The random conductance model under degenerate conditions

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Tưởng nhớ đến ông nội.

Cháu luôn tự hào về ông.

Vì ông bà cháu sẽ cố gắng chăm chỉ học hành.

To my grandparents.

Contents

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Erstens möchte ich Deutschland meinen herzlichen Dank ausdrücken! Dieser Dank sollte am Besten in der deutschen Sprache geschrieben werden! Obwohl ich nicht in Deutschland geboren wurde, ich noch kein Deutscher bin, und mein Name sowie mein Akzent sehr schwer für viele Deutsche klingt, bietet mir Deutschland seit mehr als 10 Jahren sehr großzügig eine sehr gute Umgebung zum Leben und zum Studium und mehrere Chancen zur Selbstentwicklung meiner persönlichen Kompetenz. Ich habe Bafög erhalten und konnte daher sehr auf meinem Studium konzentrieren. Als Bachelor- und Master-Student hatte ich an der Universität den Job als Mathematik-Tutor, der mir viel Erfahrungen, Freude, und Geld gebracht hat. Als Doktorand erhalte ich bis heute noch die Förderung der Deutschen Forschungsgemeinschaft. Und nach der Promotion habe ich auch einen Job und kann Mathematik weiter studieren.

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Summary in English

Random motions in random media is an interesting topic that has been studied intensively since several decades. Although these models are relatively simple mathematical objects, they have a wide variety of interesting properties from the theoretical point of view.

In this dissertation, we study an important branch within this topic, namely reversible random walks moving among nearest neighbour random conductances on $\mathbb{Z}^d$ – the random conductance model. Reversibility provides the model a variety of interesting connections with other fields in mathematics, for instance, percolation theory, $\nabla\phi$-interface models, and especially stochastic homogenization. Many questions coming from this model have been answered by techniques from partial differential equations and harmonic analysis.

As seen in the name of the thesis, we would like to consider this model under ”degenerate conditions”. Here, ”degenerate” has essentially two meanings. First, the conductances are not assumed to be bounded from above and below and stochastically independent. Second, we also consider the case of zero conductances, where the random walk can only move on a subgraph of $\mathbb{Z}^d$. Since there are percolation clusters, where the existence of the infinite cluster does not rely on stochastic independence, it is reasonable to accept the lack of stochastic independence.

In Chapter II we study quenched invariance principles. We assume that the positive conductances have some certain moment bounds, however, not bounded from above and below, and give rise to a unique infinite cluster and prove a quenched invariance principle for the continuous-time random walk among random conductances under relatively mild conditions on the structure of the infinite cluster. An essential ingredient of our proof is a new anchored relative isoperimetric inequality.

In Chapter III we study Liouville principles. As in Chapter II, we also assume some moment bounds and prove a first order Liouville property for this model. Using the corrector method introduced by Papanicolaou and Varadhan, Chapters II and III are closely related to each other at the technical level. Chapter IV proves a discrete analogue of the Dirichlet-to-Neumann estimate, which compares the tangential and normal derivatives of a harmonic function on the boundary of a domain. This result is used in Chapter III and perhaps useful for numerical analysis.

Zusammenfassung in Deutsch

Zufällige Bewegungen in zufälligen Medien ist ein interessantes Thema, das seit mehreren Jahrzehnten intensiv studiert wurde. Obwohl diese Modelle relativ einfache mathematische Objekte sind, haben sie aus theoretischer Sicht eine Vielzahl von interessanten Eigenschaften.

In dieser Dissertation betrachten wir einen wichtigen Zweig in diesem Thema, nämlich das zufällige Leitfähigkeitsmodell. Es konzentriert sich auf reversible Irrfahrten, die sich durch zufällige Leitfähigkeiten von Nächsten-Nachbarn-Kanten auf $\mathbb{Z}^d$ bewegen. Reversibilität bietet dem Modell eine Vielzahl von interessanten Verbindungen mit anderen Feldern in Mathematik, zum Beispiel Perkolationstheorie, $\nabla\phi$-interface Modelle, und ins besondere stochastische Homogenisierung. Viele Fragen aus diesem Modell wurden durch Techniken aus partiellen Differentialgleichungen und harmonischen Analysis beantwortet.

Wie der Name der Arbeit zeigt, möchten wir dieses Modell unter ”degenerierten Bedingungen” betrachten. Hier hat ”degeneriert” im Wesentlichen zwei Bedeutungen. Ersten nehmen wir nicht an, dass die Leitfähigkeiten von oben und unten beschränkt und stochastisch unabhängig sind. Zweitens betrachten wir auch den Fall von Null-Leitfähigkeiten, wo sich der Irrfahrt nur auf einem Untergraphen von $\mathbb{Z}^d$ bewegen kann. Da es Perkolationsclusters gibt, wo die Existenz des unendlichen Clusters nicht auf stochastische Unabhängigkeit beruht, ist es vernünftig, den Mangel an stochastischer Unabhängigkeit zu akzeptieren.

Im Kapitel II studieren wir fast sichere Invarianzprinzipien. Wir nehmen an, dass die positiven Leitfähigkeiten einige gewisse Momentbedingungen erfüllen, die jedoch nicht von oben und unten beschränkt sind und einen eindeutigen unendlichen Cluster erzeugen und ein fast sicheres Invarianzprinzip für eine Irrfahrt unter relativ milden Bedingungen für die Struktur des unendlichen Clusters beweisen. Ein wesentlicher Bestandteil unseres Beweises ist eine neue verankerte relative isoperimetrische Ungleichung.

In Kapitel III studieren wir Liouville-Eigenschaften. Wie in Kapitel II, nehmen wir auch einige Momentenbedingungen an und zeigen eine Liouville-Eigenschaft erster Ordnung für dieses Modell. Da sie die Korrektor-Methode von Papanicolaou und Varadhan benutzen, sind Kapitel II und III eng miteinander verknüpft auf technischer Ebene. Kapitel IV beweist ein diskretes Analogon der Dirichlet-to-Neumann-Abschätzung, die die tangentialen und normalen Ableitungen einer harmonischen Funktion auf dem Rand einer Gebiete vergleicht. Dieses Ergebnis wird in Kapitel III benutzt und eventuell nützlich für numerische Analysis.

Chapter I

Introduction

Random motions in random media is an interesting topic that has been studied intensively since several decades. Although these models are relatively simple mathematical objects, they have a wide variety of interesting properties from the theoretical point of view.

In this dissertation, we study an important branch within this topic, namely reversible random walks in random environments – the random conductance model. Reversibility provides the model a variety of interesting connections with other fields in mathematics, for instance, percolation theory, $\nabla\phi$ interface models, and especially stochastic homogenization. Many questions coming from this model have been answered by techniques from partial differential equations and harmonic analysis.

The thesis focuses on two different aspects in the random conductance model, namely *quenched invariance principles* (Chapter II) and *Liouville properties* (Chapter III), which are, however, related closely to each other at the technical level. Chapter IV proves a discrete analogue of the Dirichlet-to-Neumann estimate needed for Chapter III that seemingly has not appeared in the liturature yet. It seems to be the most technical part of the thesis and it is perhaps useful for numerical analysis. Chapter II is a joint work with Deuschel and Slowik [27] and has been already published online. Chapters III and IV are motivated by several ideas given by Bella, Fehrman, and Otto [12]. Each chapter is written quite independently from the others and can be considered as an independent paper, which contains an introduction and its own notations.

I.1 The model and assumptions

I.1.1 The model

The model can be simply described as follows. Consider the lattice $\mathbb{Z}^d$, equip each nearest neighbour bond $e = \{x, y\} = \{y, x\}$ on $\mathbb{Z}^d$ with a random conductance $\omega(e) = \omega(\{x, y\}) \in [0, \infty)$ and call ω a random environment, and assume that the probability distribution of the conductances is ergodic. Roughly speaking, it is nothing but the law of large number, meaning that averaging a large number of conductances gives us their expectation. A very natural example is the case where the conductances are mutually independent and identical distributed (i.i.d.). However, we do not restrict ourselves to this special case. For each fixed sample ω of a random environment, we consider two types of random walks: a discrete time random walk and a continuous time random walk. The discrete time random walk $\{Z_n : n \in \mathbb{N}\}$ jumps from x to y with probability $\omega(\{x, y\})/\mu^\omega(x)$ where

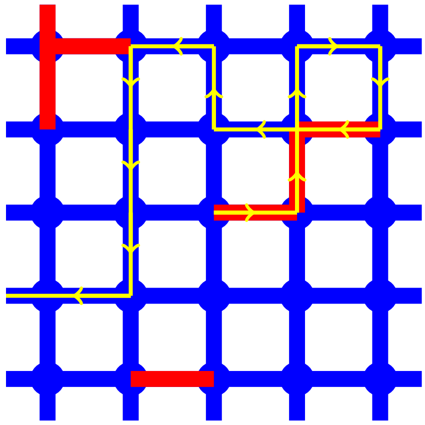

Figure I.1: A trajectory (*yellow*) of a random walk moving among conductances on $\mathbb{Z}^d$. *Red*: very high or low conductances

$$\mu^\omega(x) = \sum_{y:|x-y|_1=1} \omega(\{x, y\}),$$

while the continuous time random walk $\{X_t : t \geq 0\}$ waits at x an exponential time with means $\mu^\omega(x)^{-1}$ and jumps to a nearest neighbour y of x with probability $\omega(\{x, y\})/\mu^\omega(x)$. As a Markov process it has the following generator:

$$\mathcal{L}^\omega u(x) = \sum_{y \sim x} \omega(\{x, y\}) \left(u(y) - u(x)\right).$$

In the literature, this continuous time random walk is called *the variable speed random walk*, since the holding times depend on the space variable. The discrete time random walk can also be considered as a continuous time random walk by setting up waiting times at all points on the lattice, which are i.i.d. exponentially distributed with mean 1. This continuous time random walk is called the *constant speed random walk*, since the holding times do not depend on the space variable: it waits at x an exponential time with mean 1 and jumps to $y \sim x$ with probability $\omega(\{x, y\})/\mu^\omega(x)$. It has the following generator

$$\mathcal{P}^\omega u(x) = \mu^\omega(x)^{-1} \sum_{y \sim x} \omega(\{x, y\}) \left(u(y) - u(x)\right).$$

Let us give some simple examples to illustrate the above definition. More interesting examples can be found in the main text.

Example I.1.1. In the simplest case, $\omega(e) = 1$ for all e, the discrete time random walk is nothing but a simple random walk on $\mathbb{Z}^d$.

Example I.1.2 (Simple random walks)**.** Let $\omega(e)$ be independently sampled from $\{0,1\}$. We speak of an open edge if $\omega(e) = 1$ and a closed edge if $\omega(e) = 0$. This problem can be simulated easily (Figure I.2) where an edge e is coloured blue if $\omega(e) = 1$ and not coloured if $\omega(e) = 0$. The blue edges give us a random subgraph of $\mathbb{Z}^d$. The discrete time random walk can only jumps through an open edge and therefore can only stay in a connected component (cluster). We call it a simple random walk in the sense that in each step it jumps from its current position x to a nearest neighbour in this subgraph with probability $1/\mu^\omega(x)$ where $\mu^\omega(x)$ is the degree of the node x.

In percolation theory, it is well-known that if

$$\mathfrak{p} := \mathbb{P}[e \text{ is open}] > \mathfrak{p}_c(d) \tag{I.1.1}$$

for some $\mathfrak{p}_c \in (0,1)$, the open edges percolate and form a unique infinite connected component, a so-called supercritical percolation cluster. In this case, the discrete time random walk Z_n becomes a *simple random walk on a supercritical percolation cluster*. This random walk (or its continuous time version, the constant speed random walk) can be simulated easily (see Figure I.2) and has been studied in several papers e.g. [9, 14, 13].

Example I.1.3 (i.i.d. conductances)**.** Example I.1.1 can be easily generalized as follows. Let $\omega(e)$ be independently sampled from $[0,\infty)$. We coloured an edge e blue if $\omega(e) > 0$ and white if $\omega(e) = 0$. The random walk can only jump through a cluster of blue edges. However, the probability of jumping from x to y now depends proportionally on the conductance $\omega(\{x,y\})$. We always assume (I.1.1), where the random walk moves on an infinite connected component. Further, we are interested in several assumptions on the conductances. If $\mathfrak{p} = 1$ the cluster becomes the whole $\mathbb{Z}^d$ and the conductances are called *elliptic* or *degenerate*. In some papers, they are even assumed to be *bounded from above and below*, in other words, *uniformly elliptic* or *strongly elliptic*. If $\mathfrak{p} \in (p_c, 1)$ where the cluster forms a proper subgraph of $\mathbb{Z}^d$, similar assumptions can also be made for the conductances on the cluster, for instance, elliptic or uniformly elliptic.

Example I.1.4 (Gaussian free fields)**.** Let $d \geq 3$. Define the conductances by

$$\omega(\{x,y\}) = \exp(h(x) + h(y)), \quad |x-y|_1 = 1 \tag{I.1.2}$$

where $\{h(x) : x \in \mathbb{Z}^d\}$ is the Gaussian free field i.e. a Gaussian process indexed by $x \in \mathbb{Z}^d$ with $\mathbb{E}h(x) = 0$ and $\operatorname{cov}(h(x), h(y)) = G(x,y)$ where $G(x,y)$ is the Green function in $\mathbb{Z}^d$. This example has been considered in [2].

I.1.2 Degenerate conditions

Although Examples I.1.1, I.1.3 and I.1.3 are also very interesting cases, which are objects of several researches, as seen the name of the thesis, we are interested in more general cases. Here, ”degenerate” has essentially two meanings. First, the conductances are not assumed to be bounded from above and below and stochastically independent (cf. Example I.1.4). Second, we also consider the case of zero conductances where the random walk can only move in a subgraph of $\mathbb{Z}^d$, a similar scenario as that in Example I.1.3. Since there are percolation clusters where the existence of the infinite cluster does not rely on stochastic independence [28, 57], it is reasonable to accept the lack of stochastic independence.

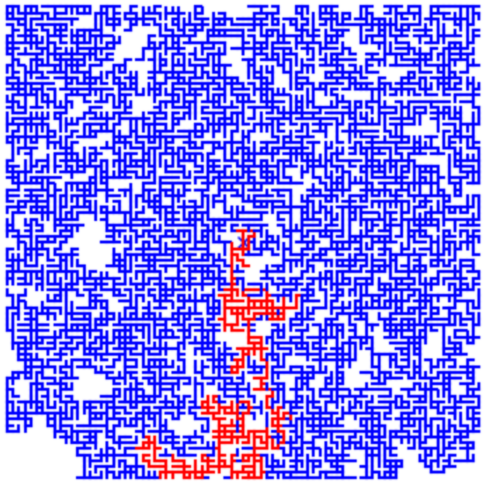

Figure I.2: A simple random walk on a Bernoulli supercritical percolation cluster for $p = 0.55$ and $d = 2$ obtained by simulation with Python. *Blue*: the percolation cluster; *red*: the trajectory of the random walk.

In degenerate cases, we have to classify assumptions on the environment so that our result applies to a possibly large class of models. First, we have to assume some certain integrability conditions for the conductances, since, due to an example by Barlow, Burdzy and Timár [8, 7], the first moment and inverse moment is necessary for the quenched invariance principle to hold. Second, in the case of zero conductances (Chapter II), we need assumptions on the infinite cluster. The novelty is to encode them in isoperimetric inequalities (cf. Definition II.1.2 and Assumption II.1.3).

I.2 The main results

We discuss the main results, quenched invariance principles and Liouville properties and provide some links between them.

I.2.1 Quenched invariance principles

In Chapter II, we are interested in a typical question in probability theory, namely quenched invariance principles, which provide, roughly speaking, long-term properties of the random walk. More rigorously, we discuss whether the sequence of processes $\{X^{(n)} : n \in \mathbb{N}\}$ where $X_t^{(n)} := \frac{1}{n} X_{tn^2}$ converges to a Brownian motion for almost every environment ω. Here, ”quenched” means nothing but almost surely. The main result, Theorem II.1.7, answers this question. Further, it applies to the constant speed random walk (cf. Remark II.1.9). Besides Assumption II.1.6 on the integrability of the conductances, we set up large-scale properties of the infinite cluster (Assumption II.1.3), encoded in the notation "θ-very regularity" (cf. [9, 57]), which basically means volume regularity and a weak relative isoperimetric inequality for large sets. These conditions apply to a large class of random conductance models (see Examples II.1.11–II.1.13) not necessarily relying on stochastic dependence.

I.2.2 Liouville properties

In Chapter III, we restrict ourselves on the case of positive conductances, meaning that the conductances are living now on the full lattice $\mathbb{Z}^d$. We make Assumption III.1.1, which is similar to Assumption II.1.6 on integrability we made before in Chapter II except the fact that we now require that the law of the conductaces is invariant under reflections on $\mathbb{Z}^d$. The last condition implies that the homogenized matrix is a diagonal matrix, which later allows us to implement the idea of *perturbing around the homogenized coefficients* [33, 12] in the discrete case. In Chapter III, we are interested in solutions to the discrete elliptic equation

$$\mathcal{L}^\omega u(x) = 0, \quad x \in \mathbb{Z}^d \tag{I.2.1}$$

which are called ω-harmonic functions. The main result is a Liouville-type property for ω-harmonic functions (Theorem III.1.2) claiming that the space of sub-quadratic (growing at most like $|x|^{1+\alpha}$, $\alpha \in (0,1)$) ω-harmonic function has dimension $d+1$. This is a consequence of a regularity estimate which is called the excess decay (Theorem III.1.3).

I.2.3 Discussion on the methods

Connections between Chapters II and III at the technical level The problem studied by Chapter III comes mainly from an analytic point of view, for instance, theory of partial differential equations and homogenization. However, at the technical level, it is closely related to Chapter II by the fact that

"(I.2.1) is true if and only if $u(X_t)$ (and $u(Z_n)$) is a martingale."

The central object studied through the dissertation, which appears in both Chapters II and III, is the corrector $\phi = (\phi_1, \dots, \phi_d) : \Omega \times \mathbb{Z}^d \to \mathbb{R}^d$. It is defined so that

$$\Psi_i(\omega, x) := x_i + \phi_i(\omega, x), \tag{I.2.2}$$

is ω-harmonic for each $i \in \{1, \dots, d\}$, meaning

$$\mathcal{L}^\omega \Psi_i(\omega, x) = 0. \tag{I.2.3}$$

We call $\Psi_1, \ldots, \Psi_d$ the harmonic coordinates, meaning $\Psi_i(\omega, \cdot)$ is a "ω-harmonized" version of the coordinate fields $\Pi_i(\omega, x) := x_i$.

The use of the corrector is based on the classical work by Papanicolaou and Varadhan [53] studying random elliptic differential operators in divergence form in the continuum setting,

$$\mathcal{L}^{\mathbf{a}} u(x) = \nabla \cdot \mathbf{a}(x) \nabla u(x) = \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i} \left(\mathbf{a}_{ij}(x) \frac{\partial u}{\partial x_j}(x) \right) \tag{I.2.4}$$

where $\mathbf{a}(x)$ is a symmetric positive definite random matrix defined on some probability space. The random diffusion generated $\mathcal{L}^{\mathbf{a}}$ can be considered as a continuum counterpart of the random conductance model. The crucial idea in their construction of the corrector is instead of solving (I.2.9) for all positions x in the physical space, they require that the gradient of ϕ is stationary in the sense

$$u(\omega, x+e) - u(\omega, x) = u(\tau_x \omega, e) \tag{I.2.5}$$

solve (I.2.9) for almost every environment ω but for only one position, the origin, and exploit (I.2.5) to extend the solution to the whole space. In both Chapters II and III we have to control the sublinear growth of the corrector informally written as

$$\phi(\omega, x) = o(|x|), \quad |x| \to \infty. \tag{I.2.6}$$

Methods of Chapter II We prove the quenched invariance principle in two step. First, we establish that for the process $\{\Psi(\omega, X_t) : t \geq 0\}$, which is a martingale by (I.2.3). Second, we prove that

$$\sup_{0 \leq t \leq T} \frac{1}{n} \left| \chi(\omega, X_{tn^2}) \right| \underset{n \to \infty}{\longrightarrow} 0 \quad \text{in } \mathrm{P}_0^{\omega}\text{-probability}, \tag{I.2.7}$$

almost surely, where P_0^{ω} is the law of the random walk starting at the origin in the environment ω. Since invariance principles for martingales are well-known [39], it suffices to show (I.2.5) which is a consequence of the ℓ^{∞}-sublinearity of the corrector,

$$\max_{|x|<n} \frac{1}{n} |\phi(\omega, x)| \overset{n \to \infty}{\longrightarrow} 0, \quad \mathbb{P}\text{-a.s.} \tag{I.2.8}$$

This is basically a maximal inequality for a solution to the Poisson-type equation

$$\mathcal{L}^{\omega} \phi_i = -\mathcal{L}^{\omega} \Pi_i, \tag{I.2.9}$$

and therefore can be obtained, appealing to the Moser iteration schema by Andres, Deuschel, and Slowik [2], once we control the corresponding ℓ^1 norm. The novelty of Chapter II is an anchored Sobolev inequality on graphs, Lemma II.3.7, which provides an elegant argument for the ℓ^1-sublinearity of the corrector, Proposition II.2.9. This inequality is proved under Assumption II.3.1 which contains essentially volume regularity and two relative isoperimetric inequalities for large sets.

Methods in Chapter III The Liouville property is proved by monitoring the excess, a natural distance between a given ω-harmonic function u and the space generated by the harmonic coordinates $\Psi_1, \ldots, \Psi_d$ in different scales (Theorem III.1.3). In contrast to Chapter II working on arbitrary infinite graphs, in Chapter III, borrowing the idea of Bella, Fehrman, and Otto [12], we only focus on the lattice $\mathbb{Z}^d$. Following [12], we construct the second order corrector σ in the discrete case (Proposition III.2.3) with stationary gradients (I.2.5). This second order corrector allows us to have an energy estimate for the homogenization error (Proposition III.2.3), which is an important step to obtain the excess decay. Since a main part of [12] contains estimates on a boundary term, when implementing the schema given in [12] into the discrete case of $\mathbb{Z}^d$, we mainly have to work on the discrete surfaces of a fixed box, while Chapter II we consider a sequence of growing balls. It is therefore not surprising that Chapter III has a flavour of numerical analysis. Indeed, we need to make an excursion to finite element to install suitable "numerical tricks" which help us to translate the idea in [12] in our setting smoothly. The most technical part is to establish the Dirichlet-to-Neumann estimate in the discrete case which is written as an independent part, Chapter IV, which may be useful for numerical analysis.

I.3 Discussion on previous works

I.3.1 Quenched invariance principles

Since the annealed quenched invariance principle was already proved in the 1980s by De Masi, Ferrari, Goldstein and Wick [24] and Kipnis and Varadhan [41] for stationary and ergodic laws $\mathbb{P}$ with $\mathbb{E}[\omega(e)] < \infty$, it took quite some time in the last two decades to obtain quenched results especially for stationary and ergodic laws under degenerate conditions (see Table I.1). This can be explained, in my opinion, by the following facts. First, almost all the papers [59, 14, 48, 17, 10, 1, 28] proved the sub-linearity of the corrector ϕ by estimates on heat kernels, which rely on an a priori knowledge in percolation theory on the distribution of *the size of holes* in the cluster or *the chemical distance*, which we do not have in the stationary and ergodic case. Second, perhaps they did not really consider the corrector ϕ as a solution to (I.2.9) (actually *the solution*, when we consider (I.2.9) at the level of the probability space). Nevertheless, it also has an advantage, since those papers

Year	Works and authors	Conditions on the environment
	The case of i.i.d. conductances	
2004	Sznitman and Sidoravicius [59]	**i.i.d and uniformly elliptic** $\lambda \leq \omega(e) \leq 1$, fixed $\lambda > 0$
2007	Berger and Biskup [14] Mathieu and Piatnitski [48]	**Simple random walks** on percolation clusters ($d \geq 2$)
2007 2008	Biskup and Prescott [17] Mathieu [47]	**i.i.d. bounded from above** $0 \leq \omega(e) \leq 1$ and $\mathbb{P}[\omega(e) > 0] > p_{\mathrm{c}}$
2010	Barlow and Deuschel [10]	**i.i.d. bounded from below** $\lambda \leq \omega(e) < \infty$, fixed $\lambda > 0$ including $\mathbb{E}[\omega(e)] = \infty$
2013	Andres, Barlow, Deuschel, and Hambly [1]	**General i.i.d.** $\mathbb{P}[\omega(e) > 0] > p_{\mathrm{c}}$ including $\mathbb{E}[\omega(e)] = \infty$
	The case of stationary ergodic conductances	
2011	Biskup [16]	**First moment and inverse moment** $\omega(e) > 0$, $d = 2$, $\mathbb{E}[\omega(e)] + \mathbb{E}[\omega(e)^{-1}] < \infty$
2015	Andres, Deuschel, and Slowik [2]	(p, q)**-moment condition** $\omega(e) > 0$, $\mathbb{E}[\omega(e)^p] + \mathbb{E}[\omega(e)^{-q}] < \infty$ and $1/p + 1/q < 2/d$
2016	Procaccia, Rosenthal, and Sapozhnikov [28]	**Simple random walks** on clusters in correlated percolation models
2017	Deuschel, Ng., and Slowik [27]	(p, q, θ)**-moment condition** $\mathbb{E}[\omega(e)] + \mathbb{E}[\omega(e)^{-q}\mathbf{1}_{e \text{ is open}}] < \infty$, $1/p + 1/q < 2(1-\theta)/(d-\theta)$ θ**-very regularity of the cluster**

Table I.1: The quenched invariance principle – previous results.

give us a more general notion of correctors, which is sometimes very useful, for instance when dealing with large deviation principles, see [15, Remark 2, p11].

As mentioned before, Chapter II is motivated by Andres, Deuschel, and Slowik [2] considering the corrector as solution to (I.2.9) and proving the ℓ^∞-sublinearity (I.2.9) with the maximal inequality obtained by Moser's iteration. This work is motivated by Fannjiang and Komorovski [30] whose proved a quenched invariance principle for random diffusions generated by (I.2.4) (under the condition $\mathbb{E}[|a_{ij}(x)|^p] < \infty$ where $p > d$). Recently, using the idea of Moser's iteration, Chiarini and Deuschel [22, 23] have proved a quenched invariance principle for diffusions under the following (p, q)-moment condition

$$\mathbb{E}[\mu(\mathbf{a})^p] + \mathbb{E}[|\lambda(\mathbf{a})|^{-q}] < \infty, \quad \frac{1}{p} + \frac{1}{q} < \frac{2}{d}, \tag{I.3.1}$$

where $\mathbb{E}$ denotes the expectation of the random coefficient $\mathbf{a}$ and

$$\lambda(\mathbf{a}) := \inf_{\xi \in \mathbb{R}^d} \frac{\xi \cdot \mathbf{a}\xi}{|\xi|^2}, \quad \text{and} \quad \mu(a) := \sup_{\xi \in \mathbb{R}^d} \frac{|\mathbf{a}\xi|^2}{\xi \cdot \mathbf{a}\xi}.$$

I.3.2 Liouville properties

Compared to quenched invariance principles, not so many researches have drawn their attention to Liouville properties for random environments. While Liouville properties are well-understood [6] for $\mathbf{a}$-harmonic functions in "periodic media", i.e. for those which make (I.2.4) equal to zero where $\mathbf{a}$ is periodic, $\mathbf{a}(x+L) = \mathbf{a}(x)$, that for random media are still objects of recent researches (see Table I.2). Questions on Liouville properties for random environments were first asked by Benjamini, Duminil-Copin, Kozma, and Yadin [13]. In their paper, they proved that the space of ω-harmonic functions on the supercritical percolation cluster, satisfying $\mathcal{L}^\omega u = 0$ with ω given in Example I.1.2, has dimension $d+1$, and their idea of using the entropy can be easily extended to the model given by Sapozhnikov [57], which is the same model as that given in his joint work on quenched invariance principles [28] with Procaccia and Rosenthal. Recently, Armstrong and Dario [5] have answered a question given by Benjamini et. al. on Liouville properties. Indeed, they consider a more general model, namely the conductances $\omega(e)$ are i.i.d. bounded from above and below and live on a supercritical percolation cluster. They have achieved a more general result which states that the space of ω-harmonic functions growing at most like $o(|x|^{k+1})$ has the same dimension as the space of harmonic polynomials of degree at most k.

In the continuum setting, several results on Liouville properties have been obtained recently. Gloria, Neukamm, and Otto have proved a $O(|x|^{1+\alpha})$ Liouville for $\mathbf{a}$-harmonic functions, where $\mathbf{a}$ is a stationary and ergodic matrix bounded from above and below. This Liouville property is a consequence of a regularity estimate

Year	Works and authors	Assumptions	Order
	THE CASE OF I.I.D. CONDUCTANCES		
2015	Benjamini, Duminil-Copin, Kozma, and Yadin [13]	**Simple random walks** on supercritical cluster	$O(\lvert x\rvert)$
2017	Armstrong and Dario [5]	**i.i.d and uniformly elliptic** on supercritical cluster $\lambda \leq \omega(e) \leq 1$, fixed $\lambda > 0$.	$o(\lvert x\rvert^{k+1})$ for $k \geq 1$
	THE CASE OF STATIONARY ERGODIC CONDUCTANCES		
2015	Marahrens and Otto [46]	**uniformly elliptic and mixing** on $\mathbb{Z}^d$ $\lambda \leq \omega(e) \leq 1$, fixed $\lambda > 0$	$O(\lvert x\rvert^{\alpha})$
2016	Sapozhnikov [57]	**Simple random walks** on clusters in correlated percolation models	$O(\lvert x\rvert)$
2017	Ng. (see Chapter III)	(p,q)**-moment condition** $\omega(e) > 0$, $\mathbb{E}[\omega(e)^p] + \mathbb{E}[\omega(e)^{-q}] < \infty$ $\mathbb{P}$ **reflection invariant**	$O(\lvert x\rvert^{1+\alpha})$

Table I.2: Known results on Liouville properties for the random conductance model called the excess decay. Fischer and Otto [31] have extended this to a $O(|x|^{k+\alpha})$ Liouville property and Fischer and Raithel [32] to the case of the haft space, both required mixing conditions on the environment. Lately, Bella, Fehrman, and Otto have shown the excess decay and therefore the Liouville property introduced by [33] under the (p, q)-condition (I.3.1).

I.3.3 Open problems

Although the idea of using Moser's iteration by Andres, Deuschel, and Slowik [2] is quite robust in the sence that it can be indeed extended to the case of random graphs, their (p, q)-condition cannot cover the "optimal case" in $d = 2$ by [16] (see Table I.2). It is conjectured by Biskup [16] that the quenched invariance principle holds true in all dimensions $d \geq 2$ under the assumption on the first moment and inverse moment,

$$\mathbb{E}[\omega(e)] + \mathbb{E}[\omega(e)^{-1}] < \infty. \tag{I.3.2}$$

Further, comparing Table I.1 and Table I.2 we can ask a natural question concerning Liouville properties: Can we fill in Table I.2 so that it becomes quite similar to Table I.1? Let us state the question more precisely. First, consider the i.i.d. case. Does a Liouville hold true under the most general i.i.d. condition i.e. under the setting of Andres, Deuschel, Barlow, and Hambly [1] (Table I.1), where it can happen that $\mathbb{E}[\omega(e)] = \infty$. Maybe it does not make sense to ask this question, since in this case we cannot construct the corrector ϕ for the original environment and therefore we do not have the harmonic coordinates.

In my opinion (it may be wrong), before addressing more "degenerate" cases, for instance, the stationary and ergodic case with moment conditions, relaxing a bit the uniform elliptic condition of Armstrong and Dario [5] would be more realistic. For instance, asking this question in the setting of Biskup and Prescott [17] and Mathieu [47] in Table I.1 (i.i.d conductances bounded from above and living on a supercritical percolation cluster) would be quite reasonable, since in this case, we can construct the corrector and therefore the harmonic coordinates, which are candidates for a basis of the space of $d + 1$ harmonic functions. The same question for the i.i.d. case, bounded from below, unbounded from above, however, with the first moment $\mathbb{E}[\omega(e)] < \infty$, where the harmonic coordinates can also be constructed. In this case, quenched invariance principles are often proved by "peeling off" bad conductances, which are too high or to low. However, when dealing with Liouville properties, "peeling off" may destroy the ω-harmonicity.

Note that as in the case of Biskup's conjecture, the (p, q)-condition (III.1.7) in Chapter III inherited from (I.3.1) used by Bella, Fehrman, Otto [12] does not cover the case $p = 1$ or $q = 1$ even in the simplest case $d = 2$ (and i.i.d.) – the reason is that we appeal to a Calderón-Zymund-type estimate when constructing the corrector σ. The use of the homogenization error defined by a harmonic extension given by Bella, Fehrman, and Otto [12] exploits too many aspects which are only true on $\mathbb{R}^d$ and $\mathbb{Z}^d$, for instance, the construction of the corrector σ and the Dirichlet-to-Neumann estimate (the main result of Chapter IV). It is very interesting to know whether it is possible to improve this idea so that it works with ”volumes” rather than with ”surfaces”.

Recently, Bella, Fehrman, Chiarini [11] have achieved a Liouville property for parabolic equations in the continuum setting. It is interesting to extend it to the discrete case and under degenerate conditions. This is perhaps the first step to achieve the idea of Jean-Dominique Deuschel: approximate the parabolic Green function by the homogenized one and apply it for the Ginzburg-Landau model.

Chapter II

Quenched invariance principles

II.1 Introduction

II.1.1 The model

Consider the d-dimensional Euclidean lattice, $(\mathbb{Z}^d, \mathbb{E}^d)$, for $d \geq 2$, where the edge set, $\mathbb{E}^d$, is given by the set of all non-oriented nearest neighbor bonds. Let $(\Omega, \mathcal{F}) = ([0,\infty)^{\mathbb{E}^d}, \mathcal{B}([0,\infty))^{\otimes \mathcal{E}^d})$ be a measurable space equipped with the Borel-σ-algebra. For $\omega \in \Omega$, we refer to $\omega(\{x,y\})$ as the *conductance* of the corresponding edge $\{x,y\}$. Henceforth, we consider a probability measure $\mathbb{P}$ on $(\Omega, \mathcal{F})$, and we write $\mathbb{E}$ to denote the expectation with respect to $\mathbb{P}$. Further, a *translation* or *shift* by $z \in \mathbb{Z}^d$ is a map $\tau_z \colon \Omega \to \Omega$,

$$(\tau_z \omega)(\{x,y\}) := \omega(\{x+z, y+z\}), \qquad \{x,y\} \in \mathbb{E}^d. \tag{II.1.1}$$

The set $\{\tau_x : x \in \mathbb{Z}^d\}$ together with the operation $\tau_x \circ \tau_y := \tau_{x+y}$ defines the group of space shifts.

For any $\omega \in \omega$, the induced set of *open* edges is denoted by

$$\mathcal{O} \;\equiv\; \mathcal{O}(\omega) \;:=\; \big\{e \in \mathbb{E}^d \mid \omega(e) > 0\big\} \;\subset\; \mathbb{E}^d.$$

We also write $x \sim y$ if $\{x,y\} \in \mathcal{O}(\omega)$. Further, we denote by $\mathcal{C}_\infty(\omega)$ the subset of vertices of $\mathbb{Z}^d$ that are in infinite connected components.

Throughout the paper, we will impose assumptions both on the law $\mathbb{P}$ and on geometric properties of the infinite cluster.

Assumption II.1.1. Assume that $\mathbb{P}$ satisfies the following conditions:

(i) The law $\mathbb{P}$ is stationary and ergodic with respect to translations of $\mathbb{Z}^d$.

(ii) $\mathbb{E}[\omega(e)] < \infty$ for all $e \in \mathbb{E}^d$.

(iii) For $\mathbb{P}$-a.e. ω, the set $\mathcal{C}_\infty(\omega)$ is connected, i.e. there exists a unique infinite connected component – also called infinite open cluster – and $\mathbb{P}[0 \in \mathcal{C}_\infty] > 0$.

Let $\Omega_0 = \big\{\omega \in \Omega \,:\, 0 \in \mathcal{C}_\infty(\omega)\big\}$ and introduce the conditional measure

$$\mathbb{P}_0[\,\cdot\,] \;:=\; \mathbb{P}[\,\cdot\,|\, 0 \in \mathcal{C}_\infty], \tag{II.1.2}$$

and we write $\mathbb{E}_0$ to denote the expectation with respect to $\mathbb{P}_0$. We denote by d^ω the natural graph distance on $(\mathcal{C}_\infty(\omega), \mathcal{O}(\omega))$, in the sense that for any $x, y \in \mathcal{C}_\infty(\omega)$, $d^\omega(x,y)$ is the minimal length of a path between x and y that consists only of edges in $\mathcal{O}(\omega)$. For $x \in \mathcal{C}_\infty(\omega)$ and $r \geq 0$, let $B^\omega(x,r) := \{y \in \mathcal{C}_\infty(\omega) : d^\omega(x,y) \leq \lfloor r \rfloor\}$ be the closed ball with center x and radius r with respect to d^ω, and we write $B(x,r) := \{y \in \mathbb{Z}^d \,:\, |y-x|_1 \leq \lfloor r \rfloor\}$ for the corresponding closed ball with respect to the ℓ^1-distance on $\mathbb{Z}^d$. Further, for a given subset $B \subset \mathbb{Z}^d$ we denote by $|B|$ the cardinality of B, and we define the *relative* boundary of $A \subset B$ by

$$\partial_B^\omega A \;:=\; \big\{\{x,y\} \in \mathcal{O}(\omega) \;:\; x \in A \text{ and } y \in B \setminus A\big\}$$

and we simply write $\partial^\omega A$ if $B \equiv \mathcal{C}_\infty(\omega)$. The corresponding boundary on $(\mathbb{Z}^d, \mathbb{E}^d)$ is denoted by $\partial_B A$ and ∂A, respectively.

Definition II.1.2 (regular balls)**.** Let $C_\mathrm{V} \in (0,1]$, $C_\mathrm{riso} \in (0,\infty)$ and $C_\mathrm{W} \in [1,\infty)$ be fixed constants. For $x \in \mathcal{C}_\infty(\omega)$ and $n \geq 1$, we say a ball $B^\omega(x,n)$ is *regular* if it satisfies the following conditions:

i) volume regularity of order d:

$$C_\mathrm{V}\, n^d \;\leq\; |B^\omega(x,n)| \tag{II.1.3}$$

ii) (weak) relative isoperimetric inequality: There exists $\mathcal{S}^\omega(x,n) \subset \mathcal{C}_\infty(\omega)$ connected such that $B^\omega(x,n) \subset \mathcal{S}^\omega(x,n) \subset B^\omega(x, C_\mathrm{W} n)$ and

$$|\partial^\omega_{\mathcal{S}^\omega(x,n)} A| \;\geq\; C_\mathrm{riso}\, n^{-1}\, |A| \tag{II.1.4}$$

for every $A \subset \mathcal{S}^\omega(x,n)$ with $|A| \leq \frac{1}{2}\, |\mathcal{S}^\omega(x,n)|$.

Assumption II.1.3 (θ-very regular balls)**.** For some $\theta \in (0,1)$ assume that for $\mathbb{P}_0$-a.e. ω there exists $N_0(\omega) < \infty$ such that for all $n \geq N_0(\omega)$ the ball $B^\omega(0,n)$ is *θ-very regular*, that is, the ball $B^\omega(x,r)$ is regular for every $x \in B^\omega(0,n)$ and $r \geq n^{\theta/d}$.

Remark II.1.4. (i) The lattice $(\mathbb{Z}^d, \mathbb{E}^d)$ satisfies the assumption above with $\theta = 0$.
(ii) The notion of θ-very regular balls is particularly useful in the context of random graphs, e.g. supercritical Bernoulli percolation clusters [9] or clusters in percolation models with long range correlations [57] (see the examples below for more details). Such random graphs have typically a local irregular behaviour, in the sense that the conditions of volume growth and relative isoperimetric inequality fail on small scales. Roughly speaking, Assumption II.1.3 provides a uniform lower bound on the radius of regular balls.

(iii) In contrast to the (weak) relative isoperimetric inequality (II.1.4), the (standard) isoperimetric inequality on $\mathbb{Z}^d$ reads

$$|\partial^\omega A| \;\geq\; C_{\mathrm{iso}}\, |A|^{(d-1)/d}, \qquad \forall\, A \subset \mathbb{Z}^d. \tag{II.1.5}$$

On random graphs, however, such an inequality is true only for large enough sets. However, under the assumption that the ball $B^\omega(x,n)$ is θ-very regular, the isoperimetric inequality (II.1.5) holds for all $A \subset B^\omega(x,n)$ with $|A| > n^\theta$; cf. Lemma II.2.10 below.

For any fixed realization $\omega \in \Omega$, we are interested in a continuous-time Markov chain, $X = \{X_t : t \geq 0\}$, on $\mathcal{C}_\infty(\omega)$. We refer to X as *random walk among random conductances* or *random conductance model* (RCM). Set

$$\mu^\omega(x) = \sum_{y \sim x} \omega(\{x,y\}),$$

X is the process that waits at the vertex $x \in \mathcal{C}_\infty(\omega)$ an exponential time with mean $1/\mu^\omega(x)$ and then jumps to a vertex y that is connected to x by an open edge with probability $\omega(\{x,y\})/\mu^\omega(x)$. Since the holding times are space dependent, this process is also called *variable speed random walk* (VSRW). The process X is a Markov process with generator, $\mathcal{L}^\omega$, acting on bounded functions as

$$\big(\mathcal{L}^\omega f\big)(x) \;=\; \sum_{y \in \mathbb{Z}^d} \omega(\{x,y\})\,\big(f(y) - f(x)\big), \qquad x \in \mathcal{C}_\infty(\omega). \tag{II.1.6}$$

We denote by P_x^ω the quenched law of the process starting at the vertex $x \in \mathcal{C}_\infty(\omega)$. The corresponding expectation will be denoted by E_x^ω. Notice that X is a *reversible* Markov chain with respect to the counting measure.

II.1.2 Main result

We are interested in the long time behavior of the random walk among random conductances for $\mathbb{P}_0$-almost every realization ω. In particular, we are aiming at obtaining a quenched functional central limit theorem (QFCLT) for the process X in the following sense.

Definition II.1.5. Set $X_t^{(n)} := \frac{1}{n} X_{tn^2}$, $t \geq 0$. We say that the *quenched functional CLT* or *quenched invariance principle* holds for X, if for every $T > 0$ and every bounded continuous function F on the Skorohod space $D([0,T],\mathbb{R}^d)$, it holds that $\mathrm{E}_0^\omega[F(X^{(n)})] \to \mathrm{E}_0^{\mathrm{BM}}[F(\Sigma \cdot W)]$ as $n \to \infty$ for $\mathbb{P}_0$-a.e. ω, where $(W, \mathrm{P}_0^{\mathrm{BM}})$ is a Brownian motion on $\mathbb{R}^d$ starting at 0 with covariance matrix $\Sigma^2 = \Sigma \cdot \Sigma^T$.

Our main result relies on the following integrability condition.

Assumption II.1.6 (Integrability condition). For some $p, q \in [1, \infty]$ and $\theta \in (0, 1)$ with

$$\frac{1}{p} + \frac{1}{q} < \frac{2(1-\theta)}{d-\theta}, \tag{II.1.7}$$

assume that the following integrability condition holds

$$\mathbb{E}\big[\omega(e)^p\big] < \infty \qquad \text{and} \qquad \mathbb{E}\big[\omega(e)^{-q}\mathbf{1}_{e\in\in\mathcal{O}}\big] < \infty, \tag{II.1.8}$$

where we used the convention that $0/0 = 0$.

Theorem II.1.7 (Quenched invariance principle). *For $d \geq 2$ suppose that $\theta \in (0, 1)$ and $p, q \in [1, \infty]$ satisfy Assumptions II.1.1, II.1.3 and II.1.6. Then, the QFCLT holds for the process X with a deterministic and non-degenerate covariance matrix Σ^2.*

Remark II.1.8. If the law $\mathbb{P}$ of the conductances is invariant under reflection and rotation of $\mathbb{Z}^d$ by $\pi/2$, the limiting Brownian motion is isotropic in the sense that its covariance matrix Σ^2 is of the form $\Sigma^2 = \sigma^2 I$ for some $\sigma > 0$. (Here $I \in \mathbb{R}^{d\times d}$ denotes the identity matrix.)

Remark II.1.9. Consider the Markov process $Y = \{Y_t : t \geq 0\}$ on $\mathcal{C}_\infty(\omega)$ that waits at the vertex $x \in \mathcal{C}_\infty(\omega)$ an exponential time with mean 1 and then jumps to a neighboring vertex y with probability $\omega(\{x,y\})/\mu^\omega(x)$. This process is also called *constant speed random walk* (CSRW). Notice that the process Y can be obtained from the process X by a time change, that is $Y_t := X_{a_t}$ for $t \geq 0$, where

$$a_t := \inf\{s \geq 0 : A_s > t\}$$

denotes the right continuous inverse of the functional

$$A_t := \int_0^t \mu^\omega(X_s)\, ds, \qquad t \geq 0.$$

By the ergodic theorem and Lemma II.2.4, we have that $\lim_{t\to\infty} A_t/t = \mathbb{E}_0[\mu^\omega(0)]$ for $\mathbb{P}_0$-a.e ω. Hence, under the assumptions of Theorem II.1.7, the rescaled process Y converges to a Brownian motion on $\mathbb{R}^d$ with deterministic and non-degenerate covariance matrix $\Sigma_Y^2 = \mathrm{E}_0[\mu^\omega(0)]^{-1}\Sigma^2$, see [1, Section 6.2].

Remark II.1.10. Notice that Assumption II.1.1 and the remark above implies that $\mathbb{P}_0$-a.s. the process X does not explode in finite time.

Random walks among random conductances is one of the most studied examples of random walks in random environments. Since the pioneering works of De Masi, Ferrari, Goldstein and Wick [24] and Kipnis and Varadhan [41] which proved a *weak FCLT* for stationary and ergodic laws $\mathbb{P}$ with $\mathbb{E}[\omega(e)] < \infty$, in the last two decades much attention has been devoted to obtain a *quenched* FCLT.

For i.i.d. environments ($\mathbb{P}$ is a product measure), it turns out that no moment conditions are required. Based on the previous works by Mathieu [47], Biskup and Prescott [17], Barlow and Deuschel [10] (for similar results for simple random walks on supercritical Bernoulli percolation clusters see also Sidoravious and Sznitman [59], Berger and Biskup [14], Mathieu and Piatniski [48]) it has been finally shown by Andres, Barlow, Deuschel and Hambly [1] that a QFCLT for i.i.d. environments holds provided that $\mathbb{P}_0[\omega(e) > 0] > p_c$ with $p_c \equiv p_c(d)$ being the bond percolation threshold. Recently, Procaccia, Rosenthal and Sapozhnikov [54] have studied a quenched invariance principle for simple random walks on a certain class of percolation models with long range correlations including random interlacements and level sets of the Gaussian Free Field (both in $d \geq 3$).

For general ergodic, elliptic environments, $\mathbb{P}[0 < \omega(e) < \infty] = 1$, where the infinite connected component $\mathcal{C}_\infty(\omega)$ coincides with $\mathbb{Z}^d$, the first moment condition on the conductances, $\mathbb{E}[\omega(e)] < \infty$ and $\mathbb{E}[\omega(e)^{-1}] < \infty$, is necessary for a QFCLT to hold, see Barlow, Burdzy and Timár [8, 7]. The uniformly elliptic situation, treated by Boivin [19], Sidoravious and Sznitman [59] (cf. Theorem 1.1 and Remark 1.3 therein), Barlow and Deuschel [10], has been relaxed by Andres, Deuschel and Slowik [2] to the condition in Assumption II.1.6 with $\theta = 0$. As it turned out, for the constant speed random walk Y as defined above, this moment condition is optimal for a *quenched local limit theorem* to hold, see [4]. In dimension $d = 2$, Biskup proved a QFCLT under the (optimal) first moment condition, and it is an open problem if this remains true in dimensions $d \geq 3$.

In Chapter II, we are interested in the random conductance model beyond the elliptic setting. We prove a quenched invariance principle in the case of stationary and ergodic laws under mild assumptions on geometric properties of the resulting clusters and on the integrability of $\mathbb{P}$. This framework includes the models considered in [2] and [54]. The main novelty is a new anchored relative isoperimetric inequality (Lemma II.3.7) that is used to show in a robust way the ℓ^1-sublinearity of the corrector (for more details see below). Another important aspect is that neither an a priori knowledge on the distribution of the size of holes in the connected components nor on properties of the chemical distance is needed. In particular, our proof does not rely on the directional sublinearity of the corrector.

In the sequel, we give a brief list of motivating examples of probability measures on $[0, \infty)^{\mathbb{E}^d}$ for the conductances.

Example II.1.11 (Supercritical percolation cluster)**.** Consider a supercritical bond percolation $\{\omega(e) : e \in \mathbb{E}^d\}$, that is, $\omega(e) \in \{0, 1\}$ are i.i.d. random variables with $\mathbb{P}[\omega(e) = 1] > p_c$. The almost sure existence of a unique infinite cluster is guaranteed by Burton–Keane's theorem, while Assumption II.1.3 on θ-very regular balls for any $\theta \in (0, 1)$ follows from a series of results in [9]: Theorem 2.18 a), c) together with Lemma 2.19, Proposition 2.11 (combined with Lemma 1.1), and Proposition 2.12 a). More precisely, we choose $\mathcal{S}^\omega(0, n)$ as the largest cluster

$\mathcal{C}^\omega(Q_1)$ where Q_1 is the smallest special cube appearing in the proof of [9, Theorem 2.18]. In this case, our result on the quenched invariance principle Theorem II.1.7 contains the ones in [59, 48, 17, 14].

Example II.1.12 (Models with long-range correlations)**.** Consider a family of distribution $\mathbb{P}^u$ on $\{0,1\}^{\mathbb{Z}^d}$ indexed by $u \in (a,b)$ that satisfies the assumptions **P1–P3**, **S1** and **S2** in [57]. For a given sample $\{\eta(x) : x \in \mathbb{Z}^d\}$ of $\mathbb{P}^u$, we set

$$\omega(\{x,y\}) \;=\; \eta(x)\cdot\eta(y) \qquad \forall\,\{x,y\} \in \mathbb{E}^d.$$

For any fixed $u \in (a,b)$, set $\mathbb{P} = \mathbb{P}^u \circ \omega^{-1}$. Obviously, $\mathbb{P}$ is ergodic with respect to translations of $\mathbb{Z}^d$. In view of [57, Remark 1.9 (2)], there exists $\mathbb{P}$-a.s. a unique infinite cluster. Hence, Assumption II.1.1 is satisfied. Moreover, Assumption II.1.3 on θ-very regular balls for any $\theta \in (0,1)$ follows from [57, Proposition 4.3] with $\varepsilon = 1/d$. Therefore, the QFCLT for the simple random walk on percolation clusters given by ω holds true. In particular, the strategy used in showing Theorem II.1.7 provides an alternative proof of [54, Theorem 1].

Let us consider a more general model in which random walks move on percolation clusters with arbitrary jump rates.

Example II.1.13 (RCM by level sets of the Gaussian Free Field)**.** Consider the discrete Gaussian Free Field $\phi = \{\phi(x) : x \in \mathbb{Z}^d\}$ for $d \geq 3$, i.e. ϕ is a Gaussian field with mean zero and covariances given by the Green function of the simple random walk on $\mathbb{Z}^d$. The excursion set of the field ϕ above level h is defined as $V_{\geq h}(\phi) := \{x \in \mathbb{Z}^d : \phi(x) \geq h\}$, which can be considered as vertex set of the random graph of with edge set $E_{\geq h}(\phi) := \{\{x,y\} : \phi(x) \wedge \phi(y) \geq h\}$. It is well known [21, 55] that there exists a threshold $h_* = h_*(d) \in [0,\infty)$ such that almost surely the graph $\big(V_{\geq h}(\phi), E_{\geq h}(\phi)\big)$ contains

(i) for $h < h_*$, a unique infinite connected component;

(ii) for $h > h_*$, only finite connected components.

We are interested in the first case, where the family $\{\mathbb{P}^{h_*-h},\ h \in (a,b)\}$, with $\mathbb{P}^u$ denoting the law of the site percolation process $\{\mathbf{1}_{\phi(x)\geq u} : x \in \mathbb{Z}^d\}$, satisfies for some $0 < a < b < \infty$ the assumptions **P1–P3**, **S1** and **S2** in [57] (for more details, see Subsection 1.1.2 therein). For $h \in (a,b)$, define

$$\omega(\{x,y\}) \;=\; \exp\big(\phi(x)+\phi(y)\big)\,\mathbf{1}_{|\phi(x)|\wedge|\phi(y)|\,\geq\, h_*-h} \qquad \forall\,\{x,y\} \in \mathbb{E}^d,$$

and denote by $\mathbb{P}$ the corresponding law. In view of [57, Proposition 4.3], Assumptions II.1.1 and II.1.3 are satisfied. Since $\mathbb{E}[\omega(e)^p] < \infty$ and $\mathbb{E}[\omega(e)^{-q}\mathbf{1}_{\omega(e)>0}] < \infty$ for every $p,q \in (0,\infty)$, Theorem II.1.7 holds for this random conductance model.

II.1.3 The method

We follow the most common approach to prove a QFCLT that is based on *harmonic embedding*, see [16] for a detailed exposition of this method. A key ingredient of this approach is the *corrector*, a random function, $\chi\colon \Omega \times \mathbb{Z}^d \to \mathbb{R}^d$ satisfying $\mathbb{P}_0$-a.s. the following *cocycle property*

$$\chi(\omega, x+y) - \chi(\omega, x) \;=\; \chi(\tau_x\omega, y), \qquad x, y \in \mathcal{C}_\infty(\omega).$$

such that $|\chi(\omega, x)| = o(|x|)$ as $|x| \to \infty$ and

$$\Phi(\omega, x) \;=\; x - \chi(\omega, x)$$

is an $\mathcal{L}^\omega$-harmonic function in the sense that $\mathbb{P}_0$-a.s.

$$\mathcal{L}^\omega \Phi(\omega, x) \;=\; \sum_y \omega(\{x, y\})\big(\Phi(\omega, y) - \Phi(\omega, x)\big) \;=\; 0, \qquad \forall\, x \in \mathcal{C}_\infty(\omega).$$

This can be rephrased by saying that χ is a solution of the Poisson equation

$$\mathcal{L}^\omega u \;=\; \mathcal{L}^\omega \Pi$$

where Π denotes the identity mapping on $\mathbb{Z}^d$. The existence of χ is guaranteed by Assumption II.1.1. Further, the $\mathcal{L}^\omega$-harmonicity of Φ implies that

$$M_t \;=\; X_t - \chi(\omega, X_t)$$

is a martingale under P_0^ω for $\mathbb{P}_0$-a.e. ω, and a QFCLT for the martingale part M can be easily shown by standard arguments. In order to obtain a QFCLT for the process X, by Slutsky's theorem, it suffices to show that for any $T > 0$ and $\mathbb{P}_0$-a.e ω

$$\sup_{0 \le t \le T} \frac{1}{n} \big|\chi(\omega, X_{tn^2})\big| \underset{n \to \infty}{\longrightarrow} 0 \quad \text{in } \mathrm{P}_0^\omega\text{-probability}, \tag{II.1.9}$$

which can be deduced from ℓ^∞-sublinearity of the corrector:

$$\lim_{n \to \infty} \max_{x \in B^\omega(0,n)} \frac{1}{n} \big|\chi(\omega, x)\big| \;=\; 0 \qquad \mathbb{P}_0\text{-a.s.} \tag{II.1.10}$$

The main challenge in the proof of the QFCLT is to show (II.1.10). In a first step we show that the rescaled corrector converges to zero $\mathbb{P}_0$-a.s. in the space averaged norm $\|\cdot\|_{1,B^\omega(0,n)}$ (see Proposition II.2.9 below). A key ingredient in the proof is a new anchored relative isoperimetric inequality (Lemma II.3.7) and an extension of Birkhoff's ergodic theorem, see Appendix II.A for more details. In a second step, we establish a maximal inequality for the solution of a certain class of Poisson equations using a Moser iteration scheme. As an application, the maximum of the rescaled corrector in the ball $B^\omega(0, n)$ can be controlled by the corresponding $\|\cdot\|_{1,B^\omega(0,n)}$-norm. In the case of elliptic conductances such a Moser iteration has already been implemented in order to show a QFCLT [2], a local limit theorem and elliptic and parabolic Harnack inequalities [3] as well as upper Gaussian estimates on the heat kernel [4]. The Moser iteration is based on a Sobolev inequality for functions with compact support which follows in the case of elliptic conductances ($\mathcal{C}_\infty(\omega) \equiv \mathbb{Z}^d$) from the isoperimetric inequality (II.1.5) on $\mathbb{Z}^d$. Since such an isoperimetric inequality on random graphs is true only for sufficiently large sets (Lemma II.2.10), the present proof of the Sobolev inequality relies on an interpolation argument in order to deal with the small sets (see Lemma II.3.3 below).

The paper is organized as follows: In Section II.2, we prove our main result. After recalling the construction of the corrector and proving the convergence of the martingale part, we show the ℓ^1- and ℓ^∞-sublinearity of the corrector. The proof of the ℓ^1-sublinearity is based on an anchored Sobolev inequality that we show in a more general context in Section II.3. Finally, the Appendix contains an ergodic theorem that is needed in the proofs.

Throughout the paper, we write c to denote a positive constant that may change on each appearance, whereas constants denoted by C_i will be the same through each argument.

II.2 Quenched invariance principle

Throughout this section we suppose that Assumption II.1.1 holds.

II.2.1 Harmonic embedding and the corrector

In this subsection, we first construct a corrector to the process X such that $M_t = X_t - \chi(\omega, X_t)$ is a martingale under P_0^ω for $\mathbb{P}$ a.e. ω. Second, we prove an invariance principle for the martingale part.

Definition II.2.1. A measurable function, also called a random field, $\Psi : \Omega \times \mathbb{Z}^d \to \mathbb{R}$ satisfies the cocycle property if for $\mathbb{P}_0$-a.e. ω, it hold that

$$\Psi(\tau_x\omega, y - x) \;=\; \Psi(\omega, y) - \Psi(\omega, x), \qquad \text{for } x, y \in \mathcal{C}_\infty(\omega).$$

We denote by L^2_{cov} the set of functions $\Psi : \Omega \times \mathbb{Z}^d \to \mathbb{R}$ satisfying the cocycle property such that

$$\|\Psi\|^2_{L^2_{\mathrm{cov}}} \;:=\; \mathbb{E}_0\Bigg[\sum_{x\sim 0} \omega(\{0,x\})\, |\Psi(\omega,x)|^2\Bigg] \;<\; \infty.$$

In the following lemma we summerize some properties of functions in L^2_{cov}.

Lemma II.2.2. *For all $\Psi \in L^2_{\mathrm{cov}}$, we have* *(i)* $\Psi(\omega,0)=0$ *and* $\Psi(\tau_x\omega,-x)=\Psi(\omega,x)$ *for any* $x\in\mathcal{C}_\infty(\omega)$ *and* $\omega\in\omega_0$*,*

(ii) $\|\Psi\|_{L^2_{\mathrm{cov}}}=0$*, if and only if,* $\Psi(\omega,x)=0$ *for all* $x\in\mathcal{C}_\infty(\omega)$ *and* $\mathbb{P}_0$*-a.e.* $\omega\in\Omega_0$*.*

Proof. (i) follows from the definition.
(ii) "$\Leftarrow$" The assertion follows immediately from the definition of $\|\cdot\|_{L^2_{\mathrm{cov}}}$.
"$\Rightarrow$" Suppose that $\|\Psi\|_{L^2_{\mathrm{cov}}}=0$. By using the stationarity of $\mathbb{P}$ and the cocycle property, we obtain that, for any $y\in\mathbb{Z}^d$,

$$\begin{aligned} 0 &= \mathbb{E}\left[\sum_{x\sim 0}(\tau_y\omega)(\{0,x\})\,\Psi(\tau_y\omega,x)^2\,\mathbf{1}_{0\in\mathcal{C}\infty(\tau_y\omega)}\right] \\ &= \mathbb{E}\left[\sum_{x\sim 0}\omega(\{y,y+x\})\,\big|\Psi(\omega,y+x)-\Psi(\omega,y)\big|^2\,\mathbf{1}_{y\in\mathcal{C}_\infty(\omega)}\right]. \end{aligned} \tag{II.2.1}$$

Hence, for any $y\in\mathbb{Z}^d$ there exists $\Omega_y^*\subset\Omega$ such that $\mathbb{P}[\Omega_y^*]=1$ and for all $\omega\in\Omega_*$

$$\omega(\{y,y+x\})\,\big|\Psi(\omega,y+x)-\Psi(\omega,y)\big|^2\,\mathbf{1}_{y\in\mathcal{C}_\infty(\omega)} \;=\; 0 \quad \forall\,|x|=1. \tag{II.2.2}$$

Set $\Omega^*:=\bigcap_{y\in\mathbb{Z}^d}\Omega_y^*$. Obviously, $\mathbb{P}[\Omega^*]=1$, and for any $\omega\in\Omega^*$, (II.2.2) holds true for all $y\in\mathbb{Z}^d$. In particular, for any $\omega\in\Omega^*\cap\Omega_0$ and $z\in\mathcal{C}_\infty(\omega)$, there exist $z_0=0,\dots,z_k=z$ with $\{z_i,z_{i+1}\}\in\mathcal{O}(\omega)$ for all $0\le i\le k-1$ such that

$$\Psi(\omega,z_i) \;=\; \Psi(\omega,z_{i+1}) \qquad \forall\,i=0,\dots,k-1.$$

Hence, $\Psi(\omega,z)=\Psi(\omega,0)=0$. This completes the proof. $\square$

In particular, it can be checked that L^2_{cov} is a Hilbert space (cf. [17, 48]).

We say a function $\varphi:\Omega\to\mathbb{R}$ is *local* if it only depends on the value of ω at a finite number of edges. We associate to φ a (horizontal) gradient $\mathrm{D}\varphi\colon\omega\times\mathbb{Z}^d\to\mathbb{R}$ defined by

$$\mathrm{D}\varphi(\omega,x) \;=\; \varphi(\tau_x\omega)-\varphi(\omega), \qquad x\in\mathbb{Z}^d.$$

Obviously, if the function φ is bounded, $\mathrm{D}\varphi$ is an element of L^2_{cov}. Following [48], we introduce an orthogonal decomposition of the space L^2_{cov}. Set

$$L^2_{\mathrm{pot}} \;=\; \mathrm{cl}\big\{\mathrm{D}\varphi \mid \varphi\colon\Omega\to\mathbb{R}\ \text{ local}\big\}\ \text{ in }\ L^2_{\mathrm{cov}},$$

being the closure in L^2_{cov} of the set gradients and let L^2_{sol} be the orthogonal complement of L^2_{pot} in L^2_{cov}, that is

$$L^2_{\mathrm{cov}} \;=\; L^2_{\mathrm{pot}}\oplus L^2_{\mathrm{sol}}.$$

In order to define the corrector, we introduce the *position field* $\Pi:\omega\times\mathbb{Z}^d\to\mathbb{R}^d$ with $\Pi(\omega,x)=x$. We write Π_j for the j-th coordinate of Π. Since $\Pi_j(\tau_x\omega,y-x)=$

$\Pi_j(\omega, y) - \Pi_j(\omega, x)$ for all $x, y \in \mathbb{Z}^d$, the j-th component of the position field Π_j satisfies the cocycle property for every $\omega \in \Omega_0$. Moreover,

$$\|\Pi_j\|^2_{L^2_{\mathrm{cov}}} \;=\; \mathbb{E}_0\Big[\textstyle\sum_{x\sim 0} \omega(\{0,x\})\,|x_j|^2\Big] \;=\; 2\mathbb{E}_0\big[\omega(\{0,e_j\})\big] \;<\; \infty, \tag{II.2.3}$$

where e_j denotes the j-th coordinate unit vector. Hence, $\Pi_j \in L^2_{\mathrm{cov}}$. So, we can define $\chi_j \in L^2_{\mathrm{pot}}$ and $\Phi_j \in L^2_{\mathrm{sol}}$ as follows

$$\Pi_j \;=\; \chi_j \,+\, \Phi_j \;\in\; L^2_{\mathrm{pot}} \oplus L^2_{\mathrm{sol}}.$$

This defines the corrector $\chi = (\chi_1, \dots, \chi_d) : \Omega \times \mathbb{Z}^d \to \mathbb{R}^d$. Further, we set

$$M_t \;=\; \Phi(\omega, X_t) \;=\; X_t - \chi(\omega, X_t). \tag{II.2.4}$$

The following proposition summarizes the properties of χ, Φ and M; see, for example, [1], [10] or [16] for detailed proofs.

Proposition II.2.3. *For $\mathbb{P}_0$-a.e. ω, we have*

$$\mathcal{L}^\omega \Phi(x) \;=\; \sum_{y\sim x} \omega(\{x,y\})\big(\Phi(\omega,y) - \Phi(\omega,x)\big) \;=\; 0 \qquad \forall\, x \in \mathcal{C}_\infty(\omega). \tag{II.2.5}$$

In particular, for $\mathbb{P}_0$-a.e. ω and for every $v \in \mathbb{R}^d$, M and $v \cdot M$ are P_0^ω-martingales with respect to the filtration $\mathcal{F}_t = \sigma(X_s, s \le t)$. The quadratic variation process of the latter is given by

$$\langle v\cdot M\rangle_t \;=\; \int_0^t \sum_{x\sim 0} (\tau_{X_s}\omega)(\{0,x\})\,\big(v \cdot \Phi(\tau_{X_s}\omega, x)\big)^2\, ds. \tag{II.2.6}$$

In the sequel, we prove a quenched invariance principle for the martingale part. This is standard and follows from the ergodicity of the process of the *environment as seen from the particle* $\{\tau_{X_t}\omega : t \ge 0\}$ which is a Markov process taking values in the environment space Ω_0 with generator

$$\widehat{\mathcal{L}}\,\varphi(\omega) \;=\; \sum_{x\sim 0} \omega(\{0,x\})\,\big(\varphi(\tau_x\omega) - \varphi(\omega)\big)$$

acting on bounded functions $\varphi\colon \Omega_0 \to \mathbb{R}$. The following result is a generalization of Kozlov's theorem [42] in the case that the underlying random walk is reversible.

Lemma II.2.4. *The measure $\mathbb{P}_0$ is reversible, invariant and ergodic for the environment process $\{\tau_{X_t}\omega : t \ge 0\}$.*

Proof. The reversibility of $\{\tau_{X_t}\omega : t \ge 0\}$ with respect to $\mathbb{P}_0$ follows directly from Assumption II.1.1. The proof of the ergodicity of the environmental process relies on the ergodicity of $\mathbb{P}$ with respect to shifts of $\mathbb{Z}^d$ and the fact that for $\mathbb{P}$-a.e. ω the infinite cluster, $\mathcal{C}_\infty(\omega)$, is unique. See [24, Lemma 4.9] for a detailed proof. □

In the next proposition we show both the convergence of the martingale part and the non-degeneracy of the limiting covariance matrix. The proof of the latter, inspired by the argument given in [54] (see also [14]), relies on the ℓ^1-sublinearity of the corrector that we will show below in Proposition II.2.9.

Proposition II.2.5 (QFCLT for the martingale part). *For $\mathbb{P}_0$-a.e. ω, the sequence of processes $\{\frac{1}{n}M_{tn^2} : t \geq 0\}$ converges in P_0^ω-probability to a Brownian motion with a deterministic covariance matrix Σ^2 given by*

$$\Sigma^2_{ij} \;=\; \mathbb{E}_0\Big[\sum\nolimits_{x\sim 0} \omega(\{0,x\})\, \Phi_i(\omega,x)\, \Phi_j(\omega,x)\Big].$$

Additionally, if $\theta \in (0,1)$ satisfies Assumption II.1.3 and $\mathbb{E}[(1/\omega(e))\mathbf{1}_{e\in\mathcal{O}}] < \infty$ for any $e \in \mathbb{E}^d$, then the limiting covariance matrix Σ^2 is non-degenerate.

Proof. The proof follows from the martingale convergence theorem by Helland, cf. [39, Theorem 5.1a)]; see also [1] or [48] for details. The argument is based on the fact that the quadratic variation of $\{\frac{1}{n}M_{tn^2} : t \geq 0\}$ converges, for which the ergodicity of the environment process in Lemma II.2.4 is needed.

It remain to show that the limiting Brownian motion is non-degenerate. The argument is similar to the one in [54], but avoids the use of the ℓ^∞-sublinearity of the corrector. Assume that $(v, \Sigma^2 v) = 0$ for some $v \in \mathbb{R}^d$ with $|v| = 1$. First, we deduce from Lemma II.2.2 that, for $\mathbb{P}_0$-a.e. ω, $v \cdot \Phi(\omega, x) = 0$ for all $x \in \mathcal{C}_\infty(\omega)$. Since $x = \chi(\omega, x) + \Phi(\omega, x)$, this implies that, for $\mathbb{P}_0$-a.e. ω, $|v \cdot x| = |v \cdot \chi(\omega, x)|$ for all $x \in \mathcal{C}_\infty(\omega)$. In particular,

$$\frac{1}{|B^\omega(n)|} \sum_{x\in B^\omega(n)} \big|v \cdot \tfrac{1}{n}x\big| \;=\; \frac{1}{|B^\omega(n)|} \sum_{x\in B^\omega(n)} \big|v \cdot \tfrac{1}{n}\chi(\omega,x)\big|. \tag{II.2.7}$$

In view of Proposition II.2.9, the right-hand side of (II.2.7) vanishes for $\mathbb{P}_0$-a.e. ω as n tends to infinity. On the other hand, for any $\delta \in (0,1)$ we have that

$$\begin{aligned}
\frac{1}{n^d} \sum_{x\in B^\omega(n)} \big|v \cdot \tfrac{1}{n}x\big| \;&\geq\; \frac{\delta^2}{n^d} \sum_{\substack{x\in B^\omega(n)\\ x\neq 0}} \mathbf{1}_{|x|>\delta n}\, \mathbf{1}_{|v\cdot x/|x||>\delta} \\
&\geq\; \frac{\delta^2}{n^d}\Bigg(|B^\omega(n)| \,-\, |B(\delta n)| \,-\, \sum_{\substack{x\in B(n)\\ x\neq 0}} \mathbf{1}_{|v\cdot x/|x||\leq\delta}\Bigg)
\end{aligned}$$

Due to (II.1.3), $|B^\omega(n)| \geq C_{\mathrm{V}} n^d$ for all $n \geq N_1(\omega)$ and $\mathbb{P}_0$-a.e. ω. Moreover, the other two terms in the bracket above are of order δn^d. Hence, by choosing δ sufficiently small, there exists $c > 0$ such that

$$\varliminf_{n\to\infty} \frac{1}{|B^\omega(n)|} \sum_{x\in B^\omega(n)} \big|v \cdot \tfrac{1}{n}x\big| \;\geq\; c \;>\; 0.$$

Thus, we proved that $(v, \Sigma^2 v) > 0$ for all $0 \neq v \in \mathbb{R}^d$, which completes the proof. $\square$

II.2.2 Sublinearity of the corrector

Recall that we denote by $B^\omega(x,r)$ and $B(x,r)$ a closed ball with center $x \in \mathcal{C}_\infty(\omega)$ and radius $r \geq 0$ with respect to the graph distance d^ω and usual ℓ^1-distance on $\mathbb{Z}^d$, respectively. To lighten notation, we write $B^\omega(r) \equiv B^\omega(0,r)$ and $B(r) \equiv B(0,r)$. Further, for any non-empty $A \subset \mathbb{Z}^d$, we define a locally space-averaged norm for functions $f \colon \mathbb{Z}^d \to \mathbb{R}$ by

$$\|f\|_{p,A} \;:=\; \Bigg(\frac{1}{|A|}\sum_{x\in A}|f(x)|^p\Bigg)^{1/p}, \qquad p \in [1,\infty).$$

Our main objective in this subsection is to prove the ℓ^∞-sublinearity of the corrector.

Proposition II.2.6 (ℓ^∞-sublinearity)**.** *Suppose that $\theta \in (0,1)$ and $p,q \in [1,\infty]$ satisfy Assumptions II.1.3 and II.1.6. Then, for any $j = 1,\ldots,d$,*

$$\lim_{n\to\infty}\max_{x\in B^\omega(n)}\big|\tfrac{1}{n}\chi_j(\omega,x)\big| \;=\; 0, \qquad \mathbb{P}_0\text{-a.s.} \tag{II.2.8}$$

The proof is based on both ergodic theory and purely analytic tools. In a first step, we show the ℓ^1-sublinearity of the corrector, that is the convergence of $\frac{1}{n}\chi$ to zero in the $\|\cdot\|_{1,B^\omega(n)}$-norm. This proof uses the spatial ergodic theorem as well as the anchored S_1-Sobolev inequality that we established in Proposition II.3.9. In a second step, we use the maximum inequality in order to bound from above the maximum of $\frac{1}{n}\chi$ in $B^\omega(n)$ by $\frac{1}{n}\|\chi\|_{1,B^\omega(n)}$.

Let us start with some consequences from the ergodic theorem. To simplify notation let us define the following measures μ^ω and ν^ω on $\mathbb{Z}^d$

$$\mu^\omega(x) \;=\; \sum_{x\sim y}\omega(\{x,y\}) \qquad \text{and} \qquad \nu^\omega(x) \;=\; \sum_{x\sim y}\frac{1}{\omega(\{x,y\})}\,\mathbf{1}_{\{x,y\}\in\mathcal{O}(\omega)},$$

where we still use the convention that $0/0 = 0$.

Lemma II.2.7. *Suppose that for $\mathbb{P}_0$-a.e. ω there exists $N_1(\omega) < \infty$ such that the ball $B^\omega(n)$ satisfies the volume regularity* (II.1.3) *for all $n \geq N_1(\omega)$. Further, assume that $\mathbb{E}[\omega(e)^p] < \infty$ and $\mathbb{E}[(1/\omega(e))^q\mathbf{1}_{e\in\mathcal{O}}] < \infty$ for some $p,q \in [1,\infty)$. Then, for $\mathbb{P}_0$-a.s. ω there exists $c < \infty$ such that*

$$\varlimsup_{n\to\infty}\|\mu^\omega\|_{p,B^\omega(n)}^p \;\leq\; c\,\mathbb{E}_0[\mu^\omega(0)^p] \quad \textit{and} \quad \varlimsup_{n\to\infty}\|\nu^\omega\|_{q,B^\omega(n)}^q \;\leq\; c\,\mathbb{E}_0[\nu^\omega(0)^q]. \tag{II.2.9}$$

Proof. The assertions follows immediately from the spatial ergodic theorem. For instance, we have for $\mathbb{P}_0$-a.s.

$$\varlimsup_{n\to\infty}\|\mu^\omega\|_{p,B^\omega(n)}^p \;\overset{\text{(II.1.3)}}{\leq}\; \varlimsup_{n\to\infty}\frac{C_{\mathrm{V}}^{-1}}{n^d}\sum_{x\in B(n)}\mu^{\tau_x\omega}(0)^p\,\mathbf{1}_{0\in\mathcal{C}_\infty(\tau_x\omega)} \;\leq\; c\,\mathbb{E}_0\big[\mu^\omega(0)^p\big],$$

where we exploit the fact that $B^\omega(n) \subset B(n)\cap\mathcal{C}_\infty(\omega)$ for every $n \geq 1$. $\square$

The next lemma relies on an extension of Birkhoff's ergodic theorem that we show in the appendix.

Lemma II.2.8. *Let $w_n \colon \mathbb{E}^d \to (0,\infty)$ be defined by*

$$w_n(\{x,y\}) = (n/\max\{|x|_1, |y|_1\})^{d-\varepsilon}$$

for some $\varepsilon \in (0,1)$, and assume that $\mathbb{E}[(1/\omega(e))\mathbf{1}_{e\in\mathcal{O}}] < \infty$ for all $e \in \mathbb{E}^d$. Then, there exists $C_5 < \infty$ such that for any $\Psi \in L^2_{\mathrm{cov}}$ and $\mathbb{P}_0$-a.e. ω,

$$\varlimsup_{n\to\infty} \frac{1}{n^d} \sum_{\substack{x,y\in B^\omega(n)\\ x\sim y}} w_n(\{x,y\})\,|\Psi(\omega,x)-\Psi(\omega,y)| \;\le\; \frac{C_5}{\varepsilon}\,\mathbb{E}_0[\nu^\omega(0)]^{1/2}\,\|\Psi\|_{L^2_{\mathrm{cov}}}. \tag{II.2.10}$$

Proof. First, an application of the Cauchy-Schwarz inequality yields

$$\begin{aligned}
&\mathbb{E}_0\Big[\textstyle\sum_{0\sim y}|\Psi(\omega,y)|\,\mathbf{1}_{\{0,y\}\in\mathcal{O}}\Big]\\
&\quad\le\; \mathbb{E}_0\Big[\textstyle\sum_{0\sim y}\big(1/\omega(\{0,y\})\big)\,\mathbf{1}_{\{0,y\}\in\mathcal{O}}\Big]^{1/2}\,\mathbb{E}_0\Big[\textstyle\sum_{0\sim y}\omega(\{0,y\})\,|\Psi(\omega,y)|^2\Big]^{1/2}\\
&\quad=\; \mathbb{E}_0[\nu^\omega(0)]^{1/2}\,\|\Psi\|_{L^2_{\mathrm{cov}}}.
\end{aligned} \tag{II.2.11}$$

which is finite since $\Psi \in L^2_{\mathrm{cov}}$ and $\mathbb{E}[(1/\omega(e))\mathbf{1}_{e\in\mathcal{O}}] < \infty$ by assumption. Recall that Ψ satisfies the cocycle property, that is $\Psi(\omega,x)-\Psi(\omega,y) = \Psi(\tau_x\omega, y-x)$ for $\mathbb{P}_0$-a.e. ω and for every $x,y \in \mathcal{C}_\infty(\omega)$. Since $B^\omega(n) \subset B(n)\cap\mathcal{C}_\infty(\omega)$ for every $n \ge 1$, we obtain that, for any $\omega \in \Omega_0$,

$$\begin{aligned}
&\frac{1}{n^d}\sum_{\substack{x,y\in B^\omega(n)\\ x\sim y}} w_n(\{x,y\})\,\big|\Psi(\tau_x\omega,y-x)\big|\\
&\qquad\le\; \frac{1}{n^d}\sum_{x,y\in B(n)} w_n(\{x,y\})\,\big|\Psi(\tau_x\omega,y-x)\big|\,\mathbf{1}_{0\in\mathcal{C}_\infty(\tau_x\omega)}\mathbf{1}_{\{0,y-x\}\in\mathcal{O}(\tau_x\omega)}\\
&\qquad\le\; \frac{\psi(\omega)}{n^\varepsilon} + \frac{1}{n^d}\sum_{\substack{x\in B(n)\\ x\neq 0}} \frac{\psi(\tau_x\omega)}{|x/n|^{d-\varepsilon}}
\end{aligned}$$

where we introduced $\psi(\omega) = \sum_{y\sim 0}|\Psi(\omega,y)|\mathbf{1}_{0\in\mathcal{C}_\infty(\omega)}\mathbf{1}_{\{0,y\}\in\mathcal{O}(\omega)}$ to lighten notation. Further, an application of (II.A.1) yields

$$\begin{aligned}
&\varlimsup_{n\to\infty}\frac{1}{n^d}\sum_{\substack{x,y\in B^\omega(n)\\ x\sim y}} w_n(\{x,y\})\,\big|\Psi(\tau_x\omega,y-x)\big|\\
&\qquad\le\; \frac{C_5}{\varepsilon}\,\mathbb{E}\Big[\textstyle\sum_{0\sim y}|\Psi(\omega,y)|\,\mathbf{1}_{0\in\mathcal{C}_\infty}\mathbf{1}_{\{0,y\}\in\mathcal{O}}\Big] \overset{\text{(II.2.11)}}{\le} \frac{C_5}{\varepsilon}\,\mathbb{E}_0[\nu^\omega(0)]^{1/2}\,\|\Psi\|_{L^2_{\mathrm{cov}}},
\end{aligned}$$

which concludes the proof. $\square$

Proposition II.2.9 (ℓ^1-sublinearity)**.** *Suppose $\theta \in (0,1)$ satisfies Assumption II.1.3, and assume that $\mathbb{E}\big[(1/\omega(e))\mathbf{1}_{e\in\mathcal{O}}\big] < \infty$ for all $e \in \mathbb{E}^d$. Then, for any $j = 1, \dots, d$,*

$$\lim_{n\to\infty} \frac{1}{|B^\omega(n)|} \sum_{x\in B^\omega(n)} \big|\tfrac{1}{n}\,\chi_j(\omega,x)\big| \;=\; 0, \qquad \mathbb{P}\text{-a.s.} \tag{II.2.12}$$

Proof. Since $\chi_j \in L^2_{\mathrm{pot}}$, there exists a sequence of bounded functions $\varphi_{j,k} : \omega \to \mathbb{R}$ such that $\mathrm{D}\varphi_{j,k} \to \chi_j$ in L^2_{cov} as $k \to \infty$. Thus, for any fixed $k \geq 1$ we obtain

$$\frac{1}{n^{d+1}} \sum_{x\in B^\omega(n)} |\chi_j(\omega,x)| \;\leq\; \frac{c\|\varphi_{j,k}\|_{L^\infty(\omega)}}{n} + \frac{1}{n^{d+1}} \sum_{x\in B^\omega(n)} \big|(\chi_j - \mathrm{D}\varphi_{j,k})(\omega,x)\big|. \tag{II.2.13}$$

In order to bound from above the second term on the right hand-side of (II.2.13) we consider the deterministic edge weight $w_n : \mathbb{E}^d \to (0,\infty)$ that is defined by $w_n(\{x,y\}) = (n/\max\{|x|_1,|y|_1\})^{d-\varepsilon}$ for some $\varepsilon \in (0,1)$. Since $d^\omega(x,y) \geq |x-y|_1$ for any $x, y \in \mathcal{C}_\infty(\omega)$, the w_n satisfies the assumption in Proposition II.3.9. By applying (II.3.15) and the cocycle property, we find for any $\omega \in \Omega_0$ that

$$\begin{aligned} &\frac{1}{n^{d+1}} \sum_{x\in B^\omega(n)} \big|(\chi_j - \mathrm{D}\varphi_{j,k})(\omega,x)\big| \\ &\quad\leq\; \frac{\overline{C}_{\mathrm{S}_1}}{n^d} \sum_{x,y\in B^\omega(C_{\mathrm{W}}n)} w_n(\{x,y\})\,\big|(\chi_j - \mathrm{D}\varphi_{j,k})(\tau_x\omega, y-x)\big|\,\mathbf{1}_{\{0,y-x\}\in\mathcal{O}(\tau_x\omega)}. \end{aligned}$$

Hence, by combining the estimate above with (II.2.13), we get

$$\begin{aligned} &\frac{1}{n^{d+1}} \sum_{x\in B^\omega(n)} |\chi(\omega,x)| \\ &\quad\leq\; \frac{c\|\varphi_{j,k}|_{L^\infty(\omega)}}{n} + \frac{\overline{C}_{\mathrm{S}_1}}{n^d} \sum_{\substack{x,y\in B^\omega(C_{\mathrm{W}}n)\\ \{0,y-x\}\in\mathcal{O}(\tau_x\omega)}} w_n(\{x,y\})\,\big|(\chi_j - \mathrm{D}\varphi_{j,k})(\tau_x\omega, y-x)\big|. \end{aligned} \tag{II.2.14}$$

In view of Lemma II.2.8, we obtain that there exists $c < \infty$ such that for $\mathbb{P}_0$-a.e. ω,

$$\begin{aligned} \lim_{k\to\infty}\lim_{n\to\infty} \|\tfrac{1}{n}\,\chi_j(\omega,\cdot)\|_{1,B^\omega(n)} &\overset{\text{(II.1.3)}}{\leq} \lim_{k\to\infty}\lim_{n\to\infty} \frac{C_{\mathrm{V}}^{-1}}{n^{d+1}} \sum_{x\in B^\omega(n)} |\chi(\omega,x)| \\ &\overset{\text{(II.2.10)}}{\leq} \lim_{k\to\infty} \frac{c}{\varepsilon}\,\mathbb{E}_0[\nu^\omega(0)]^{1/2}\,\|\chi_j - \mathrm{D}\varphi_{j,k}\|_{L^2_{\mathrm{cov}}} \;=\; 0, \end{aligned}$$

which concludes the proof. □

In the following lemma we show that under the assumption that the ball $B^\omega(n)$ is θ-very regular, the random graph $(\mathcal{C}_\infty(\omega), \mathcal{O}(\omega))$ satisfies $\mathbb{P}_0$-a.s. an isoperimetric inequality for large sets.

Lemma II.2.10 (isoperimetric inequality for large sets)**.** *Suppose that $\theta \in (0,1)$ satisfies Assumption II.1.3. Then, for any $\omega \in \Omega_0$ and $n \geq N_0(\omega)$, there exists $C_{\mathrm{iso}} \in (0,\infty)$ such that*

$$|\partial^\omega A| \;\geq\; C_{\mathrm{iso}}\, |A|^{(d-1)/d} \tag{II.2.15}$$

for all $A \subset B^\omega(n)$ with $|A| \geq n^\theta$.

Proof. First, note that (II.2.15) follows trivially from (II.1.4) for sets with $|A| \geq cn^d$.

Consider $A \subset B^\omega(n)$ with $|A| \geq n^\theta$ and set $r^d := (2/C_{\mathrm{V}})\,|A|$. Since $r \geq n^{\theta/d}$, the Assumption II.1.3 implies that any ball $B^\omega(y, 3r)$ with $y \in B^\omega(n)$ is regular. Further, there exists a finite sequence $\{y_i \in B^\omega(n) : i \in I\}$ such that

$$B^\omega(y_i, r) \cap B^\omega(y_j, r) \;=\; \emptyset \qquad \forall\, i,j \in I \text{ with } i \neq j$$

and $B^\omega(x,r) \cap \bigcup_{i\in I} B^\omega(y_i, r) \neq \emptyset$ for all $x \in B^\omega(n) \setminus \bigcup_{i\in I} B^\omega(y_i, r)$. Clearly, the sets $B^\omega(y_i, 3r)$ cover the ball $B^\omega(n)$, that is, $B^\omega(n) \subset \bigcup_{i\in I} B(y_i, 3r)$. We claim that there exists $M < \infty$, independent of n, such that every $x \in B^\omega(n)$ is contained in at most M different balls $B^\omega(y_i, 3C_{\mathrm{W}} r)$. To prove this claim, set

$$I_x \;:=\; \big\{ i \in I : x \in B^\omega(y_i, 3C_{\mathrm{W}} r) \big\}.$$

Notice that for any $i \in I_x$ we have that $B^\omega(y_i, r) \subset B^\omega(x, 4C_{\mathrm{W}} r)$. By the fact that the sets $B^\omega(y_i, r)$ are disjoint and regular, (II.1.3) hold. In particular,

$$C_0 (4C_{\mathrm{W}})^d r^d \;\geq\; |B^\omega(x, 4C_{\mathrm{W}} r)| \;\geq\; \sum_{x \in I_x} |B^\omega(y_i, r)| \;\geq\; |I_x|\, C_{\mathrm{V}} r^d,$$

where $C_0 := \max_{k\geq 1} |B(k)|/k^d < \infty$. Hence, $M \leq (4C_{\mathrm{W}})^d c_0 / C_{\mathrm{V}}$ which completes the proof of the claim. Further, set $A_i := A \cap \mathcal{S}(y_i, 3r)$. Since $B^\omega(y_i, 3r)$ is regular and $|A_i| \leq |A| \leq \frac{1}{2} |\mathcal{S}(y_i, 3r)|$ for any $i \in I$, (II.1.4) implies that

$$|\partial^\omega A| \;\geq\; \frac{1}{M} \sum_i |\partial^\omega_{\mathcal{S}(y_i,r)} A_i| \;\geq\; \frac{C_{\mathrm{riso}}}{Mr} \sum_i |A_i| \;\geq\; \frac{C_{\mathrm{riso}}}{Mr}\, |A| \;\geq\; \frac{C_{\mathrm{riso}} C_V}{2M}\, |A|.$$

By setting $C_{\mathrm{iso}} := C_{\mathrm{riso}} C_{\mathrm{V}} / (2M)$, the assertion (II.2.15) follows. □

The next proposition relies on the application of the Moser iteration scheme that has been established for general graphs in [2]. A key ingredient in this approach is the following Sobolev inequality

$$\|u\|_{d'/(d'-1), B^\omega(n)} \;\leq\; C_{\mathrm{S}_1} \frac{n}{|B^\omega(n)|} \sum_{x \vee y \in B^\omega(n)} |u(x) - u(y)|\, \mathbf{1}_{\{x,y\} \in \mathcal{O}(\omega)} \tag{II.2.16}$$

for some suitable d' that we will prove in Proposition II.3.5.

Proposition II.2.11 (maximal inequality). *Suppose $\theta \in (0,1)$ and $p, q \in [1,\infty]$ satisfies Assumption II.1.3 and II.1.6. Then, for every $\alpha > 0$ there exist $\gamma' > 0$ and $\kappa' > 0$ and $c(p,q,\theta,d) < \infty$ such that for any $\omega \in \Omega_0$ and $j = 1, \ldots, d$,*

$$\max_{x \in B^\omega(n)} \left|\tfrac{1}{n}\, \chi_j(\omega, x)\right| \;\le\; c\, \Big(1 \vee \|\mu^\omega\|_{p,B^\omega(n)}\, \|\nu^\omega\|_{q,B^\omega(n)}\Big)^{\kappa'}\, \|\tfrac{1}{n}\, \chi_j(\omega,\cdot)\|^{\gamma'}_{\alpha,B^\omega(2n)}.$$

Proof. In view of Lemma II.2.10 and Assumption II.1.3, for any $\omega \in \Omega_0$ and $n \geq N_0(\omega)$ the assumptions of Proposition II.3.5 are satisfied. Further, let $\zeta = (1-\theta)/(1-\theta/d)$ and set $d' = (d-\theta)/(1-\theta)$. Then, Proposition II.3.5 implies that

$$\|u\|_{d'/(d'-1),B^\omega(n)} \;=\; \|u\|_{d/(d-\zeta),B^\omega(n)} \;\le\; C_{\mathrm{S}_1}\, \frac{n}{|B^\omega(n)|} \sum_{\substack{x \vee y \in B^\omega(n)\\ \{x,y\} \in \mathcal{O}(\omega)}} |u(x) - u(y)|$$

for any $u\colon \mathcal{C}_\infty(\omega) \to \mathbb{R}$ with $\mathrm{supp}(u) \subset B^\omega(n)$. By taking this inequality as a starting point and using the fact that by definition $\chi(\omega,x) = 0$ for any $x \in \mathbb{Z}^d \setminus \mathcal{C}_\infty(\omega)$, the assertion for $\frac{1}{n}\chi_j(\omega,\cdot)$ follows directly from [2, Corollary 3.9] with $f(x) = \frac{1}{n}x_j$, $x_0 = 0$, $\sigma = 1$, $\sigma' = 1/2$, n replaced by $2n$ and d replaced by d'. □

Proposition II.2.6 follows immediately from Proposition II.2.11 with the choice $\alpha = 1$, combined with Proposition II.2.9 and Lemma II.2.7.

Proof of Theorem II.1.7. Proceeding as in the proof of [2, Proposition 2.13] (with the minor modification that the exit time $T_{L,n}$ of the rescaled process $X^{(n)}$ from the cube $[-L,L]^d$ is replaced by $T^\omega_{L,n} := \inf\{t \geq 0 : X_{tn^2} \notin B^\omega(n)\}$), the ℓ^∞-sublinearity of the corrector that we have established in Proposition II.2.6 implies that for any $T > 0$ and $\mathbb{P}_0$-a.e. ω

$$\sup_{t \le T} \left|\tfrac{1}{n}\, \chi(\omega, X_{tn^2})\right| \;\underset{n\to\infty}{\longrightarrow}\; 0, \qquad \text{in } \mathbb{P}^\omega_0\text{-probability.}$$

Thus, the assertion of Theorem II.1.7 now follows from Proposition II.2.5. □

II.3 Sobolev inequalities on graphs

As seen in the previous section, both the Sobolev and the anchored Sobolev inequality turned out to be a crucial tool in order to prove the ℓ^1- and ℓ^∞-sublinearity of the corrector. In this section we will prove these inequalities for general graphs.

II.3.1 Setup and preliminaries

Let us consider an infinite, connected, locally finite graph $G = (V,E)$ with vertex set V and edge set E. Let d be the natural graph distance on G. We denote by $B(x,r)$ the closed ball with center x and radius r, i.e. $B(x,r) := \{y \in V \mid d(x,y) \le \lfloor r \rfloor\}$.

The graph is endowed with the counting measure, i.e. the measure of $A \subset V$ is simply the number $|A|$ of elements in A. Given a non-empty subset $B \subseteq V$, we define for any $A \subset B$ the *relative boundary* of A with respect to B by

$$\partial_B A \;:=\; \big\{\{x,y\} \in E \;:\; x \in A \text{ and } y \in B \setminus A\big\},$$

and we simply write ∂A instead of $\partial_V A$. We impose the following assumption on the properties of the graph G.

Assumption II.3.1. For some $d \geq 2$, there exist constants $c_{\mathrm{reg}}, C_{\mathrm{reg}}, C_{\mathrm{riso}}, C_{\mathrm{iso}} \in (0,\infty)$, $C_{\mathrm{W}} \in [1,\infty)$ and $\theta \in (0,1)$ such that for all $x \in V$ it holds

(i) volume regularity of order d for large balls: there exists $N_1(x) < \infty$ such that for all $n \geq N_1(x)$,

$$c_{\mathrm{reg}}\, n^d \;\leq\; |B(x,n)| \;\leq\; C_{\mathrm{reg}}\, n^d. \tag{II.3.1}$$

(ii) (weak) relative isoperimetric inequality: there exists $N_2(x) < \infty$ and an increasing sequence $\{\mathcal{S}(x,n) \subset V : n \in \mathbb{N}\}$ of connected sets such that for all $n \geq N_2(x)$,

$$B(x,n) \;\subset\; \mathcal{S}(x,n) \;\subset\; B(x, C_{\mathrm{W}} n) \tag{II.3.2}$$

and

$$|\partial_{\mathcal{S}(x,n)} A| \;\geq\; C_{\mathrm{riso}}\, n^{-1}\, |A| \tag{II.3.3}$$

for all $A \subset \mathcal{S}(x,n)$ with $|A| \leq \frac{1}{2}|\mathcal{S}(x,n)|$.

(iii) isoperimetric inequality for large sets: there exists $N_3(x) < \infty$ such that for all $n \geq N_3(x)$,

$$|\partial A| \;\geq\; C_{\mathrm{iso}}\, |A|^{(d-1)/d}, \tag{II.3.4}$$

for all $A \subset B(x,n)$ with $|A| \geq n^\theta$.

Remark II.3.2. Suppose that a graph G satisfies the relative isoperimetric inequality (II.3.3) and $C_1 \in (0, 1/2)$. Then, for all $n \geq N_2(x)$ and any $A \subset \mathcal{S}(x,n)$ such that $\frac{1}{2}|\mathcal{S}(x,n)| < |A| < (1 - C_1)|\mathcal{S}(x,n)|$, we have that

$$|\partial_{\mathcal{S}(x,n)} A| \;=\; |\partial_{\mathcal{S}(x,n)}(\mathcal{S}(x,n) \setminus A)| \;\geq\; C_{\mathrm{riso}}\, n^{-1}\, |\mathcal{S}(x,n) \setminus A| \;\geq\; C_{\mathrm{riso}}\, C_1\, n^{-1}\, |A|.$$

Thus, any such set A also satisfies the relative isoperimetric inequality however with a smaller constant.

II.3.2 Sobolev inequality for functions with compact support

By introducing an effective dimension, we first prove a (weak) isoperimetric inequality that holds for all subsets $A \subset B(x,n)$ provided that n is large enough.

Lemma II.3.3. *Suppose that Assumption II.3.1 (i) and (iii) hold for some $\theta \in [0,1)$ and let $\zeta \in \big[0, \frac{1-\theta}{1-\theta/d}\big]$. Then, for all $x \in V$ and $n \geq N_1(x) \vee N_3(x)$,*

$$\frac{|\partial A|}{|A|^{(d-\zeta)/d}} \;\geq\; \frac{C_{\mathrm{iso}}/C_{\mathrm{reg}}^{(1-\zeta)/d} \wedge 1}{n^{1-\zeta}}, \qquad \forall\, A \subset B(x,n). \tag{II.3.5}$$

Remark II.3.4. Let $d \geq 2$ and $\zeta \in [0, \frac{1-\theta}{1-\theta/d}]$ for some $\theta \in [0,1)$. By setting $d' := d/\zeta$, we have that $(d'-1)/d' = (d-\zeta)/d$. Thus, (II.3.5) corresponds to a (weak) isoperimetric inequality with d replaced by d'.

Proof. Consider $\zeta \in \big[0, (1-\theta)/(1-\theta/d)\big]$ and $x \in V$. For any $n \geq N_1(x) \vee N_3(x)$, let $A \subset B(x,n)$ be non-empty. In the sequel, we proceed by distinguish two cases: $|A| \geq n^\theta$ and $|A| < n^\theta$. If $|A| \geq n^\theta$, we have

$$\frac{|\partial A|}{|A|^{(d-\zeta)/d}} \overset{\text{(II.3.4)}}{\geq} \frac{C_{\mathrm{iso}}}{|A|^{(1-\zeta)/d}} \;\geq\; \frac{C_{\mathrm{iso}}}{|B(x,n)|^{(1-\zeta)/d}} \overset{\text{(II.3.1)}}{\geq} \frac{C_{\mathrm{iso}}/C_{\mathrm{reg}}^{(1-\zeta)/d}}{n^{1-\zeta}}.$$

On the other hand, in case $|A| < n^\theta$, due to the choice of ζ we obtain that

$$\frac{|\partial A|}{|A|^{(d-\zeta)/d}} \;\geq\; \frac{1}{n^{\theta(1-\zeta/d)}} \;\geq\; \frac{1}{n^{1-\zeta}}.$$

This completes the proof. $\square$

Proposition II.3.5 (Sobolev inequality)**.** *Suppose Assumption II.3.1 (i) and (iii) are true for some $\theta \in [0,1)$. Then, for any $\zeta \in \big[0, \frac{1-\theta}{1-\theta/d}\big]$, there exists*

$$C_{\mathrm{S}_1} \equiv C_{\mathrm{S}_1}(\theta, d) \in (0,\infty)$$

such that, for any $x \in V$ and $n \geq N_1(x) \vee N_3(x)$,

$$\Bigg(\frac{1}{|B(x,n)|}\sum_{y\in B(x,n)} |u(y)|^{\frac{d}{d-\zeta}}\Bigg)^{\frac{d-\zeta}{d}} \;\leq\; \frac{C_{\mathrm{S}_1} n}{|B(x,n)|} \sum_{\substack{y \vee y' \in B(x,n)\\ \{y,y'\}\in E}} \big|u(y)-u(y')\big| \tag{II.3.6}$$

for every function $u\colon V \to \mathbb{R}$ with $\operatorname{supp}(u) \subset B(x,n)$.

Proof. The assertion follows by an application of the co-area formula as in [63, Proposition 3.4]. Nevertheless, we will repeat it here for the readers' convenience.

Let $u : V \to \mathbb{R}$ be a function with $\operatorname{supp} u \subset B(x,n)$. Note that it is enough to consider $u \geq 0$. Moreover, by Jensen's inequality it suffices to prove (II.3.6) for

$\zeta = \frac{1-\theta}{1-\theta/d}$. Define for $t \geq 0$ the super-level sets of u by $A_t = \{y \in V : u(y) > t\}$. Obviously, $A_t \subset B(x, n)$ for any $t \geq 0$. Thus,

$$\sum_{\{y,y'\}\in E} \big|u(y) - u(y')\big| \;=\; \int_0^\infty |\partial A_t| \,\mathrm{d}t \overset{\text{(II.3.5)}}{\geq} \frac{C_{\text{iso}}/C_{\text{reg}}^{(1-\zeta)/d} \wedge 1}{n^{1-\zeta}} \int_0^\infty |A_t|^{1-\zeta/d} \,\mathrm{d}t.$$

Now, consider a function $g\colon B(x_0, n) \to [0, \infty)$ with $\|g\|_{L^{\alpha_*}(V)} = 1$ where $\alpha_* = d/\zeta$ and $\alpha = d/(d-\zeta)$. Notice, that $1/\alpha + 1/\alpha_* = 1$ and $1 - \zeta/d = 1/\alpha$. Since $|A_t|^{1/\alpha} \geq \langle \mathbf{1}_{A_t}, g\rangle$ by Hölder's inequality, we obtain that

$$\int_0^\infty |A_t|^{1-\zeta/d} \,\mathrm{d}t \;\geq\; \int_0^\infty \langle \mathbf{1}_{A_t}, g\rangle \,\mathrm{d}t \;=\; \langle u, g\rangle\,.$$

Here, $\langle\cdot,\cdot\rangle$ denotes the scalar product in $\ell^2(V)$. Finally, taking the supremum over all $g\colon B(x_0, n) \to [0, \infty)$ with $\|g\|_{L^{\alpha_*}(V)} = 1$ implies the assertion (II.3.6). $\square$

Remark II.3.6. It is well known, see [56, Lemma 3.3.3], that for functions $u : V \to \mathbb{R}$ that are not compactly supported, the following (weak) Poincaré inequality follows from the (weak) relative isoperimetric inequality: For $x \in V$ and $n \geq N_2(x)$

$$\inf_{m\in\mathbb{R}} \sum_{y\in B(x,n)} |u(y) - m| \;\leq\; C_{\text{riso}}^{-1}\, n \sum_{\substack{y,y'\in B(x, C_{\mathrm{W}} n)\\ \{y,y'\}\in E}} \big|u(y) - u(y')\big|.$$

II.3.3 Anchored Sobolev inequality

As a second result, we prove a Sobolev inequality for functions with unbounded support that vanishes at some point $x \in V$. The proof is based on an *anchored* relative isoperimetric inequality. For this purpose, let $w : E \to (0, \infty)$ be an edge weight and for any $A \subset B$ non-empty we write

$$|\partial_B A|_w \;:=\; \sum_{e\in\partial_B A} w(e).$$

Lemma II.3.7 (anchored relative isoperimetric inequality)**.** *Suppose that the graph G satisfies Assumption II.3.1 (i) and (ii). For any $x \in V$ and $\varepsilon \in (0, 1)$, choose n large enough such that $\lfloor n^{(1-\varepsilon)/(d-\varepsilon)} \rfloor \geq N_1(x) \vee N_2(x)$. Further, assume that*

$$w_n(\{y, y'\}) \;\geq\; (n/\max\{d(x, y), d(x, y')\})^{d-\varepsilon} \qquad \forall\, \{y, y'\} \in E. \tag{II.3.7}$$

Then, there exists $C_2 \in (0, \infty)$ such that

$$|\partial_{\mathcal{S}(x,n)} A|_{w_n} \;\geq\; \frac{C_2}{n}\, |A|, \qquad \forall\, A \subset \mathcal{S}(x, n) \setminus \{x\}. \tag{II.3.8}$$

Remark II.3.8. On the Euclidean lattice, $(\mathbb{Z}^d, \mathbb{E}^d)$, the anchored relative isoperimetric inequality (II.3.8) holds for all $n \geq 1$, if

$$w_n(\{y, y'\}) \geq c \left(\frac{n}{\max\{d(x, y), d(x, y')\}}\right)^{d-1}$$

for some $c < \infty$ and for all $\{y, y'\} \in \mathbb{E}^d$.

Proof. Set $C_1 = 2^{-(d+1)} c_{\mathrm{reg}} C_{\mathrm{reg}}^{-1} C_{\mathrm{W}}^{-d}$ and $\beta = (1-\varepsilon)/(d-\varepsilon)$. In view of Remark II.3.2, it holds that for any $n \geq N_1(x) \vee N_2(x)$,

$$|\partial_{\mathcal{S}(x,n)} A| \;\geq\; C_1 C_{\mathrm{riso}}\, n^{-1}\, |A| \;=:\; C_3\, n^{-1} |A| \tag{II.3.9}$$

for all $A \subset \mathcal{S}(x,n)$ with $|A| \leq (1 - C_1)|\mathcal{S}(x,n)|$. Suppose that n is chosen in such a way that $\lfloor n^\beta \rfloor \geq N_1(x) \vee N_2(x)$ and let $A \subset \mathcal{S}(x,n) \setminus \{x\}$ be non-empty. Since $\mathcal{S}(x,n) \subset B(x, C_{\mathrm{W}} n)$, we have that $w_n(\{y,y'\}) \geq C_{\mathrm{W}}^{-d}$ and so

$$|\partial_{\mathcal{S}(x,n)} A|_{w_n} \geq C_{\mathrm{W}}^{-d} |\partial_{\mathcal{S}(x,n)} A|.$$

Thus, (II.3.8) follows from (II.3.9) for any $A \subset \mathcal{S}(x,n) \setminus \{x\}$ with $|A| \leq (1 - C_1)|\mathcal{S}(x,n)|$.

It remains to consider the case $|A| > (1 - C_1)|\mathcal{S}(x,n)|$. We proceed by distinguishing two different cases. First, assume that $A \cap \mathcal{S}(x, \lfloor n^\beta \rfloor) \neq \emptyset$. Due to the fact that A does not contain the vertex x, there exists at least one edge $\{y, y'\} \in E$ with $y \in A \subset \mathcal{S}(x, \lfloor n^\beta \rfloor)$ and $y' \in \mathcal{S}(x, \lfloor n^\beta \rfloor) \setminus A$. This implies

$$\begin{aligned}
|\partial_{\mathcal{S}(x,n)} A|_{w_n} \;&\geq\; \max\big\{ w_n(e) : e \in \partial_{\mathcal{S}(x,n)} A \big\} \\
&\overset{\text{(II.3.2)}}{\geq} \frac{n^{d-1}}{C_{\mathrm{W}}^{d}} \overset{\text{(II.3.1)}}{\geq} \frac{C_{\mathrm{reg}}^{-1}}{C_{\mathrm{W}}^{2d}}\, n^{-1} |B(x, C_{\mathrm{W}} n)| \overset{\text{(II.3.2)}}{\geq} \frac{C_{\mathrm{reg}}^{-1}}{C_{\mathrm{W}}^{2d}}\, n^{-1} |A|.
\end{aligned} \tag{II.3.10}$$

Consider now the case that $|A| > (1 - C_1)|\mathcal{S}(x,n)|$ and $A \cap \mathcal{S}(x, \lfloor n^\beta \rfloor) = \emptyset$. Set

$$k \;:=\; \min\big\{ 0 \leq j \leq n \;:\; |A \cap \mathcal{S}(x,j)| > (1 - C_1)\, |\mathcal{S}(x,j)| \big\}. \tag{II.3.11}$$

Obviously, $\lfloor n^\beta \rfloor < k \leq n$. Since $|A \cap \mathcal{S}(x, k-1)| \leq (1 - C_1)\, |\mathcal{S}(x, k-1)|$ by definition of k, we obtain by exploiting the monotonicity of the sets $\mathcal{S}(x,k)$

$$\begin{aligned}
|\partial_{\mathcal{S}(x,n)} A|_{w_n} \;&\geq\; \big| \partial_{\mathcal{S}(x,k-1)} \big(A \cap \mathcal{S}(x, k-1) \big) \big|_{w_n} \\
&\geq\; \frac{1}{C_{\mathrm{W}}^{d}} \left(\frac{n}{k-1} \right)^{d-\varepsilon} \big| \partial_{\mathcal{S}(x,k-1)} \big(A \cap \mathcal{S}(x, k-1) \big) \big| \\
&\overset{\text{(II.3.9)}}{\geq} \frac{C_3}{C_{\mathrm{W}}^{d}}\, \frac{n^{d-\varepsilon}}{k^{d+1-\varepsilon}}\, |A \cap \mathcal{S}(x, k-1)|.
\end{aligned} \tag{II.3.12}$$

On the other hand, since

$$\begin{aligned}
|\mathcal{S}(x, k-1)| &\overset{\text{(II.3.2)}}{\geq} |B(x, k-1)| \\
&\overset{\text{(II.3.1)}}{\geq} \frac{c_{\mathrm{reg}}\, C_{\mathrm{reg}}^{-1}}{C_{\mathrm{W}}^{d}}\, 2^{-d}\, |B(x, C_{\mathrm{W}} k)| \;\geq\; \frac{c_{\mathrm{reg}}\, C_{\mathrm{reg}}^{-1}}{C_{\mathrm{W}}^{d}}\, 2^{-d}\, |\mathcal{S}(x,k)|,
\end{aligned}$$

we get that $|\mathcal{S}(x,k) \setminus \mathcal{S}(x,k-1)| \leq (1 - 2C_1)|\mathcal{S}(x,k)|$. Hence,

$$|A \cap \mathcal{S}(x, k-1)| \overset{\text{(II.3.11)}}{\geq} (1 - C_1)\, |\mathcal{S}(x,k)| - |\mathcal{S}(x,k) \setminus \mathcal{S}(x,k-1)| \;\geq\; C_1 c_{\mathrm{reg}}\, k^d. \tag{II.3.13}$$

By combining (II.3.12) and (II.3.13), we find that

$$|\partial_{\mathcal{S}(x,n)} A|_{w_n} \overset{\text{(II.3.1)}}{\geq} \frac{C_1 C_3 c_{\text{reg}}}{C_{\text{W}}^{2d} C_{\text{reg}}} n^{-1} |B(x_0, C_{\text{W}} n)| \geq \frac{C_1 C_3 c_{\text{reg}}}{C_{\text{W}}^{2d} C_{\text{reg}}} n^{-1} |A|. \tag{II.3.14}$$

By setting $C_2 := \min\{C_3, C_{\text{reg}}^{-1} C_{\text{W}}^{-2d}, C_1 C_3 c_{\text{reg}} C_{\text{reg}}^{-1} C_{\text{W}}^{-2d}\}$, the assertion (II.3.8) follows. □

Proposition II.3.9 (anchored Sobolev inequality)**.** *Let $x \in V$ and suppose that the assumptions of Lemma II.3.7 are satisfied. Then, there exists $\overline{C}_{\text{S}_1} \in (0, \infty)$ such that*

$$\sum_{y \in B(x,n)} |u(y)| \leq \overline{C}_{\text{S}_1} n \sum_{\substack{y,y' \in B(x, C_{\text{W}} n) \\ \{y,y'\} \in E}} w_n(\{y, y'\}) \big| u(y) - u(y') \big| \tag{II.3.15}$$

for every function $u \colon V \to \mathbb{R}$ with $u(x) = 0$.

Proof. The proof is based on an application of the co-area formula and the anchored relative isoperimetric inequality as derived in Lemma II.3.7. For some $x \in V$, let $u : V \to \mathbb{R}$ be a function with $u(x) = 0$. It suffices to consider $u \geq 0$. Define for $t \geq 0$ the super-level sets of u by $A_t = \{y \in \mathcal{S}(x,n) : u(y) > t\}$. Then,

$$\begin{aligned} \sum_{\substack{y,y' \in \mathcal{S}(x,n) \\ \{y,y'\} \in E}} w_n(\{y, y'\}) \big| u(y) - u(y') \big| &= \int_0^\infty |\partial_{\mathcal{S}(x,n)} A_t|_{w_n} \, \mathrm{d}t \\ &\overset{\text{(II.3.8)}}{\geq} \frac{C_2}{n} \int_0^\infty |A_t| \, \mathrm{d}t = \frac{C_2}{n} \sum_{y \in B(x,n)} |u(y)|. \end{aligned}$$

Since $\mathcal{S}(x,n) \subset B(x, C_{\text{W}} n)$ by Assumption II.3.1(ii), (II.3.15) follows. □

II.A Ergodic theorem

In this appendix we provide an extension of the Birkhoff ergodic theorem that generalises the result obtained in [20, Theorem 3]. Consider a probability space $(\omega, \mathcal{F}, \mathbb{P})$ and a group of measure preserving transformations $\tau_x : \omega \to \omega$, $x \in \mathbb{Z}^d$ such that $\tau_{x+y} = \tau_x \circ \tau_y$. Further, let $B_1 := \{x \in \mathbb{R}^d : |x| \leq 1\}$.

Theorem II.A.1. *Let $\varphi \in L^1(\mathbb{P})$ and $\varepsilon \in (0, d)$. Then, for $\mathbb{P}$-a.e. ω,*

$$\lim_{n \to \infty} \frac{1}{n^d} \sum_{x \in B(0,n)}' \frac{\varphi(\tau_x \omega)}{|x/n|^{d-\varepsilon}} = \left(\int_{B_1} |x|^{-(d-\varepsilon)} \, dx \right) \mathbb{E}[\varphi], \tag{II.A.1}$$

where the summation is taken over all $x \in B(0,n) \setminus \{0\}$.

Proof. To start with, notice that the ergodic theorem, see [20, Theorem 3], implies that for $\mathbb{P}$-a.e. ω

$$\begin{aligned}\lim_{k\to\infty}\lim_{n\to\infty}\frac{1}{n^d}&\sum_{x\in B(0,n)}{}^{\prime}\Big(k\wedge|x/n|^{-(d-\varepsilon)}\Big)\,\varphi(\tau_x\omega)\\ &=\lim_{k\to\infty}\bigg(\int_{B_1}k\wedge|x|^{-(d-\varepsilon)}\,dx\bigg)\,\mathbb{E}[\varphi]\;=\;\bigg(\int_{B_1}|x|^{-(d-\varepsilon)}\,dx\bigg)\,\mathbb{E}[\varphi].\end{aligned}\tag{II.A.2}$$

On the other hand, by means of Abel's summation formula, we have that

$$\begin{aligned}\bigg|\frac{1}{n^\varepsilon}\sum_{x\in B(0,n)}{}^{\prime}\frac{\varphi(\tau_x\omega)}{|x|^{d-\varepsilon}}\bigg|\;&=\;\bigg|\frac{1}{n^\varepsilon}\sum_{j=1}^{n}\frac{1}{j^{d-\varepsilon}}\sum_{|x|=j}\varphi(\tau_x\omega)\bigg|\\ &\le\;\bigg|\frac{1}{n^d}\sum_{x\in B(0,n)}{}^{\prime}\varphi(\tau_x\omega)\bigg|\;+\;\frac{d-\varepsilon}{n^\varepsilon}\sum_{j=1}^{n-1}\frac{1}{j^{1-\varepsilon}}\bigg|\frac{1}{j^d}\sum_{x\in B(0,j)}{}^{\prime}\varphi(\tau_x\omega)\bigg|,\end{aligned}$$

where we used that $j^{-(d-\varepsilon)}-(j+1)^{-(d-\varepsilon)}\le(d-\varepsilon)j^{-(d+1-\varepsilon)}$. From this estimate we deduce that for any $\varphi\in L^1(\mathbb{P})$ and $\mathbb{P}$-a.e. ω

$$\sup_{n\ge1}\bigg|\frac{1}{n^\varepsilon}\sum_{x\in B(0,n)}{}^{\prime}\frac{\varphi(\tau_x\omega)}{|x|^{d-\varepsilon}}\bigg|\;\le\;\frac{Cd}{\varepsilon}\sup_{n\ge0}\frac{1}{|B(0,n)|}\sum_{x\in B(0,n)}\big|\varphi(\tau_x\omega)\big|\;<\;\infty,\tag{II.A.3}$$

where $C:=\sup_{n\ge1}|B(0,n)|/n^d<\infty$. On the other hand, for any $k\ge1$

$$\frac{1}{n^d}\bigg|\sum_{x\in B(0,n)}{}^{\prime}\bigg(\frac{1}{|x/n|^{d-\varepsilon}}-k\wedge\frac{1}{|x/n|^{d-\varepsilon}}\bigg)\,\varphi(\tau_x\omega)\bigg|\;\le\;\frac{C}{k^{\varepsilon/d}}\sup_{n\ge1}\bigg|\frac{1}{n^\varepsilon}\sum_{x\in B(0,n)}{}^{\prime}\frac{\varphi(\tau_x\omega)}{|x|^{d-\varepsilon}}\bigg|.\tag{II.A.4}$$

Since the last factor on the right-hand side of (II.A.4) is finite due to (II.A.3), we conclude that $\mathbb{P}$-a.s.

$$\lim_{k\to\infty}\frac{1}{n^d}\bigg|\sum_{x\in B(0,n)}{}^{\prime}|x/n|^{-(d-\varepsilon)}\,\varphi(\tau_x\omega)\;-\;\sum_{x\in B(0,n)}{}^{\prime}\Big(k\wedge|x/n|^{-(d-\varepsilon)}\Big)\,\varphi(\tau_x\omega)\bigg|\;=\;0\tag{II.A.5}$$

uniformly in n. The assertion follows by combining (II.A.2) and (II.A.5). $\square$

Chapter III

A Liouville principle

III.1 Introduction and the main results

III.1.1 Motivation

The classical Liouville theorem, a direct consequence of Cauchy's integral formula, is one of the most beautiful results in mathematics. It states that the space of harmonic functions on $\mathbb{R}^2$ which grow not faster than $|x|^k$ contains only polynomials of degree k. In the view point of probability theory a function u is harmonic if $u(B_t)$ is a martingale where B_t is the standard Brownian motion on $\mathbb{R}^2$.

We are interested in studying Liouville-type properties when replacing the standard Brownian motion on $\mathbb{R}^2$ by several types of random motions, for instance, the random conductance model (random walks among random conductances) and diffusions on random media. The random conductance model is defined by equipping each nearest neighbour bond $e = \{x, y\} = \{y, x\}$ on $\mathbb{Z}^d$ with a random conductance $\omega(e) = \omega(\{x, y\}) \in [0, \infty)$. In this model, we are interested in two types of random walks: a discrete time random walk and a continuous time random walk. For each realization ω, the discrete time random walk $\{Z_n : n \in \mathbb{N}\}$ jumps from x to y with probability $\omega(\{x, y\})/\mu^\omega(x)$ where

$$\mu^\omega(x) = \sum_{y:|x-y|_1=1} \omega(\{x, y\}),$$

while the continuous time random walk $\{X_t : t \geq 0\}$ waits at x an exponential time with means $\mu^\omega(x)^{-1}$ and jumps to a nearest neighbour y of x with probability $\omega(\{x, y\})/\mu^\omega(x)$. As a Markov process it has the following generator:

$$\mathcal{L}^\omega u(x) = \sum_{y\sim x} \omega(\{x, y\}) \left(u(y) - u(x)\right). \tag{III.1.1}$$

A function $u : \mathbb{Z}^d \to \mathbb{R}$ is said to be ω-harmonic if

$$\mathcal{L}^\omega u(x) = 0.$$

In other words, u is ω-harmonic if $\{u(X_t) : t \geq 0\}$ is a martingale, $\{X_t : t \geq 0\}$ being the continuous time random walk, or equivalently, if $\{u(Z_n) : n \in \mathbb{N}\}$ is a martingale, $\{Z_n : n \in \mathbb{N}\}$ being the discrete time random walk.

In the case of diffusions on random media generated by random elliptic differential operators in divergence form in the continuum setting,

$$\mathcal{L}^{\mathbf{a}}u(x) = \nabla \cdot \mathbf{a}(x)\nabla u(x) = \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\left(\mathbf{a}_{ij}(x)\frac{\partial u}{\partial x_j}(x)\right) \tag{III.1.2}$$

where $\mathbf{a}(x)$ is a symmetric positive definite random matrix defined on some probability space, a function u is called $\mathbf{a}$-harmonic if

$$\mathcal{L}^{\mathbf{a}}u = 0.$$

In the language of probability, a function u is $\mathbf{a}$-harmonic if $u(Y_t)$ is a martingale where Y_t is the diffusion generated by the operator $\mathcal{L}^{\mathbf{a}}$.

The main result in this chapter, Theorem III.1.2, is motivated by several results obtained recently in this topic. First, Benjamini, Duminil-Copin, Kozma, and Yadin [13] studied the case of simple random walks on Bernoulli supercritical percolation clusters. In this case, $\omega(e)$ are i.i.d. and have values in $\{0, 1\}$. We speak of an open edge if $\omega(e) = 1$ and a closed edge if $\omega(e) = 0$. Further, it is well-known that if $\mathbb{P}(\omega(e) = 1) > \mathfrak{p}_c$ for some $\mathfrak{p}_c \in (0, 1)$, the open edges percolate and form a unique infinite connected component, a so-called percolation cluster. In this case, it is showed that the space of ω-harmonic functions ($\mathcal{L}^{\omega}u = 0$) which grow at most linearly has dimension $d+1$, the same dimension as the space of harmonic functions with linear growth in $\mathbb{R}^d$ or $\mathbb{Z}^d$.

A quite similar result has been obtained by Gloria, Neukamm, and Otto [33] in the continuum setting of random differential operators $\mathcal{L}^{\mathbf{a}}$ where $\mathbf{a}$ is a stationary ergodic matrix bounded from above and below. Exploiting the idea of perturbing around the homogenized coefficients by using the two-scale homogenization error in a new fashion, they obtained a first order Liouville property as a consequence of a Schauder-type estimate called the excess decay which monitors the distance between a harmonic function and the space generated by the harmonic coordinates. Indeed, under a pure assumption on ergodicity they proved that the space of harmonic functions which grows not faster than $|x|^{1+\alpha}$ for some $\alpha \in (0, 1)$ has dimension $d + 1$.

Lately, a work by Armstrong and Dario [5] has extended the work by Benjamini, Duminil-Copin, Kozma, and Yadin [13] to higher order Liouville-type results on the random conductance model on a supercritical percolation cluster under the assumption that the conductances are uniformly elliptic, meaning bounded from above and below.

As in Chapter II we are interested in going beyond the uniformly elliptic condition. Recently, Bella, Fehrman, and Otto [12] have extended the first order Liouville property by Gloria, Neukamm, and Otto to degenerate random differential operators $\mathcal{L}^{\mathbf{a}}$ under the assumption that the random matrix $\mathbf{a}$ satisfies

$$\langle \mu(\mathbf{a})^p \rangle + \langle |\lambda(\mathbf{a})|^{-q} \rangle < \infty, \quad \frac{1}{p} + \frac{1}{q} < \frac{2}{d},$$

where $\langle \cdot \rangle$ denotes the expectation with respect to $\mathbb{P}$ and

$$\lambda(\mathbf{a}) := \inf_{\xi \in \mathbb{R}^d} \frac{\xi \cdot \mathbf{a}\xi}{|\xi|^2}, \quad \text{and} \quad \mu(a) := \sup_{\xi \in \mathbb{R}^d} \frac{|\mathbf{a}\xi|^2}{\xi \cdot \mathbf{a}\xi},$$

which is very similar to the integrability condition we assumed in Chapter II. This work also implemented the idea of perturbing around the homogenized coefficient by using the two-scale homogenization error, however, with more subtle ideas to deal with the lack of the uniform ellipticity.

In this chapter we discuss how to implement the ideas given by Bella, Fehrman, and Otto [12] to prove a similar result in the random conductance model on $\mathbb{Z}^d$.

III.1.2 The notations

In the whole chapter let $d \geq 2$. Denote by $|\cdot|_p$, $p \in [1, \infty]$ the ℓ^p-norm in $\mathbb{Z}^d$. Two points $x, y \in \mathbb{Z}^d$ are called nearest neighbours, denoted by $x \sim y$, if $|x - y|_1 = 1$. Let $\mathbb{E}^d$ denote the set of unoriented nearest neighbour edges formally defined by

$$\mathbb{E}^d := \{\{x, y\} : x, y \in \mathbb{Z}^d, x \sim y\}$$

where $\{x, y\} = \{y, x\}$. Consider the measurable space of random environments

$$(\Omega, \mathcal{F}) := ((0, \infty)^{\mathbb{E}^d}, \mathcal{B}((0, \infty))^{\otimes \mathbb{E}^d}).$$

Denote by $\omega \in \Omega$ a random environment and $\omega(e)$ the conductance of e. Further, let $\mathbb{P}$ be a probability measure on $(\Omega, \mathcal{F})$ with expectation denoted by $\langle \cdot \rangle$.

Consider the following family of translations $\{\tau_x : x \in \mathbb{Z}^d\}$:

$$[\tau_x \omega](\{y, z\}) := \omega(x + y, x + z), \quad x, y, z \in \mathbb{Z}^d. \tag{III.1.3}$$

The *random walks among random conductances* generated by (III.1.1) can be considered as a discrete counterpart of *diffusions in random media* generated by (III.1.2). To see the connection between them, let us define the random matrix

$$\mathbf{a}(x) := \mathbf{a}(\omega, x) = \operatorname{diag}\left(\{\omega(x, x + \mathbf{e}_1\}), \ldots, \omega(\{x, x + \mathbf{e}_d\})\right), \quad \omega \in \Omega, x \in \mathbb{Z}^d.$$

Further, we mimic the notation in the continuum setting by introducing the discrete forward and backward partial derivatives for $u : \mathbb{Z}^d \to \mathbb{R}$ as follows:

$$\nabla_i u := u(\cdot + \mathbf{e}_i) - u \quad \text{and} \quad \nabla_i^* u := u(\cdot - \mathbf{e}_i) - u, \ 1 \leq i \leq d,$$

respectively. The full gradient ∇u (or $\nabla^* u$) is defined as a vector of d components:

$$\nabla u := \begin{bmatrix} \nabla_1 u \\ \vdots \\ \nabla_d u \end{bmatrix} = \begin{bmatrix} u(\cdot + \mathbf{e}_1) - u \\ \vdots \\ u(\cdot + \mathbf{e}_d) - u \end{bmatrix}, \quad \nabla^* u := \begin{bmatrix} \nabla_1^* u \\ \vdots \\ \nabla_d^* u \end{bmatrix} = \begin{bmatrix} u(\cdot - \mathbf{e}_1) - u \\ \vdots \\ u(\cdot - \mathbf{e}_d) - u \end{bmatrix}. \quad \text{(III.1.4)}$$

The backward derivative ∇_i^* is the dual of the forward derivative ∇_i in the sense that for $r, r^* \in [1, \infty]$ satisfy $1/r + 1/r^* = 1$, we have the discrete partial integration

$$\sum_{\mathbb{Z}^d} \nabla_i f g = \sum_{\mathbb{Z}^d} f \nabla_i^* g \quad \text{(III.1.5)}$$

for $f \in L^r(\mathbb{Z}^d)$ and $g \in L^{r^*}(\mathbb{Z}^d)$. We will mimic the notation in vector analysis in the continuum case, e.g. the backward divergence is defined by

$$\nabla^* \cdot F := \sum_{i=1}^{d} \nabla_i^* F_i,$$

for $F = (F_1, \ldots, F_d) : \mathbb{Z}^d \to \mathbb{R}^d$. By a simple calculation we can show that

$$\mathcal{L}^\omega(x) := \sum_{y \sim x} \omega(\{x, y\})(u(y) - u(x)) = -\nabla^* \cdot \mathbf{a}(x) \nabla u(x) =: \mathcal{L}^{\mathbf{a}},$$

which is a discrete analogue of the operator in (III.1.2). Then, saying that u is ω-harmonic in the sense that $\mathcal{L}^\omega u = 0$ is equivalent to saying that u is $\mathbf{a}$-harmonic in the sense that $\mathcal{L}^{\mathbf{a}} u = 0$, where $\mathcal{L}^{\mathbf{a}}$ is defined as above. *We use both ways of speaking in the whole chapter.*

The above notation used by Gloria and Otto [34, 35] is very helpful when we mimic the calculation in the continuum setting. In contrast to (III.1.4) defining gradients as functions of vertices, it is sometimes more convenient to consider gradients as functions of edges. Consider the set of oriented nearest neighbour edges

$$\mathbb{E}_\pm^d := \{[x, y] : x, y \in \mathbb{Z}^d, x \sim y\}$$

where we mean $[x, y] \neq [y, x]$. For $u : \mathbb{Z}^d \to \mathbb{R}$ and $e = [x, y] \in \mathbb{E}_\pm^d$ we write

$$\nabla u(e) = \nabla u([x, y]) = u(y) - u(x).$$

Further, for $e' = \{x, y\} \in \mathbb{E}^d$ being an unoriented edge we write

$$|\nabla u(e')| = |\nabla u(\{x, y\}) = |u(y) - u(x)| \quad \text{(III.1.6)}$$

which does not depend on the choice of the direction. In the whole chapter we use all types of gradients. Whether ∇u is a function depending on vertices or oriented or unoriented edges will be clear in each context.

Further, denote the continuum open boxes and closed boxes and boundary by

$$C_R := \{x : |x|_\infty < R\}, \quad \overline{C}_R := \{x : |x|_\infty \le R\}, \quad \partial C_R := \{x : |x|_\infty = R\}.$$

The discrete boxes with boundary are denoted by

$$D_R := C_R \cap \mathbb{Z}^d, \quad \overline{D}_R := \overline{C}_R \cap \mathbb{Z}^d, \quad \partial D_R = \partial C_R \cap \mathbb{Z}^d.$$

For discrete boxes we only consider $R \in \{1, 2, \ldots\}$. Define the edge sets

$$\overline{E}_R := \left\{ \{x, y\} \in \mathbb{E}^d : x, y \in \overline{D}_R \right\}, \quad E_R = \left\{ \{x, y\} \in \mathbb{E}^d : |\tfrac{1}{2}(x+y)|_\infty < R \right\}.$$

In the continuum (or discrete, respectively) we denote the average by

$$\fint_A := \frac{1}{|A|} \int_A \quad \text{or} \quad \overline{\sum_A} := \frac{1}{|A|} \sum_A$$

where $|A|$ is the Lebesgue or the counting measure, the L^p-norm, $p \in [1, \infty)$ by

$$\|f\|^p_{L^p(A)} = \int_A |f|^p \quad \text{or} \quad \|f\|^p_{L^p(A)} = \sum_A |f|^p,$$

and the average L^p-norm, $p \in [1, \infty)$ by

$$\|f\|^p_{\overline{L}^p(A)} = \fint_A |f|^p \quad \text{or} \quad \|f\|^p_{\overline{L}^p(A)} = \overline{\sum_A} |f|^p.$$

In this chapter, $A \lesssim_{\alpha,\beta,\ldots} B$ means $A \le cB$ where $c \in (0, \infty)$ is a constant depending on parameters $\alpha, \beta, \ldots$ The notation $A \gtrsim_{\alpha,\beta,\ldots} B$ is defined analogously, and we write $A \sim_{\alpha,\beta} B$ if both inequalities happen. Finally, to lighten the notations, we use *Einstein's notation* summing over repeated indices.

III.1.3 Main results

Assumption III.1.1. Let $d \ge 2$. Assume the following conditions.

(i) $\mathbb{P}$ is stationary and ergodic in the sense that $\mathbb{P}$ is invariant under the family $\{\tau_x : x \in \mathbb{Z}^d\}$ defined by (III.1.3) and if $f \in L^1(\Omega, \mathbb{P})$ satisfies $f(\tau_x \omega) = f(\omega)$ for all $x \in \mathbb{Z}^d$ and $\mathbb{P}$-a.e. ω, then $f = \langle f \rangle$.

(ii) the following (p, q)-moment condition holds:

$$\langle \omega(e)^p \rangle + \langle \omega(e)^{-q} \rangle < \infty \text{ for some } p, q \in (1, \infty] \text{ with } \frac{1}{p} + \frac{1}{q} < \frac{2}{d}. \qquad \text{(III.1.7)}$$

(iii) $\mathbb{P}$ is invariant under reflections on $\mathbb{Z}^d$.

In this case, the assumption that $\mathbb{P}$ is invariant under reflections on $\mathbb{Z}^d$ is a technical assumption which implies that the covariance matrix of the limiting Brownian motion is a diagonal matrix (see [24, Theorem 4.6 (iii), p823]). Therefore, we can define the "homogenized conductances" as

$$\omega_{\mathrm{h}}(\{x, x+\mathbf{e}_i\}) := (\mathbf{a}_{\mathrm{h}})_{i,i}$$

This fact implies that both the heterogeneous conductance ω and the homogenized conductance ω_{h} live on $\mathbb{Z}^d$ which allows us to apply the idea of "perturbing around the homogenized coefficients".

Examples. The following examples satisfy Assumption III.1.1.

(a) The conductances $\omega(e), e \in \mathbb{E}^d$ are i.i.d. random variables satisfying the integrability condition (III.1.7).

(b) The conductances are given by

$$\omega(\{x, y\}) = \exp\left(h(x) + h(y)\right), \quad \{x, y\} \in \mathbb{E}^d$$

where $\{h(x) : x \in \mathbb{Z}^d\}$ (for $d \geq 3$) is the discrete Gaussian free field.

Theorem III.1.2 (First order Liouville principle). *Under Assumption III.1.1 the following is true $\mathbb{P}$-almost surely. If an ω-harmonic function u on $\mathbb{Z}^d$ satisfies the following sub-quadratic growth*

$$\lim_{R\to\infty} R^{-(1+\alpha)} \|u\|_{\overline{L}^{\frac{2p}{p-1}}(D_R)} = 0 \tag{III.1.8}$$

where $\alpha \in (0, 1)$, then it is necessarily of the form $u(x) = c + \xi_i(x_i + \phi_i(\omega, x))$ for some $c \in \mathbb{R}$ and $\xi \in \mathbb{R}^d$. In other word, the linear space of ω-harmonic functions u satisfying (III.1.8) *is $(d+1)$-dimensional.*

The field $\phi = \phi(\omega, x)$ is called the corrector which was introduced in Chapter II so that $x \mapsto x + \phi(\omega, x)$ is ω-harmonic, meaning $X_t + \phi(\omega, X_t)$ is a martingale. Theorem III.1.2 is a consequence of the following Theorem III.1.3, which is a discrete analogue of the excess decay by Bella, Fehrman, and Otto [12].

Theorem III.1.3 (Excess decay). *Let $\alpha \in (0, 1)$ and suppose that Assumption III.1.1 holds. Then, there exist a deterministic constant $C = C(d, \alpha, p, q, \mathbb{P}) > 0$ such that the following is true: For $\mathbb{P}$-a.e. ω there exists a minimal radius $r_*(\omega) > 0$ such that*

$$\mathrm{Exc}(r) \leq C \left(\frac{r}{R}\right)^{2\alpha} \mathrm{Exc}(R), \quad r_* \leq r \leq R \tag{III.1.9}$$

for any $\mathbf{a}$*-harmonic function u in D_R where the excess is defined as*

$$\mathrm{Exc}(r) := \inf_{\xi\in\mathbb{R}^d} \|\mathbf{a}(\nabla u - \xi_i(\mathbf{e}_i + \nabla\phi_i)) \cdot (\nabla u - \xi_i(\mathbf{e}_i + \nabla\phi_i))\|_{\overline{L}^1(D_r)}, \quad r \geq 1. \tag{III.1.10}$$

In order to prove Theorem III.1.3 we first consider the case the radii are comparable. The general case immediately follows by using an iteration.

Theorem III.1.4. *Let $\alpha \in (0,1)$ and suppose that Assumption III.1.1 is true. Then, there exist a deterministic integer $K = K(d,\alpha,p,q,\mathbb{P}) \geq 64$ such that the following holds. For $\mathbb{P}$-a.e. ω there exists a minimal radius $r^*(\omega) > 0$ such that*

$$\mathrm{Exc}(r) \leq K^{-2\alpha}\mathrm{Exc}(Kr), \quad r \geq r^*. \tag{III.1.11}$$

for any $\mathbf{a}$-harmonic function u in D_{Kr}.

Proof of Theorem III.1.3 from Theorem III.1.4. We follow [12, Step 5 pp32–3]. We only need to notice that we now consider integer radii. Choose $C = K^{d+2\alpha}$. If $Kr \geq R$, the claim follows from

$$\mathrm{Exc}(r) \leq \left(\tfrac{R}{r}\right)^d \mathrm{Exc}(R) = \left(\tfrac{R}{r}\right)^{d+2\alpha}\left(\tfrac{r}{R}\right)^{2\alpha}\mathrm{Exc}(R) \leq K^{d+2\alpha}\left(\tfrac{r}{R}\right)^{2\alpha}\mathrm{Exc}(R).$$

If $Kr < R$, choose n so that $K^{n-1}r \leq R < K^n r$. Then, applying (III.1.11) $(n-1)$ times and using the choice of C, we have

$$\begin{aligned}\mathrm{Exc}(r) &\leq K^{-2(n-1)\alpha}\mathrm{Exc}(K^{n-1}r) \leq K^{-2(n-1)\alpha}\left(\tfrac{R}{K^{n-1}r}\right)^d \mathrm{Exc}(R)\\ &\leq K^{-2(n-1)\alpha}K^d\mathrm{Exc}(R) \leq C\left(\tfrac{r}{R}\right)^{2\alpha}\mathrm{Exc}(R).\end{aligned}$$

The proof is complete. □

III.2 Ingredients of the proof

III.2.1 A discrete Cacciopolli inequality

Note that it is well-known that the Cacciopolli inequality also holds true on the discrete setting of the random conductance model on $\mathbb{Z}^d$ (even on random graphs) in the case the conductances are bounded from above and below. To deal with degenerate condition, we need to improve this inequality a bit. The Cacciopolli-type inequality in Lemma III.2.1 is used to prove Theorem III.1.2 from Theorem III.1.3. Its proof is the same as the classical Caccioppoli's inequality except in the last step we need to apply the Hölder inequality. For convenience, we provide its proof in Section III.A.

Lemma III.2.1. *For large integer radii ρ and R satisfying $1 \ll \rho \leq R/2$ the following holds. Let u be an $\mathbf{a}$-harmonic function on D_R. Further, suppose that for some exponents $p,q \in (1,\infty)$ we have*

$$\|\omega\|_{\overline{L}^p(E_R)} + \|\omega^{-1}\|_{\overline{L}^q(E_R)} \leq \Lambda. \tag{III.2.1}$$

Then, for any $c \in \mathbb{R}$, we have

$$\|\nabla u\|^2_{\overline{L}^{\frac{2q}{q+1}}(D_{R-\rho})} \lesssim_d \Lambda\|\mathbf{a}\nabla u\cdot\nabla u\|_{\overline{L}^1(D_{R-\rho})} \lesssim_d \frac{1}{\rho^2}\Lambda\|u-c\|^2_{\overline{L}^{\frac{2p}{p-1}}(D_R)} \tag{III.2.2}$$

Proof of Theorem III.1.2 from Theorem III.1.3 using Lemma III.2.1. Let

$$r^*(\omega) \leq r \leq R$$

where r^* is the minimal radius in Theorem III.1.3. By the excess decay (III.1.9) with $\xi := 0$ in the infimum and the second inequality of (III.2.2) we have

$$\operatorname{Exc}(r) \leq C\left(\frac{r}{R}\right)^{2\alpha} \|\mathbf{a}\nabla u \cdot \nabla u\|_{\overline{L}^1(D_R)} \lesssim_d C\left(\frac{r}{R}\right)^{2\alpha} \frac{1}{R^2}\|u\|^2_{\overline{L}^{\frac{2p}{p-1}}(D_{2R})}.$$

Letting R tend to infinity and using (III.1.8) we have $\operatorname{Exc}(r) = 0$. Since r is arbitrarily chosen, the claim follows. □

III.2.2 Construction of the correctors

In the continuum case, besides the corrector ϕ in Theorem III.1.3 and Section III.2, which is a classical object in homogenization, a second order corrector σ with stationary gradient was introduced in [33, 12]. The advantage of using σ is that we can write the two-scale homogenization error

$$w = u - v - \eta\phi_i\partial_i v \tag{III.2.3}$$

where u is an $\mathbf{a}$-harmonic function, v is an $\mathbf{a}_\mathrm{h}$-harmonic function and η is a cut-off function as solution to a Poisson-type equation with the right-hand side in divergence form which is useful for energy estimates

$$-\nabla \cdot \mathbf{a}\nabla w = \nabla \cdot ((1-\eta)(\mathbf{a} - \mathbf{a}_\mathrm{h})\nabla v) + \nabla \cdot [\phi_i\mathbf{a} - \sigma_i)(\eta\partial_i\nabla v)] \tag{III.2.4}$$

(see [33, eq. (79), p27] and [12, eq. (26), p10]). In the discrete case we have to construct the corrector so that we essentially have the same equation as (III.2.4), which is the task of Proposition III.2.2 below.

Note that in the discrete case, saying a field has a stationary gradient is equivalent to saying it is co-cycle. Here, a field $u = u(\omega, x)$ is co-cycle if

$$u(\omega, 0) = 0 \quad \text{and} \quad u(\omega, x+e) - u(\omega, x) = u(\tau_x\omega, e), \qquad \text{for all } x \in \mathbb{Z}^d, e \in \{\pm\mathbf{e}_i : 1 \leq i \leq d\} \tag{III.2.5}$$

Proposition III.2.2 (Existence of (ϕ, σ)). *Suppose that Assumption III.1.1 is satisfied. Then, there exist co-cycle functions ϕ_i and σ_{ijk} where $1 \leq i, j, k \leq d$ such that:*

i) $\mathbb{P}$*-a.s. on* $\mathbb{Z}^d$*, the following equations are true*

$$-\nabla^* \cdot \mathbf{a}(\mathbf{e}_i + \nabla\phi_i) = 0, \tag{III.2.6}$$

$$-\nabla^* \cdot \sigma_i = q_i, \tag{III.2.7}$$

$$-\Delta\sigma_{ijk} = \nabla_j q_{ik} - \nabla_k q_{ij}, \tag{III.2.8}$$

where

$$q_i := \mathbf{a}(\nabla\phi_i + e_i) - \mathbf{a}_{\mathrm{h}}\mathbf{e}_i \quad \textit{and} \quad \mathbf{a}_{\mathrm{h}}\mathbf{e}_i = \langle \mathbf{a}(\nabla\phi_i + \mathbf{e}_i)\rangle\,, \tag{III.2.9}$$

and Δ denotes the standard discrete Laplacian in $\mathbb{Z}^d$.

ii) The tensor σ is anti-symmetric in the last indices,

$$\sigma_{ijk} = -\sigma_{ikj} \quad \textit{on } \mathbb{Z}^d, \quad \mathbb{P}\textit{-a.s.} \tag{III.2.10}$$

iii) The gradients of ϕ and σ have bounded moments and zero expectations:

$$\langle \nabla\phi_i \cdot \mathbf{a}\nabla\phi_i\rangle < \infty,\ \left\langle |\nabla\phi_i|^{\frac{2q}{q+1}}\right\rangle < \infty,\ \left\langle |\nabla\sigma_{ijk}|^{\frac{2p}{p+1}}\right\rangle < \infty, \tag{III.2.11}$$

$$\langle \nabla\phi_i\rangle = \langle \nabla\sigma_{ijk}\rangle = 0. \tag{III.2.12}$$

iv) ϕ and σ are sub-linear:

$$\lim_{R\to\infty} \tfrac{1}{R}\|\phi\|_{\overline{L}^{2p/(p-1)}(\overline{D}_R)} = \lim_{R\to\infty} \tfrac{1}{R}\|\sigma\|_{\overline{L}^{2q/(q-1)}(\overline{D}_R)} = 0.$$

The idea for the construction of the corrector σ is essentially a projection argument on the probability space similar to the continuum case [33, 12]. First, we need the forward and backward horizontal derivatives (for short, ω-derivatives) defined for functions $\zeta : \Omega \to \mathbb{R}$ as follows:

$$\mathrm{D}\zeta := \begin{bmatrix} \mathrm{D}_1\zeta \\ \vdots \\ \mathrm{D}_d\zeta \end{bmatrix} := \begin{bmatrix} \zeta\circ\tau_{\mathbf{e}_1} - \zeta \\ \vdots \\ \zeta\circ\tau_{\mathbf{e}_d} - \zeta \end{bmatrix}, \quad \mathrm{D}^*\zeta := \begin{bmatrix} \mathrm{D}_1^*\zeta \\ \vdots \\ \mathrm{D}_d^*\zeta \end{bmatrix} := \begin{bmatrix} \zeta\circ\tau_{-\mathbf{e}_1} - \zeta \\ \vdots \\ \zeta\circ\tau_{-\mathbf{e}_d} - \zeta \end{bmatrix}. \tag{III.2.13}$$

Note that if a field $u(\omega, x)$ is stationary in the sense that

$$u(\omega,x) = u(\tau_x\omega, 0) \quad \forall x \in \mathbb{Z}^d, \quad \mathbb{P}\text{-a.e. } \omega$$

then

$$\nabla u(\omega,x) = \mathrm{D}u(\omega,x) \quad \forall x \in \mathbb{Z}^d, \quad \mathbb{P}\text{-a.e. } \omega.$$

The stationarity of the gradient and the flux allows us to replace the x-derivative by the ω-derivative in (III.2.8):

$$\mathrm{D}\cdot\nabla^*\sigma_{ijk} = \mathrm{D}\cdot(q_{ik}\mathbf{e}_j - q_{ij}\mathbf{e}_k). \tag{III.2.14}$$

Testing with some $\zeta \in L^\infty(\Omega)$ we have the following weak formulation:

$$\langle \nabla^*\sigma_{ijk}\cdot \mathrm{D}^*\zeta\rangle = \langle (q_{ik}\mathbf{e}_j - q_{ij}\mathbf{e}_k)\cdot \mathrm{D}^*\zeta\rangle\,. \tag{III.2.15}$$

Let us first discuss the standard case where $\mathbf{a}$ is bounded from above and below.

Construction of σ in the uniformly elliptic case. We exploit the following orthogonal decomposition:

$$L^2(\Omega, \mathbb{R}^d) = L^2_{\nabla^*} \oplus (L^2_{\nabla^*})^\perp \tag{III.2.16}$$

where

$$\begin{aligned} L^2_{\nabla^*} &:= \text{ closure of } \{\mathrm{D}^*\zeta : \zeta \in L^\infty(\Omega)\} \quad \text{in} \quad L^2(\Omega, \mathbb{R}^d) \\ (L^2_{\nabla^*})^\perp &= \{\psi \in L^2(\Omega, \mathbb{R}^d) : \mathrm{D} \cdot \psi = 0\}. \end{aligned} \tag{III.2.17}$$

Since (III.2.15) holds, we should define the stationary gradient of σ as follows:

$$\nabla^*\sigma_{ijk} := \mathrm{proj}\Big|_{L^2_{\nabla^*}} (q_{ik}\mathbf{e}_j - q_{ij}\mathbf{e}_k). \tag{III.2.18}$$

By (III.2.17), this projection obviously gives us (III.2.15), therefore (III.2.14) and (III.2.8). Further, we have

$$\begin{aligned} \Delta(\nabla^* \cdot \sigma_{ij} + q_{ij}) &= \nabla^*_k \Delta\sigma_{ijk} + \Delta q_{ij} \\ &= -\nabla^*_k \nabla_j q_{ik} \underbrace{-\nabla^*_k \nabla_k q_{ij} + \Delta q_{ij}}_{=0} = 0, \end{aligned}$$

where the second equality is due to (III.2.8) and the third (III.2.6) and (III.2.9). This immediately gives us (III.2.7) and (III.2.10) by the fact that a *stationary* field $u \in L^2(\Omega)$ satisfying $\Delta u(\omega, x) = 0$ must be a constant. Indeed, stationarity allows us to write

$$\mathrm{D}^* \cdot \mathrm{D}u(\omega, 0) = \Delta u(\omega, 0) = 0.$$

By partial integration on the probability space, we have

$$\langle \mathrm{D}u(\omega, 0) \cdot \mathrm{D}u(\omega, 0)\rangle = \langle u(\omega, 0)\mathrm{D}^* \cdot \mathrm{D}u(\omega, 0)\rangle = 0$$

which means u is invariant, therefore constant, by ergodicity. □

In the degenerate case (Assumption III.1.1) we have to deal with a lack of integrability since $q_i = \mathbf{a}(\nabla\phi_i + \mathbf{e}_i) \in L^{2p/(p+1)}$. The solution to this is to define an approximating sequence $\nabla^*\sigma^{(n)}_{ijk}$ obtained by replacing the right-hand side of (III.2.18) by its truncation sequence $(q_{ik}\mathbf{e}_j - q_{ij}\mathbf{e}_k)\mathbf{1}_{|q|\le n}$. As in [12] the convergence of $\nabla^*\sigma^{(n)}_{ijk}$ in $L^{2p/(p+1)}(\Omega)$ norm is obtained by lifting things onto the physical space and using a Calderón-Zymund estimate on this level.

The discussion on the correctors is organized in Section III.3 as follows. In Subsection III.3.1 we see that defining σ via (III.2.7), (III.2.8) and (III.2.10) is really useful for obtaining a discrete analogue of (III.2.4). Then, Subsection III.3.2 reminds some basic facts on horizontal derivatives and co-cycle fields which are important for the proof. For convenience, in Subsection III.3.3 we recall briefly the construction of the first order corrector ϕ, which is already well-known – that of the corrector σ is discussed in Subsection III.3.4.

III.2.3 Construction of the harmonic extension

Roughly speaking, the idea of *perturbing around the homogenized field* [33, 12] is to construct an $\omega_{\rm h}$-harmonic function v which approximates the given ω-harmonic function u and use the two-scale error (III.2.3) to estimate how good the approximation is. This idea can also be fully adapted to the discrete case: The following auxiliary result Proposition III.2.3, mimicking [12, (54), p28], is a crucial step to obtain the excess decay (Theorem III.1.4).

In fact, by the sublinearity of the corrector, estimate Proposition III.2.3 below yields that in a very large scale the energy of the homogenization error is very small compared to that of the given ω-harmonic function.

Note that as in the continuum case this is a deterministic result and we only need to assume (III.2.19) instead of (III.1.7).

Proposition III.2.3. *For large R the following is true. Assume that*

$$\|\omega^{-1}\|_{\overline{L}^q(\overline{E}_R)} + \|\omega\|_{\overline{L}^p(\overline{E}_R)} \le \Lambda \ \text{ where } 1/p + 1/q \le 2/d. \tag{III.2.19}$$

Consider $u : \overline{D}_R \to \mathbb{R}$ be $\mathbf{a}$-harmonic in D_R. Then, there exist an $\mathbf{a}_{\rm h}$-harmonic function v in $D_{\lfloor R/2 \rfloor}$ satisfying

$$\|\nabla v \cdot \mathbf{a}_{\rm h} \nabla v\|_{\overline{L}^1(\overline{D}_{\lfloor R/2 \rfloor})} \lesssim_{d,p,q} \Lambda \|\nabla u \cdot \mathbf{a} \nabla u\|_{\overline{L}^1(\overline{D}_R)} =: \overline{\Lambda} \tag{III.2.20}$$

such that the homogenization error $w = u - v - \nabla_i v \phi_i$ satisfies

$$\begin{aligned} \|\nabla w \cdot \mathbf{a} \nabla w\|_{\overline{L}^1(\overline{D}_{\lfloor R/4 \rfloor})} \lesssim_{d,p,q} & \ \overline{\Lambda} \varepsilon^{1-(d-1)(\frac{1}{2p}+\frac{1}{2q})} \\ & + \Lambda \overline{\Lambda} \left(\frac{\rho}{R}\right)^{\min(\frac{p-1}{2p}, \frac{q-1}{2q})} \varepsilon^{-(d-1)\min(\frac{q+1}{q}, \frac{p+1}{p})} \\ & + \Lambda \overline{\Lambda} \left(\frac{R}{\rho}\right)^{d+2} \left[\|\phi\|_{\overline{L}^{\frac{2p}{p-1}}(D_R)} + \|\sigma\|_{\overline{L}^{\frac{2q}{q-1}}(D_R)} \right] \end{aligned} \tag{III.2.21}$$

for all $\varepsilon \in (0, 1/2)$ and $\rho \in (20, R/8)$.

Let us discuss the proof of Proposition III.2.3 (Section III.5). Recall that in the continuum setting [12, p24], from (III.2.3) and (III.2.4) we have the energy of w as follows:

$$\int_{B_{R'}} \mathbf{a} \nabla w \cdot \nabla w = \int_{\partial B_{R'}} (u - v) \nu \cdot (\mathbf{a} \nabla u - \mathbf{a}_{\rm h} \nabla v) \tag{III.2.22a}$$

$$+ \int_{B_{R'}} (1 - \eta) \nabla w \cdot (\mathbf{a}_{\rm h} - \mathbf{a}) \nabla v \tag{III.2.22b}$$

$$- \int_{B_{R'}} \nabla w \cdot (\phi_i \mathbf{a} - \sigma_i) \nabla(\eta \partial_i v), \tag{III.2.22c}$$

where η is a cut-off function

$$\eta = 0 \quad \text{in} \quad (B_{R'-\rho})^{\mathrm{c}}, \quad \eta = 1 \quad \text{in} \quad B_{R'-2\rho}, \quad \text{and} \quad |\nabla\eta| \lesssim 1/\rho.$$

By the fact that v is a "very good" function (in fact, harmonic) and ϕ and σ are "small", the estimates for the near boundary (III.2.22b) and the corrector term (III.2.22c) are quite robust, when adapted to the discrete case. We therefore mainly focus on the boundary term (III.2.22a). Recall that in the continuum case, Bella, Fehrman, and Otto [12] construct the function v using the boundary condition of u. If $q \geq p$ (called the Dirichlet case), they use the Dirichlet boundary condition:

$$\nabla \cdot \mathbf{a}_{\mathrm{h}} \nabla v = 0 \text{ in } B_{R'} \quad \text{and} \quad v := u_\varepsilon \text{ on } \partial B_{R'} \tag{III.2.23}$$

If $p \geq q$ (called the Neumann case), they use the Neumann boundary condition:

$$\nabla \cdot \mathbf{a}_{\mathrm{h}} \nabla v = 0 \text{ in } B_{R'} \quad \text{and} \quad \mathbf{a}_{\mathrm{h}} \nabla v = (\mathbf{a}\nabla u)_\varepsilon \cdot \nu - \fint_{\partial B_{R'}} (\mathbf{a}\nabla u)_\varepsilon \cdot \nu \text{ on } \partial B_{R'}. \tag{III.2.24}$$

where $\nu = \nu(x)$ is the normal vector on the sphere and the last one is a surface integral. Here, the index ε denotes a smoothed version defined by taking convolution on $\partial B_{R'}$ (on the sphere) which has better integrability than the original function. In the our case, the first task is to find "a discrete smoothed version" that is the aim of the following result. Consider the tangential edges:

$$E_R^{\tan} = \left\{\{x, y\} \in \mathbb{E}^d : x, y \in \partial D_R, x \sim y\right\}. \tag{III.2.25}$$

Lemma III.2.4. *For $\varepsilon > 0$ there exists a linear operator which maps u to u_ε, both defined on ∂D_R, with the following properties. For $1 \leq s \leq r \leq \infty$ we have*

$$\|u_\varepsilon\|_{\overline{L}^r(\partial D_R)} \lesssim \varepsilon^{-(d-1)(\frac{1}{s}-\frac{1}{r})} \|u\|_{\overline{L}^s(\partial D_R)}, \tag{III.2.26}$$

$$\|\nabla u_\varepsilon\|_{\overline{L}^r(E_R^{\tan})} \lesssim \varepsilon^{-(d-1)(\frac{1}{s}-\frac{1}{r})} \|\nabla u\|_{\overline{L}^s(E_R^{\tan})}. \tag{III.2.27}$$

$$\|u - u_\varepsilon\|_{\overline{L}^s(\partial D_R)} \lesssim \varepsilon R \|\nabla u\|_{\overline{L}^s(E_R^{\tan})}. \tag{III.2.28}$$

Further, the multiplicative constants in (III.2.26)–(III.2.28) *only depend on the dimension d and the exponents r and s. Especially, they do not depend on R.*

This gives us a natural definition of v in the Dirichlet case $q \geq p$:

$$v = u_\varepsilon \quad \text{on } \partial D_{R'} \tag{III.2.29}$$

where u_ε is the "discrete smoothed version" of u provided by Lemma III.2.4. We postpone the idea of constructing u_ε in Lemma III.2.4 until Subsection III.2.4. Let us first give a comment on it. Although this way of smoothing gives us a very good way to adapt convolution estimates to the discrete setting, we still have to struggle some obstacles which appear in the discrete analogue of the boundary term (III.2.22a):

$$\sum_{[x,y]\in E_{R'}^{\mathrm{nor}}} [u(x)-v(x)]\left[\omega\nabla u([x,y])-\omega_{\mathrm{h}}\nabla v([x,y])\right], \tag{III.2.30}$$

where the sum is taken over all normal edges (see Figure III.1),

$$E_R^{\mathrm{nor}} = \left\{[x,y]\in\mathbb{E}_{\pm}^d : x\in\partial D_R, y\in D_R\right\}, \tag{III.2.31}$$

or equivalently, over their endpoints lying on the boundary,

$$\widetilde{\partial} D_R = \{x\in\partial D_R : \exists y \text{ s.t. } [x,y]\in E_R^{\mathrm{nor}}\}. \tag{III.2.32}$$

The main challenge is that the harmonic extension does not remember the normal derivatives in the Dirichlet case or the tangential derivatives in the Neumann case. This happened in the continuum case as well. As a consequence, *we need to compare the discrete tangential and normal derivatives of discrete harmonic functions in the sense of L^r norm*. This is the task of the following interesting result that will be proved separately in Chapter IV which is the most technical part of the thesis.

Theorem III.2.5 (Dirichlet-to-Neumann). *Let $R\gg 1$ and v be harmonic in D_R. Then, we have*

$$\|\nabla v\|_{L^r(E_R^{\mathrm{nor}})} \lesssim_{d,r} \|\nabla v\|_{L^r(E_R^{\mathrm{tan}})}, \quad r\in(1,\infty).$$

Further, there exists a modification $\widetilde{v}$ of v in the sense that $v=\widetilde{v}$ on $\widetilde{\partial} D_R\cup D_R$ (i.e. v is only modified at the corners) such that

$$\|\nabla \widetilde{v}\|_{L^r(E_R^{\mathrm{tan}})} \lesssim_{d,r} \|\nabla v\|_{L^r(E_R^{\mathrm{nor}})}, \quad r\in(1,\infty).$$

In the Neumann case, we still have some minor difficulties. The first difficulty is that the Neumann condition does not define the values of v in the set of the "corners" $\partial D_{R'}\setminus\widetilde{\partial} D_{R'}$ while all the terms in estimates (III.2.26)–(III.2.28) in Lemma III.2.4 contain the values at the corners. Also note that because of the "corners" we can only speak of "a modification" in Theorem III.2.5. The second difficulty is much more serious: there is a lack of symmetry of the convolution in the discrete case, namely the "discrete smoothed version" in Lemma III.2.4 does not have the property $\int f_\varepsilon g = \int f g_\varepsilon$ as in the continuum case, which has been exploited by Bella, Fehrman, and Otto [12] in the Neumann case as follows:

$$\begin{aligned}
&\int_{\partial B_{R'}} (u-v)\cdot(\mathbf{a}\nabla u\cdot\nu-\mathbf{a}_{\mathrm{h}}\nabla v\cdot\nu)dS \\
\text{[by (III.2.24)]} \quad &= \int_{\partial B_{R'}} (u-v)\left[\mathbf{a}\nabla u\cdot\nu-(\mathbf{a}\nabla u\cdot\nu)_\varepsilon\right]dS \qquad \text{(III.2.33)}\\
\text{[symmetry of convolution]} \quad &= \int_{\partial B_{R'}} \left[(u-v)-(u-v)_\varepsilon\right]\mathbf{a}\nabla u\cdot\nu dS,
\end{aligned}$$

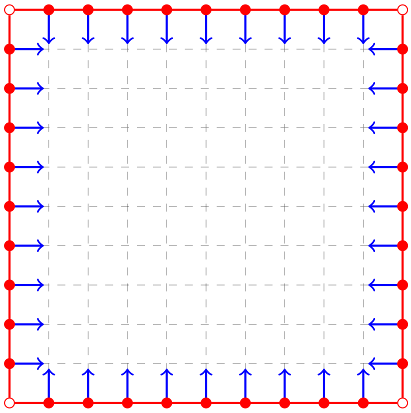

Figure III.1: The tangential (*red*) and inner normal edges (*blue*)

where the difference on the right-hand side gives us a small ε. Also note that the sum (III.2.30) does not contain the corners. So, it is definitely not clear whether we should smooth $\omega\nabla u$ on the full boundary $\partial D_{R'}$ or the boundary without the corners $\widetilde{\partial} D_{R'}$, or on each face without the corners etc., and whether we can still use the smoothed version in Lemma III.2.4 or not.

Fortunately, we can overcome these by natural solutions. First, we introduce an operator M which fills in or modifies the values of a given function at each corner. We may perhaps use the modification produced by Theorem III.2.5. However, this result looks like a black box and in order to know what really this modification is, the reader needs to start with Chapter IV first. Therefore, to make the argument clear, we do not want to reveal too many things in this chapter.

Lemma III.2.6 (Modification operator)**.** *There exists a linear operator M that maps u defined on $\partial D_{R'}$ to $Mu = \bar{u}$ also defined on $\partial D_{R'}$ such that $\bar{u} = u$ on $\widetilde{\partial} D_{R'}$ and*

$$\|\nabla \bar{u}\|_{L^r(E^{\tan}_{R'})} \lesssim_{d,r} \|\nabla u\|_{L^r(E^{\tan}_{R'})} \quad \textit{and} \quad \|\bar{u}\|_{L^r(\partial D_{R'})} \lesssim_{d,r} \|u\|_{L^r(\widetilde{\partial} D_{R'})}, \qquad r \in [1, \infty].$$

Idea of the proof. Think about $d = 2$ (Figure III.1). Define the value at each corner by copying that from an arbitrary point near them which is not a corner. The complete proof for the general case $d \geq 2$ has the same spirit (see Subsection III.5.1). □

The novelty to overcome the second difficulty is to introduce a new definition of the Neumann condition:

$$\omega_{\mathrm{h}} \nabla v := (\omega\nabla u)^*_\varepsilon - \overline{\sum_{\widetilde{\partial} D_{R'}}} (\omega\nabla u)^*_\varepsilon \quad \text{on } \widetilde{\partial} D_{R'} \tag{III.2.34}$$

where $(\cdot)^*_\varepsilon$ is defined so that we have the duality in the following sense.

Lemma III.2.7 (Duality)**.** *Given $h : \widetilde{\partial} D_R \to \mathbb{R}$ there exists $h^*_\varepsilon : \widetilde{\partial} D_{R'} \to \mathbb{R}$ such that*

$$\sum_{x \in \widetilde{\partial} D_{R'}} h^*_\varepsilon g = \sum_{x \in \widetilde{\partial} D_{R'}} h(Mg)_\varepsilon \tag{III.2.35}$$

for any $g : \partial D_{R'} \to \mathbb{R}$ where M is the modification operator in Lemma III.2.6.

Remember that we have to take care for the norm of the normal gradients.

Lemma III.2.8. *For any $h : \widetilde{\partial} D_R \to \mathbb{R}$ and $1 < s \le r \le \infty$ we have*

$$\|h^*_\varepsilon\|_{\overline{L}^r(\widetilde{\partial} D_{R'})} \lesssim_{d,r,s} \varepsilon^{-(d-1)(\frac{1}{s} - \frac{1}{r})} \|h\|_{\overline{L}^s(\widetilde{\partial} D_{R'})}.$$

This allows us to mimic (III.2.33):

$$\begin{aligned} \text{(III.2.30)} &= \sum_{E^{\mathrm{nor}}_{R'}} (u - v) \Big[\omega \nabla u - (\omega \nabla u)^*_\varepsilon \Big] && \text{[by (III.2.34)]} \\ &= \sum_{E^{\mathrm{nor}}_{R'}} \omega \nabla u \Big\{ [M(u-v)] - [M(u-v)]_\varepsilon \Big\} && \text{[by (III.2.35)].} \end{aligned}$$

In Subsection III.5.1 we rigorously prove Lemmas III.2.6–III.2.8 with very simple arguments. With the above ”tricks”, the idea of Bella, Fehrman, and Otto [12] can be completely implemented in the discrete case: In Subsection III.5.2 we calculate the energy of the homogenization error by a simple discrete Gauss-type formula, Lemma III.5.1. In Subsection III.5.3 we deal with the boundary term. The arguments for the corrector term and the near boundary term, which are quite similar to that in the continuum setting and do not require very new ideas, are presented in Subsections III.5.4 and III.5.5, respectively. Finally, Subsection III.5.6 shows the proof of Theorem III.1.4 from Proposition III.2.3.

III.2.4 Smoothing boundary conditions in the discrete case

In Section III.4 we prove Lemma III.2.4 constructing ”the discrete smoothed version” in three steps by a short excursion to Sobolev spaces on Lipschitz surfaces and to finite element theory. In the first step (Subsection III.4.1), using any type of interpolation we embed u into the continuum surface ∂C_R. The second step (Subsection III.4.2) is to smooth it there by decomposing u by a partition of unity into ”small pieces” compactly supported on local charts and lifting them to the Euclidean space. Compared to the first and second step, the third one (Subsection III.4.3), obtaining u_ε defined on the discrete surface ∂D_R from the smoothed version in the continuum surface ∂C_R, requires some details. A natural way to define a discrete function f from a given continuum function $\widetilde{f}$ is local integration:

$$f(x) := \int_{\Gamma_x} \widetilde{f} \tag{III.2.36}$$

where Γ_x is, for instance, an $(d-1)$-dimensional unit box on the surface with vertex x. However, this idea has a disadvantage: *after interpolating a discrete function and then projecting it to the discrete again, we may not get the same function*. The solution for this is to insert an integrator ψ_x into (III.2.36):

$$f(x) := \int_{\Gamma_x} \psi_x \widetilde{f}. \tag{III.2.37}$$

From the requirement that interpolating a discrete function and then projecting it to the discrete again do not change the original function, this ansatz must satisfy

$$\delta_z(x) = \int_{\Gamma_x} \psi_x \varphi_z. \tag{III.2.38}$$

where φ_z's are the interpolations of the Dirac functions on the lattice, $\delta_z(x) = \mathbf{1}_{x=z}$, which are called the nodal functions in finite element theory. If we choose ψ_x *as a linear combination of* φ_z where z are vertices of Γ_x, we immediately have (III.2.38) for all points $z \neq \Gamma_x$, for which $\varphi_z = 0$ on Γ_x. In this case, ψ_x *is uniquely determined by* (III.2.38) *restricted to* z *which are vertices of* Γ_x. This elegant idea is due to the work by Scott and Zhang [58] in numerical analysis.

III.3 The correctors and their construction

III.3.1 Equation for the homogenization error

In the following we use the correctors given in Proposition III.2.2 to write the equation for the homogenization error as a Poisson-type equation in divergence form. As in the continuum case, this calculation is useful for energy estimates in Subsection III.5.2.

Our first task is to introduce a notation which helps us to avoid lengthy calculations. Observe that for functions $f, g : \mathbb{Z}^d \to \mathbb{R}$ we can write the product rule as follows:

$$\nabla_i(fg) = (f\nabla_i)g + g\nabla_i f$$

(the first r.h.s. term *with* brackets, the second *without* brackets!) where we define $(f\nabla_i)g$ (*with* brackets) and $g\nabla_i f$ (*without* brackets) as follows:

$$[(f\nabla_i)g](x) := f(x+\mathbf{e}_i)\nabla_i g(x) = f(x+\mathbf{e}_i)(g(x+\mathbf{e}_i) - g(x)), \tag{III.3.1}$$

and

$$[g\nabla_i f](x) = g(x)\nabla_i f(x) = g(x)(f(x+\mathbf{e}_i) - f(x)).$$

Similarly, we also use this notation for the backward differential operator ∇^*, say

$$\nabla_i^*(fg) = (f\nabla_i^*)g + g\nabla_i^* f \tag{III.3.2}$$

where

$$(f\nabla_i^*)g(x) := f(x-\mathbf{e}_i)\nabla_i^* g(x) \quad \text{and} \quad g\nabla_i^* f(x) = g(x)(f(x-\mathbf{e}_i) - f(x)).$$

Finally, we write

$$(f\nabla)g := [(f\nabla_1)g, \dots, (f\nabla_d)g]^\top$$

and

$$(F\cdot\nabla)g := \sum_1^d (F_i\nabla_i)g$$

for $F : \mathbb{Z}^d \to \mathbb{R}^d$. We also use similar notations for the backward operator ∇^*. This notation is motivated by differential geometry where it is common to consider differential forms as operators acting on functions. By using it, the product rule in the discrete case looks almost the same as that the continuum case. Recall that to lighten notations we make use of *Einstein's convention* summing over repeated indexes.

Lemma III.3.1. *Let (ϕ, σ) be given as in Proposition III.2.2 and $\eta : \mathbb{Z}^d \to \mathbb{R}$. Consider*

$$w = u - v - \eta\phi_i\nabla_i v.$$

Then, we have

$$\nabla^*\cdot\mathbf{a}\nabla w = -\nabla^*\cdot((1-\eta)(\mathbf{a}-\mathbf{a}_{\mathrm{h}})\nabla v) \\ - \nabla^*\cdot[(\sigma_i\cdot\nabla^*)(\eta\nabla_i v)] - \nabla^*\cdot\mathbf{a}[(\phi_i\nabla)(\eta\nabla_i v)], \quad \text{(III.3.3)}$$

whenever the following is true

$$\nabla^*\cdot\mathbf{a}\nabla u = \nabla^*\cdot\mathbf{a}_{\mathrm{h}}\nabla v = 0. \quad \text{(III.3.4)}$$

If we write (III.3.3) with the usual notation for the chain rule, it may contain four or five lines with full of unnecessary terms.

Proof of Lemma III.3.1. The proof contains purely algebraic calculations. By the product rule (III.3.1), we have

$$\nabla w = \nabla u - \nabla v - \eta\nabla_i v\nabla\phi_i - (\phi_i\nabla)(\eta\nabla_i v).$$

Applying $\nabla^*\cdot\mathbf{a}$ yields

$$\nabla^*\cdot\mathbf{a}\nabla w = -\nabla^*\cdot\underbrace{(\mathbf{a}\nabla v + \eta\nabla_i v\mathbf{a}\nabla\phi_i)}_{=:A} - \nabla^*\cdot\mathbf{a}[(\phi_i\nabla)(\eta\nabla_i v)] \quad \text{(III.3.5)}$$

by assumption (III.3.4). Split A as follows:

$$\begin{aligned} A &= (1-\eta)\mathbf{a}\nabla v + \eta\nabla_i v\mathbf{a}(\mathbf{e}_i + \nabla\phi_i) \\ &= (1-\eta)(\mathbf{a}-\mathbf{a}_{\mathrm{h}})\nabla v + (1-\eta)\mathbf{a}_{\mathrm{h}}\nabla v + \eta\nabla_i v\mathbf{a}(\mathbf{e}_i+\nabla\phi_i) \quad \text{(III.3.6)} \\ &= A_1 + A_2 + A_3. \end{aligned}$$

By assumption (III.3.4) and the product rule (III.3.2) we have

$$
\begin{aligned}
-\nabla^* \cdot A_2 &= -\nabla^* \cdot ((1-\eta)\mathbf{a}_{\mathrm{h}}\nabla v) \\
&= \nabla^* \cdot (\eta \nabla_i v \mathbf{a}_{\mathrm{h}} \mathbf{e}_i) \\
&= \eta \nabla_i v \underbrace{\nabla^* \cdot \mathbf{a}_{\mathrm{h}} \mathbf{e}_i}_{=0} + (\mathbf{a}_{\mathrm{h}} \mathbf{e}_i \cdot \nabla^*)(\eta \nabla_i v) \\
&= (\mathbf{a}_{\mathrm{h}} \mathbf{e}_i \cdot \nabla^*)(\eta \nabla_i v)
\end{aligned}
$$

By the product rule (III.3.2) and the fact that $\nabla^* \cdot \mathbf{a}(\mathbf{e}_i + \nabla \phi_i) = 0$, we have

$$
\begin{aligned}
-\nabla^* \cdot A_3 &= -\nabla^* \cdot \{\eta \nabla_i v \mathbf{a}(\mathbf{e}_i + \nabla \phi_i)\} \\
&= -\eta \nabla_i v \underbrace{\nabla^* \cdot \mathbf{a}(\nabla \phi_i + \mathbf{e}_i)}_{=0} - \{\mathbf{a}(\mathbf{e}_i + \nabla \phi_i) \cdot \nabla^*\}(\eta \nabla_i v) \\
&= -\{\mathbf{a}(\mathbf{e}_i + \nabla \phi_i) \cdot \nabla^*\}(\eta \nabla_i v)
\end{aligned}
$$

Adding together we get

$$
\begin{aligned}
-\nabla^* \cdot (A_2 + A_3) \quad &= \quad -\{[\mathbf{a}(\mathbf{e}_i + \nabla \phi_i) - \mathbf{a}_{\mathrm{h}} \mathbf{e}_i] \cdot \nabla^*\}(\eta \nabla_i v) \\
&\overset{(\mathrm{III.2.7})}{=} \quad \{[\nabla^* \cdot \sigma_i] \cdot \nabla^*\}(\eta \nabla_i v) \\
&= \quad -\nabla^* \cdot [(\sigma_i \cdot \nabla^*)(\eta \nabla_i v)]
\end{aligned}
\tag{III.3.7}
$$

where the last equality can be explained as follows. To lighten notation $\xi_i := \eta \nabla_i v$. Applying the product rule (III.3.2) we have

$$
\begin{aligned}
\{[\nabla^* \cdot \sigma_i] \cdot \nabla^*\}\xi_i &= [(\nabla_k^* \sigma_{ijk}) \cdot \nabla_j^*]\xi_i \\
&= \nabla_j^*[\xi_i \nabla_k^* \sigma_{ijk}] - \underbrace{\xi_i \nabla_j^* \nabla_k^* \sigma_{ijk}}_{=0} = \nabla_j^*[\xi_i \nabla_k^* \sigma_{ijk}]
\end{aligned}
\tag{III.3.8}
$$

where the zero term is due to the antisymmetry of σ. Applying the product rule (III.3.2) again we write the term inside the brackets as follows:

$$
\xi_i \nabla_k^* \sigma_{ijk} = \nabla_k^*(\xi_i \sigma_{ijk}) - (\sigma_{ijk} \nabla_k^*)\xi_i,
$$

which implies that

$$
\begin{aligned}
(\mathrm{III.3.8}) &= \underbrace{\nabla_j^* \nabla_k^* (\xi_i \sigma_{ijk})}_{=0} - \nabla_j^*[(\sigma_{ijk} \nabla_k^*)\xi_i] = -\nabla_j^*[(\sigma_{ij} \cdot \nabla^*)\xi_i] \\
&= -\nabla^* \cdot [(\sigma_i \cdot \nabla^*)\xi_i]
\end{aligned}
$$

where the zero term is due to the anti-symmetry. Combining (III.3.5)–(III.3.7) yields (III.3.3). The proof is complete. □

III.3.2 Co-cycle fields and horizontal derivatives

We recall some basic facts which are useful for constructing the correctors. First, being co-cycle (III.2.5) is equivalent to having a gradient field. In other words, a field $u : \Omega \times \mathbb{Z}^d \to \mathbb{R}$ is co-cycle if and only if

$$u(\omega, 0) = 0, \quad \nabla u(\omega, x) = \nabla u(\tau_x \omega, 0),$$
$$\text{and} \quad \nabla^* u(\omega, x) = \nabla^* u(\tau_x \omega, 0), \qquad x \in \mathbb{Z}^d,\ \mathbb{P}\text{-a.e. } \omega \quad \text{(III.3.9)}$$

where ∇ and ∇^* are the forward and backward gradient defined in (III.1.4). It is easy to see that a co-cycle function $u : \Omega \times \mathbb{Z}^d \to \mathbb{R}^d$ is uniquely determined $\mathbb{P}$-a.s. by $\nabla u(\omega, 0)$ or $\nabla^* u(\omega, 0)$. Indeed, by (III.3.9), knowing $\nabla u(\omega, 0)$ means knowing all gradients with respect to the positive directions, therefore also that with respect to the negative directions. For this reason, for co-cycle fields u, we often write ∇u and $\nabla^* u$ instead of $\nabla u(\omega, 0)$ and $\nabla^* u(\omega, 0)$, when considering them at the level of the probability space.

Second, note that in general, the ω-derivative (III.2.13) are different from the x-derivatives defined by (III.1.4). However, for stationary fields u in the sense that

$$u(\omega, x) = \hat{u}(\tau_x \omega), \quad x \in \mathbb{Z}^d, \quad \mathbb{P}\text{-a.e } \omega$$

for some $\hat{u} : \Omega \to \mathbb{R}$, it is easy to see that

$$\nabla u(\omega, x) = \mathrm{D} u(\omega, x) \quad \text{and} \quad \nabla^* u(\omega, x) = \mathrm{D}^* u(\omega, x).$$

III.3.3 Construction of the first order corrector

The corrector ϕ is an important object in the literature of the random conductance model (see e.g. [16]). For convenience, we briefly recall the construction of ϕ which relies on a projection argument.

Lemma III.3.2. *Assume $\langle \omega(e)^p \rangle < \infty$ for some $p \in [1, \infty]$. Then, there exist uniquely co-cycle function ϕ_i satisfying* (III.2.6). *Additionally if $\langle \omega(e)^{-q} \rangle < \infty$ for some $q \in [1, \infty]$, the integrability of $\nabla \phi_i$ in* (III.2.11) *is true.*

Let L^2_{cov} be the Hilbert space of random vectors $b : \Omega \to \mathbb{R}^d$ with $\langle \mathbf{a}(0) b \cdot b \rangle < \infty$, equipped with the scalar product $(b, \widetilde{b}) \mapsto \langle \mathbf{a}(0) b \cdot \widetilde{b} \rangle$ for $b, \widetilde{b} \in L^2_{\mathrm{cov}}$. Define

$$L^2_{\nabla} := \text{the closure of } \{\mathrm{D}\zeta : \zeta \in L^\infty(\Omega)\} \quad \text{in } L^2_{\mathrm{cov}}. \qquad \text{(III.3.10)}$$

By Hilbert space theory, $L^2_{\mathrm{cov}} = L^2_{\nabla} \oplus (L^2_{\nabla})^\perp$ and it is easy to see that

$$(L^2_{\nabla})^\perp = \{\psi \in L^2_{\mathrm{cov}} : \mathrm{D}^* \cdot \mathbf{a}\psi = 0\}.$$

Remark III.3.3. The way to define the L^2_{cov}-norm in [16, 48]

$$\|b\|_{L^2_{\mathrm{cov}}} := \mathbb{E}\left(\sum_{|e|_1=1} \omega(0,e) b(\omega)^2\right)$$

is a bit different from that given here. However, they are equivalent, since $\mathbb{P}$ is stationary.

Proof of Lemma III.3.2. By the co-cycle property, discussed in Subsection III.3.2, it suffices to construct $\nabla\phi_i \equiv \nabla\phi_i(\omega, 0)$. By stationarity we can replace ∇^* by D^* in the equation (III.2.6) defining ϕ_i:

$$\mathrm{D}^* \cdot \mathbf{a}(\nabla\phi_i + \mathbf{e}_i) = 0 \tag{III.3.11}$$

Define

$$\nabla\phi_i = \mathrm{proj}\Big|_{L^2_\nabla}(-\mathbf{e}_i) \tag{III.3.12}$$

Then, $-\mathbf{e}_i - \nabla\phi_i \in (L^2_\nabla)^\perp$ from that (III.3.11) follows. The integrability of $\nabla\phi_i$ follows directly from this construction by applying Hölder's inequality, and integrability of $\mathbf{a}(0)$:

$$\begin{aligned}\|\nabla\phi_i\|_{L^{2q/(q+1)}} &\leq \|\mathbf{a}^{1/2}\nabla\phi_i\|_{L^2}\|\mathbf{a}^{1/2}\|_{L^{2q}} \\ &= \langle \mathbf{a}(0)\nabla\phi_i \cdot \nabla\phi_i\rangle \|\mathbf{a}(0)\|_{L^q} < \infty. \end{aligned} \tag{III.3.13}$$

The proof is complete. □

III.3.4 Construction of the second order corrector

As mentioned before in the discussion behind Proposition III.2.2, the construction of σ is straightforward, since we have the orthogonal decomposition (III.2.16) and (III.2.17). Since we have to deal with a weaker integrability condition, we need the following Meyer-type estimate.

Lemma III.3.4. *Let $f \in L^\infty(\Omega, \mathbb{R}^d)$ and u be the co-cycle field generated by*

$$\nabla^* u := \mathrm{proj}\Big|_{L^2_{\nabla^*}} f. \tag{III.3.14}$$

Then for any $r \in (1, 2]$,

$$\|\nabla^* u\|_{L^r(\Omega,\mathbb{R}^d)} \lesssim_{r,d} \|f\|_{L^r(\Omega,\mathbb{R}^d)}.$$

Let us delay its proof until Subsection III.3.6 to continue the construction of σ.

Existence of σ from Lemma III.3.4. Set

$$\sigma_{ijk}^{(n)} := \operatorname{proj}\Big|_{L^2_{\nabla^*}} \left[\left(q_{ik}\mathbf{e}_j - q_{ij}\mathbf{e}_k \right) \mathbf{1}_{|q|\le n} \right], \tag{III.3.15}$$

which means by the decomposition (III.2.17) that

$$\mathrm{D} \cdot \nabla^* \sigma_{ijk}^{(n)} = \mathrm{D} \cdot \left[(q_{ik}\mathbf{e}_j - q_{ij}\mathbf{e}_k)\mathbf{1}_{|q|\le n} \right]. \tag{III.3.16}$$

By (III.3.16) and Lemma III.3.4, the sequence $\{\nabla^* \sigma_{ijk}^{(n)}\}$ is a Cauchy sequence in $L^{2p/(p+1)}(\Omega)$. Therefore, we have

$$\nabla^* \sigma_{ijk}^{(n)} \to \nabla^* \sigma_{ijk} \quad \text{in} \quad L^{2p/(p+1)}(\Omega) \tag{III.3.17}$$

for some $\nabla^* \sigma_{ijk} \in L^{2p/(p+1)}(\Omega)$ defining a unique co-cycle field σ_{ijk}. Further,

$$\mathrm{D} \cdot \nabla^* \sigma_{ijk} = \mathrm{D} \cdot (q_{ik}\mathbf{e}_j - q_{ij}\mathbf{e}_k) \tag{III.3.18}$$

which follows by writing (III.3.16) in the weak formulation (III.2.15) and using the convergence (III.3.17). By stationarity, we can replace D by ∇ in (III.3.18) to obtain (III.2.8). The claim on the integrability of $\nabla^*\sigma$ is obvious by applying Lemma III.3.4 to $\sigma_{ijk}^{(n)}$, letting n tend to infinity and using (III.3.17). Let us check the anti-symmetry (III.2.10). Swapping the indices j and k in (III.3.16) we have

$$\mathrm{D} \cdot \nabla^* \left(\sigma_{ijk}^{(n)} + \sigma_{ikj}^{(n)} \right) = \mathrm{D} \cdot \Big[\underbrace{(q_{ik}\mathbf{e}_j - q_{ij}\mathbf{e}_k + q_{ij}\mathbf{e}_k - q_{ik}\mathbf{e}_j)}_{=0} \mathbf{1}_{|q|\le n} \Big] = 0.$$

By the decomposition (III.2.17), this trivially implies that

$$\nabla^* \left(\sigma_{ijk}^{(n)} + \sigma_{ikj}^{(n)} \right) \in L^2_{\nabla^*} \cap \left(L^2_{\nabla^*} \right)^{\perp} = \{0\}.$$

Therefore, letting n tend to infinity, using (III.3.17) and noting that a co-cycle field is defined to be zero at the origin, we obtain (III.2.10). Finally, we check (III.2.7). As a first step we claim that taking ∇_k^* on both sides of (III.3.18) yields

$$\mathrm{D} \cdot \mathrm{D}^* \nabla_k^* \sigma_{ijk} = \mathrm{D}_k^* \mathrm{D}_j q_{ik} - \mathrm{D}_k^* \mathrm{D}_k q_{ij}. \tag{III.3.19}$$

Let us check this carefully. Note that obviously we are allowed to switch differential operators of the same type and by stationarity we can also replace D^* by ∇^* or vice verse. The left-hand side becomes

$$\begin{aligned}
\nabla_k^* \mathrm{D} \cdot \nabla^* \sigma_{ijk} &= \mathrm{D}_k^* \mathrm{D} \cdot \nabla^* \sigma_{ijk} \quad \text{(stationarity)} \\
&= \mathrm{D} \cdot \mathrm{D}_k^* \nabla^* \sigma_{ijk} \\
&= \mathrm{D} \cdot \nabla_k^* \nabla^* \sigma_{ijk} \quad \text{(stationarity)} \\
&= \mathrm{D} \cdot \nabla^* \nabla_k^* \sigma_{ijk} \\
&= \mathrm{D} \cdot \mathrm{D}^* \nabla_k^* \sigma_{ijk} \quad \text{(stationarity)}
\end{aligned}$$

while the right-hand side becomes exactly the right-hand side of (III.3.19) by replacing ∇_k^* by D_k^*. Summing (III.3.19) over k and noting that $\mathrm{D}^* \cdot q_i = 0$ we get

$$\mathrm{D} \cdot \mathrm{D}^* u = 0 \tag{III.3.20}$$

where $u = \nabla^* \cdot \sigma_{ij} + q_{ij}$. *Heuristically*, testing this with u and using the partial integration yield

$$\langle |\mathrm{D}^* u|^2 \rangle = \langle u \mathrm{D} \cdot \mathrm{D}^* u \rangle = 0 \tag{III.3.21}$$

which implies that $\mathrm{D}^* u = 0$ for $\mathbb{P}$-a.e. ω. Therefore, u is translation invariant. By ergodicity, we have $u = \langle u \rangle = 0$, which is (III.2.7). However, we still have a minor difficulty, namely the integrability of u does not allow (III.3.21). Fortunately, there is a solution to this. Lift u to the physical space, i.e. define $\bar{u}(\omega, x) := u(\tau_x \omega)$. Then, (III.3.20) becomes

$$\Delta \bar{u}(\omega, x) = 0 \quad x \in \mathbb{Z}^d, \quad \mathbb{P}\text{-a.e.} \quad \omega.$$

By the maximal inequality for harmonic functions (Andres, Deuschel, and Slowik [3, Corollary 3.9], which is in fact a much stronger result than our purpose here – we need here only the most standard case: the discrete Laplacian without weights!) there exists a constant $\gamma(d, q) > 0$ such that

$$|\bar{u}(\omega, 0)| \lesssim_{d,q} \|\bar{u}(\omega, \cdot)\|^{\gamma}_{\overline{L}^{2q/(q+1)}(D_{2R})}.$$

By the ergodic theorem, letting R tend to infinity yields that $u = \bar{u}(\cdot, 0) \in L^\infty(\Omega)$. Therefore, we do not have to worry about (III.3.21). The proof is complete. $\square$

III.3.5 Proof of the sublinearity

We show that $u = \sigma_{ijk}$ is sublinear in the sense that

$$\tfrac{1}{R} \|u(\omega, \cdot)\|_{\overline{L}^{\frac{2q}{q-1}}(\overline{D}_R)} \to 0, \quad \text{as} \quad R \to \infty, \quad \mathbb{P}\text{-a.e. } \omega. \tag{III.3.22}$$

As a first step, instead of (III.3.22) let us prove that

$$Fu := \left\langle \varlimsup_{R \to \infty} \tfrac{1}{R} \inf_{c \in \mathbb{R}} \|u - c\|_{\overline{L}^{\frac{2q}{q-1}}(\overline{D}_R)} \right\rangle = 0. \tag{III.3.23}$$

Indeed, by Sobolev's inequality and the ergodic theorem, we have

$$Fu \lesssim_{d,p,q} \left\langle \varlimsup_{R \to \infty} \|\nabla u\|_{\overline{L}^{\frac{2p}{p+1}}(\overline{E}_R)} \right\rangle \sim_{d,p} \|\nabla u\|_{L^{\frac{2p}{p+1}}(\Omega, \mathbb{R}^d)}. \tag{III.3.24}$$

where we write $\nabla u = \nabla u(\omega, 0)$, when considering the stationary gradient at the level of the probability space. Recall that $u_n \to u$ in $L^{2p/(p+1)}(\Omega)$ where $u_n \in L^2_{\nabla^*}$ is defined by (III.3.15). Further, by (III.2.17) there exists $\zeta_n \in L^\infty(\Omega)$ such that

$$\|\nabla u_n - \mathrm{D}^* \zeta_n\|_{L^2(\Omega)} \to 0.$$

Therefore, by the triangle and Jensen inequality, we have

$$\mathrm{D}^*\zeta_n \to \nabla^* u \quad \text{in} \quad L^{\frac{2p}{p+1}}(\Omega, \mathbb{R}^d). \tag{III.3.25}$$

Set $u_n(\omega, x) = \zeta_n(\tau_x\omega) - \zeta_n(\omega)$. This is obviously a co-cycle field. Further, it satisfies

$$\nabla u_n(\omega, 0) = \mathrm{D}\zeta_n(\omega).$$

Combining this with (III.3.24) and (III.3.25) we have

$$F(u - u_n) \lesssim \|\nabla u - \nabla u_n\|_{L^{\frac{2p}{p+1}}(\Omega,\mathbb{R}^d)} = \|\nabla u - \mathrm{D}\zeta_n\|_{L^{\frac{2p}{p+1}}(\Omega,\mathbb{R}^d)} \to 0.$$

From the fact that

$$Fu_n \le \left\langle \varlimsup_{R\to\infty} \tfrac{1}{R}\|u_n\|_{L^{\frac{2q}{q-1}}(\overline{D}_R)} \right\rangle \lesssim \left\langle \varlimsup_{R\to\infty} \tfrac{1}{R}\|\zeta_n\|_{L^\infty(\Omega)} \right\rangle = 0$$

for each n we have (III.3.23).

In the remaining part of the proof we discuss how to get rid of the constant in (III.3.23). We introduce the following dyadic argument similar to the proof in the continuum [12]. Fix $\delta > 0$ and $R_0 > 0$ such that for all $R > R_0$ there exists $c_R > 0$ satisfying

$$\|u - c_R\|_{\overline{L}^s(\overline{D}_R)} < \delta R. \tag{III.3.26}$$

where we write $s = 2q/(q-1)$. Then, for $R_0 \le R \le R' \le 2R$, by the triangle inequality and (III.3.26) we have

$$\begin{aligned} |c_R - c_{R'}| &\le \|u - c_R\|_{\overline{L}^s(\overline{D}_R)} + \|u - c_{R'}\|_{\overline{L}^s(\overline{D}_R)} \\ &\lesssim_d \|u - c_R\|_{\overline{L}^s(\overline{D}_R)} + \|u - c_{R'}\|_{\overline{L}^s(\overline{D}_{R'})} \lesssim_d \delta R. \end{aligned} \tag{III.3.27}$$

Further, fix $R \in [2^n R_0, 2^{n+1}R_0]$ and define

$$R_j := 2^j R_0, j \le n, \quad R_{n+1} := R. \tag{III.3.28}$$

By the triangle inequality, (III.3.27) and (III.3.28), we have

$$|c_R - c_{R_0}| \;\le\; \sum_{j=0}^{n} |c_{R_{j+1}} - c_{R_j}| \;\lesssim_d\; \sum_{j=0}^{n} \delta 2^j R_0 \;\lesssim_d\; \delta R. \tag{III.3.29}$$

Therefore, by the triangle inequality and (III.3.26) and (III.3.29) we have

$$\tfrac{1}{R}\|u\|_{\overline{L}^s(\overline{D}_R)} \;\le\; \underbrace{\tfrac{1}{R}\|u - c_R\|_{\overline{L}^s(\overline{D}_R)}}_{\lesssim\delta} + \underbrace{\tfrac{1}{R}|c_R - c_{R_0}|}_{\lesssim\delta} + \tfrac{1}{R}|c_{R_0}| \;\lesssim_d\; \delta + \tfrac{1}{R}|c_{R_0}|.$$

Letting R tend to infinity and δ to zero we finish the proof of the sublinearity of σ. That for ϕ also exploits a density argument: use (III.3.10) and (III.3.12) and application of Hölder's inequality similar to (III.3.13) we have (III.3.25) for $u = \phi_i$ and p replaced by q where ζ_n is some approximating sequence in $L^{2q/(q+1)}(\Omega)$, and we only need to repeat the proof for the part σ.

III.3.6 Finishing the Meyer-type estimate

In the following, we give the proof for Lemma III.3.4 that is almost the same as in the continuum case [12]. The only difficulty is that we have to prepare estimates on Green function in the discrete setting (III.3.32) and (III.3.33).

Before going into the main part of the proof let us change the notation a bit. Define the following random vector

$$\widetilde{f}(\omega) = -\left[f(\tau_{-\mathbf{e}_1}\omega), \ldots, f(\tau_{-\mathbf{e}_d}\omega)\right]$$

which has the same integrability as f and satisfies $\mathrm{D} \cdot \widetilde{f} = \mathrm{D}^* \cdot f$. Further, note that if u is co-cycle, then

$$\mathrm{D} \cdot \nabla^* u = \nabla \cdot \nabla^* u = \nabla^* \cdot \nabla u = \mathrm{D}^* \cdot \nabla u$$

where we obtained the first and third equality by replacing the x-derivative by the ω-derivative, which is allowed by stationarity of ∇u and $\nabla^* u$. Also notice that ∇u and $\nabla^* u$ have the same integrability. Therefore, Lemma III.3.4 can be formulated in a slightly different way:

Lemma III.3.5. *Let $f \in L^\infty(\Omega, \mathbb{R}^d)$ and u be a co-cycle field satisfying $\nabla u \in L^2(\Omega, \mathbb{R}^d)$ and $\mathrm{D}^* \cdot \nabla u = \mathrm{D}^* \cdot f$. Then for any $r \in (1, 2]$,*

$$\|\nabla u\|_{L^r(\Omega,\mathbb{R}^d)} \lesssim_{r,d} \|f\|_{L^r(\Omega,\mathbb{R}^d)}.$$

The first ingredient of the proof is the following approximation:

Claim. *For each $\varepsilon > 0$ there exists a unique stationary field $u_\varepsilon \in L^2(\Omega)$ such that*

$$(\varepsilon + \mathrm{D}^* \cdot \mathrm{D})\, u_\varepsilon = \mathrm{D}^* \cdot f. \tag{III.3.30}$$

Further, as ε tends to zero,

$$\mathrm{D}u_\varepsilon \to \nabla u, \quad \textit{weakly in } L^2(\Omega, \mathbb{R}^d). \tag{III.3.31}$$

The second ingredient is the following Green function estimate:

Claim. *The Green function $G_\varepsilon(x, y) =: G_\varepsilon(x - y)$ of $(\varepsilon - \Delta)$ on $\mathbb{Z}^d$ satisfies*

$$|\nabla_x \nabla_y G_\varepsilon(x, y)| \lesssim_d (1 \vee |x|)^{-d} e^{-c\varepsilon|x-y|}, \tag{III.3.32}$$

$$\nabla^3 G_\varepsilon(x) \lesssim_d (1 \vee |x|)^{-d-1}. \tag{III.3.33}$$

Estimate (III.3.33) mainly states that the kernel $K(x, y) := \nabla_x \nabla_y G_\varepsilon(x, y)$ is a Calderon–Zymund kernel in the sense that the associated linear operator

$$Tg(x) := \sum_{y \in \mathbb{Z}^d} K(x, y) g(y)$$

is a bounded linear operator from $L^s(\mathbb{Z}^d)$ to $L^s(\mathbb{Z}^d)$ for any $s \in (1, \infty)$. This result is well-known in the continuum setting (see e.g. Stein [60, p29]). For an argument in the discrete case using a decomposition with triadic boxes see e.g. Biskup, Salvi, Wolff [18].

Argument for (III.3.32). See Mourrat [51, Proposition 3.7]. □

Argument for (III.3.33). See Biskup, Salvi, and Wolff [18, Theorem 4.4]. □

Proof of (III.3.30) *and* (III.3.31). It is a standard argument, and was perhaps written by Künnemann [43]. For convenience and completeness, let us repeat it. Equip the space $L^2(\Omega)$ with the norm $\|u\|_\varepsilon^2 = \varepsilon\langle u^2\rangle + \langle|\mathrm{D}u|^2\rangle$ for $u \in L^2(\Omega)$. Denote by $(\cdot,\cdot)_\varepsilon$ the corresponding scalar product. Then, by the Riesz lemma, there exists a unique $u_\varepsilon \in L^2(\Omega)$ such that

$$(u,v)_\varepsilon := \varepsilon\langle u_\varepsilon v\rangle + \langle \mathrm{D}u_\varepsilon \cdot \mathrm{D}v\rangle = \langle f\cdot \mathrm{D}v\rangle, \quad \forall v \in L^2(\Omega). \tag{III.3.34}$$

By partial integration,

$$\varepsilon\langle u_\varepsilon v\rangle + \langle(\mathrm{D}^*\cdot \mathrm{D}u_\varepsilon)v\rangle = \langle(\mathrm{D}^*\cdot f)v\rangle, \quad \forall v \in L^2(\Omega).$$

This implies (III.3.30). Further, by the Cauchy–Schwarz inequality, (III.3.34) with $v = u_\varepsilon$ implies that

$$\|u_\varepsilon\|_\varepsilon^2 = \langle f\cdot \mathrm{D}u_\varepsilon\rangle \le \|f\|_{L^2(\Omega)}\|\mathrm{D}u_\varepsilon\|_{L^2(\Omega)} \le \|f\|_{L^2(\Omega)}\|u_\varepsilon\|_\varepsilon.$$

It follows that

$$\|u_\varepsilon\|_\varepsilon^2 = \varepsilon\|u_\varepsilon\|_{L^2(\Omega)}^2 + \|\mathrm{D}u_\varepsilon\|_{L^2(\Omega)}^2 \le \|f\|_{L^2(\Omega)}^2. \tag{III.3.35}$$

Especially, $\mathrm{D}u_\varepsilon$ converges weakly to some $\xi \in L^2(\Omega,\mathbb{R}^d)$ as ε tends to zero. Noting that $\varepsilon\langle u_\varepsilon v\rangle \le \varepsilon\|u_\varepsilon\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)} \to 0$ by (III.3.35), we have by (III.3.34) with ε tending to zero that $\langle\xi\cdot \mathrm{D}v\rangle = \langle f\cdot \mathrm{D}v\rangle$. In other words, $\xi = \nabla u$ and we have (III.3.31). □

With (III.3.30)–(III.3.33) we are in a position to show Lemma III.3.4.

Proof of Lemma III.3.5. By stationary assumption we can replace D by ∇. Therefore, (III.3.30) implies

$$(\varepsilon-\Delta)u_\varepsilon = \nabla^*\cdot f.$$

Recall that $G_\varepsilon(x,y)$ denotes the Green function of $(\varepsilon-\Delta)$. We have

$$u_\varepsilon = \sum_{y\in\mathbb{Z}^d} G_\varepsilon(x,y)\nabla^*\cdot f(y) = \sum_{y\in\mathbb{Z}^d}\nabla_y G_\varepsilon(x,y)\cdot f(y)$$

and therefore

$$\begin{aligned}\nabla u_\varepsilon(x) &= \sum_{y\in\mathbb{Z}^d}\nabla_x\nabla_y G_\varepsilon(x,y)\cdot f(y)\\ &= \underbrace{\sum_{y\in\mathbb{Z}^d}\nabla_x\nabla_y G_\varepsilon(x,y)\cdot\eta_R f(y)}_{=:g_\varepsilon(x)} + \sum_{y\in\mathbb{Z}^d}\nabla_x\nabla_y G_\varepsilon(x,y)\cdot(1-\eta_R)f(y)\end{aligned}$$

where η_R is a cut-off function so that $\eta_R = 1$ in D_{2R} and $\eta_R = 0$ in $\mathbb{Z}^d \setminus D_{4R}$. By (III.3.32) the second derivative of the Green function $\nabla_x \nabla_y G_\varepsilon(x, y)$ is a Calderon-Zymund singular integral operator. Therefore,

$$\|g_\varepsilon\|_{L^r(\mathbb{Z}^d)} \lesssim_{d,r} \|\eta_R f\|_{L^r(\mathbb{Z}^d)} \leq \|f\|_{L^r(D_{4R})}$$

in the same fashion as in Biskup, Salvi, and Wolff [18]. Then, by (III.3.32) we have

$$|\nabla u_\varepsilon\|^r_{L^r(D_R)} \lesssim_{d,r} \|f\|^r_{L^r(D_{4R})} + R^{-d} e^{-c\varepsilon R} \|f\|^r_\infty. \tag{III.3.36}$$

Letting R tend to infinity we have

$$\|\nabla u_\varepsilon\|_{L^r(\Omega,\mathbb{R}^d)} \lesssim_{d,r} \|f\|_{L^r(\Omega,\mathbb{R}^d)} \tag{III.3.37}$$

by the ergodic theorem. Fix $G \in L^\infty(\Omega, \mathbb{R}^d)$ with $\|G\|_{L^{r'}(\Omega,\mathbb{R}^d)} = 1$ where r' is the Hölder conjugate of r. By the weak convergence (III.3.31) and (III.3.37) we have

$$\langle \nabla u \cdot G \rangle = \lim_{\varepsilon \to 0} \langle \nabla u_\varepsilon \cdot G \rangle \leq \varlimsup_{\varepsilon \to 0} \|\nabla u_\varepsilon\|_{L^r(\Omega,\mathbb{R}^d)} \lesssim_{d,r} \|f\|_{L^r(\Omega,\mathbb{R}^d)}.$$

Taking the supremum over G yields the claim. The proof is complete. □

III.4 Defining the discrete smoothed version

The construction of the discrete function u_ε in Lemma III.2.4 can be summarized as follows. First, interpolate u to get a continuum function $\widetilde{u}$ defined on ∂C_R (see Subsection III.4.1). Then, smooth $\widetilde{u}$ on the Lipschitz surface ∂C_R to get $\widetilde{u}_\varepsilon$ (making use of Proposition III.4.5 in Subsection III.4.2 by rescaling everything to ∂C_1). Finally, borrowing the idea of Scott and Zhang [58] we define u_ε as a projection of $\widetilde{u}_\varepsilon$ to the space of discrete functions embedded to the continuum (see Subsection III.4.3).

In this section the multiplicative constants in each estimate only depend on the dimension d and the exponents there. Especially, they do not depend on the size of the box.

III.4.1 Embedding discrete functions to the continuum setting

There are several ways to interpolate discrete functions on $\mathbb{Z}^d$ to get continuum functions on $\mathbb{R}^d$. A possibility coming from finite element theory is to decompose the given domain into simplexes and and interpolate discrete functions to get functions which are continuous and piecewise affine linear, i.e. affine linear on each simplex. However, when choosing this interpolation, we have to define a triangulation and piecewise linear functions, which takes time, and makes the dissertation much longer. For this reason, we choose that given by Deuschel, Giacomin, and

Ioffe [26]. We extend a discrete function on $\mathbb{Z}^d$ to a continuum function on $\mathbb{R}^d$ by an explicit formula:

$$u(x) = \sum_{a\in\{0,1\}^d} \left[\prod_{i=1}^{d} (a_i\{x_i\} + (1-a_i)(1-\{x_i\}))\right] u([x]+a), \quad x \in \mathbb{R}^d \tag{III.4.1}$$

where *we abuse the notation denoting the extension also by* u. Then, $u \in \mathcal{C}(\mathbb{R}^d)$.

Lemma III.4.1. *Let* $u : \overline{D}_R \to \mathbb{R}$ *extended to* $\overline{C}_R$ *by* (III.4.1). *Then, for any* $p \in [1,\infty]$,

$$\|u\|_{L^p(\overline{D}_R)} \sim_{p,d} \|u\|_{L^p(\overline{C}_R)}, \tag{III.4.2}$$

$$\|\nabla u\|_{L^p(\overline{E}_R)} \sim_{p,d} \|\nabla u\|_{L^p(\overline{C}_R)}. \tag{III.4.3}$$

This idea can be applied to functions defined on the boundary. In this case, a function u defined on ∂D_R can be extended trivially to $\mathbb{Z}^d$ by defining it to be zero everywhere outside its original definition set. This allows us to use (III.4.1). Notice that the interpolation on ∂C_R does not depend on the way we choose to extend u from ∂D_R to $\mathbb{Z}^d$. Indeed, for the reference box $[0,1]^d$, formula (III.4.1) becomes

$$u(x) = \sum_{a\in\{0,1\}^d} \left[\prod_{i=1}^{d} (a_i x_i + (1-a_i)(1-x_i))\right] u(a), \quad x \in [0,1]^d. \tag{III.4.4}$$

Restricted to a face of $[0,1]^d$, for instance, $[0,1]^{d-1}\times\{0\}$, it becomes

$$u(x) = \sum_{a\in\{0,1\}^{d-1}\times\{0\}} \left[\prod_{i=1}^{d-1} (a_i x_i + (1-a_i)(1-x_i))\right] u(a), \quad x \in [0,1]^{d-1}\times\{0\} \tag{III.4.5}$$

that depends only on the values of u at $\{0,1\}^{d-1}\times\{0\}$. In other words, to define the interpolation on the boundary we only need to know the values at vertices on the boundary. Our method in this subsection applies to any type of interpolations with this property.

Lemma III.4.2. *Let* $u : \partial D_R \to \mathbb{R}$ *extended to* ∂C_R *by* (III.4.1). *Then, for* $p \in [1,\infty]$,

$$\|u\|_{L^p(\partial D_R)} \sim_{p,d} \|u\|_{L^p(\partial C_R)}, \tag{III.4.6}$$

$$\|\nabla u\|_{L^p(E_R^{\tan})} \sim_{p,d} \|\nabla^{\tan} u\|_{L^p(\partial C_R)}. \tag{III.4.7}$$

For completeness let us finish the proofs before going ahead.

Proof of Lemmas III.4.1 and III.4.2. We only give the proof of Lemma III.4.1. That of the other is similar. Tile $\overline{C}_R$ by boxes of the form $B := a + [0,1]^d \subset \overline{C}_R$ where $a \in \mathbb{Z}^d$. Fix B and denote $V = a + \{0,1\}^d \subset \overline{D}_R$. Since all norms there are equivalent on the space of discrete functions defined on V extended to B via (III.4.1), which is of finite dimension, we have

$$\sum_{x\in V}|u(x)|^p \sim_{d,p} \int_B |u|^p.$$

Summing over B contained in $\overline{C}_R$ yields (III.4.2). Further, applying the triangle inequality and again the equivalence of norms we get

$$\sum_{x,y\in V, x\sim y}|u(x)-u(y)|^p \lesssim_{p,d} \sum_{x\in V}|u(x)|^p \sim_{p,d} \int_B |u|^p + |\nabla u|^p.$$

Replacing u by $u - \fint_B u$ and using Poincaré 's inequality we get rid of $\int_B |u|^p$ on the right-hand side Then, summing over B yields "$\lesssim$" of (III.4.3). To see "$\gtrsim$", use again equivalent norms:

$$\int_B |u|^p + |\nabla u|^p \sim_{p,d} \sum_{x,y\in V, x\sim y}|u(x)-u(y)|^p + |u(x_0)|^p$$

where x_0 is arbitrarily chosen in V. Then, replacing u by $u - u(x_0)$ and summing over B contained in $\overline{C}_R$ yield the claim. Finally, to get the proof for $p=\infty$, replace the sum or integral by the supremum. □

The following inequalities are standard results (perhaps folklore) and we just state them again to make the dissertation self-contained.

Lemma III.4.3 (Poincaré's inequality)**.** *Let $p\in[1,\infty]$, u be a discrete function extended to $\mathbb{R}^d$ by* (III.4.1) *and $R\geq 1$. Then,*

$$\inf_{c\in\mathbb{R}}\|u-c\|_{L^p(\overline{D}_R)} \lesssim_{p,d} R\|\nabla u\|_{L^p(\overline{E}_R)} \tag{III.4.8}$$

$$\inf_{c\in\mathbb{R}}\|u-c\|_{L^p(\partial D_R)} \lesssim_{p,d} R\|\nabla u\|_{L^p(E_R^{\tan})} \tag{III.4.9}$$

Proof. The claim follows by Lemmas III.4.1 and III.4.2 and the Poincaré inequality in the continuum setting for a cube or for the surface of a cube. For more details, see Theorem III.B.6. □

Denote p_d^* is the Sobolev conjugate of p w.r.t the dimension, meaning $\frac{1}{p^*}=\frac{1}{p}-\frac{1}{d}$.

Lemma III.4.4 (Sobolev's inequality)**.** *Let $p\in[1,\infty)$ and u be defined on $\mathbb{Z}^d$. Then,*

$$\inf_{c\in\mathbb{R}}\|u-c\|_{\overline{L}^q(\overline{D}_R)} \lesssim_{p,q,d} R\|\nabla u\|_{\overline{L}^p(\overline{E}_R)}, \quad q\in[1,p_d^*],$$

$$\inf_{c\in\mathbb{R}}\|u-c\|_{\overline{L}^q(\partial D_R)} \lesssim_{p,q,d} R\|\nabla u\|_{\overline{L}^p(E_R^{\tan})}, \quad q\in[1,p_{d-1}^*].$$

Proof. By Lemmas III.4.1 and III.4.2 the claim follows by the Sobolev inequality in the continuum Euclidean case or in the case of Lipschitz surfaces. For more details, see Theorem III.B.7. □

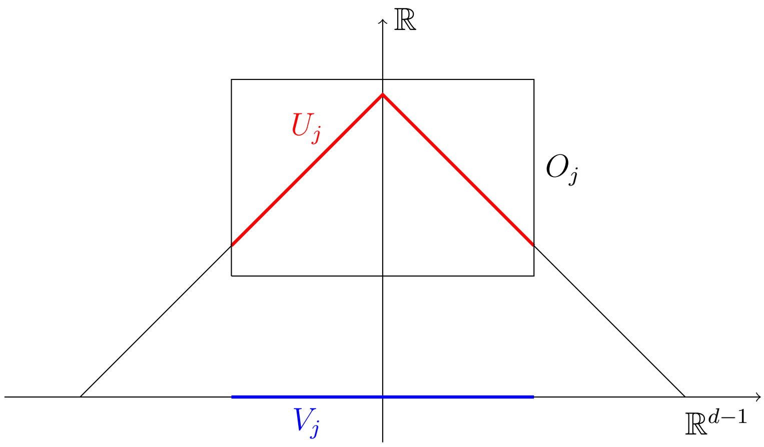

Figure III.2: A chart of M

III.4.2 Smoothing functions on Lipschitz surfaces

We think of a Lipschitz surface M as an object which can be locally (up to an isometry) represented as a graph of a Lipschitz function and on which we can define Lebesgue and Sobolev spaces $L^r(M)$ and $W^{1,r}(M)$. For more details see Section III.B. Note that by [36, Corollary 1.2.2.3, p12] the boundary of every convex subset of $\mathbb{R}^d$ (e.g. ∂C_1) is a Lipschitz surface.

Proposition III.4.5. *Let M be a compact Lipschitz surface. For $0 < \varepsilon \ll 1$ there exists a linear operator which maps $u : M \to \mathbb{R}$ to $u_\varepsilon : M \to \mathbb{R}$ and satisfies the following properties. If $u \in L^s(M)$, $s \in [1, \infty]$ then $u_\varepsilon \in L^\infty(M)$ and we have*

$$\|u_\varepsilon\|_{L^r(M)} \lesssim \varepsilon^{-(d-1)(\frac{1}{s}-\frac{1}{r})} \|u\|_{L^s(M)}, \quad r \in [s, \infty]. \tag{III.4.10a}$$

Further, if $u \in W^{1,s}(M)$, $s \in [1, \infty]$ then

$$\|u - u_\varepsilon\|_{L^s(M)} \lesssim \varepsilon \|\nabla^{\tan} u\|_{L^s(M)}, \tag{III.4.10b}$$

and in this case, $u_\varepsilon \in W^{1,\infty}(M)$ satisfies

$$\|\nabla^{\tan} u_\varepsilon\|_{L^r(M)} \lesssim \varepsilon^{-(d-1)(\frac{1}{s}-\frac{1}{r})} \|\nabla^{\tan} u\|_{L^s(M)}, \quad r \in [s, \infty]. \tag{III.4.10c}$$

Here, the multiplicative constants only depend on the structure of M, the exponents s, r and the dimension d.

The proof of Proposition III.4.5 requires some basic knowledges about Sobolev spaces on Lipschitz surfaces in Section III.B. First, compactness ensures the existence of finitely many charts $\{O_j, U_j, V_j, \varphi_j\}_{j=1}^N$ covering M. More precisely, there exist finitely many bounded open subsets O_j, $1 \le j \le N$ of $\mathbb{R}^d$ covering M, bounded open subsets V_j of $\mathbb{R}^{d-1}$ and Lipschitz continuous functions $\varphi_j : V_j \to \mathbb{R}$, $1 \le j \le N$ such that

$$U_j := O_j \cap M = \{(x', \varphi_j(x')) : x' \in V_j\}$$

(Subsection III.4.2). Second, the Lebesgue space $L^p(M)$ and the Sobolev space $W^{1,p}(M)$ for every $p \in [1,\infty]$ are defined by pulling things back to the Euclidean case (see Section III.B) so that equivalence of norms holds true on each chart $\{O_j, U_j, V_j, \varphi_j\}$:

$$\|f\|_{L^p(U_j)} \sim \|f(\cdot,\varphi_j(\cdot))\|_{L^p(V_j)} \quad \text{and} \quad \|\nabla^{\tan} f\|_{L^p(U_j)} \sim \|\nabla f(\cdot,\varphi_j(\cdot))\|_{L^p(V_j)} \tag{III.4.11}$$

(see Proposition III.B.3) where the multiplicative constants only depend on the structure of M. Third, there exists a Lipschitz partition of unity $\{\alpha_j\}_{j=1}^N$ subordinate to $\{U_j\}_{j=1}^N$ in the sense that all $\alpha_j : U_j \to \mathbb{R}$ are Lipschitz continuous and supported in a compact subset of U_j and $\sum_j \alpha_j = 1$ on M (for more details, see Proposition III.B.8). Then, $\alpha_j \in W^{1,\infty}(M)$ (see [29, Theorem 4, p279]). With the above ingredients in hand we are in a position to prove Proposition III.4.5.

Proof of Proposition III.4.5. If u is constant, we just need to define $u_\varepsilon = u$. Then, it suffices to define u_ε for u satisfying $\int_M u dS = 0$. With help of the partition of unity $\{\alpha_j\}_{j=1}^N$ we can decompose $u \in W^{1,p}(M)$ into $u_j := \alpha_j u$. Set

$$\varepsilon_0 := \tfrac{1}{2} \min_{1\le j\le N} \operatorname{dist}(\partial V_j, K_j) > 0 \quad \text{where} \quad K_j = \operatorname{supp}(\alpha_j(\cdot,\varphi_j(\cdot))) \subset\subset V_j.$$

For $0 < \varepsilon < \varepsilon_0$ and $1 \le j \le N$ define $v_j^\varepsilon := v_j * \eta_\varepsilon \in \mathcal{C}_c^\infty(V_j)$ where $v_j := u_j(\cdot,\varphi_j(\cdot))$ satisfies $\operatorname{supp}(v_j) \subset\subset V_j$ and η_ε is the standard sequence of mollifiers approximating the unity in $\mathbb{R}^{d-1}$. Define

$$u_j^\varepsilon(x',x_d) = v_j^\varepsilon(x')\theta(x_d - \varphi_j(x')) \quad \text{where} \quad \theta \in \mathcal{C}_c^\infty(-\tfrac{1}{2}\varepsilon_0, \tfrac{1}{2}\varepsilon_0) \quad \text{and} \quad \theta(0) = 1.$$

Then, we have

$$\operatorname{supp}(u_j^\varepsilon \restriction_M) \subset\subset O_j \quad \text{and} \quad u_j^\varepsilon(\cdot,\varphi_j(\cdot)) = v_j^\varepsilon.$$

Finally, set $u_\varepsilon = \sum_{j=1}^N u_j^\varepsilon$. We are going to show that the mapping $u \mapsto u_\varepsilon$ is linear and satisfies (III.4.10a)–(III.4.10c).

Linearity of $(\cdot)_\varepsilon$ follows from that of $(\cdot)_j^\varepsilon$ which is obvious by definition, since both u_j and u_j^ε are supported in U_j.

Now comes the argument for (III.4.10a). By the triangle inequality and standard estimates for convolution in the Euclidean case, we have

$$\begin{aligned}\|u_\varepsilon\|_{L^r(M)} &\le \sum_{j=1}^N \|u_j^\varepsilon\|_{L^r(U_j)} \overset{\text{(III.4.11)}}{\lesssim} \sum_{j=1}^N \|v_j^\varepsilon\|_{L^r(V_j)} \lesssim \sum_{j=1}^N \varepsilon^{-(d-1)(\frac{1}{s}-\frac{1}{r})}\|v_j\|_{L^s(V_j)} \\ &\overset{\text{(III.4.11)}}{\lesssim} \sum_{j=1}^N \varepsilon^{-(d-1)(\frac{1}{s}-\frac{1}{r})}\|u_j\|_{L^s(U_j)} \overset{|\alpha_j|\le 1}{\le} N\varepsilon^{-(d-1)(\frac{1}{s}-\frac{1}{r})}\|u\|_{L^s(M)}.\end{aligned}$$

Here and in the remaining part of the proof note that all the multiplicative constants, the number N, and the function α_j depend only on the structure of M, and for our later purpose M is in fact the unit box.

Concerning (III.4.10b), we have

$$\|u-u_\varepsilon\|_{L^s(M)} \le \sum_{j=1}^N \|u_j-u_j^\varepsilon\|_{L^s(U_j)} \overset{\text{(III.4.11)}}{\lesssim} \sum_{j=1}^N \|v_j-v_j^\varepsilon\|_{L^s(V_j)} \lesssim \sum_{j=1}^N \varepsilon\|\nabla v_j\|_{L^s(V_j)}$$

$$\overset{\text{(III.4.11)}}{\lesssim} \sum_{j=1}^N \varepsilon\|\nabla^{\tan} u_j\|_{L^s(U_j)} \lesssim N\varepsilon\left[\max_{1\le j\le N}\|\alpha_j\|_{W^{1,\infty}(M)}\right]\|u\|_{W^{1,s}(M)}$$

where the last inequality follows from $\nabla^{\tan}u_j = \alpha_j\nabla^{\tan}u + u\nabla^{\tan}\alpha_j$ (for more details see Lemma III.B.4 (Product rule) and (III.B.4)). By Theorem III.B.6 (Poincaré's inequality) this implies (III.4.10b).

Finally, (III.4.10c) is proven in the same way:

$$\|\nabla^{\tan}u_\varepsilon\|_{L^r(M)} \le \sum_{j=1}^N \|\nabla^{\tan}u_j^\varepsilon\|_{L^r(U_j)} \overset{\text{(III.4.11)}}{\lesssim} \sum_{j=1}^N \|\nabla v_j^\varepsilon\|_{L^r(V_j)}$$

$$\lesssim \sum_{j=1}^N \varepsilon^{-(d-1)(\frac{1}{s}-\frac{1}{r})}\|\nabla v_j\|_{L^s(V_j)} \overset{\text{(III.4.11)}}{\lesssim} \sum_{j=1}^N \varepsilon^{-(d-1)(\frac{1}{s}-\frac{1}{r})}\|\nabla^{\tan}u_j\|_{L^s(U_j)}$$

$$\le N\varepsilon^{-(d-1)(\frac{1}{s}-\frac{1}{r})}\left[\max_{1\le j\le N}\|\alpha_j\|_{W^{1,\infty}(M)}\right]\|u\|_{W^{1,s}(M)},$$

which implies (III.4.10c) by Poincaré's inequality. The proof is complete. □

III.4.3 From continuum to discrete

Tile $\overline{C}_R$ by the family $\mathcal{T}_R$ of closed unit boxes $B_a := a+[0,1]^d \subset \overline{C}_R$ where $a\in\mathbb{Z}^d$. Then, ∂C_R is covered by the family $\mathcal{S}_R$ of $(d-1)$-dimensional boxes being faces of boxes in $\mathcal{T}_R$. Pick $x\in\overline{D}_R$ and choose a box Γ_x from $\mathcal{T}_R$ if $x\in D_R$ or from $\mathcal{S}_R$ if $x\in\partial D_R$ such that x is a vertex of Γ_x, i.e. $x\in\Gamma_x\cap\mathbb{Z}^d$. So, for each x there are several possibilities to choose Γ_x (some illustrated in Figure III.3 in the case $x\in\partial D_R$) and we do not have any other restrictions. However, we will see that the multiplicative constants in all the estimates we derive in this subsection can be chosen independently from the way we choose Γ_x. Let φ_x be the Dirac function $\delta_x(y) := \mathbf{1}_{x=y}$ defined for $y\in\mathbb{Z}^d$ and extended to $\mathbb{R}^d$ by the interpolation given in (III.4.1). In numerical analysis, each φ_x is called a nodal function. Let $\mathcal{H}_x$ be the finite-dimensional space of discrete functions defined on the vertices of Γ_x and extended to Γ_x by (III.4.1). Since $\{\varphi_z\restriction_{\Gamma_x}: z\in\Gamma_x\cap\mathbb{Z}^d\}$ is a basis of $\mathcal{H}_x$ there exists uniquely $\psi_x\in\mathcal{H}_x$ satisfying $\int_{\Gamma_x}\psi_x\varphi_z=\delta_{xz}$ for $z\in\sigma_x\cap\mathbb{Z}^d$ and $\|\psi_x(\cdot)\|_\infty\lesssim_d 1$. By

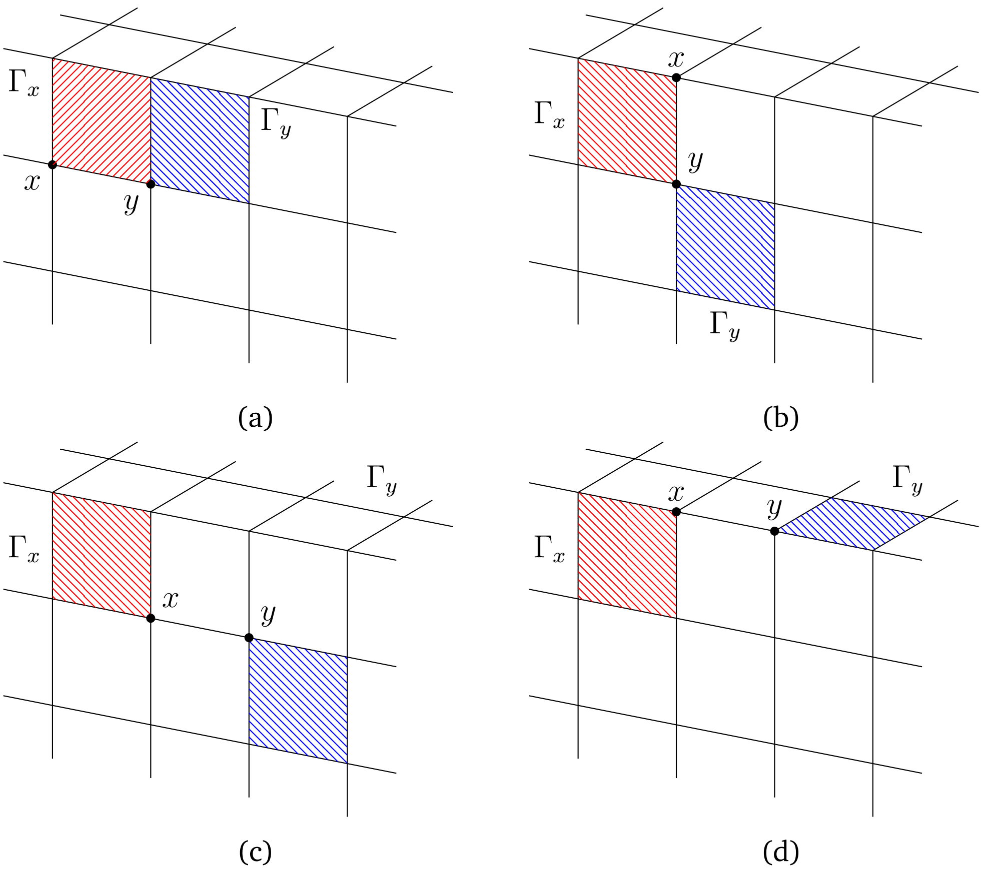

Figure III.3: Some examples for Γ_x on the surface of the box in $d = 3$

the fact that $\varphi_z \restriction_{\Gamma_x} \equiv 0$ for $z \in \mathbb{Z}^d \cap (\Gamma_x)^c$ we have

$$\int_{\Gamma_x} \psi_x \varphi_z = \delta_{xz}, \quad z \in \overline{D}_R \tag{III.4.12}$$

for all $x \in \overline{D}_R$. Finally, for $f \in W^{1,1}(\overline{C}_R)$ define

$$\Pi f(x) := \int_{\Gamma_x} \psi_x f, \quad x \in \overline{D}_R. \tag{III.4.13}$$

This operator behaves like a projection i.e. it maps discrete functions to discrete functions and does not increase the L^p norm of the function and its gradients. Further, the discrete boundary condition remains under this projection. A similar projection is introduced by Scott and Zhang [58] working with piecewise linear functions on some triangulation. For our purpose we only use Lemma III.4.7 and (III.4.14) and (III.4.15) of Lemma III.4.8 which are estimates for surfaces.

Lemma III.4.6. *Let $f : \overline{D}_R \to \mathbb{R}$. Extend f to a continuum function defined on $\mathbb{R}^d$ still denoted by f by the interpolation* (III.4.1). *Then, $\Pi f = f$ on $\overline{D}_R$.*

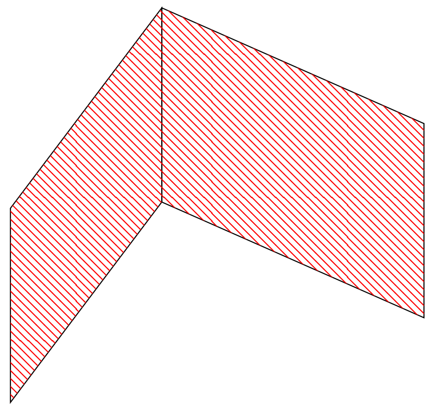

Figure III.4: An abstract roof is the union of two neighouring faces of the unit box

Proof of Lemma III.4.6. Write $f = \sum_{y \in \overline{D}_R} f(y)\varphi_y$. Then,

$$\Pi f(x) \overset{\text{(III.4.13)}}{=} \sum_{y \in \overline{D}_R} f(y) \int_{\Gamma_x} \psi_x \varphi_y \overset{\text{(III.4.12)}}{=} \sum_{y \in \overline{D}_R} f(y)\delta_{xy} = f(x).$$

The proof is complete. □

Lemma III.4.7. *Assume that $f \restriction_{\partial C_R}$ coincides with the interpolation of* (III.4.1) *of a discrete function on ∂D_R. Then, $\Pi f = f$ on ∂D_R.*

Proof. The same as the proof of Lemma III.4.6 except that we have to consider the sum over $x \in \partial D_R$. □

In the following (III.4.15) and (III.4.16), only for consistency, the gradient is understood as functions depending on unoriented edges, which means $|\nabla u|(\{x, y\}) = |u(y) - u(x)|$.

Lemma III.4.8. *Let $p \in [1, \infty)$. For $f \in W^{1,p}(\partial C_R)$, we have*

$$\|\Pi f\|_{L^p(\partial D_R)} \lesssim_{p,d} \|f\|_{L^p(\partial C_R)}, \tag{III.4.14}$$

$$\|\nabla \Pi f\|_{L^p(E_R^{\tan})} \lesssim_{p,d} \|\nabla^{\tan} f\|_{L^p(\partial C_R)}. \tag{III.4.15}$$

For $f \in W^{1,p}(\overline{C}_R)$, we have

$$\|\nabla \Pi f\|_{L^p(\overline{E}_R)} \lesssim_{p,d} \|\nabla f\|_{L^p(\overline{C}_R)}. \tag{III.4.16}$$

Remark. Although we do not use it in the dissertation, estimate (III.4.16) is useful for proving the discrete analogue of the following result in the continuum setting which plays a crucial role in the proof of the excess decay in the uniformly elliptic case i.e. $\lambda \leq \mathbf{a} \leq 1$ for fixed $\lambda > 0$. Let $u : \partial C_R \to \mathbb{R}$. Then, u can be extended to $\widetilde{u} : \overline{C}_R \to \mathbb{R}$ such that

$$\|\nabla \widetilde{u}\|_{L^{2d/(d-1)}} \lesssim_d \|\nabla^{\tan} u\|_{L^2(\partial C_R)}. \tag{III.4.17}$$

Indeed, the discrete analogue of this result can be obtained as follows. Let u now be a discrete function on ∂D_R. Interpolate u to get a continuum function still denoted by u and call $\widetilde{u}$ a continuum extension satisfying (III.4.17). Then,

$$\begin{aligned}\|\nabla\Pi\widetilde{u}\|_{L^{2d/(d-1)}(\overline{E}_R)} &\overset{\text{(III.4.16)}}{\lesssim_d} \|\nabla\widetilde{u}\|_{L^{2d/(d-1)}(C_R)} \\ &\overset{\text{(III.4.17)}}{\lesssim_d} \|\nabla^{\tan}u\|_{L^2(\partial C_R)} \overset{\text{(III.4.7)}}{\sim_d} \|\nabla^{\tan}u\|_{L^2(E_R^{\tan})}.\end{aligned}$$

By Lemma III.4.7 we have $\Pi\widetilde{u} = u$ on ∂D_R.

Proof of Lemma III.4.8. Since the upper bound of ψ_x is controlled, we have

$$|\Pi f(x)|^p \lesssim_p \int_{\Gamma_x} |f|^p.$$

by Jensen's inequality. Summing over all x yields (III.4.14). We turn to (III.4.15). First, we have

$$\Pi f(x) - \Pi f(y) = \int_{\Gamma_x} \psi_x(f-c) - \int_{\Gamma_y} \psi_y(f-c) \tag{III.4.18}$$

for all $c \in \mathbb{R}$, since $\int_{\Gamma_x} \psi_x = 1$ by combining (III.4.13) with Lemma III.4.6 for $f \equiv 1$. By Jensen's and the boundedness of ψ_x, we have

$$|\Pi f(x) - \Pi f(y)|^p \lesssim_{p,d} \inf_{c\in\mathbb{R}} \int_{\Gamma_x\cup\Gamma_y} |f-c|^p. \tag{III.4.19}$$

We want to bound the right-hand side using Poincaré's inequality and sum over all edges $\{x,y\}$. Let us give an illustrated proof for the case $d=3$. If x,y are as in Figure III.3a we only need the Poincaré's inequality for a rectangle. However, it is not always the case, since Figures III.3b–III.3c can happen. To deal with Figures III.3b and III.3c we just need to choose a larger square M_{xy} of side length 4 to cover Γ_x and Γ_y. The worst case happens in Figure III.3d where Γ_x and Γ_y are not on the same face. However, we can cover them by a "roof" M_{xy} of side 4 (see Figure III.4). Rigorously, for $\{x,y\} \in E_{R'}^{\tan}$ define M_{xy} as the intersection of the face of ∂C_R containing x, the face of ∂C_R containing y and the d-dimensional box of side length 4 and center x. Obviously, the covering M_{xy} is locally finite in the sense that there exists a constant $K = K(d)$ such that each point on the boundary of the box is covered by at most K rectangles or "roofs". Then, applying the Poincaré inequality (Lemma III.B.5) on the roof M_{xy}, which is a Lipschitz surface:

$$\inf_{c\in\mathbb{R}} \int_{\Gamma_x\cup\Gamma_y} |f-c|^p \lesssim_{p,d} \int_{M_{xy}} |\nabla f|^p,$$

and summing over $x,y \in \partial D_R$ and $x \sim y$ yield the claim (III.4.15). To prove (III.4.16) we also start with (III.4.18). For each $x \in \overline{D}_R$ define $\widehat{\Gamma}_x$ as follows: if

$x \in D_R$ then $\widehat{\Gamma}_x := \Gamma_x$ and if $x \in \partial D_R$ then choose $\widehat{\Gamma}_x$ the d-dimensional box in $\overline{C}_R$ such that Γ_x is a face of $\widehat{\Gamma}_x$. Then, by the boundedness of Ψ_x, Jensen's and Poincaré's inequality and *the trace theorem*, (III.4.18) implies that

$$|\Pi f(x) - \Pi f(y)|^p \lesssim_{p,d} \inf_{c\in\mathbb{R}} \|f - c\|^p_{L^p(\Gamma_x\cup\Gamma_y)} \lesssim_{p,d} \inf_{c\in\mathbb{R}} \|f - c\|^p_{W^{1,p}(\widehat{\Gamma}_x\cup\widehat{\Gamma}_y)}$$

$$\lesssim_{p,d} \|\nabla f\|^p_{L^p(\widehat{M}_{xy})}$$

where $\widehat{M}_{xy}$ is the d-dimensional box with center x and side length 4 intersecting $\overline{C}_R$. Since $\{\widehat{M}_{xy} : x, y \in D_R, x \sim y\}$ is a locally finite covering of $\overline{C}_R$, the claim follows by summing over all $\widehat{M}_{xy}$. □

Proof of Lemma III.2.4. Denote by $\widetilde{u} \in \partial C_R$ the extension of u, when interpolated by (III.4.1). Call $\widetilde{u}_\varepsilon \in \mathcal{C}(\partial C_R)$ the function obtained after the following process: Rescale $\widetilde{u}$ to ∂C_1, smooth it there by Proposition III.4.5 and rescale it back to ∂C_R. Then, by estimates (III.4.10a)–(III.4.10c) we have

$$\|\widetilde{u}_\varepsilon\|_{\overline{L}^r(\partial C_R)} \lesssim \varepsilon^{-(d-1)(\frac{1}{s}-\frac{1}{r})}\|u\|_{\overline{L}^s(\partial C_R)} \tag{III.4.20}$$

$$\|u - u_\varepsilon\|_{\overline{L}^s(\partial C_R)} \lesssim \varepsilon R\|\nabla^{\tan} u\|_{\overline{L}^s(\partial C_R)}, \tag{III.4.21}$$

$$\|\nabla^{\tan} u_\varepsilon\|_{\overline{L}^r(\partial C_R)} \lesssim \varepsilon^{-(d-1)(\frac{1}{s}-\frac{1}{r})}\|\nabla^{\tan} u\|_{\overline{L}^s(\partial C_R)}, \quad 1 \le s \le r \le \infty. \tag{III.4.22}$$

where the multiplicative constants only depend on d and the exponents r and s. Note that there is a rescaling factor R on the right-hand side of (III.4.21) that is typical when comparing functions with the corresponding gradients. Now, define $u_\varepsilon = \Pi\widetilde{u}_\varepsilon$ where Π the discrete projection defined by (III.4.13). Linearity of $u \mapsto u_\varepsilon$ is then obvious. By the projection properties of Π (Lemmas III.4.7 and III.4.8) we obtain (III.2.26)–(III.2.28) from (III.4.20)–(III.4.22), respectively. □

III.5 Energy estimate for the homogenization error

Recall that in Subsections III.5.1–III.5.5 we prove Proposition III.2.3, where we construct the harmonic extension v, calculate the energy of the homogenization and estimate the boundary, corrector, and near boundary term. The boundary term is the most important part of the proof – the estimates for the others are quite routine. Finally, in Subsection III.5.6 we prove Theorem III.1.4 from Proposition III.2.3.

First of all, in Subsections III.5.1–III.5.5, $\lesssim$ means $\lesssim_{d,p,q}$.

III.5.1 Construction of the harmonic extension

Following [12] we also consider the Dirichlet case $q \ge p$ the Neumann case $p \ge q$, respectively. In both cases $\mathbf{a}_\mathrm{h}$-harmonic function v in Proposition III.2.3 is constructed via the boundary condition of u on $\partial D_{R'}$ where $\lfloor R/2 \rfloor \le R' \le R$ such that

$$\|\nabla u\|_{\underline{L}^{\frac{2q}{q+1}}\left(E_{R'}^{\tan}\cup E_{R'}^{\mathrm{nor}}\right)}+\|\omega\nabla u\|_{\underline{L}^{\frac{2p}{p+1}}\left(E_{R'}^{\tan}\cup E_{R'}^{\mathrm{nor}}\right)}\lesssim\overline{\Lambda}^{1/2}. \tag{III.5.1}$$

Argument for the existence of R'. By Hölder's inequality, (III.2.19) and (III.2.20),

$$\|\nabla u\|_{\underline{L}^{\frac{2q}{q+1}}(\overline{D}_R)}+\|\mathbf{a}\nabla u\|_{\underline{L}^{\frac{2p}{p+1}}(\overline{D}_R)}\lesssim\overline{\Lambda}^{1/2}.$$

Indeed, to lighten the notation, define

$$\alpha_1=2q/(q+1),\quad \alpha_2=2p/(p+1),\quad A=\overline{\Lambda}^{1/2}$$

and

$$f_1(x)=\sum_{|y|_1=1}|u(x+y)-u(x)|,\quad f_2(x)=\sum_{|y|_1=1}\omega(\{x,x+y\})|u(x+y)-u(x)|.$$

Further, set

$$R'=\underset{r\in[R/2,R]\cap\mathbb{Z}}{\operatorname{argmin}}\sum_{i=1,2}A^{-\alpha_i}\|f_i\|^{\alpha_i}_{L^{\alpha_i}(\partial D_r)}.$$

By assumption,

$$\begin{aligned}\sum_{i=1,2}RA^{-\alpha_i}\|f_i\|^{\alpha_i}_{L^{\alpha_i}(\partial D_{R'})}&\lesssim\sum_{\rho=0}^{R}\sum_{i=1,2}A^{-\alpha_i}\|f_i\|^{\alpha_i}_{L^{\alpha_i}(\partial D_\rho)}\\&=\sum_{i=1,2}A^{-\alpha_i}\|f_i\|^{\alpha_i}_{L^{\alpha_i}(D_R)}\overset{\text{(III.5.1)}}{\lesssim_d}R^d\end{aligned}$$

which implies the claim. □

Recall that we define the boundary condition in the Dirichlet case ($q\ge p$) by (III.2.29) and in the Neumann case ($p\ge q$) by duality (III.2.34) for that we need Lemmas III.2.6–III.2.8. Let us finish these auxiliary results before continuing.

Proof of Lemma III.2.6. For each $\bar{x}\in\partial D_{R'}\setminus\widetilde{\partial}D_{R'}$ pick $x\in\widetilde{\partial}D_{R'}$ with $|x-\bar{x}|_\infty=1$ and set $\bar{u}(\bar{x})=u(x)$. For each $x\in\partial D_{R'}$ let $Q_x=\{\xi\in\mathbb{Z}^d:|\xi-x|_\infty\le4\}\cap\partial D_{R'}$. The covering $\mathcal{R}:=\{Q_x:x\in\partial D_{R'}\}$ of $\partial D_{R'}$ is locally finite in the sense that

$$\sup_{y\in\partial C_{R'}}|\{Q\in\mathcal{R}:y\in Q\}|\lesssim_d 1.$$

Let $\{\bar{x},\bar{y}\}\in E_{R'}^{\tan}$. By construction $\bar{u}(\bar{x})=u(x)$ and $\bar{u}(\bar{y})=u(y)$ for some $x,y\in\partial D_{R'}$ with $\sup\{|x-\bar{x}|_\infty;|y-\bar{y}|_\infty\}\le1$. Therefore, $|x-y|_\infty\le3$. By the triangle inequality

$$|\bar{u}(\bar{x})-\bar{u}(\bar{y})|^r=|u(x)-u(y)|^r\lesssim_{d,r}\sum_{x',y'\in Q_x,x'\sim y'}|u(x')-u(y')|^r.$$

Summing over all $\{\bar{x},\bar{y}\}\in E_{R'}^{\tan}$ yields the claim. □

Proof of Lemma III.2.7. For $x \in \widetilde{\partial} D_{R'}$ define $h^*_\varepsilon(x)$ plugging $g(y) := \mathbf{1}_{x=y}, y \in \partial D_{R'}$ into (III.2.35). By linearity we get (III.2.35) for all $g : \partial D_{R'} \to \mathbb{R}$ satisfying $g = 0$ at the corner points $\partial D_{R'} \setminus \widetilde{\partial} D_{R'}$. Since Mg only depends on the values of g on $\widetilde{\partial} D_{R'}$, (III.2.35) is true for any $g : \partial D_{R'} \to \mathbb{R}$. □

Proof of Lemma III.2.8. By Hölder's inequality, (III.2.26) and Lemma III.2.6,

$$\begin{aligned}\left| \overline{\sum_{x \in \widetilde{\partial} D_R}} h^*_\varepsilon g \right| &\lesssim_{d,r,s} \|h\|_{\overline{L}^s(\widetilde{\partial} D_R)} \|(Mg)_\varepsilon\|_{\overline{L}^{s'}(\widetilde{\partial} D_R)} \\ &\lesssim_{d,r,s} \|h\|_{\overline{L}^s(\widetilde{\partial} D_R)} \varepsilon^{-(d-1)\left(\frac{1}{r'} - \frac{1}{s'}\right)} \|Mg\|_{\overline{L}^{r'}(\partial D_R)} \\ &\lesssim_{d,r,s} \|h\|_{\overline{L}^s(\widetilde{\partial} D_R)} \varepsilon^{-(d-1)(\frac{1}{s} - \frac{1}{r})} \|g\|_{\overline{L}^{r'}(\widetilde{\partial} D_R)}\end{aligned}$$

for $g : \widetilde{\partial} D_R \to \mathbb{R}$ and $1/s + 1/s' = 1/r + 1/r' = 1$. From this the claim follows. Note that in the limit case $r = \infty$ we only need to choose g the Dirac functions. □

III.5.2 Energy of the homogenization error

Let $\rho \in (0, R/256)$ and let η be a cut off function satisfying

$$\eta = 1 \text{ on } D_{R'-32\rho} \text{ and } \eta = 0 \text{ on } D_{R'} \setminus D_{R'-16\rho} \text{ and } |\nabla \eta| \lesssim \frac{1}{\rho}. \tag{III.5.2}$$

Define the homogenization error

$$w = u - v - \eta \phi_i \nabla_i v. \tag{III.5.3}$$

The energy of w can be obtained by testing the equation of w in Lemma III.3.1 with w itself. In order to do this, we need the following Lemma III.5.1.

Lemma III.5.1. *Let $g : \mathbb{E}^d_\pm \to \mathbb{R}$ and $h, f : \mathbb{Z}^d \to \mathbb{R}$ satisfy*

(a) $g([x, y]) = -g([y, x])$,

(b) $h_i(x) = g([x, x + \mathbf{e}_i]),\ x \in \mathbb{Z}^d$,

(c) $\nabla^* \cdot h = \nabla^* \cdot f$ *in* D_R,

(d) $\mathrm{supp}(f) \subseteq \overline{D}_{R-2}$.

Then, for any functions w defined on $\mathbb{Z}^d$ we have

$$\sum_{\{x,y\} \in E_R} (g \nabla w)(\{x, y\}) = \sum_{[x,y] \in E_R^{\mathrm{nor}}} g(x, y) w(x) + \sum_{x \in \overline{D}_R} f(x) \cdot \nabla w(x). \tag{III.5.4}$$

The notation on the left-hand side of (III.5.4) is explained as follows: By assumption (a) the product $g\nabla u$ does not depend on the direction of the edge:

$$(g\nabla u)([x, y]) = (g\nabla u)([y, x])$$

and we can therefore consider it as a function acting on unoriented edges.

Formula (III.5.4) looks a bit weird! However, it can be seen as a discrete Gauss-type formula. For simplicity, let $f = 0$ and pick $g = \nabla\beta$ for some function β. Then, assumption (c) says that $\Delta\beta = 0$. Then, the continuum version of (III.5.4) is simply the following

$$\int_{B_R} \nabla\beta \cdot \nabla w = \int_{B_R} \nabla \cdot (w\nabla\beta) - \int_{B_R} w \underbrace{\nabla \cdot \nabla\beta}_{=\Delta\beta=0} = \int_{\partial B_R} w\nabla\beta \cdot \nu dS$$

where we apply the Gauss divergence theorem. Later, we choose f to be the corrector term, which lies "far away from the boundary".

Proof of Lemma III.5.1. Define

$$w_0 = \begin{cases} w & \text{in} \quad \mathbb{Z}^d \setminus D_R \\ 0 & \text{in} \quad D_R \end{cases} \qquad \text{and} \quad \alpha = w - w_0 \quad \text{so that} \quad \operatorname{supp}(\alpha) \subseteq D_R.$$

The support of α and assumption (c) yield that

$$\sum_{\mathbb{Z}^d} \alpha\nabla^* \cdot h = \sum_{\mathbb{Z}^d} \alpha\nabla^* \cdot f.$$

By the partial integration (III.1.5) and assumption (b), we have

$$\sum_{\mathbb{Z}^d} f \cdot \nabla\alpha = \sum_{\mathbb{Z}^d} \sum_{i=1}^{d} h_i \nabla\alpha = \sum_{[x,y]\in\mathbb{E}^d_+} \{g\nabla\alpha\}([x,y]) \tag{III.5.5}$$

where

$$\mathbb{E}^d_+ := \{[x, x + \mathbf{e}_i] : x \in \mathbb{Z}^d, 1 \le i \le d\}.$$

Noting that $g\nabla\alpha$ is, in fact, a function acting on $e \in \mathbb{E}^d$ and that $\operatorname{supp}(g\nabla\alpha) \subseteq E_R$, which follows from the fact that $\operatorname{supp}(\alpha) \subseteq D_R$, we continue (III.5.5) as follows:

$$\begin{aligned} \sum_{\mathbb{Z}^d} f \cdot \nabla\alpha \ &= \sum_{\{x,y\}\in\mathbb{E}^d} \{g\nabla\alpha\}(\{x,y\}) \ = \sum_{E_R} g\nabla\alpha \\ &= \sum_{E_R} g\nabla w - \sum_{E_R} g\nabla w_0 \end{aligned}$$

where we use the definition of α for the last equality. Therefore, we have

$$\sum_{E_R} g\nabla w \ = \sum_{E_R} g\nabla w_0 \ + \sum_{\mathbb{Z}^d} f \cdot \nabla\alpha$$

By assumption (d) and the definition of u_0, this is exactly (III.5.4). □

In order to use Lemma III.5.1 we set

$$h = \mathbf{a}\nabla w + (1-\eta)(\mathbf{a} - \mathbf{a}_{\mathrm{h}})\nabla v$$

$$g([x,y]) = \omega(\{x,y\})\nabla w([x,y]) + (1-\eta(x))(\omega - \omega_{\mathrm{h}})(\{x,y\})\nabla v([x,y])$$

$$f = -(\sigma_i \cdot \nabla^*)(\eta \nabla_i v) - \mathbf{a}[(\phi_i \nabla)(\eta \nabla_i v)].$$

Here, h and f are functions of vertices and g is a function of oriented edges. By Lemma III.3.1 and especially the fact that both $\mathbf{a}$ and $\mathbf{a}_{\mathrm{h}}$ are diagonal, they satisfy Lemma III.5.1. To lighten notations in the following we will not write the arguments, i.e. ω, ∇f, and f should be read as $\omega(\{x,y\})$, $\nabla f([x,y])$, and $f(x)$, $f \in \{u, v, w\}$. First, we have

$$\sum_{E_{R'}} g\nabla w = \sum_{E_{R'}} \omega(\nabla w)^2 + (1-\eta)(\omega - \omega_{\mathrm{h}})\nabla v \nabla w$$

Further, by definition (III.5.3) of w and (III.5.2) of η,

$$\begin{aligned}
\sum_{[x,y]\in E_{R'}^{\mathrm{nor}}} w(x)g(x,y) &= \sum_{E_{R'}^{\mathrm{nor}}} (u-v)\{\omega\nabla w + (1-\eta)(\omega-\omega_{\mathrm{h}})\nabla v\} \\
&= \sum_{E_{R'}^{\mathrm{nor}}} (u-v)\{\omega(\nabla u - \nabla v) + (\omega - \omega_{\mathrm{h}})\nabla v\} \\
&= \sum_{E_{R'}^{\mathrm{nor}}} (u-v)\{\omega\nabla u - \omega_{\mathrm{h}}\nabla v\}
\end{aligned}$$

Therefore, Lemma III.5.1 gives us

$$\begin{aligned}
\sum_{E_{R'}} \omega(\nabla w)^2 = &-\sum_{E_{R'}} (1-\eta)(\omega-\omega_{\mathrm{h}})\nabla v\nabla w \\
&+ \sum_{E_{R'}^{\mathrm{nor}}} (u-v)\{\omega\nabla u - \omega_{\mathrm{h}}\nabla v\} \\
&- \sum_{\overline{D}_{R'}} \Big\{(\sigma_i \cdot \nabla^*)(\eta\nabla_i v) + \mathbf{a}[(\phi_i\nabla)(\eta\nabla_i v)]\Big\} \cdot \nabla w.
\end{aligned}$$

By Young's and Hölder's inequality and definition (III.5.2) of η,

$$\sum_{E_{R'}} \omega(\nabla w)^2$$

$$\lesssim \sum_{E_{R'} \setminus E_{R'-32\rho}} (\omega + \omega^{-1})(\nabla v)^2 \tag{III.5.6a}$$

$$+ \Big| \sum_{E_{R'}^{\mathrm{nor}}} (u - v)(\omega \nabla u - \omega_{\mathrm{h}} \nabla v) \Big| \tag{III.5.6b}$$

$$+ \Lambda R^d \left[\|\sigma\|^2_{\overline{L}^{\frac{2q}{q-1}}(\overline{D}_R)} + \|\phi\|^2_{\overline{L}^{\frac{2p}{p-1}}(\overline{D}_R)} \right] \sup_{D_{R'-8\rho}} \left\{ |\nabla^2 v|^2 + \frac{1}{\rho^2} |\nabla v|^2 \right\}. \tag{III.5.6c}$$

We call (III.5.6a)–(III.5.6c) the near boundary, boundary, and corrector term, respectively. The proposition is proved once we can show that

$$\text{(III.5.6a)} \lesssim \Lambda \overline{\Lambda} R^d \left(\frac{\rho}{R} \right)^{\min(\frac{p-1}{2p}, \frac{q-1}{2q})} \varepsilon^{-(d-1)\min\left(\frac{q+1}{q}, \frac{p+1}{p}\right)},$$

$$\text{(III.5.6b)} \lesssim \overline{\Lambda} R^d \varepsilon^{1-(d-1)\left(\frac{1}{2p} + \frac{1}{2q}\right)},$$

$$\text{(III.5.6c)} \lesssim \Lambda \overline{\Lambda} R^d \left(\frac{R}{\rho} \right)^{d+2} \left[\|\phi\|_{\overline{L}^{\frac{2p}{p-1}}(D_R)} + \|\sigma\|_{\overline{L}^{\frac{2q}{q-1}}(D_R)} \right].$$

III.5.3 The boundary term

As mentioned before, the most difficult part of the proof is the boundary term, which we want to consider first to make sure that our ideas really work.

Lemma III.5.2. *In both Dirichlet and Neumann case,*

$$\text{(III.5.6b)} \lesssim \overline{\Lambda} R^d \varepsilon^{1-(d-1)\left(\frac{1}{2p} + \frac{1}{2q}\right)}.$$

Proof for the Dirichlet case. Recall that $v := u_\varepsilon$. With the smoothed discrete version and the Dirichlet-to-Neumann estimate we can almost repeat the argument by Bella, Fehrman, Otto [12]. By the triangle inequality we decompose the left-hand side into two terms:

$$R^{-(d-1)} \text{(III.5.6b)} \lesssim \|(u - v)\omega \nabla u\|_{\overline{L}^1(E_{R'}^{\mathrm{nor}})} + \|(u - v)\omega_{\mathrm{h}} \nabla v\|_{\overline{L}^1(E_{R'}^{\mathrm{nor}})} \tag{III.5.7}$$

where each term is estimated by Hölder's inequality as follows:

$$\begin{aligned} \|(u - v)\omega \nabla u\|_{\overline{L}^1(E_{R'}^{\mathrm{nor}})} &\lesssim \|u - v\|_{\overline{L}^{\frac{2p}{p-1}}(\partial D_{R'})} \|\omega \nabla u\|_{\overline{L}^{\frac{2p}{p+1}}(E_{R'}^{\mathrm{nor}})}, \\ \|(u - v)\omega_{\mathrm{h}} \nabla v\|_{\overline{L}^1(E_{R'}^{\mathrm{nor}})} &\lesssim \|u - v\|_{\overline{L}^{\frac{2q}{q-1}}(\partial D_{R'})} \|\nabla v\|_{\overline{L}^{\frac{2q}{q+1}}(E_{R'}^{\mathrm{nor}})}. \end{aligned} \tag{III.5.8}$$

Here we need to exploit the Dirichlet-to-Neumann estimate, Theorem III.2.5, to bound the discrete normal derivatives:

$$\|\nabla v\|_{\underline{L}^{\frac{2q}{q+1}}(E^{\mathrm{nor}}_{R'})} \lesssim \|\nabla v\|_{\underline{L}^{\frac{2q}{q+1}}(E^{\tan}_{R'})} \lesssim \|\nabla u\|_{\underline{L}^{\frac{2q}{q+1}}(E^{\tan}_{R'})} \tag{III.5.9}$$

where the last inequality follows from (III.2.27) with $r = s = 2q/(q+1)$. Further, (III.2.28) reads

$$\|u - v\|_{\underline{L}^{\frac{2p}{p-1}}(\partial D_{R'})} \lesssim \varepsilon R \|\nabla u\|_{\underline{L}^{\frac{2p}{p-1}}(E^{\tan}_{R'})}, \tag{III.5.10}$$

$$\|u - v\|_{\underline{L}^{\frac{2q}{q-1}}(\partial D_{R'})} \lesssim \varepsilon R \|\nabla v\|_{\underline{L}^{\frac{2q}{q-1}}(E^{\tan}_{R'})}. \tag{III.5.11}$$

Further, by the fact that the constant function is invariant under $(\cdot)_\varepsilon$, the triangle inequality, estimate (III.2.26) with $r = s = 2p/(p+1)$ and finally, the Sobolev inequality (Lemma III.4.4), we have

$$\begin{aligned}
\|u - v\|_{\underline{L}^{\frac{2p}{p-1}}(\partial D_{R'})} &= \|u - u_\varepsilon\|_{\underline{L}^{\frac{2p}{p-1}}(\partial D_{R'})} = \|(u-c) - (u-c)_\varepsilon\|_{\underline{L}^{\frac{2p}{p-1}}(\partial D_{R'})} \\
&\lesssim \|(u-c)\|_{\underline{L}^{\frac{2p}{p-1}}(\partial D_{R'})} + \|(u-c)_\varepsilon\|_{\underline{L}^{\frac{2p}{p-1}}(\partial D_{R'})} \\
&\lesssim \|(u-c)\|_{\underline{L}^{\frac{2p}{p-1}}(\partial D_{R'})} \\
&\lesssim R\|\nabla u\|_{\underline{L}^{s}(E^{\tan}_{R'})}
\end{aligned} \tag{III.5.12}$$

with

$$\frac{1}{s} = \frac{p-1}{2p} + \frac{1}{d-1}.$$

Interpolating between (III.5.10) and (III.5.12) yields

$$\|u - v\|_{\underline{L}^{\frac{2p}{p-1}}(\partial D_{R'})} \lesssim \varepsilon^{1-(d-1)\left(\frac{1}{2p}+\frac{1}{2q}\right)} R \|\nabla u\|_{\underline{L}^{\frac{2q}{q+1}}(E^{\tan}_{R'})}, \tag{III.5.13}$$

and between (III.5.11) and (III.5.12) yields

$$\|u - v\|_{\underline{L}^{\frac{2q}{q-1}}(\partial D_{R'})} \lesssim \varepsilon^{1-\frac{d-1}{q}} R \|\nabla u\|_{\underline{L}^{\frac{2q}{q+1}}(E^{\tan}_{R'})}. \tag{III.5.14}$$

Plugging all (III.5.9), (III.5.13) and (III.5.14) into (III.5.8) and then (III.5.7), using (III.5.1) and noting that

$$\varepsilon^{1-(d-1)\left(\frac{1}{2p}+\frac{1}{2q}\right)} \geq \varepsilon^{1-\frac{d-1}{q}}$$

which follows from the case assumption $q \geq p$ we finish the argument for the Dirichlet case. $\square$

Proof for the Neumann case. Exploiting duality produced by (III.2.35) and using M defined by Lemma III.2.6 are main ideas of the proof. The remaining part is very routine. First of all, we can assume that

$$\sum_{\widetilde{\partial} D_{R'}} u - v = 0$$

by adding a constant to v so that the contribution of the mean in definition (III.2.34) is zero. Therefore, by duality (III.2.35), we have

$$\begin{aligned} R^{-(d-1)}\text{(III.5.6b)} &\sim \left| \overline{\sum_{E^{\mathrm{nor}}_{R'}}} (u-v)\left(\omega\nabla u - (\omega\nabla u)^*_\varepsilon\right) \right| \\ &= \left| \overline{\sum_{E^{\mathrm{nor}}_{R'}}} \omega\nabla u \left\{[M(u-v)] - [M(u-v)]_\varepsilon\right\} \right|. \end{aligned}$$

By Hölder's inequality,

$$\begin{aligned} R^{-(d-1)}\text{(III.5.6b)} \lesssim{}& \|\omega\nabla u\|_{\underline{L}^{\frac{2p}{p+1}}(E^{\mathrm{nor}}_{R'})} \times \\ &\times \left[\|Mu - (Mu)_\varepsilon\|_{\underline{L}^{\frac{2p}{p-1}}(\partial D_{R'})} + \|Mv - (Mv)_\varepsilon\|_{\underline{L}^{\frac{2p}{p-1}}(\partial D_{R'})} \right]. \end{aligned} \tag{III.5.15}$$

First of all, we have

$$\begin{aligned} \|Mu - (Mu)_\varepsilon\|_{\underline{L}^{\frac{2p}{p-1}}(\partial D_{R'})} &\lesssim \varepsilon^{1-(d-1)\left(\frac{1}{2p}+\frac{1}{2q}\right)} R \|\nabla Mu\|_{\underline{L}^{\frac{2q}{q+1}}(E^{\tan}_{R'})} \\ &\lesssim \varepsilon^{1-(d-1)\left(\frac{1}{2p}+\frac{1}{2q}\right)} R \|\nabla u\|_{\underline{L}^{\frac{2q}{q+1}}(E^{\tan}_{R'})} \end{aligned} \tag{III.5.16}$$

where we obtain the first inequality using the same argument as that for (III.5.13) and the last exploiting a property of M. Similarly,

$$\begin{aligned} \|Mv - (Mv)_\varepsilon\|_{\underline{L}^{\frac{2p}{p-1}}(\partial D_{R'})} &\lesssim \varepsilon^{1-\frac{d-1}{p}} R \|\nabla Mv\|_{\underline{L}^{\frac{2p}{p+1}}(E^{\tan}_{R'})} \lesssim \varepsilon^{1-\frac{d-1}{p}} R \|\nabla v\|_{\underline{L}^{\frac{2p}{p+1}}(E^{\tan}_{R'})} \\ &\lesssim \varepsilon^{1-\frac{d-1}{p}} R \|\nabla v\|_{\underline{L}^{\frac{2p}{p+1}}(E^{\mathrm{nor}}_{R'})} \\ &\lesssim \varepsilon^{1-\frac{d-1}{p}} R \|\omega\nabla u\|_{\underline{L}^{\frac{2p}{p+1}}(E^{\mathrm{nor}}_{R'})} \end{aligned} \tag{III.5.17}$$

where we obtain the first inequality using the same argument as that for (III.5.14), the second using a property of M, the third exploiting the Dirichlet-to-Neumann estimate, Theorem III.2.5, and the last applying (III.2.34) and Lemma III.2.8. Combining (III.5.1) and (III.5.15)–(III.5.17) and the fact that

$$\varepsilon^{1-(d-1)\left(\frac{1}{2p}+\frac{1}{2q}\right)} \geq \varepsilon^{1-\frac{d-1}{p}},$$

which follows from the case assumption $p \geq q$ we finish the argument for the Neumann case. □

III.5.4 The corrector term

From now on until the end of the section the argument is quite routine. We continue with the corrector term, which can be estimated by combining (III.5.18) and (III.5.21) below.

Claim. *We have*

$$\sup_{D_{R'-8\rho}} |\nabla^2 v|^2 + \frac{1}{\rho^2}|\nabla v|^2 \lesssim \frac{1}{\rho^{d+2}}\|\nabla v\|^2_{L^2(D_{R'})}. \tag{III.5.18}$$

Proof. Pick $x \in D_{R'-8\rho}$. Note that each components of $\nabla^2 v$ and ∇v is $\mathbf{a}_{\mathrm{h}}$-harmonic in the box $D_{R'-2\rho}$, which contains $D_{4\rho}(x)$. By the mean value inequality (for a reference see [25, Lemma 3.4]) and Cacciopoli's inequality (Lemma III.2.1), we have

$$|\nabla^2 v(x)|^2 \lesssim \|\nabla^2 v\|_{\overline{L}^2(D_{2\rho}(x))} \lesssim \frac{1}{\rho^2}\|\nabla v\|_{\overline{L}^2(D_{4\rho}(x))} \sim \frac{1}{\rho^{2+d}}\|\nabla v\|_{L^2(D_{4\rho}(x))}. \tag{III.5.19}$$

By the mean value inequality, we have

$$\frac{1}{\rho^2}|\nabla v(x)| \lesssim \frac{1}{\rho^2}\|\nabla v\|_{\overline{L}^2(D_{4\rho}(x))} \lesssim \frac{1}{\rho^{d+2}}\|\nabla v\|_{L^2(D_{4\rho}(x))}. \tag{III.5.20}$$

Adding (III.5.19) and (III.5.20) and noting that $D_{4\rho}(x) \subset D_{R'}$ we get (III.5.18). □

Claim. *In both the Dirichlet and the Neumann case,*

$$\|\omega_{\mathrm{h}}(\nabla v)^2\|_{\overline{L}^1(E_R)} \lesssim \overline{\Lambda}. \tag{III.5.21}$$

Proof. By Lemma III.5.1 (with $u = v - c$, $g = \omega_{\mathrm{h}}\nabla v$, $f = 0$) and Hölder's inequality,

$$\text{LHS (III.5.21)} = \overline{\sum_{E^{\mathrm{nor}}_{R'}}}\{\omega_{\mathrm{h}}\nabla v\}(v-c) \leq \|\nabla v\|_{\overline{L}^{\frac{2q}{q+1}}(E^{\mathrm{nor}}_{R'})}\|v-c\|_{\overline{L}^{\frac{2q}{q-1}}(\widetilde{\partial} D_{R'})} \tag{III.5.22}$$

for any $c \in \mathbb{R}$. Applying Lemma III.4.4 (Sobolev's inequality) and using the modification M in Lemma III.2.8 we have

$$\begin{aligned}\inf_{c\in\mathbb{R}}\|v-c\|_{\overline{L}^{\frac{2q}{q-1}}(\widetilde{\partial} D_{R'})} &\lesssim \inf_{c\in\mathbb{R}}\|Mv-c\|_{\overline{L}^{\frac{2q}{q-1}}(\partial D_{R'})} \lesssim \|\nabla Mv\|_{\overline{L}^{\frac{2p}{p+1}}(E^{\tan}_R)} \\ &\lesssim \|\nabla v\|_{\overline{L}^{\frac{2p}{p+1}}(E^{\tan}_R)}.\end{aligned} \tag{III.5.23}$$

Note that in order to apply the Sobolev inequality we need

$$\frac{p+1}{2p} < \frac{q-1}{2q} + \frac{1}{d-1},$$

which is satisfied since $1/p + 1/q \leq 2/d$. By (III.5.22) and (III.5.23) we have

$$\text{LHS (III.5.21)} \lesssim \|\nabla v\|_{\overline{L}^{\frac{2q}{q+1}}(E^{\mathrm{nor}}_{R'})} \|\nabla v\|_{\overline{L}^{\frac{2p}{p+1}}(E^{\mathrm{tan}}_{R'})}.$$

Then, the claim follows from the fact that

$$p \leq q \Leftrightarrow \frac{2p}{p+1} \leq \frac{2q}{q+1}$$

and the following two estimates

$$\|\nabla v\|_{\overline{L}^{\frac{2q}{q+1}}\left(E^{\mathrm{tan}}_{R'} \cup E^{\mathrm{nor}}_{R'}\right)} \lesssim \overline{\Lambda}^{1/2}, \quad \text{in the Dirichlet case,} \tag{III.5.24}$$

$$\|\nabla v\|_{\overline{L}^{\frac{2p}{p+1}}\left(E^{\mathrm{tan}}_{R'} \cup E^{\mathrm{nor}}_{R'}\right)} \lesssim \overline{\Lambda}^{1/2}, \quad \text{in the Neumann case} \tag{III.5.25}$$

which are direct consequences of (III.5.1) and definitions (III.2.29) and (III.2.34) of v in each case. Indeed, the tangential part in (III.5.24) is estimated by using (III.2.27) with $r = s = 2q/(q+1)$ and the normal part in (III.5.25) by using Lemma III.2.8 with $r = 2p/(p+1)$. In both cases, we need to apply the Dirichlet-to-Neumann estimate, Theorem III.2.5. □

III.5.5 The near boundary term

We finally turn to the near boundary term. The only difficulty is (III.5.29). However, it is much easier than the Dirichlet-to-Neumann estimate. Set

$$m = \max\left\{\tfrac{4p}{p-1}, \tfrac{4q}{q-1}\right\} \tag{III.5.26}$$

Claim. *We have*

$$\text{(III.5.6a)} \lesssim \Lambda R^d \|\nabla v\|_{\overline{L}^m(E_{R'})} (\tfrac{\rho}{R})^{\frac{2}{m}}. \tag{III.5.27}$$

Proof. By Hölder inequality,

$$\|\omega(\nabla v)^2\|_{\overline{L}_1(E_{R'} \setminus E_{R'-32\rho})} \lesssim \|\omega\|_{\overline{L}^p(E_{R'})} \left[1 - \left(\tfrac{R'-32\rho}{R}\right)^d\right]^{1-\frac{1}{p}-\frac{2}{m}} \|(\nabla v)^2\|_{\overline{L}^{m/2}(E_{R'})}$$

$$\lesssim \Lambda \left(\frac{\rho}{R}\right)^{1-\frac{1}{p}-\frac{2}{m}} \|\nabla v\|^2_{\overline{L}^m(E_{R'})} \tag{III.5.28a}$$

and

$$\|\omega^{-1}(\nabla v)^2\|_{\overline{L}_1(E_{R'}\setminus E_{R'-32\rho})} \lesssim \|\omega^{-1}\|_{\overline{L}^q(E_{R'})}\left[1-\left(\tfrac{R'-32\rho}{R}\right)^d\right]^{1-\frac{1}{q}-\frac{2}{m}}\|(\nabla v)^2\|_{\overline{L}^{m/2}(E_{R'})}$$

$$\lesssim \Lambda\left(\frac{\rho}{R}\right)^{1-\frac{1}{q}-\frac{2}{m}}\|\nabla v\|^2_{\overline{L}^m(E_{R'})}. \tag{III.5.28b}$$

Summing (III.5.28a) and (III.5.28b) and noting that

$$\min\{1-\frac{1}{p}-\frac{2}{m}, 1-\frac{1}{q}-\frac{2}{m}\} \le \frac{2}{m}$$

yield the claim. □

Claim. *In both Dirichlet and Neumann case,*

$$\|\nabla v\|_{\overline{L}^m(E_{R'})} \lesssim \|\nabla v\|_{\overline{L}^m(E^{\tan}_{R'})}. \tag{III.5.29}$$

Proof. This is Corollary IV.2.9 in Chapter IV. □

Claim. *In the Dirichlet case, we have*

$$\|\nabla v\|_{\overline{L}^m(E_{R'})} \lesssim \varepsilon^{-(d-1)\frac{q+1}{2q}}\overline{\Lambda}^{\frac{1}{2}}. \tag{III.5.30}$$

Proof. By (III.5.29) we have

$$\|\nabla v\|_{\overline{L}^m(E_{R'})} \lesssim \|\nabla v\|_{\overline{L}^m(E^{\tan}_{R'})} \lesssim \|\nabla v\|_{\overline{L}^\infty(E^{\tan}_{R'})}.$$

Recall that $v := u_\varepsilon$. By (III.2.27) with $r=\infty$ and $s = 2q/(q+1)$, we have

$$\|\nabla v\|_{\overline{L}^\infty(E^{\tan}_{R'})} \lesssim \varepsilon^{-(d-1)\frac{q+1}{2q}}\|\nabla u\|_{\overline{L}^{\frac{2q}{q+1}}(E^{\tan}_{R'})} \lesssim \varepsilon^{-(d-1)\frac{q+1}{2q}}\overline{\Lambda}^{\frac{1}{2}}$$

where the last inequality is due to (III.5.1). The proof is complete. □

Claim. *In the Neumann case, we have*

$$\|\nabla v\|_{\overline{L}^m(E_{R'})} \lesssim \varepsilon^{-(d-1)\frac{p+1}{2p}}\overline{\Lambda}^{\frac{1}{2}}. \tag{III.5.31}$$

Proof. We argue as follows:

$$\|\nabla v\|_{\overline{L}^m(E_{R'})} \lesssim \|\nabla v\|_{\overline{L}^m(E^{\tan}_{R'})} \lesssim \|\nabla v\|_{\overline{L}^m(E^{\mathrm{nor}}_{R'})} \lesssim \|\omega_{\mathrm{h}}\nabla v\|_{\overline{L}^\infty(E^{\mathrm{nor}}_{R'})}$$

$$\lesssim \varepsilon^{-(d-1)\frac{p+1}{2p}}\|\omega\nabla u\|_{\overline{L}^{\frac{2p}{p+1}}(E^{\mathrm{nor}}_{R'})} \lesssim \varepsilon^{-(d-1)\frac{q+1}{2q}}\overline{\Lambda}^{\frac{1}{2}}.$$

where the first inequality is due to (III.5.29), the second is nothing but the Dirichlet-to-Neumann estimate, Theorem III.2.5, the fourth follows from definition (III.2.34) in the Neumann case and Lemma III.2.8 (for $h = \omega\nabla u$, $r=\infty$ and $s = 2p/(p+1)$, and finally, for the last we use (III.5.1). □

III.5.6 From the energy estimate to the excess decay

Proof of Theorem III.1.4 from Proposition III.2.3. The proof is almost the same as that in the continuum case [12, Step 4, p14]. We only have to take care make sure that the radii ρ and R are integer, especially the scale of the cut off must satisfy $\rho \gg 1$. It suffices to show that

$$\mathrm{Exc}(r) \le K^{-2\alpha} \|\mathbf{a}\nabla u \cdot \nabla u\|_{L^1(D_R)} \tag{III.5.32}$$

where $R := Kr$. Indeed, replacing u by $u - \xi_i(\phi_i + x_i)$ where ξ is chosen so that

$$\|\mathbf{a}\left(\nabla u - \xi_i(\nabla\phi_i + \mathbf{e}_i)\right) \cdot \left(\nabla u - \xi_i(\nabla\phi_i + \mathbf{e}_i)\right)\|_{L^1(D_R)} \to \min$$

we obtain (III.1.11) from (III.5.32). Since $r \le R/64$ due to $K \ge 64$, by definition of the cutoff function η, we have

$$w = u - v - \phi_i \nabla_i v \quad \text{in } D_r$$

Taking the discrete gradient yields

$$\begin{aligned}
\nabla w &= \nabla u - \nabla v - (\phi_i \nabla)\nabla_i v - \nabla_i v \nabla\phi_i \\
&= \nabla u - \nabla_i v\,(\nabla\phi_i + \mathbf{e}_i) - (\phi_i\nabla)\nabla_i v \\
&= \nabla u - \nabla_i v(0)\,(\nabla\phi_i + \mathbf{e}_i) - (\nabla_i v - \nabla_i v(0))\,(\nabla\phi_i + \mathbf{e}_i) - (\phi_i\nabla)\nabla_i v.
\end{aligned}$$

Setting $\xi = \nabla_i v(0)$, we obtain

$$\nabla u - \xi_i(\nabla\phi_i + \mathbf{e}_i) = \nabla w + (\nabla_i v - \nabla_i v(0))\,(\nabla\phi_i + \mathbf{e}_i) + (\phi_i\nabla)\nabla_i v.$$

Recall that $R := Kr$. By the triangle inequality,

$$\begin{aligned}
&\|\mathbf{a}\left((\nabla u - \xi_i(\nabla\phi_i + \mathbf{e}_i)\right) \cdot \left((\nabla u - \xi_i(\nabla\phi_i + \mathbf{e}_i)\right)\|_{\overline{L}^1(D_r)} \\
&\quad\lesssim K^d \|\mathbf{a}\nabla w \cdot \nabla w\|_{\overline{L}^1(D_{\lfloor R/64 \rfloor})} \\
&\qquad + \left\{\sup_{D_{2r}} |\nabla v - \nabla v(0)|^2\right\} \|\mathbf{a}(\nabla\phi + \mathbf{e}_i)\cdot(\nabla\phi_i + \mathbf{e}_i)\|_{\overline{L}^1(D_{2r})} \\
&\qquad + \left\{\sup_{D_{2r}} |\nabla\nabla v|^2\right\} \|\mathbf{a}\phi_i \cdot \phi_i\|_{\overline{L}^1(D_{2r})}.
\end{aligned} \tag{III.5.33}$$

The following is also true in the discrete case:

$$\sup_{D_{2r}} |\nabla v - \nabla v(0)|^2 \lesssim r^2 \sup_{D_{4r}} |\nabla\nabla v|^2. \tag{III.5.34}$$

Since $r \le R/64$, applying the mean value and Cacciopoli inequality (Lemma III.2.1) we obtain

$$\sup_{D_{4r}} |\nabla\nabla v|^2 \lesssim \sup_{D_{\lfloor R/8\rfloor}} |\nabla\nabla v|^2 \lesssim \|\nabla\nabla v\|_{\overline{L}^2(D_{\lfloor R/4\rfloor})}$$

$$\lesssim \frac{1}{R^2}\|\nabla v\|_{\overline{L}^2(D_{\lfloor R/2\rfloor})} \overset{\text{(III.5.21)}}{\lesssim} \frac{1}{R^2}\|\mathbf{a}\nabla u\cdot\nabla u\|_{\overline{L}^1(D_R)}. \tag{III.5.35}$$

By Cacciopoli's inequality (Lemma III.2.1) and Hölder's inequality, we have

$$\|\mathbf{a}(\nabla\phi_i + \mathbf{e}_i)\cdot(\nabla\phi_i + \mathbf{e}_i)\|_{\overline{L}^1(D_{2r})} \lesssim \frac{1}{r^2}\|\phi + x\|_{\overline{L}^{\frac{2p}{p-1}}(D_{4r})} \lesssim 1, \tag{III.5.36}$$

and

$$\|\mathbf{a}\phi\cdot\phi\|_{\overline{L}^1(D_r)} \le \|\mathbf{a}\|_{\overline{L}^p(D_r)}\|\phi\|^2_{\overline{L}^{\frac{2p}{p-1}}(D_r)} \lesssim r^2, \tag{III.5.37}$$

for large r, where the last inequalities in (III.5.36) and (III.5.37) are due to the sublinearity of ϕ and the ergodicity of $\mathbb{P}$. Plugging (III.5.34)–(III.5.37) into (III.5.33) yields that for large r we have

$$\|\mathbf{a}\left(\nabla u - \xi_i(\nabla\phi_i + \mathbf{e}_i)\right)\cdot\left(\nabla u - \xi_i(\nabla\phi_i + \mathbf{e}_i)\right)\|_{\overline{L}^1(D_r)}$$

$$\le C_1\left(K^d\|\mathbf{a}\nabla w\cdot\nabla w\|_{\overline{L}^1(D_{\lfloor R/64\rfloor})} + \frac{1}{K^2}\|\mathbf{a}\nabla u\cdot\nabla u\|_{L^1(D_R)}\right) \tag{III.5.38}$$

where C_1 is a deterministic constant. We obtain (III.5.32) choosing the right constant K and r^* as follows. First, choose $K \ge 64$ and $\delta_0 > 0$ such that

$$C_1\delta_0 K^d \le \frac{1}{2}K^{-2\alpha} \quad \text{and} \quad C_1\frac{1}{K^2} \le \frac{1}{2}K^{-2\alpha}, \tag{III.5.39}$$

where we first choose K so that the second is true, then choose δ_0 so that the first holds. By Proposition III.2.3 for each $\delta_0 > 0$ there exists $r^*(\omega) > 0$ large enough such that the following is true. Given $R \ge r^*$ and an ω-harmonic function u defined in $\overline{D}_R$ there exists an ω_{h}-harmonic function v in $D_{\lfloor R/2\rfloor}$ such that for $w = u - v - \phi_i\nabla_i v$ we have

$$\|\nabla w\cdot\mathbf{a}\nabla w\|_{\overline{L}^1(D_{\lfloor R/4\rfloor})} \le \delta_0\overline{\Lambda}. \tag{III.5.40}$$

Indeed, in (III.2.3) choose ε so that the first term is small, then choose the ratio $\gamma := \rho/R \ll 1$ so that the second term is small. Finally, we exploit the sublinearity of (ϕ,σ) to rebalance $\gamma^{-(d+2)} \gg 1$ in the last term. Note that this procedure only works if $\rho = \gamma R$ large enough. However, we can overcome this small difficulty multiplying both R and ρ with a very large number i.e. by choosing $r^*(\omega) \gg 1$.

This is the only difference between the discrete and continuum case. Combining (III.5.38)–(III.5.40) we obtain the claim (III.5.32) as follows:

$$\begin{aligned}\|\nabla u - \xi_i(\nabla\phi_i + \mathbf{e}_i)) \cdot \mathbf{a}\,(\nabla u - \xi_i(\nabla\phi_i + \mathbf{e}_i)\|_{\overline{L}^1(D_r)}& \\ &\le C_1\left\{\delta_0 K^d + K^{-2}\right\}\|\mathbf{a}\nabla u\cdot\nabla u\|_{L^1(D_R)} \\ &\le K^{-2\alpha}\|\mathbf{a}\nabla u\cdot\nabla u\|_{L^1(D_R)}.\end{aligned}$$

The proof of Theorem III.1.4 is complete. □

III.A Proof of Cacciopoli's inequality

Proof of Lemma III.2.1. The first inequality is only application of Hölder's inequality. To show the second one, we can assume $c = 0$. Choose a cut off function η with the following properties:

$$\operatorname{supp}(\eta) \subset D_{R-8}, \quad \eta = 1 \quad \text{in} \quad D_{R-\rho+8}, \quad \text{and} \quad |\nabla\eta| \lesssim 1/\rho. \tag{III.A.1}$$

Further, we have

$$\nabla_i(fg)(x) = \mathrm{av}_i[f](x)\nabla_i g(x) + \mathrm{av}_i[g](x)\nabla_i f(x)$$

where

$$\mathrm{av}_i[f](x) = \frac{f(x+\mathbf{e}_i) + f(x)}{2}.$$

Hence,

$$\nabla(fg)(x) = \mathrm{av}[f](x)\nabla g(x) + \mathrm{av}[g](x)\nabla f(x)$$

where

$$\mathrm{av}[f] := \operatorname{diag}\left(\mathrm{av}_1[f], \dots, \mathrm{av}_d[f]\right).$$

Using this notation we have

$$\nabla(\eta^2 u) = \mathrm{av}[\eta^2]\nabla u + \mathrm{av}[u]\nabla(\eta^2) = \mathrm{av}[\eta^2]\nabla u + 2\mathrm{av}[u]\mathrm{av}[\eta]\nabla\eta$$

Testing $\nabla^* \cdot \mathbf{a}\nabla u = 0$ with $\eta^2 u$ we have

$$0 = \overline{\sum_{D_{R-4}}}\nabla(\eta^2 u)\cdot\mathbf{a}\nabla u = \overline{\sum_{D_{R-4}}}\mathrm{av}[\eta^2]\nabla u\cdot\mathbf{a}\nabla u + \overline{\sum_{D_{R-4}}}2\mathrm{av}[u]\mathrm{av}[\eta]\nabla\eta\cdot\mathbf{a}\nabla u,$$

where $\overline{\sum}$ denotes the average. Here, the fact that the support of η is far away from ∂D_{R-4} allows us to apply partial integration. From this equality we have

$$\begin{aligned}\left\|\mathrm{av}[\eta^2]\mathbf{a}\nabla u\cdot\nabla u\right\|_{\overline{L}^1(D_{R-4})} &= \left|\overline{\sum_{B_R}} 2\mathrm{av}[u]\mathrm{av}[\eta]\nabla\eta\cdot a\nabla u\right| \\ &\lesssim \left\|\mathrm{av}[u]^2\mathbf{a}\nabla\eta\cdot\nabla\eta\right\|^{1/2}_{\overline{L}^1(D_{R-4})}\left\|\mathrm{av}[\eta]^2\mathbf{a}\nabla u\cdot\nabla u\right\|^{1/2}_{\overline{L}^1(D_{R-4})} \\ &\lesssim \left\|\mathrm{av}[u^2]\mathbf{a}\nabla\eta\cdot\nabla\eta\right\|^{1/2}_{\overline{L}^1(D_{R-4})}\left\|\mathrm{av}[\eta^2]\mathbf{a}\nabla u\cdot\nabla u\right\|^{1/2}_{\overline{L}^1(D_{R-4})}.\end{aligned}$$

Absorb the last term into the left-hand side and apply Hölder's inequality:

$$\begin{aligned}\left\|\mathrm{av}[\eta^2]\mathbf{a}\nabla u\cdot\nabla u\right\|_{\overline{L}^1(D_{R-4})} &\lesssim \left\|\mathrm{av}[u^2]\mathbf{a}\nabla\eta\cdot\nabla\eta\right\|_{\overline{L}^1(D_{R-4})} \\ &\lesssim \|\mathbf{a}\|_{\overline{L}^p(D_{R-4})}\|\mathrm{av}[u]^2|\nabla\eta|^2\|_{\overline{L}^{\frac{p}{p-1}}(D_{R-4})}.\end{aligned}$$

By (III.A.1) and assumption (III.2.1), the claim follows. □

III.B Sobolev spaces on Lipschitz surfaces

This appendix provides some basic knowledges on Sobolev spaces on Lipschitz surfaces, which we need, for example, in the following situations: First, in Section III.4 we have to smooth continuum functions defined on the surface of a box. Second, we need the Poincaré inequality on a ”roof” (Figure III.4) to bound the right-hand side of (III.4.19), when implementing the idea of Scott and Zhang [58].

The reader may think that since the surface of the unit box is just a union of open faces, it is not necessary to rigorously deal with Sobolev spaces on Lipschitz surfaces. However, when mimicking Evan's proof [29, p275] to show the Poincaré inequality on a ”roof”, we construct a limiting object g being a constant on each face and having the L^p-norm 1. It is not enough to deduce that g is constant on the whole ”roof”.

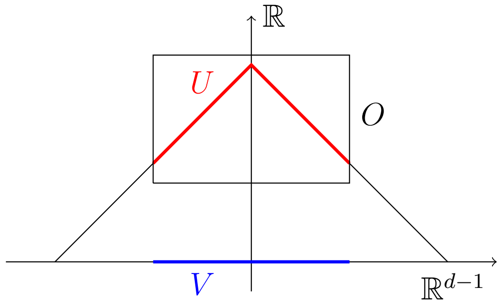

Figure III.5: A chart of M

Unfortunately, there is seemingly no literature stating this result directly. Some books consider Lipschitz manifolds, for instance, Wloka [62] but their settings are much more general than what we need. Therefore, it is worth showing it here from scratch. In Subsections III.B.1 and III.B.2 we mainly follow Chapter 2 in the lecture note by Mitrea and Mitrea [50]. However, in order to make this appendix more compact, the definitions and arguments we introduce here may be a bit different from that by Mitrea and Mitrea.

III.B.1 Basics on compact Lipschitz surfaces

Let $d \geq 2$ and assume in the whole section that M is a Lipschitz surface in the following sense. For any $a \in M$, there exists an open neighbourhood $O = O_a$ of a in $\mathbb{R}^d$, a Lipschitz function

$$\varphi = \varphi_a : \mathbb{R}^{d-1} \to \mathbb{R}$$

and an open subset $V = V_a$ in $\mathbb{R}^{d-1}$ such that up to an isometry in $\mathbb{R}^d$ we have

$$U := U_a := M \cap O = \{(x', x_d) : x' \in V, x_d = \varphi(x')\}.$$

We call $\{O_a, U_a, V_a, \varphi_a\}$ a chart for M at the point a. The normal vectors and the surface integrals on a chart $\{O, U, V, \varphi\}$ are well-defined almost everywhere:

$$\nu(x', \varphi(x')) = \frac{(\nabla\varphi(x'), -1)}{\sqrt{1 + |\nabla\varphi(x')|^2}}, \tag{III.B.1}$$

$$\int_U f dS = \int_V f(x', \varphi(x'))\sqrt{1 + |\nabla\varphi(x')|^2} dx' \tag{III.B.2}$$

As in the Euclidean case when defining Sobolev spaces we should start by choosing test functions. In the case of Lipschitz surfaces, it is reasonable to choose them to be $\mathcal{C}^1$ in a neighbourhood of the boundary M. First of all, this choice ensures the fundamental lemma of variational calculus:

Lemma III.B.1. *If $f \in L^1_{\mathrm{loc}}(M)$ satisfies*

$$\int_M fg = 0 \quad \textit{for all} \quad g \in \mathcal{C}_c^\infty(\mathbb{R}^d),$$

then $f = 0$.

Proof. The proof given by Mitrea and Mitrea [50] is very general. We introduce another argument which may be simpler to read. Let $\{O, U, V, \varphi\}$ be a chart and $\psi \in \mathcal{C}_c^\infty(V)$. Define $\Psi : V \times \mathbb{R} \to \mathbb{R}$ by

$$\Psi(x', x_d) := \psi(x')\theta(x_d - \varphi(x'))$$

where $\theta \in \mathcal{C}_c^\infty(-\frac{1}{4}\varepsilon_0, \frac{1}{4}\varepsilon_0)$ satisfying $\theta(0) = 1$ and $\varepsilon_0 = \mathrm{dist}(\partial V, \mathrm{supp}(\psi)) > 0$. Let η be the standard d-dimensional mollifier and $\eta_\varepsilon(\cdot) = \varepsilon^{-d}\eta(\cdot/\varepsilon)$ for fixed $\varepsilon \in (0, \varepsilon_0)$. Define the convolution

$$\Psi_\varepsilon := \Psi * \eta_\varepsilon \in \mathcal{C}_c^\infty(O).$$

By assumption, we obtain

$$0 = \int_M f\Psi_\varepsilon dS \overset{\text{(III.B.2)}}{=} \int_V f(x', \varphi(x'))\Psi_\varepsilon(x', \varphi(x'))\sqrt{1 + |\nabla\varphi(x')|^2}dx'.$$

Since $\Psi_\varepsilon(\cdot, \varphi(\cdot)) \to \psi$ in $L^\infty(V)$ as ε tends to zero, we have

$$\int_V f(x', \varphi(x'))\psi(x')\sqrt{1 + |\nabla\varphi(x')|^2}dx' = 0.$$

Since $\psi \in \mathcal{C}_c^\infty(V)$ is arbitrary, the fundamental theorem of variational calculus in the Euclidean case provides $f(x', \varphi(x')) = 0$ for $x' \in V$. The claim follows. $\square$

We consider the following tangential derivative operators $\partial_{\tau_{jk}}$ acting on test functions, being compactly supported functions ψ of class $\mathcal{C}^1$ in a neighbourhood of M:

$$\partial_{\tau_{jk}}\psi := \nu_j(\partial_k\psi) - \nu_k(\partial_j\psi), \quad 1 \le j, k \le d. \tag{III.B.3}$$

The tangential gradient is defined by

$$\nabla^{\tan}\psi := \nabla\psi - (\nu \cdot \nabla\psi)\nu$$

Lemma III.B.2. *For any $\mathcal{C}^1$ function ψ defined in a neighbourhood of M, we have*

$$(\nabla^{\tan}\psi)_j = \sum_{k=1}^d \nu_k\partial_{\tau_{kj}}\psi, \tag{III.B.4}$$

$$\partial_{\tau_{jk}}\psi = \nu_j(\nabla^{\tan}\psi)_k - \nu_k(\nabla^{\tan}\psi)_j. \tag{III.B.5}$$

Further, all the tangential operators in (III.B.4) *and* (III.B.5) *depend only on $\psi \restriction_M$. In other words, if $\psi \in \mathcal{C}_c^1(\mathbb{R}^d)$ satisfies $\psi \equiv 0$ on M, then $\partial_{\tau_{jk}}\psi = 0$ on M.*

Proof. The arguments for (III.B.4) and (III.B.5) contains purely elementary calculations. We completely follow Mitrea and Mitrea [50]. First, we show (III.B.4):

$$\nabla^{\tan}\psi := \nabla\psi - (\nu \cdot \nabla\psi)\nu = \sum_{j=1}^d (|\nu|\partial_j\psi - (\nu \cdot \nabla\psi)\nu_j)\mathbf{e}_j$$

$$= \sum_{j,k=1}^d \nu_k(\nu_k\partial_j\psi - \nu_j\partial_k\psi)\mathbf{e}_j = \sum_{j,k=1}^d \nu_k\partial_{\tau_{kj}}\psi\mathbf{e}_j$$

We turn to (III.B.5):

$$
\begin{aligned}
&\nu_j(\nabla^{\tan}\psi)_k - \nu_k(\nabla^{\tan}\psi)_j \\
&\qquad\overset{\text{(III.B.4)}}{=} \nu_j \sum_{\ell=1}^{d} \nu_\ell \partial_{\tau_{\ell k}} - \nu_\ell \sum_{r=1}^{d} \nu_r \partial_{\tau_{rj}} \psi \\
&\qquad\overset{\text{(III.B.5)}}{=} \nu_j \sum_{\ell=1}^{d} \nu_\ell^2 \partial_k \psi - \nu_j \nu_k \sum_{\ell=1}^{d} \nu_\ell \partial_\ell \psi - \nu_k \sum_{r=1}^{d} \nu_r^2 \partial_j \psi + \nu_j \nu_k \sum_{r=1}^{d} \nu_r \partial_r \psi \\
&\qquad\overset{|\nu|=1}{=} \nu_j \partial_k \psi - \nu_k \partial_j \psi = \partial_{\tau_{jk}} \psi.
\end{aligned}
$$

To prove the last claim, test ψ with an arbitrary function $\varphi \in \mathcal{C}^1_{\mathrm{c}}(\mathbb{R})$

$$\int_M \varphi(\partial_{\tau_{jk}}\psi) = 0.$$

and use Lemma III.B.1. The proof is complete. □

III.B.2 Sobolev spaces on compact Lipschitz surfaces

Let $p \in [1,\infty]$. We define $W^{1,p}(M)$ as the closure of $\{\psi \restriction_M \colon \psi \in \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^d)\}$ with respect to the norm

$$\|f\|_{W^{1,p}(M)} := \|f\|_{L^p(M)} + \sum_{j,k=1}^{d} \|\partial_{\tau_{jk}} f\|_{L^p(M)} \sim_{p,M} \|f\|_{L^p(M)} + \|\nabla^{\tan} u\|_{L^p(M)} \tag{III.B.6}$$

where the last equivalence of norms is due to (III.B.4) and (III.B.5). The Sobolev norm for a chart $\|\cdot\|_{W^{1,p}(U)}$ can be defined in a similar way.

Proposition III.B.3. *Let $\{O,U,V,\varphi\}$ be a chart. Then, for $p \in [1,\infty]$*

$$f \in W^{1,p}(U) \ \Leftrightarrow\ f(\cdot,\varphi(\cdot)) \in W^{1,p}(V)$$

with equivalence of norms

$$\|f(\cdot,\varphi(\cdot))\|_{L^p(V)} \sim \|f\|_{L^p(U)}, \quad \|\nabla f(\cdot,\varphi(\cdot))\|_{L^p(V)} \sim \sum_{j,k=1}^{d} \|\partial\tau_{jk} f\|_{L^p(U)} \tag{III.B.7}$$

the multiplicative constant depending on $\|\nabla\varphi\|_{L^\infty(V)}$.

Proof. "$\Rightarrow$ and $\lesssim$": By definition, there exists a sequence $\{F_n\} \subset \mathcal{C}^\infty(\mathbb{R}^d) \restriction_U$ converging to f in $W^{1,p}(U)$. By the chain rule and (III.B.1), we have

$$\partial_j[F_n(\cdot,\varphi(\cdot))] = [\partial_j F_n](\cdot,\varphi(\cdot)) + [\partial_d F_n](\cdot,\varphi(\cdot))\partial_j\varphi = \partial_{\tau_{jd}} F_n \sqrt{1+|\nabla\varphi|^2}. \tag{III.B.8}$$

Since $\{F_n\}$ and $\{\partial_{\tau_{jd}} F_n\}$ are Cauchy sequences in $L^p(U)$ and $\nabla\varphi$ is uniformly bounded, $\{F_n(\cdot,\varphi(\cdot))\}$ and $\{\partial_j F_n(\cdot,\varphi(\cdot))\}$ are Cauchy sequence in $L^p(V)$. Therefore $\{F_n(\cdot,\varphi(\cdot))\}$ converges to some $g \in W^{1,p}(V)$. Since

$$F_n(\cdot,\varphi(\cdot)) \to f(\cdot,\varphi(\cdot)) \quad \text{in} \quad L^p(V)$$

we have $f(\cdot,\varphi(\cdot)) = g$ a.e. and $f \in W^{1,p}(V)$. By (III.B.8) and the boundedness of $\nabla\varphi$ we have "$\lesssim$" of (III.B.7) with f replaced by F_n. By letting n tend to infinity the claim follows.

"$\Leftarrow$ and $\gtrsim$": There exists a sequence $\{f_n\} \subset \mathcal{C}_c^\infty(V)$ such that

$$f_n \to f(\cdot,\varphi(\cdot)) \quad \text{in} \quad W^{1,p}(V). \tag{III.B.9}$$

Extend f_n to $\mathbb{R}^d$ by defining

$$F_n(x', x_d) := f_n(x')\theta(x_d - \varphi(x')) \quad \text{where} \quad \theta \in \mathcal{C}_c^\infty(\mathbb{R}) \quad \text{s.t.} \quad \theta(0) = 1.$$

Obviously $F_n \in \mathrm{Lip}_c(\mathbb{R}^d)$ and $F_n(\cdot,\varphi(\cdot)) = f_n$. Further, by a standard approximation using convolution there exists $\widetilde{F}_n \in \mathcal{C}_c^\infty(\mathbb{R}^d)$ such that

$$\|F_n - \widetilde{F}_n\|_{L^\infty(\mathbb{R}^d)} \le \frac{1}{n}. \tag{III.B.10}$$

Then, (III.B.9) and (III.B.10) imply that

$$\widetilde{F}_n(\cdot,\varphi(\cdot)) \to f(\cdot,\varphi(\cdot)) \quad \text{in} \quad W^{1,p}(V). \tag{III.B.11}$$

As a first consequence we have $\widetilde{F}_n \to f$ in $L^p(U)$. Further, by (III.B.8) and the fact that $\nabla\varphi$ is bounded, $\{\partial_{\tau_{jd}} \widetilde{F}_n\}$ is a Cauchy sequence in $L^p(U)$. Fix $1 \le j,k \le d-1$. Let us show that

$$\{\partial_{\tau_{jk}} \widetilde{F}_n\} \text{ is a Cauchy sequence in } L^p(U). \tag{III.B.12}$$

Indeed, by the chain rule we have

$$\partial_k[\widetilde{F}_n(\cdot,\varphi(\cdot))] = [\partial_k \widetilde{F}_n](\cdot,\varphi(\cdot)) - [\partial_d \widetilde{F}_n](\cdot,\varphi(\cdot))\partial_k\varphi \tag{III.B.13}$$

$$\partial_j[\widetilde{F}_n(\cdot,\varphi(\cdot))] = [\partial_j \widetilde{F}_n](\cdot,\varphi(\cdot)) - [\partial_d \widetilde{F}_n](\cdot,\varphi(\cdot))\partial_j\varphi, \tag{III.B.14}$$

which implies

$$\begin{aligned} \partial_{\tau_{jk}} \widetilde{F}_n(\cdot,\varphi(\cdot))\sqrt{1+|\nabla\varphi|^2} &= \partial_j\varphi[\partial_k \widetilde{F}_n](\cdot,\varphi(\cdot)) - \partial_k\varphi[\partial_j \widetilde{F}_n](\cdot,\varphi(\cdot)), \\ &= \partial_j\varphi\partial_k[\widetilde{F}_n(\cdot,\varphi(\cdot))] - \partial_k\varphi\partial_j[\widetilde{F}_n(\cdot,\varphi(\cdot))]. \end{aligned} \tag{III.B.15}$$

Therefore (III.B.12) simply follows by the fact that $\{\nabla \widetilde{F}_n(\cdot,\varphi(\cdot))\}$ is a Cauchy sequence in $L^p(V)$. By (III.B.11) and (III.B.12), $\widetilde{F}_n$ converges to f in $W^{1,p}(U)$. Finally, by (III.B.2), (III.B.15) and the boundedness of $\nabla\varphi$ we have "$\gtrsim$" of (III.B.7) with f replaced by $\widetilde{F}_n$. Letting n tend to infinity we finish the proof. $\square$

Lemma III.B.4 (Product rule). *Let $\{O, U, V, \varphi\}$ be a chart. Further, let $\psi \in \mathrm{Lip}_c(U)$ and $f \in W^{1,p}(U)$. Then, $\psi f \in W^{1,p}(U)$ and $\partial_{\tau_{jk}}(\psi f) = \psi \partial_{\tau_{jk}} f + f \partial_{\tau_{jk}} \psi$ S-a.e. on U.*

Proof. Extend ψ to the following function:

$$\Psi(x', x_d) := \psi(x', \varphi(x'))\theta(x_d - \varphi(x')) \quad \text{where} \quad \theta \in \mathcal{C}_c^\infty(\mathbb{R}) \quad \text{s.t.} \quad \theta(0) = 1.$$

Obviously, we have

$$\Psi \in \mathrm{Lip}_c(\mathbb{R}^d) \quad \text{and} \quad \Psi \restriction_U = \psi.$$

By [29, Theorem 4, p279] this implies that $\Psi \in W^{1,\infty}(\mathbb{R}^d)$. By a standard approximation using convolution as in the proof of this theorem we construct $\Psi_n = \Psi * \eta_n$ satisfying $\Psi_n \to \Psi$ uniformly, $\nabla \Psi_n \to \nabla \Psi$ a.e. and $\|\nabla \Psi_n\|_\infty \leq \|\nabla \Psi\|_\infty$. Further, there exists a sequence $\{F_n\} \subset C^\infty(\mathbb{R}^d) \restriction_U$ converging to f in $W^{1,p}(U)$. Then, by the triangle inequality, we have

$$\|\Psi_n F_n - \psi f\|_{L^p(U)} \leq \|\Psi_n\|_{L^\infty(U)} \|F_n - f\|_{L^p(U)} + \|\Psi_n - \psi\|_{L^\infty(U)} \|f\|_{L^p(U)} \overset{n\to\infty}{\longrightarrow} 0. \tag{III.B.16}$$

The proof is now very routine and is almost the same as in the Euclidean case. By the product rule in the Euclidean case together with definition (III.B.3) of $\partial_{\tau_{jk}}$, we obtain

$$\partial_{\tau_{jk}}(\Psi_n F_n) = \Psi_n \partial_{\tau_{jk}} F_n + F_n \partial_{\tau_{jk}} \Psi_n. \tag{III.B.17}$$

By the triangle inequality, we have

$$\begin{aligned} &\|\partial_{\tau_{jk}}(\Psi_n F_n) - \partial_{\tau_{jk}}(\Psi_m F_m)\|_{L^p(U)} \\ &\qquad \leq \|\Psi_n \partial_{\tau_{jk}} F_n - \Psi_m \partial_{\tau_{jk}} F_m\|_{L^p(U)} + \|F_n \partial_{\tau_{jk}} \Psi_n - F_m \partial_{\tau_{jk}} \Psi_m\|_{L^p(U)} \end{aligned} \tag{III.B.18}$$

We continue splitting by means of the triangle inequality:

$$\begin{aligned} &\|\Psi_n \partial_{\tau_{jk}} F_n - \Psi_m \partial_{\tau_{jk}} F_m\|_{L^p(U)} \\ &\quad \leq \|\Psi_n\|_{L^\infty(U)} \|\partial_{\tau_{jk}} F_n - \partial_{\tau_{jk}} F_m\|_{L^p(U)} + \|\Psi_n - \Psi_m\|_{L^\infty(U)} \|\partial_{\tau_{jk}} F_m\|_{L^p(U)} \end{aligned} \tag{III.B.19}$$

and

$$\begin{aligned} &\|F_n \partial_{\tau_{jk}} \Psi_n - F_m \partial_{\tau_{jk}} \Psi_m\|_{L^p(U)} \\ &\qquad\qquad \leq \|(F_n - F_m) \partial_{\tau_{jk}} \Psi_n\|_{L^p(U)} + \|F_m(\partial_{\tau_{jk}} \Psi_n - \partial_{\tau_{jk}} \Psi_m)\|_{L^p(U)} \end{aligned} \tag{III.B.20}$$

Since $\Psi_n \to \Psi$ uniformly and $\{F_n\}$ is a Cauchy sequence in $W^{1,p}(U)$, the right-hand side of (III.B.19) tends to zero as m, n tend to infinity. Since $\|\nabla \Psi_n\|_\infty \leq \|\nabla \Psi\|_\infty$, we can apply the Dominated Convergence Theorem to get that the right-hand side of (III.B.20) also tends to zero. Therefore, $\{\Psi_n F_n\}$ is a Cauchy sequence in $W^{1,p}(U)$. Knowing (III.B.16) we have that $\Psi_n F_n \to \psi f$ in $W^{1,p}(U)$, especially

$$\partial_{\tau_{jk}}(\Psi_n F_n) \to \partial_{\tau_{jk}}(\psi f), \quad \text{in } L^p(U). \tag{III.B.21}$$

Letting m and then n tend to infinity in (III.B.19) and (III.B.20) yields that

$$\Psi_n \partial_{\tau_{jk}} F_n \to \Psi \partial_{\tau_{jk}} F, \quad F_n \partial_{\tau_{jk}} \Psi_n \to F \partial_{\tau_{jk}} \Psi, \quad \text{in } L^p(U). \tag{III.B.22}$$

Combining (III.B.21) and (III.B.22) with (III.B.17) we finish the proof. □

III.B.3 Sobolev's and Poincaré's inequality

Let us first start with the Poincaré inequality on a "roof" (see Figure III.4 and Subsection III.B.3). This object can be considered as the graph of the function

$$\varphi(x') = 1 - |x_1|, \quad x' = (x_1, \dots, x_{d-1}).$$

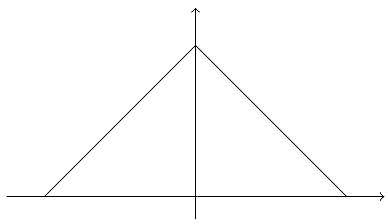

Figure III.6: A two dimensional roof

In other words, we need a Poincaré inequality on a local chart $\{O, U, V, \varphi\}$ of M. In the Euclidean case, the Poincaré inequality relies on the so-called Rellich compact embedding $W^{1,p} \subset\subset L^p, p \in [1, \infty]$ (see Evans [29]). Since a local chart looks like a Euclidean space (Proposition III.B.3), we also have

$$W^{1,p}(U) \subset\subset L^p(U). \tag{III.B.23}$$

Then, the fact that Poincaré's inequality is true on a local chart is obvious.

Lemma III.B.5 (Poincaré's inequality on local charts)**.** *Poincaré's inequality is true on an arbitrary chart $\{O, U, V, \varphi\}$ of M in the following sense. For any $p \in [1, \infty]$ and for any $f \in W^{1,p}(U)$ we have*

$$\|f - (f)_U\|_{L^p(U)} \lesssim_U \|\nabla^{\tan} f\|_{L^p(U)}.$$

Proof. The proof follows the same spirit as that for the Euclidean case (see e.g. Evans [29]). Assume there were s sequence $\{f_n\}$ such that

$$\|f_n - (f_n)_U\|_{L^p(U)} \geq n \|\nabla^{\tan} f_n\|_{L^p(U)}.$$

Normalizing

$$g_n = \frac{f_n - (f_n)_U}{\|f_n - (f_n)_U\|_{L^p(U)}}$$

we obtain

$$1 = \|g_n\|_{L^p(U)} \geq n \|\nabla^{\tan} g_n\|_{L^p(U)} \quad \text{and} \quad (g)_U = 0. \tag{III.B.24}$$

As a first consequence, $\{g_n\}$ is bounded in $W^{1,p}(U)$. Therefore, by (III.B.23) there exists a subsequence $\{g_{n_m}\}$ such that

$$g_{n_m} \to g \quad \text{in} \quad L^p(U). \tag{III.B.25}$$

On the other hand, combining (III.B.24) with the equivalent of norms (III.B.6) yields that

$$\|\partial_{\tau_{jk}} g_n\|_{L^p(U)} \to 0. \tag{III.B.26}$$

By (III.B.25) and (III.B.26), $\{g_{n_m}\}$ converges to g in $W^{1,p}(U)$, and $\partial_{\tau_{jk}} g = 0$. The equivalence of norms (III.B.7) in Proposition III.B.3 yields that $\nabla g(\cdot, \varphi(\cdot)) = 0$, meaning g is constant on U. However, it cannot happen, since $\|g\|_{L^p(U)} = 1$ and $(g)_U = 0$, which follow from (III.B.24). □

We turn to the Poincaré inequality on the whole boundary.

Theorem III.B.6 (Poincaré's inequality)**.** *Assume that M is connected. Then,*

$$\|f - (f)_M\|_{L^p(M)} \lesssim_M \|\nabla^{\tan} f\|_{L^p(M)}$$

where

$$(f)_M = \frac{\int_M f dS}{\int_M 1 dS}$$

denotes the average of f on M.

As in the Euclidean case, for Theorem III.B.6, we need the following result.

Theorem III.B.7 (Rellich)**.** *For any $p \in [1, \infty]$ we have*

$$W^{1,p}(M) \subset\subset L^p(M).$$

Proof of Theorem III.B.6 from Theorem III.B.7. The proof is almost the same as that for local charts (Lemma III.B.5). We obtain $g \in W^{1,p}(M)$ with

$$(g)_M = 0, \quad \|g\|_{L^p(M)} = 1, \quad \text{and} \quad \partial_{\tau_{jk}} = 0 \quad \text{in} \quad M.$$

By the equivalent of norms (III.B.7) in Proposition III.B.3, g is constant on each chart. Since M is connected, we get a contradiction. □

The strategy to prove Theorem III.B.7 is "divide and conquer": using a partition of unity we decompose a function defined on M into functions supported on each local charts and lift them to the Euclidean space.

Proposition III.B.8. *There exist finitely many charts $\{O_j, U_j, V_j, \varphi_j\}$, $1 \le j \le N$ such that $\{U_j\}_{j=1}^N$ covers M. Further, there exists a partition of unity $\{\alpha_j\}_{j=1}^N$ subordinate to $\{U_j\}_{j=1}^N$ in the sense that* *i)* $0 \leq \alpha_j \leq 1$ *are all Lipschitz continuous functions defined on* M*,*

ii) $\sum_{j=1}^{N} \alpha_j = 1$, *and*

iii) $\operatorname{supp}(\alpha_j)$ *is a compact subset of* U_j.

Proof. The proof is standard and can be found in every textbook on differential manifolds, for instance, Tu [61, Proposition 13.9]. In our case, we only need to make sure that the standard construction also works for the Lipschitz case. For each $a \in M$ let $\{O'_a, U'_a, V'_a, \varphi_a\}$, $\{O_a, U_a, V_a, \varphi_a\}$ be charts for M at a satisfying $O'_a \subset O_a$ and $\operatorname{dist}(O'_a, \partial O_a)\} > 0$. Further, let β_a be a smooth function satisfying $\operatorname{supp}(\beta_a) \subset\subset V_a$, $\alpha_a = 1$ in V'_a. By compactness, there exists $\{a_j\}_{j=1}^N$ such that $\{O'_{a_i}\}_{j=1}^N$ covers M. For simplicity the index a_i is replaced by i. Define

$$\widetilde{\alpha}_j : M \to \mathbb{R}, \quad \widetilde{\alpha}_j(x', \varphi_j(x')) = \beta_j(x'), \quad x' \in V_j, \quad \widetilde{\alpha}_j = 0 \quad \text{in} \quad M \setminus U_j.$$

By definition $\alpha(x) := \sum_j \widetilde{\alpha}_j > 0$ for any $x \in M$, since $x \in V'_j$ for some j. By normalizing $\alpha_j := \widetilde{\alpha}_j / \alpha$ we have $\sum_j \alpha_j = 1$. Further, by construction $\widetilde{\alpha}_j$ are Lipschitz. Therefore α_j is also Lipschitz, which can be seen by writing

$$\frac{\widetilde{\alpha}_j(x)}{\alpha(x)} - \frac{\widetilde{\alpha}_j(y)}{\alpha(y)} = \frac{(\widetilde{\alpha}_j(x) - \widetilde{\alpha}_j(y))\, \alpha(y) - (\alpha(x) - \alpha(y))\, \widetilde{\alpha}_j(y)}{\alpha(x)\alpha(y)}$$

and using the fact that both α and $\widetilde{\alpha}_j$ are Lipschitz and bounded from above, and in particular, by compactness, α is bounded from below. □

Proof of Theorem III.B.7. Assume $\{f_m\}$ is a bounded sequence in $W^{1,p}(M)$. Proposition III.B.8 compactness there exist finitely many charts $O_j, U_j, V_j, \varphi_j, 1 \leq j \leq N$ covering M and a partition of unity $\{\alpha_j\}_{j=1}^N$ subordinate to them. Note that by defining an extension as done for ψ in Lemma III.B.4 we can assume $\alpha_j \in \operatorname{Lip_c}(\mathbb{R}^d)$. Therefore, by [29, Theorem 4, p279], we have $\alpha_j \in W^{1,\infty}(\mathbb{R}^d)$ and

$$A := \max_{1 \leq j \leq N} \|\alpha_j\|_{W^{1,\infty}(\mathbb{R}^d)} < \infty \tag{III.B.27}$$

depends only on the structure of M. Noting that by Lemma III.B.4 (Product rule)

$$\partial_{\tau_{kl}}(\alpha_j f_m) = \alpha_j(\partial_{\tau_{kl}} f_m) + f_m(\partial_{\tau_{kl}} \alpha_j) \tag{III.B.28}$$

we have

$$\|\alpha_j f_m\|_{W^{1,p}(U_j)} \lesssim \|f_m\|_{W^{1,p}(U_j)} \leq \|f_m\|_{W^{1,p}(M)},$$

the multiplicative constant depending only on A. By Proposition III.B.3 and the fact that $\operatorname{supp}(\alpha_j \restriction_{U_j}) \subset U_j$ the sequence $\{[\alpha_j f_m](\cdot, \varphi(\cdot))\}_{m=1}^\infty$ is bounded in $W^{1,p}(V_j)$. Applying the embedding $W^{1,p}(V_j) \subset\subset L^p(V_j)$ (see the remark in [29, p274]) successively for $1 \leq j \leq N$ yields the existence of a subsequence $\{m_k\}$ such that $\{[\alpha_j f_{m_k}](\cdot, \varphi(\cdot))\}$ converges in $L^p(V)$ for all $1 \leq j \leq N$. Since $\nabla\varphi$ is bounded, lifting things to M yields that $\{\alpha_j f_{m_k}\}$ converges in $L^p(M)$. By the triangle inequality and the fact that $\sum_{j=1}^N \alpha_j = 1$ on M we have

$$\varlimsup_{k,k'\to\infty} \|f_{m_k} - f_{m_{k'}}\|_{L^p(M)} \leq \varlimsup_{k,k'\to\infty} \sum_j \|\alpha_j(f_{m_k} - f_{m_{k'}})\|_{L^p(M)} = 0.$$

Therefore, $\{f_{m_k}\}$ is a Cauchy sequence in $L^p(M)$. The proof is complete. $\square$

Chapter IV

The Dirichlet-to-Neumann estimate

IV.1 Statement of the main result

The aim of this part is to prove Theorem III.2.5 in Chapter III which is a discrete analogue of a well-known result [45, 49] claiming that the tangential and normal component of the gradient of a harmonic function on the boundary of a domain, for instance, the unit ball, are comparable in the following sense:

Theorem. *Let $p \in (1, \infty)$ and $u : \overline{B}_1 \to \mathbb{R}$ be harmonic in B_1 and $u|_{\partial B_1} \in W^{1,p}(\partial B_1)$. Then, there exist $c_1, c_2 \in (0, \infty)$ depending on p and d such that*

$$\int_{\partial B_1} |\nabla^{\mathrm{nor}} u|^p \leq c_1 \int_{\partial B_1} |\nabla^{\tan} u|^p \quad \textit{and} \quad \int_{\partial B_1} |\nabla^{\tan} u|^p \leq c_2 \int_{\partial B_1} |\nabla^{\mathrm{nor}} u|^p. \tag{IV.1.1}$$

By a simple scaling argument we get the estimates with the same constants c_1, c_2 for harmonic functions defined on $\overline{B}_r$ for all $r > 0$. In the proof given by Maergoiz [45] and Mikhlin [49] the tangential and the normal component are L^p-bounded from above by the density of the single and double layer potential by a Calderon-Zymund theorem. The lower bound also follows, since the densities can be represented as solutions of singular integral equations, in which the tangential and normal component are considered as inputs.

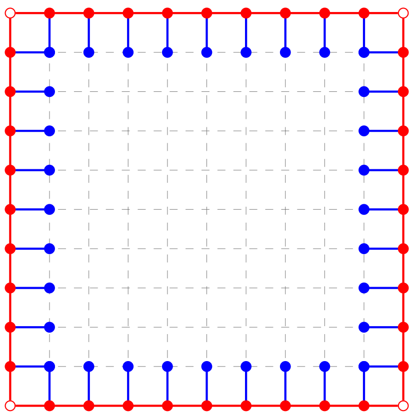

Figure IV.1: Tangential (*red*) and normal edges (*blue*)

We call $u : A \subset \mathbb{Z}^d \to \mathbb{R}$ harmonic at $x \in A$ if

$$u(x) = \frac{1}{2d} \sum_{y:|x-y|_1=1} u(y),$$

and harmonic in $B \subset A$ if it is harmonic at every $x \in B$. Consider a discrete box $[0, N]^d \cap \mathbb{Z}^d$ for $N \gg 1$. Then, the discrete analogues of the tangential and normal component are defined as follows. We denote by $E_N^{\tan}$ the set of all nearest neighbor edges $\{x,y\}$ with x,y on the boundary

$$\{0,\ldots,N\}^d \setminus \{1,\ldots,N-1\}^d$$

of the box, and by E_N^{nor} the set of all nearest neighbor edges $\{x,y\}$ with x in the interior $\{1,\ldots,N-1\}^d$ and y on the boundary

$$\{0,\ldots,N\}^d \setminus \{1,\ldots,N-1\}^d.$$

Note that in Chapter III, it was defined as the set of inner normal edges. However, for the main result below, we do not have to worry about this – we consider unoriented edges just for consistency.

For $u : \{0,\ldots,N\}^d \to \mathbb{R}$ define

$$\|\nabla u\|_{L^p\left(E_N^*\right)} := \left[\sum_{\{x,y\}\in E_N^*} |u(x)-u(y)|^p\right]^{\frac{1}{p}}, \quad * \in \{\tan, \mathrm{nor}\}. \tag{IV.1.2}$$

In the following and throughout the chapter, "$F \lesssim_{a,b} G$" means "$F \leq C(a,b)G$" where C is a positive constant depending on a and b.

Theorem IV.1.1. *Let $d \geq 2$ and $N \gg 1$. Let $u : \{0,\ldots,N\}^d \to \mathbb{R}$ be harmonic in $\{1,\ldots,N-1\}^d$. Then, for any $p \in (1,\infty)$,*

$$\|\nabla u\|_{L^p\left(E_N^{\mathrm{nor}}\right)} \lesssim_{p,d} \|\nabla u\|_{L^p\left(E_N^{\tan}\right)}, \qquad \textit{(Dirichlet case)}.$$

Further, there exists a harmonic modification $\widetilde{u}$ of u, in the sense that $\widetilde{u}$ is harmonic in $\{1,\ldots,N-1\}^d$ and $\widetilde{u} = u$ also in this set, such that the inverse estimate is true:

$$\|\nabla \widetilde{u}\|_{L^p\left(E_N^{\tan}\right)} \lesssim_{p,d} \|\nabla u\|_{L^p\left(E_N^{\mathrm{nor}}\right)}, \qquad \textit{(Neumann case)}.$$

Here, we speak of a modification, since the Neumann condition does not define the values of the harmonic function at the "corners". In other words, the values of u given in Theorem IV.1.1 can be modified freely at those points without damaging the harmonicity. Here, we use the word "corners", since for $d = 2$ those points are really the four corners (Figure IV.1). The reader can easily define what a corner is for $d \geq 3$.

Remark. Our method is robust in the sense that Theorem IV.1.1 is still true for d-dimensional rectangles $\prod_{j=1}^d \{0,\ldots,N_j\}$ with $1/a \leq N_j/N_1 \leq a$, fixed $a \in (1,\infty)$.

IV.2 Outline of the proof

Recently, Bella, Fehrman, and Otto [12] have introduced an elementary proof for (IV.1.1), where the unit ball is replaced by the unit box. Especially, their argument only relies on reflections and Fourier analysis. We are going to implement this idea for the discrete case.

IV.2.1 A reflection argument

First, mimicking their idea of reflections, we decompose u into d harmonic functions $w_1, \dots, w_d$ with periodic boundary values as in the following Propositions IV.2.1 and IV.2.2.

Proposition IV.2.1 (Dirichlet decomposition). *Given u in Theorem IV.1.1. Then, there exists a unique decomposition $u = w_1 + \ldots + w_d$ on $\{0, \dots, N\}^d$ with the following properties.*

(i) w_i is defined in the infinite strip $\mathbb{Z}^{i-1} \times \{0, \dots, N\} \times \mathbb{Z}^{d-i}$, $2N$-periodic along its definition domain, i.e.

$$w_i = w_i(\cdot + 2N\mathbf{e}_j), \quad \forall j \neq i,$$

and harmonic in $\mathbb{Z}^{i-1} \times \{1, \dots, N-1\} \times \mathbb{Z}^{d-i}$.

(ii) It holds

$$w_i = u - \sum_{\nu=1}^{i-1} w_\nu \quad \text{on} \quad \{0, \dots, N\}^{i-1} \times \{0, N\} \times \{0, \dots, N\}^{d-i}. \tag{IV.2.1}$$

(iii) It holds

$$w_i = 0 \quad \text{on} \quad \{0, \dots, N\}^{j-1} \times \{0, N\} \times \{0, \dots, N\}^{d-j}, \quad 0 \leq j \leq i-1. \tag{IV.2.2}$$

Before discussing the decomposition for the Neumann case let us introduce the notation for the inner normal derivatives. The rule for the formula is *in minus out*:

$$\nabla_j^{\mathrm{nor}} f(x) := \begin{cases} f(x + \mathbf{e}_j) - f(x), & x_j = 0, \\ f(x - \mathbf{e}_j) - f(x), & x_j = N, \end{cases} \quad x \in \mathbb{Z}^{j-1} \times \{0, N\} \times \mathbb{Z}^{d-j}. \tag{IV.2.3}$$

Proposition IV.2.2 (Neumann decomposition). *Given u in Theorem IV.1.1. Then, there exists a decomposition $u = w_1 + \ldots + w_d$ up to a constant on $\{1, N-1\}^d$ with the following properties.*

(i) w_i is defined in the infinite strip $\mathbb{Z}^{i-1} \times \{0, \dots, N\} \times \mathbb{Z}^{d-i}$, $2(N-1)$-periodic along its definition domain, i.e.

$$w_i = w_i(\cdot + 2(N-1)\mathbf{e}_j) \quad \forall j \neq i,$$

and harmonic in $\mathbb{Z}^{i-1} \times \{1, \dots, N-1\} \times \mathbb{Z}^{d-i}$.

(ii) It holds

$$\nabla_i^{\mathrm{nor}} w_i = \nabla_i^{\mathrm{nor}} u - \sum_{\nu=1}^{i-1} \nabla_i^{\mathrm{nor}} w_\nu \quad \text{on} \quad \{1, \dots, N-1\}^{i-1} \times \{0, N\} \times \{1, \dots, N-1\}^{d-i}. \tag{IV.2.4}$$

(iii) It holds

$$\nabla_j^{\mathrm{nor}} w_i = 0 \quad \textit{on} \quad \{1,\dots,N-1\}^{j-1} \times \{0,N\} \times \{1,\dots,N-1\}^{d-j}, \qquad 0 \le j \le i-1. \tag{IV.2.5}$$

Uniqueness holds only up to a constant.

The fact that the period in Proposition IV.2.2 is $2(N-1)$ instead of $2N$ is due to the discreteness. In Propositions IV.2.1 and IV.2.2, each function w_i is constructed first on the boundary $\mathbb{Z}^{j-1} \times \{0,N\} \times \mathbb{Z}^{d-j}$ with periodic values and harmonically extended into $\mathbb{Z}^{j-1} \times \{0,\dots,N\} \times \mathbb{Z}^{d-j}$. This harmonic extension can be done due to the following standard result.

Assumption IV.2.3 (harmonic functions on infinite strips)**.** Let $d \ge 2$, $N, L \gg 1$. Let $u : \mathbb{Z}^{d-1} \times \{0,\dots,N\} \to \mathbb{R}$ be $2L$-periodic in the first $(d-1)$ arguments, i.e.

$$u = u(\cdot + 2L\mathbf{e}_j), \quad 1 \le j \le d-1,$$

and harmonic in $\mathbb{Z}^{d-1} \times \{1,\dots,N-1\}$.

Theorem IV.2.4 (harmonic extension – periodic case)**.** *The Dirichlet problem "find u satisfying Assumption IV.2.3 given $u(\cdot,0)$ and $u(\cdot,N)$" is uniquely solvable. The Neumann problem "find u satisfying Assumption IV.2.3 given $\nabla_d^{\mathrm{nor}} u$ (defined in* (IV.2.3)*) with $\langle \nabla_d^{\mathrm{nor}} u \rangle = 0$", where the mean is taken on $\mathbf{I}_L^{d-1} \times \{0,N\}$, is uniquely solvable (up to a constant).*

Proof of Theorem IV.2.4 (uniqueness). In both cases uniqueness follows from the maximum principle. In the Neumann case, since the gradient $\nabla^{(2)} u := u(\cdot,\cdot+1) - u$ is defined in $\mathbb{Z}^{d-1} \times \{0,\dots,N-1\}$ and harmonic in $\mathbb{Z}^{d-1} \times \{1,\dots,N-2\}$, we get

$$\|\nabla^{(2)} u(\cdot,\cdot)\|_\infty \le \|\nabla_d^{\mathrm{nor}} u\|_\infty$$

that means the solution to zero Neumann condition is a constant function. □

In fact, the Dirichlet and Neumann problem given in Theorem IV.2.4 (Subsection IV.3.2) can be solved explicitly by using the Fourier transform taken on each horizontal line

$$\mathbf{I}_L^{d-1} \times \{y\}, \quad 0 \le y \le N$$

with

$$\mathbf{I}_L := \{-L+1,\dots,L\}, \tag{IV.2.6}$$

which is an advantage of the periodic case. Setting up Fourier analysis step by step we prove Theorem IV.2.4 in Subsections IV.3.1 and IV.3.2. Although this theorem is not our goal, explicit Fourier-type calculations using to prove it are the main part of

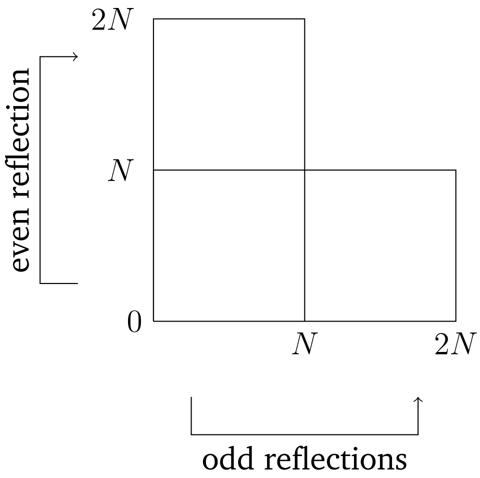

Figure IV.2: Dirichlet decomposition, $d = 2$

the proof of the main result, Theorem IV.1.1. The reader should read Section IV.3 completely and linearly.

Before going ahead, it is worth giving an argument for Propositions IV.2.1 and IV.2.2 in the simplest case $d = 2$, illustrated by figures. The proof for the general case (Section IV.A) is longer, however, it follows the spirit of that in $d = 2$. It is written in the appendix and can be skipped at the first reading.

Proof of Proposition IV.2.1 for $d = 2$. The proof is illustrated in Figure IV.2. We construct the functions w_1 and w_2 as follows. Set

$$w_1 = u \quad \text{on} \quad \{0, N\} \times \{0, \dots, N\}. \tag{IV.2.7}$$

Extend w_1 to $\{0, N\} \times \{0, \dots, 2N\}$ by an even reflection, i.e. set

$$w_1(x_1, 2N - x_2) := w_1(x_1, x_2), \quad x_1 \in \{0, N\}, \quad x_2 \in \{0, \dots, N\}.$$

Then, extend w_1 $2N$-periodically to $\{0, N\} \times \mathbb{Z}$, and to $\{1, \dots, N-1\} \times \mathbb{Z}$ so that it is harmonic in there (use Theorem IV.2.4). Define

$$w_2 = u - w_1 \quad \text{in} \quad \{0, \dots N\} \times \{0, N\}. \tag{IV.2.8}$$

By (IV.2.7), this construction implies

$$w_2 = 0 \quad \text{in} \quad \{0, N\} \times \{0, N\}. \tag{IV.2.9}$$

Extend w_2 to $\{0, \dots 2N\} \times \{0, N\}$ by an odd reflection:

$$w_2(2N - x_1, x_2) := -w_2(x_1, x_2), \quad x_1 \in \{0, \dots, N\}, \quad x_2 \in \{0, N\}.$$

Note that this extension is consistent with (IV.2.9). Then, extend w_2 $2N$-periodically to $\mathbb{Z} \times \{0, N\}$ and to $\mathbb{Z} \times \{1, \dots, N\}$ so that it is harmonic in there. Because of the

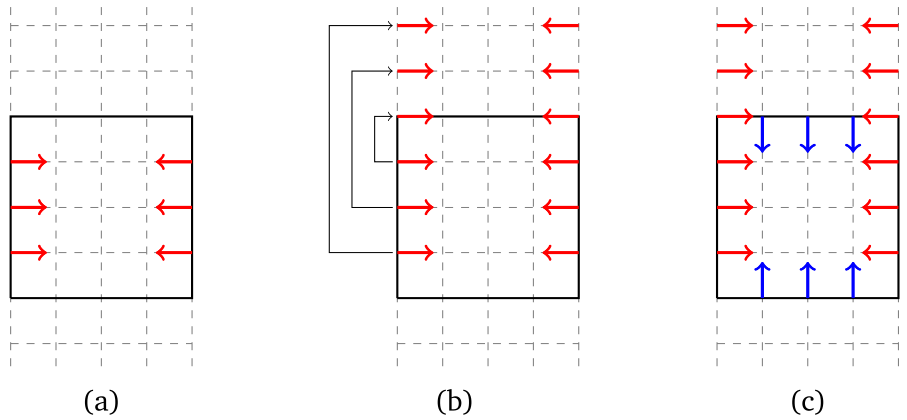

Figure IV.3: Construction of w_1 in the Neumann case, $d = 2$ and $N = 4$

odd reflection on the boundary, applying the uniqueness part of Theorem IV.2.4, we extend (IV.2.9) to the inner values:

$$w_2 = 0 \quad \text{in} \quad \{0, N\} \times \{0, \dots, N\}. \tag{IV.2.10}$$

Combining (IV.2.7), (IV.2.8) and (IV.2.10) yields

$$w_1 + w_2 = u \quad \text{in} \quad \underbrace{(\{0, N\} \times \{0, \dots, N\}) \cup (\{0, \dots, N\} \times \{0, N\})}_{\text{discrete boundary of } \{0,\dots,N\}^2}$$

Therefore $w_1 + w_2 = u$ in $\{0, \dots, N\}^2$, since these functions are harmonic. The proof of Proposition IV.2.1 for $d = 2$ is complete. $\square$

Sketch of the proof of Proposition IV.2.1 ($d \geq 2$). We construct $\{w_i : 1 \leq i \leq d\}$ successively as follows: Given $w_1, \dots, w_{i-1}$, then w_i is a priori defined via (IV.2.1) and extended in two steps: first, from the set in (IV.2.1) to its periodic boundary using reflections, and second, from its periodic boundary to its domain given in (IV.2.2) using Theorem IV.2.4. By the uniqueness part of Theorem IV.2.4, property (IV.2.5) is then a consequence of the odd reflections inherited from the periodic boundary conditions. The complete version of the proof for $d = 3$ is written in Subsection IV.A.1, illustrated by figures, and in Subsection IV.A.2 for the general case $d \geq 2$. $\square$

Proof of Proposition IV.2.2 for $d = 2$. The argument for the decomposition in the Neumann case is different from that in the Dirichlet case by the fact that we switch the role of the odd and even reflections, i.e. in contrast to Figure IV.2 (first even, then odd) we now have *first odd, then even*. This is due to a minor difficulty: in any case (peridic or box) *the Neumann condition of a harmonic function must be of mean zero*. The fact that the Neumann condition of a harmonic function defined in a box is of zero mean is standard and can be explained, for instance, by (III.5.22).

Roughly speaking, an odd reflection creates a zero mean, which is not a priori available, and an even reflection produces an odd reflection for the derivatives, i.e. it "kills" all derivatives of normal edges on the previous faces.

We constructed w_1 as follows. Set

$$v_1 = \nabla_1^{\mathrm{nor}} u \quad \text{on} \quad \{0, N\} \times \{1, \dots, N-1\} \tag{IV.2.11}$$

with ∇_i^{nor} defined in (IV.2.3), which are the derivatives with respect to the red edges in Figure IV.3a. Further, extend v_1 to $\{0, N\} \times \{1, \dots, 2N-2\}$ by an odd reflection, meaning that we set

$$v_1(x_1, x_2) = -v_1(x_1, 2N - x_2 - 1), \quad x_1 \in \{0, N\}, \quad x_2 \in \{N, \dots, 2N-2\}.$$

This is illustrated by Figure IV.3b, where the value at the sharp endpoint of an arrow is defined to be the *negative* of that at the other endpoint. Then, extend v_1 $(2N-2)$-periodically along x_2-direction to $\{0, N\} \times \mathbb{Z}$. Finally, call w_1 a function defined in $\{0, \dots, N\} \times \mathbb{Z}$, harmonic in $\{1, \dots, N-1\} \times \mathbb{Z}$ and satisfying

$$\nabla_1^{\mathrm{nor}} w_1 = v_1 \quad \text{in} \quad \{0, N\} \times \mathbb{Z} \tag{IV.2.12}$$

where we use the notation (IV.2.3). Here, we are allowed to apply Theorem IV.2.4 since *by the odd reflection, the Neumann condition immediately has zero mean.* Now set

$$v_2 = \nabla_2^{\mathrm{nor}} u - \nabla_2^{\mathrm{nor}} w_1 \quad \text{on} \quad \{1, \dots, N-1\} \times \{0, N\}, \tag{IV.2.13}$$

which are the blue edges, and extend v_2 to $\{1, \dots, 2N-2\} \times \{0, N\}$ by an even reflection,

$$v_2(x_1, x_2) = v_2(2N - x_1 - 1, x_2), \quad x_1 \in \{N, \dots, 2N-2\}, \quad x_2 \in \{0, N\},$$

and extend it $(2N-2)$-periodically to $\mathbb{Z} \times \{0, N\}$. We want to use v_2 as a Neumann condition. For this purpose, we have to check the mean zero condition carefully. We have

$$\langle v_2 \rangle_{\{1,\dots,N-1\}\times\{0,N\}} = \langle \nabla_2^{\mathrm{nor}} (u - w_1) \rangle_{\{1,\dots,N-1\}\times\{0,N\}} \tag{IV.2.14}$$

$$= - \langle \nabla_1^{\mathrm{nor}} (u - w_1) \rangle_{\{0,N\}\times\{1,\dots,N-1\}} \tag{IV.2.15}$$

$$= 0 \tag{IV.2.16}$$

where the first equality is definition (IV.2.13), the second is due to the fact that $u - w_1$, being harmonic in $\{1, \dots, N-1\}^2$, has a Neumann condition with zero mean, and the third is only definition (IV.2.11) and (IV.2.12). Now define w_2 so that it is harmonic in $\mathbb{Z} \times \{1, \dots, N-1\}$ and has the Neumann condition

$$\nabla_2^{\mathrm{nor}} w_2 = v_2. \tag{IV.2.17}$$

Since by construction w_2 is an even function in x_1, its derivative $\nabla^{(1)}w_2(\cdot,\cdot)$ is an odd function in x_1. We explain it carefully:

$$\begin{aligned}
\nabla^{(1)}w_2(x_1,x_2) &= w_2(x_1+1,x_2) - w_2(x_1,x_2) \\
\text{[even reflection]} \qquad &= -w_2(2N-(x_2+1)-1,x_2) - w_2(2N-x_1-1,x_2) \\
&= -\nabla^{(2)}w_2(x_1,2N-x_2-2) \\
\text{[}(2N-2)\text{-periodic]} \qquad &= -\nabla^{(1)}w_2(x_1,-x_2),
\end{aligned}$$

where the even reflection and the periodicity are inherited from that of the boundary condition. The last two inequalities tell us that $\nabla^{(1)}w_2$ as a function of x_2 is oddly reflected through 0 and $N-1$. Therefore, $\nabla^{(1)}w_2(\cdot,0) = \nabla^{(1)}w_2(\cdot,N-1) = 0$. In other words, the normal derivatives of w_2 at the red edges inside the box are "killed":

$$\nabla_1^{\text{nor}} w_2 = 0 \quad \text{in} \quad \{0,N\}\times\mathbb{Z}.$$

Combining this with (IV.2.12), (IV.2.13) and (IV.2.17) we have that the Neumann condition of w_1+w_2 is equal to that of u. The claim follows. □

Sketch of the proof of Proposition IV.2.1 ($d \geq 2$). We construct $\{w_k : 1 \leq k \leq d\}$ successively as follows: Given $w_1,\ldots,w_{k-1}$, then w_k is a priori defined via its Neumann condition (IV.2.4). In contrast to the Dirichlet case, we exploit odd reflections to produce Neumann conditions of zero mean, required for application of Theorem IV.2.4, and even reflections to kill normal derivatives on the previous faces. The complete version of the proof is written in Subsection IV.A.3. □

IV.2.2 Comparing edges on the periodic boundary

After getting the decomposition Propositions IV.2.1 and IV.2.2 we focus on each function w_i, satisfying Assumption IV.2.3 after changing the coordinates. An important ingredient of the proof of the main theorem (Theorem IV.1.1) is to compare the tangential derivatives and the normal derivatives of those functions on their periodic boundary. It is seemingly the most challenging issue in the part. For the proof of Theorem IV.1.1 we only need to consider $N = L$ or $N = L+1$. However, we want to set up a more general assumption:

$$N/L \geq \underline{r} \quad \text{for some } \underline{r} > 0. \tag{IV.2.18}$$

Before stating the results, let us introduce some notations. The discrete partial derivatives of

$$u : (x,y) \in \mathbb{Z}^{d-1}\times\mathbb{Z} \mapsto \mathbb{R}$$

are denoted by

$$\nabla_i^{(1)} u(x,y) = u(x+\mathbf{e}_i, y) - u(x,y), \quad \nabla^{(2)} u(x,y) = u(x,y+1) - u(x,y).$$

The full gradient is denoted by

$$\nabla u := \begin{bmatrix} \nabla^{(1)} u \\ \nabla^{(2)} u \end{bmatrix} := \left[\nabla_1^{(1)} u, \ldots, \nabla_{d-1}^{(1)} u, \nabla^{(2)} u\right]^\top. \tag{IV.2.19}$$

Further, recall $\mathbf{I}_L$ in (IV.2.6).

Proposition IV.2.5 (periodic Dirichlet)**.** *Suppose that Assumption IV.2.3 are satisfied. Further, assume* (IV.2.18) *and*

$$u(\cdot, N) = 0 \quad \textit{and} \quad \sum_{\mathbf{I}_L^{d-1}} u(\cdot, 0) = 0.$$

Then, for any $p \in (1,\infty)$, *we have*

$$\|\nabla^{(2)}(\cdot,0)\|_{L^p\left(\mathbf{I}_L^{d-1}\right)} \lesssim_{p,d,\underline{r}} \|\nabla^{(1)} u(\cdot,0)\|_{L^p\left(\mathbf{I}_L^{d-1}\right)}, \tag{IV.2.20}$$

$$\|\nabla^{(2)} u(\cdot, N-1)\|_{L^p\left(\mathbf{I}_L^{d-1}\right)} \lesssim_{p,d,\underline{r}} \|\nabla^{(1)} u(\cdot,0)\|_{L^p\left(\mathbf{I}_L^{d-1}\right)}. \tag{IV.2.21}$$

Remark. Proposition IV.2.5 is not true if the assumption $\sum_{\mathbf{I}_L^{d-1}} u(\cdot,0) = 0$ fails. A counter example is $u \equiv 1$.

Proposition IV.2.6 (periodic Neumann)**.** *Suppose that Assumption IV.2.3 are satisfied. Further, assume* (IV.2.18). *Then, for any* $p \in (1,\infty)$, *we have*

$$\|\nabla^{(1)} u(\cdot,0)\|_{L^p\left(\mathbf{I}_L^{d-1}\right)} \lesssim_{p,d,\underline{r}} \|\nabla^{(2)} u(\cdot,0)\|_{L^p\left(\mathbf{I}_L^{d-1}\times\{0,N\}\right)},$$

$$\|\nabla^{(1)} u(\cdot,N)\|_{L^p\left(\mathbf{I}_L^{d-1}\right)} \lesssim_{p,d,\underline{r}} \|\nabla^{(2)} u(\cdot,0)\|_{L^p\left(\mathbf{I}_L^{d-1}\times\{0,N\}\right)}.$$

As in the continuum case [12] the proof of Propositions IV.2.5 and IV.2.6 exploits the fact that the tangential and the normal derivatives are related by means of Fourier multipliers that can be calculated explicitly. Note that because a full tangential gradient has $(d-1)$ components, in the Dirichlet case, there are $(d-1)$ multipliers. We therefore need *a vector-valued Marcinkiewicz-type multiplier theorem*, namely Theorem IV.B.1 in the appendix, which is slightly improved from Theorem 2.49 in the book [40] by Jovanović and Süli, which is perhaps the only Marcinkiewicz-type result in the discrete case, and in fact, a scalar version.

Here, the main challenge is that Marcinkiewicz-type theorems are always requiring controls on the higher derivatives of the multipliers, which unfortunately

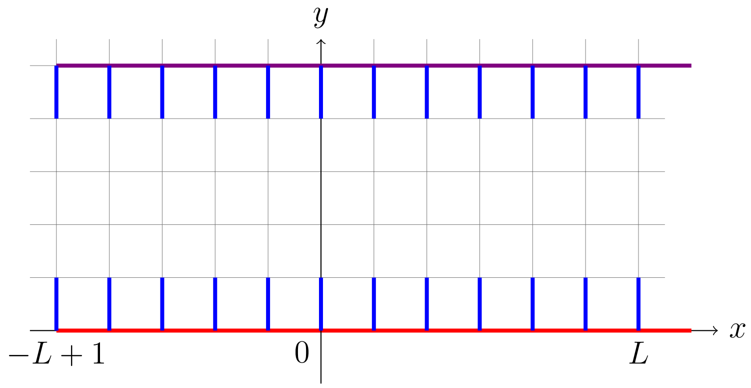

Figure IV.4: Comparing the tangential and normal derivatives $d = 2$. Proposition IV.2.5 (periodic Dirichlet) bounds the derivatives with respect to the blue edges by that with respect to the red edges assuming the function is zero on the top (violet). In constrast to this, Proposition IV.2.6 bounds the derivatives with respect to the red and violet edges by that with respect to the blue edges.

cannot be calculated explicitly due to the discreteness. We overcome this challenge by a simple application of Cauchy's integral formula, which allows us to estimate the higher derivatives elegantly without explicit calculations.

Also note that in the continuum case [12], although everything can perhaps be calculated explicitly, using Cauchy's integral formula still makes the arguments less tedious.

IV.2.3 An inner regularity estimate

Another important ingredient of the proof is the inner regularity estimate Proposition IV.2.7 for that we need to set up a new assumption:

$$N/L \leq \overline{r} \quad \text{for some } \overline{r} > 0. \tag{IV.2.22}$$

Proposition IV.2.7 (inner face – Figure IV.5)**.** *Suppose that Assumption IV.2.3 are satisfied. Further, assume* (IV.2.18) *and* (IV.2.22). *Then, for any* $p \in (1, \infty)$, *we have*

$$\|u\|_{L^p\left(\mathbf{I}_L^{i-1}\times\{0\}\times\mathbf{I}_L^{d-i-1}\times\{1,\ldots,N-1\}\right)} \lesssim_{p,d,\overline{r},\underline{r}} \|u\|_{L^p\left(\mathbf{I}_L^{d-1}\times\{0,N\}\right)}. \tag{IV.2.23}$$

Outline of the proof of Proposition IV.2.7. The proof is separated into two steps. In the first step, instead of Assumption IV.2.3, we consider harmonic functions on the haft space $\mathbb{Z}^{d-1}\times\{1,2,\ldots\}$ with $2L$-periodic boundary values on $\mathbb{Z}^{d-1}\times\{0\}$. In this case, exploiting estimates on the Poisson kernel and its derivatives, we establish (IV.2.23) under the assumption (IV.2.22). In the second step, we decompose a harmonic function on an infinite strip into an infinite sum of harmonic functions u_i on haft spaces applying Lemma IV.2.8 (this is the reason why we need to assume (IV.2.18)). Estimate (IV.2.23) for u follows then by adding that for u_i together. □

Lemma IV.2.8 (inner layer)**.** *Let $d \geq 2$ and $L \gg 1$. Let u be bounded on the haft space $\mathbb{Z}^{d-1} \times \{0, 1, \dots\}$, $2L$-periodic on the boundary, i.e.*

$$u = u(\cdot + 2L\mathbf{e}_j), \quad 1 \leq j \leq d-1,$$

and harmonic in $\mathbb{Z}^{d-1} \times \{1, 2, \dots\}$. Then, for all $y \in \{1, 2, \dots\}$,

$$\|u(\cdot, y)\|_{L^p\left(\mathbf{I}_L^{d-1}\right)} \leq \|u(\cdot, 0)\|_{L^p\left(\mathbf{I}_L^{d-1}\right)}. \tag{IV.2.24}$$

Further, assume (IV.2.18) *and $\sum_{\mathbf{I}_L^{d-1}} u(\cdot, 0) = 0$. Then, for any $p \in (1, \infty)$ there exists a constant $\alpha(p, \underline{r}) \in (0, 1)$ such that*

$$\|u(\cdot, N)\|_{L^p\left(\mathbf{I}_L^{d-1}\right)} \leq \alpha(p, \underline{r}) \|u(\cdot, 0)\|_{L^p\left(\mathbf{I}_L^{d-1}\right)}. \tag{IV.2.25}$$

Remark. For (IV.2.25) in Lemma IV.2.8 it is necessary to assume $\sum_{\mathbf{I}_L^{d-1}} u(\cdot, 0) = 0$. A counterexample is the constant function $u \equiv 1$.

The following result is not a surprise. Indeed, its periodic version follows directly from Lemma IV.2.8. We do not use it to prove the main result, Theorem IV.1.1, but it is useful for an estimate in Chapter III.

Corollary IV.2.9. *Let u be as in Theorem IV.1.1. Then, for any $p \in (1, \infty)$, we have*

$$\|u\|_{L^p(Q_N^d)} \lesssim_{d,p} N \|u\|_{L^p(\partial Q_N^d)} \tag{IV.2.26}$$

where $Q_N^d = \{0, \dots, N\}^d$ and $\partial Q_N^d = Q_N^d \setminus \{1, \dots, N\}^d$. Further,

$$\|\nabla u\|_{L^p(E_N^d)} \lesssim_{d,p} N \|\nabla u\|_{L^p(E_N^{\tan})} \tag{IV.2.27}$$

where E_N^d denotes the set of all nearest neighbour bonds $\{x, y\}$ with at least a vertex in $\{1, \dots, N-1\}^d$ and $|\nabla u(\{x, y\}) = |u(x) - u(y)|$.

Idea of the proof. Use the Dirichlet decomposition in Proposition IV.2.1. □

Structure of the proofs The complete proof for $d \geq 3$ of the Dirichlet decomposition (Proposition IV.2.1) are presented in Section IV.A. As mentioned before, the proof for the general case is longer, however, follows the spirit of the case $d = 2$. The reader can skip it at the first reading.

Main calculations for the periodic case are carried out in Sections IV.3 and IV.4. We calculate solutions to Dirichlet and Neumann problems with periodic boundary condition by means of discrete Fourier analysis (Subsections IV.3.1 and IV.3.2). Using these results, we prove Propositions IV.2.5 and IV.2.6 and Lemma IV.2.8, in Subsections IV.3.3.1, IV.3.3.2 and IV.3.5, respectively.

The inner regularity estimate Proposition IV.2.7 is shown in Section IV.4 for $d \geq 3$ by estimates on the Poisson kernel in the discrete case, based on an idea

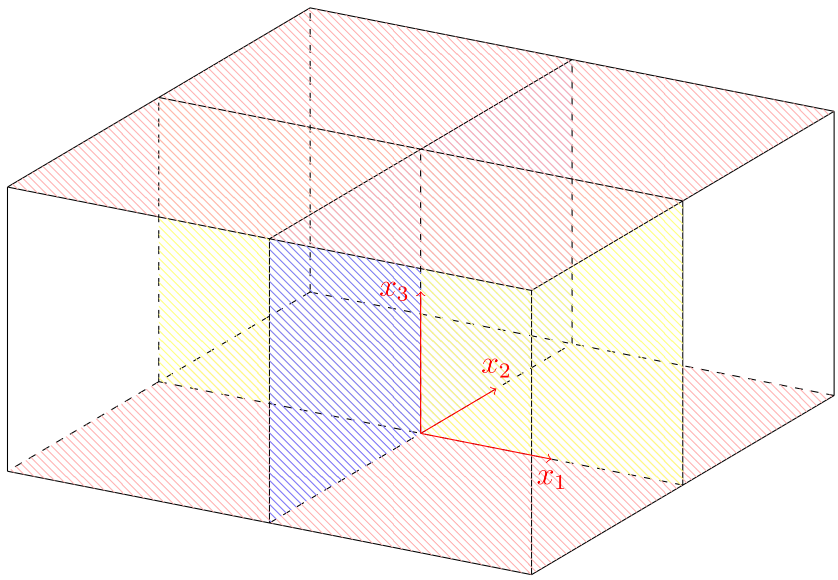

Figure IV.5: Bounding values on inner faces by periodic boundary conditions, (IV.2.23) in the case $d = 3$: the set in the right-hand side of contains the two red squares of side length $2L$. The set in the left-hand side is the blue (or yellow, respectively) rectangle for $i = 1$ (or $i = 2$, respectively).

learnt from Felix Otto in Oberwohlfach in December 2016, and written here with more ”probabilistic flavour”. For $d = 2$, the proof is much simpler and is conducted in Subsection IV.3.4. Having obtained all the ingredients we finish the proof of the main theorem and of Corollary IV.2.9 in Section IV.5.

The main challenge is the estimates on the higher derivatives on the multipliers Subsections IV.3.3.1 and IV.3.3.2.

IV.3 Boundary problems solved by Fourier analysis

IV.3.1 Discrete Fourier analysis

For convenience, we use very similar notations as in the book by Jovanović and Süli [40, Section 2.5.1, pp176–184]. Let $d \geq 1$ and consider the rescaled lattice $\mathbb{R}^d_h = h\mathbb{Z}^d$ where the mesh is defined by

$$h = \pi/L \ll 1 \Longleftrightarrow L \gg 1. \tag{IV.3.1}$$

Let v^h be a 2π-periodic function on $\mathbb{R}^d_h$ in the sense that

$$v^h(x^h + 2Lh\mathbf{e}_i) = v^h(x^h), \quad x^h \in \mathbb{R}^d_h$$

(here, $2Lh = 2\pi$). The superscript h reminds that the variables or functions are defined on the lattice with mesh h. Following [40, Eq. (2.143-4)], we define the

Fourier transform of v^h by

$$\widehat{v^h}(k) := [\mathcal{F}v^h](k) := h^d \sum_{x^h \in \omega_h^d} v^h(x^h) e^{-\mathbf{i}k\cdot x^h}, \quad k \in \mathbf{I}_L^d, \tag{IV.3.2}$$

as a $2L$-periodic function on $\mathbb{Z}^d$, and its inverse by

$$\mathcal{F}^{-1}a(x^h) := \frac{1}{(2\pi)^d} \sum_{k \in \mathbf{I}_L^d} a(k) e^{\mathbf{i}k\cdot x^h}, \quad x^h \in \omega_h^d, \tag{IV.3.3}$$

where

$$\mathbf{I}_L^d := \{-L+1, \dots, L\}^d \quad \text{and} \quad \omega_h^d = h\mathbf{I}_L^d. \tag{IV.3.4}$$

Further, define the norm and the mean

$$\|v^h\|_{L^p\left(\omega_h^d\right)} := \left(h^d \sum_{x^h \in \omega_h^d} |v^h(x^h)|^p \right)^{1/p}, \quad \langle v^h \rangle_{\omega_h^d} := \frac{1}{(2\pi)^d} \sum_{x^h \in \omega_h^d} v^h(x^h) \tag{IV.3.5}$$

as a discretization of

$$\|v\|_{L^p(\omega^d)} := \left(\int_{\omega^d} |u|^p \right)^{1/p} \quad \text{and} \quad \frac{1}{(2\pi)^d} \int_{\omega^d} v dx.$$

When showing Lemma IV.2.8 and Propositions IV.2.5 and IV.2.6, we rescale everything there into the lattice $\mathbb{R}_h^d = h\mathbb{Z}^d$, where $h = \pi/L$ and $L \gg 1$. The only reason is that we want to follow Jovanović and Süli [40] and it is convenient to have similar notations as theirs. Set

$$u^h(x^h, y^h) = u(x, y), \quad (x^h, y^h) = (hx, hy) \in \mathbb{R}_h^{d-1} \times \mathbb{R}_h^1. \tag{IV.3.6}$$

Here and in the sequel, we often denote a point in $\mathbb{R}_h^d$ by (x^h, y^h) with x^h the first $(d-1)$ components and y^h the d-th component. Then, u is harmonic at (x, y) if and only if u^h is harmonic at (x^h, y^h) in the sense that

$$2du^h(x^h, y^h) = \sum_{i=1}^{d-1} u(x^h \pm h\mathbf{e}_i, y^h) + u(y^h + h) + u(y^h - h). \tag{IV.3.7}$$

The discrete derivatives rescaled onto the lattice $\mathbb{R}_h^d$ are defined as follows:

$$\begin{aligned} \nabla_{h,i}^{(1)} u^h(x^h, y^h) &= \frac{u(x^h + h\mathbf{e}_i, y^h) - u(x^h, y^h)}{h}, \\ \nabla_h^{(2)} u^h(x^h, y^h) &= \frac{u(x^h, y^h + h) - u(x^h, y^h)}{h}. \end{aligned} \tag{IV.3.8}$$

IV.3.2 Harmonic functions with periodic boundary conditions

IV.3.2.1 Harmonic functions on the haft space

To prepare for the case of harmonic functions on infinity strips, it is worth considering a simpler case, namely the case of *harmonic functions on the haft space*. The calculations we are doing here are indeed useful for the proof of the interior estimates in Section IV.4. Let u be a *bounded* function defined on the haft space $\mathbb{Z}^{d-1} \times \{0, 1, \ldots\}$, $2L$-periodic in the $(d-1)$-first arguments and harmonic in $\mathbb{Z}^{d-1} \times \{1, 2, \ldots\}$. Rescaled to $\mathbb{R}^d_h$ by (IV.3.6)–(IV.3.8), the haft space and its boundary become:

$$\mathcal{H}_h := \mathbb{R}^{d-1}_h \times \{h, 2h, \ldots\}, \quad \partial\mathcal{H}_h := \mathbb{R}^{d-1}_h \times \{0\} \simeq \mathbb{R}^{d-1}_h, \quad \overline{\mathcal{H}}_h := \mathcal{H}_h \cup \partial\mathcal{H}_h.$$

The function u becomes $u^h : \overline{\mathcal{H}}_h \to \mathbb{R}$, which is 2π-periodic in the first $(d-1)$ arguments in the sense that

$$u(x^h, y^h) = u(x^h + 2Lh\mathbf{e}_i, y^h), \quad (x^h, y^h) \in \overline{\mathcal{H}}_h, \quad 1 \le i \le d-1, \tag{IV.3.9}$$

and is harmonic in $\mathcal{H}^h$ meaning (IV.3.7) is true for $(x^h, y^h) \in \mathcal{H}_h$. Define

$$\phi_y(k) = \widehat{u^h(\cdot, y^h)}(k), \quad k \in \mathbf{I}^{d-1}_L, \quad y = y^h/h \tag{IV.3.10}$$

that is the Fourier transform (IV.3.2) where d replaced by $d-1$. Then, we claim that given the boundary data $u(\cdot, 0)$ (or equivalently, given ϕ_0) the solution to (IV.3.13) is

$$\phi_y(k) = \phi_0(k) q^{-y}(hk), \quad \text{with} \quad y := y^h/h, \tag{IV.3.11}$$

where the function $q = q(t)$ is defined as follows:

$$q = \lambda + \sqrt{\lambda^2 - 1}, \quad \lambda = \lambda(t) = d - \sum_{i=1}^{d-1} \cos(t_i), \quad t \in [-\pi, \pi]^{d-1}. \tag{IV.3.12}$$

Argument for (IV.3.11). Taking the Fourier transform on each side of (IV.3.7) we write it in the following form

$$2d\phi_y(k) = (\phi_{y+1}(k) + \phi_{y+1}(k)) + 2\sum_{i=1}^{d-1} \phi_y(k)\cos(hk_i), \quad k \in \mathbf{I}^{d-1}_L, \tag{IV.3.13}$$

which can be easily seen as follows. By definitions (IV.3.2) and (IV.3.10), the Fourier transform of $u^h(x^h, y^h)$ and $u(x^h, y^h \pm h)$ are ϕ_y and $\phi_{y\pm1}$, respectively, and the Fourier transform of $u(x^h \pm h\mathbf{e}_i)$ is $\phi_y(k)e^{\mp \mathbf{i}hk_i}$. Summing together and noting that $e^{\mathbf{i}hk_i} + e^{-\mathbf{i}hk_i} = 2\cos hk_i$ yield (IV.3.13) which implies

$$2\lambda(hk)\phi_y(k) = \phi_{y+1}(k) + \phi_{y-1}(k). \tag{IV.3.14}$$

Noting that $2\lambda \overset{\text{(IV.3.12)}}{=} q+q^{-1}$ we find (IV.3.11) satisfies (IV.3.14).

However, (IV.3.11) gives us only *a solution*. To show that this is the unique solution we need to exploit the *boundedness*. Let us return to the lattice $\mathbb{Z}^d$ and work with the original function u. Let $\{S_n : n \geq 1\}$ be the simple random walk on $\mathbb{Z}^d$. Since u is bounded and harmonic, $u(S_n)$ is a bounded martingale. By the optional stopping theorem, we have

$$u(x,y) = \mathbb{E}\left[u(S_T)|S_0 = (x,y)\right], \quad T = \inf\{j \geq 0 : S_j \in \mathbb{Z}^{d-1} \times \{0\}\}. \tag{IV.3.15}$$

which implies that the values of u are fixed by its boundary values on $\mathbb{Z}^{d-1} \times \{0\}$. This yields uniqueness. The representation (IV.3.15) is very useful and we still exploit this later. □

IV.3.2.2 Harmonic functions on infinite strips with periodic boundary conditions

Rescaling Assumption IV.2.3 by means of (IV.3.6)–(IV.3.8) we obtain $u^h : \overline{\mathcal{S}}_N^h \to \mathbb{R}$ where

$$\mathcal{S}_N^h := \mathbb{R}_h^{d-1} \times \{h, \dots, (N-1)h\}, \quad \partial\mathcal{S}_N^h := \mathbb{R}_h^{d-1} \times \{0, Nh\}, \quad \overline{\mathcal{S}}_N^h = \mathcal{S}_N^h \cup \partial\mathcal{S}_N^h.$$

Further, u^h is 2π-periodic in the first $(d-1)$ arguments, meaning (IV.3.9) is true for $(x^h, y^h) \in \overline{\mathcal{S}}_N$ and is harmonic in $\mathcal{S}_N^h$, meaning (IV.3.7) is true for all $(x^h, y^h) \in \mathcal{S}_N^h$.

We claim that the Fourier transform of u^h satisfies

$$\begin{aligned} &\phi_y(k) = \phi_0(k)\gamma_{N-y} + \phi_N(k)\gamma_y, \\ &\qquad \text{where} \quad \gamma_y = (q^y - q^{-y})/(q^N - q^{-N})\Big|_{q=q(hk)}, \quad \forall k \in \mathbf{I}_L^{d-1} \setminus \{0\} \end{aligned} \tag{IV.3.16}$$

where q is defined in (IV.3.12). Similar calculations has been done by Guadie [37] for $L^2(\mathbb{Z}^d)$ instead of periodic boundary conditions.

Argument for (IV.3.16). The proof contains simple calculations. As in the argument for (IV.3.11) in the case of the haft space we have (IV.3.13) and therefore (IV.3.14). By (IV.3.14) and the fact that $2\lambda \overset{\text{(IV.3.12)}}{=} q+q^{-1}$ we have

$$q\phi_y - \phi_{y-1} = \phi_{y+1} - q^{-1}\phi_y.$$

Multiplying with q^{y-1}, we have

$$q^y\phi_y - q^{y-1}\phi_{y-1} = q^{-2}(q^{y+1}\phi_{y+1} - q^y\phi_y).$$

By setting

$$u_y = q^y\phi_y - q^{y-1}\phi_{y-1}, \tag{IV.3.17}$$

we get $q^2 u_y = u_{y+1}$ for $1 \le y \le N-1$. By induction, we have

$$u_y = q^{2(y-1)} u_1. \tag{IV.3.18}$$

Summing together yields

$$q^N \phi_N - \phi_0 \overset{\text{(IV.3.17)}}{=} \sum_{y=1}^{N} u_y \overset{\text{(IV.3.18)}}{=} \sum_{y=1}^{N} q^{2(y-1)} u_1 = \frac{q^{2N}-1}{q^2-1} u_1 \overset{\text{(IV.3.17)}}{=} \frac{q^{2N}-1}{q^2-1}(q\phi_1 - \phi_0).$$

This gives us ϕ_0 exactly as in (IV.3.16). Then, we can easily find $\phi_1, \phi_2, \ldots$ Without further calculation, the reader can convince himself that (IV.3.16) is really the solution to (IV.3.14) by the fact that $2\lambda\gamma_y \overset{\text{(IV.3.12)}}{=} (q+q^{-1})\gamma_y = \gamma_{y+1} + \gamma_{y-1}$, and the solution must be unique, due to the above calculation. $\square$

IV.3.2.3 Solving the Dirichlet problem on the finite strip

We solve the Dirichlet problem given by Theorem IV.2.4. By the uniqueness part of Theorem IV.2.4, already proved, we can decompose u into 3 harmonic functions u_1, u_2, u_3 satisfying Assumption IV.2.3 and

$$\begin{aligned} u_1(\cdot,0) &= u(\cdot,0) - \langle u(\cdot,0)\rangle_{\mathbf{I}_L^{d-1}}, & u_1(\cdot,N) &= 0, \\ u_2(\cdot,N) &= u(\cdot,N) - \langle u(\cdot,N)\rangle_{\mathbf{I}_L^{d-1}}, & u_2(\cdot,0) &= 0, \\ u_3(\cdot,0) &= \langle u(\cdot,0)\rangle_{\mathbf{I}_N^{d-1}}, & u_3(\cdot,N) &= \langle u(\cdot,N)\rangle_{\mathbf{I}_L^{d-1}}, \end{aligned} \tag{IV.3.19}$$

where the bracket denotes the mean

$$\langle f\rangle_{\mathbf{I}_L^{d-1}} := \frac{1}{|\mathbf{I}_L^{d-1}|} \sum_{\mathbf{I}_L^{d-1}} f.$$

Observe that u_3 is the following linear combination of its boundary conditions:

$$u_3(\cdot,y) \equiv \frac{N-y}{N} \langle u(\cdot,0)\rangle_{\mathbf{I}_N^{d-1}} + \frac{N-y}{N} \langle u(\cdot,N)\rangle_{\mathbf{I}_N^{d-1}}. \tag{IV.3.20}$$

Indeed, observe that u_3 defined by (IV.3.20) is harmonic and satisfies (IV.3.19). By the uniqueness part of Theorem IV.2.4 it must be given by (IV.3.20). Therefore, for simplicity we can assume that $\langle u(\cdot,0)\rangle_{\mathbf{I}_L^{d-1}} = 0$ and $u(\cdot,N) = 0$. In this case, given the boundary conditions ϕ_0 and ϕ_N applying (IV.3.16) we obtain $\phi_y(k)$ for all $k \neq 0$. At $k=0$ we just need to set $\phi_y(0) = 0$ for all y. Obviously, this choice satisfies (IV.3.16).

IV.3.2.4 Solving the Neumann problem on the infinite strip

By the uniqueness part of Theorem IV.2.4 already proved we can decompose u in to u_1, u_2, u_3 due to the boundary conditions:

$$\begin{aligned} \nabla^{(2)}u_1(\cdot,0) &= \nabla^{(2)}u(\cdot,0)-c, & \nabla^{(2)}u_1(\cdot,N-1) &= 0,\\ \nabla^{(2)}u_2(\cdot,N-1) &= \nabla^{(2)}u(\cdot,N-1)-c, & \nabla^{(2)}u_2(\cdot,0) &= 0, \\ \nabla^{(2)}u_3(\cdot,0) &= \nabla^{(2)}u_3(\cdot,N-1)=c. \end{aligned} \tag{IV.3.21}$$

where

$$c = \left\langle \nabla^{(2)}u(\cdot,0)\right\rangle_{\mathbf{I}_L^{d-1}} = \left\langle \nabla^{(2)}u(\cdot,N-1)\right\rangle_{\mathbf{I}_L^{d-1}},$$

in which the second equation is due to the assumption $\langle \nabla_d^{\mathrm{nor}} u\rangle = 0$ (recall definition of $\nabla_d^{\mathrm{nor}} u$ in (IV.2.3)). Here, similar to that in (IV.3.19), u_3 can be calculated explicitly (up to a constant) and is constant on each layer $\mathbb{Z}^{d-1}\times\{y\}$. Therefore, for simplicity we can assume that

$$\left\langle \nabla^{(2)}u(\cdot,0)\right\rangle_{\mathbf{I}_L^{d-1}} = 0 \quad \text{and} \quad \nabla^{(2)}u_{(}\cdot,N-1)=0. \tag{IV.3.22}$$

Rescaling to $\mathbb{R}_h^d$ by (IV.3.6)–(IV.3.8) and applying Fourier transform (IV.3.10) yield that

$$\phi_N = \phi_{N-1} \overset{\text{(IV.3.16)}}{=} \phi_0\gamma_1 + \phi_N\gamma_{N-1},$$

we have

$$\phi_N = \phi_0\frac{\gamma_1}{1-\gamma_{N-1}}, \quad k\in\mathbf{I}_L^{d-1}\setminus\{0\}. \tag{IV.3.23}$$

This implies

$$\begin{aligned} \phi_1-\phi_0 &\overset{\text{(IV.3.16)}}{=} \phi_0\gamma_{N-1}+\phi_N\gamma_1-\phi_0 \\ &\overset{\text{(IV.3.23)}}{=} \phi_0+\phi_0\frac{\gamma_1^2}{1-\gamma_{N-1}}-\phi_0 \\ &= \phi_0\frac{\gamma_1^2-(1-\gamma_{N-1})^2}{1-\gamma_{N-1}} \\ &\overset{\text{(IV.3.23)}}{=} \phi_N\frac{\gamma_1^2-(1-\gamma_{N-1})^2}{\gamma_1}, \qquad k\in\mathbf{I}_L^{d-1}\setminus\{0\}. \end{aligned} \tag{IV.3.24}$$

This gives us ϕ_0 and ϕ_N in terms of the Fourier transform $\phi_1-\phi_0$ of the Neumann condition $\nabla_h^{(2)}u^h(\cdot,0)$, therefore ϕ_y for all $1\le y\le N-1$ by (IV.3.16) for $k\in\mathbf{I}_L^{d-1}\setminus\{0\}$. For $k=0$ we just set $\phi_y(0)=c$ arbitrarily which clearly satisfies (IV.3.13) as well. Note that for the Neumann problem uniqueness only holds up to addictive constants.

IV.3.3 Estimates on the edges on the periodic boundaries

IV.3.3.1 The Dirichlet case

In the following, we provide the proof of Proposition IV.2.5. Rescaling everything onto $\mathbb{R}_h^d$ (see (IV.3.6) and (IV.3.8)) we have to show that for any $p \in (1, \infty)$,

$$\|\nabla_h^{(2)} u^h(\cdot, 0)\|_{L^p\left(\omega_h^{d-1}\right)} \lesssim_{p,d} \|\nabla_h^{(1)} u^h(\cdot, 0)\|_{L^p\left(\omega_h^{d-1}\right)}, \tag{IV.3.25}$$

$$\|\nabla_h^{(2)} u(\cdot, Nh - h)\|_{L^p\left(\omega_h^{d-1}\right)} \lesssim_{p,d} \|\nabla_h^{(1)} u(\cdot, 0)\|_{L^p\left(\omega_h^{d-1}\right)}, \tag{IV.3.26}$$

where the norm is defined in (IV.3.5). By (IV.3.16), the fact that $\phi_N = 0$, and the zero mean assumption $\phi_0(0) = 0$, the left- and right-hand side of (IV.3.25) satisfy

$$\widehat{\nabla_h^{(2)} u^h(\cdot, 0)}(k) = M_i^{\mathrm{Dir}}(k) \widehat{\nabla_{h,i}^{(1)} u^h(\cdot, 0)}(k), \quad k \in \mathbf{I}_L^{d-1} \tag{IV.3.27}$$

where

$$M_i^{\mathrm{Dir}}(k) = \frac{f}{e^{-\mathbf{i}hk_i} - 1}, \qquad k \in \mathbf{I}_L^{d-1} \setminus \{0\}, \tag{IV.3.28}$$

$$M_i^{\mathrm{Dir}}(k) = 0, \qquad k = 0, \tag{IV.3.29}$$

for $1 \leq i \leq d-1$ and

$$f = 1 - \gamma_1 = \left.\frac{(q^{2N-1} + 1)(q - 1)}{q^{2N} - 1}\right|_{q = q(kh)}, \quad k \in \mathbf{I}_L^{d-1} \setminus \{0\}. \tag{IV.3.30}$$

As mentioned before, our main tool is *a vector-valued version of the Marcinkiewicz Multiplier Theorem in the discrete case*, namely Theorem IV.B.1 in the appendix, which has been slightly improved from Theorem 2.49 in the book [40] by Jovanović and Süli. Let us point out that since all the multipliers are unbounded, there is no hope to bound the normal component by only the j-th component of the tangential derivative even in the sense of L^2 norm that is the simplest case.

Applying Theorem IV.B.1, where the multipliers can be continuously extended to $k \in [-L+1, L]^d$ by using (IV.3.28) for $k \in [-L+1, L]^d \setminus [-1, 1]^d$ and using a linear interpolation for the remaining interval, we then have to show that there exists $M_0 > 0$ depending only on d such that for any dyadic rectangle $\mathcal{R}$ there is at least one multiplier $m \in \{M_i^{\mathrm{Dir}} : 1 \leq i \leq d\}$ such that

$$\sup_{k \in \mathcal{R}} |m| \lesssim_{p,d} M_0, \quad \sup_{k \in \mathcal{R}} |k_{\alpha_1} \cdots k_{\alpha_\nu} m_{k_{\alpha_1} \cdots k_{\alpha_\nu}}| \lesssim_{p,d} M_0,$$
$$\text{for all } 1 \leq \alpha_1 < \ldots < \alpha_\nu \leq d-1.$$

By substitution

$$t := hk \overset{\text{(IV.3.1)}}{=} \pi k / L \tag{IV.3.31}$$

and using the chain rule $m_{t_i} = m_{k_i} k_i/t_i = m_{k_i} h$, this is equivalent to

$$\sup_{t\in h\mathcal{R}} |m| \lesssim_{p,d} M_0, \quad \sup_{t\in h\mathcal{R}} |t_{\alpha_1}\cdots t_{\alpha_\nu} m_{t_{\alpha_1}\cdots t_{\alpha_\nu}}| \lesssim_{p,d} M_0, \qquad \text{for all } 1\le \alpha_1 < \ldots < \alpha_\nu \le d-1.$$

By the product rule and the fact that $G(t) := t/\left(e^{-\mathrm{i}t}-1\right)$ satisfies $|G(t)| + |t\dot{G}| \lesssim 1$ uniformly in t we can replace M_i^{Dir} in (IV.3.28) by

$$m_i^{\mathrm{Dir}} = f/t_i, \quad i\in\{1,\ldots,d-1\}. \tag{IV.3.32}$$

Taking the derivatives of m_i^{Dir} in (IV.3.32) yields that for $i\notin\{\alpha_1,\ldots,\alpha_\nu\}$

$$t_{\alpha_1}\cdots t_{\alpha_\nu}\left[m_i^{\mathrm{Dir}}\right]_{t_{\alpha_1}\cdots t_{\alpha_\nu}} = \frac{t_{\alpha_1}\cdots t_{\alpha_\nu}}{t_i}\cdot f_{t_{\alpha_1}\cdots t_{\alpha_\nu}}, \tag{IV.3.33}$$

and for $i\in\{\alpha_1,\ldots,\alpha_\nu\}$, for example, for $i=\alpha_1$

$$\begin{aligned} t_{\alpha_1}\cdots t_{\alpha_\nu}\left[m_{\alpha_1}^{\mathrm{Dir}}\right]_{t_{\alpha_1}\cdots t_{\alpha_\nu}} &= t_{\alpha_1}\cdots t_{\alpha_\nu}\left(\frac{f_{t_{\alpha_2}\cdots t_{\alpha_\nu}}}{t_{\alpha_1}}\right)_{t_{\alpha_1}} \\ &= t_{\alpha_1}\cdots t_{\alpha_\nu}\frac{f_{t_{\alpha_1}\cdots t_{\alpha_\nu}}\cdot t_{\alpha_1} - f_{t_{\alpha_2}\cdots t_{\alpha_\nu}}}{t_{\alpha_1}^2} \\ &= t_{\alpha_2}\cdots t_{\alpha_\nu} f_{t_{\alpha_1}\cdots t_{\alpha_\nu}} - \frac{t_{\alpha_2}\cdots t_{\alpha_\nu}}{t_{\alpha_1}}\cdot f_{t_{\alpha_2}\cdots t_{\alpha_\nu}} \end{aligned} \tag{IV.3.34}$$

For each dyadic rectangle $\mathcal{R}$ there is an index $i = i(\mathcal{R})\in\{1,\ldots,d-1\}$ such that

$$\max_{1\le\mu\le d-1}|t_\mu| \le 4t_i, \quad t\in h\mathcal{R}. \tag{IV.3.35}$$

By (IV.3.33)–(IV.3.35), it suffices to show that

$$f_{t_{\alpha_1}\cdots t_{\alpha_\nu}} \lesssim_d |t|^{-(\nu-1)} \quad \text{for all} \quad \nu\ge 0 \qquad \text{and for all tuples} \quad 1\le\alpha_1<\ldots<\alpha_\nu\le d-1 \tag{IV.3.36}$$

with the convention that for $\nu=0$ the left-hand side becomes f. By symmetry and to lighten the notation, it suffices to consider the case $\alpha_i = i$. By the general chain rule, we have

$$f_{t_1\cdots t_\nu} = \sum_\pi \frac{\partial^{|\pi|} f}{\partial q^{|\pi|}}\cdot\prod_{B\in\pi}\frac{\partial^{|B|} q}{\prod_{j\in B}\partial t_j}. \tag{IV.3.37}$$

The notation in (IV.3.37) is explained as follows. The sum is taken over all partitions π of $\{1, \dots, \nu\}$. Then, for each fixed π, the product is taken over all mutually disjoint subsets B in this partition. Further, $|\cdot|$ denotes the cardinality. For instance,

$$\pi = \{\{1,2\},\{3,4\}\}$$

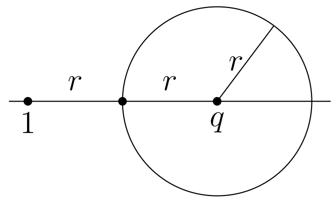

Figure IV.6

is a partition of $\{1,2,3,4\}$ and $B = \{1,2\}$ is a block of π. Here, $|\pi| = |B| = 2$. This general chain rule is called Faà di Bruno's formula. See [38, Proposition 1] for a reference.

Then, to get (IV.3.36) we need

$$f^{(\nu)}(q) \lesssim |t|^{-(\nu-1)}, \quad \forall \nu \geq 0, \tag{IV.3.38}$$
$$|q_{t_1 \cdots t_\nu}| \lesssim |t|^{-(\nu-1)}, \quad \forall \nu \geq 0. \tag{IV.3.39}$$

Indeed, plugging them into (IV.3.37) yields (IV.3.36). Recall Cauchy's formula in complex analysis:

$$f^{(\nu)}(q) = \frac{\nu!}{2\pi \mathbf{i}} \oint_{|\zeta - q| = r} \frac{f(\zeta)}{(\zeta - q)^{\nu+1}} d\zeta, \quad r := |q-1|/2 \tag{IV.3.40}$$

where $\mathbf{i} = \sqrt{-1}$ (see Figure IV.6). By the triangle inequality,

$$|f^{(\nu)}(q)| \lesssim \oint_{|\zeta - q| = r} \frac{|f(\zeta)|}{|\zeta - q|^{\nu+1}} d|\zeta| \lesssim r^{-\nu} \sup_{\zeta : |\zeta - q| = r} |f(\zeta)|. \tag{IV.3.41}$$

To estimate the supremum, apply the triangle inequality to definition (IV.3.30) of f:

$$|f(\zeta)| \leq \frac{\left(|\zeta|^{2N-1} + 1\right)|\zeta - 1|}{|\zeta|^{2N} - 1} = \frac{1}{|\zeta|} \left(1 + \frac{|\zeta| + 1}{|\zeta|^{2N} - 1}\right) |\zeta - 1| \tag{IV.3.42}$$

For ζ on the circle in Figure IV.6, we have

$$|\zeta - 1| \lesssim r, \quad 1 \lesssim |\zeta| \lesssim 1, \quad |\zeta^{2N} - 1| \gtrsim 1, \tag{IV.3.43}$$

where we get the third estimate combining application of Bernoulli's inequality,

$$|\zeta|^{2N} - 1 \geq (1+r)^{2N} - 1 \geq 2rN, \tag{IV.3.44}$$

the fact that

$$2r = q - 1 \overset{\text{(IV.3.12)}}{\sim} \sqrt{\lambda - 1} \sim |t| \tag{IV.3.45}$$

the assumption $N/L \gtrsim 1$, and finally the fact that t is bounded away from the origin, say $|t| \geq h = 1/L$. Recall that in order to apply Theorem IV.B.1, we extended the multipliers continuously for $k \in [-1,1]^d$ using a linear interpolation. Therefore, it suffices to consider $|k| \geq 1 \overset{\text{(IV.3.31)}}{\Longleftrightarrow} |t| \geq \pi/L$. To see (IV.3.45) recall (IV.3.12) and the fact that $t \mapsto (1-\cos t)/t^2$ (continuously extended at 0) has a positive maximum and minimum in $[0, \pi]$. Plugging (IV.3.43) into (IV.3.42) yields that the supremum of f on the circle is $O(r)$ which, together with (IV.3.41), finishes the argument for (IV.3.38). Note that (IV.3.38) is true even for $\nu = 0$, since (IV.3.45) holds in both directions.

We continue with the argument for (IV.3.39), which contains purely calculations. By (IV.3.45), the case $\nu = 0$ is trivial. Since $\lambda_{t_i} = \sin t_i$ depends only on t_i, we have

$$q_{t_1 \cdots t_\nu} = q_{t_1 \cdots t_{\nu-1}\lambda} \cdot \sin t_\nu = \ldots = q_\lambda^{(\nu)} \cdot \sin t_1 \sin t_2 \cdots \sin t_\nu, \quad \nu \geq 1 \qquad \text{(IV.3.46)}$$

where $q_\lambda^{(\nu)}$ denotes the ν-th λ-derivative of q. Further,

$$q_\lambda^{(1)} = 1 + \frac{\partial}{\partial\lambda}\Big\{(\lambda^2-1)^{1/2}\Big\}, \quad q_\lambda^{(2)} = \frac{\partial^2}{\partial\lambda^2}\Big\{(\lambda^2-1)^{1/2}\Big\}, \ldots,$$
$$q_\lambda^{(\nu)} = \frac{\partial^\nu}{\partial\lambda^\nu}\Big\{(\lambda^2-1)^{1/2}\Big\}.$$

Applying Leibniz's formula to $(fg)^{(n)}$ with $f = (\lambda-1)^{\frac{1}{2}}$ and $g = (\lambda+1)^{\frac{1}{2}}$ yields

$$q_\lambda^{(\nu)} \lesssim_d \frac{\partial^\nu}{\partial\lambda^\nu}\left\{\sqrt{\lambda-1}\right\} \lesssim_d (\lambda-1)^{-\frac{(2\nu-1)}{2}} \lesssim |t|^{-(2\nu-1)}, \qquad \text{(IV.3.47)}$$

which together with (IV.3.46) and the fact that $|\sin(t_i)| \lesssim t_i$ implies (IV.3.39). Therefore, we finish the argument for (IV.3.25). Another way to show (IV.3.47) without using Leibniz's formula is again application of Cauchy's integral formula to $\zeta \mapsto \sqrt{\zeta^2-1}$ where the contour integral is taken on the circle in Figure IV.6 with q replaced by λ. Here, the square root can be defined as the inverse mapping of

$$\{\zeta : \mathfrak{Re}(\zeta) > 0\} \to \mathbb{C} \setminus \{\mathbf{i}b : b \leq 0\}, \quad \zeta \mapsto \zeta^2$$

and therefore analytic by the inverse mapping theorem. We do not go into details.

The argument for (IV.3.26) is almost the same. We apply the Marcinkiewicz Multiplier Theorem (Theorem IV.B.1) with the following multipliers that can also be calculated explicitly:

$$\widetilde{M}_i^{\mathrm{Dir}} = \begin{cases} \dfrac{\widetilde{f}}{e^{-\mathbf{i}t_i}-1}, & k \in \mathbf{I}_L^{d-1} \setminus \{0\} \\ 0, & k = 0, \end{cases} \qquad \text{(IV.3.48)}$$

where

$$\widetilde{f} = \frac{q-q^{-1}}{q^N - q^{-N}} = q^{N-1}\frac{q^2-1}{q^{2N}-1}\bigg|_{q=q(hk)}, \quad k \in \mathbf{I}_L^{d-1} \setminus \{0\}. \qquad \text{(IV.3.49)}$$

We can repeat all steps in the proof of (IV.3.25). The only thing we need to check is that the supremum of $\widetilde{f}$ on the circle in Figure IV.6 is $O(r)$. However, it is also obvious by doing some simple calculations:

$$\begin{aligned}
|\widetilde{f}(\zeta)| &\leq \frac{|\zeta|^{N-1}}{|\zeta|^{2N}-1} \cdot |\zeta+1| \cdot |\zeta-1| \\
&= \frac{1}{|\zeta|}\left(\frac{|\zeta|^N-1}{|\zeta|^{2N}-1} + \frac{1}{|\zeta|^{2N}-1}\right) \cdot |\zeta+1| \cdot |\zeta-1| \\
&= \frac{1}{|\zeta|}\left(\frac{1}{|\zeta|^N+1} + \frac{1}{|\zeta|^{2N}-1}\right) \cdot |\zeta+1| \cdot |\zeta-1| \\
&\leq \frac{1}{|\zeta|}\left(\frac{1}{|\zeta|^N-1} + \frac{1}{|\zeta|^{2N}-1}\right) \cdot |\zeta+1| \cdot |\zeta-1|,
\end{aligned}$$

and repeating the argument (IV.3.43)–(IV.3.45).

IV.3.3.2 The Neumann case

Now comes the proof of Proposition IV.2.6. It suffices to assume (IV.3.22) since otherwise we decompose u into u_1, u_2, u_3 satisfying Assumption IV.2.3 and (IV.3.21). Note that u_3 is a constant on each layer $\mathbb{Z}^{d-1} \times \{y\}$, $0 \leq y \leq N$ meaning g_3 plays no role to the tangential derivatives on the periodic boundary. Rescaling into $\mathbb{R}^d_h$ (see (IV.3.6), (IV.3.8), and (IV.3.5)) we have to show that for any $p \in (1, \infty)$,

$$\|\nabla_h^{(1)} u^h(\cdot, 0)\|_{L^p\left(\omega_h^{d-1}\right)} \lesssim_{p,d} \|\nabla_h^{(2)} u^h(\cdot, 0)\|_{L^p\left(\omega_h^{d-1}\right)}, \tag{IV.3.50}$$

$$\|\nabla_h^{(1)} u^h(\cdot, Nh)\|_{L^p\left(\omega_h^{d-1}\right)} \lesssim_{p,d} \|\nabla_h^{(2)} u^h(\cdot, 0)\|_{L^p\left(\omega_h^{d-1}\right)}, \tag{IV.3.51}$$

where the left-hand and right-hand side are related by means of Fourier multipliers:

$$\widehat{\nabla_{h,i}^{(1)} u^h(\cdot, 0)}(k) = M_i^{\mathrm{Neu}}(k) \widehat{\nabla_h^{(2)} u^h(\cdot, 0)}(k),$$

$$\widehat{\nabla_{h,i}^{(1)} u^h(\cdot, Nh)}(k) = \widetilde{M}_i^{\mathrm{Neu}}(k) \widehat{\nabla_h^{(2)} u^h(\cdot, 0)}(k).$$

Recall the substitution (IV.3.31). Using (IV.3.24) we easily get the multipliers:

$$\begin{aligned}
M_i^{\mathrm{Neu}} &= \frac{1-\gamma_{N-1}}{\gamma_1^2 - (1-\gamma_{N-1})^2}(e^{-\mathrm{i}t_i} - 1), \\
\widetilde{M}_i^{\mathrm{Neu}} &= \frac{\gamma_1}{\gamma_1^2 - (1-\gamma_{N-1})^2}(e^{-\mathrm{i}t_i} - 1), \quad t \in \omega_h^{d-1} \setminus \{0\},
\end{aligned} \tag{IV.3.52}$$

with $\mathbf{i} = \sqrt{-1}$ and γ_y defined in (IV.3.16). By (IV.3.22) we just take $M_i^{\mathrm{Neu}} = \widetilde{M}_i^{\mathrm{Neu}} = 0$ for $t = 0$. With the same argument as in the Dirichlet case, we replace M_i^{Neu} by

$$m_i^{\mathrm{Neu}} = t_i g \quad \text{with } g = \frac{1-\gamma_{N-1}}{\gamma_1^2 - (1-\gamma_{N-1})^2}, \quad t \in \omega_h^{d-1} \setminus \{0\} \tag{IV.3.53}$$

and $\widetilde{M}_i^{\mathrm{Neu}}$ by

$$\widetilde{m}_i^{\mathrm{Neu}} = t_i \widetilde{g} \quad \text{with } \widetilde{g} = \frac{\gamma_1}{\gamma_1^2 - (1-\gamma_{N-1})^2}, \quad t \in \omega_h^{d-1} \setminus \{0\}. \tag{IV.3.54}$$

In this case, we only need the one-dimensional version of the Marcinkiewicz theorem (see [40, Theorem 2.49, p180]). Let us start with m_i^{Neu}. For $i \notin \{\alpha_1, \dots, \alpha_\nu\}$,

$$t_{\alpha_1} \dots t_{\alpha_\nu} \left[m_i^{\mathrm{Neu}}\right]_{t_{\alpha_1} \dots t_{\alpha_\nu}} = t_i t_{\alpha_1} \dots t_{\alpha_\nu} \cdot g_{t_{\alpha_1} \dots t_{\alpha_\nu}}.$$

For $i \in \{\alpha_1, \dots, \alpha_\nu\}$, e.g. $i = \alpha_1$,

$$\begin{aligned} t_{\alpha_1} \dots t_{\alpha_\nu} \left[m_i^{\mathrm{Neu}}\right]_{t_{\alpha_1} \dots t_{\alpha_\nu}} &= t_{\alpha_1} \dots t_{\alpha_\nu} \left[t_{\alpha_1} g\right]_{t_{\alpha_1} \dots t_{\alpha_\nu}} \\ &= t_{\alpha_1} \dots t_{\alpha_\nu} \left[g_{t_{\alpha_2} \dots t_{\alpha_\nu}} + t_{\alpha_1} g_{t_{\alpha_1} \dots t_{\alpha_\nu}}\right]. \end{aligned}$$

Therefore, it suffices to show that for all $\nu \geq 0$ and all $\{\alpha_1, \dots, \alpha_\nu\} \subset \{1, \dots, d-1\}$,

$$g_{t_{\alpha_1} \dots t_{\alpha_\nu}} \lesssim |t|^{-(\nu+1)}. \tag{IV.3.55}$$

By the chain rule (IV.3.37) and estimate (IV.3.39) on derivatives of q, this follows from the following

$$g^{(\nu)}(q) \lesssim |t|^{-(\nu+1)}. \tag{IV.3.56}$$

Repeating the argument (IV.3.40), (IV.3.41) and (IV.3.45) with Cauchy's integral formula, we only need to control the supremum of g on the circle in Figure IV.6. In this case, we have to check that

$$\sup_{\zeta : |\zeta - q| = r} |g(\zeta)| \lesssim r^{-1} \quad \text{with } r = |q-1|/2. \tag{IV.3.57}$$

Write g as follows:

$$g = -\frac{1}{1-\gamma_{N-1}-\gamma_1} \times \frac{1-\gamma_{N-1}}{1-\gamma_{N-1}+\gamma_1} =: -g_1 \times g_2, \tag{IV.3.58}$$

where γ_1 and γ_{N-1} are defined in (IV.3.16). Note that

$$1-\gamma_{N-1} = \frac{\left(q^{2N+1}+1\right)(q-1)}{q^{2N}-1}, \quad 1-\gamma_{N-1} \pm \gamma_1 = \frac{(q-1)\left(q^{N-1} \pm 1\right)}{q^N+1}. \tag{IV.3.59}$$

By (IV.3.58) and (IV.3.59) and the triangle inequality we have

$$
\begin{aligned}
|g_1(\zeta)| &= \left| \frac{\zeta^N + 1}{(\zeta-1)(\zeta^{N-1}-1)} \right| \\
&\leq \frac{|\zeta^N - \zeta| + |\zeta - 1|}{|\zeta^{N-1}-1|} \cdot \frac{1}{|\zeta-1|} = \left[|\zeta| + \frac{|\zeta-1|}{|\zeta|^{N-1}-1} \right] \frac{1}{|\zeta-1|} \lesssim r^{-1},
\end{aligned}
$$

and

$$
\begin{aligned}
|g_2(\zeta)| &= \left| \frac{\left(\zeta^{2N+1}+1\right)(\zeta-1)}{\zeta^{2N}-1} \cdot \frac{\zeta^N+1}{(\zeta-1)\left(\zeta^{N-1}+1\right)} \right| \\
&\leq \left| \frac{\zeta^{2N-1}+1}{\left(\zeta^N+1\right)\left(\zeta^{N-1}+1\right)} \right| = \left| 1 - \frac{\zeta^{N-1}(\zeta+1)}{\left(\zeta^N+1\right)\left(\zeta^{N-1}+1\right)} \right| \\
&\leq 1 + \left| \frac{\zeta^{N-1}}{\zeta^{N-1}+1} \right| \times \frac{|\zeta+1|}{|\zeta^N+1|} \\
&= 1 + \left| 1 - \frac{1}{\zeta^{N+1}+1} \right| \times \frac{|\zeta+1|}{|\zeta^N+1|} \qquad \text{(IV.3.60)} \\
&= 2 + \frac{1}{|\zeta^{N+1}+1|} \times \frac{|\zeta+1|}{|\zeta^N+1|} \\
&= 2 + \frac{1}{|\zeta|^{N+1}-1} \times \frac{|\zeta+1|}{|\zeta|^N-1} \lesssim 1.
\end{aligned}
$$

where we bound the terms

$$
|\zeta|^{N-1}-1, \quad |\zeta|^N-1, \quad |\zeta|^{N+1}-1
$$

from below by Bernoulli's inequality as done for (IV.3.43). This completes the argument for (IV.3.57).

Concerning $\widetilde{m}_i^{\mathrm{Neu}}$ in (IV.3.54) we write $\widetilde{g} = -g_1\widetilde{g}_2$ with g_1 defined in (IV.3.58) and

$$
\widetilde{g}_2(\zeta) = \frac{\gamma_1}{1-\gamma_{N-1}+\gamma_1} = \frac{\zeta^{N-1}(\zeta+1)}{\left(\zeta^N+1\right)\left(\zeta^{N-1}+1\right)}
$$

which is exactly the product in (IV.3.60). The argument for $\widetilde{m}_i^{\mathrm{Neu}}$ is therefore the same as that for m_i^{Neu}.

IV.3.4 Interior regularity for the two-dimensional case

In the following, we prove Proposition IV.2.7 for $d = 2$. By decomposing as in (IV.3.19) it suffices to consider the case $u(\cdot, N) = 0$ and $\sum_{\mathbf{I}_L} u(\cdot, 0) = 0$. The claim follows by interpolating between the weak L^1 and L^∞ estimate (see e.g. [60, Theorem 5, p21]). Since the L^∞ estimate is obvious due to the maximum principle, we only have to prove the weak L^1 estimate,

$$
|u(0,y)| \lesssim \frac{1}{y}\|u(\cdot,0)\|_{L^1(\mathbf{I}_L)}.
$$

By rescaling to $\mathbb{R}_h^2$ (see (IV.3.5)–(IV.3.8)), this is equivalent to

$$|u^h(0,y^h)| \lesssim \frac{1}{y^h}\|u^h(\cdot,0)\|_{L^1(\omega^h)}.$$

By the Fourier inverse transform (IV.3.3) and the fact that $\phi_0(0)=0$ we have

$$u^h(0,y^h) = \frac{1}{2\pi}\sum_{k\in\mathbf{I}_L\setminus\{0\}} \phi_y(k).$$

Since $\phi_0(0)=0$ and $\phi_N(\cdot)=0$, in (IV.3.16) we have $\phi_y=\phi_0\gamma_{N-y}$. Further, γ_{N-y} is bounded by q^{-y} and ϕ_0 is bounded by $\|u^h(\cdot,0)\|_{L^1(\omega^h)}$ by applying the triangle inequality to the definition (IV.3.2). Therefore,

$$|u^h(0,y^h)| \lesssim \frac{1}{2\pi}\sum_{k\in\mathbf{I}_L\setminus\{0\}} q^{-y}(kh)|\phi_0(k)| \le \frac{1}{2\pi}\sum_{k\in\mathbf{I}_L\setminus\{0\}} q^{-y}(kh)\|u^h(\cdot,0)\|_{L^1(\omega^h)}.$$

By symmetry and the fact that $\phi_0(0)=0$, we only need to consider the sum taken from 1 to L, and it suffices to show that

$$\frac{1}{2\pi}\sum_{k=1}^{L}\frac{1}{q^y(kh)} \lesssim \frac{1}{y^h} = \frac{1}{hy}.$$

Indeed, since $q(\cdot)$ is increasing, we have

$$\sum_{k\in\mathbf{I}_L}\frac{h}{q^y(kh)} \le \int_0^\pi q^{-y}(t)dt,$$

which can be obtained by summing up the following estimates for $k=1,\dots,L$:

$$\frac{h}{q^y(kh)} \le \int_{[(k-1)h,kh]}\frac{dt}{q^y(t)}$$

Since there exists $a>0$ such that $q(t)-1\ge at$ (see (IV.3.45)), we have

$$\int_0^\pi \frac{dt}{q(t)^y} \le \int_0^\pi \frac{dt}{(1+at)^y} = \frac{(at+1)^{-y+1}}{-y+1}\frac{1}{a}\bigg|_{t=0}^{\pi} \le \frac{1}{a(y-1)}, \quad y\ge 2.$$

Therefore, when we choose C large enough, we have

$$\int_0^\pi \frac{dt}{q(t)^y} \le \frac{C}{y}, \quad y\ge 1$$

which completes the proof.

IV.3.5 Proof of the inner layer lemma

In this subsection, we prove Lemma IV.2.8. The proof is structured as follows. First, we prove that (IV.2.24) and (IV.2.25) are true for $p \in \{1, \infty\}$ for $\alpha = 1$. Then, we prove that (IV.2.25) is true for $p = 2$ for some $\alpha \in (0, 1)$. The claim then follows by interpolation. Let us start with $p \in \{1, \infty\}$. Recall the representation (IV.3.15):

$$u(x, y) = \mathbb{E}\left[u(S_T)|S_0 = (x, y)\right], \quad T = \inf\{j \geq 0 : S_j \in \mathbb{Z}^{d-1} \times \{0\}\}.$$

which implies

$$u(x, y) = \mathbb{E}\left[u\left(S_T + (x, 0)\right)|S_0 = (0, y)\right].$$

From this, we easily have (IV.2.24) and (IV.2.25) (with $\alpha = 1$) for $p \in \{1, \infty\}$, which we explain precisely as follows. First,

$$\sum_{x \in \mathbf{I}_L^{d-1}} |u(x, N)| \leq \mathbb{E}\left[\sum_{x \in \mathbf{I}_L^{d-1}} |u\left(S_T + (x, 0)\right)| \middle| S_0 = (0, N)\right] = \sum_{x \in \mathbf{I}_L^{d-1}} |u(x, 0)|,$$

where we obtained the last inequality simply by taking expectation of

$$\sum_{x \in \mathbf{I}_L^{d-1}} |u(S_T + (x, 0))| = \sum_{x \in \mathbf{I}_L^{d-1}} |u(x, 0)|$$

which is true for each realization of (S_n). Second,

$$\max_{x \in \mathbf{I}_N^{d-1}} |u(x, N)| \leq \mathbb{E}\left[\max_{x \in \mathbf{I}_N^{d-1}} |u\left(S_T - (x, 0)\right)| \middle| S_0 = (0, N)\right] \leq \max_{x \in \mathbf{I}_N^{d-1}} |u(x, 0)|$$

also by taking expectation of an inequality which is true for all realizations.

In the remaining part we consider the case $p = 2$. Then, interpolating between 1 and between 2 and 2 and ∞ we finish the proof. Rescale everything into $\mathbb{R}_h^d$ (see (IV.3.7), (IV.3.8), and (IV.3.5)). By (IV.3.11)

$$\widehat{u^h(\cdot, Nh)}(k) = \ell_N(k)\widehat{u^h(\cdot, 0)}(k), \tag{IV.3.61}$$

with the multiplier

$$\ell_N(k) = q^{-N}, \quad q = q(kh), \quad k \in \mathbf{I}_L^{d-1}.$$

Since $q \geq 1$ meaning $\ell_N \leq 1$, we get (IV.2.24) easily. To get $\alpha \in (0, 1)$ in (IV.2.25) we need to exploit the assumption $N/L \geq \underline{r} > 0$ and $\sum_{\mathbf{I}_L^{d-1}} u(\cdot, 0) = 0$. By (IV.3.45) and Bernoulli's inequality, we have

$$q^N(kh) \geq \left(1 + |k|\frac{\pi}{L}\right)^N \geq 1 + |k|\pi\underline{r} > 1 + \underline{r}, \quad k \in \mathbf{I}_L^{d-1} \setminus \{0\}$$

that yields

$$\ell_N(k) \le (1+\underline{r})^{-1}, \quad k \in \mathbf{I}_L^{d-1} \setminus \{0\}.$$

Since $\sum_{\mathbf{I}_L^{d-1}} u(\cdot, 0) = 0$ by assumption, we have $\widehat{u^h(\cdot,0)}(0) = 0$, which implies the existence of $\alpha \in (0,1)$ for $p = 2$:

$$\|u^h(\cdot, y^h)\|_{L^2\left(\omega_h^{d-1}\right)} \le (1+\underline{r})^{-1} \|u^h(\cdot, 0)\|_{L^2\left(\omega_h^{d-1}\right)}$$

by Plancherel's identity applied to (IV.3.61).

IV.4 Interior regularity

In this section, we prove Proposition IV.2.7 for $d \ge 3$. For $d = 2$, see Subsection IV.3.4.

IV.4.1 The case of the haft space

Let us forget for a while about the fact that u is harmonic on the infinite strip with top and bottom boundary conditions as mentioned before in Assumption IV.2.3. Indeed, a first step, we prove (IV.2.23) under Assumption IV.4.1 below where u is supposed to be harmonic on the whole haft space which is somehow a bit easier to deal with.

Assumption IV.4.1. Let $N, L \gg 1$. Assume that there exist $\underline{r}, \overline{r} > 0$ such that

$$\underline{r} \le N/L \le \overline{r}.$$

Further, let $u : \mathbb{Z}^{d-1} \times \{0, 1, \ldots\} \to \mathbb{R}$ be $2L$-periodic in the first $(d-1)$ arguments, meaning $u = u(\cdot + 2L\mathbf{e}_j)$ for $1 \le j \le d-1$, and harmonic in $\mathbb{Z}^{d-1} \times \{1, 2, \ldots\}$.

In this subsection we denote a point $\xi \in \mathbb{Z}^d$ by

$$\xi = (x, y, z), \quad x \in \mathbb{Z}, \quad y \in \mathbb{Z}^{d-2}, \quad z \in \mathbb{Z}.$$

As said at the beginning of the section we have to show for any $p \in (1, \infty)$

$$\left(\sum_{y \in \mathbf{I}_L^{d-2}} \sum_{z=1}^{N} |u(0,y,z)|^p \right)^{1/p} \lesssim_{p,d,\underline{r}} \left(\sum_{x \in \mathbf{I}_L} \sum_{y \in \mathbf{I}_L^{d-2}} |u(x,y,0)|^p \right)^{1/p}. \tag{IV.4.1}$$

This follows from the following estimate: for any $p \in (1, \infty)$

$$\left(\sum_{y \in \mathbf{I}_L^{d-2}} \sum_{z=1}^{N} |u(0,y,z) - m(y,z)|^p \right)^{1/p} \lesssim_{p,d,\underline{r}} \left(\sum_{x \in \mathbf{I}_L} \sum_{y \in \mathbf{I}_L^{d-2}} |u(x,y,0)|^p \right)^{1/p} \tag{IV.4.2}$$

with

$$m(y,z) = \frac{1}{|\mathbf{I}_L|}\sum_{x\in\mathbf{I}_L} u(x,y,z).$$

Indeed, the means can be absorbed into the right-hand side by Lemma IV.2.8:

$$\sum_{y\in\mathbf{I}_L^{d-2}}\sum_{z=1}^{N}|m(y,z)|^p \le \frac{1}{|\mathbf{I}_L|}\sum_{x\in\mathbf{I}_L}\sum_{y\in\mathbf{I}_L^{d-2}}\sum_{z=1}^{N}|u(x,y,z)|^p \le (\text{RHS (IV.4.1)})^p,$$

Here, we need the upper bound $N/L \le \overline{r}$ for the second inequality. To get (IV.4.2), we interpolate the following weak L^1 and the strong L^∞ estimate:

$$\left(\sum_{y\in\mathbf{I}_N^{d-2}}|u(0,y,z)-m(y,z)|^p\right)^{1/p} \lesssim_{p,d,\underline{r}} \frac{1}{z}\sum_{x\in\mathbf{I}_N}\left(\sum_{y\in\mathbf{I}_N^{d-2}}|u(x,y,0)|\right)^{1/p}, \tag{IV.4.3}$$

$$\left(\sum_{y\in\mathbf{I}_N^{d-2}}|u(0,y,z)-m(y,z)|^p\right)^{1/p} \lesssim_{p,d,\underline{r}} \max_{x\in\mathbf{I}_N}\left(\sum_{y\in\mathbf{I}_N^{d-2}}|u(x,y,0)|^p\right)^{1/p}. \tag{IV.4.4}$$

In order to have a clear argument for (IV.4.3) and (IV.4.4) we come back again to the representation (IV.3.15), which reads

$$u(x,y,z) = \mathbb{E}\Big[u(S_0)\Big|S_0=(x,y,z)\Big] = \mathbb{E}\Big[u(S_T)\Big|S_0=(x,y,z)\Big] \tag{IV.4.5}$$

where (S_n) is the simple random walk on $\mathbb{Z}^d$ and $T := \inf\{j\ge 0 : S_j \in \mathbb{Z}\times\{0\}\}$. Using the Poisson kernel defined as

$$P_z(x,y) = \mathbb{P}\Big[S_T=(x,y,0)\Big|S_0=(0,0,z)\Big],$$

and the translation invariance we have

$$u(x,y,z) = \sum_{(x',y')\in\mathbb{Z}^{d-1}} u(x',y',0)P_z(x'-x,y'-y). \tag{IV.4.6}$$

Argument for the L^∞ estimate (IV.4.4). Note that the means $m(y,z)$ can be easily absorbed into the right-hand side of (IV.4.4):

$$\begin{aligned}
\sum_{y\in\mathbf{I}_L^{d-2}}|m(y,z)|^p &= \sum_{y\in\mathbf{I}_L^{d-2}}\left|\frac{1}{|\mathbf{I}_L|}\sum_{x\in\mathbf{I}_L}u(x,y,z)\right|^p \\
\text{[Jensen's ineq.]}\qquad &\le \frac{1}{|\mathbf{I}_L|}\sum_{x\in\mathbf{I}_L}\sum_{y\in\mathbf{I}_L^{d-2}}|u(x,y,z)|^p \\
\text{[Lemma IV.2.8]}\qquad &\le \sum_{x\in\mathbf{I}_L}\sum_{y\in\mathbf{I}_L^{d-2}}|u(x,y,0)|^p \le (\text{RHS (IV.4.4)})^p.
\end{aligned} \tag{IV.4.7}$$

Consider the left-hand side of (IV.4.4) without the mean:

$$\begin{aligned}\sum_{y\in\mathbf{I}_N^{d-2}} |u(0,y,z)|^p &\le \sum_{y\in\mathbf{I}_N^{d-2}} \mathbb{E}\Big[|u(S_T)|^p\Big|S_0=(0,y,z)\Big]\\ \text{(translation inv.)} &= \sum_{y\in\mathbf{I}_N^{d-2}} \mathbb{E}\Big[|u(S_T+(0,y,0))|^p\Big|S_0=(0,0,z)\Big]\\ &= \mathbb{E}\left[\sum_{y\in\mathbf{I}_N^{d-2}} |u(S_T+(0,y,0))|^p\Big|S_0=(0,0,z)\right]\\ &\le (\mathrm{RHS}\ (\text{IV.4.4}))^p\,. \end{aligned} \tag{IV.4.8}$$

Combining (IV.4.7) and (IV.4.8) we finish the argument for (IV.4.4). □

Argument for the L^1 weak estimate (IV.4.3). We have

$$\begin{aligned}|u(0,y,z)-m(y,z)| &= \left|u(0,y,z)-\frac{1}{|\mathbf{I}_L|}\sum_{x\in\mathbf{I}_L}u(x,y,z)\right|\\ \text{[triangle ineq.]} \quad &\le \frac{1}{|\mathbf{I}_L|}\sum_{x\in\mathbf{I}_L}|u(0,y,z)-u(x,y,z)|\\ \text{[telescope sum]} \quad &= \frac{1}{|\mathbf{I}_L|}\sum_{x\in\mathbf{I}_L}\left|\sum_{x'=0}^{x-1}\nabla^{(1)}u(x',y,z)\right| \le \sum_{x\in\mathbf{I}_L}|\nabla^{(1)}u(x,y,z)|.\end{aligned}$$

Applying the triangle inequality to the norm $\left(\sum_y |\cdot|^p\right)^{1/p}$ we get

$$\left(\sum_{y\in\mathbf{I}_L^{d-2}}|u(0,y,z)-m(y,z)|^p\right)^{1/p} \le \sum_{x\in\mathbf{I}_L}\left(\sum_{y\in\mathbf{I}_L^{d-2}}|\nabla^{(1)}u(x,y,z)|^p\right)^{1/p} \tag{IV.4.9}$$

where

$$\begin{aligned}\nabla^{(1)}u(x,y,z) &\overset{(\text{IV.4.6})}{=} \sum_{(x',y')\in\mathbb{Z}^{d-1}} \nabla^{(1*)}P_z(x'-x,y'-y)u(x',y',0)\\ &= \sum_{(x',y')\in\mathbb{Z}^{d-1}} \nabla^{(1*)}P_z(x',y')u(x'+x,y'+y,0)\end{aligned}$$

and

$$\nabla^{(1*)}P_z(\widetilde{x},\widetilde{y}) = P_z(\widetilde{x}-1,\widetilde{y})-P_z(\widetilde{x},\widetilde{y}).$$

Then, (IV.4.4) follows from (IV.4.9) and the following two facts. First, we have

$$\sum_{x\in\mathbf{I}_L}\left(\sum_{y\in\mathbf{I}_L^{d-2}}|\nabla^{(1)}u(x,y,z)|^p\right)^{1/p}$$
$$\leq\left[\sum_{(x,y)\in\mathbb{Z}^{d-1}}|\nabla^{(1*)}P_z(x,y)|\right]\left[\sum_{x\in\mathbf{I}_L}\left(\sum_{y\in\mathbf{I}_L^{d-2}}|u(x,y,0)|^p\right)^{1/p}\right]. \quad \text{(IV.4.10)}$$

Second, the following estimate on the Poisson kernel is true:

$$\sum_{(x,y)\in\mathbb{Z}^d}|\nabla^{1*}P_z(x,y)|\lesssim_{p,d}\frac{1}{z}. \quad \text{(IV.4.11)}$$

Hence, the proof for (IV.4.4) is complete *after proving* (IV.4.10) *and* (IV.4.11). □

In the remaining part of this subsection, we verify (IV.4.10) and (IV.4.11).

Argument for (IV.4.11). This results from the mean value theorem and the following approximation of the Poisson kernel by its continuum counterpart (see Lawler [44, Theorem 8.1.2, p227]):

$$P_z(x,y)=\frac{2z}{\omega_d|\xi|^d}\left[1+\mathcal{O}\left(\frac{z}{|\xi|^2}\right)\right]+\mathcal{O}\left(\frac{1}{|\xi|^{d+1}}\right), \quad \xi=(x,y,z).$$

Another way to get (IV.4.11) is to exploit the fact that the Poisson kernel can be represented by the Green function with respect to the simple random walk on $\mathbb{Z}^d$ as follows:

$$P_z(x,y)=G(\xi-\mathbf{e}_d)-G(\xi+\mathbf{e}_d)$$

(see Lawler [44, Proposition 8.1, p226]). Therefore, the Poisson kernel has the order of the first derivative of the Green function, and the first derivative of the Poisson kernel has the order of the second derivative of the Green function:

$$|\nabla^{1*}P_z(x,y)|\lesssim|\xi|^{-d}.$$

Then, by the following elementary estimate:

$$\sum_{(x,y)\in\mathbb{Z}^{d-1}\setminus\{(0,0)\}}\frac{1}{\{|x|^2+|y|^2+|z|^2\}^{d/2}}\lesssim_d\frac{1}{z},$$

we also obtain (IV.4.11). □

Argument for (IV.4.10). The argument contains purely calculations.

$$
\begin{aligned}
\text{LHS (IV.4.10)} &= \sum_{x\in\mathbf{I}_L}\left[\sum_{y\in\mathbf{I}_L^{d-2}}\left|\sum_{x'\in\mathbb{Z}}\sum_{y'\in\mathbb{Z}^{d-2}} u(x+x',y+y')\nabla^{(1*)}P_z(x',y')\right|^p\right]^{\frac{1}{p}} \\
[\Delta\text{-ineq.}] \quad &\le \sum_{x\in\mathbf{I}_L}\sum_{x'\in\mathbb{Z}}\sum_{y'\in\mathbb{Z}^{d-2}}\left[\sum_{y\in\mathbf{I}_L^{d-2}}|u(x+x',y+y',0)\nabla^{(1*)}P_z(x',y')|^p\right]^{\frac{1}{p}} \\
[u \text{ periodic in } y] \quad &= \sum_{x\in\mathbf{I}_L}\sum_{x'\in\mathbb{Z}}\sum_{y'\in\mathbb{Z}^{d-2}}\left[\sum_{y\in\mathbf{I}_L^{d-2}}|u(x+x',y,0)|^p\right]^{\frac{1}{p}}|\nabla^{(1*)}P_z(x',y')| \\
[\text{push sum } y' \text{ in}] \quad &= \sum_{x\in\mathbf{I}_L}\sum_{x'\in\mathbb{Z}}\left[\sum_{y\in\mathbf{I}_L^{d-2}}|u(x+x',y,0)|^p\right]^{\frac{1}{p}}\left[\sum_{y'\in\mathbb{Z}^{d-2}}|\nabla^{(1*)}P_z(x',y')|\right] \\
[\text{swap sums } x,x'] \quad &= \sum_{x'\in\mathbb{Z}}\sum_{x\in\mathbf{I}_L}\left[\sum_{y\in\mathbf{I}_L^{d-2}}|u(x+x',y,0)|^p\right]^{\frac{1}{p}}\left[\sum_{y'\in\mathbb{Z}^{d-2}}|\nabla^{(1*)}P_z(x',y')|\right] \\
[u \text{ periodic in } x] \quad &= \sum_{x'\in\mathbb{Z}}\sum_{x\in\mathbf{I}_L}\left[\sum_{y\in\mathbf{I}_L^{d-2}}|u(x,y,0)|^p\right]^{\frac{1}{p}}\left[\sum_{y'\in\mathbb{Z}^{d-2}}|\nabla^{(1*)}P_z(x',y')|\right] \\
&= \mathrm{RHS}(\text{IV.4.10})
\end{aligned}
$$

The argument for (IV.4.10) is complete. □

IV.4.2 From the haft space to the infinite strip

We completely follow the original argument by [12]. Using the decomposition (IV.3.19) we can assume $u(0,N) = 0$ and $\langle u(\cdot,0)\rangle_{\mathbf{I}_L^{d-1}} = 0$, where the brackets denote the mean. Now set $u_0 = u$ and define inductively u_{2k+1} the solution of

$$\Delta u_{2k+1} = 0 \quad \text{in} \quad \mathbb{Z}^{d-1}\times\{1,2,\ldots\}, \quad u_{2k+1}(\cdot,0) = u_{2k}(\cdot,0), \quad k\ge 0.$$

and u_{2k} the bounded solution of

$$\Delta u_{2k} = 0 \quad \text{in} \quad \mathbb{Z}^{d-1}\times\{N-1,N-2,\ldots\}, \quad u_{2k}(\cdot,N) = u_{2k-1}(\cdot,N), \quad k\ge 1.$$

Then, $u = \sum_{k=1}^{\infty}(-1)^{k+1}u_k$. To see this, note that by construction and (IV.2.25) in Lemma IV.2.8,

$$\|u_{2k}(\cdot, N)\|_{L^p(\mathbf{I}_L^{d-1})} = \|u_{2k-1}(\cdot, N)\|_{L^p(\mathbf{I}_L^{d-1})} \le \alpha\|u_{2k-1}(\cdot, 0)\|_{L^p(\mathbf{I}_L^{d-1})}$$

and

$$\|u_{2k}(\cdot, 0)\|_{L^p(\mathbf{I}_L^{d-1})} \le \alpha\|u_{2k-1}(\cdot, N)\|_{L^p(\mathbf{I}_L^{d-1})} \le \alpha^2\|u_{2k-1}(\cdot, 0)\|_{L^p(\mathbf{I}_L^{d-1})},$$

which imply the convergence of the sum on the top and the bottom boundary:

$$\sum_{k=1}^{\infty} \|u_k\|_{L^p(\mathbf{I}_L^{d-1}\times\{y\})} \lesssim_\alpha 1, \quad y \in \{0, N\}. \tag{IV.4.13}$$

By (IV.2.24) in Lemma IV.2.8, (IV.4.13) is true for every $y \in \{0, \ldots, N\}$. Observe that this sum defines a harmonic function which coincides with u on the boundary. Adding the estimate for the case of the haft space applied to u_k for $k \ge 1$ and noting (IV.4.13) we get the estimate for the case of the infinite strip (IV.2.23).

IV.5 Finishing the proof

IV.5.1 Combining the ingredients together

Proposition IV.5.1 (Neumann case). *Under Assumption IV.2.3 with $L = N - 1 \gg 1$,*

$$\|\nabla u\|_{L^p(E_N^{\tan})} \lesssim_{p,d} \|\nabla^{(2)}u(\cdot, \cdot)\|_{L^p(\mathbf{I}_{N-1}^{d-1}\times\{0,N-1\})}.$$

Proof. In the following, we prove Proposition IV.5.1. We decompose $\|\nabla u\|_{L^p(E_N^{\tan})}$ into three types of derivatives: those taken over edges

i) on the periodic boundary,

ii) within and parallel to the periodic boundary, $\nabla_i^{(1)}u(\cdot, \cdot)$, $i \in \{1, \ldots, d-1\}$ and

iii) within and perpendicular, $\nabla^{(2)}u(\cdot, \cdot) := u(\cdot, \cdot + 1) - u$ to the periodic boundary.

The first type can be controlled by Proposition IV.2.6. To bound the second type, note that $\nabla_i^{(1)}u(\cdot, \cdot)$, $i \in \{1, \ldots, d-1\}$ are also harmonic. By the inner regularity estimate Proposition IV.2.7 (with $N = L+1$), they can be controlled in terms of the gradients $\nabla^{(1)}u(\cdot, \cdot)$ on the periodic boundary $\mathbf{I}_{N-1}^{d-1} \times \{0, N\}$. Therefore, applying Proposition IV.2.6 yields the bound of the second type. We turn to the third type. Note that the derivatives $\nabla^{(2)}u(\cdot, \cdot)$ are defined in $\mathbb{Z}^{d-1}\times\{0, \ldots, N-1\}$ and harmonic in $\mathbb{Z}^{d-1} \times \{1, \ldots, N-2\}$. Applying Proposition IV.2.7 (with L replaced by $N-1$ and N replaced by $N-1$) yields the bound $\nabla^{(2)}u(\cdot, \cdot) := u(\cdot, \cdot + 1) - u$. □

Proposition IV.5.2 (Dirichlet case). *Under Assumption IV.2.3 with $N = L \gg 1$,*

$$\|\nabla u\|_{L^p\left(E_N^{\mathrm{nor}}\right)} + \|\nabla u\|_{L^p\left(E_N^{\mathrm{tan}}\right)} \lesssim_{p,d} \|\nabla^{(1)} u(\cdot,\cdot)\|_{L^p\left(\mathbf{I}_N^{d-1}\times\{0,N\}\right)} + \frac{1}{N}\|u\|_{L^p\left(\mathbf{I}_N^{d-1}\times\{0,N\}\right)}.$$

Proof of Proposition IV.5.2. Decompose u into 3 harmonic functions u_1, u_2, u_3 satisfying (IV.3.19). By the triangle inequality,

$$\|\nabla u\|_{L^p\left(E_N^{\mathrm{nor}}\right)} \lesssim \|\nabla u_1\|_{L^p\left(E_N^{\mathrm{nor}}\right)} + \|\nabla u_2\|_{L^p\left(E_N^{\mathrm{nor}}\right)} + \|\nabla u_3\|_{L^p\left(E_N^{\mathrm{nor}}\right)}.$$

Splitting E^{nor} into edges perpendicular and parallel to the top and the bottom boundary $\mathbb{Z}^{d-1}\times\{0,N\}$, respectively, we have

$$\begin{aligned}\|\nabla u_\nu\|_{L^p\left(E_N^{\mathrm{nor}}\right)} \lesssim_{p,d} & \|\nabla^{(2)} u_\nu(\cdot,0)\|_{L^p\left(\mathbf{I}_N^{d-1}\right)} + \|\nabla^{(2)} u_\nu(\cdot,N-1)\|_{L^p\left(\mathbf{I}_N^{d-1}\right)} \\ & + \sum_{i=1}^{d-1} \|\nabla_i^{(1)} u_\nu(\cdot,\cdot)\|_{L^p\left(\mathbf{I}_N^{i-1}\times\{0\}\times\mathbf{I}_N^{d-i-1}\right)} \\ & + \sum_{i=1}^{d-1} \|\nabla_i^{(1)} u_\nu(\cdot,\cdot)\|_{L^p\left(\mathbf{I}_N^{i-1}\times\{N-1\}\times\mathbf{I}_N^{d-i-1}\right)}, \quad \nu\in\{1,2,3\}.\end{aligned}$$

By Proposition IV.2.5 (with u replaced by u_1) and Proposition IV.2.7 (with u replaced by $\nabla_i^{(1)} u(\cdot,\cdot)$ being also harmonic and $N = L$) and $\nabla^{(1)} u_1(\cdot,0) = \nabla^{(1)} u(\cdot,0)$,

$$\|\nabla u_1\|_{L^p\left(E_N^{\mathrm{nor}}\right)} \lesssim_{p,d} \|\nabla^{(1)} u(\cdot,0)\|_{L^p\left(\mathbf{I}_N^{d-1}\right)}.$$

Similarly,

$$\|\nabla u_2\|_{L^p\left(E_N^{\mathrm{nor}}\right)} \lesssim_{p,d} \|\nabla^{(1)} u(\cdot,N)\|_{L^p\left(\mathbf{I}_N^{d-1}\right)}.$$

Finally, observing that u_3 is the linear interpolation between the top and the bottom boundary value we have $\nabla^{(1)} u(\cdot,\cdot) = 0$ and $\nabla^{(2)} u_3(\cdot,\cdot) = \frac{1}{N}|\langle u(\cdot,0) - u(\cdot,N)\rangle|$. By the triangle and Jensen inequality, the last one implies

$$\|\nabla u_3\|_{L^p\left(E_N^{\mathrm{nor}}\right)} \lesssim_{p,d} \frac{1}{N}\|u\|_{L^p\left(\mathbf{I}_N^{d-1}\times\{0,N\}\right)}.$$

Summing up the estimates of u_i, $i\in\{1,2,3\}$ we get

$$\|\nabla u\|_{L^p\left(E_N^{\mathrm{nor}}\right)} \lesssim_{p,d} \|\nabla^{(1)} u(\cdot,\cdot)\|_{L^p\left(\mathbf{I}_N^{d-1}\times\{0,N\}\right)} + \frac{1}{N}\|u\|_{L^p\left(\mathbf{I}_N^{d-1}\times\{0,N\}\right)}.$$

To bound the tangential derivatives we repeat the proof of Proposition IV.5.1 with two small modifications: apply Proposition IV.2.7

(i) with $L = N$ to bound the second type, and

(ii) with N replaced by $N-1$ and L replaced by N (i.e. the second case) to bound the third type.

The proof of Proposition IV.5.2 is complete. □

IV.5.2 Proof of the main theorem

Proof of Theorem IV.1.1. We first start with the Dirichlet case, i.e. showing the tangential derivatives bound the normal derivative. It suffices to show that

$$\|\nabla u\|_{L^p\left(E_N^{\mathrm{nor}}\right)} \lesssim_{p,d} \|\nabla u\|_{L^p\left(E_N^{\tan}\right)} + \frac{1}{N}\|u\|_{L^p\left(\partial Q_N^d\right)} \tag{IV.5.1}$$

where $Q_N^d := \{0,\dots,N\}^d$. Indeed, applying (IV.5.1) with u replaced by $u-c$, where $c \in \mathbb{R}$ is chosen arbitrarily, noting that $\nabla(u-c) = \nabla u$, and applying the Poincaré inequality, we easily get the claim:

$$\|\nabla u\|_{L^p\left(E_N^{\mathrm{nor}}\right)} \lesssim \|\nabla u\|_{L^p\left(E_N^{\tan}\right)} + \inf_{c\in\mathbb{R}} \frac{1}{N}\|u-c\|_{L^p\left(\partial Q_N^d\right)} \lesssim \|\nabla u\|_{L^p\left(E_N^{\tan}\right)}. \tag{IV.5.2}$$

Now comes the argument for (IV.5.1). Applying Proposition IV.5.2 to the function w_i from Proposition IV.2.1 yields

$$\begin{aligned}\|\nabla u\|_{L^p\left(E_N^{\mathrm{nor}}\right)} &\leq \sum_{i=1}^{d} \|\nabla w_i\|_{L^p\left(E_N^{\mathrm{nor}}\right)} \\ &\lesssim \sum_{i=1}^{d} \underbrace{\|\nabla w_i\|_{L^p\left(\mathbf{I}_N^{i-1}\times\{0,N\}\times\mathbf{I}_N^{d-i}\right)} + \frac{1}{N}\|w_i\|_{L^p\left(\mathbf{I}_N^{i-1}\times\{0,N\}\times\mathbf{I}_N^{d-i}\right)}}_{=:\|w_i\|_{W^{1,p}_{\mathrm{dis}}\left(\mathbf{I}_N^{i-1}\times\{0,N\}\times\mathbf{I}_N^{d-i}\right)}}. \end{aligned} \tag{IV.5.3}$$

We show by induction that the sum (IV.5.3) is bounded by the right-hand side of (IV.5.1). First of all,

$$\|w_1\|_{W^{1,p}_{\mathrm{dis}}\left(\{0,N\}\times\mathbf{I}_N^{d-1}\right)}$$

can be absorbed into the right-hand side of (IV.5.1) by construction. Assume that

$$\|w_i\|_{W^{1,p}_{\mathrm{dis}}\left(\mathbf{I}_N^{i-1}\times\{0,N\}\times\mathbf{I}_N^{d-i}\right)} \lesssim \mathrm{RHS}(\text{IV.5.1}), \quad \forall i \in \{1,\dots,k\}$$

for some $1 \leq k \leq d-1$. By Proposition IV.2.1 (IV.2.1),

$$w_{k+1} = u - \sum_{i=1}^{k} w_i, \quad \text{in} \quad \mathbf{I}_N^k \times \{0,N\} \times \mathbf{I}_N^{d-k-1}.$$

By the triangle inequality,

$$\|w_{k+1}\| \leq \|u\| + \sum_{i=1}^{k} \|w_i\|, \quad \text{with } \|\cdot\| := \|\cdot\|_{W^{1,p}_{\mathrm{dis}}\left(\mathbf{I}^k\times\{0,N\}\times\mathbf{I}^{d-k-1}\right)}. \tag{IV.5.4}$$

Here, for all $i \in \{1,\dots,k\}$ the discrete Sobolev $W^{1,p}_{\mathrm{dis}}\left(\mathbf{I}_N^k \times \{0,N\} \times \mathbf{I}_N^{d-k-1}\right)$ norm of w_i (taken on faces within the periodic boundary) can be estimated by its discrete Sobolev norm taken on its peridic boundary, i.e. its $W^{1,p}_{\mathrm{dis}}\left(\mathbf{I}^{i-1} \times \{0,N\} \times \mathbf{I}^{d-i}\right)$

norm (see Propositions IV.2.7 and IV.5.2). By the induction hypothesis and (IV.5.4), the claim is also true for $k+1$. Therefore, (IV.5.3) is bounded by the right-hand side of (IV.5.1). Finally, let us make a remark that the instead of applying the Poincaré inequality on the whole surface, we can apply it on each face for each $w_i, 2 \leq i \leq d$, which have zero mean (by odd reflections), and therefore we get rid of the last term in (IV.5.1). However, w_1 does not have zero mean and we still have to use the "tricks" subtracting a constant (IV.5.2) to bound it. We finish the argument for the Dirichlet case.

In the Neumann case, we also add the estimates from Proposition IV.5.2 applied to each w_i in the Neumann decomposition and use an induction argument similar to that in the Dirichlet case (IV.5.4). □

Proof of Corollary IV.2.9. We only sketch the proof. We use the Dirichlet decomposition $u = w_1 + \ldots + w_d$, Proposition IV.2.1, and a similar induction argument to bound the norm of w_i or ∇w_i on the boundary by the norm of u or ∇u, respectively. Therefore, it suffices to consider the periodic case. In this case, (IV.2.26) is obvious by exploiting the fact that u has periodic boundary and the inner face lemma, Proposition IV.2.7, to take the sum "over N faces". The argument for (IV.2.27) in the periodic case is less obvious, however, is very similar to that of Propositions IV.5.1 and IV.5.2: we control 2 type of edges in the interior of the box: those which are parallel and those which are perpendicular to the periodic boundary. Both types can be estimated by exploiting the fact that the derivatives of a harmonic function is still harmonic and applying (IV.2.26). □

IV.A Decompositions

We prove Propositions IV.2.1 and IV.2.2 for $d \geq 3$. The case $d = 2$ has been done before (see Page 111). We first start with the Dirichlet case in $d = 3$ where the proof can be still illustrated. The proof for the general case (Subsections IV.A.2 and IV.A.3) follows the spirit of that for $d = 2, 3$, however, we will check every step carefully.

IV.A.1 Dirichlet decomposition in the three-dimensional case

The argument is illustrated in Figure IV.7.

Construction of w_1. Define

$$w_1 = u \quad \text{in} \quad \{0, N\} \times \{0, \ldots, N\}^2 \tag{IV.A.1}$$

which is the two red sides perpendicular to x_1-axis in Figure IV.7a. Extend w_1 into $\{0, N\} \times \{0, \ldots, 2N\}^2$ (the red surface on Figure IV.7e) by using two even reflections: first extend it along x_2-direction to $\{0, N\} \times \{0, \ldots, 2N\} \times \{0, \ldots, N\}$

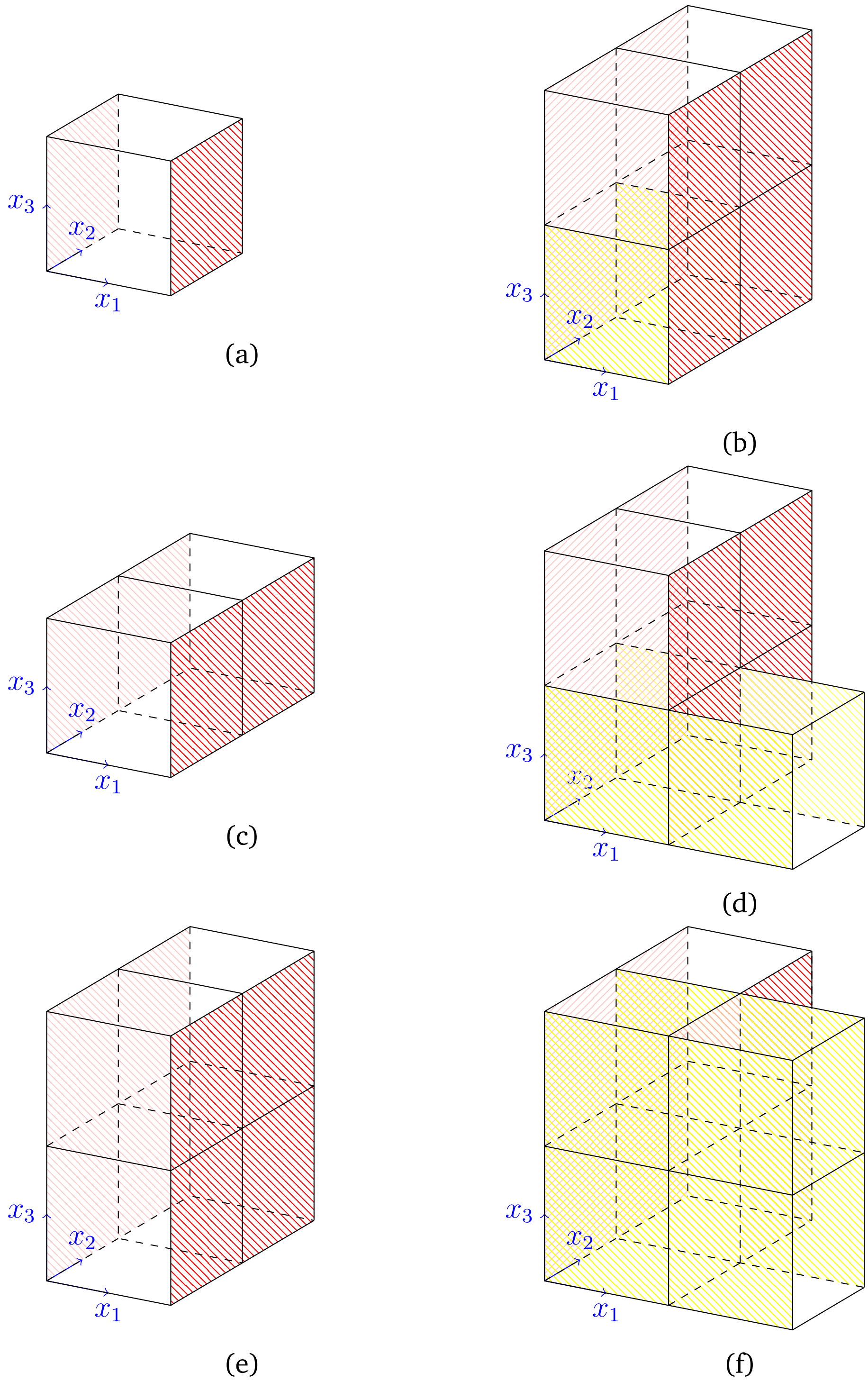

Figure IV.7: Construction of w_1 (left) and w_2 (right) – the case $d = 3$

(the red surface in Figure IV.7c), then to the two squares of side length $2N$. Then, extend w_1 to $\{0, N\}\times\mathbb{Z}^2$ $2N$-periodically and to $\{1, \ldots, N\}\times\mathbb{Z}^2$ so that it is harmonic in there.

Construction of w_2. Set

$$w_2 = u - w_1 \quad \text{in} \quad \{0, \ldots, N\} \times \{0, N\} \times \{0, \ldots, N\} \tag{IV.A.2}$$

which is the two yellow faces in Figure IV.7b). This construction implies

$$w_2 = 0 \quad \text{in} \quad \{0, N\} \times \{0, N\} \times \{0, \ldots, N\}, \tag{IV.A.3}$$

which is the intersection of the yellow and the red faces in Figure IV.7d. Then, extend w_2 to the two yellow squares in Figure IV.7f,

$$\{0, \ldots, 2N\} \times \{0, N\} \times \{0, \ldots, 2N\}$$

as follows: extend it along to

$$\{0, \ldots, 2N\} \times \{0, N\} \times \{0, \ldots, N\},$$

the two yellow rectangles in Figure IV.7d, using an odd reflection through

$$\{N\} \times \{0, N\} \times \{0, \ldots, N\}$$

and to the two squares of side length $2N$ using an even reflection through

$$\{0, \ldots, 2N\} \times \{0, N\} \times \{N\}.$$

Note that the use of an odd reflection is consistent with (IV.A.3). Extend w_2 $2N$-periodically to $\mathbb{Z} \times \{0, N\} \times \mathbb{Z}$ and to $\mathbb{Z} \times \{1, \ldots, N-1\} \times \mathbb{Z}$ so that it is harmonic in there. Because of the odd reflection of the boundary values, (IV.A.3) can be extended to the inner values

$$w_2 = 0 \quad \text{in} \quad \{0, N\} \times \{0, \ldots, N\}^2, \tag{IV.A.4}$$

which is the red faces of the box. Combining (IV.A.1)–(IV.A.3) yields

$$\begin{aligned} u = w_1 + w_2 \quad &\text{in} \\ &\Big[\{0, N\} \times \{0, \ldots, N\} \times \{0, \ldots, N\}\Big] \cup \Big[\{0, \ldots, N\} \times \{0, N\} \times \{0, \ldots, N\}\Big] \end{aligned} \tag{IV.A.5}$$

Construction of w_3. Set

$$w_3 = u - w_1 - w_2 \quad \text{in} \quad \{0, \ldots N\} \times \{0, \ldots, N\} \times \{0, N\}$$

the two faces of the box perpendicular to w_3. By this construction and (IV.A.5),

$$w_3 = 0 \quad \text{in} \quad \Big[\{0,N\} \times \{0,\dots,N\} \times \{0,N\}\Big] \cup \Big[\{0,\dots,N\} \times \{0,N\} \times \{0,N\}\Big] \tag{IV.A.6}$$

being the set of all points on the boundary of the two faces $\{x_3 \in \{0,N\}\}$. Extend w_3 to

$$\{0,\dots,2N\}^2 \times \{0,N\}$$

by using two odd reflections: along the x_1-direction to

$$\{0,\dots,2N\} \times \{0,\dots,N\} \times \{0,N\},$$

and along x_2-direction to

$$\{0,\dots,2N\}^2 \times \{0,N\},$$

where the use of odd reflections is consistent with (IV.A.6). Then, extend w_3 $2N$-periodically to $\mathbb{Z}^2 \times \{0,N\}$ and to $\mathbb{Z}^2 \times \{1,\dots,N-1\}$ so that it is harmonic in there. Because of the odd reflections of the boundary values, (IV.A.6) can be extended to the inner values:

$$w_3 = 0 \quad \text{in} \quad \Big[\{0,N\} \times \{0,\dots,N\} \times \{0,\dots,N\}\Big] \cup \Big[\{0,\dots,N\} \times \{0,N\} \times \{0,\dots,N\}\Big], \tag{IV.A.7}$$

which are the red and yellows faces of $\{0,\dots,N\}^d$ in Figure IV.7. Combining (IV.A.5) and (IV.A.7) we get $u = w_1 + w_2 + w_3$ on all faces of $\{0,\dots,N\}^d$, therefore also on $\{0,\dots,N\}^d$.

IV.A.2 Dirichlet decomposition in higher dimensions

The argument is similar to the proof in the continuum setting [12]. For convenience we repeat it here.

Construction of w_1. Set

$$w_1 = u \quad \text{in} \quad \{0,N\} \times \{0,\dots,N\}^{d-1}. \tag{IV.A.8}$$

By successively using $(d-1)$ even reflections extend w_1 to $\{0,N\} \times \{0,\dots,2N\}^{d-1}$. Then, extend w_1 to $\{0,N\} \times \mathbb{Z}^{d-1}$ $2N$-periodically, and to $\{1,\dots,N-1\} \times \mathbb{Z}^{d-1}$ so that it is harmonic in there.

Construction of w_2. Set

$$w_2 = u - w_1 \quad \text{on} \quad \{0,\dots,N\} \times \{0,N\} \times \{0,\dots,N\}^{d-2}. \tag{IV.A.9}$$

By this construction and (IV.A.8)

$$w_2 = 0 \quad \text{in} \quad \{0, N\} \times \{0, N\} \times \{0, \dots, N\}^{d-2}. \tag{IV.A.10}$$

By successively using an odd reflection consistent to (IV.A.10) in the first coordinate and even reflections in $d-2$ last coordinates extend w_2 to $\{0, 2N\} \times \{0, N\} \times \{0, \dots, 2N\}^{d-2}$. Then, extend w_1 to $\mathbb{Z} \times \{0, N\} \times \mathbb{Z}^{d-2}$ $2N$-periodically and to

$$\mathbb{Z} \times \{1, \dots, N-1\} \times \mathbb{Z}^{d-2}$$

so that it is harmonic in there. Because of the odd reflection on the boundary, (IV.A.10) can be extended to the inner values:

$$w_2 = 0 \quad \text{in} \quad \{0, N\} \times \{0, \dots, N\}^{d-1}. \tag{IV.A.11}$$

Combining (IV.A.8), (IV.A.9), and (IV.A.11) yields

$$u = w_1 + w_2 \quad \text{in} \quad \bigcup_{j=1,2} \{0, \dots, N\}^{j-1} \times \{0, N\} \times \{0, \dots, N\}^{d-j}. \tag{IV.A.12}$$

Construction of $w_k,\ k \geq 2$ *by induction.* Assume that for some $1 \leq k \leq d-1$, for every $1 \leq j \leq k$, w_j has been constructed in $\mathbb{Z}^{j-1} \times \{0, \dots, N\} \times \mathbb{Z}^{d-j}$ so that it is harmonic in $\mathbb{Z}^{j-1} \times \{1, \dots, N-1\} \times \mathbb{Z}^{d-j}$ and

$$u = \sum_{j=1}^{k} w_j \quad \text{in} \quad \bigcup_{j=1}^{k} \{0, \dots, N\}^{j-1} \times \{0, N\} \times \{0, \dots, N\}^{d-j} \tag{IV.A.13}$$

Note that (IV.A.12) is (IV.A.13) with $k = 2$. Then, define

$$w_{k+1} = u - \sum_{j=1}^{k} w_j \quad \text{in} \quad \{0, \dots, N\}^{k} \times \{0, N\} \times \{0, \dots, N\}^{d-k-1}. \tag{IV.A.14}$$

By this construction and (IV.A.13),

$$\begin{aligned} w_{k+1} = 0 \quad \text{in} \quad & \bigcup_{j=1}^{k} \{0, \dots, N\}^{j-1} \times \{0, N\} \times \{0, \dots, N\}^{k-j} \\ & \times \{0, N\} \times \{0, \dots, N\}^{d-k-1}. \end{aligned} \tag{IV.A.15}$$

By making use of odd reflections in the first k coordinates consistent to (IV.A.15) and even reflections in the last $d-k-1$ coordinates, extend w_{k+1} to

$$\{0, \dots, 2N\}^{k} \times \{0, N\} \times \{0, \dots, 2N\}^{d-k-1}.$$

Finally, extend w_{k+1} to

$$\{0, \dots, 2N\}^{k} \times \{1, \dots, N-1\} \times \{0, \dots, 2N\}^{d-k-1}$$

so that it is harmonic in there. Because of the odd reflections of the boundary conditions, (IV.A.15) can be extended to the inner values:

$$w_{k+1} = 0 \quad \text{in} \quad \bigcup_{j=1}^{k} \{0, \ldots, N\}^{j-1} \times \{0, N\} \times \{0, \ldots, N\}^{d-j}. \tag{IV.A.16}$$

Combining (IV.A.13)–(IV.A.15) we get (IV.A.13) with k replaced by $k+1$. Continue the construction we get $w_1, \ldots, w_d$ and $u = w_1 + \ldots + w_d$ on every face of the box, therefore also on the box $\{0, \ldots, N\}^d$.

IV.A.3 The Neumann decomposition in higher dimensions

Define $v_1 = \nabla^{\mathrm{nor}} u \quad \text{in} \quad \{0, N\} \times \{1, \ldots, N-1\}^{d-1}$. Then, using $(d-1)$ odd reflections extend v_1 to $\{0, N\} \times \{1, \ldots, 2N-2\}^{d-1}$, and then $(2(N-1)$-periodically to $\{0, N\} \times \mathbb{Z}^{d-1}$. Define w_1 in $\{0, \ldots, N\} \times \mathbb{Z}^{d-1}$ (up to a constant) such that w_1 is harmonic in $\{1, \ldots, N-1\} \times \mathbb{Z}^{d-1}$ and $\nabla_1^{\mathrm{nor}} w_1 = v_1$ on $\{0, N\} \times \mathbb{Z}^{d-1}$.

Now, having defined v_j and w_j for $1 \leq j \leq k$ such that each w_j is defined on $\mathbb{Z}^{j-1} \times \{0, \ldots, N\} \times \mathbb{Z}^{d-j}$ and harmonic in $\mathbb{Z}^{j-1} \times \{1, \ldots, N-1\} \times \mathbb{Z}^{d-j}$ with the Neumann condition $v_j = \nabla_j^{\mathrm{nor}}$ on $\mathbb{Z}^{j-1} \times \{0, N\} \times \mathbb{Z}^{d-j}$. Then, define w_{k+1} as follows. First, define

$$v_{k+1} = \nabla_{k+1}^{\mathrm{nor}} u - \sum_{j=1}^{k} \nabla_{k+1}^{\mathrm{nor}} w_j$$
$$\{1, \ldots, N-1\}^k \times \{0, N\} \times \{1, \ldots, N-1\}^{d-j-1} \tag{IV.A.17}$$

and extend it using k odd reflections in the first k coordinates and $d-k-1$ in the last to $\mathbb{Z}^j \times \{0, N\} \times \mathbb{Z}^{d-j-1}$ and use it as a Neumann condition to define w_{k+1}, which is harmonic in $\mathbb{Z}^j \times \{1, \ldots, N-1\} \times \mathbb{Z}^{d-j-1}$ and

$$\nabla_{k+1}^{\mathrm{nor}} w_{k+1} = v_{k+1} \quad \text{on} \quad \mathbb{Z}^j \times \{0, N\} \times \mathbb{Z}^{d-j-1}.$$

When proceeding as above we obtain (IV.2.4) and (IV.2.5) for $1 \leq i \leq k$ after $k+1$ steps, this can be proved as done in the Dirichlet case. We do not give details. Only note that (IV.2.5) follows, since even reflections ”kill” all normal derivatives on the previous faces – the same way as in the case $d = 2$.

Finally, we have to make sure that in each step we have a Neumann condition of zero mean to apply Theorem IV.2.4. In the k-th step $1 \leq k \leq d-1$, since we use at least an odd reflection, this is obvious. In the last (d-th) step we only use even reflections. However, we still have that v_d has zero mean on the last two faces $\{1, \ldots, N-1\}^{d-1} \times \{0, N\}$. This can be seen as follows. Note that we define $v_d = \nabla_d^{\mathrm{nor}} u - \sum_{j=1}^{d-1} \nabla_d^{\mathrm{nor}} w_j$ on the last two faces. Remember that $\nabla_d^{\mathrm{nor}} u - \sum_{j=1}^{d-1} \nabla_d^{\mathrm{nor}} w_j$ is zero on the first $(2d-2)$ faces and has zero mean on all $2d$ faces, since it is the

Neumann condition of $u - w_1 - \ldots - w_{d-1}$, being harmonic in the box. Therefore, it must be of zero mean on the last 2 faces. In other words v_d has zero mean. The proof is complete.

IV.B The Marcinkiewicz multiplier theorem

IV.B.1 Periodic discrete functions

In the proof of Proposition IV.2.5 we need to bound the normal component by the $(d-1)$ tangential components that requires a vector-valued multiplier theorem. Therefore, we improve the Marcinkiewicz theorem [40, Theorem 2.49] proved by Jovanović and Süli to Theorem IV.B.1 below.

Before stating the result, let us recall the notations in Subsection IV.3.1.

Theorem IV.B.1. *Let $a^{(\beta)} = \{a^{(\beta)}(k) : k \in \mathbb{Z}^d\}$, $\beta \in \{1, \ldots, d\}$ be $2L$-periodic functions defined on $\mathbb{Z}^d$. Suppose that one of the following two conditions holds.*

a) There exists a constant M such that for each $k = (k_1, \ldots, k_d)$ and each collection $1 \le j_1 < \ldots < j_m \le d$, there exists an index $\beta_0 = \beta_0(k)$ such that

$$|a^{(\beta_0)}(k)| \le M,$$

$$\sum_{\nu_{j_1} \in \left[\pm 2^{|k_1|-1}; \pm 2^{|k_1|} - 1\right] \cap \mathbf{I}_L} \cdots \sum_{\nu_{j_m} \in \left[\pm 2^{|k_m|-1}; \pm 2^{|k_m|} - 1\right] \cap \mathbf{I}_L} |\nabla_{j_1} \ldots \nabla_{j_m} a^{(\beta_0)}(\nu)| \le M, \tag{IV.B.1}$$

where $+$ or $-$ is chosen according to $k_j > 0$ or $k_j < 0$, for $k_j = 0$ the sum is extended only to $\nu_j = 0$, and

$$\nabla_i f(\nu) := f(\nu + \mathbf{e}_i) - f(\nu), \quad \nu \in \mathbb{Z}^d. \tag{IV.B.2}$$

b) For $1 \le \beta \le d$, $a^{(\beta)}$ can be extended to a function, still denoted by $a^{(\beta)}$, which is defined and continuous on $[-L+1, L]^d$, whose derivatives satisfy

$$\partial^\alpha a \in C\left([-L+1, L]^d \setminus \mathbf{I}_L^d\right), \quad \forall \alpha \in \{0,1\}^d,$$

such that for each $k \in \mathbb{Z}^d$ there exists $\beta_0 = \beta_0(k)$ satisfying

$$\sup_{\alpha \in \{0,1\}^d} \sup_{\xi \in D_k} \left|\xi^\alpha \partial^\alpha a^{(\beta_0)}(\xi)\right| \le M_0,$$

where

$$D_k = \left\{\prod_{\beta=1}^{d} \left[\pm 2^{|k_\beta|-1}; \pm 2^{|k_\beta|} - 1\right]\right\} \cap [-L+1, L]^d.$$

Further, let $V, v^{(1)}, \ldots, v^{(d)}$ be 2π-periodic functions on ω_h^d related by means of the Marcinkiewicz multipliers:

$$\widehat{V}(k) = a^{(\beta)}(k)\widehat{v^{(\beta)}}(k), \quad k \in \mathbf{I}_L^d, \quad \beta \in \{1, \ldots, d\}.$$

Then, for any $p \in (1, \infty)$, we have

$$\|V\|_{L^p\left(\omega_h^d\right)} \lesssim_{p,d} M_0 \|v\|_{L^p\left(\omega_h^d\right)}$$

where $\|v\|_{L^p\left(\omega_h^d\right)}$ denotes the $L^p\left(\omega_h^d\right)$ norm of

$$|v| := \left(\sum_{\beta=1}^{d} |v^{(\beta)}|^2\right)^{1/2}.$$

Proof. Recall that in the proof of [40, Theorem 2.49] by Jovanović and Süli one returns to the case of 2π-periodic continuum functions [40, Theorem 1.59] by using piecewise constant extensions and exploiting [40, Lemmas 2.48 and 2.50]. By using this idea, the claim then follows from Theorem IV.B.2 below dealing with the continuum periodic case. □

IV.B.2 Periodic continuum functions

The following theorem is a very simple improvement of the multiplier theorem in the book by Nikol'skiǐ [52, p52].

Theorem IV.B.2 (Multiplier theorem for vector valued functions)**.** *Let $d \geq 1$. Suppose that the functions $\lambda^{(\beta)} = \{\lambda_k^{(\beta)} : k \in \mathbb{Z}^d\}$, $\beta \in \{1, \ldots, d\}$ satisfy the following assumptions: There exists a constant $M > 0$ such that for each $k = (k_1, \ldots, k_d) \in \mathbb{Z}^d$ and each collection $1 \leq j_1 < \ldots < j_m \leq d$, there exists $\beta_0) = \beta_0(k)$ such that*

$$|\lambda_k^{(\beta_0)}| \leq M, \qquad \sum_{\nu_{j_1}=\pm 2^{|k_1|-1}}^{\pm 2^{|k_1|}-1} \cdots \sum_{\nu_{j_m}=\pm 2^{|k_m|-1}}^{\pm 2^{|k_m|}-1} |\nabla_{j_1} \ldots \nabla_{j_m} \lambda_\nu^{(\beta_0)}| \leq M, \tag{IV.B.3}$$

where $+$ or $-$ is chosen according to $k_j > 0$ or $k_j < 0$, for $k_j = 0$ the sum is extended only to $\nu_j = 0$, and (IV.B.2)*. Further, let $F, f^{(1)}, \ldots, f^{(d)}$ be 2π-periodic functions represented in terms of Fourier series and related by means of the Marcinkiewicz multipliers:*

$$f^{(1)}(x) = \sum_{k \in \mathbb{Z}^d} c_k^{(1)} e^{\mathbf{i}k \cdot x}, \ldots, f^{(d)}(x) = \sum_{k \in \mathbb{Z}^d} c_k^{(d)} e^{\mathbf{i}k \cdot x},$$

$$F(x) = \sum_{k \in \mathbb{Z}^d} \lambda_k^{(1)} c_k^{(1)} e^{\mathbf{i}k \cdot x} = \ldots = \sum_{k \in \mathbb{Z}^d} \lambda_k^{(d)} c_k^{(d)} e^{\mathbf{i}k \cdot x}.$$

Then, for any $p \in (1, \infty)$, *we have*

$$\left(\int_{[-\pi,\pi]^d} |F(x)|^p dx\right)^{1/p} \lesssim_{p,d} \left(\int_{[-\pi,\pi]^d} |f(x)|^p dx\right)^{1/p}.$$

Proof. The proof is almost the same as in [52]. However, we check every step carefully to make sure that it works in our new setting. It suffices to consider $d = 2$. Further, by [52, (11) p49] and the paragraph thereafter, we also can assume that the Fourier series here are extended over $k_i \geq 0$, $1 \leq i \leq d$. The formulas (IV.B.4) and (IV.B.5) below are almost the same as (5) and (6) in the proof in [52], except the fact that we have now d multipliers distinguished by the superscript (β). Set

$$\sum_{\mu=2^{k-1}}^{s} \sum_{\nu=2^{l-1}}^{t} c_{\mu\nu}^{(\beta)} e^{\mathbf{i}(\mu x+\nu y)} = r_{st}^{(\beta)} = r_{s,t,k,l}^{(\beta)}, \quad \beta \in \{1, \dots, d\}. \tag{IV.B.4}$$

Applying the Abel transformation yields

$$\begin{aligned}
\delta_{kl}(F) &= \sum_{\mu=2^{k-1}}^{2^k-1} \sum_{\nu=2^{l-1}}^{2^l-1} \lambda_{\mu\nu}^{(\beta)} c_{\mu\nu}^{(\beta)} e^{\mathbf{i}(\mu x+\nu y)} \\
&= \sum_{\mu=2^{k-1}}^{2^k-2} \sum_{\nu=2^{l-1}}^{2^l-2} r_{ij}^{(\beta)} \Big\{ \lambda^{(\beta)}(i,j) - \lambda^{(\beta)}(i,j+1) \\
&\qquad\qquad - \lambda^{(\beta)}(i+1,j) + \lambda^{(\beta)}(i+1,j+1) \Big\} \\
&\qquad + \sum_{2^{l-1}}^{2^l-2} r_{2^k-1,j}^{(\beta)} \left[\lambda(2^k-1,j) - \lambda^{(\beta)}(2^k-1,j+1)\right] \\
&\qquad + \sum_{2^{k-1}}^{2^k-2} r_{i,2^l-1}^{(\beta)} \left[\lambda_{i,2^l-1}^{(\beta)} - \lambda_{i+1,2^l-1}^{(\beta)}\right] + r_{2^k-1,2^l-1}^{(\beta)} \lambda_{2^k-1,2^l-1}^{(\beta)}. \\
&=: \sum_{ij} r_{ij}^{(\beta)} \gamma_{ij}^{(\beta)}, \quad \beta \in \{1, \dots, d\}.
\end{aligned} \tag{IV.B.5}$$

By the Cauchy-Schwartz inequality,

$$\begin{aligned}
|\delta_{kl}(F)|^2 &\leq \left(\sum_{ij} r_{ij}^{(\beta_0)} \gamma_{ij}^{(\beta_0)}\right)^2 \leq \left\{\sum_{ij} |\gamma_{ij}^{(\beta_0)}|\right\} \left\{\sum_{ij} \left(r_{ij}^{(\beta_0)}\right)^2 |\gamma_{ij}^{(\beta_0)}|\right\} \\
&\leq M \left\{\sum_{ij} \left(r_{ij}^{(\beta_0)}\right)^2 |\gamma_{ij}^{(\beta_0)}|\right\},
\end{aligned}$$

for β_0 depending on k, l satisfying (IV.B.3). Applying [52, (13) p50] with $n_k = 2^k$ we get

$$\iint_{[-\pi,\pi]^2} |F|^p \lesssim \iint_{[-\pi,\pi]^2} \left(\sum_{k,l} |\delta_{kl}(F)|^2 \right)^{p/2}$$

$$\lesssim M^{p/2} \iint_{[-\pi,\pi]^2} \left(\sum_{k,l} \sum_{i,j} \left(r_{ij}^{(\beta_0)} \sqrt{|\gamma_{ij}^{(\beta_0)}|} \right)^2 \right)^{p/2}$$

Applying the result (6) p52 in the same book (more precisely, with f_n replaced by the Fourier serie $\delta_{kl}(f^{(\beta_0)})\sqrt{|\gamma_{ij}^{(\beta_0)}|}$, their partial sums $r_{ij}^{(\beta_0)}\sqrt{|\gamma_{ij}^{(\beta_0)}|}$ playing the role of S_{n,k_n}, and the sum over n replaced by the sum over k, l, i, j) yields that

$$\iint_{[-\pi,\pi]^2} |F|^p \lesssim M^{p/2} \iint_{[-\pi,\pi]^2} \left(\sum_{k,l} \sum_{i,j} \left(\delta_{kl}(f^{(\beta_0)}) \sqrt{|\gamma_{ij}^{(\beta_0)}|} \right)^2 \right)^{p/2}$$

$$= M^{p/2} \iint_{[-\pi,\pi]^2} \left(\sum_{k,l} \delta_{kl}(f^{(\beta_0)})^2 \sum_{ij} |\gamma_{ij}^{(\beta_0)}| \right)^{p/2}$$

$$= M^{p} \iint_{[-\pi,\pi]^2} \left(\sum_{k,l} \delta_{kl}(f^{(\beta_0)})^2 \right)^{p/2}$$

$$\leq M^{p} \sum_{\beta=1}^{d} \iint_{[-\pi,\pi]^2} \left(\sum_{k,l} \delta_{kl}(f^{(\beta)})^2 \right)^{p/2}$$

$$\leq M^{p} \sum_{\beta=1}^{d} \iint_{[-\pi,\pi]^2} |f^{(\beta)}|^p.$$

The proof is complete. $\square$

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