Model uncertainty, improved Fréchet-Hoeffding bounds and applications in mathematical finance

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Abstract

In this thesis, we establish novel approaches to quantify model uncertainty and associated risks. We are specifically concerned with model risk that is expressed in terms of an upper and lower bound on the expectation of an aggregation functional of a real-valued random vector, whose law is only partially known. In applications, the aggregation functional can be thought of as an option payoff or a risk measure depending on asset prices or risk factors modelled by the random vector.

We establish several methods to derive bounds on expectations and show that these methods enjoy features which are appealing for many practical applications, namely (i) they allow to incorporate partial model information, which can often be obtained from data or is available in the form of expert views, (ii) they apply to a large class of aggregation functionals, including typical option payoffs or risk measures, (iii) they are tractable and can be implemented using numerical integration techniques or linear optimization procedures.

Our approach is based on improvements of the well-known Fréchet–Hoeffding bounds in the presence of partial information on the model of the underlying random vector. We begin with the derivation of improved Fréchet–Hoeffding bounds that incorporate information such as e.g. the distance to a reference model or properties of lower-dimensional marginals of the random vector. These results extend the existing body of literature on improved bivariate Fréchet–Hoeffding bounds to the higher-dimensional case and, moreover, they pertain to types of partial information that have not been considered in the literature thus far. Furthermore, we show that the improved Fréchet–Hoeffding bounds fail to be distribution functions under very weak conditions, which constitutes a surprising difference to the bivariate case.

In order to translate the improved Fréchet–Hoeffding bounds into bounds on expectations, we develop an appropriate integration theory so as to make sense of integrals with respect to integrators that do not induce measures. This in turn allows us to extend existing results on multivariate stochastic dominance and prove an integral characterization of orthant orders on a suitable extension of the set of distribution functions. Using that the improved Fréchet–Hoeffding bounds are extremal in the sense of orthant orders, we can then translate them into bounds on expectations over a class of distributions that comply with available information. We demonstrate this approach in numerical applications related to model-free derivatives pricing.

A fundamentally different approach is required in order to derive robust estimates for the Value-at-Risk of aggregations, since Value-at-Risk is not amenable to our characterization of orthant orders. In this work, we establish two distinct methods to address this problem. First, we derive Value-at-Risk bounds when extreme value information on the underlying risk vector is available. This is achieved by transforming the corresponding optimization problem into a standard Fréchet-problem that can be solved by means of the Rearrangement Algorithm or the well-known Improved Standard Bounds. Second, we develop a method to translate improved Fréchet–Hoeffding bounds into Value-at-Risk estimates. To this end, we resort to a mass transport approach and apply the Monge–Kantorovich Duality Theory to obtain sharp dual bounds on Value-at-Risk in the presence of partial information. Based on this dual formulation we derive a tractable optimization scheme to compute robust risk estimates. Moreover, we show that the well-known Improved Standard Bounds can be recovered as special instances from our optimization procedure. Finally, numerical illustrations suggest that our method typically yields Value-at-Risk bounds that are substantially narrower than the Improved Standard Bounds using the same information.

Zusammenfassung

In dieser Arbeit entwickeln wir Ansätze zur Quantifizierung von Modellunsicherheit und den damit verbundenen Risiken. Zentraler Betrachtungsgegenstand sind Modellrisiken in Form von oberen und unteren Schranken an den Erwartungswert einer Aggregationsfunktion, die von einem reellwertigen Zufallsvektor mit partiell bekannter Verteilungsfunktion abhängt. In Anwendungen beschreibt die Aggregationsfunktion beispielsweise das Auszahlungsprofil eines Derivats oder ein Risikomaß, das von mehreren, durch den Zufallsvektor modellierten Basiswerten beziehungsweise Risikofaktoren abhängt.

Hauptbestandteil der Arbeit ist die Entwicklung verschiedener Methoden zur Abschätzung von Erwartungswerten. Überdies verdeutlichen wir die praktische Bewandtnis der Verfahren, die (i) es ermöglichen, partielle Modellinformationen – die in Anwendungen meist durch Datenauswertung oder Expertenwissen gewonnen werden – in die Berechnung des Modellrisikos zu integrieren, (ii) für eine relativ große Klasse von Aggregationsfunktionen gültig sind und die (iii) rechnerisch handhabbar sind, sodass sie mittels numerischer Integrationsmethoden oder linearer Optimierungsverfahren realisiert werden können.

Ausgangspunkt der Arbeit sind Verbesserungen der generischen Fréchet–Hoeffding Schranken unter Berücksichtigung partieller Modellinformationen, wie beispielsweise der Distanz zu einem Referenzmodell. Hierdurch erweitern wir zum einen den umfangreichen Literaturkorpus zu verbesserten bivariaten Fréchet–Hoeffding Schranken auf den höherdimensionalen Fall und zum anderen erlauben unsere Ergebnisse die Einbeziehung von Informationstypen, die bisher nicht in der Literatur betrachtet wurden. Ferner beweisen wir unter schwachen Voraussetzungen, dass die verbesserten Fréchet–Hoeffding Schranken keine Verteilungen sind, was einen bemerkenswerten Unterschied zum zweidimensionalen Fall markiert.

Um Erwartungswerte mittels verbesserter Fréchet–Hoeffding Schranken abzuschätzen, entwickeln wir anschließend einen Integralbegriff für Integratoren, die kein Maß induzieren. Hiervon ausgehend erweitern wir vorhandene Resultate über stochastische Ordnungen und beweisen eine Integral-Charakterisierung der Orthantenordung auf einer geeigneten Erweiterung der Menge von Wahrscheinlichkeitsverteilungen. Schließlich verknüpfen wir die Integral-Charakterisierung der Ortanthenordnung mit den verbesserten Fréchet–Hoeffding Schranken zur Abschätzung von Erwartungswerten bezüglich einer Klasse von Wahrscheinlichkeitsverteilungen, die mit verfügbaren Modellinformationen kompatibel sind. Dieses Verfahren illustrieren wir in numerischen Anwendungen zur modellfreien Optionsbewertung.

Im zweiten Teil der Arbeit entwickeln wir zwei Ansätze zur Risikobewertung unter Modellunsicherheit mittels Value-at-Risk basierter Risikomaße. Zunächst nutzen wir Extremwertinformationen über den Risikovektor zur Berechnung robuster Value-at-Risk Schranken. Dies gelingt durch eine Transformation der entsprechenden Optimierungsaufgabe in ein Standard-Fréchet-Problem, welches durch Verfahren wie beispielsweise den Rearrangement Algorithmus oder die verbesserten Standardschranken gelöst werden kann. Unser zweiter Ansatz ermöglicht die Anwendung verbesserter Fréchet–Hoeffding Schranken zur Abschätzung von Value-at-Risk. Hierzu benutzen wir die Monge–Kantorovich Dualitätstheorie zur Herleitung scharfer dualer Risikoschranken mit partieller Modellinformation. Auf Basis der dualen Formulierung entwickeln wir ferner ein numerisch handhabbares Optimierungsschema zur Abschätzung von Value-at-Risk. Darüber hinaus zeigen wir, dass die verbesserten Standardschranken einen Spezialfall unseres Schemas darstellen.

To Regev

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תודה לך כל כך הרבה אהובי על האהבה הרבה שלך והתמיכה בלי תנאים

Contents

1 Introduction

In recent years, there has been a surge of interest in model risk and uncertainty quantification in many areas of applied mathematics. While traditionally the focus was on computing quantities of interest given a certain model, one faces today more frequently the challenge of estimating quantities in the absence of a fully specified model. This paradigm shift arises from the observation that even sophisticated modelling techniques and validation procedures cannot entirely eliminate the risk of model misspecification and the associated uncertainty about the accuracy of a model. The consideration of model risk is relevant in many applications ranging from engineering and hydrology to finance, since the consequences of choosing an inaccurate model may be severe. The subprime mortgage crisis that emerged in 2008 and which continues to burden economies until this day is a striking example, demonstrating the possible implications of the widespread use of inaccurate models. Although it would be presumptuous to attribute the cause of this global economic crisis to modelling practices alone, it is undeniably the case that inaccurate models created a breeding ground for its emergence. Such negative implications are the main reason for practitioners and regulators to integrate the quantification of risks arising from a lack of model validity as a key element in their activities. These risks are usually referred to as *model risks*. Especially in finance, industry-wide regulations such as the Basel Accord or the Solvency Directive are increasingly focusing on provisions for model uncertainty, which in turn calls for new mathematical approaches and methods to compute the respective risks.

The separation of risk and model uncertainty goes back to the economist Frank Knight who observed that

> *"[t]he essential fact is that risk means in some cases a quantity susceptible of measurement, while at other times it is something distinctly not of this character; and there are far reaching and crucial differences in the bearings of the phenomenon depending on which of the two is really operating."*

c.f. [29]. *Risk* refers to unknown future outcomes or events whose probabilities are known with accuracy, whereas *uncertainty* pertains to ambiguity about the probabilities themselves.

In a probabilistic setting, a model typically refers to the law of a random variable. In many applications, individual risks are modelled by real valued random variables $X_1, ..., X_d$ on a probability space $(\Omega, \mathcal{B}, \mathbb{P})$, e.g. one can consider $X_1, ..., X_d$ modelling the prices of financial assets at a future point in time or the size of prospective claims against an insurer. Then $\mathbf{X} := (X_1, ..., X_d)$ is an $\mathbb{R}^d$-valued random vector with distribution $F(x_1, ..., x_d) = \mathbb{P}(X_1 \leq x_1, ..., X_d \leq x_d)$. Model uncertainty is expressed in terms of ambiguity about the probability $\mathbb{P}$. Specifically, instead of fixing one particular measure, one considers a set of probability measures $\mathcal{P}$ on the measurable space $(\Omega, \mathcal{B})$ and each $\mathbb{P} \in \mathcal{P}$ corresponds to a different distribution function for $\mathbf{X}$. Therefore we can express model uncertainty equivalently in terms of a class of distribution functions $\mathcal{F}$ for the random vector $\mathbf{X}$. The associated model risk is then quantified by means of the expectation of $\varphi(\mathbf{X})$ for a functional $\varphi\colon \mathbb{R}^d \to \mathbb{R}$. More specifically, since different models for the vector $\mathbf{X}$ may lead to different values of the expectation $\mathbb{E}_F[\varphi(\mathbf{X})]$, where $\mathbb{E}_F$ denotes the expectation with respect to the distribution $F \in \mathcal{F}$, the model risk is quantified by the minimal and maximal value of $\mathbb{E}_F[\varphi(\mathbf{X})]$ where F ranges over the class $\mathcal{F}$ of admissible distributions. The choice of admissible distributions is typically such that they comply with assumptions about the model that are considered accurate or sufficiently reliable. The difference between the maximal and the minimal value of the expectation indicates the magnitude of model risk, i.e. a lower spread between the minimal and maximal expectation implies less model risk; see e.g. Cont [13].

A classical example of this approach from option valuation are sub- and super-hedging prices of call options on a single asset. These prices correspond to $\inf_{F \in \mathcal{F}} \mathbb{E}_F[(X-K)^+]$ and $\sup_{F \in \mathcal{F}} \mathbb{E}_F[(X-K)^+]$ respectively where $\mathcal{F}$ is the set of all distributions for the random variable X such that $\mathbb{E}_F[X] = c$ for some constant c. Here, the condition $\mathbb{E}_F[X] = c$ is derived from the no-arbitrage hypothesis and thus it can be viewed as a reliable constraint on models for X. Then it follows that the expectation of $(X-K)^+$ is bounded by the universal arbitrage bounds

$$(c-K)^+ \leq \mathbb{E}_F[(X-K)^+] \leq c \quad \text{for all } F \in \mathcal{F},$$

and in many situations the bounds are in fact attained. This implies that the price of the call option with payoff $(X-K)^+$ lies in the interval $[(c-K)^+, c]$ when no assumptions about the model for X are made except for it being arbitrage-free.

In a wider sense, the quantification of model uncertainty involves the optimization over a possibly constrained set of distribution functions, which is a delicate task in general. The complexity of the problem depends on the properties of φ, the class of admissible distributions, as well as on the dimension d, and solutions to the problem have evolved along these aspects. In the literature, a significant demarcation line runs between the one-dimensional ($d = 1$) and the high-dimensional ($d > 1$) case. This is due to the fact that when $d = 1$ uncertainty stems from a single univariate random variable, whereas when $d > 1$ uncertainty arises in the models for each of the d components as well as the dependence structure between them. The dependence structure between constituents of a random vector can be modelled separately from the univariate components by means of a copula. Therefore, most of the literature focuses either on the one-dimensional case or exclusively on uncertainty about the dependence structure (copula) between multiple variables whose individual distributions are known. The latter is in the literature referred to as *dependence uncertainty*, which will be the central framework of this thesis.

Dependence uncertainty arises naturally in many financial applications and the results in this thesis are inspired by and apply to problems in derivatives pricing and risk management, although the scope of our results is not limited to these areas of applications. In the first part of this thesis, our concern is with the valuation of options that depend on multiple underlyings at maturity modelled by $\mathbf{X} = (X_1, ..., X_d)$. We assume that the univariate risk-neutral distributions of each individual asset $X_1, ..., X_d$ can be inferred from market prices of liquidly traded European options. Moreover, we suppose that prices of traded options on two or more assets reveal partial information about the risk-neutral dependence structure of $\mathbf{X}$, e.g. the correlation between assets. We then seek to establish bounds on prices of options with payoffs of the form $\varphi(\mathbf{X})$ over the class of risk-neutral distribution functions for $\mathbf{X}$, that are compatible with the given market information. Distributions are thus admissible when they comply with market data. Therefore the corresponding price bounds are considered robust, or model-free, since they depend entirely on information that is derived from market prices and no underlying model for $\mathbf{X}$ is assumed. In this thesis we develop a novel approach to compute model-free bounds on option prices using partial information about the dependence structure of the underlyings. To this end, we formulate the problem in terms of copulas and establish several improvements of the well-known Fréchet–Hoeffding bounds that allow us to account for partial dependence information. Furthermore, we derive an integral characterization of multivariate stochastic orders for quasi-copulas which relates the improved Fréchet–Hoeffding bounds to option price bounds. Our approach is widely applicable and the bounds on option prices can be computed via straight-forward numerical integration.

The second part of this thesis is devoted to the study of robust risk measurement in the presence of dependence uncertainty. In the context of risk evaluation, dependence uncertainty emerges, on the one hand, from the limited amount of data and historical observations available to estimate models for risk factors and, on the other hand, from the configuration of risk management structures in financial institutions. Regarding the former, practitioners in risk management are often required to develop models for risk factors of interest. The model development involves the estimation of distributions for risk factors based on e.g. historical observations. In particular, when $\mathbf{X} = (X_1, ..., X_d)$ is assumed to model an $\mathbb{R}^d$-valued vector of risk factors, the problem amounts to the estimation of a d-dimensional distribution function. The number of observations needed to estimate a d-dimensional distribution appropriately increases however rapidly with respect to the dimension d. Hence, the limited amount of data mostly allows for a rather accurate estimation of the univariate distributions of the individual risks $X_1, ..., X_d$, whereas at best partial information about the joint distribution of $\mathbf{X}$ can be retrieved from data. The second source of dependence uncertainty is inherent to hierarchical risk management structures of financial institutions. On a corporate level, it is common to consider each department as a single risk entity. An insurance company may e.g. consider its life, health and composite departments as individual risk units. Each unit then reports the distribution of its aggregated risk to the corporate risk management which then joins this information to evaluate the company's total or corporate exposure. In this case, $X_1, ..., X_d$ model the individual risks of each unit, whose distributions are given, while their dependence structure is at most partially known. The corporate risk is usually quantified by means of a risk measure taking the form $\mathbb{E}_F[\varphi(\mathbf{X})]$ and higher values of the expectation indicate higher risk. Robust risk estimates are then obtained by computing the maximal risk (expectation) over all joint distributions F for $\mathbf{X}$ that are compatible with the available information. In this case, the set of admissible distributions is derived from statistically reliable information.

Probably the most prominent and widespread risk measure in practice is the portfolio Value-at-Risk which corresponds to the generalised inverse of the distribution function of $X_1 + \cdots + X_d$. Deriving robust Value-at-Risk estimates thus involves the computation of the minimal and maximal probability $\mathbb{P}(X_1 + \cdots + X_d < \cdot)$ over a class of admissible distributions for $\mathbf{X}$. In the presence of dependence uncertainty, this task turns out to be highly non-trivial and even in the situation where only the marginal distributions of the risk factors are given and no dependence information at all is prescribed, solutions have been obtained only under very strong assumptions on the marginal distributions. Therefore, a large part of the literature concentrates on the derivation of Value-at-Risk bounds which may not be attainable but are reasonably narrow and sufficiently tractable as to be of relevance in practical applications. In this work, we develop two methods to compute bounds on Value-at-Risk in the presence of dependence uncertainty. The methods provide tractable ways to compute risk bounds which account for different types of partial dependence information that are often available in practice. The first approach is based on a transformation of the set of admissible distributions to a Fréchet class of distributions with given marginals. This transformation makes the problem amenable to optimization techniques for Fréchet classes, such as the Improved Standard Bounds or the Rearrangement Algorithm, in order to compute Value-at-Risk estimates. In our second approach we derive a dual characterization of the risk bounds and show that strong duality holds using the Monge–Kantorovich Transport Theory. Based on the dual formulation, we then develop an optimization scheme that allows us to compute Value-at-Risk bounds using partial dependence information. Moreover, we show in numerical illustrations that our approach may produce significantly narrower risk estimates when compared to bounds that are available in the literature.

Mathematical results and outline

In Section 1.1 we introduce the basic mathematical notions used in this thesis and present some preliminary results. Essential in our framework of dependence uncertainty is the concept of a copula, which allows us to separate the individual behaviour of the constituents of an $\mathbb{R}^d$-valued random vector from the dependence structure between them. Specifically, Sklar's Theorem states that each d-dimensional distribution function F of a random vector $\mathbf{X} = (X_1, ..., X_d)$ can be written as

$$F(x_1, ..., x_d) = C(F_1(x_1), ..., F_d(x_d)) \quad \text{for all } (x_1, ..., x_d) \in \mathbb{R}^d, \tag{1.1}$$

where $C\colon [0,1]^d \to [0,1]$ is a copula and $F_1, ..., F_d$ are the marginal distributions of $X_1, ..., X_d$. This implies that dependence uncertainty is in fact uncertainty about the copula of $\mathbf{X}$. Moreover, the expectation $\mathbb{E}_F[\varphi(\mathbf{X})]$ for $\varphi\colon \mathbb{R}^d \to \mathbb{R}$ can be expressed as

$$\int_{\mathbb{R}^d} \varphi(x_1, ..., x_d) \, \mathrm{d}C(F_1(x_1), ..., F_d(x_d)). \tag{1.2}$$

Bounding $\mathbb{E}_F[\varphi(\mathbf{X})]$ in the presence of dependence uncertainty is thus tantamount to the computation of bounds on the integral in (1.2) over an admissible class of copulas. In this work we develop novel approaches to derive bounds on the integral in (1.2) using partial dependence information under different assumptions on the characteristics of φ. For the informal discussion we omit the notation $\mathbb{E}_F$ and use the term *bounds on the expectation* $\mathbb{E}[\varphi(\mathbf{X})]$, referring tacitly to bounds on $\mathbb{E}_F[\varphi(\mathbf{X})]$ for F ranging over a certain set of distribution functions.

Our first approach relates to the theory of multivariate stochastic dominance and integral stochastic orders. Here a reference to the seminal work of Lehmann [31] and the first comprehensive volumes on the theory of multivariate stochastic orders by Stoyan [62] and Marshall and Olkin [38] is in order. A prominent concept for ordering random vectors is that of orthant orders. For two random vectors $\mathbf{X} = (X_1, ..., X_d)$ and $\mathbf{Y} = (Y_1, ..., Y_d)$ with distribution functions $F_{\mathbf{X}}$ and $F_{\mathbf{Y}}$ respectively, we say that $\mathbf{Y}$ dominates $\mathbf{X}$ in lower orthant order if

$$F_{\mathbf{X}}(x_1, ..., x_d) \leq F_{\mathbf{Y}}(x_1, ..., x_d) \quad \text{for all } (x_1, ..., x_d) \in \mathbb{R}^d.$$

Similarly, $\mathbf{Y}$ dominates $\mathbf{X}$ in upper orthant order when

$$F_{-\mathbf{X}}(x_1, ..., x_d) \leq F_{-\mathbf{Y}}(x_1, ..., x_d) \quad \text{for all } (x_1, ..., x_d) \in \mathbb{R}^d,$$

where $F_{-\mathbf{X}}$ is the distribution of $(-X_1, ..., -X_d)$ and $F_{-\mathbf{Y}}$ is defined analogously. These orders admit a useful representation in terms of integrals and a complete characterization of the lower and upper orthant order is presented in Müller and Stoyan [39]. In particular, it holds that $\mathbf{Y}$ dominates $\mathbf{X}$ in lower orthant order if and only if $\mathbb{E}[\varphi(\mathbf{X})] \leq \mathbb{E}[\varphi(\mathbf{Y})]$ for all Δ-antitonic φ and similarly $\mathbf{Y}$ dominates $\mathbf{X}$ in the upper orthant order when $\mathbb{E}[\varphi(\mathbf{X})] \leq \mathbb{E}[\varphi(\mathbf{Y})]$ for all Δ-monotonic φ. We defer the definition of Δ-antitonic and Δ-monotonic functions to a subsequent chapter and confine ourselves to the remark that many functions that are relevant in mathematical finance are Δ-antitonic or Δ-monotonic. The integral characterization of orthant orders relates the problem of deriving bounds on the expectation $\mathbb{E}[\varphi(\mathbf{X})]$, for Δ-antitonic or Δ-monotonic φ, to the computation of bounds on the distribution function of $\mathbf{X}$ and $-\mathbf{X}$, respectively. Moreover, it is evident from (1.1) that in our setting of dependence uncertainty this is equivalent to bounding the copula of $\mathbf{X}$ and $-\mathbf{X}$. Generic bounds on the set of all copulas are given by the well-known Fréchet–Hoeffding bounds. These bounds ignore however all the available information about the dependence structure between the individual risk factors, such as correlations or orthant dependence. Moreover, the corresponding bounds on the expectation are typically very wide, so that the respective model risk estimates are of little practical relevance. The Fréchet–Hoeffding bounds can however be improved when additional dependence information is available.

In Chapter 2 we derive several improvements of the Fréchet–Hoeffding bounds which account for different types of partial dependence information. First, we establish improved Fréchet–Hoeffding bounds on the set of copulas that coincide with a reference copula on a compact subset of their domain. This subset can be thought of as a region where one has accurate or reliable knowledge about the copula. In the bivariate case ($d = 2$), similar improvements were previously obtained by Rachev and Rüschendorf [48], Nelsen [40], Nelsen, Quesada-Molina, Rodriguez-Lallena, and Ubeda-Flores [41], as well as Tankov [64]. Our results thus represent a natural extension of this line of research to higher-dimensional copulas. Moreover, using this initial result we establish several other improvements of the Fréchet–Hoeffding bounds, accounting for types of dependence information that have not been considered in the literature to date. An interesting question then is, whether the improved Fréchet–Hoeffding bounds are again copulas. In the bivariate case, Tankov [64] and Bernard, Jiang, and Vanduffel [8] answer this question in the affirmative for the improved Fréchet–Hoeffding bounds, including information on a subset, under fairly general conditions. In stark contrast to this, we show that in higher-dimensions the improved bounds with regional prescription fail to be copulas. The bounds are hence proper quasi-copulas and therefore other improvements which we derive therefrom are not expected to be copulas in general. This entails that the improved Fréchet–Hoeffding bounds are not amenable to results on stochastic dominance and hence they cannot be translated into bounds on the expectation. Even more delicate is the fact that the integral

$$\int_{\mathbb{R}^d} \varphi(x_1, ..., x_d) \, \mathrm{d}Q(F_1(x_1), ..., F_d(x_d)), \tag{1.3}$$

with respect to a quasi-copula Q is not well-defined, since Q does not induce a measure in general.

To remedy this problem, we develop in Chapter 3 an alternative representation of the integral in (1.3) that allows for quasi-copulas as integrators. We then use this representation in order to establish an integral characterization of orthant orders on the set of quasi-copulas, analogous to previous results on multivariate stochastic orders. Specifically, we show that our representation of multivariate integrals is monotonic with respect to the lower and upper orthant order on the set of quasi-copulas when φ is Δ-antitonic or Δ-monotonic respectively. This in turn allows us to relate the improved Fréchet–Hoeffding bounds in Chapter 2 to bounds on the expectation $\mathbb{E}[\varphi(\mathbf{X})]$, using additional dependence information. En passant, we also provide a means to relate the lower Fréchet–Hoeffding bound – which is not a copula in general – to bounds on the expectation when no dependence information at all is prescribed. We illustrate our approach in applications related to model-free option pricing. Specifically, we compute bounds on option prices of the form $\mathbb{E}[\varphi(\mathbf{X})]$, where φ is a payoff depending on multiple underlyings at maturity. We assume that information about the risk-neutral marginal distributions of $\mathbf{X}$ as well as partial information about its dependence structure can be inferred form market prices of traded derivatives. This information is then used to derive improved Fréchet–Hoeffding bounds which we translate into option price bounds applying our integral characterization of orthant orders. The computational results show that the inclusion of additional dependence information typically leads to a substantial improvement of the price bounds when compared to the marginals-only case.

Chapters 2 and 3 are largely based on Lux and Papapantoleon [34] and the results appeared in [34, 35].

In Chapters 4 and 5 we study Value-at-Risk bounds in the presence of dependence uncertainty and develop a second approach to relate copula information to bounds on expectations. We suppose that $\mathbf{X}$ models d risk factors with given marginal distributions. The individual factors are aggregated by a componentwise increasing function $\varphi\colon \mathbb{R}^d \to \mathbb{R}$ to a portfolio whose risk is quantified by means of Value-at-Risk. In applications, φ is typically the sum of the individual risk factors. The Value-at-Risk of $\varphi(\mathbf{X})$ is given by the inverse of the left-continuous quantile function $\mathbb{P}(\varphi(\mathbf{X}) < s)$, which can be expressed alternatively in terms of the expectation $\mathbb{E}[\mathbb{1}_{\varphi(\mathbf{X})<s}]$. In the situation of dependence uncertainty, the quest is once more to compute estimates on the expectation $\mathbb{E}[\mathbb{1}_{\varphi(\mathbf{X})<s}]$ over a set of admissible copulas for $\mathbf{X}$. In general, the function $\mathbf{x} \mapsto \mathbb{1}_{\varphi(\mathbf{x})<s}$ is neither Δ-monotonic nor Δ-antitonic – especially when φ is the sum of the individual risk factors – so that our approach via stochastic orders does not apply.

The search for bounds on $\mathbb{E}[\mathbb{1}_{\varphi(\mathbf{X})<s}]$ under dependence uncertainty has an extensive history and several approaches to the problem have been developed. In Chapter 4, which appeared in Lux and Papapantoleon [35], we make use of the *Improved Standard Bounds* from Embrechts, Höing, and Juri [18] and Embrechts and Puccetti [16]. Their results allow us to translate lower pointwise bounds on the copula of $\mathbf{X}$ and $-\mathbf{X}$ into bounds on the Value-at-Risk of $\varphi(\mathbf{X})$. Using the Improved Standard Bounds in conjunction with the lower Fréchet–Hoeffding bound – which holds for all copulas – yields estimates for the Value-at-Risk in the absence of dependence information. We show in Chapter 4 that these risk bounds can be improved when additional dependence information is available. To this end we make use of the improved Fréchet–Hoeffding bounds presented in Chapter 2. Moreover, we develop a novel approach to account for extreme value information in the computation of Value-at-Risk estimates for the sum $X_1+\cdots+X_d$. Specifically, we assume that the distributions of $\max\{X_{i_1},...,X_{i_n}\}$ or $\min\{X_{i_1},...,X_{i_n}\}$ are prescribed for several subvectors of $\mathbf{X}$ indexed by $0 \le i_1 \le \cdots \le i_n \le d$. Such additional information can

be obtained using methods from extreme value theory and is therefore often available in practice. In order to compute risk estimates which account for extreme value information we translate the optimization problem over the class of distributions for $\mathbf{X}$ that comply with the laws of the partial maxima or minima into an optimization over a Fréchet class of distributions with given marginals. The idea is hence to incorporate the extreme value information in terms of a modified marginals-only problem that can be solved by means of the Improved Standard Bounds or the Rearrangement Algorithm presented in Embrechts, Puccetti, and Rüschendorf [19].

Finally, in Chapter 5 we develop an approach to translate copula information into bounds on the Value-at-Risk of $\varphi(\mathbf{X})$. We suppose that an upper and a lower bound on the copula of $\mathbf{X}$ are provided. This formulation differs from the Improved Standard Bounds which only use one-sided information. We then apply the Monge–Kantorovich Transport Theory to derive a sharp dual characterization of the minimal and maximal Value-at-Risk over admissible copulas. The dual characterization holds for all lower semicontinuous aggregation functions φ and its solution corresponds to an infinite dimensional optimization problem which is both, analytically and numerically intractable. We therefore develop a numerically tractable reduction scheme based on the dual problem which allows us to compute Value-at-Risk bounds using the copula information. In particular, the scheme can be solved by means of linear optimization procedures. Moreover, we show that our scheme yields asymptotically sharp bounds in the certainty limit, i.e. when the upper and the lower copula bounds converge to a mutual limit copula. The computational ease is however achieved by omitting the sharpness of the bounds, thus our reduction scheme does not produce sharp bounds on Value-at-Risk in general. We show furthermore that the Improved Standard Bounds from [18, 16] are particular instances of our reduction scheme and numerical illustrations demonstrate that – given the same information – our scheme may produce significantly narrower risk estimates. Finally, we illustrate how different types of dependence information can be used in the computation of Value-at-Risk estimates by means of the improved Fréchet–Hoeffding bounds derived in Chapter 2.

1.1 Notation and preliminary results

In this section we introduce the notation and some basic results that will be used throughout this work. The section is largely based on Lux and Papapantoleon [34, Section 2].

Let $d \geq 2$ be an integer. In the following, $\mathbb{I}$ denotes the unit interval $[0,1]$, while boldface letters, e.g. $\mathbf{u}$, $\mathbf{v}$ or $\mathbf{x}$, denote vectors in $\mathbb{I}^d$, $\mathbb{R}^d$ or $\overline{\mathbb{R}}^d = [-\infty, \infty]^d$ with generic indexes. Random vectors with values in $\mathbb{R}^d$ are denoted by $\mathbf{X} = (X_1, ..., X_d)$ or $\mathbf{S} = (S_1, ..., S_d)$ and we use the latter typically in applications to refer to prices of financial assets. By $\mathbb{1}_A$ we denote the indicator function of a set $A \subseteq \overline{\mathbb{R}}^d$ and $\mathbf{1}$ denotes the d-dimensional unit vector $(1, ..., 1) \in \mathbb{R}^d$. Analogously, $\mathbf{0}$ refers to $(0, ..., 0) \in \mathbb{R}^d$. Moreover, $\subseteq$ denotes the inclusion between sets and $\subset$ the proper inclusion. We refer to functions as increasing when they are non-decreasing. For functions $f, g \colon \mathbb{R}^d \to \mathbb{R}$ we write $f \leq g$, when $f(\mathbf{x}) \leq g(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^d$ whereas we write $\mathbf{u} \leq \mathbf{v}$ for $\mathbf{u} = (u_1, ..., u_d), \mathbf{v} = (v_1, ..., v_d) \in \mathbb{R}^d$, when $u_1 \leq v_1, ..., u_d \leq v_d$ as well as $\mathbf{u} < \mathbf{v}$ when $u_1 < v_1, ..., u_d < v_d$. Finally, for a set $\mathcal{S}$ equipped with a partial order $\leq$ and a function $f \colon \mathcal{S} \to \mathbb{R}$, we say that f is monotonic with respect to $\leq$ if $f(s) \leq f(s')$ for all $s, s' \in \mathcal{S}$ with $s \leq s'$.

The finite difference operator Δ will be used frequently. It is defined for a function $f \colon \mathbb{R}^d \to \mathbb{R}$ and $a, b \in \mathbb{R}$ with $a \leq b$ via

$$\Delta_{a,b}^i \, f(x_1, \ldots, x_d) = f(x_1, \ldots, x_{i-1}, b, x_{i+1}, \ldots, x_d) - f(x_1, \ldots, x_{i-1}, a, x_{i+1}, \ldots, x_d).$$

Definition 1.1.1. A function $f \colon \mathbb{R}^d \to \mathbb{R}$ is called *d-increasing* if for all rectangular subsets $H = (a_1, b_1] \times \cdots \times (a_d, b_d] \subset \mathbb{R}^d$ it holds that

$$V_f(H) := \Delta_{a_d,b_d}^d \circ \cdots \circ \Delta_{a_1,b_1}^1 \, f \geq 0. \tag{1.4}$$

We call $V_f(H)$ the *f-volume* of H.

Definition 1.1.2. A function $Q \colon \mathbb{I}^d \to \mathbb{I}$ is a *d-quasi-copula* if the following properties hold:

$(\mathbf{QC1})$ Q satisfies, for all $i \in \{1, \ldots, d\}$, the boundary conditions

$$Q(u_1, \ldots, u_i = 0, \ldots, u_d) = 0 \quad \text{and} \quad Q(1, \ldots, 1, u_i, 1, \ldots, 1) = u_i.$$

$(\mathbf{QC2})$ Q is increasing in each argument.

$(\mathbf{QC3})$ Q is Lipschitz continuous, i.e. for all $\mathbf{u}, \mathbf{v} \in \mathbb{I}^d$

$$|Q(u_1, \dots, u_d) - Q(v_1, \dots, v_d)| \leq \sum_{i=1}^{d} |u_i - v_i|.$$

Moreover, Q is a d-*copula* if

$(\mathbf{QC4})$ Q is d-increasing.

We denote the set of all d-quasi-copulas by $\mathcal{Q}^d$ and the set of all d-copulas by $\mathcal{C}^d$. Obviously $\mathcal{C}^d \subset \mathcal{Q}^d$. In the following, we simply refer to a d-(quasi-)copula as (quasi-)copula if the dimension is clear from the context. Furthermore, we refer to elements in $\mathcal{Q}^d \setminus \mathcal{C}^d$ as proper quasi-copulas.

For univariate distribution functions $F_1, \dots, F_d$ we denote by $\mathcal{F}(F_1, ..., F_d)$ the Fréchet class of d-dimensional distribution functions F with marginal distributions $F_1, \dots, F_d$, i.e.

$$F(\infty, ..., \infty, x_i, \infty, ..., \infty) = F_i(x_i) \quad \text{for all } x_i \in \mathbb{R} \text{ and } i = 1, ..., d.$$

Let C be a d-copula and consider univariate probability distribution functions $F_1, \dots, F_d$. Then $F(x_1, \dots, x_d) := C(F_1(x_1), \dots, F_d(x_d))$, for all $\mathbf{x} \in \mathbb{R}^d$, defines a d-dimensional distribution function with univariate margins $F_1, \dots, F_d$, i.e. $F \in \mathcal{F}(F_1, ..., F_d)$. The converse also holds by Sklar's Theorem, cf. Sklar [60]. That is, for each d-dimensional distribution function $F \in \mathcal{F}(F_1, ..., F_d)$, there exists a copula C such that $F(x_1, \dots, x_d) = C(F_1(x_1), \dots, F_d(x_d))$ for all $\mathbf{x} \in \mathbb{R}^d$. In this case, the copula C is unique if the marginals are continuous. A simple proof of Sklar's Theorem based on the distributional transform can be found in Rüschendorf [58]. Sklar's Theorem establishes a fundamental link between copulas and multivariate distribution functions. Thus, given a random vector we refer to its copula, i.e. the copula corresponding to the distribution function of this random vector. Since the copula specifies the dependence properties of an $\mathbb{R}^d$-valued random we refer synonymously to the *dependence structure* or the copula of the random vector.

Let Q be a (quasi-)copula. We define its *survival function* as follows:

$$\widehat{Q}(u_1, \dots, u_d) := V_Q((u_1, 1] \times \cdots \times (u_d, 1]), \quad \mathbf{u} \in \mathbb{I}^d.$$

The survival function is illustrated for $d = 3$ below:

$$\begin{aligned}\widehat{Q}(u_1,u_2,u_3) = 1 &- Q(u_1,1,1,) - Q(1,u_2,1) - Q(1,1,u_3) \\ &+ Q(u_1,u_2,1) + Q(u_1,1,u_3) + Q(1,u_2,u_3) - Q(u_1,u_2,u_3).\end{aligned}$$

It is well-known, that if C is a copula then the function $\mathbb{I}^d \ni \mathbf{u} \mapsto \widehat{C}(\mathbf{1}-\mathbf{u})$ is again a copula, namely the *survival copula* of C; see e.g. Georges, Lamy, Nicolas, Quibel, and Roncalli [24]. In contrast, if Q is a quasi-copula then $\mathbb{I}^d \ni \mathbf{u} \mapsto \widehat{Q}(\mathbf{1}-\mathbf{u})$ is not a quasi-copula in general; see Example 1.1.1. Therefore we introduced the notion of a quasi-survival function.

Definition 1.1.3. A function $\widehat{Q}\colon \mathbb{I}^d \to \mathbb{I}$ is a *d-quasi-survival function* if the map

$$\mathbb{I}^d \ni \mathbf{u} \mapsto \widehat{Q}(\mathbf{1}-\mathbf{u})$$

is a d-quasi-copula.

We denote the set of all d-quasi-survival functions by $\widehat{\mathcal{Q}}^d$. In the following we also use the notation $\widehat{Q}(\mathbf{1}-\cdot)$ to refer to the map $\mathbf{u}\mapsto \widehat{Q}(\mathbf{1}-\mathbf{u})$ for a quasi-survival function $\widehat{Q}$.

Note that for a distribution function F of a random vector $\mathbf{X}=(X_1,\dots,X_d)$ with marginals $F_1,\dots,F_d$ and a corresponding copula C, such that $F(x_1,\dots,x_d)=C(F_1(x_1),\dots,F_d(x_d))$ for all $\mathbf{x}\in\mathbb{I}^d$, it holds that

$$\mathbb{P}(X_1 > x_1,\dots,X_d > x_d) = \widehat{C}(F_1(x_1),\dots,F_d(x_d)). \tag{1.5}$$

Definition 1.1.4. Let Q_1, Q_2 be d-quasi-copulas. Q_2 is larger than Q_1 in the *lower orthant order* if $Q_1 \le Q_2$. Similarly, for copulas $C_1, C_2 \in \mathcal{C}^d$, we say that C_2 is larger than C_1 in *upper orthant order* if $\widehat{C}_1 \le \widehat{C}_2$.

The well-known Fréchet–Hoeffding Theorem establishes the minimal and maximal bounds on the set of copulas or quasi-copulas in the lower orthant order. Specifically, for each $Q\in\mathcal{C}^d$ or $Q\in\mathcal{Q}^d$, it holds that

$$W_d(\mathbf{u}) := \max\left\{0, \sum_{i=1}^d u_i - d + 1\right\} \le Q(\mathbf{u}) \le \min\{u_1,\dots,u_d\} =: M_d(\mathbf{u}), \tag{1.6}$$

for all $\mathbf{u}\in\mathbb{I}^d$, i.e. $W_d \le Q \le M_d$, where W_d and M_d are the lower and upper Fréchet–Hoeffding bounds respectively. The upper bound is a copula for all $d \ge 2$, whereas the lower bound is a copula only if $d = 2$ and a proper quasi-copula otherwise. A proof of this theorem can be found in Genest, Molina, Lallena, and Sempi [23].

A bound over a set of copulas, resp. quasi-copulas, is called *sharp* if it belongs again to this set. Thus, the upper bound is sharp for the set of copulas and quasi-copulas. Although the lower bound is not sharp for the set of copulas unless $d = 2$, it is (pointwise) *best-possible* for all $d \in \mathbb{N}$ in the following sense:

$$W_d(\mathbf{u}) = \inf_{C \in \mathcal{C}^d} C(\mathbf{u}), \quad \mathbf{u} \in \mathbb{I}^d;$$

cf. Theorem 6 in Rüschendorf [53].

Since the properties of the Fréchet–Hoeffding bounds carry over to the set of survival functions in a straightforward way, one obtains similarly for any survival function $\widehat{Q} \in \widehat{\mathcal{Q}}^d$ that

$$W_d(1-u_1,\dots,1-u_d) \le \widehat{Q}(u_1,\dots,u_d) \le M_d(1-u_1,\dots,1-u_d), \quad \text{for all } \mathbf{u} \in \mathbb{I}^d.$$

Example 1.1.1. Consider the lower Fréchet-Hoeffding bound in dimension 3, i.e. W_3. Then W_3 is a quasi-copula but $\widehat{W_3}(\mathbf{1} - \cdot)$ is not a quasi-copula again. To this end, notice that quasi-copulas take values in the unit interval $\mathbb{I}$, while

$$\widehat{W}_3\Big(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\Big) = -\frac{1}{2}.$$

♦

2 Improved Fréchet–Hoeffding bounds

In this chapter we derive improved Fréchet–Hoeffding bounds on copulas in the presence of additional dependence information. The Fréchet–Hoeffding bounds in (1.6) are the best-possible bounds on the set of all copulas $\mathcal{C}^d$ with respect to the lower orthant order. It thus follows immediately from Sklar's Theorem that every distribution function F in the Fréchet class $\mathcal{F}(F_1, ..., F_d)$ is bounded by

$$W_d(F_1(x_1), ..., F_d(x_d)) \leq F(x_1, ..., x_d) \leq M_d(F_1(x_1), ..., F_d(x_d)) \quad \text{for all } \mathbf{x} \in \mathbb{R}^d,$$

and in the absence of dependence information these bounds cannot be narrowed further.

In the presence of additional dependence information the Fréchet–Hoeffding bounds can be improved and several results in this respect for bivariate copulas ($d = 2$) are available in the literature. Nelsen [40] derived improved Fréchet–Hoeffding bounds on the set of 2-copulas that are known at a single point of their domain. Similar improvements of the bivariate Fréchet–Hoeffding bounds were provided by Rachev and Rüschendorf [48] when the copula is known on an arbitrary set and by Nelsen et al. [41] for the case in which a measure of association such as Kendall's τ or Spearman's ρ is prescribed. Tankov [64] recently generalised these results, by improving the bivariate Fréchet–Hoeffding bounds if the copula is known on a compact set or when the value of a monotonic functional of the copula is prescribed. Since the bounds are in general not copulas but proper quasi-copulas, Tankov also provided sufficient conditions under which the improved bounds are copulas.

In section 2.1 we establish novel improvements of the Fréchet–Hoeffding bounds on the set of d-dimensional copulas and survival copulas, accounting for different types of dependence information. Specifically, we derive bounds on d-(survival-)copulas whose values are known on an arbitrary compact subset of $\mathbb{I}^d$. Moreover, we provide analogous improvements when the value of a monotonic functional of the copula is prescribed. Thereby monotonic refers to the lower orthant order on the set of copulas or survival-copulas. Furthermore, we establish bounds on the set of copulas which lie in the vicinity of a reference copula as measured by a statistical distance. We also obtain an explicit representation of these bounds when the distance to the reference copula is measured in terms of the Kolmogorov–Smirnov distance. Finally, we provide bounds on copulas given information on their lower-dimensional marginals.

These results give rise to the natural question whether the improved Fréchet–Hoeffding bounds are copulas and hence distribution functions. In section 2.2 we answer this question in the negative, showing that the improved bounds are quasi-copulas but fail to be copulas under fairly general conditions. This constitutes a surprising difference between the high-dimensional and the bivariate case, in which Tankov [64] and Bernard et al. [8] showed that the improved Fréchet–Hoeffding bounds are copulas under quite relaxed conditions.

The results presented in this chapter appeared in Lux and Papapantoleon [34, 35].

2.1 Improved Fréchet–Hoeffding bounds under partial dependence information

In this section we derive bounds on d-copulas and d-survival copulas that improve the universal Fréchet–Hoeffding bounds when partial information on the dependence structure is available. Our first result provides improved Fréchet–Hoeffding bounds, assuming knowledge of the copula on a subset of $\mathbb{I}^d$. Specifically, we derive bounds on copulas C that coincide with a reference copula C^* on a compact subset $\mathcal{S} \subseteq \mathbb{I}^d$, i.e. it holds that $C(\mathbf{x}) = C^*(\mathbf{x})$ for all $\mathbf{x} \in \mathcal{S}$. In practice, the set $\mathcal{S}$ may correspond to a region that contains enough historical observations to estimate C with sufficient accuracy from data, so that we can assume that C is 'known' on $\mathcal{S}$. This relates to the definition of the *trusted region* in Bernard and Vanduffel [7], who also present several techniques and criteria to select such regions in practice. If $\mathcal{S}$ is not equal to the entire domain of the copula, then dependence uncertainty stems from the fact that C remains unknown on $\mathbb{I}^d \setminus \mathcal{S}$. Similar results in the case $d = 2$ were obtained previously by Rachev and Rüschendorf [48], Nelsen [40] and Tankov [64].

Theorem 2.1.1. *Let $\mathcal{S} \subset \mathbb{I}^d$ be a compact set and Q^* be a d-quasi-copula. Consider the set*

$$\mathcal{Q}^{\mathcal{S},Q^*} := \Big\{Q \in \mathcal{Q}^d \colon Q(\mathbf{x}) = Q^*(\mathbf{x}) \text{ for all } \mathbf{x} \in \mathcal{S}\Big\}.$$

Then, for all $Q \in \mathcal{Q}^{\mathcal{S},Q^*}$, it holds that

$$\begin{aligned} \underline{Q}^{\mathcal{S},Q^*}(\mathbf{u}) \leq Q(\mathbf{u}) \leq \overline{Q}^{\mathcal{S},Q^*}(\mathbf{u}) \quad & \textit{for all } \mathbf{u} \in \mathbb{I}^d, \\ \underline{Q}^{\mathcal{S},Q^*}(\mathbf{u}) = Q(\mathbf{u}) = \overline{Q}^{\mathcal{S},Q^*}(\mathbf{u}) \quad & \textit{for all } \mathbf{u} \in \mathcal{S}, \end{aligned} \tag{2.1}$$

where the bounds $\underline{Q}^{\mathcal{S},Q^*}$ and $\overline{Q}^{\mathcal{S},Q^*}$ are provided by

$$\begin{aligned} \underline{Q}^{\mathcal{S},Q^*}(\mathbf{u}) &= \max\Big(0, \sum_{i=1}^d u_i - d + 1, \max_{\mathbf{x}\in\mathcal{S}} \Big\{Q^*(\mathbf{x}) - \sum_{i=1}^d (x_i - u_i)^+\Big\}\Big), \\ \overline{Q}^{\mathcal{S},Q^*}(\mathbf{u}) &= \min\Big(u_1, \dots, u_d, \min_{\mathbf{x}\in\mathcal{S}} \Big\{Q^*(\mathbf{x}) + \sum_{i=1}^d (u_i - x_i)^+\Big\}\Big). \end{aligned} \tag{2.2}$$

Furthermore, the bounds $\underline{Q}^{\mathcal{S},Q^*}$, $\overline{Q}^{\mathcal{S},Q^*}$ are d-quasi-copulas, hence they are sharp.

Proof. We start by considering a prescription at a single point, i.e. we let $\mathcal{S} = \{\mathbf{x}\}$ for $\mathbf{x} \in \mathbb{I}^d$, and show that the statement is true for $\mathcal{Q}^{\{\mathbf{x}\},Q^*}$. In this case, analogous results were provided by Rodríguez-Lallena and Úbeda-Flores [49]. Below we present a simpler alternative proof. So let $Q \in \mathcal{Q}^{\{\mathbf{x}\},Q^*}$ be arbitrary and $(u_1, \dots, u_d)$, $(u_1, \dots, u_{i-1}, x_i, u_{i+1}, \dots, u_d) \in \mathbb{I}^d$, then it follows from the Lipschitz property of Q and the fact that Q is increasing in each coordinate that

$$-(u_i - x_i)^+ \leq Q(u_1, \dots, u_{i-1}, x_i, u_{i+1}, \dots, u_d) - Q(u_1, \dots, u_d) \leq (x_i - u_i)^+.$$

Using the telescoping sum

$$\begin{aligned} Q(x_1, \dots, x_d) - Q(u_1, \dots, u_d) = Q(x_1, \dots, x_d) &- Q(u_1, x_2, \dots, x_d) + Q(u_1, x_2, \dots, x_d) \\ - Q(u_1, u_2, x_3, \dots, x_d) &+ \cdots + Q(u_1, \dots, u_{d-1}, x_d) - Q(u_1, \dots, u_d) \end{aligned}$$

we arrive at

$$-\sum_{i=1}^d (u_i - x_i)^+ \leq Q(x_1, \dots, x_d) - Q(u_1, \dots, u_d) \leq \sum_{i=1}^d (x_i - u_i)^+$$

which is equivalent to

$$Q(x_1, \dots, x_d) - \sum_{i=1}^d (x_i - u_i)^+ \leq Q(u_1, \dots, u_d) \leq Q(x_1, \dots, x_d) + \sum_{i=1}^d (u_i - x_i)^+.$$

The prescription yields further that $Q(x_1, \dots, x_d) = Q^*(x_1, \dots, x_d)$, from which follows that

$$Q^*(x_1,\dots,x_d) - \sum_{i=1}^d (x_i-u_i)^+ \le Q(u_1,\dots,u_d) \le Q^*(x_1,\dots,x_d) + \sum_{i=1}^d (u_i-x_i)^+,$$

while incorporating the Fréchet-Hoeffding bounds yields

$$\begin{aligned}\max\left\{0, \sum_{i=1}^d u_i - d+1, Q^*(x_1,\dots,x_d) - \sum_{i=1}^d (x_i-u_i)^+\right\} &\le Q(u_1,\dots,u_d)\\ &\le \min\left\{u_1,\dots,u_d, Q^*(x_1,\dots,x_d) + \sum_{i=1}^d (u_i-x_i)^+\right\}, \quad (2.3)\end{aligned}$$

showing that the inequalities in (2.1) are valid for $\mathcal{S} = \{\mathbf{x}\}$. Moreover, since $W_d(\mathbf{x}) \le Q^*(\mathbf{x}) \le M_d(\mathbf{x})$ it holds that

$$\begin{aligned}\underline{Q}^{\{\mathbf{x}\},Q^*}(\mathbf{x}) &= \max\left\{0, \sum_{i=1}^d x_i - d+1, Q^*(x_1,\dots,x_d)\right\} = Q^*(\mathbf{x}),\\ \overline{Q}^{\{\mathbf{x}\},Q^*}(\mathbf{x}) &= \min\left\{x_1,\dots,x_d, Q^*(x_1,\dots,x_d)\right\} = Q^*(\mathbf{x}),\end{aligned}$$

showing that the equalities in (2.1) are valid for $\mathcal{S} = \{\mathbf{x}\}$.

Next, let $\mathcal{S}$ be a compact set which is not a singleton and $Q \in \mathcal{Q}^{\mathcal{S},Q^*}$. We know from the argument above that $Q(\mathbf{u}) \ge \underline{Q}^{\{\mathbf{x}\},Q^*}(\mathbf{u})$ for all $\mathbf{x} \in \mathcal{S}$, therefore

$$Q(\mathbf{u}) \ge \max_{\mathbf{x}\in\mathcal{S}} \left\{\underline{Q}^{\{\mathbf{x}\},Q^*}(\mathbf{u})\right\} = \underline{Q}^{\mathcal{S},Q^*}(\mathbf{u}). \quad (2.4)$$

Analogously we get for the upper bound that

$$Q(\mathbf{u}) \le \min_{\mathbf{x}\in\mathcal{S}} \left\{\overline{Q}^{\{\mathbf{x}\},Q^*}(\mathbf{u})\right\} = \overline{Q}^{\mathcal{S},Q^*}(\mathbf{u}), \quad (2.5)$$

hence the inequalities in (2.1) are valid. Moreover, if $\mathbf{u} \in \mathcal{S}$ then $Q(\mathbf{u}) = Q^*(\mathbf{u})$ for all $Q \in \mathcal{Q}^{\mathcal{S},Q^*}$ and using the Lipschitz property of quasi-copulas we obtain

$$\max_{\mathbf{x}\in\mathcal{S}} \left\{Q^*(\mathbf{x}) - \sum_{i=1}^d (x_i-u_i)^+\right\} = Q^*(\mathbf{u}) \quad\text{and}\quad \min_{\mathbf{x}\in\mathcal{S}} \left\{Q^*(\mathbf{x}) + \sum_{i=1}^d (u_i-x_i)^+\right\} = Q^*(\mathbf{u}),$$

hence using again that Q^* satisfies the Fréchet–Hoeffding bounds we arrive at

$$\underline{Q}^{\mathcal{S},Q^*}(\mathbf{u}) = Q(\mathbf{u}) = \overline{Q}^{\mathcal{S},Q^*}(\mathbf{u}).$$

Finally, it remains to show that both bounds are d-quasi-copulas.

- In order to show that $(\mathbf{QC1})$ holds, first consider the case $\mathcal{S} = \{x\}$. Let $(u_1, \ldots, u_d) \in \mathbb{I}^d$ with $u_i = 0$ for one $i \in \{1, \ldots, d\}$. Then $\overline{Q}^{\mathcal{S},Q^*}(\mathbf{u})$ is obviously zero, and $\underline{Q}^{\mathcal{S},Q^*}(\mathbf{u}) = \max\left(0, Q^*(\mathbf{x}) - x_i - \sum_{j\neq i}(x_j - u_j)^+\right) = 0$ because $Q^*(\mathbf{x}) \leq M_d(\mathbf{x})$, i.e. $Q^*(\mathbf{x}) - x_i - \sum_{j\neq i}(x_j - u_j)^+ \leq 0$ for all $\mathbf{x} \in \mathcal{S}$. Moreover for $(u_1, \ldots, u_d) \in \mathbb{I}^d$ with $u_i = 1$ for all $i \in \{1, \ldots, d\} \setminus \{j\}$, the upper bound equals $\overline{Q}^{\mathcal{S},Q^*}(\mathbf{u}) = \min\left\{u_j, Q^*(\mathbf{x}) + \sum_{i=1}^{d}(u_i - x_i)^+\right\}$ and since

$$
\begin{aligned}
Q^*(\mathbf{x}) + \sum_{i=1}^{d}(u_i - x_i)^+ &= Q^*(\mathbf{x}) + \sum_{i\in\{1,\ldots,d\}\setminus\{j\}} (1 - x_i) + (u_j - x_j)^+ \\
&= Q^*(\mathbf{x}) + d - 1 - \sum_{i\in\{1,\ldots,d\}\setminus\{j\}} x_i + (u_j - x_j)^+ \\
&\geq W_d(\mathbf{x}) + d - 1 - \sum_{i\in\{1,\ldots,d\}\setminus\{j\}} x_i + (u_j - x_j)^+ \\
&\geq x_j + (u_j - x_j)^+ \geq u_j,
\end{aligned}
$$

  it follows that $Q^*(\mathbf{u}) = u_j$, hence $\overline{Q}^{\mathcal{S},Q^*}(\mathbf{u}) = u_j$. Similarly, the lower bound amounts to $\underline{Q}^{\mathcal{S},Q^*}(\mathbf{u}) = \max\left(0, u_j, Q^*(\mathbf{x}) - (x_j - u_j)^+\right)$ which equals u_j because $Q^*(\mathbf{x}) - (x_j - u_j)^+ \leq M_d(\mathbf{x}) - (x_j - u_j)^+ \leq u_j$. The boundary conditions hold analogously for $\mathcal{S}$ containing more than one element due to the continuity of the maximum and minimum functions and relationships (2.4) and (2.5).

- Both bounds are obviously increasing in each variable, thus $(\mathbf{QC2})$ holds.

- Finally, taking the pointwise minimum and maximum of Lipschitz functions preserves the Lipschitz property, thus both bounds satisfy $(\mathbf{QC3})$.

□

Remark 2.1.1. Note that the bounds in Theorem 2.1.1 hold analogously for prescriptions on copulas, i.e for all copulas

$$C \in \{C \in \mathcal{C}^d \colon C(\mathbf{x}) = Q^*(\mathbf{x}) \text{ for all } \mathbf{x} \in \mathcal{S}\} =: \mathcal{C}^{\mathcal{S},Q^*},$$

where Q^* and $\mathcal{S}$ are as above it holds that $\underline{Q}^{\mathcal{S},Q^*}(\mathbf{u}) \leq C(\mathbf{u}) \leq \overline{Q}^{\mathcal{S},Q^*}(\mathbf{u})$ for all $\mathbf{u} \in \mathbb{I}^d$. Let us point out that the set $\mathcal{C}^{\mathcal{S},Q^*}$ may possibly be empty, depending on the prescription. We do not investigate the requirements on the prescription for $\mathcal{C}^{\mathcal{S},Q^*}$ to be non-empty. A detailed discussion of this issue in the two-dimensional case can be found in Mardani-Fard, Sadooghi-Alvandi, and Shishebor [37]. ♦

The following corollary establishes analogous bounds on quasi-survival functions with prescribed values on a subset.

Corollary 2.1.2. *Let $\mathcal{S} \subset \mathbb{I}^d$ be a compact set and $\widehat{Q}^* \in \widehat{\mathcal{Q}}^d$ be a quasi-survival function. Consider the set*

$$\widehat{\mathcal{Q}}^{\mathcal{S},\widehat{Q}^*} := \Big\{Q \in \widehat{\mathcal{Q}}^d \colon \widehat{Q}(\mathbf{x}) = \widehat{Q}^*(\mathbf{x}) \text{ for all } \mathbf{x} \in \mathcal{S}\Big\}.$$

Then, for all $\widehat{Q} \in \widehat{\mathcal{Q}}^{\mathcal{S},\widehat{Q}^*}$, it holds that

$$\begin{aligned} \underline{\widehat{Q}}^{\mathcal{S},\widehat{Q}^*}(\mathbf{u}) \leq \widehat{Q}(\mathbf{u}) \leq \overline{\widehat{Q}}^{\mathcal{S},\widehat{Q}^*}(\mathbf{u}) \quad \text{for all } \mathbf{u} \in \mathbb{I}^d, \\ \underline{\widehat{Q}}^{\mathcal{S},\widehat{Q}^*}(\mathbf{u}) = \widehat{Q}(\mathbf{u}) = \overline{\widehat{Q}}^{\mathcal{S},\widehat{Q}^*}(\mathbf{u}) \quad \text{for all } \mathbf{u} \in \mathcal{S}, \end{aligned} \tag{2.6}$$

where the bounds are provided by

$$\underline{\widehat{Q}}^{\mathcal{S},\widehat{Q}^*}(\mathbf{u}) := \underline{Q}^{\widehat{\mathcal{S}},\widehat{Q}^*(\mathbf{1}-\cdot)}(\mathbf{1}-\mathbf{u}) \quad \text{and} \quad \overline{\widehat{Q}}^{\mathcal{S},\widehat{Q}^*}(\mathbf{u}) := \overline{Q}^{\widehat{\mathcal{S}},\widehat{Q}^*(\mathbf{1}-\cdot)}(\mathbf{1}-\mathbf{u})$$

with $\widehat{\mathcal{S}} = \{(1-x_1,\dots,1-x_d) \colon (x_1,\dots,x_d) \in \mathcal{S}\}$.

Proof. Let $\widehat{Q} \in \widehat{\mathcal{Q}}^{\mathcal{S},\widehat{Q}^*}$. Since $\widehat{Q}(\mathbf{1}-\cdot)$ is a quasi-copula that coincides with $\widehat{Q}^*(\mathbf{1}-\cdot)$ on the set $\widehat{\mathcal{S}}$ we obtain, by an application of Theorem 2.1.1, that

$$\underline{Q}^{\widehat{\mathcal{S}},\widehat{Q}^*(\mathbf{1}-\cdot)}(\mathbf{u}) \leq \widehat{Q}(\mathbf{1}-\mathbf{u}) \leq \overline{Q}^{\widehat{\mathcal{S}},\widehat{Q}^*(\mathbf{1}-\cdot)}(\mathbf{u}) \quad \text{for all } \mathbf{u} \in \mathbb{I}^d$$

which by a transformation of variables equals

$$\underline{Q}^{\widehat{\mathcal{S}},\widehat{Q}^*(\mathbf{1}-\cdot)}(\mathbf{1}-\mathbf{u}) \leq \widehat{Q}(\mathbf{u}) \leq \overline{Q}^{\widehat{\mathcal{S}},\widehat{Q}^*(\mathbf{1}-\cdot)}(\mathbf{1}-\mathbf{u}).$$

□

Next, we derive improved bounds on d-quasi-copulas when values of real-valued functionals of the quasi-copulas are prescribed. Examples of such functionals are the multivariate generalisations of Spearman's rho and Kendall's tau given in Taylor [65]. Moreover, in the context of multi-asset option pricing, examples of such functionals are prices of spread options. Similar results for $d = 2$ are provided by Nelsen, Quesada-Molina, Rodriguez-Lallena, and Ubeda-Flores [41] and Tankov [64].

Remark 2.1.2. In the following, by slightly abusing notation, we sometimes write $\underline{Q}^{\{\mathbf{u}\},\alpha}$ and $\overline{Q}^{\{\mathbf{u}\},\alpha}$ with $\alpha \in [W_d(\mathbf{u}), M_d(\mathbf{u})]$, instead of a quasi-copula function Q^*, and mean that $Q^*(\mathbf{u}) = \alpha$. ♦

Theorem 2.1.3. *Let $\rho\colon \mathcal{Q}^d \to \mathbb{R}$ be increasing with respect to the order $\leq$ on $\mathcal{Q}^d$ and continuous with respect to the pointwise convergence of quasi-copulas. Define*

$$\mathcal{Q}^{\rho,\theta} := \{Q \in \mathcal{Q}^d \colon \rho(Q) = \theta\}.$$

for $\theta \in (\rho(W_d), \rho(M_d))$. Then the following bounds hold

$$\underline{Q}^{\rho,\theta}(\mathbf{u}) := \min\Big\{Q(\mathbf{u})\colon Q \in \mathcal{Q}^{\rho,\theta}\Big\} = \begin{cases} \rho_+^{-1}(\mathbf{u},\theta), & \theta \in \Big[\rho\Big(\overline{Q}^{\{\mathbf{u}\},W_d(\mathbf{u})}\Big), \rho(M_d)\Big], \\ W_d(\mathbf{u}), & \textit{else}, \end{cases}$$

and

$$\overline{Q}^{\rho,\theta}(\mathbf{u}) := \max\Big\{Q(\mathbf{u})\colon Q \in \mathcal{Q}^{\rho,\theta}\Big\} = \begin{cases} \rho_-^{-1}(\mathbf{u},\theta), & \theta \in \Big[\rho(W_d), \rho\Big(\underline{Q}^{\{\mathbf{u}\},M_d(\mathbf{u})}\Big)\Big], \\ M_d(\mathbf{u}), & \textit{else}, \end{cases}$$

and these are again quasi-copulas. Here

$$\rho_-^{-1}(\mathbf{u},\theta) = \max\Big\{r\colon \rho\Big(\underline{Q}^{\{\mathbf{u}\},r}\Big) = \theta\Big\} \qquad \textit{and} \qquad \rho_+^{-1}(\mathbf{u},\theta) = \min\Big\{r\colon \rho\Big(\overline{Q}^{\{\mathbf{u}\},r}\Big) = \theta\Big\},$$

while the quasi-copulas $\underline{Q}^{\{\mathbf{u}\},r}$ and $\overline{Q}^{\{\mathbf{u}\},r}$ are given in Theorem 2.1.1 for $r \in \mathbb{I}$.

Proof. We show that the upper bound is valid, while the proof for the lower bound follows analogously. First, note that due to the continuity of ρ w.r.t. the pointwise convergence of quasi-copulas and the compactness of $\mathcal{Q}^d$, we get that the set $\Big\{Q(\mathbf{u})\colon Q \in \mathcal{Q}^{\rho,\theta}\Big\}$ is compact, hence

$$\sup\Big\{Q(\mathbf{u})\colon Q \in \mathcal{Q}^{\rho,\theta}\Big\} = \max\Big\{Q(\mathbf{u})\colon Q \in \mathcal{Q}^{\rho,\theta}\Big\}.$$

Next, let $\theta \in \Big[\rho(W_d), \rho\Big(\underline{Q}^{\{\mathbf{u}\},M_d(\mathbf{u})}\Big)\Big]$. It follows from the continuity of $r \mapsto \rho\Big(\underline{Q}^{\{\mathbf{u}\},r}\Big)$ that the set $\Big\{r\colon \rho\Big(\underline{Q}^{\{\mathbf{u}\},r}\Big) = \theta\Big\}$ is closed and thus admits a maximum

$$r^* \in \Big\{r\colon \rho\Big(\underline{Q}^{\{\mathbf{u}\},r}\Big) = \theta\Big\}.$$

Since r^* is such that $\rho\Big(\underline{Q}^{\{\mathbf{u}\},r^*}\Big) = \theta$, which entails $\underline{Q}^{\{\mathbf{u}\},r^*} \in \mathcal{Q}^{\rho,\theta}$, it follows that $r^* \leq \overline{Q}^{\rho,\theta}(\mathbf{u})$. We show furthermore by contradiction that $r^* = \overline{Q}^{\rho,\theta}(\mathbf{u})$. To this end, assume that $r^* < \overline{Q}^{\rho,\theta}(\mathbf{u})$. Then there exists a quasi-copula $Q \in \mathcal{Q}^{\rho,\theta}$ with $r^* < Q(\mathbf{u}) =: \alpha$. It follows from the properties of the improved Fréchet–Hoeffding bound $\underline{Q}^{\{\mathbf{u}\},\alpha}$ (c.f. Theorem 2.1.1) that $\underline{Q}^{\{\mathbf{u}\},\alpha} \leq Q$ and thus, due to the monotonicity of ρ w.r.t. the lower orthant order, we have that $\rho\Big(\underline{Q}^{\{\mathbf{u}\},\alpha}\Big) \leq \rho(\underline{Q}) = \theta$. Using again the continuity of $r \mapsto \rho\Big(\underline{Q}^{\{\mathbf{u}\},r}\Big)$, there exists $\alpha^* \in [\alpha, M_d(\mathbf{u})]$ with $\rho\Big(\underline{Q}^{\{\mathbf{u}\},\alpha}\Big) = \theta$, which contradicts the maximality of r^*.

Now, let $\theta > \rho\Big(\underline{Q}^{\{\mathbf{u}\},M_d(\mathbf{u})}\Big)$, then $\theta \in \Big(\rho\Big(\underline{Q}^{\{\mathbf{u}\},M_d(\mathbf{u})}\Big), \rho(M_d)\Big]$. Consider $Q^\alpha = \alpha M_d + (1-\alpha)\underline{Q}^{\{\mathbf{u}\},M_d(\mathbf{u})}$, for $\alpha \in [0,1]$, then $\rho(Q^0) < \theta$ and $\rho(Q^1) \geq \theta$. Since $\alpha \mapsto \rho(Q^\alpha)$ is continuous there exists an α with $\rho(Q^\alpha) = \theta$. Since $Q^\alpha(\mathbf{u}) = M_d(\mathbf{u})$ for all $\alpha \in [0,1]$ it follows that $M_d(\mathbf{u}) \leq \max\Big\{Q(\mathbf{u})\colon Q \in \mathcal{Q}^{\rho,\theta}\Big\}$, while the reverse inequality holds due to the upper Fréchet–Hoeffding bound.

Finally, using Theorem 2.1 in Rodríguez-Lallena and Úbeda-Flores [49] we get immediately that the bounds are again quasi-copulas. ☐

Remark 2.1.3. The bounds $\underline{Q}^{\rho,\theta}$ and $\overline{Q}^{\rho,\theta}$ are not in the class $\mathcal{Q}^{\rho,\theta}$ in general. Consider e.g. the case $d = 2$, then it follows from Tankov [64, Theorem 2] that

$$
\begin{aligned}
\underline{Q}^{\rho,\theta}(\mathbf{u}) &= \min\Big\{Q(\mathbf{u})\colon Q \in \mathcal{Q}^{\rho,\theta} \cap \mathcal{C}^2\Big\},\\
\overline{Q}^{\rho,\theta}(\mathbf{u}) &= \max\Big\{Q(\mathbf{u})\colon Q \in \mathcal{Q}^{\rho,\theta} \cap \mathcal{C}^2\Big\}.
\end{aligned}
$$

Moreover, for ρ being Kendall's tau, i.e.

$$
\rho\colon \mathcal{C}^2 \ni C \mapsto \int_{\mathbb{I}^2} C(u_1,u_2)\,\mathrm{d}u_1\mathrm{d}u_2,
$$

it follows from Nelsen et al. [41, Theorem 4] that

$$
\begin{aligned}
\underline{Q}^{\rho,\theta}(u_1,u_2) &= \max\left\{0, u_1+u_2-1, \frac{1}{2}\left[(u_1+u_2) - \sqrt{(u_1-u_2)^2+1-\theta}\right]\right\},\\
\overline{Q}^{\rho,\theta}(u_1,u_2) &= \min\left\{u_1, u_2, \frac{1}{2}\left[(u_1+u_2-1) + \sqrt{(u_1+u_2-1)^2+1+\theta}\right]\right\}.
\end{aligned}
$$

and Corollary 3(h) in [41] states that neither of the bounds is in the class $\mathcal{Q}^{\rho,\theta}$ when $\rho \in (-1,1)$. ♦

Again, the bounds in Theorem 2.1.3 hold analogously for copulas, i.e. for ρ and θ as in Theorem 2.1.3 we have $\underline{Q}^{\rho,\theta} \leq C \leq \overline{Q}^{\rho,\theta}$ for all $C \in \{C \in \mathcal{C}^d\colon \rho(C) = \theta\}$. Moreover, we obtain similar bounds for quasi-survival functions if the value of a functional which is increasing with respect to $\leq$ on $\widehat{\mathcal{Q}}^d$ is prescribed. The proof is analogous to the proof of Theorem 2.1.3 and is therefore omitted.

Corollary 2.1.4. *Let $\rho\colon \widehat{\mathcal{Q}}^d \to \mathbb{R}$ be increasing with respect to $\preceq$ on the set of quasi-survival functions $\widehat{\mathcal{Q}}^d$ and continuous with respect to the pointwise convergence on $\widehat{\mathcal{Q}}^d$. Define*

$$\widehat{\mathcal{Q}}^{\rho,\theta} := \Big\{\widehat{Q} \in \widehat{\mathcal{Q}}^d \colon \rho(\widehat{Q}) = \theta\Big\}.$$

Then for all $\widehat{Q} \in \widehat{\mathcal{Q}}^{\rho,\theta}$ it holds that

$$\underline{\widehat{Q}}^{\rho,\theta}(\mathbf{u}) \leq \widehat{Q}(\mathbf{u}) \leq \overline{\widehat{Q}}^{\rho,\theta}(\mathbf{u}) \quad \text{for all } \mathbf{u} \in \mathbb{I}^d,$$

where the bounds are provided by

$$\underline{\widehat{Q}}^{\rho,\theta}(\mathbf{u}) := \begin{cases} \rho_+^{-1}(\mathbf{u},\theta), & \theta \in \Big[\rho\Big(\overline{\widehat{Q}}^{\{\mathbf{u}\},W_d(\mathbf{1}-\mathbf{u})}\Big), \rho(M_d(\mathbf{1}-\cdot))\Big] \\ W_d(\mathbf{1}-\mathbf{u}), & \textit{else}, \end{cases}$$

and

$$\overline{\widehat{Q}}^{\rho,\theta}(\mathbf{u}) := \begin{cases} \rho_-^{-1}(\mathbf{u},\theta), & \theta \in \Big[\rho(W_d(\mathbf{1}-\cdot)), \rho\Big(\underline{\widehat{Q}}^{\{\mathbf{u}\},M_d(\mathbf{1}-\mathbf{u})}\Big)\Big] \\ M_d(\mathbf{1}-\mathbf{u}), & \textit{else}, \end{cases}$$

where

$$\rho_-^{-1}(\mathbf{u},\theta) = \max\Big\{r\colon \rho\Big(\underline{\widehat{Q}}^{\{\mathbf{u}\},r}\Big) = \theta\Big\} \qquad \textit{and} \qquad \rho_+^{-1}(\mathbf{u},\theta) = \min\Big\{r\colon \rho\Big(\overline{\widehat{Q}}^{\{\mathbf{u}\},r}\Big) = \theta\Big\},$$

while the quasi-copulas $\underline{\widehat{Q}}^{\{\mathbf{u}\},r}$ and $\overline{\widehat{Q}}^{\{\mathbf{u}\},r}$ for $r \in \mathbb{I}$ are given in Proposition 2.1.2.

We proceed with an improvement of the Fréchet–Hoeffding bounds on the set of (quasi-) copulas that lie in the vicinity of a reference model as measured by a statistical distance. More formally, we establish bounds on (quasi-)copulas C in the δ-neighbourhood of the reference copula C^*, i.e. such that $\mathcal{D}(C, C^*) \leq \delta$ for a distance $\mathcal{D}$. Our method applies to a large class of statistical distances such as the Cramér-von-Mises or the L^p distances. Having a bound on the distance to a reference model corresponds to a situation that arises naturally in practice, when one tries to estimate or calibrate a copula to empirical data. The estimation typically involves the minimization of a distance to the empirical copula over a parametric family of copulas, i.e. $\mathcal{D}(C_\theta, C^*) \to \min_\theta$ where C^* is an empirical copula and $\{C_\theta\}_\theta$ is a family of parametric copulas. This is in the literature often referred to as *minimal-distance* or *minimal-contrast* estimation. Kole et al. [30] e.g. present several distance-based techniques for selecting copulas in risk management. These estimation procedures lend themselves immediately to the methodology we propose, as typically one arrives at $\delta := \min_\theta \mathcal{D}(C_\theta, C^*) > 0$, due to the fact that none of the models in the parametric class $\{C_\theta\}_\theta$ matches the empirical observations exactly, thus dependence uncertainty remains. In this case, δ can be viewed as the inevitable risk of model misspecification due to the choice of the parametric family $\{C_\theta\}_\theta$.

Let us first define the minimal and maximal convolution between two quasi-copulas Q, Q' as the pointwise minimum and maximum between them, i.e. $(Q \wedge Q')(\mathbf{u}) = Q(\mathbf{u}) \wedge Q'(\mathbf{u})$ and $(Q \vee Q')(\mathbf{u}) = Q(\mathbf{u}) \vee Q'(\mathbf{u})$ respectively.

Definition 2.1.5. A function $\mathcal{D}\colon \mathcal{Q}^d \times \mathcal{Q}^d \to \mathbb{R}_+$ is called a *statistical distance* if for $Q, Q' \in \mathcal{Q}^d$

$$\mathcal{D}(Q, Q') = 0 \quad \Longleftrightarrow \quad Q(\mathbf{u}) = Q'(\mathbf{u}) \quad \text{for all } \mathbf{u} \in \mathbb{I}^d.$$

Definition 2.1.6. A statistical distance $\mathcal{D}$ is *monotonic* with respect to the order $\leq$ on $\mathcal{Q}^d$, if for $Q, Q', Q'' \in \mathcal{Q}^d$ it holds

$$Q \leq Q' \leq Q'' \quad \Longrightarrow \quad \mathcal{D}(Q', Q'') \leq \mathcal{D}(Q, Q'') \text{ and } \mathcal{D}(Q'', Q') \leq \mathcal{D}(Q'', Q).$$

A statistical distance $\mathcal{D}$ is *min*- respectively *max-stable* if for $Q, Q' \in \mathcal{Q}^d$ it holds

$$\begin{aligned} \mathcal{D}(Q, Q') &\geq \max\{\mathcal{D}(Q \wedge Q', Q), \mathcal{D}(Q, Q \wedge Q')\} \\ \mathcal{D}(Q, Q') &\geq \max\{\mathcal{D}(Q \vee Q', Q), \mathcal{D}(Q, Q \vee Q')\}. \end{aligned}$$

The following theorem establishes pointwise bounds on the set of quasi-copulas that are in the δ-vicinity of a reference copula C^* as measured by a statistical distance $\mathcal{D}$.

Theorem 2.1.7. *Let C^* be a d-copula and $\mathcal{D}$ be a statistical distance which is continuous with respect to the pointwise convergence of quasi-copulas, monotonic with respect to $\leq$ on $\mathcal{Q}^d$ and and min/max-stable. Consider the set*

$$\mathcal{Q}^{\mathcal{D},\delta} := \Big\{Q \in \mathcal{Q}^d \colon \mathcal{D}(Q, C^*) \leq \delta\Big\}$$

for $\delta \in \mathbb{R}_+$. Then

$$\begin{aligned} \underline{Q}^{\mathcal{D},\delta}(\mathbf{u}) &:= \min\Big\{\alpha \in \mathbb{S}(\mathbf{u}) \colon \mathcal{D}\Big(\overline{Q}^{\{\mathbf{u}\},\alpha} \wedge C^*, C^*\Big) \leq \delta\Big\} = \min\Big\{Q(\mathbf{u}) \colon Q \in \mathcal{Q}^{\mathcal{D},\delta}\Big\}, \\ \overline{Q}^{\mathcal{D},\delta}(\mathbf{u}) &:= \max\Big\{\alpha \in \mathbb{S}(\mathbf{u}) \colon \mathcal{D}\Big(\underline{Q}^{\{\mathbf{u}\},\alpha} \vee C^*, C^*\Big) \leq \delta\Big\} = \max\Big\{Q(\mathbf{u}) \colon Q \in \mathcal{Q}^{\mathcal{D},\delta}\Big\}, \end{aligned}$$

where $\mathbb{S}(\mathbf{u}) := [W_d(\mathbf{u}), M_d(\mathbf{u})]$, and both bounds are quasi-copulas.

Proof. We show that the statement holds for the lower bound, while the proof for the upper bound follows along the same lines. Fix an $\alpha \in [W_d(\mathbf{u}), M_d(\mathbf{u})]$ and a $\mathbf{u} \in \mathbb{I}^d$, then the map $\mathbf{v} \mapsto \Big(\overline{Q}^{\{\mathbf{u}\},\alpha} \wedge C^*\Big)(\mathbf{v})$ is a quasi-copula; this follows by straightforward calculations using the definition of the minimal convolution, see also Rodríguez-Lallena and Úbeda-Flores [49, Theorem 2.1]. By definition, $\mathcal{D}$ is monotonic with respect to $\leq$ on $\mathcal{Q}^d$, thus it follows for $\underline{\alpha}, \overline{\alpha} \in [W_d(\mathbf{u}), M_d(\mathbf{u})]$ with $\underline{\alpha} < \overline{\alpha}$ that

$$\mathcal{D}\Big(\overline{Q}^{\{\mathbf{u}\},\overline{\alpha}} \wedge C^*, C^*\Big) \leq \mathcal{D}\Big(\overline{Q}^{\{\mathbf{u}\},\underline{\alpha}} \wedge C^*, C^*\Big),$$

due to the fact that $\overline{Q}^{\{\mathbf{u}\},\underline{\alpha}} \leq \overline{Q}^{\{\mathbf{u}\},\overline{\alpha}}$, which readily implies

$$\Big(\overline{Q}^{\{\mathbf{u}\},\underline{\alpha}} \wedge C^*\Big) \leq \Big(\overline{Q}^{\{\mathbf{u}\},\overline{\alpha}} \wedge C^*\Big) \leq C^*.$$

Hence, the map

$$[W_d(\mathbf{u}), M_d(\mathbf{u})] \ni \alpha \mapsto \mathcal{D}\Big(\overline{Q}^{\{\mathbf{u}\},\alpha} \wedge C^*, C^*\Big)$$

is decreasing. Moreover, as a consequence of the Arzelà–Ascoli Theorem, it follows that for every sequence $(\alpha_n)_n \subset [W_d(\mathbf{u}), M_d(\mathbf{u})]$ with $\alpha_n \to \alpha$,

$$\Big(\overline{Q}^{\{\mathbf{u}\},\alpha_n} \wedge C^*\Big) \xrightarrow[n\to\infty]{} \Big(\overline{Q}^{\{\mathbf{u}\},\alpha} \wedge C^*\Big)$$

uniformly and, since $\mathcal{D}$ is continuous with respect to the pointwise convergence of quasi-copulas, it follows that $\alpha \mapsto \mathcal{D}\Big(\overline{Q}^{\{\mathbf{u}\},\alpha} \wedge C^*, C^*\Big)$ is continuous. In addition, we have that

$$\mathcal{D}\Big(\overline{Q}^{\{\mathbf{u}\},M_d} \wedge C^*, C^*\Big) = \mathcal{D}\Big(M_d \wedge C^*, C^*\Big) = \mathcal{D}\Big(C^*, C^*\Big) = 0, \tag{2.7}$$

due to the fact that $C^* \leq M_d$. We now distinguish between two cases:

(i) Let $\delta \leq \mathcal{D}\Big(\overline{Q}^{\{\mathbf{u}\},W_d} \wedge C^*, C^*\Big)$. Then, due to the monotonicity and continuity of the map $[W_d(\mathbf{u}), M_d(\mathbf{u})] \ni \alpha \mapsto \mathcal{D}\Big(\overline{Q}^{\{\mathbf{u}\},\alpha} \wedge C^*, C^*\Big)$ and (2.7) it holds that the set

$$\mathcal{O} := \Big\{\alpha \colon \mathcal{D}\Big(\overline{Q}^{\{\mathbf{u}\},\alpha} \wedge C^*, C^*\Big) = \delta\Big\}$$

is non-empty and compact. Define $\alpha^* := \min\{\alpha \colon \alpha \in \mathcal{O}\}$. We show that

$$\min\Big\{Q(\mathbf{u}) \colon Q \in \mathcal{Q}^{\mathcal{D},\delta}\Big\} = \alpha^*.$$

On the one hand, it holds that $\min\Big\{Q(\mathbf{u})\colon Q \in \mathcal{Q}^{\mathcal{D},\delta}\Big\} \leq \alpha^*$. Indeed, consider $\overline{Q}^{\{\mathbf{u}\},\alpha^*} \wedge C^*$ which is a quasi-copula and belongs to $\mathcal{Q}^{\mathcal{D},\delta}$ since $\alpha^* \in \mathcal{O}$. Then, we have that

$$\Big(\overline{Q}^{\{\mathbf{u}\},\alpha^*} \wedge C^*\Big)(\mathbf{u}) = \min\{\alpha^*, C^*(\mathbf{u})\} = \alpha^*,$$

using again that $\alpha^* \in \mathcal{O}$ and (2.7). Hence the inequality holds. On the other hand, we show now that the inequality cannot be strict by contradiction. Assume there exists a quasi-copula $Q' \in \mathcal{Q}^{\mathcal{D},\delta}$ with $Q'(\mathbf{u}) < \alpha^*$. Then it follows that

$$\begin{aligned} \mathcal{D}(Q', C^*) &\geq \mathcal{D}\Big(Q' \wedge C^*, C^*\Big) \geq \mathcal{D}\Big(\overline{Q}^{\{\mathbf{u}\},Q'} \wedge C^*, C^*\Big) \\ &\geq \mathcal{D}\Big(\overline{Q}^{\{\mathbf{u}\},\alpha^*} \wedge C^*, C^*\Big) = \delta, \end{aligned} \tag{2.8}$$

where the first inequality follows from the min-stability of $\mathcal{D}$, and the second and third ones from its monotonicity properties. However, since $Q'(\mathbf{u}) \notin \mathcal{O}$ it follows that

$$\mathcal{D}\Big(\overline{Q}^{\{\mathbf{u}\},Q'} \wedge C^*, C^*\Big) \neq \delta,$$

hence (2.8) yields that $\mathcal{D}\Big(\overline{Q}^{\{\mathbf{u}\},Q'} \wedge C^*, C^*\Big) > \delta$. This contradicts the assumption that $Q' \in \mathcal{Q}^{\mathcal{D},\delta}$, showing that indeed $\min\Big\{Q(\mathbf{u})\colon Q \in \mathcal{Q}^{\mathcal{D},\delta}\Big\} = \alpha^*$. Hence, the lower bound holds for $\delta \leq \mathcal{D}\Big(\overline{Q}^{\{\mathbf{u}\},W_d} \wedge C^*, C^*\Big)$.

(ii) Now, let $\delta > \mathcal{D}\Big(\overline{Q}^{\{\mathbf{u}\},W_d} \wedge C^*, C^*\Big)$, then it follows that

$$\min\Big\{\alpha \in [W_d(\mathbf{u}), M_d(\mathbf{u})]\colon \mathcal{D}\Big(\overline{Q}^{\{\mathbf{u}\},\alpha} \wedge C^*, C^*\Big) \leq \delta\Big\} = W_d(\mathbf{u}).$$

Moreover, since $\Big(\overline{Q}^{\{\mathbf{u}\},W_d} \wedge C^*\Big) \in \mathcal{Q}^{\mathcal{D},\delta}$ and every element in $\mathcal{Q}^{\mathcal{D},\delta}$ is bounded from below by W_d, it follows that $\min\Big\{Q(\mathbf{u})\colon Q \in \mathcal{Q}^{\mathcal{D},\delta}\Big\} = W_d(\mathbf{u})$. Hence, the lower bound holds in this case as well.

Finally, it follows again from [49, Theorem 2.1] that the bounds are quasi-copulas, which completes the proof. $\square$

Note, the bounds from Theorem 2.1.7 also apply to the set of copulas $\mathcal{C}^{\mathcal{D},\delta} := \{C \in \mathcal{C}^d\colon \mathcal{D}(C, C^*) \leq \delta\}$, assuming that $\mathcal{C}^{\mathcal{D},\delta} \neq \emptyset$, that is

$$\underline{Q}^{\mathcal{D},\delta} \leq C \leq \overline{Q}^{\mathcal{D},\delta}, \tag{2.9}$$

for all $C \in \mathcal{C}^{\mathcal{D},\delta}$, due to the fact that $\mathcal{C}^{\mathcal{D},\delta} \subseteq \mathcal{Q}^{\mathcal{D},\delta}$.

Remark 2.1.4. If $\mathcal{D}$ is not symmetric, the set $\{Q \in \mathcal{Q}^d : \mathcal{D}(Q, C^*) \leq \delta\}$ might not coincide with the set $\{Q \in \mathcal{Q}^d : \mathcal{D}(C^*, Q) \leq \delta\}$. In this case the bounds on $\{Q \in \mathcal{Q}^d : \mathcal{D}(C^*, Q) \leq \delta\}$ are provided by

$$\underline{Q}^{\mathcal{D},\delta}(\mathbf{u}) = \min\Big\{\alpha \in \mathbb{S}(\mathbf{u}) : \mathcal{D}\Big(C^*, \overline{Q}^{\{\mathbf{u}\},\alpha} \wedge C^*\Big) \leq \delta\Big\},$$
$$\overline{Q}^{\mathcal{D},\delta}(\mathbf{u}) = \max\Big\{\alpha \in \mathbb{S}(\mathbf{u}) : \mathcal{D}\Big(C^*, \underline{Q}^{\{\mathbf{u}\},\alpha} \vee C^*\Big) \leq \delta\Big\}.$$

♦

Many well-known statistical distances satisfy the requirements of Theorem 2.1.7. Typical examples are the Kolmogorov–Smirnov and the Cramér–von Mises distances, where

$$\mathcal{D}_{\text{KS}}(Q, Q') := \sup_{\mathbf{u} \in \mathbb{I}^d} |Q(\mathbf{u}) - Q'(\mathbf{u})| \quad \text{and} \quad \mathcal{D}_{\text{CM}}(Q, Q') := \int_{\mathbb{I}^d} |Q(\mathbf{u}) - Q'(\mathbf{u})|^2 \mathrm{d}\mathbf{u}.$$

The same holds for all L^p distances with $p \geq 1$, where

$$\mathcal{D}_{L^p}(Q, Q') := \Big(\int_{\mathbb{I}^d} |Q(\mathbf{u}) - Q'(\mathbf{u})|^p \mathrm{d}\mathbf{u}\Big)^{\frac{1}{p}}.$$

Distances with these properties are of particular interest in the theory of minimum distance and minimum contrast estimation, where – as opposed to maximum likelihood methods – parameters of distributions are estimated based on a statistical distance between the empirical and the estimated distribution. These estimators have favorable properties in terms of efficiency and robustness; see e.g. Spokoiny and Dickhaus [61, Chapter 2.8].

The computation of the bounds $\underline{Q}^{\mathcal{D},\delta}$ and $\overline{Q}^{\mathcal{D},\delta}$ in Theorem 2.1.7 involves the solution of optimization problems, which can be computationally intricate depending on the distance $\mathcal{D}$. An explicit representation of the bounds is thus highly valuable for applications. The following result shows that in the particular case of the Kolmogorov–Smirnov distance the bounds can be computed explicitly.

Lemma 2.1.8. *Let C^* be a d-copula, $\delta \in \mathbb{R}_+$, and consider the Kolmogorov–Smirnov distance $\mathcal{D}_{\text{KS}}$. Then*

$$\underline{Q}^{\mathcal{D}_{\text{KS}},\delta}(\mathbf{u}) = \max\Big\{C^*(\mathbf{u}) - \delta, W_d(\mathbf{u})\Big\} \quad \textit{and} \quad \overline{Q}^{\mathcal{D}_{\text{KS}},\delta}(\mathbf{u}) = \min\Big\{C^*(\mathbf{u}) + \delta, M_d(\mathbf{u})\Big\}.$$

Proof. We show that the statement holds for the lower bound $\underline{Q}^{\mathcal{D}_{\text{KS}},\delta}$ and omit the proof for the upper bound $\overline{Q}^{\mathcal{D}_{\mathrm{KS}},\delta}$ as it follows along the same lines. Due to $\left(\overline{Q}^{\{\mathbf{u}\},\alpha} \wedge C^*\right) \leq C^*$ for all $\alpha \in [W_d(\mathbf{u}), M_d(\mathbf{u})]$, it holds that

$$\mathcal{D}_{\mathrm{KS}}\Big(\overline{Q}^{\{\mathbf{u}\},\alpha} \wedge C^*, C^*\Big) = \sup_{\mathbf{x}\in\mathbb{I}^d}\Big|\Big(\overline{Q}^{\{\mathbf{u}\},\alpha} \wedge C^*\Big)(\mathbf{x}) - C^*(\mathbf{x})\Big| = \sup_{\mathbf{x}\in\mathbb{I}^d}\Big\{C^*(\mathbf{x}) - \overline{Q}^{\{\mathbf{u}\},\alpha}(\mathbf{x})\Big\}.$$

Since $\sup_{\mathbf{x}\in\mathbb{I}^d}\Big\{C^*(\mathbf{x}) - \overline{Q}^{\{\mathbf{u}\},\alpha}(\mathbf{x})\Big\} = 0$ when $\alpha > C^*(\mathbf{u})$, we can assume w.l.o.g. that the minimum is attained for $\alpha \leq C^*(\mathbf{u})$. Hence

$$\begin{aligned}\min\Big\{\alpha \in [W_d(\mathbf{u}), M_d(\mathbf{u})]\colon \mathcal{D}_{\mathrm{KS}}\Big(\overline{Q}^{\{\mathbf{u}\},\alpha} \wedge C^*, C^*\Big) \leq \delta\Big\}& \\ = \min\Big\{\alpha \in [W_d(\mathbf{u}), C^*(\mathbf{u})]\colon \sup_{\mathbf{x}\in\mathbb{I}^d}\Big\{C^*(\mathbf{x}) - \overline{Q}^{\{\mathbf{u}\},\alpha}(\mathbf{x})\Big\} \leq \delta\Big\}&.\end{aligned}$$

Then, using the definition of $\overline{Q}^{\{\mathbf{u}\},\alpha}$ in (2.2), we obtain

$$\begin{aligned}\sup_{\mathbf{x}\in\mathbb{I}^d}\Big\{C^*(\mathbf{x}) - \overline{Q}^{\{\mathbf{u}\},\alpha}(\mathbf{x})\Big\} &= \sup_{\mathbf{x}\in\mathbb{I}^d}\Big\{C^*(\mathbf{x}) - \min\Big\{M_d(\mathbf{x}), \alpha + \sum_{i=1}^d (x_i - u_i)^+\Big\}\Big\} \\ &= \sup_{\mathbf{x}\in\mathbb{I}^d}\Big\{C^*(\mathbf{x}) - \alpha - \sum_{i=1}^d (x_i - u_i)^+\Big\} \\ &= \sup_{\mathbf{x}\in\mathbb{I}^d}\Big\{C^*(\mathbf{x}) - \sum_{i=1}^d (x_i - u_i)^+\Big\} - \alpha = C^*(\mathbf{u}) - \alpha,\end{aligned}$$

where the second equality holds due to the fact that $C^*(\mathbf{x}) - M_d(\mathbf{x}) \leq 0$ for all $\mathbf{x} \in \mathbb{I}^d$. Hence, we conclude that

$$\begin{aligned}\underline{Q}^{\mathcal{D}_{\mathrm{KS}},\delta}(\mathbf{u}) &= \min\Big\{\alpha \in [W_d(\mathbf{u}), C^*(\mathbf{u})]\colon C^*(\mathbf{u}) - \alpha \leq \delta\Big\} \\ &= \min\Big\{\alpha \in [W_d(\mathbf{u}), C^*(\mathbf{u})]\colon C^*(\mathbf{u}) - \delta \leq \alpha\Big\} = \max\Big\{C^*(\mathbf{u}) - \delta, W_d(\mathbf{u})\Big\}.\end{aligned}$$

□

Analogously to Theorem 2.1.7, one can also consider the situation where information on the quasi-survival function is available. Note that each statistical distance that measures the discrepancy between quasi-copulas can easily be translated into a distance on quasi-survival functions, i.e. if $\mathcal{D}$ is a statistical distance on $\mathcal{Q}^d \times \mathcal{Q}^d$, then

$$(\widehat{Q}, \widehat{Q}') \mapsto \mathcal{D}\Big(\widehat{Q}(\mathbf{1} - \cdot), \widehat{Q}'(\mathbf{1} - \cdot)\Big)$$

defines a distance on the set of quasi-survival functions.

Corollary 2.1.9. *Let $\widehat{Q}^* \in \widehat{\mathcal{Q}}^d$ and $\mathcal{D}$ be a statistical distance which is continuous with respect to the pointwise convergence of quasi-survival functions, monotonic with respect to $\leq$ on $\widehat{\mathcal{Q}}^d$ and min/max-stable. Consider the set $\widehat{\mathcal{Q}}^{\mathcal{D},\delta} = \left\{\widehat{Q} \in \widehat{\mathcal{Q}}^d \colon \mathcal{D}(\widehat{Q}, \widehat{Q}^*) \leq \delta\right\}$ for $\delta \in \mathbb{R}_+$. Then*

$$\underline{\widehat{Q}}^{\mathcal{D},\delta}(\mathbf{u}) := \min\left\{\alpha \in \mathbb{S}(\mathbf{u}) \colon \mathcal{D}\left(\overline{\widehat{Q}}^{\{\mathbf{u}\},\alpha} \wedge \widehat{Q}^*, \widehat{Q}^*\right) \leq \delta\right\} = \min\left\{\widehat{Q}(\mathbf{u}) \colon \widehat{Q} \in \widehat{\mathcal{Q}}^{\mathcal{D},\delta}\right\}$$
$$\overline{\widehat{Q}}^{\mathcal{D},\delta}(\mathbf{u}) := \max\left\{\alpha \in \mathbb{S}(\mathbf{u}) \colon \mathcal{D}\left(\underline{\widehat{Q}}^{\{\mathbf{u}\},\alpha} \vee \widehat{Q}^*, \widehat{Q}^*\right) \leq \delta\right\} = \max\left\{\widehat{Q}(\mathbf{u}) \colon \widehat{Q} \in \widehat{\mathcal{Q}}^{\mathcal{D},\delta}\right\}.$$

The proof is analogous to the proof of Theorem 2.1.7 and is therefore omitted.

In the next theorem we derive improved Fréchet–Hoeffding bounds assuming that information only on some lower-dimensional marginals of a quasi-copula is available. This result corresponds to the situation when one is interested in a high-dimensional random vector, however information on the dependence structure is only available for lower-dimensional vectors thereof. As an example, in mathematical finance one is interested in options on several assets, however information on the dependence structure – stemming e.g. from prices of liquidly traded options – is typically available only on pairs of those assets.

Let us introduce a convenient subscript notation for the lower-dimensional marginals of a quasi-copula. Consider a subset $I = \{i_1, \ldots, i_n\} \subset \{1, \ldots, d\}$ and define the *projection* of a vector $\mathbf{u} \in \mathbb{R}^d$ to the lower-dimensional space $\mathbb{R}^n$ via $\mathbf{u}_I := (u_{i_1}, \ldots, u_{i_n}) \in \mathbb{R}^n$. Moreover, define the *lift* of the vector $\mathbf{u}_I \in \mathbb{R}^n$ to the higher-dimensional space $\mathbb{R}^d$ by $\mathbf{u}'_I =: \mathbf{v} \in \mathbb{R}^d$ where $v_i = u_i$ if $i \in I$ and $v_i = 1$ if $i \notin I$. Then, we can define the *I-margin* of the d-quasi-copula Q via $Q_I : \mathbb{I}^n \to \mathbb{I}$ with $\mathbf{u}_I \mapsto Q(\mathbf{u}'_I)$.

Remark 2.1.5. Let $\mathbf{u} \in \mathbb{I}^d$ and $I \subset \{1, \ldots, d\}$. Then, by first projecting $\mathbf{u}$ and then lifting it back, we get that $\mathbf{u} \leq \mathbf{u}'_I$. Hence, by $(\mathbf{QC2})$ we get that $Q(\mathbf{u}) \leq Q_I(\mathbf{u}_I) = Q(\mathbf{u}'_I)$. ♦

Theorem 2.1.10. *Let $I_1, \ldots, I_k$ be subsets of $\{1, \ldots, d\}$ with $|I_j| \geq 2$ for $j \in \{1, \ldots, k\}$ and $|I_i \cap I_j| \leq 1$ for $i, j \in \{1, \ldots, k\}$, $i \neq j$. Let $\underline{Q}_j, \overline{Q}_j$ be $|I_j|$-quasi-copulas with $\underline{Q}_j \leq \overline{Q}_j$ for $j = 1, \ldots, k$, and consider the set*

$$\mathcal{Q}^I = \left\{Q \in \mathcal{Q}^d \colon \underline{Q}_j \leq Q_{I_j} \leq \overline{Q}_j,\ j = 1, \ldots, k\right\},$$

where Q_{I_j} are the I_j-margins of Q. Then $\mathcal{Q}^I$ is non-empty and the following bounds hold

$$\begin{aligned}\underline{Q}^I(\mathbf{u}) &:= \min\left\{Q(\mathbf{u})\colon Q \in \mathcal{Q}^I\right\}\\ &= \max\left(\max_{j\in\{1,\dots,k\}}\left\{\underline{Q}_j(\mathbf{u}_{I_j}) + \sum_{l\in\{1,\dots,d\}\setminus I_j}(u_l-1)\right\}, W_d(\mathbf{u})\right),\\ \overline{Q}^I(\mathbf{u}) &:= \max\left\{Q(\mathbf{u})\colon Q \in \mathcal{Q}^I\right\} = \min\left(\min_{j\in\{1,\dots,k\}}\left\{\overline{Q}_j(\mathbf{u}_{I_j})\right\}, M_d(\mathbf{u})\right).\end{aligned}$$

Moreover $\underline{Q}^I, \overline{Q}^I \in \mathcal{Q}^I$*, hence the bounds are sharp.*

Proof. Let $Q \in \mathcal{Q}^I$ and $\mathbf{u} \in \mathbb{I}^d$. We first show that the upper bound $\overline{Q}^I$ is valid. It follows directly from Remark 2.1.5 that

$$Q(\mathbf{u}) \leq Q(\mathbf{u}'_{I_j}) = Q_{I_j}(\mathbf{u}_{I_j}) \leq \overline{Q}_j(\mathbf{u}_{I_j}), \quad \text{for all } j = 1,\dots,k,$$

hence $Q(\mathbf{u}) \leq \min_{j\in\{1,\dots,k\}}\left\{\overline{Q}_j(\mathbf{u}_{I_j})\right\}$. Incorporating the upper Fréchet–Hoeffding bound yields $\overline{Q}^I$. Moreover, $(\mathbf{QC1})$ and $(\mathbf{QC2})$ follow immediately since $\overline{Q}_j$ are quasi-copulas for $j = 1,\dots,k$, while $\overline{Q}^I$ is a composition of Lipschitz functions and hence Lipschitz itself, i.e. $(\mathbf{QC3})$ also holds. Thus $\overline{Q}^I$ is indeed a quasi-copula.

As for the lower bound, using once more the projection and lift operations and the Lipschitz property of quasi-copulas we have

$$\begin{aligned}Q(\mathbf{u}) &\geq Q(\mathbf{u}'_{I_j}) + \sum_{l\in\{1,\dots,d\}\setminus I_j}(u_l-1) = Q_{I_j}(\mathbf{u}_{I_j}) + \sum_{l\in\{1,\dots,d\}\setminus I_j}(u_l-1)\\ &\geq \underline{Q}_j(\mathbf{u}_{I_j}) + \sum_{l\in\{1,\dots,d\}\setminus I_j}(u_l-1), \quad \text{for all } j = 1,\dots,k.\end{aligned}$$

Therefore,

$$Q(\mathbf{u}) \geq \max_{j\in\{1,\dots,k\}}\left\{\underline{Q}_j(\mathbf{u}_{I_j}) + \sum_{l\in\{1,\dots,d\}\setminus I_j}(u_l-1)\right\}, \tag{2.10}$$

and including the lower Fréchet–Hoeffding bound yields $\underline{Q}^I$. In order to verify that $\underline{Q}^I$ is a quasi-copula, first consider $\mathbf{u} \in \mathbb{I}^d$ with $u_i = 0$ for at least one $i \in \{1,\dots,d\}$. Then $W_d(\mathbf{u}) = 0$,

$$\underline{Q}_j(\mathbf{u}_{I_j}) + \sum_{l\in\{1,\dots,d\}\setminus I_j}(u_l-1) \leq \underline{Q}_j(\mathbf{u}_{I_j}) - 1 \leq 0 \quad \text{if } i \in \{1,\dots,d\}\setminus I_j,$$

and

$$\underline{Q}_j(\mathbf{u}_{I_j}) + \sum_{l\in\{1,\dots,d\}\setminus I_j} (u_l - 1) = \sum_{l\in\{1,\dots,d\}\setminus I_j} (u_l - 1) \le 0 \quad \text{if } i \in I_j,$$

for all $j = 1, \dots, k$. Hence $\underline{Q}^I(\mathbf{u}) = 0$. In addition, for $\mathbf{u} \in \mathbb{I}^d$ with $\mathbf{u} = \mathbf{u}'_{\{i\}}$, it follows that $W_d(\mathbf{u}) = u_i$ and

$$\underline{Q}_j(\mathbf{u}_{I_j}) + \sum_{l\in\{1,\dots,d\}\setminus I_j} (u_l - 1) = 1 + (u_i - 1) = u_i \quad \text{if } i \in \{1, \dots, d\} \setminus I_j,$$

while clearly $\underline{Q}_j(\mathbf{u}_{I_j}) = u_i$ if $i \in I_j$, for all $j = 1, \dots, d$. Hence $\underline{Q}^I(\mathbf{u}) = u_i$, showing that $\underline{Q}^I$ fulfills $(\mathbf{QC1})$. $(\mathbf{QC2})$ is immediate, while noting that $\underline{Q}^I$ is a composition of Lipschitz functions and hence Lipschitz itself shows that the lower bound is also a d-quasi-copula.

Finally, knowing that $\underline{Q}^I, \overline{Q}^I$ are quasi-copulas it remains to show that both bounds are in $\mathcal{Q}^I$, i.e. we need to show that $\underline{Q}_j \le \left(\underline{Q}^I\right)_{I_j}, \left(\overline{Q}^I\right)_{I_j} \le \overline{Q}_j$ for all $j = 1, \dots, k$. For the upper bound it holds by definition that $\left(\overline{Q}^I\right)_{I_j} \le \overline{Q}_j$ for $j = 1, \dots, k$. Moreover since $|I_i \cap I_j| \le 1$ it follows that $\left(\overline{Q}^I\right)_{I_j} = \overline{Q}_j$, hence $\underline{Q}_j \le \left(\overline{Q}^I\right)_{I_j} \le \overline{Q}_j$ for $j = 1, \dots, k$ and $\overline{Q}^I \in \mathcal{Q}^I$. By the same argument it holds for the lower bound that $\left(\underline{Q}^I\right)_{I_j} = \underline{Q}_j$ for $j = 1, \dots, d$, thus $\underline{Q}_j \le \left(\underline{Q}^I\right)_{I_j} \le \overline{Q}_j$ for $j = 1, \dots, k$, showing that $\underline{Q}_j \le \left(\underline{Q}^I\right)_{I_j}, \left(\overline{Q}^I\right)_{I_j} \le \overline{Q}_j$ holds indeed. □

The bounds in Theorem 2.1.10 hold analogously for copulas. That is, for subsets $I_1, \dots, I_k$ and quasi-copulas $\underline{Q}_j, \overline{Q}_j$ as in Theorem 2.1.10 and defining

$$\mathcal{C}^I := \left\{ C \in \mathcal{C}^d \colon \underline{Q}_j \le C_{I_j} \le \overline{Q}_j,\ j = 1, \dots, k \right\},$$

it follows that $\underline{Q}^I \le C \le \overline{Q}^I$ for all $C \in \mathcal{C}^I$.

Remark 2.1.6. Theorem 2.1.10 allows us to mix the different types of dependence information considered so far on separate regions of the copula. Consider for example the set of 3-copulas C whose 2-margins $C_{\{1,2\}}$ coincide with a reference 2-copula C^* on the subset $\mathcal{S} = [a_1, b_1] \times [a_2, b_2]$ whilst $C_{\{2,3\}}$ lies in the δ-neighbourhood of a 2-copula C'. Then, by first deriving bounds on the 2-margins $C_{\{1,2\}}$ and $C_{\{2,3\}}$ using Theorems 2.1.1 and 2.1.7 and then applying Theorem 2.1.10 with $I_1 = \{1, 2\}$ and $I_2 = \{2, 3\}$ one obtains bounds on the 3-copula C. ♦

Finally, the subsequent corollary is a version of Theorem 2.1.10 for quasi-survival functions. Its proof is analogous to the proof of Theorem 2.1.10 and is therefore also omitted.

Corollary 2.1.11. *Let $I_1, \ldots, I_k$ be subsets of $\{1, \ldots, d\}$ with $|I_j| \geq 2$ for $j \in \{1, \ldots, k\}$ and $|I_i \cap I_j| \leq 1$ for $i, j \in \{1, \ldots, k\}$, $i \neq j$. Let $\underline{\widehat{Q}}_j, \widehat{\overline{Q}}_j$ be $|I_j|$-quasi-survival functions with $\underline{\widehat{Q}}_j \leq \widehat{\overline{Q}}_j$ for $j = 1, \ldots, k$ and consider the set*

$$\widehat{\mathcal{Q}}^I = \left\{ \widehat{Q} \in \widehat{\mathcal{Q}}^d \colon \underline{\widehat{Q}}_j \leq \widehat{Q}_{I_j} \leq \widehat{\overline{Q}}_j,\ j = 1, \ldots, k \right\},$$

where the I_j-margin of the quasi-survival function $\widehat{Q}$ is defined by

$$\mathbb{I}^d \ni \mathbf{u} \mapsto \widehat{Q}_{I_j}(u_1, ..., u_d) := \widehat{Q}(u'_1, ..., u'_d)$$

with $u'_i = u_i$ when $i \in I_j$ and $u'_i = 0$ if $i \in \{1, ..., d\} \setminus I_j$. Then it holds for all $\widehat{Q} \in \widehat{\mathcal{Q}}^I$

$$\widehat{\overline{Q}}^I(\mathbf{u}) \leq \widehat{Q}(\mathbf{u}) \leq \underline{\widehat{Q}}^I(\mathbf{u}) \quad \text{for all } u \in \mathbb{I}^d,$$

where

$$\widehat{\overline{Q}}^I(\mathbf{u}) := \overline{Q}^I(\mathbf{1} - \mathbf{u}) \quad \text{and} \quad \underline{\widehat{Q}}^I(\mathbf{u}) := \underline{Q}^I(\mathbf{1} - \mathbf{u}),$$

while $\underline{Q}^I$ and $\overline{Q}^I$ are provided in Theorem 2.1.10, for the quasi-copulas $\widehat{\overline{Q}}_j(\mathbf{1} - \cdot)$ and $\underline{\widehat{Q}}_j(\mathbf{1} - \cdot)$ respectively, corresponding to I_j for $j = 1, ..., k$.

2.2 Are the improved Fréchet–Hoeffding bounds copulas?

An interesting question arising now is under what conditions the improved Fréchet–Hoeffding bounds are copulas and not merely quasi-copulas. Such conditions would allow us, for example, to translate those bounds on the copulas into bounds on the expectations with respect to the underlying random variables. Tankov [64] showed that if $d = 2$, then $\underline{Q}^{\mathcal{S},Q^*}$ and $\overline{Q}^{\mathcal{S},Q^*}$ are copulas under certain constraints on the set $\mathcal{S}$. Specifically, if $\mathcal{S}$ is increasing (also called comonotone), that is if $(u_1, u_2), (v_1, v_2) \in \mathcal{S}$ then $u_1 \leq v_1$ and $u_2 \leq v_2$ or $u_1 \geq v_1$ and $u_2 \geq v_2$ hold, then the lower bound $\underline{Q}^{\mathcal{S},Q^*}$ is a copula. Conversely, if $\mathcal{S}$ is decreasing (also called countermonotone), that is if $(u_1, u_2), (v_1, v_2) \in \mathcal{S}$ then $u_1 \leq v_1$ and $u_2 \geq v_2$ or $u_1 \geq v_1$ and $u_2 \leq v_2$ hold, then the upper bound $\overline{Q}^{\mathcal{S},Q^*}$ is a copula. Bernard et al. [8] relaxed these constraints and provided minimal conditions on $\mathcal{S}$ such that the bounds are copulas. The situation however is more complicated for $d > 2$. On the one hand, the notion of a decreasing set is not clear. On the other hand, the following counterexample shows that the condition of $\mathcal{S}$ being an increasing set is not sufficient for $\underline{Q}^{\mathcal{S},Q^*}$ to be a copula.

Example 2.2.1. Let $\mathcal{S} = \left\{(u,u,u)\colon u \in \left[0,\frac{1}{2}\right] \cup \left[\frac{3}{5},1\right]\right\} \subset \mathbb{I}^3$ and Q^* be the independence copula, i.e. $Q^*(u_1,u_2,u_3) = u_1u_2u_3$ for $(u_1,u_2,u_3) \in \mathbb{I}^3$. Then $\mathcal{S}$ is clearly an increasing set, however $\underline{Q}^{\mathcal{S},Q^*}$ is not a copula. To this end, it suffices to show that the $\underline{Q}^{\mathcal{S},Q^*}$-volume of some subset of $\mathbb{I}^3$ is negative. Indeed, for $\left(\frac{56}{100},\frac{3}{5}\right]^3 \subset \mathbb{I}^3$ after some straightforward calculations we get that

$$\begin{aligned} V_{\underline{Q}^{\mathcal{S},Q^*}}\left(\left(\frac{56}{100},\frac{3}{5}\right]^3\right) = \left(\frac{3}{5}\right)^3 - 3\left[\left(\frac{3}{5}\right)^3 - \left(\frac{3}{5}-\frac{56}{100}\right)\right] \\ + 3\left[\left(\frac{3}{5}\right)^3 - 2\left(\frac{3}{5}-\frac{56}{100}\right)\right] - \left(\frac{1}{2}\right)^3 = -0.029 < 0. \end{aligned}$$

♦

In the trivial case where $\mathcal{S} = \mathbb{I}^d$ and Q^* is a d-copula, then both bounds from Theorem 2.1.1 are copulas for $d > 2$ since they equate to Q^*. Moreover, the upper bound is a copula for $d > 2$ if it coincides with the upper Fréchet–Hoeffding bound. The next result shows that essentially only in these trivial situations are the bounds copulas for $d > 2$. Out of instructive reasons we first discuss the case $d = 3$, and then generalise the result for $d > 3$.

Theorem 2.2.1. *Consider the compact subset $\mathcal{S}$ of $\mathbb{I}^3$*

$$\mathcal{S} = \Big([0,1] \setminus (s_1, s_1+\varepsilon_1)\Big) \times \Big([0,1] \setminus (s_2, s_2+\varepsilon_2)\Big) \times \Big([0,1] \setminus (s_3, s_3+\varepsilon_3)\Big), \tag{2.11}$$

for $\varepsilon_i > 0$, $i = 1,2,3$ and let C^* be a 3-copula (or a 3-quasi-copula) such that

$$\sum_{i=1}^{3} \varepsilon_i > C^*(\mathbf{s}+\boldsymbol{\varepsilon}) - C^*(\mathbf{s}) > 0, \tag{2.12}$$

$$C^*(\mathbf{s}) \geq W_3(\mathbf{s}+\boldsymbol{\varepsilon}), \tag{2.13}$$

where $\mathbf{s} = (s_1,s_2,s_3), \boldsymbol{\varepsilon} = (\varepsilon_1,\varepsilon_2,\varepsilon_3)$. Then $\underline{Q}^{\mathcal{S},C^*}$ is a proper quasi-copula.

Proof. Assume that C^* is a d-copula and choose $\mathbf{u} = (u_1,u_2,u_3) \in (\mathbf{s}, \mathbf{s}+\boldsymbol{\varepsilon})$ such that

$$C^*(\mathbf{s}+\boldsymbol{\varepsilon}) - C^*(\mathbf{s}) < \sum_{i=1}^{3}(s_i + \varepsilon_i - u_i) \quad \text{and} \tag{2.14}$$

$$C^*(\mathbf{s}+\boldsymbol{\varepsilon}) - C^*(\mathbf{s}) > \sum_{i\in J}(s_i + \varepsilon_i - u_i) \quad \text{for } J = (1,2),(2,3),(1,3); \tag{2.15}$$

such a $\mathbf{u}$ exists due to (2.12). In order to show that $\underline{Q}^{\mathcal{S},C^*}$ is not 3-increasing, and thus not a (proper) copula, it suffices to prove that $V_{\underline{Q}^{\mathcal{S},C^*}}((\mathbf{u},\mathbf{s}+\boldsymbol{\varepsilon}]) < 0$. By the definition of $V_{\underline{Q}^{\mathcal{S},C^*}}$ we have

$$\begin{aligned} V_{\underline{Q}^{\mathcal{S},C^*}}((\mathbf{u},\mathbf{s}+\boldsymbol{\varepsilon}]) = {} & \underline{Q}^{\mathcal{S},C^*}(\mathbf{s}+\boldsymbol{\varepsilon}) - \underline{Q}^{\mathcal{S},C^*}(u_1, s_2+\varepsilon_2, s_3+\varepsilon_3) \\ & - \underline{Q}^{\mathcal{S},C^*}(s_1+\varepsilon_1, u_2, s_3+\varepsilon_3) - \underline{Q}^{\mathcal{S},C^*}(s_1+\varepsilon_1, s_2+\varepsilon_2, u_3) \\ & + \underline{Q}^{\mathcal{S},C^*}(u_1, u_2, s_3+\varepsilon_3) + \underline{Q}^{\mathcal{S},C^*}(u_1, s_2+\varepsilon_2, u_3) \\ & + \underline{Q}^{\mathcal{S},C^*}(s_1+\varepsilon_1, u_2, u_3) - \underline{Q}^{\mathcal{S},C^*}(\mathbf{u}). \end{aligned}$$

Analysing the summands we see that:

- $\underline{Q}^{\mathcal{S},C^*}(\mathbf{s}+\boldsymbol{\varepsilon}) = C^*(\mathbf{s}+\boldsymbol{\varepsilon})$ because $(\mathbf{s}+\boldsymbol{\varepsilon}) \in \mathcal{S}$.
- The expression $\max_{\mathbf{x}\in\mathcal{S}}\left\{C^*(\mathbf{x}) - \sum_{i=1}^{3}(x_i - v_i)^+\right\}$ where $\mathbf{v} = (u_1, s_2+\varepsilon_2, s_3+\varepsilon_3)$ attains its maximum either at $\mathbf{x} = \mathbf{s}$ or at $\mathbf{x} = \mathbf{s}+\boldsymbol{\varepsilon}$, thus equals $\max\{C^*(\mathbf{s}), C^*(\mathbf{s}+\boldsymbol{\varepsilon}) - (s_1+\varepsilon_1-u_1)\}$, while (2.15) yields that $C^*(\mathbf{s}+\boldsymbol{\varepsilon}) - (s_1+\varepsilon_1-u_1) > C^*(\mathbf{s})$. Moreover, (2.13) yields $C^*(\mathbf{s}) \geq W_3(\mathbf{s}+\boldsymbol{\varepsilon}) \geq W_3(\mathbf{v})$, since $\mathbf{u} \in (\mathbf{s},\mathbf{s}+\boldsymbol{\varepsilon})$. Hence,

  $$\underline{Q}^{\mathcal{S},C^*}(u_1, s_2+\varepsilon_2, s_3+\varepsilon_3) = C^*(\mathbf{s}+\boldsymbol{\varepsilon}) - (s_1+\varepsilon_1-u_1),$$

  while the expressions for the terms involving $(s_1+\varepsilon_1, u_2, s_3+\varepsilon_3)$ and $(s_1+\varepsilon_1, s_2+\varepsilon_2, u_3)$ are analogous.
- Using the same argumentation, it follows that

  $$\underline{Q}^{\mathcal{S},C^*}(u_1, u_2, s_3+\varepsilon_3) = C^*(\mathbf{s}+\boldsymbol{\varepsilon}) - \sum_{i=1,2}(s_i+\varepsilon_i-u_i),$$

  while the expressions for the terms involving $(u_1, s_2+\varepsilon_2, u_3)$ and $(s_1+\varepsilon_1, u_2, u_3)$ are analogous.
- Moreover, $\underline{Q}^{\mathcal{S},C^*}(\mathbf{u}) = C^*(\mathbf{s})$, which follows from (2.14).

Therefore, putting the pieces together and using (2.14) we get that

$$\begin{aligned} V_{\underline{Q}^{\mathcal{S},C^*}}((\mathbf{u}, \mathbf{s}+\boldsymbol{\varepsilon}]) &= C^*(\mathbf{s}+\boldsymbol{\varepsilon}) - 3\,C^*(\mathbf{s}+\boldsymbol{\varepsilon}) + \sum_{i=1}^{3}(s_i+\varepsilon_i-u_i) \\ &\quad + 3\,C^*(\mathbf{s}+\boldsymbol{\varepsilon}) - 2\sum_{i=1}^{3}(s_i+\varepsilon_i-u_i) - C^*(\mathbf{s}) \\ &= C^*(\mathbf{s}+\boldsymbol{\varepsilon}) - C^*(\mathbf{s}) - \sum_{i=1}^{3}(s_i+\varepsilon_i-u_i) < 0. \end{aligned}$$

Hence, $\underline{Q}^{\mathcal{S},C^*}$ is indeed a proper quasi-copula. □

The following result shows that the requirements in Theorem 2.2.1 are minimal, in the sense that if the prescription set $\mathcal{S}$ is contained in a set of the form (2.11) then the lower bound is a proper quasi-copula.

Corollary 2.2.2. *Let C^* be a 3-copula and $\mathcal{S} \subset \mathbb{I}^3$ be compact. If there exists a compact set $\mathcal{S}' \supset \mathcal{S}$ such that $\mathcal{S}'$ and $Q^* := \underline{Q}^{\mathcal{S},C^*}$ satisfy the assumptions of Theorem 2.2.1, then $\underline{Q}^{\mathcal{S},C^*}$ is a proper quasi-copula.*

Proof. Since Q^* and $\mathcal{S}'$ fulfill the requirements of Theorem 2.2.1, it follows that $\underline{Q}^{\mathcal{S}',Q^*}$ is a proper quasi-copula. Now, in order to prove that $\underline{Q}^{\mathcal{S},C^*}$ is also a proper quasi-copula we show that $\underline{Q}^{\mathcal{S}',Q^*} = \underline{Q}^{\mathcal{S},C^*}$. Note first that $\underline{Q}^{\mathcal{S},C^*}$ is the pointwise lower bound on the set $\mathcal{Q}^{\mathcal{S},C^*}$, i.e.

$$\begin{aligned} \underline{Q}^{\mathcal{S},C^*}(\mathbf{u}) &= \min\Big\{Q(\mathbf{u})\colon Q \in \mathcal{Q}^{\mathcal{S},C^*}\Big\} \\ &= \min\Big\{Q(\mathbf{u})\colon Q \in \mathcal{Q}^3, Q(\mathbf{x}) = C^*(\mathbf{x}) \text{ for all } \mathbf{x} \in \mathcal{S}\Big\} \end{aligned}$$

for all $\mathbf{u} \in \mathbb{I}^3$. Analogously, $\underline{Q}^{\mathcal{S}',Q^*}$ is the pointwise lower bound of $\mathcal{Q}^{\mathcal{S}',Q^*}$. Using the properties of the bounds and the fact that $\mathcal{S} \subset \mathcal{S}'$, it follows that $\underline{Q}^{\mathcal{S}',Q^*}(\mathbf{x}) = Q^*(\mathbf{x}) = \underline{Q}^{\mathcal{S},C^*}(\mathbf{x}) = C^*(\mathbf{x})$ for all $\mathbf{x} \in \mathcal{S}$. Hence, $\underline{Q}^{\mathcal{S}',Q^*} \in \mathcal{Q}^{\mathcal{S},C^*}$, therefore it holds that $\underline{Q}^{\mathcal{S},C^*}(\mathbf{u}) \le \underline{Q}^{\mathcal{S}',Q^*}(\mathbf{u})$ for all $\mathbf{u} \in \mathbb{I}^3$. For the reverse inequality, note that for all $\mathbf{x} \in \mathcal{S}'$ it follows from the definition of Q^* that $\underline{Q}^{\mathcal{S},C^*}(\mathbf{x}) = Q^*(\mathbf{x})$, hence $\underline{Q}^{\mathcal{S},C^*} \in \mathcal{Q}^{\mathcal{S}',Q^*}$ such that $\underline{Q}^{\mathcal{S},C^*}(\mathbf{u}) \ge \underline{Q}^{\mathcal{S}',Q^*}(\mathbf{u})$ for all $\mathbf{u} \in \mathbb{I}^3$. Therefore, $\underline{Q}^{\mathcal{S},C^*} = \underline{Q}^{\mathcal{S}',Q^*}$ and $\underline{Q}^{\mathcal{S},C^*}$ is indeed a proper quasi-copula. □

The next example illustrates Corollary 2.2.2 in the case when $\mathcal{S}$ is a singleton.

Example 2.2.2. Let $d = 3$ and C^* be the independence copula, i.e. $C^*(u_1, u_2, u_3) = u_1u_2u_3$, and $\mathcal{S} = \{\frac{1}{2}\}^3$. Then, the bound $\underline{Q}^{\mathcal{S},C^*}$ is a proper quasi-copula since its volume is negative, for example $V_{\underline{Q}^{\mathcal{S},C^*}}\Big(\big(\frac{5}{10}-\frac{1}{20},\frac{5}{10}\big]^3\Big)=-\frac{1}{40}<0$, however Theorem 2.2.1 does not apply since $\mathcal{S}$ is not of the form (2.11). Nevertheless, using Corollary 2.2.2, we can embed $\mathcal{S}$ in a compact set $\mathcal{S}'$ such that $\mathcal{S}'$ and $Q^*:=\underline{Q}^{\mathcal{S},C^*}$ fulfill the conditions of Theorem 2.2.1. To this end let $\mathcal{S}'=\Big([0,1]\setminus(s,s+\varepsilon)\Big)^3=\Big([0,1]\setminus(\frac{4}{10},\frac{5}{10})\Big)^3$, then it follows

$$\begin{aligned}
&\sum_{i=1}^{3}\varepsilon=\frac{3}{10}>Q^*\left(\frac{5}{10},\frac{5}{10},\frac{5}{10}\right)-Q^*\left(\frac{4}{10},\frac{4}{10},\frac{4}{10}\right)=\left(\frac{5}{10}\right)^3>0\\
\text{and}\quad &Q^*\left(\frac{4}{10},\frac{4}{10},\frac{4}{10}\right)=0\geq W_3\left(\frac{5}{10},\frac{5}{10},\frac{5}{10}\right)=0.
\end{aligned}$$

Hence, Q^* and $\mathcal{S}'$ fulfill conditions (2.12) and (2.13) of Theorem 2.2.1, thus it follows from Corollary 2.2.2 that $\underline{Q}^{\mathcal{S},C^*}$ is a proper quasi-copula. ♦

Remark 2.2.1. Analogously to Theorem 2.2.1 and Corollary 2.2.2 one obtains that the upper bound $\overline{Q}^{\mathcal{S},C^*}$ is a proper quasi-copula if the set $\mathcal{S}$ is of the form (2.11) and the copula C^* satisfies

$$\sum_{i=1}^{3}\varepsilon_i>C^*(\mathbf{s}+\boldsymbol{\varepsilon})-C^*(\mathbf{s})>0\quad\text{and}\quad C^*(\mathbf{s}+\boldsymbol{\varepsilon})\leq M_3(\mathbf{s}),$$

or if $\mathcal{S}$ is contained in a compact set $\mathcal{S}'$ for which the above hold. The respective details and proofs are provided in Theorem 2.2.5. ♦

We are now in the position to state the general result for $d>3$. The proof follows the same lines as the proof of 2.2.1. However, a more intricate notation is required for the formulation of the result in higher dimensions.

Theorem 2.2.3. *Consider the compact subset $\mathcal{S}$ of $\mathbb{I}^d$*

$$\begin{aligned}
\mathcal{S}=[0,1]\times\cdots\times[0,1]\times&\underbrace{\Big([0,1]\setminus(s_i,s_i+\varepsilon_i)\Big)}_{i-\text{th component}}\times[0,1]\times\cdots\times[0,1]\\
&\times\underbrace{\Big([0,1]\setminus(s_j,s_j+\varepsilon_j)\Big)}_{j-\text{th component}}\times[0,1]\times\cdots\times[0,1]\times\underbrace{\Big([0,1]\setminus(s_k,s_k+\varepsilon_k)\Big)}_{k-\text{th component}}\\
&\times[0,1]\times\cdots\times[0,1],
\end{aligned}\tag{2.16}$$

for $\mathbf{s}=(s_i,s_j,s_k),\overline{\mathbf{s}}=(s_i+\varepsilon_i,s_j+\varepsilon_j,s_k+\varepsilon_k)\in\mathbb{I}^3$ *and* $\varepsilon_i,\varepsilon_j,\varepsilon_k>0$. *Moreover, let* C^*

be a d-copula (or a d-quasi-copula) such that

$$\sum_{i=1}^{3} \varepsilon_i > C^*(\overline{\mathbf{s}}'_I) - C^*(\mathbf{s}'_I) > 0, \tag{2.17}$$
$$C^*(\mathbf{s}'_I) \geq W_d(\overline{\mathbf{s}}'_I), \tag{2.18}$$

where $I := \{i, j, k\}$ and $\mathbf{s}'_I$, $\overline{\mathbf{s}}'_I$ are defined by the lift operation. Then $\underline{Q}^{\mathcal{S},C^*}$ is a proper quasi-copula.

Proof. From Theorem 2.2.1 we know already that the statement holds if $d = 3$. For the general case, i.e. $d > 3$, choose $u_l \in [0,1]$ with $u_l \in (s_l, s_l + \varepsilon_l)$ for $l \in I = \{i, j, k\}$, such that

$$\begin{aligned}
C^*(\overline{\mathbf{s}}'_I) - C^*(\mathbf{s}'_I) &< \sum_{l \in I}(s_l + \varepsilon_l - u_l) \quad \text{and} \\
C^*(\overline{\mathbf{s}}'_I) - C^*(\mathbf{s}'_I) &> \sum_{l \in J}(s_l + \varepsilon_l - u_l) \quad \text{for } J = (i,j), (j,k), (i,k);
\end{aligned}$$

this exists due to (2.17). Then, considering the set

$$\begin{aligned}
H = (0,1] \times \cdots \times (0,1] \times (u_i, s_i + \varepsilon_i] \times (0,1] \times \cdots \times (0,1] \times (u_j, s_j + \varepsilon_j] \times (0,1] \times \cdots \\
\times (0,1] \times (u_j, s_j + \varepsilon_j] \times (0,1] \times \cdots \times (0,1]
\end{aligned}$$

and using a similar argumentation as in the case $d = 3$ together with property ($\mathbf{QC1}$), it follows that

$$\begin{aligned}
V_{\underline{Q}^{\mathcal{S},C^*}}(H) &= \underline{Q}^{\mathcal{S},C^*}\Big(\overline{\mathbf{s}}'\Big) - \underline{Q}^{\mathcal{S},C^*}\Big((u_i, s_j + \varepsilon_j, s_k + \varepsilon_k)'\Big) - \dots \\
&\quad + \underline{Q}^{\mathcal{S},C^*}\Big((u_i, u_j, s_k + \varepsilon_k)'\Big) + \cdots - \underline{Q}^{\mathcal{S},C^*}\Big(\mathbf{u}'_I\Big) \\
&= C^*\Big(\overline{\mathbf{s}}'_I\Big) - 3\,C^*\Big(\overline{\mathbf{s}}'_I\Big) + \sum_{l \in I}(s_l + \varepsilon_l - u_l) \\
&\quad + 3\,C^*\Big(\overline{\mathbf{s}}'_I\Big) - 2\sum_{l \in I}(s_l + \varepsilon_l - u_l) - C^*\Big(\mathbf{s}'_I\Big) \\
&= C^*\Big(\overline{\mathbf{s}}'_I\Big) - C^*\Big(\mathbf{s}'_I\Big) - \sum_{l \in I}(s_l + \varepsilon_l - u_l) < 0.
\end{aligned}$$

Hence, $\underline{Q}^{\mathcal{S},C^*}$ is a proper-quasi-copula. □

The following corollary shows that the requirements in Theorem 2.2.3 are minimal, in the sense that if the prescription set $\mathcal{S}$ is contained in a set of the form (2.16) then the lower bound is a proper quasi-copula. Its proof is analogous to the proof of Corollary 2.2.2 and therefore omitted.

Corollary 2.2.4. *Let C^* be a d-copula and $\mathcal{S} \subset \mathbb{I}^d$ be compact. If there exists a compact set $\mathcal{S}' \supset \mathcal{S}$ such that $\mathcal{S}'$ and $Q^* := \underline{Q}^{\mathcal{S},C^*}$ satisfy the assumptions of Theorem 2.2.3, then $\underline{Q}^{\mathcal{S},C^*}$ is a proper quasi-copula.*

Finally, we establish that the upper improved Fréchet–Hoeffding from Theorem 2.1.1 is essentially only a copula in degenerate cases.

Theorem 2.2.5. *Consider the compact subset $\mathcal{S}$ of $\mathbb{I}^d$ in (2.16) for $\mathbf{s} = (s_i, s_j, s_k), \overline{\mathbf{s}} = (s_i + \varepsilon_i, s_j + \varepsilon_j, s_k + \varepsilon_k) \in \mathbb{I}^3$ and $\varepsilon_i, \varepsilon_j, \varepsilon_k > 0$. Let C^* be a d-copula (or d-quasi-copula) such that*

$$\sum_{i=1}^{3} \varepsilon_i > C^*(\overline{\mathbf{s}}'_I) - C^*(\mathbf{s}'_I) > 0 \quad \text{and} \tag{2.19}$$

$$C^*(\overline{\mathbf{s}}'_I) \leq M_d(\mathbf{s}'_I), \tag{2.20}$$

where $I = \{i, j, k\}$ and $\mathbf{s}'_I$ and $\overline{\mathbf{s}}'_I$ are defined by the lift operation. Then $\overline{Q}^{\mathcal{S},C^*}$ is a proper quasi-copula.

Proof. We show that the result holds for $d = 3$. The general case for $d > 3$ follows as in the proof of Theorem 2.2.3. Let C^* be a 3-copula and $\mathcal{S} = \mathbb{I}^3 \setminus (\mathbf{s}, \mathbf{s} + \boldsymbol{\varepsilon})$ for some $\mathbf{s} \in [0, 1]^3$ and $\varepsilon_i > 0$, $i = 1, 2, 3$. Moreover, choose $\mathbf{u} = (u_1, u_2, u_3) \in (\mathbf{s}, \mathbf{s} + \boldsymbol{\varepsilon})$ such that

$$C^*(\mathbf{s} + \boldsymbol{\varepsilon}) - C^*(\mathbf{s}) < \sum_{i=1}^{3} (s_i + \varepsilon_i - u_i) \quad \text{and} \tag{2.21}$$

$$C^*(\mathbf{s} + \boldsymbol{\varepsilon}) - C^*(\mathbf{s}) > \sum_{i \in I} (s_i + \varepsilon_i - u_i) \quad \text{for } I = (1, 2), (2, 3), (1, 3); \tag{2.22}$$

such a $\mathbf{u}$ exists due to (2.19). Now, in order to show that $\overline{Q}^{\mathcal{S},C^*}$ is not d-increasing, and thus a proper quasi-copula, it suffices to prove that $V_{\overline{Q}^{\mathcal{S},C^*}}((\mathbf{s}, \mathbf{u}]) < 0$. By the definition of $V_{\overline{Q}^{\mathcal{S},C^*}}$ we have

$$\begin{aligned} V_{\overline{Q}^{\mathcal{S},C^*}}((\mathbf{s}, \mathbf{u}]) = \overline{Q}^{\mathcal{S},C^*}(\mathbf{u}) - \overline{Q}^{\mathcal{S},C^*}(s_1, u_2, u_3) - \overline{Q}^{\mathcal{S},C^*}(u_1, s_2, u_3) - \overline{Q}^{\mathcal{S},C^*}(u_1, u_2, s_3) \\ + \overline{Q}^{\mathcal{S},C^*}(s_1, s_2, u_3) + \overline{Q}^{\mathcal{S},C^*}(s_1, u_2, s_3) + \overline{Q}^{\mathcal{S},C^*}(u_1, s_2, s_3) - \overline{Q}^{\mathcal{S},C^*}(\mathbf{s}). \end{aligned}$$

Analysing the summands we see that

- $\overline{Q}^{\mathcal{S},C^*}(\mathbf{u}) = \min_{x\in\mathcal{S}}\Big\{C^*(\mathbf{x})+\sum_{i=1}^{3}(x_i-u_i)^+\Big\} = C^*(\mathbf{s}+\boldsymbol{\varepsilon})$, where the first equality holds due to (2.20) and the second one due to (2.21).
- $\overline{Q}^{\mathcal{S},C^*}(s_1,u_2,u_3) = \min_{x\in\mathcal{S}}\Big\{C^*(\mathbf{x}) + (s_1-x_1)^+ + (u_2-x_2)^+ + (u_3-x_3)^+\Big\} = \min_{x\in\mathcal{S}}\Big\{C^*(\mathbf{s}+\boldsymbol{\varepsilon}), C^*(\mathbf{s}) + (u_2-x_2)^+ + (u_3-x_3)^+\Big\} = C^*(\mathbf{s}) + (u_2-s_2) + (u_3-s_3)$, where the first equality holds due to (2.20) and the third one due to (2.22). The second equality holds since the minimum is only attained at either $\mathbf{s}$ or $\mathbf{s}+\boldsymbol{\varepsilon}$. Analogously it follows that $\overline{Q}^{\mathcal{S},C^*}(u_1,s_2,u_3) = C^*(\mathbf{s}) + (u_1-s_1) + (u_3-s_3)$ and $\overline{Q}^{\mathcal{S},C^*}(u_1,u_2,s_3) = C^*(\mathbf{s}) + (u_1-s_1) + (u_2-s_2)$.
- Similarly, it follows that $\overline{Q}^{\mathcal{S},C^*}(s_1,s_2,u_3) = C^*(\mathbf{s}) + (u_3-s_3)$, $\overline{Q}^{\mathcal{S},C^*}(u_1,s_2,s_3) = C^*(\mathbf{s}) + (u_1-s_1)$ and $\overline{Q}^{\mathcal{S},C^*}(s_1,u_2,s_3) = C^*(\mathbf{s}) + (u_2-s_2)$.
- In addition, $\overline{Q}^{\mathcal{S},C^*}(\mathbf{s}) = C^*(\mathbf{s})$ because $s\in\mathcal{S}$.

Therefore, putting the pieces together and using (2.21), we get

$$\begin{aligned} V_{\underline{Q}^{\mathcal{S},C^*}}((\mathbf{s},\mathbf{u}]) &= C^*(\mathbf{s}+\boldsymbol{\varepsilon}) - 3C^*(\mathbf{s}) - 2\sum_{i=1}^{3}(u_i-s_i) + 3C^*(\mathbf{s}) + \sum_{i=1}^{3}(u_i-s_i) - C^*(\mathbf{s}) \\ &= C^*(\mathbf{s}+\boldsymbol{\varepsilon}) - C^*(\mathbf{s}) - \sum_{i=1}^{3}(u_i-s_i) < 0. \end{aligned}$$

Thus $\underline{Q}^{\mathcal{S},C^*}$ is indeed a proper quasi-copula. □

The following Corollary shows that the requirements in Theorem 2.2.5 are minimal in the sense that if the prescription set is contained in a set of the form (2.16) then the upper bound is a proper-quasi-copula. Its proof is analogous to the proof of Corollary 2.2.2 and therefore omitted.

Corollary 2.2.6. *Let C^* be a d-copula and $S\subset\mathbb{I}^d$ be compact. If there exists a compact set $\mathcal{S}'\supset\mathcal{S}$ such that $\mathcal{S}'$ and $Q^* := \overline{Q}^{\mathcal{S},C^*}$ satisfy the assumptions of Theorem 2.2.5, then $\overline{Q}^{\mathcal{S},C^*}$ is a proper quasi-copula.*

The results presented in this section show that the improved Fréchet–Hoeffding bounds on $\mathcal{C}^{\mathcal{S},C^*}$, for a reference copula C^*, fail to be copulas in virtually all situations of interest; for the definition of $\mathcal{C}^{\mathcal{S},C^*}$ see Remark 2.1.1. In particular, they are not sharp. This motivates the question whether the bounds are at least pointwise best-possible in the sense that

$$\underline{Q}^{\mathcal{S},C^*}(\mathbf{u}) = \inf\Big\{C(\mathbf{u})\colon C\in\mathcal{C}^{\mathcal{S},C^*}\Big\} \quad \text{and} \quad \overline{Q}^{\mathcal{S},C^*}(\mathbf{u}) = \sup\Big\{C(\mathbf{u})\colon C\in\mathcal{C}^{\mathcal{S},C^*}\Big\},$$

for all $\mathbf{u} \in \mathbb{I}^d$, under mild conditions on the set $\mathcal{S}$. In our ongoing research we address this question by means of a mass transport approach and a corresponding dual formulation of the improved Fréchet–Hoeffding bounds.

3 Stochastic dominance for quasi-copulas and applications in model-free option pricing

In this chapter we relate the improved Fréchet–Hoeffding bounds from Section 2.1 to bounds on the expectation $\mathbb{E}[\varphi(\mathbf{X})]$ for $\varphi\colon \mathbb{R}_+^d \to \mathbb{R}$ and an $\mathbb{R}_+^d$-valued random vector $\mathbf{X} = (X_1, ..., X_d)$, whose probability distribution is partially unknown. Specifically, we assume that the marginal distributions F_i of X_i are provided, while only partial information on the copula of $\mathbf{X}$ is available.

To this end we resort to the theory of stochastic orders and stochastic dominance. However, since the improved Fréchet–Hoeffding bounds are merely quasi-copulas, existing results from stochastic order theory which translate bounds on the copula of $\mathbf{X}$ into bounds on the expectation of $\varphi(\mathbf{X})$ do not apply. Even worse, the integrals with respect to quasi-copulas are not well-defined. Therefore, we derive in Section 3.1 an alternative representation of multivariate integrals with respect to copulas, which admits also quasi-copulas as integrators, and establish integrability and continuity properties of this representation. Moreover, we provide an integral characterization of the lower orthant order on the set of quasi-copulas and quasi-survival functions in terms of generators consisting of Δ-antitonic or Δ-monotonic functions, analogous to existing results on integral stochastic orders for copulas. This enables us to compute bounds on the expectation of $\varphi(\mathbf{X})$ that account for the available information on the marginal distributions and the copula of $\mathbf{X}$.

We then apply our results in order to compute model-free bounds on the prices of European options in the presence of dependence uncertainty in Section 3.2. We assume that $\mathbf{X}$ models the terminal value of financial assets whose risk-free marginal distributions can be inferred from market prices of traded vanilla options on the individual constituents. Moreover, we suppose that additional information on the dependence structure of $\mathbf{X}$ can be obtained from prices of traded derivatives on $\mathbf{X}$ or a subset of its components. This could be e.g.

information about the pairwise correlations of the components or prices of traded spread options. Then, the improved Fréchet–Hoeffding bounds and the integral characterization of the lower orthant order allow us to efficiently compute bounds on the set of arbitrage-free prices of $\varphi(\mathbf{X})$ that are compatible with the available information on the distribution of $\mathbf{X}$. Moreover, we illustrate in numerical applications that the improved Fréchet–Hoeffding bounds typically lead to a significant improvement of the associated option price estimates compared to the ones obtained from the 'standard' Fréchet–Hoeffding bounds.

The results in this chapter appeared in Lux and Papapantoleon [34].

3.1 Stochastic dominance for quasi-copulas

The aim of this section is to relate the lower orthant order on the set of quasi-copulas or quasi-survival functions to expectations of the form $\mathbb{E}[\varphi(\mathbf{X})]$, for an $\mathbb{R}^d_+$-valued random vector $\mathbf{X} = (X_1, \ldots, X_d)$ with marginals $F_1, \ldots, F_d$ and a function $\varphi\colon \mathbb{R}^d_+ \to \mathbb{R}$. Using Sklar's Theorem, there exists a copula C such that $F(x_1, \ldots, x_d) = C(F_1(x_1), \ldots, F_d(x_d))$ for all $(x_1, \ldots, x_d) \in \mathbb{R}^d_+$. The expectation hence becomes a function of the copula C. Define the *expectation operator* via

$$\begin{aligned}\pi_\varphi(C) := \mathbb{E}[\varphi(\mathbf{X})] &= \int_{\mathbb{R}^d} \varphi(x_1, \ldots, x_d)\, \mathrm{d}C(F_1(x_1), \ldots, F_d(x_d)) \\ &= \int_{\mathbb{I}^d} \varphi(F_1^{-1}(u_1,), \ldots, F_d^{-1}(u_d))\, \mathrm{d}C(u_1, \ldots, u_d). \end{aligned} \tag{3.1}$$

This definition however no longer applies when C is merely a quasi-copula since the integral in (3.1), and in particular the term $\mathrm{d}C$, are no longer well defined. This is due to the fact that a quasi-copula C does not necessarily induce a (signed) measure $\mathrm{d}C$ to integrate against. Therefore, we establish a multivariate integration-by-parts formula which allows for an alternative representation of $\pi_\varphi(C)$ that is suitable for quasi-copulas. Similar representations were obtained by Rüschendorf [52] for a Δ-monotonic function φ fulfilling certain boundary conditions, and by Tankov [64] for general Δ-monotonic $\varphi\colon \mathbb{R}^2_+ \to \mathbb{R}$. In addition, we establish properties of the function φ such that the extended map $\mathcal{Q}^d \ni Q \mapsto \pi_\varphi(Q)$ is monotonic with respect to the lower orthant order on the sets $\mathcal{Q}^d$ and $\widehat{\mathcal{Q}}^d$.

Rüschendorf [52] and Müller and Stoyan [39] show that for φ being Δ-antitonic or Δ-monotonic the map $\mathcal{C}^d \ni C \mapsto \pi_\varphi(C)$ is increasing with respect to $\leq$ on $\mathcal{C}^d$ and $\widehat{\mathcal{C}}^d$ respectively. We define Δ-antitonic and Δ-monotonic functions as follows.

Definition 3.1.1. A function $\varphi\colon \mathbb{R}_+^d \to \mathbb{R}$ is called Δ-*antitonic* if for every subset $\{i_1, \dots, i_n\} \subseteq \{1, \dots, d\}$ with $n \geq 2$ and every hypercube $\times_{j=1}^n (a_j, b_j] \subset \mathbb{R}_+^n$ it holds that

$$(-1)^n \Delta_{a_1,b_1}^{i_1} \circ \cdots \circ \Delta_{a_n,b_n}^{i_n} \varphi(\mathbf{x}) \geq 0 \quad \text{for all } \mathbf{x} \in \mathbb{R}_+^d.$$

Analogously, φ is called Δ-*monotonic* if for every subset $\{i_1, \dots, i_n\} \subseteq \{1, \dots, d\}$ with $n \geq 2$ and every hypercube $\times_{j=1}^n (a_j, b_j] \subset \mathbb{R}_+^n$ it holds

$$\Delta_{a_1,b_1}^{i_1} \circ \cdots \circ \Delta_{a_n,b_n}^{i_n} \varphi(\mathbf{x}) \geq 0 \quad \text{for all } \mathbf{x} \in \mathbb{R}_+^d.$$

As a consequence of Theorem 3.3.15 in Müller and Stoyan [39] we have for $\underline{C}, \overline{C} \in \mathcal{C}^d$ with $\underline{C} \leq \overline{C}$ that $\pi_\varphi(\underline{C}) \leq \pi_\varphi(\overline{C})$ for all bounded Δ-antitonic functions φ. Moreover, if for the corresponding survival functions it holds that $\widehat{\underline{C}} \leq \widehat{\overline{C}}$, then it follows that $\pi_\varphi(\underline{C}) \leq \pi_\varphi(\overline{C})$ for all bounded Δ-monotonic functions φ. In order to formulate analogous results for the case when $\underline{C}, \overline{C}$ are quasi-copulas, we first note that each right-continuous Δ-monotonic or Δ-antitonic function $\varphi\colon \mathbb{R}_+^d \to \mathbb{R}$ induces a possibly signed measure on the Borel σ-Algebra of $\mathbb{R}_+^d$, which we denote by μ_φ; see Lemma 3.5 and Theorem 3.6 in Gaffke [22]. It holds in particular that

$$\mu_\varphi((a_1, b_1] \times \cdots \times (a_d, b_d]) = V_\varphi((a_1, b_1] \times \cdots \times (a_d, b_d]), \tag{3.2}$$

for hypercubes $[a_1, b_1] \times \cdots \times [a_d, b_d] \subset \mathbb{R}_+^d$. Next we define for a subset $I = \{i_1, \dots, i_n\} \subset \{1, \dots, d\}$ the *I-margin* of φ via

$$\varphi_I\colon \mathbb{R}_+^n \ni (x_{i_1}, \dots, x_{i_n}) \mapsto \varphi(x_1, \dots, x_d), \text{ with } x_k = 0 \text{ for all } k \notin I,$$

and the associated *I-marginal measure* μ_{φ_I}, for which

$$\mu_{\varphi_I}((a_{i_1}, b_{i_1}] \times \cdots \times (a_{i_n}, b_{i_n}]) = V_{\varphi_I}((a_{i_1}, b_{i_1}] \times \cdots \times (a_{i_n}, b_{i_n}]).$$

Note that if $I = \{1, \dots, d\}$ then μ_{φ_I} equals μ_φ, while if $I \subset \{1, \dots, d\}$ then μ_{φ_I} can be viewed as a marginal measure of μ_φ. Now define iteratively

$$\begin{aligned}
\text{for } |I| = 1: \quad & \mathcal{L}_\varphi^I(C) := \int_{\mathbb{R}_+} \varphi_{\{i_1\}}(x_{i_1}) \, \mathrm{d}F_{i_1}(x_{i_1}); \\
\text{for } |I| = 2: \quad & \mathcal{L}_\varphi^I(C) := -\varphi(0, ..., 0) + \mathcal{L}_\varphi^{\{i_1\}}(C) + \mathcal{L}_\varphi^{\{i_2\}}(C) \\
& \qquad + \int_{\mathbb{R}_+^2} \widehat{C}_I(F_{i_1}(x_{i_1}), F_{i_2}(x_{i_2})) \, \mathrm{d}\mu_{\varphi_I}(x_{i_1}, x_{i_2});
\end{aligned}$$

$$\text{for } |I| = n > 2: \quad \mathcal{L}_\varphi^I(C) := \int_{\mathbb{R}_+^{|I|}} \widehat{C}_I(F_{i_1}(x_{i_1}), \dots, F_{i_n}(x_{i_n}))\, \mathrm{d}\mu_{\varphi_I}(x_{i_1}, \dots, x_{i_n}) + \sum_{\substack{J \subset \{1,\dots,n\} \\ J \neq \emptyset}} (-1)^{|J|+1} \mathcal{L}_\varphi^J(C),$$

where $\widehat{C}_I$ denotes the I-margin of the survival function of C. Moreover, in order to simplify the notation in later applications we define for a survival function $\widehat{C}$ of a d-copula C the dual operator $\widehat{\mathcal{L}}_\varphi^J(\widehat{C}) := \mathcal{L}_\varphi^J(C)$. The following Proposition shows that $\mathcal{L}_\varphi^{\{1,\dots,d\}}$ and $\widehat{\mathcal{L}}_\varphi^{\{1,\dots,d\}}$ are alternative representations of the map π_φ, in the sense that

$$\pi_\varphi(C) = \mathcal{L}_\varphi^{\{1,\dots,d\}}(C) = \widehat{\mathcal{L}}_\varphi^{\{1,\dots,d\}}(\widehat{C})$$

for all copulas C.

Proposition 3.1.2. *Let $\varphi\colon \mathbb{R}_+^d \to \mathbb{R}$ be right-continuous, Δ-monotonic or Δ-antitonic and C be a d-copula. Then $\pi_\varphi(C) = \mathcal{L}_\varphi^{\{1,\dots,d\}}(C) = \widehat{\mathcal{L}}_\varphi^{\{1,\dots,d\}}(\widehat{C})$.*

Proof. Note first that the equality $\mathcal{L}_\varphi^{\{1,\dots,d\}}(\widehat{C}) = \widehat{\mathcal{L}}_\varphi^{\{1,\dots,d\}}(\widehat{C})$ follows immediately from the definition of $\mathcal{L}_\varphi^{\{1,\dots,d\}}$ and $\widehat{\mathcal{L}}_\varphi^{\{1,\dots,d\}}$. It thus suffices to show that $\pi_\varphi(C) = \mathcal{L}_\varphi^{\{1,\dots,d\}}(C)$. To this end, assume first that $\varphi(x_1, \dots, x_d) = V_\varphi((0, x_1] \times \cdots \times (0, x_d])$ for all $(x_1, \dots, x_d) \in \mathbb{R}_+^d$. An application of Fubini's Theorem yields directly that

$$\begin{aligned}
\pi_\varphi(C) &= \int_{\mathbb{R}_+^d} \varphi(x_1, \dots, x_d)\, \mathrm{d}C(F_1(x_1), \dots, F_d(x_d)) \\
&= \int_{\mathbb{R}_+^d} V_\varphi((0, x_1] \times \cdots \times (0, x_d])\, \mathrm{d}C(F_1(x_1), \dots, F_d(x_d)) \\
&= \int_{\mathbb{R}_+^d} \mu_\varphi((0, x_1] \times \cdots \times (0, x_d])\, \mathrm{d}C(F_1(x_1), \dots, F_d(x_d)) \\
&= \int_{\mathbb{R}_+^d} \left(\int_{\mathbb{R}_+^d} \mathbb{1}_{x_1' < x_1} \cdots \mathbb{1}_{x_d' < x_d} \mathrm{d}\mu_\varphi(x_1', \dots, x_d') \right) \mathrm{d}C(F_1(x_1), \dots, F_d(x_d)) \\
&= \int_{\mathbb{R}_+^d} \left(\int_{\mathbb{R}_+^d} \mathbb{1}_{x_1' > x_1} \cdots \mathbb{1}_{x_d' > x_d} \mathrm{d}C(F_1(x_1'), \dots, F_d(x_d')) \right) \mathrm{d}\mu_\varphi(x_1, \dots, x_d) \\
&= \int_{\mathbb{R}_+^d} \widehat{C}(F_1(x_1), \dots, F_d(x_d))\, \mathrm{d}\mu_\varphi(x_1, \dots, x_d),
\end{aligned} \tag{3.3}$$

where the last equality follows from (1.5). Next, we drop the assumption $\varphi(x_1, \dots, x_d) = V_\varphi((0, x_1] \times \cdots \times (0, x_d])$ and show that the general statement holds by induction over the dimension d. By Proposition 2 in [64] we know that the statement is valid for $d = 2$. Now, assume it holds true for $d = n-1$, then for $d = n$ we have that

$$\varphi(x_1,\dots,x_n) = V_\varphi((0,x_1]\times\cdots\times(0,x_n]) - [V_\varphi((0,x_1]\times\cdots\times(0,x_n]) - \varphi(x_1,\dots,x_n)].$$

Moreover, the term $-[V_\varphi((0,x_1]\times\cdots\times(0,x_n]) - \varphi(x_1,\dots,x_n)]$ can be expressed as

$$\begin{aligned}
&-[V_\varphi((0,x_1]\times\cdots\times(0,x_n]) - \varphi(x_1,\dots,x_n)]\\
&\quad= \sum_{(y_1,\dots,y_d)\in\times_{i=1}^n\{0,x_i\}} -(-1)^{|\{i\in\mathbb{N}\colon\, y_i=0\}|}\varphi(y_1,...,y_n) + \varphi(x_1,...,x_n)\\
&\quad= \sum_{(y_1,\dots,y_d)\in\times_{i=1}^n\{0,x_i\}} (-1)^{|\{i\in\mathbb{N}\colon\, y_i=0\}|+1}\varphi(y_1,...,y_n) + \varphi(x_1,...,x_n)\\
&\quad= \sum_{J\subseteq\{1,\dots,n\}} (-1)^{|J|+1}\varphi_J(x_1,...,x_d) + \varphi(x_1,...,x_n)\\
&\quad= \sum_{\substack{J\subseteq\{1,\dots,n\}\\ J\neq\emptyset}} (-1)^{|J|+1}\varphi_J(x_1,...,x_n),
\end{aligned}\tag{3.4}$$

It is hence the sum of functions φ_J, each with domain $\mathbb{R}_+^{n-|J|}$ and $n-|J|\le n-1$ due to $J\neq\emptyset$. It follows

$$\begin{aligned}
\pi_\varphi(C) &= \int \varphi(x_1,\dots,x_n)\,\mathrm{d}C(F_1(x_1),\dots,F_n(x_n))\\
&= \int V_\varphi((0,x_1]\times\cdots\times(0,x_n])\,\mathrm{d}C(F_1(x_1),\dots,F_n(x_n))\\
&\quad - \int [V_\varphi((0,x_1]\times\cdots\times(0,x_n]) - \varphi(x_1,\dots,x_n)]\,\mathrm{d}C(F_1(x_1),\dots,F_n(x_n))\\
&= \int \widehat{C}(F_1(x_1),\dots,F_n(x_n))\,\mathrm{d}\mu_\varphi(x_1,\dots,x_n)\\
&\quad + \int [-V_\varphi((0,x_1]\times\cdots\times(0,x_n]) + \varphi(x_1,\dots,x_n)]\,\mathrm{d}C(F_1(x_1),\dots,F_n(x_n))\\
&= \int \widehat{C}(F_1(x_1),\dots,F_n(x_n))\,\mathrm{d}\mu_\varphi(x_1,\dots,x_n)\\
&\quad + \int \Big[\sum_{\substack{J\subseteq\{1,\dots,n\}\\ J\neq\emptyset}} (-1)^{|J|+1}\varphi_J(x_1,...,x_n)\Big]\,\mathrm{d}C(F_1(x_1),\dots,F_n(x_n))\\
&= \int \widehat{C}(F_1(x_1),\dots,F_n(x_n))\,\mathrm{d}\mu_\varphi(x_1,\dots,x_n) + \sum_{\substack{J\subseteq\{1,\dots,n\}\\ J\neq\emptyset}} (-1)^{|J|+1}\mathcal{L}_\varphi^J(C),
\end{aligned}$$

where we apply equation (3.3) to $\int V_\varphi((0,x_1]\times\cdots\times(0,x_n])\,\mathrm{d}C(F_1(x_1),\dots,F_n(x_n))$ in order to obtain the third equality, and the fourth equality follows from (3.4). Finally we use the induction hypothesis to obtain the last equality as for each $J\subseteq\{1,\dots,n\}$ with $J\neq\emptyset$ the domain of φ_J is $\mathbb{R}^{n-|J|}$ with $n-|J|\le n-1$. □

Proposition 3.1.2 enables us to extend the notion of the expectation operator π_φ to quasi-copulas and establish monotonicity properties for the generalised mapping.

Definition 3.1.3. Let $\varphi\colon \mathbb{R}_+^d \to \mathbb{R}$ be right-continuous and Δ-antitonic or Δ-monotonic. Then, the *quasi-expectation operator* for $Q \in \mathcal{Q}^d$ is defined via

$$\pi_\varphi(Q) := \int_{\mathbb{R}_+^d} \widehat{Q}(F_1(x_1), \ldots, F_d(x_d))\,\mathrm{d}\mu_\varphi(x_1, \ldots, x_d) + \sum_{\substack{J\subseteq\{1,\ldots,n\}\\ J\neq\emptyset}} (-1)^{|J|+1}\mathcal{L}_\varphi^J(Q).$$

Moreover, we define for a quasi-survival function $\widehat{Q} \in \widehat{\mathcal{Q}}^d$ the dual operator by

$$\widehat{\pi}_\varphi(\widehat{Q}) := \int_{\mathbb{R}_+^d} \widehat{Q}(F_1(x_1), \ldots, F_d(x_d))\,\mathrm{d}\mu_\varphi(x_1, \ldots, x_d) + \sum_{\substack{J\subseteq\{1,\ldots,n\}\\ J\neq\emptyset}} (-1)^{|J|+1}\widehat{\mathcal{L}}_\varphi^J(\widehat{Q}).$$

Lemma 3.1.4. *Let $\varphi(x_1, ..., x_d) = \mathbb{1}_{x_1\leq y_1,\ldots,x_d\leq y_d}$ for $(y_1, ..., y_d) \in \mathbb{R}_+^d$. Then it holds for every quasi-copula $Q \in \mathcal{Q}^d$ that $\pi_\varphi(Q) = Q(F_1(y_1), ..., F_d(y_d))$.*

Proof. For ease of notation we first recall the notion of the lift operator $\mathbf{u}_J$ for a subset $J \subseteq \{1, ..., d\}$ and $\mathbf{u} \in [0, 1]^d$: $\mathbf{u}_J = (v_1, ..., v_d)$ with $v_i = u_i$ when $i \in J$ and $v_i = 1$ for $i \notin J$. We now show by induction that for every $J \subset \{1, ..., d\}$ it holds that $\mathcal{L}_\varphi^J(Q) = Q\big((F_1(y_1), ..., F_d(y_d))_J\big)$. The assertion then follows immediately by choosing $J = \{1, ..., d\}$.

When $|J| = 2$ we have

$$\begin{aligned}
\pi_\varphi(Q) &= -\varphi(0, ..., 0) + \mathcal{L}_\varphi^{\{i_1\}}(Q) + \mathcal{L}_\varphi^{\{i_2\}}(Q) + \int_{\mathbb{R}_+^2} \widehat{Q}_J(F_{i_1}(x_{i_1}), F_{i_2}(x_{i_2}))\,\mathrm{d}\mu_{\varphi_J}(x_{i_1}, x_{i_2}) \\
&= -1 + Q(1, ..., 1, F_{i_1}(y_{i_1}), 1, ..., 1) + Q(1, ..., 1, F_{i_2}(y_{i_2}), 1, ..., 1) \\
&\quad + \widehat{Q}_J(F_{i_1}(u_{i_1}), F_{i_2}(y_{i_2})) \\
&= -1 + F_{i_1}(y_{i_1}) + F_{i_2}(y_{i_2}) + \widehat{Q}(0, ..., 0, F_{i_1}(y_{i_1}), 0, ..., 0, F_{i_2}(y_{i_2}), 0, ..., 0) \\
&= Q\big((F_1(y_1), ..., F_d(y_d))_J\big),
\end{aligned} \tag{3.5}$$

where we apply the fact that $\varphi(0, ..., 0) = 1$ together with the definition of the survival function $\widehat{Q}$, as well as

$$\begin{aligned}
\mathcal{L}_\varphi^{\{i_1\}}(Q) &= \int_{\mathbb{R}_+} \varphi(0, ..., 0, x_{i_1}, 0, ..., 0)\,\mathrm{d}F_{i_1}(x_{i_1}) = \int_{\mathbb{R}_+} \mathbb{1}_{x_{i_1}\leq y_{i_1}}\,\mathrm{d}F_{i_1}(x_{i_1}) \\
&= F_{i_1}(y_{i_1}) = Q(1, ..., 1, F_{i_1}(y_{i_1}), 1, ..., 1),
\end{aligned}$$

and analogously for $\mathcal{L}_\varphi^{\{i_2\}}(Q)$. Hence, the assertion holds for $|J| = d-1$. When $|J| = d$ we obtain

$$\begin{aligned}
\pi_\varphi(Q) &= \int_{\mathbb{R}_+^d} \widehat{Q}(F_1(x_1), \dots, F_d(x_d))\, \mathrm{d}\mu_\varphi(x_1, \dots, x_d) + \sum_{\substack{J \subseteq \{1,\dots,d\} \\ J \neq \emptyset}} (-1)^{|J|+1} \mathcal{L}_\varphi^J(Q) \\
&= (-1)^d \Big\{ \widehat{Q}(F_1(y_1), ..., F_d(y_d)) - [V_Q((F_1(y_1), 1] \times \cdots \times (F_d(y_d), 1]) \\
&\quad + (-1)^d Q(F_1(y_1), ..., F_d(y_d))] \Big\} \\
&= (-1)^d \Big\{ (-1)^d Q(F_1(y_1), ..., F_d(y_d)) \Big\} = Q(F_1(y_1), ..., F_d(y_d)),
\end{aligned}$$

where we apply the induction hypothesis and the definition of the Q-volume V_Q to obtain the second equality and the third equality follows immediately from the definition of the survival function, i.e. $\widehat{Q}(F_1(y_1), ..., F_d(y_d)) = V_Q((F_1(y_1), 1] \times \cdots \times (F_d(y_d), 1])$. The proof is hence complete. $\square$

Theorem 3.1.5. *Let $\underline{Q}, \overline{Q} \in \mathcal{Q}^d$ and $\widehat{\underline{Q}}, \widehat{\overline{Q}} \in \widehat{\mathcal{Q}}^d$, then it holds*

$$(i) \quad \underline{Q} \leq \overline{Q} \quad \Longrightarrow \quad \pi_\varphi(\underline{Q}) \leq \pi_\varphi(\overline{Q}) \qquad \text{for all } \Delta\text{-antitonic } \varphi\colon \mathbb{R}_+^d \to \mathbb{R} \text{ s.t. the integrals exist;}$$

$$(ii) \quad \widehat{\underline{Q}} \leq \widehat{\overline{Q}} \quad \Longrightarrow \quad \widehat{\pi}_\varphi(\widehat{\underline{Q}}) \leq \widehat{\pi}_\varphi(\widehat{\overline{Q}}) \qquad \text{for all } \Delta\text{-monotonic } \varphi\colon \mathbb{R}_+^d \to \mathbb{R} \text{ s.t. the integrals exist.}$$

Moreover, if $F_1, \dots, F_d$ are continuous then the converse statements are also true.

Proof. We prove the statements assuming that the condition $\varphi(x_1, \dots, x_d) = V_\varphi((0, x_1] \times \cdots \times (0, x_d])$ holds. The general case follows then by induction as in the proof of Proposition 3.1.2. Let φ be Δ-antitonic and $\underline{Q} \leq \overline{Q}$, then it follows

$$\begin{aligned}
\pi_\varphi(\underline{Q}) &= \int_{\mathbb{R}_+^d} \widehat{\underline{Q}}(F_1(x_1), \dots, F_d(x_d))\, \mathrm{d}\mu_\varphi(x_1, \dots, x_d) \\
&= \int_{\mathbb{R}_+^d} V_{\underline{Q}}\Big((F_1(x_1), 1] \times \cdots \times (F_d(x_d), 1] \Big)\, \mathrm{d}\mu_\varphi(x_1, \dots, x_d) \\
&= \int_{\mathbb{R}_+^d} \Big\{ \underline{Q}(1, \dots, 1) - \underline{Q}(F_1(x_1), 1, \dots, 1) - \cdots - \underline{Q}(1, \dots, 1, F_d(x_d)) \\
&\quad + \underline{Q}(F_1(x_1), F_2(x_2), 1, \dots, 1) + \cdots + \underline{Q}(1, \dots, 1, F_{d-1}(x_{d-1}), F_d(x_d)) \\
&\quad - \cdots + (-1)^d \underline{Q}(F_1(x_1), \dots., F_d(x_d)) \Big\}\, \mathrm{d}\mu_\varphi(x_1, \dots, x_d)
\end{aligned}$$

$$
\begin{aligned}
&= \int_{\mathbb{R}_+^d} \Big\{ \underline{Q}(1,\dots,1) + \underline{Q}(F_1(x_1),1,\dots,1) + \dots + \underline{Q}(1,\dots,1,F_d(x_d)) \\
&\quad + \underline{Q}(F_1(x_1),F_2(x_2),1,\dots,1) + \dots + \underline{Q}(1,\dots,1,F_{d-1}(x_{d-1}),F_d(x_d)) \\
&\quad + \dots + \underline{Q}(F_1(x_1),\dots,F_d(x_d)) \Big\} \,\mathrm{d}|\mu_\varphi|(x_1,\dots,x_d),
\end{aligned}
$$

where for the last equality we use that φ is Δ-antitonic, hence μ_φ has alternating signs. A similar representation holds for $\pi_\varphi(\overline{Q})$, thus

$$
\begin{aligned}
&\pi_\varphi(\overline{Q}) - \pi_\varphi(\underline{Q}) \\
&= \int_{\mathbb{R}^d} \Big\{ \Big[\overline{Q}(F_1(x_1),1,\dots,1) - \underline{Q}(F_1(x_1),1,\dots,1)\Big] + \cdots \\
&\quad + \Big[\overline{Q}(1,\dots,1,F_d(x_d) - \underline{Q}(1,\dots,1,F_d(x_d))\Big] \\
&\quad + \Big[\overline{Q}(F_1(x_1),F_2(x_2),1,\dots,1) - \underline{Q}(F_1(x_1),F_2(x_2),1,\dots,1)\Big] + \cdots \\
&\quad + \Big[\overline{Q}(1,\dots,1,F_{d-1}(x_{d-1}),F_d(x_d)) - \underline{Q}(1,\dots,1,F_{d-1}(x_{d-1}),F_d(x_d))\Big] + \cdots \\
&\quad + \Big[\overline{Q}(F_1(x_1),\dots,F_d(x_d)) - \underline{Q}(F_1(x_1),\dots,F_d(x_d))\Big\} \,\mathrm{d}|\mu_\varphi|(x_1,\dots,x_d) \ \geq 0,
\end{aligned}
$$

since $\underline{Q} \leq \overline{Q}$. Hence assertion (i) is true. Regarding (ii), we have directly that

$$
\begin{aligned}
&\widehat{\pi}_\varphi(\widehat{\overline{Q}}) - \widehat{\pi}_\varphi(\widehat{\underline{Q}}) \\
&\quad = \int_{\mathbb{R}^d} \widehat{\overline{Q}}(F_1(x_1),\dots,F_d(x_d)) - \widehat{\underline{Q}}(F_1(x_1),\dots,F_d(x_d)) \,\mathrm{d}\mu_\varphi(x_1,\dots,x_d) \geq 0,
\end{aligned}
$$

where we used that φ is Δ-monotonic, thus μ_φ is a positive measure, as well as $\widehat{\underline{Q}} \leq \widehat{\overline{Q}}$.

As for the converse statements, assume that $F_1,\dots,F_d$ are continuous. If $\pi_\varphi(\underline{Q}) \leq \pi_\varphi(\overline{Q})$ holds for all Δ-antitonic φ, then it holds in particular for all φ of the form $\varphi(x_1,\dots,x_d) = \mathbb{1}_{x_1\leq u_1,\dots,x_d\leq u_d}$ for arbitrary $(u_1,\dots,u_d) \in (0,\infty]^d$. An application of Lemma 3.1.4 then yields

$$
\pi_\varphi(\underline{Q}) \leq \pi_\varphi(\overline{Q}) \quad \Longrightarrow \quad \underline{Q}(F_1(u_1),\dots,F_d(u_d)) \leq \overline{Q}(F_1(u_1),\dots,F_d(u_d)),
$$

while from the fact that $\pi_\varphi(\underline{Q}) \leq \pi_\varphi(\overline{Q})$ holds for all choices of $(u_1,\dots,u_d)$ and the continuity of the marginals it follows that (i) holds. Assertion (ii) follows by an analogous argument. To this end, note that if $\widehat{\pi}_\varphi(\widehat{\underline{Q}}) \leq \widehat{\pi}_\varphi(\widehat{\overline{Q}})$ holds for all Δ-monotonic φ, then it holds in particular for φ of the form $\varphi(x_1,\dots,x_d) = \mathbb{1}_{x_1\geq u_1,\dots,x_d\geq u_d}$ for arbitrary $(u_1,\dots,u_d) \in (0,\infty]^d$, so that $\widehat{\underline{Q}}(F_1(u_1),\dots,F_d(u_d)) \leq \widehat{\overline{Q}}(F_1(u_1),\dots,F_d(u_d))$. $\square$

Finally, we provide an integrability condition for the extended map $\pi_\varphi(\cdot)$ based on the marginals $F_1, \ldots, F_d$ and the properties of the function φ. In particular, the finiteness of $\pi_\varphi(C)$ is independent of C being a copula or a proper quasi-copula.

Proposition 3.1.6. *Let $\varphi\colon \mathbb{R}_+^d \to \mathbb{R}$ be right-continuous, Δ-antitonic or Δ-monotonic such that*

$$\sum_{J\subset\{1,\ldots,d\}} \sum_{i=1}^{d} \left\{ \int_{\mathbb{R}_+} |\varphi_J(x,\ldots,x)| \,\mathrm{d}F_i(x) \right\} < \infty. \tag{3.6}$$

Then the maps π_φ and $\widehat{\pi}_\varphi$ are well-defined and continuous with respect to the pointwise convergence of quasi-copulas.

Proof. First, we show that for $C \in \mathcal{C}^d$ the expectation

$$\int_{\mathbb{R}_+^d} \varphi(x_1, \ldots, x_d) \,\mathrm{d}C(F_1(x_1), \ldots, F_d(x_d))$$

is finite by induction over the dimension d. By [64, Proposition 2] we know that the statement is true for $d = 2$. Assume that the statement holds for $d = n - 1$, then for $d = n$ we have

$$\begin{aligned}
&|\varphi(x_1, \ldots, x_n)| \\
&\quad = |V_\varphi((0,x_1] \times \cdots \times (0,x_n]) - [V_\varphi((0,x_1] \times \cdots \times (0,x_n]) - \varphi(x_1, \ldots, x_n)]| \\
&\quad \leq |V_\varphi((0,x_1] \times \cdots \times (0,x_n])| + |V_\varphi((0,x_1] \times \cdots \times (0,x_n]) - \varphi(x_1, \ldots, x_n)| \\
&\quad \leq |V_\varphi((0,x_1]^n)| + \cdots + |V_\varphi((0,x_n]^n)| + |V_\varphi((0,x_1] \times \cdots \times (0,x_n]) - \varphi(x_1, \ldots, x_n)| \\
&\quad \leq \sum_{i=1}^{n} \sum_{J\subset\{1,\ldots,n\}} |\varphi_J(x_i, \ldots, x_i)| + |V_\varphi((0,x_1] \times \cdots \times (0,x_n]) - \varphi(x_1, \ldots, x_n)| \\
&\quad \leq \sum_{i=1}^{n} \sum_{J\subset\{1,\ldots,n\}} |\varphi_J(x_i, \ldots, x_i)| + \text{const} \cdot \sum_{J\subset\{1,\ldots,n\}} |\varphi_J(x_1, \ldots, x_n)|,
\end{aligned} \tag{3.7}$$

where the second inequality follows from the definition of V_φ and $\times_{i=1}^n (0, x_i] \subseteq \bigcup_{i=1}^n (0, x_i]^n$. Now, note that for $J \subset \{1, \ldots, n\}$, φ is a function with domain $\mathbb{R}_+^{|J|}$ where $|J| < n$, hence by the induction hypothesis and (3.6) we get that

$$\int_{\mathbb{R}_+^{|J|}} |\varphi_J(x_1, \ldots, x_n)| \,\mathrm{d}C_J(F_1(x_1), \ldots, F_n(x_n)) < \infty$$

for each $J \subset \{1, \ldots, n\}$, where $|J| \leq n-1$. Hence

$$b := \text{const} \cdot \sum_{J \subset \{1,\ldots,n\}} \left\{ \int_{\mathbb{R}_+^{|J|}} |\varphi_J(x_1, \ldots, x_n)| \, \mathrm{d}C_J(F_1(x_1), \ldots, F_n(x_n)) \right\} < \infty.$$

Finally, from (3.6) and (3.7) we obtain

$$\int_{\mathbb{R}_+^n} |\varphi(x_1, \ldots, x_n)| \, \mathrm{d}C(F_1(x_1), \ldots, F_n(x_n)) \leq \sum_{J \subset \{1,\ldots,n\}} \sum_{i=1}^n \left\{ \int_{\mathbb{R}_+} |\varphi_J(x, \ldots, x)| \, \mathrm{d}F_i(x) \right\} + b < \infty.$$

Hence the assertion is true for $\mathcal{C}^d \ni C \mapsto \pi_\varphi(C)$. Now for the extended map, let Q be a proper quasi-copula and assume that φ is Δ-antitonic. Then it follows from Theorem 3.1.5 and the properties of the upper Fréchet–Hoeffding bound that $0 \leq \pi_\varphi(Q) \leq \pi_\varphi(M_d) < \infty$, where the finiteness of $\pi_\varphi(M_d)$ follows from the fact that $M_d \in \mathcal{C}^d$. By the same token, since all quasi-copulas are bounded from above by the upper Fréchet–Hoeffding bound M_d and the integrals with respect to M_d exist, the dominated convergence theorem yields that π_φ is continuous with respect to the pointwise convergence of quasi-copulas. The well-definedness of $\widehat{\pi}_\varphi$ for Δ-monotonic φ follows analogously. □

3.2 Applications in model-free finance

A direct application of our stochastic dominance results for quasi-copulas is the computation of bounds on the prices of multi-asset options assuming that the marginal distributions of the assets fully known, while the dependence structure between them is only partially known. Partial information on the joint behaviour of several underlyings can be inferred from market prices of traded multi-asset derivatives such as spread or basket options. This information then translates into improved Fréchet–Hoeffding bounds by the methods presented in Section 2.1. Finally, by an application of the stochastic dominance results in section 3.1 we can relate the improved Fréchet–Hoeffding bounds to estimates on expectations, and hence option prices. The resulting option price bounds are *model-free* in the sense that they only depend on available market information and no model for the joint risk-free distribution of the assets is assumed. The literature on model-free bounds for multi-asset option prices focuses almost exclusively on basket options, see e.g. Hobson et al. [27, 28], d'Aspremont and El Ghaoui [14] and Peña et al. [42], while Tankov [64] considers general payoff functions in a two-dimensional setting.

We consider European-style options whose payoffs depend on a positive random vector $\mathbf{S} = (S_1, \dots, S_d)$. The constituents of $\mathbf{S}$ represent the values of the option's underlyings at the time of maturity. In the absence of arbitrage opportunities, the existence of a risk-neutral probability measure $\mathbb{Q}$ for $\mathbf{S}$ is guaranteed by the fundamental theorem of asset pricing. Then, the price of an option on $\mathbf{S}$ equals the discounted expectation of its payoff under a risk-neutral probability measure. We assume that all information about the risk-neutral distribution of $\mathbf{S}$ or its constituents comes from prices of traded derivatives on these assets, and that single-asset European call options with payoff $(S_i - K)^+$ for $i = 1, \dots, d$ and for all strikes $K > 0$ are liquidly traded in the market. Assuming zero interest rates, the prices of these options are given by $\Pi_K^i = \mathbb{E}_{\mathbb{Q}}[(S_i - K)^+]$. Using these prices, one can fully recover the risk neutral marginal distributions F_i of S_i as shown by Breeden and Litzenberger [11].

Let $\varphi\colon \mathbb{R}_+^d \to \mathbb{R}$ be the payoff of a European-style option on $\mathbf{S}$. Given the marginal risk-neutral distributions $F_1, \dots, F_d$ of $S_1, \dots, S_d$, the price of $\varphi(\mathbf{S})$ becomes a function of the copula C of $\mathbf{S}$ and is provided by the expectation operator as defined in (3.1), i.e.

$$\mathbb{E}_{\mathbb{Q}}[\varphi(S_1, \dots, S_d)] = \pi_\varphi(C).$$

Assuming that the only available information about the risk-neutral distribution of $\mathbf{S}$ is the marginal distributions, the set of all arbitrage-free prices for $\varphi(\mathbf{S})$ equals

$$\Pi := \{\pi_\varphi(C) \colon C \in \mathcal{C}^d\}.$$

Moreover, if additional information on the copula C is available, one can narrow the set of arbitrage-free prices by formulating respective constraints on the copula. Let therefore $\mathcal{C}^*$ represent any of the constrained sets of copulas from Section 2.1, and define the set of arbitrage-free prices compatible with the respective constraints via

$$\Pi^* := \{\pi_\varphi(C) \colon C \in \mathcal{C}^*\}.$$

Since $\mathcal{C}^* \subset \mathcal{C}$ we have immediately that $\Pi^* \subset \Pi$.

Theorem 3.1.5 yields that if the payoff φ is Δ-antitonic, then $\pi_\varphi(C)$ is monotonically increasing in the copula C with respect to the order $\leq$. Conversely if φ is Δ-monotonic, then $\pi_\varphi(C)$ is monotonically increasing in the survival-copula $\widehat{C}$ with respect to $\leq$. In the following result, we exploit this fact to compute bounds on the sets Π and Π^*.

Proposition 3.2.1. *Let φ be Δ-antitonic and $\mathcal{C}^*$ be a constrained set of copulas. Then if $\underline{Q}^*, \overline{Q}^* \in \mathcal{Q}^d$ are bounds on $\mathcal{C}^*$ in the sense that $\underline{Q}^* \leq C \leq \overline{Q}^*$ for all $C \in \mathcal{C}^*$, it follows that*

$$\pi_\varphi(W_d) \leq \pi_\varphi(\underline{Q}^*) \leq \inf \Pi^* \leq \pi_\varphi(C) \leq \sup \Pi^* \leq \pi_\varphi(\overline{Q}^*) \leq \pi_\varphi(M_d) = \sup \Pi$$

for all $C \in \mathcal{C}^*$ when the respective integrals exist. Moreover, it holds that $\inf \Pi = \pi_\varphi(W_d)$ *when $d = 2$. If φ is Δ-monotonic and $\underline{\widehat{Q}}^*, \overline{\widehat{Q}}^* \in \widehat{\mathcal{Q}}^d$ are bounds on $\mathcal{C}^*$ in the sense that $\underline{\widehat{Q}}^* \leq \widehat{C} \leq \overline{\widehat{Q}}^*$ for all $C \in \mathcal{C}^*$, then it follows that*

$$\widehat{\pi}_\varphi(W_d(\mathbf{1}-\cdot)) \leq \widehat{\pi}_\varphi(\underline{\widehat{Q}}^*) \leq \inf \Pi^* \leq \pi_\varphi(C) \leq \sup \Pi^* \leq \widehat{\pi}_\varphi(\overline{\widehat{Q}}^*) \leq \widehat{\pi}_\varphi(M_d(\mathbf{1}-\cdot)) = \sup \Pi$$

for all $C \in \mathcal{C}^*$ if the respective integrals exist and $\inf \Pi = \widehat{\pi}_\varphi(W_d(\mathbf{1} - \cdot))$ *holds if* $d = 2$.

Proof. Let $C \in \mathcal{C}^*$, then it holds that

$$W_d \leq \underline{Q}^* \leq C \leq \overline{Q}^* \leq M_d,$$

and the result follows from Theorem 3.1.5(i) for a Δ-antitonic function φ. Note that $\sup \Pi = \pi_\varphi(M_d)$ since the upper Fréchet–Hoeffding bound is a copula. The second statement follows analogously from the properties of the improved Fréchet–Hoeffding bounds on survival functions and an application of Theorem 3.1.5(ii). □

Remark 3.2.1. Let us point out that $\pi_\varphi(M_d)$ is an upper bound on the set of prices Π even under weaker conditions on φ than Δ-antitonicity. This is due to the fact that the upper Fréchet–Hoeffding bound is a copula and thus a sharp bound on the set of all copulas as well as the properties of the support of the measure $\mathrm{d}M_d$. Hobson et al. [27] e.g. derive upper bounds on basket options and show that they are attained by a comonotonic random vector having copula M_d. Moreover, Carlier [12] obtain bounds on Π for φ being monotonic of order 2 using an optimal transport approach. Moreover, he shows that the bounds are attained for a monotonic rearrangement of the a random vector which in turn leads to the upper Fréchet–Hoeffding bound. ♦

Remark 3.2.2. Let $\underline{Q}^*$ be any of the improved Fréchet–Hoeffding bounds from Section 2.1. Then the inequality

$$\inf \Pi \leq \pi_\varphi(\underline{Q}^*) \tag{3.8}$$

does not hold in general. In particular, the sharp bound $\inf \Pi$ without additional dependence information might exceed the price bound obtained using $\underline{Q}^*$. A sufficient condition for (3.8) to hold is the existence of a copula $C \in \mathcal{C}^d$ such that $C \leq \underline{Q}^*$. However, this condition is difficult to verify in practice. Moreover, in many cases $\inf \Pi$ cannot be computed analytically so that a direct comparison of the bounds is mostly not possible. On the other hand, one can resort to computational approaches in order to check whether (3.8) is satisfied. A numerical method to compute $\inf \Pi$ for continuous payoff functions φ, fulfilling a minor growth condition, based on assignment problems is presented in Preischl [43]. This approach thus lends itself for a direct comparison of the bounds. ♦

Let us recall that by Proposition 3.1.2 the computation of π_φ or $\widehat{\pi}_\varphi$ amounts to an integration with respect to the measure μ_φ that is induced by the function φ. Table 3.1 below provides some examples of payoff functions φ, such that φ or $-\varphi$ is Δ-antitonic or Δ-monotonic, along with explicit representations of the integrals $\int \widehat{Q}_I(F_{i_1}(x_{i_1}), \ldots, F_{i_n}(x_{i_n}))\, \mathrm{d}\mu_{\varphi_I} =: \int g(x_{i_1}, ..., x_{i_n})\, \mathrm{d}\mu_{\varphi_I}$ in π_φ and $\widehat{\pi}_\varphi$ for $I = \{i_1, ..., i_n\}$; see again Definition 3.1.3. An important observation here is that the multidimensional integrals with respect to the copula reduce to one-dimensional integrals, which makes the computation of option prices very fast and efficient.

Remark 3.2.3 (Differentiable payoffs). Assume that the payoff function is differentiable, i.e. the partial derivatives of the function φ exist, then we obtain the following representation for the integral with respect to μ_φ:

$$\int_{\mathbb{R}_+^d} g(x_1, \ldots, x_d)\, \mathrm{d}\mu_\varphi(x_1, \ldots, x_d) = \int_{\mathbb{R}_+^d} g(x_1, \ldots, x_d)\, \frac{\partial^d \varphi(x_1, \ldots, x_d)}{\partial x_1 \cdots \partial x_d}\, \mathrm{d}x_1 \cdots \mathrm{d}x_d.$$

The formula holds since from the definition of the volume V_φ we get that

$$V_\varphi(H) = \int_H \frac{\partial^d \varphi(x_1, \ldots, x_d)}{\partial x_1 \cdots \partial x_d}\, \mathrm{d}x_1 \cdots \mathrm{d}x_d,$$

for every H-box in $\mathbb{R}_+^d$. Differentiable Δ-antitonic functions occur in problems related to utility maximization; see e.g. the definition of Mixex Utility Functions in Tsetlin and Winkler [67]. ♦

Remark 3.2.4 (Basket and spread options). Although basket options on two underlyings are Δ-monotonic, their higher-dimensional counterparts, i.e. $\varphi\colon \mathbb{R}_+^d \ni (x_1, ..., x_d) \mapsto \left(\sum_{i=1}^d \alpha_i x_i - K\right)^+$ for $\alpha_i, ..., \alpha_d \in \mathbb{R}_+$, are neither Δ-monotonic nor Δ-antitonic in general. However, from the monotonicity of bivariate basket options it follows that their expectation is monotonic with respect to the lower and upper orthant order on the set of 2-copulas. Therefore, prices of bivariate basket options provide information that can

Payoff $\varphi(x_1,\dots,x_d)$	Δ-anti-/monotonic	$\int g(x_{i_1},\dots,x_{i_n})\,\mathrm{d}\mu_{\varphi_I}$
Digital put on maximum $\mathbb{1}_{\max\{x_1,\dots,x_d\}\le K}$	φ antitonic	$\begin{cases} g(K,\dots,K), & \lvert I\rvert \text{ even} \\ -g(K,\dots,K), & \lvert I\rvert \text{ odd} \end{cases}$
Digital call on minimum $\mathbb{1}_{\min\{x_1,\dots,x_d\}\le K}$	$-\varphi$ monotonic	$\begin{cases} -g(K,\dots,K), & I=\{1,\dots,d\} \\ 0, & \text{else} \end{cases}$
Call on minimum $(\min\{x_1,\dots,x_d\}-K)^+$	φ monotonic	$\begin{cases} \int_K^\infty g(x,\dots,x)\mathrm{d}x, & I=\{1,\dots,d\} \\ 0, & \text{else} \end{cases}$
Put on minimum $(K-\min\{x_1,\dots,x_d\})^+$	$-\varphi$ monotonic	$\begin{cases} \int_0^K g(x,\dots,x)\mathrm{d}x, & I=\{1,\dots,d\} \\ 0, & \text{else} \end{cases}$
Call on maximum $(\max\{x_1,\dots,x_d\}-K)^+$	$-\varphi$ antitonic	$\begin{cases} -\int_K^\infty g(x,\dots,x)\mathrm{d}x, & \lvert I\rvert \text{ even} \\ \int_K^\infty g(x,\dots,x)\mathrm{d}x, & \lvert I\rvert \text{ odd} \end{cases}$
Put on maximum $(K-\max\{x_1,\dots,x_d\})^+$	φ antitonic	$\begin{cases} -\int_0^K g(x,\dots,x)\mathrm{d}x, & \lvert I\rvert \text{ even} \\ \int_0^K g(x,\dots,x)\mathrm{d}x & \lvert I\rvert \text{ odd} \end{cases}$

Table 3.1: Examples of payoff functions for multi-asset options and the respective representation of the integral with respect to the measure μ_φ. The formulas for the digital call on the maximum and the digital put on the minimum can be obtained by a put-call parity.

be accounted for by Theorems 2.1.3 or 2.1.10. In particular, if $\varphi\colon \mathbb{R}_+^2 \ni (x_1,x_2) \mapsto (\alpha_1 x_1+\alpha_2 x_2-K)^+$ then φ is Δ-monotonic for $\alpha_1\alpha_2>0$, thus $\rho(C):=\pi_\varphi(C)$ is increasing with respect to the lower orthant order on $\mathcal{C}^2$. Analogously, if φ is a spread option, i.e. $\alpha_1\alpha_2<0$, then $\rho(C):=-\pi_\varphi(C)$ is increasing with respect to the lower orthant order on $\mathcal{C}^2$. Thus, by means of Theorem 2.1.3 one can translate market prices of basket or spread options into improved Fréchet–Hoeffding bounds for 2-copulas which may then serve as information to compute higher-dimensional bounds by means of Theorem 2.1.10. ♦

An interesting question arising naturally is under what conditions the bounds in Proposition 3.2.1 are *sharp*, in the sense that

$$\inf \Pi^* = \pi_\varphi(\underline{Q}^*) \quad \text{and} \quad \sup \Pi^* = \pi_\varphi(\overline{Q}^*), \tag{3.9}$$

and similarly for $\widehat{\pi}_\varphi(\underline{\widehat{Q}}^*)$ and $\widehat{\pi}_\varphi(\widehat{\overline{Q}}^*)$. In Section 2.2 we show that generally the improved Fréchet–Hoeffding bounds fail to be copulas, hence they are not sharp. However, introducing rather strong conditions on the function φ we still obtain sharpness of the price bounds in the sense of (3.9) when $\underline{Q}^*$ and $\overline{Q}^*$ are the improved Fréchet–Hoeffding bounds provided in Theorem 2.1.1. In order to formulate such conditions we introduce the notion of an increasing d-track as defined by Genest et al. [23].

Definition 3.2.2. Let $G_1, \dots, G_d$ be continuous, univariate distribution functions on $\overline{\mathbb{R}}$, such that $G_i(-\infty) = 0$ and $G_i(\infty) = 1$ for $i = 1, \dots, d$. Then

$$T^d := \{(G_1(x), \dots, G_d(x)) \colon x \in \overline{\mathbb{R}}\} \subset \mathbb{I}^d$$

is an (increasing) d-track in $\mathbb{I}^d$.

The following result establishes sharpness of the option price bounds, under conditions which are admittedly rather strong for practical applications.

Proposition 3.2.3. *Let $\varphi \colon \mathbb{R}^d_+ \to \mathbb{R}$ be right-continuous, Δ-monotonic and such that $\varphi(x_1, \dots, x_d) = V_\varphi((0, x_1] \times \cdots \times (0, x_d])$. Assume that*

$$\mathcal{B} := \{(F_1(x_1), \dots, F_d(x_d)) \colon \mathbf{x} \in \operatorname{supp} \mu_\varphi\} \subset T^d,$$

for some d-track T^d. Moreover, consider the upper and lower bounds $\underline{\widehat{Q}}^{\mathcal{S},\widehat{C}^*}, \widehat{\overline{Q}}^{\mathcal{S},\widehat{C}^*}$ from Corollary 2.1.2, for a copula C^*. Then, if $\mathcal{S} \subset T^d$ it follows that

$$\inf\{\pi_\varphi(C) \colon C \in \widehat{\mathcal{C}}^{\mathcal{S},\widehat{C}^*}\} = \widehat{\pi}_\varphi\Big(\underline{\widehat{Q}}^{\mathcal{S},\widehat{C}^*}\Big) \quad \text{and} \quad \sup\{\pi_\varphi(C) \colon C \in \widehat{\mathcal{C}}^{\mathcal{S},\widehat{C}^*}\} = \widehat{\pi}_\varphi\Big(\widehat{\overline{Q}}^{\mathcal{S},\widehat{C}^*}\Big),$$

where $\widehat{\mathcal{C}}^{\mathcal{S},\widehat{C}^*} := \{C \in \mathcal{C}^d \colon \widehat{C}(\mathbf{x}) = \widehat{C}^*(\mathbf{x}), \text{ for all } \mathbf{x} \in \mathcal{S}\}$.

Proof. Since $\mathbf{u} \mapsto \underline{\widehat{Q}}^{\mathcal{S},\widehat{C}^*}(\mathbf{1}-\mathbf{u})$ and $\mathbf{u} \mapsto \widehat{\overline{Q}}^{\mathcal{S},\widehat{C}^*}(\mathbf{1}-\mathbf{u})$ are quasi-copulas and $\mathcal{B}$ is a subset of a d-track T^d, it follows from the properties of a quasi-copula (c.f. Rodriguez-Lallena and Ubeda-Flores [50]) that there exist survival copulas $\underline{\widehat{C}}^{\mathcal{S},\widehat{C}^*}$ and $\widehat{\overline{C}}^{\mathcal{S},\widehat{C}^*}$ which coincide with $\underline{\widehat{Q}}^{\mathcal{S},\widehat{C}^*}$ and $\widehat{\overline{Q}}^{\mathcal{S},\widehat{C}^*}$ respectively on T^d. Hence, it follows for the lower bound

$$\begin{aligned}
\widehat{\pi}_\varphi\Big(\underline{\widehat{Q}}^{\mathcal{S},\widehat{C}^*}\Big) &= \int_{\mathbb{R}^d} \underline{\widehat{Q}}^{\mathcal{S},\widehat{C}^*}(F_1(x_1), \dots, F_d(x_d))\, \mathrm{d}\mu_\varphi(x_1, \dots, x_d) \\
&= \int_{\mathcal{B}} \underline{\widehat{Q}}^{\mathcal{S},\widehat{C}^*}(u_1, ..., u_d)\, \mathrm{d}\mu_\varphi(F_1^{-1}(u_1), \dots, F_d^{-1}(u_d))
\end{aligned}$$

$$= \int_{\mathcal{B}} \underline{\widehat{C}}^{\mathcal{S},\widehat{C}^*}(u_1,...,u_d)\,\mathrm{d}\mu_\varphi(F_1^{-1}(u_1),\dots,F_d^{-1}(u_d)) = \widehat{\pi}_\varphi\Big(\underline{\widehat{C}}^{\mathcal{S},\widehat{C}^*}\Big),$$

where we use the fact that $\varphi(x_1,\dots,x_d) = V_\varphi((0,x_1]\times\cdots\times(0,x_d])$ for the first equality and the second equality follows from the definition of $\mathcal{B}$ as well as $\underline{\widehat{Q}}^{\mathcal{S},\widehat{C}^*}$ and $\underline{\widehat{C}}^{\mathcal{S},\widehat{C}^*}$ being equal on T^d, and thus also on $\mathcal{B}$.

In addition, using that $\underline{\widehat{C}}^{\mathcal{S},\widehat{C}^*}$ is a copula that coincides with $\underline{\widehat{Q}}^{\mathcal{S},\widehat{C}^*}$ on T^d and $\underline{\widehat{Q}}^{\mathcal{S},\widehat{C}^*}(\mathbf{u}) = \widehat{C}^*(\mathbf{u})$ for $\mathbf{u}\in\mathcal{S}\subset T^d$, it follows that $\underline{\widehat{C}}^{\mathcal{S},\widehat{C}^*}\in\widehat{\mathcal{C}}^{\mathcal{S},\widehat{C}^*}$, hence by the Δ-monotonicity of φ we get that $\widehat{\pi}_\varphi(\underline{\widehat{C}}^{\mathcal{S},\widehat{C}^*}) = \inf\{\pi_\varphi(C)\colon C\in\widehat{\mathcal{C}}^{\mathcal{S},\widehat{C}^*}\}$. The proof for the upper bound can be obtained in the same way. $\square$

Finally, we are ready to apply our results in order to compute bounds on prices of multi-asset options when additional information on the dependence structure of $\mathbf{S}$ is available. The following examples illustrate this approach for different payoff functions and different kinds of additional information.

Example 3.2.1. Consider an option $\varphi(\mathbf{S})$ on three assets $\mathbf{S} = (S_1,S_2,S_3)$. We are interested in computing bounds on the price of $\varphi(\mathbf{S})$ assuming that partial information on the dependence structure of $\mathbf{S}$ is available. In particular, we assume that the marginal distributions $S_i\sim F_i$ are implied by the market prices of European call options. Moreover, we assume that partial information on the dependence structure stems from market prices of liquidly traded digital options of the form $\mathbb{1}_{\max\{S_i,S_j\}<K}$ for $(i,j) = (1,2),(1,3),(2,3)$ and $K\in\mathbb{R}_+$. The prices of such options are immediately related to the copula C of $\mathbf{S}$ since

$$\mathbb{E}_{\mathbb{Q}}[\mathbb{1}_{\max\{S_1,S_2\}<K}] = \mathbb{Q}(S_1<K,S_2<K,S_3<\infty) = C(F_1(K),F_2(K),1),$$

and analogously for $(i,j) = (1,3),(2,3)$, for some martingale measure $\mathbb{Q}$.

Considering a set of strikes $\mathcal{K} := \{K_1,\dots,K_n\}$, one can recover the values of the copula of $\mathbf{S}$ at several points. Let $\Pi_K^{(i,j)}$ denote the market price of a digital option on (S_i,S_j) with strike K. These market prices imply then the following prescription on the copula of $\mathbf{S}$:

$$\begin{aligned} C(F_1(K),F_2(K),1) &= \Pi_K^{(1,2)},\\ C(F_1(K),1,F_3(K)) &= \Pi_K^{(1,3)},\\ C(1,F_2(K),F_3(K)) &= \Pi_K^{(2,3)}, \end{aligned} \tag{3.10}$$

for $K\in\mathcal{K}$. Therefore, the collection of strikes induces a prescription on the copula on a compact subset of $\mathbb{I}^3$ of the form

$$\mathcal{S} = \bigcup_{K\in\mathcal{K}} \Big(F_1(K), F_2(K), 1\Big) \cup \Big(F_1(K), 1, F_3(K)\Big) \cup \Big(1, F_2(K), F_3(K)\Big).$$

The set of copulas that are compatible with this prescription is provided by

$$\mathcal{C}^{\mathcal{S},\Pi} = \Big\{C \in \mathcal{C}^3 \colon C(\mathbf{x}) = \Pi_K^{(i,j)} \text{ for all } \mathbf{x} \in \mathcal{S}\Big\};$$

see again (3.10). Hence, we can now employ Theorem 2.1.1 in order to compute the improved Fréchet–Hoeffding bounds on the set $\mathcal{C}^{\mathcal{S},\Pi}$ as follows:

$$\overline{Q}^{\mathcal{S},\Pi}(\mathbf{u}) = \min\Big(u_1, u_2, u_3, \min_{(i,j),K}\Big\{\Pi_K^{(i,j)} + \sum_{l=i,j}\Big(u_l - F_l(K)\Big)^+\Big\}\Big)$$

$$\underline{Q}^{\mathcal{S},\Pi}(\mathbf{u}) = \max\Big(0, \sum_{i=1}^{3} u_i - 2, \max_{\substack{(i,j),K,\\ q\in\{1,2,3\}\setminus\{i,j\}}}\Big\{\Pi_K^{(i,j)} - \sum_{l=i,j}\Big(F_l(K) - u_l\Big)^+ + (1-u_q)\Big\}\Big).$$

Observe that the minimum and maximum in the equations above are taken over the set $\mathcal{S}$, using simply a more convenient parametrization. Using these improved Fréchet–Hoeffding bounds, we can now apply Proposition 3.2.1 and compute bounds on the price of an option $\varphi(\mathbf{S})$ depending on all three assets. That is, we can compute bounds on the set of arbitrage-free option prices $\Big\{\pi_\varphi(C) \colon C \in \mathcal{C}^{\mathcal{S},\Pi}\Big\}$ which are compatible with the information stemming from pairwise digital options.

In order to illustrate our results, we consider a digital option depending on all three assets, i.e. $\varphi(\mathbf{S}) = \mathbb{1}_{\max\{S_1,S_2,S_3\}<K}$. In order to generate prices of pairwise digital options, we use the multivariate Black–Scholes model without drift, thus $\mathbf{S} = (S_1, S_2, S_3)$ is multivariate log-normally distributed with $S_i = s_i \exp(-\frac{1}{2} + X_i)$ where $(X_1, X_2, X_3) \sim \mathcal{N}(\mathbf{0}, \Sigma)$ with

$$\Sigma = \begin{pmatrix} 1 & \rho_{1,2} & \rho_{1,3} \\ \rho_{1,2} & 1 & \rho_{2,3} \\ \rho_{1,3} & \rho_{2,3} & 1 \end{pmatrix}.$$

The following figures show the improved price bounds on the 3-asset digital option as a function of the strike K, as well as the prices using the 'standard' Fréchet–Hoeffding bounds where we fix the value, $s_i = 10$. As a benchmark, we also include the prices in the Black–Scholes model. We consider two scenarios for the pairwise correlations: in the left plot $\rho_{i,j} = 0.3$ and in the right plot $\rho_{1,2} = 0.5$, $\rho_{1,3} = -0.5$, $\rho_{2,3} = 0$.

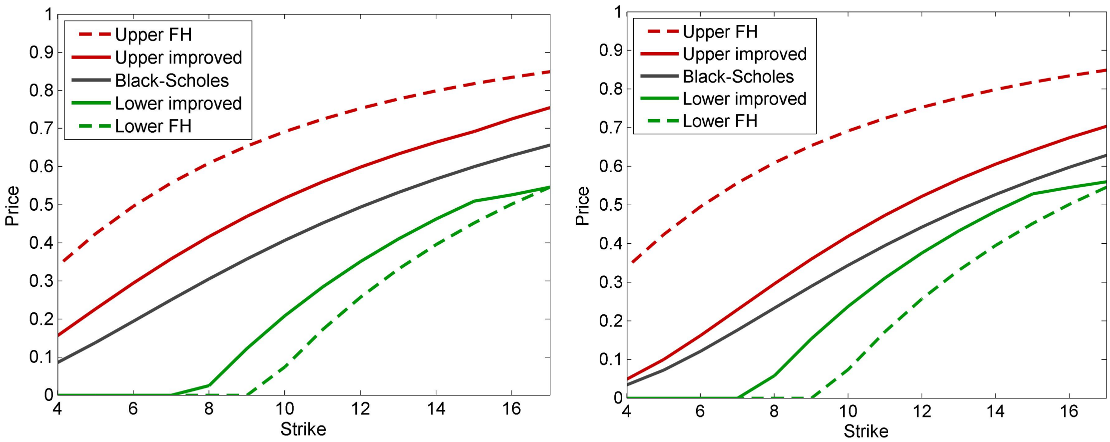

Figure 3.1: Bounds on the prices of 3-asset digital options as functions of the strike.

Observe that the improved Fréchet–Hoeffding bounds that account for the additional information from market prices of pairwise digital options lead in both cases to a considerable improvement of the option price bounds compared to the ones obtained with the 'standard' Fréchet–Hoeffding bounds. The improvement seems to be particularly pronounced if there are negative and positive correlations among the constituents of $\mathbf{S}$; see the right plot. ♦

Example 3.2.2. As a second example, we assume that digital options on $\mathbf{S} = (S_1, S_2, S_3)$ of the form $\mathbb{1}_{\min\{S_1,S_2,S_3\}\geq K_i}$ for only two strikes $K_1, K_2 \in \mathbb{R}_+$ are observed in the market. Their market prices are denoted by Π_1, Π_2, and immediately imply a prescription on the survival copula $\widehat{C}$ of $\mathbf{S}$ as follows:

$$\Pi_i = \mathbb{Q}(S_1 \geq K_i, S_2 \geq K_i, S_3 \geq K_i) = \widehat{C}(F_1(K_i), F_2(K_i), F_3(K_i))$$

for $i = 1, 2$. This is a prescription on two points, hence

$$\mathcal{S} = \{(F_1(K_i), F_2(K_i), F_3(K_i)) \colon i = 1, 2\} \subset \mathbb{I}^3,$$

and we can employ Proposition 2.1.2 to compute the lower and upper bounds $\underline{\widehat{Q}}^{\mathcal{S},\Pi}$ and $\overline{\widehat{Q}}^{\mathcal{S},\Pi}$ on the set of copulas $\widehat{\mathcal{C}}^{\mathcal{S},\Pi} = \{C \in \mathcal{C}^3 \colon \widehat{C}(\mathbf{x}) = \Pi_i,\ \mathbf{x} \in \mathcal{S}\}$ which are compatible with this prescription. We have that

$$\overline{\widehat{Q}}^{\mathcal{S},\Pi}(\mathbf{u}) = \min\Big(1 - u_1, 1 - u_2, 1 - u_3, \min_{i=1,2}\Big\{\Pi_i + \sum_{l=1}^{3}\big(F_l(K_i) - u_l\big)^+\Big\}\Big),$$

$$\underline{\widehat{Q}}^{\mathcal{S},\Pi}(\mathbf{u}) = \max\Big(0, \sum_{i=1}^{3}(1 - u_i) - 2, \max_{i=1,2}\Big\{\Pi_i - \sum_{l=1}^{3}\big(u_l - F_l(K_i)\big)^+\Big\}\Big).$$

Using these bounds, we can now apply Proposition 3.2.1 and compute improved bounds on the set of arbitrage-free prices for a call option on the minimum of $\mathbf{S}$, whose payoff is $\varphi(\mathbf{S}) = (\min\{S_1, S_2, S_3\} - K)^+$. The set of prices for $\varphi(\mathbf{S})$ that are compatible with the market prices of given digital options is denoted by $\Pi^* = \left\{\widehat{\pi}_\varphi(\widehat{C})\colon C \in \widehat{\mathcal{C}}^{\mathcal{S},\Pi}\right\}$, and since φ is Δ-monotonic it holds that $\widehat{\pi}_\varphi(\underline{\widehat{Q}}^{\mathcal{S},\Pi}) \leq \pi \leq \widehat{\pi}_\varphi(\overline{\widehat{Q}}^{\mathcal{S},\Pi})$ for all $\pi \in \Pi^*$. The computation of $\widehat{\pi}_\varphi(Q)$ reduces to

$$\widehat{\pi}_\varphi(Q) = \int_K^\infty Q(F_1(x), F_2(x), F_3(x))\,\mathrm{d}x,$$

see Table 3.1, which is an integral over a subset of the 3-track

$$\{(F_1(x), F_2(x), F_3(x))\colon x \in \overline{\mathbb{R}}_+\} \supset \{(F_1(x), F_2(x), F_3(x))\colon x \in [K, \infty)\} \supset \mathcal{S}.$$

Hence, Lemma 3.2.3 yields that the price bounds $\widehat{\pi}_\varphi(\underline{\widehat{Q}}^{\mathcal{S},\Pi})$ and $\widehat{\pi}_\varphi(\overline{\widehat{Q}}^{\mathcal{S},\Pi})$ are sharp, that is

$$\widehat{\pi}_\varphi\Big(\underline{\widehat{Q}}^{\mathcal{S},\Pi}\Big) = \inf\{\pi\colon \pi \in \Pi^*\} \quad \text{and} \quad \widehat{\pi}_\varphi\Big(\overline{\widehat{Q}}^{\mathcal{S},\Pi}\Big) = \sup\{\pi\colon \pi \in \Pi^*\}.$$

Analogous to the previous example we assume, for the sake of the numerical illustration, that $\mathbf{S}$ follows the multivariate Black–Scholes model and the pairwise correlations are denoted by $\rho_{i,j}$. We then use this model to generate prices of digital options that determine the prescription. The following figures depict the bounds on the prices of a call on the minimum of $\mathbf{S}$ stemming from the improved Fréchet–Hoeffding bounds as a function of the strike K, as well as those from the 'standard' Fréchet–Hoeffding bounds. The price from the multivariate Black–Scholes model is also included as a benchmark. Again we consider two scenarios for the pairwise correlations: in the left plot $\rho_{i,j} = 0$ and in the right one $\rho_{i,j} = 0.5$. We can observe once again, that the use of the additional information leads to a significant improvement of the bounds relative to the 'standard' situation, although in this example the additional information is merely two prices.

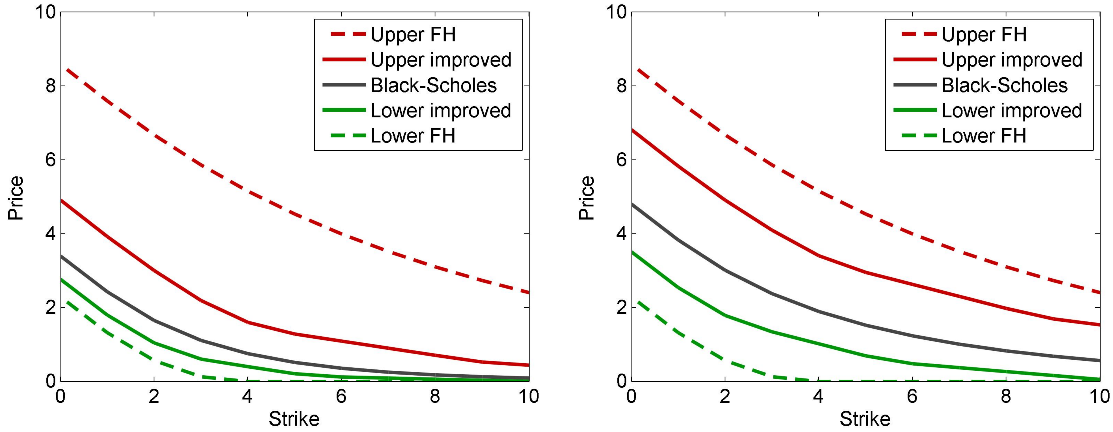

Figure 3.2: Bounds on the prices of options on the minimum of $\mathbf{S}$ as functions of the strike.

♦

4 Dependence uncertainty in risk aggregation

In this chapter we study the evaluation of multivariate risks under dependence uncertainty. The quantification of model ambiguity in risk management has become a central issue in financial applications due to tightening regulatory requirements (e.g. Basel III) regarding provisions for model risks. Measuring risk under uncertainty often relates to the computation of bounds on probabilities of the form $\mathbb{P}(\psi(\mathbf{X}) < s)$, where $\mathbf{X} = (X_1, \ldots, X_d)$ is an $\mathbb{R}^d$-valued random vector and $\psi \colon \mathbb{R}^d \to \mathbb{R}$ an aggregation function. Here $\mathbf{X}$ can be thought of as a vector modelling d risks in a portfolio and ψ as a function to aggregate these risks. The most commonly used aggregation function is the sum of the individual risks, while the minimum and maximum are also possible choices.

Resuming the framework of dependence uncertainty, we assume that the marginal distributions of the constituents $X_i \sim F_i$ for $i = 1, \ldots, d$ are known, while the dependence structure between the components of $\mathbf{X}$ is at most partially known. We then derive bounds on the quantile function of $\psi(\mathbf{X})$ using the available information on the distribution of $\mathbf{X}$. By inversion, the bounds on the distribution of $\psi(\mathbf{X})$ can be translated immediately into estimates on the Value-at-Risk (VaR) of $\psi(\mathbf{X})$.

Note that the quantile function $\mathbb{P}(\psi(\mathbf{X}) < s)$ can be expressed analogously in terms of the expectation $\mathbb{E}[\mathbb{1}_{\psi(\mathbf{X})<s}]$. Thus for $\varphi(\mathbf{x}) = \mathbb{1}_{\psi(\mathbf{x})<s}$ the problem relates to the discussion in Chapter 3, where we use concepts from stochastic dominance to derive bounds on expectations. These arguments apply in the same way to $\mathbb{E}[\mathbb{1}_{\psi(\mathbf{X})<s}]$ whenever $\mathbf{x} \mapsto \mathbb{1}_{\psi(\mathbf{x})<s}$ is Δ-monotonic or Δ-antitonic. For the most common aggregation function $\psi(x_1, ..., x_d) = x_1 + \cdots + x_d$ it follows however that $\mathbf{x} \mapsto \mathbb{1}_{\psi(\mathbf{x})<s}$ is neither Δ-monotonic nor Δ-antitonic when $d > 2$. This calls for a different approach to compute VaR bounds in the presence of dependence uncertainty.

A significant part of the related literature focuses on the situation where only the marginals $F_1, \ldots, F_d$ are known and no information at all on the dependence structure of $\mathbf{X}$ is available. In this case, explicit bounds on the distribution function of the sum of two random variables, i.e. $\psi(\mathbf{X}) = X_1 + X_2$, were derived by Makarov [36] and for more general ψ and $d = 2$ by Rüschendorf [53] in the early 1980's. These results were later generalised for functions of more than two variables, for instance by Denuit, Genest, and Marceau [15] for the sum of finitely many risks, and by Embrechts, Höing, and Juri [18] and Embrechts and Puccetti [16] for more general aggregation functions. These bounds however may fail to be sharp. Therefore, numerical schemes to compute sharp bounds on quantile functions of aggregations have become increasingly popular. The Rearrangement Algorithm, which was introduced by Puccetti and Rüschendorf [44] and Embrechts, Puccetti, and Rüschendorf [19], represents an efficient method to approximate sharp bounds on the VaR of aggregations $\psi(\mathbf{X})$, under additional requirements on the marginal distributions $F_1, \dots, F_d$. Alternatively, Wang et al. [70] used the concept of joint and complete mixability to derive sharp bounds in the homogeneous case under some monotonicity requirements on the marginals. However, the complete absence of information on the dependence structure typically leads to very wide bounds that are not sufficiently informative for practical applications. In addition, a complete lack of information about the dependence structure of $\mathbf{X}$ is often unrealistic, since quantities such as correlations or regions of the distribution function of $\mathbf{X}$ can be estimated with a sufficient degree of accuracy. Therefore, the quest for methods to improve the marginals-only bounds by including additional dependence information has turned into a thriving area of mathematical research in recent years.

Several analytical and numerical approaches to derive risk bounds using additional dependence information have been developed recently. Analytical bounds were derived by Embrechts, Höing, and Juri [18] and Embrechts and Puccetti [16] for the case that a lower bound on the copula of $\mathbf{X}$ is given. Moreover, Embrechts and Puccetti [17] and Puccetti and Rüschendorf [46] establish bounds when the laws of some lower dimensional marginals of $\mathbf{X}$ are known. Analytical bounds that account for positive or negative dependence assumptions were presented in Embrechts, Höing, and Juri [18] and Rüschendorf [56]. Bernard, Rüschendorf, and Vanduffel [9] derive risk bounds when an upper bound on the variance of $\psi(\mathbf{X})$ is prescribed, and they present a numerical scheme to efficiently compute these bounds. In addition, Bernard and Vanduffel [7] consider the case where the distribution of $\mathbf{X}$ is known only on a subset of its domain and establish a version of the Rearrangement Algorithm to account for this type of dependence information. A detailed account of this literature can be found in the overview article Rüschendorf [59].

Continuing this line of research, in this chapter we develop alternative approaches to compute VaR bounds for aggregations of multiple risks in the presence of dependence uncertainty. In Section 4.1 we first review the *Improved Standard Bounds* on VaR which relate a lower bound on the copula of the risk vector to an upper bound on the aggregated VaR for all componentwise increasing aggregation functions. The computation of the Improved Standard Bounds typically involves the solution of a high-dimensional optimization problem. For ψ being either the minimum or the maximum of the individual risks we derive an explicit representation of the bounds that allows us to circumvent the optimization problem. We then use the improved Fréchet–Hoeffding bounds presented in Section 2.1 in conjunction with the Improved Standard Bounds in order to account for partial dependence information in the computation of VaR estimates.

We proceed in Section 4.2 with the development of a reduction principle to incorporate extreme value information, such as the distribution of partial minima or maxima of the risk vector $\mathbf{X}$, in the computation of risk bounds for the sum $X_1 + \cdots + X_d$. The term partial maxima hereby refers to the maximum of lower dimensional marginals of $\mathbf{X}$, i.e. $\max\{X_{j_1}, \dots, X_{i_n}\}$ for $1 \leq i_1 \leq \cdots \leq i_n \leq d$, and analogously for the minimum. We thereby interpolate between the marginals-only case and the situation where the distributions of the lower-dimensional marginals of $\mathbf{X}$ are completely specified, as in [17, 46]. Extreme value information can typically be inferred form empirical data, with an appropriate degree of accuracy, using tools from extreme value theory and is thus available in many practical applications.

The results in this chapter appeared in Lux and Papapantoleon [35].

4.1 Standard and Improved Standard Bounds on Value-at-Risk

We consider a vector of d risks $\mathbf{X} = (X_1, \dots, X_d)$ and an aggregation function $\psi\colon \mathbb{R}^d \to \mathbb{R}$, and want to quantify the risk of $\psi(\mathbf{X})$ by means of VaR. This corresponds to the quantile function of $\psi(\mathbf{X})$, i.e. when $\psi(\mathbf{X}) \sim F_{\psi(\mathbf{X})}$ then the VaR of $\psi(\mathbf{X})$ for a certain confidence level $\alpha \in (0,1)$ is given by the quantity

$$\mathrm{VaR}_\alpha(\psi(\mathbf{X})) := \left(F_{\psi(\mathbf{X})}\right)^{-1}(\alpha) = \inf\{x \in \mathbb{R}\colon F_{\psi(\mathbf{X})}(x) > \alpha\}.$$

Typical levels of α are close to 1 in practice, assuming that risks (or losses) correspond to the right tail of the distribution. Once the distribution of $\psi(\mathbf{X})$ is specified, the determination of VaR amounts to a simple inversion of the quantile function. If the distribution

$F_{\psi(\mathbf{X})}$ is not known, but instead the joint law of $\mathbf{X}$ is given, then the problem lies in the computation of the quantile function of $\psi(\mathbf{X})$ from the law of $\mathbf{X}$. In order to solve this problem, one can resort to numerical integration techniques or Monte Carlo methods. For the important case $\psi(\mathbf{X}) = X_1 + \cdots + X_d$, two efficient algorithms to determine the law of the aggregated risk given the joint distribution of $\mathbf{X}$ are presented in Arbenz, Embrechts, and Puccetti [1, 2].

In the situation of dependence uncertainty it follows that for most functionals ψ of interest, neither the distribution of $\psi(\mathbf{X})$ can be determined completely, nor can its risk be calculated exactly. Indeed, each model for $\mathbf{X}$ that is consistent with the available information can produce a different risk estimate. Therefore, one is interested in deriving upper and lower bounds on the risk of $\psi(\mathbf{X})$ over the set of distributions that comply with the given information. These bounds are then considered best or worst case estimates for the VaR of $\psi(\mathbf{X})$, given the available information about the distribution of $\mathbf{X}$.

In the situation of complete dependence uncertainty, where only the marginals $F_1, \ldots, F_d$ are known and one has no information about the copula of $\mathbf{X}$, bounds for the quantiles of the sum $X_1 + \cdots + X_d$ were derived in a series of papers, starting with the results by Makarov [36] and Rüschendorf [54] for $d = 2$, and their extensions for $d > 2$ in [21, 15, 18]. These bounds are in the literature referred to as *Standard Bounds* and they are given by

$$\begin{aligned} \max\left\{\sup_{\mathcal{U}(s)}\left(F_1^-(u_1) + \sum_{i=2}^d F_i(u_i)\right) - d + 1, 0\right\} &\leq \mathbb{P}(X_1 + \cdots + X_d < s) \\ &\leq \min\left\{\inf_{\mathcal{U}(s)} \sum_{i=1}^d F_i^-(u_i), 1\right\}, \end{aligned} \tag{4.1}$$

where $\mathcal{U}(s) = \{(u_1, \ldots, u_d) \in \mathbb{R}^d \colon u_1 + \cdots + u_d = s\}$ and F_i^- denotes the left-continuous version of F_i. These bounds hold for all random variables $\mathbf{X}$ with marginals $F_1, \ldots, F_d$, and the corresponding bounds on the VaR of the sum $X_1 + \cdots + X_d$ are given by the respective inverse functions.

Embrechts et al. [18] and Embrechts and Puccetti [16] established improvements of the Standard Bounds that account for a lower bound on the copula of $\mathbf{X}$ or its survival function. These bounds are in the literature referred to as *Improved Standard Bounds* and they relate the problem of computing improved VaR estimates in the presence of additional dependence information to the task of improving the Fréchet–Hoeffding bounds on copulas. Specifically, they consider for an $\mathbb{R}^d$-valued random vector $\mathbf{X}$ with marginals $F_1, \ldots, F_d$

the following problem

$$m_{C_0,\psi}(s) := \inf\Big\{\mathbb{P}_C(\psi(\mathbf{X}) < s) \colon C \in \mathcal{C}^d, C_0 \leq C\Big\},$$
$$M_{\widehat{C}_1,\psi}(s) := \sup\Big\{\mathbb{P}_C(\psi(\mathbf{X}) < s) \colon C \in \mathcal{C}^d, \widehat{C}_1 \leq \widehat{C}\Big\},$$

for some copulas C_0 and C_1, where

$$\mathbb{P}_C(\psi(\mathbf{X}) < s) := \int_{\mathbb{R}^d} \mathbb{1}_{\{\psi(x_1,\ldots,x_d)<s\}}\, \mathrm{d}C(F_1(x_1),\ldots,F_d(x_d)).$$

The following bounds on the quantities $m_{C_0,\psi}(s)$ and $M_{\widehat{C}_1,\psi}(s)$ were presented in [18, 16].

Theorem 4.1.1 (Improved Standard Bounds). *Let $\mathbf{X} = (X_1, ..., X_d)$ be an $\mathbb{R}^d$-valued random vector with copula C and marginals $F_1,\ldots,F_d$ and let $\psi \colon \mathbb{R}^d \to \mathbb{R}$ be increasing in each coordinate. Assume that there exist copulas C_0, C_1 such that $C_0 \leq C$ and $\widehat{C}_1 \leq \widehat{C}$. Then it holds that*

$$m_{C_0,\psi}(s) \geq \sup_{x_1,\ldots,x_{d-1}\in\mathbb{R}} C_0\Big(F_1(x_1),\ldots,F_{d-1}(x_{d-1}), F_d^-(\psi^*_{x_{-d}}(s))\Big) =: \underline{m}_{C_0,\psi}(s),$$
$$M_{\widehat{C}_1,\psi}(s) \leq \inf_{x_1,\ldots,x_{d-1}\in\mathbb{R}} 1 - \widehat{C}_1\Big(F_1(x_1),\ldots,F_{d-1}(x_{d-1}), F_d(\psi^*_{x_{-d}}(s))\Big) =: \overline{M}_{\widehat{C}_1,\psi}(s),$$

where $x_{-d} = (x_1,\ldots,x_{d-1})$ and $\psi^*_{x_{-d}}(s) := \sup\{x_d \in \mathbb{R} \colon \psi(x_{-d}, x_d) < s\}$.

A careful examination of the proof of Theorem 3.1 in Embrechts and Puccetti [16] reveals that the result holds also when C_0, resp. $\widehat{C}_1$, is just increasing, resp. decreasing, in each coordinate. Hence, they hold in particular when C_0 is a quasi-copula and $\widehat{C}_1$ a quasi-survival function. The above bounds relate to the VaR of $\psi(\mathbf{X})$ in the following way.

Remark 4.1.1. Let $\psi \colon \mathbb{R}^d \to \mathbb{R}$ be increasing in each component and the copula C of $\mathbf{X}$ be such that $Q_0 \leq C$ and $\widehat{Q}_1 \leq \widehat{C}$, for a quasi-copula Q_0 and a quasi-survival function $\widehat{Q}_1$. Then

$$\overline{M}^{-1}_{\widehat{Q}_1,\psi}(\alpha) \leq \mathrm{VaR}_\alpha(\psi(\mathbf{X})) \leq \underline{m}^{-1}_{Q_0,\psi}(\alpha).$$

♦

Besides the aggregation function $\psi(x_1,\ldots,x_d) = x_1+\cdots+x_d$, the operations $\psi(x_1,\ldots,x_d) = \max\{x_1,\ldots,x_d\}$ and $\psi(x_1,\ldots,x_d) = \min\{x_1,\ldots,x_d\}$ are also of particular interest in risk management, however fewer methods to handle dependence uncertainty for these operations exist; cf. Embrechts et al. [20]. In the following proposition we establish bounds for the minimum and maximum operations in the presence of copula bounds using straightforward computations, and further we show that these bounds coincide with the ones from Theorem 4.1.1.

Proposition 4.1.2. *Let* $\mathbf{X} = (X_1, ..., X_d)$ *be an* $\mathbb{R}^d$*-valued random vector with copula* C *and marginals* $F_1, \dots, F_d$*, and let* $\underline{Q}, \overline{Q}$ *be quasi-copulas. Then we have that*

$$\begin{aligned} m_{\underline{Q},\max}(s) &:= \inf\left\{\mathbb{P}_C(\max(X_1, ..., X_d) < s) \colon \underline{Q} \le C\right\} \\ &\qquad \ge \underline{Q}(F_1^-(s), \dots, F_d^-(s)) = \underline{m}_{\underline{Q},\max}(s) \\ M_{\overline{Q},\max}(s) &:= \sup\left\{\mathbb{P}_C(\max(X_1, ..., X_d) < s) \colon C \le \overline{Q}\right\} \le \overline{Q}(F_1^-(s), \dots, F_d^-(s)), \end{aligned}$$

Where $\underline{m}_{\underline{Q},\max}(s)$ *is given in Theorem 4.1.1. Analogously, if* $\underline{\widehat{Q}}$ *and* $\overline{\widehat{Q}}$ *are quasi-survival functions we obtain*

$$\begin{aligned} m_{\overline{\widehat{Q}},\min}(s) &:= \inf\left\{\mathbb{P}_C(\min(X_1, ..., X_d) < s) \colon \widehat{C} \le \overline{\widehat{Q}}\right\} \ge 1 - \overline{\widehat{Q}}(F_1(s), \dots, F_d(s)) \\ M_{\underline{\widehat{Q}},\min}(s) &:= \sup\left\{\mathbb{P}_C(\min(X_1, ..., X_d) < s) \colon \underline{\widehat{Q}} \le \widehat{C}\right\} \\ &\qquad \le 1 - \underline{\widehat{Q}}(F_1(s), \dots, F_d(s)) = \overline{M}_{\underline{\widehat{Q}},\min}(s), \end{aligned}$$

where $\overline{M}_{\underline{\widehat{Q}},\min}(s)$ is given in Theorem 4.1.1.

Proof. Let $\psi(x_1, \dots, x_d) = \max\{x_1, \dots, x_d\}$, then for any copula C we have that

$$\mathbb{P}_C(\max\{X_1, \dots, X_d\} < s) = \mathbb{P}_C(X_1 < s, \dots, X_d < s) = C(F_1^-(s), \dots, F_d^-(s)),$$

using Sklar's Theorem for the last equality. Hence, it follows immediately that

$$\begin{aligned} m_{\underline{Q},\max}(s) &= \inf\left\{C(F_1^-(s), \dots, F_d^-(s)) \colon \underline{Q} \le C\right\} \ge \underline{Q}(F_1^-(s), \dots, F_d^-(s)) \\ \text{and} \quad M_{\overline{Q},\max}(s) &= \sup\left\{C(F_1^-(s), \dots, F_d^-(s)) \colon C \le \overline{Q}\right\} \le \overline{Q}(F_1^-(s), \dots, F_d^-(s)). \end{aligned}$$

Moreover, since

$$\begin{aligned} \psi^*_{x_{-d}}(s) &= \sup\{x_d \in \mathbb{R} \colon \psi(x_{-d}, x_d) < s\} \\ &= \sup\{x_d \in \mathbb{R} \colon \max\{x_1, \dots, x_d\} < s\} = \begin{cases} s, & x_1, \dots, x_{d-1} < s \\ -\infty, & \text{otherwise}, \end{cases} \end{aligned}$$

we get from Theorem 4.1.1 that

$$\begin{aligned}\underline{m}_{\underline{Q},\max}(s) &= \sup_{x_1,\dots,x_{d-1}<s} \underline{Q}\Big(F_1(x_1),\dots,F_{d-1}(x_{d-1}),F_d^-(\psi^*_{x_{-d}}(s))\Big)\\ &= \sup_{x_1,\dots,x_{d-1}<s} \underline{Q}\Big(F_1(x_1),\dots,F_{d-1}(x_{d-1}),F_d^-(s)\Big) = \underline{Q}(F_1^-(s),\dots,F_d^-(s)),\end{aligned}$$

where the last equality follows from the fact that $\underline{Q}$ is a quasi-copula, hence it is increasing in each component such that the supremum is attained at $(F_1^-(s),\dots,F_d^-(s))$.

Similarly, we have for $\psi(x_1,\dots,x_d)=\min\{x_1,\dots,x_d\}$ and any copula C that

$$\begin{aligned}\mathbb{P}_C(\min\{X_1,\dots,X_d\}<s) &= 1-\mathbb{P}_C(\min\{X_1,\dots,X_d\}\geq s)\\ &= 1-\widehat{C}(F_1(s),\dots,F_d(s)),\end{aligned}$$

where the last equality follows from the definition of $\widehat{C}$. Hence it follows

$$m_{\widehat{\underline{Q}},\min}(s) = \inf\Big\{1-\widehat{C}(F_1(s),\dots,F_d(s))\colon \widehat{C}\leq\widehat{\underline{Q}}\Big\} \geq 1-\widehat{\underline{Q}}(F_1(s),\dots,F_d(s))$$

and analogously $M_{\widehat{\underline{Q}},\min}(s)\leq 1-\widehat{\underline{Q}}(F_1(s),\dots,F_d(s))$. Moreover, it follows from Theorem 4.1.1 again that

$$\begin{aligned}\overline{M}_{\widehat{\underline{Q}},\min}(s) &= \inf_{x_1,\dots,x_{d-1}<s} 1-\widehat{\underline{Q}}\Big(F_1(x_1),\dots,F_{d-1}(x_{d-1}),F_d(\psi^*_{x_{-d}}(s))\Big)\\ &= \inf_{x_1,\dots,x_{d-1}<s} 1-\widehat{\underline{Q}}\Big(F_1(x_1),\dots,F_{d-1}(x_{d-1}),F_d(s))\Big)\\ &= 1-\widehat{\underline{Q}}\Big(F_1(s),\dots,F_d(s)\Big).\end{aligned}$$

The proof is hence complete. □

Combining the Improved Standard Bounds on VaR with the improved Fréchet–Hoeffding bounds in Section 2.1 allows us to use different types of dependence information in the computation of risk estimates. In the following we illustrate this procedure and derive VaR bounds using the information that the copula of the risk vector lies the proximity of a reference model.

Approaches to compute robust risk estimates over a class of models that lie in the vicinity of a reference model have been proposed earlier in the literature. Glasserman and Xu [25] derive robust bounds on the portfolio variance, the conditional VaR and the CVA over the class of models within a relative entropy distance of a reference model. Barrieu and Scandolo [3] establish bounds on the VaR of a univariate random variable given that its distribution is close to a reference distribution in the sense of the Kolmogorov–Smirnov or Lévy distance. In a multivariate setting, Blanchet and Murthy [10] use an optimal transport approach to derive robust bounds on quantities such as ruin probabilities, over a neighbourhood of models defined in terms of the Wasserstein distance.

Example 4.1.1. We consider a portfolio of three risks (X_1, X_2, X_3) with $X_1 \sim \mathrm{GP}(0, 2, 0.5)$, where $\mathrm{GP}(\mu, \xi, \sigma)$ denotes the Generalised Pareto distribution with location, scale and shape parameters equal to μ, ξ and σ respectively, and $X_2 \sim \mathcal{LN}(2, 0.5)$ as well as $X_3 \sim \mathcal{LN}(1, 1)$, where $\mathcal{LN}(\mu, \sigma)$ refers to the log-normal distribution with parameters μ and σ. We assume that the reference model C^* is a Student-t copula with equicorrelation matrix and two degrees of freedom, and we are interested in computing bounds on the VaR over the class of models in the δ-neighbourhood of C^* as measured by the Kolmogorov–Smirnov distance. In other words, we consider the class of copulas

$$\mathcal{C}^{\mathcal{D}_{\mathrm{KS}},\delta} := \Big\{ C \in \mathcal{C}^d \colon \mathcal{D}_{\mathrm{KS}}(C, C^*) \leq \delta \Big\},$$

and using Theorem 2.1.7 and Lemma 2.1.8 we obtain the bounds $\underline{Q}^{\mathcal{D}_{\mathrm{KS}},\delta}$ and $\overline{Q}^{\mathcal{D}_{\mathrm{KS}},\delta}$ on the copulas in $\mathcal{C}^{\mathcal{D}_{\mathrm{KS}},\delta}$.

Then, we apply Proposition 4.1.2 using $\underline{Q}^{\mathcal{D}_{\mathrm{KS}},\delta}$ and $\overline{Q}^{\mathcal{D}_{\mathrm{KS}},\delta}$ in order to compute estimates on the VaR of $\max\{X_1, X_2, X_3\}$ over the class of models in the vicinity of C^*. The following table shows the confidence level α and the upper and lower Standard Bounds in the first two columns. The third, fourth and fifth column contain the upper and lower improved bounds based on the information on the distance from C^* for different levels of the threshold δ. Moreover we indicate the improvement over the Standard Bounds in percentage terms. For the computation of these estimates we assumed that the pairwise correlations of the t-copula C^* equal 0.5. The results are rounded to one decimal number for the sake of legibility.

α	(lower : upper)	$\delta = 0.005$ (lower : upper)	impr. %	$\delta = 0.01$ (lower : upper)	impr. %	$\delta = 0.02$ (lower : upper)	impr. %
90%	(14.0 : 18.0)	(15.4 : 15.9)	87	(15.2 : 16.3)	72	(14.8 : 16.9)	47
95%	(16.8 : 23.0)	(19.3 : 20.8)	75	(18.8 : 21.8)	51	(17.8 : 21.7)	37
98%	(24.2 : 32.6)	(26.3 : 32.6)	25	(24.2 : 32.6)	0	(24.2 : 32.6)	0

Table 4.1: Standard and improved VaR bounds for $\max\{X_1, X_2, X_3\}$ given a threshold on the distance from the reference t-copula C^* with pairwise correlation equal to 0.5.

The next table is analogous to Table 4.1, but this time higher dependence is prescribed, assuming that the pairwise correlations of the t-copula C^* are equal to 0.9.

α	(lower : upper)	$\delta = 0.005$ (lower : upper)	impr. %	$\delta = 0.01$ (lower : upper)	impr. %	$\delta = 0.02$ (lower : upper)	impr. %
90%	(14.0 : 18.0)	(14.5 : 15.0)	87	(14.3 : 15.3)	75	(14.0 : 15.8)	55
95%	(16.8 : 23.0)	(18.0 : 19.4)	77	(17.4 : 20.1)	56	(16.8 : 22.5)	8
98%	(24.2 : 32.6)	(24.2 : 31.0)	20	(24.2 : 32.6)	0	(24.2 : 32.6)	0

Table 4.2: Standard and improved VaR bounds for $\max\{X_1, X_2, X_3\}$ given a threshold on the distance from the reference t-copula C^* with pairwise correlation equal to 0.9.

The following observations ensue from this example: (i) The addition of partial information reduces significantly the spread between the upper and lower bounds. This reduction is more pronounced as the threshold δ decreases; in other words, the more reliable the reference model, the more pronounced the reduction of model risk. (ii) The level of improvement decreases, sometimes dramatically, with increasing confidence level α. In particular, for $\alpha = 99\%$ there was no improvement found, at least for the values of δ used above. (iii) The improvement is more pronounced in the high-dependence scenario, with improvements over the Standard Bounds of up to 87%. ♦

4.2 Improved bounds for Value-at-Risk using extreme value information

In this section we improve the Standard Bounds on the VaR of the sum $X_1 + \cdots + X_d$ in the situation where, besides the marginal distributions, the laws of the minima and maxima of some subsets of the risks $X_1, \ldots, X_d$ are known. Specifically, we assume that for a system $J_1, \ldots, J_m \subset \{1, \ldots, d\}$ the distributions of $\max_{j \in J_n} X_j$ or $\min_{j \in J_n} X_j$ for $n = 1, \ldots, m$ are given. This setting can be viewed as an interpolation between the marginals-only case and the situation where the lower-dimensional marginals of the vectors $(X_j)_{j \in J_n}$ are completely specified. The latter setting has been studied extensively in the literature, and risk bounds for aggregations of $\mathbf{X}$ given some of its lower-dimensional marginals were obtained e.g. by Rüschendorf [55], Embrechts and Puccetti [17] and Puccetti and Rüschendorf [46]. In practice however, it is often difficult to determine the distributions of the lower-dimensional vectors $(X_j)_{j \in J_n}$. In particular for large dimensions of the subsets, a vast amount of data is required to estimate the distribution of $(X_j)_{j\in J_n}$ with sufficient accuracy. Thus, having complete information about lower-dimensional marginals of $(X_1,\ldots,X_d)$ turns out to be a rather strong assumption. Therefore, methods that interpolate between this scenario and the marginals-only case are of practical interest. Extending the reduction principle in Puccetti and Rüschendorf [46], we develop in the following a method to improve the Standard Bounds when instead of the distribution of $(X_j)_{j\in J_n}$, only the distribution of the maximum $\max_{j\in J_n} X_j$ or minimum $\min_{j\in J_n} X_j$ is known. Let us point out that obtaining information about the maximum or minimum of a sequence of random variables is the central theme of extreme value theory, which provides a rich collection of methods for their estimation; see e.g. Beirlant, Goegebeur, Teugels, and Segers [6, Chapter 9].

Let us denote $\mathcal{I} := \{1,\ldots,d\}$ and $\mathcal{J} := \{1,\ldots,m\}$.

Theorem 4.2.1. *Let $(X_1,\ldots,X_d)$ be a random vector with marginals $F_1,\ldots,F_d$, and consider a collection $\mathcal{E} = \{J_1,\ldots,J_m\}$ of subsets $J_n \subset \mathcal{I}$ for $n\in\mathcal{J}$ with $\bigcup_{n\in\mathcal{J}} J_n = \mathcal{I}$. Denote by G_n the distribution of $Y_n = \max_{j\in J_n} X_j$. Then it follows that*

$$\begin{aligned}\inf\Big\{\mathbb{P}(X_1+\cdots+X_d\le s)\colon X_i\sim F_i, i\in\mathcal{I}, \max_{j\in J_n} X_j\sim G_n, n\in\mathcal{J}\Big\}\\ \ge \sup_{(\alpha_1,\ldots,\alpha_m)\in\underline{\mathcal{A}}} \inf\Big\{\mathbb{P}(\alpha_1 Y_1+\cdots+\alpha_m Y_m\le s)\colon Y_n\sim G_n, n\in\mathcal{J}\Big\} =: \underline{m}_{\mathcal{E},\max}(s),\end{aligned}$$

where

$$\underline{\mathcal{A}} = \Big\{(\alpha_1,\ldots,\alpha_m)\in\mathbb{R}^m_+\colon \sum_{n=1}^m \alpha_n \max_{j\in J_n} x_j \ge \sum_{i=1}^d x_i, \text{ for all } (x_1,\ldots,x_d)\in\mathbb{R}^d\Big\}\neq\emptyset.$$

Moreover if $(X_1,\ldots,X_d)$ is $\mathbb{R}^d_+$-valued, then

$$\begin{aligned}\sup\Big\{\mathbb{P}(X_1+\cdots+X_d\le s)\colon X_i\sim F_i, i\in\mathcal{I}, \max_{j\in J_n} X_j\sim G_n, n\in\mathcal{J}\Big\}\\ \le \inf_{(\alpha_1,\ldots,\alpha_m)\in\overline{\mathcal{A}}} \sup\Big\{\mathbb{P}(\alpha_1 Y_1+\cdots+\alpha_m Y_m\le s)\colon Y_n\sim G_n, n\in\mathcal{J}\Big\} =: \overline{M}_{\mathcal{E},\max}(s),\end{aligned}$$

where

$$\overline{\mathcal{A}} = \Big\{(\alpha_1,\ldots,\alpha_m)\in\mathbb{R}^m_+\colon \sum_{n=1}^m \alpha_n \max_{j\in J_n} x_j \le \sum_{i=1}^d x_i, \text{ for all } (x_1,\ldots,x_d)\in\mathbb{R}^d_+\Big\}\neq\emptyset.$$

Proof. We first show that the lower bound $\underline{m}_{\mathcal{E},\max}$ is valid. It follows from $\bigcup_{n=1}^m J_n =$

$\{1,\dots,d\}$ that $\underline{\mathcal{A}} \neq \emptyset$. Indeed, choosing for instance $\alpha_n = |J_n|$ we get that $\sum_{j\in J_n} x_j \leq \alpha_n \max_{j\in J_n} x_j$, for all $(x_1,\dots,x_d)\in\mathbb{R}^d$ and $n=1,\dots,m$. Hence

$$\sum_{n=1}^m \alpha_n \max_{j\in J_n} x_j \geq \sum_{n=1}^m \sum_{j\in J_n} x_j \geq \sum_{i=1}^d x_i \quad \text{for all } (x_1,\dots,x_d)\in\mathbb{R}^d.$$

Then, it follows for arbitrary $(\alpha_1,\dots,\alpha_m)\in\underline{\mathcal{A}}$ that

$$\left\{\sum_{n=1}^m \alpha_n \max_{j\in J_n} X_j \leq s\right\} \subseteq \left\{\sum_{i=1}^d X_i \leq s\right\},$$

hence

$$\begin{aligned}
&\inf\Big\{\mathbb{P}(X_1+\cdots+X_d\leq s)\colon X_i\sim F_i, i\in\mathcal{I}, \max_{j\in J_n} X_j \sim G_n, n\in\mathcal{J}\Big\}\\
&\quad\geq \inf\left\{\mathbb{P}\left(\sum_{n=1}^m \alpha_n \max_{j\in J_n} X_j \leq s\right)\colon X_i\sim F_i, i\in\mathcal{I}, \max_{j\in J_n} X_j\sim G_n, n\in\mathcal{J}\right\}\\
&\quad= \inf\Big\{\mathbb{P}(\alpha_1 Y_1+\cdots+\alpha_m Y_m\leq s)\colon Y_n\sim G_n, n\in\mathcal{J},\Big\}.
\end{aligned}$$

Now, since $(\alpha_1,\dots,\alpha_m)\in\underline{\mathcal{A}}$ was arbitrary, it follows that the lower bound holds by taking the supremum over all elements in $\underline{\mathcal{A}}$.

Likewise for the upper bound, we note that since $(X_1,\dots,X_d)$ is $\mathbb{R}^d_+$-valued, $(0,\dots,0)$ and $(1,\dots,1)$ belong to $\overline{\mathcal{A}}$, hence it is not empty. Moreover, for arbitrary $(\alpha_1,\dots,\alpha_m)\in\overline{\mathcal{A}}$, it follows that

$$\left\{\sum_{n=1}^m \alpha_n \max_{j\in J_n} X_j \leq s\right\} \supseteq \left\{\sum_{i=1}^d X_i \leq s\right\},$$

due to the fact that $(X_1,\dots,X_d)$ is non-negative and $\sum_{j\in J_n} x_j \geq \sum_{n=1}^m \alpha_n \max_{j\in J_n} x_j$. Hence, we get that

$$\begin{aligned}
&\sup\Big\{\mathbb{P}(X_1+\cdots+X_d\leq s)\colon X_i\sim F_i, i\in\mathcal{I}, \max_{j\in J_n} X_j \sim G_n, n\in\mathcal{J}\Big\}\\
&\quad\leq \sup\left\{\mathbb{P}\left(\sum_{n=1}^m \alpha_n \max_{j\in J_n} X_j \leq s\right)\colon X_i\sim F_i, i\in\mathcal{I}, \max_{j\in J_n} X_j\sim G_n, n\in\mathcal{J}\right\}\\
&\quad= \sup\Big\{\mathbb{P}(\alpha_1 Y_1+\cdots+\alpha_m Y_m\leq s)\colon Y_n\sim G_n, n\in\mathcal{J}\Big\}.
\end{aligned}$$

Since $(\alpha_1,\dots,\alpha_m)\in\overline{\mathcal{A}}$ was arbitrary, it follows that the upper bound holds indeed. □

Remark 4.2.1. The assumption $\bigcup_{n=1}^m J_n = \{1,\dots,d\}$ can always be met by adding singletons to $\mathcal{E}$, i.e. $J_n = \{i_n\}$ for $i_n \in \{1,\dots,d\}$, since the marginal distributions of

$(X_1, \dots, X_d)$ are known. However, the bounds are valid even when the marginal distributions are not known. ♦

By the same token, the following result establishes bounds on the quantile function of $X_1 + \cdots + X_d$ when distributions of some minima are known.

Theorem 4.2.2. *Consider the setting of Theorem 4.2.1 and denote by H_n the distribution of $Z_n = \min_{j \in J_n} X_j$. Then it follows that*

$$\begin{aligned}
&\sup\Big\{\mathbb{P}(X_1 + \cdots + X_d \le s)\colon X_i \sim F_i, i \in \mathcal{I}, \min_{j \in J_n} X_j \sim H_n, n \in \mathcal{J}\Big\} \\
&\quad \le \inf_{(\alpha_1,\dots,\alpha_m) \in \overline{\mathcal{B}}} \sup\Big\{\mathbb{P}(\alpha_1 Z_1 + \cdots + \alpha_m Z_m \le s)\colon Z_n \sim H_n, n \in \mathcal{J}\Big\} =: \overline{M}_{\mathcal{E},\min}(s),
\end{aligned}$$

where

$$\overline{\mathcal{B}} = \Big\{(\alpha_1, \dots, \alpha_m) \in \mathbb{R}_+^m \colon \sum_{n=1}^m \alpha_n \min_{j \in J_n} x_j \le \sum_{i=1}^d x_i, \text{ for all } (x_1, \dots, x_d) \in \mathbb{R}^d\Big\} \ne \emptyset.$$

Moreover if $(X_1, \dots, X_d)$ is $\mathbb{R}_-^d$-valued, then

$$\begin{aligned}
&\inf\Big\{\mathbb{P}(X_1 + \cdots + X_d \le s)\colon X_i \sim F_i, i \in \mathcal{I}, \min_{j \in J_n} X_j \sim H_n, n \in \mathcal{J}\Big\} \\
&\quad \ge \sup_{(\alpha_1,\dots,\alpha_m) \in \underline{\mathcal{B}}} \inf\Big\{\mathbb{P}(\alpha_1 Z_1 + \cdots + \alpha_m Z_m \le s)\colon Z_n \sim H_n, n \in \mathcal{J}\Big\} =: \underline{m}_{\mathcal{E},\min}(s),
\end{aligned}$$

where

$$\underline{\mathcal{B}} = \Big\{(\alpha_1, \dots, \alpha_m) \in \mathbb{R}_+^m \colon \sum_{n=1}^m \alpha_n \min_{j \in J_n} x_j \ge \sum_{i=1}^d x_i, \text{ for all } (x_1, \dots, x_d) \in \mathbb{R}_-^d\Big\} \ne \emptyset.$$

The proof follows along the same lines of argumentation as the proof of Theorem 4.2.1, and is therefore omitted.

The computation of the bounds presented in Theorems 4.2.1 and 4.2.2 can be cumbersome for two reasons. Firstly, for fixed $(\alpha_1, \dots, \alpha_m)$ there does not exist a method to compute sharp analytical bounds on the set $\big\{\mathbb{P}(\alpha_1 Y_1 + \cdots + \alpha_m Y_m \le s)\colon Y_n \sim G_n, n = 1, \dots, m\}$, except when $m = 2$. This problem can however be circumvented either by using the Standard Bounds in (4.1), or numerically, by an application of the Rearrangement Algorithm of Embrechts et al. [19]. Using the Rearrangement Algorithm, we are able to approximate upper and lower bounds on the set in an efficient way. Secondly, the determination of the sets $\underline{\mathcal{A}}$, $\overline{\mathcal{A}}$ and $\underline{\mathcal{B}}$, $\overline{\mathcal{B}}$ depends on the system $J_1, \dots, J_m$ and is, in general, not straightforward. However, the following application demonstrates that, even for possibly non-optimal elements in $\underline{\mathcal{A}}$, $\overline{\mathcal{A}}$ and $\underline{\mathcal{B}}$, $\overline{\mathcal{B}}$, the bounds in Theorems 4.2.1 and 4.2.2 yield a significant improvement over the Standard Bounds.

To this end we estimate the probability $\mathbb{P}(\alpha_1 Y_1 + \dots + \alpha_m Y_m \leq s)$ for fixed $(\alpha_1, \dots, \alpha_m)$ by means of the Standard Bounds given in (4.1), i.e.

$$\max\left\{0, \sup_{\mathcal{U}(s)} \sum_{i=1}^m G_i^-\left(\frac{u_i}{\alpha_i}\right) - m + 1\right\} \leq \mathbb{P}(\alpha_1 Y_1 + \dots + \alpha_m Y_m \leq s)$$
$$\leq \min\left\{1, \inf_{\mathcal{U}(s)} \sum_{i=1}^m G_i^-\left(\frac{u_i}{\alpha_i}\right)\right\},$$

where $\mathcal{U}(s) = \{(u_1, \dots, u_m) \in \mathbb{R}^m \colon u_1 + \dots + u_m = s\}$ and G_i^- denotes the left-continuous version of G_i. Thus, the bounds $\underline{m}_{\mathcal{E},\max}$ and $\overline{M}_{\mathcal{E},\max}$ are estimated by

$$\begin{aligned} \underline{m}_{\mathcal{E},\max}(s) &\geq \sup_{(\alpha_1,\dots,\alpha_m)\in\underline{\mathcal{A}}} \max\left\{0, \sup_{\mathcal{U}(s)} \sum_{i=1}^m G_i^-\left(\frac{u_i}{\alpha_i}\right) - m + 1\right\}, \\ \overline{M}_{\mathcal{E},\max}(s) &\leq \inf_{(\alpha_1,\dots,\alpha_m)\in\overline{\mathcal{A}}} \min\left\{1, \inf_{\mathcal{U}(s)} \sum_{i=1}^m G_i^-\left(\frac{u_i}{\alpha_i}\right)\right\}. \end{aligned} \tag{4.2}$$

The following example illustrates the improvement achieved by the VaR bounds in (4.2), that account for extreme value information.

Example 4.2.1. We consider a portfolio $\mathbf{X} = (X_1, \dots, X_6)$, where

$$X_1, X_2, X_3 \sim \text{GP}(0, 2, 0.5) \quad \text{and} \quad X_4, X_5, X_6 \sim \text{GP}(0, 4, 0.5),$$

for GP(μ, ξ, σ) denoting the Generalised Pareto distribution with location, scale and shape parameters equal to μ, ξ and σ respectively. We then analyse the improvement over the Standard Bounds when additional information on the dependence structure is taken into account. In particular, we assume that the distributions G_n of the maxima $\max_{j\in J_n} X_j$ are known for $J_1 = \{1, 2, 3\}$ and $J_2 = \{4, 5, 6\}$. In this case, it follows from Theorem 4.2.1 and equation (4.2), that

$$\begin{aligned} &\sup_{(\alpha_1,\dots,\alpha_8)\in\underline{\mathcal{A}}} \max\left\{0, \sup_{\mathcal{U}(s)} \left(G_1^-\left(\frac{u_1}{\alpha_1}\right) + G_2^-\left(\frac{u_2}{\alpha_2}\right) + \sum_{i=1}^6 F_i^-\left(\frac{u_{i+2}}{\alpha_{i+2}}\right)\right) - 7\right\} \\ &\quad \leq \inf\Big\{\mathbb{P}(X_1 + \dots + X_6 \leq s) \colon X_1, X_2, X_3 \sim \text{Pareto}_2, X_4, X_5, X_6 \sim \text{Pareto}_4, \\ &\qquad\qquad \max_{j\in J_n} X_n \sim G_n, n = 1, 2\Big\} \end{aligned} \tag{4.3}$$

and analogously

$$\inf_{(\alpha_1,\dots,\alpha_8)\in\overline{\mathcal{A}}} \min\left\{1, \inf_{\mathcal{U}(s)}\left(G_1^-\left(\frac{u_1}{\alpha_1}\right)+G_2^-\left(\frac{u_2}{\alpha_2}\right)+\sum_{i=1}^{6}F_i^-\left(\frac{u_{i+2}}{\alpha_{i+2}}\right)\right)\right\}$$
$$\geq \sup\Big\{\mathbb{P}(X_1+\cdots+X_6\leq s)\colon X_1,X_1,X_3\sim \text{Pareto}_2, X_4,X_5,X_6\sim\text{Pareto}_4, \qquad (4.4)$$
$$\max_{j\in J_n} X_n\sim G_n, n=1,2\Big\},$$

where $\mathcal{U}(s)=\{(u_1,\dots,u_8)\in\mathbb{R}^8\colon u_1+\cdots+u_8=s\}$. Note that the marginals $F_1,\dots,F_d$ appear in the optimization since the distribution of the maximum of every individual variable is known and equals the respective marginal distribution; i.e. $\max\{X_i\}=X_i\sim F_i$ for $i=1,\dots,d$; see again Remark 4.2.1. The marginal distributions are thus accounted for in the computation of the bounds.

The solution of the optimization problems in (4.3) and (4.4) yields estimates on the VaR of the sum $X_1+\cdots+X_6$ when the distributions of the partial maxima are taken into account. Table 4.3 shows the α confidence level in the first column and the Standard Bounds, without additional information, in the second column. The third and fourth columns contain the improved VaR bounds that incorporate the extreme value information, as well as the improvement over the Standard Bounds in percentage terms. In order to illustrate our method, we need to know the distribution of the partial maxima. To this end, we assume that the vectors (X_1,X_2,X_3) and (X_4,X_5,X_6) have Student-t copulas with equicorrelation matrices and two degrees of freedom, and numerically determine the distribution of $\max\{X_1,X_2,X_3\}$ and $\max\{X_4,X_5,X_6\}$. In the third column it is assumed that the pairwise correlations of (X_1,X_2,X_3) are equal to 0.9 and the pairwise correlations of (X_4,X_5,X_6) are equal to 0.6. In the fourth column the pairwise correlations amount to 0.5 and 0.2 respectively.

α	lower	upper	lower impr.	upper impr.	impr. %	lower impr.	upper impr.	impr. %
90%	17.3	251.3	22.2	168.6	37.4	27.8	201.6	25.7
95%	27.8	372.7	34.9	256.2	35.8	43.2	303.4	24.5
99%	72.1	884.2	88.4	626.7	33.7	106.5	732.2	23.0

Table 4.3: Standard and improved VaR bounds for the sum $X_1+\cdots+X_6$ with known distribution of partial maxima for different confidence levels.

The observations made following Example 4.1.1 are largely valid also in the present one, namely (i) the addition of partial dependence information allows to significantly reduce the spread between the upper and lower bounds. However, the model risk is still not negligible. (ii) The level of improvement decreases with increasing confidence level α. This phenomenon has been observed already in the related literature; see e.g. [7]. (iii) The improvement is more pronounced in the high-correlation scenario. The phenomenon that the Improved Standard Bounds perform better when high dependence information is prescribed has been encountered previously, see e.g. [47], and is due to the fact that higher dependence leads to a more significant improvement of the lower bound on the copula, which is then employed in the computation of the Improved Standard Bounds in Theorem 4.1.1. ♦

5 An optimal transport approach to Value-at-Risk bounds with partial dependence information

In this chapter we develop an optimal transport approach to derive bounds on the expectation $\mathbb{E}[\varphi(\mathbf{X})]$ over a class of constrained distributions for the $\mathbb{R}^d$-valued random vector $\mathbf{X} = (X_1, ..., X_d)$. We work in the setting of dependence uncertainty and assume that $\mathbf{X}$ has marginal distributions $F_1, ..., F_d$. Moreover, we suppose that an upper and a lower bound on the copula of $\mathbf{X}$ are provided. Using this information, we derive in Section 5.1 sharp dual bounds on the expectation $\mathbb{E}[\varphi(\mathbf{X})]$ for lower semicontinuous φ. This extends the results in Embrechts, Höing, and Juri [18] and Embrechts and Puccetti [16] regarding the Improved Standard Bounds in two ways: (i) we show that the dual bounds are sharp for rather general aggregation functions φ and (ii) we include two-sided information on the copula of the random vector $\mathbf{X}$. The proof of the dual characterization is based on the Monge–Kantorovich Duality Theory and the corresponding bounds amount to an infinite-dimensional optimization problem which is typically analytically and numerically intractable. They however lend themselves to the development of a numerical scheme for the computation of VaR estimates that account for the upper and lower copula bounds. This is made rigorous in Section 5.2, where we develop a tractable optimization scheme to compute bounds on the VaR of aggregations using the copula information. The scheme corresponds to an optimization over a subset of admissible functions for the respective dual problems. Moreover, we show that the reduced scheme produces asymptotically sharp bounds in the certainty limit, i.e. when the bounds on the copula converge to a mutual limit copula. However, considering only a tractable subclass of admissible functions we obtain VaR bounds which are not sharp in general. Furthermore, we show that the Improved Standard Bounds presented in Theorem 4.1.1 can be recovered as special cases of our reduced scheme. Finally, we illustrate that our approach may produce significantly narrower VaR estimates compared to the Improved Standard Bounds.

5.1 Dual bounds on expectations using copula information

In this section we derive sharp dual bounds on the expectation $\mathbb{E}[\varphi(\mathbf{X})]$ when a lower and an upper bound on the copula C of $\mathbf{X}$ are provided. Let us first re-define the notation in (3.1) and introduce

$$\mathbb{E}_C[\varphi(\mathbf{X})] := \pi_\varphi(C) = \int_{\mathbb{R}^d} \varphi(x_1, \dots, x_d)\, \mathrm{d}C(F_1(x_1), \dots, F_d(x_d)),$$

for marginal distributions $F_1, ..., F_d$ and a copula C. We use $\mathbb{E}_C[\varphi(\mathbf{X})]$ to refer to the expectation here since this notation is rather customary in the risk and optimal transport literature and also because it proves convenient for some of the computations in this chapter.

We then consider the following primal problems:

$$\underline{P}_\varphi := \inf\left\{\mathbb{E}_C[\varphi] \colon C \in \mathcal{C}^d, \underline{Q} \leq C \leq \overline{Q}\right\}, \tag{5.1}$$

$$\overline{P}_\varphi := \sup\left\{\mathbb{E}_C[\varphi] \colon C \in \mathcal{C}^d, \underline{Q} \leq C \leq \overline{Q}\right\}, \tag{5.2}$$

for quasi-copulas $\overline{Q}$ and $\underline{Q}$, with $\underline{Q} \leq \overline{Q}$. Note that here – as opposed to previous chapters – we do not assume that φ is of a particular form. In fact, the results in this section apply to all lower semicontinuous functions φ fulfilling a minor growth condition.

Allowing quasi-copulas as bounds, there might not exist a copula that complies with the constraints and so $\mathcal{C}_b := \left\{C \in \mathcal{C}^d \colon \underline{Q} \leq C \leq \overline{Q}\right\} = \emptyset$. Consider e.g. $\overline{Q} = \underline{Q} = W_d$, then $\mathcal{C}_b$ is empty whenever $d > 2$. In this case, we set $\overline{P}_\varphi = \infty$ and $\underline{P}_\varphi = -\infty$.

Remark 5.1.1. When the copula bounds are equal to the lower and upper Fréchet–Hoeffding bounds, i.e. $\underline{Q} = W_d$ and $\overline{Q} = M_d$, then the optimization corresponds to a standard Fréchet problem where no information at all about the dependence structure is prescribed. ♦

In the remainder of this section, we establish a sharp dual characterization of the primal problems $\underline{P}_\varphi$ and $\overline{P}_\varphi$. To this end, let us introduce the class

$$\mathcal{R} := \left\{h = \sum_{n=1}^{k} \alpha_n \Lambda_{\mathbf{u}^n} \colon k \in \mathbb{N}, \alpha_1, ..., \alpha_k \geq 0, \mathbf{u}^1, ..., \mathbf{u}^k \in \overline{\mathbb{R}}^d\right\},$$

where the functions $\Lambda_{\mathbf{u}}$ are of the form

$$\Lambda_{\mathbf{u}} \colon \mathbb{R}^d \ni (x_1, ..., x_d) \mapsto \mathbb{1}_{x_1 \leq u_1, ..., x_d \leq u_d}.$$

Hence, the elements in $\mathcal{R}$ are positive, linear combinations of indicator functions of rectangles of the form $(-\infty, u_1] \times \cdots \times (-\infty, u_d]$. Analogously we denote the lower semi-continuous version of $\Lambda_{\mathbf{u}}$ by

$$\Lambda_{\mathbf{u}}^{-} : \mathbb{R}^d \ni (x_1, ..., x_d) \mapsto \mathbb{1}_{x_1 < u_1, ..., x_d < u_d},$$

and for $h = \sum_{n=1}^{k} \alpha_n \Lambda_{\mathbf{u}^n} \in \mathcal{R}$ we define $h^- := \sum_{n=1}^{k} \alpha_n \Lambda_{\mathbf{u}^n}^-$.

Note that for a copula C it holds that

$$\mathbb{E}_C[\Lambda_{\mathbf{u}}] = \int_{\mathbb{R}^d} \Lambda_{\mathbf{u}}(x_1, ..., x_d) \, \mathrm{d}C(F_1(x_1), ..., F_d(x_d)) = C(F_d(x_1), ..., F_d(x_d)), \tag{5.3}$$

and analogously we obtain that $\mathbb{E}_C[\Lambda_{\mathbf{u}}^-] = C(F_d^-(u_1), ..., F_d^-(u_d))$, where F_i^- is the left-continuous version of F_i for $i = 1, ..., n$.

Moreover, let us define, for a quasi-copula Q and a function $h = \sum_{n=1}^{k} \alpha_n \Lambda_{\mathbf{u}^n} \in \mathcal{R}$,

$$Q(h) := \sum_{n=1}^{k} \alpha_n Q\Big(F_1(u_1^n), ..., F_d(u_d^n)\Big); \quad Q(h^-) := \sum_{n=1}^{k} \alpha_n Q\Big(F_1^-(u_1^n), ..., F_d^-(u_d^n)\Big).$$

If $Q = C$ for a copula C, we have that $Q(h) = \mathbb{E}_Q[h]$ as well as $Q(h^-) = \mathbb{E}_Q[h^-]$.

This leads us to the dual problem corresponding to $\underline{P}_\varphi$:

$$\begin{aligned} \underline{D}_\varphi = \sup \Bigg\{ \underline{Q}(h) - \overline{Q}(g^-) + \sum_{i=1}^{d} \mathbb{E}_i[\nu_i] \colon \nu_i \in L(F_i), i = 1, ..., d, \\ h, g \in \mathcal{R} \text{ s.t. } h - g^- + \sum_{i=1}^{d} \nu_i \leq \varphi \Bigg\}, \end{aligned} \tag{5.4}$$

where $\mathbb{E}_i[\nu_i] = \int \nu_i \, \mathrm{d}F_i$ and $L(F_i)$ is the class of F_i-integrable functions ν_i, i.e. $\mathbb{E}_i[\nu_i] < \infty$, for $i = 1, ..., d$. Analogously, for the upper bound $\overline{P}_\varphi$, the corresponding dual is given by

$$\begin{aligned} \overline{D}_\varphi = \inf \Bigg\{ \overline{Q}(h^-) - \underline{Q}(g) + \sum_{i=1}^{d} \mathbb{E}_i[\nu_i] \colon \nu_i \in L(F_i), i = 1, ..., d, \\ h, g \in \mathcal{R} \text{ s.t. } h^- - g + \sum_{i=1}^{d} \nu_i \geq \varphi \Bigg\}. \end{aligned} \tag{5.5}$$

Note, that the roles of $\underline{Q}$ and $\overline{Q}$ are reversed in $\underline{D}_\varphi$, i.e. we subtract the sum w.r.t. $\overline{Q}$ from the sum w.r.t. $\underline{Q}$ in the formulation of $\underline{D}_\varphi$ and vice versa for $\overline{D}_\varphi$. In the rest of this section, we show that strong duality between the primal and the dual problem holds under mild conditions on the function φ, so that:

$$\underline{P}_\varphi = \underline{D}_\varphi \quad \text{and} \quad \overline{P}_\varphi = \overline{D}_\varphi.$$

Several approaches to proving duality results of this type have been established in the literature. For instance, Rüschendorf [53] and Rüschendorf and Gaffke [51] establish duality results for functionals of multivariate random variables with given marginals using a Hahn-Banach separation argument. More recently, a duality result for the martingale optimal transport problem was established by Beiglböck, Henry-Labordère, and Penkner [5] using a minmax argument, and Bartl, Cheridito, Kupper, and Tangpi [4] derive a general duality result for convex functionals with countably many marginal constraints using the Daniell-Stone Theorem. An account of the history of the Monge–Kantorovich Duality Theory and associated references can be found in the survey by Rüschendorf [57] or in the book by Villani [68].

The proof of our Duality Theorem 5.1.4 below is based on the following auxiliary results.

Lemma 5.1.1 (Kantorovich Duality for copulas)**.** *Let $\varphi\colon \mathbb{R}^d \to \mathbb{R}$ be lower semicontinuous and such that*

$$\sum_{i=1}^{d} \varrho_i(x_i) \geq |\varphi(x_1, ..., x_d)| \quad \textit{for all } (x_1, ..., x_d) \in \mathbb{R}^d, \tag{5.6}$$

for some continuous functions $\varrho_i \in L(F_i)$, $i = 1, ..., d$. Then the following duality holds

$$\inf\left\{\mathbb{E}_C[\varphi]\colon C \in \mathcal{C}^d\right\} = \sup\left\{\sum_{i=1}^{d} \mathbb{E}_i[\nu_i]\colon \nu_i \in L(F_i),\ i = 1, ..., d,\ \sum_{i=1}^{d} \nu_i \leq \varphi\right\}. \tag{5.7}$$

Proof. The statement follows immediately by an application of Sklar's Theorem. In particular, we have that

$$\inf\left\{\mathbb{E}_C[\varphi]\colon C \in \mathcal{C}^d\right\} = \inf\left\{\int_{\mathbb{R}^d} \varphi(\mathbf{x})\, \mathrm{d}F(\mathbf{x})\colon F \in \mathcal{F}(F_1, ..., F_d)\right\}.$$

Then, due to (5.6) we can apply the classical Kantorovich Duality Theorem (see e.g. [68, Theorem 5.10] or [57, Theorem 3.1]) to the right-hand side of the equation and obtain (5.7). □

The following Lemma links the uniform convergence of copulas to the weak convergence of the associated random vectors. The proof is an immediate consequence of Theorem 2.1 in Lindner and Szimayer [32] and therefore omitted.

Lemma 5.1.2. *Let $(X_1^n, ..., X_d^n)_{n\in\mathbb{N}}$ be a sequence of $\mathbb{R}^d$-valued random vectors with marginals $X_i^n \sim F_i$ for $i = 1, ..., d$ and all $n \in \mathbb{N}$. Moreover, denote by C_n the copula of $(X_1^n, ..., X_d^n)$ for $n \in \mathbb{N}$. If the sequence of copulas C_n converges uniformly to a copula C then $(X_1^n, ..., X_d^n)$ converges weakly to a random vector $(X_1, ..., X_d)$ with marginals $F_1, ..., F_d$ and copula C.*

The following Minmax Theorem is presented as Corollary 2 in Terkelsen [66].

Lemma 5.1.3 (Minmax Theorem). *Let B_1 be a compact convex subset of a topological vector space V_1 and B_2 be a convex subset of a vector space V_2. If $f\colon B_1 \times B_2 \to \mathbb{R}$ is such that*

1. *$f(\cdot, b_2)$ is lower semicontinuous and convex on B_1 for all $b_2 \in B_2$,*
2. *$f(b_1, \cdot)$ is concave on B_2 for all $b_1 \in B_1$,*

then

$$\inf_{b_1\in B_1} \sup_{b_2\in B_2} f(b_1, b_2) = \sup_{b_2\in B_2} \inf_{b_1\in B_1} f(b_1, b_2).$$

With these results, we are now in the position to establish our main duality theorem.

Theorem 5.1.4. *Let $\varphi\colon \mathbb{R}^d \to \mathbb{R}$ be such that*

$$\sum_{i=1}^{d} \varrho_i(x_i) \geq |\varphi(x_1, ..., x_d)| \quad \text{for all } (x_1, ..., x_d) \in \mathbb{R}^d, \tag{5.8}$$

for some continuous functions $\varrho_i \in L(F_i)$, $i = 1, ..., d$. Moreover, assume that there exists a copula $C \in \mathcal{C}^d$ with $\underline{Q} \leq C \leq \overline{Q}$. Then if φ is lower semicontinuous the following duality holds:

$$\underline{P}_\varphi = \underline{D}_\varphi.$$

Conversely, when φ is upper semicontinuous the following duality holds:

$$\overline{P}_\varphi = \overline{D}_\varphi.$$

Moreover, the primal values are attained, i.e. there exist copulas $\underline{C}, \overline{C}$ such that $\mathbb{E}_{\underline{C}}[\varphi] = \underline{P}_\varphi$ and $\mathbb{E}_{\overline{C}}[\varphi] = \overline{P}_\varphi$.

Proof. We show that the statement holds for the lower bound, i.e. $\underline{D}_\varphi = \underline{P}_\varphi$. The proof for the upper bound can be derived by applying analogous arguments to the function $-\varphi$.

First, assume that φ is bounded and continuous. By ν_i we refer to functions in $L(F_i)$ for $i = 1, ..., d$. It follows that

$$\underline{D}_\varphi = \sup_{h,g\in\mathcal{R}} \sup_{\substack{\nu_1,...,\nu_d \\ h-g^-+\sum_{i=1}^d \nu_i \le \varphi}} \left\{ \underline{Q}(h) - \overline{Q}(g^-) + \sum_{i=1}^d \mathbb{E}_i[\nu_i] \right\} \tag{5.9}$$

$$= \sup_{h,g\in\mathcal{R}} \sup_{\substack{\nu_1,...,\nu_d \\ \sum_{i=1}^d \nu_i \le \varphi-h+g^-}} \left\{ \underline{Q}(h) - \overline{Q}(g^-) + \sum_{i=1}^d \mathbb{E}_i[\nu_i] \right\} \tag{5.10}$$

$$= \sup_{h,g\in\mathcal{R}} \inf_{C\in\mathcal{C}^d} \left\{ \underline{Q}(h) - \overline{Q}(g^-) + \mathbb{E}_C[\varphi - h + g^-] \right\} \tag{5.11}$$

$$= \sup_{h,g\in\mathcal{R}} \inf_{C\in\mathcal{C}^d} \left\{ \Big(\underline{Q}(h) - C(h)\Big) - \Big(\overline{Q}(g^-) - C(g^-)\Big) + \mathbb{E}_C[\varphi] \right\} \tag{5.12}$$

$$= \inf_{C\in\mathcal{C}^d} \left\{ \sup_{h,g\in\mathcal{R}} \left\{ \Big(\underline{Q}(h) - C(h)\Big) - \Big(\overline{Q}(g^-) - C(g^-)\Big) \right\} + \mathbb{E}_C[\varphi] \right\} \tag{5.13}$$

$$= \inf_{\substack{C\in\mathcal{C}^d \\ \underline{Q}(h)\le C(h)\le \overline{Q}(h),\ \forall h\in\mathcal{R}}} \mathbb{E}_C[\varphi] \tag{5.14}$$

$$= \inf_{\underline{Q}\le C\le \overline{Q}} \mathbb{E}_C[\varphi] = \underline{P}_\varphi. \tag{5.15}$$

Equation (5.11) follows from an application of Lemma 5.1.1 to the function $\varphi' := \varphi - h + g^-$. Note, that the application of Lemma 5.1.1 is justified since φ' is lower semicontinuous, being the sum of the lower semicontinuous functions φ, $-h$ and g^-. Moreover, since h and g are of the form

$$h = \sum_{n=1}^k \alpha_n \Lambda_{\mathbf{u}^n}, \quad g = \sum_{n=1}^m \beta_n \Lambda_{\mathbf{v}^n}$$

for $\alpha_1, ..., \alpha_k, \beta_1, ..., \beta_m \in \mathbb{R}_+$, we obtain

$$|(\varphi + h - g^-)(x_1, ..., x_d)| \le \sum_{i=1}^d \varrho_i(x_i) + \sum_{n=1}^k \alpha_n + \sum_{n=1}^m \beta_n.$$

Equation (5.12) then follows by rearranging the terms, using the linearity of the expectation and the definition of the operator $C(h)$ for $h \in \mathcal{R}$. Now, applying the Minmax Theorem 5.1.3 to the function

$$f\colon \mathcal{C}^d \times \mathcal{R}^2 \ni \Big(C, (h, g)\Big) \mapsto \Big(\underline{Q}(h) - C(h)\Big) - \Big(\overline{Q}(g^-) - C(g^-)\Big) + \mathbb{E}_C[\varphi]$$

yields equation (5.13). Note, that the requirements of Theorem 5.1.3 are fulfilled, since

$$\mathcal{C}_b = \Big\{C \in \mathcal{C}^d \colon \underline{Q} \leq C \leq \overline{Q}\Big\}$$

is a closed, bounded and equicontinuous subset of the topological space of all continuous functions on $\mathbb{I}^d$, equipped with the uniform metric. Hence, it follows from the Arzelà-Ascoli Theorem that $\mathcal{C}_b$ is compact. Moreover, $\mathcal{C}_b$ and $\mathcal{R}^2$ are convex sets. On the other hand, it follows from Lemma 5.1.2 that for all $h, g \in \mathcal{R}$ the map $f\Big(\cdot\,, (h,g)\Big)$ is continuous w.r.t. the uniform convergence of copulas since we assume φ to be bounded and continuous. Furthermore, we have that $f\Big(\cdot\,, (h,g)\Big)$ is convex on $\mathcal{C}^d$. Also, for all $C \in \mathcal{C}^d$ it holds that $f(C, \cdot)$ is linear on $\mathcal{R}^2$. To verify (5.14), assume that $\underline{Q}(h) \leq C(h) \leq \overline{Q}(h)$ does not hold for one $h \in \mathcal{R}$, i.e. let w.l.o.g. $C(h) < \underline{Q}(h)$, then for each $\alpha > 0$ it follows that

$$\Big(\underline{Q}(\alpha h) - C(\alpha h)\Big) = \alpha\Big(\underline{Q}(h) - C(h)\Big) > 0$$

and thus, by scaling α, the supremum is ∞ and C can be disregarded in the infimum in (5.13). Hence, it holds that

$$\underline{Q}(h) \leq C(h) \leq \overline{Q}(h), \text{for all } h \in \mathcal{R}.$$

This entails that $\underline{Q}(h) - C(h) \leq 0$ and $-(\overline{Q}(g) - C(g)) \leq 0$ for all $(g,h) \in \mathcal{R}^2$ and thus the supremum is attained for $h, g \equiv 0$. Finally, (5.15) holds due to the fact that $\underline{Q}(h) \leq C(h) \leq \overline{Q}(h)$ for all $h \in \mathcal{R}$ implies

$$\underline{Q}(F_1(x_1), ..., F_d(x_d)) \leq C(F_1(x_1), ..., F_d(x_d)) \leq \overline{Q}(F_1(x_1), ..., F_d(x_d))$$

for all $(x_1, ..., x_d) \in \mathbb{R}^d$ and $\mathbb{E}_C[\varphi] = \mathbb{E}_{C'}[\varphi]$ for all copulas C and C' with

$$C(F_1(x_1), ..., F_d(x_d)) = C'(F_1(x_1), ..., F_d(x_d)).$$

We proceed by relaxing the condition of φ being bounded and continuous. So let φ merely be lower semicontinuous. We can w.l.o.g. assume that $\varphi \geq 0$ as otherwise there exist, due to condition (5.8), functions $\varrho_1, ...\varrho_d$ with $\varphi + \sum_{i=1}^d \varrho_i \geq 0$ and

$$\overline{P}_{\varphi} = \overline{P}_{\varphi + \sum_{i=1}^d \varrho_i} - \sum_{i=1}^d \mathbb{E}_i[\varrho_i].$$

Now, since φ is lower semicontinuous there exists a sequence of positive, bounded, continuous functions $\varphi_1 \le \varphi_2 \le \cdots$ with $\varphi = \lim_n \varphi_n$ pointwise and $\underline{P}_{\varphi_n} \le \underline{P}_{\varphi}$. Furthermore, due to the compactness of $\mathcal{C}_b$ there exist optimizers $C_1, C_2, ...$ of $\underline{P}_{\varphi_1}, \underline{P}_{\varphi_2}, ...$ and we can, by passing to a subsequence, assume that $C_1, C_2, ...$ converges to some $C^* \in \mathcal{C}_b$. Then it follows by monotone convergence that

$$\underline{P}_{\varphi} \le \mathbb{E}_{C^*}[\varphi] = \lim_n \mathbb{E}_{C^*}[\varphi_n] = \lim_n \lim_j \mathbb{E}_{C_j}[\varphi_n] \le \lim_j \mathbb{E}_{C_j}[\varphi_j] = \lim_j \underline{P}_{\varphi_j} = \lim_j \underline{D}_{\varphi_j}.$$

Lastly, note that the optimizers of the primal problems are attained due to the compactness of $\mathcal{C}_b$ which completes the proof. □

Remark 5.1.2. Assuming the existence of a copula $C \in \mathcal{C}^d$ with $\underline{Q} \le C \le \overline{Q}$ in Theorem 5.1.4 rules out the degenerate situation where no probabilistic model exists which is compatible with the prescribed information. Verifying this assumption however is a delicate task in general. The existence of a copula C with $\underline{Q} \le C$ follows immediately from the fact that the upper Fréchet–Hoeffding bound M_d is a copula and hence $\underline{Q} \le M_d$. The difficulty thus lies in verifying $C \le \overline{Q}$ which fails e.g. when $\overline{Q} = W_d$ and $d > 2$, where W_d is the lower Fréchet–Hoeffding bound. Nevertheless, when $\underline{Q}$ and $\overline{Q}$ are improved Fréchet–Hoeffding bounds it is often straight-forward to verify that $\{C \in \mathcal{C}^d : \underline{Q} \le C \le \overline{Q}\} \neq \emptyset$. ♦

The following counter-example shows that the dual optimizers are not attained in general.

Example 5.1.1. Consider the case $d = 2$ and let F_1 and F_2 be uniform marginal distribution on $\mathbb{I}$. Moreover, let $\underline{Q}(u_1, u_2) = \overline{Q}(u_1, u_2) = \Pi(u_1, u_2) = u_1 u_2$ for all $(u_1, u_2) \in \mathbb{I}^2$ and consider $\varphi\colon \mathbb{R}^2 \to \mathbb{R}\colon (u_1, u_2) \mapsto \mathbb{1}_{\psi(u_1,u_2)<1}$ where $\psi(u_1, u_2) = \sqrt{u_1^2 + u_2^2}$, i.e. φ is the characteristic function of the circular segment of the unit circle on $\mathbb{I}^2$. It then follows from $\underline{Q} = \overline{Q} = \Pi$, that

$$\underline{P}_{\varphi} = \overline{P}_{\varphi} = \int_{\mathbb{I}^2} \mathbb{1}_{\sqrt{u_1^2+u_2^2}<1} \, \mathrm{d}u_1 \mathrm{d}u_2 = \frac{\pi}{4}.$$

Now, assume the dual optimizer for $\underline{D}_{\varphi}$ is attained. Then it is of the form

$$f^* := h - g + \nu_1 + \nu_2 = \sum_{n=1}^{k} \alpha_n \Lambda_{\mathbf{u}^n} + \sum_{n=1}^{m} -\beta_n \Lambda_{\mathbf{v}^n} + \nu_1 + \nu_2$$

and since $f^*(u_1, u_2) \le \mathbb{1}_{\psi(u_1,u_2)<1}$ for all $(u_1, u_2) \in \mathbb{I}^2$ and $\mathbb{E}_{\Pi}[f^*] = \frac{\pi}{4}$, we have that

$$f^*(u_1, u_2) = (h - g + \nu_1 + \nu_2)(u_1, u_2) = \mathbb{1}_{\psi(u_1,u_2)<1} \quad \lambda\text{-a.s.} \tag{5.16}$$

Moreover, we can assume w.l.o.g. that $\nu_1 \equiv \nu_2 \equiv 0$ λ-a.s. since it follows from equation

(5.16) that

$$(h-g)(u_1,1) = \mathbb{1}_{\psi(u_1,1)<1} - \nu_1(u_1) - \nu_2(1) = -\nu_1(u_1) - c \quad \lambda\text{-a.s.}$$

where the last inequality is due to $\mathbb{1}_{\psi(u_1,1)<1} = 0$ λ-a.s. and $c := \nu_2(1)$. Now, by the same argument it follows that

$$(h-g)(1,u_2) = -\nu_2(u_2) - c' \quad \lambda\text{-a.s.},$$

for $c' := \nu_1(1)$, and thus we obtain

$$(h-g)(u_1,u_d) = \left(\sum_{n=1}^{k} \alpha_n \Lambda_{\mathbf{u}^n} + \sum_{n=1}^{m} -\beta_n \Lambda_{\mathbf{v}^n}\right)(u_1,u_2) = \mathbb{1}_{\psi(u_1,u_2)<1} \quad \lambda\text{-a.s.}$$

This however, corresponds to a construction of the characteristic function of the circular segment by a finite number of rectangular characteristic functions which contradicts the impossibility of the squaring of the circle. ♦

5.2 A reduction scheme to compute bounds on Value-at-Risk

The dual characterizations of $\underline{P}_\varphi$ and $\overline{P}_\varphi$ in Section 5.1 lend themselves to the development of a scheme to compute VaR estimates, accounting for an upper and a lower bound on the copula of the risks. In general, the dual problems are intractable and closed form solutions have only been obtained in the situation where $\underline{Q} = W_d$, $\overline{Q} = M_d$ with homogeneous marginals $F_1 = \cdots = F_d$ fulfilling additional constraints; c.f. Puccetti and Rüschendorf [45] and Wang and Wang [69]. We therefore develop in this section a scheme that corresponds to an optimization over a tractable subset of admissible functions for the duals $\underline{D}_\varphi$ and $\overline{D}_\varphi$ and produces narrow VaR bounds. Furthermore, we show that the scheme produces asymptotically sharp bounds in the certainty limit, i.e. when $\underline{Q}$ and $\overline{Q}$ converge to some copula C.

5.2.1 A reduction scheme for the lower bound

Consider the function $\varphi(x_1, ..., x_d) = \mathbb{1}_{\psi(x_1,...,x_d)<s}$ for componentwise increasing $\psi\colon \mathbb{R}^d \to \mathbb{R}$ and recall from (5.1) that our primal problem of interest reads

$$\underline{P}_\varphi := \inf\Big\{\mathbb{E}_C[\varphi]\colon C \in \mathcal{C}^d, \underline{Q} \leq C \leq \overline{Q}\Big\},$$

for quasi-copulas $\underline{Q}$ and $\overline{Q}$, whereas the corresponding dual problem is given in (5.4) by

$$\underline{D}_\varphi = \sup\Bigg\{\underline{Q}(h) - \overline{Q}(g^-) + \sum_{i=1}^d \mathbb{E}_i[\nu_i]\colon \nu_i \in L(F_i),\ i = 1, ..., d,\\ h, g \in \mathcal{R} \text{ s.t. } h - g^- + \sum_{i=1}^d \nu_i \leq \varphi\Bigg\}.$$

In the following, we identify admissible functions for the dual $\underline{D}_\varphi$ by $(d+2)$-tuples in the class

$$\underline{\mathcal{A}} := \Bigg\{(h, g, \nu_1, ..., \nu_d)\colon \nu_i \in L(F_i),\ i = 1, ..., d,\ h, g \in \mathcal{R} \text{ s.t. } h - g^- + \sum_{i=1}^d \nu_i \leq \varphi\Bigg\}$$

and for each admissible tuple the corresponding value of the objective function amounts to

$$\underline{Q}(h) - \overline{Q}(g^-) + \sum_{i=1}^d \mathbb{E}_i[\nu_i].$$

Regarding the Improved Standard Bounds in [18, 16], we note that when the copula C of $\mathbf{X}$ is bounded from below by $\underline{Q}$, i.e. $\underline{Q} \leq C$, then the lower Improved Standard Bound is given by

$$\mathbb{E}_C[\mathbb{1}_{\psi(\mathbf{X})<s}] \geq \sup_{u_1,...,u_{d-1}\in\mathbb{R}} \underline{Q}\Big(F_1(u_1), \dots, F_{d-1}(u_{d-1}), F_d^-(\psi^*_{u_{-d}}(s))\Big) = \underline{m}_{\underline{Q},\psi}(s),$$

which corresponds, in the case of continuous marginals, to the maximization of $\underline{Q}(h)$ over functions $h = \Lambda_{\mathbf{u}} \in \mathcal{R}$ with $\mathbf{u} \in \Big\{\Big(u_1, ..., u_{d-1}, \psi^*_{u_{-d}}(s)\Big)\colon (u_1, ...u_{d-1}) \in \mathbb{R}^{d-1}\Big\}$. Hence, $\underline{m}_{\underline{Q},\psi}(s)$ can be viewed as an optimization over a – rather small – subset of admissible elements in $\underline{\mathcal{A}}$, i.e. tuples of the form $(h, 0, ..., 0) \in \underline{\mathcal{A}}$.

Leveraging this observation, we develop an optimization scheme over a larger subset of admissible functions. To this end, let us first consider admissible $(h, g, \nu_1, ..., \nu_d)$ with

$(\mathbf{A1})$ $\nu_1, ..., \nu_d \equiv 0$,

$(\mathbf{A2})$ $h, g \in \mathcal{R}^\square$, where

$$\mathcal{R}^\square := \left\{ \sum_{n=1}^{k} \Lambda_{\mathbf{u}^n} \colon k \in \mathbb{N}, \mathbf{u}_1, ..., \mathbf{u}_k \in \mathcal{U}_\psi(s) \right\},$$

and $\mathcal{U}_\psi(s) = \{(x_1, ..., x_d) \in \mathbb{R}^d \colon \psi(x_1, ..., x_d) < s\}$. We thus obtain a subset of projections of admissible functions in $\underline{\mathcal{A}}$ given by

$$\underline{\mathcal{A}}^\square := \Big\{(h, g) \colon h, g \in \mathcal{R}^\square \text{ s.t. } h - g^- \leq \varphi\Big\}.$$

The optimization over the subset $\underline{\mathcal{A}}^\square$ remains however intractable due to the constraint $h - g^- \leq \varphi$. Moreover, optimizing over $\underline{\mathcal{A}}^\square$ requires a truncation of k and an appropriate choice for such a truncation is not obvious. We therefore proceed with the development of an unconstrained optimization scheme over a finite number of elements in $\mathcal{U}_\psi(s)$. An informal description and illustration of the scheme and the idea of the proofs is provided in Section 5.4. We make use of the following notion of multisets (c.f. Syropoulos [63, Definition 2]).

Definition 5.2.1. Let $\mathcal{B}$ be some set. A multiset over $\mathcal{B}$ is a pair $\langle \mathcal{B}, f \rangle$ where $f \colon \mathcal{B} \to \mathbb{N}$ and f is called *multiplicity function*.

Multisets generalise the notion of a set so as to allow for finite but multiple occurrences of elements. The following example illustrates this feature.

Example 5.2.1. By the conventional notion of a set we have that $\mathcal{B} := \{1, 1, 2\} = \{1, 2\}$. Using the notion multisets we refer to $\{1, 1, 2\}$ as $\langle \mathcal{B}, f \rangle$ with $f(1) = 2$ and $f(2) = 1$. The multiplicity function f hence counts the number of occurrences of each element of $\mathcal{B}$. ♦

We establish the following Inclusion-Exclusion Principle for multisets as an auxiliary result for our scheme.

Lemma 5.2.2 (Multiset Inclusion-Exclusion Principle)**.** *Let $B_1, ..., B_k \subset \mathbb{R}^d$ and define for $m = 1, ..., k$ the multisets*

$$\begin{aligned} &\langle \mathcal{B}^o, f^o \rangle, \quad \mathcal{B}^o := \{B_{i_1} \cap \cdots \cap B_{i_m} \colon 0 \leq i_1 \leq \cdots \leq i_m \leq k,\ m \text{ odd}\}, \\ &\langle \mathcal{B}^e, f^e \rangle, \quad \mathcal{B}^e := \{B_{i_1} \cap \cdots \cap B_{i_m} \colon 0 \leq i_1 \leq \cdots \leq i_m \leq k,\ m \text{ even}\}, \end{aligned} \tag{5.17}$$

where

$$f^o(B) = |\{(i_1, ..., i_m) \colon 0 \leq i_1 \leq \cdots \leq i_m \leq k,\ m \text{ odd},\ B = B_{i_1} \cap \cdots \cap B_{i_m}\}|,$$

for $B \in \mathcal{B}^o$ *and* f^e *is defined analogously. Then*

$$\mathbb{1}_{B_1 \cup \cdots \cup B_k} = \sum_{B \in \mathcal{B}^o} \Big(f^o(B) - f^e(B)\Big)^+ \mathbb{1}_B - \sum_{B \in \mathcal{B}^e} \Big(f^e(B) - f^o(B)\Big)^+ \mathbb{1}_B.$$

Proof. Applying the classical Inclusion-Exclusion Principle (see e.g. Loera et al. [33, Lemma 6.1.2]) to $\mathbb{1}_{B_1 \cup \cdots \cup B_k}$ yields

$$\mathbb{1}_{B_1 \cup \cdots \cup B_k} = \sum_{B \in \mathcal{B}^o} f^o(B)\, \mathbb{1}_B - \sum_{B \in \mathcal{B}^e} f^e(B)\, \mathbb{1}_B.$$

Then, by rearranging the terms and using the fact that $f^e(B) = 0$ when $B \in \mathcal{B}^o \setminus \mathcal{B}^e$ and $f^o(B) = 0$ when $B \in \mathcal{B}^e \setminus \mathcal{B}^o$ we obtain

$$\begin{aligned}\sum_{B \in \mathcal{B}^o} f^o(B)\, \mathbb{1}_B - \sum_{B \in \mathcal{B}^e} f^e(B) \mathbb{1}_B = &\sum_{B \in \mathcal{B}^o \setminus \mathcal{B}^e} f^o(B)\, \mathbb{1}_B - \sum_{B \in \mathcal{B}^e \setminus \mathcal{B}^o} f^e(B)\, \mathbb{1}_B \\ &+ \sum_{B \in \mathcal{B}^o \cap \mathcal{B}^e} \Big(f^o(B) - f^e(B)\Big)\, \mathbb{1}_B.\end{aligned}$$

The statement then follows from

$$\Big(f^o(B) - f^e(B)\Big) = \Big(f^o(B) - f^e(B)\Big)^+ - \Big(f^e(B) - f^o(B)\Big)^+.$$

which completes the proof. □

Remark 5.2.1. Lemma 5.2.2 establishes a non-redundant version of the classical Inclusion-Exclusion Principle. To illustrate this, consider $B_1, B_2, B_3 \in \mathbb{R}^d$ such that $B_1 \cap B_2 \cap B_3 = B_1 \cap B_2$ and $B_1 \neq B_2$. Then applying the classical Inclusion-Exclusion Principle to $B_1 \cup B_2 \cup B_3$ yields

$$\mathbb{1}_{B_1 \cup B_2 \cup B_3} = \mathbb{1}_{B_1} + \mathbb{1}_{B_2} + \mathbb{1}_{B_3} - \mathbb{1}_{B_1 \cap B_2} - \mathbb{1}_{B_1 \cap B_3} - \mathbb{1}_{B_2 \cap B_3} + \mathbb{1}_{B_1 \cap B_2 \cap B_3},$$

where the terms $-\mathbb{1}_{B_1 \cap B_2}$ and $+\mathbb{1}_{B_1 \cap B_2 \cap B_3}$ cancel each other out. This superfluous subtraction and addition of terms is avoided using the multisets $\langle \mathcal{B}^o, f^o \rangle$ and $\langle \mathcal{B}^e, f^e \rangle$ as in Lemma 5.2.2. Due to $f^o(B_1 \cap B_2) = f^e(B_1 \cap B_2 \cap B_3)$ we then obtain

$$\mathbb{1}_{B_1 \cup B_2 \cup B_3} = \mathbb{1}_{B_1} + \mathbb{1}_{B_2} + \mathbb{1}_{B_3} - \mathbb{1}_{B_1 \cap B_3} - \mathbb{1}_{B_2 \cap B_3},$$

and thus a more parsimonious representation of $\mathbb{1}_{B_1 \cup B_2 \cup B_3}$. ♦

In the following we denote the componentwise minimum of vectors $\mathbf{u}^1, ..., \mathbf{u}^k \in \mathbb{R}^d$ by

$$\min(\mathbf{u}^1, ..., \mathbf{u}^k) = \Big(\min_{n=1,...,k} \{u_1^n\}, ..., \min_{n=1,...,k} \{u_d^n\} \Big).$$

Moreover, let us define the sets

$$\begin{aligned} \mathcal{M}^o(\mathbf{u}^1, ..., \mathbf{u}^k) &:= \{\min(\mathbf{u}^{i_1}, ..., \mathbf{u}^{i_m}) \colon 0 \leq i_1 \leq \cdots \leq i_m \leq k \;\; m \text{ odd}\}, \\ \mathcal{M}^e(\mathbf{u}^1, ..., \mathbf{u}^k) &:= \{\min(\mathbf{u}^{i_1}, ..., \mathbf{u}^{i_m}) \colon 0 \leq i_1 \leq \cdots \leq i_m \leq k \;\; m \text{ even}\}. \end{aligned} \tag{5.18}$$

We refer to the multiplicity function

$$l^o(\mathbf{u}) := |\{(i_1, ..., i_m) \colon 0 \leq i_1 \leq \cdots \leq i_m \leq k, \; m \text{ odd}, \; \mathbf{u} = \min(\mathbf{u}^{i_1}, ..., \mathbf{u}^{i_m})\}|$$

for $\mathbf{u} \in \mathcal{M}^o(\mathbf{u}^1, ..., \mathbf{u}^k)$ as the *multiplicity function associated to* $\mathcal{M}^o(\mathbf{u}^1, ..., \mathbf{u}^k)$ and define the multiplicity function associated to $\mathcal{M}^e(\mathbf{u}^1, ..., \mathbf{u}^k)$, denoted by l^e, analogously. The normalization of a vector $\mathbf{u} \in \mathbb{R}^d$ by the marginals $F_1, ..., F_d$ is denoted by $F(\mathbf{u}) := \big(F_1(u_1), ..., F_d(u_d)\big)$ as well as the left-continuous version $F^-(\mathbf{u}) := \big(F_1^-(u_1), ..., F_d^-(u_d)\big)$. Finally, for $\boldsymbol{\varepsilon} := (\varepsilon, ..., \varepsilon) \in \mathbb{R}^d$ and $\mathbf{u} \in \mathbb{R}^d$ we denote $\mathbf{u} + \boldsymbol{\varepsilon} = (u_1 + \varepsilon, ..., u_d + \varepsilon)$.

Lemma 5.2.3. *Let $k \in \mathbb{N}$ and $\mathbf{u}^1, ..., \mathbf{u}^k \in \mathcal{U}_\psi(s)$. Define the functions*

$$h := \sum_{\mathbf{u} \in \mathcal{M}^o(\mathbf{u}^1, ..., \mathbf{u}^k)} \Big(l^o(\mathbf{u}) - l^e(\mathbf{u})\Big)^+ \Lambda_{\mathbf{u}}; \quad g_{\boldsymbol{\varepsilon}} := \sum_{\mathbf{u} \in \mathcal{M}^e(\mathbf{u}^1, ..., \mathbf{u}^k)} \Big(l^e(\mathbf{u}) - l^o(\mathbf{u})\Big)^+ \Lambda_{\mathbf{u}+\boldsymbol{\varepsilon}},$$

for l^o and l^e being the multiplicity functions associated to $\mathcal{M}^o(\mathbf{u}^1, ..., \mathbf{u}^k)$ and $\mathcal{M}^e(\mathbf{u}^1, ..., \mathbf{u}^k)$ respectively. Then for every $\boldsymbol{\varepsilon} > \mathbf{0}$ it holds that $(h, g_{\boldsymbol{\varepsilon}})$ is admissible for the dual problem $\underline{D}_\varphi$, i.e. $(h, g_{\boldsymbol{\varepsilon}}) \in \underline{\mathcal{A}}^\square$, and the value of the objective function is given by

$$\sum_{\mathbf{u} \in \mathcal{M}^o(\mathbf{u}^1, ..., \mathbf{u}^k)} \Big(l^o(\mathbf{u}) - l^e(\mathbf{u})\Big)^+ \underline{Q}\Big(F(\mathbf{u})\Big) - \sum_{\mathbf{u} \in \mathcal{M}^e(\mathbf{u}^1, ..., \mathbf{u}^k)} \Big(l^e(\mathbf{u}) - l^o(\mathbf{u})\Big)^+ \overline{Q}\Big(F^-(\mathbf{u} + \boldsymbol{\varepsilon})\Big).$$

Proof. It suffices to show that $h - g_{\boldsymbol{\varepsilon}}^- \leq \varphi$ for any $\boldsymbol{\varepsilon} > \mathbf{0}$. Recalling the notion of the sublevel set

$$\mathcal{U}_\psi(s) = \Big\{(u_1, ...u_d) \in \mathbb{R}^d \colon \psi(u_1, ..., u_d) < s\Big\},$$

we have for every $(u_1, ..., u_d) \in \mathcal{U}_\psi(s)$ that

$$\Big\{(x_1, ..., x_d) \in \mathbb{R}^d \colon x_1 \leq u_1, ..., x_d \leq u_d\Big\} \subset \Big\{(x_1, ..., x_d) \in \mathbb{R}^d \colon \psi(x_1, \cdots, x_d) < s\Big\},$$

due to the fact that ψ is increasing in each coordinate.

Hence, for $\mathbf{u}^1, ..., \mathbf{u}^k \in \mathcal{U}_\psi(s)$ and $B_n := \Big\{(x_1, ..., x_d) \in \mathbb{R}^d \colon x_1 \leq u_1^n, ..., x_d \leq u_d^n\Big\}$ for $n = 1, ..., k$ we have that

$$\bigcup_{n=1}^{k} B_n \subset \Big\{(x_1, ..., x_d) \in \mathbb{R}^d \colon \psi(x_1, \cdots, x_d) < s\Big\}.$$

Now applying the Inclusion-Exclusion Principle for multisets (Lemma 5.2.2) to $\bigcup_{n=1}^k B_n$ we obtain

$$\mathbb{1}_{B_1 \cup \cdots \cup B_k} = \sum_{B \in \mathcal{B}^o} \Big(f^o(B) - f^e(B)\Big)^+ \mathbb{1}_B - \sum_{B \in \mathcal{B}^e} \Big(f^e(B) - f^o(B)\Big)^+ \mathbb{1}_B,$$

where $\mathcal{B}^o$ and $\mathcal{B}^e$ are as in (5.17) and f^o, f^e are the respective multiplicity functions. Moreover, we have for $\bigcap_{l=1}^m B_{n_l} \in \mathcal{B}^o \cup \mathcal{B}^e$ that

$$\begin{aligned}
\bigcap_{l=1}^{m} B_{n_l} &= \bigcap_{l=1}^{m} \Big\{(x_1, ..., x_d) \in \mathbb{R}^d \colon x_1 \leq u_1^{n_l}, ..., x_d \leq u_d^{n_l}\Big\} \\
&= \Big\{(x_1, ..., x_d) \in \mathbb{R}^d \colon x_1 \leq \min(u_1^{n_1}, ..., u_1^{n_m}), ..., x_d \leq \min(u_d^{n_1}, ..., u_d^{n_m})\Big\} \\
&= \{\mathbf{x} \in \mathbb{R}^d \colon \mathbf{x} \leq \min(\mathbf{u}^{n_1}, ..., \mathbf{u}^{n_m})\},
\end{aligned}$$

and thus $\mathbb{1}_{B_{n_1} \cap \cdots \cap B_{n_m}} = \Lambda_{\min(\mathbf{u}^{n_1}, ..., \mathbf{u}^{n_m})}$ for $\min(\mathbf{u}^{n_1}, ..., \mathbf{u}^{n_m}) \in \mathcal{M}^o \cup \mathcal{M}^e$. Also, if $B = \bigcap_{l=1}^m B_{n_l} \in \mathcal{B}^o$ we have that

$$f^o(B) = l^o(\min(\mathbf{u}^{n_1}, ..., \mathbf{u}^{n_m}))$$

and $f^e(B) = l^e(\min(\mathbf{u}^{n_1}, ..., \mathbf{u}^{n_m}))$ for $B = \bigcap_{l=1}^m B_{n_l} \in \mathcal{B}^e$. In particular, it follows for any $\boldsymbol{\varepsilon} > \mathbf{0}$ that

$$\begin{aligned}
h(\mathbf{x}) - g_{\boldsymbol{\varepsilon}}^-(\mathbf{x}) &\leq h(\mathbf{x}) - g_{\mathbf{0}}(\mathbf{x}) \\
&\leq \sum_{\mathbf{u} \in \mathcal{M}^o(\mathbf{u}^1, ..., \mathbf{u}^k)} \Big(l^o(\mathbf{u}) - l^e(\mathbf{u})\Big)^+ \Lambda_{\mathbf{u}}(\mathbf{x}) - \sum_{\mathbf{u} \in \mathcal{M}^e(\mathbf{u}^1, ..., \mathbf{u}^k)} \Big(l^e(\mathbf{u}) - l^o(\mathbf{u})\Big)^+ \Lambda_{\mathbf{u}}(\mathbf{x}) \\
&= \sum_{B \in \mathcal{B}^o} \Big(f^o(B) - f^e(B)\Big)^+ \mathbb{1}_B(\mathbf{x}) - \sum_{B \in \mathcal{B}^e} \Big(f^e(B) - f^o(B)\Big)^+ \mathbb{1}_B(\mathbf{x})
\end{aligned}$$

$$= \mathbb{1}_{B_1 \cup \cdots \cup B_k}(\mathbf{x}) \le \mathbb{1}_{\psi(\mathbf{x})<s} = \varphi(\mathbf{x})$$

for all $\mathbf{x} \in \mathbb{R}^d$ and so $(h, g_{\boldsymbol{\varepsilon}}) \in \underline{\mathcal{A}}^{\square}$ which completes the proof. $\square$

We are now in the position to state the reduced optimization problem for $\underline{D}_{\varphi}$.

Corollary 5.2.4. *Let $\varphi(x_1, ..., x_d) = \mathbb{1}_{\psi(x_1,...,x_d)<s}$ for $\psi\colon \mathbb{R}^d \to \mathbb{R}$ increasing in each coordinate and let*

$$\underline{D}_{\varphi}^{\square}(k) := \sup\Bigg\{ \sum_{\mathbf{u}\in\mathcal{M}^o(\mathbf{u}^1,...,\mathbf{u}^k)} \Big(l^o(\mathbf{u}) - l^e(\mathbf{u})\Big)^+ \underline{Q}\Big(F(\mathbf{u})\Big) - \sum_{\mathbf{u}\in\mathcal{M}^e(\mathbf{u}^1,...,\mathbf{u}^k)} \Big(l^e(\mathbf{u}) - l^o(\mathbf{u})\Big)^+ \overline{Q}\Big(F(\mathbf{u})\Big) \colon \mathbf{u}^1, ..., \mathbf{u}^k \in \mathcal{U}_{\psi}(s)\Bigg\}, \tag{5.19}$$

where l^o and l^e are the canonical multiplicity functions associated to $\mathcal{M}^o(\mathbf{u}^1, ..., \mathbf{u}^k)$ and $\mathcal{M}^e(\mathbf{u}^1, ..., \mathbf{u}^k)$ respectively. Then

$$\underline{D}_{\varphi}^{\square}(k) \le \underline{D}_{\varphi}^{\square}(k+1) \le ... \le \underline{D}_{\varphi}.$$

Proof. We first show that $\underline{D}_{\varphi}^{\square}(k) \le \underline{D}_{\varphi}^{\square}(k+1)$ for $k \in \mathbb{N}$. Therefore note that when $\mathbf{u}^k = \mathbf{u}^{k+1} \in \mathcal{U}_{\psi}(s)$ it follows that $\mathcal{M}^o(\mathbf{u}^1, ..., \mathbf{u}^{k+1}) = \mathcal{M}^o(\mathbf{u}^1, ..., \mathbf{u}^k)$ and $\mathcal{M}^e(\mathbf{u}^1, ..., \mathbf{u}^{k+1}) = \mathcal{M}^e(\mathbf{u}^1, ..., \mathbf{u}^k)$. Moreover, by straight-forward calculations we obtain

$$\begin{aligned}
&\sum_{\mathbf{u}\in\mathcal{M}^o(\mathbf{u}^1,...,\mathbf{u}^{k+1})} \Big(l^o_{k+1}(\mathbf{u}) - l^e_{k+1}(\mathbf{u})\Big)^+ \underline{Q}\Big(F(\mathbf{u})\Big) \\
&\qquad - \sum_{\mathbf{u}\in\mathcal{M}^e(\mathbf{u}^1,...,\mathbf{u}^{k+1})} \Big(l^e_{k+1}(\mathbf{u}) - l^o_{k+1}(\mathbf{u})\Big)^+ \overline{Q}\Big(F(\mathbf{u})\Big) \\
&= \sum_{\mathbf{u}\in\mathcal{M}^o(\mathbf{u}^1,...,\mathbf{u}^k)} \Big(l^o_k(\mathbf{u}) - l^e_k(\mathbf{u})\Big)^+ \underline{Q}\Big(F(\mathbf{u})\Big) - \sum_{\mathbf{u}\in\mathcal{M}^e(\mathbf{u}^1,...,\mathbf{u}^k)} \Big(l^e_k(\mathbf{u}) - l^o_k(\mathbf{u})\Big)^+ \overline{Q}\Big(F(\mathbf{u})\Big),
\end{aligned}$$

where l^o_j, l^e_j are the canonical multiplicity functions associated to $\mathcal{M}^o(\mathbf{u}^1, ..., \mathbf{u}^j)$ and $\mathcal{M}^e(\mathbf{u}^1, ..., \mathbf{u}^j)$, respectively for $j = k, k+1$. This implies in particular that $\underline{D}_{\varphi}^{\square}(k) \le \underline{D}_{\varphi}^{\square}(k+1)$ for all $k \in \mathbb{N}$.

Furthermore, the inequality $\underline{D}_{\varphi}^{\square}(k) \le \underline{D}_{\varphi}$ for all $k \in \mathbb{N}$ follows by an application of Lemma 5.2.3 to $(h, g_{\boldsymbol{\varepsilon}})$ where $\boldsymbol{\varepsilon} > \mathbf{0}$ and

$$h = \sum_{\mathbf{u}\in\mathcal{M}^o(\mathbf{u}^1,...,\mathbf{u}^k)} \Big(l^o(\mathbf{u}) - l^e(\mathbf{u})\Big)^+ \Lambda_{\mathbf{u}}; \quad g_{\boldsymbol{\varepsilon}} = \sum_{\mathbf{u}\in\mathcal{M}^e(\mathbf{u}^1,...,\mathbf{u}^k)} \Big(l^e(\mathbf{u}) - l^o(\mathbf{u})\Big)^+ \Lambda_{\mathbf{u}+\boldsymbol{\varepsilon}}.$$

This yields that $(h, g_{\boldsymbol{\varepsilon}}) \in \underline{\mathcal{A}}^{\square}$ and the respective value of the objective function is given by

$$\sum_{\mathbf{u}\in\mathcal{M}^o(\mathbf{u}^1,\ldots,\mathbf{u}^k)} \big(l^o(\mathbf{u})-l^e(\mathbf{u})\big)^+ \,\underline{Q}\big(F(\mathbf{u})\big) - \sum_{\mathbf{u}\in\mathcal{M}^e(\mathbf{u}^1,\ldots,\mathbf{u}^k)} \big(l^e(\mathbf{u})-l^o(\mathbf{u})\big)^+ \,\overline{Q}\big(F^-(\mathbf{u}+\boldsymbol{\varepsilon})\big).$$

In particular, we have for all $k \in \mathbb{N}$ that

$$\begin{aligned}
&\sup\Bigg\{ \sum_{\mathbf{u}\in\mathcal{M}^o(\mathbf{u}^1,\ldots,\mathbf{u}^k)} \big(l^o(\mathbf{u})-l^e(\mathbf{u})\big)^+ \,\underline{Q}\big(F(\mathbf{u})\big) \\
&\quad - \sum_{\mathbf{u}\in\mathcal{M}^e(\mathbf{u}^1,\ldots,\mathbf{u}^k)} \big(l^e(\mathbf{u})-l^o(\mathbf{u})\big)^+ \,\overline{Q}\big(F^-(\mathbf{u}+\boldsymbol{\varepsilon})\big) \colon \boldsymbol{\varepsilon} > \mathbf{0};\ \mathbf{u}^1,\ldots,\mathbf{u}^k \in \mathcal{U}_\psi(s) \Bigg\} \\
&=: c^* \le \underline{D}_\varphi.
\end{aligned} \tag{5.20}$$

Moreover, it holds for all $\boldsymbol{\varepsilon} > \mathbf{0}$ that

$$\begin{aligned}
&\sum_{\mathbf{u}\in\mathcal{M}^o(\mathbf{u}^1,\ldots,\mathbf{u}^k)} \big(l^o(\mathbf{u})-l^e(\mathbf{u})\big)^+ \,\underline{Q}\big(F(\mathbf{u})\big) \\
&\qquad\qquad - \sum_{\mathbf{u}\in\mathcal{M}^e(\mathbf{u}^1,\ldots,\mathbf{u}^k)} \big(l^e(\mathbf{u})-l^o(\mathbf{u})\big)^+ \,\overline{Q}\big(F^-(\mathbf{u}+\boldsymbol{\varepsilon})\big) \\
&\le \sum_{\mathbf{u}\in\mathcal{M}^o(\mathbf{u}^1,\ldots,\mathbf{u}^k)} \big(l^o(\mathbf{u})-l^e(\mathbf{u})\big)^+ \,\underline{Q}\big(F(\mathbf{u})\big) - \sum_{\mathbf{u}\in\mathcal{M}^e(\mathbf{u}^1,\ldots,\mathbf{u}^k)} \big(l^e(\mathbf{u})-l^o(\mathbf{u})\big)^+ \,\overline{Q}\big(F(\mathbf{u})\big)
\end{aligned}$$

as well as $\lim_{\boldsymbol{\varepsilon}\to\mathbf{0}} \overline{Q}\big(F^-(\mathbf{u}+\boldsymbol{\varepsilon})\big) = \overline{Q}\big(F(\mathbf{u})\big)$ due to the Lipschitz continuity of $\overline{Q}$. Hence using (5.20) it follows that

$$\begin{aligned}
c^* = \sup\Bigg\{ &\sum_{\mathbf{u}\in\mathcal{M}^o(\mathbf{u}^1,\ldots,\mathbf{u}^k)} \big(l^o(\mathbf{u})-l^e(\mathbf{u})\big)^+ \,\underline{Q}\big(F(\mathbf{u})\big) \\
&- \sum_{\mathbf{u}\in\mathcal{M}^e(\mathbf{u}^1,\ldots,\mathbf{u}^k)} \big(l^e(\mathbf{u})-l^o(\mathbf{u})\big)^+ \,\overline{Q}\big(F(\mathbf{u})\big) \colon \mathbf{u}^1,\ldots,\mathbf{u}^k \in \mathcal{U}_\psi(s) \Bigg\}
\end{aligned}$$

which completes the proof. □

Corollary 5.2.4 establishes a tractable optimization problem yielding a lower bound on $\underline{D}_\varphi$ and thus also on $\underline{P}_\varphi$. The optimization takes place over vectors in the sublevel set $\mathcal{U}_\psi(s)$ and the trade-off between the computational effort and the quality of the bound is moderated by the variable k. For fixed k, $\underline{D}^{\square}_\varphi(k)$ amounts to a $k \cdot d$ dimensional optimization that can be solved with standard optimization packages. Note, that most mathematical programming environments also provide efficient built-in procedures to compute the multiplicity functions l^o and l^e.

5.2.2 A reduction scheme for the upper bound

We proceed with the development of a similar reduction scheme based on the dual $\overline{D}_\varphi$. Recall from (5.5) that

$$\overline{D}_\varphi = \inf\Bigg\{\overline{Q}(h^-) - \underline{Q}(g) + \sum_{i=1}^{d} \mathbb{E}_i[\nu_i] \colon \nu_i \in L(F_i),\ i = 1, ..., d, \\ h, g \in \mathcal{R} \text{ s.t. } h^- - g + \sum_{i=1}^{d} \nu_i \geq \varphi\Bigg\}.$$

We refer to the class of admissible functions for $\overline{D}_\varphi$ by

$$\overline{\mathcal{A}} := \Bigg\{(h, g, \nu_1, ..., \nu_d) \colon \nu_i \in L(F_i),\ i = 1, ..., d,\ h, g \in \mathcal{R} \text{ s.t. } h^- - g + \sum_{i=1}^{d} \nu_i \geq \varphi\Bigg\}$$

and for each admissible function the corresponding value of the objective function is given by

$$\overline{Q}(h^-) - \underline{Q}(g) + \sum_{i=1}^{d} \mathbb{E}_i[\nu_i].$$

Again, for our reduction scheme, we consider a subclass of admissible pairs (h, g) such that $h, g \in \mathcal{R}^\square$, i.e.

$$\overline{\mathcal{A}}^\square := \Big\{(h, g) \colon h, g \in \mathcal{R}^\square \text{ s.t. } h^- - g \geq \varphi\Big\}$$

We now turn to the formal construction of admissible functions in $\overline{\mathcal{A}}^\square$ with an auxiliary version of the multiset Inclusion-Exclusion Principle for intersections.

Lemma 5.2.5. *Let $B_1^n, ..., B_d^n \subset \mathbb{R}^d$ for $n = 1, ..., k$ and $k \in \mathbb{N}$ and define*

$$G_{(i_1,...,i_k)} := (B_{i_1}^1 \cap \cdots \cap B_{i_k}^k),\ \textit{for}\ (i_1, ..., i_k) \in \{1, ..., d\}^k; \quad \textit{and} \quad \mathcal{B} := \bigcap_{n=1}^{k} \bigcup_{l=1}^{d} B_l^n.$$

Moreover, for an enumeration $\{\mathbf{i}^1, ..., \mathbf{i}^{dk}\}$ *of the set* $\{1, ..., d\}^k$ *define the multisets*

$$\begin{aligned} &\langle \mathcal{G}^o, f^o \rangle, \quad \mathcal{G}^o := \{G_{\mathbf{i}^{n_1}} \cap \cdots \cap G_{\mathbf{i}^{n_m}} : 0 \le n_1 \le \cdots \le n_m \le k,\ m \text{ odd}\} \\ &\langle \mathcal{G}^e, f^e \rangle, \quad \mathcal{G}^e := \{G_{\mathbf{i}^{n_1}} \cap \cdots \cap G_{\mathbf{i}^{n_m}} : 0 \le n_1 \le \cdots \le n_m \le k,\ m \text{ even}\} \end{aligned} \tag{5.21}$$

where

$$f^o(G) = |\{(\mathbf{i}^{n_1}, ..., \mathbf{i}^{n_m}) \colon 0 \le n_1 \le \cdots \le n_m \le k,\ m \text{ odd},\ G = G_{\mathbf{i}^{n_1}} \cap \cdots \cap G_{\mathbf{i}^{n_m}}\}|$$

and f^e *is defined analogously. Then it holds that*

$$\mathbb{1}_{\mathcal{B}} = \sum_{G \in \mathcal{G}^o} \Big(f^o(G) - f^e(G) \Big)^+ \mathbb{1}_G - \sum_{G \in \mathcal{G}^e} \Big(f^e(G) - f^o(G) \Big)^+ \mathbb{1}_G.$$

Proof. Since the union and the intersection of sets commute we have that $\mathcal{B} = \mathcal{G}_{\mathbf{i}^1} \cup \cdots \cup \mathcal{G}_{\mathbf{i}^{kd}}$ and hence the statement follows by a straight-forward application of Lemma 5.2.2. □

We are now ready to establish an explicit construction of admissible pairs $(h, g) \in \overline{\mathcal{A}}^{\square}$. To this end, let us denote for $\mathbf{u}^1, ..., \mathbf{u}^k \in \mathbb{R}^d$ and an enumeration $\{\mathbf{i}^1, ..., \mathbf{i}^{dk}\}$ of $\{1, ..., d\}^k$

$$\mathrm{U}_{\mathbf{i}^n} := \min(\mathrm{pr}_{i_1}(\mathbf{u}^1), ..., \mathrm{pr}_{i_k}(\mathbf{u}^k)) \quad \text{for } (i_1, ..., i_k) = \mathbf{i}^n, n = 1, ..., dk,$$

where $\mathrm{pr}_i(\mathbf{u}) := (\infty, ..., \infty, u_i, \infty, ..., \infty)$ for $i \in \{1, ..., d\}$. Moreover, we define

$$\begin{aligned} \mathcal{W}^o(\mathbf{u}^1, ..., \mathbf{u}^k) &:= \{\min(\mathrm{U}_{\mathbf{i}^{n_1}}, ..., \mathrm{U}_{\mathbf{i}^{n_m}}) \colon 0 \le n_1 \le \cdots \le n_m \le k \ \ m \text{ odd}\} \\ &= \mathcal{M}^o(\mathrm{U}_{\mathbf{i}^1}, ..., \mathrm{U}_{\mathbf{i}^{dk}}) \\ \mathcal{W}^e(\mathbf{u}^1, ..., \mathbf{u}^k) &:= \{\min(\mathrm{U}_{\mathbf{i}^{n_1}}, ..., \mathrm{U}_{\mathbf{i}^{n_m}}) \colon 0 \le n_1 \le \cdots \le n_m \le k \ \ m \text{ even}\} \\ &= \mathcal{M}^e(\mathrm{U}_{\mathbf{i}^1}, ..., \mathrm{U}_{\mathbf{i}^{dk}}). \end{aligned} \tag{5.22}$$

Lemma 5.2.6. *Let* $k \in \mathbb{N}$ *and* $\mathbf{u}^1, ..., \mathbf{u}^k \in \mathcal{U}^c_\psi(s)$. *Define the functions*

$$h_{\boldsymbol{\varepsilon}} := \sum_{\mathbf{u} \in \mathcal{W}^o(\mathbf{u}^1, ..., \mathbf{u}^k)} \Lambda_{\mathbf{u}+\boldsymbol{\varepsilon}}; \quad g := \sum_{\mathbf{u} \in \mathcal{W}^o(\mathbf{u}^1, ..., \mathbf{u}^k)} \Lambda_{\mathbf{u}},$$

for l^o *and* l^e *being the multiplicity functions associated to* $\mathcal{W}^o(\mathbf{u}^1, ..., \mathbf{u}^k)$ *and* $\mathcal{W}^e(\mathbf{u}^1, ..., \mathbf{u}^k)$ *respectively. Then for every* $\boldsymbol{\varepsilon} > \mathbf{0}$ *it holds that* $(h_{\boldsymbol{\varepsilon}}, g)$ *is admissible for the dual problem*

$\overline{D}_\varphi$, *i.e.* $(h_{\boldsymbol{\varepsilon}}, g) \in \overline{\mathcal{A}}^{\square}$, *and the value of the objective function is given by*

$$\sum_{\mathbf{u} \in \mathcal{W}^o(\mathbf{u}^1,\dots,\mathbf{u}^k)} \Big(l^o(\mathbf{u}) - l^e(\mathbf{u})\Big)^+ \overline{Q}\Big(F^-(\mathbf{u}+\boldsymbol{\varepsilon})\Big) - \sum_{\mathbf{u} \in \mathcal{W}^e(\mathbf{u}^1,\dots,\mathbf{u}^k)} \Big(l^e(\mathbf{u}) - l^o(\mathbf{u})\Big)^+ \underline{Q}\Big(F(\mathbf{u})\Big).$$

Proof. We need to show that $h^-_{\boldsymbol{\varepsilon}} - g \geq \varphi$. Note, that for every $(u_1, ..., u_d) \in \mathcal{U}^c_\psi(s)$ we have that

$$\Big\{(x_1, ..., x_d) \in \mathbb{R}^d \colon x_1 \geq u_1, ..., x_d \geq u_d\Big\}^c \supset \Big\{(x_1, ..., x_d) \in \mathbb{R}^d \colon \psi(x_1, \cdots, x_d) < s\Big\},$$

due to the fact that ψ is increasing in each coordinate. Hence, for $\mathbf{u}^1, ..., \mathbf{u}^k \in \mathcal{U}^c_\psi(s)$ and $B_n := \Big\{(x_1, ..., x_d) \in \mathbb{R}^d \colon x_1 \geq u^n_1, ..., x_d \geq u^n_d\Big\}$ for $n = 1, ..., k$ it follows that

$$\bigcap_{n=1}^k B^c_n \supset \Big\{(x_1, ..., x_d) \in \mathbb{R}^d \colon \psi(x_1, \cdots, x_d) < s\Big\}. \tag{5.23}$$

Moreover, for $n = 1, ..., k$ we have that

$$B^c_n = \bigcup_{i=1}^d \Big\{(x_1, ..., x_d) \in \mathbb{R}^d \colon \mathbf{x} < \mathrm{pr}(u^n)_i\Big\},$$

which follows from

$$\begin{aligned} \Big\{(x_1, ..., x_d) \in \mathbb{R}^d \colon x_1 \geq u_1, ..., x_d \geq u_d\Big\}^c &= \Big([u_1, \infty) \times \cdots \times [u_d, \infty)\Big)^c \\ &= \bigcup_{i=1}^d \mathbb{R} \times \cdots \times \mathbb{R} \times (-\infty, u_i) \times \mathbb{R} \times \cdots \times \mathbb{R} \\ &= \bigcup_{i=1}^d \Big\{(x_1, ..., x_d) \in \mathbb{R}^d \colon \mathbf{x} < \mathrm{pr}(u^n)_i\Big\}. \end{aligned}$$

Hence, we obtain

$$\bigcap_{n=1}^k B^c_n = \bigcap_{n=1}^k \bigcup_{i=1}^d \Big\{(x_1, ..., x_d) \in \mathbb{R}^d \colon \mathbf{x} < \mathrm{pr}(u^n)_i\Big\}. \tag{5.24}$$

Now, defining

$$\begin{aligned} &H^n_i := \Big\{(x_1, ..., x_d) \in \mathbb{R}^d \colon \mathbf{x} < \mathrm{pr}(u^n)_i)\Big\}, \text{ for } i = 1, ..., d, n = 1, ..., k \text{ and} \\ &\mathcal{G}_{(i_1,\dots,i_k)} := H^1_{i_1} \cap \cdots \cap H^k_{i_k}, \ (i_1, ..., i_k) \in \{1, ..., d\}^k, \end{aligned}$$

and applying Lemma 5.2.5 to equation (5.24) we arrive at

$$\mathbb{1}_{B_1^c\cap\cdots\cap B_k^c} = \sum_{G\in\mathcal{G}^o} \Big(f^o(G) - f^e(G)\Big)^+ \mathbb{1}_G - \sum_{G\in\mathcal{G}^e} \Big(f^e(G) - f^o(G)\Big)^+ \mathbb{1}_G,$$

where $\mathcal{G}^o, \mathcal{G}^e$ and f^o, f^e are defined in 5.21. Finally, note that for $1 \le n_1 \le \cdots \le n_m \le dk$

$$\mathcal{G}_{(i_1,\ldots,i_k)} = \Big\{(x_1, ..., x_d) \in \mathbb{R}^d \colon \mathbf{x} < \min\Big(\operatorname{pr}_{i_1}(\mathbf{u}^1), ..., \operatorname{pr}_{i_k}(\mathbf{u}^k)\Big)\Big\},$$

so that with the definition of $\mathrm{U}_{\mathbf{i}}$ for $\mathbf{i} \in \{1, ..., d\}^k$ it follows for every $1 \le n_1, ..., n_m \le dk$ that

$$\begin{aligned}\bigcap_{l=1}^m \mathcal{G}_{\mathbf{i}^{n_l}} &= \bigcap_{l=1}^m \Big\{(x_1, ..., x_d) \in \mathbb{R}^d \colon \mathbf{x} < \mathrm{U}_{\mathbf{i}^{n_l}}\Big\}\\ &= \Big\{(x_1, ..., x_d) \in \mathbb{R}^d \colon \mathbf{x} < \min(\mathrm{U}_{\mathbf{i}^{n_1}}, ..., \mathrm{U}_{\mathbf{i}^{n_m}})\Big\},\end{aligned}$$

and thus $\mathbb{1}_{\mathcal{G}_{\mathbf{i}^{n_1}}\cap\cdots\cap\mathcal{G}_{\mathbf{i}^{n_m}}} = \Lambda^-_{\min(\mathrm{U}_{\mathbf{i}^{n_1}},...,\mathrm{U}_{\mathbf{i}^{n_m}})}$ for $\min(\mathrm{U}_{\mathbf{i}^{n_1}}, ..., \mathrm{U}_{\mathbf{i}^{n_m}}) \in \mathcal{W}^o(\mathbf{u}^1, ..., \mathbf{u}^k) \cup \mathcal{W}^e(\mathbf{u}^1, ..., \mathbf{u}^k)$. In particular, using equation (5.23) and the fact that

$$l^o(\min(\mathrm{U}_{\mathbf{i}^{n_1}}, ..., \mathrm{U}_{\mathbf{i}^{n_m}})) = f^o(G)$$

for $G = \bigcap_{l=1}^m G_{\mathbf{i}^{n_l}} \in \mathcal{G}^o$ and vice versa for l^e, we conclude that

$$h_{\boldsymbol{\varepsilon}}^-(\mathbf{x}) - g(\mathbf{x}) \ge h_{\mathbf{0}}^-(\mathbf{x}) - g^-(\mathbf{x}) = \mathbb{1}_{B_1^c\cap\cdots\cap B_k^c}(\mathbf{x}) \ge \mathbb{1}_{\psi(\mathbf{x})<s} = \varphi(\mathbf{x})$$

for all $\mathbf{x} \in \mathbb{R}^d$, which completes the proof. $\square$

We are now in the position to establish our reduction scheme based on $\overline{D}_\varphi$. The proof of the following corollary is analogous to the proof of Corollary 5.2.4 and therefore omitted.

Corollary 5.2.7. *Let $\varphi(x_1, ..., x_d) = \mathbb{1}_{\psi(x_1,...,x_d)<s}$ for $\psi\colon \mathbb{R}^d \to \mathbb{R}$ increasing in each coordinate and let*

$$\begin{aligned}\overline{D}_\varphi^{\square}(k) := \inf\Bigg\{ &\sum_{\mathbf{u}\in\mathcal{W}^o(\mathbf{u}^1,...,\mathbf{u}^k)} \Big(l^o(\mathbf{u}) - l^e(\mathbf{u})\Big)^+ \overline{Q}\Big(F(\mathbf{u})\Big)\\ &- \sum_{\mathbf{u}\in\mathcal{W}^o(\mathbf{u}^1,...,\mathbf{u}^k)} \Big(l^e(\mathbf{u}) - l^o(\mathbf{u})\Big)^+ \underline{Q}\Big(F(\mathbf{u})\Big) \colon \mathbf{u}^1, ..., \mathbf{u}^k \in \mathcal{U}_\psi^c(s)\Bigg\},\end{aligned} \tag{5.25}$$

where l^o and l^e are the canonical multiplicity functions associated to $\mathcal{W}^o(\mathbf{u}^1, ..., \mathbf{u}^k)$ and

$\mathcal{W}^e(\mathbf{u}^1, ..., \mathbf{u}^k)$ *respectively. Then*

$$\overline{D}_\varphi^\square(k) \geq \overline{D}_\varphi^\square(k+1) \geq ... \geq \overline{D}_\varphi.$$

5.2.3 Sharp asymptotic bounds in the certainty limit

In general, the schemes $\underline{D}_\varphi^\square(k)$ and $\overline{D}_\varphi^\square(k)$ do not approximate the dual bounds $\underline{D}_\varphi$ and $\overline{D}_\varphi$ respectively for $k \to \infty$. In the homogeneous, complete dependence uncertainty case, i.e. $F_1 = \cdots = F_d = F$ and $\underline{Q} = W_d$ and $\overline{Q} = M_d$, Puccetti and Rüschendorf [45] derived an explicit solution to the dual $\underline{D}_\varphi$ under additional requirements on the marginals. They showed, that the optimizer is of the form $d \cdot \nu$ for a piecewise linear function $\nu \in L(F)$ which cannot be represented by the linear combinations in $\mathcal{R}^\square$.

The counterpart to the situation of complete dependence uncertainty is the case of certainty, i.e. the limit when $\underline{Q}$ and $\overline{Q}$ converge from below and above respectively to a copula C. A natural feature of any bound on the expectation of φ using the information from $\underline{Q}$ and $\overline{Q}$ should be that it converges to $\mathbb{E}_C[\varphi]$ as $\underline{Q}, \overline{Q} \to C$. The following theorem shows that for $k \to \infty$ the reduced bounds $\underline{D}_\varphi^\square(k)$ and $\overline{D}_\varphi^\square(k)$ indeed converge to the desired object in the certainty limit.

Theorem 5.2.8. *Let* $\varphi(x_1, ..., x_d) = \mathbb{1}_{\psi(x_1,...,x_d)<s}$ *for* $\psi\colon \mathbb{R}^d \to \mathbb{R}$ *increasing in each coordinate. Moreover, define for* $k \in \mathbb{N}$

$$\Big[\underline{D}_\varphi^\square(k)\Big](\underline{Q}, \overline{Q}) := \sup\Bigg\{ \sum_{\mathbf{u}\in\mathcal{M}^o(\mathbf{u}^1,...,\mathbf{u}^k)} \Big(l^o(\mathbf{u}) - l^e(\mathbf{u})\Big)^+ \underline{Q}\Big(F(\mathbf{u})\Big) - \sum_{\mathbf{u}\in\mathcal{M}^e(\mathbf{u}^1,...,\mathbf{u}^k)} \Big(l^e(\mathbf{u}) - l^o(\mathbf{u})\Big)^+ \overline{Q}\Big(F(\mathbf{u})\Big)\colon \mathbf{u}^1, ..., \mathbf{u}^k \in \mathcal{U}_\psi(s)\Bigg\}$$

where $\mathcal{M}^o(\mathbf{u}^1, ..., \mathbf{u}^k)$ *and* $\mathcal{M}^e(\mathbf{u}^1, ..., \mathbf{u}^k)$ *are defined in* (5.18) *and* l^o, l^e *are the associated multiplicity functions. Analogously, let*

$$\Big[\overline{D}_\varphi^\square(k)\Big](\underline{Q}, \overline{Q}) := \inf\Bigg\{ \sum_{\mathbf{u}\in\mathcal{W}^o(\mathbf{u}^1,...,\mathbf{u}^k)} \Big(f^o(\mathbf{u}) - f^e(\mathbf{u})\Big)^+ \underline{Q}\Big(F(\mathbf{u})\Big) - \sum_{\mathbf{u}\in\mathcal{W}^o(\mathbf{u}^1,...,\mathbf{u}^k)} \Big(f^e(\mathbf{u}) - f^o(\mathbf{u})\Big)^+ \overline{Q}\Big(F(\mathbf{u})\Big)\colon \mathbf{u}^1, ..., \mathbf{u}^k \in \mathcal{U}_\psi^c(s)\Bigg\}.$$

where $\mathcal{W}^o(\mathbf{u}^1, ..., \mathbf{u}^k)$ *and* $\mathcal{W}^e(\mathbf{u}^1, ..., \mathbf{u}^k)$ *as in* (5.22) *with associated multiplicity func-*

tions f^o, f^e. Then it holds for any copula C and sequences of quasi-copulas $\left(\underline{Q}^j\right)_{j=1,2...}$ and $\left(\overline{Q}^j\right)_{j=1,2...}$ with $\underline{Q}^j \leq C \leq \overline{Q}^j$ for all $j \in \mathbb{N}$ and $\underline{Q}^j, \overline{Q}^j \to_j C$ pointwise, that

$$\lim_j \inf_k \left[\overline{D}^{\square}_{\varphi}(k)\right](\underline{Q}^j, \overline{Q}^j) = \lim_j \sup_k \left[\underline{D}^{\square}_{\varphi}(k)\right](\underline{Q}^j, \overline{Q}^j) = \mathbb{P}_C(\psi(X_1, ..., X_d) < s)$$

Proof. We show that

$$\lim_j \sup_k \left[\underline{D}^{\square}_{\varphi}(k)\right](\underline{Q}^j, \overline{Q}^j) = \mathbb{P}_C(\psi(X_1, ..., X_d) < s).$$

The proof for $\overline{D}^{\square}_{\varphi}(k)$ follows along similar lines.

First, note that there exists a sequence $\mathbf{u}^1, \mathbf{u}^2, ... \in \mathcal{U}_\psi(s)$ such that for

$$h^k := \sum_{\mathbf{u} \in \mathcal{M}^o(\mathbf{u}^1,...,\mathbf{u}^k)} \left(l^o(\mathbf{u}) - l^e(\mathbf{u})\right)^+ \Lambda_{\mathbf{u}}; \quad g^k := \sum_{\mathbf{u} \in \mathcal{M}^e(\mathbf{u}^1,...,\mathbf{u}^k)} \left(l^e(\mathbf{u}) - l^o(\mathbf{u})\right)^+ \Lambda_{\mathbf{u}},$$

we have that $h^k - g^k \to_k \varphi$ pointwise. To verify the existence of such a sequence one can choose as $(\mathbf{u}^n)_{n=1,...,k}$, any discretization of the set $\mathcal{U}_\psi(s)$, whose mesh converges to zero for $k \to \infty$.

From the fact that $\mathbf{u}^1, \mathbf{u}^2, ... \in \mathcal{U}_\psi(s)$ and the proof of Lemma 5.2.3 it follows that $h^k - g^k \leq \varphi$ for all $k \in \mathbb{N}$, and the corresponding value of the objective function is given by

$$\sum_{\mathbf{u} \in \mathcal{M}^o(\mathbf{u}^1,...,\mathbf{u}^k)} \left(l^o(\mathbf{u}) - l^e(\mathbf{u})\right)^+ \underline{Q}\left(F(\mathbf{u})\right) - \sum_{\mathbf{u} \in \mathcal{M}^e(\mathbf{u}^1,...,\mathbf{u}^k)} \left(l^e(\mathbf{u}) - l^o(\mathbf{u})\right)^+ \overline{Q}\left(F(\mathbf{u})\right) \leq \left[\underline{D}^{\square}_{\varphi}(k)\right](\underline{Q}^j, \overline{Q}^j). \tag{5.26}$$

Using the fact that C is a copula and applying the Dominated Convergence Theorem yields

$$\mathbb{E}_C\left[h^k - g^k\right] \to_k \mathbb{E}_C[\varphi] = \mathbb{P}_C(\psi(X_1, ..., X_d) < s). \tag{5.27}$$

Therefore we can w.l.o.g. assume that for fixed j, there exists an $N \in \mathbb{N}$ such that

$$\mathbb{E}_C[h^k - g^k] \geq \left[\underline{D}^{\square}_{\varphi}(k)\right](\underline{Q}^j, \overline{Q}^j) \quad \text{for all } k \geq N,$$

since otherwise

$$\left[\underline{D}^{\square}_{\varphi}(k_m)\right](\underline{Q}^j, \overline{Q}^j) \geq \mathbb{E}_C[h^{k_m} - g^{k_m}] \to_m \mathbb{E}_C[\varphi],$$

along a subsequence $(k_m)_m$ and we are done.

We proceed by showing that the convergence

$$\lim_j \sup_k \Big[\underline{D}_\varphi^\square(k)\Big](\underline{Q}^j, \overline{Q}^j) = \mathbb{P}_C(\psi(X_1, ..., X_d) < s)$$

holds. To this end, fix an arbitrary $\varepsilon > 0$. Due to (5.27) we can choose $k \geq N$ such that

$$\Big|\mathbb{P}_C(\psi(X_1, ..., X_d) < s) - \mathbb{E}_C\Big[h^k - g^k\Big]\Big| < \frac{\varepsilon}{2}. \tag{5.28}$$

Moreover, the fact that quasi-copulas are Lipschitz continuous yields, by an application of the Arzelà-Ascoli Theorem, that $\underline{Q}^j \to_j C$ and $\overline{Q}^j \to_j C$ uniformly. Thus, for

$$p := \sum_{\mathbf{u} \in \mathcal{M}^o \mathbf{u}^1, ..., \mathbf{u}^k)} l^o(\mathbf{u}) \; + \sum_{\mathbf{u} \in \mathcal{M}^e(\mathbf{u}^1, ..., \mathbf{u}^k)} l^e(\mathbf{u})$$

we can choose an $j \in \mathbb{N}$ so that

$$\|C - \underline{Q}^j\|_\infty + \|C - \overline{Q}^j\|_\infty < \frac{\varepsilon}{2p}. \tag{5.29}$$

With this choice of k and j we arrive at

$$\begin{aligned}
&\Big|\mathbb{P}_C(\psi(X_1, ..., X_d) < s) - \Big[\underline{D}_\varphi^\square(k)\Big](\underline{Q}^j, \overline{Q}^j)\Big| \\
&\leq \Big|\mathbb{P}_C(\psi(X_1, ..., X_d) < s) - \mathbb{E}_C\Big[h^k - g^k\Big] + \mathbb{E}_C\Big[h^k - g^k\Big] - \Big[\underline{D}_\varphi^\square(k)\Big](\underline{Q}^j, \overline{Q}^j)\Big| \\
&\leq \Big|\mathbb{P}_C(\psi(X_1, ..., X_d) < s) - \mathbb{E}_C\Big[h^k - g^k\Big]\Big| \\
&+ \Big|\mathbb{E}_C\Big[h^k - g^k\Big] - \Big(\sum_{\mathbf{u} \in \mathcal{M}^o(\mathbf{u}^1, ..., \mathbf{u}^k)} \Big(l^o(\mathbf{u}) - l^e(\mathbf{u})\Big)^+ \underline{Q}^j\Big(F(\mathbf{u})\Big) \\
&\qquad\qquad - \sum_{\mathbf{u} \in \mathcal{M}^e(\mathbf{u}^1, ..., \mathbf{u}^k)} \Big(l^e(\mathbf{u}) - l^o(\mathbf{u})\Big)^+ \overline{Q}^j\Big(F(\mathbf{u})\Big)\Big)\Big| \\
&< \frac{\varepsilon}{2} + \Big|\sum_{\mathbf{u} \in \mathcal{M}^o(\mathbf{u}^1, ..., \mathbf{u}^k)} \Big(l^o(\mathbf{u}) - l^e(\mathbf{u})\Big)^+ \Big(C(F(\mathbf{u})) - \underline{Q}^j(F(\mathbf{u}))\Big) \\
&\qquad\qquad - \sum_{\mathbf{u} \in \mathcal{M}^e(\mathbf{u}^1, ..., \mathbf{u}^k)} \Big(l^o(\mathbf{u}) - l^e(\mathbf{u})\Big)^+ \Big(C(F(\mathbf{u})) - \overline{Q}^j(F(\mathbf{u}))\Big)\Big| \\
&< \frac{\varepsilon}{2} + p\Big(\|C - \underline{Q}^j\|_\infty + \|C - \overline{Q}^j\|_\infty\Big) < \frac{\varepsilon}{2} + p\frac{\varepsilon}{2p} = \varepsilon.
\end{aligned}$$

The second inequality is a consequence of equation (5.26) and the fact that $\mathbb{E}_C[h^k - g^k] \geq \Big[\underline{D}_\varphi^\square(k)\Big](\underline{Q}^j, \overline{Q}^j)$. The third inequality follows from equation (5.28) and last inequality holds due to equation (5.29).

Finally, since ε was arbitrary we have shown that

$$\lim_j \sup_k \left[\underline{D}_\varphi^\square(k)\right](\underline{Q}^j, \overline{Q}^j) = \mathbb{P}_C(\psi(X_1, ..., X_d) < s)$$

holds and hence the proof is complete. □

5.3 Using information on the survival copula

In this section we show that the reduction schemes in Section 5.2.1 and 5.2.2 can be applied similarly when information on the survival copula is provided. Specifically, we assume that the copula of $\mathbf{X}$ is such that $\underline{\widehat{Q}} \leq \widehat{C} \leq \widehat{\overline{Q}}$ where $\widehat{C}$ is the survival-function of C and $\underline{\widehat{Q}}, \widehat{\overline{Q}}$ are quasi-survival functions. We hence consider the primal problems

$$\underline{\widehat{P}}_\varphi := \inf\left\{\mathbb{E}_C[\varphi] \colon C \in \mathcal{C}^d, \underline{\widehat{Q}} \leq \widehat{C} \leq \widehat{\overline{Q}}\right\}, \tag{5.30}$$

$$\widehat{\overline{P}}_\varphi := \sup\left\{\mathbb{E}_C[\varphi] \colon C \in \mathcal{C}^d, \underline{\widehat{Q}} \leq \widehat{C} \leq \widehat{\overline{Q}}\right\}. \tag{5.31}$$

Note that due to

$$\widehat{C}(F_1(u_1), ..., F_d(u_d)) = \mathbb{P}(X_1 > u_1, ..., X_d > u_d) = \mathbb{P}(-X_1 < -u_1, ..., -X_d < -u_d),$$

for all $\mathbf{u} \in \mathbb{I}^d$, the condition $\underline{\widehat{Q}} \leq \widehat{C} \leq \widehat{\overline{Q}}$ is equivalent to $\underline{Q} \leq C_{-\mathbf{X}} \leq \overline{Q}$, where $\underline{Q}(\mathbf{u}) := \underline{\widehat{Q}}(\mathbf{1} - \mathbf{u})$, $\overline{Q}(\mathbf{u}) := \widehat{\overline{Q}}(\mathbf{1} - \mathbf{u})$ and $C_{-\mathbf{X}}$ is the copula of $-\mathbf{X}$. In particular, since $\underline{Q}$ and $\overline{Q}$ are quasi-copulas it follows from our Duality Theorem 5.1.4 and a transformation of variables that the sharp dual bound corresponding to $\underline{\widehat{P}}_\varphi$ is given by

$$\underline{\widehat{P}}_\varphi = \underline{\widehat{D}}_\varphi = \sup\left\{\underline{\widehat{Q}}(h) - \widehat{\overline{Q}}(g^-) + \sum_{i=1}^d \mathbb{E}_i[\nu_i] \colon (h, g, \nu_1, ..., \nu_d) \in \underline{\widehat{\mathcal{A}}}\right\}, \tag{5.32}$$

where

$$\underline{\widehat{\mathcal{A}}} := \left\{(h, g, \nu_1, ..., \nu_d) \colon \nu_i \in L(F_i),\ i = 1, ..., d,\ h, g \in \widehat{\mathcal{R}} \text{ s.t. } h - g^- + \sum_{i=1}^d \nu_i \leq \varphi\right\}.$$

and

$$\widehat{\mathcal{R}} := \left\{h = \sum_{n=1}^k \alpha_n \widehat{\Lambda}_{\mathbf{u}^n} \colon k \in \mathbb{N}, \alpha_1, ..., \alpha_k \geq 0, \mathbf{u}^1, ..., \mathbf{u}^k \in \overline{\mathbb{R}}^d\right\},$$

for $\widehat{\Lambda}_{\mathbf{u}}$ of the form

$$\widehat{\Lambda}_{\mathbf{u}}\colon \mathbb{R}^d \ni (x_1,...,x_d) \mapsto \mathbb{1}_{x_1\geq u_1,...,x_d\geq u_d}.$$

Moreover, we denote $\widehat{\Lambda}^-_{\mathbf{u}}(x_1,...,x_d) := \mathbb{1}_{x_1>u_1,...,x_d>u_d}$ and h^- for $h \in \widehat{\mathcal{R}}$ is defined accordingly. Finally, for quasi-survival functions $\widehat{Q}$ and $h = \sum_{n=1}^k \alpha_n \widehat{\Lambda}_{\mathbf{u}^n} \in \mathcal{R} \in \widehat{\mathcal{R}}$ we define,

$$Q(h) := \sum_{n=1}^k \alpha_n \widehat{Q}\Big(F_1(u_1^n),...,F_d(u_d^n)\Big);\quad Q(h^-) := \sum_{n=1}^k \alpha_n \widehat{Q}\Big(F_1^-(u_1^n),...,F_d^-(u_d^n)\Big).$$

Analogously, the sharp dual bound associated to $\widehat{\overline{P}}_\varphi$ is equal to

$$\widehat{\overline{P}}_\varphi = \widehat{\overline{D}}_\varphi = \inf\left\{\widehat{\overline{Q}}(h^-) - \underline{\widehat{Q}}(g) + \sum_{i=1}^d \mathbb{E}_i[\nu_i]\colon (h,g,\nu_1,...,\nu_d) \in \widehat{\overline{\mathcal{A}}}\right\}, \tag{5.33}$$

where

$$\widehat{\overline{\mathcal{A}}} := \left\{(h,g,\nu_1,...,\nu_d)\colon \nu_i \in L(F_i),\ i=1,...,d,\ h,g \in \widehat{\mathcal{R}} \text{ s.t. } h^- - g + \sum_{i=1}^d \nu_i \geq \varphi\right\}.$$

Based on these dual characterizations the following corollaries establish the corresponding reduction schemes. Using the fact that

$$\mathbb{P}(\psi(X_1,...,X_d) < s) = 1 - \mathbb{P}(\psi(X_1,...,X_d) \geq s),$$

the proofs involve similar arguments as the proofs of Corollary 5.2.4 and 5.2.7 and therefore they are omitted. We denote the componentwise maximum of vectors $\mathbf{u}^1,...,\mathbf{u}^k \in \mathbb{R}^d$ by

$$\max(\mathbf{u}^1,...,\mathbf{u}^k) = \Big(\max_{n=1,...,k}\{u_1^n\},...,\max_{n=1,...,k}\{u_d^n\}\Big).$$

Corollary 5.3.1. *Let $\varphi(x_1,...,x_d) = \mathbb{1}_{\psi(x_1,...,x_d)<s}$ for $\psi\colon \mathbb{R}^d \to \mathbb{R}$ increasing in each coordinate and let*

$$\begin{aligned}\widehat{\overline{D}}^{\square}_\varphi(k) := \inf\Bigg\{1 - &\sum_{\mathbf{u}\in\widehat{\mathcal{M}}^o(\mathbf{u}^1,...,\mathbf{u}^k)} \Big(l^o(\mathbf{u}) - l^e(\mathbf{u})\Big)^+ \underline{\widehat{Q}}\Big(F(\mathbf{u})\Big)\\ &- \sum_{\mathbf{u}\in\widehat{\mathcal{M}}^e(\mathbf{u}^1,...,\mathbf{u}^k)} \Big(l^e(\mathbf{u}) - l^o(\mathbf{u})\Big)^+ \widehat{\overline{Q}}\Big(F(\mathbf{u})\Big)\colon \mathbf{u}^1,...,\mathbf{u}^k \in \mathcal{U}^c_\psi(s)\Bigg\},\end{aligned} \tag{5.34}$$

where

$$\widehat{\mathcal{M}}^o(\mathbf{u}^1, ..., \mathbf{u}^k) := \{\max(\mathbf{u}^{i_1}, ..., \mathbf{u}^{i_m})\colon 0 \le i_1 \le \cdots \le i_m \le k \ \ m \text{ odd}\},$$
$$\widehat{\mathcal{M}}^e(\mathbf{u}^1, ..., \mathbf{u}^k) := \{\max(\mathbf{u}^{i_1}, ..., \mathbf{u}^{i_m})\colon 0 \le i_1 \le \cdots \le i_m \le k \ \ m \text{ even}\}.$$

and

$$\begin{aligned} l^o(\mathbf{u}) &:= |\{(i_1, ..., i_m)\colon 0 \le i_1 \le \cdots \le i_m \le k, \ m \text{ odd}, \mathbf{u} = \max(\mathbf{u}^{i_1}, ..., \mathbf{u}^{i_m})\}|, \\ l^e(\mathbf{u}) &:= |\{(i_1, ..., i_m)\colon 0 \le i_1 \le \cdots \le i_m \le k, \ m \text{ odd}, \mathbf{u} = \max(\mathbf{u}^{i_1}, ..., \mathbf{u}^{i_m})\}|, \end{aligned} \tag{5.35}$$

for $\mathbf{u} \in \widehat{\mathcal{M}}^o(\mathbf{u}^1, ..., \mathbf{u}^k)$ *and* $\mathbf{u} \in \widehat{\mathcal{M}}^e(\mathbf{u}^1, ..., \mathbf{u}^k)$ *respectively. Then it holds that*

$$\widehat{\overline{D}}^{\square}_{\varphi}(k) \ge \widehat{\overline{D}}^{\square}_{\varphi}(k+1) \ge ... \ge \widehat{\overline{D}}_{\varphi}.$$

Remark 5.3.1. Corollary 5.3.1 extends the upper Improved Standard Bound from Embrechts et al. [18] presented in Theorem 4.1.1 in the sense that

$$\inf_{h \in \widehat{\mathcal{R}}} 1 - \underline{\widehat{Q}}(h) = \inf_{\mathbf{u} \in \mathcal{U}^c_{\psi}(s)} 1 - \underline{\widehat{Q}}(F(\mathbf{u})) = \overline{M}_{\underline{\widehat{Q}}, \psi}(s).$$

♦

The following corollary establishes a similar reduction scheme for $\underline{\widehat{D}}$. To this end, let us denote for $\mathbf{u}^1, ..., \mathbf{u}^k \in \mathbb{R}^d$ and an enumeration $\{\mathbf{i}^1, ..., \mathbf{i}^{dk}\}$ of $\{1, ..., d\}^k$

$$\widehat{\mathrm{U}}_{\mathbf{i}^n} := \max(\widehat{\mathrm{pr}}_{i_1}(\mathbf{u}^1), ..., \widehat{\mathrm{pr}}_{i_k}(\mathbf{u}^k)) \quad \text{for } (i_1, ..., i_k) = \mathbf{i}^n, n = 1, ..., dk,$$

where $\widehat{\mathrm{pr}}_i(\mathbf{u}) := (-\infty, ..., -\infty, u_i, -\infty, ..., -\infty)$ for $i \in \{1, ..., d\}$.

Corollary 5.3.2. *Let* $\varphi(x_1, ..., x_d) = \mathbb{1}_{\psi(x_1, ..., x_d) < s}$ *for* $\psi\colon \mathbb{R}^d \to \mathbb{R}$ *increasing in each coordinate and let*

$$\begin{aligned} \underline{\widehat{D}}^{\square}_{\varphi}(k) := \sup\Bigg\{ 1 - &\sum_{\mathbf{u} \in \widehat{\mathcal{W}}^o(\mathbf{u}^1, ..., \mathbf{u}^k)} \Big(l^o(\mathbf{u}) - l^e(\mathbf{u})\Big)^+ \,\widehat{\overline{Q}}\Big(F(\mathbf{u})\Big) \\ &- \sum_{\mathbf{u} \in \widehat{\mathcal{W}}^o(\mathbf{u}^1, ..., \mathbf{u}^k)} \Big(l^e(\mathbf{u}) - l^o(\mathbf{u})\Big)^+ \,\underline{\widehat{Q}}\Big(F(\mathbf{u})\Big)\colon \mathbf{u}^1, ..., \mathbf{u}^k \in \mathcal{U}_{\psi}(s) \Bigg\}, \end{aligned} \tag{5.36}$$

where

$$\widehat{\mathcal{W}}^o(\mathbf{u}^1,...,\mathbf{u}^k) := \widehat{\mathcal{M}}^o(\widehat{\mathrm{U}}_{\mathbf{i}^1},...,\widehat{\mathrm{U}}_{\mathbf{i}^{dk}}),$$
$$\widehat{\mathcal{W}}^e(\mathbf{u}^1,...,\mathbf{u}^k) := \widehat{\mathcal{M}}^e(\widehat{\mathrm{U}}_{\mathbf{i}^1},...,\widehat{\mathrm{U}}_{\mathbf{i}^{dk}}), \tag{5.37}$$

for an enumeration $\{\mathbf{i}^1,...,\mathbf{i}^{dk}\}$ *of* $\{1,...,d\}^k$ *and* l^o, l^e *are given in* (5.35). *Then it holds that*

$$\widehat{\underline{D}}^{\square}_{\varphi}(k) \geq \widehat{\underline{D}}^{\square}_{\varphi}(k+1) \geq ... \geq \widehat{\underline{D}}_{\varphi}.$$

5.4 Illustrations and numerical examples

In this section we provide an informal description of the reduction schemes in section 5.2.1 and 5.2.2 in order to illustrate the underlying ideas. Furthermore, we provide several numerical examples comparing the performance of our reduction scheme to the Improved Standard Bounds from [18, 16].

A graphical illustration of $\underline{D}^{\square}_{\varphi}(k)$

For a graphical illustration of the scheme $\underline{D}^{\square}_{\varphi}(k)$ let us consider $\psi(x_1, x_2) = x_1 + x_2$ and F_1, F_2 uniform distributions on $[0,1]$. Due to assumption $(\mathbf{A1})$ and $(\mathbf{A2})$, we consider admissible functions which are sums of indicator functions of rectangular regions in $\mathcal{U}_\psi(s)$, as in

$$h - g = \sum_{n=1}^{k} \Lambda_{\mathbf{u}^k} - \sum_{n=1}^{m} \Lambda_{\mathbf{v}^n} \leq \mathbb{1}_{x_1+x_2<s}$$

for $\mathbf{u}^1,...,\mathbf{u}^k,\mathbf{v}^1,...,\mathbf{v}^m \in \mathcal{U}_\psi(s)$ and $k, m \in \mathbb{N}$. The corresponding value of the objective function to be maximized is given by $\underline{Q}(h) - \overline{Q}(g)$ for each pair $h, g \in \mathcal{R}^{\square}$. Figure 5.1 illustrates the structure of admissible functions considered in the reduction scheme.

The gray triangular region in figure 5.1, corresponds to the area

$$\mathcal{U}_\psi(s) = \{(u_1,u_2) \in \mathbb{I}^2 \colon u_1 + u_2 < s\}$$

for $s = 0.8$. The upper LHS simply depicts the region $\mathcal{U}_\psi(s)$. The green area in the upper RHS figure corresponds to the rectangle $[0, 0.4]^2 \subset \mathcal{U}_\psi(s)$ as induced by the function $h = \Lambda_{(0.4,0.4)}$, i.e. $\Lambda_{(0.4,0.4)}(u_1,u_2) = \varphi(u_1,u_2) = 1$ for all $(u_1,u_2) \in [0,0.4]^2$. The value

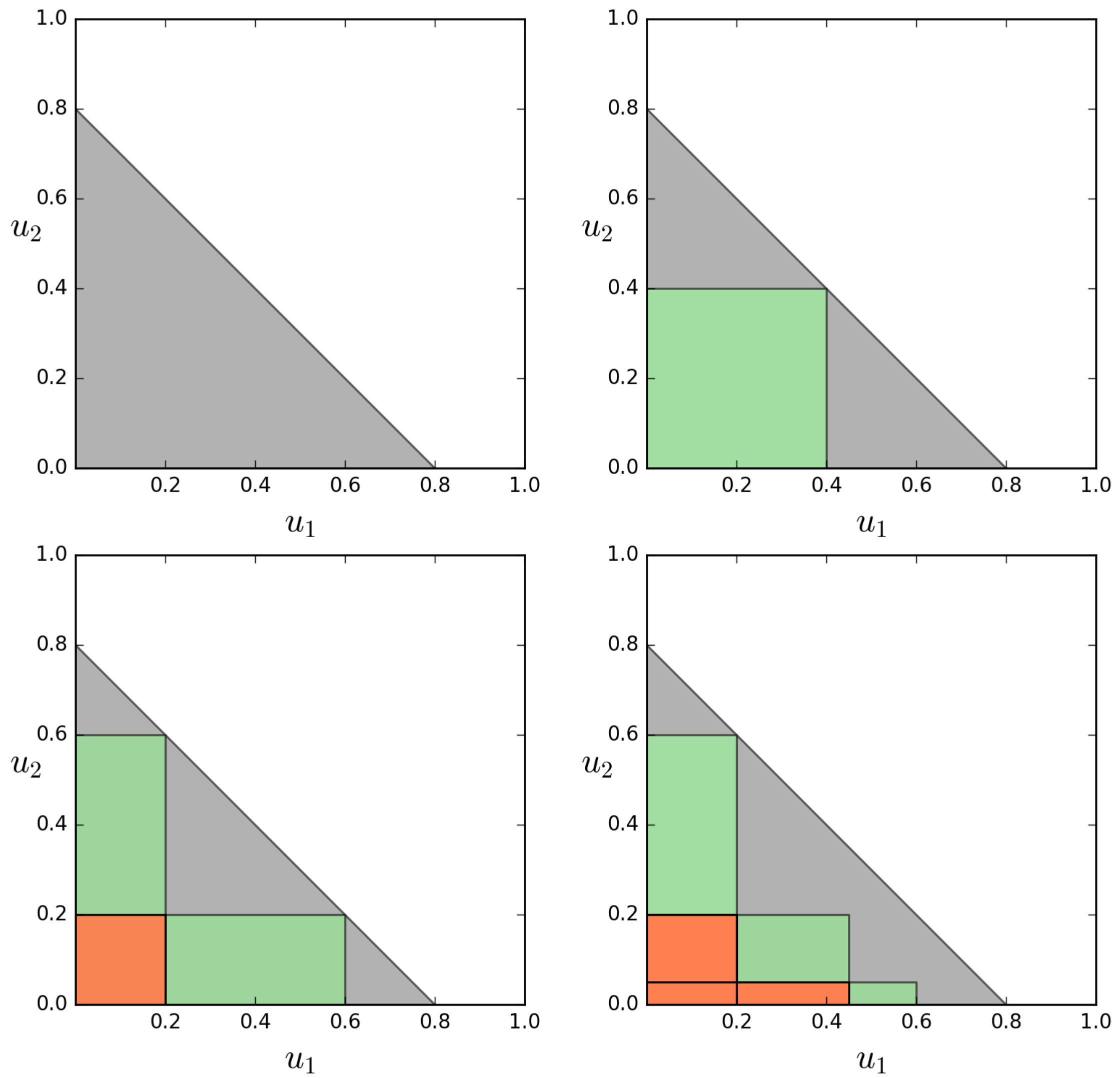

Figure 5.1: Constrained set of admissible functions

of the objective function for h is given by

$$\underline{Q}(\Lambda_{(0.4,0.4)}) = \underline{Q}(F_1(0.4), F_2(0.4)) = \underline{Q}(0.4, 0.4).$$

Similarly, the lower LHS represents the rectangles $[0, 0.2] \times [0, 0.6]$ and $[0, 0.6] \times [0, 0.2]$ induced by $\Lambda_{(0.2,0.6)}$ and $\Lambda_{(0.6,0.2)}$. The red area corresponds to $[0, 0.2] \times [0, 0.2]$ where an overlap occurs due to

$$h(\mathbf{u}) = \Lambda_{(0.2,0.6)}(\mathbf{u}) + \Lambda_{(0.6,0.2)}(\mathbf{u}) = 2 > \varphi(\mathbf{u}) \quad \text{for all } \mathbf{u} \in [0, 0.2] \times [0, 0.2].$$

This overlap is then compensated by applying the Inclusion-Exclusion Principle and subtracting $g = \Lambda_{(0.2,0.2)}$, yielding the admissible function

$$h - g = \Lambda_{(0.2,0.6)} + \Lambda_{(0.6,0.2)} - \Lambda_{(0.2,0.2)}.$$

The respective value of the objective function is equal to $\underline{Q}(0.2, 0.6) + \underline{Q}(0.6, 0.2) - \overline{Q}(0.2, 0.2)$. Finally, the lower RHS represents the function constructed by

$$h = \Lambda_{(0.2,0.6)} + \Lambda_{(0.45,0.2)} + \Lambda_{(0.6,0.05)}$$

and an appropriate compensation of the overlap by $g = \Lambda_{(0.2,0.2)} + \Lambda_{(0.45,0.05)}$ so that $(h, g) \in \underline{\mathcal{A}}^{\Box}$ and the corresponding value of the objective function is equal to

$$\underline{Q}(0.2, 0.6) + \underline{Q}(0.45, 0.2) + \underline{Q}(0.6, 0.05) - \overline{Q}(0.2, 0.2) - \overline{Q}(0.45, 0.05).$$

Note, that the construction of (h, g) depends entirely on the choice of $\mathbf{u}_1, ..., \mathbf{u}_k \in \mathbb{R}^d$. Specifically, maximizing over all (h, g) that are constructed in this way amounts to an optimization over $\mathbf{u}_1, ..., \mathbf{u}_k \in \mathcal{U}_\psi(s)$, leading to our reduction scheme presented in Corollary 5.2.4.

A graphical illustration of $\overline{D}_\varphi^{\Box}(k)$

Using again $\psi(x_1, x_2) = x_1 + x_2$ and uniform distributions F_1, F_2 on $[0, 1]$, let us illustrate the idea of the scheme $\overline{D}_\varphi^{\Box}(k)$. This time, (h, g) with $h, g \in \mathcal{R}^{\Box}$ are admissible when

$$h - g = \sum_{n=1}^{k} \Lambda_{\mathbf{u}^k} - \sum_{n=1}^{m} \Lambda_{\mathbf{v}^n} \geq \mathbb{1}_{x_1 + x_2 < s}.$$

Figure 5.2 illustrates two possible constructions of admissible pairs (h, g). Again the green area corresponds to $\{\mathbf{x} \in [0, 1]^2 : h(\mathbf{x}) = 1\}$ whereas the red shaded area marks overlaps $\{\mathbf{x} \in [0, 1]^2 : h(\mathbf{x}) > 1\}$ which we compensate using the Inclusion Exclusion principle. The LHS corresponds to

$$h - g = \Lambda_{(0.8,0.4)} + \Lambda_{(0.4,0.8)} - \Lambda_{(0.4,0.4)}$$

and the respective value of the objective function amounts to $\overline{Q}(0.8, 0.4) + \overline{Q}(0.4, 0.8) - \underline{Q}(0.4, 0.4)$. The RHS represents the admissible function given by $h - g$ for

$$h = \Lambda_{(0.8,0.2)} + \Lambda_{(0.2,0.8)} + \Lambda_{(0.6,0.6)}; \quad g = \Lambda_{(0.2,0.6)} + \Lambda_{(0.6,0.2)}$$

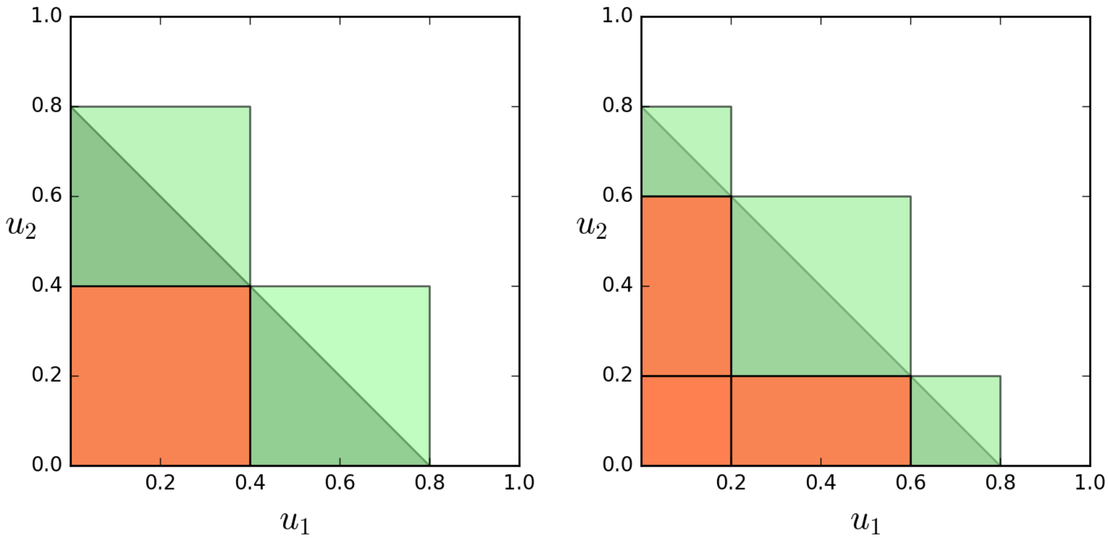

Figure 5.2: Constrained set of admissible functions

with corresponding value of the objective function

$$\overline{Q}(0.8, 0.2) + \overline{Q}(0.2, 0.8) - \underline{Q}(0.2, 0.6) - \underline{Q}(0.6, 0.2).$$

Note, that in contrast to $\underline{D}^{\square}_{\varphi}(k)$ it does not suffice to consider $h = \sum_{n=1}^{k} \Lambda_{\mathbf{u}^k}$ for $\mathbf{u}^1, ..., \mathbf{u}^k \in \mathcal{U}_{\psi}(s)$. A construction of admissible functions in the spirit of section 5.2.1 is however possible if we formulate it in terms of characteristic functions of upper level sets of the form

$$\{(x_1, x_2) \colon x_1 \geq u_1,\ x_2 \geq u_2\},$$

for $\mathbf{u}$ in the complement $\mathcal{U}^c_{\psi}(s)$. Returning to the LHS of figure 5.2 we then note that the region

$$\{\mathbf{x} \in \mathbb{R}^2 \colon \Lambda_{(0.8,0.4)}(\mathbf{x}) + \Lambda_{(0.4,0.8)}(\mathbf{x}) - \Lambda_{(0.4,0.4)}(\mathbf{x}) = 1\}$$

can be expressed in terms of complements of upper level sets via

$$\Big([0.4, 1] \times [0.4, 1]\Big)^c \cap \Big([0, 1] \times [0.8, 1]\Big)^c \cap \Big([0.8, 1] \times [0, 1]\Big)^c,$$

and a similar representation holds for the RHS. Moreover, we can represent the complements via

$$\Big([u_1, 1] \times [u_2, 1]\Big)^c = ([0, u_1] \times [0, 1]) \cup ([0, 1] \times [0, u_2]),$$

where the right-hand side of the equation is the union of sets that can be evaluated by means of the quasi-copulas $\underline{Q}, \overline{Q}$. This construction is made rigorous in Lemma 5.2.6 and the resulting optimization is provided in Corollary 5.2.7.

Numerical examples

The following numerical example shows, how the reduction schemes can be applied in order to account for copula information in the computation of VaR estimates. Our results are compared to the Improved Standard Bounds using the same information.

Example 5.4.1. Consider an $\mathbb{R}^2$-valued risk vector (X_1, X_2) with copula C and Pareto$_2$-marginals. We assume that the copula C lies in the vicinity of a reference copula C^* as measured by the Kolmogorov–Smirnov distance, i.e.

$$\mathcal{D}_{\text{KS}}(C, C^*) \leq \delta$$

for some $\delta > 0$. Hereby, C^* is assumed to be a Gaussian copula with correlation ρ. We then compute VaR estimates on the sum X_1+X_2, using the copula information via the reduction schemes presented in Section 5.2.1 and 5.2.2. Applying Theorem 2.1.7 in conjunction with the explicit representation of the improved Fréchet–Hoeffding bounds given in Corollary 2.1.8, we obtain

$$\begin{aligned} \underline{Q}^{\mathcal{D}_{\text{KS}},\delta}(\mathbf{u}) = \max\{C^*(\mathbf{u}) - \delta, W_2(\mathbf{u})\} &\leq C(\mathbf{u}) \\ &\leq \min\{C^*(\mathbf{u}) + \delta, M_2(\mathbf{u})\} = \overline{Q}^{\mathcal{D}_{\text{KS}},\delta}(\mathbf{u}), \end{aligned} \tag{5.38}$$

for all $\mathbf{u} \in \mathbb{I}^d$. Note that because of $d = 2$ the bounds apply analogously to the survival function of C, i.e.

$$\widehat{\underline{Q}}^{\mathcal{D}_{\text{KS}},\delta} \leq \widehat{C} \leq \widehat{\overline{Q}}^{\mathcal{D}_{\text{KS}},\delta}. \tag{5.39}$$

Our reduction schemes now allow us to translate these improved Fréchet–Hoeffding bounds into VaR estimates. As a benchmark to demonstrate the quality of our estimates, we compare them to the Improved Standard Bounds, which are given by the inverses of the following bounds on the quantile function of $X_1 + X_2$:

$$\begin{aligned} \mathbb{P}(X_1 + X_2 < s) &\geq \sup_{x\in\mathbb{R}} \underline{Q}^{\mathcal{D}_{\text{KS}},\delta}\Big(F_1(x), F_2^-(s-x)\Big) \\ \mathbb{P}(X_1 + X_2 < s) &\leq \inf_{x\in\mathbb{R}} 1 - \widehat{\underline{Q}}^{\mathcal{D}_{\text{KS}},\delta}\Big(F_1(x), F_2(s-x)\Big). \end{aligned}$$

The following tables show the values of the Improved Standard Bounds on the VaR of the sum $X_1 + X_2$ for different confidence levels α. These are compared to the VaR bounds obtained by inverting $\underline{D}^{\square}_{\varphi}(k)$ and $\overline{D}^{\square}_{\varphi}(k)$, for $k = 3$ and $\varphi(x_1, x_2) = \mathbb{1}_{x_1+x_2<s}$, along the variable s. Analogously, we compute VaR estimates by inverting $\widehat{\underline{D}}^{\square}_{\varphi}(k)$ and $\widehat{\overline{D}}^{\square}_{\varphi}(k)$ with $k = 3$, using the bounds on the survival copula in (5.39). We thus obtain two lower and two upper VaR estimates for each α, of which the largest lower bound and the lowest upper bound respectively are reported in each of the tables. For $k \geq 4$ no further improvement of the bounds was obtained. For the sake of legibility, the results are rounded to one decimal place. Table 5.1 shows the VaR estimates for different levels of the correlation of the reference copula and $\delta = 0.0001$, while for Table 5.2 we assume that $\delta = 0.0005$.

	$\rho = -0.5$			$\rho = 0$			$\rho = 0.5$		
α	i. standard (low : up)	scheme (low : up)	impr. %	i. standard (low : up)	scheme (low : up)	impr. %	i. standard (low : up)	scheme (low : up)	impr. %
0.95	(1.5 : 10.7)	(5.5 : 6.7)	86	(3.5 : 10.7)	(5.5 : 7.1)	77	(3.5 : 10.1)	(5.6 : 8.2)	61
0.99	(2.3 : 27)	(13.1 : 16.2)	87	(4.3 : 27)	(13.5 : 16.8)	85	(7.7 : 26.5)	(14.1 : 20.5)	66
0.995	(2.8 : 38)	(18.3 : 20.7)	93	(5.5 : 38)	(19.5 : 23.8)	87	(10.4 : 38)	(19.5 : 29.7)	63

Table 5.1: Improved Standard Bounds on VaR of $X_1 + X_2$ and VaR estimates via reduction schemes using $\underline{Q}^{\mathcal{D}_{\mathrm{KS}},\delta}$ and $\underline{Q}^{\mathcal{D}_{\mathrm{KS}},\delta}$ for $\delta = 0.0001$.

	$\rho = -0.5$			$\rho = 0$			$\rho = 0.5$		
α	i. standard (low : up)	scheme (low : up)	impr. %	i. standard (low : up)	scheme (low : up)	impr. %	i. standard (low : up)	scheme (low : up)	impr. %
0.95	(1.5 : 10.7)	(5.5 : 7.1)	82	(2.2 : 10.7)	(5.5 : 7.9)	72	(3.4 : 10.2)	(5.5 : 8.6)	54
0.99	(2.3 : 27)	(13.1 : 16.6)	85	(4.3 : 27)	(13.1 : 18.8)	74	(7.3 : 27)	(14 : 22)	60
0.995	(2.8 : 38)	(18.3 : 23.4)	85	(5.3 : 38)	(19.2 : 27)	76	(9.9 : 38)	(19.5 : 33.3)	50

Table 5.2: Improved Standard Bounds on VaR of $X_1 + X_2$ and VaR estimates via reduction schemes using $\underline{Q}^{\mathcal{D}_{\mathrm{KS}},\delta}$ and $\underline{Q}^{\mathcal{D}_{\mathrm{KS}},\delta}$ for $\delta = 0.0005$.

The improvement obtained by including two-sided information via the reduction scheme ranges from 54% in the case of positive correlation and $\delta = 0.0005$, up to a considerable 93% in the case of negative correlation and $\delta = 0.0001$. Overall, the improvement is more pronounced when negative correlation is prescribed. Moreover, the improvement is particularly strong for high levels of the confidence threshold α, except for the case of positive correlation and $\delta = 0.0005$. ♦

Example 5.4.2. We now consider an $\mathbb{R}^4$-valued risk vector (X_1, X_2, X_3, X_4) with copula C and Pareto$_2$-marginals. Moreover, we assume that

$$C^{\underline{\Sigma}} \leq C \leq C^{\overline{\Sigma}}$$

where $C^{\underline{\Sigma}}$ and $C^{\overline{\Sigma}}$ denote 4-dimensional Gaussian copulas with correlation matrices $\underline{\Sigma} = (\underline{\rho}_{ij})_{i,j=1,\dots,4}$ and $\overline{\Sigma} = (\overline{\rho}_{ij})_{i,j=1,\dots,4}$ respectively. Also, we assume that $\underline{\rho}_{ij} \leq \overline{\rho}_{ij}$ for $i,j = 1,\dots,4$, which by Slepian's Lemma guarantees non-degeneracy in the sense that $C^{\underline{\Sigma}} \leq C^{\overline{\Sigma}}$; c.f. also Gupta, Eaton, Olkin, Perlman, Savage, and Sobel [26, Theorem 5.1].

This corresponds to a situation of correlation uncertainty which occurs naturally in many practical applications. Whenever correlation is estimated from data one obtains, rather than an exact estimate, a confidence interval for the pairwise correlations $(\underline{\rho}_{ij}, \overline{\rho}_{ij}) \subset [-1, 1]$, in which the parameters lie with a certain probability. Moreover, we assume that bounds on the survival function $\widehat{C}$ are given by the respective survival functions of $C^{\underline{\Sigma}}$ and $C^{\overline{\Sigma}}$, i.e.

$$C^{\underline{\Sigma}}(\mathbf{1} - \cdot) \leq \widehat{C} \leq C^{\overline{\Sigma}}(\mathbf{1} - \cdot).$$

We then relate the bounds on C and $\widehat{C}$ respectively to the VaR of $X_1 + \cdots + X_4$, using our reduction schemes and again we compare the results to the Improved Standard Bounds obtained from $C^{\underline{\Sigma}}$ and $C^{\underline{\Sigma}}(\mathbf{1} - \cdot)$. Table 5.3 shows the results for different confidence levels α, assuming that $\underline{\Sigma}$ and $\overline{\Sigma}$ are equicorrelation matrices with correlation parameters $\underline{\rho}$ and $\overline{\rho}$ respectively. The VaR estimates were obtained by inverting $\underline{D}^{\square}_{\varphi}(5)$ and $\overline{D}^{\square}_{\varphi}(5)$ as well as $\widehat{\underline{D}}^{\square}_{\varphi}(5)$ and $\widehat{\overline{D}}^{\square}_{\varphi}(5)$ for $\varphi(x_1, \dots, x_4) = \mathbb{1}_{x_1 + \cdots + x_4 < s}$ along the variable s. Thus, we obtain two upper and two lower VaR estimates of which the largest lower bound and the lowest upper bound are reported. No further improvement of the bounds was obtained for $k > 5$. For the sake of legibility the results are rounded to full integers.

	$\underline{\rho} = -0.1, \overline{\rho} = 0.2$			$\underline{\rho} = 0.3, \overline{\rho} = 0.5$		
α	i. standard (low : up)	scheme (low : up)	impr. %	i. standard (low : up)	scheme (low : up)	impr. %
0.95	(3 : 32)	(8 : 26)	38	(1 : 30)	(7 : 29)	24
0.99	(9 : 74)	(20 : 52)	51	(2 : 74)	(18 : 63)	37
0.995	(13 : 104)	(26 : 70)	52	(3 : 104)	(25 : 86)	40

Table 5.3: Improved Standard Bounds on VaR of $X_1 + \cdots + X_4$ and VaR estimates computed via reduction schemes using $C^{\underline{\Sigma}}$ and $C^{\overline{\Sigma}}$.

The improvement of the spread reaches from 24% in the case of moderate positive correlation up to 52% in the case of low correlation. Moreover, the improvement is particularly pronounced for high levels of the confidence threshold α. ♦

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