Macroeconomics and Imperfect Information

Uniqueness and Calculation of Dynamic Equilibria

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To Ruth

When one hundred millions, or more, of the circulation we now have shall be withdrawn, who can contemplate without terror the distress, ruin, bankruptcy, and beggary that must follow?. . .

The general distress thus created will, to be sure, be temporary, because, whatever change may occur in the quantity of money in any community, time will adjust the derangement produced. . . .

—A. Lincoln; Springfield, Illinois; December 20, 1839

Contents

Figures

Tables

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Summary

This study presents three essays covering restrictions on monetary policy to deliver a unique equilibrium and computational methods for the recursive formulation and estimation of such an equilibrium. The study is linked by the common theme of informational imperfections, in the first and third chapters preventing the realization of a fully flexible equilibrium and in the second preventing the use of standard recursive solution and estimation methods.

Chapter 2 derives restrictions on monetary policy to deliver a unique bounded equilibrium for a three equation New Keynesian model with sticky information à la Mankiw and Reis (2002). The analysis finds tighter bounds on the coefficients in the Taylor rule than in sticky-price models, irrelevance of the degree of output-gap targeting for determinacy, and independence of determinacy regions from parameters outside monetary policy. The long-run verticality of the Phillips curve plays the decisive role in explaining the differences to sticky-price models. Consequences for optimal monetary policy and difficulties presented by the infinite-dimensional sticky-information model are discussed.

Chapter 3 contains a solution and an estimation method for linear rational-expectations models with lagged expectations. The solution method is a synthetic approach, combining state-space and infinite-MA representations with a simple system of linear equations. The advantage lies in the particular combination of methods from the literature, providing faster execution, more general applicability, and more straightforward usage than existing algorithms. Bayesian estimation methods are employed without the Kalman filter using a recursive algorithm to evaluate the likelihood function and are used to compare small-scale sticky-information and sticky-price DSGE models. Standard truncation methods are shown to not generally be innocuous.

Chapter 4 reiterates that the monetary authority can reasonably be held responsible for inflation. The bounds on monetary policy to ensure determinacy in a class of models that satisfy Lucas's (1972a) natural rate hypothesis (NRH) are shown to be identical for all supply specifications, save isolated singularities. This follows, as is argued, from determinacy being a criterion of the long run when all NRH supply specifications coincide. Thus, no specific knowledge of the supply side beyond its fulfillment of the NRH is necessary to assess whether a particular monetary policy will ensure determinacy and, under the standard dynamic IS-equation, determinacy is solely a function of the parameters in the interest rate rule. Cochrane's (2007) criticism of determinacy for selecting equilibrium is verified and shown to be associated with reckless money growth accommodating the associated explosive inflation. Monetary policy's inability to control the nominal interest rate in the long run is to blame and appending policy with a credible commitment to stable long-run money growth suffices to rule out these otherwise accommodated nominal explosions.

Zusammenfassung

In dieser Studie werden drei Aufsätze vorgelegt, die sich auf geldpolitische Beschränkungen zur Herbeiführung eines eindeutigen Gleichgewichtes und auf numerische Methoden für die rekursive Formulierung und Schätzung eines solchen Gleichgewichtes konzentrieren. Das Leitthema der Studie ist imperfekte Information, die im ersten und im dritten Kapitel die Realisation eines vollkommen flexiblen Gleichgewichtes verhindert und die im zweiten Kapitel die Nutzung üblicher rekursiver Lösungs- und Schätzmethoden erschwert.

In Kapitel 2 werden geldpolitische Beschränkungen für die Herbeiführung eines eindeutigen Gleichgewichtes in einem Neu-Keynesianischen Dreigleichungsmodell mit starrer Information nach Mankiw and Reis (2002) hergeleitet. In der Analyse wird festgestellt, dass engere Schranken für die Koeffizienten der Taylorregel als in herkömmlichen Modellen mit starren Preisen vorhanden sind, dass das Gewicht auf die Produktionslücke für die Determiniertheit des Modells irrelevant ist und dass die Determiniertheitsschranken von Parametern außerhalb der Geldpolitik unabhängig sind. Dabei spielt die langfristige Vertikalität der Phillipskurve die entscheidene Rolle bei der Erklärung der Unterschiede zu herkömmlichen Modellen mit starren Preisen. Ferner werden die Konsequenzen für optimale Geldpolitik sowie die Schwierigkeiten diskutiert, die durch das unendlich-dimensionale Modell mit starrer Information entstehen.

Kapital 3 stellt eine Lösungs- und Schätzmethode für lineare Modelle rationaler Erwartungen mit verzögerten Erwartungen vor. Die Lösugsmethode ist ein synthetischer Ansatz, der übliche Zustandsraumabbildungen und Abbildungen durch unendliche gleitende Durchschnitte mit einem einfachen System linearer Gleichungen kombiniert. Die spezielle Kombination von Methoden aus der Literatur ermöglicht eine schnellere Ausführung, einen höheren Grad an Allgemeinheit und eine unkompliziertere Anwendung als die jeweiligen Methoden für sich genommen. Bayesianische Schätzmethoden werden mit Hilfe eines rekursiven Algorithmus für die Berechnung der logarithmierten Likelihood-Funktion ohne den Kalmanfilter angewandt; als Beispiel werden einfache DSGE-Modelle mit starrer Information und starren Preisen verglichen. Überdies wird gezeigt, dass die Anwendung herkömmlicher Trunkierungsmethoden durchaus problematische Folgen zeitigen kann.

Kapital 4 schließlich betont, dass die Geldpolitik für Inflation verantwortlich gemacht werden kann. Gegenstand der Analyses ist eine Klasse von Modellen, welche die Hypothese der natürlichen Arbeitslosenquote (HnA) erfüllen. Es wird gezeigt, dass für alle Angebotsspezifikationen in dieser Modellklasse, mit Ausnahme isolierter Singularitäten, die geldpolitischen Beschränkungen für die Herbeiführung eines determinierten Gleichgewichtes identisch sind. Dies ergibt sich aus der Tatsache, dass Determiniertheit ein Kriterium der langen Frist ist, bei welcher alle HnA-Angebotsspezifikationen zusammenfallen. Demzufolge ist außer der Einhaltung der HnA kein weiteres Wissen bezüglich der Angebotsspezifikation notwendig, um feststellen zu können, ob eine spezifische Geldpolitik ein determiniertes Gleichgewicht gewährleistet. Ferner ist die Determiniertheit unter der Annahme einer üblichen dynamischen IS-Gleichung ausschließlich eine Funktion der Parameter in der Zinsregel. Die Kritik von Cochrane (2007) bezüglich Determiniertheit als Gleichgewichtsauswahlmechanismus wird bestätigt. Die zusätzlichen Gleichgewichte werden durch maßloses Geldmengenwachstum erzeugt, das die assozierte explosive Inflation akkommodiert. Ursächlich dafür ist die Unfähigkeit der Geldpolitik, den nominalen Zinssatz in der langen Frist kontrollieren zu können. Die Erweiterung der Geldpolitik um eine glaubwürdige Bindung zum stabilen langfristigen Geldmengenwachstum reicht aus, um diese sonst akkommodierten nominalen Explosionen auszuschließen.

Chapter 1

Introduction

1.1 Scope and Outline of the Study

This study is concerned with the uniqueness and calculation of equilibrium paths in dynamic macroeconomic models: specifically, on the bounds on monetary policy to ensure determinacy, the numerical calculation, simulation, and estimation of such a determinate equilibrium path, and on the limitations of determinacy as a selection mechanism. First, I examine the role of monetary policy and determinacy with Mankiw and Reis's (2002) sticky-information Phillips curve. Then, I extend and combine numerical solution and estimation methods to be able to efficiently and accurately treat models with lagged expectations including methods to solve the infinite regress of lagged expectations found in the sticky-information model. Finally, I expand the determinacy analysis to encompass an entire class of models that satisfy the natural rate hypothesis before proceeding to verify the validity of the explosive equilibria that determinacy rules out—as noted by Cochrane (2007)—and offer a monetarist solution to overcome this additional multiplicity.

Chapter 2 derives the bounds on parameters in monetary policy's interest rate rule to ensure a determinate equilibrium in a three-equation New Keynesian model with Mankiw and Reis's (2002) sticky information. I examine a variety of interest rate targeting rules common in the literature, covering current and forecasted inflation targeting, exogenous interest rate rules, price-level targeting, interest rate smoothing, and output gap targeting. I find that the bounds associated with determinacy in the sticky-information model are tighter than in the literature-standard sticky-price model. The key to the results lies in the the sticky-information Phillips curve's asymptotic transition to verticality—which, for example, removes the ability of the monetary authority to substitute reactions to the output gap for reactions to inflation.

Due to the infinite dimensionality of Mankiw and Reis's (2002) sticky-information Phillips curve, standard eigenvalue counting methods cannot be applied. The analysis proceeds by reformulating the problem as a system of deterministic time-varying (or non-autonomous) difference equations in the variables' unrestricted moving-average coefficients. The resulting bounds to ensure determinacy are identical to those that would have been obtained from eigenvalue counting á la Blanchard and Kahn (1980) using the asymptotic relations of the model, as the coefficients of non-autonomous system converge quickly enough (i.e., the system is almost time invariant or autonomous) to their limiting, asymptotic values.

Chapter 3 is concerned with the numerical solution and estimation of variables on a determinate path. Thus, in contrast to the first chapter, this chapter asks not whether such a unique path exists, but rather which numerical values do magnitudes of interest take along such a path. Following current practice, the solution method provides a recursive representation of the solution that resolves the rational expectations in the model into functions of variables present in the information set. Aiming to provide such a method for infinite-dimensional problems such as those in chapter 2, the method first provides a solution method for intermediate cases. That is, it incorporates any finite collection of lagged (i.e., outdated) expectations on variables into a canonical vector-valued second-order expectational system.

While an exact method for the infinite-dimensional problem is not provided, the method develops the insight from chapter 2 regarding quick-enough convergence to provide for a truncation criterion that justifies a particular truncation method and can be used to enable arbitrary accuracy (up to machine precision) of such a truncation. The particular truncation method, armed with the truncation criterion, provides a reference point that is used to evaluate other solution methods in the literature in terms of accuracy and required computing time. Finally, a recursive method to evaluate the log-likelihood function for estimation is derived without use of the Kalman filter, enabling the truncated solution method to maintain its numerical efficiency while bringing such models to the data. Highlighting the use of the combined solution and estimation methods is a comparison of sticky-information and sticky-price models, in which an arbitrary truncation method from the literature is shown to reverse the likelihood ordering of the two models.

Chapter 4 returns to the question of equilibrium uniqueness. Instead of examining a particular model, like the analysis in chapter 2, this chapter seeks to analyze the role of monetary policy in ensuring a unique equilibrium for an entire class of models. Applying the solution method for intermediate cases developed in chapter 3, I prove a general equivalence of the determinacy bounds among all admissible models that satisfy a version of the natural rate hypothesis, saving for isolated singularities. This extends current determinacy analyses by examining an entire class of models rather than a single specific model and highlights some misleading conclusions on the bounds on monetary policy that stem from non-verticalities in standard New Keynesian models' Phillips curves.

With a set of general bounds on monetary policy to ensure determinacy, the analysis proceeds to question whether determinacy itself is sufficient to ensure a unique equilibrium. Following the criticism of Cochrane (2007), I explore the explosive paths ruled out by determinacy to deduce their permissibility. Constraining the analysis to the set of models that satisfy the natural rate hypothesis examined for determinacy, these explosive paths display explosiveness only in nominal variables. Thus, in essence, Cochrane's (2007) criticism is linked to earlier studies of speculative hyperinflation and, having added money to the analysis following these studies, I confirm that these alternate paths with explosive inflation are indeed permissible: along such a path, the monetary authority is engaged in reckless money creation, accommodating the nominal explosions. This conclusion brings more monetarist criticisms to bear: it is a commitment to monetary restraint á la Friedman (1960) that restores monetary stability, be the commitment only in the long run following Nelson (2008).

1.2 Placing the Study within the Literature

The literature of recursive solutions to rational-expectations problems can be traced back to Muth (1961), who solves for the rational distributed lag expectation by relating endogenous variables to exogenous disturbances through an infinite moving average. Having applied rational expectations in a problem-specific manner—as in Muth (1961), Lucas (1972b), Sargent (1973), Sargent and Wallace (1973), and Brock (1975), both the issue of the possibility of multiple equilibria and the potential for a systematic method to solve general models under rational expectations became apparent. Numerous methods emerged to select among multiple equilibria: Taylor (1977) introduces a minimum variance criterion, Blanchard (1979) explores the requirement of stationarity as a selection criterion, McCallum (1983) proposes the minimum state variable method, and Evans (1985) expectational stability. Blanchard and Kahn (1980) introduces the familiar practice of eigenvalue counting in search of saddle-point stability, limiting the solution to unit-root stationarity. The first (1979) edition of Sargent (1987a) as well as Hansen and Sargent (1980) and Hansen and Sargent (1981) lay out a solution method using the Wiener-Kolmogorov prediction formula, laying the groundwork for the Z-transform and frequency-domain methods employed by Futia (1981), Whiteman (1983), and Sargent (1987a). Whiteman (1983) provides a detailed comparison of this method with the infinite moving average of Muth (1961), state-space methods used by Lucas (1972a) and other frequency- as well as time-domain methods. Additionally, Gourieroux, Laffont, and Monfort (1982) and Broze, Gourieroux, and Szafarz (1985) provide solution characterizations in terms of martingales and Evans and Honkapohja (1986) in terms of mixed autoregressions and moving averages. Of particular importance for chapters 2 and 3 of this work is Taylor (1986), who provides a detailed exposition on the solution of rational-expectations models from the perspective of an infinite moving average, following Muth (1961).

In their seminal real business cycle article, Kydland and Prescott (1982) approximate their nonlinear model with a quadratic objective function and linear constraints leading to linear first-order conditions. The linear quadratic framework places a direct relationship between "transversality" conditions and restricting solutions to be stationary or non-explosive.[1] Uhlig (1999, pp. 32 & 46) partially avoids the issue of stability and transversality in a linear setup, stating: "the reasons for concentrating on stable solutions are not discussed,"[2] but continues later to say, "[t]he literature on solving linear rational expectations equilibria typically assumes [the stability] condition to hold or shows it to hold in social planning problems under reasonable conditions." In the real-business-cycle paradigm, one could generally rule out explosive solutions[3] and the task at hand was, computationally, automating the solution procedure to deliver Lucas's (1980, pp. 709–710) FORTRAN program relating economic policy as inputs to economic outcomes, i.e., time series, as outputs.[4] Chapter 3 continues in this spirit, developing an automated solution and estimation method, expanding on the generality of current methods.

Blanchard and Kahn (1980) develops and McCallum (1983) extends a set of procedures to calculate the unique stable equilibrium path—should it exist—for variables of interest, but the software associated with Anderson and Moore (1985) provides an set of computer algorithms that automates the procedure.[5] Subsequently, Binder and Pesaran (1995), King

[1]See, e.g., Hansen and Sargent (1980, pp. 11–12). Though, as the authors note, the restriction of stationarity can actually be relaxed somewhat.

[2]In the extended version available on his webpage, however, he clearly states on page 9 that "[i]t is the transversality condition which (essentially) rules out explosive solutions."

[3]In a model with secular growth, of course, these would be explosive relative to a balanced growth path.

[4]This computational automation was a great feat in and of itself: noting the state-of-the-art in the mid '70s, Brunner and Meltzer (1976, p. 153 of the same volume) dismiss the criticism that a complete dynamic model is the appropriate means to assess the system, as "economists are so far from developing, estimating and comparing nonlinear dynamic models that [such criticism] denies the usefulness of economics for most contemporary macro policy issues."

[5]Anderson (2010, p. 472) noted, a quarter-century after their development, that these algorithms remained

and Watson (1998), Uhlig (1999), Klein (2000), Sims (2001), and Christiano (2002) have all provided automated algorithms to solve dynamic rational-expectations models. The algorithms vary in the canonical form of the model they solve, the manner in which they treat non-singularities in certain coefficient matrices,[6] and in their numerical properties.[7] For the range of linear(ized) stochastic models in the literature above, the suite of programs *DYNARE*[8] combines the automated solution method of Sims (2001), Klein (2000), or Anderson and Moore (1985) with estimation routines to automate the entire process of solving a model and taking it to the data. The algorithms associated with chapter 3 extend this approach to encompass models with lagged expectations or outdated information in a computationally efficient manner.[9]

Even if attention is restricted to stable solutions, there need not be a unique stable solution. This form of indeterminacy is the focus of the current strand of literature examining multiple equilibria in mainstream macroeconomics. In fact, the alternate selection criteria mentioned previously—Taylor's (1977) minimum variance criterion, McCallum's (1983) minimum state variable method, and Evans's (1985) expectational stability—have primarily found use in selecting *among* stable solutions.[10] McCallum (2003) provides a more recent overview, examining some of the topics explored here in chapters 2 and 4 such as the Taylor Principle—requiring a more than one-for-one increase in the nominal interest rate in response to an

essentially unknown outside of the Federal Reserve, despite their having proven "very durable and usable" for economists there.

[6]For example, Klein (2000) uses a QZ method, Uhlig (1999) a Moore-Penrose pseudoinverse, Anderson and Moore (1985) a shuffle method.

[7]See Anderson (2008) for a detailed comparison.

[8]See http://www.dynare.org for programs and documentation.

[9]Wang and Wen (2006) provide a solution method for models of this type and both Mankiw and Reis's (2007) and Trabandt (2007) develop model-specific solutions for models with an infinite regress of lagged expectations. An extensive comparison can be found in chapter 3.

[10]Exceptions abound, though—e.g., McCallum (2004b) provides a model whose MSV solution and unique stable solution differ and McCallum (2009b) uses LS learnability (a related concept of expectational stability, see Evans and Honkapohja (2001)) to select an equilibrium among stable and unstable equilibria. McCallum (2009d) argues that determinacy is neither necessary nor sufficient and that some form of learnability ought to be preferred.

increase in inflation, see Taylor (2001)—and forward-looking monetary policy along with other topics related to indeterminacy not examined here such as the fiscal theory of the price level and the zero lower bound on nominal interest rates.

In chapter 2, determinacy—or existence of a unique stable solution—is examined within a canonical New Keynesian model with aggregate supply given by Mankiw and Reis's (2002) sticky-information Phillips curve. This equation possesses an infinite regress of lagged expectations leading to a infinite state vector that prevents the model from being cast into the first-order form of Blanchard and Kahn (1980) necessary for the eigenvalue-counting method of ascertaining determinacy. This infinite dimensionality can be recast, as shown by Mankiw and Reis's (2002), into a nonstochastic problem by examining the infinite moving average representation of Muth (1961) instead of the state-space representation of Lucas (1972a). In models that can be cast into Blanchard and Kahn's (1980) form, this is redundant, as the underlying homogenous system of difference equations is the same with stability a question of the size of the eigenvalues common to both specifications. In the infinite dimensional model, however, the associated nonstochastic system of difference equations is nonautonomous or time-varying. Although this frustrates the eigenvalue approach key to the solution and hence stability analysis of constant coefficient or autonomous systems,[11] stability can frequently still be assessed. So long as the model is "close" to being time invariant, the eigenvalues of the asymptotic system are sufficient to ascertain stability—a result established by Perron (1929).[12]

The observation of Phillips (1958) that there had been a seemingly stable negative relationship between the rate of change of money wages and unemployment in the UK over a century was celebrated for its contribution to the discussion at the time. As summarized by

[11]See Elaydi (2005, Ch. 3) for an introduction and overview and Ludyk (1985) for a compact collection of theorems.

[12]Indeed, the analysis in chapter 2 can be performed entirely using Perron's (1929) results as is done parallelly to the approach here in an earlier working paper version of chapter 2 entitled "The Natural Rate Hypothesis and Real Determinacy: A Sticky-Information Perspective."

Brunner and Meltzer (1993, p. 32), the Keynesian model of the day was incomplete and this paper “filled the gap.” Friedman (1976, pp. 215–216) not only highlights that Fisher (1926) had already noted the empirical relation between unemployment and inflation, but also interprets that—in contrast to Phillips (1958), the Keynesian analysis of the day, and the New Keynesian analysis of today as examined in detail in chapter 4 of this work—Fisher (1926) clearly had the important distinction between anticipated and unanticipated changes in mind. That is, Fisher’s (1926) original work anticipated the natural rate hypothesis, whereas Phillips’s (1958) re-discovery did not.[13]

Friedman (1968) and Phelps (1967) independently proposed what became known as the natural rate hypothesis. This hypothesis advances the notion that there exists some rate of unemployment that would prevail under neutral monetary policy and to which unemployment will converge in the long run regardless of monetary policy. The consequence for stabilization policy as summarized by Friedman (1976, p. 232), “you cannot achieve an unemployment target other than the natural rate by any fixed rule. The only way you can do so is by continually being cleverer than all the people, by continually making up new rules [...] This is not a very promising possibility.” By the late ’70s, Friedman (1977, p. 459) was able to note that his and Phelps’s (1967) hypothesis was a widely accepted consensus.

Lucas (1980, p. 705) considers Muth’s (1961) rational expectations to be the natural way to formalize the natural rate hypothesis of Friedman (1968) and Phelps (1967). Friedman (1976, p. 230) goes further, calling the application of rational expectations to the natural rate hypothesis “a more fundamental criticism” of the Phillips curve.[14] In either case, Lucas (1972b) and Sargent (1973) introduce models of rational expectations under the natural rate hypothesis, in accords with the definition that “different time paths of the general price level

[13]Friedman (1976, p. 215) also remarks that the complete circuit from “truth” in 1926 to “error” thirty years later and back to the original “truth” in the ’70s took about 50 years. One can only speculate as to whether the circuit will be completed again before another 50 years are out.

[14]Certainly Lucas’s (1976) Phillips curve example highlights just such “a more fundamental criticism.”

will be associated with time paths of real output that do not differ on average." (Lucas 1972a, p. 50) Succinctly, Friedman (1976, p. 231), "since you can't fool all the people all the time, the true long-run Phillips curve is vertical."

A pertinent implication of the natural rate hypothesis for chapters 2 and 4 is a sort of policy invariance reminiscent of the policy evaluation critique of Lucas (1976) and policy ineffectiveness critique of Lucas (1972b).[15] Evoking the natural rate hypothesis in a dynamic model necessarily places restriction on properties of the model in the long-run[16] where the evaluation of stability for a unique path becomes decisive. As is examined in chapters 2 and 4, such an evocation renders some policy measures—like output-gap targeting—ineffective for contributing to a unique equilibrium and highlights that the effectiveness of others—inflation-forecast targeting, for example—follows from the implausible continuance into the long-run of their empirically plausible short-run Phillips curves.[17] A non-vertical Phillips curve in the long-run violates Friedman's (1976, p. 227) "absence of any long-run money illusion" that led to long-run restrictions of the natural rate hypothesis.[18]

Rational-expectations macroeconomics went on to extend the long-run verticality to the short run with the real business cycle literature, exemplified by Kydland and Prescott (1982).[19] There is a complete separation between real and nominal magnitudes. King and Plosser (1984) provide a detailed treatment of money in a prototypical model of real business cycles: only after having fully determined the equilibrium of the real side of the economy is attention turned to the nominal side where the sole purpose of the role of money and the institutional provision thereof is to determine the price level. Sims (1980) reexamines the monetary

[15] See McCallum (1980) for an overview.

[16] I.e., the long-run Phillips curve must be vertical.

[17] Reminiscent of Lucas's (1976, p. 39) charge that such a "'long-run' output-inflation relationship as calculated or simulated in the conventional way has no bearing on the actual consequences of [...] a policy."

[18] For the standard New Keynesian model, McCallum (2003, p. 1157) notes that its non-vertical Phillips curve imbues the model with a form of "dynamic money illusion."

[19] This is not to say that the monetarist insistence on the influence of money and prices on output over the business cycle had disappeared, see, e.g., Brunner, Cukierman, and Meltzer (1983).

causality of business cycles as advanced by Milton Friedman and Anna J. Schwartz[20] and finds that the role of money is substantially diminished with the inclusion of nominal interest rates into his VAR, lending support to the reverse causality role assigned to money in real business cycles. As noted by Brunner and Meltzer (1993, p. 59), the real business cycle literature reached conclusions similar to those in the radical Keynesian literature regarding the relation of output and money.

Cooley and Hansen (1995) conclude in their real business cycle model with a motive for holding money and money wage rigidities that monetary shocks do play an important role in the business cycle, restoring "the need to assign money an important role in a full theory of business cycles." (Friedman and Schwartz 1963, p. 49) Yun (1996) and Woodford (1996) introduce Calvo (1983) contracts for price setting as a nominal rigidity into real business cycle models, laying the foundation for the standard New Keynesian Phillips curve and restoring non-verticality not only to the short-run but to the long-run Phillips curve as well. This non-verticality—Wolman's (2007, p. 1366) 'awkward situation in monetary economics"—is noted in textbook treatments of the New Keynesian model[21] as well as in the seminal study of Woodford (2003b)[22]. It has been criticized by McCallum (2004a, p. 21) as overturning a neutrality proposal that "by 1980 even self-styled Keynesian economists were agreeing to." While Andrés, López-Salido, and Nelson (2005) examine some consequences of the hypothesis for models' dynamics and Carlstrom and Fuerst (2002) examine determinacy for two special interest rate rules in their own ad-hoc model that fulfills the hypothesis, the consequences of this violation remain largely unknown and the violation itself has gone relatively unnoticed.[23] That there might be general consequences of violating the hypothesis for determinacy—a

[20] See Friedman and Schwartz (1963).

[21] See Galí (2008, p. 78).

[22] See, for example, Woodford (2003b, p. 254). As a consequence, as is discussed in detail in chapter 2, Woodford (2003b, pp. 254–255), "a large enough [response to] *either* [the output gap or inflation] suffices to guarantee determinacy." (Emphasis in the original)

[23] Relative, that is, to the attention given to such violations a few decades ago.

subject of not insignificant interest[24]—has gone almost entirely unexplored and certainly unproven.[25]

Whereas the cornerstone of aggregate supply in the standard New Keynesian model is a mechanism whereby producers' prices may be fixed for a period of time, Mankiw and Reis's (2002) sticky-information model postulates that it is not their prices but rather the information upon which they condition their pricing decisions that may remain fixed. Presupposing the standard Calvo (1983) mechanism[26] and a simple quantity-equation demand side with exogenous money growth, Mankiw and Reis (2002) show that this mechanism can replicate three features of the data that the standard New Keyensian model cannot: the contractionary nature of disinflation, the delayed response of inflation to monetary shocks, and the positive correlation of the second derivative of prices with the level of economic activity. While some later studies have confirmed these results, others have shown them to be more fragile. Korenok and Swanson (2007) and Trabandt (2007) yield positive results in models with firm-specific labor markets, yielding strategic complementarities in firms' pricing decisions,[27] while Keen (2007) and Andrés, López-Salido, and Nelson (2005), using common labor markets that fail to yield strategic complementarities, deliver much more negative results regarding the robustness of Mankiw and Reis (2002). Coibion (2006) also demonstrates the sensitivity of the results to whether monetary policy is defined over the

[24]As McCallum (2009d, p. 26) notes, the subject appears on "75 different pages in Michael Woodford's hugely influential treatise Interest and Prices (Woodford 2003b). In addition, the number of new writings (books, articles, and working papers) with both of the phrases 'indeterminacy' and 'monetary policy' appearing in their text was 166 over the time span January 1995 through June 2008."

[25]For example, Woodford (2003b, Ch. 2) examines determinacy within a model that satisfies the natural rate hypothesis, but his analysis is concerned with price-level determination in a fully flexible model. Carlstrom and Fuerst (2002) explicitly make the connection between price-level determination in fully flexible models and determinacy in models with rigidities that satisfy the natural rate hypothesis. Unfortunately, their analysis is limited to their specific model and no attempt is made to provide proofs for more general conclusions. Chapter 4 takes this issue up in more detail and provides the missing proofs—but with a proviso.

[26]A Poisson process stochastically controlling the rate of arrival of price-change signals in the standard sticky-price setup or of information-update signals in Mankiw and Reis's (2002) sticky-information model, resulting in an exponential distribution of vintages of prices or information sets.

[27]See Woodford (2003b, Ch. 3).

money supply or the nominal interest rate. Reis (2006) provides microfoundations for the underlying mechanism of sticky information, including Mankiw and Reis's (2002) assumed exponential distribution of information sets following from the Calvo mechanism. Chapter 2 presents the first—to my knowledge—analysis of determinacy in the sticky-information framework,[28] enabling a comparison of the policy recommendations from sticky information with those from sticky prices for parameter bounds on interest rate rules.

The empirical evaluation of sticky information DSGE models has focused on their likelihood performance relative to a sticky-price benchmark and the literature has not yet come to a definitive conclusion.[29] Andrés, López-Salido, and Nelson (2005) favor both of their sticky-information models for price setting over a fully dynamically indexed sticky-price model, though they stress the tentativeness of their results due to their neglection of wage rigidities. Paustian and Pytlarczyk (2006) strongly favor a sticky-price and -wage model with partial indexation over their model with sticky information in price and wage setting, reversing the results of Andrés, López-Salido, and Nelson (2005) with the inclusion of wage rigidities. Laforte (2007) compares several specifications of sticky prices and sticky information with a pricing model that abandons the exponential distribution of pricing/information updates, noting that although the sticky information model is perhaps more consistent with micro data on contract duration, it is almost wholly rejected both in in terms of posterior odds and of one-period forecast errors. Coibion and Gorodnichenko (Forthcoming), in contrast to Andrés, López-Salido, and Nelson (2005), favor sticky prices over sticky information

[28] Fries (2007) briefly examines determinacy in his sticky-information model, but his examination is predicated on his assuming a supposition of Wang and Wen (2006) for determinacy in finite dimensional models be valid in infinite settings. Chapter 2 emphasizes that this cannot be true in general for infinite-dimensional problems, as eigenvalue analyses do not function in the associated time-varying difference system. And chapter 4 proves that this supposition of Wang and Wen (2006) holds even in finite-dimensional cases only with a proviso.

[29] A notable exception is Reis (2009), whose focus is the effects of sticky information in consumption and wage decisions along with price decisions. Reis (2009, pp. 19–20) also reviews the results from partial equilibrium estimates of sticky information, exemplified by Klenow and Willis (2007)—who find evidence in support of sticky information, Kiley (2007)—whose results are more mixed, and Korenok (2008)–who favors the sticky-price approach. These analyses usually examine either professional forecast data to capture expectations or estimate a reduced form VAR to capture the missing restrictions from general equilibrium.

in a model without wage rigidities but with strategic complementarities in price setting absent from Andrés, López-Salido, and Nelson (2005).[30] Coibion and Gorodnichenko's (Forthcoming) main focus, however, is a model with multiple types of firms, including both sticky-price and sticky-information firms, and their main result is that these two types of firms—sticky-price and sticky-information—are necessary to match the data. In this vein, Dupor, Kitamura, and Tsuruga (Forthcoming) examine a model of hybrid sticky information and sticky prices, where—in contrast to Coibion and Gorodnichenko's (Forthcoming)—the firm space is not divided into sticky-price and sticky-information firms, but rather one unified sector facing both types of rigidities simultaneously is examined. Importantly, they note that this combination achieves the backward-looking behavior of inflation—emphasized by Christiano, Eichenbaum, and Evans (2005) and introduced exogenously in many sticky-price models through indexation—endogenously through the interaction of these two rigidities. The analysis of chapter 3 shows that it would be premature in any case to draw definitive conclusions in models with sticky information, as the methodology currently in widespread use can suffer from bias resulting from an arbitrary truncation of the infinite regress in lagged expectations. That chapter provides an automated method[31] that overcomes this bias and other difficulties that stem from such truncations.

Reintroducing a role for money and other nominal variables—be it through sticky-prices, sticky-information, or some other rigidity—in business cycles alters the determinacy analysis. With a complete dichotomy, as in the real business cycle literature, determinacy—a unique stable solution—pertains only to real variables. Without any mention of nominal variables, a standard real business cycle model cannot determine any nominal magnitude: that is, it

[30]It is noteworthy, I believe, that the presence of strategic complementarities, so important to replicating the the delayed response of inflation to monetary shocks that favors sticky information in the impulse-response analyses of the foregoing paragraph, would reverse the favoring of sticky information in full-information likelihood analyses.

[31]At the risk of appearing vain, I should note that, in Reis's (2009, p. 17) assessment, the programs implementing the method of chapter 3 "hold the promise of further advancing [the sticky-information] literature."

displays nominal indeterminacy. Nominal indeterminacy has a long tradition that predates the real business cycle literature, from Wicksell (1898)[32] to Patinkin (1965, p. 43) to Sargent and Wallace (1975) to Obstfeld and Rogoff (1983). The criticism of Cochrane (2007) in the context of the models that satisfy the natural rate hypothesis examined in chapter 4 provides another example. The literature has an equally long-tradition in providing a means to overcome this problem: a nominal anchor is necessary. Indeed, Wicksell (1898, pp. 68–73) certainly implies that the pure credit economy has the difficult task of explaining how the price level is determined, whereas in the pure cash and simple credit systems the quantity theory provides the answer. Patinkin (1965, Ch. VIII:3) discusses at length invalid dichotomies that arise from the confusion between accounting prices (in an abstract unit of account) and money prices (in the medium of exchange): fixing the quantity of money fixes the absolute level of money prices.

McCallum (2001a, pp. 27–28) and McCallum (2003, pp. 1156–1157) discuss nominal indeterminacy and nonuniqueness, emphasizing the difference between the former with its lack of a nominal anchor and the latter with its emphasis on stability. The indeterminacy investigated by Obstfeld and Rogoff (1983), for example, is concerned with the determination of the price level, yet this does not fall into McCallum's category of nominal indeterminacy, as the monetary authority provides a nominal anchor—a uniquely defined path for the money supply. Obstfeld and Rogoff's (1983) analysis is concerned with nonuniqueness: a real variable—real balances—is not necessarily uniquely defined by requiring stability in their model. In the standard New Keynesian world, this distinction has minimal content. Without a complete dichotomy, as is the usual case in the New Keynesian literature today, it would be

[32] E.g., on the independence of the absolute level of money prices from the real economy on page 21: "Es ergiebt sich hieraus die wichtige Thatsache, welche allerdings als selbstverständlich gelten könnte, deren Nichtbeachtung aber immer und immer zu fehlerhaften Schlussfolgerungen verleitet hat, dass nämlich der Warentausch als solcher und seine Vorbedingungen in den Verhältnissen der Produktion und Konsumtion der Güter, nur für die Tauschwerte oder relativen Preise der Waren massgebend sein, auf die absolute Höhe des Geldpreises aber gar keinen direkten Einfluss ausüben können."

the exception that a unique stable solution would exist for all real variables but no nominal variables or vice versa. By the very nature of there not being a complete dichotomy, the nominal and real sides are connected and interdependent.

While the determinacy (or in McCallum's words nonuniqueness) literature is generally concerned with the possibility of the existence of multiple *stable* solutions, Cochrane (2007) has emphasized the possibility of unstable solutions being admissible. Cochrane's (2007) critique notes that while there are good reasons to focus on stable solutions to real variables due to transversality conditions, there is less of a case to do so with regards to nominal variables. In the transition from real business cycle models to New Keynesian models, the literature kept the computational technique of linearizing and focusing on the stable solution, but paid little attention to whether this focus was appropriate when a nominal sector is readmitted into the model. Cochrane (2007) criticizes the determinacy analysis in New Keynesian models with interest rate rules as designing monetary policy so as to induce *instability*, rendering all equilibrium paths but one unstable. The monetary authority produces a unique equilibrium by committing to hyperinflationary policy, which is then ruled out by the assumption of stability. In terms of expectations management, Cochrane (2007, p. 15) states

> If you are observing an unstable dynamic system, and you see a small change today, the fact of that change causes a large change in your expectations of the future. If you see the waiter trip, it's a good bet the stack of plates he is carrying will crash. [...T]here is nothing that prevents agents from seeing [a] disturbance, knowing the Fed will feed back on its own past mistakes, thinking "oh no, here we go," and radically changing their expectations of the future.

One could interpret Cochrane's (2007) criticism of rules for nominal interest rates in the New Keynesian literature as Friedman's (1960, p. 23) "monetary passenger [...who] lean[s] over and giv[es] the steering wheel a jerk that threatens to send the car off the road." In essence, monetary policy in the New Keynesian world delivers a unique equilibrium by firstly assuming the car cannot go off the road and secondly ensuring that the monetary passenger is constantly jerking at the wheel, and on that basis concluding that the car must not be moving as it would otherwise tumble off the road.

The reasons for the possible admissibility and type of indeterminacy or nonuniqueness in Cochrane's (2007) critique are considered in chapter 4. The introduction of standard money demand functions into the otherwise "cashless" model common to New Keynesian analysis allows for the assessment of the explosive nominal paths in the context of Friedman's (2008) famous dictum, "inflation is always and everywhere a monetary phenomenon in the sense that it is and can be produced only by a more rapid increase in the quantity of money than in output. Many phenomena can produce temporary fluctuations in the rate of inflation, but they can have lasting effects only insofar as they affect the rate of monetary growth." To assess sustained (or in Cochrane's (2007) case a sustained explosion in) inflation, one cannot avoid this incorporation of money, contradicting what McCallum (2001b, p. 145) has termed the "virtually standard practice" of conducting "monetary policy analysis in models that include no reference to any monetary aggregate." Reinstating a more monetarist perspective on the critique of Cochrane (2007) allows the analysis to confirm the validity of the alternate explosive paths as they are being accommodated by reckless money growth and to diagnose the cause as being the definition of monetary policy solely over the nominal interest rate.

The difficulties of using the nominal interest rate alone to judge the stance of monetary policy was a central critique of Friedman's (1968) pivotal presidential address emphasizing that the Fisher effect, which attributes high nominal interest rates to loose monetary policy, obscures the standard practice of viewing the operation of nominal interest rate control solely through the liquidity effect, by which higher nominal interest rates are associated with tighter monetary policy. To complete monetary policy, some relation of monetary policy to money is deemed necessary, extending the determinacy result of Sargent and Wallace (1975) to interest rate feedback rules. The proposal in chapter 4, based on Nelson (2008), notes that some form of a commitment to stable money growth at the very latest in the long run—as can be attributed to many central banks in practice—provides the missing anchor to rule out the unstable additional equilibria from Cochrane (2007).

The addition of an interest rate rule to a longer term commitment to stable money growth captures what Nelson and Schwartz (2008, p. 848) label the *faute de mieux* of Friedman's (1960) constant money growth rule within the flexibility of an interest rate target rule. As Friedman (1960, p. 98) made clear, the case for a constant money growth rule is entirely that it would work in practice: central banks can fulfill a policy regarding a monetary aggregate over the longer run, while deviating from a constant money supply rule over the shorter run. Although we have made much progress in understanding the monetary mechanism in the past 50 years, chapter 4 emphasizes that the core of Friedman's (1960) policy analysis is still fully pertinent and a central bank would be ill-advised in practice to wholly abandon its monetary pillars.

Chapter 2

Monetary Policy and Determinacy
A Sticky-Information Perspective

2.1 Introduction

Considerable attention has been given in the literature to the role of monetary policy in determining a unique equilibrium,[1] because the model is otherwise potentially subject to welfare-reducing sunspots with arbitrary volatility. Mankiw and Reis's (2002) sticky-information model is particularly challenging with respect to determinacy, as the dynamic relationships change with the horizon. In the near term, the model posits a trade-off between inflation and output, yet transitions in the infinite horizon to a model without such a trade-off. This relationship in the infinite horizon allows the model to satisfy Lucas's (1972a) natural rate hypothesis (NRH), which postulates that monetary policy, however formulated, cannot indefinitely keep output above its natural level. Thus, the output gap must be zero on average, regardless of monetary policy, and the long-run Phillips curve is accordingly vertical. This long-run property is important, even for the analysis of short-run dynamics, as an equilibrium is an entire path for a variable and such long-run relationships between variables are relevant for determining whether a single or many locally bounded equilibrium paths for variables exist. If monetary policy must be restricted so as to deliver a unique equilibrium, long-run relationships cannot be ignored when considering bounds on monetary policy.

This chapter provides the relevant bounds on monetary policy to ensure a unique equilibrium in a simple New Keynesian model with Mankiw and Reis's (2002) sticky-information

[1] McCallum (2009a, p. 26) notes the subject is "ubiquitous in the literature" and can be found on no fewer than 75 pages in Woodford (2003b).

Phillips curve. For the purposes here, the relevant difference to Calvo (1983)-style sticky-price models is "that it survives the McCallum critique" (Mankiw and Reis 2002, p. 1300); that is, it fulfills the NRH. Unlike the sticky-information model, the sticky-price model imposes a systematic relationship between inflation and output, stable even in the long run.[2] This trade off can be exploited by monetary policy to widen the parameter spaces associated with unique equilibria. For example, a reaction of the nominal interest rate to the output gap serves as a substitute for a reaction to inflation, allowing the (direct) response to inflation to be less than one while still adhering to the Taylor principle. Woodford (2003b, pp. 254–255), "... indeed, a large enough [response to] *either* [the output gap or inflation] suffices to guarantee determinacy."[3] As is shown here, the long-run verticality of Mankiw and Reis's (2002) sticky-information Phillips curve precludes such a substitutability.

The analysis here is most closely related to that of Carlstrom and Fuerst (2002), who analyze determinacy and E-stability in an ad-hoc sticky-price model that satisfies the NRH in a finite horizon. Their determinacy analysis is extended here in two main aspects. First, the sticky-information model satisfies the NRH only asymptotically. Thus, Mankiw and Reis's (2002) sticky-information model cannot be analyzed directly with the finite-horizon approach of Carlstrom and Fuerst (2002, p. 80). Second, the set of interest rate rules is expanded to allow for output-gap targeting, price-level targeting, and interest rate feedback. The latter is shown here to restore determinacy by its inclusion of history-dependence for inflation-targeting rules that Carlstrom and Fuerst (2002) had found to be indeterminate.[4]

The degree of output-gap targeting is irrelevant for determinacy in the sticky-information model. Simply put: via the NRH, the output gap must be zero asymptotically regardless of inflation and monetary policy. With the demand side defined by a dynamic

[2] See, e.g., Woodford (2003b, p. 254) or Galí (2008, p. 78).

[3] Emphasis in the original.

[4] Due to a different timing convention, Carlstrom and Fuerst's (2002) backward-looking Taylor rule is analogous to the contemporaneous rule and their current-looking one to the forward-looking rule in the analysis here.

IS curve, the convergence of the output gap implies convergence of the real interest rate regardless of monetary policy. Thus, a permanent increase in inflation will not yield a long-run change in the nominal interest rate through feedback from output-gap targeting. Determinacy then rests on the determinacy of nominal variables through a Fisher-type equation, with no relation to parameter values in the dynamic IS or aggregate supply equations.[5] This has a further implication: determinacy is independent of parameter values outside the interest rate rule.

The specific results for the sticky-information model follow from the derivation of conditions for saddle-path stability in the system of time-varying homogenous linear difference equations that describe the dynamic response of the model to an endogenous fluctuation. The form of the time-variance is shown to be such that the familiar eigenvalue accounting of Blanchard and Kahn (1980) can be applied to the system using the asymptotic coefficients to ascertain determinacy. For comparison with the sticky-price literature,[6] an inventory of rules comprising inflation-forecast and contemporaneous inflation targeting, price-level targeting, output-gap targeting, interest-rate smoothing, as well as exogenous interest rate rules is examined. A strict interpretation (i.e., the irrelevance of output-gap targeting) of the Taylor Principle is shown to be a necessary condition for determinacy in all the rules considered. A pure inflation-forecast rule is shown to be indeterminate everywhere with the inclusion of interest-rate smoothing opening a small window for determinacy, clearly demonstrating the need for history dependence in monetary policy advocated by Woodford (2000). For the set of rules examined here, the bounds on monetary policy are shown to be (weakly) tighter in the sticky-information model than in the sticky-price model, with the latter's violation of the NRH being the driving force behind this result.

[5] Carlstrom and Fuerst (2002) reach the same conclusion in their finite-horizon NRH model.

[6] E.g., Bullard and Mitra (2002) analyze inflation and output-gap targeting at different horizons, Woodford (2003b, Ch. 4) provides the literature-standard inventory of determinacy results, or Lubik and Marzo (2007) presents a more recent compendium of determinacy results in a standard sticky-price model.

The rest of the chapter is organized as follows: in section 2.2, I shall discuss the basic sticky-price and sticky-information models to motivate the modeling framework. In section 2.3, conditions for determinacy in the sticky-information model for various interest rate rules will be presented. Section 2.4 discusses the results along with alternative equilibrium selections and optimal monetary policy and section 2.5 concludes.

2.2 A Sticky-Information Model

Abstracting from exogenous driving processes, the basic sticky-price New Keynesian model is given (in log-deviations) by [7]

$$y_t = E_t\left[y_{t+1}\right] - a_1 R_t + a_1 E_t\left[\pi_{t+1}\right] \tag{2.1}$$

$$\pi_t = \beta E_t\left[\pi_{t+1}\right] + \kappa y_t \tag{2.2}$$

where y_t is the output gap, π_t inflation , and R_t the nominal interest rate. Equation (2.1) is an expectational IS-curve derived from the first-order conditions of the household for intertemporal utility maximization and equation (2.2) is the New Keynesian Phillips curve derived from Dixit-Stiglitz aggregators of individual firms' intertemporal discounted profit maximization with Calvo (1983)-style price stickiness.

Both Woodford (2003b, pp. 243 & 245) and Bullard and Mitra (2002, p. 1110) restrict both κ and a_1 to be strictly positive and a positive a_1 is assumed here throughout. Lubik and Marzo (2007, p. 17) emphasize that the derivation of these parameters from first principles is essential due to "cross-equation restrictions". A main result of this chapter is that the specific parameter values in the sticky-information model are irrelevant for determinacy.

In Mankiw and Reis's (2002) variant of the New Keynesian model, equation (2.2) is

[7] See, e.g., Woodford (2003b, p. 246), equations (1.12) and (1.13).

replaced by the sticky-information Phillips curve

$$\pi_t = \frac{1-\lambda}{\lambda}\xi y_t + (1-\lambda)\sum_{i=0}^{\infty}\lambda^i E_{t-i-1}\left[\pi_t + \xi\left(y_t - y_{t-1}\right)\right] \tag{2.3}$$

where ξ is Woodford's (2003b, pp. 160–161) measure of strategic complementarities, and $1-\lambda$ is the probability that a firm receives an information update. Equation (2.3) results from firms' pricing decisions being the expectation of the optimal price conditional on their (potentially) out-dated information set. A derivation of (2.1) and (2.3) based on first principles analogous to Woodford (2003b, Ch. 4) can be found in Trabandt (2007). In any case, despite the similarities to the model examined by Carlstrom and Fuerst (2002), their model lacks the infinite regress in the information structure of the Phillips curve found in (2.3). This makes their model isomorphic to a flex-price model in finite time such that $E_{t-N}\left[y_t\right] = 0$, whereas this only holds asymptotically in the sticky-information model—i.e., $lim_{N\to\infty}E_{t-N}\left[y_t\right] = 0$.

The dynamics of inflation as presented by Mankiw and Reis (2002) have been criticized by, e.g., Keen (2007) as the assumption of $0 < \xi < 1$ drives the results of the former and the latter find a specification larger than unity to be more plausible. Though the degree of strategic complementarities may be crucial for the dynamics of the model, I find that it will be irrelevant for determinacy. So long as it can be accepted that, *ceteris paribus*, an increase in the deviation of aggregate output from its "natural" level induces firms to want to raise their prices ($\xi > 0$), no further restriction is necessary for the results that follow. This is certainly a mild assumption and covers the entire parameter space considered by Woodford (2003b, pp. 162–164).

A specification of monetary policy is needed to close the model. "With the interest rate as the policy instrument, the central bank adjusts the money supply to hit the interest rate target." (Clarida, Galí, and Gertler 1999, p. 1667) Thus, following Woodford (2003b) among many, I shall focus on interest-rate setting rules. Monetary policy will be initially described by the following rule

$$R_t = \phi_R R_{t-1} + \phi_\pi \left[\left(1-\psi_\pi\right) E_t \left[\pi_{t+1}\right] + \psi_\pi \pi_t \right] + \phi_y y_t$$
$$0 \leq \phi_\pi < \infty,\ 0 \leq \phi_y < \infty,\ 0 \leq \phi_R < 1,\ 0 \leq \psi_\pi \leq 1 \tag{2.4}$$

where ϕ_R describe the degree of interest-rate smoothing, ϕ_π of inflation targeting, and ϕ_y of output-gap targeting. The coefficient ψ_π nests contemporaneous inflation targeting ($\psi_\pi = 1$) and inflation forecast targeting ($\psi_\pi = 0$) into the rule.

2.3 Indeterminacy and the Nominal Interest Rate

After introducing the methods of the analysis and the results for the interest-rate rule (2.4), a set of special cases found in Woodford (2003b, Ch. 4) will be examined in the context of equilibrium determinacy. After first analyzing output-gap targeting rules (with exogenous rules presented as a special case), I shall proceed to inflation targeting. Finally, I shall replace (2.4) with an rule that allows for price-level targeting.

2.3.1 Endogenous Fluctuations and Determinacy

Following, e.g., Theorem 3.15 of Elaydi (2005, p. 130), the solution to a system of difference equations can be split into a particular and a homogenous solution. Only the homogenous solution of the system of difference equations is relevant for the examination of determinacy.[8] Following Taylor (1986), the bounded solution will be unique for any given bounded exogenous sequence of shocks if and only if the homogenous solution is uniquely determined by the boundedness conditions on the endogenous variables.[9] For that reason, I abstract from exogenous driving forces without loss of generality.

[8]See also the analysis of Lubik and Marzo (2007).

[9]Analogous conclusions can be found in, e.g., Woodford (2003b, pp. 252, & 636).

By examining the infinite moving average representation of the model in response to endogenous fluctuations (i.e., to sunspot shocks), the system of difference equations originating from the model of sticky information yields a non-autonomous or time-varying system of homogenous difference equations. Appendix 2.A shows the derivation of the system of difference equations that arise from the infinite moving average representation of the model's variables to sunspot shocks and Appendix 2.B provides the necessary theorems for the analysis of determinacy. With the system of difference equations, the theorems in Appendix 2.B essentially show that the stability of the original system can be determined from associated system using the asymptotic coefficients.[10]

Equation (2.14) in Appendix 2.A gives the time-varying difference equation described by the sticky-information Phillips curve (2.3). This equation can also be interpreted as a perfect-foresight version of the model with all expectations formed before the initial period equal to zero. That is, consider (2.3) for $t = 0, 1, ...$ with $E_s[x_t] = 0$, $\forall s < 0$ and $E_s[x_t] = x_t$, $\forall s = 0, 1, ...$, this yields

$$\lambda^{t+1}\pi_t = \left(1-\lambda^{t+1}\right))\xi y_t - \xi\lambda\left(1-\lambda^t\right)y_{t-1} \tag{2.5}$$

As $t \to \infty$, the foregoing converges to the "unrestricted" perfect-foresight version of the model, given by $y_t = \lambda y_{t-1}$ as all outdated information sets are updated. The lagged expectations in Mankiw and Reis's (2002) sticky-information serve to "transition" the Phillips curve from having a positive trade-off at time 0, given by $\lambda\pi_0 = (1-\lambda))\xi y_0$, to being vertical with no trade-off in the limit. This is starkly contrasted with the the standard sticky-price Phillips curve in (2.2), which always posits the same dynamic trade-off between inflation and output: $\pi_t - \beta\pi_{t+1} = \kappa y_t$ under perfect foresight. Focusing on the impulse responses of the sticky-information model to sunspot shocks, makes explicit that the model itself is time-invariant, but that the response of a variable to a shock is time-varying. That is, the equilibrium relationships between the responses of endogenous variables to a shock change

[10]See, analogously, chapter 3.

as the shock becomes more outdated. The model will be determinate (sunspots can be ruled out), if the only sequence of impulse responses to a sunspot shock that remains bounded is the trivial sequence of zeros for all variables at all horizons; i.e., if the only bounded response of endogenous variables to sunspots is no response at all.

Assume without loss of generality that a sunspot shock occurs at time t and denote with δ_i^x the response of the variable x i periods after the sunspot. The impulse response of the model, defined by (2.1), (2.3), and (2.4), to a sunspot shock can be summarized by the system of deterministic time-varying difference equations found in Appendix 2.A. Determinacy of the models is established by the uniqueness of this representation through the following

Lemma 2.3.1. *The uniqueness of the impulse response is determined by the existence of unique bounded sequences* $\left\{\delta_i^y, \delta_i^R\right\}_{i=0}^{\infty}$ *that solve the following non-autonomous recursion:*[11]

$$\begin{bmatrix} \lambda^{i+2}(1-a_1\xi)+a_1\xi & -\lambda^{i+2}a_1 \\ -\phi_\pi\left(1-\psi_\pi\right)\xi(1-\lambda) & \lambda^{i+2}+a_1\xi\left(\lambda-\lambda^{i+2}\right)\left[1-\phi_\pi\left(1-\psi_\pi\right)\right] \end{bmatrix} \begin{bmatrix} \delta_{i+1}^y \\ \delta_i^R \end{bmatrix}$$
$$= \left(\lambda^{i+2}(1-a_1\xi)+\lambda a_1\xi\right) \begin{bmatrix} 1 & 0 \\ \phi_y + \frac{\phi_\pi\psi_\pi\xi(1-\lambda)}{\lambda^{i+1}(1-a_1\xi)+\lambda a_1\xi} & \phi_R + \frac{\phi_\pi\psi_\pi\xi a_1\left(\lambda-\lambda^{i+1}\right)}{\lambda^{i+1}(1-a_1\xi)+\lambda a_1\xi} \end{bmatrix} \begin{bmatrix} \delta_i^y \\ \delta_{i-1}^R \end{bmatrix} \qquad (2.6)$$
$$i = 0, 1, 2, \ldots$$
$$\delta_{-1}^R = 0$$

Proof. See Appendix 2.C.1.

The foregoing system has one initial condition, $\delta_{-1}^R = 0$, requiring that the nominal interest rate not be a function of future innovations (here, sunspots), but two variables. Certainly, one solution is $\delta_i^y, \delta_i^R = 0,\ i = 0, 1, \ldots$, but it may not be the only bounded solution. There is one "missing" initial condition, δ_0^y; if the system (2.6) is stable, then it will remain bounded for any finite δ_0^y and, thus, the sunspots cannot be ruled out. If the system, however, is unstable, then the boundedness requirement will provide the missing initial condition in

[11] Equation (2.20) gives the response of inflation given the responses of the other two variables.

terms of a linear relationship between δ_0^y and δ_{-1}^R, admitting only the trivial, zero solution without sunspots.

Proposition 2.3.2. *The model given by (2.1), (2.3), and (2.4) is determinate iff* $|\frac{\phi_R+\phi_\pi\psi_\pi}{1-\phi_\pi(1-\psi_\pi)}| > 1$.[12]

Proof. See Appendix 2.C.2.

The restriction on monetary policy given by (2.3.2), ensures that the system is unstable, yielding the missing initial condition and ruling out sunspot equilibria. Figure (2.1) plots the regions of determinacy and indeterminacy that follow from (2.3.2). With this proposition, I can proceed to analyze specific cases of the interest-rate rule (2.4) as derived for sticky-prices models in, e.g., Woodford (2003b, Ch. 4).

2.3.2 Output-Gap Targeting and Exogenous Interest Rates

In this section, I shall examine interest-rate rules with feedback solely from the output gap. As a special case, a constant interest rate (i.e., no feedback) is considered.

Consider the model defined by (2.1) and (2.3) with an output-gap targeting interest-rate rule:

$$R_t = \phi_y y_t, \ \ 0 \leq \phi_y < \infty \tag{2.7}$$

Using proposition 2.3.2, with $\phi_R,\ \phi_\pi = 0$, leads to the following

Proposition 2.3.3. *The model given by (2.1), (2.3), and (2.7) is indeterminate for all* $0 \leq \phi_y < \infty$.

Proof. Following (2.3.2), for $\phi_R,\ \phi_\pi = 0$, it follows immediately that $|\frac{\phi_R+\phi_\pi\psi_\pi}{1-\phi_\pi(1-\psi_\pi)}| = 0 < 1$.

[12] It is to be understood that the analysis will be abstracting from cases where the relevant eigenvalues lie on the unit circle. Following Woodford (2003b, p. 254), in such a case, the linearized models examined here are insufficient to address the question of local determinacy.

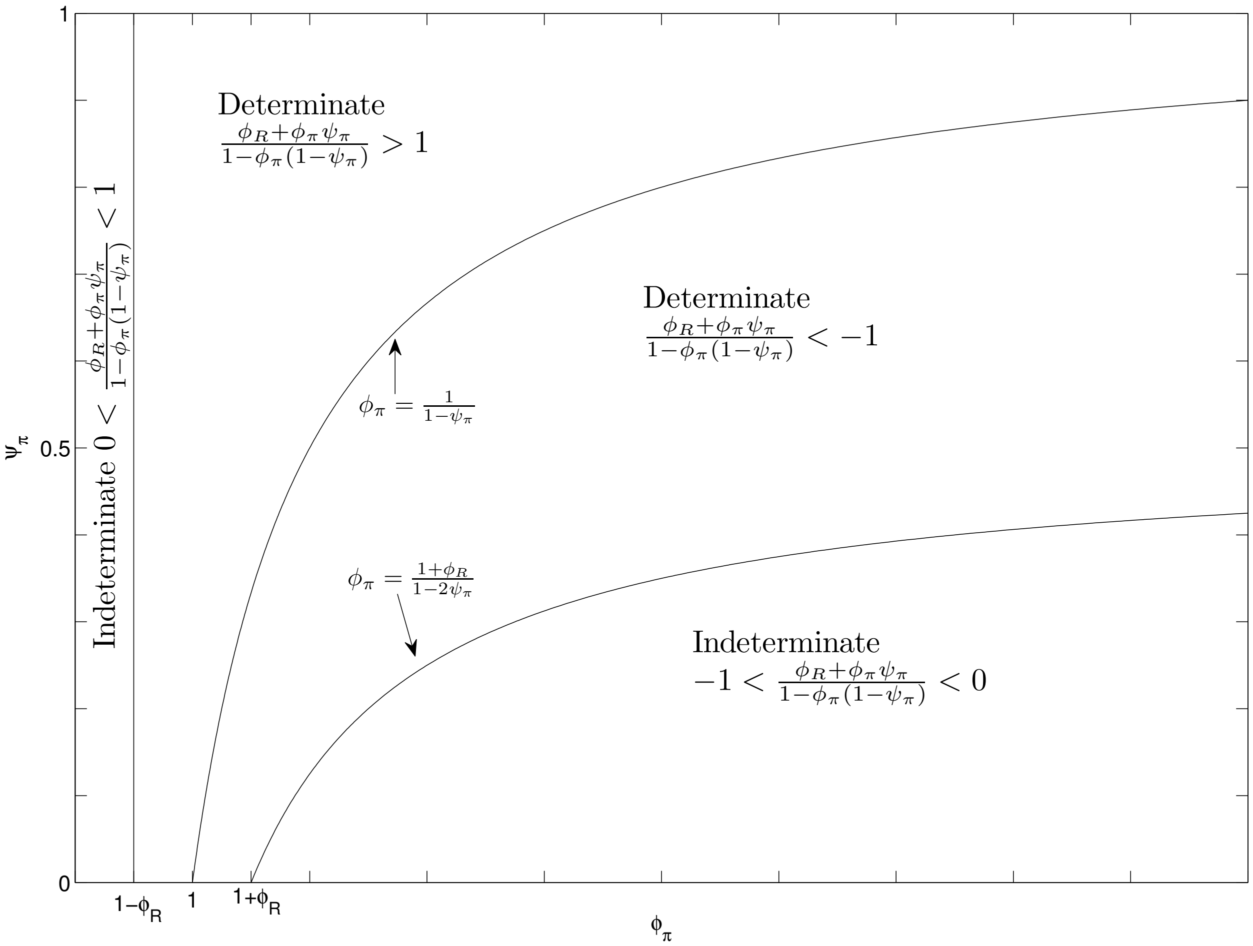

Figure 2.1: Determinacy Regions from Proposition 2.3.2

Thus, contrary to Woodford (2003b, p. 254), if the feedback from endogenous variables is limited to the output gap, no degree of output-gap targeting will suffice to ensure real determinacy. This difference between the sticky-information and sticky-price models is due to the long-run slope of the Phillips curve. The non-verticality in the latter allows monetary policy to substitute output-gap targeting for inflation targeting so as to satisfy the Taylor Principle, a possibility not available in the former, a model without a systematic, long-rung relationship between inflation and output.[13]

[13] Fries (2007) also notes the independence of determinacy from the degree of output-gap targeting in a sticky-information model, but his conclusion is based on Wang and Wen's (2006) conjecture for finite lagged expectations and not directly applicable to the true, infinite specification of the sticky information model.

With a bounded, exogenous interest rate, the system defined by (2.1) and (2.3) is extended by a bounded exogenous process for R_t. As determinacy is related solely to the homogenous part of the system of difference equations, the addition of any bounded stochastic process in the interest rate rule will not affect the results. Thus, determinacy with a bounded exogenous interest rate will be obtained under the same conditions as for a constant interest rate. Therefore, without loss of generality, the model is closed by the following interest rate rule:

$$R_t = 0 \tag{2.8}$$

This is simply a special case of proposition 2.3.3 and, thus, any constant or bounded exogenous interest rate rule is necessarily associated with indeterminacy. This corresponds to Woodford (2003b, p. 253) and confirms that a nominal interest rate rule must involve feedback from endogenous variables, if sunspot equilibria are to be avoided. This extends Sargent and Wallace's (1975, p. 251) conclusion in their model, where their "Phillips curve is not vertical [in the short run], but Wicksell's indeterminacy [i.e., indeterminacy with an exogenous interest rate] still arises," to hold in the model here where the Phillips curve is vertical only asymptotically.

2.3.3 Forward-Looking Inflation Targeting

Consider the model defined by (2.1) and (2.3), with an extended inflation-forecast Taylor-type rule:

$$R_t = \phi_R R_{t-1} + \phi_\pi E_t\left[\pi_{t+1}\right] + \phi_y y_t \tag{2.9}$$

Using proposition 2.3.2, with $\psi_\pi = 0$, leads to the following

Proposition 2.3.4. *The model given by (2.1), (2.3), and (2.9) is determinate iff* $1 - \phi_R < \phi_\pi < 1 + \phi_R$.

Proof. Following (2.3.2), for $\psi_\pi = 0$, it follows that $|\frac{\phi_R+\phi_\pi\psi_\pi}{1-\phi_\pi(1-\psi_\pi)}| = |\frac{\phi_R}{1-\phi_\pi}| > 1$ must be satisfied. But this is equivalent to $1-\phi_R < \phi_\pi < 1+\phi_R$.

It is instructive to begin with the special case $\phi_y = \phi_R = 0$, the case of pure inflation-forecast targeting. Note that according to proposition 2.3.4, the determinacy region collapses to an empty set: a pure inflation-forecast targeting rule is necessarily indeterminate.

Thus, despite the fact that the sticky-information does not satisfy Carlstrom and Fuerst's (2002) more stringent natural-rate hypothesis (i.e., the model is not isomorphic to its flexible-price equivalent in finite time), the same result (except for the alternate timing-convention) for indeterminacy is obtained. Contrary to sticky-price models,[14] there is no region of determinacy for pure inflation-forecast targeting rules. In Carlstrom and Fuerst's (2002) world of finite stickiness, the model displays real determinacy only if it possesses nominal determinacy. The latter is fulfilled only if the inflation rate is uniquely determined at the dawn of the flexible-price world, which itself cannot hold if inflation forecasts are the sole feedback variable for nominal interest rate rules. Here, the flexible-price world is relevant only at the end of time, yet the asymptotic vertically of the Phillips curve suffices to prevent determinacy under the rule considered here. This would certainly seem to be more consistent with Woodford's (2000) discussion of the non-optimality of purely forward-looking monetary policy rules than the analogous analysis in sticky-price models: the purely forward-looking rule considered here will always be indeterminate and, thus, opens the model to potentially welfare-reducing arbitrary fluctuations.

The driving force behind this result can be seen by first examining the sticky price model. Lubik and Marzo (2007, pp. 23–24) derive a lower bound for ϕ_π corresponding to the Taylor Principle, which rules out monotonic sunspot behavior, and an upper bound which rules out non-monotonic sunspot dynamics. A key insight from their analysis is: "that the determinacy

[14]See, e.g., Bullard and Mitra (2002) or Woodford (2003b).

region disappears as [...] prices become perfectly flexible."(Lubik and Marzo 2007, p. 23) As the Phillips curve becomes perfectly vertical in the long run, the upper bound converges to the lower bound.

Like in sticky-price models, the sticky information models posits both a lower bound and an upper bound on the determinacy region, likewise associated with monotonic and non-monotonic sunspot paths.[15] The lower bound requires the interest rate to follow the Taylor Principle, necessitating an active interest rate. The upper bound, however, requires that the interest rate not be overly aggressive, lest "the output gap and inflation [be] projected to converge back to the steady state *regardless* of their values in the current period."[16] (Levin, Wieland, and Williams 2003, p. 628) The difference is that the two bounds collapse, meaning every interest rate rule of this type is either too aggressive or not aggressive enough.

Turning to the general case, the history dependence in the interest-rate rule induced by interest-rate smoothing ($\phi_R > 0$) is enough to open a window of determinacy for a forward-looking Taylor-type rule. That there exists an upper and a lower bound on the elasticity of the nominal interest rate with respect to expected inflation is consistent with sticky-price models as discussed above. The lower bound conforms to Woodford's (2003b, p. 96) inertial modification of the Taylor Principle: the cumulative response of the nominal interest rate must react more than one-to-one to a sustained deviation in inflation (saving for the irrelevance of output-gap targeting as discussed previously). Woodford (2003b, p. 259) remarks that with "coefficients in the range that is likely to be of practical interest, [the upper bound does] not seem likely to be a problem"; likewise Galí (2008, p. 79). These assurances are far from convincing in the sticky-information model. Indeed, allowing for interest rate smoothing ($0 < \phi_R < 1$), the coefficient on inflation must be less than two: a value which is certainly not far above the reasonable range.

[15] Examining figure (2.1) with $\psi_\pi = 0$ and $\phi_R = 0$, the eigenvalue is positive left of one and negative to the right.
[16] Emphasis in the original.

2.3.4 Contemporaneous Inflation Targeting

Consider again the model (2.1) and (2.3). If monetary policy pursues an extended contemporaneous inflation target, the model will be closed by the following Taylor-type rule:

$$R_t = \phi_R R_{t-1} + \phi_\pi \pi_t + \phi_y y_t \tag{2.10}$$

Using proposition 2.3.2, with $\psi_\pi = 1$, leads to the following

Proposition 2.3.5. *The model given by (2.1), (2.3), and (2.10) is determinate iff* $1 - \phi_R < \phi_\pi$.

Proof. Following (2.3.2), for $\psi_\pi = 1$, it follows that $|\frac{\phi_R + \phi_\pi \psi_\pi}{1 - \phi_\pi(1-\psi_\pi)}| = |\phi_R + \phi_\pi| > 1$ must be satisfied. But this is equivalent to $1 - \phi_R < \phi_\pi$.

In the special case of pure inflation targeting $\phi_y = \phi_R = 0$, the same result for determinacy as in Carlstrom and Fuerst (2002) is obtained, even though the sticky-information model does not satisfy their more stringent, "finite-time" natural-rate hypothesis. The celebrated Taylor Principle is a necessary (as discussed in the previous section) and, now, sufficient condition for determinacy.

Thus, contrary to Woodford (2003b, p. 255), determinacy is independent of the degree of output-gap targeting, as discussed in section 2.3.2. Examining the case $\phi_R = 0$, this condition reduces to that of a pure inflation target, and, for $\phi_R \neq 0$, determinacy requires the nominal interest rate to move cumulatively more than one-to-one in response to a permanent change in inflation, Woodford's (2003b, pp. 95–96) "eventual" Taylor Principle.

The equivalence (up to cumulative effects) of the two interest-rate feedback rules examined here, reiterate the conclusion from previous sections that output-gap targeting is irrelevant for determinacy. The absence of parameters besides those of monetary policy has the convenient attribute that determinacy can be evaluated solely on the merits of the interest rate rule. Thus, Woodford's (2003b, p. 255) slightly complicated interpretation of Taylor

(2001), requiring parameter estimations of the sticky-price Phillips curve is not applicable in the sticky-information world. Indeed, if one is to take Woodford's (2003b, p. 255) analysis seriously, indeterminacy pre-Volker era was not necessarily due to too weak of a reaction to inflation, a higher reaction to the output gap would have also sufficed; a conclusion which cannot be reached here.

2.3.5 Price-Level Targeting

As an alternative to inflation targeting, monetary policy could pursue a price-level target. Incorporating feedback from the output gap into such a rule replaces (2.4) with the following

$$R_t = \phi_p p_t + \phi_y y_t, \ 0 \leq \phi_p, \phi_y < \infty \tag{2.11}$$

Lemma 2.3.6. *The uniqueness of the impulse response under price-level targeting is determined by the existence of unique bounded sequences* $\left\{\delta_i^y, \delta_i^R\right\}_{i=0}^{\infty}$ *that solve the following non-autonomous recursion:*

$$\begin{bmatrix} \xi + \lambda^{i+2}\left(\frac{1}{a_1} - \xi\right) & 0 \\ \frac{\phi_p}{a_1} - \phi_y & 1 \end{bmatrix} \begin{bmatrix} \delta_{i+1}^y \\ \delta_{i+1}^R \end{bmatrix} = \begin{bmatrix} \xi\lambda + \lambda^{i+2}\left(\frac{1}{a_1} - \xi\right) & -\lambda^{i+2}\frac{\phi_R}{\phi_p} \\ \frac{\phi_p}{a_1} & \left(1+\phi_p\right) \end{bmatrix} \begin{bmatrix} \delta_i^y \\ \delta_i^R \end{bmatrix}, \ i = 0, 1, 2, \ldots$$

$$\delta_0^R = \left(\frac{1-\lambda}{\lambda}\xi\phi_p + \phi_y\right)\delta_0^y \tag{2.12}$$

Proof. See Appendix 2.C.3.

As before, the requirement of boundedness will provide an additional restriction on the recursion only if the system in (2.12) is unstable.

Proposition 2.3.7. *The model given by (2.1), (2.3), and (2.11) is determinate iff* $\phi_p > 0$.

Proof. See Appendix 2.C.4.

Any non-zero response to the price level will ensure determinacy. This corresponds to sticky-price models and follows from the Taylor Principle. With a zero response to the price level, however, the the model is indeterminate regardless of the degree of output-gap targeting. This stands again in contrast to Woodford (2003b, p. 261) and is an obvious consequence of the irrelevance of the degree of output-gap targeting as discussed in the previous sections. Woodford (2003b, p. 261) notes the attractiveness of a price-level target on the basis that determinacy is independent of the degree of output-gap targeting. This feature holds not only under price-level target but throughout this chapter.

2.4 Discussion

Woodford's (2003b, pp. 252–259) conclusion that an interest-rate setting rule which does not directly (i.e., with respect to inflation only) satisfy the Taylor principle need not be associated with indeterminacy does not carry over to the sticky-information model. Whereas the standard sticky-price model exhibits a non-vertical long-run Phillips curve that allows for both "substitution" of output-gap targeting for inflation targeting and a pure inflation-forecast target to ensure uniqueness, the sticky-information model's vertical long-run Phillips curve precludes both possibilities. This yields tighter bounds (lower and, in the case of forward-looking policy rules, upper) on the coefficients of interest-rate setting rules for the sticky-information model. The bounds on interest-rate rules as derived here are juxtaposed in table 2.1 with the bounds derived in Woodford (2003b, Ch. 4)[17] for the sticky-price model.

These tighter bounds have two important features relative to the looser ones derived by Woodford (2003b). Firstly, they are independent of model-specific parameter values. Regardless of the calibration, the Taylor principle is only satisfied if the direct (cumulative) reaction of the nominal interest rate to a (permanent) deviation in inflation is greater than one: $\phi_\pi > 1 - \phi_R$ is necessary under sticky information. This corresponds neatly to the conclusions of Taylor (2001) with regards to the pre-Volcker and the Volcker-Greenspan eras

[17]Note that the output gap is expressed quarterly here, thus Woodford's (2003b, Ch. 4) $\frac{\phi_x}{4}$ corresponds to my ϕ_y.

Table 2.1: Determinacy Regions: Comparison of Sticky Information and Sticky Prices

Interest Rate Rule	Sticky Prices	Sticky Information
	Non-Price-Related Feedback	
$R_t = 0$	$\emptyset$	$\emptyset$
$R_t = \phi_y y_t$	$\phi_y > \frac{\kappa}{1-\beta}$	$\emptyset$
	Inflation-Forecast Feedback	
$R_t = \phi_\pi E_t[\pi_{t+1}]$	$1 < \phi_\pi < 1 + 2\frac{1+\beta}{a_1\kappa}$	$\emptyset$
$R_t = \phi_R R_{t-1} + \phi_\pi E_t[\pi_{t+1}] + \phi_y y_t$	$\phi_\pi > 1 - \phi_R - \frac{1-\beta}{\kappa}\phi_y$ $\phi_\pi < 1 + \phi_R + \frac{1+\beta}{\kappa}\left(\phi_y + 2\frac{1+\phi_R}{a_1}\right)$	$\phi_\pi > 1 - \phi_R$ $\phi_\pi < 1 + \phi_R$
	Contemporaneous Inflation Feedback	
$R_t = \phi_\pi \pi_t$	$\phi_\pi > 1$	$\phi_\pi > 1$
$R_t = \phi_R R_{t-1} + \phi_\pi \pi_t + \phi_y y_t$	$\phi_\pi > 1 - \phi_R - \frac{1-\beta}{\kappa}\phi_y$	$\phi_\pi > 1 - \phi_R$
	Price-Level Feedback	
$R_t = \phi_p p_t + \phi_y y_t$	$\phi_p > 0$ or $\phi_y > \frac{\kappa}{1-\beta}$	$\phi_p > 0$

that the elasticity of the nominal interest rate with respect to inflation was less than one in the former implying an instability in inflation and greater than one in the latter implying stable inflation. Secondly, the tighter bounds derived here are more robust, as they would also deliver determinacy in the comparable sticky-price model.

The bounds for determinacy derived in section 2.3 coincide with those for nominal determinacy in the associated RBC model, as was also shown by Carlstrom and Fuerst (2002) for their finite NRH model. To see this, letting $y_t = 0$, $\forall t$ (i.e., the output gap is always closed) yields from (2.1) $R_t = E_t[\pi_{t+1}]$ and the condition $|\frac{\phi_R+\phi_\pi\psi_\pi}{1-\phi_\pi(1-\psi_\pi)}| > 1$ using (2.4) for nominal determinacy follows straightforwardly from Blanchard and Kahn (1980). This equivalence has nothing to do with a "folk theorem" difference between finite and infinite games as suggested by Carlstrom and Fuerst (2002, p. 81), as the sticky-information mechanism uses the same, infinite Calvo mechanism as in the standard sticky-price model. The decisive factor is the fulfillment of the NRH, be it in finite time as in Carlstrom and Fuerst (2002) or asymptotically as here. Both models converge to their respective RBC counterparts regardless of monetary policy and boundedness of equilibria does not require this convergence at any finite horizon.

Discomforting is the conclusion that the upper bound, present under a forward-looking interest-rate setting rule, is significantly lower than would be concluded from a sticky-price model. Indeed, without interest-rate smoothing, the model is necessarily indeterminate.[18] The estimated values from Clarida, Galí, and Gertler (2000), however, are such that the period 1982-96 would be associated with a unique equilibrium in the sticky-information model, as in a sticky-price model of Woodford (2003b, p. 260). The sticky-information model serves to strengthen conclusions drawn from empirical examinations of monetary policy, as determinacy results directly and solely from the form and parameter values of monetary policy.

The upper bound on determinacy with forward-looking rules has been criticized as not being consistent with McCallum's (1983) "minimum state variable" solution[19] or Evans and Honkapohja's (2001) E-stability[20]. As the sticky-information model has a state vector of infinite dimension, E-stability would seem difficult to ascertain; the analysis of E-stability by Carlstrom and Fuerst (2002, p. 83), though, indicates that forward-looking monetary policy can result in learnable sunspots. The MSV approach proposes to select a solution using the minimum number of state variables, but this is necessarily infinite with sticky-information. Additionally, however, "the MSV solution involves a procedure that makes it unique by construction." (McCallum 1999, p. 627) The bubble-free solution in the model considered here would be the trivial solution zero for all variables, as no exogenous forces were postulated. That this solution is readily identifiable here is not very useful when confronted with a model containing such exogenous forces. Investigation of the MSV solution and E-stability in sticky-information models, though clearly desirable, will have to be left for further research.

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[18]As can be seen in figure (2.1), a region of determinacy without interest-rate smoothing also exists when both contemporaneous and forecasted inflation are targeted. There continues to be an upper bound on the reaction coefficients so long as $\psi_\pi < \frac{1}{2}$, but not for $\psi_\pi \geq \frac{1}{2}$. The upper bound will only completely disappear if contemporaneous inflation is given more weight.

[19]See, e.g., McCallum (2001b).

[20]See, e.g., Bullard and Mitra (2002) or McCallum (2009a).

Woodford's (2003b, p. 258) Figure 4.1 shows, for his parameter set, that the upper bound in sticky-price models is so high that the discussion of, e.g., McCallum (2003) and Woodford (2003a) regarding this upper bound is empirically irrelevant. Yet, this discussion is of pressing importance in the sticky-information model, as its upper bound $1+\phi_R$ is not far from the range of relevant parameter values for ϕ_π. Should the upper bound not be found to be "of dubious merit" (McCallum 2003, p. 1154), a pure inflation-forecast rule should be avoided rather generally by monetary policy and implemented only with caution and some form of history dependence such as interest-rate smoothing. Even then, the lower bound derived here would still prescribe a more stringent interpretation of the Taylor Principle than in sticky-price models due to the irrelevance of output-gap targeting for determinacy.

Some conclusions about optimal monetary policy can be derived from the foregoing determinacy analysis despite the lack of a first-principles derivation. Equations (2.3) and (2.2) possess Blanchard and Galí's (2007, p. 36) "divine coincidence" property; that is, stabilizing inflation also stabilizes output. In the absence of exogenous forces, an equilibrium exists where inflation and the output gap are always zero. If policy makers are concerned with variations in these variables, this equilibrium is clearly attractive. The desirability of determinacy would be that it renders this the only equilibrium. Even if one were to append (2.1) with some stationary stochastic variable z_t, the interest rate given by $R_t^{new} = \frac{1}{a_1} z_t + R_t^{old}$ with R_t^{old} given by (2.4), would reduce the system to the one analyzed here where one equilibrium for inflation and the output gap has both always equal to zero. Following Svensson and Woodford (2004, p. 44), the component $\frac{1}{a_1} z_t$ is the "decision procedure that is consistent with an optimal equilibrium" and the R_t^{old} component ensures, so long as the bounds are fulfilled, that this equilibrium is the only consistent one. For any common IS specification, "the decision procedure" component will be the same regardless of whether (2.3) or (2.2) describes the supply side. When the divine coincidence property breaks down and additional—e.g., cost-push—exogenous components leak into the supply side, "the decision procedure" components will surely differ. Nonetheless, the restrictions on the "determinacy" component that follow from this model are preferable from the point of view of Svensson and Woodford's (2004, p. 25) robustness.

2.5 Conclusion

Inflation targeting does work in bringing about a unique equilibrium, even with Mankiw and Reis's (2002) sticky-information Phillips curve. Yet this analysis shows that some conclusions reached thus far on the basis of the sticky-price model might be premature. As argued in Sargent (1973, p. 480), "right or wrong, the long-run natural rate [hypo]thesis has immediate relevance because it says something important about the impact of systematic and predictable changes on the economic system." The sticky-information fulfills the natural rate hypothesis and, if one is unwilling to accept the systematic, long-run relationship between inflation and output in the standard sticky-price model, the tighter parameter bounds for interest rate rules derived here ought to be heeded.

Appendix

2.A Model Appendix

In this appendix, I derive the non-autonomous system of difference equations that characterizes the response of (2.1) and (2.3) to a sunspot.[21]

Equation (2.1) can be rewritten as

$$\delta_i^y = \delta_{i+1}^y - a_1 \delta_i^R + a_1 \delta_{i+1}^\pi \tag{2.13}$$

As the system's response to a perturbation at time t from equilibrium is being examined, the response of all variables and all expectations dated before t are equal to zero (i.e., the model is starting from the equilibrium steady-state solution).[22] Equation (2.3) can thus be rewritten as

$$\delta_i^\pi = \frac{1-\lambda}{\lambda} \xi \delta_i^y + (1-\lambda) \sum_{j=0}^{i-1} \lambda^j \left[\delta_i^\pi + \xi \left(\delta_i^y - \delta_{i-1}^y \right) \right]$$

collecting terms yields

$$\lambda^{i+1} \delta_i^\pi = \left(1 - \lambda^{i+1}\right)) \xi \delta_i^y - \xi \lambda \left(1 - \lambda^i\right) \delta_{i-1}^y \tag{2.14}$$

The system defined by (2.13) and (2.14) is closed by the equation

$$\delta_i^R = \phi_R \delta_{i-1}^R + \phi_\pi \left[\left(1 - \psi_\pi\right) \delta_{i+1}^\pi + \psi_\pi \delta_i^\pi \right] + \phi_y \delta_i^y \tag{2.15}$$

2.B Time-Varying Difference Equations

In this appendix, I shall present the necessary theorems for (locally) unique bounded solutions to the model in the text. The systems of difference equations that describe

[21]Compare Taylor's (1986) method for solving for the infinite moving average coefficients of endogenous variables.

[22]See Mankiw and Reis's (2002) Appendix.

the solutions to the sticky-information Phillips curve are non-autonomous (time-varying) difference equations.[23] Unfortunately, following Elaydi (2005, p. 191), “eigenvalues do not generally provide any information about the stability of nonautonomous difference equations.” This would appear to preclude the standard eigenvalue counting method of Blanchard and Kahn (1980). However, the time-variance in coefficient matrices disappears in the limit—i.e., the coefficient matrices converge to limiting matrices—and the variation in the coefficient matrices is bounded. This convergence and boundedness will prove sufficient to allow the familiar eigenvalue accounting of Blanchard and Kahn (1980).

2.B.1 Stability of Nearly Time-Invariant Systems

Here, I shall present necessary conditions for the stability (and therefore boundedness)[24] of nearly time-invariant linear systems by repeating Theorem 3-29 in Ludyk (1985, p. 61).

Consider the system given by $x_{k+1} = [C + D(k)]\, x_k$

Theorem 2.B.1. *Assume (1) C is stable and (2) $\sum_{k=k_0}^{\infty} ||D(k)|| < \infty$.*

Then $x_{k+1} = [C + D(k)]\, x_k$ is stable.

Proof. See Ludyk (1985, pp. 61–62).

This theorem suffices to establish that the unstable manifold needed for determinacy is absent. A system that is stable will remain bounded for any bounded initial conditions. Requiring the system to remain bounded will fail to provide any additional restrictions and, hence, will be unable to determine the “missing” initial condition. If the system is not stable (i.e., C is not stable), then the requirement of boundedness will provide the missing restriction, as is shown through the one-to-one correspondence in the following theorem.

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[23]See also chapter 3.

[24]See Theorem 3-12 of Ludyk (1985, p. 39).

2.B.2 Asymptotically Constant Systems of Difference Equations

Here, I shall present an application of a variation of constants formula to establish the correspondence between bounded solutions of the the system of interest and its diagonalized constant-coefficient counterpart.

Consider the system given by $x_{k+1} = A(k)x_k$, where $A(k) = C + D(k)$

Theorem 2.B.2. *Assume (1) C contains as many distinct, non-zero eigenvalues as its dimension and (2)* $\sum_{k=k_0}^{\infty} ||D(k)|| < \infty$.

Then there exists a one-to-one correspondence between bounded solutions of $x_{k+1} = A(k)x_k$, $\Theta_{k+1} = \Lambda\Theta_k + \tilde{D}(k)\Theta_k$, *and* $\Xi_{k+1} = \Lambda\Xi_k$, *where* $\Lambda = P^{-1}CP$ *is the matrix of eigenvalues,* P *the corresponding right eigenvectors,* $\tilde{D}(k) = P^{-1}D(k)P$, *and* $\Theta_{k+1} = P^{-1}x_{k+1}$.

Proof. Given the assumption of distinct eigenvalues, the set of eigenvectors is linearly independent and there is a one-to-one correspondence between bounded solutions of x_t and Θ_t given by $\Theta_{k+1} = P^{-1}x_{k+1}$. Following Theorem 8.19 of Elaydi (2005, pp. 360–361), there exists a one-to-one correspondence between bounded solutions of $\Theta_{k+1} = \Lambda\Theta_k + \tilde{D}(k)\Theta_k$ and $\Xi_{k+1} = \Lambda\Xi_k$ if the eigenvalues of the diagonal matrix Λ are all non-zero given by

$$\Theta_k = \Xi_k + \sum_{j=k_0}^{k-1} \Phi_1(k)\Phi^{-1}(j+1)\tilde{D}(j)\Theta_j - \sum_{j=k}^{\infty} \Phi_2(k)\Phi^{-1}(j+1)\tilde{D}(j)\Theta_j \tag{2.16}$$

where $\Phi(k)$ is the solution map of Ξ_k, $\Phi_2(k)$ is the solution map of Ξ_k with regards to unstable eigenvalues, and $\Phi_1(k)$ with regards to stable eigenvalues. □

2.C Proofs

2.C.1 Proof of Lemma 2.3.1

Proof. Equation (2.13) can be rewritten as

$$\delta_{i-1}^{y} = \delta_i^{y} - a_1\delta_{i-1}^{R} + a_1\delta_i^{\pi}, \text{ for } i = 1, 2, ... \tag{2.17}$$

Equation (2.14) is

$$\begin{aligned}\lambda\delta_0^\pi &= (1-\lambda)\xi\delta_0^y \\ \lambda^{i+1}\delta_i^\pi &= \left(1-\lambda^{i+1}\right))\xi\delta_i^y - \xi\lambda\left(1-\lambda^i\right)\delta_{i-1}^y, \text{ for } i=1,2,\ldots\end{aligned} \tag{2.18}$$

Inserting (2.17) into (2.18) and solving for δ_i^π yields[25]

$$\begin{aligned}\delta_0^\pi &= \frac{1-\lambda}{\lambda}\xi\delta_0^y \\ \delta_i^\pi &= \frac{\xi(1-\lambda)}{\lambda^{i+1}(1-\xi a_1)+\lambda a_1\xi}\delta_i^y + \frac{a_1\xi\left(\lambda-\lambda^{i+1}\right)}{\lambda^{i+1}(1-\xi a_1)+\lambda a_1\xi}\delta_{i-1}^R, \text{ for } i=1,2,\ldots\end{aligned} \tag{2.19}$$

Which is the same as,

$$\delta_i^\pi = \frac{\xi(1-\lambda)}{\lambda^{i+1}(1-\xi a_1)+\lambda a_1\xi}\delta_i^y + \frac{a_1\xi\left(\lambda-\lambda^{i+1}\right)}{\lambda^{i+1}(1-\xi a_1)+\lambda a_1\xi}\delta_{i-1}^R, \text{ for } i=0,1,\ldots \tag{2.20}$$

Replacing δ_i^π and δ_{i+1}^π in (2.15) and in (2.13) with the foregoing yields (2.6). A solution for (2.6) yields $\left\{\delta_i^y, \delta_i^R\right\}_{i=0}^\infty$, which delivers $\left\{\delta_i^\pi\right\}_{i=0}^\infty$ from (2.20).

2.C.2 Proof of Proposition 2.3.2

Proof. The system of difference equations in (2.6) can be inverted to yield

$$\begin{bmatrix}\delta_{i+1}^y \\ \delta_i^R\end{bmatrix} = (C+D(i))\begin{bmatrix}\delta_i^y \\ \delta_{i-1}^R\end{bmatrix} \tag{2.21}$$

[25] Note that as a_1, $\xi>0$ and $0<\lambda<1$, $\lambda^{i+1}(1-\xi a_1)+\lambda a_1\xi = \lambda^{i+1}+\lambda a_1\xi\left(1-\lambda^i\right)>0$, for $i\geq 0$.

so long as $\phi_\pi\left(1-\psi_\pi\right) \neq 1+\frac{\lambda^{i+2}}{\left(1-\lambda^{i+2}\right)\xi a_1}$, $\forall i \geq 0$. Where

$$
\begin{aligned}
C &= \begin{bmatrix} \lambda & 0 \\ \frac{a_1\lambda\phi_y+\phi_\pi\left(1-\psi_\pi\right)(1-\lambda)\lambda+\phi_\pi\psi_\pi(1-\lambda)}{a_1\lambda\left(1-\phi_\pi\left(1-\psi_\pi\right)\right)} & \frac{\phi_R+\phi_\pi\psi_\pi}{1-\phi_\pi\left(1-\psi_\pi\right)} \end{bmatrix} \\
D(i) &= \alpha_1(i)D_1+\alpha_2(i)D_2+\alpha_3(i)D_3 \\
\alpha_1(i) &= \frac{\lambda^{i+2}}{\left(1-\lambda^{i+2}\right)a_1\xi\left(1-\phi_\pi\left(1-\psi_\pi\right)\right)+\lambda^{i+2}} \\
\alpha_2(i) &= \frac{\alpha_1(i)}{\lambda^{i+1}(1-a_1\xi)a_1\xi+\lambda a_1\xi} \\
\alpha_3(i) &= \lambda^{i+1}\alpha_2(i) \\
D_1 &= \begin{bmatrix} (1-\lambda)\left[1-a_1\xi\left(1-\phi_\pi\left(1-\psi_\pi\right)\right)\right]+a_1\phi_y\phi_R & a_1 \\ -\frac{\phi_y\phi_\pi\left(1-\psi_\pi\right)+\left[\phi_\pi\left(1-\psi_\pi\right)\lambda+\phi_\pi\psi_\pi\right]\frac{1-\lambda}{\lambda}\frac{1}{a_1}\left[1-a_1\xi\left(1-\phi_\pi\left(1-\psi_\pi\right)\right)\right]}{1-\phi_\pi\left(1-\psi_\pi\right)} & (1-a_1\xi)\phi_R+\left(\phi_R+\phi_\pi\psi_\pi\right)\frac{1-a_1\xi\left(1-\phi_\pi\left(1-\psi_\pi\right)\right)}{1-\phi_\pi\left(1-\psi_\pi\right)} \end{bmatrix} \\
D_2 &= \begin{bmatrix} \phi_\pi\psi_\pi\xi(1-\lambda)a_1 & a_1\phi_\pi\psi_\pi a_1\xi\lambda \\ -\frac{1-\lambda^2}{\lambda^2}\phi_\pi\psi_\pi\xi(1-\lambda)(1-a_1\xi) & \frac{\phi_\pi\psi_\pi}{\lambda}\left[\lambda^2(1-a_1\xi)-(a_1\xi)^2-(1-a_1\xi)a_1\xi\right] \end{bmatrix} \\
D_3 &= \begin{bmatrix} 0 & -\phi_\pi\psi_\pi a_1\xi \\ 0 & -\phi_\pi\psi_\pi(1-a_1\xi) \end{bmatrix}
\end{aligned}
$$

Using the ratio test, $\sum_{i=0}^{\infty}|\alpha_1(i)|<\infty$, $\sum_{i=0}^{\infty}|\alpha_2(i)|<\infty$, and $\sum_{i=0}^{\infty}|\alpha_3(i)|<\infty$, so

$$
\sum_{i=0}^{\infty}||D(i)|| \leq ||D_1||\sum_{i=0}^{\infty}|\alpha_1(i)|+||D_2||\sum_{i=0}^{\infty}|\alpha_2(i)|+||D_3||\sum_{i=0}^{\infty}|\alpha_3(i)|<\infty
$$

Thus, $D(i)$ satisfies the second assumption of theorems 2.B.1 and 2.B.2.

Examining the eigenvalues of C, $z_1=\lambda$, $z_2=\frac{\phi_R+\phi_\pi\psi_\pi}{1-\phi_\pi\left(1-\psi_\pi\right)}$, the first of which is necessarily inside the unit circle.

If $|z_2|<1$, the system satisfies theorem 2.B.1 and is stable for any set of bounded initial conditions. In this case, the boundedness condition will be insufficient to pin down the missing initial condition and one cannot rule out sunspot equilibria (i.e., the model is indeterminate).

Should, however, z_2 be outside the unit circle, then z_1 and z_2 are necessarily distinct and, as $0<z_1<1$, are both non-zero. Applying theorem 2.B.2, there is a one-to-one correspondence between bounded solutions of the system in question and its diagonalized constant-coefficient counterpart. The latter is saddle-path stable $|z_1| < 1$ and $|z_2| > 1$ and hence the requirement of boundedness will provide the system with an additional restriction to pin down the missing initial condition and rule out sunspot equilibria (i.e., the model is determinate).

Should $\phi_\pi\left(1-\psi_\pi\right) = 1 + \frac{\lambda^{i+2}}{\left(1-\lambda^{i+2}\right)\xi a_1}$ for some i (say i_τ), then the system does not fulfill the second assumption of theorem (2.B.2) for $k_0 = 0$. The singularity of the coefficient matrix at $i = i_\tau$ combined with the initial condition $\delta^R_{-1} = 0$ implies that $\delta^R_{i-1} = \delta^y_i = 0,\ i \le i_\tau$. Using either of the two equations in the recursion delivers a new initial condition, $\left(1-\lambda^{i_\tau+2}\right)\xi\delta^y_{i_\tau+1} = \frac{\lambda^{i_\tau+2}}{\phi_\pi}\delta^R_{i_\tau}$, which results in a non-singular recursion for $i = i_\tau+1, i_\tau+2, ...$, with the same stability characteristics as in the recursion without the singularity.

Therefore, the requirement of a locally bounded solution will provide the "missing" initial condition (i.e., the system is determinate) when $|\frac{\phi_R+\phi_\pi\psi_\pi}{1-\phi_\pi\left(1-\psi_\pi\right)}| > 1$.

2.C.3 Proof of Lemma 2.3.6

Proof. The system defined by (2.13) and (2.14) is now closed by a monetary policy described by

$$\delta^R_i = \phi_p\delta^p_i + \phi_y\delta^y_i \tag{2.22}$$

First differencing the foregoing, recalling that $\delta^\pi_i = \delta^p_i - \delta^p_{i-1}$, and substituting appropriately yields the recursion in (2.12). Noting that $\delta^p_{-1} = 0$, equations (2.13) and (2.22) for $i = 0$ yield the initial condition in (2.12).

2.C.4 Proof of Proposition 2.3.7

Proof. Consider first the case $\phi_p = 0$ and $\phi_y \geq 0$. This is the same as the special case examined in section (2.3.2) and, thus, the system is indeterminate for all $\phi_y \geq 0$.

Consider now the case that $\phi_p > 0$. The system of difference equations in (2.12) can be inverted, as both a_1 and ξ are finite and positive and $0 < \lambda < 1$, to yield

$$\begin{aligned}
\begin{bmatrix} \delta^y_{i+1} \\ \delta^R_{i+1} \end{bmatrix} &= (C + D(i)) \begin{bmatrix} \delta^y_i \\ \delta^R_i \end{bmatrix} \\
C &= \begin{bmatrix} \lambda & 0 \\ \phi_y \lambda + \frac{\phi_p}{a_1}(1-\lambda) & 1+\phi_p \end{bmatrix} \\
D(i) &= \alpha(i) \begin{bmatrix} (1-\lambda)\left(\frac{1}{a_1} - \xi\right) & 1 \\ (1-\lambda)\left(\frac{1}{a_1} - \xi\right)\left(\phi_y - \frac{\phi_p}{a_1}\right) & \phi_y - \frac{\phi_p}{a_1} \end{bmatrix} \\
\alpha(i) &= \frac{\lambda^{i+2}}{\xi + \lambda^{i+2}\left(\frac{1}{a_1} - \xi\right)}
\end{aligned} \tag{2.23}$$

Using the ratio test, $\sum_{i=0}^{\infty} |\alpha_1(i)| < \infty$ and thus $\sum_{i=0}^{\infty} ||D(i)|| < \infty$, so $D(i)$ satisfies the second assumption of theorems 2.B.1 and 2.B.2.

Turning to the the eigenvalues of C, z_1 and z_2, $z_1 = \lambda$, $z_2 = 1 + \phi_p$. As attention here is restricted to the case $\phi_p > 0$, z_2 is necessarily outside and z_1 is necessarily is inside the unit circle. As such, z_1 and z_2 are necessarily distinct and, as $0 < z_1 < 1$, are both non-zero. Applying theorem 2.B.2, there is a one-to-one correspondence between bounded solutions of the system in question and its diagonalized constant-coefficient counterpart. The latter is saddle-path stable $|z_1| < 1$ and $|z_2| > 1$ and hence the requirement of boundedness will provide the system with an additional restriction to pin down the missing initial condition and rule out sunspot equilibria (i.e., the model is determinate).

Combining the two cases, the requirement of a locally bounded solution will provide the "missing" initial condition (i.e., the system is determinate) when $\phi_p > 0$.

2.D Translating the Given Initial Condition

Following Klein (2000), even if there are exactly as many backward-looking variables as given initial conditions, it need not be the case that the arbitrary initial condition associated with the stable manifold can be "translated" into the given initial condition. With potentially singular time-varying coefficient matrices, a definitive answer to this translatability cannot be given analytically. Thus, although regions of indeterminacy greater than those in standard sticky-price models are analytically proven to exist, regions of determinacy are contingent upon translatability, confirmed for broad parameter spaces numerically.

The missing initial condition that ensures the system lies on the stable manifold can be found by applying theorem 2.B.2 to two-dimensional systems.

Proposition 2.D.1. *For a two-dimensional system $x_{k+1} = A(k)x_k$ that satisfies the assumptions of theorem 2.B.2 with eigenvalues $|z_1| < 1 < |z_2|$, then x_k is bounded iff*

$$[P^{-1}]_2 \left(I_{2x2} + \sum_{j=0}^{\infty} z_2^{-(j+1)} D(j) \prod_{i=0}^{j-1} A(j-i-1) \right) x_0 = 0 \tag{2.24}$$

where P_2 is the second row of the matrix of (right) eigenvectors of C.

Proof. Let $A(k)$ satisfy the assumptions of theorem 2.B.2. Then C can be diagonalized to yield the system $\Theta_{k+1} = \Lambda\Theta_k + \tilde{D}(k)\Theta_k$ with constant-coefficient diagonal counterpart $\Xi_{k+1} = \Lambda\Xi_k$. Assume the lower right entry in Λ lies outside the unit circle. The only bounded solution to Ξ_k^2 (the second element of Ξ_k) is thusly $\Xi_k^2 = 0 \forall k$. Using (2.16),

$$\Theta_0 = \Xi_0 - \sum_{j=0}^{\infty} diag(0, z_2^{-(j+1)}) \tilde{D}(j)\Theta_j \tag{2.25}$$

As the solution of Ξ_k^1 (the first element of Ξ_k) was stable by construction, Ξ_k^1 is indeterminate with respect to boundedness, and, therefore, does not provide a restriction. Thus

$$\Theta_0 = \begin{bmatrix} c \\ 0 \end{bmatrix} - \sum_{j=0}^{\infty} \begin{bmatrix} 0 & 0 \\ 0 & z_2^{-(j+1)} \end{bmatrix} \tilde{D}(j)\Theta_j \tag{2.26}$$

for some initial condition c. With the definitions $P\Theta_k = x_k$ and $x_{k+1} = A(k)x_k$,

$$P^{-1}x_0 = \begin{bmatrix} c \\ 0 \end{bmatrix} - \sum_{j=0}^{\infty} \begin{bmatrix} 0 & 0 \\ 0 & z_2^{-(j+1)} \end{bmatrix} P^{-1}D(j)\prod_{i=0}^{j-1} A(j-i-1)x_0 \tag{2.27}$$

The second row of which is

$$[P^{-1}]_2 \left(I_{2x2} + \sum_{j=0}^{\infty} z_2^{-(j+1)} D(j) \prod_{i=0}^{j-1} A(j-i-1) \right) x_0 = 0 \tag{2.28}$$

For $|z_2| > 1$ from Appendix 2.C to be a sufficient condition, the additional restriction resulting from boundedness must be "translatable" into the missing initial condition.[26] This requires (2.28) to be linearly independent from the other initial condition. Numerical calculations confirm that (2.28) provides a linearly independent relationship generically. Thus, only the trivial, sunspot-free equilibrium will satisfy the boundedness condition (i.e., the model is determinate) when $|z_2| > 1$.

[26]Klein (2000) shows this in the standard case, requiring his Z_{11} matrix to be invertible. Unfortunately, this necessity is often overlooked. For example, Woodford's (2003b, p. 255) Proposition 4.4 implies that an interest rate rule with no feedback on inflation or output ($\phi_\pi = \phi_y = 0$) can still result in determinacy so long as the interest rate process itself is explosive ($\phi_R > 1$). This is analogous to the example worked through by Klein (2000, p. 1419): that the backward-looking interest rate in this case is explosive is insofar useless as this explosiveness cannot be translated away from the backward-looking variable to either of the forward-looking variables.

Chapter 3

Linear Rational-Expectations Models with Lagged Expectations
A Synthetic Method[1]

3.1 Introduction

This chapter presents a method for solving and estimating linear rational-expectations models with lagged expectations. Though the method itself contains little novelty, it contributes to the literature by combining several different methods established in the literature into one coherent approach. The resulting algorithm performs at least as well as each individual method while maintaining generality. The freely available software[2] strives to minimize computing and preprocessing time. I estimate simple sticky-information and sticky-price models using Bayesian methods, evaluating the likelihood function with an alternative to the Kalman filter. Two new contributions of this chapter are the explicit consideration of models with infinite lagged expectations and the examination of truncation methods from the literature for such cases. The method and software should be of special interest to those interested in sticky information à la Mankiw and Reis (2002).

The solution method starts with the method of Taylor (1986), analyzing an infinite moving average solution. The undetermined coefficients approach yields a deterministic non-autonomous system of difference equations. After the largest expectational lag has been included, the system of difference equations becomes autonomous. Standard algorithms for

[1] Published as "Linear Rational-Expectations Models with Lagged Expectations: A Synthetic Method," *Journal of Economic Dynamics and Control,* 34(5), May 2010, pp. 984–1002.

[2] MATLAB® software and examples are available on the author's website.

solving potentially singular systems of difference equations are employed for the coefficients thereafter. Using the infinite moving average solution, the method will provide the unique, stable solution of the problem should it exist. The software provides the option of using the QZ method of Klein (2000) or the shuffle and eigenvalue method of Anderson and Moore (1985). The remaining coefficients are then determined by solving a block version of Mankiw and Reis's (2007) tridiagonal system. This particular synthesis eliminates the need for any manual reformulation and provides a computationally efficient algorithm that draws on preexisting algorithms with established properties.

For models with an infinite number of lagged expectations, e.g., models with the sticky-information Phillips curve of Mankiw and Reis (2002), the method of this chapter uses an explicit convergence criterion to determine when and how to truncate. This is advantageous as current analyses of models containing infinite lagged expectations generally truncate either arbitrarily or through a process of trial and error. The appropriate truncation point will depend not only on the specific model, but also on the specific choice of parameter values. Using an arbitrary truncation point can provide potentially misleading results and using the truncation point derived from one particular parameter combination is unlikely to be suitable when parameter values are changed.

I estimate a simple New Keynesian model with Bayesian likelihood methods, comparing a sticky-information Phillips curve with its sticky-price equivalent. I treat the entire sample as a single draw from a multivariate normal distribution and obtain the covariance matrix using spectral methods. Evaluating the likelihood function requires the determinant and inverse of this matrix, which are calculated recursively using Akaike's (1973) Levinson method for block Toeplitz matrices. I am able to avoid the use of the Kalman filter, which is desirable due to the potentially prohibitive size of the state space underlying the filter when many lagged expectations are present. A similar Levinson algorithm, familiar in the time series literature for the solution of the Yule–Walker equations and other aspects of ARMA estimation, (for a review, see Morettin 1984) was also used by Leeper and Sims (1994, p. 99) to evaluate the likelihood function. The resulting estimates show that the sticky-information model can fare favorably in comparison with the standard sticky-price model, especially in reproducing the empirical lead of the output gap over inflation, and that arbitrary truncation can reverse this assessment.

Many solution methods for solving linear rational-expectation models can be found in the literature. For the analysis here, they can be split into two groups: those that explicitly allow for lagged expectations and those that do not. An incomplete list of the latter includes Blanchard and Kahn (1980), McCallum (1983), Anderson and Moore (1985), Binder and Pesaran (1995), King and Watson (1998), Uhlig (1999), Anderson (2010), Klein (2000), Sims (2001), and Christiano (2002). Although these methods can solve models with a finite number of lagged expectations, this requires the manual definition of dummy variables (see Binder and Pesaran (1995) or Sims (2001)) to bring the system into canonical form.[3] The disadvantage is twofold. First, defining dummy variables is tedious and prone to user error. Second, the computational burden from the increased number of variables can become prohibitive.

There are several solution methods that operate directly with lagged expectations. Taylor (1986) analyzes solutions that take the form of an infinite moving average and Mankiw and Reis (2002) demonstrate how this solution form can be applied to models with lagged expectations in the absence of forward-looking variables. Zadrozny (1998) provides a general method for solving systems with a finite number of lagged expectations, but the absence of a software implementation, as noted by Anderson (2008, p. 96), would require substantial work on behalf of the modeler to use his method. Wang and Wen's (2006) method solves models with lagged expectations by combining standard state-space

[3] Christiano (2002, p. 23) does allow for the information set to vary across equations, but dummy definitions would still be necessary to have varying information sets within one equation.

techniques with a fixed-point approach for expectational errors. Requiring the modeler to manually reformulate lagged expectations as expectational errors reintroduces the potential for user error. Their fixed-point approach is unnecessarily complicated and computationally burdensome. Finally, Mankiw and Reis (2007) provide a method that works entirely on the infinite moving average representation. By reducing the system of equations to a scalar second-order non-autonomous difference equation and imposing a boundary condition at a finite horizon, they reduce the problem to solving a tridiagonal system. While the method could be altered to avoid the reformulation into a scalar system, it is unclear how and when the boundary conditions for a vector of variables should be imposed in more general settings. None of these methods give an explicit criterion for how to proceed when lagged expectations reach back into the infinite past.

The chapter is organized as follows. Section 3.2 presents the model to be analyzed. Section 3.3 derives the solution method. Section 3.4 examines the dangers associated with truncations. Section 3.5 compares the method and its performance with alternate methods. Section 3.6 presents the method used for estimation and section 3.7 examines the importance of lagged expectations using estimated sticky-information and sticky-price New Keynesian models. Finally, section 3.8 concludes.

3.2 Statement of the Problem

Log-linearized economic models can typically be represented by a system of linear expectational difference equations:

$$\begin{aligned} 0 = & \sum_{i=0}^{I} A_i E_{t-i}[Y_{t+1}] + \sum_{i=0}^{I} B_i E_{t-i}[Y_t] + \sum_{i=0}^{I} C_i E_{t-i}[Y_{t-1}] \\ & + \sum_{i=0}^{I} F_i E_{t-i}[W_{t+1}] + \sum_{i=0}^{I} G_i E_{t-i}[W_t] \end{aligned} \tag{3.1}$$

$$W_t = \sum_{j=0}^{\infty} N_j \epsilon_{t-j},\ \epsilon_t \sim i.i.d.\mathcal{N}(0,\Omega) \tag{3.2}$$

$$\lim_{j\to\infty} \xi^{-j} E_t \left[Y_{t+j}\right] = 0,\ \forall \xi \in \mathbb{R}\ s.t.\ \xi > g^u,\ \text{where } g^u \geq 1 \tag{3.3}$$

where Y_t is a $k \times 1$ vector of endogenous variables of interest, W_t an $n \times 1$ vector of exogenous processes stable[4] with respect to ξ and with given moving average coefficients $\left\{N_j\right\}_{j=0}^{\infty}$, and where $I \in \mathbb{N}_0$. It is assumed that there are as many equations, k, as endogenous variables in (3.1). Variables dated t are in the information set dated t.

Equation (3.1) represents the collection of log-linearized equilibrium equations. Equation (3.2) specifies the exogenous process W_t as a vector $MA(\infty)$ process. Equation (3.3) may be interpreted as a transversality condition derived as a condition from intertemporal maximization, where g^u is the maximal growth rate of endogenous variables (see Sims (2001) or Burmeister (1980) for discussions on the limitations of this interpretation).

3.3 Solution of the Problem

In the following, I shall differentiate between three cases: (i) $I = 0$, (ii) $0 < I < \infty$, and (iii) $I \to \infty$. The distinction between the first two serves to compare the solution here with methods in the literature for standard (i.e., without lagged expectations) formulations. The infinite case will need to be handled separately and provides a justification and criterion for appropriate truncations.

Following Muth (1961) and Taylor (1986), the solution will take the form

$$Y_t = \sum_{j=0}^{\infty} \Theta_j \epsilon_{t-j}. \tag{3.4}$$

[4]E.g., for $g^u = 1$, unit roots in both the endogenous and exogenous variables are allowed.

Inserting (3.4) and (3.2) into (3.1) yields

$$0=\sum_{j=0}^{\infty}\left(\sum_{i=0}^{min(I,j)}A_i\right)\Theta_{j+1}\epsilon_{t-j}+\sum_{j=0}^{\infty}\left(\sum_{i=0}^{min(I,j)}B_i\right)\Theta_j\epsilon_{t-j}$$
$$+\sum_{j=0}^{\infty}\left(\sum_{i=0}^{min(I,j+1)}C_i\right)\Theta_j\epsilon_{t-j-1}+\sum_{j=0}^{\infty}\left(\sum_{i=0}^{min(I,j)}F_i\right)N_{j+1}\epsilon_{t-j}$$
$$+\sum_{j=0}^{\infty}\left(\sum_{i=0}^{min(I,j)}G_i\right)N_j\epsilon_{t-j} \tag{3.5}$$

Defining $\tilde{M}_j=\sum_{i=0}^{min(I,j)}M_i$, for $M=A,B,C,F,G$, one can rewrite the foregoing as

$$0=\sum_{j=0}^{\infty}\tilde{A}_j\Theta_{j+1}\epsilon_{t-j}+\sum_{j=0}^{\infty}\tilde{B}_j\Theta_j\epsilon_{t-j}+\sum_{j=0}^{\infty}\tilde{C}_{j+1}\Theta_j\epsilon_{t-j-1}$$
$$+\sum_{j=0}^{\infty}\tilde{F}_jN_{j+1}\epsilon_{t-j}+\sum_{j=0}^{\infty}\tilde{G}_jN_j\epsilon_{t-j} \tag{3.6}$$

Comparing coefficients, this yields the non-stochastic linear recursion

$$0=\tilde{A}_j\Theta_{j+1}+\tilde{B}_j\Theta_j+\tilde{C}_j\Theta_{j-1}+\tilde{F}_jN_{j+1}+\tilde{G}_jN_j \tag{3.7}$$

with initial conditions, $\Theta_{-1}=0$, and terminal conditions from (3.3), $\lim_{j\to\infty}\xi^{-j}\Theta_j=0$. The initial conditions require Y_{t-1} to be independent of ϵ_t (i.e., variables from yesterday cannot respond to innovations today), leaving an additional k restrictions for the terminal conditions to determine.

3.3.1 Case 1: $I=0$

This is the standard case without lagged expectations. Here $\tilde{M}_j=M_0$, for $M=A,B,C,F,G$, and thus (3.7) reduces to a recursion with constant coefficients

$$0=A_0\Theta_{j+1}+B_0\Theta_j+C_0\Theta_{j-1}+F_0N_{j+1}+G_0N_j \tag{3.8}$$

This system of deterministic difference equations can be solved using standard methods. In 3.A, I follow Klein (2000) and note how his approach can be adapted to the deterministic and potentially non-stationary system.

The QZ method of Klein (2000) or Anderson and Moore's (1985) AIM method yields

$$\Theta_j = \alpha\Theta_{j-1} + \beta\Xi_j^u \tag{3.9}$$

a recursive form, along with the initial conditions, for the MA-coefficients of Y_t.[5]

Following Blanchard and Kahn (1980), the solution will be unique if the system has exactly k eigenvalues greater than g^u in modulus, non-unique if there are more than k such eigenvalues, and non-existent if there are fewer than k such eigenvalues. With exactly k eigenvalues greater than g^u in modulus, the terminal conditions (3.2) should provide the missing k linear restrictions needed to complete the recursion.[6]

The algorithm uses standard recursive methods for potentially singular difference equation systems. Whereas current standard methods (e.g., Uhlig (1999) or Klein (2000)) solve for a recursive form for the endogenous variables themselves, the solution form here yields a recursive form for the infinite-MA coefficients, following the representations proposed by Muth (1961) and Taylor (1986). This approach transforms the stochastic system of difference equations into a deterministic system in the impulse responses of endogenous variables to exogenous shocks ϵ_t. Both non-existence and non-uniqueness of the fundamental solution will be indicated by the non-existence or non-uniqueness of the stable manifold. The algorithm is mute on the form of the solution(s), should the MA representation be non-unique or non-existent. The infinite-MA representation and associated deterministic system of difference equations avoids any expansion of the state space (see Mankiw and Reis 2002, p. 1325) and still admits the use of standard methods when lagged expectations are present. The following section will develop this case.

[5]Where Ξ_j^u is a convolution of the unstable manifold and the exogenous moving average, see appendix 3.A.

[6]A caveat is noted by Klein (2000): the explosive eigenvalues have to be "translatable" to the missing initial conditions. If the upper-left $k \times k$ block of Z from the QZ decomposition in 3.A is invertible, however, this will not be a problem.

3.3.2 Case 2: $0 < I < \infty$

This is a deviation of the standard case examined in the literature. Here $\tilde{M}_j = \tilde{M}_I$, for $M = A, B, C, F, G$ and $\forall j \geq I$, and thus (3.7) reduces to a recursion with constant coefficients $\forall j \geq I$.

$$0 = \tilde{A}_I \Theta_{j+1} + \tilde{B}_I \Theta_j + \tilde{C}_I \Theta_{j-1} + \tilde{F}_I N_{j+1} + \tilde{G}_I N_j \tag{3.10}$$

Analogously to the previous section, if $s = k$ and if the k restrictions can be associated with the "missing" initial conditions,

$$\Theta_j = \alpha(I)\Theta_{j-1} + \beta(I)\Xi_j^u(I),\ \forall j \geq I \tag{3.11}$$

a recursive solution for all MA-coefficients from I onward.[7]

The remaining coefficients can then be obtained as the solutions to

$$\begin{bmatrix} \tilde{B}_0 & \tilde{A}_0 & 0 & & & \dots & & 0 \\ \tilde{C}_1 & \tilde{B}_1 & \tilde{A}_1 & 0 & & \dots & & 0 \\ 0 & \tilde{C}_2 & \tilde{B}_2 & \tilde{A}_2 & 0 & \dots & & 0 \\ \vdots & & & & & & & \vdots \\ 0 & & \dots & & 0 & \tilde{C}_{I-1} & \tilde{B}_{I-1} & \tilde{A}_{I-1} \\ 0 & & & \dots & & 0 & -\alpha(I) & I \end{bmatrix} \begin{bmatrix} \Theta_0 \\ \Theta_1 \\ \Theta_2 \\ \vdots \\ \Theta_{I-1} \\ \Theta_I \end{bmatrix} = \begin{bmatrix} -\tilde{F}_0 N_1 - \tilde{G}_0 N_0 \\ -\tilde{F}_1 N_2 - \tilde{G}_1 N_1 \\ -\tilde{F}_2 N_3 - \tilde{G}_2 N_2 \\ \vdots \\ -\tilde{F}_{I-1} N_I - \tilde{G}_{I-1} N_{I-1} \\ \beta(I)\Xi_I^u(I) \end{bmatrix} \tag{3.12}$$

The left-hand side is a block extension of Mankiw and Reis's (2007) tridiagonal structure, readily exploitable numerically. (See Golub and van Loan 1989, p. 170)

As in the case when $I = 0$, the method here provides a linear recursion for the infinite MA coefficients for $j \geq I$. So long as I is finite, the inclusion of lagged expectations extends standard solution methods by a sparse system of equations for all coefficients up to I. Standard state-space methods, however, would extend the state space with dummy variables to capture the effects of lagged expectations.

[7] Where $\Xi_j^u(I)$ is given by (3.36), see 3.A to compare with the solution from section 3.3.1.

3.3.3 Case 3: $I \to \infty$

Unlike the previous two cases, (3.7) cannot be reduced to a linear recursion with constant coefficients for $j \geq I$. Assuming that (where l and m denote row and column)

$$\lim_{j\to\infty} \left(\tilde{M}_j\right)_{l,m} = \left(\tilde{M}_\infty\right)_{l,m}, \text{ for } M = A, B, C, F, G; \text{ and } \forall l, m \tag{3.13}$$

exists and is finite, then there exists, by the definition of a limit in $\mathbb{R}^1$, some $I(\delta)_{M,l,m}$ for each M, l, and m, such that

$$\forall \delta > 0,\ \exists I(\delta)_{M,l,m}\ s.t.\ n > I(\delta)_{M,l,m} \Rightarrow |\left(\tilde{M}_n\right)_{l,m} - \left(\tilde{M}_\infty\right)_{l,m}| < \delta \tag{3.14}$$

and, thus, there exists some maximal $I(\delta)_{max} = max\{I(\delta)_{M,l,m}\}$ such that

$$\forall \delta > 0,\ \exists I(\delta)_{max}\ s.t.\ n > I(\delta)_{max} \Rightarrow |\left(\tilde{M}_n\right)_{l,m} - \left(\tilde{M}_\infty\right)_{l,m}| < \delta;\ \forall M, l, m \tag{3.15}$$

Therefore, (3.7) can be approximated as

$$0 = \tilde{A}_j\Theta_{j+1} + \tilde{B}_j\Theta_j + \tilde{C}_j\Theta_{j-1} + \tilde{F}_jN_{j+1} + \tilde{G}_jN_j,\ 0 \leq j < I(\delta)_{max} \tag{3.16}$$

$$0 = \tilde{A}_\infty\Theta_{j+1} + \tilde{B}_\infty\Theta_j + \tilde{C}_\infty\Theta_{j-1} + \tilde{F}_\infty N_{j+1} + \tilde{G}_\infty N_j,\ j \geq I(\delta)_{max} \tag{3.17}$$

This system is now analogous to the system in the previous section where I now equals $I(\delta)_{max}$ and can be solved using the methods presented there.

The main distinction is that the autonomous recursion is defined by the limiting coefficients ($I \to \infty$) and not by the finite $I = I(\delta)_{max}$ coefficients. As the behavior of the system in the limit is decisive for the application of (3.3) to ascertain whether additional restrictions to determine the system exist, the use of coefficients other than those of the limiting case can produce spurious results regarding asymptotic stability.

The existence and uniqueness of the stable manifold now depends on the eigenvalues of the system of these limiting coefficients. The assumption of element-wise limits in the coefficient matrices rules out the possibility of asymptotically periodic coefficients and ensures that any desired degree of accuracy, through an appropriate choice of δ, can be achieved without endangering the asymptotic behavior of the recursion.

3.3.4 A Recursive Law of Motion

For a recursive law of motion, the infinite-MA solution can be rewritten as

$$Y_t = \sum_{j=0}^{\infty} \Theta_j \epsilon_{t-j} = \sum_{j=0}^{I-1} \Theta_j \epsilon_{t-j} + \sum_{j=I}^{\infty} \Theta_j \epsilon_{t-j} \tag{3.18}$$

Assuming the MA coefficients of the exogenous process W_t follow the simple recursion $N_{j+1} = NN_j$ with all eigenvalues of N less than or equal to g^u,[8] a recursive law of motion can be derived as all MA coefficients after I follow an autonomous recursion.[9] From equations (3.11) and (3.36), as well as Klein (2000, p. 1423),

$$\Theta_j = \alpha(I)\Theta_{j-1} + \beta(I)M(I)N^j,\ j \geq I \tag{3.19}$$

where $vec(M(I)) = (N' \otimes S_{22}(I) - I \otimes T_{22}(I))^{-1}\, vec\left(Q_2(I)\begin{bmatrix} \tilde{F}_I N + \tilde{G}_I \\ 0 \end{bmatrix}\right)$.

Defining $U_t = \sum_{j=I}^{\infty} \Theta_j \epsilon_{t+I-j}$, the solution with a VAR(1) exogenous process is

$$\begin{aligned} Y_t &= \sum_{j=0}^{I-1} \Theta_j \epsilon_{t-j} + U_{t-I} \\ U_{t-I} &= \alpha(I)(U_{t-I-1} + \Theta_{I-1}\epsilon_{t-I}) + \beta(I)M(I)N^I W_{t-I} \\ W_t &= NW_{t-1} + \epsilon_t \end{aligned} \tag{3.20}$$

with U_{t-I} being the same as $E_{t-I}[Y_t]$. Note that if $I = 0$, the foregoing reduces to

$$\begin{aligned} Y_t &= \alpha(0)Y_{t-1} + \beta(0)M(0)W_t \\ W_t &= NW_{t-1} + \epsilon_t \end{aligned} \tag{3.21}$$

a standard form for the recursive law of motion.

[8] I.e., W_t is a stable VAR(1) process.

[9] Recalling from the foregoing section that $I = I(\delta)_{max}$ with infinitely lagged expectations.

3.4 The Perils of Premature Truncation

Solving linear rational-expectations models when many lagged expectations appear in the structural equations generally entails a truncation of the number of lagged expectations included in the model. In this section, I shall explore the implications of two methods of truncation to show that truncation is not generally innocuous.

The sticky-information model of Mankiw and Reis (2002), incorporating an infinite sum of lagged expectations, has presented the literature with an alternative to the sticky-price Phillips curve. Andrés, López-Salido, and Nelson (2005), Keen (2007), and Wang and Wen (2006) are a few examples of models that combine forward-looking agents and an infinity of lagged expectations: all of them truncate this infinity with the truncation point ranging from 3, Andrés, López-Salido, and Nelson (2005), to 50, Wang and Wen (2006). Kiley (2007, p. 112) compares sticky prices and sticky information empirically and notes,“ [i]n practice, the longest information lag is truncated at four quarters.” Using a truncated version to draw inference on the infinite specification requires that the former yield results that do not differ substantially from the latter. The methods in the previous section allow for a clear picture (to machine precision) of the infinite specification, allowing for the assessment of the generally arbitrary truncations found in the literature. These arbitrary truncations can distort the dynamics of the model, even to an extent apparent by visual inspection.

The sticky-information Phillips curve of Mankiw and Reis (2002) is[10]

$$\pi_t = \frac{\lambda\alpha}{1-\lambda} y_t + \lambda \sum_{j=0}^{\infty} (1-\lambda)^j \, E_{t-j-1}\left[\pi_t + \alpha \Delta y_t\right] \tag{3.22}$$

with π_t, Δy_t, and y_t being the gross inflation rate, the growth of the output gap, and the output gap itself. Equation (3.22) is the sticky-information Phillips curve and, as it is the only equation here to contain lagged expectations, will be the focus of the examination of

[10]The system is closed by an AR(1) process for the growth of money and the quantity equation in first-difference form.

the consequences of truncation. Note that this model does not contain any forward-looking variables. As Mankiw and Reis (2002) show, this allows the MA coefficients to be solved for directly. This yields an analytical solution that facilitates comparison of truncation methods.

Andrés, López-Salido, and Nelson (2005, p. 1033) note that to make the model tractable, "[they] approximate it by truncating [lagged expectations in the Phillips curve] at three quarters."[11] Using this truncation[12] alters equation (3.22) to

$$\pi_t = \frac{\lambda\alpha}{1-\lambda} y_t + \lambda \sum_{j=0}^{3} (1-\lambda)^j E_{t-j-1}\left[\pi_t + \alpha\Delta y_t\right] \tag{3.23}$$

Kiley (2007, p. 112) follows a different truncation technique and states, "the probabilities of information arrival are constant in each period up to the truncation period, with the remaining mass of the probability distribution placed on the last period." Following this truncation, equation (3.22) is rewritten as

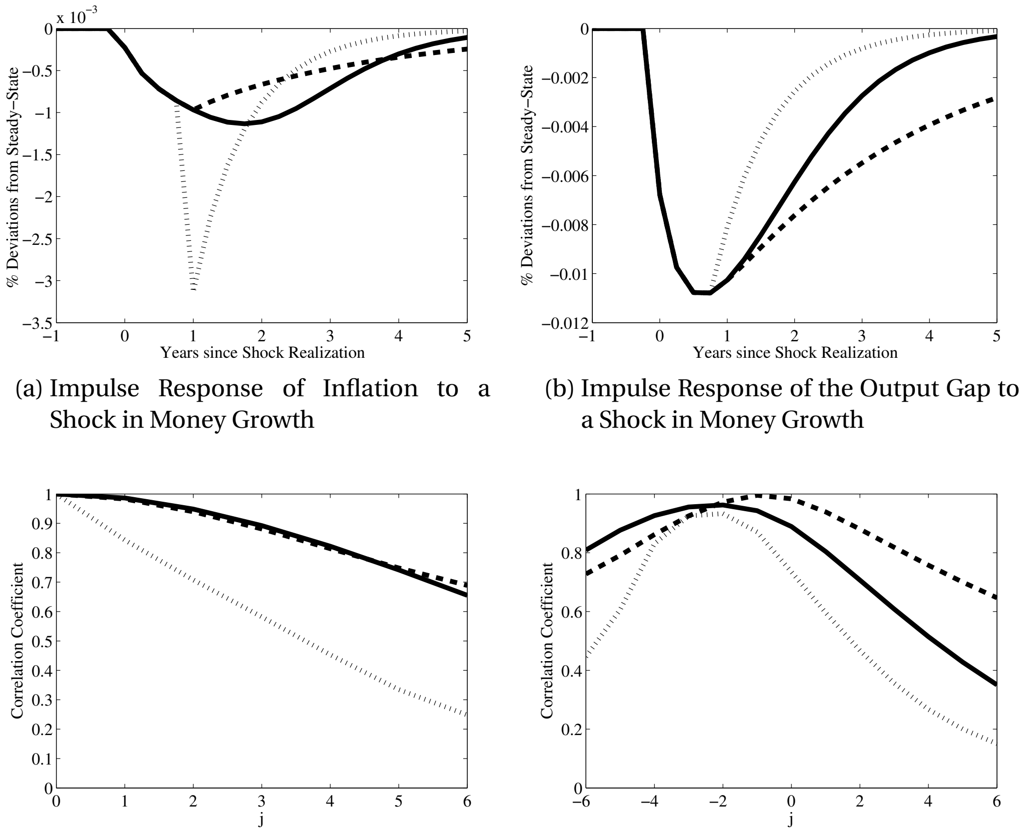
$$\pi_t = \frac{\lambda\alpha}{1-\lambda} y_t + \lambda \left(\sum_{j=0}^{2} (1-\lambda)^j E_{t-j-1}\left[\pi_t + \alpha\Delta y_t\right] + \frac{(1-\lambda)^3}{\lambda} E_{t-4}\left[\pi_t + \alpha\Delta y_t\right] \right) \tag{3.24}$$

Figure 3.1, clockwise from the upper-left panel, shows the impulse responses of inflation to a negative shock to the money growth rate, the impulse responses of the output gap to the same, the cross-correlations of the output gap with contemporary inflation, and the autocorrelation of inflation for the two approximations and the original specification of Mankiw and Reis (2002).[13] As the model does not contain any forward-looking behavior, the initial responses of inflation and the output gap are the same in all three versions. The second truncation, equation (3.24), displays a sharp jump in the response of inflation four periods

[11] Note that Andrés, López-Salido, and Nelson (2005, p. 1038) interpret the parameter for the probability of the arrival of new information according to the non-truncated version and concludes that its estimated value "leads to an average duration slightly higher than six quarters" despite their truncation point.

[12] I extend the truncation point to four quarters as in Kiley (2007) for comparability.

[13] The solutions in this section not labeled as a truncation are implemented using the method developed here with δ (see section 3.3.3) set to floating point accuracy. This level of tolerance implies that the computer is no longer capable of distinguishing between the autonomous recursion from the limiting coefficients and the non-autonomous recursion continued past $I(\delta)_{max}$, see equation (3.16). As an anonymous referee pointed out, the results derived thusly for Mankiw and Reis's (2002) model are indistinguishable from their original results.



(a) Impulse Response of Inflation to a Shock in Money Growth

(b) Impulse Response of the Output Gap to a Shock in Money Growth

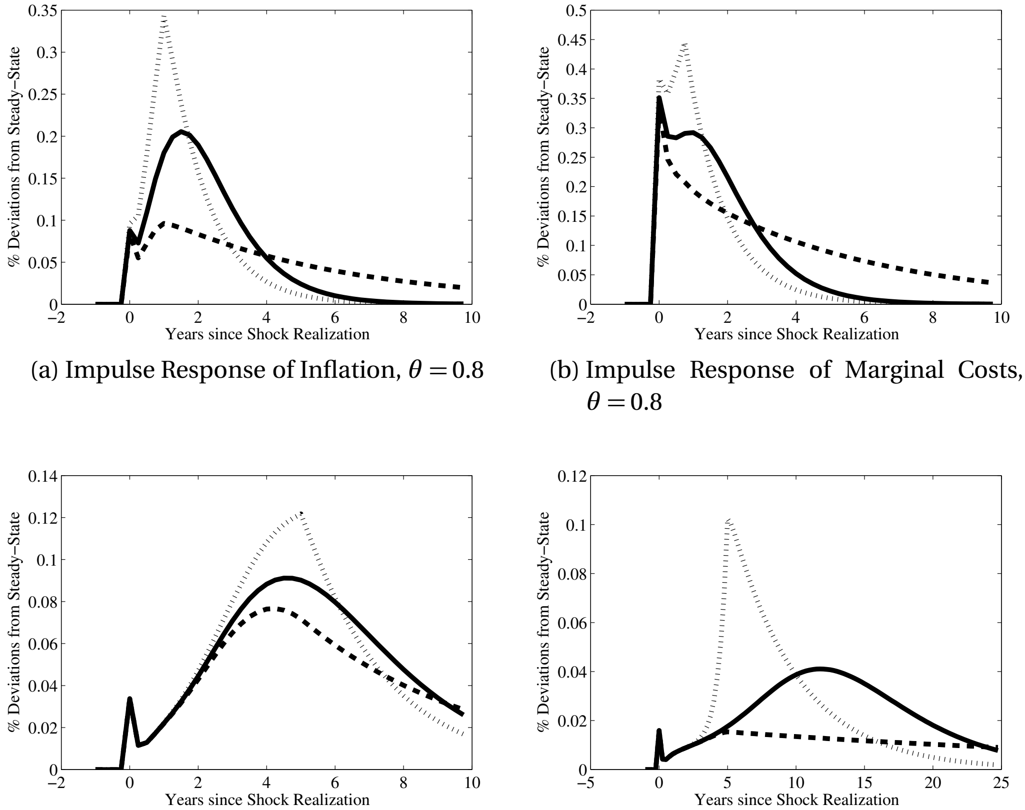
(c) Autocorrelation of Inflation

(d) Cross-correlation of the Output Gap at $t+j$ with Inflation at t

Figure 3.1: Consequences of Truncation in the Model of Mankiw and Reis (2002); Solid Line - Method Here, Dashed - Truncation Type 1, Dotted - Truncation Type 2

after the shock, due to the large weight attached to lagged expectations at this horizon. Neither of the truncations can reproduce the maximal response of inflation at seven quarters. The impulse response of the output gap shows the transition of the rate of convergence of the output gap from the first truncation, equation (3.23), to the second. The second truncation underestimates the cross-correlation of the output gap with inflation and the autocorrelation of the latter. The first specification matches the autocorrelation of inflation within the displayed horizon remarkably well, though it misses the horizon of the lead of the output gap in the cross-correlation.

When forward-looking behavior is added to the model, responses will depend on future trajectories. The first example of Wang and Wen (2006) will serve to illustrate the issue. They present a simple model with sticky information and monopolistic competition on the supply side (leading to a sticky-information Phillips curve) and capital accumulation, a cash-in-advance constraint, bond holdings, and labor and consumption decisions maximized intertemporally on the demand side.



(a) Impulse Response of Inflation, $\theta = 0.8$

(b) Impulse Response of Marginal Costs, $\theta = 0.8$

(c) Impulse Response of Inflation, $\theta = 0.9$

(d) Impulse Response of Inflation, $\theta = 0.95$

Figure 3.2: Consequences of Truncation (I=20) in the First Model of Wang and Wen (2006); Impulse Responses to a Shock in Money Growth; Solid Line - Method Here, Dashed - Truncation Type 1, Dotted - Truncation Type 2

Using the truncations presented above, the first row of figure 3.2 shows the impulse responses of inflation and marginal costs, replacing the output gap in (3.22), to a positive unit innovation in the money growth rate. Both truncation methods[14] fail to match the peak response of inflation, with responses differing now both before and after truncation. The impulse responses of marginal costs demonstrate similar short-comings.

Wang and Wen (2006, p. 10) note, for their sticky-information model, that a truncation point of 20 provides "very good results." Trabandt (2007) draws the same conclusion in his model—though with an explicit convergence criterion, see section 3.5. There is, of course, nothing intrinsically special about a truncation point of 20. The second row of figure 3.2 shows the consequences of the two truncation methods with truncation points of 20 for the impulse response of inflation to a shock in money growth in the sticky-information model of Wang and Wen (2006) when θ[15] is increased from 0.8 to 0.9, figure 3.2c, and to 0.95, figure 3.2d. Both truncations miss the peak response of inflation and either over- or underestimate the autocorrelation of inflation.

Mankiw and Reis (2007) analyze a DSGE model with pervasive inattention. Their aggregate supply curve is given in terms of the price level instead of inflation:

$$p_t = \lambda \sum_{j=0}^{\infty} (1-\lambda)^j \, E_{t-j}\left[p_t + x_t\right] \tag{3.25}$$

where p_t is the price level and x_t is a composite term comprising real marginal costs and desired markups. In their appendix, they show that the price level displays unit-root behavior in the limit. Figure 3.3 shows the impulse response of the price level to a shock in aggregate demand for the two truncation types with differing truncation points. Notice that first type of truncation fails to deliver the unit root, whereas the second type does not. Examining (3.25), if the infinite sum is simply truncated (i.e., type-one truncation) without correcting for the remaining probability mass (i.e., type-two truncation), the supply curve would imply a long-run relation between the price level and the composite term. As both real marginal costs and desired markups are stationary, this forces the price level itself to become stationary. Using

[14]Again, truncation is imposed at four quarters.

[15]Their equivalent to $1-\lambda$ from (3.22)—the probability of not receiving an information update.

type-two truncation delivers the unit root in prices regardless of the truncation point, as the limiting coefficients which contain the unit root are imposed after truncation. Premature truncation with the type-two method, however, can still lead to an erroneous new steady state of the price level.

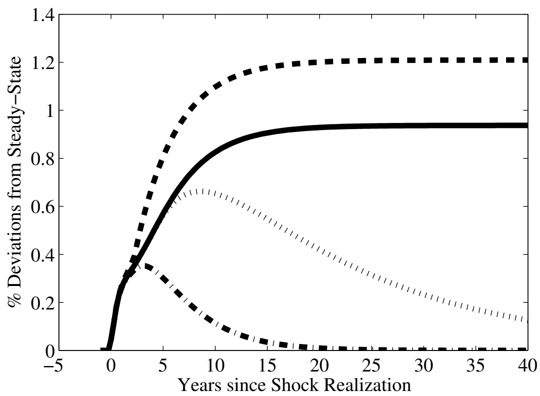


Figure 3.3: Impulse Response of the Price Level in the Model of Mankiw and Reis (2007); Truncation Type 2 (Solid Line - I=100, Dashed - I=10), Truncation Type 1 (Dotted - I=20, Dash-Dotted - I=10)

Different truncation methods can have different consequences, which themselves might differ when applied to different models or parameter settings. Both of the truncation methods presented above, as found in the literature, will converge to the true model as the truncation point is extended towards infinity. Knowing a priori when and how to truncate would seem difficult to ascertain. The method presented in this chapter automatically calculates the truncation point given the tolerance parameter δ. As the setting changes, so too will the truncation point, eliminating the need for arbitrary truncations or processes of trial and error to deliver a clear picture of the model's dynamics.

3.5 Comparison of Solution Methods

In this section, I compare the solution method presented in section 3.3 with three alternate methods in the literature for solving models with lagged expectations. The method dominates all three alternative solution methods in terms of computation time and/or implementation time on behalf of the user for given error tolerances. In the absence of lagged expectations, the method here collapses to the method of Klein (2000)[16] and a comparison of its performance with other methods can be found in Anderson (2008).

For the comparison that follows, computation times and relative errors, following Golub and van Loan (1989, p. 54) and using the Euclidean norm, of impulse responses between the alternative solution methods and the method here are reported. Trabandt (2007) provides computation times but his software is not publicly available, so the comparison is restricted to comparing computation time across platforms. For both Wang and Wen (2006) and Mankiw and Reis (2007), software is publicly available from the authors' websites and is used to compare computation times and calculated impulse responses on the same platform.[17]

Trabandt (2007) uses the QZ implementation of Uhlig (1999) to solve a model with sticky information by expanding the state vector. That is, a variable $E_{t-1}[x_t]$ is modeled by defining $x^1_{t-1} = E_{t-1}[x_t]$ and adding the additional equation $x^1_t = E_t[x_{t+1}]$. While this method has the advantage of using standard methods, it requires the definition of an ever-increasing state vector by including additional variables and equations. This increases not only the dimension of the problem but also the programming time, as these lagged expectations must be manually redefined using additional variables. Notably, Trabandt's (2007) method is the only method besides the one presented here that works with an explicit convergence criterion for imposing truncation. The criterion adds additional lagged expectations one-by-one, re-solving the model until the solution does not change more than a specified tolerance. While preferable to an arbitrary truncation, this is computationally expensive compared with the criterion in section 3.3.3, which determines the truncation point before solving the model, and it is not clear whether his method can be applied in other settings. Trabandt (2007,

[16] Or of Anderson and Moore (1985) if the option to solve using AIM is chosen.

[17] Platform used: Pentium® IV 3 GHz machine with 2 GB of RAM running MATLAB® version R2007a under Windows® XP 2002 SP 2.

p. 18) requires three minutes to solve his model with twenty lagged expectations included; the method here requires about one-and-a-half hundredths of a second to do the same.

The computational disadvantage of methods based on state-vector expansion is due to the computation costs associated with a QZ decomposition, a function of the cube of the state vector. (Golub and van Loan 1989, p. 404) Anderson and Moore's (1985) AIM method presents an alternative method and their method generally entails significant reductions in computation times.[18] Mathias Trabandt[19] notes that using the AIM method reduces his computation time to 1.75 seconds. Though an improvement, this is still more than two orders of magnitude slower than the method developed here. One advantage of the method presented in this chapter, as with Wang and Wen (2006) and Mankiw and Reis (2007), comes from its division of the problem into an autonomous and a non-autonomous part. The dimensions of the autonomous recursion of section 3.3.3 are invariant to the inclusion of lagged expectations.

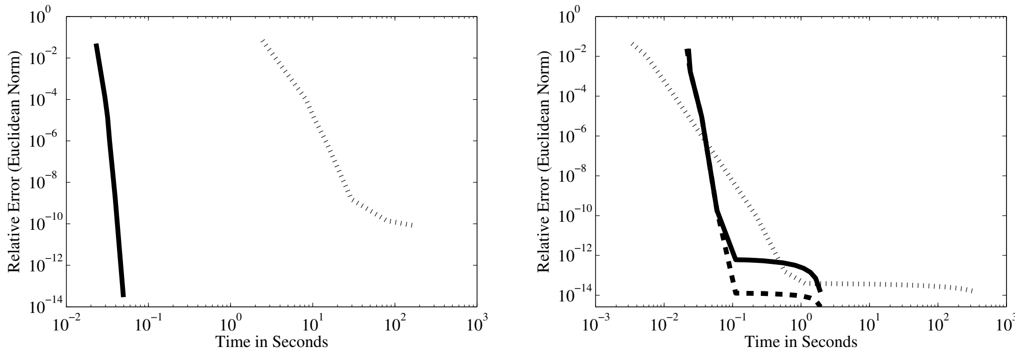
Wang and Wen (2006) present a method for solving linear rational-expectations models with lagged expectations that is similar to the solution presented here in several ways. In contrast to the method here, however, the authors work directly with a recursion in state variables and solve for the forecast errors induced by lagged expectations. By approximating models with lagged expectations reaching back into the infinite past with a finite number of forecast errors, Wang and Wen (2006) impose the same condition that is imposed in the method presented here.[20] The method in their paper, however, requires the modeler to reformulate lagged expectations into expectation errors, opening a window for user error. Furthermore, the combination of the recursion in state variables with forecast errors poses a more complicated fix-point problem than the tridiagonal problems posed by Mankiw and

[18] See Anderson (2008, p. 102).

[19] Personal communication.

[20] As pointed out by an anonymous referee, the software of Wang and Wen (2006) uses a different method than presented in their paper, truncating the lagged expectations themselves rather than the expectation errors.

Reis (2007) working with the innovations representation. The fix-point problem seems to limit Wang and Wen's (2006) method in terms of accuracy and is also a likely culprit for the rather excessive increase in computation time as $nlag$, the authors' parameter for the number of lagged expectations included, is increased.



(a) Solid Line - Method Here (QZ), Dotted - Wang and Wen (2006)

(b) Solid Line - Method Here (QZ), Dashed - Method Here (AIM), Dotted Mankiw and Reis (2007)

Figure 3.4: Computation Time versus Accuracy, Log Scale

In figure 3.4a, for varying values of δ (see section 3.3.3) and $nlag$, the computation time and relative errors associated with the first example in Wang and Wen's (2006) paper are compared. The relative errors contrast the impulse responses of the model's variables to a shock to the rate of money growth with varying truncation points against the impulses obtained when using δ equal to floating-point accuracy and $nlag = 252$. This value of $nlag$ is used, as Wang and Wen's (2006) method required more than one hour with this value to calculate the solution. As can be seen in the figure, the method proposed here solves the model for a given relative error at least 100 times more quickly than the method of Wang and Wen (2006).

The methods proposed by Trabandt (2007) and Wang and Wen (2006) differ subtly. Wang and Wen (2006) propose a solution for a finite number of lagged expectation errors, whereas Trabandt's (2007) algorithm applies to a finite number of lagged expectations. When a truncation is necessary, Trabandt's (2007) solution is equivalent to (3.23) and Wang and Wen's (2006) to (3.24).[21] Truncating the lagged expectations themselves does not preserve the asymptotic qualities of the recursion. The methods proposed by Wang and Wen (2006) and implemented by this chapter, by contrast, do preserve these asymptotic qualities. Although, in the limiting case when the truncation points go to infinity, all three approaches are theoretically equivalent, a method that preserves these asymptotic qualities would seem more appropriate for numerical application.

Mankiw and Reis (2007) develop a solution method from the MA representation that differs in two major respects from the method presented here. Firstly, they reduce the problem to a univariate second-order non-autonomous difference equation. Secondly, Mankiw and Reis (2007) solve the model by imposing a boundary condition prematurely (i.e., in finite time). Reducing the system to a scalar system requires considerable work on behalf of the modeler and, as such, is liable to user error. Furthermore, it is not clear that every model can be reduced to a scalar second-order difference equation and, absent such a reduction, it is not obvious that all the boundary conditions ought to be imposed at the same point. The method developed here neither requires manual reduction nor does it impose a univariate structure. Both the method of Mankiw and Reis (2007) and the method here exploit readily available and fast implementations of Gaussian elimination to solve a (block) tridiagonal system. That the autonomous recursion consistent with the limiting coefficients is imposed instead of the boundary conditions themselves allows fewer non-autonomous coefficients to be added to achieve a given relative error.

In figure 3.4b, for varying values of δ (see section 3.3.3) and N, the authors' parameter for the number of MA coefficients included before the boundary conditions are imposed, the computation time and relative errors associated with solving the model in Mankiw and

[21]The method actually implemented by Wang and Wen (2006) in their software, however, is equivalent to (3.23), see footnote 20.

Reis (2007) are compared. The method of this chapter entirely avoids the several pages of "tedious algebra" from the technical appendix of Mankiw and Reis (2007) to arrive at their solution. That Mankiw and Reis's (2007) method solves the model more quickly than the method presented here for large relative errors is likely due to the initial fixed costs of the higher level of generality of the method here. The method presented in this chapter, however, requires a smaller increase in computational time for a given increase in the level of accuracy; at some level of accuracy, the method here surpasses that of Mankiw and Reis (2007) in terms of computation time. Numerical limitations on the QZ decomposition, as discussed by Anderson (2008, p. 103), can be reduced by solving for the limiting recursion using Anderson and Moore's (1985) AIM method, as can be seen in figure 3.4b. Less than two seconds are needed to solve the model using the convergence criterion $\delta = eps(0)$, including thrice as many lagged expectations in half as much time. As the method derived here is not model specific and does not require any manual reformulation on behalf of the user, it appears to be at an advantage.

When the model to be analyzed possesses no lagged expectations, the method here fits within the class of solution methods used throughout the literature. For models with lagged expectations, the method derived in this chapter is superior to current models with respect to computation and/or implementation times. The method here is non-model-specific and can be readily applied to existing and new DSGE models both with and without lagged expectations efficiently.

3.6 Likelihood Estimation

The use of Bayesian likelihood methods to explore the unknown posterior distribution of DSGE models' parameters given the data and the researcher's prior beliefs has been gaining popularity. (See An and Schorfheide, (2007), for an overview) The iterative nature of the methods highlights the advantages of having a fast solution method: the solution will need to be recalculated thousands if not millions of times.

One difficulty in implementing likelihood methods, aside from the calculation of a recursive solution as presented in section 3.3, lies in the evaluation of the likelihood function. The Kalman filter, frequently used to obtain a prediction error decomposition of the likelihood function, is undesirable here. Using the Kalman filter would require the state space of the recursive laws of motion in section 3.3.4 to be expanded to accommodate a first-order form. The resulting recursion would calculate products and inverses on a dimension equal to the expanded state space. An alternative approach is developed here based on the Toeplitz structure of the covariance of stationary time series to yield an iterative method for evaluating the likelihood function by treating the sample as a single draw from a multivariate normal distribution.[22] It should be noted that as the standard model (i.e., without lagged expectations) is included as a special case of the class of models considered here, the method can be used as an alternative to the Kalman filter for the likelihood estimation of standard DSGE models.[23]

The recursive laws of motion in section 3.3.4 will provide the basis for estimation. The block-Levinson-type algorithm in 3.B relies on there being a stationary innovations representation of the observables as discussed in Anderson and Moore (1979, Ch. 9), so I restrict the eigenvalues of $\alpha(I)$ and N in the recursive law of motion (3.20) to lie inside the unit circle. Given this assumption of stationarity and that of $n \geq p$ mean-zero, serially independent, normal innovations ϵ_t, T observations on X_t, a $p \times 1$ linear function of Y_t given by $X_t = HY_t$, are normally distributed with mean zero and non-singular block Toeplitz

[22] Similarly to Leeper and Sims (1994), though the authors truncate their calculations. Additionally, details on the algorithm are provided here to make it more accessible to the literature.

[23] Whether or under what conditions this would actually be desirable remains a subject of future research. In a preliminary work, Schmitt-Grohé and Uribe (2007) express the likelihood of a standard DSGE model in a format compatible with this section - but without guidance as to how it might be evaluated numerically.

covariance matrix

$$\Psi = \begin{bmatrix} \Gamma_0 & \Gamma_1' & \dots & \Gamma_{T-2}' & \Gamma_{T-1}' \\ \Gamma_1 & \Gamma_0 & \dots & \Gamma_{T-3}' & \Gamma_{T-2}' \\ \vdots & & \ddots & & \vdots \\ \Gamma_{T-2} & \Gamma_{T-3} & \dots & \Gamma_0 & \Gamma_1' \\ \Gamma_{T-1} & \Gamma_{T-2} & \dots & \Gamma_1 & \Gamma_0 \end{bmatrix} \tag{3.26}$$

where Γ_k is the kth autocovariance matrix given by $E\left[X_t X_{t-k}'\right] = E\left[X_{t+k} X_t'\right]$.[24]

The log-likelihood of a vector of underlying parameters ϑ given the data is thus

$$\mathscr{L}(\vartheta|X) = -0.5pTln(2\pi) - 0.5ln(det(\Psi(\vartheta))) - 0.5X'\Psi(\vartheta)^{-1}X \tag{3.27}$$

where $X = [X_1' X_2' ... X_T']'$.

If a prior density $\mathscr{P}(\vartheta)$ for the underlying parameters is given, the log of the posterior density of the underlying parameters given the data is

$$ln(\mathscr{P}(\vartheta|X)) \propto \mathscr{L}(\vartheta|X) + ln(\mathscr{P}(\vartheta)) \tag{3.28}$$

Given (3.26) and $\mathscr{P}(\vartheta)$, only two potentially challenging quantities need to be calculated: $ln(det(\Psi(\vartheta)))$ and $X'\Psi(\vartheta)^{-1}X$. 3.B provides details of the recursive algorithm used to calculate these two quantities. The algorithm incorporates the calculations into Akaike's (1973) iterative method for the inversion of block Toeplitz matrices. Neither Ψ nor its inverse is either stored or calculated in full and the dimensions of the calculations are invariant to the size of the underlying state space.

All that remains, then, is a method of deriving the sequence of autocovariance matrices needed for the likelihood calculations. Under the stationarity assumptions, (3.20) can be rewritten using the lag operator L as

$$\begin{aligned} Y_t &= \left[\mathbf{\Theta}(L) + (I_k - \alpha L)^{-1}\left(\alpha\Theta_{I-1} + \beta M N^I (I_n - NL)^{-1}\right) L^I\right]\epsilon_t \\ W_t &= (I_n - NL)^{-1}\epsilon_t \end{aligned} \tag{3.29}$$

[24]See Hamilton's (1994, pp. 261–262) equations 10.2.1 and 10.2.2.

where $\mathbf{\Theta}(L) = \sum_{j=0}^{I-1} \Theta_j L^j$, I_k and I_n are $k \times k$ and $n \times n$ identity matrices, and α, β, and M refer to $\alpha(I)$, $\beta(I)$, and $M(I)$.[25] Thus, the autocovariance-generating function of $X_t = HY_t$ is given by

$$G_X(z) = H\left[\mathbf{\Theta}(z) + (I_k - \alpha z)^{-1}\left(\alpha\Theta_{I-1} + \beta M N^I (I_n - Nz)^{-1}\right) z^I\right]\Omega \\ \times \left[\mathbf{\Theta}\left(z^{-1}\right) + \left(I_k - \alpha z^{-1}\right)^{-1}\left(\alpha\Theta_{I-1} + \beta M N^I \left(I_n - Nz^{-1}\right)^{-1}\right) z^{-I}\right]' H' \tag{3.30}$$

whence the spectrum (Hamilton 1994, pp. 268–269) and, through an inverse Fourier transform, the autocovariance matrices can be calculated. (Uhlig 1999, p. 49)

3.7 Estimating Sticky Information and Sticky Prices

A simple example will illustrate the estimation method.[26] As a baseline, consider the following basic New Keynesian model[27]

$$\pi_t = \frac{1-\lambda}{\lambda}\left(\xi y_t + w_t^{pc}\right) + (1-\lambda)\sum_{j=0}^{\infty} \lambda^j E_{t-j-1}\left[\pi_t + \xi \Delta y_t + \Delta w_t^{pc}\right] \tag{3.31}$$

$$y_t = E_t\left[y_{t+1}\right] + a_1\left(R_t - E_t\left[\pi_{t+1}\right]\right) + w_t^{is} \tag{3.32}$$

$$R_t = \phi_R R_{t-1} + \left(1 - \phi_R\right)\left(\phi_\pi \pi_t + \phi_y y_t\right) + w_t^{mp} \tag{3.33}$$

where π_t denotes inflation, y_t the output gap and R_t the nominal interest rate. The first equation is Mankiw and Reis's (2002) sticky-information Phillips curve, the second a standard dynamic IS-curve derived from the Euler equation associated with household intertemporal optimization, and the third a Taylor rule with interest rate smoothing. w_t^{pc} and w_t^{is} are AR(1)

[25] It is to be understood here and in the following that all matrices are potentially functions of the underlying parameters ϑ.

[26] Comparing estimation methods and results for sticky information from the literature is beyond the scope of this chapter- The interested reader is directed to the overview provided by Reis (2009, pp. 18–19).

[27] See Trabandt (2007) for a first principles' derivation; here, monetary policy is defined here through control of the nominal interest rate and a markup shock has been added to the Phillips curve.

processes with persistence parameters ρ_{pc} and ρ_{is} and innovation variances σ^2_{pc} and σ^2_{is}. w_t^{mp} is serially uncorrelated with variance σ^2_{mp}. All three of the exogenous processes are assumed to be mutually independent and normally distributed.

The parameters outside the exogenous processes are λ the probability for price setters of not receiving an information update, ξ the degree of strategic complementarities in price setting, a_1 whose inverse for the purposes here can be interpreted as the coefficient of relative risk aversion, ϕ_R the degree of interest rate smoothing, ϕ_π the degree of inflation targeting, and ϕ_y the degree of output-gap targeting.

Two additional variants of the model will also be estimated. First, a version of (3.31) truncated according to the first type of truncation in section 3.4 after three lagged expectations, following Andrés, López-Salido, and Nelson (2005, p. 1033). Second, a sticky-price model will also be estimated, replacing (3.31) with the sticky-price Phillips curve

$$\pi_t = \frac{(1-\lambda)(1-\beta\lambda)}{\lambda}\left(\xi y_t + w_t^{pc}\right) + \beta E_t\left[\pi_{t+1}\right] \tag{3.34}$$

where λ now refers to the probability for price setters of not being able to update their prices and β the discount factor. (E.g., Woodford, (2003b)) As the latter is present only in the sticky-price model, it is fixed at 0.99 and not included in the estimation procedure. The baseline version uses the solution method from section 3.3.3 with the tolerance parameter δ set to 1e-5 and the truncated variant and sticky-price model will serve the investigation of the importance of sticky information and lagged expectations.

The priors used are identical in all specifications, coincide primarily with those found in Smets and Wouters (2007), and can be found in table 3.1. λ is centered at 0.5, implying an average information or price update every two quarters; the mean of ξ is set at 0.25, conservative with respect to the value set by Mankiw and Reis (2002) and those discussed by Woodford (2003b, Ch. 3), and is restricted to imply strategic complementarities;[28] and a_1

[28]As noted by Keen (2007), this is not a wholly innocuous assumption, as the hump-shaped behavior of

is set with mean one (log utility) and a wide variance.

The data used is taken from Smets and Wouters (2005) for the United States for 1970:1-2002:2. The observables are: the effective federal funds rate expressed on a quarterly basis (R_t), 100 times the log difference of the implied deflator of GDP (π_t) and the HP-filtered series of 100 times the log of real GDP divided by the civilian population over 16 (y_t). All series are demeaned.

The estimation procedure follows the Random-Walk Metropolis Algorithm from An and Schorfheide (2007, p. 131) with the likelihood evaluated with the algorithm in section 3.6 instead of the Kalman filter. For each model following Mankiw and Reis (2007), I generate five chains of 50,000 draws discarding the first 30,000. This gives, for each model, a total of 250,000 draws calculated and 100,000 draws kept and mixed after checking the chains for convergence.[29] The truncation point varies in the baseline model with the minimum and maximum over all the draws being 17 and 184, respectively.

The algorithms presented here are able to obtain these 250,000 draws in just under seven hours for the baseline sticky-information model, requiring less than one-tenth of a second to solve the model, calculate the sequence of autocovariance matrices, and calculate the log-likelihood for each draw. These seven hours were split roughly equally between these three operations. Obtaining the draws for the sticky-price model required five hours. More computational time is required with lagged expectations, but not prohibitively so.

The differences in terms of computation times presented in section 3.5 are starkly highlighted by the estimation example here. From section 3.5, the method of, e.g., Wang and Wen (2006) is two orders of magnitude slower than the method of this chapter. This would translate to well over a week to replicate the estimation and, as the computation time required

inflation in Mankiw and Reis (2002) disappears if price-setting decisions are strategic substitutes. See Woodford (2003b, Ch. 3) for a defense of the assumption of strategic complementarities in price setting.

[29]The chains for the truncation did not appear to have converged. The covariance matrix was recalculated using the 100,000 draws and five new chains that indicated convergence were generated.

by their method increases quickly in the truncation point, likely much longer. Trabandt (2007, p. 22), in a robustness exercise, recalculates his solution for some 5,000 draws from uniform parameter spaces and notes that "somewhat more than two weeks" are required. When facing such computational burdens, the option of truncating after just a few lagged expectations would seem appealing. However, as the comparison of the baseline results with those from the truncated model will show, this can bias the estimates and is, with the methods here, unnecessary.

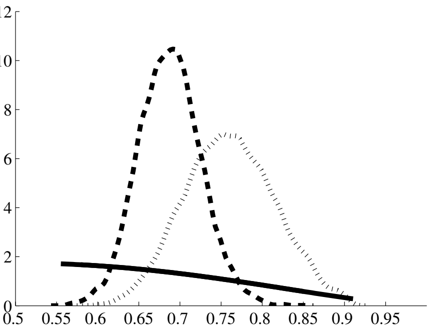
As was shown in section 3.4, the accuracy of a truncation can deteriorate as λ is increased. The methods provided in this chapter automatically adjust the truncation point as needed based on the tolerance parameter. Estimating the truncated model with a higher but still fixed truncation point might still be inaccurate for some parameter combination, producing an erroneous likelihood calculation; and might use too high a truncation point for other combinations, unnecessarily burdening the computations.

The estimates can be found in table 3.1. The data is generally informative about the parameters, the main exception being ξ. The primary differences between the two sticky-information models occur in the estimates in the Phillips curve: λ, ρ_{pc}, and σ_{pc}. This is not at all surprising given the potentially altered persistence of variables due to truncation, as shown in section 3.4, and that it is the Phillips curve that is being truncated here. The priors and posteriors under the three specifications can be found in figure 3.5 for these parameters. The posterior mean of the truncated model implies that firms, on average, receive an information update about once every four quarters, whereas the estimate in the baseline model is about once every three quarters. The posterior distribution of truncated model is skewed to the right with many accepted draws with a high degree of information rigidity; exactly the region for this parameter where the truncation is most likely to give inaccurate results. Truncation induces varying degrees of inaccuracy as different regions are explored by the estimation procedure.

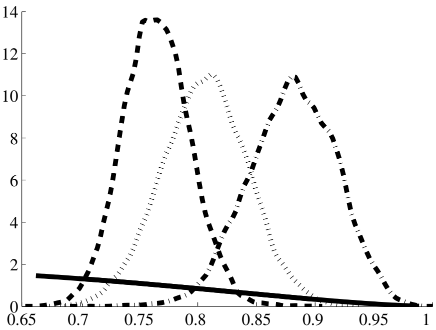
Table 3.1: Priors and Posteriors of Parameters

	Prior			Posterior (Baseline)				Posterior (Truncation)				Posterior (Sticky Prices)			
	Type	Mean	Std.	Mean	5%	Median	95%	Mean	5%	Median	95%	Mean	5%	Median	95%
λ	$\mathscr{B}$	0.5	0.2	0.69	0.63	0.69	0.75	0.76	0.67	0.76	0.85	0.74	0.66	0.75	0.81
ξ	$\mathscr{B}$	0.25	0.05	0.23	0.16	0.22	0.31	0.25	0.17	0.24	0.33	0.22	0.14	0.22	0.30
$\frac{1}{a_1}$	$\mathscr{N}$	1	0.75	3.40	2.60	3.38	4.27	3.39	2.57	3.37	4.26	3.49	2.70	3.48	4.33
ϕ_π	$\mathscr{N}$	1.5	0.125	1.32	1.16	1.32	1.49	1.31	1.14	1.31	1.48	1.33	1.16	1.33	1.50
ϕ_y	$\mathscr{N}$	0.125	0.05	0.25	0.18	0.25	0.31	0.25	0.19	0.25	0.32	0.26	0.19	0.26	0.32
ρ_{is}	$\mathscr{B}$	0.5	0.2	0.88	0.84	0.89	0.93	0.88	0.82	0.88	0.94	0.84	0.78	0.84	0.89
ρ_{pc}	$\mathscr{B}$	0.5	0.2	0.76	0.72	0.76	0.81	0.81	0.75	0.81	0.87	0.88	0.82	0.88	0.94
ρ_R	$\mathscr{B}$	0.5	0.2	0.76	0.72	0.76	0.80	0.77	0.72	0.77	0.81	0.78	0.73	0.78	0.81
σ^{ϵ}_{is}	$\mathscr{G}^{-1}$	0.1	2	0.16	0.12	0.16	0.21	0.16	0.11	0.16	0.22	0.20	0.15	0.20	0.27
σ^{ϵ}_{pc}	$\mathscr{G}^{-1}$	0.1	2	0.66	0.49	0.64	0.89	1.03	0.62	0.95	1.74	0.52	0.23	0.49	0.94
σ^{ϵ}_{mp}	$\mathscr{G}^{-1}$	0.1	2	0.27	0.24	0.26	0.30	0.26	0.24	0.26	0.30	0.26	0.24	0.26	0.29
Log Marginal Data Density				-230.398				-234.0005				-233.1548			

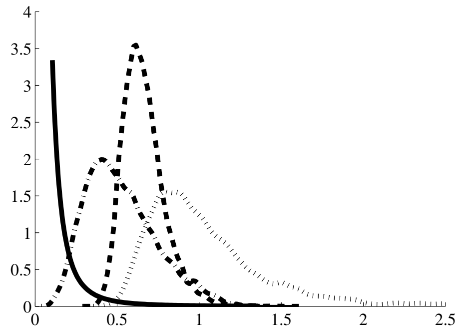
Note: Data comprises 1970:1-2002:2; $\mathscr{N}$ - Normal, $\mathscr{B}$ - Beta, $\mathscr{G}^{-1}$ - Inverse Gamma; Std. - Standard Deviation; The effective prior is truncated at the determinacy boundary and appropriately normalized. The log marginal densities are calculated using Geweke's (1999) modified harmonic mean with the 100,000 posterior draws.



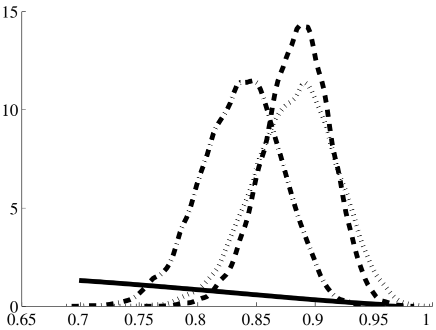
(a) Prior and Posterior Densities of λ



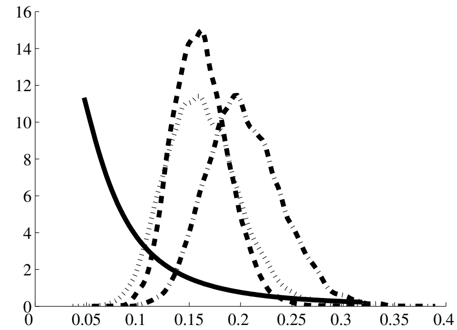
(b) Prior and Posterior Densities of ρ_{pc}



(c) Prior and Posterior Densities of σ_{pc}



(d) Prior and Posterior Densities of ρ_{is}



(e) Prior and Posterior Densities of σ_{is}

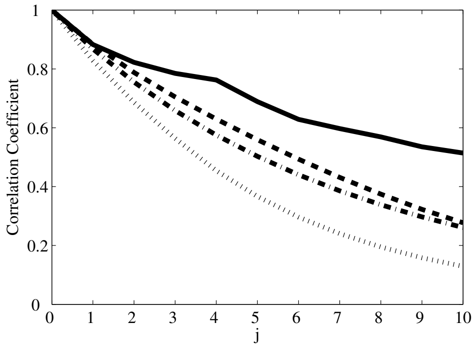
Figure 3.5: Selected Priors and Posterior Densities; Solid - Prior, Dashed - Baseline Model, Dotted - Truncation, Dash-Dotted - Sticky Prices

The notable differences in the estimates for the sticky-price model are ρ_{is} and ρ_{pc}, with the sticky-price model placing more persistence into the PC shocks and less into the IS shocks. The priors and posteriors for these parameters are shown in figure 3.5. The estimate of λ

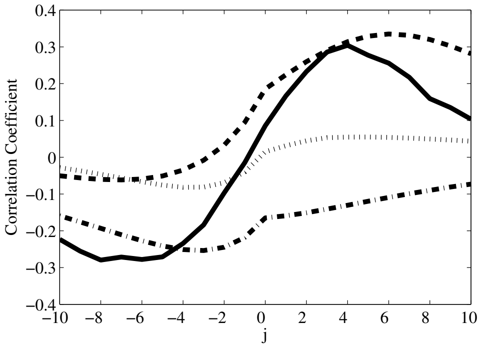
implies that firms update their prices once every four quarters on average.

Overall, the posterior estimates for all three specifications are similar. Relative risk aversion, the inverse of a_1, is consistently estimated to be about 3.4, between the estimates of, say, Rabanal and Rubio-Ramirez (2005) and Smets and Wouters (2005). The degree of interest-rate smoothing is consistent with the estimates of Smets and Wouters (2005) and Clarida, Galí, and Gertler (2000), with the elasticities with respect to inflation and the output gap within standard estimates. The models predict a degree of persistence in the exogenous processes similar to Rabanal and Rubio-Ramirez (2005), but without any of near unit-roots in Smets and Wouters (2005) or Smets and Wouters (2007).

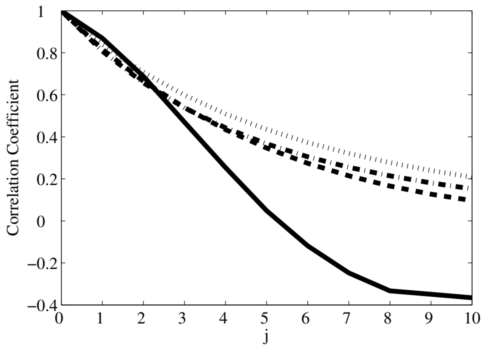
Turning to the second moments, the autocorrelations of inflation, the output gap, and the nominal interest rate can be found in figure 3.6. The baseline sticky-information model fairs best, but not appreciably so, in replicating the autocorrelations of inflation and the output gap, but substantially overestimates the autocorrelation of the nominal interest rate. The truncated sticky-information and sticky-price models deliver very similar results in this dimension. In terms of the cross-correlations, only the baseline sticky-information model replicates the observed lead of the output gap over inflation, with the truncated and sticky-price models failing to generate the hump shape in the cross-correlelogram. The sticky-price model is the only model that generates the negative correlation between current inflation and future output gaps, but it maintains this prediction at all horizons. All three models replicate the shape of the the cross-correlelogram of inflation and the nominal interest rate, but the baseline model overestimates the degree of cross-correlation throughout. Both the baseline and the truncated sticky-information models do a reasonable job replicating the cross-correlation of the output gap with the nominal interest rate, but they miss the negative lead of the interest rate over the output gap. The sticky price model predicts much less correlation than is found in the data.



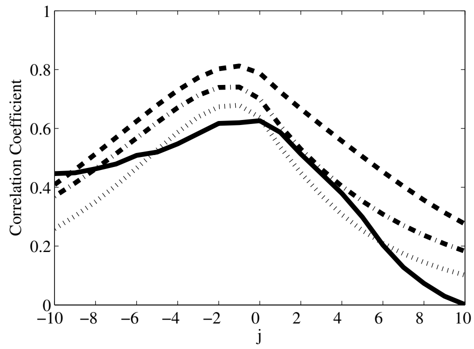
(a) Autocorrelation of Inflation



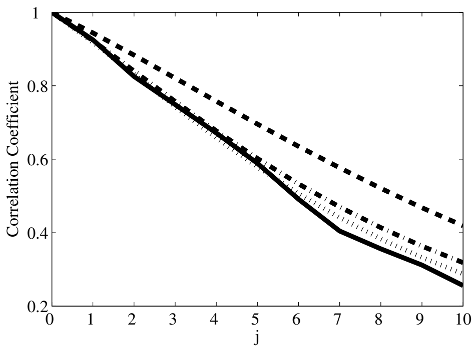
(b) Cross-correlation of Inflation at $t+j$ with the Output Gap at t



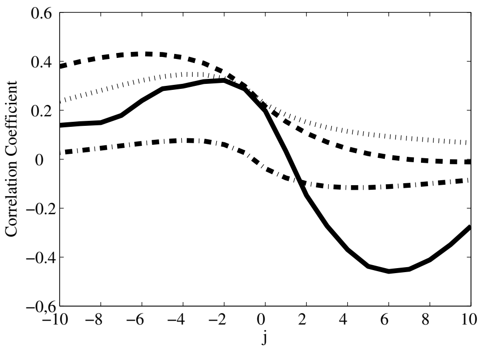
(c) Autocorrelation of the Output Gap



(d) Cross-correlation of Inflation at $t+j$ with the Nominal Interest Rate at t

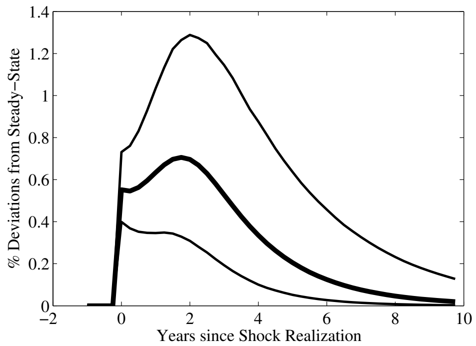


(e) Autocorrelation of the Nominal Interest Rate

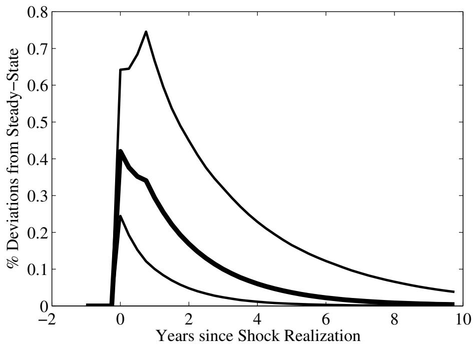


(f) Cross-correlation of the Output Gap at $t+j$ with the Nominal Interest Rate at t

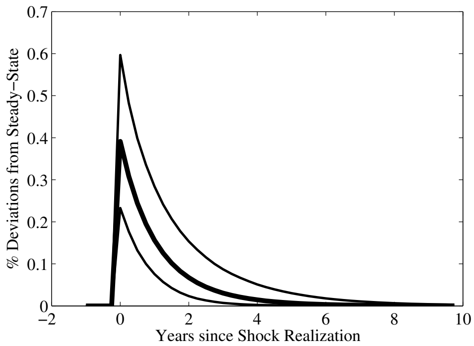
Figure 3.6: Selected Empirical and Posterior Statistics; Solid - Data, Dashed - Baseline Model, Dotted - Truncation, Dash-Dotted - Sticky Prices



(a) Impulse Response - Baseline Model



(b) Impulse Response - Truncation



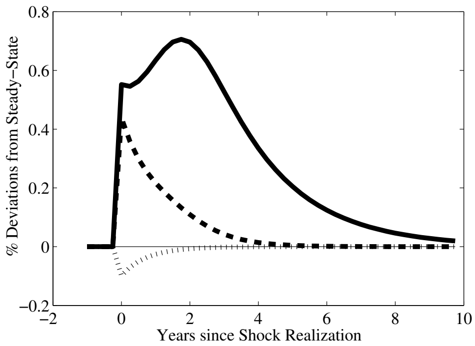
(c) Impulse Response - Sticky Prices

Figure 3.7: Impulse Responses of Inflation to a Unit IS Shock; Heavier Weighted Lines - Evaluated at the Posterior Mean, Lighter Weighted Lines - 10% and 90% Bounds on the Credible Set of Impulse Responses

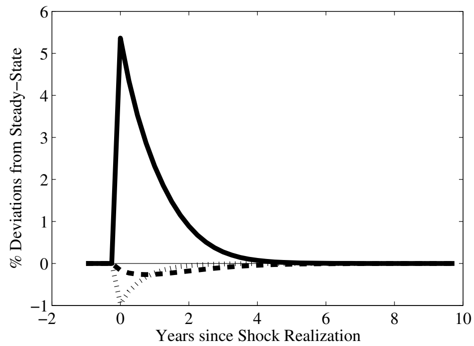
The source of the lead of the output gap over inflation can be seen in figure 3.7, the impulse response of inflation to a unit IS shock. In the baseline model, the maximal impact of a demand shock on inflation occurs about eight quarters after impact.[30] With truncation after three lagged expectations, the truncated model is not even capable of creating such a delay despite the higher degree of information persistence implied by its estimates. The posterior estimates put the maximal response on impact of the shock and the upper bound of the credible set displays a sharp peak at the truncation point, a visual cue to a distortive truncation. The internal propagation method of the sticky-information model relies on the

[30]The maximal response of the output gap for all three specifications occurs on impact.

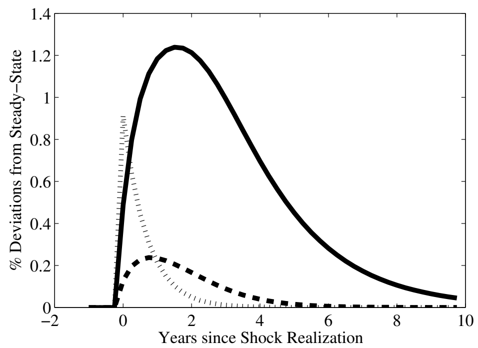
moving average terms generated by the lagged expectations. On impact of the shock, those firms receiving an information update set their pricing plans realizing that as time progresses more firms will become aware of the persistent shock—it is the interplay between the shock's persistence and the degree of information rigidity that produces the hump shape. Instead of front-loading future desired price increases, as in the sticky-price model, firms fix a plan for future prices given current information. In the truncated model, current information includes the fact that inflation will begin returning to its zero steady-state at an exponential rate after the truncation point, severely limiting the potential inertia in inflation.



(a) Inflation



(b) The Output Gap



(c) The Nominal Interest Rate

Figure 3.8: Impulse Responses to Unit Shocks in the Baseline Model; Solid - IS, Dashed - PC, Dotted - MP; Evaluated at the Posterior Mean

Apart from the IS shock, all three models display qualitatively identical impulse responses; the impulse responses for the baseline model can be found in figure 3.8. The maximal response of inflation to a monetary shock occurs on impact. All three models are unable to reproduce the hump-shaped responses to a monetary shock with output leading inflation as found by Christiano, Eichenbaum, and Evans (2005). As Coibion (2006) shows, the hump-shaped response of inflation to a monetary shock in the sticky-information model found by Mankiw and Reis (2002) is sensitive to whether monetary policy is defined over the money supply or the nominal interest rate. In their model with pervasive information stickiness, Mankiw and Reis (2007) find that a monetary shock can produce a hump-shaped response of inflation even under an interest-rate rule. As Reis (2009) illustrates, however, this requires substantial persistence in the monetary shock itself, and a much greater degree of interest-rate smoothing than was estimated would be needed here.[31] The failure to reproduce the hump shapes should not be taken as too serious a failure, as Christiano, Eichenbaum, and Evans (2005, pp. 2–3) note that wage rigidity is the critical nominal friction and the inclusion of variable capital utilization is crucial for this result—neither of which are present in the estimated models. Trabandt (2007) notes that Mankiw and Reis's (2002) monetary shock can be interpreted as a nominal income shock and, in that sense, the estimates here confirm the results of Mankiw and Reis (2002), albeit not directly through the monetary shock.

By and large, the baseline sticky-information and sticky-price models agree as to the relative contribution of shocks in the forecast variances. The variance decompositions can be found in table 3.2 and the PC shock is the primary driver of inflation, the IS shock of the output gap, and the MP shock of the interest rate at lower forecast horizons. Through the high persistence of the PC and IS shocks, the MP shock loses importance as the horizon is increased. In the sticky-price model, the PC shock is more persistent than the IS shock and gains importance relative to the latter at higher horizons for the output gap and the nominal interest rate. In the sticky-information model, the internal propagation of IS shocks

[31] Though a higher degree of interest-rate smoothing would counteract the momentum in the response to the other shocks. Likewise for the PC shock, a higher degree of persistence in the exogenous process than was estimated would produce a humped-shaped response in inflation.

Table 3.2: Variance Decompositions

Sticky Information (Baseline Model)									
	Inflation			Output Gap			Nominal Interest Rate		
Horizon	IS	PC	MP	IS	PC	MP	IS	PC	MP
1	8.59	90.68	0.74	90.87	1.33	7.80	8.57	9.86	81.57
	(5.1,14)	(85,94)	(0.4,1.3)	(88,93)	(0.5,2.5)	(5.9,11)	(5.8,12)	(7.2,12)	(77,86)
2	10.03	89.31	0.66	90.59	2.43	6.98	16.98	17.82	65.21
	(5.9,16)	(83,94)	(0.4,1.2)	(87,93)	(1.1,4.1)	(5.2,9.6)	(12,23)	(13,21)	(59,73)
5	16.05	83.46	0.49	88.76	5.61	5.63	36.21	29.41	34.38
	(8.6,27)	(72,91)	(0.3,0.9)	(84,92)	(3,8.7)	(4.1,7.9)	(27,46)	(22,35)	(28,43)
10	28.00	71.63	0.37	86.36	8.58	5.06	51.99	28.86	19.15
	(13,47)	(53,87)	(0.2,0.8)	(80,91)	(4.6,13)	(3.6,7.1)	(38,66)	(19,40)	(14,27)
15	34.03	65.64	0.33	85.68	9.33	4.99	58.94	25.71	15.35
	(16,57)	(43,84)	(0.2,0.7)	(79,91)	(5,15)	(3.6,7)	(42,74)	(14,39)	(10,23)
∞	36.63	63.06	0.32	85.57	9.45	4.98	62.66	23.51	13.83
	(16,64)	(36,83)	(0.1,0.7)	(79,91)	(5,15)	(3.5,7)	(43,80)	(11,37)	(7.7,21)

Sticky Prices									
	Inflation			Output Gap			Nominal Interest Rate		
Horizon	IS	PC	MP	IS	PC	MP	IS	PC	MP
1	5.92	93.79	0.29	90.77	1.50	7.73	7.72	10.77	81.52
	(2.9,13)	(86,97)	(0.1,0.8)	(87,93)	(0.3,3.0)	(5.8,11)	(5.2,11)	(7.1,12)	(78,87)
2	5.52	94.24	0.25	89.82	2.94	7.24	14.01	20.34	65.64
	(2.6,12)	(87,97)	(0.1,0.7)	(86,93)	(1,5.0)	(5.5,10)	(9.9,20)	(14,22)	(61,74)
5	4.73	95.08	0.19	85.94	7.80	6.26	24.12	38.90	36.99
	(2.1,11)	(89,98)	(0.1,0.6)	(81,91)	(3.5,12)	(4.7,9.0)	(18,33)	(28,42)	(32,47)
10	4.21	95.64	0.16	81.32	13.06	5.62	26.74	49.08	24.18
	(1.8,10)	(89,98)	(0.0,0.5)	(74,89)	(6.2,19)	(4.1,8.2)	(20,39)	(35,55)	(20,33)
15	4.05	95.81	0.15	79.56	14.99	5.45	26.46	52.05	21.49
	(1.7,10)	(89,98)	(0.0,0.5)	(71,88)	(7.0,23)	(3.9,8)	(19,40)	(36,60)	(17,30)
∞	3.98	95.88	0.15	78.77	15.84	5.39	25.98	53.44	20.58
	(1.5,10)	(89,98)	(0.0,0.5)	(68,88)	(7.3,26)	(3.8,7.8)	(17,41)	(36,65)	(15,29)

Note: Entries are given in percent. The main entries were evaluated at the posterior mean and may not add up to 100 due to rounding. The entries in parentheses give the 5% and 95% bounds for the posterior credible set and were calculated cell-by-cell.

in inflation, and thereby in the nominal interest rate, drives the difference to the sticky-price model. At the posterior mean, the IS shock is more important relative to the sticky-price model and its relative importance is increasing as the the horizon is increased. The width of the credible sets indicate, however, that there is a high degree of variability in this internal propagation method with results much closer to those of the sticky-price model contained within the 90% credible set.

Log marginal data densities can be found in the last line of table 3.1. Though not conclusive, the results are striking, with the ranking of sticky information and sticky prices reversed by truncation. Although the baseline model reproduced the lead of the output gap over inflation, it was less successful than the sticky-price model in other dimensions. It is not surprising that the baseline model does not fare overwhelmingly better than the sticky-price model in terms of posterior odds. Additionally, the sticky-information model reveals a greater degree of uncertainty in the dynamics, reflected in the wider credible sets for the responses to an IS shock and in the variance decompositions.

Truncation can alter the dynamics predicted by a model with lagged expectations and estimates that try to match these dynamics are vulnerable to making biased inferences. The estimates here have shown that taking the sticky-information model seriously can lead to a different assessment of the model's ability to match the data and that, using the methods developed here, sticky-information models can be readily analyzed.

3.8 Conclusion

I have derived a method for solving linear rational-expectations models with lagged expectations using standard methods for the autonomous recursion of the MA coefficients and a sparse block tridiagonal system of equations for the non-autonomous coefficients, thus combining several known methods into one coherent approach. The method explicitly allows for models with lagged expectations reaching back into the infinite past, providing a formal justification for when and how to truncate based on a convergence criterion. The method performs favorably in comparison with existing methods, minimizing computing and preprocessing time, while avoiding the problems associated with an arbitrary truncation.

I have also presented likelihood methods for estimation that avoid the Kalman filter. This is desirable as lagged expectations makes the state space necessary for the use of the Kalman filter prohibitively large, negating the gains made with the solution method. The comparison of simple New Keynesian models with sticky information and prices favored sticky information marginally with truncation reversing the ordering.

The solution and estimation methods derived here allow researchers to analyze models with lagged expectations quickly and easily. As such, they should facilitate the further investigation of models such as the sticky-information model.

Appendix

3.A Application of Klein's (2002) QZ Method

Use the QZ decomposition to find unitary matrices Q and Z and upper-triangular matrices S and T such that $\left[\begin{smallmatrix} Q_1 \\ Q_2 \end{smallmatrix}\right] \left[\begin{smallmatrix} 0 & -A_0 \\ I & 0 \end{smallmatrix}\right] \left[\begin{smallmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{smallmatrix}\right] = S$ and $\left[\begin{smallmatrix} Q_1 \\ Q_2 \end{smallmatrix}\right] \left[\begin{smallmatrix} C_0 & B_0 \\ 0 & I \end{smallmatrix}\right] \left[\begin{smallmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{smallmatrix}\right] = T$. The decomposition will be arranged such that the first s eigenvalues are those less than or equal to g^u, satisfying (3.3), and the remaining ones, given by the generalized eigenvalues of $S_{22}z - T_{22}$ following Klein (2000, p. 1415), are greater than g^u. Assuming $s = k$ and Z_{11} is of full rank,

$$\Xi_j^u = -\sum_{k=0}^{\infty} \left[T_{22}^{-1} S_{22}\right]^k T_{22}^{-1} Q_2 \left[\begin{smallmatrix} F_0 N_{j+1+k} + G_0 N_{j+k} \\ 0 \end{smallmatrix}\right] \tag{3.35}$$

and, following Theorem 5.1 of Klein (2000, p. 1417) where $\alpha = Z_{21} Z_{11}^{-1}$ and $\beta = Z_{22} - Z_{21} Z_{11}^{-1} Z_{12}$, this delivers (3.9).

For section 3.3.2, the system (3.8) is replaced by the system (3.10). This yields, analogous to the foregoing, (3.11), where

$$\Xi_j^u(I) = -\sum_{k=0}^{\infty} \left[T_{22}(I)^{-1} S_{22}(I)\right]^k T_{22}(I)^{-1} Q_2(I) \left[\begin{smallmatrix} \tilde{F}_I N_{j+1+k} + \tilde{G}_I N_{j+k} \\ 0 \end{smallmatrix}\right] \tag{3.36}$$

Klein (2000) demonstrates, with a stationary VAR(1) exogenous process, that (3.35) can be reduced to a geometric sum. To meet the assumption in his appendix, the exogenous process need not be stationary. To see this, assume that $N_{j+1} = N N_j$ and define

$$H_k = \Phi^k \Delta N^k,\ \Phi = T_{22}^{-1} S_{22},\ \Delta = T_{22}^{-1} Q_2 \left[\begin{smallmatrix} \tilde{F}_I N^{j+1} + \tilde{G}_I N^j \\ 0 \end{smallmatrix}\right] \tag{3.37}$$

then $H_{k+1} = \Phi H_k N \Rightarrow vec(H_{k+1}) = (N' \otimes \Phi) vec(H_k)$. The stability of this recursion is determined by $eig(N' \otimes \Phi) = vec\left(eig(N')' eig(\Phi)\right)$. As, by definition, $|eig(\Phi)| < \frac{1}{g^u}$ then $|eig(N' \otimes \Phi)| < 1$ so long as $|eig(N')| \leq g^u$. Thus, if the moving-average coefficients of the exogenous process follow a recursion that itself satisfies the uniform growth restriction, then (3.35) and (3.36) meet Klein's (2000) assumption and are well defined.

3.B Recursive Algorithm for Computing the Log-Likelihood

Akaike (1973) provides a recursive algorithm for inverting block Toeplitz matrices. As Ψ, defined in (3.26), is additionally symmetric, the first block column contains all the information necessary for the calculations. Following Akaike (1973, p. 237), define

$$\Psi_T = \begin{bmatrix} \Psi_{T-1} & \hat{a}_{T-1} \\ \tilde{\hat{r}}_{T-1} & \Gamma_0 \end{bmatrix} = \begin{bmatrix} \Gamma_0 & \tilde{a}_{T-1} \\ r_{T-1} & \Psi_{T-1} \end{bmatrix} \tag{3.38}$$

where $\tilde{\hat{r}}_{T-1} = [\Gamma_{T-1}\ \Gamma_{T-2}\ \ldots\ \Gamma_1]$, $\hat{a}_{T-1} = \tilde{\hat{r}}'_{T-1}$, $\tilde{a}_{T-1} = [\Gamma'_1\ \ldots\ \Gamma'_{T-2}\ \Gamma'_{T-1}]$, $r_{T-1} = \tilde{a}'_{T-1}$. The inverse of Ψ_T can be likewise defined recursively, noting symmetry, as

$$\Psi_T^{-1} = \begin{bmatrix} \Psi_{T-1}^{-1} + \hat{j}_{T-1} s_{T-1}^{-1} \tilde{\hat{i}}_{T-1} & \hat{j}_{T-1} \\ \tilde{\hat{i}}_{T-1} & s_{T-1} \end{bmatrix} = \begin{bmatrix} \Psi_{T-1}^{-1} + \tilde{\hat{i}}'_{T-1} s_{T-1}^{-1} \tilde{\hat{i}}_{T-1} & \tilde{\hat{i}}'_{T-1} \\ \tilde{\hat{i}}_{T-1} & s_{T-1} \end{bmatrix} \tag{3.39}$$

From Akaike (1973, p. 239), the determinant of $\Psi \equiv \Psi_T$ is calculated recursively with

$$ln(det(\Psi_T)) = ln(det(\Psi_{T-1})) + ln\left(det\left(s_{T-1}^{-1}\right)\right) \tag{3.40}$$

and the quadratic form $X'\Psi^{-1}X \equiv \tilde{X}'_T \Psi_T^{-1} \tilde{X}_T$, using (3.39), with

$$\tau_T \equiv \tilde{X}'_T \Psi_T^{-1} \tilde{X}_T = [\tilde{X}'_{T-1}\ X'_T] \begin{bmatrix} \Psi_{T-1}^{-1} + \tilde{\hat{i}}'_{T-1} s_{T-1}^{-1} \tilde{\hat{i}}_{T-1} & \tilde{\hat{i}}'_{T-1} \\ \tilde{\hat{i}}_{T-1} & s_{T-1} \end{bmatrix} \begin{bmatrix} \tilde{X}_{T-1} \\ X_T \end{bmatrix}$$

where $\tilde{X}_T = [X'_1 X'_2 \ldots X'_T]'$. Multiplying out yields

$$\tau_T = \tau_{T-1} + \tilde{X}'_{T-1} \tilde{\hat{i}}'_{T-1} s_{T-1}^{-1} \tilde{\hat{i}}_{T-1} \tilde{X}_{T-1} + X'_T \tilde{\hat{i}}_{T-1} \tilde{X}_{T-1} + \tilde{X}'_{T-1} \tilde{\hat{i}}'_{T-1} X_T + X'_T s_{T-1} X_T \tag{3.41}$$

Equations (3.40) and (3.41) along with the recursions from Akaike (1973, p. 238)[32] provide a recursive algorithm for calculating the log-likelihood, requiring as input only the data and the sequence of autocovariances.

[32]Note that Akaike (1973, p. 238) provides recursions for and using $s_T^{-1}\tilde{i}_T$ and $q_T^{-1}\tilde{\hat{f}}_T$. Postmultiplying his equations 3.18 and 3.20 with E_T, defined by his equation 2.2, and redefining his equations 3.23 and 3.24 in terms of $s_T^{-1}\tilde{\hat{i}}_T$ and $q_T^{-1}\tilde{f}_T$ provide the necessary recursions.

Chapter 4

A Natural Rate Perspective on Equilibrium Selection and Monetary Policy

4.1 Introduction

This chapter has two main results. First, in extending the determinacy analysis to Lucas's (1972a) natural rate hypothesis (NRH)—the proposition that monetary policy cannot permanently induce a non-zero output gap— following Carlstrom and Fuerst (2002), I establish that all supply equations that satisfy the NRH for a given demand function yield the same bounds on determinacy, save isolated singularities. Second, I provide a monetarist interpretation for the admissibility of Cochrane's (2007) explosive nominal equilibria; namely that the monetary authority is accommodating these equilibria with exploding money growth rates.

The first result implies that one can reasonably expect the monetary authority to know when its interest rate policy will admit many stable equilibria (indeterminacy) or a single stable equilibrium (determinacy), even if it has no specific knowledge regarding the supply side beyond that it satisfies the NRH. The analysis attempts to provide the proofs missing from the general claim of Carlstrom and Fuerst (2002) that there is a one-to-one correspondence between determinacy in models that satisfy the NRH and their corresponding frictionless counterparts. Specifically, I prove the necessity of determinacy in the latter for determinacy in the former, but disprove the sufficiency. Fortunately, the cases of insufficiency can be characterized as singular parameterizations. Applying the result to the standard dynamic IS curve with monetary policy defined as any finite linear relationship between the nominal interest rate, inflation, and the output gap, I prove that indeterminacy is solely a function of the parameters in monetary policy.

I provide further insight into Cochrane's (2007) criticism of determinacy as being an arbitrary elimination of explosive nominal equilibria by demonstrating that a tenet of the quantity theory provides support to his critique. Adding a standard money-demand specification, I find that the explosive paths for inflation are being accommodated by the money supply. I.e., the hyperinflationary paths are consistent with the monetarist view that, "sizable changes in the rate of change in the money stock are a necessary and sufficient condition for sizable changes in the rate of change in money income," (Friedman and Schwartz 1963, p. 63) and thus that monetary restraint is a necessary and sufficient condition for controlling inflation. (See Nelson and Schwartz (2008, p. 838)) Note that this does not rule out hyperinflation *per se*: it rules out hyperinflations that are speculative—i.e., non-fundamental to the money supply. The alternate, explosive equilibria of Cochrane (2007) are indeed fully valid monetary equilibria, with the monetary authority increasing the money *growth rate* exponentially commensurate with the explosive path for inflation.

The standard sticky-price New Keynesian model with Calvo (1983) contracts is known to violate the NRH. This violation is "an awkward situation in monetary economics" (Wolman 2007, p. 1366) and contradicts the consensus widely accepted by the late '70s (Friedman 1977, p. 459). My first result implies that the standard New Keynesian model's determinacy results and violation of the NRH are inextricably linked. This has immediate consequences as determinacy is concerned with the admissibility of multiple short-run equilibria. Determinacy analyses in standard New Keynesian models[1] must either disavow the relevance of their bounds on monetary policy or defend their models' violation of the NRH. Additionally, the sticky-price model's violation of the NRH actually frees it from Cochrane's (2007) critique: nominal explosions go hand in hand with real explosions that Cochrane (2007) admits

[1] See Bernanke and Woodford (1997), Clarida, Galí, and Gertler (1999), Bullard and Mitra (2002) and Woodford (2003b), among many others.

economics can rule out. Yet this result, as it too rests on the violation of the NRH, is dubious.

The main focus for Cochrane's (2007) analysis, however, is a frictionless model—i.e., a model that satisfies the NRH in the most extreme sense and that forms the basis for my determinacy analysis. In this model, pinning down the inflation rate when monetary policy controls the nominal interest rate requires a particular constellation for the interest rate rule and the elimination of explosive paths. This constellation is one that ensures determinacy and, from my first result, these are the same across a class of NRH models with a common demand specification. Remaining is then the elimination of explosive paths and thusly, for models that satisfy the NRH, Cochrane's (2007) critique does apply. The analysis of the frictionless model is directly applicable as it behaves identically in the long run to the rest of the class of NRH models, where the uniqueness of a path for real variables (should supply pose a short-run tradeoff between inflation and output) depends on the uniqueness of a path for inflation. This coincidence of determinacy follows from the frictionless model being a model where there is no liquidity effect, only the Fisher effect; and for a model to satisfy the NRH, the liquidity effect must eventually disappear,[2] leaving only the Fisher effect in the long run. Thus, monetary policy has the same effects in all NRH models in the long run, impacting the economy only through the Fisher effect, where "high interest rates are a sign that monetary policy has been easy." (Friedman 1968, p. 7)

Aside from Cochrane's (2007) non-Ricardian fiscal solutions along the lines of Benhabib, Schmitt-Grohé, and Uribe (2001a) and Sims (1994), McCallum (2009b) and Minford and Srinivasan (2009) have attempted to answer the Cochrane's (2007) critique. The analysis here provides an answer similar in vein to Minford and Srinivasan's (2009) by examining money. Minford and Srinivasan (2009, p. 15) examine the question illustratively within an unrelated Cagan model and ultimately "appeal to an optimizing government [...] that sets the inflation tax" to rule out explosions in inflation. I show this to be an unnecessary and misleading

[2]See Nelson (2008, p. 1804).

detour: the underlying NRH model *reduces* to a specific Cagan model, viz. that of Sargent and Wallace (1973), and the speculative hyperinflation literature—e.g., Obstfeld and Rogoff (1983) and Gray (1984)—links Cochrane's (2007) explosive equilibria unequivocally to reckless money growth. McCallum (2009b) rules these equilibria out by appealing to LS learning. The interpretation of LS learning in the context of my analysis is not very satisfying: McCallum (2009b) rules out hyperinflation caused by an ever-increasing growth rate of money supply as the associated inflation is increasing too quickly for it to be learnable in a least-squares sense. Additionally, I argue that McCallum (2009b) misintreprets his model with money within the speculative hyperinflation literature. Upon closer inspection, his model confirms my claim: the explosive paths of inflation that his model cannot rule out are necessarily associated with explosive paths of money growth.

The quantity theory also provides the way out of these hyperinflationary equilibria for the monetary authority: do not accommodate such equilibria and commit credibly to not do so beforehand. But this assumption is already implicit in the New Keynesian analysis as defended by Nelson (2008). If one defines monetary policy solely over the nominal interest rate, it should come as no surprise that this opens up the potential for problems in the long run, as "the monetary authority cannot treat the nominal interest rate as an instrument in the long run [—a position] widely shared." (Nelson 2008, p. 1805) Yet, monetary policy can be completed through the specification of a steady-state money growth rate, and the steady-state inflation rate specified in most New Keynesian models can be interpreted as such an average money growth rate. The off-equilibrium "threat" of the monetary authority to rule out the explosive equilibria of Cochrane (2007), therefore, is nothing more than to keep money growth finite.

The importance of monetary aggregates for monetary policy has found support recently in Nelson (2003), Svensson (2003), McCallum and Nelson (2005), Nelson (2008), McCallum (2008), McCallum and Nelson (2009b). Woodford (2008) presents the case for interest rate feedback rules, as opposed to the pegging of interest rates criticized by Sargent and Wallace (1975), in the spirit of Taylor (1993). Taking nonlinearities seriously, Benhabib, Schmitt-Grohé, and Uribe (2001b) show the dangers of assessing Taylor rules in a linear framework. Additional support of interest rate rules using Evans and Honkapohja's (2001) E-Stability can be found in Bullard and Mitra (2002) and McCallum (2003). McCallum and Nelson (2009a) provide a recent overview of the money in current analyses of inflation without dismissing the lessons of the quantity theory.

Both Woodford (2002) and Woodford (2003b) acknowledge the nonverticality of the standard New Keynesian Phillips curve in the long run, which McCallum (2004a) formulates into a critique of the model's violation of the NRH. Andrés, López-Salido, and Nelson (2005) examine the NRH and New Keynesian models both theoretically and empirically. Levin and Yun (2007) bring the standard model closer to the NRH by endogenizing the contract length.

This chapter is organized as follows, section 4.2 sets the stage intuitively, section 4.3 assesses determinacy in a class of models that satisfy the NRH, section 4.4 links Cochrane's (2007) critique to money, section 4.5 examines several nonlinear money-demand specifications, section 4.6 provides a monetarist context for interpretation, and section 4.7 concludes.

4.2 Linking the NRH, the Long-Run, and Determinacy

To establish the necessary intuition for the mechanisms at work in the analysis and specific results, I shall build a conceptual link between the NRH, the long run, and determinacy in this section. First, I shall review the two different forms of the NRH emphasized by McCallum (2004a, pp. 21–22) and argue that the stricter, or Lucas version, ought to be used in analyzing determinacy. Requiring the NRH to hold imposes restrictions only in the long-run—a stable short-run Phillips curve tradeoff does not contradict the hypothesis. Finally, I shall argue that ascertaining whether many (indeterminacy) or only one (determinacy) equilibrium paths are non-explosive is an inherently long-run exercise, though with short-run consequences (i.e., which equilibrium path prevails). Thus, intuitively, the long-run restrictions imposed by the NRH should be relevant for analyzing determinacy and, therefore, the NRH is pertinent for the short-run despite its long-run nature.

In bringing attention to the standard New Keynesian Phillips curve's violation of the NRH, McCallum (2004a, pp. 21–22) draws a distinction between "Friedman's weaker version" and the "stronger Lucas version" of the NRH. The former states that a higher, but constant, rate of inflation cannot permanently affect output and the latter that no path for prices, inflation, inflation growth, etc., can permanent keep output above its natural level. The association of the weaker version with Milton Friedman is unfortunate in any case, as Friedman (1977, p. 274) made his preference for the stronger version clear by noting the discrepancy between these two versions: "some substitute a stable relation between the acceleration of inflation and unemployment for a stable relationship between inflation and unemployment—aware of but not concerned about the possibility that the same logic that drove them to a second derivative will drive them to even higher derivatives." It would take an infinite number of steps to get a weaker-version-NRH model to satisfy the stronger version, incorporating all possible higher derivatives. Adapting the New Keynesian Phillips curve with indexation, to either steady state or lagged inflation, is subject to Friedman's criticism above: neither adaptation brings the model in line with the Lucas version.

This workhorse of the literature, the standard New Keynesian sticky-price model with Calvo (1983)-style overlapping contracts in general equilibrium, is given (in log-deviations and abstracting from exogenous driving processes) by [3]

$$y_t = E_t\left[y_{t+1}\right] - a_1 R_t + a_1 E_t\left[\pi_{t+1}\right] \tag{4.1}$$

$$\pi_t = \beta E_t\left[\pi_{t+1}\right] + \kappa y_t \tag{4.2}$$

[3]Cf. McCallum (2001b, p. 152), equations (2.7) and (2.14), Woodford (2003b, p. 246), or Galí (2008, p. 49).

and an as of yet unspecified rule for monetary policy, where y_t is the output gap, π_t inflation , and R_t the nominal interest rate. Equation (4.1) is a dynamic IS-curve resulting from the Euler equation of households' intertemporal maximization and equation (4.2) is the New Keynesian Phillips curve derived from Dixit-Stiglitz aggregators of individual firms' intertemporal profit maximization reflecting the probability that prices set today remain in effect into the future.

First, one can confirm that (4.2) does not satisfy Lucas's (1972a) NRH by taking unconditional expectations

$$E\left[y_t\right] = \frac{1}{\kappa}\left(E\left[\pi_t\right] - \beta E\left[\pi_{t+1}\right]\right) \neq 0 \tag{4.3}$$

Note that even in the extreme parameterization $\beta = 1$, $E\left[y_t\right] \neq 0$ should inflation be nonstationary. Requiring inflation to be stationary *a priori* precludes the possibility of an entire class of potential monetary policies, including pernicious hyperinflationary policies. As made explicit by McCallum (1998), the NRH requires that "[, o]n average, output should be equal to potential output, for any monetary policy." Nothing in this statement excludes nonstationary policies. The only way for this Phillips curve to satisfy the NRH, is if $\kappa \to \infty$, making the Phillips curve always[4] vertical.

The sticky-price Phillips curve has been indexed, either to steady-state inflation,[5]

$$\pi_t - \bar{\pi} = \beta E_t\left[\pi_{t+1} - \bar{\pi}\right] + \kappa y_t \tag{4.4}$$

or past inflation[6]

$$\pi_t = \frac{\gamma}{1+\gamma\beta}\pi_{t-1} + \frac{\beta}{1+\gamma\beta}E_t\left[\pi_{t+1}\right] + \frac{\kappa}{1+\gamma\beta}y_t \tag{4.5}$$

but both of these modifications still fail to satisfy the strict version of the NRH,[7] for the same reason above. Only those monetary policies that lead to a stationary path for inflation allow

[4]I.e., at every expectational horizon.

[5]See Yun (1996).

[6]See Christiano, Eichenbaum, and Evans (2005) for $\gamma = 1$ and Smets and Wouters (2003) for $0 < \gamma \leq 1$.

[7]See McCallum (2004a, pp. 21–22) and McCallum and Nelson (2009a, pp. 6–7).

the the output gap to be equal, on average, to zero. Certainly, indexation to steady-state inflation is meaningless, should inflation be nonstationary. As pointed out by Nelson (2008), it is monetary policy that determines steady-state inflation, or indeed whether it should exist, and without having specified monetary policy, it is almost vacuous to speak of such a value. As above, these Phillips curves can be made to satisfy the NRH, but this requires $\kappa \to \infty$, making them always vertical.

Consider a definition of the NRH, due to Carlstrom and Fuerst (2002), that holds in finite time[8]

$$E_{t-k}\left[y_t\right] = 0 \; \forall t \tag{4.6}$$

This allows us to trivially express any supply function that fulfills this hypothesis as

$$y_t = \sum_{j=0}^{k-1} \left(E_{t-j}\left[y_t\right] - E_{t-j-1}\left[y_t\right] \right) \tag{4.7}$$

Non-zero output gaps can be represented wholly as innovations or forecast errors without making any conjecture as to admissible solutions, in the words of Friedman (1977, p. 456), "[o]nly surprises matter." Note that the effect of a surprise need not disappear immediately after impacting the output gap, it can have a lasting—but not permanent—effect. That is, there can be a stable short-run tradeoff between the output gap and inflation, but this tradeoff cannot be permanent if the model is to satisfy the NRH.

[8]The list of models that satisfy this version of the NRH include: Andrés, López-Salido, and Nelson's (2005, p. 1034) "Sticky information, staggered á la Taylor"; the Mussa-McCallum-Barro-Grossmann "P-bar model"—see McCallum (1994) and McCallum and Nelson (2001); models of staggered predetermined prices such as Fischer (1977) and Blanchard and Fischer (1989, pp. 390–394); Carlstrom and Fuerst's (2002, p 81-82) model in the spirit of Fischer (1977); as well as the expectational Phillips curve of Lucas (1973)—see also Sargent and Wallace (1975)—that formalized the rational expectations revolution. Though one might argue that a NRH in finite time is overly restrictive, this subset covers every model to my knowledge that purports to satisfy Lucas's (1972a) NRH with one exception: Mankiw and Reis's (2002) sticky-information model staggered á la Calvo, whose determinacy properties are examined separately here in chapter 2 and coincide with those of this analysis for the demand and monetary policy specifications examined there. In any case, k is completely arbitrary here, it makes no difference for the conclusions whether the long run sets in after four quarters, four years, or four millennia.

In the frictionless counterpart model, there is no impediment to firms' setting the optimal, full-information price every period. It follows by definition that the output gap is always zero, which can be viewed as an extreme version of the NRH

$$y_t = 0 \; \forall t \tag{4.8}$$

the special case of $k = 0$ in (4.6). In this case, (4.1) reduces to

$$R_t = E_t\left[\pi_{t+1}\right] \tag{4.9}$$

this is identical to the Fisher-type equation in Woodford's (2003b, Ch. 2) analysis of nominal (price-level) determinacy in a frictionless economy, as well as the simple model found in the discussion of Cochrane (2007) and McCallum (2009b) regarding the appropriateness of determinacy as an equilibrium criterion in monetary models. After k periods have passed since some disturbance from equilibrium, the supply side described by (4.6) behaves identically to that of (4.8), i.e., applying the conditional expectations operator to the LHS of both supply sides yields zero—$E_{t-k}\left[(4.8)\right] = E_{t-k}\left[(4.6)\right] = 0$. Hence, given a common specification for the remainder of the model, any two models that satisfy (4.6) for some k are identical in the long-run (or indeed, after k).

Determinacy is most frequently ascertained by the eigenvalue counting method of Blanchard and Kahn (1980). Roughly speaking, a model is brought into first-order form $E_t\left[G_{t+1}\right] = HG_t$, where some variables in G_t might be predetermined, and is said to be determinate if the number of stable eigenvalues in H is exactly equal to the number of these predetermined variables. Thus, the instantaneous reaction of G_t to some disturbance is sufficient to ascertain whether some equilibrium path will lead to explosive or stable behavior. While this remains technically true of the models that satisfy (4.6),[9] it is easy to forget that the explosiveness being ruled out need not occur instantaneously in the variables of interest

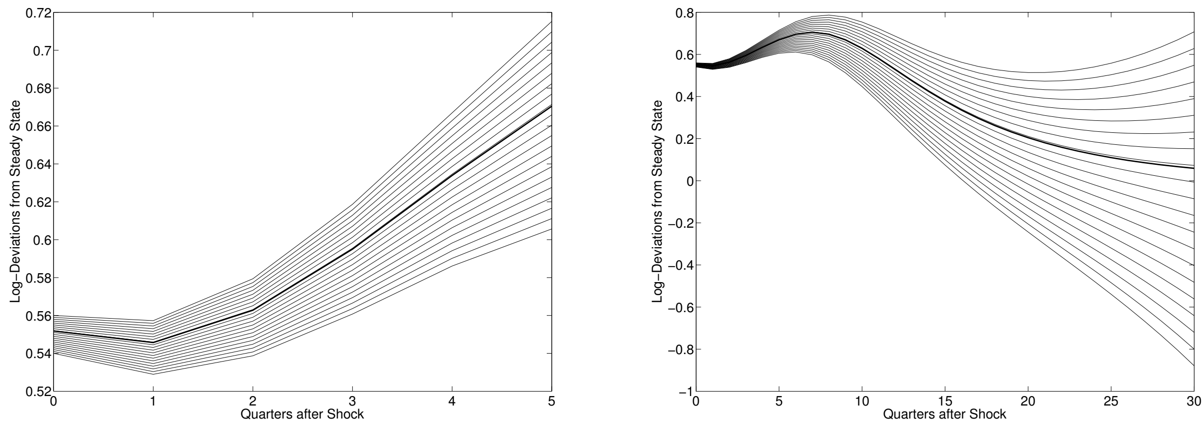
[9]By defining sufficient dummy variables to capture the information structure. See, e.g., Sims (2001).

(i.e., a subset of G_t) and, in general, any finite value at any finite horizon is permissable.[10] Determinacy rules out paths that would lead to explosive, i.e., infinite values of, variables of interest.

To illustrate, assume that inflation is required to be stable. That is, inflation must converge back to equilibrium following any disturbance. Examining figure 4.1a, all the paths pictured (here with initial log-deviations of inflation to some unit exogenous disturbance in demand between 0.54 and 0.56) certainly appear to be uniformly explosive (within five periods, inflation on all paths exceeds the initial deviations), violating the required stability. Yet this is deceiving: there is nothing that violates the requirement of stability for inflation in the figure but for one's own imagined extrapolation of the behavior depicted into the infinite future. To see this, examine figure 4.1b, the same picture as before, but now extended out to thirty periods after the initial disturbance. The initial common explosiveness dissipates rather swiftly as some variables are below their initial values and some are above. One could imagine now that some path, here highlighted as a more heavily weighted line, is uniquely convergent, with all paths that started above diverging to positive infinity and all that started below diverging to negative infinity. Again, this is the result of one's extrapolation of the first thirty periods on out into the infinite future, the same "shift" that occurred between figures 4.1a and 4.1b could certainly occur again at a more distant horizon. It is the behavior in the long run that establishes whether a particular path is diverging, yet the particular path chosen by the long-run is associated with specific short run reactions of variables. That is, the long run is decisive for the short run through the selection of valid equilibrium paths.

The eigenvalue counting method, by bringing the model into a first order form, ensures that the system described by $E_t\left[G_{t+1}\right] = HG_t$ behaves instantaneously exactly as it would asymptotically. This is convenient, but upon reflection highlights an important shortcoming

[10]Exceptions would be, e.g., finite but negative values for prices, but assuming that variables are transformed, as they usually are, appropriately to allow the range of the transformed variable to encompass the reals, e.g., for prices, the log of the price would be included in the system.



(a) Short Horizon (b) Longer Horizon

Figure 4.1: Impulse Responses of Inflation to a Demand Shock for Different Initial Conditions

of the standard New Keynesian model: it behaves in the short run as is does in the long run with the same stable tradeoff between output and inflation.[11] Allowing a short-run tradeoff in this model implies a long-run tradeoff that can impact determinacy, admitting a single path (determinacy) for some parameter regions where a NRH model might admit many (indeterminacy). Of course and as was seen in figure 4.1, different paths are usually associated with different instantaneous reactions of variables to disturbances as well. NRH models of the class satisfying (4.6) can also be brought into first-order form by defining dummy variables. The variables of interest, like inflation in the illustration, are a subset of G_t and may differ in their behavior before and after k. Thus, all models that differ only in their supply side and that satisfy the NRH will display the same behavior after k and thusly ought to have identical determinacy regions in parameters, regardless of their behavior in the short run.

[11]To see this, simply note that (4.2) is already in first-order form. The relation between the output gap and current and future inflation is the same no matter what horizon is examined, a very stable tradeoff indeed.

4.3 Determinacy in Natural Rate Models

Here, I shall establish an equivalence between nominal determinacy in the frictionless model with (4.8) and real determinacy in the general model with any supply side satisfying (4.6). This equivalence was asserted, but without proof, by Carlstrom and Fuerst (2002) to be one-to-one. I shall prove that nominal determinacy in the frictionless model is a necessary but not a sufficient condition: one must guard against singular cases. Saving for such cases, the equivalence alluded to intuitively in the previous section is established and this intuition is extended. Additionally, the questionability of existing determinacy analyses using the standard New Keynesian model is highlighted.

In what follows, I will analyze linear rational-expectations models of the following class:

$$0 = \sum_{i=0}^{p} \sum_{j=-m}^{n} Q(i,j) E_{t-i} X_{t+j}, \ X_t = \begin{bmatrix} R_t \\ \pi_t \\ y_t \end{bmatrix}, \ 0 \le p, m, n < \infty \tag{4.10}$$

where the $Q(i,j)$'s are matrices of dimensions 3×3. That is, the model is composed of three structural equations determining the supply side, demand side, and monetary policy. The class encompasses all linear rational-expectations models in the three variables of interest that (i) have a finite number of leads (given by n), (ii) have a finite number of lags (given by m), and (iii) have expectations formed at differing horizons from t into the finite past $t-p$.[12] This, of course, encompasses the models discussed in section 4.2.

To close out any of the models of the previous section, monetary policy needs to be specified. The only restriction I shall impose on monetary policy is that it fits into the class defined in (4.10). Accordingly, let monetary policy be the third equation of (4.10),[13] given by

$$0 = \sum_{i=0}^{p} \sum_{j=-m}^{n} Q_{3,\cdot}(i,j) E_{t-i} \left[X_{t+j} \right] \tag{4.11}$$

[12]Note that the absence of exogenous driving forces in (4.10) is not restrictive. The conditions for determinacy remain the same if (4.10) is appended with stationary driving forces—i.e., I am investigating the properties of the homogenous component of the system of difference equations, but one has the additional task of associating the exogenous driving forces with the expectation errors—see Sims (2001).

[13]Where $Q_{3,\cdot}(i,j)$ is the row vector given by the the third row of $Q(i,j)$.

This captures a wide range of interest rate rules found in the literature, including the current and forward-looking inflation targeting, interest rate smoothing, and output-gap targeting as examined in Woodford (2003b) and all the rules of Bullard and Mitra (2002).

Lemma 4.3.1. *For the system (4.10) to be determinate, i.e., to have a unique stationary solution,*

1. *The model*

$$0 = \sum_{j=-m}^{n} \tilde{Q}_j X_{t+j} \tag{4.12}$$

where $\tilde{Q}_j = \sum_{i=0}^{p} Q(i,j)$*, must have a unique saddle-point stable solution.*

2. *The square matrix*

$$\begin{bmatrix} \mathbf{Q} \\ \mathbf{B} \end{bmatrix} \tag{4.13}$$

must be non-singular. $\mathbf{Q}$ *and* $\mathbf{B}$ *are block matrices of dimensions* $3p \times 3(p+n)$ *and* $3n \times 3(p+n)$ *respectively with blocks of dimension* 3×3*. The* s^{th} *block row of* $\mathbf{Q}$ *is given by*

$$\begin{bmatrix} 0_{max(0,s-1-m)} & \tilde{Q}(s-1, -min(s-1,m), n) & 0_{p-s} \end{bmatrix} \tag{4.14}$$

where 0_i *is a* $3\times 3i$ *block vector of zeros and* $\tilde{Q}(a,b,c) = \begin{bmatrix} \tilde{Q}(a,b) & \tilde{Q}(a,b+1) & \ldots \tilde{Q}(a,c) \end{bmatrix}$ *with* $\tilde{Q}(a,b) = \sum_{i=0}^{min p,a} Q(i,b)$*. The* s^{th} *block row of* $\mathbf{B}$ *is given by*

$$\begin{bmatrix} 0_{max(0,s+p-m-1)} & -\tilde{B}\left(min(p+s-1,m)\right) & I & 0_{n-s} \end{bmatrix} \tag{4.15}$$

where I *is a* 3×3 *identity matrix and* $\tilde{B}(a)$ *being the last* $3 \times 3a$ *elements of the* $3 \times 3m$ *matrix* B *that forms Anderson's (2010, p. 7) convergent autoregressive solution to (4.12).*

Proof. See Appendix □

The first condition requires that the model be determinate if all lagged expectations are replaced with time t expectations and the second condition requires additionally that one can uniquely resolve the lagged expectations. Whiteman (1983, pp. 29–36) shows that resolving lagged expectations, "withholding" constraints in his language, is not generally a trivial task.

Carlstrom and Fuerst (2002, p. 82) make a quite general claim, without proof, regarding the conditions under which a model that satisfies the NRH is determinate: "[I]n a model that satisfies the NRH, there is real determinacy if and only if there is nominal determinacy in the corresponding flexible-price economy." In the two propositions that follow, I will substantiate the necessity component of their claim but refute the sufficiency component.

Proposition 4.3.2. *Consider a model in (4.10) that satisfies the NRH defined in (4.6). The model is determinate only if the corresponding frictionless model—i.e., one that satisfies (4.8)—is determinate.*

Proof. See Appendix □

Thus, a necessary condition for determinacy in any model that satisfies the NRH is that the corresponding frictionless model is determinate. In the latter, real variables are determinate by definition, so the question of determinacy pertains only to nominal variables. In the former, the output gap is jointly determined with nominal variables and thus determinacy relates to real as well as nominal variables. So the foregoing proposition corroborates the "only if" component of Carlstrom and Fuerst's (2002) claim, showing essentially that the eigenvalue counting method of Blanchard and Kahn (1980) is the same regardless of actual value of k.

Proposition 4.3.3. *Consider a determinate frictionless model—i.e. one that satisfies (4.8) in (4.10). There exist corresponding NRH models—i.e., that satisfy (4.6) for $k > 0$—that are not determinate.*

Proof. See Appendix □

Therefore, it does not necessarily follow that a model that satisfies the NRH is determinate when its frictionless counterpart is, refuting the sufficiency component of Carlstrom and Fuerst (2002). Lemma (4.3.1) shows that, while necessary, Blanchard and Kahn's (1980) saddle-point property of the underlying matrix polynomial is insufficient to conclusively establish determinacy. As Whiteman (1983, p. 33) points out, "the conditions for existence and uniqueness of solutions to withholding equations are quite different from those for the general expectational difference equation." The class of models in (4.10) combines the latter—i.e., forward looking difference equations—with withholding equations—i.e., lagged expectations—and, thus, it is not surprising that one has to take both the standard—i.e., saddle-point—and these quite different conditions into account.

Fortunately, it should be more the exception than the rule that the "if" is not fulfilled. This pertains to the non-singularity of the matrix $\begin{bmatrix}\mathbf{Q}' & \mathbf{B}'\end{bmatrix}'$, which cannot be guaranteed due to the generality of the class of models specified in (4.10). Yet, there is nothing in the class of models to induce this matrix to be singular in general. Even should one encounter a particular model parameterization leading to singularity, it should be expected that a minor pertubation of the model or its parameterization will lead to non-singularity. This is reminiscent of King and Watson's (1998, p. 1017) "mundane source" of indeterminacy, requiring here the structure of the model to be such that it leaves no linear combination of forecast errors unrestricted.

Moving past this additional source of mundanity, the close relationship between determinacy under the NRH and determinacy in the corresponding frictionless model has some strong implications. Indeed, if one restricts attention to models that satisfy the condition of non-singularity in (4.3.1), the following proposition can be made

Proposition 4.3.4. *Consider a model in (4.10) restricted to rule out the singularity of (4.13) and fix the demand equation and monetary policy.*

1. *If the model is determinate under one supply equation that satisfies (4.6), it is determinate*

under all supply equations that satisfy (4.6).

2. *If the model is not determinate under one supply equation that satisfies (4.6), it is not determinate under all supply equations that satisfy (4.6).*

In other words, for any given demand specification, the bounds on monetary policy to ensure determinacy are same for all supply equations that satisfy (4.6).

Proof. See Appendix □

With demand given by (4.1), restricting supply equations to satisfy the NRH (4.6), but leaving monetary policy still generically specified as in (4.11), a more specific statement can be made

Corollary 4.3.5. *Consider a model in (4.10) with demand given by (4.1) and any supply equation satisfying (4.6) and restricted to rule out the singularity of (4.13). Determinacy is a function solely of the parameters in the interest rate rule (4.11) pertaining to inflation and the interest rate.*

Proof. See Appendix. □

If the model satisfies the NRH, then the output gap must on average be equal to zero independent of monetary policy (see McCallum (1998, p. 359)). From (4.1):

$$E\left[y_t - y_{t+1}\right] = a_1 E\left[\pi_{t+1} - R_t\right] \tag{4.16}$$

which posits a relationship between the average output gap and monetary policy (defined over the nominal interest rate R_t). One could certainly specify a process for the nominal interest rate such that the average output gap would be equal to zero, but the NRH requires that this hold *regardless* of monetary policy. Thus, that the output gap on average is equal to zero must follow from the supply side equation and must hold independently of (4.1).

The NRH delivers, then, the existence but not necessarily the uniqueness of a bounded path for the output gap irrespective of the existence and uniqueness of bounded paths for inflation and the nominal interest rate. However, from (4.16) it must then be the case that the real interest rate $R_t - E_t [\pi_{t+1}]$ also converges. Furthermore, if the bounded path for the real interest rate is uniquely determined, then so is the bounded path for the output gap and vice-versa.

The uniqueness of a bounded path for inflation and the nominal interest rate is, thus, given by the rule for monetary policy and (4.9).[14] Determinacy, therefore, corresponds to nominal determinacy in the frictionless counterpart.

Were $k = 0$, there would be complete separation between the real and nominal sides of the economy and monetary policy (through a rule for the nominal interest rate) would serve only to establish nominal determinacy. Otherwise, if $k > 0$, the lack of a complete separation but fulfillment of the NRH by assumption links nominal and real determinacy: without a unique path for the nominal side, the link between the output gap and the nominals at horizons less than k implies that although every path for the output gap be bounded, a unique path for the output gap cannot be pinned down. If a unique path for the nominal side can be determined by (4.9) and monetary policy, this path selects, through the link at horizons less than k, a single path for the output gap.

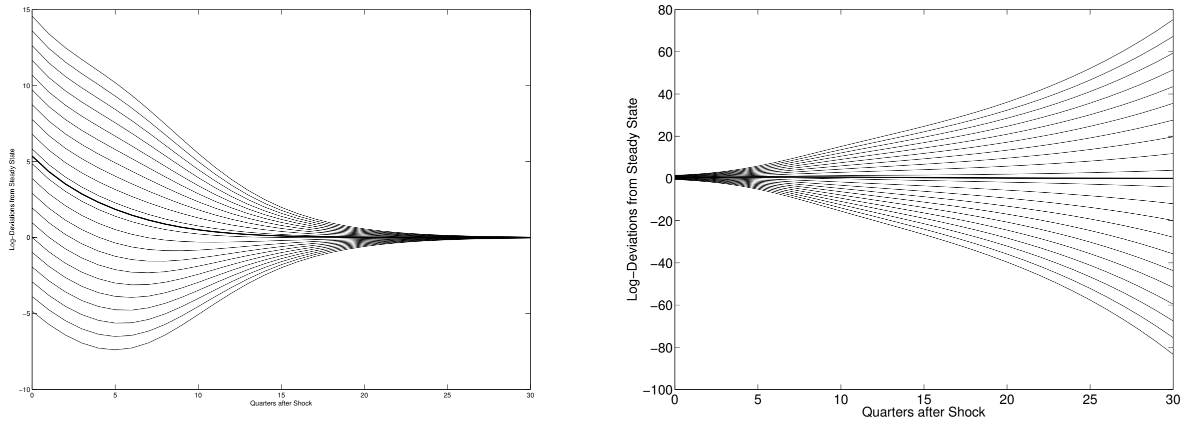
Therefore, there is a unique convergent path for the output gap if and only if there is a unique convergent path for inflation and the nominal interest rate in the counterpart model (4.9).[15]

The situation is exemplified graphically in figure 4.2. All the different paths of the output gap in figure 4.2a converge even though all but one of the paths for inflation, depicted in

[14] I.e., the Fisher-type equation with the real interest rate normalized to zero or as derived from the dynamic IS equation (4.16) with the output gap always closed.

[15] Saving, of course, for the caveat of the singularity of (4.13).

figure 4.2b, diverge. If one has reason, say by requiring inflation to be stable, to select among the different paths for inflation, the selected path for inflation corresponds to a particular path for the output gap, thus determining both through consideration solely over inflation.



(a) Output Gap (b) Inflation

Figure 4.2: Impulse Responses to a Demand Shock for Different Initial Conditions

A few comments are in order here. Real business cycle models are generally of the type that the NRH holds but does so already at $k = 0$, as complete flexibility in prices is assumed.[16] In the sticky-price New Keynesian model, the NRH does not hold at any horizon. As a consequence, the sticky-price model is not even asymptotically isomorphic to its frictionless equivalent, and there is no reason to expect a general equivalence between determinacy conditions in the two models. With there being a permanent link between the nominal and real side of the economy, nominal and real determinacy must be simultaneously ascertained. As discussed previously, many modifications of the standard sticky-price model do satisfy the NRH assuming inflation is stationary. Since we are only interested in stationary equilibria, there would not seem to be a contradiction. This is mistaken as establishing determinacy requires one to look at all possible equilibria, including explosive equilibria, in the hope that only one is non-explosive. Thus, in assessing determinacy in the standard New Keynesian

[16] See Woodford (2003b, p. 6).

model, one is forced to look at paths along which both the NRH is violated and its violation is consequential for the ensuing path.

When the NRH does not hold at every horizon (i.e., $k > 0$), nominal and real determinacy are linked as in standard sticky-price models. That the NRH holds at all, however, ensures that this link dissolves such that conditions necessary to determine this determinacy are identical to the conditions for nominal determinacy that would prevail were the NRH to hold at all horizons. This conceptual link between nominal determinacy in RBC models and both real and nominal determinacy in NRH models provides for a simple means to establish nominal and real determinacy: one need only to examine the conditions for nominal determinacy in the corresponding frictionless equivalent. This is generally a much simpler task.

Table 2.1 in chapter 2 juxtaposes the bounds on several standard interest rate rules both with the standard sticky-price Phillips curve and Mankiw and Reis's (2002) sticky-information Phillips curve.[17] As noted by its authors, the latter satisfies the NRH—but only asymptotically as opposed to the $k < \infty$ assumed here following Carlstrom and Fuerst (2002). The bounds derived in chapter 2 for determinacy coincide with those required for nominal determinacy in the frictionless model for the set of standard interest rate rules examined.

Thus a broad class of models, those satisfying Lucas's (1972a) NRH, achieve determinacy under the same conditions and do so independently of parameters outside of the monetary policy rule. Wherefore, the bounds derived under the NRH pass the additional criticism of Cochrane (2007, p. 27) that the bounds for determinacy ought to not be complex functions of the entire parameter space of the model. This follows from the reduction of the system to the interest rate rule and the Fisher-type equation, which makes no reference to any parameters in either the demand or the supply side. The common trait is a long-run vertical Phillips

[17]It is astounding that Cochrane (2007, p. 24) claims, "Mankiw and Reis (2002) argue for a return to mechanical or adaptive expectations, [...] though this means throwing out economic microfoundations." Mankiw and Reis's (2002, p. 1297) model has fully rational expectations and is microfounded (see Reis (2006)).

curve,[18] that "by 1980 even self-styled Keynesian economists were agreeing to." (McCallum 2004a, p. 21)

The analysis here should make one wary of the conclusions from determinacy analysis in the New Keynesian literature: its policy recommendations or restrictions in terms of bounds on monetary policy are a consequence of the New Keynesian Phillips curve's violation of the NRH. This does not mean that the literature-standard sticky-price model ought to be rejected, merely that we should not ask it to perform tasks for which it was not intended. Among these is the assessment of determinacy, a long-run question[19] that requires the examination of explosive paths, and when addressing it, we should use models whose long-run properties are defensible.

4.4 Determinacy and the Cochrane (2007) Critique

With the general results for determinacy of models that satisfy the NRH, I shall confront the issue, raised by Cochrane (2007), of whether determinacy is an appropriate means to justify a unique equilibrium. The equilibria ruled out by determinacy are in fact legitimate monetarist equilibria resulting from the deficiency of defining monetary policy solely over the nominal interest rate. Interpreting steady-state inflation as a long run monetary target provides the missing mechanism to select the determinate equilibrium as the only permissible one.

Cochrane (2007) has challenged the determinacy analysis in the New Keynesian literature. It notes that explosive paths are ruled out for both nominal and for real variables. One can generally rule out explosive paths for real variables by appealing to a transversality condition, but such a condition is lacking for nominal variables. In the previous section, I imposed saddle-point stability on a real variable, the output gap, and two nominal variables,

[18] The NRH and vertical Phillips curves are central to the rational expectations revolution, see Lucas (1972a) and Sargent (1973), with Sargent (1987b, p. 7) calling Friedman's (1968) address its "opening shot".

[19] Emphasized also in chapter 2.

the nominal interest rate and inflation. Cochrane's (2007) critique is directly relevant for the analysis of the foregoing section: if the model is determinate, there is one stable path and a continuum of explosive paths; if the model is indeterminate there is a continuum of stable paths. Conveniently, the model of dialogue between Cochrane (2007), Cochrane (2009) and McCallum (2009b), McCallum (2009c) is the frictionless model of the previous section, upon whose stability the determinacy analysis of all NRH models with the standard dynamic IS equation depends.

Interestingly, Cochrane's (2007) critique, however, does not actually apply to the standard three equation New Keynesian model. As was laid out in previous section,[20] the lack of a long-run vertical Phillips curve implies quite generally that an explosive path for inflation implies an explosive path for the output gap. Thus, if "[e]conomics can rule out real explosions", then a supply schedule that violates the NRH, by associating explosive paths for inflation with explosive paths for the output gap, will give one the means to rule out the nominal explosions as well. This situation, depicted by Cochrane (2007, p. 28), is reproduced in figure 4.3: the explosiveness for the nominals is associated with explosiveness for the real variables. Cochrane (2007, p. 25) admits that the output gap, a real variable, explodes in all equilibria except for the equilibrium chosen in standard New Keynesian analysis, but softens his distinction between real and nominal variables with the statement, "[n]o economic consideration rules out the explosive solutions." I believe he is mistaken with the claim that the situation here is exactly the same as in the frictionless case. In the frictionless case, the problem was the legitimate one of a nominal explosion without a real explosion, whereas here the two go hand-in-hand. This permanent tradeoff makes the New Keynesian Phillips curve ill-suited to examine or even exclude hyperinflationary paths, reiterating the analysis of the foregoing sections. Long-run questions—like determinacy as well—require a model whose

[20]And, of course, within the linear(ized) framework of the previous section.

long-run properties are defensible.[21]

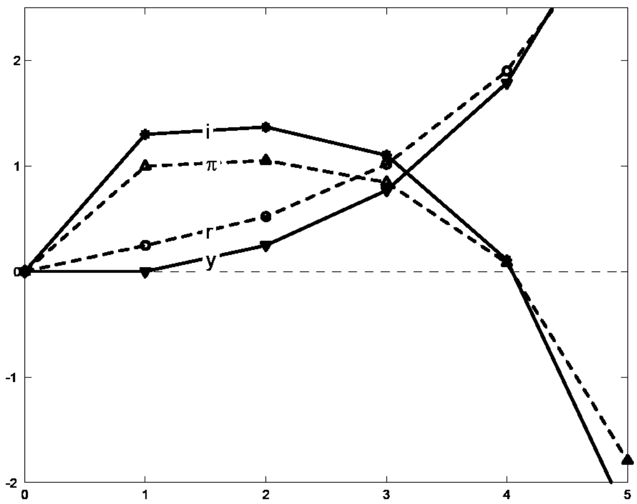


Figure 4.3: Response of the Three-Equation New-Keynesian Model to a One-Percent Off-Equilibrium Inflation Innovation, with No Change in Output. From Cochrane (2007, p. 28)

Returning to the NRH model of the previous sections to address Cochrane's (2007) critique within the NRH, temporarily replace the assumption of a Taylor rule with monetary policy defined as control over the money supply. Append the model with a standard money demand function[22] in first difference form

$$\mu_t - \pi_t = \eta_y \gamma_t - \eta_R \Delta R_t + \Delta \epsilon_t^m \tag{4.17}$$

where μ_t is the money growth rate, γ_t the growth rate of output, and ϵ_t^m a money demand shock. The output gap is necessarily stationary due to the NRH being fulfilled so we can neglect both γ_t and ϵ_t^m for the purposes of asymptotic behavior if it can be assumed that

[21]McCallum (2003, p. 1157) actually anticipates this discussion: "the [Calvo] form of sticky prices [...] is such that the model continues to include nominal variables even when monetary policy supplies no nominal anchor, because private behavior involves a type of dynamic money illusion [as the model violates the NRH.]"

[22]I adopt the notation of Woodford (2008) for ease. Note that as discussed in, e.g., Woodford (2008), McCallum (2008), and Nelson (2008), adding a money demand relation does not alter the previous analysis. It adds one variable and one equation and is 'superfluous' according to McCallum (2008, p. 1785) with monetary policy defined over the interest rate or the previous analysis was 'self-contained' in its absence according to Nelson (2008, p. 1799). The nonlinear origin of this standard equation is of importance only insofar as it provides transversality conditions to rule out particular paths of variables or insofar as its linearization leads to spurious artifacts. In the next section, some specific origins will be examined and an artifact of linearization will be addressed.

the natural rate of output and the money demand shock are at least difference stationary[23]

$$\mu_t - \pi_t = -\eta_R \Delta R_t \tag{4.18}$$

Thus, (4.18), the Fisher equation $R_t = E_t [\pi_{t+1}]$, and a process for the money supply constitute a specification for inflation, money growth, and the nominal interest rate. This is identical to the Cagan model under rational expectations of Sargent and Wallace (1973),[24] but the focus here—due to Cochrane (2007)—is on potentially explosive inflation and not just the price level.

Consider the case of a constant money supply constant ($\mu_t = 0$), reducing the system to

$$R_t = \eta_R E_t [\Delta R_{t+1}] = \frac{\eta_R}{1+\eta_R} E_t [R_{t+1}] \tag{4.19}$$

One solution is $R_t = \pi_t = 0$. McCallum (2001a, p. 26) labels this the "monetarist solution". But a whole continuum of solutions exists with R_t and π_t diverging to positive or negative infinity. These hyperinflations and hyperdeflations are speculative in nature, as they are not accompanied by equivalent movements in the money supply. Although Sargent and Wallace (1973) rule them out with an arbitrary terminal condition, this continuum of additional solutions *can* be ruled out by economic theory. I shall address this in the next section by, e.g., postulating that money is essential.[25]

But this, of course, does not mean that the model is incompatible with hyperinflation. Assume that the monetary authority follows an extraordinary money creation scheme,

[23] As emphasized by McCallum and Nelson (2009a, pp. 13–15), the key element for the quantity theory is the unitary relation between money and prices—a stability of the money demand function with respect to other parameters and variables is not necessary for the theory's relations.

[24] See their Equation (4), where the only difference is the first difference of a "stochastic term with central tendency equal to zero" that I have omitted here.

[25] Note that the essentiality of money rules out speculative hyperinflation. Speculative hyperdeflation can typically be ruled out under weaker restrictions and I, like McCallum (2009b, pp. 1106–1107), will not dwell on them in the following. Gray (1984) shows that such paths can always be ruled out in the class of money-in-the-utility models she examines as they provide households with an open-ended arbitrage opportunity. Obstfeld and Rogoff (1986, pp. 355–358) demonstrate that such paths can be ruled out in a transactions-technology model and provide some intuition for off-equilibrium threats that can rule out speculative hyperdeflation even in some extreme cases.

whereby *the growth rate* of the money supply is increasing exponentially ($\mu_t = \lambda\mu_{t-1}$, $1 < \lambda < \frac{1+\eta_R}{\eta_R}$).[26] Thus,

$$R_t = \eta_R E_t [\Delta R_{t+1}] + E_t [\mu_{t+1}] = \frac{\eta_R}{1+\eta_R} E_t [R_{t+1}] + \frac{1}{1+\eta_R} E_t [\mu_{t+1}] \tag{4.20}$$

One equilibrium has R_t and π_t increasing at the same rate as μ_t—the monetarist solution

$$\pi_t = \frac{1}{1-\eta_R(\lambda-1)}\mu_t,\ R_t = \frac{\lambda}{1-\eta_R(\lambda-1)}\mu_t \tag{4.21}$$

Defining $\tilde{\pi}_t$ and $\tilde{R}_t$ as the difference of inflation and the nominal interest rate from their values in the monetarist solution, the system can be reduced to the case of a constant money supply in the redefined variables and, thus, there is a whole continuum of solutions with R_t and π_t diverging from μ_t. All the paths off of the monetarist hyperinflation path can be ruled out under the same conditions as before—e.g., the essentiality of money.

Cochrane's (2007) critique need not, therefore, be referring to speculative abberations. What, then, goes awry with interest rate rules? Define monetary policy over the nominal interest rate,

$$R_t = \phi_\pi \pi_t \tag{4.22}$$

Given this rule, and the Fisher equation ($\pi_t = \frac{1}{\phi_\pi} E_t [\pi_{t+1}]$) one solution is $\pi_t = R_t = 0$, which implies through (4.18) that $\mu_t = 0$. But a whole continuum of solutions satisfying

$$\pi_t = \phi_\pi \pi_{t-1} \tag{4.23}$$

are also potential equilibria. Despite the similarity of the approach to the case of money supply rules, it does not immediately follow that the first solution is the monetarist solution,

[26] The restriction on the growth rate of the money growth rate is required for "process consistency" reasons, see Flood and Garber (1980a) and McCallum (1983). Essentially, the rate of money growth would be growing too quickly to be commensurate with the (linearized) money demand function. Taking, e.g., Ball's (2001) estimate for the interest semi-elasticity of money demand, $\eta_R = 0.05$, the process consistency limit is equal to 21—limiting the period-over-period change in the *growth rate* to a fantastical 2000%. The next section will show this to be an artifact of linearization.

and the remaining continuum can be ruled out as speculative. This classification rests decisively on the relation of the behavior of prices to that of money, which has not yet been introduced for the analysis with interest rate rules. Combining (4.18) with (4.22) and (4.23)

$$\mu_t = \left[1 - \eta_R\left(\phi_\pi - 1\right)\right]\pi_t \tag{4.24}$$

In the context of determinacy, one would require $\phi_\pi > 1$.[27] A $\phi_\pi > 1$ means the potential equilibria are characterized by explosive paths for inflation and the nominal interest rate. But this implies that the money supply growth rate is increasing proportionally with the inflation rate. Monetary policy accommodates these hyperinflationary equilibria, making the continuum of explosive paths of inflation consistent "moneterist solutions" through extraordinary money supply growth.[28]

This highlights where the New Keynesian sticky-price model breaks down: monetary policy cannot pursue the aggressively inflationary money-supply growth associated with these explosive equilibria, as this policy, through the violation of the NRH, would be associated with an explosion in the output gap, which can be ruled out by appealing to a transversality argument. Cochrane (2007, p. 25) states, "sensible economic models work in hyperinflation or deflation. If they don't, it usually reveals something wrong with the model." This statement needs to be tempered, I believe, with the assessment that the New Keynesian model was never intended as an explanation of hyperinflation. However, this certainly does mean that one must be wary of drawing any conclusions that implicitly rest on the analysis of hyperinflation, such as determinacy, in such models.

The sticky-price model was conceived as a model for short-term fluctuations. In the background and in the back of the modelers' minds is an RBC model with full neutrality in the long run.[29] Woodford (2008) shows that the standard sticky-price model fulfills a list of

[27] Assuming the interest rate react positively to inflation.

[28] As explained in footnote 26, a process consistency constraint is present here as well: $\phi_\pi < \frac{1+\eta_R}{\eta_R}$.

[29] See Woodford (2003b, Ch. 3, esp. p. 142).

neutrality properties. What has not garnered attention is that these properties may only be fulfilled by the determinate solution itself. Indeed, the examination of determinacy—though short-run in its consequences through potential sunspot equilibrium—is an examination of the long-run: does a particular equilibrium path converge asymptotically to the steady state or does it diverge? The New Keynesian model through its violation of the NRH and inability to give an accurate picture of equilibria on divergent (i.e., hyperinflationary) paths is not suitable for such long-run analyses as determinacy.

Thus, Cochrane's (2007) critique is wholly valid in the set of models examined in the previous section. Should the model be associated with determinacy, all of the explosive paths constitute fully valid equilibria. But the reasoning of Cochrane (2007)—the absence of transversality conditions for nominal variables—obfuscates the real reason for the validity of these equilibria. An equilibrium with inflation diverging towards infinity is valid precisely because the monetary authority keeps increasing the growth rate of the money supply, accommodating the ever increasing inflation rates.

McCallum (2009b) offers LS learning as a means to "select" the determinate solution. If Cochrane's (2007) explosive equilibria are legitimate, McCallum's (2009b) argument must have some defect. Reinterpreting the explosive equilibria in terms of an exogenous process for the money growth as I have done, Cochrane's (2007) explosive equilibria are associated with explosive processes for money growth. But McCallum (2009b, p. 1103), following Evans and Honkapohja (2001, pp. 198& 229), requires the exogenous processes to be stationary. That is, McCallum's (2009b) LS learning rules out Cochrane's (2007) explosive equilibria by assumption. With *least-squares (LS)* learning, agents' expectation formation progresses too slowly for explosive money growth: this is not a reassuring mechanism to prevent hyperinflation.

With monetary policy defined solely over control of the short-run nominal interest rate, there is, therefore, an entire continuum of valid equilibrium paths in the absence of any fundamental shock ranging from hyperinflation to hyperdeflation when the determinacy conditions of the previous section are satisfied.[30] That is, there must be some defect in defining monetary policy solely in terms of the short-run nominal interest rate. This is precisely the point made by Nelson (2008, p. 1805): "the monetary authority cannot treat the nominal interest rate as an instrument in the long run." What is his proposed solution? "Long-run money growth determines long-run inflation,"

> Though they no longer affect real interest rates, and no longer can affect nominal rates via *a liquidity effect*, the central bank's open market operations continue in the long run to affect nominal money growth. So nominal money growth is unambiguously and undeniably susceptible to central bank influence even in the long run... Reaching [an] inflation target means a specified quantity of open market operations in the steady state; specifically, open market operations that deliver a steady-state money growth [consistent with the inflation target and the secular growth]. There it is: the sense in which steady-state inflation can be regarded as pinned down by steady-state money growth. Nelson (2008, p. 1805)[emphasis in the original]

Let monetary policy be fully specified by adding a steady-state inflation rate, which can be sensibly interpreted as an average growth rate for the money supply. Thus the main result:

Proposition 4.4.1. *Consider the NRH model of the foregoing section appended with (4.18). Monetary policy is specified by an interest rate rule and an average money growth rate. If the interest-rate rule is associated with a determinate equilibrium, this equilibrium is the unique equilibrium.*

[30] When they are not, there is an additional dimension of indeterminacy.

Proof. See Appendix □

Cochrane's (2007) "threat" of monetary policy is not hyperinflation, hyperdeflation, or "to blow up the world", but rather to simply keep money growth constant. All that is needed here is the commitment on behalf of the central bank to ensure the unconditional expectation of the money growth rate be equal to the steady-state value it selects. Note that this still allows for the multiple equilibria in case of indeterminacy, not curing all the ills of interest rate policy. All of the multiple equilibria in case of indeterminacy converge back to the steady state allowing the average money growth rate to be satisfied and thusly cannot be ruled out.

Monetary policy is not bound by any restriction to accommodate the hyperinflationary or hyperdeflationary paths. The threat that monetary policy will not keep increasing [decreasing] the rate of money growth boundlessly would seem credible and is already incorporated in the framework of several central banks. Most notably the monetary analysis pillar of the ECB, but also Section 2a of the Federal Reserve Act requiring that the Federal Reserve "shall maintain long run growth of the monetary and credit aggregates [...] so as to promote effectively [...] stable prices."[31]

Both of these central banks have committed, implicitly or explicitly, to keeping the rate of growth of the money supply at very least finite. So long as this commitment is credible, no explosive path for inflation can be an equilibrium. Following, e.g., Friedman and Schwartz (1963), monetary restraint is necessary and sufficient for controlling inflation, at least in the long-run. And, as emphasized by Nelson (2008), monetary policy if defined soley over control of the nominal interest rate is incomplete, as the monetary authority cannot control this variable in the long run. There is thusly, no contradiction between monetary policy being defined over control of the nominal interest rate at all finite horizons and over the rate of

[31] Paraphrasing the Chairman of the Board of Governors slightly: Bernanke (2008, pp. 317 & 319) emphasizes that although they have not played a central role in recent times, monetary data is and will continue to be monitored by the Federal Reserve as a sensible part of the framework of monetary policy.

money growth asymptotically.[32]

One immediately appealing equivalent measure to the average growth rate of the money supply in proposition 4.4.1 is a direct inflation target. One could interpret proposition 4.4.1 as wholly consistent with such a form of direct inflation targeting: if the inflation target is credible, any equilibrium path that diverges from the target contradicts the target's credibility.[33] However, this is the "high-level assumption" that Nelson (2008, p. 1803) argues is deceiving, as it assumes a permanent liquidity effect. It is exactly this permanent liquidity effect that imbues the nominal interest rate with an always and everywhere stabilizing effect, which Cochrane (2007) criticizes as the New Keynesian literature's intuitive reliance on "old Keynesian" thinking. Likewise Meltzer (1999, p. 268) notes that the reliance on the nominal interest rate to indicate the expansiveness of monetary policy has misled the Federal Reserve on a number of occasions. As one should not neglect the NRH and its short-run implications in assessing determinacy, one should not neglect that monetary policy has no direct control over the nominal interest rate or inflation in the long run.

However, keeping the foregoing reservations in mind, the notion of an inflation target for the long run as being a key element of a well-formulated monetary policy is germane to the "constrained discretion" interpretation of inflation targeting by Bernanke, Laubach, Mishkin, and Posen (1999, p. 22), under which "inflation targets keep the economic ship in the right area in the long term," but where the interpretation of inflation targeting as a strict rule is rejected. In sum, a particular inflation rate in the long-run is the target and the commitment to keeping the money growth rate finite, monitoring mid-term developments in the monetary aggregates, and/or a commitment to an average money growth rate consistent

[32]Indeed, Friedman (1960, p. 35) states, "[t]he sufficiency of open market operations as a tool for monetary policy is not, of course, a decisive reason for relying on this tool alone." Likewise, Brunner and Meltzer's (1976, pp. 98–99) analysis differentiates between the "accumulated effects of past policies" and one-off impulses.

[33]Such a policy was rejected half a century ago by Friedman (1960, p. 88): "[W]e will [...] further the ultimate end of achieving a reasonably stable price level better by specifying the role of the monetary authorities in terms of magnitudes they effectively control and for whose behavior they can properly be held responsible[...] In this as in so many human activities what seems the long way round may be the short way home."

with the inflation target is the rule.[34]

4.5 Nonlinear Money Demand and the Monetarist Equilibrium

In this section, I wish to justify the selection of the monetarist equilibrium in the previous section that validated Cochrane's (2007) explosive inflation by reckless money growth. For the sake of brevity, I would only note that significant price level movements in the absence of corresponding movements in the money supply are inconsistent with the empirical evidence. Yet, as Meltzer (1999, p. 262) notes, "[e]conomists are rarely satisfied with evidence that something works in practice. They are inclined to be more interested in whether it works in theory." So despite the compelling reasons to dismiss speculative inflation and deflation *a priori*, as conceded by Obstfeld and Rogoff (1986), I shall also offer formal arguments in the context of the models presented by Cochrane (2007) and McCallum (2009b) that an explosive equilibrium for inflation is only admissible with an associated explosive money growth rate.

As to practice, Friedman (1958, p. 172) noted,"[t]here is perhaps no empirical regularity among economic phenomena that is based on so much evidence for so wide a range of circumstances as the connection between substantial changes in the stock of money and in the level of prices." Flood and Garber (1980b) reject the hypothesis of a bubble in the German hyperinflation of the '20s and, in the face of such empirical evidence, Flood and Garber (1980b, p. 760) state that "this artifact of dynamic models is unimportant; a special case of these models adequately predicts behavior, and further elaboration of the model to explain unobserved phenomena is unnecessary." More recently, McCallum and Nelson (2009a, p. 37)

[34]Such a rule is easily implemented here as there are no impediments to the central bank committing to set policy according to the interest rate rule along a determinate equilibrium and or by keeping money growth equal to the target on off-determinate equilibrium paths—the "threat" from above. True welfare- or loss-function-based assessments as to the credibility of such an immediate switch is beyond the analysis here. However, with all off-determinate equilibrium paths associated with infinite divergence of inflation, there would seem to be a great a priori incentive for the central bank to avoid such paths. Nelson (2008, p. 1806) also notes that "what needs to be kept in mind is that such an approach is a shortcut or an abstraction that takes for granted the underlying operations involving money on the part of the central bank."

conclude, "[n]ominal homogeneity of money demand is not rejected irrespective of the inflation series used, the definition of money chosen, or sample period considered."

Theoretically, explosive paths of inflation *can* be associated with explosive paths of the money growth rate. The question at hand from the foregoing section is whether this *must* be the case. Cochrane (2007, p. 22) mentions and McCallum (2009b, p. 1106) discusses the literature that addresses this question, that of speculative hyperinflations, but both fail to note the decisive role of money. This literature does not purport to address whether explosions in inflation can be ruled out in general, as my reading of McCallum (2009b, p. 1106) might lead one to believe, but seeks to address whether those explosions can be ruled out that are "unrelated to monetary growth." (Obstfeld and Rogoff 1983, p. 675) The question of whether explosive price paths can exist *without* monetary growth cannot be equated to the credibility of the "threat of the government to take the economy to a configuration (hyperinflation or deflation) in which the [sic] we all know the economy will blow up on its own." (Cochrane 2007, pp. 22–23) In a nutshell, Scheinkman (1980), Obstfeld and Rogoff (1983), Gray (1984) and Woodford (1994) demonstrate that the speculative hyperinflations in separable money-in-the-utility-function, medium-of-exchange, and cash-in-advance setups can be ruled out by requiring money to be essential or have intrinsic value. Intuitively, if real balances are necessary or necessarily of worth, a hyperinflationary path initiated by a whim and not accompanied by money growth would rob utility maximizers of this necessity, bringing the rational origin of such a whim into question.

Turning to the specific discussion of Cochrane (2007) and McCallum (2009b), Cochrane (2007, p. 22) lays out a two-equation nonlinear model under perfect foresight to address the issue, whose necessary conditions are

$$1+i_t = \beta^{-1}\Pi_{t+1}\frac{u_c(Y, M_t/P_t)}{u_c(Y, M_{t+1}/P_{t+1})} \tag{4.25}$$

$$M_t/P_t = L(Y, i_t) \tag{4.26}$$

along with a specification of monetary policy. In his appendix, Cochrane (2007) solves for the latter of the foregoing using a first-order condition relating bond and money holdings,

$$\frac{i_t}{1+i_t} u_c(Y, M_t/P_t) = u_m(Y, M_t/P_t) \tag{4.27}$$

The foregoing, or more generally (4.26), can be linearized and first-differenced to yield (4.17).

Let us eliminate the only possibility mentioned by Cochrane (2007, pp. 21–23) to rule out explosions with the extension to money, namely the possibility of the real interest rate going to infinity due to monetary distortions—i.e., the passive "blow-up threat by the government." One easy way to do this is to assume separability ($u_{cm} = 0$), reducing the model to $1 + i_t = \beta^{-1}\Pi_{t+1}$ and $\frac{i_t}{1+i_t} u_c(Y) = u_m(M_t/P_t)$. From the latter, it follows immediately that $lim_{i_t \to \infty} u_m(M_t/P_t) = u_c(Y)$. Thus, real balances must be constant if $i_t \to \infty$, necessarily requiring the growth rate of money to be equal to inflation. If the interest rate follows an active Taylor rule (i.e., $i_t = \Phi(\Pi_t)$, with $\Phi' > 0$), explosive inflation leads to an explosive nominal interest rate. In terms of the process consistency requirement of the preceding section, there is no upper bound on the elasticity of the nominal interest rate with respect to inflation here. More generally, the assumption that money is essential, following Obstfeld and Rogoff (1983, p. 681) and Gray (1984, p. 100),

$$\lim_{m_t \to 0} m_t u_m(Y, m_t) > 0 \tag{4.28}$$

would suffice to ensure that any hyperinflation or deflation is *necessarily* associated with a corresponding path of money.

Using a standard money-in-the-utility function from Galí's (2008, p. 27),

$$U(C_t, \frac{M_t}{P_t}, ...) = \frac{C_t^{1-\sigma}}{1-\sigma} + \frac{(M_t/P_t)^{1-\nu}}{1-\nu} \tag{4.29}$$

the foregoing condition holds for all $\nu > 1$: i.e., for elasticities of utility with respect to real balances greater than unity—not a severe restriction.[35] With these preferences, optimality requires,

$$\frac{M_t}{P_t} = C_t^{\frac{\sigma}{\nu}} \left(1 - \frac{1}{R_t}\right)^{-\frac{1}{\nu}} \tag{4.30}$$

which can be linearized, combined with market-clearing, and first-differenced to yield (4.17). The process-consistency restrictions come from the interest elasticity of money demand, η_R, which is a constant after linearizing. In the nonlinear version, however, it is equal to $\frac{1}{\nu}\frac{1}{R_t-1}$ and with an active interest rate rule, this elasticity will approach zero as inflation explodes, again confirming the process-consistency restrictions to be an artifact of linearization.

The essentiality of money required by (4.28) might seem too much to require of a model. Indeed, McCallum (2009b, p. 1106) cites Obstfeld and Rogoff's (1983, p. 675) conclusion in their money-in-the-utility framework that this constitutes an extreme restriction on preferences and goes on to claim that, "a model specification that drives consumption to zero (as real money holdings decrease) implies that a barter economy would necessarily feature zero consumption. That should be regarded as an inadmissible assumption." (McCallum 2009b, p. 1107) Yet, McCallum (2009b, p. 1107) adopts a transaction function that does just this.

Gray (1984, p. 106) requires the limit of real balances times the marginal transaction cost to be negative infinity as real balances approach zero. McCallum (2009b) mistakenly states that Gray's (1984) analysis lacks an extension of a transaction-costs function dependant on the quantity of transactions. Gray (1984), however, does not address the case that combines this extended transaction-costs function with convex utility. Yet, her results extend straightforwardly to this case, as I show in the appendix, and with the sufficiency conditions fulfilled by McCallum's (2009b) transaction function, Gray's (1984, p. 113) requirement is

[35] Of course, this does not contradict Obstfeld and Rogoff's (1983) assessment that this is an extreme restriction on preferences. The functional form itself of preferences over real balances is what here might justifiably be called "extreme."

necessarily fulfilled

$$\lim_{m \to 0} m\Phi_2(C,m) = \lim_{m \to 0} -a_2 a_1 C^{1+a_2} m^{-a_2} = -\infty < 0 \tag{4.31}$$

as a_1, a_2 are both positive.[36] So McCallum (2009b) does rule out speculative hyperinflations and hyperdeflations. Implicitly, McCallum (2009b, p. 1107) finds the monetarist hyperinflation: "as inflation explodes, [... real balances do] not approach zero." If inflation explodes, the price level explodes at an exploding rate. With real balances approaching a constant, money is exploding at the same exploding rate as prices. That is, inflation and the rate of money growth are exploding together.

Beyond essentiality of money, Obstfeld and Rogoff (1983) show that if money has some intrinsic value, however fleetingly small, speculative hyperinflations will be impossible. Despite having technically ruled out the possibility, McCallum (2009b, p. 1107) reasons for the existence of speculative hyperinflation, as their impossibility would require a barter economy to be associated with zero consumption. Yet the return to a barter economy along such paths is not an inexorable conclusion, as the transactions-chain approach to the medium-of-exchange explanation of money in Brunner and Meltzer (1971, p. 801) demonstrates.[37] Friedman and Schwartz (1970, p. 108) too, in their discussion of money as a medium of exchange, note an "irreducible minimum [real-value quantity of money] necessary for transactions purposes" that make the necessity of money absolute. One could, alternatively, assert that there is a discontinuous difference between approaching a barter economy through rampant hyperinflation and actually being in a functional barter economy.

Theoretical and empirical considerations aside, ruling out speculative hyperinflations is necessary for maintaining the proposition that "the central bank can reasonably be held accountable for controlling inflation." (Woodford 2008, p. 1563) The central bank would

[36] Additionally, note that in McCallum's (2009b, p. 1107) model, money demand is given by $\Phi_2(C_t, m_t) = -\frac{i_t}{1+i_t}$ which, again, can be linearized, combined with market-clearing, and first-differenced to yield (4.17).

[37] Though this approach simultaneously appears to rule out all hyperinflations, speculative or monetary, as new mediums of exchange are sought.

certainly be relieved of this accountability if it were—at any moment of time—probable (or even if it were merely possible) that the price-level or inflation could go spiralling out of control despite a constant money supply or growth rate thereof.

Thus, in any sensible monetary description, it ought to hold that "[t]here is a one-to-one relation between monetary changes and changes in [...] prices"—at very least in the long run or for "major economic fluctuations". (Friedman and Schwartz 1963, p. 50) This requires ruling out non-monetary divergences. At the same time, accepting this one-to-one relation and the empirical evidence that hyperinflations have occurred forces one to dismiss specifications or equilibrium-selection devices that would rule out *fundamental* divergences. The associated skepticism applies not only to the analyses where monetary policy drives the real interest rate to infinity to rule out hyperinflations as argued in Cochrane (2007, pp. 22–23) or to a transversality-based argument on real variables in a model like the standard New Keynesian model with a non-vertical long-run Phillips curve, but also to the LS-learnability analysis of McCallum (2009b, pp.3–13) that would rule out explosive money supply growth rates as inflation in the associated equilibria would accelerate more quickly than could be learned by least-squares agents.

4.6 The Nominal Interest Rate

From the previous sections, it should be clear that some mention of money is advantageous in a monetary model. Monetary policy should make some reference, implicit or explicit, to the money supply as a monetary policy defined solely over the nominal interest rate is insufficient to control inflation. As Cochrane (2007, p. 42) rightfully criticizes, one cannot "use old Keynesian stabilizing logic" to describe the mechanism of inflation control at work with an interest rate rule in a New Keynesian model.[38] The old Keynesian stabilizing logic

[38]Perhaps, the alternative nomenclature "New Neoclassical" noted again by McCallum (2009b, p. 1102) is indeed more appropriate with the NRH supply of this chapter, reserving "New Keynsian" for those models that

focuses on the liquidity effect and neglects the Fisher effect,[39] leading to difficulties for New Keynesian explanations of Friedman's (1968, p. 7) observation that "low interest rates are a sign that monetary policy has been tight—in the sense that the quantity of money has grown slowly; high interest rates are a sign that monetary policy has been easy—in the sense that the quantity of money has grown rapidly." Raising the nominal interest rate once is associated with tight monetary policy via the liquidity effect, but raising the nominal interest rate continually must certainly be associated with easy monetary policy:

> Add only one wrinkle to Wicksell—the Irving Fisher distinction between the nominal and the real rate of interest. Let the monetary authority keep the nominal market rate for a time below the natural rate by inflation. That in turn will raise the nominal natural rate itself, once anticipations of inflation become widespread, thus requiring still more rapid inflation to hold down the market rate. (Friedman 1968, p. 8)

Cochrane's (2007) explosive equilibria under an active interest rate rule (i.e., $\partial R_t/\partial \pi_t > 0$), though caused by some exogenous shift in belief, can be brought into the reasoning of the foregoing statement: (1) Let anticipations of inflation become widespread ($E_t[\pi_{t+1}] > 0$), (2) this raises the nominal natural rate itself ($R_t = E_t[\pi_{t+1}]$), (3) meaning that monetary policy kept the nominal market rate for a time below the natural rate by inflation ($\pi_t = \frac{1}{\partial R_t/\partial \pi_t} R_t > R_{t-1} \equiv 0$), (4) requiring now still more rapid inflation ($E_t[\pi_{t+2}] = E_t[R_{t+1}] = \frac{\partial R_t}{\partial \pi_t} E_t[\pi_{t+1}] > E_t[\pi_{t+1}]$) to hold down the market rate. Thus, the multiple equilibria of Cochrane (2007) can be interpreted as the Fisher effect of monetarism rearing its ugly head in the New Keynesian model.

According to Bordo and Schwartz (1999, p. 193), "[t]he dangers of operating with an interest rate instrument became clear when rising interest rates from the mid-1960s on

possess the stable long-run tradeoff.

[39] See, e.g., Nelson and Schwartz (2008, p. 844).

reflected growing fears of inflation, not restrictive monetary policy. Rising interest rates were accompanied by high money growth." With the ameliorative policies of the Great Moderation having dulled the memory of the Great Inflation, Issing (2008, p. 266) surmised, "[i]t is not surprising that in a world of low inflation, the interest in 'money' in central banks as well as in academia has declined, if not disappeared. I do, however, hope that the world does not have to go through the same process of pathological learning as at the end of the last century." With the apparent end of the Great Moderation, Leijonhufvud (2009, p. 6) reiterates that "[i]t is a dangerous illusion that you can always control the price level in an economy where the money stock however measured is left to vary in purely endogenous fashion."

Though monetary restraint is necessary for monetary policy to control inflation, the framework of interest rate rules need not be discarded. Nelson (2008) has given a very appealing justification for the use of an interest rate rule by appending the rule with steady state money growth. Combining this with the determinacy bounds of section 4.3 provides clear guidance to the monetary authority on the interest rate independent of the actual short-run mechanism at work on the supply side.

4.7 Conclusion

It should be clear that Cochrane's (2007) critique is substantially correct: there are explosive nominal paths associated with interest rate rules that cannot be ruled out. The requirement that the economy ought to fulfill Lucas's (1972a) NRH means, through determinacy, that Cochrane's (2007) critique applies—for a given demand specification—to all non-degenerate models at the same policy specifications. As a consequence, the monetary authority needs no knowledge of the supply side to ascertain whether its policy will ensure determinacy. Indeed, in the case of the literature-standard dynamic IS equation, no parameters of the model other than those in the interest rate rule can affect whether determinacy is achieved.

Asserting additionally that monetary policy can be held reasonably accountable for inflation demands monetary restraint and thus the hyperinflations or hyperdeflations of Cochrane (2007) can only occur if the monetary authority allows them to. These explosive equilibrium paths are admissible not for lack of LS-learnability (McCallum 2009b) nor of a non-Ricardian fiscal regime (Cochrane 2007), but simply because the monetary authority is increasing or decreasing the growth rate of money commensurate with accelerating inflation or deflation. Monetary policy associated with a determinate equilibrium, therefore, must additionally credibly commit to "prevent[ing] money itself from being a major source of economic disturbance," (Friedman 1968, p. 12) and a commitment to an average money growth rate following Nelson (2008) is offered as a means to that end. Thus money still plays a decisive role for the short run even when relegated to the very long run for monetary policy.

Appendix

4.A Appendix

4.A.1 Proof of Lemma 4.3.1

By the Wold theorem,[40] any stationary process can be represented as

$$X_t = \sum_{l=0}^{\infty} \theta_l \epsilon_{t-l} + \Xi_t, \text{ where } E\epsilon_t = 0 \text{ and } E\epsilon_t \epsilon'_{t+j} = 0, \ \forall j \neq 0 \tag{4.32}$$

and Ξ_t is an orthogonal linearly deterministic process, forecastable perfectly from its own history. Starting with the indeterministic part,[41] and inserting into (4.10)

$$0 = \sum_{j=0}^{n} \left[\sum_{l=0}^{\infty} \left(\sum_{i=0}^{min(p,l)} Q(i,j) \right) \theta_{l+j} \epsilon_{t-l} \right] + \sum_{j=1}^{m} \left[\sum_{l=0}^{\infty} \left(\sum_{i=0}^{min(p,l+j)} Q(i,j) \right) \theta_l \epsilon_{t-l-j} \right] \tag{4.33}$$

Using the definition of $\tilde{Q}(i,j)$ yields

$$0 = \sum_{j=0}^{n} \left[\sum_{l=0}^{\infty} \tilde{Q}(l,j) \theta_{l+j} \epsilon_{t-l} \right] + \sum_{j=1}^{m} \left[\sum_{l=0}^{\infty} \tilde{Q}(l+j,j) \theta_l \epsilon_{t-l-j} \right] \tag{4.34}$$

This must hold for all realizations of ϵ_t. Comparing coefficients yields

$$0 = \sum_{j=0}^{n} \tilde{Q}(l,j) \theta_{l+j} + \sum_{j=1}^{m} \tilde{Q}(l,j) \theta_{l-j} \tag{4.35}$$

a time-varying system of difference equations with initial conditions $\sum_{j=1}^{m} \theta_{-j} = 0$. But as $\tilde{Q}(p+i,j) = \tilde{Q}(p,j), \ \forall i \geq 0$, the system of difference equations has constant coefficients, after and including p. This system can be written as (4.12) and coincides with Anderson's (2010) canonical form. If the solution to this system is unique, its stable solution can be written as

$$\theta_l = B \begin{bmatrix} \theta_{l-m} \\ \vdots \\ \theta_{l-1} \end{bmatrix}, \ \forall l \geq p \tag{4.36}$$

[40] See, e.g., Sargent (1987a, pp. 286–290), as well as Priestley (1981, pp. 756–758).

[41] Whittle (1983, p. 31) and Sargent (1987a, p. 290) focus primarily on the purely indeterministic case. This forms the basis for the time-domain solution methods of Muth (1961) and Taylor (1986).

The first p (block) equations—remembering the initial conditions—can be gathered into

$$\mathbf{Q}\begin{bmatrix}\theta_0\\ \vdots\\ \theta_{n+p-1}\end{bmatrix}=0 \tag{4.37}$$

giving $3p$ equations in $3(p+n)$ variables. (4.36) yields $3n$ more equations that can be gathered into

$$\mathbf{B}\begin{bmatrix}\theta_0\\ \vdots\\ \theta_{n+p-1}\end{bmatrix}=0 \tag{4.38}$$

stacking the two yields (4.13).[42]

The system (4.35) is homogenous. Thus, one stationary solution is given by $\theta_l = 0$, $\forall i$, the fundamental solution in the absence of exogenous driving forces. If (4.13) is invertible and if (4.12) is saddle-point stable, then this is the only solution.

Only Ξ_t remains. Inserting it into (4.10), it follows that this can also be written as (4.12). If there is a unique solution in past values of Ξ_t, the solution can be written in the same form as (4.36), which must be zero when taken to its remote past from the stability of (4.36).

4.A.2 Proof of Proposition 4.3.2

Assume the opposite is true. Thus, the NRH model is determinate and the frictionless model is not. From the former, according to lemma 4.3.1, (4.13) is invertible and the system (4.12) is saddle-point stable. But the system (4.12) is the same for both models and (4.13) is lower triangular for the frictionless model. Thus, the frictionless model that satisfies (4.8) is determinate, a contradiction.

4.A.3 Proof of Proposition 4.3.3

As the frictionless model is determinate, the system (4.12) is saddle-point stable. This system is the same for the NRH (4.6) model. The second requirement (4.13) is lower triangular for the

[42] This extends 3.12 in chapter 3 to Anderson's (2010) higher leads and lags.

frictionless model, but is unrestricted for the NRH model. Thus, there exist NRH models with a singular (4.13) that are thusly indeterminate, even though the corresponding frictionless model is determinate.

4.A.4 Proof of Proposition 4.3.4

Ruling out the singularity of (4.13), proposition 4.3.3 has been ruled out by assumption. Thus, a model in this class that satisfies the NRH defined in (4.6) is determinate if and only if the corresponding model that satisfies (4.8) is determinate. This must hold for all k and thus holds for all $\tilde{k} < k$. Any supply equation that satisfies the NRH at a horizon $\tilde{k} < k$, necessarily satisfies it at the horizon k as well. Thus, for a given k, all supply equations that satisfy the NRH are determinate if and only if the corresponding frictionless model is determinate.

4.A.5 Proof of Corollary 4.3.5

It follows from proposition (4.3.4) that one may choose any supply equation to establish determinacy. Choosing (4.8) reduces the demand equation to (4.9), thus eliminating the parameters in the demand equation. Additionally, (4.8) removes the parameters in monetary policy pertaining to the output gap. Furthermore, from proposition (4.3.4), it follows that the parameters in the supply equation are irrelevant. Thus the only parameters in the model remaining that can affect determinacy are those in the interest rate rule pertaining to inflation and the interest rate.

4.A.6 Proof of Proposition 4.4.1

If the interest-rate rule induces determinacy, all nominal paths but one diverge. Thus, the money growth rate diverges for all paths but one. All divergent paths for the money growth rate contradict the assumption that monetary policy chose the average money growth rate.

Therefore, the only consistent path is the non-divergent one, which is unique following from determinacy.

4.A.7 Extension of Gray (1984)

Gray (1984, pp. 101–116) provides criteria to rule out speculative hyperinflation and hyperdeflation with a transactions cost model of money assuming linear utility from consumption and a transaction cost function that depends solely on real balances. Gray (1984, p. 118) relaxes the two assumptions individually, but not jointly. In the following, I will allow for diminishing marginal utility and the generalization of the transaction cost function to include the quantity of transactions—i.e., the level of consumption. This entails neither great difficulty nor significant insight and is thusly relegated to the appendix here.

Following Gray (1984, p. 102), the representative household seeks to maximize its lifetime discounted utility

$$Z = \int_0^\infty e^{-\rho t} U(c_t)\, dt \tag{4.39}$$

subject to

$$P_t y = P_t c_t + P_t \phi(c_t, m_t) + \dot{M}_t \tag{4.40}$$

where $\phi(c_t, m_t)$ is McCallum's (2009b) transaction cost function with $\phi_c > 0$, $\phi_{cc} < 0$, $\phi_m < 0$ and $\phi_{mm} > 0$. Additionally, $U_c > 0$ and $U_{cc} \leq 0$. Finally, c_t is consumption, ρ the rate of time preference, P_t the price level, y real income "rain[ing] from heaven at a fixed rate of y units per period" (Gray 1984, p. 97), and M_t nominal and $m_t = M_t/P_t$ real money balances.

The resulting optimization problem produces the following Euler equation

$$\frac{U_{cc}(c_t)}{U_c(c_t)}\dot{c}_t - \rho = \phi_m(c_t, m_t) + \frac{\dot{P}_t}{P_t} + \frac{\phi_{cc}(c_t, m_t)\dot{c}_t + \phi_{cm}(c_t, m_t)\dot{m}_t}{1 + \phi_c(c_t, m_t)} \tag{4.41}$$

Holding nominal balances constant[43] yields

$$\dot{m}_t = -m_t \frac{\dot{P}_t}{P_t} \tag{4.42}$$

and subsequently differentiating the budget constraint with respect to time yields,

$$\dot{c}_t = -\frac{\phi_m(c_t, m_t)}{1+\phi_c(c_t, m_t)}\dot{m}_t \tag{4.43}$$

Combining the foregoing three yields

$$\frac{\dot{P}_t}{P_t} = -\frac{\rho + \phi_m}{1 + \frac{m_t}{1+\phi_c}\left[\frac{\phi_m \phi_{cc}}{1+\phi_c} - \frac{\phi_m U_{cc}}{U_c} - \phi_{cm}\right]} \tag{4.44}$$

For Gray's (1984, p. 106) condition on the transaction function to rule out speculative hyperinflation under her assumptions of linear utility and transaction costs only dependent on real balances

$$lim_{m_t \to 0} m_t \phi_m < 0 \tag{4.45}$$

to carry over to this more general case, it suffices that the denominator of (4.44) not be negative.[44]

Note, firstly, that setting $\phi_c = 0$ yields

$$\frac{\dot{P}_t}{P_t} = -\frac{\rho + \phi_m}{1 - m_t \frac{\phi_m U_{cc}}{U_c}} \tag{4.46}$$

i.e., the special case of transaction costs independent of the level of transactions, but with nonlinear utility. This corresponds to Gray's (1984, p. 118) Equation (36). As she notes, a condition to ensure the denominator always be positive is for $-\phi_m U_{cc}$ to be positive, which is contradicted by assumption. Gray (1984, p. 118) interprets this compound term as, "the effect

[43]This is the assumption maintained throughout Gray (1984). The issue at hand is whether *speculative* hyperinflation and hyperdeflation can be ruled out.

[44]See Gray (1984, pp. 117–118). The numerator in (4.44) gives the rate of inflation for Gray's (1984, p. 103) simple model, and if the denominator remains positive, "[t]he dynamics of the more general model will be the same as the dynamics of the simpler model." (Gray 1984, p. 118)

on the marginal utility of consumption of the change in consumption generated by a change in real balances."

Also examined by Gray (1984, p. 118) is the special case of linear utility, but with the general transaction function. Setting U_{cc} to zero yields

$$\frac{\dot{P}_t}{P_t} = -\frac{\rho + \phi_m}{1 + \frac{m_t}{1+\phi_c}\left[\frac{\phi_m \phi_{cc}}{1+\phi_c} - \phi_{cm}\right]} \tag{4.47}$$

which corresponds to the equation in Gray's (1984, p. 118) Footnote 30. As interpreted there, a sufficient condition is now $\phi_{cm} < 0$. McCallum (2001b, p. 148) argues for setting this "cross partial derivative negative, so that the marginal benefit of holding money—i.e., the reduction in transaction costs—increases with the volume of consumption spending." And indeed the transaction function in both McCallum (2001b) and McCallum (2009b) does this.[45]

Using the foregoing two special cases, a sufficient condition would be that the marginal benefit of holding money increase *sufficiently* with an increase in consumption spending to outweigh the associated decrease in marginal utility from such a consumption spending increase. I.e., $-\phi_{cm}U_c > \phi_m U_{cc}$. Thus, the cross partial derivative being sufficiently negative constitutes a sufficient condition.

As this cross partial derivative is in no way constrained by the general transaction function of McCallum (2009b), assume the foregoing condition is fulfilled, and hence it suffices that

$$lim_{m_t \to 0} m_t \phi_m < 0 \tag{4.48}$$

for speculative hyperinflation to be ruled out.[46] As noted in the main text, this assumption is fulfilled by McCallum's (2009b, p. 1107) specific transaction function.

[45]Nevermind that this function does not satisfy McCallum's (2009b, p. 1106) own requirement that $\phi_{cc} < 0$ as Gray (1984, p. 118) too requires.

[46]Note that speculative hyperdeflation is ruled out with a transversality condition that would be violated along such a path given that the saddle-point property is ensured by the assumption $-\phi_{cm}U_c > \phi_m U_{cc}$.

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