Particular Timelike Flows in Global Lorentzian Geometry

vorgelegt von

Diplom-Physiker
Alexander Dirmeier

aus Lindau

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Zusammenfassung

Spezielle zeitartige Flüsse in der globalen Lorentzgeometrie

von Alexander Dirmeier

Diese Arbeit untersucht topologische und kausale Eigenschaften Lorentz'scher Mannigfaltigkeiten (M, g), die als zusätzlich Struktur ein vollständiges, zeitartiges Einheitsvektorfeld V, oder in anderen Worten einen globalen zeitartigen Fluss, aufweisen. Diese Lorentz'schen Mannigfaltigkeiten sind in natürlicher Weise Raumzeiten. Es werden allgemeine geometrische Bedingungen hergeleitet, die dazu führen, dass diese Raumzeiten eine Produktstruktur $\mathbb{R} \times S$ aufweisen, wobei das Vektorfeld V entlang des Faktors $\mathbb{R}$ zeigt und S der Raum der Integralkurven von V ist. Die möglichen Kausalstufen für diese Produktraumzeiten werden analysiert und eine vollständige kausale Klassifikation wird angegeben. Durch eine Klassifikation bezüglich einer Zerlegung der kovarianten Ableitung ∇V des gegebenen zeitartigen Vektorfelds können Unterklassen dieser Produktraumzeiten gewonnen werden. Die speziellen Unterklassen der geblätterten Raumzeiten und der stationären Raumzeiten werden hinsichtlich ihrer globalen Hyperbolizität analysiert und mehrere neue Beziehungen werden gewonnen. Für stationäre und homothetische Raumzeiten wird eine neue Version der Lorentz'schen Bochnertechnik hergeleitet. Schließlich werden konforme Lorentz'sche Submersionen und insbesondere Hubble-isotrope Raumzeiten analysiert und Bedingungen für deren globale Hyperbolizität und geodätische Vollständigkeit werden gewonnen.

School for Mathematics and Natural Sciences
of Technische Universität Berlin

Abstract

Particular Timelike Flows in Global Lorentzian Geometry

by Alexander Dirmeier

This work investigates the topological and causal characteristics of Lorentzian manifolds (M, g), which possess a complete and timelike unit vector field V, or in other words a global timelike flow, as an additional structure. Naturally, these Lorentzian manifolds are spacetimes. General geometric requirements for these spacetimes to split diffeomorphically as a product $\mathbb{R} \times S$, with the vector field V along the $\mathbb{R}$-factor and S the space of flow lines of V, are derived. The possible causality conditions for these splitting spacetimes are analyzed and a complete causal classification is given. Sub-classes of these splitting spacetimes can be obtained by a classification according to a decomposition of the covariant derivative ∇V of the given timelike vector field. The specific sub-classes of sliced spacetimes and stationary spacetimes are analyzed with regard to global hyperbolicity and several new relations are obtained. For stationary and homothetic spacetimes, a new version of the Lorentzian Bochner technique is derived. Finally, conformal Lorentzian submersions, and particularly Hubble-isotropic spacetimes, are analyzed and conditions for their global hyperbolicity and geodesic completeness are obtained.

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DEDICATION

For Catharina.

Chapter 1

INTRODUCTION

After the Golden Age of General Relativity in the 1960s and early 1970s (cf. [Tho03]), when the theory and the mathematics of Lorentzian manifolds were developed intensively by theoretical physicists, Lorentzian geometry took great leaps forward in recent years by progress from a pure mathematical, though physically inspired, perspective. A selection of these developments, which are relevant to this work, include: the Lorentzian splitting theorem, which has started to emerge already in the 1980s, based on the simplified proof of the Riemannian Cheeger–Gromoll splitting theorem by J.-H. Eschenburg and E. Heintze (see [EH84]) and was then developed and constantly improved by J.K. Beem, P.E. Ehrlich, J.-H. Eschenburg, E. Heintze, R. Bartnik, G.J. Galloway and others (see, e.g., [Gal89] and the references therein for a reasonably strong version of the theorem), the recent progress in the notion of global hyperbolicity of spacetimes, particularly in the work of A. Bernal and M. Sánchez (see, e.g., [BS07] and [BS05] or [Ś11] for an overview), as well as the further development of the causal ladder (see [MS08]), especially in the work of E. Minguzzi (e.g., [Min08c] [Min08d] [Min09a] [Min09b] [Min09c]), where a particular highlight is the connection of Lorentzian causality theory to order theory (see [Min10]). Other important, more specific, recent developments in global Lorentzian geometry, on which this work is based, are: the question of a diffeomorphic splitting of (conformally) stationary spacetimes analyzed by M.A. Javaloyes and M. Sánchez (see [JS08]), the correspondence of global hyperbolicity to the completeness of Finslerian metrics of Randers type for stationary spacetimes, devised in the seminal work [CJM11] by E. Caponio, M.A. Javaloyes and A. Masiello and developed further in [CJS11] by the first two authors and M. Sánchez, the concept of a Lorentzian Bochner technique established by A. Romero and M. Sánchez (see [RS96] and [RS98]), as well as the notion of regularly sliced spacetimes (see [CB09] for the introduction of this terminology) developed in [CBC02] and [Cot04] by Y. Choquet-Bruhat and S. Cotsakis.

This work is concerned with global Lorentzian geometry, i.e., questions of the topological and causal structure of manifolds admitting a Lorentzian metric. Local considerations, particularly questions of curvature, will be of secondary importance, but will for example come into play in section 5.3 in connection with the Lorentzian Bochner technique. We will analyze Lorentzian manifolds which carry the additional structure of a global and timelike, complete unit vector field or, in other words admit a global timelike flow. Naturally, these Lorentzian manifolds are spacetimes. This will allow us to prove general theorems on the topological and causal structure of these spacetimes in chapter 4 and, subsequently, analyze particular classes of these spacetimes in chapters 5 and 6.

Particularly, this work is structured along the following train of thought. In chapter 2, we first establish the basic mathematical concepts used in the later chapters, concerning the differential geometry of semi-Riemannian manifolds, and fix the notational conventions we will use. Then we provide an overview of the existing theory of $\mathbb{R}$-actions on manifolds, generalized principal $\mathbb{R}$-bundles and dynamical systems, connect these concepts and cast them into a form applicable to a global timelike flow on a Lorentzian manifold in the later chapters. In chapter 3, we will establish the basic theory on specific Lorentzian geometry used in this work. Particularly, we define in that chapter the basic geometric objects that we analyze in this work, i.e., kinematical spacetimes (cf. Def. 3.15). Having at hand, not only a Lorentzian metric, but also a specific vector field on a spacetime, allows us to define some derived geometric quantities (called kinematical quantities; see section 3.1), give a classification of spacetimes according to these quantities (section 3.2) and analyze their evolution, which leads to the Raychaudhuri equations (section 3.3). The basic Lorentzian causality theory will also be established in that chapter in section 3.4.

The main new results are contained in chapters 4, 5 and 6. In chapter 4, after an introduction to different splitting philosophies, we establish in section 4.1 conditions for a kinematical spacetime to split as a product manifold and derive various propositions on the geometric structure of the resulting manifold. In section 4.2 we analyze the possible causality conditions of kinematical spacetimes in the splitting case. A main result of those two sections can, for example, be summarized as follows (cf. Thm. 4.13, Thm. 4.19, Thm. 4.29 and Prop. 4.31):

Main Result. *Let (M, g, V) be a kinematical spacetime, i.e., a Lorentzian manifold (M, g) with a timelike, complete unit vector field V. If the integral curves of V are non–partially-imprisoned and the space S of integral curves of V is a manifold (particularly, this is the case if there is a Lorentzian metric $\tilde{g}$ in the conformal class of g, such that V is $\tilde{g}$-geodesic), then M is diffeomorphic to $\mathbb{R} \times S$ and V is mapped to a vector field along the factor $\mathbb{R}$ by a map realizing this diffeomorphism. Furthermore, such a spacetime is causally continuous if it is feebly distinguishing and if S is compact, it is globally hyperbolic if it is non-imprisoning.*

Besides the known splitting results and causality properties in the case of a Killing vector field V established in [JS08], which are included as special cases in the results of this work, there are several other examples of related works with a similar aim. Classical articles that deal with similar splitting questions in the (semi-)Riemannian case are for example [Wad75], [Her60] or [PR93]. But these have always the aim to achieve “more” than only a diffeomorphic splitting by establishing, for example, the structure of a twisted or warped product manifold. An approach to the splitting of stably causal spacetimes was taken in [GRK96], while paying attention to possible connections to timelike completeness. Ideas most similar to the approach in this work were pursued in [GO03] and [GO09], although only for a vector field V with an integrable vertical distribution. Various results in these two references can be derived as special cases from the propositions established in this work.

In section 4.3, we will analyze the causality of splitting spacetimes in the particular case of spacetime dimension smaller or equal to four. In this situation there exists a specifically close connection between the topology of the spacetime and the causality conditions. Furthermore, the propositions established in that section provide interesting (and maybe surprising) connections between Lorentzian causality theory and Poincaré–Bendixson theory of dynamical systems and between Lorentzian causality theory and contact geometry.

Chapter 5 contains three sections, each of which deals with a different sub-class of splitting spacetimes. We use the results about their topological and causal structure, derived in the chapter before, as a basis to establish more specific results about (regularly) sliced spacetimes in section 5.1, global hyperbolicity of stationary spacetimes in section 5.2 and the Lorentzian Bochner technique in section 5.3. The notion of regularly sliced spacetimes was introduced in [CBC02], to put bounds on the components of a Lorentzian metric of a spacetime and derive simple conditions for global hyperbolicity in this case. In Thm. 5.11, we develop this concept further to infer conditions for the existence of a regularly sliced metric in the conformal class of a globally hyperbolic metric. Section 5.2 contains the following proposition on the completeness of Riemannian metrics as a main result (see Thm. 5.22):

Main Result. *The conformal transformation $\frac{g}{A^2}$ of a complete Riemannian metric g by a positive function A on a non-compact manifold M is complete if and only if the function $\frac{1}{A}$ is not an L^1-function along any curve escaping to infinity on M with respect to g.*

The remainder of that section deals with the application of this result to stationary spacetimes in connection to the correspondence of global hyperbolicity of stationary spacetimes to the completeness of Finslerian metrics of Randers type established in [CJS11]. We are able to derive various propositions on the global hyperbolicity of stationary spacetimes based on Riemannian completeness and the growth of specific functions (cf. Thm. 5.26, Prop. 5.28, Prop. 5.29 and Thm. 5.31). In section 5.3, we first establish connections between the formulas of Lorentzian Bochner technique developed in [RS96] and [RS98] and the Raychaudhuri equations from section 3.3, and then we are able to establish a new version of the Lorentzian Bochner technique for particular non-compact stationary spacetimes, by regarding them as Lorentzian submersions. Furthermore, we are able to extend this technique to the case of a homothetic vector field instead of a Killing vector field. The main Bochner-like results of this section are contained in the Thms. 5.39, 5.45 and 5.50. The first theorem deals with the stationary case and a compact base manifold of the submersion, the second theorem deals with the stationary case and a base manifold of the submersion with certain conditions at infinity, and the third theorem includes the homothetic case.

In chapter 6, we identify kinematical spacetimes obeying a certain condition on the kinematical quantities (namely vanishing shear; cf. section 3.1) with conformal Lorentzian submersions. This gives the possibility to describe large classes of spacetimes, which are inspired by constraints on the kinematical quantities from physics, by purely geometric conditions. Hence, the remainder of that chapter deals with the global structure and the causality conditions of kinematical spacetimes which are, at the same time, particular conformal Lorentzian submersions with a Riemannian base manifold and totally geodesic fibers. It turns out that these are the so-called Hubble-isotropic spacetimes (cf. e.g. [HP99]; the idea for the analysis of these metrics is also inspired by [GPS$^+$10]). We introduce these spacetimes in section 6.1 and analyze their global properties, particularly their metric structure with respect to a diffeomorphic splitting (see Thm. 6.8), and conditions for their global hyperbolicity (see Thm. 6.18 and Thm. 6.20) in section 6.2. Section 6.3 contains two propositions on the time- and spacelike completeness of Hubble-isotropic spacetimes.

Finally, in chapter 7, we will discuss some interesting open problems arising from the theorems and propositions established in this work. We will also give some preliminary ideas for directions of further research about these problems and state a conjecture together with ideas towards its proof.

Chapter 2

NOTATION, CONVENTIONS AND PRELIMINARIES

In this chapter, we fix some notation and conventions from differential geometry that will be used throughout this work. In section 2.1, we give some definitions and well-known results on semi-Riemannian manifolds, but will also introduce some concepts which are non-standard. In section 2.2, we will establish some facts about Finslerian geometry, particularly about Finslerian metrics of Randers type, which will be needed for the analysis in subsequent chapters. In sections 2.3 and 2.4, we introduce some details about $\mathbb{R}$-actions on manifolds, i.e., global flows, which lead to a connection with the theory of dynamical systems. Most results in these sections are well-known, but we give a different comprehensive viewpoint on these matters that will be suitable for the purposes of this work in the following chapters. The propositions and theorems established in those two sections are the groundwork for the splitting results which will be derived in chapter 4. Standard references for this first chapter are [Con01], [O'N83] and [KN63], additional references will be cited where applicable.

Basic Essentials

The subset relation $\subset$ is assumed to be reflexive, i.e., $A \subset A$ for all sets A. Let $A \subset T$, where T is some topological space. We will denote the interior of A by $\mathring{A}$, its closure by $\overline{A}$ and its complement with respect to T by $T \setminus A$. For subsets of the real numbers, we use $\mathbb{R}_{\geq a} = [a, \infty)$, $\mathbb{R}_{\leq a} = (\infty, a]$ and $\mathbb{R}_{>a} = (a, \infty)$, $\mathbb{R}_{<a} = (\infty, a)$ for all fixed $a \in \mathbb{R}$ as abbreviations.

All (topological) manifolds M of dimension $\dim(M) = n$ are assumed to be connected, second countable, locally homeomorphic to $\mathbb{R}^n$ and Hausdorff. Occasionally, we will indicate the dimension of a manifold by a superscript, i.e., M^n is the manifold M with dimension n indicated. This definition also implies that all manifolds have empty boundary. Manifolds with boundary can be included in this definition by considering them locally homeomorphic to the half-space $\mathbb{H}^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \,|\, x_1 \geq 0\}$ (cf. [Con01, Sec. 1.6]). A space $\tilde{M}$ fulfilling all conditions of a topological manifold except for being Hausdorff, but is a T_1-space only instead, will be called a near-manifold. Then one can readily define differentiable structures on near-manifolds in the same way as on ordinary manifolds (see e.g. [Hic65]). As for manifolds, we will assume that they carry a fixed differential structure of degree C^r for some $r \in \mathbb{N} \cup \{\infty\}$. We will take smooth and diffeomorphic to mean of the same class of differentiability as the underlying differentiable structure of the manifold. If not indicated otherwise, all geometric objects (curves, maps, fields, etc.) are considered smooth, and we will usually assume that the manifolds we consider are sufficiently smooth, i.e., for most applications C^3 is enough. In some places, where we consider manifolds or geometric objects of a lower differentiability class—or even continuous objects or topological manifolds—this will be stated explicitly. The group of diffeomorphisms of a manifold M of class $r > 0$ to itself will be denoted by $\mathrm{Diff}^r(M)$ and the class of C^s-functions $M \to \mathbb{R}$ will be denoted by $C^s(M)$ for some $0 \leq s \leq r$. Usually, we will consider smooth functions $C^r(M)$ for a C^r-manifold M.

We call a function $d\colon T \times T \to \mathbb{R}_{\geq 0}$, with T some topological space, a generalized distance if it satisfies the conditions $d(x, y) = 0 \Leftrightarrow x = y$, $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in T$. A function $d\colon T \times T \to \mathbb{R}_{\geq 0}$ which is a generalized distance and additionally fulfills $d(x, y) = d(y, x)$ for all $x, y \in T$ is called a distance. We reserve the term metric for metric tensors on manifolds. Nevertheless, we call a topological space T together with a (generalized) distance d, which is compatible with the topology, a (generalized) metric space (T, d). Generalized metric spaces will arise in the context of Finslerian metrics (cf. [BCS00, Sec. 6.2] for details).

A metric space (T, d) will be called complete if every Cauchy sequence converges, i.e., for all sequences $\{x_k\}_{x\in\mathbb{N}} \subset T$ obeying $d(x_k, x_l) \to 0$ as $k, l \to \infty$, there is some $x_\infty \in T$ such that $d(x_k, x_\infty) \to 0$ as $k \to \infty$.

We refer to [Rov98] for the following

Definition 2.1. *Let J be some index set. A family $\{\sigma_i\}_{i\in J}$ of connected subsets of a manifold M^n of class C^r $(r \geq 1)$ is called a k-dimensional foliation or codimension $n-k$ foliation of class C^s $(0 \leq s \leq r)$ if*

(i) $\bigcup_{i\in J} \sigma_i = M$,
(ii) $i \neq j \;\Rightarrow\; \sigma_i \cap \sigma_j = \emptyset$,
(iii) for all $x \in M$ there exists a C^s-chart map $\phi_x\colon U_x \to \mathbb{R}^n$, with U_x some open neighborhood about x, such that the connected components of $\phi_x(U_x \cap \sigma_i)$ are given by the following parts of parallel affine subspaces

$$A_{c_1,\ldots,c_{n-k}} = \{(x_1, \ldots, x_n) \in \phi_x(U_x) \,|\, x_{k+1} = c_1, \ldots, x_n = c_{n-k}\},$$

for some constants $c_l \in \mathbb{R}$, $l = 1, \ldots, n-k$, whenever $U_x \cap \sigma_i$ is non-empty.

The pairs (U_x, ϕ_x) are called foliated charts. The elements σ_i of a foliation are called leaves. A vector field $X\colon M \to TM$ is called transversal to the (differentiable) foliation $\{\sigma_i\}_{i\in J}$ if for any foliated chart

$$\langle \frac{\partial}{\partial x_i}, \phi_{x*}X\rangle \neq 0,$$

for some $i \in \{k+1, \ldots, n\}$, i.e., the vector field X is nowhere tangential to the leaves.

Hence, naturally, we can have foliations of a lower differentiability class than the ambient manifold. For example, continuous foliations in differentiable manifolds will occur in the following chapters.

Vector and Tensor Fields

We denote vector fields on a manifold M (of class C^r) by capital letters and as sections in the tangent bundle, e.g., $X\colon M \to TM$ or $X \in \Gamma(TM)$ is a (global) vector field on M.

Single vectors in some tangent space T_yM, $y \in M$ will be denoted by a subscript X_y or by lower case letters, e.g., $v = X_y \in T_yM$. We denote differential forms and covariant vector fields by lower case letters and as sections in the co-tangent bundle or some tensor bundle, respectively, e.g., a one-form $u\colon M \to T^*M$ or $u \in \Gamma(T^*M)$. But at some places we will also deviate from these general rules and for example denote special vector fields by lower case or Greek letters. Furthermore, we denote the covariant tensor bundle of k-th order ($k \in \mathbb{N}_0$) over M by T^kM, with the natural identifications $T^0M = C^r(M)$, $T^1M = T^*M$, $T^2M = T^*M \otimes T^*M$, and so forth, with $\otimes$ denoting the tensor product. The symmetric and textcolorblueantisymmetric tensor bundles of k-th order ($k \in \mathbb{N}_0$) over a manifold M^n will be denoted by Σ^kM and Λ^kM, respectively. In the usual way, we have $\Sigma^0M = \Lambda^0M = C^r(M)$, $\Sigma^1M = \Lambda^1M = T^*M$ and $\Lambda^{j>n}M = \emptyset$. Hence, for example, we have a differential k-form w as a map $w\colon M \to \Lambda^kM$ or as a section $w \in \Gamma(\Lambda^kM)$. In the same way $\Lambda^{-1}M$ can be defined to be the set of constant functions over M if necessary.

For vector fields X, Y, we denote the Lie bracket and the Lie derivative by $[X,Y] = \pounds_XY$. We denote the symmetrized and antisymmetrized tensor product by $\vee$ and $\wedge$, respectively. The inner product of a vector field X with a covariant tensor field w will be denoted by $X\rfloor w$. We adopt the following convention of normalization of these products:

For a k-form u and an l-form w, a $(k+l)$-form $(u \wedge w)$ is given by

$$(u \wedge w)(X_1, \dots, X_{(k+l)}) = \frac{1}{(k+l)!} \sum_{\tau \in P_{k+l}} \operatorname{sgn}(\tau) u(X_{\tau(1)}, \dots, X_{\tau(k)}) w(X_{\tau(k+1)}, \dots, X_{\tau(k+l)}),$$

for $k+l$ vector fields $X_1, \dots, X_{(k+l)}$, and with P_{k+l} being the group of all permutations of $k+l$ numbers. Moreover, $u \wedge w = (-1)^{kl}(w \wedge u)$ holds. Similarly, for two totally symmetric tensor fields s and t of order k and l respectively, we have a totally symmetric tensor field $(s \vee t)$ of order $(k+l)$ given by

$$(s \vee t)(X_1, \dots, X_{(k+l)}) = \frac{1}{(k+l)!} \sum_{\tau \in P_{k+l}} s(X_{\tau(1)}, \dots, X_{\tau(k)}) t(X_{\tau(k+1)}, \dots, X_{\tau(k+l)}).$$

We adopt the inner product notation $\rfloor$ for all tensor fields, not only for differential forms. Hence for $k \in \mathbb{N}_0$

$$(X\rfloor t)(Y_1, \dots, Y_k) := (k+1) \cdot t(X, Y_1, \dots, Y_k),$$

for all vector fields $X, Y_1, \dots, Y_k\colon M \to TM$ and all tensor fields $t \in \Gamma(T^{k+1}M)$ and specifically for $t \in \Gamma(\Lambda^{k+1}M)$ or $t \in \Gamma(\Sigma^{k+1}M)$. Note that in this convention the inner product picks up a factor equal to the order of the tensor field on which it operates. This is necessary to obtain consistency with the factors in the antisymmetrized product defined above and the exterior derivative defined below.

The inner product and the wedge product obey the following distributive law for differential forms. Let u be a k-form and w an l-form, then

$$X\rfloor(u \wedge w) = (X\rfloor u) \wedge w + (-1)^k u \wedge (X\rfloor w),$$

for all $X\colon M \to TM$.

Maps Between Manifolds

Let $f\colon M^m \to N^n$ be a smooth mapping from a manifold M of dimension m to a manifold N of dimension n. Then we denote by $\mathrm{d}f_x\colon T_xM \to T_{f(x)}N$ the differential map of f at $x \in M$. This yields a bundle map $\mathrm{d}f\colon TM \to TN$, which is, in general, neither injective nor surjective. Therefore, one usually distinguishes three important special cases: If $m = n$ and f is a diffeomorphism, then $\mathrm{d}f\colon TM \to TN$ is a diffeomorphism and $\mathrm{d}f$ maps vector fields on M to vector fields on N. If $m < n$ and $\mathrm{d}f_x$ is injective for all $x \in M$, the mapping f is called an immersion. If f is additionally injective, it is called an embedding and, in this case, $\mathrm{d}f$ maps vector fields on M to vector fields tangential to the image of M under f. Specifically, for an embedding f, $\mathrm{d}f$ assigns a subbundle $\mathrm{im}_{\mathrm{d}f}(TM) \subset TN$ along $\mathrm{im}_f(M) \subset N$. This operation on vector fields $X\colon M \to TM$ is called push-forward, written f_*X, from M to N and is defined by setting

$$\mathrm{d}f(X) = f_*X\colon \mathrm{im}_f(M) \to \mathrm{im}_{\mathrm{d}f}(TM), \quad (f_*X)_{f(x)} := \mathrm{d}f_x(X_x).$$

If $m > n$ and the mapping f is a submersion, i.e., if $\mathrm{d}f_x$ is surjective for all $x \in M$, given a vector field $X \in \Gamma(TM)$, its image $\mathrm{d}f(X)$ is generally not a well-defined vector field on any subset of N. But one can now define an operation for one-forms defined on $\mathrm{im}_f(M) \subset N$, which values in the set of one-forms on M. For a one-form $u\colon \mathrm{im}_f(M) \to T^*(\mathrm{im}_f(M))$ this operation is called pull-back, written f^*u, and is defined by setting

$$f^*u\colon M \to T^*M, \quad (f^*u)_x(Y_x) := u_{f(x)}(\mathrm{d}f_x(Y_x)),$$

for any vector field Y on M. Moreover, a smooth mapping $f\colon M \to N$ between two manifolds M and N will be called a surjective submersion if it is a submersion and additionally $\mathrm{im}_f(M) = N$ holds.

If $m = n$ and $f\colon M^n \to N^n$ is a diffeomorphism, the pull-back and push-forward operations can be defined for f in the same way.

Subsequently, the definitions above carry over to the pull-back and push-forward of tensor fields of any type. Moreover, in the special case of f being a diffeomorphism one can define a push-forward for differential forms and a pull-back for vector fields in the obvious way using the inverse mapping $f^{-1}\colon N \to M$. Specifically, we can now define the differential $\mathrm{d}f$ of any smooth function $f\colon M \to \mathbb{R}$, which need not be a submersion. The mapping $x \mapsto \mathrm{d}f_x(X_x)$ for any vector field X on M values in $T_{f(x)}\mathbb{R} \simeq \mathbb{R}$ for all $x \in M$. Hence, $\mathrm{d}f(X) \in C^r(M)$, $\mathrm{d}f \in \Gamma(T^*M)$ and $\mathrm{d}f(X)$ has the obvious interpretation of a directional derivative of f along X, which we will occasionally also denote by $Xf := \mathrm{d}f(X)$.

Definition 2.2. *Let M^n be a manifold (of class C^r) with $n \geq 2$. A subset $S \subset M$ is called a submanifold of dimension $k < n$ and class C^s, $s \leq r$ if there is an embedding $j\colon P \to M$ of class C^s from a k-dimensional manifold P (necessarily of class C^s), such that $j(P) = S$. Furthermore, if j is only a topological embedding, i.e., a homeomorphism onto its image, S is called topological submanifold and if $k = n - 1$, S is called a hypersurface. Furthermore, if j is only an immersion, S is called an immersed submanifold.*

Naturally, the leaves of a foliation are submanifolds.

Curves

A (parametrized) curve is a continuous map $\gamma\colon [a,b] \to M$ with $[a,b] \subset \mathbb{R}$, $-\infty < a < b < \infty$ an interval of the real numbers and M a manifold. If $\gamma(t) = \gamma(a)$ for all $t \in [a,b]$, the curve will be called constant. The points $\gamma(a)$ and $\gamma(b)$ will be called end-points of γ. Usually, we will require curves to be smooth, i.e., $\gamma|_{(a,b)}$ is smooth and all necessary left- and right-sided derivatives at the endpoints exist, but we will also allow for curves defined on non-compact subsets of the real numbers and piecewise smooth curves. Thus, we naturally have curves with one end-point ($\gamma\colon [a,b) \to M$ with $-\infty < a < b \leq \infty$ or $\gamma\colon (a,b] \to M$ with $-\infty \leq a < b < \infty$) or no end-point ($\gamma\colon (a,b) \to M$ with $-\infty \leq a < b \leq \infty$). A continuous curve $\gamma\colon [a,b] \to M$ is called piecewise smooth if there is an $n \in \mathbb{N}$, such that a partition of $[a,b]$ by n sub-intervals $a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b$ exists, with $\gamma|_{[a,b]\setminus T_n}$ for $T_n = \{t_i\}_{i=0,\ldots,n}$ being smooth and all necessary left- and right-sided derivatives at points in T_n exist. Subsequently, a curve will be called inextendible if there is no continuous extension of γ such that its domain is compact in $\overline{\mathbb{R}} = \{-\infty\} \cup \mathbb{R} \cup \{\infty\}$. Moreover, we call a curve $\gamma\colon \mathbb{R}_{\geq a} \to M$ with $x = \gamma(a)$ a ray starting at $x \in M$.

Let $(s,x) \in \mathbb{R} \times M$ be points in a product manifold. We call ∂_t the canonical vector field along $\mathbb{R}$ on $\mathbb{R} \times M$ given by $\partial_t = \frac{\partial}{\partial s}(s,x) = (1,0) \in \Gamma(T(\mathbb{R} \times M))$. Then the tangent vector field or the velocity of a curve $\gamma\colon [a,b] \to M$ will be denoted $\dot\gamma\colon [a,b] \to TM$ with $\dot\gamma(t_0) = \gamma_*(\partial_t|_{t=t_0}) \in T_{\gamma(t_0)}M$, in the sense of a left- or right-sided derivative at the end-points. If $\gamma(a) = \gamma(b)$, the curve will be called a closed curve.

We use the designation curve here explicitly for a map $\gamma\colon [a,b] \to M$, i.e, for a fixed domain $[a,b]$ of the curve parameter. Certainly, the image $\mathrm{im}_\gamma([a,b]) \subset M$ of a curve can be obtained by different maps possessing different domains, related to $[a,b]$ by reparametrizations. Hence, in situations where we are interested in the image of a curve only, we will refer to that image as an unparametrized curve.

Derivatives

The exterior derivative $\mathrm{d}\colon \Lambda^k M \to \Lambda^{k+1} M$ will be defined as follows. Let u be a k-form, then the $(k+1)$-form $\mathrm{d}u$ is given for any $k+1$ vector fields $X_0, \ldots, X_k \in \Gamma(TM)$ by

$$\mathrm{d}u(X_0,\ldots,X_k) = \frac{1}{(k+1)!}\sum_i (-1)^i X_i\left(u(X_0,\ldots,\hat{X}_i,\ldots,X_k)\right) + \\ + \frac{1}{(k+1)!}\sum_{i<j}(-1)^{i+j}u([X_i,X_j],X_0,\ldots,\hat{X}_i,\ldots,\hat{X}_j,\ldots,X_k),$$

where the hat $\hat{}$ stands for an omitted vector. For one-forms, we have

$$\mathrm{d}u(X,Y) = \frac{1}{2}\left(Xu(Y) - Yu(X) - u([Y,Y])\right),$$

for all vector fields $X, Y \in \Gamma(TM)$. This definition assures the following properties. Assume there is a torsion-free connection with covariant derivative ∇ on a manifold M. Let u be a one-form on M, then $\mathrm{d}u$ is the antisymmetric part of the tensor ∇u. Specifically,

$$(\nabla u)(X,Y) = \mathrm{d}u(X,Y) + \mathrm{sym}(\nabla u)(X,Y),$$

for all vector fields X, Y on M, with $\mathrm{sym}(\nabla u)(X,Y) = \frac{1}{2}\left((\nabla u)(X,Y) + (\nabla u)(Y,X)\right)$. Furthermore, naturally identifying $\Lambda^0 M$ with the smooth functions on M, the exterior derivative coincides with the differential on functions, i.e., 0-forms. The exterior derivative obeys the following law of compatibility with the wedge product. Let u be a k-form and w an l-form, then

$$\mathrm{d}(u \wedge w) = (\mathrm{d}u) \wedge w + (-1)^k u \wedge (\mathrm{d}w).$$

Now Cartan's magic formula can be used to define a Lie derivative for differential forms. Let ω be a differential form, then

$$\mathcal{L}_X \omega = X \lrcorner \mathrm{d}\omega + \mathrm{d}(X \lrcorner \omega),$$

for all $X \in \Gamma(TM)$. Hence, we have a Lie derivative for all differential forms, particularly also for functions, i.e., 0-forms and

$$\mathcal{L}_X f = Xf = \mathrm{d}f(X)$$

holds for all $X \in \Gamma(TM)$ and all smooth functions $f \colon M \to \mathbb{R}$. Moreover, this yields a Lie derivative for tensor fields of any kind, using the chain rule in the obvious way. The Lie derivative commutes with the exterior derivative: $\mathcal{L}_X(\mathrm{d}u) = \mathrm{d}(\mathcal{L}_X u)$.

2.1 Semi-Riemannian Geometry

Let M^n be a manifold. A tensor field $t \colon M \to T^*M \otimes T^*M$ is called non-degenerate if $t_x(v,w) = 0$ for all $x \in M$ and all $w \in T_xM$ implies $v = 0$. Any non-degenerate, covariant 2-tensor field $t \in \Gamma(\Sigma^2 M)$ yields a non-degenerate scalar product $t_x \colon T_xM \times T_xM \to \mathbb{R}$ in every tangent space. As any vector space allows for an orthonormal basis, there is a basis $\{e_1, \ldots, e_n\} \subset T_xM$ in every tangent space, such that

$$t_x(e_i, e_j) = \varepsilon_j \delta_{ij}, \quad \varepsilon_j := t_x(e_j, e_j) = \pm 1.$$

Then $s = \sum_{i=1}^n \varepsilon_i$ is called signature of t at $x \in M$. The signature is independent of the chosen basis. Furthermore, as we assume all manifolds to be connected, the signature of a non-degenerate, symmetric, covariant 2-tensor field is constant over M.

Let $x \in M$ and denote by $U_x \subset M$ an open neighborhood about x. If U_x is sufficiently small, given a basis $\{e_1, \ldots, e_n\} \subset T_xM$, we can extended this basis to a set of vector fields $\{E_1, \ldots, E_n\} \subset \Gamma(TU_x)$, such that $(E_i)_x = e_i$ and $\{(E_i)_y\}_{i=1,\ldots,n}$ is a basis of T_yM for all $y \in U_x$, which will be called a local frame. A locally finite covering $\{U_a \subset M \,|\, a \in \mathcal{A},\ U_a \text{ open}\}$ of M with $\mathcal{A}$ some index set (i.e. $M = \bigcup_{a \in \mathcal{A}} U_a = M$), together with local frames $\{E_1, \ldots, E_n\}_a \subset \Gamma(TU_a)$ for all $a \in \mathcal{A}$ will be called a frame on M. Using a frame we can define global quantities, which are independent of the choice of the covering and the local frames. Hence, in the remainder of this work, we will denote a frame on a manifold M^n just by $\{E_1, \ldots, E_n\}$.

Definition 2.3. *A non-degenerate tensor field $g\colon M \to \Sigma^2 M$ is called semi-Riemannian metric on M^n. If the signature of g is n, it is called a Riemannian metric and if the signature is $n-2$, it is called a Lorentzian metric. The ordered pair (M, g) is called a Riemannian, Lorentzian or semi-Riemannian manifold accordingly.*

A frame $\{E_1, \ldots, E_n\}$ on M for which $g(E_i, E_j) = \varepsilon_j \delta_{ij}$ holds, will be called a pseudo-orthonormal frame or a g-orthonormal frame and just an orthonormal frame in the special case of a Riemannian metric g. It is a standard result that g-orthonormal frames exist with respect to any semi-Riemannian metric g. With the use of a frame, it is now possible to define the trace of a tensor field with respect to a semi-Riemannian metric. Let $w \in \Gamma(T^2 M)$, then $\mathrm{Tr}(w)\colon M \to \mathbb{R}$ is given by

$$\mathrm{Tr}(w) := \sum_i \varepsilon_i w(E_i, E_i).$$

The summation runs over all indices that label the elements of a (pseudo-)orthonormal frame and it can easily be shown that this definition is independent of the choice of the frame.

We will often denote the norm of a vector field X on a Riemannian manifold (M, g) by vertical double bars, i.e.,

$$\|X\|^g := \sqrt{g(X, X)}$$

is a function $M \to \mathbb{R}$. Particularly, for some point $x \in M$ and $v \in T_x M$ we will write $\|v\|^g_x := \sqrt{g_x(v, v)}$. The same notation will be used for the norm of one-forms u on M by defining

$$\|u\|^g_x := \sup_{v \in T_x M \setminus \{0\}} \frac{|b_x(v)|}{\sqrt{g_x(v, v)}},$$

which equivalent to $\|u\|^g = \sqrt{\sum_i u(E_i)^2}$ when using an orthonormal frame $\{E_i\}_{i=1,\ldots,n}$ on M. This definition can be naturally extended to covariant tensor fields of any rank. For some particular quantities defined in section 3.1, we will use single bars $|\cdot|$ to indicate the norm. We will omit the supersrcipt indicating the metric if it is no source of confusion.

Remark 2.4. *These definitions of a norm above can be extended to semi-Riemannian manifolds. If (M, g) is a semi-Riemannian manifold and $X\colon M \to TM$ a vector field, we can set*

$$\|X\|^g = \begin{cases} \sqrt{g(X, X)} & \text{if } g(X, X) \geq 0 \\ -\sqrt{|g(X, X)|} & \text{if } g(X, X) < 0. \end{cases}$$

And in the same way, for a one-form $u\colon M \to T^*M$ using a pseudo-orthonormal frame $\{E_i\}_{i=1,\ldots,n}$, we can also set $\|u\|^g = \pm\sqrt{|\sum_i \varepsilon_i u(E_i)^2|}$, depending on the sign of $\sum_i \varepsilon_i u(E_i)^2$ as above.

We will denote the spaces of Riemannian and Lorentzian metrics on a manifold M by $\mathcal{R}(M) = \{g \in \Gamma(\Sigma^2 M) \,|\, g \text{ Riemannian}\}$ and $\mathrm{Lor}(M) = \{g \in \Gamma(\Sigma^2 M) \,|\, g \text{ Lorentzian}\}$. Particularly, $\mathrm{Lor}(M)$ can be empty.

Let (M, g) be a semi-Riemannian manifold, then a vector $0 \neq v \in T_xM$, such that $g_x(v, v) = 0$ is called isotropic at $x \in M$, and a vector field $K\colon M \to TM$ is called isotropic vector field if K_x is isotropic for all $x \in M$.

For any semi-Riemannian manifold (M, g) there is a unique torsion-free connection with covariant derivative ∇ associated to g, such that $\nabla g = 0$. This connection is called Levi-Civita connection associated to g. In this case we have

$$\pounds_X Y = [X, Y] = \nabla_X Y - \nabla_Y X,$$

for all vector fields $X, Y\colon M \to TM$. The curvature associated to ∇ will be introduced in section 3.3.

The divergence of a vector field $X\colon M \to TM$ on semi-Riemannian manifold (M, g) is defined by

$$\operatorname{div}_g(X) = \sum_i \varepsilon_i g(\nabla_{E_i} X, E_i),$$

for any pseudo-orthonormal frame $\{E_1, \ldots, E_n\}$ on M and with ∇ the Levi-Civita connection associated to g. Normally, we will omit the subscript g to the divergence if it is clear with respect to which metric the divergence is computed.

Now let (M, g) be a Riemannian manifold. In this case we can assign a positive number $l(\gamma)$, called length, to any piecewise smooth curve $\gamma\colon [a, b] \to M$ by setting

$$l(\gamma) := \int_a^b \|\dot\gamma(t)\|^g \mathrm{d}t := \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \sqrt{g_{\gamma(t)}(\dot\gamma(t), \dot\gamma(t))}\mathrm{d}t,$$

where $\{t_i\}_{i=0,\ldots,n}$ are the (finitely many) values in $[a, b]$ where γ is only continuous but not differentiable. Certainly, this definition is independent of the parametrization chosen for the curve. Hence, defining the space of piecewise smooth curves connecting two points $x, y \in M$ by

$$\Omega(x, y) := \{\gamma\colon [0, 1] \to M \mid \gamma \text{ is piecewise smooth, } \gamma(0) = x, \gamma(1) = y\}$$

leads to the definition of the distance

$$d_g(x, y) := \inf_{\gamma \in \Omega(x,y)} l(\gamma)$$

associated to the Riemannian metric g on M. This definition makes (M, d_g) a metric space and the topology associated to d_g coincides with the manifold topology. We will say that a Riemannian manifold (M, g) is complete or g is a complete Riemannian metric on M if the the metric space (M, d_g) is complete. The following lemma is well known.

Lemma 2.5. *If the manifold M is compact, any Riemannian metric g on M is complete.*

Proof. See, e.g., [Pet98, Cor. 3.5, p. 116]. □

A curve $\gamma\colon [a,b] \to M$ on a Riemannian manifold (M,g) is called a geodesic if $\nabla_{\dot\gamma}\dot\gamma = 0$ holds. At least, in some appropriate sense, short geodesics are minimizers of the length of all piecewise smooth curves connecting two points, such that $d_g(\gamma(a),\gamma(b)) = b-a$ holds. As ususal, the geodesic condition fixes the parameter of a geodesic curve up to affine transformations, which also yields $g(\dot\gamma,\dot\gamma) = \text{const}$ along the curve. A geodesic is called complete if it can be extended to a geodesic defined on all of $\mathbb{R}$. The extension is unique up to affine reparametrization in this case. Then we call a Riemannian manifold (M,g) geodesically complete if all geodesics are complete. Then the famous theorem by Hopf and Rinow holds:

Theorem 2.6. *Let (M,g) be a Riemannian manifold. The following is equivalent:*

(i) *(M,g) is complete.*
(ii) *(M,g) is geodesically complete.*
(iii) *Every closed and d_g-bounded subset of M is compact.*

Proof. See, e.g., [Pet98, Thm. 7.1, p. 125]. □

The notion of geodesic completeness carries over to semi-Riemannian manifolds without substantial adjustments. Although there is no Hopf–Rinow type theorem in the semi-Riemannian case, as there is no associated metric space in general.

We will also need the following

Lemma 2.7. *Let (M,h) be a complete Riemannian manifold, η any Riemannian metric on M and $f\colon M \to \mathbb{R}_{>0}$, such that $\inf_{x\in M} f(x) > 0$. Then $g = fh + \eta$ is a complete Riemannian metric on M.*

Proof. See [FM78, Lem. 2]. □

Let M^n be a manifold. A distribution $\Delta \subset TM$ in the tangent bundle of a manifold, is a smooth assignment $x \mapsto \Delta_x \subset T_xM$ of a subspace Δ_x to all points in M, such that $\dim(\Delta_x) = m \le n$ is constant over M. Then we write $\Delta = \bigcup_{x\in M} \Delta_x$ and call m the dimension of Δ. Subsequently, a vector field X is in Δ, i.e., we write $X\colon M \to \Delta$ or $X \in \Gamma(\Delta)$ if $X_x \in \Delta_x$ for all $x \in M$. Any non-vanishing one-form $u\colon M \to T^*M$ fosters a distribution Δ^u associated to it by

$$\Delta^u_x := \{v \in T_xM \,|\, u_x(v) = 0\},$$

which is of dimension $n-1$.

Definition 2.8. *Let $\Delta \subset TM$ be a distribution and $h\colon M \to \Sigma^2 M$. The tensor field h is called a semi-Riemannian metric on Δ if for any $x \in M$ it holds that $h_x(v,w) = 0$ for all $w \in \Delta_x$ implies either $v = 0$ or $v \notin \Delta_x$.*

A distribution $\Delta \subset TM$ is a subbundle of the tangent bundle in the natural way. If there is a semi-Riemannian metric g on M, there is naturally also a perpendicular distribution $\Delta^\perp \subset TM$ given by

$$\Delta_x^\perp := \{v \in T_xM \,|\, g_x(v,w) = 0 \text{ for all } w \in \Delta_x\},$$

for all $x \in M$. Note that $\Delta_x^\perp \cap \Delta_x = \{0\}$ for all $x \in M$. This gives rise to a dual distribution $\Delta^* \subset T^*M$ by setting $u \in \Gamma(\Delta^*)$ if $u \in \Gamma(T^*M)$ and $u(X) = 0$ for all vector fields $X \in \Gamma(\Delta^\perp)$. Extending the sections over the distributions by tensor products we naturally also get tensor fields and tensor bundles over distributions.

Remark 2.9. *Note that the above definition of dual distributions is ambiguous in the case of isotropic planes Δ_x at some $x \in M$ for a distribution $\Delta \subset TM$ in a semi-Riemannian manifold (M,g). The reason is that isotropic vectors are perpendicular to one another if and only if they are parallel. In this case one is forced to consider lightlike geometry (cf. [DS10]). But in this work we will exclusively consider non-isotropic distributions and subbundles.*

Proposition 2.10. *Let (M,g) be a semi-Riemannian manifold and $X\colon M \to TM$ a nowhere vanishing vector field that is nowhere isotropic. Then the one-form $u = g(X,\cdot)$ metrically associated to V gives rise to a distribution Δ^u. The tensor field*

$$h := g - \frac{u \otimes u}{g(X,X)}$$

is a semi-Riemannian metric on Δ^u. Particularly, if (M,g) is Lorentzian and $g(X,X) < 0$, then h is Riemannian.

Proof. Obviously, h is symmetric. Let $x \in M$ be an arbitrary fixed point and assume $h_x(v,w) = 0$ for all $w \in \Delta_x^u$. Then we have

$$g_x(v,w) = \frac{u_x(v)u_x(w)}{g_x(X,X)}.$$

Furthermore, v can be g-orthogonally decomposed by $v = \lambda X_x + \tilde{v}$, where $\tilde{v} \in \Delta_x^u$ and $\lambda \in \mathbb{R}$. As $u_x(X_x) = g_x(X,X)$, we get

$$\lambda g_x(X_x,w) + g_x(\tilde{v},w) = \lambda u_x(w) \quad \Rightarrow \quad g_x(\tilde{v},w) = 0.$$

Hence, $\tilde{v} = 0$ and, therefore, either $v = 0$ or $v \notin \Delta_x^u$. Now let (M,g) be Lorentzian, $g(X,X) < 0$ and $v \in \Delta_x^u$. Then

$$g_x(v,v) = h_x(v,v),$$

and it follows that $g_x(v,X_x) = 0$, thus $h_x(v,v) = g_x(v,v) > 0$. □

Remark 2.11. *The notion of a semi-Riemannian metric on a distribution bears resemblance to sub-Riemannian geometry, and in fact it is a generalization of this concept. In sub-Riemannian geometry (see e.g. [Str86]) one considers Riemannian manifolds (M,g) together with a completely non-integrable distribution $H \subset TM$, such that $g|_H$ is a Riemannian metric* *on H in the sense of Def. 2.8. Here the complete non-integrability means that any vector field $Y\colon M \to TM$ is a finite linear combination of vector fields X_1, $[X_1, X_2]$, $[X_1, [X_2, X_3]], \ldots$ for $X_1, \ldots, X_m\colon M \to H$ with some $m \in \mathbb{N}$. Then (M, H, g) is called a sub-Riemannian manifold.*

Contact geometry formalizes the concept of complete non-integrability of a distribution in terms of a differential form. We give the following

Definition 2.12. *Let M^{2n+1} be a manifold, $n \geq 1$ and $u \in \Gamma(\Lambda^1 M)$. Then (M, u) is called contact manifold if $u \wedge (du)^n \neq 0$ everywhere, where $(du)^n = du \wedge \cdots \wedge du$ (n-times).*

We will also need the following concept on a contact manifold.

Definition 2.13. *Let (M, u) be a contact manifold. A vector field $R \in \Gamma(TM)$ is called Reeb vector field if $u(R) = 1$ and $R \rfloor du = 0$.*

It is not difficult to see that every contact manifold admits a unique associated Reeb vector field (see, e.g., [Gei08]). There is an interesting conjecture—formulated by A. Weinstein in 1979 (see [Wei79])—in contact geometry, that we will apply to causality theory of Lorentzian manifolds in section 4.3 below.

Weinstein Conjecture. *Let (M, u) be a closed and oriented contact manifold. Then the Reeb vector field has at least one closed integral curve.*

Actually, this conjecture has only recently been proven in dimension three by C.H. Taubes in [Tau07]. All higher dimensions remain open in the general case.

Theorem 2.14. *Let (M, u) be an closed and oriented contact manifold with $\dim(M) = 3$. Then the Reeb vector field has at least on closed integral curve.*

Proof. See [Tau07] or [Hut10] for a review. □

2.2 Global Randers Geometry

In this section, we briefly recall the notions of Finslerian geometry. We follow the conventions and notation in [BCS00] and [CJS11], which are the standard references for this section. For theorems on global hyperbolicity of stationary and related classes of spacetimes, we will only need a special case of Finslerian metrics, namely the so called Randers-type metrics. These metrics arise from a given Riemannian metric and a one-form on a manifold.

Definition 2.15. *A Finslerian metric on a manifold S is a function $F\colon TS \to \mathbb{R}_{\geq 0}$, which has the following properties for all $(x, y) \in TS$ with $x \in S$ and $y \in T_x S$:*

(i) F is continuous on TS, smooth on $TS \setminus \{(x, 0)\}$ and $F(x, y) = 0$ only for $y = 0$.

(ii) F is fiberwise positively homogeneous of degree one, i.e., $F(x, \lambda y) = \lambda F(x, y)$, for all $(x, y) \in TS$ and $\lambda > 0$.

(iii) $\left[\frac{1}{2}\frac{\partial^2(F^2)}{\partial y^i \partial y^j}(x,y)\right]$ *is a positive-definite matrix for any* $(x,y) \in TS \setminus (x,0)$.

In this case we call (S,F) *a Finslerian manifold.*

Since F is only positive homogeneous of degree 1 for a piecewise smooth curve $\gamma : [a,b] \to S$, the Finslerian length

$$l_F(\gamma) = \int_\gamma F(\gamma, \dot{\gamma}) = \int_a^b F(\gamma(t), \dot{\gamma}(t))\mathrm{d}t$$

depends on the orientation of the curve. As a consequence, the Finslerian distance between two points $p, q \in S$:

$$d_F(p,q) = \inf_{\gamma \in \Omega(p,q)} l_F(\gamma),$$

where $\Omega(p,q)$ is the set of all piecewise smooth curves $\gamma \colon [a,b] \to S$ with $\gamma(a) = p$ and $\gamma(b) = q$ is not symmetric in p and q, in general. Thus, two non equivalent notions of completeness make sense (cf. [BCS00, Sec. 6.2]).

Definition 2.16. *A sequence* $\{x_n\}_{n\in\mathbb{N}} \subset S$ *is called forward (resp. backward) Cauchy sequence if for all* $\varepsilon > 0$ *there is a* $\nu(\varepsilon) \in \mathbb{N}$*, such that for all* i, j *with* $\nu(\varepsilon) \le i \le j$*,* $d(x_i, x_j) \le \varepsilon$ *(resp.* $d(x_j, x_i) \le \varepsilon$*) holds. Furthermore,* (S,F) *is called forward complete (resp. backward complete) if all forward (resp. backward) Cauchy sequences are convergent in the generalized metric space* (S, d_F)*.*

Introducing forward balls $B^+(x,r) := \{y \in S \,|\, d_F(x,y) < r\}$ and backward balls $B^-(x,r) := \{y \in S \,|\, d_F(y,x) < r\}$ for $0 < r < \infty$ in a Finslerian manifold (S,F), there is a Finslerian version of the Hopf–Rinow theorem. A subset of the generalized metric space (S, d_F) is called forward (backward) bounded if it is contained in some forward (backward) ball.

Theorem 2.17. *Let* (S,F) *be a Finslerian manifold, then the following is equivalent.*

(i) The Finslerian metric F *is forward (backward) complete.*

(ii) Every closed and forward (backward) bounded subset in the generalized metric space (S, d_F) *is compact.*

Proof. See, e.g., [BCS00, Thm. 6.6.1]. □

Remark 2.18. *By an appropriate definition one can include forward and backward Finslerian geodesics in this Hopf–Rinow theorem, but we will not need that concept in this work.*

A special class of the Finslerian metrics are those which are of Randers type. These metrics consist of a given Riemannian metric g and a one-form b on a manifold S, with the g-norm of b pointwise bounded by 1:

$$\|b\|_x^g < 1,$$

for all $x \in S$. Then for all $(x,y) \in TS$, the Finslerian metric of Randers type is given by

$$F(x,y) = \sqrt{g_x(y,y)} + b_x(y).$$

The completeness of Randers-type metrics depend on the completeness of the underlying Riemannian metric and the norm of the one-form with respect to this metric. The following proposition is known and appeared, e.g., in [CJM11, Rem. 4.1]. We give a full proof for the sake of completeness. Also the proof demonstrates in detail a standard technique that will be used later on in this work and we will refer to this proof for details if necessary.

Proposition 2.19. *A Randers-type metric $F = \sqrt{g} + b$ on a manifold S is forward and backward complete if the Riemannian metric g is complete and the g-norm of b is uniformly bounded by 1, i.e.,*

$$|||b|||^g := \sup_{x\in S} \|b\|^g_x < 1.$$

Proof. In the following we will omit the superscript indicating the metric g. We denote by $B\colon S \to TS$ the vector field metrically associated to the one-form b, such that $b = g(B, \cdot)$ and the norm at $x \in S$ is given by $\|b\|_x = \sqrt{g_x(B, B)}$. Let $X \in \Gamma(TS)$ be a vector field obeying $g(X, X) = 1$. Then the angle α between B and X is given by

$$\cos(\alpha) = \frac{g(B, X)}{\sqrt{g(B, B)}}.$$

For the norm $\|b\|_x$ we get

$$\sup_{y\in T_xS\setminus 0} \frac{|g_x(B_x, y)|}{\sqrt{g_x(y, y)}} = \sup_{\|y\|_x=1} |g_x(B_x, y)| =$$

$$= \sup_{\alpha\in[-\pi,\pi]} |\cos(\alpha)|\sqrt{g_x(B_x, B_x)} = \sqrt{g_x(B_x, B_x)} = \|b\|_x.$$

Let $\dot\gamma$ be the tangential vector field at curves γ in the set $\Omega(p, q)$ of piecewise smooth curves connecting p and q, with curve parameter $s \in [0, 1]$. The Riemannian distance between p and q is, hence, given by

$$d_g(p, q) = \inf_{\gamma\in\Omega(p,q)} \int_0^1 \sqrt{g_{\gamma(s)}(\dot\gamma(s), \dot\gamma(s))}\,\mathrm{d}s$$

and is obviously symmetric in p and q. Because g is complete, every Cauchy sequence $\{x_n\}_{n\in\mathbb{N}} \subset S$ in the Riemannian distance $d_g(x_i, x_j)$ converges. The Finslerian distance between two points x_i and x_j is given by

$$d_F(x_i, x_j) = \inf_{\gamma\in\Omega(x_i,x_j)} \int_\gamma F(\gamma, \dot\gamma) = \inf_{\gamma\in\Omega(x_i,x_j)} \int_\gamma (\sqrt{g(\dot\gamma, \dot\gamma)} + g(B_\gamma, \dot\gamma)).$$

Thus, for a forward Cauchy sequence $\{x_n\}_{n\in\mathbb{N}}$ in (S, d), the Finslerian distance obeys

$$d_F(x_i, x_j) = \inf_{\gamma\in\Omega(x_i,x_j)} \int_\gamma [(1 + \frac{g(B_\gamma, \dot\gamma)}{\sqrt{g(\dot\gamma, \dot\gamma)}})\sqrt{g(\dot\gamma, \dot\gamma)}] < \varepsilon,$$

for all $\varepsilon > 0$ and for $\nu(\varepsilon) < i \leq j$. Due to the assumption

$$|||b||| = \sup_{x\in S} \sup_{v\in T_xS\setminus 0} \frac{|g_x(B_x, v)|}{\sqrt{g_x(v, v)}} < 1,$$

it follows that

$$\varepsilon > d_F(x_i, x_j) > (1 - \|b\|) \inf_{\gamma \in \Omega(x_i, x_j)} \int_\gamma \sqrt{g(\dot\gamma, \dot\gamma)} = (1 - \|b\|) d_g(x_i, x_j)$$

$$\Rightarrow \ \varepsilon' := \frac{\varepsilon}{1 - \|b\|} > d_g(x_i, x_j).$$

Hence, every forward Cauchy sequence in (S, d_F) is also a Cauchy sequence in (S, d_g), which converges to some limit point, due to the completeness of g. Now, we can make use of the fact that the topology induced by the open balls of a Riemannian metric, the topology induced by the forward (and backward) open balls of a Finslerian metric and the manifold topology coincide. See [ST51, p. 44] for the Riemannian case and [BCS00, Sec. 6.1] for the Finslerian case. Hence, a sequence converging in the distance induced by the the Riemannian metric, converges in the manifold topology and also converges forward in the distance induced by the Randers-type metric. An analogue proof holds for the backward case. □

On the other hand we can now show that the forward *or* backward completeness of a Randers-type metric implies the completeness of the underlying Riemannian metric. Compare, e.g., [CJM11] and [CJS11].

Lemma 2.20. *If a Randers-type metric $F = \sqrt{g} + b$ on a manifold S is forward or backward complete, then the Riemannian metric g is complete.*

Proof. We prove this lemma by contradiction. Assume that g is not complete. Let $\{x_n\}_{n\in\mathbb{N}}$ be a g-Cauchy sequence, which does not converge in (S, d_g). Let $i \leq j \in \mathbb{N}$ and thus we have similar to the proof of Prop. 2.19

$$d_F(x_i, x_j) = \inf_{\gamma \in \Omega(x_i, x_j)} \int_\gamma \sqrt{g(\dot\gamma, \dot\gamma)} + b(\dot\gamma) = \inf_{\gamma \in \Omega(x_i, x_j)} \int_\gamma \left(1 + \frac{b(\dot\gamma)}{\sqrt{g(\dot\gamma, \dot\gamma)}}\right) \sqrt{g(\dot\gamma, \dot\gamma)} \leq$$

$$\leq 2 \inf_{\gamma \in \Omega(x_i, x_j)} \int_\gamma \sqrt{g(\dot\gamma, \dot\gamma)} = 2 d_g(x_i, x_j).$$

The same holds for $d_F(x_j, x_i)$. Hence, whenever $d_g(x_i, x_j) < \varepsilon$, we also have $d_F(x_i, x_j) < 2\varepsilon$ and $d_F(x_j, x_i) < 2\varepsilon$, such that $\{x_n\}_{n\in\mathbb{N}}$ is a forward and backward Cauchy sequence with respect to F. But following the same reasoning as in the proof of Prop. 2.19, the sequence $\{x_n\}_{n\in\mathbb{N}}$ cannot converge in the metric space (S, d_F), either, as this would imply a limit point with respect to the manifold topology and subsequently in the metric space (S, d_g). Hence, F is forward and backward incomplete, which is the desired contradiction. □

The Prop. 2.19 and Lem. 2.20 above give only very rough estimates on completeness relations between Riemannian and Randers-type metrics. When employing these completeness considerations to Lorentzian causality theory, we will reveal much more refined connections between different completeness conditions involving growth conditions of the norm of the constituting one-form and other particular functions. See section 5.2 for this analysis.

2.3 $\mathbb{R}$-manifolds and Principal Bundles

Standard references for this section are [Pal61] and [KN63]. The results presented are mostly standard and are contained, maybe using a different language, in other references, such as [tD87]. In [Pal61] R. Palais investigated the action of non-compact Lie groups on topological spaces, and examined, particularly, the question of the existence of slices of the action. We will adapt the basic techniques, definitions and theorems established in [Pal61] to the action of $(\mathbb{R},+)$ on a Lorentzian manifold. Therefore, we specialize the notions to the Lie group $(\mathbb{R},+)$ and add results on differentiability. Essentially, what we end up with in this process is a dynamical system. So, we prepare the ground for connecting the concepts and notions in this section to the conceptions used in the topological theory of dynamical systems, which we will introduce in the following section. As we will use results from the differentiable version of Palais' non-compact Lie group theory developed in this section and from dynamical systems in our further analysis, we will prove several results which connect the concepts from both fields. There are often notions in these two areas that look different at first sight, but are essentially the same. Moreover, we recall in this section some basic results on principal bundles—with and without a Hausdorff base space—and narrow them down to principal bundles with a one-dimensional structure group, particularly to $\mathbb{R}$-principal bundles.

Definition 2.21. *Let $(\mathbb{R},+)$ be the real line furnished with the structure of an additive group. By an $\mathbb{R}$-action on a manifold M (of class C^r, $r>0$), we mean a homomorphism f of $(\mathbb{R},+)$ into $\mathrm{Diff}^r(M)$, such that the map $\mathbb{R}\times M\to M$ with $(t,p)\mapsto f(t)(p) =: t\circ p$ is C^r-smooth. Such a manifold M, together with a fixed $\mathbb{R}$-action f, is called $\mathbb{R}$-manifold.*

In the following, we will usually omit the homomorphism $f\colon(\mathbb{R},+)\to\mathrm{Diff}^r(M)$ and stick to the notation $t\circ p$, for $t\in\mathbb{R}$ and $p\in M$. We use the following standard characterization of the group action.

Definition 2.22. *Let M be an $\mathbb{R}$-manifold. The $\mathbb{R}$-action is called free on M if $t\neq 0$ implies $t\circ p\neq p$ for all $p\in M$. The $\mathbb{R}$-action is called locally free if there is a neighborhood $I\subset\mathbb{R}$ with $0\in I$, such that $t\in I\setminus\{0\}$ implies $t\circ p\neq p$ for all $p\in M$.*

Definition 2.23. *The set $\mathbb{R}p=\{t\circ p\,|\,t\in\mathbb{R}\}$ is called orbit of $p\in M$ and for any subset $S\subset M$ we define the set $\mathbb{R}S=\{t\circ p\,|\,t\in\mathbb{R},p\in S\}$, which is called the saturation of S.*

The set of orbits of an $\mathbb{R}$-manifold will be denoted by $M/_{\mathbb{R}}$ with the canonical projection $\pi_M\colon p\mapsto\mathbb{R}p$. The quotient space $M/_{\mathbb{R}}$ can be furnished with the quotient topology inherited from the manifold topology of M. But it is known that this topology may lack some of the separation properties of the manifold topology, depending on the structure of the $\mathbb{R}$-action. For example, in general, the quotient topology of $M/_{\mathbb{R}}$ is not Hausdorff (see e.g. [vQ76]) and there are cases where it is not even T_1. We will consider subsets of $\mathbb{R}$, emerging from the $\mathbb{R}$-action on subsets $S,T\subset M$, which are of the following form:

$$((S,T)) := \{t\in\mathbb{R}\,|\,(t\circ S)\cap T\neq\emptyset\}.$$

Here we denote $t\circ S := \{t\circ p\in M\,|\,p\in S\}$.

Definition 2.24. *Let $S, T \subset M$ be two subsets of an $\mathbb{R}$-manifold M. Then we call S thin relatively to T if $((S,T))$ is bounded in $\mathbb{R}$. If S is thin relative to itself, it is called thin.*

In [Pal61], some basic features of thin sets are listed. The most relevant attribute which we will use in this work is shown in the following

Lemma 2.25. *Let $S, T \subset M$ be two subsets of an $\mathbb{R}$-manifold M. If S and T are compact, then $((S,T))$ is closed in $\mathbb{R}$. If additionally S and T are relatively thin, then $((S,T))$ is compact in $\mathbb{R}$.*

Proof. Let $S, T \subset M$ be compact subsets and assume $((S,T))$ is not closed. Then there is a sequence $\{t_n\}_{n\in\mathbb{N}} \subset ((S,T))$ such that $t_n \to t_0 \notin ((S,T))$ as $n \to \infty$. Hence, for all $n \in \mathbb{N}$ we have $(t_n \circ S) \cap T \neq \emptyset$ or there is $q_n \in S$ such that $t_n \circ q_n \in T$. By passing to a subsequence of $\{q_n\}_{n\in\mathbb{N}} \subset S$, we have that $q_n \to q_0 \in S$ as $n \to \infty$ and $t_n \circ q_n \to r_0 \in T$ as $n \to \infty$, because S and T are compact. But as the $\mathbb{R}$-action $(t,p) \mapsto t \circ p$ is smooth, we have $r_0 = t_0 \circ p_0$, and thus $(t_0 \circ S) \cap T \neq \emptyset$ and $t_0 \in ((S,T))$ is the desired contradiction. Now, if S and T are relatively thin, then $((S,T))$ is closed and bounded, hence compact. $\square$

The following definition differs slightly from the convention used in [Pal61, Def. 1.1.2], but will prove very well adapted to the theory developed in this work. Recalling that a near-manifold is a topological space which fulfills all conditions of a manifold except for being Hausdorff but is a T_1-space only, we give the following

Definition 2.26. *An $\mathbb{R}$-manifold M^n will be called a Cartan $\mathbb{R}$-manifold if the following two conditions hold:*

(i) Each orbit is closed (as a point set) in M and $M/_{\mathbb{R}}$ is an $(n-1)$-dimensional near-manifold.

(ii) The map $t \mapsto t \circ p$ from $\mathbb{R}$ onto $\mathbb{R}p$ is a diffeomorphism for all $p \in M$.

Then we have the following

Lemma 2.27. *If M is a Cartan $\mathbb{R}$-manifold, the $\mathbb{R}$-action is free on M.*

Proof. Assume there is $p \in M$ and $t \in \mathbb{R}$, such that $t \circ p = p = 0 \circ p$. Obviously, in this case the map $t \mapsto t \circ p$ is not injective, hence not a diffeomorphism. $\square$

As we will see below, the idea of a Cartan $\mathbb{R}$-manifold is to have the manifold M as a generalized $\mathbb{R}$-principal bundle over $M/_{\mathbb{R}}$. The terminology *Cartan* $\mathbb{R}$-manifold is justified by the following considerations. The free $(\mathbb{R}, +)$ action on M can be characterized as follows: let $R \subset M \times M$ be the set of pairs $(p,q) \in M \times M$ such that p and q belong to the same orbit, then there is a unique element $t_{p,q} \in \mathbb{R}$, such that $q = t_{p,q} \circ p$. Based on this characterization we can introduce the notion of a Cartan principal bundle, such that the map $R \ni (p,q) \mapsto t_{p,q} \in \mathbb{R}$ is smooth, similar to the definition given by H. Cartan in [Car49]. It can be shown (see, e.g., [Pal61] for the continuous case) that such a Cartan principal bundle fulfills in fact all conditions of a Cartan $\mathbb{R}$-manifold.

The following is the standard example of an $\mathbb{R}$-manifold, which is not a Cartan $\mathbb{R}$-manifold.

Example 2.28. *Let the two-torus $\mathbb{T}^2$ be given as the coordinate patch $(x, y) \in [0,1] \times [0,1] \subset \mathbb{R}^2$ modulo the identifications $x \sim x+1$ and $y \sim y+1$. Let $(\mathbb{R}, +)$ act on $\mathbb{T}^2$ by $t \circ (x, y) = (x+t, y+at)$ with $a \in \mathbb{R} \setminus \mathbb{Q}$ an irrational constant. If we had $a \in \mathbb{Q}$, each orbit would be an embedded circle. Now, it is an application of L. Kronecker's diophantine approximation theorem (see [Kro68]) to conclude that each orbit is dense in $\mathbb{T}^2$ in this case. So, there can be no orbit which is a closed set in this case, and the orbit space $\mathbb{T}^2/_{\mathbb{R}}$ is not even a T_1-space.*

It is important to notice that under very mild assumptions, the $\mathbb{R}$-action on an $\mathbb{R}$-manifold gives rise to a one-dimensional foliation of M by the equivalence classes, that is the orbits, of the action. This is shown in the following

Lemma 2.29. *Let M^n be an $\mathbb{R}$-manifold. If the $\mathbb{R}$-action is locally free, M is foliated by the family of orbits $\mathcal{F} := \{\pi_M^{-1}(\xi)\}_{\xi \in M/_{\mathbb{R}}}$ as one-dimensional leaves.*

Proof. All points $p \in M$ are elements of precisely one orbit. To see this, assume there is a $p \in M$ such that $p \in \mathbb{R}q$ and $p \in \mathbb{R}q'$ for $\mathbb{R}q \neq \mathbb{R}q'$. By the definition of the orbits there are two numbers $t, t' \in \mathbb{R}$ such that $p = t \circ q$ and $p = t' \circ q'$. But then also $q = t^{-1} \circ t' \circ q' = (t'-t) \circ q'$ and thus $\mathbb{R}q = \mathbb{R}q'$. Moreover, for all $p \in M$ the $\mathbb{R}$-action produces an orbit $\mathbb{R}p$ with $p \in \mathbb{R}p$. This assures that

$$M = \bigsqcup_{\xi \in M/_{\mathbb{R}}} \pi_M^{-1}(\xi).$$

The foliated charts are constructed in the following way. Take any manifold chart (U, φ) about $p \in U \subset M$. Maybe by making U smaller, the locally free $\mathbb{R}$-action on M constrains to a free $\mathbb{R}$-action on U, such that the orbits in U are given by $\mathbb{R}q \cap U$ for all $q \in U$. This is always the case as any locally free homomorphism of $(\mathbb{R}, +)$ into $\mathrm{Diff}^r(M)$ yields only $0 \in \mathbb{R}$ as a *trivial* stabilizer. By the map φ, these orbits of the $\mathbb{R}$-action on U are carried to one-dimensional smooth submanifolds of $\varphi(U) \subset \mathbb{R}^n$. So, there is a diffeomorphism $\psi \colon \varphi(U) \to \mathbb{R} \times \mathbb{R}^{n-1}$ which identifies these submanifolds $\varphi(\mathbb{R}q \cap U)$ with affine subspaces $(\mathbb{R}\vec{c}) \cap (\psi \circ \varphi)(U) \subset \mathbb{R} \times \mathbb{R}^{n-1}$ for $\vec{c} = (0, c_1, \dots, c_{n-1}) \in \mathbb{R} \times \mathbb{R}^{n-1} \cap (\psi \circ \varphi)(U)$. Thus $(U, \psi \circ \varphi)$ is the desired foliated chart. $\square$

The following lemma connects the notion of thin sets to Cartan $\mathbb{R}$-manifolds. The proof of this lemma can essentially already be found in [Pal61] for the continuous case and general group actions. We reproduce the proof here along the ideas in [Pal61], but we fill the gaps to differentiability and specialize to an $\mathbb{R}$-action.

Lemma 2.30. *Let M^n be an $\mathbb{R}$-manifold. If all $p \in M$ possess a thin neighborhood, then M is a Cartan $\mathbb{R}$-manifold.*

Proof. *(i)* Assume there is a sequence $\{t_n\}_{n \in \mathbb{N}} \subset \mathbb{R}$, such that the sequence $\{t_n \circ p\}_{n \in \mathbb{N}} \subset \mathbb{R}p$ for a fixed $p \in M$ converges to a point $t_n \circ p \to q \notin \mathbb{R}p$ as $n \to \infty$. As q admits a thin neighborhood U, there is an $N_0 \in \mathbb{N}$, such that $t_n \circ p \in U$ for all $n > N_0$. Fixing $N_1 > N_0$ we have $(t_n - t_{N_1}) \circ p' = t_n \circ p$ for some $p' = t_{N_1} \circ p \in U$ such that $t_n - t_{N_1} \in ((U, U))$ for $n > N_0$. But as $((U, U))$ is bounded, we can pass to a sub-sequence of $t_n - t_{N_1}$ in $((U, U))$

that converges to some number $\tau \in \overline{((U,U))}$. Hence $(t_n - t_{N_1}) \circ p' \to q = \tau \circ p' \in \mathbb{R}p$, which yields the desired contradiction. Now that $M/_{\mathbb{R}}$ is a T_1-space is a standard result in topology for spaces admitting closed orbits (see [vQ76]). In the same way, the connectedness of M and the second countability of the topology are passed on to $M/_{\mathbb{R}}$. Let $(U, \psi \circ \varphi)$ be a foliated chart about $p \in M$ as in Lem. 2.29 and $V = \pi_M(U)$ an open neighborhood of $\xi = \pi_M(p)$ in $M/_{\mathbb{R}}$. Then by π_M^{-1} the points $\eta \in V$ are homeomorphically identified with orbit segments $\widetilde{\mathbb{R}q} := \mathbb{R}q \cap U \subset U$, such that $\eta = \pi_M(q)$. Thus, by $\mathrm{pr}_2 \circ \psi \circ \phi \colon U \to \mathbb{R}^{n-1}$ the orbit segments $\widetilde{\mathbb{R}q}$ are homeomorphically carried to the open set $\mathrm{pr}_2(\psi \circ \phi(U)) \subset \mathbb{R}^{n-1}$. Hence, $M/_{\mathbb{R}}$ is locally homeomorphic to Euclidean space, and thus it is an $(n-1)$-dimensional near-manifold.

(ii) In [Pal61] it was shown that $\mathbb{R}$ is homeomorphic to $\mathbb{R}p$ for all $p \in M$, such that there is a topological embedding $\mathbb{R} \to \mathbb{R}p$ for all $p \in M$. By *(i)* each orbit is closed in M and for each $p \in \mathbb{R}p$ there is a smooth coordinate chart (U, φ) about $p \in M$ such that $\varphi(U \cap \mathbb{R}p) \subset \varphi(U)$ is the image of the section of the orbit through p, which is a submanifold of $\varphi(U)$. Thus the topological embedding $\mathbb{R} \to \mathbb{R}p$ is also a smooth immersion. □

However, we will see in the following section that admitting a thin neighborhood about any point is a much stronger condition than being a Cartan $\mathbb{R}$-manifold. This will become clear by using the topological theory of dynamical systems.

Given a manifold M and the action of the one-dimensional Lie group $(\mathbb{R}, +)$ on M, we call the triple $(M, M/_{\mathbb{R}}, \pi)$ an $\mathbb{R}$-principal bundle if the following conditions hold: $\mathbb{R}$ acts freely on M, the canonical projection $\pi \colon M \to M/_{\mathbb{R}}$ is a surjective submersion onto a manifold $M/_{\mathbb{R}}$ and M is locally trivial, i.e., for all x in $M/_{\mathbb{R}}$ there is a neighborhood U, such that there is a bundle isomorphism between $\pi^{-1}(U) \subset M$ and the trivial bundle $(\mathbb{R} \times U, U, \mathrm{pr}_2)$. As $(\mathbb{R}, +)$ is Abelian, there is no need to distinguish between left and right actions.

Obviously, the definition above can be relaxed to include the quotient space to be a near-manifold. We will call the bundle a generalized $\mathbb{R}$-principal bundle in this case. Occasionally, we will be also interested in the compact structure group $\mathbb{R}/_{\mathbb{Z}} = U(1)$. Particularly, this will occur in the case of a non-free $\mathbb{R}$-action with isotropy subgroup $\mathbb{Z}$.

The following proposition connects $\mathbb{R}$-principal bundles to Cartan $\mathbb{R}$-manifolds.

Proposition 2.31. *An $\mathbb{R}$-manifold M^n is a generalized $\mathbb{R}$-principal bundle if and only if it is a Cartan $\mathbb{R}$-manifold. Furthermore, every Cartan $\mathbb{R}$-manifold is locally an $\mathbb{R}$-principal bundle.*

Proof. Firstly, we show that if M is a Cartan $\mathbb{R}$-manifold, then it is a generalized $\mathbb{R}$-principal bundle. We have to prove that *(i)* the projection $\pi_M \colon M \to M/_{\mathbb{R}}$ is a surjective submersion and *(ii)* for any $\xi \in M/_{\mathbb{R}}$ there is local trivialization $t \colon (\pi^{-1}(U), U, \pi_M) \to (\mathbb{R} \times U, U, \mathrm{pr}_2)$ for some neighborhood U of ξ.

(i) We note that $\mathrm{im}_{\pi_M}(M) = M/_{\mathbb{R}}$ by definition. It remains to show that $\mathrm{d}\pi_M$ has maximal rank. Let $x \in M$ and (U, ϕ) a foliated chart about x, given by Lem. 2.29. Suppose that $\phi(U) \subset \mathbb{R} \times \mathbb{R}^{n-1}$ is open and $\phi(x) = (0, \vec{0}) \in \mathbb{R} \times \mathbb{R}^{n-1}$. Now we define two vectors $a, y \in T_xM$ by $a = \phi^*(1, \vec{0})$ and $y = \phi^*(0, \vec{x})$ for some $\vec{x} \in \mathbb{R}^{n-1}$. Then we have $\mathrm{d}\pi_M(a) =$

$\varphi^*(\vec{0})$ and $\mathrm{d}\pi_M(y) = \varphi^*(\vec{x})$, with the isomorphism $\varphi^*\colon \mathbb{R}^{n-1} \to T_{\pi_M(x)}(M/_{\mathbb{R}})$ induced by the properties of the foliated chart. As this holds for all $x \in M$ and all $\vec{x} \in \mathbb{R}^{n-1}$, we conclude that $\mathrm{d}\pi_M$ has rank $n-1$.

(ii) Take any $\xi \in M/_{\mathbb{R}}$ and any $x \in \pi_M^{-1}(\xi)$ with a chart neighborhood $W \subset M$. Suppose again that (W, ϕ) is a foliated chart about $x \in M$, with $\phi(W) \subset \mathbb{R} \times \mathbb{R}^{n-1}$ being open and $\phi(x) = (0, \vec{0}) \in \mathbb{R} \times \mathbb{R}^{n-1}$. Let the $(n-1)$-disk $B \subset \mathbb{R}^{n-1}$ be defined by $B = \{(0, \vec{x}) \in \phi(W)\}$. Then $\tilde{B} = \phi^{-1}(B) \subset M$ is an $(n-1)$-dimensional submanifold with boundary in M, such that $x \in \tilde{B}$ and all orbits of the $\mathbb{R}$-action that intersect W, intersect $\tilde{B}$ only a countable number of times. Even more so, as the $\mathbb{R}$-action is free and any orbit is closed as a point set, we conclude that all orbits of the $\mathbb{R}$-action that intersect W, intersect $\tilde{B}$ only a finite number of times. Hence, by making W smaller we can achieve that all orbits that intersect W, intersect $\tilde{B}$ exactly once. Thus, the neighborhood $U = \pi_M(W)$ about ξ in $M/_{\mathbb{R}}$ has a chart mapping $\varphi\colon U \to \mathbb{R}^{n-1}$ with $varphi(U) = B$ and $\varphi(\xi) = \vec{0}$. The saturation $\mathbb{R}\tilde{B}$ in M is now diffeomorphic to $\pi_M^{-1}(U)$, so that there is a diffeomorphism $t\colon \pi_M^{-1}(U) \to \mathbb{R} \times U$, as we can identify a point $p \in \mathbb{R}\tilde{B}$ by $\varphi(\mathbb{R}p) \in B$ and by a unique number $t \in \mathbb{R}$ such that $t \circ p \in \tilde{B}$.

Secondly, suppose that $(M, M/_{\mathbb{R}}, \pi)$ is a generalized $\mathbb{R}$-principal bundle. Then the closed set property of any orbit is a straightforward consequence of the local trivialization condition. For similar reasons, every orbit is diffeomorphic to $\mathbb{R}$.

Now consider any $\xi \in M/_{\mathbb{R}}$ for a generalized $\mathbb{R}$-principal bundle $(M, M/_{\mathbb{R}}, \pi)$. Then certainly there is a neighborhood U of ξ, such that there is a local trivialization $(\mathbb{R} \times U, U, \mathrm{pr}_2)$ of the bundle over U and U is homeomorphic to $\mathbb{R}^{n-1}$ in the sense of $M/_{\mathbb{R}}$ being a locally Euclidean near-manifold. Then $(\mathbb{R} \times U, U, \mathrm{pr}_2)$ is a generalized $\mathbb{R}$-principal bundle in its own right, but it is even an $\mathbb{R}$-principal bundle, because U is homeomorphic to a Euclidean space and fulfills, therefore, all separation properties of the Euclidean space. Hence, U is a manifold. $\square$

Principal fiber bundles have the advantage to admit connections. We will work with the following notion of connections, which makes sense because the Lie algebra of the Lie group $(\mathbb{R}, +)$ is just the one-dimensional Euclidean vector space $\mathbb{R}$.

Definition 2.32. *Let $(M, M/_{\mathbb{R}}, \pi)$ be a principal fiber bundle. Let $\Phi\colon \mathbb{R} \times M \to M$ be the $\mathbb{R}$-action on M, such that $\Phi_t \in \mathrm{Diff}^r(M)$ for all $t \in \mathbb{R}$. Furthermore, we denote by $A^a \in \Gamma(TM)$ the fundamental vector field belonging to the value a in (the Lie algebra) $\mathbb{R}$. Then a one-form $u \in \Gamma(\Lambda^1 M)$ is called connection in $(M, M/_{\mathbb{R}}, \pi)$ if*

- *(i) $u(A^a) = a$ for all $a \in \mathbb{R}$ and*
- *(ii) $\Phi_t^* u = u$ for all $t \in \mathbb{R}$.*

Any principal bundle admits a connection. That is a standard result (cf. Thm. 2.1, p. 67 in [KN63]). The crucial condition in the proof of this fact is that the quotient $M/_{\mathbb{R}}$ is indeed a manifold, particularly paracompact. Moreover, it is also a standard result that an $\mathbb{R}$-principal bundle is globally trivial or trivializable, i.e., there is a principal bundle isomorphism $(M, M/_{\mathbb{R}}, \pi_M) \to (\mathbb{R} \times Q, Q = M/_{\mathbb{R}}, \mathrm{pr}_2)$ (see, e.g., Thm. 5.7, p. 58 in [KN63] or Prop. 16.14.5 in [Die69]), the proof of which requires Q to be a manifold. Thus, a Cartan $\mathbb{R}$-manifold is trivializable or admits a connection if $M/_{\mathbb{R}}$ is Hausdorff.

Connections allow for the definition of unique horizontal lifts of vector fields and curves. For an $\mathbb{R}$-principal fiber bundle $(M, M/_{\mathbb{R}}, \pi)$ with connection $u \in \Gamma(\Lambda^1 M)$ and a vector field $X \in \Gamma(T(M/_{\mathbb{R}}))$, there is a unique vector field $X^* \in \Gamma(TM)$, called the horizontal lift of X, obeying $\pi_*(X^*_p) = X_{\pi(p)}$ for all $p \in M$ and $u(X^*) = 0$. Particularly, given a curve $\gamma\colon [a,b] \to M/_{\mathbb{R}}$ with tangential vector field $\dot\gamma$, there is a unique horizontal lift $\dot\gamma^*$ on $\pi^{-1}(\gamma) \subset M$. Hence, fixing some $p \in \pi^{-1}(\gamma(t_0))$ for a fixed $t_0 \in [a,b]$ yields a unique lifted curve $\gamma^*\colon [a,b] \to M$ with $\gamma(t_0) = p$. These are standard results (cf. [KN63]).

Moreover, the notion of a connection can be extended to Cartan $\mathbb{R}$-manifolds or generalized principal bundles, by extending the definition of a fundamental vector field in a natural way. For example, let $\Phi\colon \mathbb{R} \times M \to M$ be the $\mathbb{R}$-action on a Cartan $\mathbb{R}$-manifold. Then every a in the Lie algebra $\mathbb{R}$ induces an invariant vector field $\tilde{A}^a$ on the Lie group $(\mathbb{R}, +)$ by the exponential map, hence $\tilde{A}^a \in \Gamma(T\mathbb{R})$. Now extend $\tilde{A}^a$ to a vector field $(\tilde{A}^a, 0) \in \Gamma(T(\mathbb{R} \times M)) \simeq \Gamma(T\mathbb{R} \times TM)$ on $\mathbb{R} \times M$. Then the fundamental vector field A^a on M is given by the push-forward $A^a = \Phi_*(\tilde{A}^a, 0)$. Thus, we can prove the following

Proposition 2.33. *Let M be a Cartan $\mathbb{R}$-manifold with associated generalized principal bundle $(M, M/_{\mathbb{R}}, \pi)$. If $(M, M/_{\mathbb{R}}, \pi)$ admits a (global) connection, then $M/_{\mathbb{R}}$ is Hausdorff.*

Proof. Let $\gamma_1, \gamma_2\colon [0,1] \to M/_{\mathbb{R}}$ be two curves contained in $M/_{\mathbb{R}}$. As $(M, M/_{\mathbb{R}}, \pi)$ admits a connection, given $p_1 \in \pi^{-1}(\gamma_1(0))$ and $p_2 \in \pi^{-1}(\gamma_2(0))$, there are unique horizontal lifts $\gamma_1^*, \gamma_2^*\colon [0,1] \to M$ of γ_1, γ_2, such that $\gamma_i^*(0) = p_i$ and $\gamma_i^*(1) \in \pi^{-1}(\gamma_1(1))$ for $i = 1, 2$. Now, assume that $M/_{\mathbb{R}}$ is not Hausdorff. Then there are two points $x, y \in M/_{\mathbb{R}}$ that cannot be separated by disjoint open neighborhoods. Assume U_x and U_y are open neighborhoods of x and y respectively, such that $W := U_x \setminus \{x\} = U_y \setminus \{y\}$. Assume that the curve γ_1 is closed at x, i.e., $\gamma_1(0) = \gamma_1(1) = x$ and $\gamma_1(0,1) \subset W$. Furthermore, assume that γ_2 connects x and y and coincides with γ_1 in W. That is $\gamma_2(0) = x$, $\gamma_2(1) = y$ and $\gamma_2(0,1) = \gamma_1(0,1)$. As the horizontal lift is unique for all given $p \in \pi^{-1}(x)$, we also have $\gamma_2^*(0,1) = \gamma_1^*(0,1)$ in M. By continuity, it also follows that $\gamma_1^*(1) = \gamma_2^*(1)$. Thus, $\gamma_2^*(1) \in \pi^{-1}(x)$ and as this holds for all given $p \in \pi^{-1}(x)$, we have that the fibers over x and y coincide in contradiction to the non-Hausdorff condition. □

Now, we are ready to assess the question when a Cartan $\mathbb{R}$-manifold M allows for a Hausdorff quotient $M/_{\mathbb{R}}$ and is, therefore, a (globally trivial) $\mathbb{R}$-principal bundle, from a purely topological perspective. To this end we state the following

Definition 2.34. *Let M be an $\mathbb{R}$-manifold. A subset $S \subset M$ is called small if each point $p \in M$ has a neighborhood which is thin relative to S. And an $\mathbb{R}$-manifold is called proper if each point $p \in M$ has a small neighborhood.*

The following theorem assembles important equivalent conditions for an $\mathbb{R}$-manifold to be proper.

Theorem 2.35. *Let M^n be an $\mathbb{R}$-manifold. Then the following items are equivalent.*

(i) *For all* $p, q \in M$ *there exist relatively thin neighborhoods* $S, T \subset M$ *with* $p \in S$ *and* $q \in T$.
(ii) M *is a proper* $\mathbb{R}$*-manifold.*
(iii) M *is a Cartan* $\mathbb{R}$*-manifold and* $M/_{\mathbb{R}}$ *is an* $(n-1)$*-dimensional manifold.*
(iv) *If* $K \subset M$ *is compact,* $((K, K)) \subset \mathbb{R}$ *is compact.*
(v) *The projection* π_M *is closed.*

Furthermore, every Cartan $\mathbb{R}$*-manifold is locally proper.*

Proof. We refer to [Pal61] for the proof of the equivalence of items *(i)*, *(ii)* and *(iv)*. Therein it was also shown that for a proper $\mathbb{R}$-manifold M, the quotient $M/_{\mathbb{R}}$ is Hausdorff, i.e., that *(ii)* implies *(iii)*. Item *(v)* implies *(iii)* because M is a manifold and hence a normal space. Thus, a closed projection implies that $M/_{\mathbb{R}}$ is a normal space, too, and in particular Hausdorff.

(iii)⇒(v): Suppose that $A \subset M$ is closed, but $B = \pi_M(A) \subset M/_{\mathbb{R}}$ is not closed. So, there is $x \in \overline{B} \setminus B$, such that every open neighborhood U about x has a non-empty intersection with B. Hence, because $M/_{\mathbb{R}}$ is a T_1-space, we can assume without loss of generality that A is compact. Let $\{x_n\}_{n\in\mathbb{N}} \subset B$ be a sequence with $x_n \to x$ as $n \to \infty$. As A is closed, we certainly have $A \cap \pi_M^{-1}(x) = \emptyset$. Hence, we can lift x_n to a sequence convergent in A. This is always possible because x_n induces a sequence in any local section $\sigma\colon B \cap U \to M$ that is contained in A. The local sections exist because of the local $\mathbb{R}$-principal bundle structure of the Cartan $\mathbb{R}$-manifold. Thus the sequence $p_n = \sigma(x_n)$ converges to some $p \in A$. But this implies that $\pi_M(p) \neq x$ in $M/_{\mathbb{R}}$ and $\pi_M(p)$ and x cannot be separated by disjoint open neighborhoods contradicting the Hausdorff property.

(iii)⇒(i): Suppose $(M, M/_{\mathbb{R}}, \pi_M)$ is an $\mathbb{R}$-principal bundle, which is equivalent to item *(iii)* (cf. Prop. 2.31). Suppose first that there are two points $p, q \in M$ such that $p \notin \mathbb{R}q$. Then $x = \pi_M(p)$ and $y = \pi_M(q)$ can be separated by disjoint open neighborhoods U_x and U_y in $M/_{\mathbb{R}}$. Hence, $\pi_M^{-1}(U_x)$ and $\pi_M^{-1}(U_y)$ are trivializable, open and disjoint in M. Hence, any two neighborhoods W_p and W_q of p and q are relatively thin as long as they are contained in $\pi_M^{-1}(U_x)$ or $\pi_M^{-1}(U_y)$, respectively, as we have $((W_p, W_q)) = \emptyset$. Suppose now that $p \in \mathbb{R}q$, i.e., there is $t \in \mathbb{R}$ such that $p = t \circ q$. Now, take a neighborhood U about $x = \pi_M(p) = \pi_M(q)$ in $M/_{\mathbb{R}}$, such that there is a trivialization $\mathbb{R} \times U$ of $\pi_M^{-1}(U)$ obeying $p = (0, x)$ and $q = (t, x)$. Then there is a $\tau \in \mathbb{R}$, such that $W_p = (-\tau, \tau) \times U$ and $W_q = (t - \tau, t + \tau) \times U$ are open neighborhoods of p and q in $\mathbb{R} \times U$ respectively. Then it is easily checked that $((W_p, W_q)) \subset [t - 2\tau, t + 2\tau]$, which implies that W_p and W_q are relatively thin.

It is now a straightforward application of Prop. 2.31 to show that every Cartan $\mathbb{R}$-manifold is locally proper. ☐

2.4 Flows and Dynamical Systems

We aim to apply the differentiable version of Palais' theory of $\mathbb{R}$-manifolds from the previous section to analyze the global splitting properties of Lorentzian manifolds. Therefore, we are particularly interested in $\mathbb{R}$-actions on manifolds which are generated by a given vector field on M. To connect vector fields to $\mathbb{R}$-actions as introduced above, we define in the following the notion of the flow of a vector field. The main reference for the treatment of flows is [Con01, Ch. 4].

Definition 2.36. *A global flow on a manifold M is a map*

$$\Phi\colon \mathbb{R} \times M \to M,$$

such that for all $t \in \mathbb{R}$, the stage $\Phi_t(\cdot) := \Phi(t, \cdot)$ is a diffeomorphism $M \to M$ and

(i) $\Phi_0 = \mathrm{id}_M$,

(ii) $\Phi_{t+s} = \Phi_t \circ \Phi_s$ *for all $t, s \in \mathbb{R}$.*

Definition 2.37. *Let $p \in M$ and Φ a global flow on M. The curve $\gamma_p\colon \mathbb{R} \to M$ defined by*

$$\gamma_p(t) := \Phi_t(p),$$

such that $\gamma_p(0) = p$ is called a flow line of Φ through p.

Definition 2.38. *Let Φ be a global flow on M. The vector field $X\colon M \to TM$ defined pointwise by*

$$X_p = \dot{\gamma}_p(0) \in T_pM$$

is called the generator of Φ. Conversely, any given vector field $Y\colon M \to TM$ is called complete if it is the generator of some global flow on M.

It is a standard result (see [Con01]) that the vector field as in Def. 2.38 is indeed globally well-defined and smooth, and that we have the following

Lemma 2.39. *On a compact manifold M, every vector field $X\colon M \to TM$ is complete.*

Theorem 2.40. *Let M be a manifold of class C^r, $r > 0$. There is a one-to-one correspondence between a global flow Φ on a manifold M and an $\mathbb{R}$-action on M. The orbits of the $\mathbb{R}$-action correspond to the flow lines of Φ.*

Proof. Let Φ be a global flow on M. Consider the map $f\colon t \mapsto \Phi_t(\cdot)$. Then f is the homomorphism $\mathbb{R} \to \mathrm{Diff}^r(M)$ constituting the $\mathbb{R}$-action, because due to the definition of the global flow each $f(t) = \Phi_t(\cdot) \in \mathrm{Diff}^r(M)$ and $f(0) = \mathrm{id}_M$, as well as $f(t+s) = f(t) \circ f(s)$ for all $s, t \in \mathbb{R}$. Conversely, let there be an $\mathbb{R}$-action on M generated by a homomorphism $f\colon \mathbb{R} \to \mathrm{Diff}^r(M)$. Now define $\Phi\colon \mathbb{R} \times M \to M$ by $\Phi_t(p) := f(t)(p)$. By the definition of the $\mathbb{R}$-action, this map is smooth. And as f is a homomorphism, we also have $\Phi_0 = \mathrm{id}_M$ and $\Phi_{t+s} = \Phi_t \circ \Phi_s$ for all $t, s \in \mathbb{R}$. Now due to

$$\gamma_p(t) = \Phi_t(p) = t \circ p,$$

for all $(t, p) \in \mathbb{R} \times M$, we have $\gamma_p(t) = \mathbb{R}p$ for all $p \in M$. $\square$

Remark 2.41. *Hence, obviously, a global flow coincides with the global $\mathbb{R}$-action of a generalized $\mathbb{R}$-principal bundle or a Cartan $\mathbb{R}$-manifold, respectively. This also leads to the conclusion that the vector field generating the global flow is a fundamental vector field of the $\mathbb{R}$-action.*

Hence, if we want to use the theory of $\mathbb{R}$-manifolds to analyze splitting and slicing results with respect to flows generated by a vector field, this vector field has to be complete, so that we have a global flow. Subsequently, we will call a global flow that leads to a Cartan $\mathbb{R}$-action on the manifold M a Cartan flow. Moreover, we state the following

Definition 2.42. *Let M be a manifold and $V\colon M \to TM$ a globally defined, nowhere vanishing and complete vector field. Then the ordered pair (M, V) is called a kinematical manifold. Furthermore, if V generates a Cartan flow we call (M, V) a Cartan kinematical manifold.*

Obviously, a kinematical manifold is an $\mathbb{R}$-manifold with the $\mathbb{R}$-action generated by the global flow of V. The natural connection of kinematical manifolds to Lorentzian metrics arise from the following considerations. The preconditions for the existence of a globally defined and nowhere vanishing vector field and for a Lorentzian metric on a manifold M are the same, namely, either M being non-compact or having Euler characteristic zero (see the following chapter). This is caused by the signature of a Lorentzian metric, that distinguishes a special direction in the tangent bundle at all points in M, which will be called timelike direction. A vector field, basically, provides the same distinction by a decomposition of the tangent bundle.

The main reference for the treatment of dynamical systems is [NS60], but other references will also be used and cited where applicable. What we will be concerned with here, is the so called general theory of dynamical systems or the question of the topological structure of a manifold on which a dynamical system is defined. This differs from more specific investigations of dynamical systems defined by given differential equations. In these cases one is particularly interested in stability properties of specific orbits or the whole system, or in fixed points and bifurcations. These arise for example by closed integral curves or zeros of a vector field, which defines a dynamical system on a manifold. This is not the theory fitting the applications in this work, as we will be concerned with kinematical manifolds and, subsequently, timelike vector fields on Lorentzian manifolds. Hence, the important part of the dynamical systems theory for the following chapters is about unstable or parallelizable dynamical systems (see [Mar69]).

We start out with a definition of dynamical systems adapted to our purposes.

Definition 2.43. *Let M be a non-compact manifold. A dynamical system on M is a global flow according to Def. 2.36. We will denote this by the ordered pair (M, Φ).*

Also for dynamical systems, we will use the terms *flow line* and *orbit* interchangeably.

Definition 2.44. *Let $\gamma\colon \mathbb{R} \to M$ be any orbit of a dynamical system (M, Φ). Then the* *(possibly empty) sets*

$$\omega(\gamma) = \bigcap_{t\in\mathbb{R}} \overline{\gamma(\mathbb{R}_{\geq t})} \quad \textit{and} \quad \alpha(\gamma) = \bigcap_{t\in\mathbb{R}} \overline{\gamma(\mathbb{R}_{\leq t})}$$

are called ω-limit set and α-limit set of the orbit γ, respectively. We also set $\Omega(\gamma) = \alpha(\gamma) \cup \omega(\gamma)$ for all orbits γ. In the same way, the limit sets

$$\omega(\Phi) = \bigcup_{\gamma} \omega(\gamma) \quad \textit{and} \quad \alpha(\Phi) = \bigcup_{\gamma} \alpha(\gamma),$$

as well as $\Omega(\Phi) = \omega(\Phi) \cup \alpha(\Phi)$ are defined, where the union is understood over all orbits.

Obviously, $\Omega(\Phi)$ is a closed set for any dynamical system (M, Φ), but further conclusions cannot be drawn about $\Omega(\Phi)$ in the generic case.

Remark 2.45. *Note that the notion of a limit set already makes sense for individual curves defined over the real numbers. We will use this in causality theory of Lorentzian manifolds in the sections 3.4 and 4.2.*

In the usual way, we can now define a periodic orbit of a dynamical system (M, Φ), as a flow line of Φ which is diffeomorphic to an embedding of $\mathbb{S}^1$ into M. Similarly, we call a point $x_0 \in M$ a fixed point if $\gamma_{x_0}(t) = x_0$ for all $t \in \mathbb{R}$. This is the case if and only if the generating vector field of the flow possesses a zero at x_0. Obviously, periodic orbits and fixed points are contained in $\Omega(\Phi)$ and even in $\omega(\Phi)$ *and* $\alpha(\Phi)$.

It turns out that the emptiness of the limit sets of a dynamical system is the condition corresponding precisely to a Cartan $\mathbb{R}$-action in the differentiable version of Palais' theory developed in the previous section. Hence, we have the following

Proposition 2.46. *The dynamical system (M, Φ) fulfills $\Omega(\Phi) = \emptyset$ if and only if M is a Cartan $\mathbb{R}$-manifold with $\mathbb{R}$-action corresponding to the global flow Φ.*

Proof. By Thm. 2.40 the global flow Φ corresponds to an $\mathbb{R}$-action.

"$\Rightarrow$": $\Omega(\Phi) = \emptyset$ implies that $\omega(\gamma) = \emptyset$ and $\alpha(\gamma) = \emptyset$ for all orbits γ of the dynamical system. Particularly, this implies that every orbit is closed as a point set. This can be seen as follows. Assume there is a point $x \in M$ and a flow line γ_x through x, which is not closed as a point set. Thus, there is a sequence $\{t_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$ with $|t_n| \to \infty$ and $\gamma_x(t_n)$ converging to some point $y \notin \gamma_x$. But, hence, $y \in \Omega(\gamma_x)$, which is a contradiction. Furthermore, for all $x \in M$ the map $\Phi(\cdot, x) \colon \mathbb{R} \to M$ is a diffeomorphism of $\mathbb{R}$ onto the orbit γ_x through x. So, it remains to show that the quotient $M/_{\mathbb{R}}$ is a near-manifold. But this follows in exactly the same way as in the proof of Lem. 2.30, as the orbits being closed implies that $M/_{\mathbb{R}}$ is a T_1-space, and the remaining conditions follow from investigating foliated charts.

"$\Leftarrow$": If M is a Cartan $\mathbb{R}$-manifold we conclude by a similar argument as above that $\Omega(\gamma) = \emptyset$ for every orbit γ. Assume $x \in \overline{\gamma}$ for any orbit γ in an $\mathbb{R}$-manifold or a dynamical system. Then there is a sequence $\{t_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$ such that $\gamma(t_n) \to x$ as $n \to \infty$. Then either t_n

admits a subsequence which converges to some $\tau \in \mathbb{R}$, in which case $x \in \gamma$, or a subsequence obeying $|t_n| \to \infty$ as $n \to \infty$, in which case $x \in \Omega(\gamma)$. This shows that $\overline{\gamma} = \gamma \cup \Omega(\gamma)$, which proves that $\Omega(\Phi)$ is empty if all orbits are closed as a point set. □

The following two notions are standard definitions in the general theory of dynamical systems.

Definition 2.47. *Let (M, Φ) be a dynamical system. A point $x_0 \in M$ is called wandering if there exists a neighborhood U of x_0 and a number $T \in \mathbb{R}$, such that $\Phi(t, U) \cap U = \emptyset$ for all $|t| > T$, with $\Phi(t, U) = \bigcup_{x \in U} \gamma_x(t)$. All other points are called non-wandering. Furthermore, the dynamical system is said to have an improper saddle point (also called a saddle at infinity) if there are two sequences $\{\tau_n\}_{n\in\mathbb{N}}, \{t_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$ with $0 < \tau_n < t_n$ for all $n \in \mathbb{N}$ and $\tau_n \to \infty$ as $n \to \infty$, as well as two points $x, y \in M$ and a sequence $\{x_n\}_{n\in\mathbb{N}} \subset M$ with $x_n \to x$ and $\Phi(t_n, x_n) \to y$ as $n \to \infty$, but the sequence $\{\Phi(\tau_n, x_n)\}_{n\in\mathbb{N}}$ has no limit point.*

The following lemma extends the correspondence between $\mathbb{R}$-manifolds and dynamical systems even further.

Lemma 2.48. *A point x in a dynamical system (M, Φ) is wandering if and only if it has a thin neighborhood with respect to the $\mathbb{R}$-action corresponding to the global flow Φ.*

Proof. Just compare Def. 2.47 with Def. 2.24. □

We will see below that the notion of the non-existence of an improper saddle point for a dynamical system is a relatively strong condition. Although improper saddle points do not really arrange properly according to the limit point scheme. See item *(ii)* in Lem. 2.50 below.

Following [Mar69] we give the subsequent definitions.

Definition 2.49. *A dynamical system (M, Φ) is called*

(i) unstable if no compact set $K \subset M$ contains an entire half-orbit $\gamma_x(\mathbb{R}_{>0})$ or $\gamma_x(\mathbb{R}_{<0})$ for some $x \in M$,

(ii) completely unstable if all points are wandering and

(iii) parallelizable if it is unstable and has no improper saddle point.

Item *(i)* in the following lemma is a standard result in the topological theory of dynamical systems. We can prove it now by highlighting the equivalence of certain conditions to the theory of $\mathbb{R}$-actions. Item *(ii)* is a new implication that will be useful for the analysis conducted in this work.

Lemma 2.50. *For a dynamical system (M, Φ) we have the following implications:*

(i) completely unstable $\Longrightarrow \Omega(\Phi) = \emptyset \Longrightarrow$ unstable,

(ii) no improper saddle point $\Longrightarrow \Omega(\Phi) = \emptyset$ or every half-orbit $\gamma_x(\mathbb{R}_{>0})$ (or $\gamma_x(\mathbb{R}_{<0})$) with $\omega(\gamma_x) \neq \emptyset$ (or $\alpha(\gamma_x) \neq \emptyset$) for $x \in M$ is entirely contained in some compact set $K \subset M$.

Proof. *(i)*: By Lem. 2.48 and Prop. 2.46 the first implication is equivalent to Lem. 2.30. For the second implication assume that there is a point $x \in M$ such that the half-orbit $\gamma_x(\mathbb{R}_{>0})$ is entirely contained in a compact set K. Then for any sequence $\{t_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$ with $t_n > 0$ and $t_n \to \infty$ as $n \to \infty$ the sequence $\gamma_x(t_n) \subset K$ has a subsequence in K, which converges to some $y \in K$. Hence, $y \in \omega(\gamma_x)$ and an equivalent reasoning applies if the half-orbit $\gamma_x(\mathbb{R}_{<0})$ is contained in some compact set. Thus we have non-empty $\Omega(\Phi)$ in these cases.

(ii): We will show the following: if $\Omega(\Phi)$ is not empty and there is a half-orbit $\gamma_x(\mathbb{R}_{>0})$ with $\omega(\gamma_x) \neq \emptyset$, which is not entirely contained in a compact set $K \subset M$, then there is an improper saddle point. Again, the case for the half-orbit $\gamma_x(\mathbb{R}_{<0})$ is completely analogous and will be omitted. Let $x \in M$ be a point such that $\omega(\gamma_x) \neq \emptyset$. Then the condition that $\gamma_x(\mathbb{R}_{>0})$ is not entirely contained in a compact set is characterized in the following way. Take any compact exhaustion of M based at x, i.e., a family of non-empty, compact sets $\{K_i\}_{i\in\mathbb{N}_0} \subset \mathcal{P}(M)$ (the power set of M), such that $K_0 = \{x\}$, $x \in K_i$ for all $i \in \mathbb{N}$, $K_i \subset \mathring{K}_{i+1}$ and $\bigcup_{i\in\mathbb{N}} K_i = M$. Then there is a sequence $\{t_i\}_{i\in\mathbb{N}} \subset \mathbb{R}$, such that $t_i > 0$ for all $i \in \mathbb{N}$ and $\gamma_x(t_i) \in K_{i+1} \setminus K_i$. Hence, $\Phi(t_i, x)$ has no limit point in M and $t_i \to \infty$ as $i \to \infty$. But as $\omega(\gamma_x) \neq \emptyset$, there is also a sequence $\{\tau_i\}_{i\in\mathbb{N}} \subset \mathbb{R}$ with $\tau_i > 0$ for all $i \in \mathbb{N}$ and $\tau_i \to \infty$ for $i \to \infty$, such that $\gamma_x(\tau_i) \to y \in \omega(\gamma_x)$ as $i \to \infty$. Thus we have $\Phi(\tau_i, x) \to y$ as $i \to \infty$. By switching to subsequences if necessary we can achieve that $t_i < \tau_i$, and by regarding x as a constant sequence, we conclude that there is an improper saddle point. □

Remark 2.51. *All the implications in Lem. 2.50 do not generally hold in the opposite direction (cf. [NS60]). Finding examples for dynamical systems which are unstable but not completely unstable is particularly involved. In fact, it can be shown that in $\mathbb{R}^2$ every unstable dynamical system is completely unstable (see chapter V. Thm. 10.02 in [NS60]). Examples can be constructed in $\mathbb{R}^3$ and we refer to chapter V. Ex. 10.03 in [NS60] for such a construction.*

Moreover, we have the following characterization of parallelizability. This characterization has already been shown in [Mar69], although therein, parallelizability was given as a dynamical system which is *completely* unstable and has no saddle at infinity. It was only in [BF72] that the possibility of relaxing this assumption was shown. We give here a completely different proof. To this end we first establish the following

Lemma 2.52. *If a dynamical system (M, Φ) is unstable and has no improper saddle point and if there are sequences $\{x_n\}_{n\in\mathbb{N}} \subset M$ and $\{t_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$ such that $x_n \to x$ as $n \to \infty$ and $\Phi(t_n, x_n)$ converges to some $y \in M$ as $n \to \infty$, then $\{t_n\}_{n\in\mathbb{N}}$ is bounded.*

Proof. See chapter V. Lem. 10.06 in [NS60]. □

Proposition 2.53. *A dynamical system is parallelizable if and only if the $\mathbb{R}$-action corresponding to the global flow Φ on M is proper if and only if it corresponds to an $\mathbb{R}$-principal bundle.*

Proof. By Lem. 2.50, parallelizability of a dynamical system (M, Φ) implies $\Omega(\Phi) = \emptyset$. Hence, by Prop. 2.46 and Prop. 2.31 we conclude that M is a generalized $\mathbb{R}$-principal bundle.

So, it remains to show that no improper saddle point implies that $M/_{\mathbb{R}}$ is Hausdorff. We prove this by showing that the non-Hausdorffness of the quotient space implies the existence of an improper saddle point. Assume $a, b \in M/_{\mathbb{R}}$ are two points which cannot be separated by disjoint open sets. Let U_a and U_b be two neighborhoods of a and b respectively, such that $W = U_a \setminus \{a\} = U_b \setminus \{b\}$, and such that the generalized bundle $(M, M/_{\mathbb{R}}, \pi_M)$ allows for a local trivialization over U_a and U_b respectively. Let $\{\xi_n\}_{n\in\mathbb{N}} \subset W$ be a sequence that converges to the double point a, b in $M/_{\mathbb{R}}$. Then assume two local sections $\sigma_a \colon U_a \to M$ and $\sigma_b \colon U_b \to M$ in the local trivializations over U_a and U_b respectively. We denote by $x_n = \sigma_a(\xi_n)$ and $y_n = \sigma_b(\xi_n)$ two sequences in M and by $p = \sigma_a(a)$, $q = \sigma_b(b)$ two points in the fibers over a and b, which are lifted by the local sections. Then obviously $x_n \to p$ and $y_n \to q$ as $n \to \infty$. Furthermore, as $\pi_M(x_n) = \pi_M(y_n) = \xi_n$, we conclude that the elements x_n and y_n of the respective sequences lie in the same fiber over ξ_n. So for all $n \in \mathbb{N}$, there is a $t_n \in \mathbb{R}$ such that $\Phi(t_n, x_n) = y_n$, and we can assume $t_n > 0$ without loss of generality. Moreover, we must have that $t_n \to \infty$ as $n \to \infty$. To see this we suppose that $\{t_n\}_{n\in\mathbb{N}}$, or a subsequence of $\{t_n\}_{n\in\mathbb{N}}$, had a limit point $t_0 \in \mathbb{R}$. But then we would have $\Phi(t_0, p) = q$ and hence q would be an element of the fiber over p and $M/_{\mathbb{R}}$ was Hausdorff. Now the existence of an improper saddle point follows from Lem. 2.52. □

Prop. 2.53 will be the foundation for certain diffeomorphic splitting results of Lorentzian manifolds, which will be derived in the subsequent chapters. At last, we will need the concept of invariant and minimal sets for dynamical systems, given in the following

Definition 2.54. *A subset $A \subset M$ in a dynamical system (M, Φ) is called invariant if $\Phi(t, x) \in A$ for all $x \in \mathbb{R}$ and all $x \in A$. Furthermore, A is called minimal if it is a non-empty, closed and invariant set and it does not contain an invariant proper subset.*

Remark 2.55. *Obviously, every union of orbits of a dynamical system is invariant and every individual orbit is minimal, if it is closed as a point set. Furthermore, the limit sets $\omega(\Phi)$ and $\alpha(\Phi)$ are closed, invariant sets for any dynamical system Φ and they each contain at least one minimal set (these are standard results, cf. [NS60, Ch. 5]).*

Chapter 3

LORENTZIAN MANIFOLDS AND SPACETIMES

In this chapter, we will establish some basic definitions and properties about the main geometric objects of this work, and we will also prove some propositions about them. Standard references that will be used throughout this chapter are [BEE96], [O'N83] and [HE73]. We define Lorentzian manifolds, spacetimes and Lorentzian manifolds with a fixed complete timelike vector field, i.e., a global timelike flow. We will also define and establish some specific terminology that only occurs in Lorentzian geometry. This will provide the basis for the subsequent analysis of these geometric objects, and particular cases of them, in the following chapters. The sections 3.1, 3.2 and 3.3 will then be concerned with geometric quantities constructed from the covariant derivative of the timelike vector field, the so-called kinematical quantities, and their relation to the geometry, topology, and partly also to the curvature of the Lorentzian manifold. In section 3.4 we establish some basic notions about Lorentzian causality theory, which will be needed in the following chapters.

Based on the definition of semi-Riemannian manifolds in chapter 2, we give now the following more detailed definition of a Lorentzian manifold, which will be used in the subsequent analysis.

Theorem & Definition 3.1. *Let M^{n+1} be a manifold with $n \geq 1$, either non-compact or with Euler characteristic zero. Then M admits a non-degenerate metric $g\colon M \to \Sigma^2 M$ of signature $n-1$, i.e., a Lorentzian metric. An ordered pair (M, g), consisting of a manifold M and a Lorentzian metric g, is called a Lorentzian manifold.*

Proof. See e.g. [MS08, Thm. 2.4]. □

We use standard causal classification of tangent vectors by the unique bilinear form on each tangent space induced by the metric g. For all $p \in M$, the Lorentzian metric g_p constrained to T_pM is a non-degenerate scalar product $g_p\colon T_pM \otimes T_pM \to \mathbb{R}$ on T_pM, such that T_pM admits a pseudo-orthonormal basis, cf. section 2.1.

Definition 3.2. *For all $p \in M$, a tangent vector $v \in T_pM$ is classified as:*

(i) timelike if $g_p(v, v) < 0$,
(ii) lightlike if $g_p(v, v) = 0$ and $v \neq 0$,
(iii) causal if v is timelike or lightlike, or
(iv) spacelike if $g_p(v, v) > 0$ or $v = 0$.

The same classification holds for vector fields and curves. A C^1-curve (resp. a vector field) on M is called {timelike, lightlike, causal, spacelike} if all its tangent vectors (resp. values) are {timelike, lightlike, causal, spacelike}.

At first sight it seems as if the causal classification is constrained to differentiable curves. But in the subsequent analysis, we also need the notion of a continuous causal curve. Such a notion will be given if the Lorentzian manifold is a spacetime (see Def. 3.7 and 3.8 below).

Furthermore, the causal classification of vectors yields a causal classification of hypersurfaces (i.e., submanifolds of dimension n in a spacetime (M^{n+1}, g)).

Definition 3.3. *Let $S^n \subset M^{n+1}$ be a hypersurface in the spacetime (M, g). If there is a uniquely defined vector field $\nu\colon S \to NS$ obeying $|g(\nu,\nu)| = 1$ (where we denote the normal bundle of S by NS), then S is called a non-degenerate hypersurface (i.e., the metric induced on S by g is nowhere degenerate) and S is spacelike if $g(\nu,\nu) = -1$ and timelike if $g(\nu,\nu) = 1$ everywhere on S.*

Remark 3.4. *Let (M, g) be a Lorentzian manifold. A pointwise conformal transformation of (M, g) is a change of the Lorentzian metric $g \mapsto g'$ by a function $\phi\colon M \to \mathbb{R}$ such that $g' = e^{\phi} g$. A conformal transformation of (M, g) is a map $\varphi\colon M \to M$, such that there is a pointwise conformal transformation $g' = e^{\phi} g$ obeying $\varphi^* g = g'$. Obviously any vector $v \in T_pM$ that is {timelike, lightlike, causal, spacelike} with respect to a Lorentzian metric g on M, is also {timelike, lightlike, causal, spacelike} with respect to any pointwise conformally transformed metric g'. Thus the causal classification of tangent vectors is conformally invariant.*

We name the collection of causal or lightlike tangent vectors at a point in the following way. The subset

$$\mathfrak{L}_p = \{v \in T_pM \,|\, v \text{ lightlike}\} \subset T_pM$$

is called light cone at p, the subset

$$\mathfrak{C}_p = \{v \in T_pM \,|\, v \text{ causal}\} \subset T_pM$$

is called causal cone and the subset

$$\mathfrak{T}_p = \{v \in T_pM \,|\, v \text{ timelike}\} = \mathfrak{C}_p \setminus \mathfrak{L}_p \subset T_pM$$

is called time cone at p.

Lemma 3.5. *For all $p \in M$, $\mathfrak{T}_p$ has two connected components.*

Proof. Let $B_p = \{e_0, \ldots, e_n\}$ be any pseudo-orthonormal basis of T_pM with $e_0 \in \mathfrak{T}_p$. Then any vector $v \in T_pM$ can be expanded with respect to B_p by $v = v^a e_a = v^0 e_0 + \sum_i v^i e_i$. If $v \in \mathfrak{T}_p$, we have $g_p(v,v) = -(v^0)^2 + \sum_i (v^i)^2 < 0$ and necessarily $|v^0| > 0$. Thus, all vectors $v \in \mathfrak{T}_p$ naturally come in two classes with respect to $e_0 \in B_p$, namely $\mathfrak{T}^{\uparrow}_{e_0} = \{v \in \mathfrak{T}_p \,|\, v^0 > 0\}$ and $\mathfrak{T}^{\downarrow}_{e_0} = \{v \in \mathfrak{T}_p \,|\, v^0 < 0\}$ with $\mathfrak{T}^{\uparrow}_{e_0} \cap \mathfrak{T}^{\downarrow}_{e_0} = \emptyset$. Thus, it remains to show that $\mathfrak{T}^{\uparrow}_{e_0}$ and $\mathfrak{T}^{\downarrow}_{e_0}$ are open sets. Consider $v, w \in \mathfrak{T}^{\uparrow}_{e_0}$ with $(v^0)^2 > \sum_i (v^i)^2$, $(w^0)^2 > \sum_i (w^i)^2$ and $v^0 > 0$, $w^0 > 0$. Hence, we have by the Cauchy-Schwarz inequality

$$g_p(v,w) \leq -v^0 w^0 + |\sum_i v^i w^i| \leq -v^0 w^0 + \sqrt{\sum_i (v^i)^2}\sqrt{\sum_i (w^i)^2} < -v^0 w^0 + |v^0 w^0| = 0.$$

Thus, particularly, $g_p(v+w,v+w) = g_p(v,v) + 2g_p(v,w) + g_p(w,w) < 0$ for all $v,w \in \mathfrak{T}^{\uparrow}_{e_0}$, which makes $\mathfrak{T}^{\uparrow}_{e_0}$ an open set. An analogue reasoning holds for $\mathfrak{T}^{\downarrow}_{e_0}$. As obviously $e_0 \in \mathfrak{T}^{\uparrow}_{e_0}$, we can choose any other basis $\tilde{B}_p = \{\tilde{e}_0, \dots, \tilde{e}_n\}$ as long as $\tilde{e}_0 \in \mathfrak{T}^{\uparrow}_{e_0}$, which makes the notion of the two connected components independent of the basis. □

By extension, and because the vector $v = 0$ in T_pM is spacelike, then also each causal cone and each light cone has two connected components. The concept of *future* and *past* is the unique assignment of one of the connected components of the time cone in T_pM to each point p in a Lorentzian manifold. Such an assignment of past and future to each point $p \in M$ is called time orientation on (M,g) and following [Car71] it is a map $\tau(p)\colon \mathbb{R}^{n+1} \to T_pM$, for all $p \in M$, which maps the Euclidean solid half cone

$$x^0 > \left(\sum_{1 \le i \le n} (x^i)^2 \right),$$

with $x^0, \dots, x^n$ Cartesian coordinates in $\mathbb{R}^{n+1}$, into T_pM in a non-degenerate, homogeneous and linear way. Moreover, it is required to change continuously with p. The image $\tau(p)$ will be called future cone at $p \in M$. A Lorentzian manifold which admits a time orientation is called time-oriented.

Remark 3.6. *Time orientability—its dual, space orientability—as well as orientability are logically independent concepts for a Lorentzian manifold (M,g). Generally, these can be assessed in terms of Stiefel–Whitney classes associated to a decomposition $TM = \xi \oplus \nu$ of the tangent bundle (cf. [Hus94, Ch. 17]). This is understood as the direct g-orthogonal sum of a maximal timelike subbundle ξ and a maximal spacelike subbundle ν. Such a decomposition exists for every Lorentzian manifold due to the vanishing Euler characteristic. Now, one can see that (M,g) is time-orientable if ξ is orientable, i.e., if the first Stiefel–Whitney class of ξ vanishes, it is space-orientable if ν is orientable, i.e, the first Stiefel–Whitney class of ν vanishes and it is orientable if TM is orientable, i.e., the first Stiefel–Whitney class of TM vanishes. Furthermore, one can conclude from this that, firstly, any Lorentzian manifold admits a time-orientable double cover (see [Pen72]) and, secondly, if any two kinds of orientability hold, the third one is implied. In fact, this is a simple corollary of the product property of Stiefel–Whitney classes: $w_1(TM) = w_1(\xi) + w_1(\nu)$.*

Definition 3.7. *A spacetime is a Lorentzian manifold together with a fixed time-orientation τ on M.*

The points $p \in M$ are also called events if (M,g) is a spacetime. In a spacetime, the notions of future and past directedness for causal curves and vector fields can now be defined. We say that a causal vector $v \in T_pM$ at some event p in a spacetime (M,g) is future-directed (resp. past-directed) if it is an element of the future (resp. past) component of the causal cone $\mathfrak{C}_p$, distinguished by the fixed time-orientation of (M,g). This definition naturally carries over to causal vector fields and to velocities of causal curves, i.e., we can also distinguish future- and past-directed causal curves.

Now we are in a position to extend the notion of causal curves to the continuous case. Only for causal curves, this can be conducted in a unique and sensible way. For example, there is no unique notion of continuous timelike curves (see [MS08, Rem. 3.17]). To this end we need the notion of a causally convex set. An open set $U \subset M$, with (M, g) a spacetime, is called causally convex if every future-directed causal curve with end-points in U is entirely contained in U. It is a standard result that every event in a spacetime has a causally convex neighborhood. We can now use [MS08, Prop. 3.16] as a definition for continuous causal curves.

Definition 3.8. *Let (M, g) be a spacetime and $I \subset \mathbb{R}$ a (bounded) interval. A continuous curve $\gamma\colon I \to M$ is future-directed causal if for each causally convex set U, given $t_1, t_2 \in I$, $t_1 < t_2$ with $\gamma([t_1, t_2]) \subset U$, there is a differentiable, future-directed causal curve connecting $\gamma(t_1) \in U$ to $\gamma(t_2) \in U$.*

It is also possible to assess continuous causal curves from the perspective of Sobolev spaces (cf. [MS08, Rem. 3.18] and [CS08]).

Definition 3.9. *A globally defined timelike vector field $V\colon M \to TM$ on a Lorentzian manifold (M, g) is called reference frame if it has unit length, i.e., $g(V, V) = -1$.*

Obviously, a time-orientation can be given by assigning a timelike vector V_p to every point in the Lorentzian manifold, which varies smoothly from point to point. The future cone is then given by choosing the cone of which V_p is an element. Moreover, we then have a globally defined timelike vector field, which can be chosen to be a global reference frame. But we emphasize the viewpoint of assigning a unique future cone to all points as opposed to distinguishing a particular timelike vector at each point, because in a spacetime the important feature is that such a distinguished direction is given *but a timelike vector field is not fixed*. Any timelike unit vector field chosen such that all of its vectors are elements of the future cone can be used to serve as a global reference frame. We emphasize that a fixed global timelike vector field is an additional structure on the Lorentzian manifold, and in this work, we will mostly be concerned with Lorentzian manifolds which carry this kind of additional structure. This gives rise to the following definition.

Definition 3.10. *A kinematical Lorentzian manifold (M, g, V) is a Lorentzian manifold (M, g) together with a globally defined timelike vector field $V\colon M \to TM$, which is complete.*

Obviously, following definition 2.42, a kinematical Lorentzian manifold possesses both the features of a kinematical manifold (M, V) and of a (time-orientable) Lorentzian manifold (M, g). Moreover, the Lorentzian and kinematical structures are required to be compatible in the sense that the complete vector field V has to be timelike everywhere.

Proposition 3.11. *Every kinematical Lorentzian manifold is a spacetime.*

Proof. As V is timelike for all $p \in M$, the vector $V_p \in T_pM$ is an element of a specific connected component of $\mathfrak{T}_p$ at p, say $V_p \in \mathfrak{T}_p^V$. Then set

$$\mathfrak{T}_p^{\uparrow} = \tau(p) := \mathfrak{T}_p^V,$$

for all $p \in M$, which gives the time-orientation. The future cone changes continuously with p as V is smooth. $\square$

Proposition 3.12. *Every spacetime can be made a kinematical Lorentzian manifold by an appropriate choice of a vector field $V: M \to TM$.*

Proof. Let (M, g) be a spacetime. Choose any globally defined timelike vector field $\tilde{V}: M \to TM$, which exists due to the time orientation τ on M. If $\tilde{V}$ is not complete, choose any auxiliary complete Riemannian metric g_R on M, which always exists (see, e.g., [NO61]). Then set

$$V := \frac{\tilde{V}}{|g_R(\tilde{V}, \tilde{V})|^{\frac{1}{2}}},$$

which is a complete timelike vector field on M. $\square$

Remark 3.13. *Moreover, Prop. 3.12 shows that the completeness of the vector field $V: M \to TM$ is independent of the Lorentzian metric g on M, which makes (M, g) a spacetime.*

Example 3.14. *Consider the two-dimensional Minkowski space with coordinates $(t, x) \in \mathbb{R}^2$ and the canonical flat metric $\eta = -\mathrm{d}t^2 + \mathrm{d}x^2$, which we denote as a Lorentzian manifold $(\mathbb{R}^2, \eta)$, and which is a spacetime with the usual time cone structure inherited from Minkowski space. Now remove the origin of $\mathbb{R}^2$ leading to $\mathring{\mathbb{R}}^2 := \mathbb{R}^2 \setminus \{(0,0)\}$, and $(\mathring{\mathbb{R}}^2, \eta)$ is still a spacetime. By setting $V_1 = \partial_t$, we get a spacetime $(\mathring{\mathbb{R}}^2, \eta, V_1)$ with a fixed incomplete reference frame V_1. Now*

$$g = \frac{\mathrm{d}t^2 + \mathrm{d}x^2}{t^2 + x^2}$$

is a complete Riemannian metric on $\mathring{\mathbb{R}}^2$. Then we set

$$V_2 = \frac{V_1}{\|V_1\|^g} = (t^2 + x^2)^{\frac{1}{2}} V_1,$$

such that $(\mathring{\mathbb{R}}^2, \eta, V_2)$ is obviously a kinematical Lorentzian manifold with complete timelike vector field V_2, but $\eta(V_2, V_2) \neq -1$.

Definition 3.15. *A kinematical spacetime (M, g, V) is a kinematical Lorentzian manifold, with the globally defined and complete timelike vector field $V: M \to TM$ having unit length $g(V, V) = -1$, i.e., being a reference frame.*

As Example 3.14 shows, not all spacetimes can be made into kinematical spacetimes, but it is a proper additional requirement to the metric g and the vector field V on M. In some sense the metric and the vector field have to be adjusted to the topological setting of the underlying manifold. If we allow for conformal changes of the Lorentzian metric the kinematical spacetime condition can be achieved.

Proposition 3.16. *For all kinematical Lorentzian manifolds (M, g, V), there is a conformal factor $e^\phi: M \to \mathbb{R}_{>0}$ leading to a pointwise conformal Lorentzian metric $g^* = e^\phi g$, such that (M, g^*, V) is a kinematical spacetime.*

Proof. We have $g(V,V) < 0$ and

$$-1 = g^*(V,V) = e^{\phi} g(V,V),$$

such that the choice

$$\phi = -\ln(-g(V,V))$$

gives the desired result. $\square$

Combining this result with Prop. 3.12 we can infer that any spacetime (M,g) can be made into a kinematical spacetime by an appropriate choice of a complete timelike vector field and a conformal transformation.

Example 3.17. *Recall $\mathring{\mathbb{R}}^2$ and V_2 from Example 3.14. If we set*

$$g := \frac{\eta}{t^2 + x^2},$$

then $(\mathring{\mathbb{R}}^2, g, V_2)$ is a kinematical spacetime. Furthermore, if we consider the global flow of V_2 in $(\mathring{\mathbb{R}}^2, g, V_2)$ as an $\mathbb{R}$-action on $\mathring{\mathbb{R}}^2$, this is an example of a Cartan $\mathbb{R}$-manifold which is not proper. This can be seen as follows. The set of flow lines of V_2 is given by the set of lines and half-lines

$$\mathring{\mathbb{R}}^2/_{\mathbb{R}} = \{\{(t, x_0) \,|\, t \in \mathbb{R}\} \,|\, x_0 \in \mathbb{R} \setminus \{0\}\} \cup \{(t^+, 0) \,|\, t^+ > 0\} \cup \{(t^-, 0) \,|\, t^- < 0\}.$$

We furnish this set with the quotient topology. Let O be open in $\mathring{\mathbb{R}}^2/_{\mathbb{R}}$ if $\pi^{-1}(O)$ is open in $\mathring{\mathbb{R}}^2$. Thus any open set in $\mathring{\mathbb{R}}^2/_{\mathbb{R}}$ which contains $(t^+, 0)$ also contains elements from a neighborhood of $(t^-, 0)$ and vice versa. Hence, $(t^+, 0)$ and $(t^-, 0)$ cannot be separated by disjoint open neighborhoods and $\mathring{\mathbb{R}}^2/_{\mathbb{R}}$ is not Hausdorff. In fact, $\mathring{\mathbb{R}}^2/_{\mathbb{R}}$ is homeomorphic to the line with two origins (see, e.g., [SS78]).

Proposition 3.18. *Every time-oriented and compact Lorentzian manifold is a kinematical Lorentzian manifold.*

Proof. Choose a time orientation by a timelike vector field, then the assertion is a direct consequence of Lem. 2.39. $\square$

Particularly, in chapter 4, we will assume a fixed complete timelike vector field, and will allow the Lorentzian metric to vary, in order to ensure the vector field to have unit norm, if necessary. For the topological and causal properties investigated in that chapter, considering the conformal class of metrics is sufficient.

3.1 Kinematical Quantities

In this section, we develop the theory and basic equations for the invariant parts of the covariant derivative of a reference frame in a kinematical spacetime. Given any kinematical spacetime (M, g, V), it is naturally equipped with a Lorentzian metric g and its

Levi-Civita connection ∇, as well as the reference frame V and its metric dual one-form $u = g(V, \cdot)$. Thus, in a next step, the quantities we would like to examine are the tensor fields $\nabla u = g(\nabla_{\cdot} V, \cdot) \in \Gamma(T^2M)$ and $\nabla_V(\nabla u) \in \Gamma(T^2M)$. By taking into account the decomposition $T_pM \simeq \mathbb{R} \oplus V_p^{\perp}$ of each tangent space with respect to the reference frame V, we can extract the invariant parts of ∇u—called kinematical quantities here—and $\nabla_V(\nabla u)$ with respect to that decomposition. Note that the kinematical quantities are particularly famous in the physically oriented literature on general relativity and are often referred to as kinematical invariants, as the decomposition in the tangent space at each event $p \in M$ yields the irreducible parts of the matrix $((\nabla u)_p)_{ij}$, i.e., the invariant parts in terms of linear algebra (cf., e.g., [HE73, Sec. 4.1] or [Ehl93]). For $\nabla_V(\nabla u)$ the decomposition leads to a set of geometric constraint equations for the kinematical quantities known as Raychaudhuri equations. These can be regarded as evolution equations for the kinematical quantities along the flow lines of V, and they are fulfilled because of the identities of the curvature tensor on the manifold.

Remark 3.19. *In this section, as well as in the sections 3.2 and 3.3, we state all results for kinematical spacetimes (M, g, V) to achieve consistency with the larger part of this work. But as all results in these sections are local by nature, they do also hold for spacetimes (M, g) together with a reference frame V, which is* not necessarily complete*. This will be needed at some points in section 5.3 and chapter 6.*

In the following, we will assume (M^{n+1}, g, V) to be a kinematical spacetime and we will use a pseudo-orthonormal frame $\{E_0, E_1, \dots, E_n\}$ with $E_0 = V$. Summation over vector fields E_i in the frame will run from 0 to n if not indicated otherwise.

The timelike vector field V induces an integrable distribution in the tangent bundle of codimension n, the integral manifolds of which are the flow lines of V according to section 2.3. Thus, each tangent space allows for a decomposition

$$T_pM = \operatorname{span}\{V_p\} \oplus V_p^{\perp} \simeq: V_pM \oplus H_pM.$$

Hence, V induces a horizontal distribution $HM \subset TM$ and a vertical distribution $VM \subset TM$. By Prop. 2.10, the projection

$$h = g + u \otimes u$$

is a Riemannian metric on HM. The maps $\mathcal{V} \colon TM \to VM$ and $\mathcal{H} \colon TM \to HM$ given by

$$\mathcal{V}(X) = -u(X)V$$

and

$$\mathcal{H}(X) = X + u(X)V,$$

for all $X \in \Gamma(TM)$ will be called projection operators. We will use the same notation $\mathcal{H}$ and $\mathcal{V}$ for the projection of differential forms and tensors. Hence, we have $\mathcal{H}(g) = h$ by setting

$$\mathcal{H}(g)(X, Y) := g(\mathcal{H}(X), \mathcal{H}(Y)) = g(X + u(X)V, Y + u(Y)V) =$$

$$= g(X,Y) + u(X)g(V,Y) + u(Y)g(V,X) - u(X)u(Y) = g(X,Y) + u(X)u(Y) = h(X,Y),$$

for all $X, Y : M \to TM$. This carries over to all tensors and differential forms, for example for any one-form $w \in \Gamma(T^*M)$, the one-form $\mathcal{H}(w) \in \Gamma(H^*M)$ is given by

$$\mathcal{H}(w)(X) = w(\mathcal{H}(X)),$$

for all vector fields X on M. Now we are ready for the following

Definition 3.20. *Let (M^{n+1}, g, V) be a kinematical spacetime. The expansion Θ is the function $M \to \mathbb{R}$ given by*

$$\Theta := \operatorname{div}(V) = \operatorname{Tr}(\nabla u) = \sum_i g(\nabla_{E_i} V, E_i).$$

The acceleration $\dot{u}$ is the one-form $M \to T^*M$ given by

$$\dot{u} := g(\nabla_V V, \cdot).$$

The shear σ is the symmetric trace-free tensor field $M \to \Sigma^2 M$

$$\sigma := \operatorname{sym}(\nabla u) + u \vee \dot{u} - \frac{\Theta}{n} h.$$

The rotation ω is the two-form field $M \to \Lambda^2 M$ given by

$$\omega := \mathrm{d}u + u \wedge \dot{u}.$$

Proposition 3.21. *For the kinematical quantities of a kinematical spacetime as in Def. 3.20, the following holds:*

(i) $\sigma(V, \cdot) = 0$, $\omega(V, \cdot) = 0$ *and* $\dot{u}(V) = 0$, *hence,* $\sigma \in \Gamma(H^*M \vee H^*M)$, $\omega \in \Gamma(H^*M \wedge H^*M)$ *and* $\dot{u} \in \Gamma(H^*M)$,

(ii) $\pounds_V g = 2\operatorname{sym}(\nabla u) = 2\sigma - 2u \vee \dot{u} + \frac{2}{n}\Theta h$, *and*

(iii) $\nabla u = \frac{\Theta}{n} h + \sigma + \omega - u \otimes \dot{u}$.

Proof. We use $g(V,V) = -1$ and the metricity of the Levi-Civita connection ∇.

(i): From the definition of the shear, rotation and acceleration we get:

$$2\dot{u}(V) = 2g(\nabla_V V, V) = \nabla_V(g(V,V)) = 0,$$

$$2\sigma(V, \cdot) = \nabla_V u + (\nabla.u)(V) + u(V)\dot{u} + \dot{u}(V)u = \dot{u} - \nabla.(u(V)) - \dot{u} = 0$$

and

$$2\omega(V, \cdot) = V \rfloor \mathrm{d}u + u(V)\dot{u} - \dot{u}(V)u = \dot{u} + \nabla.(u(V)) - \dot{u} = 0.$$

(ii): From the properties of the Lie derivative we get

$$(\pounds_V g)(X,Y) = g(\nabla_X V, Y) + g(\nabla_Y V, X),$$

for any $X, Y\colon M \to TM$, so that $\pounds_V g$ is indeed a symmetric tensor field $M \to \Sigma^2 M$ equal to $2\,\mathrm{sym}(g(\nabla_{\cdot} V, \cdot)) = 2\,\mathrm{sym}(\nabla u)$, and the assertion follows from the definition of the shear.

(iii): From the definition of the kinematical quantities we get

$$\frac{\Theta}{n} h + \sigma + \omega - u \otimes \dot{u} = \mathrm{sym}(\nabla u) + u \vee \dot{u} + \mathrm{d}u + u \wedge \dot{u} - u \otimes \dot{u} = \mathrm{sym}(\nabla u) + \mathrm{d}u = \nabla u.$$

□

The kinematical quantities can be regarded as unique tensorial objects that can be constructed from a given Levi-Civita connection ∇ and a vector field V. There are certain specific functions which can be formed from the Lorentzian metric g and the kinematical quantities in a kinematical spacetime.

Definition 3.22. *Let (M, g, V) be a kinematical spacetime and Θ, $\dot{u}$, σ, ω the kinematical quantities as above. We call the following set of functions $M \to \mathbb{R}$ scalar kinematical quantities.*

(i) The expansion Θ itself,
(ii) the acceleration scalar $|\dot{u}|^2 := g(\nabla_V V, \nabla_V V) = \dot{u}(\nabla_V V)$,
(iii) the rotation scalar $|\omega|^2 := \sum_{i,j} \omega(E_i, E_j)\omega(E_i, E_j)$,
(iv) the shear scalar $|\sigma|^2 := \sum_{i,j} \sigma(E_i, E_j)\sigma(E_i, E_j)$.

It is also useful to know how the kinematical quantities change under a pointwise conformal transformation $\tilde{g} = e^{2\phi} g$ of the metric g of a kinematical spacetime (M, g, V). As we would like to still have a reference frame after the conformal transformation, V has also to be changed to $\tilde{V} = e^{-\phi} V$. Although, the resulting spacetime is not necessarily a kinematical one any more as $\tilde{V}$ may be incomplete. Hence, in [Pla12] it was shown that the following holds.

Proposition 3.23. *Let (M^{n+1}, g, V) be a kinematical spacetime and $\phi\colon M \to \mathbb{R}$ induces a conformal transformation $\tilde{g} = \mathrm{e}^{2\phi} g$, as well as $\tilde{V} = \mathrm{e}^{-\phi} V$. Then the kinematical quantities $\tilde{\Theta}$, $\tilde{\omega}$, $\tilde{\sigma}$ and $\tilde{\dot{u}}$ of $(M^{n+1}, \tilde{g}, \tilde{V})$ read as follows:*

$$\mathrm{e}^{\phi} \tilde{\Theta} = \Theta + n \mathrm{d}\phi(V),$$

$$\tilde{\dot{u}} = \dot{u} + \mathcal{H}(\mathrm{d}\phi),$$

$$\mathrm{e}^{-\phi} \tilde{\omega} = \omega,$$

$$\mathrm{e}^{-\phi} \tilde{\sigma} = \sigma.$$

Proof. See [Pla12, Prop. 3.3.1]. □

3.2 Classification of Spacetimes by Kinematical Quantities

In this section and in the remainder of this work we will make use of the following symmetry properties of spacetimes.

Definition 3.24. *A spacetime (M, g) admitting a timelike conformal vector field $K\colon M \to TM$, i.e., there is a function $\varphi\colon M \to \mathbb{R}$, such that $\pounds_K g = \varphi^2 g$, is called conformally stationary. The timelike vector field K is called homothetic if φ is a constant. If (M, g) admits a timelike Killing vector field $K\colon M \to TM$, i.e., $\pounds_K g = 0$, it is called stationary, and it is called static if additionally the reference frame V parallel to K has vanishing rotation.*

There is a standard result that links symmetry properties of a kinematical spacetime (M, g, V) to the kinematical properties of the timelike unit vector field $V\colon M \to TM$. These have firstly been proven in articles by J. Ehlers, P. Geren and R. Sachs [EGS68], as well as by D.R. Oliver and W.R. Davis [OD77] and have been widely examined in [HP88], [Per90], [DS99] and [DPS08].

Theorem 3.25. *Let (M^{n+1}, g, V) be a kinematical spacetime. There is a conformal vector field ξ parallel to V, i.e., there are functions $f\colon M \to \mathbb{R}$, $\varphi\colon M \to \mathbb{R}$, such that $\xi = e^f V$ and $\pounds_\xi g = \varphi^2 g$ if and only if $\sigma = 0$ and $\dot{u} - \frac{\Theta}{n} u$ is exact, such that $\dot{u} - \frac{\Theta}{n} u = \mathrm{d}f$. Furthermore, ξ is homothetic if and only if additionally there is a constant $c \in \mathbb{R}$ such that $\Theta = c^2 e^{-f}$, it is Killing if and only if additionally $\Theta = 0$ and it is geodesically Killing if and only if additionally $\Theta = 0$ and $f = 0$.*

Proof. The first two assertions have been widely examined in the references above. So we only sketch the proofs here for the sake of completeness. Examining

$$(\pounds_\xi g)(V, V) = (\pounds_{e^f V} g)(V, V) = -\varphi^2,$$

yields after a short computation

$$\varphi^2 = 2\mathrm{d}f(V)e^f.$$

If we combine this with $\mathrm{Tr}(\pounds_\xi g) = \mathrm{Tr}(\varphi^2 g)$, we arrive at

$$\varphi^2 = \frac{2}{n} e^f \Theta. \tag{$*$}$$

Using this to manipulate the equation

$$\pounds_\xi g = \pounds_{e^f V} g = e^f \pounds_V g + 2e^f \mathrm{d}f \vee u = \varphi^2 g$$

and using item (ii) from Prop. 3.21 one gets after some tedious algebra the desired conditions

$$\sigma = 0 \quad \text{and} \quad \dot{u} - \frac{\Theta}{n} u = \mathrm{d}f.$$

On the other hand, starting from these conditions and setting $\varphi^2 := \frac{2}{n} e^f \Theta$ and $\xi := e^f V$ one can calculate $\pounds_\xi g = \varphi^2 g$, again from item (ii) in Prop. 3.21.

The vector field ξ is homothetic if φ^2 is constant. For $\Theta = c^2 e^{-f}$ we obviously get from ($*$) that $\varphi^2 = \frac{2c^2}{n}$ and if $\varphi^2 = \varphi_0^2 = const$, we arrive at $\Theta = \frac{n}{2}\varphi_0^2 e^{-f}$.

From ($*$) it is clear that $\varphi^2 = 0$ if and only if $\Theta = 0$, so ξ is Killing only in this case.

For $f = 0$, we have obviously $\xi = V$, and hence $g(\xi, \xi) = -1$. Take an arbitrary vector field $X\colon M \to TM$ such that $g(\xi, X) = 0$ everywhere. Then we get

$$0 = \nabla_\xi(g(\xi, X)) = g(\nabla_\xi \xi, X) + g(\xi, \nabla_\xi X) = g(\nabla_\xi \xi, X) + g(\pounds_\xi X + \nabla_X \xi, \xi) =$$

$$= g(\nabla_\xi \xi, X) + \pounds_\xi(g(\xi, X)) + \frac{1}{2}\nabla_X(g(\xi, \xi)) = g(\nabla_\xi \xi, X).$$

As this holds for any orthogonal vector field X, we conclude that $\nabla_\xi \xi = 0$ and therefore $\xi = V$ is geodesic. □

Remark 3.26. *Obviously, the theorem above yields that if the reference frame V in a kinematical spacetime (M, g, V) is itself Killing, then it is geodesically Killing.*

The following classification result of kinematical spacetimes is a simple consequence of Thm. 3.25 above.

Corollary 3.27. *Let (M^{n+1}, g, V) be a kinematical spacetime. The spacetime (M, g) is conformally stationary if $\sigma = 0$ and $\dot{u} - \frac{\Theta}{n}u$ is exact, it is stationary if $\sigma = 0$ and $\dot{u}$ is exact and it is static if $\sigma = 0$, $\omega = 0$ and $\dot{u}$ is exact.*

Subsequently, the kinematical spacetimes with vanishing shear $\sigma = 0$ will be of special interest, because—as we will investigate in chapter 6—they possess the structure of conformal pseudo-Riemannian submersions in many particular situations. The following definitions are standard for those spacetimes (cf. [GRK96]).

Definition 3.28. *A kinematical spacetime (M, g, V) is called spatially conformally stationary if there is a vector field $\xi\colon M \to TM$ parallel to V, a function $f\colon M \to \mathbb{R}$ such that $\xi = e^f V$ and a function $\varphi\colon M \to \mathbb{R}$, such that $\pounds_\xi h = \varphi h$ holds. It is called spatially homothetic if additionally $\mathcal{H}(\mathrm{d}\varphi) = 0$ and spatially stationary if additionally $\varphi = 0$. The vector field ξ is then called spatially conformal, spatially homothetic or spatially Killing, respectively.*

The reference frame V is also called rigid if it is parallel to a spatial Killing field.

Remark 3.29. *Certainly, one could also define a spacetime (M, g) to be spatially conformally stationary, spatially homothetic or spatially stationary if there exists a timelike vector field $\xi\colon M \to TM$, such that for the tensor field $\eta = g + g(\xi, \cdot) \otimes g(\xi, \cdot)$, $\pounds_\xi \eta = \varphi\eta$ holds with some function $\varphi\colon M \to \mathbb{R}$ (with $\mathcal{H}(\mathrm{d}\varphi) = 0$ in the spatially homothetic case and $\varphi = 0$ in the spatially stationary case). But we will use these notions only for kinematical spacetimes or for spacetimes with a fixed reference frame in this work.*

Lemma 3.30. *A kinematical spacetime (M^{n+1}, g, V) is*

(i) spatially conformally stationary if and only if $\sigma = 0$,

(ii) spatially homothetic if and only if $\sigma = 0$ *and* $\mathcal{H}(\mathrm{d}(\Theta e^f)) = 0$,

(iii) spatially stationary if and only if $\sigma = 0$ *and* $\Theta = 0$.

Proof. Using the identities $\pounds_V u = \dot{u}$, $\pounds_{\psi V} u = \psi \pounds_V u - \mathrm{d}\psi$ and $\pounds_{\psi V} g = \psi \pounds_V g + 2\mathrm{d}\psi \vee u$ for any function $\psi \colon M \to \mathbb{R}$, as well as *(ii)* from Prop. 3.21, we compute

$$\pounds_\xi h = \pounds_\xi (g + u \otimes u) = \pounds_{e^f V} g + 2(\pounds_{e^f V} u) \vee u =$$

$$= e^f [\pounds_V g + 2\mathrm{d}f \vee u + 2(\dot{u} - \mathrm{d}f) \vee u] = e^f (2\sigma + \frac{2}{n}\Theta h).$$

Hence, there is a function $\varphi \colon M \to \mathbb{R}$ such that $\pounds_\xi h = \varphi h$ holds if and only if $\sigma = 0$, which implies item *(i)*. As $\varphi = \frac{2}{n} e^f \Theta$ in this case, items *(ii)* and *(iii)* follow. □

Moreover, we have the following

Lemma 3.31. *A kinematical spacetime* (M^{n+1}, g, V) *is spatially conformally stationary if and only if the reference frame* V *is spatially conformal with* $\pounds_V h = \frac{2\Theta}{n} h$ *and it is spatially stationary if and only if* V *is spatially Killing. Furthermore, in the case of vanishing shear,* V *is spatially homothetic if and only if* $\mathcal{H}(\mathrm{d}\Theta) = 0$.

Proof. Examining the computations in Lem. 3.30 yields

$$\pounds_V h = 2\sigma + \frac{2\Theta}{n} h$$

and the results follow. □

3.3 Raychaudhuri Equations

Regarding the kinematical quantities in section 3.1, one can now ask the question how they evolve along the flow lines of the vector field V in a kinematical spacetime (M, g, V). This leads to the investigation of the quantity $\nabla_V(\nabla u)$. As this expression involves second derivatives of the metric, we expect curvature terms to become involved. Originally, the evolution of the expansion Θ was investigated first in [Ray55] by A.K. Raychaudhuri himself and it was the evolution equation of the expansion that was named after him in the first place. The evolution equations for the remaining kinematical quantities are, subsequently, summarized under the term Raychaudhuri equations as well (cf. [KS07]). Geometrically, all Raychaudhuri equations are in fact curvature identities, namely some components of the curvature tensor ordered conveniently. In this section, we will derive all Raychaudhuri equations in their full generality.

Let (M^{n+1}, g, V) be a kinematical spacetime of class C^r, $r \geq 3$ and $\{E_0, \ldots, E_n\}$ a pseudo-orthonormal frame for g, where we sum over $i, j, \ldots = 0, \ldots, n$ and set $E_0 = V$ as usual. We denote by

$$R \colon \Gamma(TM) \times \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM)$$

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

the curvature tensor associated to the Lorentzian metric g, for all vector fields $X, Y, Z \in \Gamma(TM)$, which satisfies the usual symmetries.

We denote the Ricci tensor associated to R by $Ric\colon \Gamma(TM) \times \Gamma(TM) \to C^r(M)$, $Ric \in \Gamma(\Sigma^2 M)$, which is defined by

$$Ric(X,Y) = \mathrm{Tr}(R(\cdot, X)Y) = \sum_i g(R(E_i, X)Y, E_i),$$

for all $X, Y \in \Gamma(TM)$.

Now, the Raychaudhuri equations follow from the investigation of the expression

$$X, Y \mapsto g(R(X,V)V, Y),$$

for all $X, Y \in \Gamma(TM)$ and for the reference frame V. Using

$$\nabla_V \nabla_X V = \nabla_{\nabla_V X} V + (\nabla_V(\nabla V))(X)$$

and $(\nabla \dot{u})(X,Y) = g(\nabla_X \nabla_V V, Y)$, as well as the usual $R(X,V,V,Y) := g(R(X,V)V,Y)$ yields

$$(\nabla_V(\nabla u))(X,Y) = (\nabla \dot{u})(X,Y) - (\nabla u)(\nabla_X V, Y) - R(X,V,V,Y), \tag{$*$}$$

for all $X, Y\colon M \to TM$. And we will uncover the evolution equations of Θ, σ, ω and $\dot{u}$, involving their time derivatives $\dot{\Theta} = \nabla_V \Theta = \mathrm{d}\Theta(V)$, $\dot{\sigma} = \nabla_V \sigma$, $\dot{\omega} = \nabla_V \omega$ and $\ddot{u} = \nabla_V \dot{u} = g(\nabla_V \nabla_V V, \cdot)$, by examining the invariant components of equation ($*$) with respect to $\mathcal{V}$, $\mathcal{H}$ and the trace, as well as its symmetric and antisymmetric parts.

The term $(\nabla u)(\nabla_X V, Y)$ in ($*$) is of specific relevance for the pending derivation of the Raychaudhuri identities. Therefore, we first prove the following

Lemma 3.32. *The tensor $(\nabla u)(\nabla . V, \cdot) \in \Gamma(T^2 M)$ does only depend on the kinematical quantities and the one-form u. Particularly, we have*

$$(\nabla u)(\nabla_X V, Y) = \frac{\Theta^2}{n^2} h(X,Y) + \frac{2\Theta}{n}(\sigma(X,Y) + \omega(X,Y)) + \sum_i \sigma(E_i, X)\sigma(E_i, Y) +$$

$$- \sum_i \omega(E_i, X)\omega(E_i, Y) + \sum_i \sigma(E_i, X)\omega(E_i, Y) - \sum_i \omega(E_i, X)\sigma(E_i, Y) -$$

$$- \frac{\Theta}{n} u(X)\dot{u}(Y) - u(X)\sigma(\nabla_V V, Y) - u(X)\omega(\nabla_V V, Y),$$

for all $X, Y\colon M \to TM$.

Proof. We use $\nabla u = \frac{\Theta}{n} h + \sigma + \omega - u \otimes \dot{u}$ from Prop. 3.21, as well as

$$(\nabla u)(\nabla_X V, Y) = \sum_i (\nabla_X u)(E_i)(\nabla_{E_i} u)(Y),$$

for any vector fields $X, Y\colon M \to TM$ and for a g-orthonormal frame $\{E_0, \ldots, E_n\}$ with $E_0 = V$, i.e., we sum over $i = 0, \ldots, n$. This yields

$$(\nabla u)(\nabla_X V, Y) = \sum_i \left(\frac{\Theta}{n} h(E_i, X) + \sigma(E_i, X) + \omega(X, E_i) - u(X)\dot{u}(E_i)\right) \times$$

$$\left(\frac{\Theta}{n} h(E_i, Y) + \sigma(E_i, Y) + \omega(E_i, Y) - u(E_i)\dot{u}(Y)\right) =$$

$$= \frac{\Theta^2}{n^2} h(X, Y) + \frac{2\Theta}{n}(\sigma(X, Y) + \omega(X, Y)) + \sum_i \sigma(E_i, X)\sigma(E_i, Y) + \sum_i \omega(X, E_i)\omega(E_i, Y) +$$

$$+ \sum_i \sigma(E_i, X)\omega(E_i, Y) - \sum_i \omega(E_i, X)\sigma(E_i, Y) - \frac{\Theta}{n} u(X)\dot{u}(Y) -$$

$$-u(X)\dot{u}(E_i)(\sigma(E_i, Y) + \omega(E_i, Y))$$

by using Prop. 3.21, and the result follows. □

Now, we are ready to derive the Raychaudhuri equations. To this end, we introduce the shorthand notation $\dot{V} = \nabla_V V$. Furthermore, we will denote by $\mathcal{H}\mathrm{sym}(\nabla\dot{u})$ the symmetric and by $\widetilde{\mathcal{H}\mathrm{sym}}(\nabla\dot{u})$ the symmetric trace-free part of $\mathcal{H}(\nabla\dot{u})$. Moreover, we define $\tilde{\vee}$ to be the trace-free symmetrized tensor product, given for horizontal one-forms $u, w \in \Gamma(H^*M)$ by

$$u \,\tilde{\vee}\, w = u \vee w - \frac{\sum_{i=1}^n u(E_i)w(E_i)}{n} h,$$

where we use a pseudo-orthonormal frame $\{V, E_1, \ldots, E_n\}$, such that $\{E_1, \ldots, E_n\}$ yields a basis of H_pM for all $p \in M$.

Proposition 3.33. *Let (M^{n+1}, g, V) be a kinematical spacetime. The kinematical quantities Θ, ω, σ and $\dot{u}$ and their time derivatives obey the following geometrical identities, called Raychaudhuri equations:*

(i) Raychaudhuri equation for the expansion:

$$\dot{\Theta} = -\frac{\Theta^2}{n} + |\omega|^2 - |\sigma|^2 + \mathrm{div}(\dot{V}) - Ric(V, V).$$

(ii) Raychaudhuri equation for the acceleration:

$$\ddot{u} - \pounds_V \dot{u} = |\dot{u}|^2 u - \frac{\Theta}{n}\dot{u} + \frac{1}{2}\dot{V}\rfloor\omega - \frac{1}{2}\dot{V}\rfloor\sigma.$$

(iii) Raychaudhuri equation for the rotation:

$$\dot{\omega} = -\frac{2\Theta}{n}\omega + \frac{1}{2}\sum_i (E_i\rfloor\omega) \wedge (E_i\rfloor\sigma) + \mathcal{H}(\mathrm{d}\dot{u}) + u \wedge (\dot{V}\rfloor\omega).$$

(iv) Raychaudhuri equation for the shear:

$$\dot{\sigma} = -\frac{2\Theta}{n}\sigma - \frac{1}{2}\sum_i (E_i\rfloor\sigma)\,\tilde{\vee}\,(E_i\rfloor\sigma) + \frac{1}{2}\sum_i (E_i\rfloor\omega)\,\tilde{\vee}\,(E_i\rfloor\omega) +$$

$$+\widetilde{\mathcal{H}\mathrm{sym}}(\nabla\dot{u}) + \dot{u}\,\tilde{\vee}\,\dot{u} + \frac{1}{2}u \vee (\dot{V}\rfloor\sigma) - R(\cdot, V, V, \cdot) + \frac{Ric(V,V)}{n}h.$$

Proof. *(i)* Due to equation ($*$) on page 44, we have

$$\dot{\Theta} = \sum_i (\nabla_V(\nabla u))(E_i, E_i) = \sum_i \left[(\nabla\dot{u})(E_i, E_i) - (\nabla u)(\nabla_{E_i}V, E_i) - R(E_i, V, V, E_i)\right].$$

Using Lem. 3.32, as well as $\sum_i (\nabla\dot{u})(E_i, E_i) = \mathrm{div}(\dot{V})$, $\sum_i h(E_i, E_i) = n$ and the definition of the Ricci curvature, we get

$$\dot{\Theta} = \mathrm{div}(\dot{V}) - \frac{\Theta^2}{n} + \sum_{i,j}\sigma(E_i, E_j)\sigma(E_i, E_j) - \sum_{i,j}\omega(E_i, E_j)\omega(E_i, E_j) - Ric(V, V).$$

Using Def. 3.22 for $|\sigma|^2$ and $|\omega|^2$ yields the desired equation.

(ii) The Raychaudhuri equation for the acceleration is an identity that stems from expanding the Lie derivative $\pounds_V\dot{u}$ with respect to the kinematical quantities and from the identity

$$|\dot{u}|^2 = g(\nabla_V V, \nabla_V V) = -g(\nabla_V\nabla_V V, V) = -\ddot{u}(V),$$

which is just a consequence of V being a unit vector field in a kinematical spacetime. Using Cartan's magic formula we compute for any vector field $X \in \Gamma(TM)$

$$-(\pounds_V\dot{u})(X) = -(V\rfloor\mathrm{d}\dot{u})(X) = 2\mathrm{d}\dot{u}(X, V) = (\nabla_X\dot{u})(V) - (\nabla_V\dot{u})(X) =$$

$$g(\nabla_X\nabla_V V, V) - \ddot{u}(X) = -g(\nabla_V V, \nabla_X V) - \ddot{u}(X) = -(\nabla u)(X, \dot{V}) - \ddot{u}(X) =$$

$$= -\frac{\Theta}{n}h(X, \dot{V}) - \omega(X, \dot{V}) - \sigma(X, \dot{V}) + u(X)\dot{u}(\dot{V}) - \ddot{u}(X) =$$

$$= -\frac{\Theta}{n}\dot{u}(X) + \frac{1}{2}(\dot{V}\rfloor\omega)(X) - \frac{1}{2}(\dot{V}\rfloor\sigma)(X) + |\dot{u}|^2 u(X) - \ddot{u}(X)$$

and the result follows.

(iii) The Raychaudhuri equation for the rotation follows from the anti-symmetric part of equation ($*$) on page 44, which is just given by $\nabla_V(\mathrm{d}u)$. As $\mathrm{d}u = \omega - u \wedge \dot{u}$, we have

$$\nabla_V(\mathrm{d}u) = \nabla_V\omega - \nabla_V(u \wedge \dot{u}) = \dot{\omega} - u \wedge \ddot{u}.$$

As $R(\cdot, V, V, \cdot)$ is symmetric and the anti-symmetric part of $\nabla\dot{u}$ is $\mathrm{d}\dot{u}$, it follows, using Lem. 3.32, that

$$\dot{\omega} = u \wedge \ddot{u} + \mathrm{d}\dot{u} - \frac{2\Theta}{n}\omega + \sum_i \omega(E_i, \cdot)\sigma(E_i, \cdot) - \sum_i \sigma(E_i, \cdot)\omega(E_i, \cdot) +$$

$$+\frac{\Theta}{n}u\wedge\dot{u}+u\wedge\sigma(\dot{V},\cdot)+u\wedge\omega(\dot{V},\cdot),$$

and hence

$$\dot{\omega}=u\wedge\ddot{u}+\mathrm{d}\dot{u}-\frac{2\Theta}{n}\omega+\frac{1}{2}\sum_i(E_i\rfloor\omega)\wedge(E_i\rfloor\sigma)+\frac{\Theta}{n}u\wedge\dot{u}+\frac{1}{2}u\wedge(\dot{V}\rfloor\sigma)+\frac{1}{2}u\wedge(\dot{V}\rfloor\omega).$$

Inserting the Raychaudhuri equation for the acceleration to replace $\ddot{u}$ yields

$$\dot{\omega}=-\frac{2\Theta}{n}\omega+\frac{1}{2}\sum_i(E_i\rfloor\omega)\wedge(E_i\rfloor\sigma)+u\wedge(V\rfloor\mathrm{d}\dot{u})+\mathrm{d}\dot{u}+u\wedge(\dot{V}\rfloor\omega),$$

when using $\pounds_V\dot{u}=V\rfloor\mathrm{d}\dot{u}$. Noting that for all vector fields $X,Y\in\Gamma(TM)$

$$(\mathcal{H}(\mathrm{d}\dot{u}))(X,Y)=\mathrm{d}\dot{u}(\mathcal{H}X,\mathcal{H}Y)=\mathrm{d}\dot{u}(X+u(X)V,Y+u(Y)V)=$$

$$=\mathrm{d}\dot{u}(X,Y)+u(Y)\mathrm{d}\dot{u}(X,V)+u(X)\mathrm{d}\dot{u}(V,Y)=\mathrm{d}\dot{u}(X,Y)+(u\wedge(V\rfloor\mathrm{d}\dot{u}))(X,Y)$$

holds, yields $\mathcal{H}(\mathrm{d}\dot{u})=\mathrm{d}\dot{u}+u\wedge(V\rfloor\mathrm{d}\dot{u})$ and, therefore, the desired equation.

(iv) Similar reasoning leads to the Raychaudhuri equation for the shear. In this case we have

$$\sigma=\mathrm{sym}(\nabla u)+u\vee\dot{u}-\frac{\Theta}{n}h,$$

hence using the symmetric part of $(*)$ on page 44 yields

$$\dot{\sigma}=\nabla_V(u\vee\dot{u})-\nabla_V(\frac{\Theta}{n}h)+\mathrm{sym}(\nabla\dot{u})-\mathrm{sym}[(\nabla u)(\nabla.V,\cdot)]-\mathrm{sym}[R(\cdot,V,V,\cdot)].$$

Now, we compute

$$\nabla_V(u\vee\dot{u})=u\vee\ddot{u}+\dot{u}\otimes\dot{u}$$

and

$$\nabla_V(\Theta h)=\dot{\Theta}h+\Theta\nabla_V(u\otimes u)=\dot{\Theta}h+2\Theta(u\vee\dot{u}),$$

and use Lem. 3.32 to arrive at

$$\dot{\sigma}=u\vee\ddot{u}+\dot{u}\otimes\dot{u}-\frac{\dot{\Theta}}{n}h-\frac{2\Theta}{n}(u\vee\dot{u})+\mathrm{sym}(\nabla\dot{u})-\frac{\Theta^2}{n^2}h-\frac{2\Theta}{n}\sigma-\frac{1}{2}\sum_i(E_i\rfloor\sigma)\otimes(E_i\rfloor\sigma)+$$

$$+\frac{1}{2}\sum_i(E_i\rfloor\omega)\otimes(E_i\rfloor\omega)+\frac{\Theta}{n}u\vee\dot{u}+\frac{1}{2}u\vee(\dot{V}\rfloor\sigma)+\frac{1}{2}u\vee(\dot{V}\rfloor\omega)-R(\cdot,V,V,\cdot).$$

Now, we use the Raychaudhuri equation for the expansion to replace $\dot{\Theta}$ and get

$$\dot{\sigma}=u\vee\left[\ddot{u}-\frac{\Theta}{n}\dot{u}+\frac{1}{2}(\dot{V}\rfloor\sigma)+\frac{1}{2}(\dot{V}\rfloor\omega)\right]+\dot{u}\otimes\dot{u}+\mathrm{sym}(\nabla\dot{u})-\frac{2\Theta}{n}\sigma-\frac{\mathrm{div}(\dot{V})}{n}h-$$

$$-\frac{1}{2}\sum_i(E_i\rfloor\sigma)\tilde{\vee}(E_i\rfloor\sigma)+\frac{1}{2}\sum_i(E_i\rfloor\omega)\tilde{\vee}(E_i\rfloor\omega)-R(\cdot,V,V,\cdot)+\frac{Ric(V,V)}{n}h.$$

Considerations similar to the rotation case above lead to

$$\mathcal{H}\mathrm{sym}(\nabla\dot{u}) = \mathrm{sym}(\nabla\dot{u}) + u \vee (2\ddot{u} - \pounds_V \dot{u}) - |\dot{u}|^2 u \otimes u$$

by using $(\pounds_V \dot{u})(X) = \ddot{u}(X) - (\nabla_X \dot{u})(V)$ for all vector fields X on M. Using this to replace $\mathrm{sym}(\nabla\dot{u})$, and employing the Raychudhuri equation for the acceleration to get rid of $\ddot{u}$, yields

$$\dot{\sigma} = u \vee (\dot{V}\rfloor\sigma) + \dot{u}\otimes\dot{u} + \mathcal{H}\mathrm{sym}(\nabla\dot{u}) - \frac{2\Theta}{n}\sigma - \frac{\mathrm{div}(\dot{V})}{n}h -$$
$$-\frac{1}{2}\sum_i (E_i\rfloor\sigma)\,\tilde{\vee}\,(E_i\rfloor\sigma) + \frac{1}{2}\sum_i (E_i\rfloor\omega)\,\tilde{\vee}\,(E_i\rfloor\omega) - R(\cdot, V, V, \cdot) + \frac{Ric(V,V)}{n}h.$$

Now the desired equation follows from

$$\mathrm{Tr}(\dot{u}\otimes\dot{u}) = \sum_i [\dot{u}(E_i)]^2 = |\dot{u}|^2$$

and

$$\mathrm{Tr}(\mathcal{H}\mathrm{sym}(\nabla\dot{u})) = \mathrm{Tr}(\mathrm{sym}(\nabla\dot{u})) + |\dot{u}|^2[u(V)]^2 - 2u(V)\ddot{u}(V) = \mathrm{div}(\dot{V}) - |\dot{u}|^2,$$

which implies

$$\widetilde{\mathcal{H}\mathrm{sym}}(\nabla\dot{u}) + \dot{u}\,\tilde{\vee}\,\dot{u} = \mathcal{H}\mathrm{sym}(\nabla\dot{u}) + \dot{u}\otimes\dot{u} - \frac{\mathrm{div}(\dot{V})}{n}h.$$

$\square$

Several remarks are in order. The vertical part of the Raychaudhuri equation for the rotation yields just the identity

$$V\rfloor\dot{\omega} + \dot{V}\rfloor\omega = 0,$$

which forces the horizontal part to read

$$\mathcal{H}(\dot{\omega}) = -\frac{2\Theta}{n}\omega + \frac{1}{2}\sum_i (E_i\rfloor\omega)\wedge(E_i\rfloor\sigma) + \mathcal{H}(\mathrm{d}\dot{u}).$$

The same holds for the shear equation, which yields

$$V\rfloor\dot{\sigma} + \dot{V}\rfloor\sigma = 0$$

and the corresponding horizontal equation

$$\mathcal{H}(\dot{\sigma}) = -\frac{2\Theta}{n}\sigma - \frac{1}{2}\sum_i (E_i\rfloor\sigma)\,\tilde{\vee}\,(E_i\rfloor\sigma) + \frac{1}{2}\sum_i (E_i\rfloor\omega)\,\tilde{\vee}\,(E_i\rfloor\omega) +$$
$$+\widetilde{\mathcal{H}\mathrm{sym}}(\nabla\dot{u}) + \dot{u}\,\tilde{\vee}\,\dot{u} - R(\cdot, V, V, \cdot) + \frac{Ric(V,V)}{n}h.$$

The equations for the expansion and the shear are the only ones to include curvature terms. In the case of the expansion it is $Ric(V,V)$, the Ricci curvature along the reference frame V, also called Raychaudhuri scalar, and in the case of the shear it is the traceless part of $R(\cdot, V, V, \cdot)$, which can be expressed in terms of the Weyl curvature tensor W and the traceless part of the Ricci tensor in the following way (cf. [KS07]).

Remark 3.34. *Denote by*

$$\Pi = \widetilde{\mathcal{H}\mathrm{sym}}(Ric)$$

the symmetric trace-free part of $\mathcal{H}(Ric)$. Then after some tedious, but straightforward algebra one gets

$$\frac{Ric(V,V)}{n}h - R(\cdot,V,V,\cdot) = \frac{\Pi}{n-1} - W(\cdot,V,V,\cdot).$$

In four spacetime dimensions (i.e., $n = 3$), the symmetric tracefree tensor E, defined by $E(X,Y) := W(X,V,V,Y)$ for all $X, Y : M \to TM$ is called electric part of the Weyl tensor. Besides that, there is, in arbitrary spacetime dimension, a so called magnetic part H of the Weyl tensor which depends on the kinematical quantities and their derivatives only (see formula (B12) in [HOW12, Appendix B]). Furthermore, in four spacetime dimensions, the Weyl tensor W is completely determined by H and the electric part E defined above. Hence, the Weyl tensor in four spacetime dimensions can be computed from Π and the kinematical quantities. It is worth noting that in higher spacetime dimensions (i.e., $n+1 \geq 5$) a splitting of the Weyl tensor into electric and magnetic parts is possible, too (cf. [HOW12]), but the electric part is not *only given by $W(\cdot,V,V,\cdot)$ occurring in the Raychaudhuri equation for the shear. This amounts to the fact that in higher dimensions the Weyl curvature is* not *completely determined by the kinematical quantities and Π.*

The considerations about curvature terms above, underline the insight that the rotation is an attribute of the reference frame V, which is independent of the curvature of g. Especially, since a simple computation shows that

$$\pounds_V \omega = \mathcal{H}(\mathrm{d}\dot{u}),$$

which makes the horizontal part of the Raychaudhuri equation for the rotation even independent of the acceleration. However, curvature properties enter the development of ω indirectly via the expansion Θ and its Raychaudhuri equation.

In their full generality, the Raychaudhuri equations seem to be too complicated to be useful. Therefore, we will primarily use them in special cases in the following sections and chapters. Particularly, the case of vanishing shear will be of interest. In this case the Raychaudhuri equation for the shear transforms into a constraint equation, which connects rotation, acceleration and Weyl curvature of the kinematical spacetime. Basically, this implies that these three quantities cannot be chosen independently in a spatially conformally stationary kinematical spacetime (cf. Lem. 3.30). The Raychaudhuri equation for the expansion will be of interest in the case of vanishing shear and vanishing expansion, particularly in stationary spacetimes. This will enable us to derive connections between rotation, acceleration and Ricci curvature by a Lorentzian Bochner technique in section 5.3. The Raychaudhuri equation for the rotation will become important in shear-free cases with geodesic reference frame V, where it just reads

$$\dot{\omega} = -\frac{2\Theta}{n}\omega$$

in chapter 6.

3.4 The Causal Ladder

In this section, we introduce the basic notions of causality conditions for spacetimes. Besides the standard references for this chapter, we will also base this section on [MS08]. It is important to keep in mind that all causality conditions are conformally invariant. Although we will define the specific causality conditions for a single spacetime (M, g) only, it does always hold for the whole conformal class of spacetimes $(M, [g])$ with $[g] = \{\tilde{g} \in \mathrm{Lor}(M) \,|\, \tilde{g} = e^{\varphi} g,\ \varphi\colon M \to \mathbb{R}\}$, too.

First, we fix some causal binary relations for events in a spacetime and some definitions for causal sets.

Definition 3.35. *Let (M, g) be a spacetime and $p, q \in M$. Then we say:*

(i) p is chronologically related to q, denoted $p \ll q$, if there is a future-directed timelike curve connecting p with q.
(ii) p is strictly causally related to q, denoted $p < q$, if there is a future-directed causal curve connecting p with q.
(iii) p is causally related to q, denoted $p \leq q$, if there is a future-directed causal or constant curve connecting p with q.
(iv) p is horismotically related to q, denoted $p \to q$, if there is a future-directed causal or constant but not timelike curve connecting p with q.
(v) The chronological future of p is the set $I^+(p) := \{q \in M \,|\, p \ll q\}$.
(vi) The causal future of p is the set $J^+(p) := \{q \in M \,|\, p \leq q\}$.
(vii) The future horismos of p is the set $E^+(p) := \{q \in M \,|\, p \to q\}$.

The chronological and causal past $I^-(p)$, $J^-(p)$ and the past horismos $E^-(p)$ are defined analogously. Furthermore, the binary relations $\ll, \leq, \to$ can be regarded as subsets of $M \times M$ by setting

(i) $I^+ := \{(p, q) \in M \times M \,|\, p \ll q\}$,
(ii) $J^+ := \{(p, q) \in M \times M \,|\, p \leq q\}$ and
(iii) $E^+ := \{(p, q) \in M \times M \,|\, p \to q\}$.

Moreover, we call a set $U \subset M$ achronal (acausal) if there are no two events $p, q \in U$, such that $p \ll q$ $(p < q)$ holds.

Using the notion of continuous causal curves from the beginning of this chapter, it becomes clear that the causal relation and the chronological relation are both transitive, i.e., let p, q, r be events in some spacetime (M, g), then $p \leq q$ and $q \leq r$ ($p \ll q$ and $q \ll r$) implies $p \leq r$ ($p \ll r$). This is obvious, as the causal or constant curve connecting p to r can be differentiable almost everywhere but only continuous at q, as long as the limits of the tangent vectors—when approaching q from the future or the past, respectively—are both future pointing. This carries over to timelike curves.

Definition 3.36. *Let (M, g) be a spacetime and $\gamma\colon [a, b] \to M$ a closed curve in M. Then γ is called a CTC (closed timelike curve) if it is timelike and a CCC (closed causal curve)* *if it is causal.*

Please note that a CTC (or CCC) is not necessarily periodic as $\dot{\gamma}(a) \neq \dot{\gamma}(b)$ in general.

Definition 3.37. *Let (M, g) be a spacetime and $\gamma\colon \mathbb{R} \to M$ a (possibly non-smooth) future-directed causal and inextendible curve in M and $K \subset M$ compact.*

(i) The curve γ is called future (resp. past) imprisoned in K if there is a $T \in \mathbb{R}$, such that $\gamma([T, \infty)) \subset K$ (resp. $\gamma((-\infty, T]) \subset K$).
(ii) The curve γ is called partially future (resp. past) imprisoned in K if there is a sequence $\{t_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$ with $t_n \to \infty$ as $n \to \infty$ (resp. $t_n \to -\infty$ as $n \to \infty$), such that $\gamma(t_n) \in K$ for all $n \in \mathbb{N}$.
(iii) We call a curve γ non–(partially)-imprisoned if it is neither (partially) future nor (partially) past imprisoned in any compact set.
(iv) A spacetime (M, g) is called non-imprisoning if all causal curves are non-imprisoned.
(v) A spacetime (M, g) is called non–partially-future-(resp. past)-imprisoning if it does not contain any partially future (resp. past) imprisoned causal curve. It is called non–partially-imprisoning if it is non–partially-future-imprisoning and non–partially-past-imprisoning.

Remark 3.38. *J.K. Beem has shown that a spacetime is non–future-imprisoning if and only if it is non–past-imprisoning (see [Bee76], where this is called condition N). Therefore, as opposed to partial imprisonment, we do not have to to distinguish between the future and the past cases.*

In [Min08c], E. Minguzzi showed the connection between limit sets of a curve and its imprisonment by the following

Proposition 3.39. *Let (M, g) be a spacetime and $\gamma\colon \mathbb{R} \to M$ a (maybe continuous) future-directed causal and inextendible curve in M. The curve γ is partially future (past) imprisoned in some compact $K \subset M$ if and only if $\omega(\gamma) \neq \emptyset$ ($\alpha(\gamma) \neq \emptyset$) and it is future (past) imprisoned in some compact $K \subset M$ if and only if $\omega(\gamma) \neq \emptyset$ and $\omega(\gamma)$ is compact ($\alpha(\gamma) \neq \emptyset$ and $\alpha(\gamma)$ is compact).*

Proof. See [Min08c, Prop. 3.2]. □

Definition 3.40. *A spacetime (M, g) is called totally vicious if for all $p \in M$ there is a CTC intersecting p.*

Note that for this worst causal property of a spacetime, there does not exist a simple topological reason. See Example 4.6 below.

Now we are ready to give the definitions for the steps on the causal ladder which are most important in this work. We mainly base the definitions on [MS08], but we will also use some other references. There is one condition, related in some ways to the causality conditions below, which stands, nevertheless, outside the causal ladder.

Definition 3.41. *A spacetime (M, g) is called reflecting if $I^+(p) \supset I^+(q) \Leftrightarrow I^-(p) \subset I^-(q)$ for all $p, q \in M$.*

We will see below that reflectivity helps to *jump* up the causal ladder from distinction to causal continuity.

Definition 3.42. *A spacetime (M, g) is called chronological if there is no CTC in (M, g).*

Combining chronology with reflectivity yields the existence of a so-called semi-time function, i.e., a *continuous* function $t\colon M \to \mathbb{R}$ for a spacetime (M, g), such that $p \ll q$ implies $t(p) < t(q)$ for all events $p, q \in M$. In other words, t is strictly increasing along any future-directed timelike curve. Then the following holds.

Proposition 3.43. *A spacetime (M, g) which is chronological and reflecting admits a semi-time function.*

Proof. See [Min10, Thm. 5]. □

Definition 3.44. *A spacetime (M, g) is called causal if there is no CCC in (M, g).*

Obviously, causality of a spacetime implies that the spacetime is also chronological. The next stronger condition on the causal ladder is non-imprisonment, as defined in Def. 3.37.

In [Min08d] the following new step on the causal ladder was established.

Definition 3.45. *A spacetime (M, g) is called feebly distinguishing if $(p, q) \in J^+$, $p \in \overline{I^+(q)}$ and $q \in \overline{I^-(p)}$ implies $p = q$.*

Definition 3.46. *A spacetime (M, g) is called distinguishing if $I^+(p) = I^+(q)$ implies $p = q$ and $I^-(p) = I^-(q)$ implies $p = q$.*

In [Min08d] and [Min08c] it was shown that distinction of a spacetime implies feeble distinction and feeble distinction, in turn, implies non-imprisonment.

The next stronger condition on the causal ladder is non–partial-imprisonment, which was shown to be stronger than distinction, but weaker than strong causality defined below, in [Min08c].

For the next step on the causal ladder there are several equivalent formulations. We will omit most of them here as this step of the causal ladder is scarcely needed in this work and will only give the formulation which can be cast into the shortest form, although this is non-standard.

Definition 3.47. *A spacetime (M, g) is called strongly causal if the Alexandrov topology, generated by the base sets $B_{p,q} = \{I^+(p) \cap I^-(q) \,|\, p, q \in M\}$, equals the manifold topology.*

Furthermore, there are some more subtle steps on the causal ladder around distinction and strong causality (see, e.g., [Min09b], [Min08a], [Min08b]), which will be omitted here, as they are not relevant for the causality conditions of kinematical spacetimes analyzed in chapter 4. In [MS08], it was shown that the next condition on the causal ladder is stronger than strong causality.

Theorem & Definition 3.48. *A spacetime (M, g) is called stably causal if it fulfills one of the following equivalent properties.*

(i) There is a time function on (M, g), i.e., a continuous function $t\colon M \to \mathbb{R}$, which is strictly increasing along any future-directed causal curve.

(ii) There is a temporal function on (M, g), i.e., a smooth function $\tau\colon M \to \mathbb{R}$ with past-directed timelike gradient $\nabla\tau$.

In fact, the equivalence of items *(i)* and *(ii)* above has only been rigorously established a few years ago (see, e.g., [BS05]). The following step of the causal ladder will be of special importance in this work.

Definition 3.49. *A spacetime (M, g) is called causally continuous if it is reflecting and feebly distinguishing.*

Usually (cf. [MS08]), causal continuity was defined as a spacetime being reflecting and *distinguishing*. In [Min08d], it was proven that the assumption can be relaxed to feeble distinction. Causal continuity is stronger than stable causality, as shown in [MS08].

The following two last and strongest causality conditions are usually connected to the completeness of some properly constructed Riemannian or Finslerian metric on (hypersurfaces in) the spacetime under consideration (see chapters 4 and 5).

Definition 3.50. *A spacetime (M, g) is called causally simple if it is causal and $J^+(p)$, $J^-(p)$ are closed sets for all $p \in M$.*

Definition 3.51. *A spacetime (M, g) is called globally hyperbolic if it is causal and $J^+(p) \cap J^-(q)$ are compact sets for all $p, q \in M$.*

We again refer to [MS08] for the implication global hyperbolicity $\Rightarrow$ causal simplicity $\Rightarrow$ causal continuity. We will also need the following

Theorem 3.52. *A spacetime (M, g) is globally hyperbolic if and only if it is causal and the space $C(p, q)$ of all continuous and future-directed causal curves connecting the events p and q in M is compact in the C^0-topology.*

Proof. See [MS08, Thm. 3.79]. □

The C^0-topology is the simplest topology one can impose on the space $C(p, q)$ and is given by considering open neighborhoods of the curves in the spacetime.

Moreover, we will need the characterization of global hyperbolicity in terms of Cauchy hypersurfaces. We have the following

Definition 3.53. *A Cauchy hypersurface S in a spacetime (M, g) is an achronal, topological submanifold $S \subset M$, such that every inextendible timelike curve in M intersects S.*

Often one is interested not in general Cauchy surfaces, but in smooth and spacelike ones, which are then acausal smooth hypersurfaces.

Theorem 3.54. *A spacetime (M, g) is globally hyperbolic if and only if there is a (smooth spacelike) Cauchy hypersurface in (M, g).*

Proof. This theorem has originally been proven by R. Geroch in [Ger70], without the requirements of the Cauchy hypersurface to be smooth and spacelike, and only recently the problems of smoothability and the existence of spacelike Cauchy hypersurfaces were solved (see [MS08, Thm. 3.78] for an overview and the references therein for details). □

We conclude this section with a diagram that summarizes the steps on the causal ladder which are most relevant for this work:

$$\begin{tikzcd}
\text{globally hyperbolic} \arrow[d, Rightarrow] \\
\text{causally continuous} \arrow[d, Rightarrow] \\
\text{stably causal} \arrow[d, Rightarrow] \\
\text{non–partially-imprisoning} \arrow[d, Rightarrow] \\
\text{feebly distinguishing} \arrow[d, Rightarrow] \arrow[uuu, bend left=60, "\text{reflectivity}"] \\
\text{non-imprisoning} \arrow[d, Rightarrow] \\
\text{causal} \arrow[d, Rightarrow] \\
\text{chronological} \arrow[d, Rightarrow] \\
\text{non–totally-vicious}
\end{tikzcd}$$

Chapter 4

DIFFEOMORPHIC SPLITTING OF LORENTZIAN $\mathbb{R}$-MANIFOLDS

In the first section 4.1 of this chapter, we will derive the main diffeomorphic splitting result for kinematical spacetimes by the use of the theory of $\mathbb{R}$-actions, principal bundles and flows established in chapter 2. Hence, we investigate the conditions under which a kinematical spacetime has the topology of $\mathbb{R} \times S$, i.e., it is diffeomorphic to $\mathbb{R} \times S$, with S being the space of flow lines of the reference frame. Subsequently, we can derive some interesting propositions, for example, on the existence of Lorentzian metrics that fulfill the preconditions for a splitting on particular manifolds, or on the question of the local and global existence of a spacelike codimension one foliation of the kinematical manifold. This question provides a smooth transition to section 4.2, where we will analyze the causality conditions of such splitting kinematical spacetimes in greater detail. Finally, in section 4.3 we will consider the causality conditions of some low dimensional splitting kinematical spacetimes, which possess a particularly close connection between topology and causality if the space of flow lines S is compact.

The general problem of any splitting theorem can be stated as follows: *When does a given splitting $TM = \xi \oplus \nu$ of the tangent bundle TM of a manifold M, yield a splitting of the manifold $M = X \times N$ (such that $TX = \xi$ and $TN = \nu$) and when does this, subsequently, also yield the splitting of a semi-Riemannian metric g on M, such that $g = s + h$, with s and h a (family of) semi-Riemannian metrics on X and N, respectively?* The first question only asks for a diffeomorphic splitting of the manifold, without requiring any conditions for the metric. So it is conceivable that, in general, it should be possible to formulate conditions for this type of splitting without referring to a metric at all, but specific properties of the metric will turn out to be exactly the adequate conditions for the splitting in particular cases. This is precisely what we will establish in Thm. 4.13, Prop. 4.14 and Thm. 4.19 below for the diffeomorphic splitting of kinematical spacetimes. Furthermore, the splitting of Lorentzian manifolds as $M = \mathbb{R} \times S$ is a natural condition to ask for, as all Lorentzian manifolds naturally come with a splitting of the tangent bundle $TM = \xi \oplus \nu$ (cf. Rem. 3.6). The second splitting question also asks for a global splitting of the metric, which amounts to $g = -A^2 \mathrm{d}t^2 + h_t$ in the case of a Lorentzian manifold $(\mathbb{R} \times S, g)$, with A some positive function on $\mathbb{R} \times S$ and h_t a family of Riemannian metrics on S varying with $t \in \mathbb{R}$. Usually this question is asked in spacetimes, i.e., without *fixing* a specific reference frame, and is then connected to conditions on causality or completeness. One example for such a splitting is the requirement of global hyperbolicity as in Thm. 5.4. Global hyperbolicity or geodesic completeness are exactly the conditions that are necessary in the Lorentzian version of the Cheeger–Gromoll splitting theorem, together with positive timelike Ricci curvature and the existence of a timelike line (see, e.g., [Gal89]). In this case one even gets a splitting of the metric as $g = -\mathrm{d}t^2 + h$ on $\mathbb{R} \times S$ with a fixed Riemannian metric h on S. If one adopts the viewpoint of a kinematical spacetime ($M = \mathbb{R} \times S, g, V$), certainly a splitting of g as $g = -A^2 \mathrm{d}t^2 + h_t$ requires the reference frame V to have vanishing rotation if V is parallel to ∂_t. The analysis of a diffeomorphic splitting in the case of an irrotational vector field V is indeed a special case included in the investigation in this work. This particular case was intensively examined in [GO03] and [GO09].

Hence, we will assume a diffeomorphic splitting $M = \mathbb{R} \times S$ along the—generally rotating—flow lines of a complete reference frame in a kinematical spacetime (M, g, V), and in the first step, we will not be interested in a splitting of the metric. This approach complies with the one taken in [JS08] and [Har92] for spacetimes with a complete (conformal) Killing vector field. But we will generalize it to arbitrary complete reference frames, i.e., kinematical spacetimes and will not use any assumptions on the causality of the spacetime. This in turn will allow us to analyze the causality of arbitrary splitting spacetimes in section 4.2 independently, without having to assume a specific causality condition in order to achieve the splitting.

4.1 Topology of Kinematical Spacetimes

Theorem 4.1. *Let (M, g, V) be a kinematical spacetime. Then M is a Cartan $\mathbb{R}$-manifold with $\mathbb{R}$-action induced by the flow of V if and only if the flow lines of V are non–partially-imprisoned curves.*

Proof. Applying Prop. 3.39 to the integral curves of V yields the fact that every integral curve has empty limit sets if and only if it is non–partially-imprisoned. Thus, regarding the kinematical manifold (M, V) as a dynamical system with the global flow Φ induced by V on M implies that $\Omega(\Phi) = \emptyset$ if and only if all integral curves of V are non–partially-imprisoned. Now we can apply Prop. 2.46 to this dynamical system and deduce that M is a Cartan $\mathbb{R}$-manifold if and only if all integral curves of V are non–partially-imprisoned. □

There are two interesting simple consequences of Thm. 4.1.

Corollary 4.2. *Every kinematical spacetime (M, g, V) with Cartan flow of V is non-compact.*

Proof. If M was compact, every flow line of V would be imprisoned in the compact set M. □

Corollary 4.3. *Assume there is a kinematical scalar invariant Θ, $|\sigma|^2$, $|\omega|^2$ or $|\dot{u}|^2$, which is unbounded and monotonic along all flow lines of V in a kinematical spacetime (M, g, V), then the flow is Cartan.*

Proof. Assume the flow of V is not Cartan. Thus, there is at least one $p \in M$ and a flow line $\gamma_p(t)$ through p, which is partially future or past imprisoned in a compact set K. Let $F \in \{\Theta, |\sigma|^2, |\omega|^2, |\dot{u}|^2\}$ be the scalar kinematical invariant that is unbounded and monotonic along $\gamma_p(t)$, that is the function $F \circ \gamma_p \colon \mathbb{R} \to \mathbb{R}$ is unbounded and monotonically increasing or decreasing. But as there is a sequence $\{t_n\}_{n\in\mathbb{N}}$, such that $\gamma_p(t_n) \subset K$, the sequence $(F \circ \gamma_p)(t_n)$ is bounded, which is the desired contradiction. □

Remark 4.4. *Of course, the corollary above is true for any unbounded function along the flow lines. But in a kinematical spacetime the complete vector field V provides us with a natural set of functions $\{\Theta, |\sigma|^2, |\omega|^2, |\dot{u}|^2\}$ along the flow lines to test for being unbounded.*

Thus, kinematical spacetimes (M, g, V) with a Cartan flow induced by the vector field V are exactly those kinematical spacetimes for which the flow lines of V are non–partially-imprisoned. Based on the results in chapter 2, we see that these spacetimes have a quite regular structure. They are generalized $\mathbb{R}$-principal bundles over a base $M/_{\mathbb{R}}$ (which can be non-Hausdorff), and thus, in particular, they are locally trivial. Non–partial-imprisonment is a considerably weaker assumption than chronology, which is usually made whenever the theory of $\mathbb{R}$-manifolds is used to obtain Lorentzian splitting results (see [Har92] and [JS08]). Of course, chronology ensures for any kinematical spacetime (M, g, V) to have Cartan flow. But in this case one leaves unregarded a whole interesting class of kinematical spacetimes which do possess CTCs, that are created by peculiarities of the Lorentzian metric g and not by topological obstructions of the spacetime manifold M. Thus, kinematical spacetimes admitting a Cartan flow are optimal to study CTCs that are brought about by metric properties. We could even define a CTC to be non-trivial if it occurs in a kinematical spacetime with Cartan flow. Note that this definition of non-trivial CTCs differs from the classical one given by B. Carter in [Car68]. Therein, a CTC is defined to be non-trivial if it is homotopic to a point and trivial if this is not the case. A trivial CTC, therefore, could be dissolved in this definition by passing to the universal covering spacetime. As the example below shows, our definition also includes CTCs as non-trivial that would be excluded by B. Carter's definition. In our setup of kinematical spacetimes this definition would be too limited. We want to examine the topology of a kinematical spacetime, with fixed reference frame field and not its universal covering manifold. We assume a fixed reference frame field V and a Lorentzian metric that can be altered—while keeping V a timelike reference frame—to avoid or create CTCs. Hence, the CTCs in kinematical spacetimes occur due to a specific choice of the Lorentzian metric. As the examples below show, there are indeed CTCs in a kinematical spacetime with Cartan flow that are not homotopic to a point, but which occur due to a specific choice of the Lorentzian metric. Furthermore, B. Carter's definition of non-trivial CTCs include for example CTCs occurring in a compact spacetime $(\mathbb{S}^3, g)$, as any closed curve on the 3-sphere is homotopic to a point. But from the viewpoint of Lorentzian geometry these CTCs occur *trivially* as the spacetime manifold $\mathbb{S}^3$ is compact.

Example 4.5. *Consider $\mathbb{R}^2$ with global coordinates $(t, x) \in \mathbb{R}^2$. We wind up $\mathbb{R}^2$ to a cylinder $\mathbb{S}^1 \times \mathbb{R}$ by the identification $(t, x) \sim (t + 1, x)$. Now we can make different choices of global complete vector fields and associated Lorentzian metrics. Let us consider*

(i) $V_1 = \partial_t$ *and* $g_1 = -\mathrm{d}t^2 + \mathrm{d}x^2$,

(ii) $V_2 = \partial_t + \frac{1}{2}\partial_x$, $g_{2a} = \frac{4}{3} g_1$, $g_{2b} = -4\mathrm{d}t\mathrm{d}x + 4\mathrm{d}x^2$ *and* $g_{2c} = 2\mathrm{d}t^2 - 10\mathrm{d}t\mathrm{d}x + 8\mathrm{d}x^2$ *as well as*

(iii) $V_3 = \partial_t + x^2\partial_x$ *and* $g_3 = -(1 + x^4)\mathrm{d}t^2 + \mathrm{d}x^2$.

Obviously, the vector fields given are complete and are normed to -1 *with respect to all the metrics associated to them. Thus, in all cases, we have a kinematical spacetime structure on the cylinder.*

In case (i), however, V_1 *obviously has the circles* $\mathbb{S}^1$ *as flow lines, which are CTCs and the corresponding* $\mathbb{R}$*-action on* $M = \mathbb{S}^1 \times \mathbb{R}$ *has the isotropy subgroup* $\mathbb{Z}$*. In this case, there is actually an action by the compact group* $\mathbb{R}/\mathbb{Z} = \mathbb{S}^1$ *on* M *and, of course,* M *is not a Cartan* $\mathbb{R}$*-manifold. The flow lines themselves are compact and they are, therefore, naturally future and past imprisoned.*

Things are completely different if we keep the metric conformally fixed and tilt the vector field a little, as it is the case in (ii) with the complete vector field V_2 *and the first metric* g_{2a}*. Any integral curve of the vector field* ∂_t *is still a CTC, but the vector field* V_2 *gives rise to to Cartan* $\mathbb{R}$*-action on* M*, as its flow lines wind up the cylinder towards* $\pm x$*-infinity. Thus, they leave any compact set and are future and past non-imprisoned. The same is true for the metrics* g_{2b} *and* g_{2c}*, but in the case of* g_{2b} *the vector field* ∂_t *is lightlike, so the resulting spacetime is chronological but non-causal. For the metric* g_{2c}*, the vector field* ∂_t *becomes spacelike and the spacetime is causal. These three choices of the metric show that—although the CTCs or CCCs, that may occur, are non-homotopic to a point and would in fact disappear if we unwind the cylinder, which is equal in this case to pass to the universal covering manifold—their appearance crucially depends on the choice of the Lorentzian metric on* $\mathbb{S}^1 \times \mathbb{R}$*, and it is not just a result from topological obstructions. Moreover, in these three subcases of (ii), the* $\mathbb{R}$*-action induced by* V_2 *is even proper and the space of flow lines is homeomorphic to* $\mathbb{S}^1$*.*

The case (iii) has the integral curve of $V_3|_{x=0} = \partial_t|_{x=0}$ *as a flow line of the complete vector field* V_3*. This flow line equals the orbit* $\mathbb{R}p_0$ *for any* $p_0 = (t_0, 0) \in \mathbb{S}^1 \times \mathbb{R}$*, it is compact and, therefore, future and past imprisoned. Hence, the flow of* V_3 *is not Cartan in this case. But the flow lines corresponding to the orbits* $\mathbb{R}p_-$ *and* $\mathbb{R}p_+$ *for all* $p_- = (t, -c^2)$ *and* $p_+ = (t, c^2)$ *are either only future imprisoned—in the case of* $\mathbb{R}p_-$ *in any compact subset* $\mathbb{S}^1 \times [-\varepsilon^2, 0]$*—or past imprisoned—in the case of* $\mathbb{R}p_+$ *in any compact subset* $\mathbb{S}^1 \times [0, \varepsilon^2]$*. The orbit space is not a* T_1*-space in this case, as it can be easily seen that the orbit* $\mathbb{R}p_0$ *cannot be separated from the neighborhoods of* $\mathbb{R}p_-$ *and* $\mathbb{R}p_+$*.*

In Thm. 4.32 below, we will give a complete causal classification of Lorentzian cylinder spacetimes, which will include the cases in the example above.

Example 4.6. *The standard example of a topologically trivial spacetime providing non-trivial CTCs is the Gödel spacetime. Consider* $\mathbb{R}^3$ *with global coordinates* (t, x, y) *and vector field* $V = \partial_t$*. Then, obviously,* ∂_t *is complete and its flow lines cause a proper* $\mathbb{R}$*-action on* $\mathbb{R}^3$*, with the space of orbits homeomorphic to* $\mathbb{R}^2$*. This holds for any Lorentzian metric on* $\mathbb{R}^3$*. We consider the Gödel metric on* $\mathbb{R}^3$ *given by*

$$g_{\ddot{o}} = -\mathrm{d}t^2 + 2e^x \mathrm{d}y\mathrm{d}t + \mathrm{d}x^2 - \frac{1}{2}e^{2x}\mathrm{d}y^2.$$

Thus, $(\mathbb{R}^3, g_{\ddot{o}}, \partial_t)$ *is a proper kinematical spacetime (see Def. 4.7 below). CTCs occur because the geometry induced on the global* $t = \mathrm{const}$ *slices, homeomorphic to the orbit space, is not*

Riemannian. Thus, these slices are not spacelike when regarded as embedded submanifolds in $(\mathbb{R}^3, g_{ö}, \partial_t)$ and CTCs occur.

As Example 3.17 shows, the non–partial-imprisonment of the flow lines of the vector field V in a kinematical spacetime (M, g, V) does make M a Cartan $\mathbb{R}$-manifold, but it is not enough to make it proper. Thus we give the following

Definition 4.7. *A kinematical spacetime (M, g, V) will be called Cartan kinematical spacetime if the $\mathbb{R}$-action induced by the flow of V makes M a Cartan $\mathbb{R}$-manifold and it will be called proper kinematical spacetime if the $\mathbb{R}$-action induced by the flow of V is proper on M.*

Remark 4.8. *Having at hand the definition above, and looking back on Prop. 3.12 and Prop. 3.16, we can now state the following. Given a spacetime (M, g) and a fixed timelike vector field $V : M \to TM$, there is a complete timelike vector field $V^* : M \to TM$ parallel to V and a Lorentzian metric g^* conformal to g, such that (M, g^*, V^*) is a kinematical spacetime. Hence, if the integral curves of V are non–partially-imprisoned, (M, g^*, V^*) is a Cartan kinematical spacetime. This implies that all results on Cartan kinematical spacetimes in this, and the following, chapters below, which are conformally invariant—or do only depend on the differentiable and not on the metric structure of (M, g)—hold also for spacetimes with a fixed global timelike vector field with non–partially-imprisoned integral curves.*

Below we will make use of the following lemma by A.W. Wadsley.

Lemma 4.9. *Let (M, g) be a Riemannian (Lorentzian) manifold and $K : M \to TM$ a nowhere vanishing (timelike) Killing vector field, i.e., $\pounds_K g = 0$. There is a conformal factor $f : M \to \mathbb{R}_{>0}$, given by $f(x) = [g_x(K_x, K_x)]^{-1}$ ($f(x) = [-g_x(K_x, K_x)]^{-1}$), such that for the conformally transformed metric $\tilde{g} = fg$, it holds that $\pounds_K \tilde{g} = 0$, $\tilde{g}(K, K) = 1$ ($\tilde{g}(K, K) = -1$) and K is $\tilde{g}$-geodesic.*

Proof. See [Wad75, Lem. 3.1] for the Riemannian case. The Lorentzian case is completely analogous. □

The following proposition is a generalization of known theorems for stationary and conformally stationary spacetimes [Har92] and [JS08], in which the spacetime is required to be chronological to ensure the Cartan condition. But that is not necessary and we see that the non–partial-imprisonment of the the flow lines of V is enough to ensure the Cartan property, so that spacetimes with possible chronology violations can be included in this notion. The proof in [JS08] shows that $M/_{\mathbb{R}}$ is Hausdorff if V is Killing. We will give a slightly different proof here using item *(iv)* from Thm. 2.35.

Proposition 4.10. *Let (M, g, V) be a Cartan kinematical spacetime. If V is Killing, i.e., $\pounds_V g = 0$, then (M, g, V) is a proper kinematical spacetime. Particularly, every global trivialization ψ of the $\mathbb{R}$-principal bundle associated to (M, g, V), is an isometry $\psi : (M, g) \to (\mathbb{R} \times S, \psi_* g)$ with S diffeomorphic to $M/_{\mathbb{R}}$ and $\psi_* V = \partial_t$. Furthermore, denoting by $t : \mathbb{R} \times S \to \mathbb{R}$ the projection on the first factor, which depends on the choice of ψ, and* *the canonical projection by* $\mathrm{pr}_2\colon \mathbb{R}\times S\to S$*, there is a one-form* $b\in\Gamma(\Lambda^1 S)$ *and a symmetric 2-tensor field* $\gamma\in\Gamma(\Sigma^2 S)$*, such that*

$$\psi_* g = -\mathrm{d}t\otimes \mathrm{d}t + 2\mathrm{pr}_2^*(b)\vee \mathrm{d}t + \mathrm{pr}_2^*(\gamma).$$

Proof. By setting $g_0 = g + 2g(V,\cdot)\otimes g(V,\cdot)$ we get a Riemannian metric g_0 on M with V being Killing for g_0, too. This can be seen by computing

$$\pounds_V g_0 = \pounds_V(g + 2g(V,\cdot)\otimes g(V,\cdot)) = 4g([V,V],\cdot)\vee g(V,\cdot) = 0.$$

Hence, there is a distance d_{g_0} associated to g_0 such that $d_{g_0}(p,q) = d_{g_0}(\Phi_t(p),\Phi_t(q))$ for all $p,q\in M$ and all $t\in\mathbb{R}$, where Φ is the Killing flow associated to V. We will now show that for all compact $K\subset M$, the set $((K,K))\subset\mathbb{R}$ is also compact, thus (M,g,V) is a proper kinematical spacetime by Thm. 2.35 and Def. 4.7. From Lem. 2.25 we know that $((K,K))$ is closed if K is compact, so we only have to show that $((K,K))$ is bounded. Set $c_K := \mathrm{diam}_{d_{g_0}}(K) := \max\{d_{g_0}(r,s)\,|\, r,s\in K\}$ for any compact set K. Then, as V is Killing, we have

$$c_{t\circ K} = \mathrm{diam}_{d_{g_0}}(t\circ K) = c_K,$$

for all $t\in\mathbb{R}$. Hence, it certainly holds that $((t\circ K, K)) = \emptyset$ for all t with $|t| > 2c_K$, such that we have $((K,K))\subset[-2c_K, 2c_K]$.

As a proper kinematical spacetime, (M,g,V) is in particular an $\mathbb{R}$-principal bundle over $M/_{\mathbb{R}}$, and by virtue of a diffeomorphism we can set $S = M/_{\mathbb{R}}$, i.e., there is a trivialization $\psi\colon M\to\mathbb{R}\times S$. There is a one-to-one correspondence between (global) trivializations and (global) sections of the principal bundle (M,S,π_M), i.e., we can associate a global section $\sigma_0\colon S\to M$ to any trivialization ψ, such that the saturation $\{t\circ\sigma_0(S) =: \sigma_t(S)\,|\, t\in\mathbb{R}\} = M$. Then for any $p\in M$ with $\psi(p) = (t,x)$, we have that $p\in\sigma_t(S)$ and $x = \pi_M(p)$.

Hence, the projection $t = \mathrm{pr}_1$ depends on the choice of the trivialization ψ, whereas the canonical projection pr_2 is the same for all trivializations ψ. We denote by $S_t := \{t\}\times S = \psi(\sigma_t(S))\subset\mathbb{R}\times S$ the slices associated to the global sections $\sigma_t(S)$.

Now we set $\tilde\gamma = \psi_* g|_{S_t}$ for any $t\in\mathbb{R}$ and conclude that $\pounds_{\psi_* V}\tilde\gamma = 0$, as $\psi_* V$ is transversal to S_t and Killing for $\psi_* g$, hence $\tilde\gamma$ is the pull-back of some symmetric 2-tensor field γ on S. In the same way we conclude that $\tilde b = (\psi_* g)(\psi_* V,\cdot)|_{S_t}$ is the pull-back of some one-form b on S. Furthermore, this implies that ψ_* maps V to a multiple of ∂_t, say $\tilde A^{-1}\partial_t$ with $\tilde A = A\circ\mathrm{pr}_2$ for some function $A\colon S\to\mathbb{R}_{>0}$. As $g(V,V) = (\psi_* g)(\tilde A^{-1}\partial_t, \tilde A^{-1}\partial_t) = -1$, we have that $\psi_* g = -\tilde A^2\mathrm{d}t\otimes\mathrm{d}t + 2\mathrm{pr}_2^*(b)\vee\mathrm{d}t + \mathrm{pr}_2^*(\gamma)$ on $\mathbb{R}\times S$. But as $\tilde A^{-1}\partial_t$ is still Killing for $\psi_* g$ a short computation of the formula $\pounds_{\psi_* V}\psi_* g = 0$ yields $\mathrm{d}\tilde A = 0$ and the result follows. $\square$

Using Lem. 4.9 this result can be generalized to complete timelike Killing vector fields of arbitrary norm.

Corollary 4.11. *Let* (M,g) *be a stationary spacetime with complete timelike Killing vector field* $K\colon M\to TM$*, such that no integral curve of* K *is partially imprisoned. Then* (M,g) *is isometric to a spacetime* $(\mathbb{R}\times S, g^*)$ *with the Killing vector field given by* $K^* = \partial_t$ *on* $\mathbb{R}\times S$ *and*

$$g^* = -(A\circ\mathrm{pr}_2)^2\mathrm{d}t\otimes\mathrm{d}t + 2\mathrm{pr}_2^*(b)\vee\mathrm{d}t + \mathrm{pr}_2^*(\gamma),$$

such that $b \in \Gamma(\Lambda^1 S)$ is a one-form on S, $\gamma \in \Gamma(\Sigma^2 S)$ is a symmetric 2-tensor field and $A\colon S \to \mathbb{R}_{>0}$.

Proof. Applying Lem. 4.9 to the spacetime (M, g) and the Killing vector field K, yields a Cartan kinematical spacetime (M, g', K) with $g' = [g(K,K)]^{-1} g$. Now we apply Prop. 4.10 to (M, g', K) and get a Killing vector field $K^* = \psi_* K = \partial_t$, as well as

$$\psi_* g' = -\mathrm{d}t \otimes \mathrm{d}t + 2\mathrm{pr}_2^*(b') \vee \mathrm{d}t + \mathrm{pr}_2^*(\gamma')$$

on $\mathbb{R} \times S$ for any trivialization ψ. Next we note that $\nabla_K [g(K,K)] = 0$ as K is Killing. Hence, there is a function $A\colon S \to \mathbb{R}_{>0}$, such that $\psi_*[g(K,K)] = A \circ \mathrm{pr}_2$. Thus, $\psi_* g = (A \circ \mathrm{pr}_2)^2 \psi_* g'$ and by setting $b = A^2 b'$, as well as $\gamma = A^2 \gamma'$ the result follows. $\square$

Using Rem. 4.8, the corollary above can be generalized to a complete timelike *conformal* vector field $K\colon M \to TM$ as in this case K is Killing for $g^* = -[g(K,K)]^{-1} g$ and has unit norm with respect to g^*, which can be inferred similar to Lem. 4.9.

A main generalization to the splitting results in [Har92] and [JS08] is now that the metric γ on S is allowed to be semi-Riemannian or even degenerate. With this at hand, the question of the Riemannian nature of γ is intimately tied to the question of causality of (M, g) only, whereas the question of the topological and metric splitting of the kinematical spacetime is liberated from all causality assumptions. The stationary splitting results of Prop. 4.10 and Cor. 4.11 are now naturally applicable to non-chronological spacetimes, such as the totally vicious Gödel spacetime in Example 4.6, or more sophisticated examples like the torus spacetime in the following example. This example is based on B. Carter's classical one in [HE73, Fig. 39, p. 195], and one version also appears in [JS08].

Example 4.12. *Let the two-torus $\mathbb{T}^2$ be given as the coordinate patch $(x, y) \in [0,1] \times [0,1] \subset \mathbb{R}^2$ with the identifications $(0, y) \sim (1, y)$ for all $y \in [0,1]$ and $(x, 0) \sim (x + \sqrt{2}, 1)$. As in Example 2.28 there is a vector field—$\partial_y = (0,1)^T$ in this case—the orbits of which are dense in T^2 as $\sqrt{2}$ is irrational. We will now consider the spacetime $(\mathbb{R} \times \mathbb{T}^2, g)$ with a Lorentzian metric g, such that the vector field $K := (0, \partial_y)^T$ on $\mathbb{R} \times \mathbb{T}^2$ is lightlike for some, or even all, $t \in \mathbb{R}$. Such a metric is given by*

$$g = -\mathrm{d}t^2 + 2\mathrm{d}t\mathrm{d}y + \mathrm{d}x^2.$$

Moreover, $(\mathbb{R} \times \mathbb{T}^2, g)$ is stationary with Killing vector field ∂_t, and every $t = \text{constant}$ slice $\{t\} \times \mathbb{T}^2$ contains imprisoned, but non-closed, lightlike curves, the orbits of K, i.e., the spacetime is causal but causally imprisoning. One could even modify this example to contain only one slice, say for $t = 0$, with imprisoned lightlike curves. Let $s\colon \mathbb{R} \to \mathbb{R}$ be a function with $s(0) = 1$ and $s(t) > 1$ for $t \neq 0$, e.g., $s(t) = t^2 + 1$. Then the metric

$$g' = -\mathrm{d}t^2 + 2\mathrm{d}t\mathrm{d}y + s(t)(\mathrm{d}x^2 + \mathrm{d}y^2) - \mathrm{d}y^2$$

on $\mathbb{R} \times \mathbb{T}^2$ fulfills the required conditions. The spacetime $(\mathbb{R} \times \mathbb{T}^2, g')$ is an example of a Hubble-isotropic spacetime, which we will study in section 6.1.

We are now ready to give conditions for a Cartan kinematical spacetime (M, g, V) to be proper, which are much more general and do not depend on the existence of a Killing vector field parallel to V. Hence, the theorem below can be seen as the most general splitting result along V, for a given Cartan kinematical spacetime (M, g, V).

Theorem 4.13. *Let (M, g, V) be a Cartan kinematical spacetime. Then the following statements are equivalent:*

(i) (M, g, V) *is a proper kinematical spacetime.*
(ii) *There is a one-form $\tilde{u}$ on M, such that $V \rfloor \mathrm{d}\tilde{u} = 0$ and $\tilde{u}(V) = 1$.*
(iii) *M is an $\mathbb{R}$-principal bundle over $S = M/_{\mathbb{R}}$, the space of flow lines of V.*
(iv) *There is a Riemannian metric $\tilde{g}$ on M, such that $\tilde{g}(V, V) = 1$, $\pounds_V \tilde{g} = 0$ and V is $\tilde{g}$-geodesic.*
(v) *There is a Lorentzian metric $\tilde{g}$ on M, such that $\tilde{g}(V, V) = -1$, $\pounds_V \tilde{g} = 0$ and V is $\tilde{g}$-geodesic.*

Proof. "*(iv)⇔(v)*": If $\tilde{g}$ is Riemannian, we set $\tilde{u} = \tilde{g}(V, \cdot)$, and then by $\tilde{g}(V, V) = 1$ it follows that $\tilde{h} = \tilde{g} - 2\tilde{u} \otimes \tilde{u}$ is Lorentzian. If $\tilde{g}$ is Lorentzian the same setting yields that $\tilde{h} = \tilde{g} + 2\tilde{u} \otimes \tilde{u}$ is Riemannian. Furthermore, in both cases we have

$$\pounds_V \tilde{g} = \pounds_V(\tilde{h} \pm 2\tilde{u} \otimes \tilde{u}) = \pounds_V \tilde{h} = 0$$

because of $\pounds_V \tilde{u} = \pm \dot{\tilde{u}}_{\tilde{g}} = \pm \tilde{g}(\nabla^{\tilde{g}}_V V, \cdot) = 0$, as V is geodesic. For the acceleration $\dot{\tilde{u}}$, we get in both cases

$$\dot{\tilde{u}}_{\tilde{g}} = \pm \pounds_V(\tilde{g}(V, \cdot)) = \pounds_V(\tilde{h}(V, \cdot) \pm 2\tilde{u}(V)\tilde{u}) = \pm \pounds_V(\tilde{h}(V, \cdot)) = \pm \dot{\tilde{u}}_{\tilde{h}} = 0.$$

Hence, V is $\tilde{g}$-geodesic if and only if it is $\tilde{h}$-geodesic.

"*(v)⇒(i)*": This is an application or Prop. 4.10.

"*(i)⇔(iii)*": This is an application of Prop. 2.53.

"*(iii)⇔(ii)*": Using Prop. 2.31, we conclude that M is a generalized $\mathbb{R}$-principal bundle and by Prop. 2.33 M is an $\mathbb{R}$-principal bundle if and only if it admits a connection, say $\tilde{u}$ obeying $\tilde{u}(sV) = s$ for all $s \in \mathbb{R}$ and $\Phi_t^* \tilde{u} = \tilde{u}$ for all $t \in \mathbb{R}$, where Φ is the global flow associated to the complete vector field V. Hence, these two conditions are equivalent to $\tilde{u}(V) = 1$ and $\pounds_V \tilde{u} = \lim_{t \to 0} \frac{\Phi_t^* \tilde{u} - \tilde{u}}{t} = 0$, which in turn is equivalent to $V \rfloor \mathrm{d}\tilde{u}$ and $\tilde{u}(V) = 1$ by the formula $\pounds_V \tilde{u} = V \rfloor \mathrm{d}\tilde{u} + \mathrm{d}(V \rfloor \tilde{u})$.

"*(iii)⇒(iv)*": Using Lem. 4.9 we only need to find a Riemannian metric g' on M for which V is Killing. As condition *(iii)* holds, we know that M is trivializable as a principal bundle, i.e., there is $\psi\colon M \to \mathbb{R} \times S$. So we set $\psi_* g' = \mathrm{d}t \otimes \mathrm{d}t + h$ on $\mathbb{R} \times S$ for an arbitrary Riemannian metric h on S. Then we have $\pounds_{\partial_t} \psi_* g' = 0$, as ∂_t is Killing for $\psi_* g'$. But the trivialization is an isometry for the chosen metric and the metric $g' = \psi_*^{-1} \psi_* g'$ on M, such that we have $\pounds_V g' = 0$ for $V = \psi^* \partial_t$, thus V is Killing for g'. □

Given a kinematical spacetime (M, g, V), we will now connect the diffeomorphic splitting properties above to characteristics of the dynamical system (M, Φ) associated to the spacetime by the complete vector field V. Specifically, we will consider the instability of this dynamical system and the existence of improper saddle points, hence its parallelizability. Thus, we will arrive at the same splitting result from a different perspective.

Proposition 4.14. *Let (M, g, V) be a kinematical spacetime and Φ the global flow associated to V, such that (M, Φ) is a dynamical system. If the integral curves of V are non-imprisoned, which is equivalent to (M, Φ) being unstable, and (M, Φ) has no improper saddle point, (M, g, V) is a proper kinematical spacetime.*

On the other hand, if (M, g, V) is a proper kinematical spacetime, then the associated dynamical system (M, Φ) has no improper saddle point.

Proof. By Def. 2.49 a dynamical system is parallelizable if it is unstable and has no improper saddle point. A dynamical system is unstable if no half-orbit is entirely contained in a compact set. Comparing this notion to future and past imprisonment from Def. 3.37 shows that the dynamical system (M, Φ) is unstable if and only if all integral curves of the vector field V are non-imprisoned. Hence, using Prop. 2.53, we can conclude that (M, g, V) is a proper kinematical spacetime.

Again by Prop. 2.53, we can infer that a proper kinematical spacetime (M, g, V) is associated to a parallelizable dynamical system (M, Φ), which is necessarily without an improper saddle point. □

Remark 4.15. *Inspired by Example 3.14, we would like to give some intuition of the meaning of improper saddle points in the framework of Cartan kinematical spacetimes. Obviously, the two-dimensional spacetime from Example 3.14 is not proper because of the removed origin of $\mathbb{R}^2$, which leads to a non-Hausdorff orbit space. In some sense, the removed origin is the improper saddle point in this case. Hence, the absence of improper saddle points signals the absence of holes in the spacetime. But note that improper saddle points are not always what we would intuitively consider a hole in spacetime. One could even modify Example 3.14 by removing the whole non-negative x-axis. We would get the branched line as orbit space, instead of the line with two origins, in this case. Hence, the absence of improper saddle points or the condition of properness assures that such situations as in Example 3.14 do not occur.*

This notion of hole-freeness depends not only on the spacetime (M, g), but also on the complete timelike vector field V, as it is linked with the dynamical system associated to V. Hence, it is different from other efforts to give a rigorous definition of hole-freeness, e.g., in [Man09]. Furthermore, a recent clarification of conditions for spacetimes to contain holes in [Min12], which is based on [Man09], uses Cauchy developments and causality conditions of the spacetime.

Given a kinematical spacetime (M, g, V), we can assess the splitting question from two different sets of conditions. On the one hand, based on Thm. 4.13, we need non–partial-imprisonment of the integral curves of V and some one-form on M that serves as a principal connection. On the other hand, based on Prop. 4.14, we need only non-imprisonment of the integral curves of V, which is a weaker condition than non–partial-imprisonment (cf. Def. 3.37). But additionally we need the slippery notion of improper saddle points.

Although Rem. 4.15 provides an interpretation of improper saddle points for some cases of kinematical spacetimes, the question of existence of improper saddle points is usually not solved straightforwardly for a given kinematical spacetime. So it is comforting that we can solve the question for improper saddle points in a dynamical system associated to some Cartan kinematical spacetime by differential geometric means, i.e., a principal connection one-form. As we will see in Thm. 4.19 below, such a principal connection is provided in many interesting situations by the Lorentzian metric and the reference frame V themselves.

We are now ready to give a generic formula for the Lorentzian metric of a splitting Lorentzian manifold, i.e., a proper kinematical spacetime.

Theorem 4.16. *Let (M, g, V) be a proper kinematical spacetime. Then every trivialization ψ of the $\mathbb{R}$-principal bundle associated to (M, g), is an isometry $\psi\colon (M, g) \to (\mathbb{R} \times S, \psi_* g)$, such that $\psi_* V = A^{-1}\partial_t$ for some function $A\colon \mathbb{R} \times S \to \mathbb{R}_{>0}$ and $S = M/_{\mathbb{R}}$.*

Furthermore, for every trivialization ψ there is a family of one-forms $\{b_t\}_{t\in\mathbb{R}} \subset \Gamma(\Lambda^1 S)$ and a family of Riemannian metrics $\{h_t\}_{t\in\mathbb{R}} \subset \mathcal{R}(S)$ on S, both varying smoothly with t, such that

$$\psi_* g_{(t,x)} = -A^2(t,x)\mathrm{d}t \otimes \mathrm{d}t + 2\mathrm{pr}_2^*(b_{(t,x)}) \vee \mathrm{d}t + \mathrm{pr}_2^*(h_{(t,x)}) - \frac{\mathrm{pr}_2^*(b_{(t,x)}) \otimes \mathrm{pr}_2^*(b_{(t,x)})}{A^2(t,x)},$$

with the canonical projection $\mathrm{pr}_2\colon \mathbb{R} \times S \to S$ and the projection $\mathrm{pr}_1 = t\colon \mathbb{R} \times S \to \mathbb{R}$ associated to the trivialization ψ.

Moreover, any other trivialization $\tilde{\psi}$ is given by a gauge transformation in the $\mathbb{R}$-principal bundle, i.e., by a transformation of the projection $t \mapsto \tau = t - t_0 + f(x)$ for any function $f\colon S \to \mathbb{R}$ and any fixed $t_0 \in \mathbb{R}$. The metric with respect to the trivialization $\tilde{\psi}$ is given by

$$\tilde{\psi}_* g_{(\tau,x)} = -A^2(\tau,x)\mathrm{d}\tau \otimes \mathrm{d}\tau + 2\mathrm{pr}_2^*(b_{(\tau,x)} + A^2(\tau,x)\mathrm{d}f_x) \vee \mathrm{d}\tau +$$
$$+\mathrm{pr}_2^*(h_{(\tau,x)}) - \frac{\mathrm{pr}_2^*(b_{(\tau,x)} + A^2(\tau,x)\mathrm{d}f_x) \otimes \mathrm{pr}_2^*(b_{(\tau,x)} + A^2(\tau,x)\mathrm{d}f_x)}{A^2(\tau,x)}.$$

Proof. As a trivialization ψ maps any fiber, i.e., any integral curve γ of V, to a line $\mathbb{R} \times \{x\} \subset \mathbb{R} \times S$ with $x = \pi_M(\gamma) \in S$, the push-forward $\psi_* V$ is parallel to ∂_t, i.e., there is a function $A\colon \mathbb{R} \times S \to \mathbb{R}_{>0}$, such that $\psi_* V = A^{-1}\partial_t$.

Denoting by $u = g(V, \cdot)$ the one-form metrically associated to V, there is a decomposition of $\psi_* u$ in $\mathbb{R} \times S$ given by

$$(\psi_* u)_{(t,x)} = -A(t,x)\mathrm{d}t + \frac{\beta_{(t,x)}}{A(t,x)},$$

for all $(t,x) \in \mathbb{R} \times S$, as $-1 = u(V) = (\psi_* u)(A^{-1}\partial_t)$. Here, $\beta_{(t,x)} \in T^*_{(t,x)}(\mathbb{R} \times S)$ is a one-form obeying $\beta(\partial_t) = 0$, hence β is pulled back from some one-form $b_{(t,x)} \in T^*_x S$, i.e., $\beta_{(t,x)} = \mathrm{pr}_2^*(b_{(t,x)})$. As this holds for all $(t,x) \in \mathbb{R} \times S$, there is a family of one-forms $\{b_t\}_{t\in\mathbb{R}} \subset \Gamma(\Lambda^1 S)$, such that

$$(\psi_* u) = -A\mathrm{d}t + \frac{\mathrm{pr}_2^*(b_t)}{A}.$$

Based on the analysis in section 3.1, we can decompose g using u and the projection h, i.e., we have $g = h - u \otimes u$, and thus

$$\psi_* g = \psi_* h - \psi_* u \otimes \psi_* u.$$

Certainly, we have $0 = h(V, \cdot) = (\psi_* h)(A^{-1}\partial_t, \cdot)$, and thus $(\psi_* h)_{(t,x)} \in \Sigma^2_{(t,x)}(\mathbb{R} \times S)$ is pulled back from some symmetric 2-tensor in $\Sigma^2_x S$, which we will also denote by h. As this holds for all $(t, x) \in \mathbb{R} \times S$, there is a family of symmetric 2-tensor fields $\{h_t\}_{t\in\mathbb{R}} \subset \Gamma(\Sigma^2 S)$, such that

$$\psi_* h = \mathrm{pr}_2^*(h_t).$$

Also from section 3.1, we know that h is a Riemannian metric on HM, hence $\psi_* h$ is a Riemannian metric on the bundle horizontal to ∂_t, which ensures the h_t to be Riemannian metrics on S. Hence, we have

$$(\psi_* g)_{(t,x)} = \mathrm{pr}_2^*(h_{(t,x)}) - \left(-A(t, x)\mathrm{d}t + \frac{\mathrm{pr}_2^*(b_{(t,x)})}{A(t, x)}\right) \otimes \left(-A(t, x)\mathrm{d}t + \frac{\mathrm{pr}_2^*(b_{(t,x)})}{A(t, x)}\right),$$

and the result follows. Certainly, b_t and h_t vary smoothly with t, as g is a smooth metric.

As (M, g, V) constitutes a globally trivial principal bundle, there is a one-to-one correspondence between global trivializations and gauge transformations as stated. We now only have to compute how $\psi_* g$ changes under the transformation $t \mapsto \tau(t, x) = t - t_0 + f(x)$. For computational simplicity, we will suppress pull-backs, push-forwards and dependencies on (t, x) in the following. We have

$$\mathrm{d}\tau = \mathrm{d}t + \mathrm{d}f,$$

hence

$$\mathrm{d}t \otimes \mathrm{d}t = \mathrm{d}\tau \otimes \mathrm{d}\tau - 2\mathrm{d}\tau \vee \mathrm{d}f + \mathrm{d}f \otimes \mathrm{d}f.$$

This yields

$$g = -A^2 \left(\mathrm{d}\tau \otimes \mathrm{d}\tau - 2\mathrm{d}\tau \vee \mathrm{d}f + \mathrm{d}f \otimes \mathrm{d}f\right) + 2b \vee (\mathrm{d}\tau - \mathrm{d}f) + h - \frac{b \otimes b}{A^2},$$

thus

$$g = -A^2 \mathrm{d}\tau \otimes \mathrm{d}\tau + 2\mathrm{d}\tau \vee (b + A^2 \mathrm{d}f) + h - \frac{b \otimes b}{A^2} - 2b \vee \mathrm{d}f - A^2 \mathrm{d}f \otimes \mathrm{d}f,$$

and the result follows. □

Using Rem. 4.8, the splitting form of the metric in the theorem above can be generalized to spacetimes (M, g), together with a complete timelike vector field V which induces a proper flow. Certainly, in this case, there is a Lorentzian metric g^* in the conformal class of g, such that (M, g^*, V) is a proper kinematical spacetime. Then the splitting form of the metric as above can be pulled back to (M, g) by the conformal transformation. We will make use of this in the sections and chapters below.

Remark 4.17. *In the following, if it is no source of confusion we will often omit the pull-backs when writing the metric in the splitting form, i.e., we will use the same symbol for objects on the base S and their pull-back to $\mathbb{R} \times S$, but occasionally we will indicate the dependence on $t \in \mathbb{R}$ by a subscript t. To every trivialization ψ, there corresponds a unique family of global sections $\{\sigma_t(S)\}_{t\in\mathbb{R}} \subset M$, which form a foliation of M by submanifolds diffeomorphic to S. The images of these submanifolds under ψ are given by $S_t := \{t\} \times S := \psi(\sigma_t(S))$ and we will refer to these submanifolds $S_t \subset \mathbb{R} \times S$ as slices, to the foliation $\{S_t\}_{t\in\mathbb{R}} \subset \mathbb{R} \times S$ as slicing and to a gauge transformation in the principal bundle, which corresponds to a change of the trivialization, as a change of slicing. In this sense, we can then talk about geometric objects on a slice as objects that are pulled back from S and then restricted to a specific $t \in \mathbb{R}$. Furthermore, making use of Thm. 4.16 above, it is now possible to refer to an isometry ψ as a trivialization of a proper kinematical spacetime if it is a trivialization of the associated $\mathbb{R}$-principal bundle and maps $\psi\colon (M, g, V) \to (\mathbb{R} \times S, \psi_* g, A^{-1}\partial_t)$.*

Moreover, all possible trivializations can be obtained by specific projections onto $\mathbb{R}$.

Lemma 4.18. *Let (M^{n+1}, g, V) be a proper kinematical spacetime. Every surjective function $f\colon M \to \mathbb{R}$, with $\mathrm{d}f(V) > 0$ (or $\mathrm{d}f(V) < 0$) and $\mathrm{im}_f(\gamma) = \mathbb{R}$ for all integral curves γ of V, gives rise to a trivialization ψ as in the theorem above, such that for the preimages $f^{-1}(t)$ of f*

$$\psi(f^{-1}(t)) = S_t,$$

with the slices $S_t \subset \mathbb{R} \times S$ holds.

Proof. For all $t \in \mathbb{R}$ the preimage $\sigma_t := f^{-1}(t) \subset M$ is an n-dimensional submanifold of M, transversal to V, due to $\mathrm{d}f(V) \neq 0$. Together with $\mathrm{im}_f(\gamma) = \mathbb{R}$, this also implies that all integral curves of V are mapped diffeomorphically to $\mathbb{R}$ by f. Thus $t \neq s$ implies $\sigma_t \cap \sigma_s = \emptyset$ and $\{\sigma_t\}_{t\in\mathbb{R}}$ is a foliation of M. Furthermore, because of $\mathrm{im}_f(\gamma) = \mathbb{R}$, every flow line of V intersects every hypersurface σ_t exactly once, i.e., all σ_t are diffeomorphic to $S = M/_{\mathbb{R}}$, the space of flow lines of V. Hence, the map $\psi\colon M \to \mathbb{R} \times S$ given by

$$\psi(p) = (f(p), \pi_M(p)),$$

for all $p \in M$, is a trivialization and $\psi(\sigma_t) = (t, S) = \{t\} \times S = S_t$. □

Theorem 4.19. *Let (M, g, V) be a Cartan kinematical spacetime and $\dot{u} = g(\nabla_V V, \cdot)$ the acceleration one-form associated to V. If there is a function $\phi\colon M \to \mathbb{R}$ such that $\dot{u} = \mathrm{d}\phi$, then $(M, e^{-2\phi} g, e^{\phi} V)$ is a proper kinematical spacetime. Particularly, if $\dot{u} = 0$, i.e., the vector field V is geodesic, then (M, g, V) is a proper kinematical spacetime.*

Proof. First, we show the assertion for $\dot{u} = 0$. Let $u = g(V, \cdot)$ be the one-form metrically associated to V. Then $\tilde{u} = -u$ obeys $\tilde{u}(V) = 1$. Furthermore, the rotation ω of V is given by $\omega = \mathrm{d}u$ as $\dot{u} = 0$, so that we conclude $V \lrcorner \mathrm{d}\tilde{u} = -V \lrcorner \omega = 0$ by Prop. 3.21. Hence, (M, g, V) is a proper kinematical spacetime by item *(ii)* of Thm. 4.13.

Second, we assume that $\dot{u} = \mathrm{d}\phi$ and observe that this implies $\mathrm{d}\phi(V) = 0$ as $\nabla_V V$ is horizontal (cf. Prop. 3.21). In particular, we have $\mathcal{H}(\mathrm{d}\phi) = \mathrm{d}\phi$. Hence, using Prop. 3.23 to compute the acceleration one-form $\dot{u}^*$ associated to the conformally transformed metric $g^* = e^{-2\phi}g$ and the vector field $V^* = e^{\phi}V$ yields $\dot{u}^* = 0$. Furthermore, we have $g^*(V^*, V^*) = -1$ and due to $e^{\phi} > 0$ and $\mathrm{d}\phi(V) = 0$ the vector field V^* is also complete. Thus, using the same argument as above, $(M, e^{-2\phi}g, e^{\phi}V)$ is a proper kinematical spacetime. □

Two remarks are in order concerning this theorem.

Remark 4.20. *(i) Obviously, the conclusion also holds if* $\mathrm{d}\dot{u} = 0$ *and* $B_1(M) = 0$*, i.e., the first Betti number of* M *vanishes, as this implies that there is a function* $\phi\colon M \to \mathbb{R}$*, such that* $\dot{u} = \mathrm{d}\phi$*, and again we have* $\mathrm{d}\phi(V) = 0$ *due to Prop. 3.21.*

(ii) Using Rem. 4.8, we can further infer that every Cartan kinematical spacetime with an exact acceleration one-form is of splitting type as in Thm. 4.16.

Of particular concern is now the following equivalent formulation of some results above. It gives interesting non-existence results for certain classes of manifolds.

Corollary 4.21. *Let* (M, V) *be a Cartan kinematical manifold and assume* $M/_{\mathbb{R}}$ *is not Hausdorff. Then there is no Lorentzian or Riemannian metric* g *on* M*, such that* V *is a (timelike)* g*-geodesic vector field or a (timelike) Killing vector field for* g*.*

Proof. Certainly, there is a Lorentzian metric g on M with V being timelike geodesic or Killing if and only if there is a Lorentzian metric g^* in the conformal class of g, such that V is timelike geodesic or Killing for g^* with unit norm $g^*(V, V) = -1$ (cf., e.g., Lem. 4.9).

Assume V was geodesic and timelike for some Lorentzian metric g^* on M and $g^*(V, V) = -1$. Then by Thm. 4.19, (M, g^*, V) would by a proper kinematical spacetime, and hence $M/_{\mathbb{R}}$ would be Hausdorff. In the same way, if V was Killing for some Lorentzian metric g^* on M and $g^*(V, V) = -1$, then by Prop. 4.10, (M, g^*, V) would be a proper kinematical spacetime and hence, $M/_{\mathbb{R}}$ would be Hausdorff.

Hence, by the same arguments as in the proof of Thm. 4.13, V is geodesic (or Killing) and timelike for a Lorentzian metric g^* with unit norm if and only if it is geodesic (or Killing) for the Riemannian metric $g^* + 2g^*(V, \cdot) \otimes g^*(V, \cdot)$ and the Riemannian case follows. □

We have now generalized known diffeomorphic splitting results of Lorentzian spacetimes, at the expense of allowing the restriction $h - \frac{b \otimes b}{A^2}$ of the Lorentzian metric g to the slices of a trivialization to be Lorentzian or even degenerate. Hence, the question arises in which cases we can have Riemannian slices, locally or globally. Of course, locally about a point p in the proper kinematical spacetime this is always possible, but the following proposition shows that in particular cases we have also a Riemannian slicing locally about an orbit, i.e., in a tubular neighborhood of an integral curve of V in a proper kinematical spacetime.

Proposition 4.22. *Let* (M, g, V) *be a proper kinematical spacetime. For every* $p \in M$ *there is a trivialization* ψ *and a neighborhood* U_p *about* $\psi(p) \in \mathbb{R} \times S$*, such that the slices* S_t *in* $\mathbb{R} \times S$

induced by ψ are spacelike in U_p, i.e., $h - \frac{b\otimes b}{A^2}|_{S_t\cap U_p}$ is Riemannian, whenever $S_t \cap U_p \neq \emptyset$. If the acceleration and the shear of V vanish, i.e., $\dot{u} = 0$ and $\sigma = 0$, then for every integral curve γ of V there is a trivialization ψ and a tubular neighborhood T_γ about $\psi(\gamma) \subset \mathbb{R} \times S$, i.e., some set $\mathbb{R} \times W \subset \mathbb{R} \times S$ with W some open neighborhood about $(\mathrm{pr}_2 \circ \psi)(\gamma) \in S$, such that the slices S_t in $\mathbb{R} \times S$ induced by ψ are spacelike in T_γ, i.e., $h - \frac{b\otimes b}{A^2}|_{S_t\cap T_\gamma}$ is Riemannian for all $t \in \mathbb{R}$.

Proof. Let Φ be the global flow associated to V. We show the assertions for global sections $\sigma_0\colon S \to M$ and $\sigma_t\colon S \to M$ with $\sigma_t(x) = \Phi_t(\sigma_0(x))$ for all $x \in S$ in the $\mathbb{R}$-principal bundle $(M, S = M/_{\mathbb{R}}, \pi_M)$. These sections are in one-to-one correspondence to the slices S_t by virtue of a trivialization.

Take any $p \in M$ and consider all unit spacelike vectors g-orthogonal to $V_p \in T_pM$, i.e., the set $\Sigma_p := \{E_p \in T_pM \mid g_p(V_p, E_p) = 0,\ g_p(E_p, E_p) = 1\}$. Choose some open neighborhood U' about $0 \in T_pM$, such that the exponential map $\exp\colon U' \to M$ is a diffeomorphism onto its image $U = \exp(U')$ in M. Then for all $E_p \in \Sigma_p$ we have spacelike geodesic segments $\alpha_E(t) := \exp(tE_p) \subset U$ through p, with t in some interval about $0 \in \mathbb{R}$. Denoting the tangential vector fields as $E = \dot{\alpha}_E$, these span a spacelike integral manifold $S_p \subset U$ through p. As S_p is spacelike, it is transversal to the vector field V, and hence can be written as a local section $S_p\colon \pi_M(U) \to M$. Because M is trivializable as a principal bundle, we can extend S_p to a global section $\sigma_0\colon S \to M$, such that $\sigma_0|_U = S_p$. Thus, we get a foliation of M by the global sections $\sigma_t = \Phi_t(\sigma_0)$. It remains to show that there is an $\varepsilon > 0$, such that $\sigma_t|_U$ is spacelike for all $t \in (-\varepsilon, \varepsilon)$. The tangential vector fields $E = E(0)$ at S_p are transported to the tangential vector fields $E(t) = \Phi_{t*}E(0)$ on σ_t by the flow of V as $\sigma_t = \Phi_t(\sigma_0)$. Thus, all $E(t)$ are vector fields defined in all of the saturation $t \circ U$, all tangential to the sections $\sigma_t|_{t\circ U}$, respectively. Hence, it holds that

$$\pounds_V E(t) = 0,$$

for all $t \in \mathbb{R}$. Using Prop. 3.21 for $u = g(V, \cdot)$, we compute $\pounds_V g = \frac{2}{n}\Theta g + 2\sigma - 2\dot{u} \vee u + \frac{2}{n}\Theta u \otimes u$, such that

$$\nabla_V[g(E(t), E(t))] = (\pounds_V g)(E(t), E(t)) =$$
$$= \frac{2}{n}\Theta g(E(t), E(t)) + 2\sigma(E(t), E(t)) - 2\dot{u}(E(t))u(E(t)) + \frac{2}{n}\Theta u(E(t))^2$$

holds. This is an ordinary differential equation for all fixed $x \in \pi_M(S_p)$. For every such fixed x we denote $X(t) = g(E(t), E(t))$, $\varphi(t) = \sigma(E(t), E(t)) - \dot{u}(E(t))u(E(t)) + \frac{\Theta}{n}u(E(t))^2$ and $s(t) = \int_0^t \frac{\Theta(\tau)}{n}\mathrm{d}\tau$ for all $t \in \mathbb{R}$ and arrive at the differential equation

$$\dot{X}(t) = \frac{2}{n}\Theta(t)X(t) + 2\varphi(t).$$

The solution of this ordinary differential equation is given by

$$X(t) = e^{2s(t)}\left(X(0) + \int_0^t 2\varphi(\tau)e^{-2s(\tau)}\mathrm{d}\tau\right).$$

Hence, as the vector fields $E = E(0)$ are spacelike, i.e., $X(0) > 0$, there certainly is some interval I about $0 \in \mathbb{R}$ such that $X(t) > 0$ for all $t \in I$.

Furthermore, in the case $\dot{u} = 0$ and $\sigma = 0$, we consider for every fixed $x \in \pi_M(S_p)$ the function

$$\alpha(t) = \frac{[g(V, E(t)]^2}{g(E(t), E(t))}$$

along the orbit $\gamma_x \simeq \mathbb{R}$ over $x \in S$. As such, α is a function $\alpha\colon \mathbb{R} \to [0, \infty)$ for all spacelike vector fields $E(t)$ along the orbit, and it measures the Lorentzian angle between $E(t)$ and V. It becomes infinite if $E(t)$ becomes lightlike somewhere, but is certainly defined in some neighborhood of $0 \in \mathbb{R}$, as the vector fields $E(0)$ are considered spacelike on S_p in our case. Hence, if $\alpha(t)$ is defined for all $t \in \mathbb{R}$, i.e., it remains finite for finite times, the vector fields $E(t)$ remain spacelike in the whole tubular neighborhood $t \circ U$ of the orbit, and so do the local sections $\sigma_t|_{t \circ U}$.

So we compute $\nabla_V[g(V, E(t))] = (\pounds_V g)(V, E(t)) = 0$, as $\dot{u} = 0$ and $\nabla_V[g(E(t), E(t))] = (\pounds_V g)(E(t), E(t)) = \frac{2}{n}\Theta[g(E(t), E(t)) + g(V, E(t))^2]$. This yields the following differential equation for $\alpha(t)$:

$$\dot{\alpha}(t) = -\alpha^2(t) - \frac{2}{n}\Theta\alpha(t)$$

along the orbit γ_x. As $E(0)$ is assumed spacelike and geodesic, we have $\alpha(0) = [g(V, E(0))]^2 := c^2$. Note that all vectors in the vector field $E(0)$ with $g(V, E(0)) = 0$ certainly remain spacelike and even perpendicular to V, due to $\nabla_V[g(V, E(t))] = 0$. Hence, we only have to consider initial values $c^2 > 0$ for the differential equation in α. The solution of which is, in this case, given by

$$\alpha(t) = \frac{e^{-\int_0^t \frac{2}{n}\Theta(s)\mathrm{d}s}}{\int_0^t e^{-\int_0^s \frac{2}{n}\Theta(\tau)\mathrm{d}\tau}\mathrm{d}s + \frac{1}{c^2}} < \infty,$$

for all $t \in \mathbb{R}$. □

This proposition will have interesting consequences for the causal properties of shear-free and acceleration-free proper kinematical spacetimes. These are the Hubble-isotropic spacetimes, which will be a subject in chapter 6.

4.2 Causality of Splitting Spacetimes

In this section, we assume $\dim(S) \geq 1$ for kinematical spacetimes $(\mathbb{R} \times S, g, V)$. The special low dimensional case $\dim(S) = 1$, together with particularities in the situation when $\dim(S) = 2$ or $\dim(S) = 3$, for compact S is the subject of the following section 4.3.

We analyze the possible causality conditions for proper kinematical spacetimes in detail. We will derive various conditions on the spacetimes for certain steps on the ladder of causality. We will also determine the causality conditions which cannot (generally) hold for a proper kinematical spacetime of arbitrary dimension.

Due to the considerations in the previous section, we can now view proper kinematical spacetimes in the following way: By virtue of a trivialization, a proper kinematical spacetime is diffeomorphic to $\mathbb{R} \times S$, with V parallel to ∂_t and the push-forward of the metric g to $\mathbb{R} \times S$ is given by Thm. 4.16. This push-forward is only fixed up to global gauge transformations in the $\mathbb{R}$-principal bundle, and the transformation of the metric under such a gauge transformation—a change of slicing—is again given in Thm. 4.16. These gauge transformations correspond to the choice of another trivialization of the principal bundle associated to (M, g, V). Naturally, the slices $S_t \subset \mathbb{R} \times S$ are pulled back to a foliation $\{\sigma_t\}_{t\in\mathbb{R}}$ of M, and any such foliation of M, given as the preimages of some function $t\colon M \to \mathbb{R}$, give rise to a specific trivialization according to Lem. 4.18. Furthermore, as all causality conditions are conformally invariant, it is, here and in the following section, enough to consider proper kinematical spacetimes for which $\psi_* V = \partial_t$ holds, with respect to a trivialization $\psi\colon M \to \mathbb{R} \times S$. Here and in the following, we will often omit the pull-back pr_2^* when writing the metric if this is no source of confusion. Hence, we can fix a trivialization $\psi\colon M \to \mathbb{R} \times S$, such that the proper kinematical spacetime is given as $(\mathbb{R} \times S, g, \frac{1}{A}\partial_t)$ with

$$g = -A^2 \mathrm{d}t \otimes \mathrm{d}t + 2b \vee \mathrm{d}t + h - \frac{b \otimes b}{A^2}$$

and A, b, h as in Thm. 4.16, but then analyze the conformally transformed metric $g' = \frac{1}{A^2} g$, i.e., the proper kinematical spacetime $(\mathbb{R} \times S, g', \partial_t)$ with

$$g' = -\mathrm{d}t \otimes \mathrm{d}t + 2b' \vee \mathrm{d}t + h' - b' \otimes b',$$

for which $b' = \frac{1}{A^2} b$ and $h' = \frac{1}{A^2} h$ holds. All propositions on causality conditions hold for every member in the conformal class of the Lorentzian metric g. This implies that the results on causality conditions in this section also hold for spacetimes which are conformal to a proper kinematical spacetime, i.e., spacetimes (M, g) together with a *not necessarily complete* reference frame V, such that M is diffeomorphic to $\mathbb{R} \times S$ and V points along the factor $\mathbb{R}$ with respect to the diffeomorphism.

Certainly, we have the following

Lemma 4.23. *Let $(\mathbb{R} \times S, g, \partial_t)$ be a spacetime, together with the canonical timelike vector field ∂_t, such that*

$$g_{(t,x)} = -\mathrm{d}t \otimes \mathrm{d}t + 2\mathrm{pr}_2^*(b_{(t,x)}) \vee \mathrm{d}t + \mathrm{pr}_2^*(h_{(t,x)}) - \mathrm{pr}_2^*(b_{(t,x)}) \otimes \mathrm{pr}_2^*(b_{(t,x)}),$$

for a family of one-forms $\{b_t\}_{t\in\mathbb{R}} \subset \Gamma(\Lambda^1 S)$ and a family of Riemannian metrics $\{h_t\}_{t\in\mathbb{R}} \subset \mathcal{R}(S)$, as well as the canonical projection $\mathrm{pr}_2\colon \mathbb{R} \times S \to S$ and $t\colon \mathbb{R} \times S \to \mathbb{R}$. Then ∂_t is complete, particularly $(\mathbb{R} \times S, g, \partial_t)$ is a proper kinematical spacetime. Furthermore, the induced metric

$$g_{(t,x)}|_{S_{t_0}} = g_{(t_0,x)} = \mathrm{pr}_2^*(h_{(t_0,x)}) - \mathrm{pr}_2^*(b_{(t_0,x)}) \otimes \mathrm{pr}_2^*(b_{(t_0,x)})$$

on the slice S_{t_0} is Riemannian at some point $(t_0, x) \in S_{t_0}$ if and only if the h_{t_0}-norm of the one-form b_{t_0} on S is strictly smaller than 1 at $x \in S$, i.e.,

$$\|b_{t_0}\|_x^{h_{t_0}} := \sup_{v \in T_x S \setminus \{0\}} \frac{|b_{(t_0,x)}(v)|}{\sqrt{h_{(t_0,x)}(v,v)}} < 1.$$

Proof. Certainly, the transformations $\varphi_t\colon \mathbb{R}\times S \to \mathbb{R}\times S$ given by $\varphi_t(s,x) = (s+t,x)$ exist for all $t \in \mathbb{R}$, thus as $\partial_t = \dot{\varphi}_t = (1,0)$, the vector field ∂_t is complete. Furthermore, we have $g(\partial_t,\partial_t) = -1$, and therefore, $(\mathbb{R}\times S, g, \partial_t)$ is a proper kinematical spacetime. Moreover, we have that $h_{t_0} - b_{t_0}\otimes b_{t_0}$ is positive definite at $x \in S$ if and only if

$$h_{(t_0,x)}(v,v) - b_{(t_0,x)}(v)^2 > 0,$$

for all $v \in T_xS\setminus\{0\}$, i.e., if and only if $\frac{b_{(t_0,x)}(v)^2}{h_{(t_0,x)}(v,v)} < 1$ for all $v \in T_xS\setminus\{0\}$, i.e., if and only if

$$\frac{|b_{(t_0,x)}(v)|}{\sqrt{h_{(t_0,x)}(v,v)}} < 1,$$

for all $v \in T_xS\setminus\{0\}$. This is equivalent to

$$|b_{(t_0,x)}(v)| < 1,$$

for all $v \in T_xS$ obeying $h_{(t_0,x)}(v,v) = 1$. As $\{v \,|\, h_{(t_0,x)}(v,v) = 1\} \subset T_xS$ is compact, the supremum exists and is obtained as a maximum smaller than one. $\square$

Furthermore, we will use the following

Lemma 4.24. *Let $(\mathbb{R}\times S, g, \partial_t)$ be a proper kinematical spacetime with metric*

$$g = -\mathrm{d}t\otimes\mathrm{d}t + 2b_t\vee\mathrm{d}t + h_t - b_t\otimes b_t$$

and $\{b_t\}_{t\in\mathbb{R}}$, $\{h_t\}_{t\in\mathbb{R}}$ a family of one-forms and Riemannian metrics on S, respectively. For all $(t,x)\in\mathbb{R}\times S$ and all orbits $\mathbb{R}\times\{y\}$, there is a future-directed and a past-directed timelike curve connecting (t,x) to $\mathbb{R}\times\{y\}$, i.e., for all $(t,x)\in\mathbb{R}\times S$ and all $y\in S$ there are two numbers $T,\tilde{T}\in\mathbb{R}$, as well as a future-directed timelike curve $\lambda\colon[0,1]\to\mathbb{R}\times S$ and a past-directed timelike curve $\tilde{\lambda}\colon[0,1]\to\mathbb{R}\times S$, such that $\lambda(0) = \tilde{\lambda}(0) = (t,x)$, $\lambda(1) = (T,y)$ and $\tilde{\lambda}(1) = (\tilde{T},y)$.

Proof. We will prove the future-directed case only. The past-directed case works completely analogous. Without loss of generality we can assume $t = 0$. Given $x,y\in S$, fix any curve $c\colon[0,1]\to S$ obeying $c(0) = x$ and $c(1) = y$. Consider the lift $L = \{(\tau,c(s)) \,|\, \tau \geq 0,\ s\in[0,1]\} \subset \mathbb{R}\times S$, which is a two-dimensional submanifold with boundary in $\mathbb{R}\times S$. Then the velocity $\dot{c}$ of c in S lifts to a vector field $(0,\dot{c})$ tangential to L. Set the function $\eta\colon L\simeq\mathbb{R}_{\geq 0}\times[0,1]\to\mathbb{R}$ to be

$$\eta(\tau,s) := \sqrt{h_{(\tau,c(s))}(\dot{c}(s),\dot{c}(s))} + b_{(\tau,c(s))}(\dot{c}(s)).$$

As $[0,1]$ is compact, there exists a function $\xi\colon\mathbb{R}_{\geq 0}\to\mathbb{R}$ defined by

$$\xi(\tau) = \sup_{s\in[0,1]}\eta(\tau,s) < \infty,$$

for all $\tau \geq 0$ and we set the vector field Z tangential to L to be

$$Z(\tau, s) = (\xi(\tau) + 1, \dot{c}(s)).$$

Then the integral curve $\lambda\colon [0,1] \to L$ of Z obeying $\lambda(0) = (0, x)$ eventually intersects $\mathbb{R} \times \{y\}$ for $s = 1$ at the point (T, y) with $T = \int_0^1 (\xi(\tau) + 1)\mathrm{d}\tau$. It remains to show that the vector field Z is timelike and future-directed. We compute

$$g_{(\tau,c(s))}(\partial_t, Z(\tau, s)) = -\xi(\tau) - 1 + b_{(\tau,c(s))}(\dot{c}(s)) \leq -\sqrt{h_{(\tau,c(s))}(\dot{c}(s), \dot{c}(s))} - 1 < 0$$

as $b_{(\tau,c(s))}(\dot{c}(s)) \leq \xi(\tau) - \sqrt{h_{(\tau,c(s))}(\dot{c}(s), \dot{c}(s))}$ for all $(\tau, s) \in \mathbb{R}_{\geq 0} \times [0,1]$, as well as

$$g_{(\tau,c(s))}(Z(\tau, s), Z(\tau, s)) = -\left[(\xi(\tau) + 1) - b_{(\tau,c(s))}(\dot{c}(s))\right]^2 + h_{(\tau,c(s))}(\dot{c}(s), \dot{c}(s)).$$

We observe that, due to the calculation above, $g(Z, Z) < 0$ is equivalent to

$$\sqrt{h_{(\tau,c(s))}(\dot{c}(s), \dot{c}(s))} < (\xi(\tau) + 1) - b_{(\tau,c(s))}(\dot{c}(s)),$$

for all $(\tau, s) \in \mathbb{R}_{\geq 0} \times [0, 1]$, which is always true by the construction of ξ. Hence, Z is timelike and future-directed, and so are all its integral curves. $\square$

Remark 4.25. *Due to the considerations at the beginning of this section, Lem. 4.24 certainly also holds for a general proper kinematical spacetime (M, g, V) by pull-back, and because the property of being timelike is conformally invariant. Hence, any arbitrary point p in (M, g, V) can be connected to any arbitrary integral curve of V by a future-directed and a past-directed timelike curve.*

Proposition 4.26. *A proper kinematical spacetime (M, g, V) is stably causal if and only if there is a codimension one foliation (necessarily transversal to V) $\{\sigma_t\}_{t\in\mathbb{R}}$ of M, such that all leaves are spacelike.*

Proof. "$\Leftarrow$": As (M, g, V) is a proper kinematical spacetime and we are interested in the causality condition of stable causality, we can assume, using the considerations at the beginning of this section, that there is a trivialization, such that the kinematical spacetime is given by $(\mathbb{R} \times S, g, \partial_t)$ with

$$g = -\mathrm{d}t \otimes \mathrm{d}t + 2b_t \vee \mathrm{d}t + h_t - b_t \otimes b_t.$$

The existence of the spacelike foliation transversal to V in M, then amounts to the slices $S_t \subset \mathbb{R} \times S$ with the induced metric

$$g|_{S_t} = h_t - b_t \otimes b_t$$

being Riemannian, i.e., $\|b_t\|^{h_t} < 1$ on all of $\mathbb{R} \times S$, by Lem. 4.23. In the following we suppress the index t at the objects b and h, if this is no source of confusion. Now, let $\lambda\colon I \to \mathbb{R} \times S$ be any future-directed causal curve in $\mathbb{R} \times S$, with $I \subset \mathbb{R}$ some interval. Then we have

$\lambda(s) = (t(s), x(s))$ and $\dot{\lambda} = (\dot{t}, \dot{x})$, where $s \in I$ and the dot means the derivative with respect to the curve parameter s. As λ is assumed future-directed, we have

$$g(\partial_t, \dot{\lambda}) = -\dot{t} + b(\dot{x}) < 0 \quad \Rightarrow \quad b(\dot{x}) < \dot{t}.$$

As λ is assumed causal, we have

$$g(\dot{\lambda}, \dot{\lambda}) = -\dot{t}^2 + 2b(\dot{x})\dot{t} + h(\dot{x}, \dot{x}) + b(\dot{x})^2 = -(\dot{t} - b(\dot{x}))^2 + h(\dot{x}, \dot{x}) \leq 0,$$

hence $\sqrt{h(\dot{x}, \dot{x})} \leq \dot{t} - b(\dot{x})$. Because of $\|b_t\|^{h_t} < 1$, this yields

$$0 < \sqrt{h(\dot{x}, \dot{x})} + b(\dot{x}) \leq \dot{t}.$$

Thus $\dot{t}$ could only be zero if $\dot{x} = 0$. But in these cases, $\dot{t}$ certainly cannot be zero as otherwise the curve λ would be constant. So, we have $\dot{t} > 0$ on all of λ, i.e., the function $t\colon \mathbb{R} \times S \to \mathbb{R}$ is strictly increasing along every future-directed causal curve in $\mathbb{R} \times S$ and thus the spacetime is stably causal.

"$\Rightarrow$": Recalling the definition of stable causality, Def. 3.48, we can state the following. If the proper kinematical spacetime (M, g, V) is stably causal, there is a temporal function $t\colon M \to \mathbb{R}$, i.e., a smooth function with past-directed timelike gradient ∇t. Particularly, this means $g(V, \nabla t) = \mathrm{d}t(V) > 0$ and $g(\nabla t, \nabla t) > 0$. As M is connected, $\mathrm{im}_t(M)$ is some connected and open (possibly unbounded) interval $I \subset \mathbb{R}$. Thus, by composition with a strictly increasing diffeomorphism $I \to \mathbb{R}$, we can modify t such that $\mathrm{im}_t(M) = \mathbb{R}$ holds. Employing Lem. 4.24 and Rem. 4.25, we can infer that for all integral curves γ of V it must hold that $t(\gamma) = \mathbb{R}$. Assume there is some $p \in M$, such that the integral curve γ_p through p obeys $\sup t(\gamma_p) = k < \infty$ for some $k \in \mathbb{R}$. As $\mathrm{im}_t(M) = \mathbb{R}$, there is some $q \in M$, such that $t(q) = k + 1$. But as we can connect q to γ_p by a future- directed timelike curve and t is certainly non-decreasing along this curve, there is some $r \in \gamma_p$ with $t(r) > k$, in contradiction. The same argument applies if we assume $t(\gamma_p)$ to be bounded from below and use a past-directed timelike curve. Due to Lem. 4.18, the function t then gives rise to a codimension one foliation $\{\sigma_t\}_{t\in\mathbb{R}}$ of the proper kinematical spacetime (M, g, V). Certainly, the gradient ∇t is g-orthogonal to the leaves σ_t everywhere, and as ∇t is timelike, the leaves are spacelike. $\square$

We recall Def. 3.37 and can state the following

Proposition 4.27. *A spacetime (M, g) is non-imprisoning if and only if for all relatively compact open sets $U \subset M$, the restricted spacetime $(U, g|_U)$ is stably causal.*

Proof. This proposition has already been proven by J.K. Beem in [Bee76]. See also [Min09a, Thm. 1]. $\square$

For proper kinematical spacetimes, Prop. 4.27 has the following consequence.

Proposition 4.28. *A proper kinematical spacetime (M, g, V) is non-imprisoning if and only if for all compact sets $K \subset M$ (with non-empty, open interior) there is a foliation $\{\sigma_t\}_{t\in\mathbb{R}}$ of M, such that the partial leaves $\sigma_t \cap \mathring{K}$ are spacelike for all $t \in \mathbb{R}$, for which this intersection is non-empty.*

Proof. "$\Leftarrow$": Let $U \subset M$ be a relatively compact set. Then there certainly is a compact set K with $U \subset K \subset M$ obeying the following condition. Let $\psi\colon M \to \mathbb{R} \times S$ be the trivialization of (M, g, V) associated to the foliation $\{\sigma_t\}_{t\in\mathbb{R}}$ of M, i.e., ψ maps the leaves $\sigma_t \subset M$ to the slices $S_t \subset \mathbb{R} \times S$, such that $\psi(K) = I \times L$ with $L \subset S$ compact and $I \subset \mathbb{R}$ a compact interval. Thus the partial slices $L_t := S_t \cap \psi(K)$ for $t \in \mathring{I}$ are also spacelike in their interior. Then we can regard $(\psi(\mathring{K}), \psi_* g|_{\psi(\mathring{K})}) = (\mathring{I} \times \mathring{L}, \psi_* g|_{\mathring{I}\times\mathring{L}})$ as conformal to a proper kinematical spacetime, maybe after applying an isometry, which maps $\mathring{I}$ diffeomorphically to $\mathbb{R}$ (see, e.g., the construction below for details). Applying Prop. 4.26 to this restricted spacetime shows that it is stably causal and so is its pull-back to M by ψ. Hence, $(U, g|_U)$ is also stably causal as $U \subset K$ and the result follows from Prop. 4.27.

"$\Rightarrow$": Assuming (M, g, V) to be non-imprisoning, it follows from Prop. 4.27 that for all relatively compact open sets $U \subset M$, the restricted spacetime $(U, g|_U)$ is stably causal. Particularly, we can assume U to obey the following condition. There is a trivialization $\psi\colon M \to \mathbb{R} \times S$ of (M, g, V), such that $\psi(U) = J \times W \subset \mathbb{R} \times S$ with $J \subset \mathbb{R}$ a relatively compact open interval and $W \subset S$ a relatively compact open subset. Certainly, there is a diffeomorphism $\rho\colon J \times W \to \mathbb{R} \times W$, which maps the slices $W_t = \{t\} \times W \subset J \times W$ to slices $\tilde{W}_\tau := \rho(W_t) \subset \mathbb{R} \times W$, such that $\mathbb{R} \ni \tau = (\mathrm{pr}_1 \circ \rho)(t, x)$ for all $(t, x) \in J \times W$, i.e., ρ stems from a diffeomorphism between J and $\mathbb{R}$. As $\mathbb{R}\times W$, together with the metric pushed forward by the composition $\rho \circ \psi|_U$, can be regarded as conformal to a proper kinematical spacetime, we can apply Prop. 4.26. Hence, there is a foliation $\{R_\tau\}_{\tau\in\mathbb{R}}$ of $\mathbb{R} \times W$ by spacelike leaves. Pulling back these leaves by ρ to slices in $J \times W$, i.e., considering $P_t := \rho^{-1}(R_\tau) \subset J \times W$, yields the $\{P_t\}_{t\in J}$ as a family of spacelike sections in $\mathbb{R} \times S$ over $W \subset S$. As W is relatively compact, we can consider the family of closed sections $\{\overline{P}_t\}_{t\in J}$ over the closed $\overline{W}$. The extension of P_t to $\overline{P}_t$ is always possible for all $t \in J$ as the set $U \subset M$, with which we started, can be chosen arbitrarily large, as long as it is relatively compact. Then by [KN63, Thm. 5.7, p. 58] we extend the closed sections to global ones $\{S_t\}_{t\in J}$ with $\overline{P}_t \subset S_t$. Hence, the global sections S_t are spacelike for all $(t, x) \in S_t$ for which $x \in W$ holds. Assume $J = (a, b) \subset \mathbb{R}$. Now we choose some small $\varepsilon > 0$ and set the slices S_t for $t > b - \varepsilon$ to be the saturation of the section $S_{b-\varepsilon}$ by the principal action in $\mathbb{R} \times S$ and slices S_t for $t < a + \varepsilon$ to be the saturation of the section $S_{a+\varepsilon}$ by the principal action in $\mathbb{R} \times S$. Pulling these slices back to M yields leaves $\sigma_t := \psi^{-1}(S_t)$ of a foliation $\{\sigma_t\}_{t\in\mathbb{R}}$ which are spacelike in the interior of the compact set $\psi^{-1}([a + \varepsilon, b - \varepsilon] \times \overline{W}) \subset \overline{U}$, which is only marginally smaller (depending on how small ε is chosen) than $\overline{U}$. As we can choose U arbitrarily large, as long as it is relatively compact, the result follows. □

Having this proposition at hand, we are now able to prove the following theorem, which will substantially simplify the causal ladder for proper kinematical spacetimes.

Theorem 4.29. *Let (M, g, V) be a non-imprisoning, proper kinematical spacetime. Then (M, g) is reflecting. Particularly, if (M, g) is feebly distinguishing, it is also causally continuous.*

Proof. Following the considerations at the beginning of this section, we can assume $(M, g, V) =$

$(\mathbb{R}\times S, g, \partial_t)$ with

$$g = -\mathrm{d}t\otimes \mathrm{d}t + 2b_t\vee \mathrm{d}t + h_t - b_t\otimes b_t$$

under omission of the pull-back operation pr_2^*. Furthermore, for any compact interval $I\subset\mathbb{R}$ and any compact $K\subset S$ with open interior, using Prop. 4.28, we can assume that the induced metric

$$\gamma_{(t,x)} = h_{(t,x)} - b_{(t,x)}\otimes b_{(t,x)}$$

on the slice S_t is Riemannian, i.e., $\|b_t\|_x^{h_t} < 1$ by Lem. 4.23, if $(t,x)\in\mathring{I}\times\mathring{K}$. This can always be achieved by the following steps: For a compact $L\subset\mathbb{R}\times S$ obeying $I\times K\subset\mathring{L}$, there is a foliation $\{\sigma_t\}_{t\in\mathbb{R}}$ of $\mathbb{R}\times S$ as in Prop. 4.28 with L as the compact set, as we assumed the spacetime to be non-imprisoning. Then there is a change of slicing $\psi\colon(\mathbb{R}\times S, g, \partial_t)\to(\mathbb{R}\times S, \psi_* g, \partial_t)$ obeying $\psi(\sigma_t) = S_t$ and we use the spacetime $(\mathbb{R}\times S, \psi_* g, \partial_t)$.

We will denote the principal action in $\mathbb{R}\times S$ by the global flow Φ associated to ∂_t, which operates by $\Phi_s(t,x) = (t+s,x)$ for all $s\in\mathbb{R}$ and all $(t,x)\in\mathbb{R}\times S$.

We now have to show that for any two points $(t,x)\in\mathbb{R}\times S$ and $(s,y)\in\mathbb{R}\times S$

$$I^+((t,x))\supset I^+((s,y))\Leftrightarrow I^-((t,x))\subset I^-((s,y))$$

holds. We will only prove the implication "$\Rightarrow$" explicitly, as the proof of "$\Leftarrow$" works completely analogous. We certainly have that $(s+\varepsilon,y)\in I^+((s,y))$ for all $\varepsilon>0$. Hence, assuming $I^+((s,y))\subset I^+((t,x))$, there is some future-directed timelike curve $\lambda_\varepsilon\colon[0,1]\to\mathbb{R}\times S$ with $\lambda_\varepsilon(0) = (t,x)$ and $\lambda_\varepsilon(1) = (s+\varepsilon,y)$ for all $\varepsilon>0$. Now, we choose some compact $I\times K\subset\mathbb{R}\times S$, such that $(t,x),(s,y)\in I\times K$ and also $\lambda_{\varepsilon_0}(\tau),\Phi_{-\varepsilon_0}(\lambda_{\varepsilon_0}(\tau))\in I\times K$ for all $\tau\in[0,1]$ with some fixed $\varepsilon_0>0$. This is always possible as the intervals $[-\varepsilon_0,\varepsilon_0]\subset\mathbb{R}$ and $[0,1]\subset\mathbb{R}$ are compact and so is the image $\Phi_{[-\varepsilon_0,\varepsilon_0]}(\lambda_{\varepsilon_0}([0,1]))\subset\mathbb{R}\times S$. Then we certainly have that also $\lambda_\varepsilon(\tau),\Phi_{-\varepsilon}(\lambda_\varepsilon(\tau))\in I\times K$ for all $\tau\in[0,1]$ and all $\varepsilon>0$ obeying $\varepsilon<\varepsilon_0$.

Now for all $\varepsilon\in(0,\varepsilon_0)$, we define a curve $\tilde{\mu}_\varepsilon\colon[0,1]\to I\times K$ by $\tilde{\mu}_\varepsilon(\tau) = \Phi_{-\varepsilon}(\lambda_\varepsilon(\tau))$ for all $\tau\in[0,1]$. This curve obeys $\tilde{\mu}_\varepsilon(0) = \Phi_{-\varepsilon}(t,x) = (t-\varepsilon,x)$ and $\tilde{\mu}_\varepsilon(1) = \Phi_{-\varepsilon}(s+\varepsilon,y) = (s,y)$. Thus, by setting $\mu_\varepsilon(\tau) = \tilde{\mu}_\varepsilon(1-\tau)$, we get a curve $\mu_\varepsilon\colon[0,1]\to I\times K$ obeying $\mu_\varepsilon(0) = (s,y)$ and $\mu_\varepsilon(1) = (t-\varepsilon,x)$. We will now show that μ_ε is timelike and past-directed in all its domain if ε is small enough.

We denote $\lambda_\varepsilon(\tau) = (T(\tau),c(\tau))$ for some function $T\colon[0,1]\to\mathbb{R}$ with $T(0)=t$ and $T(1) = s+\varepsilon$, as well as a curve $c\colon[0,1]\to S$ with $c(0)=x$ and $c(1)=y$. Denoting the derivative with respect to the curve parameter τ with a dot, we get $\dot{\lambda}_\varepsilon = (\dot{T},\dot{c})$ as a tangential vector field at the curve λ_ε. As λ_ε is assumed to be future-directed we get

$$g(\partial_t,\dot{\lambda}_\varepsilon) = -\dot{T} + b(\dot{c}) < 0 \quad\Leftrightarrow\quad \dot{T} > b(\dot{c})$$

and as it is timelike we get

$$g(\dot{\lambda}_\varepsilon,\dot{\lambda}_\varepsilon) = -\dot{T}^2 + 2\dot{T}b(\dot{c}) + h(\dot{c},\dot{c}) - b(\dot{c})^2 < 0.$$

These two conditions together yield $\dot{T} > \sqrt{h(\dot{c},\dot{c})} + b(\dot{c})$. Furthermore, as the slices in the compact set $I\times K$ are Riemannian, we can infer that

$$1 > \sup_{(t,x)\in I\times K}\|b_t\|_x^{h_t} > \frac{|b(\dot{c})|}{\sqrt{h(\dot{c},\dot{c})}},$$

and hence there is a constant $1 > k > 0$, such that $|b(\dot{c})| < k\sqrt{h(\dot{c},\dot{c})}$, which yields

$$\dot{T} > \sqrt{h(\dot{c},\dot{c})} + b(\dot{c}) \geq 0.$$

Therefore, $\dot{T} > 0$ and we can reparametrize λ_ε over the interval $[t, s+\varepsilon]$, which yields $\lambda_\varepsilon(\tau) = (\tau, c(\tau))$ and

$$1 > \sqrt{h_{(\tau,c(\tau))}(\dot{c}(\tau),\dot{c}(\tau))} + b_{(\tau,c(\tau))}(\dot{c}(\tau)) =: \eta(\tau),$$

for all for $\tau \in [t, s+\varepsilon]$. Hence, there is a constant η_0 with $1 > \eta_0 = \sup_{\tau\in[t,s+\varepsilon_0]} \eta(\tau)$ and we set

$$\delta := \frac{1-\eta_0}{2} > 0,$$

such that surely $\eta(\tau) + \delta < 1$ holds for all $\tau \in [t, s+\varepsilon]$. Now consider the reparametrized curves

$$\tilde{\mu}_\varepsilon(\tau) = \Phi_{-\varepsilon}(\lambda_\varepsilon(\tau)) = (\tau - \varepsilon, c(\tau)),$$

for $\tau \in [t, s+\varepsilon]$ and $\varepsilon_0 > \varepsilon > 0$. The tangential vector field is given by $\dot{\tilde{\mu}}_\varepsilon = (1, \dot{c})$, such that $\tilde{\mu}_\varepsilon$ is timelike if

$$\sqrt{h_{(\tau-\varepsilon,c(\tau))}(\dot{c}(\tau),\dot{c}(\tau))} + b_{(\tau-\varepsilon,c(\tau))}(\dot{c}(\tau)) < 1, \tag{$*$}$$

for all $\tau \in [t, s+\varepsilon]$. For all $\tau \in [t, s+\varepsilon]$ and all $\varepsilon \in [0, \varepsilon_0)$ there is some $i(\tau,\varepsilon) \geq 0$, such that

$$\sqrt{h_{(\tau-\varepsilon,c(\tau))}(\dot{c}(\tau),\dot{c}(\tau))} + b_{(\tau-\varepsilon,c(\tau))}(\dot{c}(\tau)) \leq \eta(\tau) + i(\tau,\varepsilon),$$

with $i(\tau, 0) = 0$, and thus

$$\sqrt{h_{(\tau-\varepsilon,c(\tau))}(\dot{c}(\tau),\dot{c}(\tau))} + b_{(\tau-\varepsilon,c(\tau))}(\dot{c}(\tau)) \leq \eta(\tau) + I(\varepsilon),$$

with $0 \leq I(\varepsilon) := \sup_{\tau\in[t,s+\varepsilon]} i(\tau,\varepsilon)$. The functions $i(\tau,\varepsilon)$ and $I(\varepsilon)$ can be assumed to vary continuously with the variables τ and ε, as the Riemannian metrics h_t and the one-forms b_t vary smoothly with t. Hence, as $I(0) = 0$, there is some specific $\varepsilon > 0$, such that $I(\varepsilon) < \delta$, which implies that ($*$) holds and hence $\tilde{\mu}_\varepsilon$ is timelike. Certainly, $\tilde{\mu}_\varepsilon$ is also future-directed, and as μ_ε is just $\tilde{\mu}_\varepsilon$ with reversed orientation, it is past-directed and timelike. As this holds for an arbitrarily small $\varepsilon > 0$ and the chronological relation $\ll$ is transitive (cf. Def. 3.35), we have $I^-((t,x)) \subset I^-((s,y))$.

Furthermore, the remaining assertion follows from Def. 3.49. □

The theorem above implies that for proper kinematical spacetimes the steps distinction, non–partial-imprisonment, strong causality and stable causality are not separable, i.e., either the spacetime is causally continuous or it is not even feebly distinguishing. Moreover, the appearance of causal continuity is controlled by the existence of partially imprisoned causal curves. This leads to the fact that in proper kinematical spacetimes all causal steps, up to causal continuity, are directly determined by the non-existence of specific types of “badly behaving” causal curves. As usual, the proper kinematical spacetime is chronological, causal or non-imprisoning if there is no closed timelike, closed causal or imprisoned causal curve, respectively, but, particularly, it is causally continuous if there is no partially imprisoned causal curve.

Recalling Prop. 3.43, we can deduce the following corollary from Thm. 4.29.

Corollary 4.30. *Let (M, g, V) be a non-imprisoning, proper kinematical spacetime. Then there is a homeomorphism $\varphi\colon M \to \mathbb{R} \times S$, such that the sets $\sigma_t = \varphi^{-1}(S_t)$, $t \in \mathbb{R}$ are achronal topological hypersurfaces, i.e., there is an achronal, topological foliation of (M, g).*

Proof. As (M, g) is reflecting by Thm. 4.29, we can deduce from Prop. 3.43 that there is a continuous semi-time function $t\colon M \to \mathbb{R}$, which is strictly increasing along every future-directed timelike curve. As M is connected, $\mathrm{im}_t(M)$ is, in general, some open and connected (possibly unbounded) interval $I \subset \mathbb{R}$. Modifying t by composition with a strictly increasing homeomorphism $I \to \mathbb{R}$, yields a semi-time function $t\colon M \to \mathbb{R}$ obeying $\mathrm{im}_t(M) = \mathbb{R}$. Now an argument just as in the proof of Prop. 4.26 and similar to Lem. 4.18 applies. As any integral curve γ_p of V through $p \in M$ is timelike, we certainly have $t(r) < t(s)$ for all $r \ll s \in \gamma(p)$, but we also have $t(\gamma_p) = \mathbb{R}$ for all $p \in M$. Because, assume $\sup t(\gamma_p) = k < \infty$, then there is some $q \in M$ with $t(q) = k + 1$, as $\mathrm{im}_t(M) = \mathbb{R}$, but due to Lem. 4.24 and Rem. 4.25, there is a future-directed timelike curve connecting q to γ_p, which implies the existence of some $r \in \gamma_p$ with $t(r) > k + 1$, as t is strictly increasing along timelike curves, in contradiction. The same argument applies, mutatis mutandis, if we assume $t(\gamma_p)$ to be bounded from below.

Thus for all $c \in \mathbb{R}$, the preimage $\sigma_c := t^{-1}(c)$ is the disjoint union of one and only one point from each integral curve of V in (M, g, V). As t is continuous and strictly increasing along timelike curves, the σ_c are achronal topological hypersurfaces, homeomorphic to the manifold of flow lines $S = M/_{\mathbb{R}}$. Then setting

$$\varphi(p) = (t(p), \pi_M(p))$$

yields a homeomorphism $\varphi\colon M \to \mathbb{R} \times S$ obeying $\varphi(\sigma_c) = \{c\} \times S = S_c$ for all $c \in \mathbb{R}$, and $\{\sigma_c\}_{c\in\mathbb{R}}$ is an achronal, topological foliation of (M, g). □

It is known that the higher steps on the causal ladder, causal simplicity and particularly global hyperbolicity, are controlled, in many important cases, by the completeness of Riemannian or Finslerian metrics on specific hypersurfaces (cf. chapters 5 and 6 below). If we consider the case of a compact manifold S for a proper kinematical spacetime $(\mathbb{R} \times S, g, V)$, the situation is much simpler and global hyperbolicity can be assessed without referring to the completeness of specific metrics, as the following proposition shows.

Proposition 4.31. *Let (M, g, V) be a non-imprisoning, proper kinematical spacetime with the manifold of flow lines $S = M/_{\mathbb{R}}$ of V being compact, then (M, g) is globally hyperbolic.*

Proof. Assume $\varphi\colon M \to \mathbb{R} \times S$ is a homeomorphism as in Cor. 4.30, such that the $\sigma_t = \varphi^{-1}(\{t\} \times S)$ are achronal topological hypersurfaces for all $t \in \mathbb{R}$. Recalling Def. 3.53 and Thm. 3.54, it is enough to show that every inextendible timelike curve intersects the hypersurface σ_T, for some fixed $T \in \mathbb{R}$, to infer that σ_T is a Cauchy hypersurface and hence (M, g) is globally hyperbolic.

Assume $\lambda\colon \mathbb{R} \to M$ is an inextendible timelike curve and assume that λ is future-directed with $t(0) < T$, but λ does not intersect σ_T. The proof for past-directed curves and $t(0) > T$ works completely analogous and will be omitted. As t is strictly increasing along λ we can infer that $\lambda(\mathbb{R}_{\geq 0}) \subset \varphi^{-1}([t(0), T] \times S)$. As S is compact and φ is a homeomorphism, $\varphi^{-1}([t(0), T] \times S) \subset M$ is compact, hence λ is future imprisoned in a compact set, in contradiction. □

4.3 The Low-dimensional Cases

Based on Prop. 4.31 in the previous section, we will try to answer in this section the following question: Let (M, g, V) be a proper kinematical spacetime with a compact space of flow lines $S = M/_{\mathbb{R}}$. When does the condition of (M, g) being causal already, imply the global hyperbolicity of (M, g)? Certainly, this is equivalent to establishing when such a spacetime is non-imprisoning if it is causal. We will assess this question for $\dim(S) \in \{1, 2, 3\}$.

In this section, we will often impose the condition of a smooth semi-time function on the spacetimes under consideration. Recall that a semi-time function $t\colon M \to \mathbb{R}$ for a spacetime (M, g) obeys $t(p) < t(q)$ whenever $p \ll q$. But in this case one assumes t to be, not only continuous, but also smooth. Certainly, if there exists a semi-time function then the spacetime is chronological. But the circumstances that assure the existence of smooth semi-time functions are not clear, as reflectivity plus chronology only leads to a continuous function (cf. Prop. 3.43). However, there are strong indications that in a proper kinematical spacetime (M, g, V) a smooth semi-time function exists in many important cases. Particularly, if (M, g) is chronological and if, maybe additionally, the space of flow lines $S = M/_{\mathbb{R}}$ is compact and (M, g) is reflecting (cf. Conj. 7.1, as well as Prop. 7.2 in chapter 7 and the discussion thereafter).

If $\dim(S) = 1$ and S is compact, we can assume $S = \mathbb{S}^1$, and a spacetime $(\mathbb{R} \times \mathbb{S}^1, g)$ is a Lorentzian cylinder. Two-dimensional spacetimes possess the particular feature that with g being a Lorentzian metric, also $-g$ is a Lorentzian metric, while spacelike and timelike directions are exchanged.

Theorem 4.32. *Let $(\mathbb{R} \times \mathbb{S}^1, g, \partial_t)$ be a proper kinematical Lorentzian cylinder spacetime. Then one and only one of the following three cases holds.*

(i) $(\mathbb{R} \times \mathbb{S}^1, g)$ *is totally vicious and* $(\mathbb{R} \times \mathbb{S}^1, -g)$ *is globally hyperbolic.*
(ii) $(\mathbb{R} \times \mathbb{S}^1, g)$ *and* $(\mathbb{R} \times \mathbb{S}^1, -g)$ *are non–totally-vicious but non-causal.*
(iii) $(\mathbb{R} \times \mathbb{S}^1, g)$ *is globally hyperbolic and* $(\mathbb{R} \times \mathbb{S}^1, -g)$ *is totally vicious.*

Proof. We choose global coordinates $(t, x \bmod 1) \in \mathbb{R} \times \mathbb{S}^1$, such that the canonical vector field $\partial_t \in \Gamma(T(\mathbb{R} \times \mathbb{S}^1))$ is the reference frame of the proper kinematical spacetime and the canonical vector field $\partial_x \in \Gamma(T(\mathbb{R} \times \mathbb{S}^1))$ induces closed curves, which are not homotopic to a point. Following Thm. 4.16, we can write the metric g in the following way:

$$g_{(t,x)} = -\mathrm{d}t \otimes \mathrm{d}t + 2\beta(t,x)\mathrm{d}t \vee \mathrm{d}x + (\eta(t,x) - \beta^2(t,x))\mathrm{d}x \otimes \mathrm{d}x,$$

with $\beta\colon \mathbb{R} \times \mathbb{S}^1 \to \mathbb{R}$ and $\eta\colon \mathbb{R} \times \mathbb{S}^1 \to \mathbb{R}_{>0}$ functions, periodic in x, generating one-forms $b_t = \beta(t,x)\mathrm{d}x$ and Riemannian metrics $h_t = \eta(t,x)\mathrm{d}x \otimes \mathrm{d}x$ on $\mathbb{S}^1$ for all $t \in \mathbb{R}$.

If $(\mathbb{R} \times \mathbb{S}^1, g)$ is globally hyperbolic, it is in particular stably causal and by Prop. 4.26 the slices $S_t = \{t\} \times \mathbb{S}^1$ are spacelike for all $t \in \mathbb{R}$, i.e, $\eta(t,x) > \beta^2(t,x)$ for all $(t,x) \in \mathbb{R} \times \mathbb{S}^1$ (maybe after employing a change of slicing). Hence, for the Lorentzian metric

$$-g = \mathrm{d}t \otimes \mathrm{d}t - 2\beta \mathrm{d}t \vee \mathrm{d}x - (\eta - \beta^2)\mathrm{d}x \otimes \mathrm{d}x$$

all integral curves of ∂_x, through all points, are CTCs, i.e., $(\mathbb{R} \times \mathbb{S}^1, -g)$ is totally vicious in this case. If we start from global hyperbolicity of $(\mathbb{R} \times \mathbb{S}^1, -g)$, we get total viciousness of $(\mathbb{R} \times \mathbb{S}^1, g)$ in the same way. This shows the existence of the cases *(i)* and *(iii)* and their mutual exclusion.

Furthermore, this implies that whenever $(\mathbb{R} \times \mathbb{S}^1, g)$ or $(\mathbb{R} \times \mathbb{S}^1, -g)$ is globally hyperbolic or totally vicious, the other one of both must obey the opposite causality condition. Thus, whenever neither *(i)* nor *(iii)* holds, $(\mathbb{R} \times \mathbb{S}^1, g)$ and $(\mathbb{R} \times \mathbb{S}^1, -g)$ must be both non–totally-vicious and non–globally-hyperbolic. As $\mathbb{S}^1$ is compact, employing Prop. 4.31, this implies that both spacetimes are also causally imprisoning. It remains to show that a Lorentzian cylinder spacetime is non-causal if it is causally imprisoning.

Let $\mu\colon \mathbb{R} \to \mathbb{R} \times \mathbb{S}^1$ be an inextendible, causal and future-directed curve, which is imprisoned in a compact set, say $K = I \times \mathbb{S}^1 \subset \mathbb{R} \times \mathbb{S}^1$ for some compact interval $I \subset \mathbb{R}$. Employing Prop. 3.39, we can infer that the limit set $\omega(\mu)$ is compact. Furthermore, we can assume that $(\mathbb{R} \times \mathbb{S}^1, g)$ is chronological, because otherwise there is nothing to show. This implies that we can employ Prop. 3.4 from [Min08a], which assures that in the chronological case through all $p \in \omega(\mu)$, there passes exactly one inextendible lightlike curve that is completely contained in $\omega(\mu)$, and is, therefore, also future imprisoned. Let $\lambda\colon \mathbb{R}_{\geq 0} \to \mathbb{R} \times \mathbb{S}^1$ be such a future imprisoned, lightlike curve, hence, $\lambda(\mathbb{R}_{>0}) \subset K$. This yields that $\mathrm{pr}_2(\lambda(\mathbb{R}_{>0})) = \mathbb{S}^1$, i.e., λ intersects every partial orbit $I \times \{x\}$ $(x \in \mathbb{S}^1)$ in K. This can be seen as follows: We can write $\lambda(s) = (t(s), c(s))$ for $s \geq 0$ and $c(s)$ is not constant in any finite interval of the curve parameter, as otherwise λ would be identical, in this interval, to an integral curve of ∂_t, which is timelike. As $c\colon \mathbb{R}_{\geq 0} \to \mathbb{S}^1$, we can infer that c takes all values in $\mathbb{S}^1$, eventually even infinitely often, due to the fact that $c(s) \in \mathbb{S}^1$, and thus it is periodic.

Now, we analyze the points for which λ intersects a specific partial orbit, say $I \times \{x\}$ for some $x \in \mathbb{S}^1$. Let $s_n(x) \in \mathbb{R}$ for all $n \in \mathbb{N}$ be the curve parameters for which $\mathrm{pr}_2(\lambda(s_n(x))) = x$ holds. Now we can assume $t(s_{n+1}(x)) > t(s_n(x))$ for all $n \in \mathbb{N}$, because otherwise we had already found a closed causal curve by combining $\lambda\colon [s_n(x), s_{n+1}(x)] \to K$ with the orbit piece $[t(s_{n+1}(x)), t(s_n(x))] \times \{x\}$, which would yield a closed causal curve. Thus the sequence $\{\lambda(s_n(x))\}_{n\in\mathbb{N}} \subset I \times \{x\}$ is strictly increasing and bounded from above on the fiber, such that $\lambda(s_n(x))$ converges to some point on $I \times \{x\}$, which certainly is contained in $\omega(\lambda)$. As this holds for all $x \in \mathbb{S}^1$, we can infer that $\omega(\lambda) \supset \{\lim_{n\to\infty}(t(s_n(x)), x) \,|\, x \in \mathbb{S}^1\}$. But this set inclusion is even an equality. This can be seen as follows: Assume $(t_0, y) \in \omega(\lambda)$ and let $\{\sigma_n\}_{n\in\mathbb{N}}$ $subset\mathbb{R}$ be a strictly increasing sequence, such that $\lambda(\sigma_n) \to (t_0, y)$ as $n \to \infty$. Then, as $c(s)$ is never constant, there is some $\tau_n \in \mathbb{R}$, such that $\lambda(\sigma_n - \tau_n) \in I \times \{y\}$ for all $n \in \mathbb{N}$. But certainly $\tau_n \to 0$ as $n \to \infty$, such that $\lambda(\sigma_n - \tau_n) \to (t_0, y)$ as $n \to \infty$ and $\{\lambda(\sigma_n - \tau_n)\}_{n\in\mathbb{N}}$ is the sequence in the partial fiber $I \times \{y\}$, which ensures the equality of the two sets above.

Therefore, $\omega(\lambda)$ is a section over $\mathbb{S}^1$ in $K \subset \mathbb{R} \times \mathbb{S}^1$, as for all $x \in \mathbb{S}^1$ there is exactly one point on $I \times \{x\}$ in $\omega(\lambda)$. Now, we can again employ Prop. 3.39 to the limit set $\omega(\lambda)$ and deduce that through all $p \in \omega(\lambda)$ there passes exactly one inextendible lightlike curve completely contained in $\omega(\lambda)$. This is only possible if $\omega(\lambda)$ is a continuous section generated by exactly one lightlike curve, which is, therefore, a closed causal curve, i.e, the spacetime is non-causal. This implies that the only remaining situation, which can happen if neither *(i)* nor *(iii)* holds, is item *(ii)*. □

Hence, for $\dim(S) = 1$ and compact S, causality of $(\mathbb{R}\times\mathbb{S}^1, g, \partial_t)$ does certainly imply global hyperbolicity, because the only case that involves a non-causal Lorentzian cylinder, and none of a higher level of causality, is item *(ii)* in the theorem above. Hence, a causal Lorentzian cylinder enforces items *(i)* or *(iii)* to hold and, therefore, it is a globally hyperbolic spacetime, either with metric g or $-g$.

For $\dim(S) = 2$ and compact S, the situation is a bit more involved. We can infer from Example 4.12 that there is a proper kinematical spacetime which is causal, but causally imprisoning, if S is a 2-torus. But we will show in the following that this cannot happen if S is *not* a torus. To this end we need a suitably strong version of the Poincaré–Bendixson theorem. We recall Def. 2.54, as well as Rem. 2.55 and state the following

Theorem 4.33. *Let S be a compact, 2-dimensional manifold, which is at least of class C^2 and not a torus. Let $\Phi\colon \mathbb{R} \times S \to S$ be an $\mathbb{R}$-action on S, which is also at least of class C^2 and constitutes a dynamical system on S. Then every Φ-minimal set $A \subset S$ is either a fixed point or a closed orbit homeomorphic to $\mathbb{S}^1$. Furthermore, let S be orientable and γ be an orbit of Φ, such that $\omega(\gamma)$ is non-empty and contains no fixed points, then $\omega(\gamma)$ is homeomorphic to $\mathbb{S}^1$.*

Proof. See the theorem and the corollary in [Sch63]. □

We would like to apply this version of the Poincaré–Bendixson theorem to a spacetime in the theorem below. Hence, we will assume that the spacetime is at least of class C^2 and all geometric objects are C^2-smooth in the following

Theorem 4.34. *Assume (M, g, V) is a proper kinematical spacetime of class C^2, with the space of flow lines $S = M/_{\mathbb{R}}$ compact and orientable, $\dim(S) = 2$ and admitting a smooth semi-time function. If (M, g) is causal and S is not a torus, (M, g) is globally hyperbolic.*

Proof. Let $t\colon M \to \mathbb{R}$ be the smooth semi-time function, which we can assume surjective (maybe by composition with a strictly increasing diffeomorphism). A chain of argument completely analogous to Lem. 4.18 and Prop. 4.26 yields a smooth trivialization

$\psi\colon (M,g,V) \to (\mathbb{R}\times S, \psi_* g, \psi_* V)$, such that the slices S_t are non-Lorentzian everywhere, and they are even smooth achronal hypersurfaces, due to the existence of a smooth semi-time function. Specifically, this yields that any sort of causality violation has to be generated by some lightlike curve, totally contained in one slice.

As we are only interested in causal properties we can consider the conformal class of the Lorentzian metric g and assume

$$\psi_* g = -\mathrm{d}t \otimes \mathrm{d}t + 2b_t \vee \mathrm{d}t + h_t - b_t \otimes b_t,$$

such that $\{b_t\}_{t\in\mathbb{R}}$ is a family of one-forms on S and $\{h_t\}_{t\in\mathbb{R}}$ is a family of Riemannian metrics, with the usual omittance of the pull-back pr_2^*. As the slices are achronal, we infer completely analogous to Lem. 4.23 that $\|b_t\|^{h_t} \leq 1$ for all $t \in \mathbb{R}$, and if a slice S_t is degenerate at some $x \in S$, we have $\|b_{(t,x)}\|_x^{h_t} = 1$ there.

For all $t \in \mathbb{R}$ there is a unique vector field $B_t \in \Gamma(TS)$ given by $h_t(B_t, \cdot) = b_t$. The lift of B_t to $\mathbb{R}\times S$ is a non-timelike vector field everywhere (as $\|b_t\|^{h_t} \leq 1$) and becomes lightlike at $(t,x) \in \mathbb{R}\times S$ if and only if $\|b_{(t,x)}\|_x^{h_t} = 1$. This can be seen as follows: We denote the lift of B_t to $\mathbb{R}\times S$ by $\tilde{B}_t$, which is the unique vector field tangential to the slices S_t (i.e., $\mathrm{d}t(\tilde{B}_t) = 0$), that projects to B_t on S under the action of pr_{2*}. We get

$$g(\tilde{B}_t, \tilde{B}_t) = h_t(B_t, B_t) - b_t(B_t)^2 = b_t(B_t)\left[1 - \sqrt{\|b_t\|^{h_t}}\right].$$

As S is compact, B_t is a complete vector field for all $t \in \mathbb{R}$, i.e., there is also a global flow $\Phi^t\colon \mathbb{R}\times S_t \to S_t$ associated to $\tilde{B}_t$ for all $t \in \mathbb{R}$, which constitutes a dynamical system.

Now, we show that $(\mathbb{R}\times S, \psi_* g)$ is non-causal if it is causally imprisoning and the same will hold for (M,g). Employing Prop. 4.31, this shows that (M,g) is globally hyperbolic if it is causal. Let $\mu\colon \mathbb{R} \to \mathbb{R}\times S$ be a future imprisoned causal curve, i.e., $\omega(\mu)$ is non-empty and compact. We can assume that the spacetime is chronological, as otherwise there is nothing to show, hence we get from Prop. 3.4 in [Min08a], that through every point $p \in \omega(\mu)$ there passes an inextendible lightlike curve completely contained in $\omega(\mu)$. As we have a slicing according to the semi-time function t, we can infer that t is non-decreasing along μ. Therefore, the limit points $p \in \omega(\mu)$ are all contained in one slice, say S_t. Thus, we have an inextendible lightlike curve $\lambda\colon \mathbb{R} \to \mathbb{R}\times S$ obeying $\lambda(\mathbb{R}) \subset \omega(\mu) \subset S_t$. Now, as the only vectors tangential to the slice S_t, which can become lightlike, are the elements of the vector field $\tilde{B}_t$, we can infer that λ is an orbit of Φ^t on S_t. Hence, λ is an orbit of a C^2-dynamical system on a C^2-manifold S_t. Again we can examine $\omega(\lambda)$, which is also generated by inextendible lightlike curves, due to Prop. 3.4 in [Min08a], and thus contains no fixed points. Now, we can employ Thm. 4.33 to $\omega(\lambda)$ and deduce that it is homeomorphic to $\mathbb{S}^1$, and hence it constitutes a closed causal curve, i.e., the spacetime is non-causal. □

Remark 4.35. *Following Rem. 3.6, the orientability of S, in the theorem above, can be achieved by demanding the orientability of M. The orientability of M yields the orientability of $TM = \xi \oplus \nu$, and as (M,g) is certainly time-oriented as a spacetime by V, the subbundle ξ is oriented, hence, we also have the orientability of ν. But ν is isomorphic to the tangent bundle TS, due to the splitting $M = \mathbb{R}\times S$, therefore, S is orientable.*

If we now switch to the case of $\dim(S) = 3$, we are looking for propositions similar to the Poincaré–Bendixson theorem above, that assure the existence of a closed orbit of a dynamical system, induced by the one-forms b_t on S if b_t corresponds to a causal vector field tangential to the slices S_t. Unfortunately, the structure of limit sets of dynamical systems in three dimensions is much harder to assess and there is even the possibility of chaotic behavior and fractal limit sets (see, e.g., the discussion in Part 2 of [NS60]). A solution in some special cases is provided by contact geometry, i.e., if the b_t are contact forms and the induced dynamical system on S is generated by the Reeb vector field. Then we can employ the Weinstein conjecture and, particularly, Thm. 2.14 to infer the existence of a closed orbit.

Proposition 4.36. *Let $(\mathbb{R} \times S, g, \partial_t)$ be a proper kinematical spacetime, with $\dim(S) = 3$ and S compact. As usual, there is a family of one-forms $\{b_t\}_{t\in\mathbb{R}} \subset \Gamma(\Lambda^1 S)$ and a family of Riemannian metrics $\{h_t\}_{t\in\mathbb{R}} \subset \mathcal{R}(S)$, such that*

$$g_{(t,x)} = -\mathrm{d}t \otimes \mathrm{d}t + 2\mathrm{pr}_2^*(b_{(t,x)}) \vee \mathrm{d}t + \mathrm{pr}_2^*(h_{(t,x)}) - \mathrm{pr}_2^*(b_{(t,x)}) \otimes \mathrm{pr}_2^*(b_{(t,x)}).$$

Let $B_t \in \Gamma(TS)$ be the unique vector field given by $b_t = h_t(B_t, \cdot)$. Assume there is some $T \in \mathbb{R}$, such that $\pounds_{B_T} b_T = 0$, $\|b_T\|_x^{h_T} = 1$ for all $x \in S$ and b_T is a contact form on S, then $(\mathbb{R} \times S, g)$ is not causal. Particularly, this implies that if such a spacetime is causal, then it is globally hyperbolic.

Proof. As in the proof of Thm. 4.34, the condition $\|b_T\|_x^{h_T} = 1$ implies that the lift of the vector field $\tilde{B}_T$ to $\mathbb{R} \times S$, which is tangential to S_T, is a lightlike vector field. In this case we get $g(\tilde{B}_T, \tilde{B}_T) = 0$ for all $(T, x) \in \mathbb{R} \times S$, i.e., the integral curves of $\tilde{B}_T$ are lightlike curves imprisoned in the compact slice S_T. Thus, we show that there is at least one closed orbit of B_T on S, i.e., there is a closed causal integral curve of $\tilde{B}_T$ on S_T.

As $\|b_T\|_x^{h_T} = 1$, we get $b_T(B_T) = 1$ and

$$0 = \pounds_{B_T} b_T = B_T \rfloor \mathrm{d}b_T + \mathrm{d}(B_T \rfloor b_T) = B_T \rfloor \mathrm{d}b_T,$$

such that B_T is the Reeb vector field associated to the contact form b_T. As S is three-dimensional and compact, we infer from Thm. 2.14 that there is at least one closed orbit of B_T, i.e., the spacetime is non-causal. The remaining assertion follows from Prop. 4.31. $\square$

If we now consider particular proper kinematical spacetimes, we can give a surprising condition equivalent to the Weinstein conjecture, using Lorentzian causality theory. Let $Y\colon \mathbb{R} \to \mathbb{R}_{\geq 1}$ be a smooth function which obeys $Y(T) = 1$ for exactly one $T \in \mathbb{R}$ and $Y(t) > 1$ for all $t \in \mathbb{R}$ with $t \neq T$.

Proposition 4.37. *The Weinstein conjecture holds if and only if all proper kinematical spacetimes $(\mathbb{R} \times S, g, \partial_t)$ with S compact, $\dim(S) = 2n + 1$ for $n \in \mathbb{N}$ and metric given by*

$$g = -\mathrm{d}t \otimes \mathrm{d}t + 2\mathrm{pr}_2^*(b) \vee \mathrm{d}t + Y(t)\mathrm{pr}_2^*(h) - \mathrm{pr}_2^*(b) \otimes \mathrm{pr}_2^*(b),$$

for a contact form b on S, a Riemannian metric h on S obeying $\|b\|^h = 1$ and an h-geodesic vector field $B \in \Gamma(TS)$ obeying $h(B, \cdot) = b$, are non-causal.

Proof. "$\Leftarrow$": Due to the function Y, we have that $\|b\|^{Y(t)h} < 1$ for all $t \neq T$. Hence, the only violations of causal continuity can occur, for this spacetime, by the behavior of the integral curves of B, lifted to $\mathbb{R} \times S$, on the slice S_T, where B is lightlike. Furthermore, B is the Reeb vector field for b, as obviously $b(B) = 1$ and $\nabla^h_B B = 0$, where ∇^h denotes the Levi-Civita connection associated to h on S. Similar to the kinematical quantities defined in section 3.1, we can decompose the exterior derivative $\mathrm{d}b$ into a part $\omega^\perp$, which obeys $B \rfloor \omega^\perp = 0$ and a part containing $\dot{b} := h(\nabla^h_B B, \cdot)$, such that $\mathrm{d}b = \omega^\perp + b \wedge \dot{b}$ holds. As $h(B, B) = 1$, we get $B \rfloor \mathrm{d}b = \dot{b}$, hence $\nabla^h_B B = 0$ implies $B \rfloor \mathrm{d}b = 0$ and B is indeed the Reeb vector field associated to b. Now, if $(\mathbb{R} \times S, g)$ is non-causal, there is a closed causal curve, which can only exist on the slice S_T as an integral curve of the lift of B. Thus, there is a closed orbit of the Reeb vector field B and the Weinstein conjecture holds.

"$\Rightarrow$": Assume (S, b) is a compact contact manifold and the Weinstein conjecture holds, i.e., for the Reeb vector field $B \in \Gamma(TS)$ obeying $b(B) = 1$ and $B \rfloor \mathrm{d}b = 0$, there exists a closed orbit. We will now construct a non-causal Lorentzian metric g on $\mathbb{R} \times S$ obeying the conditions in the proposition. We need a Riemannian metric h on S for which $h(B, \cdot) = b$ holds.

Such a metric can be constructed as follows: Regard S as the slice for $t = 0$ in $I \times S$, with I some (small) open interval containing 0. Then b and $\mathrm{d}b$ can be uniquely extended to $I \times S$ and $\omega = \mathrm{d}(e^t b)$ is a symplectic form on $I \times S$. It is known (cf., e.g., [McD98]) that there is a non-empty set of ω-compatible almost complex structures $J \colon T(I \times S) \to T(I \times S)$, $J^2 = -\mathrm{id}$ obeying $\omega(X, Y) = \omega(JX, JY)$ for all vector fields X, Y on $I \times S$ and $\omega(X, JX) > 0$ for all nowhere zero vector fields X on $I \times S$. This implies that $\tilde{h}(\cdot, \cdot) = \omega(\cdot, J\cdot)$ is a Riemannian metric on $I \times S$. With respect to this metric we can now choose a vector field normal to S, i.e., some $n \in \Gamma(NS)$ obeying $\tilde{h}(X, n) = 0$ for all $X \in \Gamma(TS)$, where we denote the normal bundle of S in $I \times S$ by NS. The condition of X being tangential to S, certainly implies $\mathrm{d}t(X) = 0$, so that we can compute

$$0 = \tilde{h}(X, n) = \omega(X, Jn) = e^t(\mathrm{d}t \wedge b)(X, Jn) + e^t \mathrm{d}b(X, Jn) = \mathrm{d}b(X, Jn) - \mathrm{d}t(Jn)b(X) \quad (*)$$

on the slice S, where $t = 0$ holds. This yields $\mathrm{d}t(Jn) = 0$ and $Jn \rfloor \mathrm{d}b = 0$, i.e., Jn is tangential to S and proportional to the Reeb vector field B. Hence, by normalizing n properly and changing $\tilde{h}$ by a conformal factor, we can also achieve $b(Jn) = 1$, i.e., $B = Jn$. This can be done as follows: On S we compute

$$\tilde{h}(n, n) = \omega(n, Jn) = (\mathrm{d}t \wedge b)(n, Jn) = b(Jn)\mathrm{d}t(n).$$

Hence, normalizing n so that $\mathrm{d}t(n) = 1$ holds and switching to a metric $h = \frac{1}{\tilde{h}(n,n)} \tilde{h}$ (certainly this leaves the condition $(*)$ unchanged) yields $b(Jn) = 1$. Furthermore, we compute

$$h(X, Jn) = \omega(X, -n) = (\mathrm{d}t \wedge b)(n, X) = b(X)$$

on S for $t = 0$ and all $X \in \Gamma(TS)$. Thus, restricting h to the slice S yields a Riemannian metric obeying $h(B, \cdot) = b$.

Now, we take the Riemannian metric h on S from the paragraph above and perform the following construction. From the condition $h(B, \cdot) = b$ we get $h(B, B) = \|b\|^h = 1$ on S and, furthermore, with a chain of argument analogous to the proof of "$\Leftarrow$" above, we can conclude that B is h-geodesic. Now consider the product $\mathbb{R} \times S$ and any function $Y : \mathbb{R} \times S \to \mathbb{R}_{\geq 1}$ obeying $Y(T) = 1$ for exactly one $T \in \mathbb{R}$ and $Y(t) > 1$ for all $\mathbb{R} \ni t \neq T$. Then

$$g = -\mathrm{d}t \otimes \mathrm{d}t + 2\mathrm{pr}_2^*(b) \vee \mathrm{d}t + Y(t)\mathrm{pr}_2^*(h) - \mathrm{pr}_2^*(b) \otimes \mathrm{pr}_2^*(b)$$

is a Lorentzian metric on $\mathbb{R} \times S$ (b and $\mathrm{d}b$ are nowhere vanishing on S), and $(\mathbb{R} \times S, g, \partial_t)$ is a proper kinematical spacetime, as ∂_t is complete. As the vector field B lifted to S_T is obviously lightlike and has a closed orbit by assumption, the spacetime $(\mathbb{R} \times S, g)$ is non-causal. $\square$

The metric in Prop. 4.37 defines a Hubble-isotropic spacetime, which we will analyze in more detail in chapter 6. Although the proposition above constitutes an interesting correspondence between Lorentzian causality theory and the Weinstein conjecture in arbitrary dimensions, it seems unlikely that this could provide an alternative route towards a proof of the general Weinstein conjecture (cf. the detailed discussion in chapter 7).

Chapter 5

SLICED SPACETIMES

In this chapter, we will approach spacetimes with a fixed reference frame from a perspective which is in some sense dual to to the one adopted in the previous chapters. Instead of setting out from a Lorentzian manifold M with a Cartan flow and, subsequently, determining the circumstances leading to a splitting $M = \mathbb{R} \times S$, as well as analyzing the causality properties arising in this context, we start now from a spacetime already given in a general splitting form $I \times S$, with I an open interval of the real numbers. Furthermore, we will even assume the metric induced on the slices S_t to be Riemannian (with the exception of section 5.3, where this condition is not necessary). This will lead to the analysis of various special cases of these *sliced spacetimes* and, particularly, to the investigation of their global hyperbolicity, the "nicest" condition on the causal ladder.

Definition 5.1. *Let $I = (a, b)$ be an interval of the real numbers with $-\infty \leq a < b \leq \infty$. A spacetime (M, g) is called sliced spacetime if $M = I \times S$ and*

$$g_{(t,x)} = -N^2(t, x)\mathrm{d}t \otimes \mathrm{d}t + 2\mathrm{pr}_2^*(b_{(t,x)}) \vee \mathrm{d}t + \mathrm{pr}_2^*(h_{(t,x)}),$$

with $(t, x) \in I \times S$, $\{b_t\}_{t\in I}$ a family of one-forms on S varying smoothly with t, which are called shift, $\{h_t\}_{t\in I}$ a family of Riemannian metrics on S varying smoothly with t and $N\colon I \times S \to \mathbb{R}_{>0}$ a function, which is called lapse. We denote by $\mathrm{pr}_2\colon I \times S \to S$ the canonical projection on the second factor and the coordinate t is identified with the projection on the first factor: $t = \mathrm{pr}_1\colon I \times S \to I$.

In this chapter, we will again often use the natural identification of objects on the base S and their pull-back to M, and hence simply write w for a one-form $\mathrm{pr}_2^*(w)$ on M, which is pulled back from a one-form on S if this is no source of confusion. For example, the one-form b_t on S can be identified with its pull-back $\mathrm{pr}_2^*(b_t)$ and can hence also be regarded as a one-form on the slice $S_t = \{t\} \times S$. Note that the natural reference frame $V = N^{-1}\partial_t$ is not necessarily a complete vector field, but we have the following

Proposition 5.2. *Every sliced spacetime $(I \times S, g)$ is isometric to a spacetime $(\mathbb{R} \times S, \tilde{g})$, and there is a metric g^* pointwise conformal to $\tilde{g}$, such that $(\mathbb{R} \times S, g^*, V)$, with V pointing along the factor $\mathbb{R}$, is a proper kinematical spacetime. Hence, every sliced spacetime is causally continuous.*

Proof. By virtue of a strictly increasing diffeomorphism $f\colon I \to \mathbb{R}$, we get an isometry $\psi\colon (I \times S, g) \to (\mathbb{R} \times S, \tilde{g})$, $\psi(t, x) = (f(t), x)$ by setting the Lorentzian metric

$$\tilde{g}_{(f,x)} = (\psi_* g)_{(f,x)} = -(N \circ \psi^{-1})^2(f, x)[(f^{-1})']^2\mathrm{d}f \otimes \mathrm{d}f + 2b_{\psi^{-1}(f,x)} \vee (f^{-1})'\mathrm{d}f + h_{\psi^{-1}(f,x)}$$

on $\mathbb{R}\times S$. Then setting $g^* = [(N\circ\psi^{-1})(f^{-1})']^2\tilde{g}$, we have $g^*(\partial_f, \partial_f) = -1$ and ∂_f is complete on $\mathbb{R}\times S$ by the same argument as in the proof of Lem. 4.23. Hence, $(\mathbb{R}\times S, g^*, V)$ with $V = \partial_f$ is a proper kinematical spacetime, which is causally continuous by Prop. 4.26 and Thm. 4.29, as the restriction of g^* to any slice of constant f is a Riemannian metric. Thus, $(I\times S, g)$ is causally continuous, too. □

The notion of regularly sliced spacetimes was introduced by Y. Choquet-Bruhat and S. Cotsakis in [CBC02] and further investigated in [Cot04] (cf. also [CBC03]). The terminology *regularly sliced spacetime* was introduced in [CB09]. The idea is to constrain the geometric objects lapse, shift and the Riemannian metric constituting a sliced spacetime by some bounds and regularity conditions in order to have straightforward access to a simple theorem on global hyperbolicity for these spacetimes. As can be seen from the definition below, the bounds are quite restrictive, but we will see in the following section that actually many examples of globally hyperbolic spacetimes occur in this form. We will give a definition of a regularly sliced spacetime, that is slightly more general than the one used in [Cot04] and [CB09], as we will allow for different upper and lower metric bounds γ and $\tilde{\gamma}$ (See item *(iii)* in the definition below). Indeed, it will turn out in the following section that the condition of equal upper and lower metric bounds $\gamma = \tilde{\gamma}$ is very restrictive.

Definition 5.3. *A sliced spacetime $(M = I\times S, g)$ with metric given by*

$$g_{(t,x)} = -N^2(t,x)\mathrm{d}t\otimes\mathrm{d}t + 2b_{(t,x)}\vee\mathrm{d}t + h_{(t,x)},$$

for all $(t,x)\in I\times S$ is called regularly sliced if the following three conditions hold:

(i) The lapse is bounded from below and above by constants $n_0, n_1 > 0$, such that

$$0 < n_0 \le N(t,x) \le n_1,$$

for all $(t,x)\in M$.

(ii) The h-norm of the shift is bounded by a constant $A > 0$, such that

$$\|b\|^h_{(t,x)} < A,$$

for all $(t,x)\in M$.

(iii) The Riemannian metrics h_t are bounded from above and below by Riemannian metrics $\gamma, \tilde{\gamma}$ on S. That is, there are constants $0 < B \le C < \infty$, such that

$$B\tilde{\gamma}_x(v,v) \le h_{(t,x)}(v,v) \le C\gamma_x(v,v),$$

for all $(t,x)\in M$ and all $v\in T_xS$.

Particularly, we will also allow for the following relaxation of a spacetime being regularly sliced. We call $(I\times S, g)$ regularly sliced from below (above) if (i) and (ii) holds, but (iii) is modified to possibly hold only with $C = \infty$ $(B = 0)$.

5.1 Global Hyperbolicity of Sliced Spacetimes

The following theorem by M. Sánchez and A. Bernal (cf. [BS05]) highlights the particular relevance of the step of global hyperbolicity on the causal ladder. The theorem shows that globally hyperbolic spacetimes are a particular class of sliced spacetimes, although not regularly sliced ones in general.

Theorem 5.4. *Let (M, g) be a globally hyperbolic spacetime, then it is isometric to $(\mathbb{R}\times S, g^*)$ with*

$$g^* = -N^2 \mathrm{dt} \otimes \mathrm{dt} + h_t,$$

with $N\colon \mathbb{R} \times S \to \mathbb{R}_{>0}$ and $\{h_t\}_{t\in\mathbb{R}}$ a family of Riemannian metrics on S varying smoothly with t. Hence, any globally hyperbolic spacetime is a sliced spacetime with vanishing shift and $I = \mathbb{R}$.

Proof. See [BS05] and compare to Def. 5.1. □

Theorem 5.5. *Suppose a regularly sliced spacetime obeys $\gamma = \tilde{\gamma}$, i.e., it is regularly sliced with the same Riemannian metric γ bounding the family $\{h_t\}_{t\in I}$ of Riemannian metrics from below and above. Then the spacetime has the slices $S_t = \{t\} \times S$, $t \in I$ as Cauchy surfaces—and is, therefore, globally hyperbolic—if and only if the Riemannian metric γ on S is complete. Particularly,*

(i) a spacetime regularly sliced from below is globally hyperbolic if $\tilde{\gamma}$ is complete, and,

(ii) a globally hyperbolic spacetime which is regularly sliced from above has complete γ.

Proof. See [CBC02], [CBC03] or [CB09] for *(i)*, and [Cot04] for *(ii)*. Then the first assertion follows from *(i)* and *(ii)*. □

The conditions for a spacetime to be regularly sliced are not conformally invariant, nor is the completeness condition involved in the theorem on global hyperbolicity. Particularly, not all globally hyperbolic spacetimes are regularly sliced. This is shown by the following example.

Example 5.6. *Let $\mathring{S} = \mathbb{R}^3 \setminus \{0\}$ be the 3-dimensional punctured Euclidean space and $M = \mathbb{R} \times \mathring{S}$ a sliced spacetime with metric*

$$g = -r^2 dt^2 + \sum_{i,j} \delta_{ij} dx^i dx^j,$$

with $r^2 = \sum_{i=1}^3 (x^i)^2$. It is well-known that the standard Euclidean metric δ on $\mathring{S}$ is incomplete. Moreover, the lapse $N(x^i) = r$ is unbounded on M, i.e., the spacetime is not regularly sliced. But if we pass over to the conformal metric

$$g' = \frac{1}{r^2} g = -dt^2 + \frac{1}{r^2} \sum_{i,j} \delta_{ij} dx^i dx^j,$$

we see that (M, g') is regularly sliced because now the lapse is bounded and $\frac{1}{r^2}\delta$ is complete on $\mathring{S}$. Therefore, (M, g') and (M, g) are globally hyperbolic.

However, subsequently, we will show that necessarily at least one element of the conformal class of a globally hyperbolic spacetime is regularly sliced.

Lemma 5.7. *Let the metric of a regularly sliced spacetime $(M = I \times S, g)$, with $\tilde{\gamma} = \gamma$, be given by*

$$g = -\mathrm{d}t \otimes \mathrm{d}t + h_t.$$

Then (M, g) is globally hyperbolic if and only if all Riemannian metrics h_t on the slices S_t $(t \in I)$ are complete.

Proof. This follows directly from Thm. 5.5. We have $B\gamma \leq h_t \leq C\gamma$ in this case. If (M, g) is globally hyperbolic, γ is complete, hence $B\gamma$ is complete and all h_t are complete, as $B\gamma \leq h_t$. If all h_t are complete, so is $C\gamma$ as $h_t \leq C\gamma$, hence γ is complete and (M, g) is globally hyperbolic. □

Below we will need the following lemma established by S. Born and the author in [BD12].

Lemma 5.8. *Let $\mathfrak{T}_1$ and $\mathfrak{T}_2$ be two locally compact topological spaces, which are Hausdorff and countable at infinity. Let $F \colon \mathfrak{T}_1 \times \mathfrak{T}_2 \to \mathbb{R}$ be a continuous function. Then there are two continuous functions $f_1 \colon \mathfrak{T}_1 \to \mathbb{R}$ and $f_2 \colon \mathfrak{T}_2 \to \mathbb{R}$, such that $F \leq f_1 + f_2$. Particularly, if $\mathrm{im}(F) \subset \mathbb{R}_{>0}$, there are four functions $g_1 \colon \mathfrak{T}_1 \to \mathbb{R}_{>0}$, $g_2 \colon \mathfrak{T}_2 \to \mathbb{R}_{>0}$ and $G_1 \colon \mathfrak{T}_1 \to \mathbb{R}_{>0}$, $G_2 \colon \mathfrak{T}_2 \to \mathbb{R}_{>0}$, such that $g_1 \cdot g_2 \leq F \leq G_1 \cdot G_2$, and g_1, g_2, G_1, G_2 can be chosen to be smooth if $\mathfrak{T}_1$ and $\mathfrak{T}_2$ are manifolds.*

Proof. We can assume that $F(x_1, x_2) \geq 0$ for all $x_1 \in \mathfrak{T}_1$ and $x_2 \in \mathfrak{T}_2$. If this is not the case, we just replace $F(x_1, x_2)$ by $\max\{F(x_1, x_2), 0\} \geq F(x_1, x_2)$.

The statement is obvious if $\mathfrak{T}_1$ or $\mathfrak{T}_2$ is compact: If $\mathfrak{T}_1$ is compact, we have $F(x_1, x_2) \leq \max_{x_1 \in \mathfrak{T}_1} F(x_1, x_2) =: f_2(x_2)$. The same argument applies if $\mathfrak{T}_2$ is compact. Therefore, we assume that neither $\mathfrak{T}_1$ nor $\mathfrak{T}_2$ is compact.

By assumption (locally compact and countable at infinity), there are exhaustions

$$\mathfrak{T}_1 = \bigcup_{i \in \mathbb{N}_0} K_i, \quad \mathfrak{T}_2 = \bigcup_{i \in \mathbb{N}_0} L_i,$$

with

$$\emptyset = K_0,\ K_i \subset \mathring{K}_{i+1},$$

such that the K_i's are compact subsets of $\mathfrak{T}_1$ for all $i \in \mathbb{N}_0$ and

$$\emptyset = L_0,\ L_i \subset \mathring{L}_{i+1},$$

such that the L_i's are compact subsets of $\mathfrak{T}_2$ for all $i \in \mathbb{N}_0$.

Furthermore, we set $U_i = \mathring{K}_i$ with $U_i \subset \mathfrak{T}_1$ an open set for all $i \in \mathbb{N}$, as well as $V_i = \mathring{L}_i$ with $V_i \subset \mathfrak{T}_2$ an open set for all $i \in \mathbb{N}$. Hence, $\{U_i\}_{i\in\mathbb{N}}$ is an open cover of $\mathfrak{T}_1$ and $\{V_i\}_{i\in\mathbb{N}}$ is an open cover of $\mathfrak{T}_2$. The spaces M and N are paracompact, as they are locally compact, Hausdorff and countable at infinity, hence there are partitions of unity $\{\phi_i \colon \mathfrak{T}_1 \to \mathbb{R}\}_{i\in\mathbb{N}}$ resp. $\{\chi_i \colon \mathfrak{T}_2 \to \mathbb{R}\}_{i\in\mathbb{N}}$ that are subordinate to locally finite refinements of $\{U_i\}_{i\in\mathbb{N}}$ resp. $\{V_i\}_{i\in\mathbb{N}}$.

Now we define

$$a_i := \max_{(x_1,x_2)\in K_i\times L_i} F(x_1,x_2).$$

and two functions $f_1 \colon \mathfrak{T}_1 \to \mathbb{R}$ and $f_2 \colon \mathfrak{T}_2 \to \mathbb{R}$:

$$f_1(x_1) := \sum_{i\in\mathbb{N}} \phi_i(x_1)a_i$$

and

$$f_2(x_2) := \sum_{i\in\mathbb{N}} \chi_i(x_2)a_i.$$

Note that in each series we sum only finitely many non-zero terms for every $x_1 \in \mathfrak{T}_1$ resp. $x_2 \in \mathfrak{T}_1$, and f_1 and f_2 are continuous functions.

Now let (x_1, x_2) be an arbitrary point in $\mathfrak{T}_1 \times \mathfrak{T}_2$. Then define m to be the smallest natural number such that $x_1 \in U_m$ and n to be the smallest natural number such that $x_2 \in V_n$. Hence, $x_1 \notin U_i$ for $i < m$ and $x_2 \notin V_i$ for $i < n$.

This yields

$$\begin{aligned} f_1(x_1) + f_2(x_2) &= \sum_{i\in\mathbb{N}} \phi_i(x_1)a_i + \sum_{i\in\mathbb{N}} \chi_i(x_2)a_i \\ &= \sum_{i\geq m} \phi_i(x_1)a_i + \sum_{i\geq n} \chi_i(x_1)a_i \\ &\geq a_m + a_n \geq F(x_1,x_2)\ , \end{aligned}$$

as $(x_1, x_2) \in K_k \times L_k$, where $k = \max\{m, n\}$.

Obviously, if $\mathfrak{T}_1$ and $\mathfrak{T}_2$ are manifolds, they allow for smooth partitions of unity and the construction above can be carried out with these smooth partitions of unity, such that the resulting functions f_1 and f_2 are smooth.

Moreover, by taking exponentials we get

$$F(x_1,x_2) \leq f_1(x_1) + f_2(x_2) \leq e^{f_1(x_1)+f_2(x_2)} = e^{f_1(x_1)}e^{f_2(x_2)} =: G_1(x_1)G_2(x_2).$$

Assuming $\mathrm{im}(F) \subset \mathbb{R}_{>0}$, we can analyze $\tilde{F}(x_1,x_2) = [F(x_1,x_2)]^{-1}$ and we find two functions $\tilde{G}_1$ and $\tilde{G}_2$ such that $\tilde{F}(x_1,x_2) \leq \tilde{G}_1(x_1)\tilde{G}_2(x_2)$. Setting $g_1(x_1) = [\tilde{G}_1(x_1)]^{-1}$ and $g_2(x_2) = [\tilde{G}_2(x_2)]^{-1}$ yields

$$g_1(x_1)g_2(x_2) \leq F(x_1,x_2).$$

$\square$

Applying this to sliced spacetimes yields the following

Lemma 5.9. *For any sliced spacetime $((a,b)\times S, g)$ with $-\infty \leq a < b \leq \infty$ and*

$$g = -\mathrm{d}t \otimes \mathrm{d}t + h_t,$$

there is a function $\Omega^2\colon (a,b) \to (0,\infty)$, such that the conformal metric

$$g^* = \Omega^2 g = -\Omega^2(t)\mathrm{d}t^2 + \Omega^2(t) h_t = -\Omega^2(t)\mathrm{d}t^2 + h_t^* = -\mathrm{d}\tau^2 + h_\tau^*,$$

after the transformation $\tau(t) = \int_{t_0}^t \Omega(s)\mathrm{d}s$ with arbitrary fixed $t_0 \in (a,b)$, on $(a^*,b^*)\times S$ with $a^* = \lim_{t\to a}\tau(t)$ and $b^* = \lim_{t\to b}\tau(t)$, is regularly sliced.

Proof. Fixing any $t_0 \in I = (a,b)$ we define two norms $\|\cdot\|^{\uparrow}$ and $\|\cdot\|^{\downarrow}$ for the family of Riemannian metrics h_t on S, depending on $h^0 := h_{t_0}$, by setting

$$\|h_t\|_x^{\uparrow} := \sup_{\|v\|_x^{h^0}=1} h_{(t,x)}(v,v)$$

and

$$\|h_t\|_x^{\downarrow} := \inf_{\|v\|_x^{h^0}=1} h_{(t,x)}(v,v),$$

for all $(t,x) \in I \times S$. Then we have for all Riemannian metrics h_t that

$$0 < \|h_t\|_x^{\downarrow} \leq \|h_t\|_x^{\uparrow} < \infty,$$

for all $x \in S$ and all $t \in I$.

First, we show that the condition of being regularly sliced

$$B\tilde{\gamma} \leq h_t \leq C\gamma,$$

for a family of Riemannian metrics h_t, Riemannian metrics γ, $\tilde{\gamma}$ and constants $0 < B \leq C < \infty$ is equivalent to

$$D(x) \leq \|h_t\|_x^{\downarrow} \leq \|h_t\|_x^{\uparrow} \leq E(x),$$

for all $(t,x) \in I \times S$ with $D, E\colon S \to \mathbb{R}_{>0}$ two functions, and that this equivalence is independent of the fixed t_0 chosen to define the norms $\|\cdot\|^{\uparrow}$ and $\|\cdot\|^{\downarrow}$.

Certainly, $B\tilde{\gamma} \leq h_t \leq C\gamma$ implies $D(x) \leq \|h_t\|_x^{\downarrow} \leq \|h_t\|_x^{\uparrow} \leq E(x)$ by setting $D(x) = B\|\tilde{\gamma}\|_x^{\downarrow}$ and $E(x) = C\|\gamma\|_x^{\uparrow}$. Obviously, this holds for any chosen fixed $t_0 \in I$.

Assume now that there is a function $E\colon S \to \mathbb{R}$, such that $\|h_t\|_x^{\uparrow} \leq E(x)$ holds for all $x \in S$, but there is no Riemannian metric γ on S, such that $h_{(t,x)}(v,v) \leq \gamma_x(v,v)$ for all $(t,x) \in I \times S$ and all $v \in T_xS$. Hence, there is some $x \in S$, some $v \in T_xS$ and a sequence $\{t_n\}_{n\in\mathbb{N}} \subset I$ with $t_n \to b$ or $t_n \to a$ as $n \to \infty$, such that

$$h_{(t_n,x)}(v,v) \overset{n\to\infty}{\longrightarrow} \infty.$$

It suffices to assume $v \in T_xS$ to be an element of the h^0-unit sphere in T_xS, i.e., $h^0(v,v) = 1$. So we have $\|h_{t_n}\|_x^{\uparrow} \geq h_{(t_n,x)}(v,v)$ and thus

$$\|h_{t_n}\|_x^{\uparrow} \overset{n\to\infty}{\longrightarrow} \infty$$

is the desired contradiction. If we have defined $\|\cdot\|^{\uparrow}$ by another fixed number, say, $t_1 \neq t_0 \in I$, we can rescale $v \mapsto c \cdot v$ by some $c = c(t_0,t_1)$, such that v has unit h_{t_1}-length. The same argument then holds for the norm $\|\cdot\|^{\uparrow}$ based on t_1.

For the lower bound assume that there is a function $D\colon S \to \mathbb{R}_{>0}$, such that $D(x) \leq \|h_t\|_x^{\downarrow}$ holds for all $x \in S$, but there is no Riemannian metric $\tilde{\gamma}$ on S such that $\tilde{\gamma}_x(v,v) \leq h_{(t,x)}(v,v)$ for all $(t,x) \in I \times S$ and all $v \in T_xS$. By a similar argument to the one above, there is now some $x \in S$, some $v \in T_xS$ with unit h^0-length and a sequence $\{t_n\}_{n\in\mathbb{N}} \subset I$ with $t_n \to a$ or $t_n \to b$ as $n \to \infty$, such that

$$h_{(t_n,x)}(v,v) \overset{n\to\infty}{\longrightarrow} 0.$$

Hence, as $\|h_{t_n}\|_x^{\downarrow} \leq h_{(t_n,x)}(v,v)$ we get

$$\|h_{t_n}\|_x^{\downarrow} \overset{n\to\infty}{\longrightarrow} 0,$$

the desired contradiction. The argument above for another mapping based on a number $t_1 \in I$ holds, mutatis mutandis, also in this case.

achieved by modifications of normal coordinates about each x_n. Hence we find by

Second, starting from the metric $g = -\mathrm{d}t^2 + h_t$ on $(a,b) \times S$, the norms $\|\cdot\|^{\uparrow}$ and $\|\cdot\|^{\downarrow}$ imply functions $(t,x) \mapsto \|h_t\|_x^{\uparrow}$ and $(t,x) \mapsto \|h_t\|_x^{\downarrow}$ on $(a,b) \times S$ with values in $\mathbb{R}_{>0}$, such that we can find four smooth functions $f_1, F_1\colon (a,b) \to \mathbb{R}_{>0}$ and $f_2, F_2\colon S \to \mathbb{R}_{>0}$ obeying

$$f_1(t) \cdot f_2(x) \leq \|h_t\|_x^{\downarrow} \leq \|h_t\|_x^{\uparrow} \leq F_1(t) \cdot F_2(x)$$

by applying Lem. 5.8.

We now assume that $\|h_t\|_x^{\uparrow}$ is not bounded from above as $t \to b$ and $\|h_t\|_x^{\downarrow}$ is not bounded from below as $t \to a$. If the functions were bounded, there would be nothing to show. All other cases (e.g., $\|h_t\|_x^{\uparrow}$ not bounded from *below* as $t \to b$, etc.) are shown by completely analogous arguments. Hence, we can assume that f_1 is bounded away from zero from below in $[t_0, b)$ and F_1 is bounded from above in $(a, t_0]$. Now we set

$$\Omega^2(t) := \frac{1}{f_1(t) \cdot F_1(t)},$$

such that

$$g^* = -\left(f_1(t) \cdot F_1(t)\right)^{-1} \mathrm{d}t^2 + h_t^*,$$

with

$$h^*_{(t,x)} = \frac{h_{(t,x)}}{f_1(t) \cdot F_1(t)}.$$

This yields

$$\|h_t^*\|_x^{\downarrow} = \frac{\|h_t\|_x^{\downarrow}}{f_1(t) \cdot F_1(t)} \quad \text{and} \quad \|h_t^*\|_x^{\uparrow} = \frac{\|h_t\|_x^{\uparrow}}{f_1(t) \cdot F_1(t)},$$

such that we have

$$B \cdot F_1(x) \leq \|h_t^*\|_x^{\downarrow} \quad \text{and} \quad \|h_t^*\|_x^{\uparrow} \leq C \cdot F_2(x),$$

for constants $0 < B \leq C \in \mathbb{R}$ and $t \in [t_0, b)$, $t \in (a, t_0]$, respectively. Hence, we can find functions $D(x) = B \cdot F_1(x)$ and $E(x) = C \cdot F_2(x)$ such that $D(x) \leq \|h_t^*\|_x^{\downarrow} \leq \|h_t^*\|_x^{\uparrow} \leq E(x)$ holds for all $x \in S$.

Furthermore, setting

$$\tau(t) = \int_{t_0}^{t} \frac{\mathrm{d}s}{\sqrt{f_1(s) \cdot F_1(s)}} = \int_{t_0}^{t} \Omega(s)\mathrm{d}s,$$

we find that τ is a strictly increasing function in (a, b) with values in (a^*, b^*), with

$$a^* = \int_{t_0}^{a} \Omega(s)\mathrm{d}s \quad \text{and} \quad b^* = \int_{t_0}^{b} \Omega(s)\mathrm{d}s,$$

in the sense of a limit if necessary, as $\Omega > 0$. Thus, $\tau(t)$ can be solved for $t(\tau)$ and we can set $h_\tau^* = h_{t(\tau)}^*$, such that

$$g^* = -\mathrm{d}\tau^2 + h_\tau^*$$

is a regularly sliced Lorentzian metric on $(a^*, b^*) \times S$ conformal to the original metric g on $(a, b) \times S$. $\square$

Remark 5.10. *Note that the metrics γ and $\tilde{\gamma}$ that bound a one-parameter family of Riemannian metrics $\{h_t\}_{t\in(a,b)}$, constituting the spacelike part of a Lorentzian metric g, are generally* not *obtained as some kind of limit γ "$=$" $\lim_{t\to b} h_t$ or $\tilde{\gamma}$ "$=$" $\lim_{t\to a} h_t$. These limits may not be smooth Riemannian metrics, as in example 5.13 below, but this can certainly be repaired by making the limit metrics a bit "bigger" or "smaller", respectively.*

Theorem 5.11. *Every globally hyperbolic spacetime (M, g) is conformal to a regularly sliced spacetime $(I \times S, -\mathrm{d}t^2 + h_t)$, such that the Riemannian metrics h_t are bounded by a complete Riemannian metric from above, i.e., $\gamma \geq h_t$ holds for all $t \in I$ with γ complete. Furthermore, for any globally hyperbolic spacetime $(\mathbb{R} \times S, g)$ with*

$$g = -N^2 \mathrm{d}t \otimes \mathrm{d}t + \bar{g}_t,$$

we have that all Riemannian metrics $h_{(t,x)} = \frac{\bar{g}_{(t,x)}}{N^2(t,x)}$ are complete if the regularly sliced spacetime, to which $(\mathbb{R} \times S, g)$ is conformal, obeys $\tilde{\gamma} = \gamma$.

Proof. Applying Thm. 5.4, we conclude that (M, g) is isometric to $(\mathbb{R} \times S, -N^2\mathrm{d}t^2 + h_t)$, with h_t a family of smoothly varying Riemannian metrics. Thus, g is obviously conformal to $\tilde{g} = -\mathrm{d}t^2 + \frac{h_t}{N^2}$. Hence, we can apply Lem. 5.9 to $\tilde{g}$ and get a regularly sliced metric $g^* = -\mathrm{d}t^2 + h_t^*$ on $I \times S$ with some interval $I \subset \mathbb{R}$, which is conformal to the original spacetime (M, g), and hence also globally hyperbolic. By Thm. 5.5, the Riemannian metric γ on S that bounds the family h_t from above is complete. The remaining assertion follows from Lem. 5.7 and from the fact that conformal changes by functions of the same type as $\Omega(t)$, given by Lem. 5.9, do only affect the parameter t of the Riemannian metrics h_t, hence their completeness stays unaffected by such conformal transformations. $\square$

Thm. 5.11 can be seen as a regularity result for generic globally hyperbolic spacetimes. On the one hand, it implies that in the conformal class of a globally hyperbolic metric, there is at least one element $(I \times S, -\mathrm{d}t^2 + h_t)$, such that the one parameter family $\{h_t\}_{t \in I}$ of Riemannian metrics has a complete metric γ as upper bound. This constrains the possible curves in the space of Riemannian metrics $\mathcal{R}(S) = \{g \in \Gamma(\Sigma^2 S) \,|\, g \text{ Riemannian}\}$ over a manifold S that can serve as such a one-parameter family for globally hyperbolic spacetimes (cf. [FM78] for an introduction to the notion of the space of Riemannian metrics considered here, [GMM91] for the construction of a Riemannian manifold of Riemannian metrics in particular cases, or [Bla00] for a more recent overview). On the other hand, it implies that globally hyperbolic spacetimes $(I \times S, -\mathrm{d}t^2 + h_t)$, for which every metric h_t is complete, are amongst others (cf. example 5.14), those which are regularly sliced with $\tilde{\gamma} = \gamma$. This has some interesting consequences.

Remark 5.12. *Let γ and $C\gamma$, $C > 0$ be two Riemannian metrics in the space of Riemannian metrics $\mathcal{R}(S)$. Let γ be complete, then $l\colon [1, C] \to \mathcal{R}(S)$ with $s \mapsto s\gamma$ is a line segment of complete Riemannian metrics connecting γ and $C\gamma$. If γ and $C\gamma$ are the upper and lower bounds of some family of metrics $h_t \subset \mathcal{R}(S)$, $t \in [1, C]$, constituting a regularly sliced and globally hyperbolic spacetime $((1, C) \times S, -\mathrm{d}t^2 + h_t)$, then the metrics h_t can not be "too far away" from the line segment $l([1, C])$, in some sense. In fact, suppose that $C > 1$ and for the expansion Θ of the reference frame ∂_t in the spacetime $((1, C) \times S, -\mathrm{d}t^2 + h_t)$ it holds that $\Theta > 0$, then it is straightforward that $\{h_t\}_{t \in [1,C]}$ is a graph over the line segment $l([1, C])$ in $\mathcal{R}(S)$.*

The following example shows that there are globally hyperbolic and regularly sliced spacetimes $(I \times S, -\mathrm{d}t^2 + h_t)$ with h_t an incomplete Riemannian metric for all $t \in I$.

Example 5.13. *Let g be the Lorentzian metric*

$$g = -\mathrm{d}t^2 + \frac{\mathrm{d}x^2}{(1 + |x|^{(2-t)})^2}$$

on $(0, 1) \times \mathbb{R}$. Obviously, the spacetime $((0, 1) \times \mathbb{R}, g)$ is regularly sliced. From Thm. 5.22 in the next section, we may already conclude that

$$h_t = \frac{\mathrm{d}x^2}{(1 + |x|^{(2-t)})^2}$$

is an incomplete Riemannian metric on $\mathbb{R}$ for all $t \in (0, 1)$. If we now take a look at the limit metric $h_1 = \frac{\mathrm{d}x^2}{(1+|x|)^2}$, we see that it is only continuous at $x = 0$, but complete and $h_t \leq h_1$ for all $t \in (0, 1)$. Of course, we can change h_1 in a compact set about $0 \in \mathbb{R}$ a little bit, to obtain a smooth, complete metric, which is "bigger" than any h_t, too. As the lower limit metric $h_0 = \frac{\mathrm{d}x^2}{(1+|x|^2)^2}$ is a smooth, incomplete metric on $\mathbb{R}$, the impossibility to find constants, such that $((0, 1) \times \mathbb{R}, g)$ is regularly sliced with the same upper and lower bound metrics, is already implied by the quadratic growth $(1 + |x|^2)$ towards infinity, compared to the linear growth of $(1 + |x|)$. Nevertheless, $((0, 1) \times \mathbb{R}, g)$ is globally hyperbolic. A straightforward calculation *shows that all lightlike vector fields in $T((0,1)\times\mathbb{R})$ are given by*

$$K_{(t,x)} = \begin{pmatrix} \pm 1 \\ \pm(1+|x|^{(2-t)}) \end{pmatrix}.$$

The lightlike integral curves of K are easily calculated, and we can conclude that all future-directed integral curves of K eventually intersect the line $t=1$ and all past-directed ones eventually intersect $t=0$. Hence, the sets $J^+((t,x))\cap J^-((s,y))$ are compact or empty for all $(t,x),(s,y)\in(0,1)\times\mathbb{R}$.

The following example shows that in a globally hyperbolic and regularly sliced spacetime $(I\times S, -\mathrm{d}t^2+h_t)$ with h_t a family of complete Riemannian metrics and a complete upper limit metric γ, the lower limit metric $\tilde{\gamma}$ is not necessarily complete.

Example 5.14. *Modifying example 5.6, we consider the punctured plane $S=\mathbb{R}^2\setminus\{0\}\simeq\mathbb{R}_{>0}\times\mathbb{S}^1$ with the Riemannian metrics*

$$h_t = h_t^{(c)} + \delta = \left(\frac{t}{r^2}\mathrm{d}r^2 + \mathrm{d}\phi^2\right) + \left(\mathrm{d}r^2 + r^2\mathrm{d}\phi^2\right) = \left(\frac{t}{r^2}+1\right)\mathrm{d}r^2 + (1+r^2)\mathrm{d}\phi^2,$$

for $t\in(0,1)$. Hence, $h_t^{(c)}$ is complete on S for all $t\in(0,1)$ and the Euclidean metric δ is incomplete, but h_t is complete for all $t\in(0,1)$. This can be concluded from Lem. 2.7. Thus we can construct a spacetime $((0,1)\times S, g)$ with $g=-\mathrm{d}t^2+h_t$, which is obviously regularly sliced with complete upper bound $\gamma=\left(\frac{1}{r^2}+1\right)\mathrm{d}r^2+(1+r^2)\mathrm{d}\phi^2$ and incomplete lower bound $\tilde{\gamma}=\mathrm{d}r^2+(1+r^2)\mathrm{d}\phi^2$. Analogous to the previous example, we can solve the lightlike geodesic equations for fixed $\phi\in\mathbb{S}^1$ and see that none of these geodesics meets the singularity at $r=0$ for a finite value of the affine parameter. Hence, the sets $J^+((t,x))\cap J^-((s,y))$ are compact for all $(t,x),(s,y)\in(0,1)\times S$.

These two examples correspond well to the fact that both the complete Riemannian metrics and the incomplete ones are dense in $\mathcal{R}(S)$ for some manifold S (cf. [FM78]).

5.2 Cauchy Hypersurfaces in Stationary Spacetimes

In this section, we will investigate particular classes of sliced spacetimes, namely standard stationary and standard static spacetimes. We are interested in their global hyperbolicity via the existence of Cauchy hypersurfaces. It has been established that this question is related to the completeness of Riemannian and Randers-type metrics on the spacelike slices (cf. [CJM11]). There is even a duality result that connects Lorentzian causality of stationary spacetimes with Randers-type completeness, known as stationary-to-Randers correspondence (cf. [CJS11]). With our results in this section, we refine the existing connection between Riemannian completeness and causality of stationary spacetimes, mostly by introducing growth conditions and results on pinching Randers-type metrics by Riemannian metrics. To this end we prove a new result in global Riemannian geometry yielding a rigorous condition for the completeness of a Riemannian metric under conformal transformations (see Thm. 5.22). Please note that some results in this section have already been published in [DPS12], but with the results presented in this section some errors contained in [DPS12] are corrected. Particularly, Thm. 5.22 below is a rectified version of the corresponding of Thm. 1 in [DPS12] and its subsequent implications are adjusted as well.

Definition 5.15. *A sliced spacetime $(\mathbb{R} \times S, g = -N^2 \mathrm{d}t \otimes \mathrm{d}t + 2b \vee \mathrm{d}t + h)$ is called standard stationary if the function N, the one-form b and the Riemannian metric h are pulled back from an individual function, one-form and metric on S, respectively by the canonical projection $\mathrm{pr}_2 \colon \mathbb{R} \times S \to S$, i.e., N, b and h do not depend on t. Furthermore, a standard stationary spacetime is called standard static if $b = 0$.*

Several remarks are in order.

Remark 5.16.
(i) Obviously the vector field ∂_t is a complete Killing vector field in any standard stationary or static spacetime and $V = \frac{\partial_t}{N}$ is a reference frame, which is also complete as N does not depend on t (compare to Def. 3.9 and Prop. 5.2). Hence, $(\mathbb{R} \times S, g, \frac{\partial_t}{N})$ is a proper kinematical spacetime for a standard stationary metric g as in Def. 5.15 above, particularly it is causally continuous by section 4.2 (compare to the results in [JS08]).
(ii) Note that the question if a stationary or static spacetime is standard stationary or standard static, can be assessed by the result in Cor. 4.11 and the results on causality in section 4.2.
(iii) A standard static spacetime can obviously be considered to be a Lorentzian warped product $(S, h) \times_N (\mathbb{R}, -\mathrm{d}t^2)$.

Note that not even standard static spacetimes are necessarily regularly sliced, as the function $N \colon S \to \mathbb{R}_{>0}$ does not have to be bounded. Nevertheless, standard static spacetimes—in contrast to standard stationary ones—are conformal to regularly sliced spacetimes by virtue of a conformal factor $\frac{1}{N^2}$.

Conditions on the Riemannian slices $S_t := \{t\} \times S$ to be Cauchy hypersurfaces in standard stationary spacetimes were investigated in [CJM11] and [CJS11]. A crucial point in this investigation is the construction of a Randers-type metric from the Riemannian metric h and the one-form b on S.

Definition 5.17. *Let $(\mathbb{R} \times S, g = -N^2 \mathrm{d}t \otimes \mathrm{d}t + 2b \vee \mathrm{d}t + h)$ be a standard stationary spacetime. The function $F \colon TS \to \mathbb{R}_{\geq 0}$ given by*

$$F(x, v) = \sqrt{\frac{h_x(v, v)}{N^2(x)} + \frac{b_x(v)^2}{N^4(x)}} + \frac{b_x(v)}{N^2(x)},$$

for all $x \in S$ and all $v \in T_x S$ is called Fermat metric.

Proposition 5.18. *The Fermat metric for a given standard stationary spacetime is a well defined Finslerian metric of Randers type.*

Proof. The function N is non-zero and with h being a Riemannian metric and b a one-form on S, also $\frac{h}{N^2}+\frac{b\otimes b}{N^4}$ is a Riemannian metric on S. Hence, F is well defined for all $(x,v)\in TS$. The smoothness and continuity conditions for F are fulfilled on TS, because N, b and h are smooth for all $v\neq 0$, and $F(x,v)$ is continuous at $v=0$ as it is a composition of smooth, respectively continuous, maps. Moreover, F is positively homogeneous of degree one in the second argument because we have

$$F(\cdot,\lambda v)=\sqrt{\lambda^2\frac{h(v,v)}{N^2}+\lambda^2\frac{b(v)^2}{N^4}}+\lambda\frac{b(v)}{N^2}=\lambda F(\cdot,v)$$

if $\lambda>0$. Furthermore, we have

$$\|\frac{1}{N^2}b\|_x^{h+b^2}=\sup_{v\in T_xS\setminus\{0\}}\frac{|b(v)|}{\sqrt{N^2h(v,v)+b(v)^2}}<1,$$

for all $x\in S$, which assures $F(x,v)\geq 0$ for all $(x,v)\in TS$ and makes F a Finslerian metric of Randers type. □

It is known that a slice S_t (and thus all slices) of a standard stationary spacetime $M=\mathbb{R}\times S$ is a Cauchy hypersurface if and only if the associated Fermat metric on S is forward and backward complete (see [CJM11] and [CJS11]). Therefore, this is a sufficient condition for M to be globally hyperbolic. However, it is not a necessary condition in general, since for a slice in a globally hyperbolic standard stationary spacetime with non-complete Fermat metric—which is, therefore, not a Cauchy hypersurface—there is always another slicing such that the slices are Cauchy hypersurfaces, providing a forward and backward complete Fermat metric. Nevertheless, for any globally hyperbolic standard stationary spacetime, the Riemannian metric

$$\tilde{h}=\frac{h}{N^2}+\frac{b\otimes b}{N^4}$$

is necessarily complete on S. This follows from Lem. 2.20. Moreover, this Riemannian metric is actually independent of the slicing. This is a straightforward consequence of Thm. 4.16, as $\tilde{h}$ can be identified via pull-back with the projection defined in section 3.1, and thus it is a t-independent Riemannian metric on the horizontal bundle $H(\mathbb{R}\times S)$.

Now we will establish a new result in global Riemannian geometry.

A ray $\alpha\colon[0,b)\to M$ (or a curve $\tilde{\alpha}\colon(a,b)\to M$ for some $a\leq 0<b$ such that $\alpha=\tilde{\alpha}|_{[0,b)}$ is a ray) with $0<b\leq\infty$ will be called escaping to infinity on a manifold M if there is a sequence $\{t_n\}_{n\in\mathbb{N}}\subset[0,b)$ with $t_n\to b$ as $n\to\infty$, such that $\alpha(t_n)\to\infty$ as $n\to\infty$ in the following sense: there is an exhaustion of M by compact sets $\{K_n\}_{n\in\mathbb{N}_0}\subset\mathcal{P}(M)$ (the power set of M), $M=\bigcup_{n\in\mathbb{N}_0}K_n$, $K_0=\{\alpha(0)\}$, $K_n\subset\mathring{K}_{n+1}$, such that $\alpha(t_n)\in K_n\setminus K_{n-1}$ for all $n\in\mathbb{N}$.

Remark 5.19. *Note that necessarily all non-imprisoned curves are escaping to infinity. Compare to Def. 3.37.*

Definition 5.20. *Let (M,g) be a complete Riemannian manifold.*

(i) *A function* $f\colon M \to \mathbb{R}_{>0}$ *grows at most linearly towards g-infinity on* (M,g) *if for all fixed* $x_0 \in M$ *there are constants* $c_1, c_2 > 0$*, such that for all* $x \in M$*,* $f(x) \leq c_1 d_g(x_0, x) + c_2$ *holds.*

(ii) *A function* $f\colon M \to \mathbb{R}_{>0}$ *grows superlinearly towards g-infinity on* (M,g) *if for all fixed* $x_0 \in M$ *there are constants* $\varepsilon, c_1, c_2 > 0$*, such that for all* $x \in M$*,* $f(x) \geq c_1 d_g(x_0, x)^{1+\varepsilon} + c_2$ *holds.*

(iii) *A function* $f\colon M \to \mathbb{R}_{>0}$ *will be called* L^1 *on an escaping curve w.r.t. g if there is* $x_0 \in M$ *and a ray* $\gamma\colon [0,\infty) \to M$ *with* $\gamma(0) = x_0$*, such that*

$$\int_0^\infty (f \circ \gamma)(s)\sqrt{g_{\gamma(s)}(\dot\gamma(s), \dot\gamma(s))}\mathrm{d}s < \infty.$$

This condition is independent of the parametrization of γ*, and without loss of generality, we can assume* γ *to be parametrized by arc length, such that* $g(\dot\gamma, \dot\gamma) = 1$*, and we have* $\int_0^\infty (f \circ \gamma)\mathrm{d}s < \infty$*, i.e.,* $f \circ \gamma \in L^1(\mathbb{R}_{\geq 0})$*.*

Inspecting item *(iii)*, we see that the condition is equivalent to state that the length of the escaping ray in the conformally transformed metric $f^2 g$ on M is finite. As it is known that a Riemannian metric is complete if and only if the length of any escaping curve is infinite, the connection to completeness becomes obvious. But stating the condition in terms of growth of the conformal factor, instead of in terms of conformally transformed curve lengths, allows to compare this condition to the linear growth conditions in items *(i)* and *(ii)*. The precise relation is clarified in the following

Lemma 5.21. *If* $f\colon M \to \mathbb{R}_{>0}$ *grows superlinearly towards g-infinity on a complete Riemannian manifold* (M,g)*, then* $\frac{1}{f}\colon M \to \mathbb{R}^+$ *is* L^1 *w.r.t. g on* all *escaping g-geodesic rays. If* $f\colon M \to \mathbb{R}_{>0}$ *grows at most linearly towards g-infinity on a complete Riemannian manifold* (M,g)*, then* $\frac{1}{f}\colon M \to \mathbb{R}_{>0}$ *is* not L^1 *on any escaping curve w.r.t. g.*

Proof. Let $x_0 \in M$ be a fixed point and $\gamma\colon [0,\infty) \to M$ any escaping g-geodesic ray with $\gamma(0) = x_0$, parametrized by arc length. Assume f to grow superlinearly towards g-infinity, thus there are constants $\varepsilon, c_1, c_2 > 0$ such that

$$\frac{1}{f(x)} \leq \frac{1}{c_1 d_g(x_0, x)^{1+\varepsilon} + c_2},$$

for all $x \in M$. Hence, we have for all $s \in [0,\infty)$

$$\frac{1}{(f \circ \gamma)(s)} \leq \frac{1}{c_1 d_g(x_0, \gamma(s))^{1+\varepsilon} + c_2} = \frac{1}{c_1 s^{1+\varepsilon} + c_2} \in L^1([0,\infty))$$

as $\varepsilon, c_1, c_2 > 0$.

Assume now that f grows at most linearly towards g-infinity on M. Hence, for all $x_0 \in M$, there are constants $c_1, c_2 > 0$ such that

$$\frac{1}{f(x)} \geq \frac{1}{c_1 d_g(x_0, x) + c_2},$$

for all $x \in M$. Let $\gamma\colon [0,\infty) \to M$ be any escaping ray emanating at $x_0 \in M$ and parametrized by arc length. Then we have $d_g(x_0, \gamma(s)) \leq s$ for all $s \in [0,\infty)$ and thus

$$\frac{1}{(f \circ \gamma)(s)} \geq \frac{1}{c_1 d_g(x_0, \gamma(s)) + c_2} \geq \frac{1}{c_1 s + c_2}.$$

This implies $\int_{[0,\infty)} \frac{1}{f \circ \gamma} = \infty$ as $\int_0^\infty \frac{\mathrm{d}s}{c_1 s + c_2} = \infty$ for all $c_1, c_2 > 0$. □

Theorem 5.22. *Let (M,g) be a non-compact and complete Riemannian manifold and $A\colon M \to \mathbb{R}_{>0}$ be a positive function. We denote a conformally transformed metric on M by $g' = \frac{g}{A^2}$. Then (M,g') is complete if and only if $\frac{1}{A}\colon M \to \mathbb{R}_{>0}$ is not L^1 on any escaping curve w.r.t. g. Moreover, if A grows at most linearly towards g-infinity on M, then (M,g') is complete and if A grows superlinearly towards g-infinity on M, then g' is a bounded metric on M, particularly (M,g') is incomplete.*

Proof. We will show the following statement: $\frac{1}{A}$ is L^1 on an escaping curve if and only if (M,g') is incomplete. The first remaining statement then follows easily from Lem. 5.21.

Assume first that there is $x_0 \in M$ and a ray $\gamma\colon [0,b) \to M$ with $\gamma(0) = x_0$ escaping to infinity, such that $\frac{1}{A}$ is L^1 on γ w.r.t. g. We can parametrize γ by g-arc length, i.e., we have $g(\dot\gamma,\dot\gamma) = 1$ and hence some constant $B < \infty$ such that $\int_0^\infty \frac{\mathrm{d}s}{(A\circ\gamma)(s)} = B$. Take any sequence $\{s_n\}_{n\in\mathbb{N}} \subset \mathbb{R}_{\geq 0}$, with $s_n \to \infty$ and $\gamma(s_n) \to \infty$ as $n \to \infty$. Then we compute the distance between x_0 and each $\gamma(s_n)$ in the conformally transformed metric g':

$$d_{g'}(x_0, \gamma(s_n)) \leq \int_0^{s_n} \frac{\sqrt{g(\dot\gamma(s), \dot\gamma(s))}}{A(\gamma(s))} \mathrm{d}s \leq \int_0^\infty \frac{1}{(A \circ \gamma)(s)} \mathrm{d}s = B.$$

Hence, the sequence $\{\gamma(s_n)\}_{n\in\mathbb{N}}$ is contained in a closed and bounded g'-ball of radius B about x_0. But by the definition of escaping curves the sequence $\{\gamma(s_n)\}_{n\in\mathbb{N}}$ has no convergent subsequence, thus the closed and bounded g'-ball of radius B about x_0 is not compact and, therefore, (M,g') is incomplete by the Hopf–Rinow theorem.

Assume now that (M,g') is incomplete, hence there is a point $x_0 \in M$ and a g'-geodesic ray $\gamma\colon [0,b) \to M$ emanating from x_0 that is not extendible to the parameter value b. As a g'-geodesic is parametrized to unit g'-velocity, we conclude the length of γ to be $b < \infty$. But obviously γ escapes to infinity, as there exists no point $\gamma(b) \in M$. Now reparametrize γ to unit g-velocity, i.e., $g(\dot\gamma,\dot\gamma) = 1$, then we get $\gamma\colon [0,\infty) \to M$ as g is assumed complete. We compute

$$\infty > b = \int_0^\infty \frac{\sqrt{g(\dot\gamma(s), \dot\gamma(s))}}{A(\gamma(s))} \mathrm{d}s = \int_0^\infty \frac{1}{(A \circ \gamma)(s)} \mathrm{d}s.$$

Hence, $\frac{1}{A}$ is L^1 on the escaping curve γ w.r.t. g.

As $\frac{1}{A}$ is L^1 w.r.t. g on all escaping g-geodesic rays if A grows superlinearly towards g-infinity on M by Lem. 5.21, we certainly have in this case that g' is incomplete. Furthermore, computing the distance of a fixed $x_0 \in M$ to any $x \in M$ in the g' distance we get for some finite constant $r(\varepsilon, c_1, c_2) < \infty$

$$d_{g'}(x_0, x) \leq \int_0^\infty \frac{\mathrm{d}s}{c_1 s^{1+\varepsilon} + c_2} = r(\varepsilon, c_1, c_2).$$

Hence, g' is bounded as now $d_{g'}(x,y) \leq d_{g'}(x_0,x)+d_{g'}(x_0,y) = 2r$ holds for all $x,y \in M$. $\square$

Example 5.23. *For a simple instructive example we look at the standard metric g_0 of the punctured plane $\mathbb{R}^2 \setminus \{0\} \cong \mathbb{R}_{>0} \times \mathbb{S}^1$, given in polar coordinates (r,ϕ) by*

$$g_0 = \mathrm{d}r^2 + r^2 \mathrm{d}\phi^2.$$

This metric is obviously incomplete since radial geodesics would eventually meet the removed origin. However, the conformally transformed metric

$$g = \frac{g_0}{r^2} = \frac{1}{r^2}\mathrm{d}r^2 + \mathrm{d}\phi^2$$

is isometric to the (complete) flat cylinder $\mathbb{R} \times S^1$, which may be seen by adopting the new coordinate $\rho := \ln r$:

$$g = \mathrm{d}\rho^2 + \mathrm{d}\phi^2.$$

Another complete metric g' in the conformal class of g_0 is then, for example, given by

$$g' = \frac{\mathrm{d}\rho^2 + \mathrm{d}\phi^2}{A^2},$$

with $A > 0$ growing at most linearly with respect to ρ:

$$A(\rho,\phi) \leq c_1|\rho| + c_2.$$

To come full circle, we see that the metric $g_0 = r^2 g$ is necessarily incomplete since the function $\frac{1}{r} = \exp(-\rho)$ clearly grows superlinearly.

As a consequence of Thm. 5.22 we now establish an application to static spacetimes.

Corollary 5.24. *Let $(M = \mathbb{R} \times S, g = -N^2 \mathrm{d}t^2 + h)$ be a standard static spacetime with a complete Riemannian manifold (S,h) as base and a lapse function $N \colon S \to \mathbb{R}_{>0}$. Then (M,g) is globally hyperbolic if and only if the function $\frac{1}{N} \colon S \to \mathbb{R}_{>0}$ is not L^1 on any escaping curve on S w.r.t. h. Particularly, (M,g) is globally hyperbolic if N grows at most linearly towards h-infinity on S and it is not globally hyperbolic if N grows superlinearly towards h-infinity on S.*

Proof. This follows directly from Thm. 5.22 and from Thm. 5.5 applied to the conformally transformed metric

$$\frac{g}{N^2} = -\mathrm{d}t^2 + \frac{h}{N^2}.$$

$\square$

In the remainder of this section, we will consider standard stationary spacetimes $(\mathbb{R} \times S, g)$ with metric given by

$$g = -\mathrm{d}t \otimes \mathrm{d}t + 2b \vee \mathrm{d}t + h.$$

All results on global hyperbolicity below, established for standard stationary spacetimes in this form also hold for general standard stationary spacetimes if we replace b by $\frac{b}{N^2}$ and h by $\frac{h}{N^2}$. This is due to the fact that the metric $g = -\mathrm{d}t^2 + 2b\mathrm{d}t + h$ may be seen as derived from the metric $-N^2\mathrm{d}t^2 + 2b\mathrm{d}t + h$ by a conformal transformation with factor $\frac{1}{N^2}$.

We will repeatedly make use of the following lemma.

Lemma 5.25. *Let (S, g) be a Riemannian manifold, b a one-form on S and $\|b\|_x^g < 1$ for all $x \in S$, such that $h^\pm = g \pm b \otimes b$ are Riemannian metrics on S. Then, for the corresponding norms of b the following holds:*

$$(\|b\|_x^{h^\pm})^2 = \frac{(\|b\|_x^g)^2}{1 \pm (\|b\|_x^g)^2}$$

for all $x \in S$.

Proof. We compute

$$(\|b\|_x^{g-b\otimes b})^2 = \sup_{v \in T_xS\setminus\{0\}} \frac{|b_x(v)|^2}{g_x(v,v) - |b_x(v)|^2} = \sup_{v \in T_xS\setminus\{0\}} \frac{\frac{|b_x(v)|^2}{g_x(v,v)}}{1 - \frac{|b_x(v)|^2}{g_x(v,v)}} = \frac{(\|b\|_x^g)^2}{1 - (\|b\|_x^g)^2},$$

since

$$\sup_{v \in T_xS\setminus\{0\}} \frac{|b_x(v)|^2}{g_x(v,v)} = (\|b\|_x^g)^2 < 1$$

holds for all $x \in S$. Solving for $(\|b\|_x^g)^2$ and renaming $g = h^- + b \otimes b$ yields the assertion for h^+. □

Theorem 5.26. *Let $(M = \mathbb{R} \times S, g)$ be a standard stationary spacetime with metric $g = -\mathrm{d}t^2 + 2b\mathrm{d}t + h$. Then, (M, g) has Cauchy hypersurfaces S_t if the Riemannian metric*

$$\hat{h} := \frac{h + b \otimes b}{(1 + (\|b\|^h)^2)^2}$$

is complete on S. Moreover, if (M, g) is globally hyperbolic, then S_t is a Cauchy hypersurface if $\frac{1}{1+(\|b\|_x^h)^2}$ is not L^1 w.r.t. $h + b \otimes b$ on any escaping curve on S and, particularly, if $(\|b\|_x^h)^2$ grows at most linearly towards $(h + b \otimes b)$-infinity on S.

Proof. For the standard stationary metric $g = -\mathrm{d}t^2 + 2b\mathrm{d}t + h$ on $\mathbb{R} \times S$, the Fermat metric on S is given by

$$F(x, v) = \sqrt{h_x(v,v) + (b_x(v))^2} + b_x(v),$$

for all $x \in S$ and all $v \in T_xS$. Using Lem. 5.25, we compute

$$F(x, v) = \sqrt{h_x(v,v) + (b_x(v))^2}\left(1 + \frac{b_x(v)}{\sqrt{h_x(v,v) + (b_x(v))^2}}\right) \geq$$

$$\geq \sqrt{h_x(v,v)+(b_x(v))^2}\left(1-\|b\|_x^{h+b^2}\right) = \sqrt{h_x(v,v)+(b_x(v))^2}\left(1-\sqrt{\frac{(\|b\|_x^h)^2}{1+(\|b\|_x^h)^2}}\right) \geq$$

$$\geq \frac{1}{2}\frac{\sqrt{h_x(v,v)+(b_x(v))^2}}{1+(\|b\|^h)^2} = \frac{1}{2}\sqrt{\hat{h}_x(v,v)}$$

establishing

$$d_F(x,y) \geq d_{\frac{1}{2}\hat{h}}(x,y),$$

for all $x, y \in S$, with the Finslerian distance d_F associated to the Fermat metric F and the Riemannian distance $d_{\frac{1}{2}\hat{h}}$ associated to the metric $\frac{1}{2}\hat{h}$. As $\frac{1}{2}\hat{h}$ is complete if and only if $\hat{h}$ is complete, the distance $d_{\frac{1}{2}\hat{h}}$ is a complete distance if and only if $\hat{h}$ is complete. Therefore, by an argument similar to the one in the proof of Prop. 2.19, the completeness of $\hat{h}$ implies the forward and backward completeness of the Fermat metric F and thus the slices S_t are Cauchy surfaces.

If (M, g) is globally hyperbolic, $h + b \otimes b$ is complete for any slicing $M = \mathbb{R} \times S$. Thus, by Thm. 5.22, $\hat{h}$ is complete on S if $\frac{1}{1+(\|b\|_x^h)^2}$ is not L^1 on any escaping curve on S w.r.t. $h+b\otimes b$ and, particularly, if $(\|b\|_x^h)^2$ grows at most linearly towards $(h+b\otimes b)$-infinity and, therefore, the slices S_t are Cauchy hypersurfaces. □

By virtue of the stationary-to-Randers correspondence, this theorem also encompasses a condition for the completeness of a Randers-type metric:

Corollary 5.27. *Let (M, h) be a Riemannian manifold and $b \in \Gamma(\Lambda^1 M)$ a one-form on M, obeying $\|b\|_x^h < 1$ for all $x \in M$. The Randers-type metric $R = \sqrt{h} + b$ is forward and backward complete if the Riemannian metric $\left(1 - (\|b\|^h)^2\right)^2 h$ is complete on M.*

Proof. Replacing h by $h - b \otimes b$ in Thm. 5.26 leads to the completeness of

$$\hat{h} := \frac{h}{\left(1 + (\|b\|^{h-b^2})^2\right)^2}$$

as a sufficient condition for the forward and backward completeness of R. By Lem. 5.25, the Riemannian metric $\hat{h}$ is equal to $\left(1 - (\|b\|^h)^2\right)^2 h$. □

Thm. 5.26 gives a sufficient condition for the existence of Cauchy hypersurfaces and global hyperbolicity in standard stationary spacetimes based solely on Riemannian completeness and growth conditions on the base manifold. Moreover, the single fact that

$$\hat{h} \leq h + b \otimes b$$

yields another simple proof for the necessity of the Riemannian metric $h+b\otimes b$ to be complete in globally hyperbolic standard stationary spacetimes.

Moreover, Thm. 5.26 above gives rise to the following first pinching result.

Proposition 5.28. *Let $F = \sqrt{h + b \otimes b} + b$ be the Fermat metric associated with a standard stationary spacetime $(\mathbb{R} \times S, g)$, with metric $g = -\mathrm{d}t^2 + 2b\mathrm{d}t + h$. Then we have*

$$\frac{1}{2}\sqrt{\frac{h + b \otimes b}{(1 + (\|b\|^h)^2)^2}} \leq F \leq 2\sqrt{h + b \otimes b}.$$

Furthermore, a standard stationary spacetime $(\mathbb{R} \times S, -\mathrm{d}t^2 + 2b\mathrm{d}t + h)$ with complete metric $h + b \otimes b$ on S is globally hyperbolic if $\left(\frac{\|b\|_x^{h+b\otimes b}}{\|b\|_x^h}\right)^2$ is not L^1 on any escaping curve on S w.r.t. $h + b \otimes b$.

And for a globally hyperbolic standard stationary spacetime, the Randers-type metric

$$\left(\frac{\|b\|^h}{\|b\|^{h+b\otimes b}}\right)^2 F$$

is complete on S.

Proof. All inequalities are supposed to hold pointwise, but we omit the dependencies $h_x(v, v)$ and $b_x(v)$ with $x \in S$ and $v \in T_xS$ for notational simplicity. We need to establish the right inequality $F \leq 2\sqrt{h + b \otimes b}$, as the left inequality follows from the proof of Thm. 5.26. Starting from

$$\sqrt{h + b \otimes b} + b \leq \sqrt{X^2(h + b \otimes b)},$$

we would like to find a function $X \colon S \to \mathbb{R}$ (i.e., X depends on $x \in S$ only), such that this inequality holds for all $x \in S$ and all $v \in T_xS$. Solving for X yields

$$X \geq 1 + \frac{\frac{b}{\sqrt{h}}}{\sqrt{1 + \frac{b^2}{h}}}.$$

As it is true that $\|b\|^h \geq |\frac{b}{\sqrt{h}}|$, we have that

$$\frac{\frac{b}{\sqrt{h}}}{\sqrt{1 + \frac{b^2}{h}}} \leq 1.$$

Thus, the inequality holds if $X \geq 2$, as the optimal bound, i.e., in the general case $\sup_{x\in S} \|b\|^h = \infty$ (equivalent to $\sup_{x\in S} \|b\|^{h+b\otimes b} = 1$ by Lem. 5.25), the constant function $X(x) = 2$ is the "smallest" possible function, which makes the right inequality to hold for all $x \in S$ and all $v \in T_xS$.

Now, let $(\mathbb{R} \times S, -\mathrm{d}t^2 + 2b\mathrm{d}t + h)$ be a standard stationary spacetime with complete Riemannian metric $h + b \otimes b$. Consequently, by Lem. 5.25 we have that

$$\left(\frac{\|b\|_x^{h+b\otimes b}}{\|b\|_x^h}\right)^2 = \frac{1}{1 + (\|b\|_x^h)^2},$$

and hence by Thm. 5.26, the slices are Cauchy hypersurfaces and $(\mathbb{R}\times S, g)$ is globally hyperbolic if the L^1-condition holds.

The left inequality reads

$$\frac{1}{2}\sqrt{h+b\otimes b} \leq (1+(\|b\|^h)^2 F = \left(\frac{\|b\|^h}{\|b\|^{h+b\otimes b}}\right)^2 F$$

by Lem. 5.25. Hence, the completeness of $h+b\otimes b$ in the case of global hyperbolicity yields the completeness of $\left(\frac{\|b\|^h}{\|b\|^{h+b\otimes b}}\right)^2 F$. □

Now, the question naturally arises if one may find further pinching results for Fermat metrics on S in terms of Riemannian metrics on S. By this, we mean inequalities of the form

$$\sqrt{\frac{g_0}{Y^2}} \leq F = \sqrt{h+b\otimes b}+b \leq \sqrt{X^2 g_0},$$

with g_0 a (preferably complete) Riemannian metric on S and $X, Y\colon S \to \mathbb{R}_{>0}$ functions on S. The Fermat metric $F(x,v)$ (like any Randers metric) depends on the direction of the vector argument v, since the one-form b enters its definition. Thus, depending on the norm of b (with respect to the metric g_0) the conformal factors X and Y have to alter the magnitudes of $g_0(v,v)$ for a vector v in order to produce Riemannian metrics "larger" or "smaller" than the Fermat metric. For any Riemannian metric g_0 there are *optimal estimates* of the functions X and Y, meaning the "smallest" possible functions we may introduce for a given Riemannian metric g_0, such that the inequalities above hold. For example, if we have $g_0 = g + b\otimes b$, the optimal estimate for the upper bound is the constant 2 as the proof of Prop. 5.28 above shows.

The obvious choice is $g_0 = h$.

Proposition 5.29. *Let $F = \sqrt{h+b\otimes b}+b$ be the Fermat metric associated with a standard stationary spacetime $(\mathbb{R}\times S, g)$, with metric $g = -\mathrm{d}t^2 + 2b\mathrm{d}t + h$. Then we have as optimal estimates*

$$\frac{\sqrt{h}}{\sqrt{1+(\|b\|^h)^2}+\|b\|^h} \leq F \leq \left(\sqrt{1+(\|b\|^h)^2}+\|b\|^h\right)\sqrt{h}.$$

Thus, furthermore, we have the following two results:

(i) For a standard stationary spacetime $(\mathbb{R}\times S, -\mathrm{d}t^2+2b\mathrm{d}t+h)$ with complete metric h, the slices S_t are Cauchy hypersurfaces if and only if $\frac{\|b\|^{h+b\otimes b}}{\|b\|^h(1+\|b\|^{h+b\otimes b})}$ is not L^1 on any escaping curve on S w.r.t. h. Particularly, the slices are Cauchy hypersurfaces if $\|b\|^h$ grows at most linearly towards h-infinity on S and they are not if $\|b\|^h$ grows superlinearly towards h-infinity on S.

(ii) For a standard stationary spacetime $(\mathbb{R}\times S, -\mathrm{d}t^2+2b\mathrm{d}t+h)$, for which $\frac{\|b\|^{h+b\otimes b}}{\|b\|^h(1+\|b\|^{h+b\otimes b})}$ is not L^1 on any escaping curve on S w.r.t. h, the slices S_t are Cauchy hypersurfaces if and only if h is complete.

Proof. We establish the left inequality starting from

$$\sqrt{\frac{h}{Y^2}} \leq \sqrt{h + b \otimes b} + b.$$

The right inequality follows from similar reasoning. Like above, we omit the dependencies $h_x(v,v)$ and $b_x(v)$ for simplicity. We get

$$\frac{1}{Y} \leq \sqrt{1 + \frac{b^2}{h}} + \frac{b}{\sqrt{h}}.$$

Since $|\frac{b}{\sqrt{h}}| \leq \|b\|^h$ holds, this inequality is valid if

$$Y^2 \geq \frac{1}{(\sqrt{1 + (\|b\|^h)^2} - \|b\|^h)^2} = (\sqrt{1 + (\|b\|^h)^2} + \|b\|^h)^2.$$

For the remaining assertions we first observe that

$$\frac{1}{\sqrt{1 + (\|b\|^h)^2} + \|b\|^h} = \frac{1}{\frac{\|b\|^h}{\|b\|^{h+b\otimes b}} + \|b\|^h} = \frac{\|b\|^{h+b\otimes b}}{\|b\|^h(1 + \|b\|^{h+b\otimes b})}$$

by Lem. 5.25.

For item *(i)* we conclude by Thm. 5.22 that the L^1-condition assures the completeness of F by the left inequality and hence the slices S_t are Cauchy hypersurfaces. On the other hand, if the slices are Cauchy hypersurfaces, the Fermat metric F is complete and by the right inequality so is $(\sqrt{1 + (\|b\|^h)^2} + \|b\|^h)^2 h$. As we have that h is complete by assumption, this implies the L^1-condition by using Thm. 5.22.

Furthermore, assume that $\|b\|^h$ grows at most linearly towards h-infinity on S, then so does $\sqrt{1 + (\|b\|^h)^2} + \|b\|^h$. Hence, by using Thm. 5.22, we conclude that F is complete—and thus the slices are Cauchy hypersurfaces—by the left inequality under the assumption of a complete metric h. On the other hand, assume $\|b\|^h$ to grow superlinearly towards h-infinity on S, then so does $\sqrt{1 + (\|b\|^h)^2} + \|b\|^h$. But assuming that the slices are Cauchy hypersurfaces, and hence F is complete, yields a complete metric $(\sqrt{1 + (\|b\|^h)^2} + \|b\|^h)^2 h$ by the right inequality and hence an incomplete metric h—again by Thm. 5.22—in contradiction to the assumption of h being complete.

For item *(ii)* first assume that the slices S_t are Cauchy hypersurfaces, i.e., the Fermat metric F is complete. This yields a complete metric $(\sqrt{1 + (\|b\|^h)^2} + \|b\|^h)^2 h$ by the right inequality and thus a complete metric h by the L^1-condition and Thm. 5.22. Then assuming that h is complete and using the left inequality together with the L^1-condition and Thm. 5.22, yields a complete Fermat metric and hence the slices as Cauchy hypersurfaces. □

Now, we would like to establish a pinching result for the Fermat metric using a necessarily complete Riemannian metric g_0. This will result in a purely analytic criterion for global hyperbolicity of standard stationary spacetimes based on L^1- and growth conditions only.

The results of K. Nomizu and H. Ozeki [NO61] assure the existence of a complete Riemannian metric on any manifold. Thus, for the base S of a standard stationary spacetime $(\mathbb{R} \times S, g)$ there necessarily always exists a complete Riemannian metric, generally differing from the Riemannian metric h inherited from the Lorentzian metric $g = -\mathrm{d}t^2 + 2b\mathrm{d}t + h$ on $\mathbb{R} \times S$. However, a complete Riemannian metric may be analytically constructed from any Riemannian metric on a manifold S using proper functions (see [Gor73], [Gor74]). We recall that a function $f\colon S \to \mathbb{R}$ is said to be proper if $f^{-1}(K)$ is compact whenever $K \subset \mathbb{R}$ is compact. The following theorem holds:

Theorem 5.30. *Let (S, h) be any Riemannian manifold and $f\colon S \to \mathbb{R}$ any proper function on S. Then $(S, \tilde{h})$ is a complete Riemannian manifold, where*

$$\tilde{h} = h + \mathrm{d}f \otimes \mathrm{d}f.$$

Moreover, a Riemannian manifold (S, h) is complete if and only if there is a proper function $f\colon S \to \mathbb{R}$ on S such that

$$\sup_{x \in S} \|\mathrm{d}f\|_x^h < \infty.$$

Proof. See [Gor73, Gor74] and references therein. □

Now, we use Thm. 5.30 to establish the pinching result

$$\sqrt{\frac{h + \mathrm{d}f \otimes \mathrm{d}f}{Y^2}} \leq F \leq \sqrt{X^2(h + \mathrm{d}f \otimes \mathrm{d}f)},$$

for a necessarily complete metric $h + \mathrm{d}f \otimes \mathrm{d}f$ on the base S of a standard stationary spacetime $(\mathbb{R} \times S, g)$ with $f\colon S \to \mathbb{R}$ being any proper function on S.

Theorem 5.31. *Let $(\mathbb{R} \times S, g)$ be a standard stationary spacetime with Lorentzian metric $g = -\mathrm{d}t^2 + 2b\mathrm{d}t + h$ and $f\colon S \to \mathbb{R}$ a proper function on S, such that $h + \mathrm{d}f \otimes \mathrm{d}f$ is a complete Riemannian metric on S. Then the pinching of the Fermat metric $F = \sqrt{h + b \otimes b} + b$ results in*

$$Y^2 \geq 4(1 + (\|\mathrm{d}f\|^h)^2)(1 + (\|b\|^h)^2)$$

and

$$X^2 \geq 4(1 + (\|b\|^h)^2).$$

Moreover, we have the following two results:

(i) The standard stationary spacetime $(\mathbb{R} \times S, g)$ is globally hyperbolic if there exists a proper function $f\colon S \to \mathbb{R}$ such that $\frac{\|b\|^{h+b\otimes b}}{\|b\|^h \sqrt{1 + (\|\mathrm{d}f\|^h)^2}}$ is not L^1 on any escaping curve on S w.r.t. $h + \mathrm{d}f \otimes \mathrm{d}f$ and, particularly, if there exists a proper function $f\colon S \to \mathbb{R}$ such that the product $\|\mathrm{d}f\|^h \cdot \|b\|^h$ grows at most linearly towards $(h + \mathrm{d}f \otimes \mathrm{d}f)$-infinity.

(ii) For a globally hyperbolic and standard stationary spacetime $(\mathbb{R} \times S, g)$, with the slices S_t as Cauchy hypersurfaces, the function $\frac{\|b\|^{h+b\otimes b}}{\|b\|^h}$ is not L^1 on any escaping curve on S w.r.t. $h + \mathrm{d}f \otimes \mathrm{d}f$, particularly, for all proper functions $f\colon S \to \mathbb{R}$, the function $\|b\|^h$ does not *grow superlinearly towards $(h + \mathrm{d}f \otimes \mathrm{d}f)$-infinity on S.*

Proof. We first establish the inequality for Y. It must hold that

$$\sqrt{\frac{h+\mathrm{d}f\otimes\mathrm{d}f}{Y^2}} \leq \sqrt{h+b\otimes b}+b.$$

Omitting the dependencies as in the proof of the previous proposition, this is equivalent to

$$\frac{1+\frac{\mathrm{d}f^2}{h}}{Y^2} \leq \left(\sqrt{1+\frac{b^2}{h}}+\frac{b}{\sqrt{h}}\right)^2,$$

which is fulfilled if

$$\frac{1+\frac{\mathrm{d}f^2}{h}}{Y^2} \leq (\sqrt{1+(\|b\|^h)^2}-\|b\|^h)^2.$$

This is equivalent to

$$Y^2 \geq \frac{1+\frac{\mathrm{d}f^2}{h}}{(\sqrt{1+(\|b\|^h)^2}-\|b\|^h)^2},$$

and due to $\frac{\mathrm{d}f^2}{h} \leq (\|\mathrm{d}f\|^h)^2$ and $(\sqrt{1+(\|b\|^h)^2}+\|b\|^h)^2 \leq 4(\sqrt{1+(\|b\|^h)^2})^2$, this is true if

$$Y^2 \geq 4(1+(\|\mathrm{d}f\|^h)^2)(1+(\|b\|^h)^2)$$

holds. By a similar reasoning the inequality for X is established. We get

$$X^2 \geq 4\frac{1+(\|b\|^h)^2}{1+\frac{\mathrm{d}f^2}{h}},$$

which is fulfilled if

$$X^2 \geq 4(1+(\|b\|^h)^2)$$

holds.

Now we can pinch with the complete Riemannian metric $h+\mathrm{d}f\otimes\mathrm{d}f$:

$$\frac{1}{2}\sqrt{\frac{h+\mathrm{d}f\otimes\mathrm{d}f}{(1+(\|\mathrm{d}f\|^h)^2)(1+(\|b\|^h)^2)}} \leq \sqrt{h+b\otimes b}+b \leq 2\sqrt{(1+(\|b\|^h)^2)(h+\mathrm{d}f\otimes\mathrm{d}f)}.$$

By Lem. 5.25 we observe that

$$\frac{1}{\sqrt{1+(\|\mathrm{d}f\|^h)^2}\sqrt{1+(\|b\|^h)^2}} = \frac{\|b\|^{h+b\otimes b}}{\|b\|^h\sqrt{1+(\|\mathrm{d}f\|^h)^2}}.$$

Thus, if the L^1-condition is fulfilled, the left inequality of the pinching above, yields the completeness of F by Thm. 5.22 as $h+\mathrm{d}f\otimes\mathrm{d}f$ is complete. Particularly, if $\|\mathrm{d}f\|^h\cdot\|b\|^h$ grows at most linearly towards $(h+\mathrm{d}f\otimes\mathrm{d}f)$-infinity, so does $\sqrt{1+(\|\mathrm{d}f\|^h)^2}\sqrt{1+(\|b\|^h)^2}$ and again by Thm. 5.22 and the completeness of $h+\mathrm{d}f\otimes\mathrm{d}f$, we get the completeness of F. This proves item *(i)*.

For item *(ii)* assume the slices S_t are Cauchy hypersurfaces, hence the Fermat metric is complete. By the right inequality of the pinching result we now get that

$$(1+(\|b\|^h)^2)(h+\mathrm{d}f\otimes\mathrm{d}f)=\left(\frac{\|b\|^h}{\|b\|^{h+b\otimes b}}\right)^2(h+\mathrm{d}f\otimes\mathrm{d}f)$$

is complete, using Lem. 5.25. As f is a proper function, $h+\mathrm{d}f\otimes\mathrm{d}f$ is complete and thus $\frac{\|b\|^{h+b\otimes b}}{\|b\|^h}$ is not L^1 on any escaping curve on S w.r.t. $h+\mathrm{d}f\otimes\mathrm{d}f$ by Thm. 5.22. Similarly, the superlinear growth of $\|b\|^h$ towards $(h+\mathrm{d}f\otimes\mathrm{d}f)$-infinity on S is impossible by Thm. 5.22. $\square$

Thus, we conclude that the linear growth of $\|b\|^h$ towards h-infinity on the base S of a standard stationary spacetime $(\mathbb{R}\times S,g)$ alone is neither a necessary nor a sufficient condition for $(\mathbb{R}\times S,g)$ to be globally hyperbolic—as one might guess from Prop. 5.29—but only the linear growth of $\|b\|^h$ towards $(h+\mathrm{d}f\otimes\mathrm{d}f)$-infinity for any proper function f on S is necessary. This is a weaker condition. Of course, in the case of a complete Riemannian metric h on S, the linear growth of the product $\|\mathrm{d}f\|^h\cdot\|b\|^h$ towards $(h+\mathrm{d}f\otimes\mathrm{d}f)$-infinity leads to the condition of linear growth of $\|b\|^h$ towards $(h+\mathrm{d}f\otimes\mathrm{d}f)$-infinity, which is sufficient for global hyperbolicity in this case.

Further remarkable results are the completeness of the Randers-type metric

$$G:=\left(\frac{\|b\|^h}{\|b\|^{h+b\otimes b}}\right)^2F$$

in the case of global hyperbolicity from Prop. 5.28 and the L^1-condition of $\frac{\|b\|^{h+b\otimes b}}{\|b\|^h}$ in the case of Cauchy hypersurfaces S_t from Thm. 5.31. It is worth comparing the completeness of G to the results on the stationary-to-Randers correspondence in [CJS11]. Therein, it has been established that a standard stationary spacetime $(\mathbb{R}\times S,g)$ with $g=-\mathrm{d}t^2+2b\mathrm{d}t+h$ and Fermat metric $F=\sqrt{h+b\otimes b}+b$ is globally hyperbolic if and only if there is a function $f\colon S\to\mathbb{R}$ such that $\tilde{F}=F-\mathrm{d}f$ is forward and backward complete on S. See Thms. 4.3 and 5.10 in [CJS11]. Hence, with the results above we have the following

Corollary 5.32. *Let (M,g) be a Riemannian manifold and b a one-form on M, such that $\|b\|^g<1$ and $R=\sqrt{g}+b$ is a Randers metric. If there is a function $f\colon M\to\mathbb{R}$, such that $\tilde{R}=R-\mathrm{d}f$ is a complete Randers metric on M, the Randers metric*

$$R'=\left(\frac{\|b\|^{g-b\otimes b}}{\|b\|^g}\right)^2R$$

is complete.

The L^1-condition of $\frac{\|b\|^{h+b\otimes b}}{\|b\|^h}$ gives a criterion to check for the slices S_t to be Cauchy hypersurfaces, based purely on the growth of this function along curves.

5.3 Lorentzian Bochner Technique

The Bochner technique in Riemannian geometry yields important non-existence results for Killing vector fields on compact Riemannian manifolds. For example, *a compact (without boundary), oriented Riemannian manifold* (M, g) *with non-positive Ricci curvature, Ric* ≤ 0*, admits only geodesic Killing vector fields, and if the Ricci curvature is negative, Ric* < 0*, then there are no non-trivial Killing vector fields.* See, e.g., [Wu80] for a survey. The Riemannian Bochner technique relies on the ellipticity of the Laplace–Beltrami operator; so generalizations to the semi-Riemannian setting are ambiguous. Nevertheless, A. Romero and M. Sánchez developed a Bochner technique for compact Lorentzian manifolds in [RS96] and [RS98], using an integral formula. Similar ideas were also developed by S.E. Stepanov in [Ste93], [Ste99] and [Ste00]. Already A. Lichnerowicz has obtained some of the results in this section about Ricci flat stationary Lorentzian manifolds with compact or asymptotically flat spacelike slices in [Lic55] using similar considerations.

We will need the following particular class of semi-Riemannian submersions. The standard references for the theory of submersions are [O'N66] and [Gra67], where we have to apply a straightforward generalization from the Riemannian to the Lorentzian case at some places. Some theory on Lorentzian submersions is contained in [FIP04], but we will not base this section on that reference, as our focus here is a different one.

Definition 5.33. *A mapping* $\pi\colon (M^{n+1}, g) \to (N^n, h)$ *will be called a Lorentz-to-Riemann submersion if it is a surjective submersion,* (M, g) *is a Lorentzian,* (N, h) *is a Riemannian manifold and* π_* *preserves the norms of horizontal vectors. A vector* $v \in T_pM$ *for some* $p \in M$ *is called vertical if* $v \in \ker(\mathrm{d}\pi_p)$*, i.e.,* $\mathrm{d}\pi_p(v) = 0$ *and it is called horizontal if it is in the* g*-orthocomplement* $\ker(\mathrm{d}\pi_p)^\perp$ *of* $\ker(\mathrm{d}\pi_p)$*. The manifold* N *is called the base of the Lorentz-to-Riemann submersion, the preimages* $\pi^{-1}(x) \subset M$ *are called fibers over* $x \in N$*.*

If $\pi\colon (M, g) \to (N, h)$ is a Lorentz-to-Riemann submersion, we necessarily have that all vertical vector fields are timelike and all horizontal vector fields are spacelike. Particularly, for any horizontal vector field $X \in \Gamma(TM)$ we have

$$g_p(X_p, X_p) = h_{\pi(p)}(\pi_{p_*} X_p, \pi_{p_*} X_p),$$

for all $p \in M$. It is not difficult to see that any Lorentz-to-Riemann submersion $\pi\colon (M, g) \to (N, h)$ has the splitting structure of a proper kinematical spacetime if the fibers are diffeomorphic to $\mathbb{R}$. Indeed, in this case, as N is a manifold and $\ker(\mathrm{d}\pi_p)$ is timelike for all $p \in M$, there is a free $\mathbb{R}$-action on every fiber $\pi^{-1}(x)$ for $x \in N$, as well as a global trivialization. Therefore, there naturally exists a unique vertical unit vector field V, with $V_p \in \ker(\mathrm{d}\pi_p)$ for all $p \in M$, which serves as a reference frame and there is a complete timelike vector field V^* parallel to V (cf. Thm. 6.2 for details). We adopt the designations VM and HM from section 3.1 for the horizontal and vertical distributions, as well as $\mathcal{V}$ and $\mathcal{H}$ for the projections on them.

A vector field $X \in \Gamma(HM)$ on any surjective submersion $\pi\colon M \to N$ is called basic if it projects to a unique vector field $X^- \in \Gamma(TN)$ on N, i.e., we can write $X^- = \pi_* X$. Hence, for any vector field $X \in \Gamma(TN)$, there is a unique vector field $\tilde{X} \in \Gamma(HM)$ called its horizontal lift, given by the unique basic vector field on M that projects to X. Generally, we will denote lifted objects with a tilde in this section. Particularly, we also have for functions: if $j\colon N \to \mathbb{R}$ is any function on the base N of the Lorentz-to-Riemann submersion, then $\tilde{j} := j \circ \pi\colon M \to \mathbb{R}$ is the lifted function on M. We now have the following

Lemma 5.34. *Let $\pi\colon (M, g) \to (S, h)$ be a Lorentz-to-Riemann submersion. For a function $F\colon M \to \mathbb{R}$, there is a function $f\colon S \to \mathbb{R}$, such that*

$$(\tilde{f} =)F = f \circ \pi\colon M \to \mathbb{R}$$

if and only if $\mathrm{d}F(V) = 0$ for some (and hence for all) nowhere vanishing vertical vector fields V on M.

Proof. Obviously, as $\ker(\mathrm{d}\pi_p)$ is one-dimensional for all $p \in M$ by definition, we see that $\mathrm{d}F(V) = 0 \Leftrightarrow \mathrm{d}F(\varphi V) = \varphi \mathrm{d}F(V) = 0$ for any nowhere vanishing function φ on M, shows that $\mathrm{d}F(V) = 0$ for one nowhere vanishing vector field $V \in \Gamma(VM)$ implies the same for all nowhere vanishing vertical vector fields.

"$\Rightarrow$": If $F = f \circ \pi$, then $\mathrm{d}F = \mathrm{d}(f \circ \pi) = \pi^*(\mathrm{d}f)$, and hence for any vector field $X\colon M \to TM$ we have for all $p \in M$

$$(\mathrm{d}F)(X)|_p = \pi^*(\mathrm{d}f)(X)|_p = \mathrm{d}f_{\pi(p)}(\pi_{p_*} X_p),$$

which is obviously zero if X is vertical.

"$\Leftarrow$": Let $x \in S$ and $\pi^{-1}(x) \subset M$ be the fiber over x. As $\mathrm{d}F(V) = 0$, F is constant along any fiber, i.e., $F(\pi^{-1}(x)) = F(p)$ for any $p \in \pi^{-1}(x)$. Hence, $f\colon S \to \mathbb{R}$ given by $f(x) := F(\pi^{-1}(x))$ is a well defined function on S. But this obviously implies that $F(p) = f(\pi(p)) = (f \circ \pi)(p)$ for all $p \in M$. □

The following theorem shows that Lorentz-to-Riemann submersions occur naturally as spatially stationary spacetimes.

Theorem 5.35. *Let (M, g, V) be a proper kinematical spacetime with the usual projection $\pi_M\colon M \to S = M/_{\mathbb{R}}$. π_M is a Lorentz-to-Riemann submersion if and only if (M, g, V) is spatially stationary.*

Proof. Following section 4.1 there is a trivialization, such that a proper kinematical spacetime can be written as $(M = \mathbb{R} \times S, g, V)$, with $\pi_M = \mathrm{pr}_2\colon \mathbb{R} \times S \to S$ being a surjective submersion and $\dim(S) = \dim(\mathbb{R} \times S) - 1$. Denote by X any horizontal vector field $X \in \Gamma(HM)$, i.e., $g(X, V) = 0$. Following section 3.1, the projection $\tilde{h} = g + u \otimes u$ with $u = g(V, \cdot)$ is a Riemannian metric on HM and $g(X, X) = \tilde{h}(X, X)$.

"$\Rightarrow$": If pr_2 acts as a Lorentz-to-Riemann submersion, we have $\tilde{h} = \mathrm{pr}_2^*(h)$ for some Riemannian metric h on S. In this case $\pounds_V \tilde{h} = 0$ holds and $(\mathbb{R} \times S, g, V)$ is spatially stationary by definition (cf. Def. 3.28).

"$\Leftarrow$": Suppose the vector fields $X, Y \in \Gamma(HM)$ are basic, i.e., lifted from some $X^-, Y^- \in \Gamma(TS)$. In this case we have

$$V(\tilde{h}(X,Y)) = (\pounds_V \tilde{h})(X,Y) = 0$$

as $\pounds_V X = \pounds_V Y = 0$. By the same argument as in Lem. 5.34, there is a function $\eta\colon S \to \mathbb{R}$ such that $\tilde{h}(X,Y) \circ \mathrm{pr}_2 = \eta$ and thus there is a Riemannian metric h on S such that $\eta = \tilde{h}(X,Y) \circ \mathrm{pr}_2 = h(X^-, Y^-)$. As this holds for all basic vector fields, the submersion $\mathrm{pr}_2\colon (\mathbb{R} \times S, g) \to (S,h)$ preserves the norms of horizontal vectors. $\square$

In particular, stationary spacetimes, which play a key role in this section, can be regarded as Lorentz-to-Riemann submersions, as the following corollary shows.

Corollary 5.36. *Any stationary spacetime $(\mathbb{R}\times S, g)$ with a Killing vector field ∂_t and metric*

$$g = -(A \circ \mathrm{pr}_2)^2 \mathrm{d}t \otimes \mathrm{d}t + 2\mathrm{pr}_2^*(b) \vee \mathrm{d}t + \mathrm{pr}_2^*(h) - \frac{\mathrm{pr}_2^*(b) \otimes \mathrm{pr}_2^*(b)}{(A \circ \mathrm{pr}_2)^2},$$

such that h is a Riemannian metric on S, $b \in \Gamma(\Lambda^1 S)$ and $A\colon S \to \mathbb{R}_{>0}$ is a Lorentz-to-Riemann submersion $\pi = \mathrm{pr}_2\colon (\mathbb{R} \times S, g) \to (S,h)$.

Proof. Obviously, $\pi = \mathrm{pr}_2\colon \mathbb{R} \times S \to S$ is a surjective submersion. Moreover, $(\mathbb{R} \times S, g)$ is Lorentzian and (S,h) is Riemannian with $\dim(S) = \dim(\mathbb{R}\times S) - 1$, by assumption. Now, for any $p = (t,x) \in \mathbb{R}\times S$ and a vector $v \in T_p(\mathbb{R}\times S)$, we denote by $v_0 := \mathcal{V}_p(v) \in V_p(\mathbb{R}\times S) \simeq \mathbb{R}$ its vertical part and by $\vec{v} := \pi_{p_*}(v) \in T_x S$ its projection to S. The vector v is horizontal if and only if $g_p(\partial_t|_p, v) = 0$, i.e., if and only if

$$g_p(\partial_t|_p, v) = -(A\circ\pi)^2(p) v_0 + \pi_p^*(b)(v) = -A^2(x) v_0 + b_x(\vec{v}) = 0.$$

Hence, we have

$$g_p(v,v) = -(A\circ\pi)^2(p) v_0^2 + 2\pi_p^*(b)(v) v_0 + \pi_p^*(h)(v,v) - \frac{\pi_p^*(b)(v)\pi_p^*(b)(v)}{(A\circ\pi)^2(p)} =$$

$$= -\frac{A^2(x) v_0 - b_x(\vec{v})}{A^2(x)} + h_x(\vec{v},\vec{v}) = h_x(\vec{v},\vec{v}).$$

$\square$

These two results allow proper kinematical spacetimes $(\mathbb{R} \times S, g, V)$ to be called (spatially) stationary Lorentz-to-Riemann submersions in the following if V is parallel to a Killing vector field (V is a rigid reference frame). Compare to section 3.2.

Now we would like to establish a relation between the Raychaudhuri equations introduced in section 3.3 and the Lorentzian Bochner technique, which was investigated in [RS96] and [RS98]. To this end, we assume for the moment that (M,g) is any Lorentzian manifold and

$X\colon M \to TM$ any complete timelike vector field. Defining the operator $A_X\colon \Gamma(TM) \to \Gamma(TM)$ by

$$A_X(Y) = -\nabla_Y X,$$

for all $Y \in \Gamma(TM)$ yields the equation

$$X\,\mathrm{div}(X) = -Ric(X,X) + \mathrm{div}(\nabla_X X) - \mathrm{Tr}(A_X \otimes A_X).$$

As can be checked by inspection, this is nothing more than the Raychaudhuri equation for the expansion of X (cf. item *(i)* in Prop. 3.33), generalized to non-unit vector fields X. Now, we assume (M,g) to be a compact Lorentzian manifold and use the identities $\mathrm{Tr}(A_X) = -\,\mathrm{div}(X)$, as well as $\mathrm{div}(\mathrm{div}(X)X) = X\,\mathrm{div}(X) + (\mathrm{div}(X))^2$ to obtain the integral

$$\int_M [Ric(X,X) + \mathrm{Tr}(A_X \otimes A_X) - (\mathrm{Tr}(A_X))^2]\mu_g = 0,$$

with μ_g the canonical measure induced by g on M. Unfortunately, as the metric g is Lorentzian, the difference $\mathrm{Tr}(A_X \otimes A_X) - (\mathrm{Tr}(A_X))^2$ has no fixed sign. But we can consider a reference frame $V = |g(X,X)|^{-\frac{1}{2}}X$ on (M,g), which gives rise to the orthogonal decomposition of the tangent bundle as in section 3.1. Hence, we can define an operator $A'_V\colon \Gamma(TM) \to \Gamma(HM)$ on $HM = V^\perp$ by

$$A'_V(Y) = \mathcal{H}(A_V(Y)),$$

for all vector fields Y on M. As it holds that $\mathrm{Tr}(A'_V) = \mathrm{Tr}(A_V)$, we have

$$\int_M [Ric(V,V) + \mathrm{Tr}(A'_V \otimes A'_V) - (\mathrm{Tr}(A'_V))^2]\mu_g = 0.$$

Thus, a decomposition of A'_V into a symmetric part S'_V and an anti-symmetric part H'_V, such that $A'_V = S'_V + H'_V$ yields

$$\int_M [Ric(V,V) - (\mathrm{Tr}(S'_V))^2 + \mathrm{Tr}(S'_V \otimes S'_V) + \mathrm{Tr}(H'_V \otimes H'_V)]\mu_g = 0. \tag{$*$}$$

This integral was used in [RS96] and [RS98] as a Lorentzian Bochner technique to deduce non-existence results of particular vector fields, mostly Killing vector fields, on compact Lorentzian manifolds. Our goal in this section is to broaden this argument to non-compact Lorentzian manifolds—particularly, to stationary spacetimes—and certain related classes of Lorentz-to-Riemann submersions. Therefore, let (M^{n+1}, g, V) be a proper kinematical spacetime and consider the Raychaudhuri equation for the expansion of V (cf. item *(i)* in Prop. 3.33) in the form

$$Ric(V,V) + |\sigma|^2 - |\omega|^2 - \frac{n-1}{n}\Theta^2 = -\,\mathrm{div}(\Theta V - \nabla_V V),$$

which follows from $\mathrm{div}(\Theta V) = \mathrm{d}\Theta(V) + \Theta^2$. Some implications of this form of the Raychaudhuri equation for physically motivated questions were recently also investigated in [AV11].

Remark 5.37. *Of course, the Raychudhuri equation above also holds if we assume* (M, g, V) *to be a compact kinematical spacetime. In this case, we get the integral*

$$\int_M [Ric(V,V) + |\sigma|^2 - |\omega|^2 - \frac{n-1}{n}\Theta^2]\mu_g = 0,$$

which is equivalent to $(*)$*. Hence, we can translate the Bochner quantities depending on* S'_V *and* H'_V *to the corresponding kinematical quantities appearing in the Raychaudhuri equation. We get the following Raychaudhuri-Bochner dictionary:*

Rauchudhuri		*Bochner*
$\frac{\Theta}{n}h + \sigma$	$=$	$-g(\cdot, S'_V(\cdot))$
ω	$=$	$g(\cdot, H'_V(\cdot))$
Θ	$=$	$\mathrm{Tr}(S'_V)$
$\|\omega\|^2$	$=$	$-\mathrm{Tr}(H'_V \otimes H'_V)$
$\|\sigma\|^2 + \frac{\Theta^2}{n}$	$=$	$\mathrm{Tr}(S'_V \otimes S'_V)$

Moreover, in the case of $n = 1$*, i.e., if* (M^2, g, V) *is a compact spacetime, which is necessarily a torus in the oriented case, we have that* $\omega = 0$ *and* $\sigma = 0$*, as the shear only consists of its trace* Θ*, and we recover the Lorentzian Gauss–Bonnet formula*

$$\int_{M^2} Ric(V,V)\mu_g = \int_{M^2} \mathcal{K}\mu_g = 0,$$

where $\mathcal{K} = -Ric(V,V)$ *is the Gaussian curvature of* M^2*.*

Proposition 5.38. *Let* $(\mathbb{R} \times S^n, g, V)$ *be a stationary Lorentz-to-Riemann submersion with* $\pi\colon (\mathbb{R} \times S, g) \to (S, h)$*. Then*

- *(i)* $u = g(V, \cdot) = -(A \circ \pi)\mathrm{d}t + \pi^*(\frac{b}{A})$ *and* $\dot{u} = g(\nabla_V V, \cdot) = \pi^*(\frac{\mathrm{d}A}{A})$*,*
- *(ii)* $\omega = \pi^*(\mathrm{d}(\frac{b}{A^2})A)$*, hence* $\omega = 0$ *if and only if* $\mathrm{d}(\frac{b}{A^2}) = 0$*,*
- *(iii)* $(\mathbb{R} \times S, g)$ *with reference frame* V *is standard static if* $\omega = 0$ *and the first Betti number of* S *vanishes, i.e.,* $B_1(S) = 0$*,*
- *(iv) there are functions* $r\colon S \to \mathbb{R}_{>0}$ *and* $\Omega^2\colon S \to \mathbb{R}_{>0}$*, such that* $Ric(V,V) = r \circ \pi$ *and* $|\omega|^2 = \Omega^2 \circ \pi$*,*
- *(v)* $\nabla_V V$ *is a basic vector field,*
- *(vi)* $\mathrm{div}(\nabla_V V) = [\|\nabla_V V\|_h^2 + \mathrm{div}_h(\pi_*(\nabla_V V))]\circ\pi$*, where* $\|\nabla_V V\|_h^2 = h(\pi_*(\nabla_V V), \pi_*(\nabla_V V))$ *and* div_h *is the divergence on* S *with respect to* h*.*

Proof. The spacetime $(\mathbb{R} \times S, g)$ is stationary as in Cor. 5.36, we have $V = \frac{\partial_t}{A \circ \pi}$ and ∂_t is Killing, with $\mathrm{d}t(\partial_t) = 1$, based on the metric splitting results in section 4.1, particularly Cor. 4.11.

(i)

$$u = \frac{1}{A \circ \pi} g(\partial_t, \cdot) = \frac{1}{A \circ \pi} \left(-(A \circ \pi)^2 \mathrm{d}t + \pi^*(b) + \pi^*(b)(\partial_t)\mathrm{d}t + \pi^*(\gamma)(\partial_t) \right) =$$

$$= -(A \circ \pi)\mathrm{d}t + \pi^*(\frac{b}{A}).$$

As $\omega = \mathrm{d}u + u \wedge \dot{u}$, we have $\dot{u} = V \rfloor \mathrm{d}u$. Thus,

$$\dot{u} = \frac{\partial_t}{A \circ \pi} \rfloor \left(-\mathrm{d}(A \circ \pi) \wedge \mathrm{d}t - \frac{\mathrm{d}(A \circ \pi)}{(A \circ \pi)^2} \wedge \pi^*(b) + \frac{\mathrm{d}\pi^*(b)}{(A \circ \pi)} \right) = \frac{\mathrm{d}(A \circ \pi)}{(A \circ \pi)} = \pi^*(\frac{\mathrm{d}A}{A})$$

as $\mathrm{d}(A \circ \pi)(\partial_t) = 0$ and $\partial_t \rfloor \mathrm{d}\pi^*(b) = 0$.

(ii) We compute

$$\omega = \mathrm{d}u + u \wedge \dot{u} = -\mathrm{d}(A \circ \pi) - \frac{\mathrm{d}(A \circ \pi)}{(A \circ \pi)^2} \wedge \pi^*(b) + \frac{\mathrm{d}\pi^*(b)}{A \circ \pi} + \left(-(A \circ \pi)\mathrm{d}t + \frac{\pi^*(b)}{A \circ \pi} \right) \wedge \frac{\mathrm{d}(A \circ \pi)}{A \circ \pi} =$$

$$= \frac{\mathrm{d}\pi^*(b)}{A \circ \pi} + 2\pi^*(b) \wedge \frac{\mathrm{d}(A \circ \pi)}{(A \circ \pi)^2} = \pi^* \left(\frac{\mathrm{d}b}{A} + 2b \wedge \frac{\mathrm{d}A}{A^2} \right) = \pi^*(\mathrm{d}(\frac{b}{A^2})A),$$

hence $\omega = 0$ if and only if $\mathrm{d}(\frac{b}{A^2}) = 0$ follows from $A > 0$.

(iii) By Def. 5.15, $(\mathbb{R} \times S, g)$ is standard static if $b = 0$. We show that there is an isometry (a change of slicing according to section 4.1) $\phi\colon (\mathbb{R} \times S, g) \to (\mathbb{R} \times S, g')$, such that $(\mathbb{R} \times S, g')$ is standard static if $\omega = 0$ and $B_1(S) = 0$. We showed above in *(ii)* that $\omega = 0$ implies $\mathrm{d}(\frac{b}{A^2}) = 0$ on S. Thus, as $B_1(S) = 0$, there is a function $f\colon S \to \mathbb{R}$, such that $b = A^2\mathrm{d}f$ if the rotation vanishes. As $(\mathbb{R} \times S, g)$, together with the reference frame V, constitutes a proper kinematical spacetime, we can apply Thm. 4.16 to $(\mathbb{R} \times S, g)$ using a trivialization ϕ that transforms $b \mapsto b - A^2\mathrm{d}f$. This is certainly an isometry and $\phi_*(b) = 0$ in $(\mathbb{R} \times S, g')$, hence $(\mathbb{R} \times S, g')$ is standard static.

(iv) By Lem. 5.34, it suffices to show that $\pounds_{\partial_t}[Ric(V, V)] = 0$ and $\nabla_V |\omega|^2 = 0$. As ∂_t is Killing we certainly have $\pounds_{\partial_t} g = 0$, which also implies $\pounds_{\partial_t} Ric = 0$. This together with $[V, \partial_t] = 0$ implies the first equation. For the second equation, we will use the Raychaudhuri equation for the rotation from Prop. 3.33. As $(\mathbb{R} \times S, g)$ is stationary the expansion Θ and the shear σ for the reference frame V both vanish. Moreover, we have $\mathrm{d}\dot{u} = 0$ because of *(i)*, hence the Raychaudhuri equation for the rotation reads in this case

$$\nabla_V \omega = u \wedge (\nabla_V V \rfloor \omega).$$

Thus, we have

$$\nabla_V |\omega|^2 = \nabla_V \mathrm{Tr}(\omega \otimes \omega) = \mathrm{Tr}((\nabla_V \omega) \vee \omega) = \mathrm{Tr}\left([u \wedge (\nabla_V V \rfloor \omega)] \vee \omega\right) =$$

$$= \sum_{i,j} [u \wedge (\nabla_V V \rfloor \omega)](E_i, E_j)\omega(E_i, E_j) = 0$$

as $u(E_i) = 0$ ($i \in \{1, \ldots, n\}$) for any horizontal vector field in some pseudo-orthonormal frame $\{V, E_1, \ldots, E_n\}$ and $\omega(V, \cdot) = 0$.

(v) Let $\{E_i\}_{i=1,\dots,n}$ be any h-orthonormal frame on S. Then, the E_i's lift to basic vector fields $\tilde{E}_i$ on $\mathbb{R}\times S$, such that $\{V,\tilde{E}_i\}_{i=1,\dots,n}$ is a g-pseudo-orthonormal frame on $\mathbb{R}\times S$. Following Lem. 1.2 in [Ej75], for a horizontal vector field X on $\mathbb{R}\times S$ to be basic, it suffices to show that $g_p(X,\tilde{E}_i)=g_q(X,\tilde{E}_i)$ for all $p,q\in\pi^{-1}(x)$ in the fiber over any $x\in S$. (Actually, a proof for this assertion is given in [Ej75] for Riemannian submersions only, but a generalization to the Lorentzian case, at hand here, is straightforward.) Clearly, $\nabla_V V$ is horizontal. As in our case the fibers are one-dimensional, it suffices to show that $V[g(\nabla_V V,\tilde{E}_i)]=0$ for all the $\tilde{E}_i$'s. As the $\tilde{E}_i$'s are basic we have $[V,\tilde{E}_i]=0$ and thus

$$V[g(\nabla_V V,\tilde{E}_i)]=\pounds_V(\dot{u}(\tilde{E}_i))=(\pounds_V\dot{u})(\tilde{E}_i)=(V\rfloor\mathrm{d}\dot{u})(\tilde{E}_i)=0$$

due to $\mathrm{d}\dot{u}=0$. Hence, there is a unique and well-defined vector field $\pi_*(\nabla_V V)$ on S that lifts to the acceleration of V on $\mathbb{R}\times S$.

(vi) As above, let $\{V,\tilde{E}_i\}_{i=1,\dots,n}$ be a g-pseudo-orthonormal frame on $\mathbb{R}\times S$, that results from the lift of an h-orthonormal frame $\{E_i\}_{i=1,\dots,n}$ on S. We denote by ∇^h the Levi-Civita connection associated to h on S. Following Thm. 3.2 in [Gra67], we have for all basic vector fields $X,Y\in\Gamma(H(\mathbb{R}\times S))$, related to $X^-,Y^-\in\Gamma(TS)$, that $\nabla_X Y$ is basic and related to $\nabla^h_{X^-}Y^-$ on S. Then we compute

$$\mathrm{div}(\nabla_V V)=-g(\nabla_V\nabla_V V,V)+\sum_i g(\nabla_{\tilde{E}_i}(\nabla_V V),\tilde{E}_i)=g(\nabla_V V,\nabla_V V)+$$

$$+\sum_i g(\nabla^h_{E_i}\widetilde{[\pi_*(\nabla_V V)]},\tilde{E}_i)=h(\pi_*(\nabla_V V),\pi_*(\nabla_V V))\circ\pi+\sum_i h(\nabla^h_{E_i}[\pi_*(\nabla_V V)],E_i)\circ\pi,$$

and the result follows. □

Now we are ready to state a main result of this section.

Theorem 5.39. *Let $(\mathbb{R}\times S,g,V)$ be a stationary Lorentz-to-Riemann submersion with $\pi\colon\mathbb{R}\times S\to S$ and* compact *S. Then there are two functions $r\colon S\to\mathbb{R}_{\geq 0}$ and $\Omega^2\colon S\to\mathbb{R}_{\geq 0}$ with $Ric(V,V)=r\circ\pi$ and $|\omega|^2=\Omega^2\circ\pi$, such that*

$$\int_S[r-\Omega^2-\|\nabla_V V\|_h^2]\mu_h=0,$$

as well as

$$\int_S r\mu_h\geq 0$$

holds. Thus, the timelike Ricci curvature cannot be non-positive, and negative somewhere, i.e., for any stationary Lorentz-to-Riemann submersion with compact S, $Ric(X,X)\leq 0$, and $Ric(X,X)<0$ somewhere, for all timelike $X\in\Gamma(T(\mathbb{R}\times S))$ cannot occur. Furthermore,

- *(i) if $(\mathbb{R}\times S,g)$ is Ricci flat, then it is static and V is geodesic, if additionally $B_1(S)=0$, then $(\mathbb{R}\times S,g)$ is standard static and*
- *(ii) if $Ric(X,X)\geq 0$, and $Ric(X,X)>0$ somewhere, for all timelike $X\in\Gamma(T(\mathbb{R}\times S))$ then $(\mathbb{R}\times S,g)$ is not static or V is not geodesic.*

Proof. We start from the Raychaudhuri equation for the expansion of V, which reads

$$Ric(V,V) - |\omega|^2 = \mathrm{div}(\nabla_V V)$$

in the stationary case. Based on *(iv)* and *(vi)* in Prop. 5.38, there are two functions r and Ω^2 on S as stated and we have

$$r \circ \pi - \Omega^2 \circ \pi = \|\nabla_V V\|_h^2 \circ \pi + \mathrm{div}_h(\pi_*(\nabla_V V)) \circ \pi,$$

hence

$$r - \Omega^2 - \|\nabla_V V\|_h^2 = \mathrm{div}_h(\pi_*(\nabla_V V))$$

holds on S. As S is compact, integration with respect to the measure μ_h induced by h yields

$$\int_S [r - \Omega^2 - \|\nabla_V V\|_h^2]\mu_h = 0.$$

Clearly, we have $\Omega^2 \geq 0$ and $\|\nabla_V V\|_h^2 \geq 0$ on all of S, such that $\int_S r\mu_h \geq 0$ follows. Now suppose that for all timelike $X \in \Gamma(T(\mathbb{R} \times S))$, we have $Ric(X,X) \leq 0$, and $Ric(X,X) < 0$ somewhere. This would imply $\int_S r\mu_h < 0$, a contradiction. In the Ricci flat case we have

$$\int_S [\Omega^2 + \|\nabla_V V\|_h^2]\mu_h = 0,$$

and due to $\Omega^2 \geq 0$ and $\|\nabla_V V\|_h^2 \geq 0$ this implies $\Omega^2 = 0$ and $\|\nabla_V V\|_h^2 = 0$. Hence, because of *(v)* in Prop. 5.38 we have that $\|\nabla_V V\|_h^2 = 0$ if and only if $|\dot{u}|^2 = 0$ and because of $\Omega^2 = |\omega|^2 \circ \pi$ we have that $\Omega^2 = 0$ if and only if $|\omega|^2 = 0$. This yields $\dot{u} = 0$ and $\omega = 0$ in the Ricci flat case. The standard static case follows from *(iii)* in Prop. 5.38. For the last assertion suppose that $(\mathbb{R} \times S, g)$ is static and V is geodesic, i.e., $\Omega^2 = \|\nabla_V V\|_h^2 = 0$. This implies

$$\int_S r\mu_h = 0,$$

hence $Ric(V,V) \geq 0$, and $Ric(V,V) > 0$ somewhere, yields a contradiction, as this would imply $\int_S r\mu_h > 0$.

□

Remark 5.40. *Based on Thm. 5.39, we can conclude that negative timelike Ricci curvature on spacetimes $(\mathbb{R} \times S, g)$ with g being timelike along the factor $\mathbb{R}$ and S compact is an obstruction for the existence of a timelike Killing vector field along the factor $\mathbb{R}$.*

In a next step, we will generalize the result in Thm. 5.39 in some directions. There are two directions in which the assumptions in the theorem could be relaxed. The first one is to generalize the existence of a Killing vector field parallel to V, to the vector field parallel to V being a homothetic vector field. The problem that arises in this case is that the mapping $\pi\colon (\mathbb{R} \times S, g) \to (S,h)$ will still be a submersion, but generally not a *Lorentz-to-Riemann* submersion any more. The second direction of relaxation is to consider non-compact manifolds S, while imposing some asymptotic conditions on the geometry of (S,h)

towards the ends of S. This leads to consider Riemannian manifolds (S, h) as a base which possess a specific asymptotic behavior of the metric towards infinity. This generalization, we will explore first.

A manifold S^n is said to have m $(m \in \mathbb{N})$ ends, denoted by $N^{(k)} \subset S$, $k = 1, \ldots, m$, if there is a compact set $K \subset S$, such that $S \setminus K = \bigsqcup_{k=1}^{m} N^{(k)}$ and each end $N^{(k)}$ is diffeomorphic to $\mathbb{R}_{>1} \times \mathbb{S}^{n-1}$ (or $\mathbb{R}_{>1}$ if $n = 1$).

Definition 5.41. *Let (S^n, h) be a Riemannian manifold. We say that (S^n, h) is of order p (at infinity) if S has m ends diffeomorphic to $\mathbb{R}_{>1} \times \mathbb{S}^{n-1}$ (or $\mathbb{R}_{>1}$ if $n = 1$), and in each end there exists a coordinate chart $(r, \phi) \in \mathbb{R}_{>1} \times \mathbb{S}^{n-1}$ (with ϕ being the canonical coordinates on the spheres $\mathbb{S}^{n-1}$) or $r \in \mathbb{R}_{>1}$ if $n = 1$, such that for the components of the metric h in these charts*

$$h_{ij} = \delta_{ij} + \mathcal{O}(\frac{1}{r^p})$$

holds $(i, j = 1, \ldots, n)$, for some $p \geq 0$ and with $\delta = \mathrm{d}r^2 + r^2 \mathrm{d}\phi^2$ being the flat metric on $\mathbb{R}_{>1} \times \mathbb{S}^{n-1}$ or $\delta = \mathrm{d}r^2$ being the flat metric on $\mathbb{R}_{>1}$ if $n = 1$.

Remark 5.42. *Being of order $p > 0$ is a weaker condition for a Riemannian manifold (S, h) than being p-asymptotically flat, in which case one additionally requires the first derivatives of the metric to be of order $p + 1$, i.e., $\partial_k h_{ij} = \mathcal{O}(\frac{1}{r^{p+1}})$, and the second derivatives to be of order $p + 2$, i.e, $\partial_l \partial_k h_{ij} = \mathcal{O}(\frac{1}{r^{p+2}})$, on all ends (cf. [Wal84, Ch. 11]). But hence, a p-asymptotically flat Riemannian manifold is certainly of order $p > 0$ at infinity.*

Definition 5.43. *Let $(\mathbb{R} \times S, g, V)$ be a stationary Lorentz-to-Riemann submersion. We say that the reference frame V is q-spatially asymptotically geodesic if there is some $q > 0$, such that*

$$\|\pi_*(\nabla_V V)\|_h^2 = \mathcal{O}(\frac{1}{r^q})$$

holds on all ends of S.

In the case when (S, h) is additionally of order p we have the following

Proposition 5.44. *Let $(\mathbb{R} \times S, g, V)$ be a stationary Lorentz-to-Riemann submersion with (S, h) being of order p at infinity. Then the reference frame V is q-spatially asymptotically geodesic if*

$$\frac{\partial_i A}{A} = \mathcal{O}(\frac{1}{r^{\frac{p+q}{2}}})$$

holds for the lapse function A and all indices $i = 1, \ldots, n$, on all ends of S.

Proof. If (S, h) is of order p, the components of the inverse metric $h^\sharp \in \Gamma(TS \vee TS)$ associated to h on any end of S behave as

$$h^\sharp = \delta + \mathcal{O}(r^p),$$

where δ is the flat metric. Hence, clearly

$$\|\pi_*(\nabla_V V)\|_h^2 = h^\sharp(\frac{\mathrm{d}A}{A}, \frac{\mathrm{d}A}{A}) = \delta(\frac{\mathrm{d}A}{A}, \frac{\mathrm{d}A}{A}) + \mathcal{O}(r^p)\mathcal{O}(\frac{\mathrm{d}A}{A})^2$$

is of order $\frac{1}{r^q}$ if $\frac{\partial_i A}{A}$ is of order $\frac{1}{r^{\frac{p+q}{2}}}$ on all ends of S. □

Theorem 5.45. *Let $(\mathbb{R} \times S^n, g, V)$ be a stationary Lorentz-to-Riemann submersion, such that S has $m > 0$ ends diffeomorphic to $\mathbb{R}_{>1} \times \mathbb{S}^{n-1}$ or $\mathbb{R}_{>1}$ if $n = 1$. Then there are two functions $r\colon S \to \mathbb{R}_{\geq 0}$ and $\Omega^2\colon S \to \mathbb{R}_{\geq 0}$ with $Ric(V,V) = r \circ \pi$ and $|\omega|^2 = \Omega^2 \circ \pi$, and*

$$\int_S [r - \Omega^2 - \|\nabla_V V\|_h^2]\mu_h = 0,$$

as well as

$$\int_S r\mu_h \geq 0$$

hold if

(i) (S,h) is of order p and

$$\frac{\partial_i A}{A} = \mathcal{O}(\frac{1}{r^{\frac{p+n-1+\varepsilon}{2}}})$$

holds for the lapse function A, all indices $i = 1, \ldots, n$ and some $\varepsilon > 0$ on all ends of S, or

(ii) the function $\|\pi_*(\nabla_V V)\|_h^2$ decreases faster towards infinity on all ends of S, than the $(n-1)$-volume of the embedded $(n-1)$-spheres of radius R grows towards infinity, i.e.,

$$\max_{\mathbb{S}^{n-1}(R)} \left(\|\pi_*(\nabla_V V)\|_h^2\right) \cdot \mathrm{Vol}_{n-1}^h(\mathbb{S}^{n-1}(R)) \overset{R\to\infty}{\longrightarrow} 0,$$

or

(iii) $n = 1$ and $\|\pi_*(\nabla_V V)\|_h^2 \overset{R\to\infty}{\longrightarrow} 0$ towards infinity on all ends of S.

In all these cases, the same conclusions on the Ricci curvature, and on the spacetime being (standard) static and/or V being geodesic, hold as in Thm. 5.39.

Proof. As in the proof of Thm. 5.39 above, the existence of the functions r and Ω^2 follow from *(iv)* and *(vi)* in Prop. 5.38 and we get

$$r - \Omega^2 - \|\nabla_V V\|_h^2 = \mathrm{div}_h(\pi_*(\nabla_V V)),$$

on S from the Raychaudhuri equation. Now we integrate both sides of this equation over a compact set $K_R \subset S$, the boundary of which is a disjoint union of m embedded $(n-1)$-dimensional spheres in the ends of S, all of which have radius $R > 1$ in the coordinates (r, ϕ) on the ends of S. We denote these spheres by $\mathbb{S}_k^{n-1}(R)$ for $k = 1, \ldots, m$ and get

$$\int_{K_R} [r - \Omega^2 - \|\nabla_V V\|_h^2]\mu_h = \int_{K_R} \mathrm{div}_h(\pi_*(\nabla_V V))\mu_h = \sum_{k=1}^m \int_{\mathbb{S}_k^{n-1}(R)} *^h(h(\pi_*(\nabla_V V), \cdot)),$$

where we denote by $*^h$ the Hodge operator on S with respect to h. This yields

$$\int_{K_R} [r - \Omega^2 - \|\nabla_V V\|_h^2]\mu_h \leq \sum_{k=1}^m \max_{\mathbb{S}_k^{n-1}(R)} \left(\|\pi_*(\nabla_V V)\|_h^2\right) \cdot \mathrm{Vol}_{n-1}^h(\mathbb{S}_k^{n-1}(R)).$$

As clearly $\int_{K_R} \mu_g \to \int_S \mu_g$ when $R \to \infty$, this proves the condition in item *(ii)*. Moreover, if $n = 1$ we have $\mathrm{Vol}_0^h(\mathbb{S}^0(R)) = 1$ and the condition in item *(iii)* follows.

Now assume that (S, h) is of order p at infinity and the condition for the order of $\frac{\partial_i A}{A}$ to decrease towards infinity on all ends of S holds as in item *(i)*. Using Prop. 5.44 this yields that the Lorentz-to-Riemann submersion is $(n-1+\varepsilon)$-spatially asymptotically geodesic, i.e.,

$$\|\pi_*(\nabla_V V)\|_h^2 = \mathcal{O}(\frac{1}{r^{n-1+\varepsilon}})$$

on all ends of S. As (S, h) is of order p, the volume of an embedded $(n-1)$-sphere in an end of S with radius $R > 1$ increases as $(1 + \frac{1}{R^p})R^{n-1}$ towards infinity. Hence, we have

$$\max_{\mathbb{S}^{n-1}(R)} \left(\|\pi_*(\nabla_V V)\|_h^2\right) \cdot \mathrm{Vol}_{n-1}^h(\mathbb{S}^{n-1}(R)) = \mathcal{O}(\frac{1}{R^\varepsilon}),$$

for some $\varepsilon > 0$, which approaches zero as $R \to \infty$. This proves the condition in item *(i)*. All remaining assertions follow from reasoning identical to the proof of Thm. 5.39. Note that we allow $\int_S r\mu_h = \infty$ if necessary. □

It is worth comparing the results in Thms. 5.39 and 5.45 to Thm. 0.1 by M.T. Anderson in [And00], which says that *any stationary, chronological, geodesically complete and Ricci flat spacetime, of dimension four, is (a finite spacelike quotient of) flat Minkowski space.* The crucial step in proving this theorem is to conclude from the Raychaudhuri equation $\mathrm{div}_h(\pi_*(\nabla_V V)) = \Omega^2 + \|\nabla_V V\|_h^2$ on the base (S, h) of a stationary spacetime $(\mathbb{R} \times S, g)$, and other constraint equations for the kinematical quantities that follow from $Ric = 0$, that $\omega = 0$ and $\dot{u} = 0$. If S is compact or of order p at infinity—the cases we investigated in this work—the assertion follows from the developed Lorentzian Bochner technique. In fact in the compact case we can conclude that S (if it is assumed orientable) can only be isometric to the flat 3-torus. Based on further propositions shown in this thesis, we can conclude the following about the scope of Anderson's theorem: Once the vanishing rotation and acceleration are established, the flatness of the spacetime, i.e., the vanishing of the Weyl curvature, follows from Remark 3.34. Certainly, this shows that the theorem cannot be generalized to spacetime dimensions ≥ 5, as only in dimension four the Weyl tensor is completely determined by the kinematical quantities and the trace-free part of Ricci curvature. The assumption of geodesic completeness certainly cannot be relaxed (just consider the spacetime arising from cutting pieces from Minkowski space). But we can see that the chronology condition is not necessary. In fact Anderson states in [And00] that it is unknown if the theorem holds for non-chronological stationary spacetimes. We can fill this gap based on the considerations in chapter 4. Indeed, the chronology condition is used in the proof of the theorem only for assuring the diffeomorphic splitting of the spacetime as $\mathbb{R} \times S$ along the Killing vector field, based on the considerations in [Har92]. But looking at Prop. 4.10 and its corollary, reveals that the only necessary assumptions that cannot be relaxed are the completeness of the Killing vector field and the non–partial-imprisonment of its integral curves. As a result the spacetimes can be *arbitrary* finite quotients of Minkowski space.

We will now investigate the scope of the particular version of the Lorentzian Bochner technique, which was developed in this section, for spacetimes admitting a timelike homothetic vector field. To this end we will consider spacetimes $(\mathbb{R}\times S, g)$ already given in a splitting form, together with the complete canonical vector field ∂_t which we assume timelike and homothetic for g. Certainly, there is a timelike vector field V parallel to ∂_t which is a reference frame with respect to g, but is not necessarily complete. Hence, we call a triple $(\mathbb{R}\times S, g, V)$ with the properties as stated a homothetic splitting spacetime. Subsequently, we will only consider the special case of a compact base manifold S.

Proposition 5.46. *Let $(\mathbb{R}\times S^n, g, V)$ be a homothetic splitting spacetime, such that $\pounds_{\partial_t} g = \frac{2c^2}{n} g$ for some constant $c > 0$. Then there is a metric g^* in the conformal class of g, such that $(\mathbb{R}\times S, g^*, \partial_t)$ is a proper kinematical spacetime, and the metric g can be written as*

$$g = e^{\frac{2c^2}{n}t}[-(A\circ\pi)^2\mathrm{d}t\otimes\mathrm{d}t + 2\pi^*(b)\vee\mathrm{d}t + \pi^*(h) - \frac{\pi^*(b\otimes b)}{(A\circ\pi)^2}],$$

with $\pi = \mathrm{pr}_2\colon \mathbb{R}\times S\to S$ and $t = \mathrm{pr}_1\colon \mathbb{R}\times S\to\mathbb{R}$ being the natural projections onto the factors, $b\in\Gamma(\Lambda^1 S)$, $A\colon S\to\mathbb{R}_{>0}$ and h a Riemannian metric on S. The reference frame V is given by

$$V = e^{-\frac{c^2}{n}t}\frac{\partial_t}{A\circ\pi},$$

and the expansion associated to V reads

$$\Theta = \frac{c^2}{A\circ\pi}e^{-\frac{c^2}{n}t(p)}.$$

Furthermore, the projection $\pi\colon(\mathbb{R}\times S, g)\to(S,h)$ is a surjective submersion and for horizontal vectors $X_p\in\ker(\mathrm{d}\pi_p)^\perp$ we have

$$g_p(X_p, X_p) = e^{\frac{2c^2}{n}t}h_{\pi(p)}(\pi_{p_*}X_p, \pi_{p_*}X_p)$$

for all $p\in\mathbb{R}\times S$.

Proof. We set

$$g^* = \frac{g}{|g(\partial_t,\partial_t)|},$$

which yields $g^*(\partial_t,\partial_t) = -1$ and g^* is a smooth Lorentzian metric as ∂_t is timelike by assumption. Thus ∂_t is a complete reference frame for g^* and $(\mathbb{R}\times S, g^*, \partial_t)$ is a proper kinematical spacetime as pr_2 projects on the manifold S. Hence, the splitting of the metric g^* given in Thm. 4.16 for a proper kinematical spacetime can be pulled back to the homothetic splitting spacetime $(\mathbb{R}\times S, g, V)$, which yields

$$g = -\tilde{A}^2\mathrm{d}t\otimes\mathrm{d}t + 2\pi^*(b_t)\vee\mathrm{d}t + \pi^*(h_t) - \frac{\pi^*(b_t)\otimes\pi^*(b_t)}{\tilde{A}^2}.$$

Here, $\tilde{A}\colon\mathbb{R}\times S\to\mathbb{R}_{>0}$ and the families $\{b_t\}_{t\in\mathbb{R}}$, $\{h_t\}_{t\in\mathbb{R}}$ are one-forms and Riemannian metrics respectively on S, varying smoothly with t, hence the splitting is understood to hold pointwise for all $(t,x)\in\mathbb{R}\times S$. Now we use $\pounds_{\partial_t} g = \frac{2c^2}{n}g$ together with $\pounds_{\partial_t}(\mathrm{d}t) = 0$ and get

$$-2\tilde{A}(\partial_t\tilde{A})\mathrm{d}t\otimes\mathrm{d}t + 2\pi^*(\partial_t b_t)\vee\mathrm{d}t + \pi^*(\partial_t h_t) -$$

$$-\frac{(2\pi^*(\partial_t b_t)\vee\pi^*(b_t))\tilde{A}-2\tilde{A}(\partial_t\tilde{A})(\pi^*(b_t)\otimes\pi^*(b_t))}{\tilde{A}^4}=$$

$$=\frac{2c^2}{n}[-\tilde{A}^2\mathrm{d}t\otimes\mathrm{d}t+2\pi^*(b_t)\vee\mathrm{d}t+\pi^*(h_t)-\frac{\pi^*(b_t)\otimes\pi^*(b_t)}{\tilde{A}^2}].$$

This yields

$$\partial_t\tilde{A}=\frac{c^2}{n}\tilde{A},\quad \partial_t b_t=\frac{2c^2}{n}b_t,\quad \partial_t h_t=\frac{2c^2}{n}h_t.$$

Hence, there is a function $A\colon S\to\mathbb{R}_{>0}$, such that $\tilde{A}=e^{\frac{c^2}{n}t}(A\circ\pi)$ and a one-form $b\in\Gamma(\Lambda^1 S)$ such that $b_t=e^{\frac{2c^2}{n}t}b$, as well as a Riemannian metric h on S such that $h_t=e^{\frac{2c^2}{n}t}h$, and the desired result follows.

Obviously, $g(V,V)=-1$ implies $V=e^{-\frac{c^2}{n}t}\frac{\partial_t}{A\circ\pi}$ and the formula for the expansion follows from Thm. 3.25.

The fact that a vector field X on $\mathbb{R}\times S$ is horizontal if and only if

$$g(V,X)=e^{\frac{c^2}{n}t}[-(A\circ\pi)\mathrm{d}t(X)+\frac{\pi^*(b)(X)}{A\circ\pi}]=0,$$

yields for horizontal vector fields the formula

$$\mathrm{d}t(X)=\frac{\pi^*(b)(X)}{(A\circ\pi)^2}.$$

A straightforward computation for any $p\in\mathbb{R}\times S$ yields

$$g_p(X_p,X_p)=e^{\frac{2c^2}{n}t}\pi_p^*(h_{\pi(p)})(X_p,X_p)=e^{\frac{2c^2}{n}t}h_{\pi(p)}(\pi_{p_*}X_p,\pi_{p_*}X_p).$$

□

The notions of basic vector fields and lifts carry over to homothetic splitting spacetimes. Particularly, this implies for an h-orthonormal frame $\{E_i\}_{i=1,\dots,n}$, i.e., $h(E_i,E_j)=\delta_{ij}$, on the base S of such a spacetime that its lift $\{\tilde{E}_i\}_{i=1,\dots,n}$ is horizontal and obeys $g(\tilde{E}_i,\tilde{E}_j)=e^{\frac{2c^2}{n}t}\delta_{ij}$. Hence, there is a g-orthonormal frame on $\mathbb{R}\times S$ given by $\{V,\hat{E}_i=e^{-\frac{c^2}{n}t}\tilde{E}_i\}_{i=1,\dots,n}$.

Remark 5.47. *The homothetic splitting spacetimes are special cases of conformal Lorentzian submersions, which we will analyze in greater detail in the following chapter.*

Lemma 5.48. *For a homothetic splitting spacetime $(\mathbb{R}\times S^n,g,V)$ it holds that the expansion Θ of V obeys*

$$\nabla_V\Theta+\frac{\Theta^2}{n}=0.$$

Proof. Based on the formulas for V and Θ from Prop. 5.46, we compute

$$\nabla_V\Theta+\frac{\Theta^2}{n}=\frac{e^{-\frac{c^2}{n}t}}{A\circ\pi}\partial_t\left(\frac{c^2}{A\circ\pi}e^{-\frac{c^2}{n}t}\right)+\frac{1}{n}\frac{c^4}{(A\circ\pi)^2}e^{-2\frac{c^2}{n}t}=$$

$$= \frac{c^2 e^{-\frac{c^2}{n}t}}{(A\circ\pi)^2}\left(-\frac{c^2}{n}\right)e^{-\frac{c^2}{n}t} + \frac{1}{n}\frac{c^4}{(A\circ\pi)^2}e^{-2\frac{c^2}{n}t} = 0.$$

□

Now, we establish a few facts about homothetic splitting spacetimes similar to the statements about stationary Lorentz-to-Riemann submersions shown in Prop. 5.38. All proofs below work similar to the ones in Prop. 5.38 and we refer to the proof of that proposition for missing details.

Proposition 5.49. *Let $(\mathbb{R}\times S^n, g, V)$ be a homothetic splitting spacetime. Then*

(i) $u = g(V,\cdot) = e^{-\frac{c^2}{n}t}\left(-(A\circ\pi)\mathrm{d}t + \frac{\pi^*(b)}{A\circ\pi}\right)$ *and* $\dot u = g(\nabla_V V,\cdot) = \pi^*\left(\frac{c^2}{n}\frac{b}{A^2} + \frac{\mathrm{d}A}{A}\right)$,

(ii) $\omega = e^{\frac{c^2}{n}t}\pi^*\left(\mathrm{d}\left(\frac{b}{A^2}\right)A\right)$ *and* $\mathrm{d}\dot u = \frac{\Theta\omega}{n}$,

(iii) there are functions $\Omega^2\colon S\to\mathbb{R}_{\geq 0}$ *and* $r\colon S\to\mathbb{R}$*, such that* $|\omega|^2 = e^{-\frac{2c^2}{n}t}(\Omega^2\circ\pi)$ *and* $Ric(V,V) = e^{-\frac{2c^2}{n}t}(r\circ\pi)$,

(iv) there is a vector field $W\colon S\to TS$*, such that* $\nabla_V V = e^{-\frac{2c^2}{n}t}\tilde W$ *with the lift* $\tilde W\in\Gamma(H(\mathbb{R}\times S))$*, i.e.,* $\nabla_V V$ *is proportional to a basic vector field,*

(v) with the divergence div_h *and the norm* $\|\cdot\|_h$ *associated to* h *on* S *we have* $\mathrm{div}(\nabla_V V) = e^{-2\frac{c^2}{n}t}[\|W\|_h^2 + \mathrm{div}_h(W)]\circ\pi$.

Proof. *(i)* Using $V = e^{-\frac{c^2}{n}t}\frac{\partial_t}{A\circ\pi}$, we compute

$$u = g(V,\cdot) = -(A\circ\pi)e^{\frac{c^2}{n}t}\mathrm{d}t + \frac{e^{-\frac{c^2}{n}t}}{A\circ\pi}\pi^*(b),$$

and hence

$$\dot u = V\rfloor\mathrm{d}u = V\rfloor\left(\frac{c^2}{n}e^{\frac{c^2}{n}t}\mathrm{d}t\wedge u + e^{\frac{c^2}{n}t}\mathrm{d}[-(A\circ\pi)\mathrm{d}t + \frac{\pi^*(b)}{A\circ\pi}]\right) =$$

$$= \frac{c^2}{n}\frac{\partial_t}{A\circ\pi}\rfloor\left(\mathrm{d}t\wedge\frac{\pi^*(b)}{A\circ\pi}\right) + \pi^*(\frac{\mathrm{d}A}{A}) = \frac{c^2}{n}\pi^*(\frac{b}{A^2}) + \pi^*(\frac{\mathrm{d}A}{A}).$$

(ii) In *(i)* we have already partially used

$$\mathrm{d}u = \frac{c^2}{n}e^{\frac{c^2}{n}t}\mathrm{d}t\wedge\frac{\pi^*(b)}{A\circ\pi} + e^{\frac{c^2}{n}t}\left(-\pi^*(\mathrm{d}A)\wedge\mathrm{d}t - \frac{\pi^*(\mathrm{d}A\wedge b)}{(A\circ\pi)^2} + \frac{\pi^*(\mathrm{d}b)}{A\circ\pi}\right).$$

Combining this with

$$u\wedge\dot u = e^{\frac{c^2}{n}t}\left(\frac{c^2}{n}\frac{-\mathrm{d}t\wedge\pi^*(b)}{A\circ\pi} - \mathrm{d}t\wedge\pi^*(\mathrm{d}A) + \frac{\pi^*(b\wedge\mathrm{d}A)}{(A\circ\pi)^2}\right)$$

yields

$$\omega = \mathrm{d}u + u \wedge \dot{u} = e^{\frac{c^2}{n}t}\pi^*\left(\frac{\mathrm{d}b}{A} + 2b \wedge \frac{\mathrm{d}A}{A^2}\right) = e^{\frac{c^2}{n}t}\pi^*(\mathrm{d}(\frac{b}{A^2})A).$$

Based on *(i)* we compute

$$\mathrm{d}\dot{u} = \pi^*(\frac{c^2}{n}\mathrm{d}(\frac{b}{A^2})) = \omega\frac{\frac{c^2}{n}e^{-\frac{c^2}{n}t}}{A\circ\pi}$$

and using Prop. 5.46 the result $\mathrm{d}\dot{u} = \frac{\Theta\omega}{n}$ follows.

(iii) Let $\{V, \hat{E}_i\}_{i=1,\ldots,n}$ be a g-orthonormal frame, such that $\hat{E}_i = e^{-\frac{c^2}{n}t}\tilde{E}_i$ for all $i = 1, \ldots, n$ and the $\tilde{E}_i$'s are the lifts of some h-orthonormal frame $\{E_i\}_{i=1,\ldots,n}$ on S. Then we have

$$|\omega|^2 = \sum_{i,j}\omega(\hat{E}_i, \hat{E}_j)\omega(\hat{E}_i, \hat{E}_j) = \sum_{i,j}e^{-\frac{2c^2}{n}t}\left[\pi^*(\mathrm{d}(\frac{b}{A^2})A)(\tilde{E}_i, \tilde{E}_j)\right]^2 =$$

$$= \sum_{i,j}e^{-\frac{2c^2}{n}t}[\mathrm{d}(\frac{b}{A^2})(E_i, E_j)A]^2 \circ \pi.$$

Now setting $\Omega^2 := \sum_{i,j}[\mathrm{d}(\frac{b}{A^2})(E_i, E_j)A]^2 \colon S \to \mathbb{R}_{\geq 0}$, yields the desired $|\omega|^2 = e^{-\frac{2c^2}{n}t}(\Omega^2\circ\pi)$. Furthermore, any homothetic vector field is indeed an affine vector field, i.e., the curvature tensor and hence the Ricci tensor is constant along it (cf. [KY61]). Thus, we have

$$\partial_t Ric(V, V) = \pounds_{\partial_t}(Ric(V, V)) = 2Ric(\pounds_{\partial_t}V, V).$$

Now using $\pounds_{\partial_t}V = \pounds_{\partial_t}(\frac{e^{-\frac{c^2}{n}t}}{A\circ\pi}\partial_t) = -\frac{c^2}{n}V$, yields

$$\partial_t Ric(V, V) = -2\frac{c^2}{n}Ric(V, V),$$

so there is a function $r \colon S \to \mathbb{R}$, such that $Ric(V, V) = e^{-\frac{2c^2}{n}t}(r\circ\pi)$ solves this ordinary differential equation.

(iv) Due to the formula for $\dot{u}$ in *(i)*, $u(\nabla_V V) = 0$ and the structure of the metric in Prop. 5.46, we get

$$e^{2\frac{c^2}{n}t}(\pi^*h)(\nabla_V V, \cdot) = \pi^*\left(\frac{c^2}{n}\frac{b}{A^2} + \frac{\mathrm{d}A}{A}\right).$$

Due to the right hand side of this equation being a one-form on $\mathbb{R}\times S$ pulled back from S, we infer that there is a vector field $W \colon S \to TS$, such that its lift $\tilde{W}$ to $\mathbb{R}\times S$ obeys $\tilde{W} = e^{2\frac{c^2}{n}t}\nabla_V V$, which gives the desired result.

(v) Denote by ∇^h the Levi-Civita connection associated to h on S. Then A. Gray's result (cf. Thm. 3.2 in [Gra67]) holds in the same way as it does for Lorentz-to-Riemann submersions and we have that for any two basic vector fields $X, Y \in \Gamma(H(\mathbb{R}\times S))$ related to the

vector fields $X^-, Y^- \in \Gamma(TS)$, the vector field $\nabla_X Y$ is basic and related to $\nabla^h_{X^-} Y^-$. Let $\{V, \hat{E}_i\}_{i=1,\ldots,n}$ be a g-orthonormal frame as in the proof of *(iii)*, then we compute

$$\operatorname{div}(\nabla_V V) + g(V, \nabla_V \nabla_V V) = \sum_i g(\hat{E}_i, \nabla_{\hat{E}_i} \nabla_V V) = e^{-\frac{2c^2}{n}t} \sum_i g(\tilde{E}_i, \nabla_{\tilde{E}_i} \nabla_V V) =$$

$$= e^{-\frac{4c^2}{n}t} \sum_i g(\tilde{E}_i, \nabla_{\tilde{E}_i} \tilde{W}) = e^{-\frac{2c^2}{n}t} \sum_i h(E_i, \nabla^h_{E_i} W) \circ \pi = e^{-\frac{2c^2}{n}t} \operatorname{div}_h(W) \circ \pi.$$

Furthermore,

$$g(V, \nabla_V \nabla_V V) = -g(\nabla_V V, \nabla_V V) = -e^{-\frac{4c^2}{n}t} g(\tilde{W}, \tilde{W}) = -e^{-\frac{2c^2}{n}t} h(W, W) \circ \pi$$

and the result follows. □

Now, we establish results about the Ricci curvature of homothetic splitting spacetimes, which are essentially similar to the ones in Thm. 5.39. Hence, there is a Bochner technique which works in the same way for stationary and homothetic splitting spacetimes with a compact base.

Theorem 5.50. *Let $(\mathbb{R} \times S^n, g, V)$ be a homothetic splitting spacetime and S compact. Then*

(i) the timelike Ricci curvature of $(\mathbb{R} \times S, g)$ cannot be non-positive, and negative somewhere, as $\int_S r \mu_h \geq 0$,

(ii) if $(\mathbb{R} \times S, g)$ is Ricci flat, then $\omega = 0$ and $\dot{u} = 0$, if additionally $B_1(S) = 0$ and the homothetic vector field is proper, i.e., $c \neq 0$, then $(\mathbb{R} \times S, g)$ is isometric to a warped product $(\mathbb{R}_{>0}, -\mathrm{d}\tau^2) \times_{\tau^2} (S, \frac{c^4}{a_0^2 n^2} h)$ for some $a_0 \in \mathbb{R}$,

(iii) if $Ric(X, X) \geq 0$, and $Ric(X, X) > 0$ somewhere, for all timelike vector fields X on $\mathbb{R} \times S$, then the rotation of V does not vanish or V is not geodesic.

Proof. Using Lem. 5.48 together with items *(iii)* and *(v)* in Prop. 5.49 yields the Raychaudhuri equation for the expansion Θ of V to read

$$e^{-\frac{2c^2}{n}t} \operatorname{div}_h(W) \circ \pi = e^{-\frac{2c^2}{n}t} (r \circ \pi) - e^{-\frac{2c^2}{n}t} (\Omega^2 \circ \pi) - e^{-\frac{2c^2}{n}t} \|W\|_h^2 \circ \pi.$$

Hence, as S is compact and $e^{-\frac{2c^2}{n}t} > 0$ for all $t \in \mathbb{R}$, we get

$$\int_S [r - \Omega^2 - \|W\|_h^2] \mu_h = 0. \tag{$*$}$$

Thus, $\int_S r \mu_h \geq 0$ follows, as well as the first assertion, as $r \circ \pi$ and $Ric(V, V)$ share the same sign everywhere.

Ricci flatness enforces $\int_S [\Omega^2 + \|W\|_h^2] \mu_h = 0$, thus $\Omega = 0$ and $W = 0$ and hence $\omega = 0$ and $\nabla_V V = 0$. With the same argument as in the proof of Thm. 5.39, vanishing first Betti number now ensures $b = 0$, by isometry. But the spacetime is not static in this case, we rather have a metric of the form

$$g = e^{-\frac{2c^2}{n}t}[-a_0^2 \mathrm{d}t \otimes \mathrm{d}t + \pi^*(h)]$$

on $\mathbb{R} \times S$, where we have used that vanishing acceleration now ensures the function $A \circ \pi$ to be equal to a constant, which we denote by a_0. Now, consider an isometry (which is admissible if $c \neq 0$) fostered by the transformation $t \mapsto a_0 \frac{n}{c^2} e^{-\frac{c^2}{n}t} = \tau$, which yields $\mathrm{d}\tau \otimes \mathrm{d}\tau = a_0^2 e^{-\frac{2c^2}{n}t} \mathrm{d}t \otimes \mathrm{d}t$ and the transformed metric reads

$$g = -\mathrm{d}\tau \otimes \mathrm{d}\tau + \tau^2 \pi^*(\frac{c^4}{n^2 a_0^2} h).$$

This is the desired warped product metric on $\mathbb{R}_{>0} \times S$ as $\tau > 0$.

The third assertion follows from the integral ($*$), as positive timelike Ricci curvature certainly enforces $r \geq 0$ and $r > 0$ somewhere. $\square$

To conclude this section, we state the following. Based on Thm. 5.50 we see that, for spacetimes $(\mathbb{R} \times S, g)$ with compact S and g timelike along the factor $\mathbb{R}$, negative timelike Ricci curvature is not only an obstruction to the existence of a timelike Killing vector field along the factor $\mathbb{R}$, but also to the existence of a timelike homothetic vector field along the factor $\mathbb{R}$.

Chapter 6

CONFORMAL LORENTZIAN SUBMERSIONS

In this chapter, we will consider spacetimes (M, g) which carry the structure of a surjective submersion $\pi\colon M^{n+1} \to N^n$ (cf. Def. 5.33), such that there is a function $s\colon M \to \mathbb{R}_{>0}$ and we have

$$g_p(X_p, X_p) = s^2(p) h_{\pi(p)}(\pi_* X_p, \pi_* X_p),$$

for all $p \in M$, some Riemannian metric h on N and all *horizontal* vector fields X on M. Obviously, this is a generalization of the Lorentz-to-Riemann submersions, and the homothetic splitting spacetimes, considered above. We will prove that this submersion structure enforces the spacetime to be conformal to a proper kinematical one in the most important cases (hence, they split as in Thm. 4.16), and that these submersions can be unequivocally characterized by kinematical quantities (see Thm. 6.2 below). Some theory on semi-Riemannian conformal submersions is already contained in [Gra67]. Analogues to O'Neill's fundamental tensors for submersions were developed in [OR93] and [Gud92] for the Riemannian case. In section 6.1 we will introduce the notion of Hubble-isotropic spacetimes as a particular case of these submersions, which is motivated by considerations from cosmology. Subsequently, in sections 6.2 and 6.3 we will analyze topological and causal properties of these spacetimes.

Just as in the case of Lorentz-to-Riemann submersions above, we will use the designations HM and VM for the horizontal and vertical vector bundle and H^*M and V^*M for the horizontal and vertical covector bundle, respectively, as well as $\mathcal{H}$ and $\mathcal{V}$ for the projections onto these bundles.

Definition 6.1. *A submersion $\pi\colon (M^{n+1}, g) \to (N^n, h)$, with (M, g) a spacetime and (N, h) a Riemannian manifold will be called a conformal Lorentz-to-Riemann submersion, or conformal Lorentzian submersion, if there is a function $s\colon M \to \mathbb{R}_{>0}$, such that for all horizontal vector fields $X \in \Gamma(HM)$ and all $p \in M$*

$$g_p(X_p, X_p) = s^2(p) h_{\pi(p)}(\pi_* X_p, \pi_* X_p)$$

holds. Furthermore, the conformal Lorentzian submersion will be called horizontally homothetic if $\mathcal{H}(\mathrm{d}s) = 0$.

Theorem 6.2. *Let $\pi\colon (M, g) \to (N, h)$ be a conformal Lorentzian submersion with fibers diffeomorphic to $\mathbb{R}$. Then there is a uniquely determined future-directed reference frame $V\colon M \to TM$ tangential to the fibers, as well as a future-directed, timelike and complete vector field V^* parallel to V, such that (M, g^*, V^*) is a proper kinematical spacetime with g^* being a Lorentzian metric conformal to g.*

Furthermore, a spacetime $(M = \mathbb{R} \times N, g)$ *together with a reference frame* V *parallel to* ∂_t *gives rise to* $\mathrm{pr}_2 \colon \mathbb{R} \times N \to N$ *being a conformal Lorentzian submersion if and only if it is spatially conformally stationary, i.e.,* $\sigma = 0$ *for* V*.*

Proof. Assume $\pi \colon (M, g) \to (N, h)$ is a conformal Lorentzian submersion. For all $p \in M$, let $V_pM = \ker(\pi_{p_*}) \subset T_pM$. Then V_pM is a timelike subspace, by the definition of the conformal Lorentzian submersion (the orthocomplement of V_pM with respect to g is spacelike). Moreover, $\dim(V_pM) = 1$, hence there is a uniquely defined reference frame $V \colon M \to TM$, given by $V_p \in V_pM$, $g_p(V_p, V_p) = -1$ and by the requirement of future-directedness for all $p \in M$. Certainly, (M, N, π) is a fiber bundle, with fibers diffeomorphic to $\mathbb{R}$, hence, there is a global section (see [KN63, Ch. 1, Thm. 5.7]). As N is a manifold, it remains to show that (M, N, π) is principal in order to get a global trivialization, and hence a proper kinematical spacetime, i.e., we need to construct a free $\mathbb{R}$-action on M. The vector field V constructed above is not necessarily complete, i.e., its associated flow is not necessarily global. But just as in Prop. 3.12, with the help of a complete Riemannian metric g_R on M, we can construct a complete vector field V^* parallel to V. The flow associated to V^* is global and its flow lines coincide with the fibers. Thus, (M, N, π) is a globally trivializable principal fiber bundle, i.e., (M, g^*, V^*) is a proper kinematical spacetime if we choose g^* from the conformal class of g, such that $g^*(V^*, V^*) = -1$ holds.

Now we consider the spacetime $(M = \mathbb{R} \times N, g)$ with a reference frame $V \colon M \to TM$ pointing along the factor $\mathbb{R}$, i.e., V is parallel ∂_t. Certainly, $\mathrm{pr}_2 \colon M \to N$ is a surjective submersion. Denote by X a horizontal vector field $X \in \Gamma(HM)$, i.e., $g(X, V) = 0$. Following section 3.1, the projection $\tilde{h} = g + u \otimes u$ with $u = g(V, \cdot)$ is a Riemannian metric on HM and $g(X, X) = \tilde{h}(X, X)$.

"$\Rightarrow$": If pr_2 acts as a conformal Lorentzian submersion, we have

$$\tilde{h}(X, X) = s^2[h(\mathrm{pr}_{2*}X, \mathrm{pr}_{2*}X) \circ \mathrm{pr}_2],$$

for some Riemannian metric h on N and thus

$$\tilde{h} = s^2 \mathrm{pr}_2^* h$$

as $\tilde{h}$ is purely horizontal. In this case we get

$$\pounds_V \tilde{h} = V(s^2 \mathrm{pr}_2^* h) = (2s\dot{s}) \mathrm{pr}_2^* h = \frac{2\dot{s}}{s} \tilde{h}$$

denoting $\dot{s} = \nabla_V s$. Hence, $(\mathbb{R} \times N, g, V)$ is spatially conformally stationary by definition, setting $\varphi := \frac{2\dot{s}}{s}$, and the shear vanishes (cf. Def. 3.28 and Lem. 3.30).

"$\Leftarrow$": Suppose the vector fields $X, Y \in \Gamma(HM)$ are basic, i.e., lifted from some $X^-, Y^- \in \Gamma(TN)$. In this case, we have $\pounds_V X = \pounds_V Y = 0$. Hence

$$V(\tilde{h}(X, Y)) = (\pounds_V \tilde{h})(X, Y) = \varphi \tilde{h}(X, Y),$$

for some function $\varphi \colon M \to \mathbb{R}_{>0}$. Now we have that for any two basic vector fields X, Y, the function $\tilde{h}(X, Y) \colon \mathbb{R} \times N \to \mathbb{R}$ changes according to the differential equation

$$(\partial_t \tilde{h})(X, Y)(t, x) = A(t, x)\varphi(t, x)\tilde{h}(X, Y)(t, x)$$

as V is parallel to ∂_t, i.e., there is a function $A\colon \mathbb{R}\times M \to \mathbb{R}_{>0}$, such that $A\cdot V = \partial_t$. This is an ordinary differential equation in every fiber $\pi^{-1}(x)$, which has the solution

$$\tilde{h}(X,Y)(t,x) = \tilde{h}(X,Y)(t_0,x)\exp(\int_{t_0}^{t} A(t',x)\varphi(t',x)\mathrm{d}t'),$$

for some fixed $t_0\in\mathbb{R}$. Obviously, the initial value $\tilde{h}(X,Y)(t_0,x)$ does not depend on the position in the fiber. Using an argument analogous to Lem. 5.34, this implies that there is a function $h(X^-,Y^-)\colon N\to\mathbb{R}_{>0}$, such that $\tilde{h}(X,Y)(t_0,x) = h(X^-,Y^-)\circ \mathrm{pr}_2$. But as this holds for all vector fields X^-,Y^- on N, this defines a Riemannian metric on N, and by setting $s^2(t,x) = \exp(\int_{t_0}^{t} A(t',x)\varphi(t',x)\mathrm{d}t')$, we get the desired result. □

Furthermore, this implies the following

Proposition 6.3. *Let $\pi\colon (M^{n+1} = \mathbb{R}\times N^n, g) \to (N^n,h)$ be a conformal Lorentzian submersion and $V\colon M\to TM$ the future-directed reference frame parallel to ∂_t. Then there is a function $\bar{s}\colon N\to\mathbb{R}_{>0}$, such that $\overline{\pi}\colon (M,g)\to(N,\bar{s}h)$ is a horizontally homothetic conformal Lorentzian submersion if and only if the spacetime (M,g) together with the reference frame V is spatially homothetic.*

Proof. Let Θ be the expansion associated to V, and as we have a conformal Lorentzian submersion the shear associated to V vanishes due to Thm. 6.2 above. Following Lem. 3.31, a spacetime (M,g) with reference frame V is spatially homothetic if and only if $\mathcal{H}(\mathrm{d}\Theta) = 0$, i.e., the reference frame V is spatially homothetic. Furthermore, from the computations in the proof of Thm. 6.2, we get

$$\pounds_V \tilde{h} = 2\frac{\dot{s}}{s}\tilde{h},$$

for $\tilde{h} = g + u\otimes u$ in the spatially conformally stationary case, which implies $\Theta = n\frac{\dot{s}}{s}$, with the dot denoting the derivative ∇_V. Thus, we have

$$\mathcal{H}[\mathrm{d}(\frac{\dot{s}}{s})] = 0$$

if and only if V is spatially homothetic.

Investigating this expression, we can infer that it is fulfilled if and only if s has product form $s = s_1\cdot s_2$ with $\mathcal{H}(\mathrm{d}s_1) = 0$ and $\mathcal{V}(\mathrm{d}s_2) = 0$. But this is the case if and only if there is a function $\bar{s}\colon N\to\mathbb{R}_{>0}$, such that $s_2\circ\pi = \bar{s}$. Thus, a horizontally homothetic conformal Lorentzian submersion $\overline{\pi}\colon (M,g)\to(N,\overline{h} = \bar{s}h)$ is determined by

$$g_p(X_p,X_p) = s_1^2(p)\overline{h}_{\pi(p)}(\pi_* X_p, \pi_* X_p),$$

for all horizontal vector fields $X\in\Gamma(HM)$ if and only if s has a product structure as above, i.e., if and only if V is spatially homothetic. □

Particularly, the conclusions in the theorem and the proposition above hold for kinematical spacetimes (M,g,V), but in general we do not have to require V to be complete. Moreover, we see that a spatially conformal (and even a spatially homothetic) kinematical spacetime does not uniquely determine the conformal Lorentzian submersion $\pi\colon (M,g) \to (N,h)$ associated to it. Rather, there is the freedom of choosing the Riemannian metric h up to a conformal factor $\lambda\colon N \to \mathbb{R}_{>0}$, which in turn then determines $s\colon M \to \mathbb{R}_{>0}$ up to the multiplication with a function $\Lambda\colon M \to \mathbb{R}_{>0}$ given by $\Lambda \circ \pi = \frac{1}{\lambda}$.

6.1 Hubble-isotropic Spacetimes

In the remaining sections of this chapter, we will consider a particular class of conformal Lorentzian submersions only. To this end, we perform the following

Definition 6.4. *An ordered triple (M,g,V) is called Hubble-isotropic spacetime if (M,g) is a spacetime together with a future-directed reference frame $V\colon M \to TM$, and the shear and the acceleration of V vanish, i.e., $\sigma = 0$ and $\dot{u} = 0$.*

In general, we do not require a Hubble-isotropic spacetime (M,g,V) to be a (Cartan) kinematical spacetime, as V is not necessarily complete. But below we will primarily consider Hubble-isotropic spacetimes given in a splitting form $(\mathbb{R} \times N, g, V)$ with V parallel to ∂_t. Obviously, the notion of Hubble-isotropic spacetimes do naturally include conformally stationary and stationary ones with vanishing acceleration (cf. Prop. 6.5 below). We will sometimes refer to a Hubble-isotropic spacetime which is not conformally stationary, and hence also not stationary, as being properly Hubble-isotropic.

Proposition 6.5. *A Hubble-isotropic spacetime (M,g,V) is*

(i) *a proper kinematical spacetime if (M,g,V) is a Cartan kinematical spacetime,*
(ii) *a conformal Lorentzian submersion with totally geodesic fibers if $M = \mathbb{R} \times N$ for some manifold N and V parallel to ∂_t,*
(iii) *conformally stationary if $\mathrm{d}(\Theta u) = 0$ holds for $u = g(V,\cdot)$, and*
(iv) *stationary if $\Theta = 0$.*

Proof. *(i)* This is an application of Thm. 4.19.

(ii) This follows from Thm. 6.2. As the fibers are the integral curves of V, vanishing acceleration implies totally geodesic fibers.

(iii) and *(iv)* For vanishing shear and acceleration, the conditions $\mathrm{d}(\Theta u) = 0$ resp. $\Theta = 0$ lead to conformally stationary resp. stationary spacetimes as can be deduced from Thm. 3.25. $\square$

The following definition and proposition clarify the term Hubble-isotropic for the spacetimes defined above. These spacetimes are of particular interest in physical applications, especially in cosmology. Nevertheless, their global properties have scarcely been analyzed up to now. The standard references for Hubble-isotropic spacetimes are [Has91] and [HP99]. But already J. Ehlers et al. (see [Ehl93][1]) have analyzed spacetimes of this kind.

[1] Note that this a reprint of a 1961 article.

Definition 6.6. *For a spacetime (M,g) together with a future-directed reference frame $V\colon M\to TM$, we denote by $K\in\Gamma(TM)$ a lightlike vector field, i.e., $g(K,K)=0$, such that $g(V,K)\neq 0$ and by $(z,k)\in TM$, with $z\in M$ and $k:=K_z\in T_zM$, its coordinates in the tangent bundle. Let*

$$CM:=\{(z,k)\in TM\,|\,g_z(k,k)=0,\ g_z(V_z,k)\neq 0\}$$

be the lightlike subset of the tangent bundle TM. Then, we define a functional on CM, called the Hubble functional, by

$$H\colon CM\to\mathbb{R},\quad H(z,k):=\frac{g_z((\nabla_kV)_z,k)}{g_z(V_z,k)^2}.$$

Let $P\colon TM\to M$ be the natural projection. Then the Hubble functional H is called isotropic if it is the composition of a Hubble function $\tilde{H}\colon M\to\mathbb{R}$ and the natural projection restricted to CM, such that

$$H(z,k)=\tilde{H}(z)\circ P|_{CM}.$$

The Hubble functional appears naturally in cosmology as the first coefficient H_1 in a series expansion of the relation of the redshift ζ to the distance D of galaxies, firstly given by J. Kristian and R.K. Sachs in [KS66]:

$$\zeta=H_1D+\cdots+H_nD^n+\mathcal{O}(D^{n+1}).$$

Hence, there is a strong motivation from physics to analyze these spacetimes. If H_1 is isotropic, it is usually called *Hubble constant*, but may depend on the position. Thus, the Hubble-isotropic spacetimes are exactly those which have an isotropic linear coefficient in the series expansion of the $\zeta-D$ relation (see [HP99] for a detailed analysis).

The following proposition is contained in [HP99], but has already been implicitly shown in [Ehl93]. We give a full explicit proof here for the sake of completeness.

Proposition 6.7. *(M^{n+1},g,V) is a Hubble-isotropic spacetime if and only if its Hubble functional is isotropic: $H=\tilde{H}\circ P$. Furthermore, we have in this case $\tilde{H}=\frac{\Theta}{n}$.*

Proof. From Prop. 3.21 we get

$$g_z((\nabla_kV)_z,k)=\sigma_z(k,k)-\dot{u}_z(k)g_z(V_z,k)+\frac{\Theta}{n}h_z(k,k),$$

for all $(z,k)\in CM$. As $h=g+u\otimes u$, it follows that $h_z(k,k)=(g_z(V_z,k))^2$. Hence, we have

$$H(z,k)=\underbrace{\frac{\sigma_z(k,k)}{(g_z(V_z,k))^2}-\frac{\dot{u}_z(k)}{g_z(V_z,k)}}_{=:Y(z,k)}+\frac{\Theta(z)}{n}.$$

Thus, we observe that H is independent of k if the term $Y(z,k)$ is zero for all $(z,k)\in CM$. In this case, we also get $\tilde{H}(z)=\frac{\Theta(z)}{n}$. We observe that $Y(z,k)=0$ if and only if $\sigma=0$ and $\dot{u}=0$. This can be seen as follows: suppose σ and $\dot{u}$ is such that

$$\sigma_z(k,k)=\dot{u}_z(k)g_z(V_z,k)=\dot{u}_z(k)u_z(k),$$

for all $(z,k) \in CM$. Now let $e \in T_zM$ be some spacelike vector obeying $g_z(V_z, e) = 0$ and $g_z(e,e) = 1$ and consider two lightlike vectors $k_1 = V_z + e$, as well as $k_2 = -V_z + e$ in T_zM. Then we have

$$\sigma_z(e,e) = \sigma_z(k_1,k_1) = \dot{u}_z(k_1)u_z(k_1) = -\dot{u}_z(e) = -\dot{u}_z(k_2)u_z(k_2) = -\sigma_z(k_2,k_2) = -\sigma_z(e,e).$$

As this holds for any spacelike vector $e \in T_zM$, we have $\sigma = 0$ and hence $\dot{u} = 0$. Furthermore, this also shows that $Y(z,k)$ cannot assume some non-zero value, which is constant with respect to k, hence we have that H is independent of k if and only if $Y(z,k)$ is zero for all $(z,k) \in CM$. □

6.2 Topological and Causal Properties

Having established that a Hubble-isotropic spacetime is a conformal Lorentzian submersion if the underlying manifold splits appropriately as a product, and that it is a proper kinematical spacetime if it is a Cartan kinematical spacetime, in the previous section, it is natural to ask for the splitting structure of the metric implied by Thm. 4.16 in these cases.

Theorem 6.8. *Let $(M^{n+1} = \mathbb{R} \times N^n, g, V)$ be a Hubble-isotropic spacetime of splitting type, with the reference frame V parallel to ∂_t. Then there are two functions $A, s\colon \mathbb{R} \times N \to \mathbb{R}_{>0}$ and a Riemannian metric h on N, such that $V = \frac{1}{A}\partial_t$ and the metric is given by*

$$g_{(t,x)} = -A^2(t,x)\mathrm{d}t \otimes \mathrm{d}t + 2\mathrm{pr}_2^*(b_{(t,x)}) \vee \mathrm{d}t + s^2(t,x)\mathrm{pr}_2^*(h_x) - \frac{\mathrm{pr}_2^*(b_{(t,x)}) \otimes \mathrm{pr}_2^*(b_{(t,x)})}{A^2(t,x)}$$

with $x \in N$, $t = \mathrm{pr}_1\colon \mathbb{R} \times N \to \mathbb{R}$, $\mathrm{pr}_2\colon \mathbb{R} \times N \to N$ and $\{b_t\}_{t\in\mathbb{R}}$ a family of one-forms on N obeying

$$b_{(t,x)} = A(t,x)\left(\beta_x + \int_{t_0}^{t} \mathcal{H}(\mathrm{d}A)_{(t',x)}\mathrm{d}t'\right), \tag{$*$}$$

for some $t_0 \in \mathbb{R}$ and a one-form $\beta \in \Gamma(\Lambda^1 N)$. The expansion Θ of V is given by

$$\Theta(t,x) = n\frac{\nabla_V s}{s}(t,x) = \frac{n(\partial_t s)(t,x)}{A(t,x)s(t,x)}.$$

Furthermore, the Hubble-isotropic spacetime is locally *isometric to $(I \times N, \tilde{g}, \partial_t)$, with*

$$\tilde{g} = -\mathrm{d}\tau \otimes \mathrm{d}\tau + 2\mathrm{pr}_2^*(\beta) \vee \mathrm{d}\tau + \tilde{s}^2\mathrm{pr}_2^*(h) - \mathrm{pr}_2^*(\beta) \otimes \mathrm{pr}_2^*(\beta),$$

where $s = \tilde{s} \circ \varphi$, with φ being the local isometry and $I \subset \mathbb{R}$ a, possibly unbounded, open interval.

Proof. Certainly, ∂_t is complete and there is a Lorentzian metric g^* in the conformal class of g, such that $(\mathbb{R} \times N, g^*, \partial_t)$ is a proper kinematical spacetime. Hence, the splitting of the metric as in Thm. 4.16 can be applied to g by pullback.

As usual we have $u = g(V, \cdot) = -A\mathrm{d}t + \frac{\mathrm{pr}_2^*(b)}{A}$ and the splitting yields

$$g_{(t,x)} = -A^2(t,x)\mathrm{d}t \otimes \mathrm{d}t + 2\mathrm{pr}_2^*(b_{(t,x)}) \vee \mathrm{d}t + \mathrm{pr}_2^*(\tilde{h}_{(t,x)}) - \frac{\mathrm{pr}_2^*(b_{(t,x)}) \otimes \mathrm{pr}_2^*(b_{(t,x)})}{A^2(t,x)},$$

for a family of Riemannian metrics $\{\tilde{h}_t\}_{t\in\mathbb{R}}$ on N. We will show that $\dot{u} = 0$ implies $(*)$ to hold for the one-forms b_t and $\sigma = 0$ implies the existence of a function $s\colon \mathbb{R} \times N \to \mathbb{R}_{>0}$ such that $\tilde{h}_{(t,x)} = s^2(t,x)h_x$. We have

$$\dot{u} = \nabla_V u = V \rfloor \mathrm{d}u = \frac{\partial_t}{A} \rfloor \left(-\mathrm{d}A \wedge \mathrm{d}t - \frac{\mathrm{d}A}{A^2} \wedge \mathrm{pr}_2^*(b) + \frac{\mathrm{d}[\mathrm{pr}_2^*(b)]}{A} \right) =$$

$$= -\frac{\partial_t A}{A}\mathrm{d}t + \frac{\mathrm{d}A}{A} - \frac{\partial_t A}{A^3}\mathrm{pr}_2^*(b) + \frac{\mathrm{pr}_2^*(\partial_t b)}{A^2}$$

where we denoted by $\partial_t b$ the derivative of the family of one-forms $\{b_t\}_{t\in\mathbb{R}}$ on N with respect to their parameter t. As

$$\mathrm{d}A = (\partial_t A)\mathrm{d}t + \mathcal{H}(\mathrm{d}A),$$

it follows that

$$\dot{u} = -\frac{\partial_t A}{A^3}\mathrm{pr}_2^*(b) + \frac{\mathcal{H}(\mathrm{d}A)}{A} + \frac{\mathrm{pr}_2^*(\partial_t b)}{A^2}.$$

Hence, as $\dot{u} = 0$, we have

$$\mathrm{pr}_2^*(\partial_t b) = \frac{\partial_t A}{A}\mathrm{pr}_2^*(b) + A\mathcal{H}(\mathrm{d}A).$$

But this is an ordinary differential equation in each fiber $\mathbb{R} \times \{x\} \subset \mathbb{R} \times N$, and we have for all fixed $x \in N$

$$\frac{\mathrm{d}b_x}{\mathrm{d}t}(t) = \frac{\mathrm{d}\log(A_x)}{\mathrm{d}t}(t)b_x(t) + A_x(t)\mathcal{H}(\mathrm{d}A)_x(t)$$

with $A_x(t) := A(t,x)$. The general solution of this differential equation is easily computed to be

$$b_x(t) = A_x(t) \left(\beta_x + \int_{t_0}^{t} \mathcal{H}(\mathrm{d}A)_x(t')\mathrm{d}t' \right)$$

where $b_x(t_0) = \beta_x$ is the initial value.

The proof of $\tilde{h}_{(t,x)} = s^2(t,x)h_x$ works similar to the proof of Thm. 6.2 above. Following section 3.1, we have $g = \mathrm{pr}_2^*(\tilde{h}) - u \otimes u$ and, as the acceleration vanishes, $\pounds_V u = 0$ holds. Hence, by Prop. 3.21 we get

$$\pounds_V \mathrm{pr}_2^*(\tilde{h}) = \pounds_V g = 2\sigma + \frac{2}{n}\Theta \mathrm{pr}_2^*(\tilde{h}).$$

As $\mathrm{pr}_2^*(\tilde{h})$ is horizontal, we get $\pounds_V \mathrm{pr}_2^*(\tilde{h}) = \pounds_{\frac{\partial_t}{A}} \mathrm{pr}_2^*(\tilde{h}) = \frac{1}{A}\pounds_{\partial_t}\mathrm{pr}_2^*(\tilde{h}) = \frac{1}{A}\mathrm{pr}_2^*(\partial_t \tilde{h})$. Thus, $\sigma = 0$ implies the ordinary differential equation

$$\frac{\mathrm{d}\tilde{h}_x}{\mathrm{d}t}(t) = \frac{2}{n}\Theta_x(t)A_x(t)\tilde{h}_x(t) \tag{$**$}$$

in each fiber $\mathbb{R} \times \{x\} \subset \mathbb{R} \times N$. The general solution of this differential equation is easily found to be

$$\tilde{h}_x(t) = \exp(\frac{2}{n}\int_{t_0}^{t} \Theta_x(t')A_x(t')\mathrm{d}t')h_x,$$

for some initial value $h_x = \tilde{h}_x(t_0)$. As the choice of $t_0 \in \mathbb{R}$ is arbitrary, we can define the function s as

$$s(t,x) = \exp(\frac{1}{n}\int_{t_0}^{t} \Theta(t',x)A(t',x)\mathrm{d}t'),$$

which is the desired result. Furthermore, computing

$$\frac{\partial_t s}{s} = \frac{1}{n}\Theta A$$

yields the formula for the expansion as stated.

Locally the Hubble-isotropic spacetime is certainly diffeomorphic to $\mathbb{R}^{n+1}$, hence also to $\tilde{I} \times \mathbb{R}^n$, with a (possibly unbounded) open interval $\tilde{I} \subset \mathbb{R}$ and a splitting metric given as above. Then, locally, we can always find a diffeomorphism $\varphi\colon \tilde{I} \times \mathbb{R}^n \to I \times \mathbb{R}^n$, with possibly another bounded or unbounded open interval $I \subset \mathbb{R}$, which acts by transforming the t-coordinate according to

$$t \mapsto \tau(t,x) = \int_{t_0}^{t} A(t',x)\mathrm{d}t',$$

for some fixed $t_0 \in \tilde{I}$ and all $t \in \tilde{I}$. Hence, we have

$$A\mathrm{d}t = \mathrm{d}\tau - \int_{t_0}^{t} \mathcal{H}(\mathrm{d}A)_{(t',x)}\mathrm{d}t'$$

as $\mathcal{H}(\mathrm{d}\int_{t_0}^{t} A(t',x)\mathrm{d}t') = \int_{t_0}^{t} \mathcal{H}(\mathrm{d}A)_{(t',x)}\mathrm{d}t'$ can be assumed to hold due to the local nature of this analysis by application of Lebesgue's dominated convergence theorem. This yields

$$\begin{aligned}
\tilde{g}_{(\tau,x)} = &-\left(\mathrm{d}\tau - \int_{t_0}^{t} \mathcal{H}(\mathrm{d}A)_{(t',x)}\mathrm{d}t'\right)^2 + \frac{2}{A(t,x)}\mathrm{pr}_2^*(b_{(t,x)}) \vee \left(\mathrm{d}\tau - \int_{t_0}^{t} \mathcal{H}(\mathrm{d}A)_{(t',x)}\mathrm{d}t'\right) + \\
&+ s^2(t,x)\mathrm{pr}_2^*(h_x) - \frac{\mathrm{pr}_2^*(b_{(t,x)}) \otimes \mathrm{pr}_2^*(b_{(t,x)})}{A^2(t,x)} = \\
= &-\mathrm{d}\tau \otimes \mathrm{d}\tau + 2\left(\frac{\mathrm{pr}_2^*(b_{(t,x)})}{A(t,x)} + \int_{t_0}^{t} \mathcal{H}(\mathrm{d}A)_{(t',x)}\mathrm{d}t'\right) \vee \mathrm{d}\tau + s^2(t,x)\mathrm{pr}_2^*(h_x) - \\
&- \frac{\mathrm{pr}_2^*(b_{(t,x)}) \otimes \mathrm{pr}_2^*(b_{(t,x)})}{A^2(t,x)} - (\int_{t_0}^{t} \mathcal{H}(\mathrm{d}A)_{(t',x)}\mathrm{d}t') \otimes (\int_{t_0}^{t} \mathcal{H}(\mathrm{d}A)_{(t',x)}\mathrm{d}t').
\end{aligned}$$

Using ($*$) and $s(t,x) = (\tilde{s} \circ \varphi)(t,x)$ yields

$$\tilde{g}_{(\tau,x)} = -\mathrm{d}\tau \otimes \mathrm{d}\tau + 2\mathrm{pr}_2^*(\beta_x) \vee \mathrm{d}\tau + \tilde{s}^2(\tau,x)\mathrm{pr}_2^*(h_x) - \mathrm{pr}_2^*(\beta_x) \otimes \mathrm{pr}_2^*(\beta_x).$$

□

Remark 6.9. *The investigation of Hubble-isotropic spacetimes and particularly the analytic approach taken in this section, as well as the theorem above, are inspired by [GPS+10]. Therein, a toolbox was given for constructing spacetimes of splitting type with given kinematical quantities (e.g. $\sigma = 0$, $\nabla_V V = 0$, but $\Theta \neq 0$ as in the Hubble-isotropic case), by starting from a stationary spacetime or a Riemannian metric on the spacelike factor of the spacetime. This can be seen as the complementary synthetic approach to our analysis of Hubble-isotropic spacetimes as conformal Lorentzian submersions. At last, the common ground of both approaches is the differential equation (**) in the proof of Thm. 6.8 above. The general solution of this differential equation can be regarded to hold necessarily in any Hubble-isotropic spacetime, due to the conformal submersion condition, as in our approach here. Or it can be seen as evolving from a given initial condition metric and in this process constructing a Hubble-isotropic spacetime from the initial Riemannian metric as in [GPS+10].*

It is now natural to ask under which circumstances the local isometry in the theorem above holds even globally. Certainly, this is not true in general, as the following example shows.

Example 6.10. *Let (t, x, y) be coordinates on $\mathbb{R}^3$, $s\colon \mathbb{R} \to \mathbb{R}_{>0}$ a smooth function and $(\mathbb{R}^3, g, e^{tx}\partial_t)$ the Hubble-isotropic spacetime given by*

$$g_{(t,x)} = -e^{-2tx}\mathrm{d}t^2 - 2\frac{e^{-tx}(1+tx)-1}{x^2}\mathrm{d}x\mathrm{d}t + s^2(t)(\mathrm{d}x^2 + \mathrm{d}y^2) - \frac{[e^{-tx}(1+tx)-1]^2}{x^4}\mathrm{d}x^2,$$

which is also defined and smooth at $x = 0$ in the sense of taking the limit $x \to 0$. Computing $\mathcal{H}(\mathrm{d}e^{-tx}) = -te^{-tx}\mathrm{d}x$ and

$$-\int_0^t t' e^{-t'x}\mathrm{d}t' = \begin{cases} \frac{e^{-tx}(1+tx)-1}{x^2} & \text{if } x \neq 0 \\ -\frac{t^2}{2} & \text{if } x = 0, \end{cases}$$

which is actually smooth at $x = 0$, we see that this metric is indeed Hubble-isotropic with $\beta = 0$. But now we observe that

$$\int_{t_0}^\infty e^{-t'x}\mathrm{d}t' = \begin{cases} \frac{e^{-t_0 x}}{x} & \text{if } x > 0 \\ \infty & \text{if } x \leq 0. \end{cases}$$

Hence, a time-coordinate τ as in Thm. 6.8, can never be globally defined on all of $\mathbb{R}$.

Corollary 6.11. *Let $(M = \mathbb{R} \times N, g, V)$ be a Hubble-isotropic spacetime as in Thm. 6.8. Then there is an isometry $\varphi\colon (M, g) \to (I \times N, \tilde{g})$, such that (M, g, V) is isometric to $(I \times N, \tilde{g}, \partial_\tau)$, with $\tilde{g}$ as in Thm. 6.8 and $\partial_\tau = \varphi_* V$, if there are two constants c_1, c_2 obeying $-\infty \leq c_1 < c_2 \leq \infty$, such that $\int_0^\infty A(t', x)\mathrm{d}t' = c_2$ for all $x \in N$ and $\int_{-\infty}^0 A(t', x)\mathrm{d}t' = c_1$ for all $x \in N$.*

Proof. From Thm. 6.8 we have $V = \frac{\partial_t}{A}$ on $M = \mathbb{R} \times N$. We need an isometry

$$\varphi\colon (\mathbb{R} \times N, g) \to (I \times N, \varphi_* g = \tilde{g}),$$

which acts by

$$t \mapsto \tau(t,x) = \int_{t_0}^{t} A(t',x)\mathrm{d}t',$$

for a fixed $t_0 \in \mathbb{R}$ and also maps $\frac{\partial_t}{A}$ to the canonical vector field ∂_τ on $I \times N$ $(\tau \in I)$ by push-forward. Without loss of generality we can choose $t_0 = 0$. Such an isometry exists if the range of $\tau(t,x)$ in t is equal for all $x \in N$, i.e., if the conditions in the corollary are met. As in this case we have

$$c_1 = \lim_{t\to-\infty} \tau(t,x) < \lim_{t\to\infty} \tau(t,x) = c_2,$$

for all $x \in N$, thus every fiber $\mathbb{R}\times\{x\}$ is mapped to $I\times\{x\} := (c_1,c_2)\times\{x\}$ by φ. Obviously, the choice of the metric $\tilde{g} = \varphi_* g$ and the vector field $\partial_\tau = \varphi_*(\frac{\partial_t}{A})$, on $I \times N$ assures that φ is indeed an isometry, according to the computations in the proof of Thm. 6.8. At last, we have to address the problem of interchanging the integral and the partial derivatives in the computation of $A\mathrm{d}t = \mathrm{d}\tau - \int_{t_0}^{t} \mathcal{H}(\mathrm{d}A)_{(t',x)}\mathrm{d}t'$ in the global case. We need to ensure that the integral $\int_{t_0}^{t} \mathcal{H}(\mathrm{d}A)_{(t',x)}\mathrm{d}t'$ exists for all $(t,x) \in \mathbb{R}\times N$, so that $\mathcal{H}(\mathrm{d}\int_{t_0}^{t} A(t',x)\mathrm{d}t') = \int_{t_0}^{t} \mathcal{H}(\mathrm{d}A)_{(t',x)}\mathrm{d}t'$ holds for all $(t,x) \in \mathbb{R}\times N$. But that this is indeed the case, can be seen from the condition $(*)$ in Thm. 6.8 above that holds for every Hubble-isotropic spacetime. □

Now we will examine the causal properties of Hubble-isotropic spacetimes. First we give several examples, which show that all steps on the causal ladder, that are allowed by the analysis in section 4.2 for proper kinematical spacetimes, can indeed appear for properly Hubble-isotropic spacetimes.

Example 6.12. *Consider global coordinates (t,x,y) on $\mathbb{R}^3$, the reference frame $V = \partial_t$ and the metric*

$$g^*_{\ddot{o}} = -\mathrm{d}t^2 + 2e^x\mathrm{d}y\mathrm{d}t + s^2(t)\mathrm{d}x^2 + \frac{1}{2}(s^2(t) - 2)e^{2x}\mathrm{d}y^2$$

with $s\colon \mathbb{R}^3 \to \mathbb{R}_{>0}$ a function that only depends on the t-coordinate. This metric can be considered to be a modification of the Gödel spacetime from example 4.6 with possibly non-vanishing expansion. A metric of this form was first given in [GPS+10], where it was also shown that $(\mathbb{R}^3, g^*_{\ddot{o}}, \partial_t)$ is indeed Hubble-isotropic. Now obviously, the signature of the metric induced on the slices S_t of constant t in $\mathbb{R}^3$ depends on the magnitude of s. We have that a slice $\mathbb{R}^2 \simeq S_{t_0} \subset \mathbb{R}^3$ is Riemannian if $s(t_0) > \sqrt{2}$, it is Lorentzian if $s(t_0) < \sqrt{2}$ and it is degenerate if $s(t_0) = \sqrt{2}$. Then following section 4.2, and for example considering the function $s^2(t) = \sin^2(t) + \frac{1}{2}$, leads to a properly Hubble-isotropic spacetime $(\mathbb{R}^3, g^*_{\ddot{o}}, \partial_t)$ which is totally vicious, but for example considering $s^2(t) = t^2 + 3$ leads to $(\mathbb{R}^3, g^*_{\ddot{o}}, \partial_t)$ being causally continuous. Total viciousness can be shown analogously to the Gödel spacetime by considering large enough closed curves on the slices. Moreover, if $s(t)$ is somewhere smaller and somewhere bigger than 2, for example $s^2(t) = t^2 + 1$, we have a region of $(\mathbb{R}^3, g^*_{\ddot{o}}, \partial_t)$ which is non-chronological $(-1 < t < 1)$ and two regions $(t < -1$ and $t > 1)$ which are causal. The regions are bounded by degenerate hypersurfaces. Furthermore, in the case of Riemannian slices as above, i.e., with $s(t)$ uniformly bounded away from 2, we can assert by Thm. 6.18 below that $(\mathbb{R}^3, g^*_{\ddot{o}}, \partial_t)$ is even globally hyperbolic, as the metric $h = \mathrm{d}x^2 + e^{2x}\mathrm{d}y^2$ is the complete hyperbolic Riemannian metric on $\mathbb{R}^2$.

Example 6.13. *Consider the manifold* $\mathbb{R} \times Z^2$*, with* Z^2 *being a two-dimensional cylinder obtained by using coordinates* $(x, y) \in \mathbb{R}^2$ *and the identification* $x \sim x+1$*. We use coordinates* $(t, x \bmod 1, y)$ *on* $\mathbb{R} \times Z^2$*. Consider the metric*

$$g = -\mathrm{d}t^2 + 2\sin(y)\mathrm{d}x\mathrm{d}t + (t^2+1)^2(\mathrm{d}x^2 + \mathrm{d}y^2) - \sin^2(y)\mathrm{d}x^2$$

on $\mathbb{R} \times Z^2$*, which is clearly properly Hubble-isotropic as it is of the splitting form given in Thm. 6.8 and* $\Theta = 2\frac{\partial_t(t^2+1)}{t^2+1} = \frac{4t}{t^2+1}$ *is non-zero almost everywhere. As any slice of constant* t *in this spacetime is a copy of* Z^2*, the integral curves of the canonical vector field* ∂_x *are closed. Furthermore, we have*

$$g(\partial_x, \partial_x) = (t^2+1)^2 - \sin^2(y),$$

which is zero if and only if $t = 0$ *and* y *is an odd integer multiple of* $\frac{\pi}{2}$*. Hence, only on the* $t = 0$ *slice we have CCCs, namely the integral curves of* ∂_x *at the values* $\{y_j = (2j+1)\frac{\pi}{2}\}_{j\in\mathbb{Z}}$*. As everywhere else the slices of constant* t *are Riemannian, this spacetime is an example of a properly Hubble-isotropic spacetime which is chronological, but non-causal.*

Example 6.14. *We recall the torus spacetime from example 4.12 given by the metric*

$$g' = -\mathrm{d}t^2 + 2\mathrm{d}t\mathrm{d}y + (t^2+1)(\mathrm{d}x^2 + \mathrm{d}y^2) - \mathrm{d}y^2$$

on $(\mathbb{R} \times T^2)$*, with respect to the identifications* $(0, y) \sim (1, y)$ *for all* $y \in [0, 1]$ *and* $(x, 0) \sim (x+\sqrt{2}, 1)$ *for all* $x \in [0, 1]$*, which is clearly Hubble-isotropic. In example 4.12 it has already been stated that this spacetime is causal but causally imprisoning.*

Example 6.15. *Consider coordinates* $(t, x, y) \in \mathbb{R}^3$ *with* $x < 1$*, i.e., the manifold* $\mathbb{R}\times\mathbb{R}_{<1}\times\mathbb{R}$ *and transform* x *and* y *to polar coordinates* r *and* φ *on* $\mathbb{R}^2$ *by setting* $x = r\cos\varphi$ *and* $y = r\sin\varphi$*. Then we have* $r\cos\varphi < 1$*. Consider the Euclidean metric* $h = \mathrm{d}r^2 + r^2\mathrm{d}\varphi^2$ *on* $\mathbb{R}_{<1} \times \mathbb{R}$ *and the vector field*

$$Y = B(r)\frac{\sqrt{1-c^2(r)}}{r}\partial_\varphi + c(r)\partial_r.$$

Here $B(r)$ *is a bump function which obeys* $B(0) = 0$ *and* $B(r) = 1$ *for all* $r > \frac{1}{2}$*, and* $c(r)$ *is the smooth bump function*

$$c(r) = \begin{cases} \exp(-\frac{1}{1-r^2}), & 0 \leq r < 1 \\ 0, & r \geq 1, \end{cases}$$

which obeys $c(r) < 1$ *for all* $r \leq 0$*. Hence, the vector field* Y *is globally well-defined and we even have* $h(Y, Y) = 1$ *if* $r > \frac{1}{2}$*. We observe that the integral curves of* Y *are spirals in the region* $\frac{1}{2} < r < 1$ *converging to the limit curve* $r = 1$*. But as* $r\cos\varphi < 1$*, this limit curve is not closed, i.e., there is the point* $(x, y) = (1, 0)$ *missing in* $\mathbb{R}_{<1} \times \mathbb{R}$*. This implies that the integral curves of* Y *in the region* $\frac{1}{2} < r < 1$ *are partially imprisoned but non-imprisoned. We will now construct a Hubble-istropic metric on* $\mathbb{R} \times \mathbb{R}_{<1} \times \mathbb{R}$ *for which the integral curves of* Y*, lifted to the spacetime, are lightlike on one slice, i.e., we get a partially imprisoning but non-imprisoning spacetime.*

We set the one-form $b \in \Gamma(\Lambda^1(\mathbb{R}_{>0} \times \mathbb{R}))$ to be

$$b = h(Y, \cdot) = c(r)\mathrm{d}r + rB(r)\sqrt{1 - c^2(r)}\mathrm{d}\varphi$$

and define a Hubble-isotropic metric g on $\mathbb{R} \times \mathbb{R}_{<1} \times \mathbb{R}$ by

$$g = -\mathrm{d}t \otimes \mathrm{d}t + 2b \vee \mathrm{d}t + (t^2 + 1)h - b \otimes b =$$

$$= -\mathrm{d}t^2 + 2\left[c(r)\mathrm{d}r + rB(r)\sqrt{1 - c^2(r)}\mathrm{d}\varphi\right]\mathrm{d}t +$$

$$+ \left[t^2 + 1 - c^2(r)\right]\mathrm{d}r^2 - 2rB(r)c(r)\sqrt{1 - c^2(r)}\mathrm{d}r\mathrm{d}\varphi + \left[t^2 + 1 - B^2(r)r^2(1 - c^2(r))\right]\mathrm{d}\varphi^2.$$

As $h(Y, Y) = 1$, we have that $g(\tilde{Y}, \tilde{Y}) = 0$ for the lift $\tilde{Y}$ of Y to $\mathbb{R} \times \mathbb{R}_{<1} \times \mathbb{R}$ if $r > \frac{1}{2}$ and $t = 0$, as the induced metric $(t^2+1)h - b \otimes b$ on the $t = const$ slices degenerates only for $t = 0$. Hence, the slice $t = 0$ in $\mathbb{R} \times \mathbb{R}_{<1} \times \mathbb{R}$ contains partially imprisoned lightlike curves, which are not imprisoned in some compact set, namely the integral curves of $\tilde{Y}$ and $(\mathbb{R} \times \mathbb{R}_{<1} \times \mathbb{R}, g, \partial_t)$, with g as above, is a partially imprisoning but non-imprisoning Hubble-isotropic spacetime. By the analysis in section 4.2 it is not feebly distinguishing, either.

For the further analysis of the causality conditions of Hubble-isotropic spacetimes in this section, we will limit ourselves to metrics that can be cast into the form

$$g = -\mathrm{d}t \otimes \mathrm{d}t + 2\mathrm{pr}_2^*(b) \vee \mathrm{d}t + s^2\mathrm{pr}_2^*(h) - \mathrm{pr}_2^*(b) \otimes \mathrm{pr}_2^*(b)$$

on $I \times N$, with $I \subset \mathbb{R}$ a possibly unbounded interval, $\mathrm{pr}_2 \colon I \times N \to N$ the natural projection, $s \colon I \times N \to \mathbb{R}_{>0}$, $b \in \Gamma(\Lambda^1 N)$ and $h \in \mathcal{R}(N)$, by use of Cor. 6.11. Then the reference frame V is given by the canonical vector field ∂_t along the factor I. We will call such a Hubble-isotropic spacetime specially Hubble-isotropic and we will usually omit the pull-back by pr_2 if this is no source of confusion. Note that $V = \partial_t$ is not necessarily a complete reference frame as I may be bounded, but it certainly is complete if $I = \mathbb{R}$. Furthermore, we set $\|b\|^h := \sup_{x \in N} \|b\|^h_x$, and we will usually assume $\|b\|^h > 0$, as otherwise we had $b = 0$ and the Hubble-isotropic metric would simply be a twisted product metric.

Proposition 6.16. *Let $(I \times N, g, \partial_t)$ be a specially Hubble-isotropic spacetime with*

$$g = -\mathrm{d}t \otimes \mathrm{d}t + 2b \vee \mathrm{d}t + s^2 h - b \otimes b.$$

If

$$s^2(t, x) > (\|b\|^h_x)^2$$

holds for all $(t, x) \in I \times N$, then $(I \times N, g)$ is causally continuous and the induced metric

$$\gamma = s^2 h - b \otimes b$$

on the slice $N_t = \{t\} \times N$ is Riemannian for all $t \in I$. Furthermore, if a given specially Hubble-isotropic spacetime $(\mathbb{R} \times N, g, \partial_t)$ is causally continuous, then there is a slicing (according to Thm. 4.16), such that the metric g is given as above and $s^2(t, x) > (\|b\|^h_x)^2$ holds for all $(t, x) \in \mathbb{R} \times N$.

Proof. Let $\tau\colon I \to \mathbb{R}$ be a monotonically increasing diffeomorphism, which implies an isometry $\varphi\colon (I \times N, g) \to (\mathbb{R} \times N, \tilde{g})$ by setting

$$\tilde{g} = \varphi_* g = -\frac{\mathrm{d}\tau \otimes \mathrm{d}\tau}{(\tau')^2} + 2\mathrm{pr}_2^*(b) \vee \frac{\mathrm{d}\tau}{\tau'} + (s \circ \tau^{-1})^2 \mathrm{pr}_2^*(h) - \mathrm{pr}_2^*(b) \otimes \mathrm{pr}_2^*(b).$$

Setting $V := \varphi_* \partial_t = \tau' \partial_\tau$ yields a Hubble-isotropic spacetime $(\mathbb{R} \times N, \tilde{g}, V)$, and there certainly is a vector field V^* parallel to V which is complete and a reference with respect to some Lorentzian metric g^* in the conformal class of $\tilde{g}$, such that $(\mathbb{R} \times N, g^*, V^*)$ is a proper kinematical spacetime. By the analysis in section 4.2, the spacetime $(\mathbb{R} \times N, g^*)$ is causally continuous if the slices $N_\tau = N \times \{\tau\} \subset \mathbb{R} \times N$ are spacelike for all $\tau \in \mathbb{R}$ with respect to g^*. As g^* is conformal to g, this is the case if and only if the induced metric

$$\gamma_{(t,x)} = s^2(t,x) h_x - b_x \otimes b_x$$

on all slices N_t (for all $t \in I$) is Riemannian, i.e., if and only if

$$\gamma_{(t,x)}(v,v) > 0,$$

for all $(t,x) \in I \times N$ and all $0 \neq v \in T_x N$ (This is a situation similar to Lem. 4.23). This is equivalent to the condition

$$s^2(t,x) > \frac{|b_x(v)|^2}{h_x(v,v)},$$

for all $(t,x) \in I \times N$ and all $0 \neq v \in T_x N$, as h is certainly Riemannian. Now, the supremum with respect to all $0 \neq v \in T_x N$ of the right hand side of this inequality exists and is finite for all $x \in N$, but the left hand side does not depend on v. Therefore, the condition is equivalent to

$$s^2(t,x) > \sup_{v \in T_x N \setminus \{0\}} \frac{|b_x(v)|^2}{h_x(v,v)} = (\|b\|_x^h)^2.$$

And with the spacetime $(\mathbb{R} \times N, g^*)$ being causally continuous, also the spacetime $(I \times N, g)$ is causally continuous.

Furthermore, if $(\mathbb{R} \times N, g, \partial_t)$ is specially Hubble-isotropic, the reference frame ∂_t is complete and $(\mathbb{R} \times N, g, \partial_t)$ is a proper kinematical spacetime with the metric given by

$$g = -\mathrm{d}t \otimes \mathrm{d}t + 2b' \vee \mathrm{d}t + s^2 h - b' \otimes b'.$$

By Thm. 4.16, a change of slicing acts on $(\mathbb{R} \times N, g, \partial_t)$ by $t \mapsto \tau = t + f(x)$ for a function $f\colon N \to \mathbb{R}$ and the metric reads

$$g = -\mathrm{d}\tau \otimes \mathrm{d}\tau + 2(b' + \mathrm{d}f) \vee \mathrm{d}\tau + s^2 h - (b' + \mathrm{d}f) \otimes (b' + \mathrm{d}f)$$

after the transformation. By the analysis in section 4.2, there is at least one slicing, such that $\gamma = s^2 h - (b' + \mathrm{d}f) \otimes (b' + \mathrm{d}f)$ is Riemannian on N for all $\tau \in \mathbb{R}$ and the result follows by considering $b = b' + \mathrm{d}f$ and inferring $s^2(\tau, x) > (\|b\|_x^h)^2$ from γ being Riemannian as above. $\square$

We can now proceed to give a first assessment of global hyperbolicity of specially Hubble-isotropic spacetimes by introducing uniform bounds on $s(t,x)$ and $\|b\|^h_x$ and using the completeness of the Riemannian metric h. To this end, we will need the following

Lemma 6.17. *Let (N,h) be a complete Riemannian manifold and $\alpha\colon [a,b) \to N$ a curve of finite length $l(\alpha) = C > 0$. Then α is extendible to the value b, i.e., there exists a point $x \in N$, such that $\alpha(t) \to x$ as $t \to b$.*

Proof. By the Hopf–Rinow theorem, the metric ball $K = \{y \in N \,|\, d_h(\alpha(a), y) \leq C\}$ is compact. Hence, for any strictly increasing sequence $\{t_n\}_{n\in\mathbb{N}} \subset [a,b)$ with $t_n \to b$ as $n \to \infty$, the sequence $\{\alpha(t_n)\}_{n\in\mathbb{N}} \subset K$ converges to some $x \in K$. As α has finite length, this $x \in K$ is uniquely determined for all such sequences $\{t_n\}_{n\in\mathbb{N}} \subset [a,b)$, thus $\alpha(t) \to x$ as $t \to b$ and we can uniquely extend α to the value b. □

The metric structure of specially Hubble-isotropic spacetimes, together with the uniform bound on s and $\|b\|^h$, does now imply necessary and sufficient conditions for global hyperbolicity, which can be proved similarly to the conditions for global hyperbolicity in warped product spacetimes (cf. [BEE96, Sec. 3.6]).

Theorem 6.18. *Let $(I \times N, g, \partial_t)$ be a specially Hubble-isotropic spacetime, with*

$$g = -\mathrm{d}t \otimes \mathrm{d}t + 2b \vee \mathrm{d}t + s^2 h - b \otimes b.$$

Assume

$$S_0(t) := \inf_{x\in N} s^2(t,x) > (\|b\|^h)^2 = \sup_{x\in N} (\|b\|^h_x)^2 > 0$$

holds for all $t \in I$.

(i) If the Riemannian metric h on N is complete, then $(I \times N, g)$ is globally hyperbolic.

(ii) If $(I \times N, g)$ is globally hyperbolic, then the Riemannian metrics $s^2 h$ on N are complete for all $t \in I$ and if additionally the function $s(t,\cdot)\colon N \to \mathbb{R}_{>0}$ is bounded from above on N for all fixed $t \in I$, then even h is complete. Particularly, this is the case if $\mathrm{pr}_2\colon (I \times N, g) \to (N,h)$ is a horizontally homothetic Lorentzian submersion.

Proof. *(i)* We assume $(I \times N, g)$ is not globally hyperbolic and have to show that h is not complete in this case. The condition $S_0(t) > (\|b\|^h)^2$ certainly implies that $(I \times N, g)$ is causally continuous and $s^2 h - b \otimes b$ is Riemannian for all $t \in I$ by Prop. 6.16. Hence, if $(I \times N, g)$ is not globally hyperbolic, there are two values $r < \tilde{r} \in I$ and two points $x_r, x_{\tilde{r}} \in N$, such that $J^+((r, x_r)) \cap J^-((\tilde{r}, x_{\tilde{r}}))$ is not compact. Thus, there is a future-directed causal curve $\gamma\colon [t_1, t_2) \to J^+((r, x_r)) \cap J^-((\tilde{r}, x_{\tilde{r}}))$ with $r \leq t_1 < t_2 \leq \tilde{r}$, which is not extendible to the value t_2. This follows from Thm. 3.52. As $(I \times N, g)$ is causally continuous, we can assume without loss of generality that γ is parametrized by t, i.e., $\gamma(t) = (t, c(t))$ and $\dot{\gamma} = (1, \dot{c})$ with $c\colon [t_1, t_2) \to N$ a curve in N and the dot denoting the derivative with respect to t. As we have $u = g(\partial_t, \cdot)$, the condition of γ being future-directed causal amounts to

$$u(\dot{\gamma}) = -1 + b(\dot{c}) < 0$$

and

$$g(\dot\gamma,\dot\gamma) = s^2 h(\dot c,\dot c) - (1-b(\dot c))^2 \leq 0,$$

which yields

$$s\sqrt{h(\dot c,\dot c)} \leq 1 - b(\dot c).$$

By the definition of the norm $\|b\|^h_x$, it certainly holds that $|b(\dot c)| \leq \|b\|^h \sqrt{h(\dot c,\dot c)}$, so that we get

$$s\sqrt{h(\dot c,\dot c)} \leq 1 + |b(\dot c)| \leq 1 + \|b\|^h \sqrt{h(\dot c,\dot c)}.$$

Hence, we infer

$$\sqrt{h_x(\dot c,\dot c)} \leq \frac{1}{s(t,x) - \|b\|^h_x} \leq \frac{1}{\inf_{x\in N}(s(t,x) - \|b\|^h_x)} \leq \frac{1}{\inf_{x\in N} s(t,x) - \sup_{x\in N}\|b\|^h_x} \leq$$

$$\leq \frac{1}{S_0(t) - \|\!|b|\!\|^h}.$$

As $[t_1,t_2] \subset I$ is compact, there certainly is some $k \in \mathbb{R}$ with $k = \inf_{t\in[t_1,t_2]} S_0(t) > \|\!|b|\!\|^h$, such that

$$\sqrt{h_x(\dot c,\dot c)} \leq \frac{1}{k - \|\!|b|\!\|^h}.$$

This implies that the h-length of c in N is finite. Furthermore, as γ is inextendible to the value t_2, so is c. By Lem. 6.17, this implies that h is not complete.

(ii) We will show that $(I \times N, g)$ is not globally hyperbolic if $s^2(t,x)h_x$ is an incomplete Riemannian metric on N for some $t \in I$. To this end, we first prove that s^2h incomplete for some $t \in I$, implies that the Riemannian metric $\tilde h := (\|\!|b|\!\|^h)^2 h$ is also incomplete on N. Assume $(\|\!|b|\!\|^h)^2 h$ was complete, then we would have $s^2(t,x)h_x(v,v) \geq (\|\!|b|\!\|^h)^2 h_x(v,v)$ for all $v \in T_xN$ and even all $t \in I$. Hence, the distances d_{s^2h} and $d_{\tilde h}$ for points $x,y \in N$ are also related by $d_{s^2h}(x,y) \geq d_{\tilde h}(x,y)$. But this implies (similar to Prop. 2.19) that any d_{s^2h}-Cauchy sequence is also a $d_{\tilde h}$-Cauchy sequence, which converges as $\tilde h$ is assumed complete, contradicting the incompleteness of s^2h.

Now, assume $(\|\!|b|\!\|^h)^2 h$ is an incomplete Riemannian metric on N. The Hopf–Rinow theorem assures the existence of a $(\|\!|b|\!\|^h)^2 h$-geodesic ray $c\colon [0,r) \to N$, parametrized by arc length $(\|\!|b|\!\|^h)^2 h(\dot c,\dot c) = 1$, which is not extendible to the value r. We call $x = c(0)$ and choose some $p = (t_1, x) \in I \times N$. Next, we will construct a future-directed causal curve $\alpha\colon [0,r) \to I \times N$, with $\alpha(\sigma) = (t(\sigma), c(\sigma))$ and $\alpha(0) = p$. Let $\dot\alpha = (\dot t, \dot c)$, with the dot denoting the derivative with respect to σ in this case, and similar to the proof of *(i)* the future-directedness amounts to $\dot t - b(\dot c) > 0$. We compute

$$g(\dot\alpha,\dot\alpha) = -\dot t^2 + 2\dot t b(\dot c) + h(\dot c,\dot c) - b(\dot c)^2 = -(\dot t - b(\dot c))^2 + \frac{1}{(\|\!|b|\!\|^h)^2},$$

thus we need a function $t(s)$ obeying

$$\dot t \geq \frac{1}{\|\!|b|\!\|^h} + b(\dot c),$$

for α being causal. Certainly, $\dot{t} > 0$, as by assumption the spacetime is causally continuous. Hence, as $|b(\dot{c})| \leq \|b\|_x^h \sqrt{h(\dot{c},\dot{c})} \leq \|\!|b|\!\|^h \sqrt{h(\dot{c},\dot{c})} = 1$, this can be accomplished by setting $\dot{t} = K$ with

$$K = \frac{1}{\|\!|b|\!\|^h} + 1.$$

Thus, we have $\alpha(\sigma) = (t_1 + K\sigma, c(\sigma))$. Setting $q = (t_2, x) \in I \times N$ with $t_2 > t_1$, we get by an analogous reasoning a past-directed causal curve $\beta\colon [0,r) \to I \times N$, with $\beta(0) = q$ and $\beta(\sigma) = (t_2 - K\sigma, c(\sigma))$. Then assuming $t_2 - t_1 \geq 2Kr$, implies $\alpha(\sigma) \leq \beta(\sigma)$ for all $\sigma \in [0,r)$ and particularly $\alpha \subset J^+(p) \cap J^-(q)$. As c is not extendible to the value r, α is not extendible, either. By the use of Thm. 3.52, this implies that $J^+(p) \cap J^-(q)$ is not compact and thus $(I \times N, g)$ is not globally hyperbolic. The only remaining obstacle is, if t_1 and t_2 can always be chosen such that $t_2 - t_1 \geq 2Kr$ holds. The problem is that $t_1, t_2 \in I$, which is possibly bounded. As $\|\!|b|\!\|^h$ is a fixed constant depending on g, so is K. Let L be the (possibly infinite) length of I. Then by cutting the geodesic c shorter, such that $r < \frac{L}{2K}$ and after reparametrization choosing a new initial point x, the choice $t_2 - t_1 \geq 2Kr$ is always possible.

To prove the remaining assertions we will use Lem. 2.7. If for all $t \in I$, the function $s(t,\cdot)\colon N \to \mathbb{R}_{>0}$ is bounded from above,

$$\inf_{x \in N} \frac{1}{s^2(t,x)} > 0,$$

for all $t \in I$. Hence, with $s^2 h$ being complete for all $t \in I$, also $h = \frac{1}{s^2} s^2 h$ is complete. Particularly, if we have a horizontally homothetic Lorentzian submersion at hand, then $\mathcal{H}(\mathrm{d}s) = 0$, which implies that s depends on t only and is, therefore, constant over N. $\square$

Obviously, for all $t \in I$ there is a Randers-type metric

$$R = s\sqrt{h} + b,$$

given on the factor N in a specially Hubble-isotropic spacetime $(I \times N, g, \partial_t)$, with the usual metric g if $s^2(t,x) > (\|b\|_x^h)^2$ holds for all $(t,x) \in I \times N$, because in this case

$$1 > \frac{(\|b\|_x^h)^2}{s^2(t,x)} = \sup_{v \in T_x N \setminus \{0\}} \frac{|b_x(v)|^2}{s^2(t,x) h_x(v,v)} = (\|b\|_x^{s^2 h})^2,$$

i.e., R is indeed a well-defined Finslerian metric of Randers-type. Due to Prop. 2.19, the conditions "h complete" and "$S_0(t) > \|\!|b|\!\|^h$" in the theorem above imply the forward and backward completeness of R. It is now natural to ask if, in analogy to the case of stationary spacetimes, already the weaker condition of forward or backward completeness of R for all $t \in I$, does imply global hyperbolicity of the specially Hubble-isotropic spacetime $(I \times N, g, \partial_t)$. To prove the theorem below, we need a lemma similar to Lem. 6.17 for Finslerian metrics.

Lemma 6.19. *Let (S, F) be a forward or backward complete Finslerian manifold and $\alpha\colon [a,b) \to S$ a curve of finite length*

$$l_F(\alpha) = \int_a^b F(\alpha(t), \dot{\alpha}(t))\mathrm{d}t = C > 0.$$

Then α is extendible to the value b, i.e., there exists a point $x \in S$ such that $\alpha(t) \to x$ as $t \to b$.

Proof. Using the Hopf–Rinow type theorem for Finslerian metrics (cf. Thm. 2.17), the proof is completely analogous to Lem. 6.17. We have to make use of the closed and forward bounded metric balls $K_f = \{y \in S \mid d_F(\alpha(a), y) \leq C\}$ or the closed and backward bounded metric balls $K_b = \{y \in S \mid d_F(y, \alpha(a)) \leq C\}$, which are compact in the forward or backward complete cases, respectively. And the extendibility follows. $\square$

Theorem 6.20. *Let $(I \times N, g, \partial_t)$ be a specially Hubble-isotropic spacetime with*

$$g = -\mathrm{d}t \otimes \mathrm{d}t + 2b \vee \mathrm{d}t + s^2 h - b \otimes b$$

and $s^2(t,x) > (\|b\|_x^h)^2$ for all $(t,x) \in I \times N$. If the Randers-type metric

$$R(t,x) = s(t,x)\sqrt{h_x} + b_x$$

on N is forward complete for all $t \in I$ or backward complete for all $t \in I$, then $(I \times N, g)$ is globally hyperbolic.

Proof. We assume, that $(I \times N, g)$ is not globally hyperbolic and have to show that there is some $\tau \in I$, such that $R(\tau)$ is forward and backward incomplete. Just as in the proof of Thm. 6.18 we have that $(I \times N, g)$ is causally continuous and $s^2 h - b \otimes b$ is Riemannian for all $t \in I$ by assumption. Hence, there are two values $r < \tilde{r} \in I$ and two points $x_r, x_{\tilde{r}} \in N$, such that $J^+((r, x_r)) \cap J^-((\tilde{r}, x_{\tilde{r}}))$ is not compact. Thus, by Thm. 3.52, there is a future-directed causal curve $\gamma\colon [t_1, t_2) \to J^+((r, x_r)) \cap J^-((\tilde{r}, x_{\tilde{r}}))$ with $r \leq t_1 < t_2 \leq \tilde{r}$, which is not extendible to the value t_2. As the spacetime is causally continuous, we have γ parametrized by t, i.e., $\gamma(t) = (t, c(t))$ and $\dot{\gamma} = (1, \dot{c})$, with $c\colon [t_1, t_2) \to N$ a curve in N, and

$$u = g(\partial_t, \dot{\gamma}) = u(\dot{\gamma}) = -1 + b(\dot{c}) < 0.$$

As γ is causal,

$$g(\dot{\gamma}, \dot{\gamma}) = s^2 h(\dot{c}, \dot{c}) - (1 - b(\dot{c}))^2 \leq 0$$

yields

$$s(t, c(t))\sqrt{h_{c(t)}(\dot{c}(t), \dot{c}(t))} + b_{c(t)}(\dot{c}(t)) \leq 1, \tag{$*$}$$

for all $t \in [t_1, t_2)$. Now as $[t_1, t_2] \subset I$ is compact, there is some $\tau \in [t_1, t_2]$ such that $s(\tau, x) = \min_{t \in [t_1, t_2]} s(t, x)$. Hence, for the Randers-type metric $R(\tau, x) = s(\tau, x)\sqrt{h_x} + b_x$, the inequality ($*$) still holds, and we get

$$R(\tau, c(t), \dot{c}(t)) = s(\tau, c(t))\sqrt{h_{c(t)}(\dot{c}(t), \dot{c}(t))} + b_{c(t)}(\dot{c}(t)) \leq 1,$$

for the curve $c\colon [t_1, t_2) \to N$. But this implies that the Finslerian length $l_{R(\tau)}(c)$ is finite, and c is not extendible to t_2 as γ is not extendible to t_2. Thus by Lem. 6.19, $R(\tau)$ is forward and backward incomplete. $\square$

We conclude this section with the following

Remark 6.21. *As all causality conditions of a spacetime are conformally invariant, all propositions in this section also hold for kinematical spacetimes arising from specially Hubble-isotropic ones by conformal transformations. Following Prop. 3.23, the shear of a conformally transformed Hubble-isotropic spacetime will still be zero, but the vanishing acceleration transforms to* $\mathcal{H}(\mathrm{d}\phi)$ *for a pointwise conformal transformation* $g \mapsto e^{2\phi} g$. *Hence, the spacetimes* $(\mathbb{R} \times N, g)$, *conformal to specially Hubble-isotropic ones—and thus obeying the same conditions for global hyperbolicity given by the propositions in this section—can be characterized by the existence of a function* $\phi\colon \mathbb{R} \times N \to \mathbb{R}$, *such that* $V = e^{-\phi}\partial_t$ *is a reference frame for* g *with* $\sigma = 0$ *and* $\dot{u} = \mathrm{d}\phi$, *as well as a lapse function in the splitting according to Thm. 4.16 given by* $A = e^{\phi}$.

6.3 Completeness and Singularities

We recall that in a Lorentzian manifold (M, g), geodesic completeness comes in three different types: timelike, lightlike and spacelike completeness, referring to the completeness of the geodesics of the respective causal type. The three types of geodesic completeness are logically independent, in general (cf. [BEE96, Sec. 6.2]), and do not depend on causality conditions in a direct or obvious way. Therefore, propositions that connect completeness of different type with one another or with steps on the causal ladder for specific classes of spacetimes are particularly interesting. In the case of Hubble-isotropic spacetimes we have the following

Proposition 6.22. *Let* $(\mathbb{R} \times N, g, V)$ *be a proper kinematical spacetime (with* V *parallel to* ∂_t*), which is Hubble-isotropic. If* $(\mathbb{R} \times N, g)$ *is spacelike geodesically complete, it is causally continuous.*

Proof. As $(\mathbb{R} \times N, g, V)$ is Hubble-isotropic, the shear and the acceleration of V vanish. Thus, we can employ Prop. 4.22 to $(\mathbb{R} \times N, g, V)$. Hence, for any $(t, x) \in \mathbb{R} \times N$ we set $U_{(t,x)} \subset T_{(t,x)}(\mathbb{R} \times N)$ to be

$$U_{(t,x)} = \{E_{(t,x)} \in T_{(t,x)}(\mathbb{R} \times N) \,|\, g_{(t,x)}(V_{(t,x)}, E_{(t,x)}) = 0,\; g_{(t,x)}(E_{(t,x)}, E_{(t,x)}) = 1\}.$$

Now, as $(\mathbb{R} \times N, g)$ is spacelike geodesically complete, the spacelike geodesic rays $\gamma_E(\tau) = \exp(\tau E_{(t,x)})$, $\tau \in \mathbb{R}$ are all complete, hence $\exp(U_{(t,x)}) \subset \mathbb{R} \times N$ is a spacelike submanifold diffeomorphic to N. As $\sigma = 0$ and $\nabla_V V = 0$, we can use the flow $\Phi\colon \mathbb{R} \times \mathbb{R} \times N \to \mathbb{R} \times N$ associated to V to transport $\exp(U_{(t,x)})$, such that the saturation $\{\Phi(s, \exp(U_{(t,x)})) \,|\, s \in \mathbb{R}\}$ is diffeomorphic to $\mathbb{R} \times N$ and in fact constitutes a trivialization of $(\mathbb{R} \times N, g, V)$ with slices that are spacelike everywhere, as it is a foliation of $(\mathbb{R} \times N, g)$ by spacelike hypersurfaces. By Prop. 4.26 and Thm. 4.29, this implies that $(\mathbb{R} \times N, g)$ is causally continuous. $\square$

In the light of this proposition and item *(i)* from Thm. 6.5, we can conclude that all Hubble-isotropic spacetimes, which are not causally continuous must be spacelike geodesically incomplete if the reference frame induces a Cartan flow. Particularly, this applies to the various examples given in section 6.2, which illustrate the lower steps on the causal ladder.

In a Hubble-isotropic spacetime the integral curves of the reference frame V are geodesics. Thus, we can give criteria for the timelike geodesic incompleteness of a Hubble-isotropic spacetime, i.e., it being singular, by deriving conditions for the geodesic incompleteness of the integral curves of V. As we have the Raychaudhuri equations at hand in kinematical spacetimes, this leads to a specific singularity theorem for Hubble-isotropic spacetimes.

As it is obvious that for specially Hubble-isotropic spacetimes $(I \times N, g, \partial_t)$, the integral curves of ∂_t are complete if $I = \mathbb{R}$ and incomplete if $I \subsetneq \mathbb{R}$, we will consider Hubble-isotropic spacetimes (M, g, V), which are not special for the remainder of this section only. Furthermore, we will assume $M = \mathbb{R} \times N$ with the reference frame V parallel to ∂_t, i.e., we consider Hubble-isotropic spacetimes which split as a product manifold.

Theorem 6.23. *Let $(\mathbb{R} \times N^n, g, V)$ be a Hubble-isotropic spacetime, such that the expansion of V is given by*

$$\Theta = n\frac{\dot{s}}{s},$$

where $s\colon \mathbb{R} \times N \to \mathbb{R}_{>0}$ is the function obtained from the splitting of g according to Thm. 6.8 and the dot denotes the derivative $\nabla_V s$. Then

$$W_0 := |\omega|^2 s^4 \geq 0$$

is constant along every fiber, i.e., $\nabla_V W_0 = 0$ and the function s obeys

$$\ddot{s} = \nabla_V \nabla_V s = \frac{W_0}{n}\frac{1}{s^3} - Ric(V, V)s,$$

which is an ordinary differential equation of Ermakov type in every fiber. Then we have: if

$$n \cdot Ric(V, V) > |\omega|^2$$

everywhere along a fiber or $n \cdot Ric(V, V) \geq |\omega|^2$ everywhere, and $\dot{s} \neq 0$ somewhere, along a fiber, this fiber is an incomplete geodesic of the reference frame V and the spacetime is timelike geodesically incomplete. Furthermore, assuming $|\omega|^2 > 0$ and the Ricci curvature $Ric(V, V)$ constant along a fiber, this fiber is a complete geodesic of the reference frame V.

Proof. The acceleration $\dot{u} = g(\nabla_V V, \cdot)$ and the shear σ are zero by definition in $(\mathbb{R} \times N^n, g, V)$. Then following Prop. 3.33, the Raychudhuri equations for the expansion Θ and the rotation ω are given by

$$\dot{\Theta} + \frac{\Theta^2}{n} = |\omega|^2 - Ric(V, V)$$

and

$$\dot{\omega} = -\frac{2\Theta}{n}\omega.$$

As we have $|\omega|^2 = \sum_{i,j} \omega(E_i, E_j)\omega(E_i, E_j)$, for any pseudo-orthonormal frame $\{E_i\}$ on $\mathbb{R} \times N$ by Def. 3.22, it follows that

$$\nabla_V |\omega|^2 = 2\sum_{i,j} \dot{\omega}(E_i, E_j)\omega(E_i, E_j) = -\frac{4\Theta}{n}\sum_{i,j} \omega(E_i, E_j)\omega(E_i, E_j) = -\frac{4\Theta}{n}|\omega|^2.$$

Using Thm. 6.8 yields $\Theta = n\frac{\dot{s}}{s}$, and hence for $W_0 = |\omega|^2 s^4$, we can compute

$$\nabla_V W_0 = V(|\omega|^2 s^4) = s^4 \nabla_V(|\omega|^2) + 4|\omega|^2 s^3 \dot{s} = -s^4 \frac{4\Theta}{n}|\omega|^2 + 4|\omega|^2 s^3 \cdot \frac{\Theta}{n} s = 0.$$

Furthermore, we have

$$\dot{\Theta} = \nabla_V(n\frac{\dot{s}}{s}) = n\frac{\ddot{s}s - \dot{s}^2}{s^2} \quad \text{and} \quad \frac{1}{n}\Theta^2 = n\frac{\dot{s}^2}{s^2},$$

thus the Raychaudhuri equation for the expansion reads

$$n\frac{\ddot{s}}{s} = |\omega|^2 - Ric(V,V),$$

and hence

$$\ddot{s} = \frac{W_0}{n}\frac{1}{s^3} - Ric(V,V)s.$$

Obviously, for all fixed $x \in N$, this is an ordinary differential equation for $s(\cdot, x) \colon \mathbb{R} \times \{x\} \to \mathbb{R}_{>0}$ in the fiber $\mathbb{R} \times \{x\} \subset \mathbb{R} \times N$ over x. Furthermore, this equation is of the type of an Ermakov[2] equation (see, e.g., [PZ03] or particularly [LA08] and the references therein for various applications of this type of equation). The general solution of this type of equation can be given (see [PZ03]), and will be used for the analysis in the case of constant Ricci curvature below. In the general case, we analyze this equation qualitatively. Certainly, a fiber, i.e., a geodesic integral curve of V, is incomplete if $s(t,x)$ assumes the value zero for a finite value of the parameter t (note that the parameter t does not coincide, in this case, with the first value of points $(\tau, x) \in \mathbb{R} \times N$, but is an affine parameter of the timelike geodesic integral curve of V, i.e., a proper time). Consider any initial value problem of the Ermakov-type equation above with $s(t_0, x) > 0$. Then $n \cdot Ric(V,V) > |\omega|^2$ implies

$$\ddot{s} = \frac{W_0}{n}\frac{1}{s^3} - Ric(V,V)s < \frac{W_0}{n}\frac{1}{s^3} - \frac{|\omega|^2}{n}s = 0$$

as $W_0 = |\omega|^2 s^4$. Therefore, we have $\ddot{s} < 0$ along all the fiber. Together with the initial value $s(t_0, x) > 0$, this implies that $s(r, x) = 0$ for some finite $r \in \mathbb{R}$, hence this fiber is an incomplete geodesic. If only $n \cdot Ric(V,V) \geq |\omega|^2$ holds along the fiber, we have $\ddot{s} \leq 0$ and need the additional condition, that $\dot{s} \neq 0$ somewhere, in order for $s(t,x)$ to become zero in finite proper time, because otherwise we could have $s(t,x) = s(t_0,x) = const$ along the fiber for all $t \in \mathbb{R}$.

If $Ric(V,V)$ is constant along a fiber, the solution of the Ermakov-type equation can be computed explicitly. The general solution is given in the following way (see [PZ03]). Let $w \colon \mathbb{R} \to \mathbb{R}$ be any non-trivial solution of the differential equation

$$\ddot{w} + Ric(V,V)w = 0$$

[2] sometimes also: Yermakov

in some fiber $\mathbb{R} \times \{x\} \subset \mathbb{R} \times N$. Then the general solution of the Ermakov equation $\ddot{s} + \mathit{Ric}(V,V)s = \frac{W_0}{n}\frac{1}{s^3}$ in this fiber is given by

$$c_1 s^2(t,x) = w^2(t,x)\left[\frac{W_0}{n} + \left(c_2 + c_1\int_{t_0}^{t}\frac{\mathrm{d}t'}{w^2(t',x)}\right)^2\right],$$

where the constants $c_1, c_2 \in \mathbb{R}$ are determined by the initial values $s(t_0,x)$ and $\dot{s}(t_0,x)$. If $\mathit{Ric}(V,V)$ is constant along $\mathbb{R} \times \{x\}$, a function w is easily computed explicitly. In the case $\mathit{Ric}(V,V) = 0$ along the fiber, we get

$$s(t,x) = \sqrt{\frac{na_1(x)^2W_0 + (t + a_2(x))^2}{na_1(x)}},$$

with $a_1(x), a_2(x)$ determined by the initial values. Due to the initial value $s(t_0,x) > 0$, we have $na_1^2W_0 > 0$, hence the solution exists for all $t \in \mathbb{R}$ and obeys $s(t,x) > 0$.

If $\mathit{Ric}(V,V) =: -\rho^2 < 0$ for some $\rho > 0$, we have $w(t) = e^{\rho t}$ as a solution for $\ddot{w} - \rho^2 w = 0$. Without loss of generality, we can choose $t_0 = 0$ as the initial point and the general solution can be written as

$$c_1 s^2(t) = e^{2\rho t}\left[\frac{W_0}{n} + \left(c_2 + \frac{c_1}{2\rho} - \frac{c_1}{2\rho}e^{-2\rho t}\right)^2\right].$$

Certainly, this solution exists for all $t \in \mathbb{R}$ and a positive initial value $s(0) > 0$ ensures $s(t) > 0$ for all $t \in \mathbb{R}$ as $W_0, \rho > 0$.

If $\mathit{Ric}(V,V) =: \rho^2 > 0$ for some $\rho > 0$, we can use $w(t) = \cos(\rho t)$ as a solution for $\ddot{w} + \rho^2 w = 0$. Assuming again $t_0 = 0$, the general solution is

$$c_1 s^2(t) = \cos^2(\rho t)\left[\frac{W_0}{n} + \left(c_2 + \frac{c_1}{\rho}\tan(\rho t)\right)^2\right].$$

Now we observe that

$$\lim_{t \to (2k+1)\frac{\pi}{2\rho}} \cos(\rho t) = 0 \quad \text{and} \quad \lim_{t \to (2k+1)\frac{\pi}{2\rho}} \tan(\rho t) = \infty,$$

for all $k \in \mathbb{Z}$. But computing

$$\lim_{t \to (2k+1)\frac{\pi}{2\rho}} (c_1 s^2(t)) = \left(\frac{c_1}{\rho}\right)^2 \cos^2(\rho t)\tan^2(\rho t) = \left(\frac{c_1}{\rho}\right)^2$$

shows that the general solution exists for all $t \in \{(2k+1)\frac{\pi}{2\rho}\}_{k\in\mathbb{Z}}$ and is positive everywhere as $W_0, c_1 > 0$. We also see that this only holds if $W_0 > 0$, i.e., $|\omega|^2 > 0$ along the fiber. □

Thm. 6.23 constitutes a singularity theorem for Hubble-isotropic spacetimes (cf. [HE73, Ch. 8], [Pen72]). We observe that in this case, no assumption on the causality of the spacetime is needed to ensure the appearance of a singularity. Instead, we need the timelike Ricci curvature to be (strictly) larger than the rotation scalar, which is a different (and in many cases stronger) condition than the energy and timelike convergence conditions (cf. [HE73, Sec. 4.3]) usually used in the standard singularity theorems. Moreover, this result clearly shows that a large enough rotation can avoid the formation of singularities (cf. [Obu00] and the references therein). This result becomes most evident in the case of constant Ricci curvature $Ric(V, V)$ along a fiber. As soon as we add a, possibly very small but non-vanishing, amount of rotation, the fiber is necessarily always complete.

Chapter 7

OUTLOOK

We will conclude this work with pointing out some interesting directions of further research and some open problems, based on the propositions established in this thesis. Some of the annotations below are based on remarks already given in the previous chapters.

With kinematical spacetimes, we considered free, timelike $\mathbb{R}$-actions on non-compact Lorentzian manifolds. An interesting extension of this concept could be to consider compact spacetimes, which are necessarily non-chronological, with an $\mathbb{S}^1$-action. Similar splitting questions can be asked in this case, i.e., when is a compact kinematical spacetime (M, g, V), with the integral curves of V being embedded circles, a product manifold $\mathbb{S}^1 \times N$ with N the space of flow lines of V. Certainly, such a spacetime is always an $\mathbb{S}^1$-principal bundle, which is trivial if N is simply connected, but can we find conditions for this to happen based on geometric quantities derived from g and V? And what can be derived about the causal nature of (global) sections in this bundle? As a compact manifold must have Euler characteristic zero if it admits Lorentzian metric, a natural class of manifolds to analyze, which are non-trivial $\mathbb{S}^1$-principal bundles, would be odd-dimensional Lorentzian Berger spheres. Then one can, for example, classify Hubble-isotropic metrics on these spheres or embedded Hopf tori, which are timelike submanifolds.

In the beginning of section 4.1, we introduced a possible alternative definition for the non-triviality of closed timelike curves. With example 4.5, we have already given an impression of the difference of this definition to B. Carter's classical one in [Car68]. It could be an interesting task to further determine the differences and similarities of these two notions of triviality of CTCs, for example by the construction of more, maybe physically interesting, examples.

In Rem. 4.15, we gave a possible definition of "hole freeness" of kinematical spacetimes, based on the notion of improper saddle points from the theory of dynamical systems. As mentioned in this remark, there are other notions of "hole freeness" established in [Man09] and [Min12]. It would be an interesting task to work out the detailed similarities and discrepancies of these different ideas of holes in a spacetime.

In section 4.3, the question of the existence of a smooth semi-time function in a proper kinematical spacetime was introduced. Recall from section 3.4 that a continuous semi-time function exists in every chronological and reflecting spacetime. To the best knowledge of the author there exist no examples of proper kinematical spacetimes, which are chronological, but do not admit a smooth semi-time function. Therefore, we formulate the following

Conjecture 7.1. *A proper kinematical spacetime (M, g, V) (with a compact manifold $S = M/_{\mathbb{R}}$ of flow lines of V) is reflecting and admits a smooth semi-time function if it is chrono-*

logical.

The conjecture seems in any case simpler to assess if a compact S is assumed, but we believe it to be true also in the general case. Evidence for this hypothesis is provided by the following

Proposition 7.2. *Let $\gamma\colon \mathbb{R} \to M$ be a partially future (or past) imprisoned timelike curve in a chronological, proper kinematical spacetime (M, g, V). Then $\psi \circ \gamma$ is not contained in a single slice $S_t \subset \mathbb{R} \times S$, with respect to any trivialization $\psi\colon (M, g, V) \to (\mathbb{R} \times S, \psi_* g, \psi_* V)$.*

Proof. We will show the assertion for the partially future imprisoned case. The partially past imprisoned case works completely analogous and will be omitted. From the considerations in section 4.2, we can assume

$$\psi_* g = -\mathrm{d}t \otimes \mathrm{d}t + 2b_t \vee \mathrm{d}t + h_t - b_t \otimes b_t$$

and $\psi_* V = \partial_t$ with respect to a trivialization ψ. Consider $\lambda\colon \mathbb{R}_{\geq 0} \to \mathbb{R} \times S$ given by $\lambda(s) = (\psi \circ \gamma)(s)$, to be partially future imprisoned, i.e., there is a monotonically increasing sequence $\{s_n\}_{n \in \mathbb{N}} \subset \mathbb{R}_{\geq 0}$, such that $\lambda(s_n)$ is contained in some compact set $K \subset \mathbb{R} \times S$. Hence, switching to a subsequence we can assume $\lambda(s_n) \to (t_0, x_0) \in K$ as $n \to \infty$. Assume now that all of λ is contained in some slice, i.e., $\lambda(s) = (t_0, c(s)) \in S_{t_0}$ for all $s \geq 0$, with $c\colon \mathbb{R}_{\geq 0} \to S$ the projection of λ to S. Particularly, this implies that the induced metric $h_{t_0} - b_{t_0} \otimes b_{t_0}$ on S_{t_0} is Lorentzian on $\lambda \subset S_{t_0}$, i.e., by Lem. 4.23

$$\|b_{t_0}\|^{h_{t_0}}_{\lambda(s)} > 1.$$

As the set $\{(t, x) \in \mathbb{R} \times M \mid \|b_t\|^{h_t}_x > 1\}$ is certainly open, there is a neighborhood U of λ in $\mathbb{R} \times S$ such that $\|b_t\|^{h_t}_x > 0$ for all $(t, x) \in U$. Thus, for all $n \in \mathbb{N}$, we can define a future-directed timelike curve $\mu_n\colon [s_n, s_{n+1}] \to \mathbb{R} \times S$ in the following way. Let $\tau_n \in (s_n, s_{n+1})$ be the value of the curve parameter for which λ re-enters K to approach $\lambda(s_{n+1})$. If λ is not only partially- but totally-imprisoned in K, we can just set $\tau_n = s_n$. Then we set

$$\mu_n(s) = \begin{cases} \lambda(s) = (t_0, c(s)), & s \in [s_n, \tau_n) \\ (t_0 - \varepsilon(s - \tau_n), c(s)), & s \in [\tau_n, s_{n+1}], \end{cases}$$

with some $\varepsilon > 0$. As U is open, we can choose ε so small that μ_n is future-directed and timelike in all its domain for all $n \in \mathbb{N}$. This can be achieved as follows: Denote by a dot the derivative with respect to the curve parameter, then we have $g(\partial_t, \dot{\lambda}) = b(\dot{c}) < 0$ as λ is assumed future-directed. Now we set

$$-\delta := \sup_{\{s \mid \lambda(s) \in K\}} g_{\lambda(s)}(\dot{\lambda}(s), \dot{\lambda}(s)) = \sup_{\{s \mid \lambda(s) \in K\}} \left(h_{(t_0, c(s))}(\dot{c}(s), \dot{c}(s)) - b_{(t_0, c(s))}(\dot{c}(s))^2\right) < 0$$

and

$$\alpha := \sup_{\{s \mid \lambda(s) \in K\}} |b_{(t_0, c(s))}(\dot{c}(s))| > 0.$$

Then we compute

$$g(\dot{\mu}_n, \dot{\mu}_n) \leq -\varepsilon^2 + 2\alpha\varepsilon - \delta,$$

and this expression is certainly negative if $\varepsilon = 0$ as $-\delta < 0$, and hence it is also negative in a neighborhood of $\varepsilon = 0$, from where we can pick a positive ε to assure μ_n is timelike. Then, certainly, also $g(\partial_t, \dot{\mu}_n) = -\varepsilon + b(\dot{c}) < 0$ and μ_n is future-directed.

Furthermore, by making the compact set K maybe a bit larger we can achieve that $s_{n+1} - \tau_n$ is bounded from below for all $n \in \mathbb{N}$, say by some $k > 0$, such that $s_{n+1} - \tau_n > k > 0$. This implies that

$$\mathrm{pr}_1(\mu_n(s_{n+1})) = t_0 - \varepsilon(s_{n+1} - \tau_n) < t_0 - k\varepsilon,$$

for all $n \in \mathbb{N}$. But this implies that there must be some $N \in \mathbb{N}$, for which $\lambda(s_N) \in I^+(\mu_N(s_{N+1}))$, thus there is a closed timelike curve in contradiction to the assumption of chronology. That such an N must exist can be inferred as follows: As ε is small $\mathrm{pr}_1(\mu_n(s_{n+1}))$ is also bounded from below. Since $c(s_n) \to x_0$ as $n \to \infty$, we can infer that, maybe after switching to a subsequence, also $\mu_n(s_{n+1})$ converges to a point (t_1, x_0) in the fiber over x_0 with $t_1 \leq t_0 - k\varepsilon$. Now we set $T = \frac{t_1+t_0}{2}$ and analyze the point (T, x_0) in the same fiber. Certainly, $(t_1, x_0) \in I^-((T, x_0))$ and $(t_0, x_0) \in I^+((T, x_0))$, but because the chronological future and past sets are open, there certainly is some $N \in \mathbb{N}$, such that $\mu_N(s_{N+1}) \in I^-((T, x_0))$ and $\lambda(s_N) \in I^+((T, x_0))$, as the sequences converge to (t_1, x_0) and (t_0, x_0), respectively. Then choose a future-directed timelike curve connecting $\mu_N(s_{N+1})$ to (T, x_0) and a future-directed timelike curve connecting (T, x_0) to $\lambda(s_N)$. This yields a future-directed timelike curve connecting $\mu_N(s_{N+1})$ to $\lambda(s_N)$, and hence $\lambda(s_N) \in I^+(\mu_N(s_{N+1}))$. $\square$

The proposition above could make it possible to find a trivialization $\psi\colon (M, g, V) \to (\mathbb{R} \times S, \psi_* g, \psi_* V)$, such that the slices $S_t = \{t\} \times S \subset \mathbb{R} \times S$ are achronal hypersurfaces, and hence, are the associated preimages of a smooth semi-time function, which is strictly increasing along all future-directed timelike curves. But there are various technical difficulties to overcome in order to construct such a trivialization. Another possible idea to resolve the conjecture would be to independently prove the reflectivity of chronological proper kinematical spacetimes, which would then yield a continuous semi-time function. Then one can try to apply a smoothing procedure to this continuous function, which could work similar to the smoothing of time functions established in [BS05].

Prop. 4.37, establishing a relation between the Weinstein conjecture and Lorentzian causality, is particularly intriguing. In order to use this relation to prove a general version of the Weinstein conjecture one would have to view causality from a perspective which does not directly involve CTCs. Such a perspective exists, using various topologies on the space of Lorentzian metrics on a manifold (see, e.g., [Min09c] for a recent account). Usually in this approach, metrics with a causality conditions from the lower part of the causal ladder are identified by some topological boundary construction on the space of Lorentzian metrics. In order to assess the Weinstein conjecture, one would need a topology fine enough to distinguish between non-chronological and non-causal metrics. It is unclear if such a reasonable topology exists.

In Rem. 5.12, the possibility to regard spacetimes as graphs over lines in the space of Riemannian metrics $\mathcal{R}(S)$ was established. It would be an interesting task to investigate which spacetimes can appear as such graphs and to classify them. Then one could even add a non-trivial twist to the timelike reference frame in the resulting spacetime and analyze spacetimes, which are graphs over lines in $\mathcal{R}(S) \times \Lambda^1(S)$, where $\Lambda^1(S)$ is the space of one-forms on S.

In section 5.3, we established the new version of the Lorentzian Bochner technique for stationary spacetimes in the case of a compact and, in a reasonable sense, asymptotically flat base manifold S. The homothetic case was only established for a compact base S. After performing the appropriate definitions of asymptotic flatness for the homothetic case, conditions similar to Thm. 5.45 should hold.

In theorems 6.18 and 6.20, we established conditions for the global hyperbolicity of specially Hubble-isotropic spacetimes. There are at least two ways how these results could possibly be expanded. The obvious first way is to include general Hubble-isotropic spacetimes in these theorems. In this case one has to deal with a non-trivial lapse function. The second way is to try to derive an appropriate converse proposition from Thm. 6.20, i.e., to deduce from global hyperbolicity of a Hubble-isotropic spacetime the completeness of particular Randers-type metrics.

Furthermore, as we can view Hubble-isotropic spacetimes as conformal Lorentzian submersions $(\mathbb{R} \times N, g) \to (N, h)$, it could be an interesting task to analyze generalizations of these submersions with a Finslerian manifold of Randers type as base, i.e., maps of the type $\pi\colon (\mathbb{R} \times N, g) \to (N, \sqrt{h} + b)$, where h is a Riemannian metric and b a one-form on N. Stationary spacetimes, therefore, occur as such Lorentz-to-Randers submersions by demanding, among other things, that for any $p = (t, x) \in \mathbb{R} \times N$ the bounded causal future (past) $J^+_T(p) := J^+(p) \cap ([t, t+T] \times N)$ $(J^-_T(p) := J^-(p) \cap ([t-T, t] \times N))$ is mapped by π to a forward (backward) geodesic ball of $\sqrt{h} + b$ on N. Moreover, it could be interesting to analyze such submersion structures, which could also be regarded as Lorentzian submetries, and their generalizations, e.g., for a generic Finslerian base manifold, under global viewpoints and also to consider the question of a possible relation between the curvatures of the Finslerian base manifold and the Lorentzian total space.

Another fascinating result obtained from regarding shear-free kinematical spacetimes as conformal Lorentzian submersions is the possibility to formulate a long-standing conjecture from theoretical physics in a completely geometrical way. This is the *shear-free fluid conjecture* (see, e.g., [VdB99] and the references therein for a statement of the conjecture in a physical context and the known special cases), which can be re-formulated as follows.

Conjecture 7.3. *Let $\pi\colon (M^4 = \mathbb{R} \times N^3, g) \to (N^3, h)$ be a conformal Lorentzian submersion. Denote by $\mathcal{H}$ the horizontal projection and the horizontal distribution. Assume there are two functions $\mu, p\colon M \to \mathbb{R}$, such that $\mu + p \neq 0$ and $p = p(\mu)$, the exterior curvature of the fibers is given by $-\frac{\mathcal{H}(\nabla p)}{\mu + p}$ and the Ricci curvature of (M, g) obeys*

$$Ric(V,V) = \frac{3}{2}(\mu + 2p), \quad Ric(V,E) = 0, \quad Ric(E,E) = -\frac{1}{2}(\mu + 4p),$$

for all vertical unit vector fields V and all horizontal unit vector fields E. Then $\mathcal{H}$ is integrable or there is a constant $c > 0$, such that $\pi\colon (M^4, g) \to (N^3, c \cdot h)$ is a Lorentz-to-Riemann submersion.

The problems for a general proof of this conjecture arise from the fact that the exterior curvature can vary relatively arbitrarily along the fibers. Although the exterior curvature is essentially a horizontal gradient, one has in general no control over its evolution along the fiber. Every known special case of the conjecture provides, in one way or another, such a control over the evolution of the exterior derivative and allows for a proof in this way. The geometric formulation of the shear-free fluid conjecture above, also allows for possible generalizations, for example by considering arbitrary dimensions. But it particularly points to a possible alternative direction towards a proof of the conjecture, firstly, by wielding B. O'Neill's curvature formulas for submersions ([O'N66]), as well as their generalizations to the conformal case (see, e.g., [Gud92]) and, secondly, by employing, adapting and maybe developing known results about semi-Riemannian submersions with constraints on the Ricci curvature (see [KY97]).

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