots & h_{\eta,\eta} \end{bmatrix} = \left[h_{i,j}\right]_{i,j=1,\dots,\eta} = \left[\overline{h_{j,i}}\right]_{i,j=1,\dots,\eta}.$$

Then we see that we are looking for an Hermitian X such that

$$\begin{aligned}
0 &= F^*XG + G^*XF + H \\
&= \begin{bmatrix} 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{bmatrix} \begin{bmatrix} x_{1,1} & \dots & x_{1,\eta+1} \\ \vdots & & \vdots \\ x_{\eta+1,1} & \dots & x_{\eta+1,\eta+1} \end{bmatrix} \begin{bmatrix} 0 & & \\ 1 & \ddots & \\ & \ddots & 0 \\ & & 1 \end{bmatrix} + (F^*XG)^* + H \\
&= \begin{bmatrix} x_{1,2} & \dots & x_{1,\eta+1} \\ \vdots & & \vdots \\ x_{\eta,2} & \dots & x_{\eta,\eta+1} \end{bmatrix} + \begin{bmatrix} \overline{x_{1,2}} & \dots & \overline{x_{\eta,2}} \\ \vdots & & \vdots \\ \overline{x_{1,\eta+1}} & \dots & \overline{x_{\eta,\eta+1}} \end{bmatrix} + \begin{bmatrix} h_{1,1} & \dots & h_{1,\eta} \\ \vdots & & \vdots \\ h_{\eta,1} & \dots & h_{\eta,\eta} \end{bmatrix} \\
&= \left[x_{i,j+1}\right]_{i,j=1,\dots,\eta} + \left[\overline{x_{j,i+1}}\right]_{i,j=1,\dots,\eta} + \left[h_{i,j}\right]_{i,j=1,\dots,\eta} \\
&= \left[x_{i,j+1} + \overline{x_{j,i+1}} + h_{i,j}\right]_{i,j=1,\dots,\eta} \\
&= \left[x_{i,j+1} + x_{i+1,j} + h_{i,j}\right]_{i,j=1,\dots,\eta}.
\end{aligned} \tag{A.28}$$

We construct such an X in the following recursive way. First, choose all $x_{i,i} = 0$ for $i = 1, \dots, \eta+1$ and choose $x_{i,i+1} := x_{i+1,i} := -\frac{h_{i,i}}{2} \in \mathbb{R}$ for all $i = 1, \dots, \eta$. With this choice all $x_{i,j}$ with $|i-j| \leq 1$ are fixed and all equations in (A.28) with $|i-j| \leq 0$ are fulfilled.
As induction hypothesis, assume that for some $k \in \{1, \dots, \eta-1\}$ we have that all $x_{i,j}$ with $|i-j| \leq k$ are fixed and all equations in (A.28) with $|i-j| \leq k-1$ are fulfilled.
For the inductive step, note that all equations in (A.28) with $|i-j| = k$ are given by

$$0 = x_{j+k,j+1} + x_{j+k+1,j} + h_{j+k,j}$$

for $j = 0, \dots, \eta-k$ and their complex conjugate equations, which are not really additional equations. Since $|(j+k)-(j+1)| = k-1 \leq k$, we know that all $x_{j+k,j+1}$ are already fixed but not the $x_{j+k+1,j}$, since $|(j+k+1)-j| = k+1 > k$. Thus we define

$$\overline{x_{j,j+k+1}} := x_{j+k+1,j} := -x_{j+k,j+1} - h_{j+k,j},$$

for $j = 0, \dots, \eta - k$ and thus have fixed all $x_{i,j}$ with $|i - j| \le k + 1$ while at the same time all equations in (A.28) with $|i - j| \le k$ are fulfilled. Thus the inductive argument is finished and the claim is proved. $\square$

Lemma A.26. *Let the pencils $\lambda F_1 + G_1 \in \mathbb{C}[\lambda]_1^{\eta_1+1,\eta}$ and $\lambda F_2 + G_2 \in \mathbb{C}[\lambda]_1^{\eta_2+1,\eta_2}$ both be of the form (2.8). Let $H_{12} \in \mathbb{C}^{\eta_1,\eta_2}$. Then there exists a matrix $X_{12} \in \mathbb{C}^{\eta_1+1,\eta_2+1}$ such that*

$$0 = F_1^* X_{12} G_2 + G_1^* X_{12} F_2 + H_{12}.$$

Proof. For the matrix X_{12} we introduce the notation

$$X_{12} = \begin{bmatrix} x_{1,1} & \dots & x_{1,\eta_2+1} \\ \vdots & & \vdots \\ x_{\eta_1+1,1} & \dots & x_{\eta_1+1,\eta_2+1} \end{bmatrix} = \begin{bmatrix} x_{i,j} \end{bmatrix}_{\substack{i=1,\dots,\eta_1+1 \\ j=1,\dots,\eta_2+1}} \in \mathbb{C}^{\eta_1+1,\eta_2+1},$$

and for the matrix H_{12} analogously

$$H_{12} = \begin{bmatrix} h_{1,1} & \dots & h_{1,\eta_2} \\ \vdots & & \vdots \\ h_{\eta_1,1} & \dots & h_{\eta_1,\eta_2} \end{bmatrix} = \begin{bmatrix} h_{i,j} \end{bmatrix}_{\substack{i=1,\dots,\eta_1 \\ j=1,\dots,\eta_2}} \in \mathbb{C}^{\eta_1,\eta_2}.$$

Then we see that we are looking for an X_{12} such that

$$\begin{aligned}
0 &= F_1^* X_{12} G_2 + G_1^* X_{12} F_2 + H_{12} \\
&= \begin{bmatrix} 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{bmatrix} \begin{bmatrix} x_{1,1} & \dots & x_{1,\eta_2+1} \\ \vdots & & \vdots \\ x_{\eta_1+1,1} & \dots & x_{\eta_1+1,\eta_2+1} \end{bmatrix} \begin{bmatrix} 0 & & \\ 1 & \ddots & \\ & \ddots & 0 \\ & & 1 \end{bmatrix} \\
&\quad + \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \end{bmatrix} \begin{bmatrix} x_{1,1} & \dots & x_{1,\eta_2+1} \\ \vdots & & \vdots \\ x_{\eta_1+1,1} & \dots & x_{\eta_1+1,\eta_2+1} \end{bmatrix} \begin{bmatrix} 1 & & \\ 0 & \ddots & \\ & \ddots & 1 \\ & & 0 \end{bmatrix} + H_{12} \\
&= \begin{bmatrix} x_{1,2} & \dots & x_{1,\eta_2+1} \\ \vdots & & \vdots \\ x_{\eta_1,2} & \dots & x_{\eta_1,\eta_2+1} \end{bmatrix} + \begin{bmatrix} x_{2,1} & \dots & x_{2,\eta_2} \\ \vdots & & \vdots \\ x_{\eta_1+1,1} & \dots & x_{\eta_1+1,\eta_2} \end{bmatrix} + \begin{bmatrix} h_{1,1} & \dots & h_{1,\eta_2} \\ \vdots & & \vdots \\ h_{\eta_1,1} & \dots & h_{\eta_1,\eta_2} \end{bmatrix} \\
&= \begin{bmatrix} x_{i,j+1} \end{bmatrix}_{\substack{i=1,\dots,\eta_1 \\ j=1,\dots,\eta_2}} + \begin{bmatrix} x_{i+1,j} \end{bmatrix}_{\substack{i=1,\dots,\eta_1 \\ j=1,\dots,\eta_2}} + \begin{bmatrix} h_{i,j} \end{bmatrix}_{\substack{i=1,\dots,\eta_1 \\ j=1,\dots,\eta_2}} \\
&= \begin{bmatrix} x_{i,j+1} + x_{i+1,j} + h_{i,j} \end{bmatrix}_{\substack{i=1,\dots,\eta_1 \\ j=1,\dots,\eta_2}}.
\end{aligned} \tag{A.29}$$

We construct such an X_{12} in the following recursive way. First, choose $x_{i,1} = 0$ for $i = 1, \dots, \eta_1 + 1$, choose $x_{i,2} := h_{i,1}$ for $i = 1, \dots, \eta_1$, and choose $x_{\eta_1+1,2}$ arbitrary. Then all $x_{i,j}$ with $j \le 2$ are fixed and all equations in (A.29) with $j \le 1$ are fulfilled.

As induction hypothesis, assume that for some $k \in \{2, \ldots, \eta_2\}$ we have that all $x_{i,j}$ with $j \leq k$ are fixed and all equations in (A.29) with $j \leq k-1$ are fulfilled.
For the inductive step, note that all equations in (A.29) with $j = k$ are given by

$$x_{i,k+1} + x_{i+1,k} + h_{i,k} = 0,$$

with $i = 1, \ldots, \eta_1$. Because of the induction hypothesis all $x_{i+1,k}$ are already fixed but not the $x_{i,k+1}$. Thus we define

$$x_{i,k+1} := -x_{i+1,k} - h_{i,k},$$

for $i = 1, \ldots, \eta_1$ and choose $x_{\eta_1+1,k+1}$ arbitrary. Then all $x_{i,j}$ with $j \leq k+1$ are fixed and all equations in (A.29) with $j \leq k$ are fulfilled. Thus the inductive argument is finished and the claim is proved. $\square$

Lemma A.27. *Let $w \in \mathbb{N}$ and consider the pencil*

$$\lambda F + G = \lambda \cdot \operatorname{diag}(F_1, \ldots, F_w) + \operatorname{diag}(G_1, \ldots, G_w),$$

where the pencils on the block diagonal $\lambda F_i + G_i \in \mathbb{C}^{\eta_i+1,\eta_i}$ are of the form (2.8) for $i = 1, \ldots, w$. Set $\eta := \eta_1 + \ldots + \eta_w$ and observe that $F, G \in \mathbb{C}^{\eta+w,\eta}$. Let an arbitrary $H = H^* \in \mathbb{C}^{\eta,\eta}$ be given. Then there exists a matrix $Z \in \mathbb{C}^{\eta+w,\eta}$ such that

$$\begin{aligned} F^*Z &= Z^*F, \\ 0 = G^*Z &+ Z^*G + H. \end{aligned}$$

Proof. We construct an $X = X^* \in \mathbb{C}^{\eta+w,\eta+w}$ such that $F^*XG + G^*XF + H = 0$. Then we obtain the assertion by setting $Z := XF$. Partition the matrix X according to the partition of F and G as

$$X = \begin{array}{cc} \begin{bmatrix} X_{11} & \cdots & X_{1w} \\ \vdots & & \vdots \\ X_{w1} & \cdots & X_{ww} \end{bmatrix} & \begin{array}{c} \eta_1 + 1 \\ \vdots \\ \eta_2 + 1 \end{array} \\ \begin{array}{ccc} \eta_1 + 1 & & \eta_2 + 1 \end{array} & \end{array}$$

and observe that from $X = X^*$ we obtain that $X_{ij} = X_{ji}^*$ for all $i, j = 1, \ldots, w$. We see that we are looking for an X such that

$$\begin{aligned} 0 &= F^*XG + G^*XF + H \\ &= \begin{bmatrix} F_1^* & & \\ & \ddots & \\ & & F_w^* \end{bmatrix} \begin{bmatrix} X_{11} & \cdots & X_{1w} \\ \vdots & & \vdots \\ X_{1w}^* & \cdots & X_{ww} \end{bmatrix} \begin{bmatrix} G_1 & & \\ & \ddots & \\ & & G_w \end{bmatrix} + (F^*XG)^* + H \\ &= \begin{bmatrix} F_1^*X_{11}G_1 & \cdots & F_1^*X_{1w}G_w \\ \vdots & & \vdots \\ F_w^*X_{1w}^*G_1 & \cdots & F_w^*X_{ww}G_w \end{bmatrix} + \begin{bmatrix} G_1^*X_{11}F_1 & \cdots & G_1^*X_{1w}F_w \\ \vdots & & \vdots \\ G_w^*X_{1w}^*F_1 & \cdots & G_w^*X_{ww}F_w \end{bmatrix} + \begin{bmatrix} H_{11} & \cdots & H_{1w} \\ \vdots & & \vdots \\ H_{1w}^* & \cdots & H_{ww} \end{bmatrix} \end{aligned}$$

$$= \begin{bmatrix} F_1^* X_{11} G_1 + G_1^* X_{11} F_1 + H_{11} & \cdots & F_1^* X_{1w} G_w + G_1^* X_{1w} F_w + H_{1w} \\ \vdots & & \vdots \\ F_w^* X_{1w}^* G_1 + G_w^* X_{1w}^* F_1 + H_{1w}^* & \cdots & F_w^* X_{ww} G_w + G_w^* X_{ww} F_w + H_{ww} \end{bmatrix}. \tag{A.30}$$

Using Lemma A.25 we obtain $X_{ii} = X_{ii}^* \in \mathbb{C}^{\eta_i+1,\eta_i+1}$ such that $F_i^* X_{ii} G_i + G_i^* X_{ii} F_i + H_{ii} = 0$ for all $i = 1, \ldots, w$. Using Lemma A.26 we obtain $X_{ij} \in \mathbb{C}^{\eta_i+1,\eta_j+1}$ such that $F_i^* X_{ij} G_j + G_i^* X_{ij} F_j + H_{ij} = 0$ for all $i, j = 1, \ldots, w$ with $i < j$. Since the equations in the strict block-lower-left part of (A.30) are the conjugate transpose of the equations in the strict block-upper-right part, they are also fulfilled. $\square$

Lemma A.28. *Let $\lambda F_{\mathcal{L}} + G_{\mathcal{L}} \in \mathbb{C}[\lambda]_1^{\epsilon,\epsilon+1}$ be in the form (2.5) and $\lambda F_{\mathcal{M}} + G_{\mathcal{M}} \in \mathbb{C}^{\eta+1,\eta}$ be in the form (2.8) with $\eta \leq 2$. Let $H_{14} \in \mathbb{C}^{\epsilon+1,\eta}$ be arbitrary. Then there exist $Z_{14} \in \mathbb{C}^{\epsilon,\eta}$ and $Z_{41} \in \mathbb{C}^{\eta+1,\epsilon+1}$ such that*

$$\begin{aligned} F_{\mathcal{L}}^* Z_{14} &= Z_{41}^* F_{\mathcal{M}} \\ 0 = G_{\mathcal{L}}^* Z_{14} &+ Z_{41}^* G_{\mathcal{M}} + H_{14}. \end{aligned}$$

Proof. Denote the entries of H by $[h_{i,j}]$. If $\eta = 1$ then the matrices Z_{14} and Z_{41} take the form

$$Z_{14} = \begin{bmatrix} z_1 \\ \vdots \\ z_\epsilon \end{bmatrix}, \qquad Z_{41} = \begin{bmatrix} \tilde{z}_{1,1} & \ldots & \tilde{z}_{1,\epsilon+1} \\ \tilde{z}_{2,1} & \ldots & \tilde{z}_{2,\epsilon+1} \end{bmatrix}.$$

With this we obtain

$$\begin{bmatrix} z_1 \\ \vdots \\ z_\epsilon \\ 0 \end{bmatrix} = F_{\mathcal{L}}^* Z_{14} = Z_{41}^* F_{\mathcal{M}} = \begin{bmatrix} \overline{\tilde{z}_{1,1}} \\ \vdots \\ \overline{\tilde{z}_{1,\epsilon+1}} \end{bmatrix},$$

which implies

$$Z_{41} = \begin{bmatrix} \overline{z_1} & \cdots & \overline{z_\epsilon} & 0 \\ \tilde{z}_{2,1} & \ldots & \tilde{z}_{2,\epsilon} & \tilde{z}_{2,\epsilon+1} \end{bmatrix}.$$

With this notation at hand we can verify that

$$G_{\mathcal{L}}^* Z_{14} + Z_{41}^* G_{\mathcal{M}} = - \begin{bmatrix} 0 & & \\ 1 & \ddots & \\ & \ddots & 0 \\ & & 1 \end{bmatrix} Z_{14} + \begin{bmatrix} z_1 & \overline{\tilde{z}_{2,1}} \\ \vdots & \vdots \\ z_\epsilon & \overline{\tilde{z}_{2,\epsilon}} \\ 0 & \overline{\tilde{z}_{2,\epsilon+1}} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = - \begin{bmatrix} 0 \\ z_1 \\ \vdots \\ z_\epsilon \end{bmatrix} + \begin{bmatrix} \overline{\tilde{z}_{2,1}} \\ \vdots \\ \overline{\tilde{z}_{2,\epsilon}} \\ \overline{\tilde{z}_{2,\epsilon+1}} \end{bmatrix},$$

which proves that one can choose $Z_{14} = 0$ and $\tilde{z}_{2,i} := -\overline{h_{i,1}}$ for $i = 1, \ldots, \epsilon + 1$.

If $\eta = 2$ then the matrices Z_{14} and Z_{41} take the form

$$Z_{14} = \begin{bmatrix} z_{1,1} & z_{1,2} \\ \vdots & \vdots \\ z_{\epsilon,1} & z_{\epsilon,2} \end{bmatrix}, \qquad Z_{41} = \begin{bmatrix} \tilde{z}_{1,1} & \dots & \tilde{z}_{1,\epsilon+1} \\ \tilde{z}_{2,1} & \dots & \tilde{z}_{2,\epsilon+1} \\ \tilde{z}_{3,1} & \dots & \tilde{z}_{3,\epsilon+1} \end{bmatrix}.$$

With this we obtain

$$\begin{bmatrix} z_{1,1} & z_{1,2} \\ \vdots & \vdots \\ z_{\epsilon,1} & z_{\epsilon,2} \\ 0 & 0 \end{bmatrix} = F_{\mathcal{L}}^* Z_{14} = Z_{41}^* F_{\mathcal{M}} = \begin{bmatrix} \overline{\tilde{z}_{1,1}} & \overline{\tilde{z}_{2,1}} \\ \vdots & \vdots \\ \overline{\tilde{z}_{1,\epsilon+1}} & \overline{\tilde{z}_{2,\epsilon+1}} \end{bmatrix}$$

which implies

$$Z_{41} = \begin{bmatrix} \overline{z_{1,1}} & \cdots & \overline{z_{\epsilon,1}} & 0 \\ \overline{\tilde{z}_{1,2}} & \dots & \overline{\tilde{z}_{\epsilon,2}} & 0 \\ \tilde{z}_{3,1} & \dots & \tilde{z}_{3,\epsilon} & \tilde{z}_{3,\epsilon+1} \end{bmatrix}.$$

With this notation at hand we can verify that

$$\begin{aligned} & G_{\mathcal{L}}^* Z_{14} + Z_{41}^* G_{\mathcal{M}} \\ = & -\begin{bmatrix} 0 & & \\ 1 & \ddots & \\ & \ddots & 0 \\ & & 1 \end{bmatrix} Z_{14} + \begin{bmatrix} z_{1,1} & z_{1,2} & \overline{\tilde{z}_{3,1}} \\ \vdots & \vdots & \vdots \\ z_{\epsilon,1} & z_{\epsilon,2} & \overline{\tilde{z}_{3,\epsilon}} \\ 0 & 0 & \overline{\tilde{z}_{3,\epsilon+1}} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = -\begin{bmatrix} 0 & 0 \\ z_{1,1} & z_{1,2} \\ \vdots & \vdots \\ z_{\epsilon,1} & z_{\epsilon,2} \end{bmatrix} + \begin{bmatrix} z_{1,2} & \overline{\tilde{z}_{3,1}} \\ \vdots & \vdots \\ z_{\epsilon,2} & \overline{\tilde{z}_{3,\epsilon}} \\ 0 & \overline{\tilde{z}_{3,\epsilon+1}} \end{bmatrix}. \end{aligned}$$

Thus, choosing $z_{i,1} := h_{i+1,1}$ for $i = 1, \dots, \epsilon$, $z_{1,2} := -h_{1,1}$, $z_{i,2} := 0$ for $i = 2, \dots, \epsilon$, $\tilde{z}_{3,1} := -\overline{h_{1,2}}$, $\tilde{z}_{3,2} := \overline{z_{1,2}} - \overline{h_{2,2}}$, and $\tilde{z}_{3,i} := -\overline{h_{i,2}}$ for $i = 3, \dots, \epsilon + 1$ we obtain the assertion. □

Lemma A.29. *Let $\lambda F_{\mathcal{J}} + G_{\mathcal{J}} \in \mathbb{C}[\lambda]_1^{\rho,\rho}$ be in the form (2.6) with $\lambda_j =: \mu$ and let $\lambda F_{\mathcal{M}} + G_{\mathcal{M}} \in \mathbb{C}[\lambda]_1^{\eta+1,\eta}$ is in the form (2.8). Let an arbitrary $H_{24} \in \mathbb{C}^{\rho,\eta}$ be given. Then there exist matrices $Z_{24} \in \mathbb{C}^{\rho,\eta}$ and $Z_{42} \in \mathbb{C}^{\eta+1,\rho}$ such that*

$$\begin{aligned} Z_{24} &= Z_{42}^* F_{\mathcal{M}}, \\ 0 = G_{\mathcal{J}}^* Z_{24} &+ Z_{42}^* G_{\mathcal{M}} + H_{24}. \end{aligned}$$

Proof. Let the matrices Z_{24} and Z_{42} be given in the form

$$Z_{24} = \begin{bmatrix} z_{1,1} & \dots & z_{1,\eta} \\ \vdots & & \vdots \\ z_{\rho,1} & \dots & z_{\rho,\eta} \end{bmatrix}, \qquad Z_{42} = \begin{bmatrix} \tilde{z}_{1,1} & \dots & \tilde{z}_{1,\rho} \\ \vdots & & \vdots \\ \tilde{z}_{\eta+1,1} & \dots & \tilde{z}_{\eta+1,\rho} \end{bmatrix}.$$

Since $F_{\mathcal{J}} = I$, we see that

$$Z_{24} = Z_{42}^* F_{\mathcal{M}} = Z_{42}^* \begin{bmatrix} 1 & & \\ 0 & \ddots & \\ & \ddots & 1 \\ & & 0 \end{bmatrix},$$

which means that we have to find Z_{24} and Z_{42} with

$$Z_{42} = \begin{bmatrix} & Z_{24}^* & \\ & & \\ \tilde{z}_{\eta+1,1} & \dots & \tilde{z}_{\eta+1,\rho} \end{bmatrix}.$$

Consequently, we find that with the notation $\overline{\tilde{z}_{\eta+1,i}} =: z_{i,\eta+1}$ we have

$$G_{\mathcal{J}}^* Z_{24} + Z_{42}^* G_{\mathcal{M}} = G_{\mathcal{J}}^* Z_{24} + \begin{bmatrix} & \overline{\tilde{z}_{\eta+1,1}} \\ Z_{24} & \vdots \\ & \overline{\tilde{z}_{\eta+1,\rho}} \end{bmatrix} \begin{bmatrix} 0 & & \\ 1 & \ddots & \\ & \ddots & 0 \\ & & 1 \end{bmatrix}$$

$$= - \begin{bmatrix} \mu z_{1,1} & \dots & \mu z_{1,\eta} \\ \mu z_{2,1} + z_{1,1} & \dots & \mu z_{2,\eta} + z_{1,\eta} \\ \mu z_{3,1} + z_{2,1} & \dots & \mu z_{3,\eta} + z_{2,\eta} \\ \vdots & & \vdots \\ \mu z_{\rho,1} + z_{\rho-1,1} & \dots & \mu z_{\rho,\eta} + z_{\rho-1,\eta} \end{bmatrix} + \begin{bmatrix} z_{1,2} & \dots & z_{1,\eta} & \overline{\tilde{z}_{\eta+1,1}} \\ z_{2,2} & \dots & z_{2,\eta} & \overline{\tilde{z}_{\eta+1,2}} \\ z_{3,2} & \dots & z_{3,\eta} & \overline{\tilde{z}_{\eta+1,3}} \\ \vdots & & \vdots & \vdots \\ z_{\rho,2} & \dots & z_{\rho,\eta} & \overline{\tilde{z}_{\eta+1,\rho}} \end{bmatrix}$$

$$= \begin{bmatrix} -\mu z_{1,1} + z_{1,2} & \dots & -\mu z_{1,\eta-1} + z_{1,\eta} & -\mu z_{1,\eta} + z_{1,\eta+1} \\ -z_{1,1} - \mu z_{2,1} + z_{2,2} & \dots & -z_{1,\eta-1} - \mu z_{2,\eta-1} + z_{2,\eta} & -z_{1,\eta} - \mu z_{2,\eta} + z_{2,\eta+1} \\ -z_{2,1} - \mu z_{3,1} + z_{3,2} & \dots & -z_{2,\eta-1} - \mu z_{3,\eta-1} + z_{3,\eta} & -z_{2,\eta} - \mu z_{3,\eta} + z_{3,\eta+1} \\ \vdots & & \vdots & \vdots \\ -z_{\rho-1,1} - \mu z_{\rho,1} + z_{\rho,2} & \dots & -z_{\rho-1,\eta-1} - \mu z_{\rho,\eta-1} + z_{\rho,\eta} & -z_{\rho-1,\eta} - \mu z_{\rho,\eta} + z_{\rho,\eta+1} \end{bmatrix}.$$

Using $H_{24} = [h_{i,j}]$ we can choose $z_{1,1} := 0$ and $z_{1,2} := h_{1,1}$. Defining $z_{1,j} := \mu z_{1,j-1} - h_{1,j-1}$ for $j = 3, \dots, \eta+1$ recursively we find that all $z_{1,i}$ with $i = 1, \dots, \eta+1$ are fixed and that the first row of $H_{24} + Z_{42}^* G_{\mathcal{M}} + G_{\mathcal{J}}^* Z_{24}$ vanishes. For an inductive argument assume that the first k rows of $H_{24} + Z_{42}^* G_{\mathcal{M}} + G_{\mathcal{J}}^* Z_{24}$ vanish and that all $z_{j,i}$ with $j \le k$ are already fixed. Then in the $k+1$-th row we find the equations $h_{k+1,i} + z_{k,i} - \mu z_{k+1,i} + z_{k+1,i+1} = 0$ for $i = 1, \dots, \eta$. Again, set $z_{k+1,1} = 0$, $z_{k+1,2} = -h_{k+1,1} - z_{k,1}$, and then define $z_{k+1,i+1} := \mu z_{k+1,i} - h_{k+1,i} - z_{k,i}$ recursively for $i = 2, \dots, \eta$. This fixes all $z_{j,i}$ with $j \le k+1$ and the first $k+1$ rows of $H_{24} + Z_{42}^* G_{\mathcal{M}} + G_{\mathcal{J}}^* Z_{24}$ vanish. This finishes the inductive argument and the claim is shown. $\square$

Appendix B

MATLAB codes

This Appendix contains an implementation of Algorithm 4.9 with various adaptions as described in Remark 4.10.

```
%
%   PURPOSE
%
%   Given the system
%          E x' = A x + B u
%            y  = C x + D u
%   together with the supply
%      [ y ]^* [ Q   S ] [ y ]
%      [ u ]   [ S^* R ] [ u ] ,
%   compute a slightly perturbed system
%          E x' = newA x + newB u
%            y  = newC x + newD u
%   which is cylo-dissipative with respect to the given supply.
%
%   rank_EAB has to be the rank of [ zE-A , -B ] over the field
%   of the rational functions
%
%   alpha          The scaling parameter, see Algorithm 4.9.
%                  alpha has to be greater or equal to 1.0.
%
%   tol            All eigenvalues for which the absolute value
%                  of the real part is below tol are considered
%                  to be purely imaginary. Also, purely
%                  imaginary eigenvalues which are less than tol
%                  appart are considered to be double
%                  eigenvalues. tol has to be greater or equal
%                  0.0.
%
```

```
%  maxiter          < 0       : no maximum number of iterations
%                   >=0       : specifies maximum number of
%                               iterations
%
function [newA, newB, newC, newD, iter] = ...
         enforce_dissipativity(E, A, B, C, D, Q, S, R, ...
         rank_EAB, alpha, tol, maxiter)

%
% Check parameters
%

if( alpha<1 )
   error('alpha has to be greater or equal to 1.0!');
end

if( tol < 0 )
   error('tol has to be greater or equal to 0.0!');
end

[rho, n] = size(E);
[l, m] = size(D);

if( rho~=size(A,1) || n~=size(A,2) ||...
    rho~=size(B,1) || m~=size(B,2) ||...
      l~=size(C,1) || n~=size(C,2) ||...
      l~=size(Q,1) || l~=size(Q,2) ||...
      l~=size(S,1) || m~=size(S,2) ||...
      m~=size(R,1) || m~=size(R,2) )
   error('Matrix size mismatch!');
end

%
% Step 1
%
eta = rho+n+m-2*rank_EAB;

iter=0;

while( true )
   %
   % Step 2
   %
```

```
    [N_0, N_1] = system_to_paraHermitian(E, A, B, C, D,...
                                         Q, S, R);

    %
    % Steps 3, 4, and 5
    %
    p_N_0 = perturb_paraHermitian(N_0, N_1, eta, tol);

    %
    % Step 6
    %
    if( norm(p_N_0)==0.0 )
       break;
    end

    iter = iter+1;

    if( maxiter>=0 && iter>=maxiter )
       disp('Maximum number of iterations reached!');
       break;
    end

    %
    % Step 7
    %
    [newE, newA, newB, newC, newD] = paraHermitian_to_system(...
                            N_0 + p_N_0, N_1, rho, n, m, l);

    %
    % Step 8
    %
    A = A + alpha*(newA-A);
    B = B + alpha*(newB-B);
    C = C + alpha*(newC-C);
    D = D + alpha*(newD-D);
end

newA = A;
newB = B;
newC = C;
newD = D;

function [N_0, N_1] = ...
```

```
        system_to_paraHermitian(E, A, B, C, D, Q, S, R)

[rho, n] = size(E);
[l, m] = size(D);

N_0 = [zeros(rho, rho+l+l), -A, -B;...
       zeros(l, rho+l), eye(l), -C, -D;...
       zeros(l, rho), eye(l), Q, zeros(l,n), S;...
       -A', -C', zeros(n, l+n+m);...
       -B', -D', S', zeros(m, n), R];

N_1 = [zeros(rho, rho+l+l), E, zeros(rho, m);...
       zeros(l+l, rho+l+l+n+m);...
       -E', zeros(n, l+l+n+m);...
       zeros(m, rho+l+l+n+m)];

function [E, A, B, C, D] = ...
         paraHermitian_to_system(N_0, N_1, rho, n, m, l)

E = N_1(1:rho, rho+2*l + 1:rho+2*l + n);
A = -N_0(1:rho, rho+2*l + 1:rho+2*l + n);
B = -N_0(1:rho, rho+2*l+n+1:rho+2*l+n+m);
C = -N_0(rho+1:rho+l, rho+2*l + 1:rho+2*l + n);
D = -N_0(rho+1:rho+l, rho+2*l+n+1:rho+2*l+n+m);

function PA = perturb_paraHermitian(A, E, eta, tol)
%
% Ideally, one should first split of the singular part and then
% work on the regular part only. However, we assume that the
% pencil is regular in the first place.
%
% Q1' * ( lambda E + A ) * Q1 = lambda tE + tA
%[tE,tA,Q1,rb,l,p]=matlab_stcssp(E,A);
%
tE = E;
tA = A;
Q1 = eye(size(A));
rb=0; l=size(A, 1);

%
% obtain regular index 1 block
regE = tE(rb+1:rb+l, rb+1:rb+l);
regA = tA(rb+1:rb+l, rb+1:rb+l);
n = size(regE, 1);
```

```
% compute all eigenpairs
[vecs,eigs] = eig(regA, regE);
eigs = diag(eigs);

% find "purely" imaginary eigenvalues
ieigs_mask = (abs(real(eigs))<tol);
unsorted_vectors =     vecs(:,ieigs_mask );
unsorted_omegas = imag(eigs(  ieigs_mask ));

% sort purely imaginary eigenvalues
[omegas,sort_perm] = sort( unsorted_omegas, 1, 'ascend' );
vectors = unsorted_vectors(:,sort_perm);

%
% Compute the slopes in the signsum plot
%
sigmas = zeros(size(omegas));

j=1;
processed=0;
onstack=0;
while j<=length(omegas)
   if( j==length(omegas) || abs(omegas(j)-omegas(j+1))>tol )
      act_vecs = vectors(:,processed+1:j);

      local_eigs = imag(eig(act_vecs'*regE*act_vecs));
      local_eigs = sort( local_eigs, 1, 'descend');

      for k=1:j-processed
         if( onstack>0 )
            sigmas(processed+k) = local_eigs(1);
            local_eigs = local_eigs(2:end);
            onstack = onstack-1;
         else
            sigmas(processed+k) = local_eigs(end);
            local_eigs = local_eigs(1:end-1);
            onstack = onstack+1;
         end
      end

      processed = j;
   end
```

```
   j = j+1;
end

M=length(omegas);

%
% Compute the perturbation
%
onstack = 0;
processed = 0;

D_out = [];
V_out = zeros(n,0);

for j=1:M
   if( sigmas(j)<0 )
      onstack = onstack+1;
   else
      onstack = onstack-1;

      if( onstack==0 )
         ingroup = (j-processed)/2;
         middle = sum(omegas(processed+1:j))/(2*ingroup);

         N = sqrt(-1)*middle*regE + regA;
         [V,D] = eig(N);
         D=diag(D);

         % strip all but the innermost eta eigenvalues
         to_strip = (n-eta)/2;
         [D,perm]=sort(D,1,'ascend');
         V=V(:,perm);

         V=V(:,to_strip+1:to_strip+eta);
         D=D(  to_strip+1:to_strip+eta);

         V=V(:,D<0);
         D=D(D<0);

         D_out = [D_out;D];
         V_out = [V_out,V];
```

```
            processed = j;
        end
    end
end

regPA = real(-V_out * diag(D_out) * V_out');
regPA = (regPA + regPA')/2;

tPA = zeros(size(tA));
tPA(rb+1:rb+l,rb+1:rb+l) = regPA;
PA = Q1 *   tPA * Q1';
PA = (PA+PA')/2;
```