Inconsistency of **set** theory via evaluation

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Abstract

We introduce in an axiomatic way the categorical theory **PR** of primitive recursion as the initial cartesian category with Natural Numbers Object. This theory has an extension into constructive *set* theory **S** of primitive recursion with abstraction of predicates into *subsets* and two-valued (boolean) truth algebra. Within the framework of (typical) classical, quantified **set** theory **T** we construct an evaluation of arithmetised theory **PR** via Complexity Controlled Iteration with witnessed termination of the iteration, *witnessed* termination by availability of Hilbert's iota operator in **set** theory. Objectivity of that evaluation yields inconsistency of **set** theory **T** by a liar (anti)diagonal argument.

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Introduction and overview

The **problem** with 19th/20th century mathematical **foundations**, clearly stated in SKOLEM 1919,[1] is unbounded infinitary (non-constructive) formal *quantification* '$\forall$' and '$\exists$', needed for **set** theoretical introduction of maps to *consist* (in general) out of an *actual infinity* of element pairs.

In this paper we attempt to deduce **inconsistency** of classical, *quantified* **set** theories as in particular Zermelo-Fraenkel **set** theory **ZF**.

In order to reach this result, we proceed as follows:

- Section 1 on **Primitive Recursion** first states the **axioms** of *cartesian categorical free-variables theory* **PR** of primitive recursion and deduces the *full schema of primitive recursion*, deduces it from Freyd's uniqueness axiom for the initialised iterated map, taken as last axiom of theory **PR**.

- Section 2 discusses p. r.[2] predicates, their abstraction into *subsets* of **PR**'s objects (in turn nested cartesian products of *one-object* $\mathbb{1}$ and Natural Numbers Object $\mathbb{N}$), two-elements (boolean) truth algebra $\mathbb{2}$, and states the FERMAT theorems as free-variable **PR** predicates.

[1] "Was ich nun in dieser Abhandlung zu zeigen wünsche ist folgendes: *Faßt man die allgemeinen Sätze der Arithmetik als Funktionalbehauptungen auf, und basiert man sich auf der rekurrierenden Denkweise, so läßt sich diese Wissenschaft in folgerichtiger Weise ohne Anwendung der Russel-Whitehead'schen Begriffe "always" und "sometimes" begründen.*"

[2] primitive recursive

- Section 3 on strings and polynomials introduces ordinal semiring $\mathbb{N}[\omega]$ needed as domain of complexity values for Complexity Controlled iteration; evaluation is defined as such a CCI.

- Section 4 introduces terminating Complexity Controlled Iteration CCI in general, *terminating* by availability of Hilbert's iota operator given as a weak choice operator.

- In section 5 we code, "gödelise", *arithmetise* theory **PR** – just its maps – in a very simple, litteral way into an *internal* theory PR, construct – central part of the paper – evaluation transformation ev as terminating (descending) Complexity Controlled Iteration CCC_{τ} – *terminating* because Hilbert's iota operator is available in **set** theory **T** taken as frame –, and show a characterisation theorem for the evaluation as well as evaluation objectivity.

- Section 6 is to prove – based on evaluation objectivity – inconsistency of (typical) **set** theory **T** and hence of all classical, quantified **set** theories. The argument is an (anti)diagonal construction of a "liar" truth value map, equal to its own negation.

- Final section 7 sketches a way out by weakening **set** theory into recursive theory $\pi\mathbf{R} = \mathbf{S} + (\pi)$ of *non-infinitely descending complexity-controlled iteration.*

1 Categorical theory PR of primitive recursion

Here we state the **axioms** of *cartesian categorical free-variables theory* **PR** of primitive recursion and deduce the *full schema of primitive recursion*, deduce it from Freyd's uniqueness axiom for the initialised iterated map, taken as last axiom of theory **PR**.

1.1 Cartesian category structure

Let us start with the axioms of the cartesian category **CA** with a ("naked") natural number object $\mathbb{N} = \langle \mathbb{N}, 0, \mathsf{s} \rangle$.

Categorical theory **CA**, subsystem of – typical – **set** theory **T** comes with **objects**

- $\mathbb{1}$ *terminal object,* origin of pointer maps
  to "elements" of any – countably many – objects
  $\mathbb{1} \equiv \{0\}$, one-element set of **set** theory **T**
  in particular **T** = **ZF** Zermelo/Fraenkel **set** theory[3]

- $\mathbb{N}$, *natural numbers object,* "NNO"
  $\mathbb{N} \equiv \{0, 1, 2, \ldots\}$
  natural numbers set of **set** theory **T**

- *cartesian product(s)* of objects

[3] for a catgorical version of **ZF** see Osius 1974

$$\frac{A, B \text{ objects}}{(A \times B) \text{ object},}$$

cartesian product

$A \times B \equiv \{(a, b) : a \in A,\ b \in B\}$

a set of pairs in **set** theory **T**

- **basic maps** bas of **CA** subsystem of theory **T**

  $0 : \mathbb{1} \to \mathbb{N}$ (*zero constant*)
  $\mathtt{s} = \mathtt{s}(n) : \mathbb{N} \to \mathbb{N}$ (*successor map*)
  $\mathrm{id} = \mathrm{id}(a) = a : A \to A$ (*identity map*)

  Single **free variable** a over A *interpreted* in (categorical) **theory CA** as **identity** map $a := \mathrm{id} : A \to A$
  (each term of **CA** should designate an object or a map)

- **projections**

  $\Pi = \Pi(a) : A \to \mathbb{1}$ (*terminal map*)

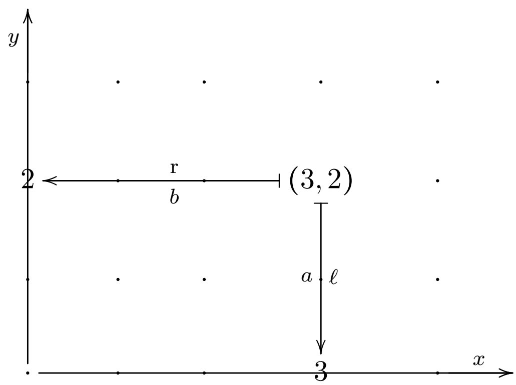
  $\ell = \ell(a, b) = a : A \times B \to A$ (*left projection*)

  $r = r(a, b) = b : A \times B \to B$ (*right projection*)

  $\Pi(a) = \Pi(\mathrm{id}) = \Pi \circ \mathrm{id} = \Pi$ – see below

  $\ell(a, b) = \ell \circ (a, b) = \ell \circ \mathrm{id}_{A \times B} = \ell$ – see below

  $r(a, b) = r \circ (a, b) = r \circ \mathrm{id}_{A \times B} = r$ – see below



Cartesian product $\mathbb{N} \times \mathbb{N}$

with *projections*
$\ell : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ and $\mathrm{r} : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$
pair of *free variables*
a over A, b over B *interpreted* as
projection *maps* $a := \ell : A \times B \to A$ and $b := r : A \times B \to B$

- associative *map composition*

$$\frac{f : A \to B,\ g : B \to C \text{ maps}}{g \circ f = (g \circ f)(a) \equiv g(f(a)) : A \xrightarrow{f} B \xrightarrow{g} C \text{ map}}$$

$a := \mathrm{id} : A \to A$ free variable

$$\frac{f : A \to B,\ g : B \to C,\ h : C \to D \text{ maps}}{h \circ (g \circ f) = (h \circ g) \circ f \equiv h(g(f(a))):}$$

$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$

up to here: *categorical theory* $\mathbf{CA} = \langle \mathrm{Obj}, \mathrm{Map}, \mathrm{id}, \circ \rangle$

objects, maps, associative map composition, identity maps neutral to composition

- *unique induced map* into product

  $f : C \to A$, $g : C \to B$ maps

$$\begin{tikzcd}
 & A & & f(c) \\
C \arrow[ur, bend left, "f"] \arrow[r, dashed, "(f,g)"] \arrow[dr, bend right, "g"'] \arrow[ur, phantom, "=" near end, shift right=6] \arrow[dr, phantom, "=" near end, shift left=6] & A \times B \arrow[u, "\ell"'] \arrow[d, "r"] & c \arrow[ur, mapsto, bend left] \arrow[r, mapsto] \arrow[dr, mapsto, bend right] & (f(c), g(c)) \arrow[u, mapsto] \arrow[d, mapsto] \\
 & B & & g(c)
\end{tikzcd}$$

  Godement's DIAGRAM[4]

- cartesian *map product*

$$\begin{tikzcd}
\boldsymbol{a} \arrow[rrr, mapsto, bend left] & A \arrow[r, "f"'] & A' & \boldsymbol{f(a)} \\
\boldsymbol{(a,b)} \arrow[u, mapsto] \arrow[d, mapsto] \arrow[rrr, mapsto, bend right] & A \times B \arrow[u, "\ell"'] \arrow[d, "r"'] \arrow[r, "f \times g"] \arrow[ur, phantom, "="] \arrow[dr, phantom, "="] & A' \times B' \arrow[u, "\ell"'] \arrow[d, "r"'] & \boldsymbol{(f(a), g(b))} \arrow[u, mapsto] \arrow[d, mapsto] \\
\boldsymbol{b} \arrow[rrr, mapsto, bend right] & B \arrow[r, "g"] & B' & \boldsymbol{g(b)}
\end{tikzcd}$$

[4] this is the very beginning of – maps based – category theory, started by EILENBERG, MAC LANE 1945

Cartesian category **CA** as cartesian *theory*

$$\mathbf{CA} = \langle \mathrm{Obj}, \mathrm{Map}, =, \mathrm{id}, \circ, \mathbb{1}, \Pi, \mathbb{N}, 0, \mathsf{s}, \times, \ell, r \rangle \text{ with}$$

objects, maps, map equality, identic maps, composition, one object, terminal maps,
(*natural numbers object, zero map, successor map*),
cartesian product, projections to the factor objects.

1.2 Endomap iteration

$$(\S)\quad \frac{\text{endo } f = f(a) : A \to A}{f^{\S} = f^{\S}(a,n) \equiv f^{n}(a) : A \times \mathbb{N} \to A}$$

such that

$$f^{\S}(a,0) \equiv f^{0}(a) = a = \mathrm{id}(a)$$

$$f^{\S} \circ (\mathrm{id} \times \mathrm{s})(a,n) \equiv f^{\mathsf{s}\,n}(a)$$

$$= (f \circ f^{\S})(a,n)$$

[5]

as a commutative diagram:

$$\begin{tikzcd}
 & A \times \mathbb{N} \arrow[r, "A \times \mathsf{s}"] \arrow[d, "f^{\S}"] & A \times \mathbb{N} \arrow[d, "f^{\S}"] \\
A \arrow[ur, "(\mathrm{id}, 0\,\Pi)"] \arrow[dr, "\mathrm{id}"'] \arrow[r, phantom, "="] & A \arrow[r, "f"'] \arrow[r, phantom, shift left=8, "="] & A
\end{tikzcd} \qquad (\S)$$

[5] cf. Lawvere 1964 as well as Eilenberg, Elgot 1970

Example addition $+ = \mathbf{s}^\S$

$$\begin{tikzcd}
 & \mathbb{N}\times\mathbb{N} \arrow[r, "\mathbb{N}\times\mathbf{s}"] \arrow[dd, "\mathbf{s}^\S"', "+"] & \mathbb{N}\times\mathbb{N} \arrow[dd, "\mathbf{s}^\S"', "+"] \\
\mathbb{N} \arrow[ur, "{(\mathrm{id},0\,\Pi)}"] \arrow[dr, "\mathrm{id}"'] \arrow[r, phantom, "="] & {} \arrow[r, phantom, "="] & {} \\
 & \mathbb{N} \arrow[r, "\mathbf{s}"] & \mathbb{N}
\end{tikzcd} \qquad (+)$$

$$a + 0 = a, \ \ a + \mathbf{s}\,n = \mathbf{s}(a + n)$$

1.3 Full schema of primitive recursion

The *full schema of primitive recursion* **reads**:

A map $f = f(a, n) : A \times \mathbb{N} \to B$ can be uniquely be p. r. **defined** by the following – characteristic – inference

$$(\mathrm{pr}) \quad \frac{\begin{array}{l} g = g(a) : A \to B \quad \text{(anchor)} \\ h = h((a, n), b) : (A \times \mathbb{N}) \times B \to B \quad \text{(step)} \end{array}}{f = f(a, n) = \mathrm{pr}[f, g] : A \times \mathbb{N} \to B}$$

such that

(**anchor**) $f(a, 0) = g(a)$ **and**

(**step**) $f(a, n + 1) = h((a, n), f(a, n))$

$+$ (pr!) *uniqueness* of such f

Example multiplication $f(a, n) = a \cdot n : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$

$$
(\mathrm{pr}) \quad \frac{\begin{array}{l} g = g(a) = 0\,\Pi : \mathbb{N} \to \mathbb{1} \to \mathbb{N} \\ h = h((a,n),b) = b + a : (\mathbb{N}\times\mathbb{N})\times\mathbb{N} \to \mathbb{N} \end{array}}{f(a,n) = a \cdot n : \mathbb{N}\times\mathbb{N} \to \mathbb{N}}
$$

unique such that

(anchor) $f(a,0) = a \cdot 0 = g(a) = 0$ **and**

(step) $f(a,n+1) = a \cdot (n+1)$

$= h((a,n), f(a,n)) = a \cdot n + a$

Full schema (pr) is a consequence of

- **iteration schema** (§)

  and

- **uniqueness**
  of the initialised iterated map
  this taken as (last) **axiom** (FR!) for the (categorical) theory **PR** of primitive recursion:

1.4 Freyd's (commuting) diagram (FR!)

Freyd's **axiom**[6] of *uniqueness of the initialised iterated* **reads** as uniqueness of commutative fill-in $h : A \times \mathbb{N}$ into the following DIAGRAM:

[6] see FREYD 1972

$$
\begin{tikzcd}
 & A \times \mathbb{N} \arrow[rr, "A\times\mathrm{s}"] \arrow[d, dashed, bend right, "h"'] \arrow[d, "g^{\S}\circ(f\times\mathbb{N})"] & & A \times \mathbb{N} \arrow[d, dashed, bend right, "h"'] \arrow[d, "g^{\S}\circ(f\times\mathbb{N})"] \\
A \arrow[ur, "(\mathrm{id}_A, 0\,\Pi_A)"] \arrow[dr, "f"'] \arrow[r, phantom, "="] & B \arrow[rr, "g"] \arrow[r, phantom, "="] & & B
\end{tikzcd}
\qquad \text{(FR!)}
$$

(FR!) **diagram** chase

$$
\begin{tikzcd}
 & (a,n) \arrow[rr, mapsto, "A\times\mathrm{s}"] \arrow[d, dashed, bend right, "h"'] \arrow[d, mapsto, "g^{\S}\circ(f\times\mathbb{N})"] & & (a, \mathbf{s}\,n) \arrow[d, dashed, bend right, "h"'] \arrow[d, mapsto, "g^{\S}\circ(f\times\mathbb{N})"] \\
a \arrow[ur, mapsto, "(\mathrm{id}_A, 0\,\Pi_A)"] \arrow[dr, mapsto, "f"'] \arrow[r, phantom, "="] & f(a) = h(a,n) = g^n(f(a)) \arrow[rr, mapsto, "g"] \arrow[r, phantom, "="] & & g(h(a,n)) = g^{n+1}(f(a))
\end{tikzcd}
$$

All together the **axioms** of *cartesian category/theory* **CA** extended by (constructing) iteration **axiom** (§) and Freyd's uniqueness **axiom** (FR!) **define** the

categorical free-variables cartesian *category/theory*

$$\mathbf{PR} = \langle \mathrm{Obj}_{\mathbf{CA}}, \mathrm{PR}, =, \mathrm{id}, \circ, \mathbb{1}, \Pi, \mathbb{N}, 0, \mathbf{s}, \times, \ell, r, ^{\S} \rangle$$

with

- *objects, maps, map equality, identic maps, composition*
- *one object, terminal maps*

- *cartesian product, projections to the factor objects*
- *natural numbers object, zero map, successor map*

  and
- *iteration operation* §

of **categorical Primitive Recursion**, **subsystem** of **set** theory **T**.

Proof of **availability** of initialised iterated map $h : A \times \mathbb{N} \to B$:

Consider commuting DIAGRAM

$$\begin{tikzcd}
A \arrow[r, "(\mathrm{id}_A{,}0_A)"] \arrow[d, "f"] \arrow[dr, "(f{,}0_A)"] \arrow[ddr, bend right, "f"'] & A \times \mathbb{N} \arrow[r, "A\times \mathrm{s}"] \arrow[d, "f\times\mathbb{N}"] \arrow[dr, "f\times s"] \arrow[dd, bend left, "h"] & A \times \mathbb{N} \arrow[d, "f\times\mathbb{N}"] \arrow[dd, bend left, "h"] \\
B \arrow[r, "(\mathrm{id}_B{,}0_B)"'] \arrow[dr, "\mathrm{id}_B"'] & B \times \mathbb{N} \arrow[r, "B\times \mathrm{s}"'] \arrow[d, "g^{\S}"] \arrow[dr, phantom, "="] & B \times \mathbb{N} \arrow[d, "g^{\S}"] \\
 & B \arrow[r, "g"] & B
\end{tikzcd}$$

2 PR predicates

This section discusses p. r. predicates, their abstraction into *subsets* of **PR**'s objects (in turn nested cartesian products of *one-object* $\mathbb{1}$ and Natural Numbers Object $\mathbb{N}$), two-elements (boolean) truth algebra $\mathbb{2}$, and states the FERMAT theorems as free-variable **PR** predicates.

Definition

A **PR** *predicate* – on object A – is a **PR** map

$\chi = \chi(a) : A \to \mathbb{N}$ such that
PR $\vdash \chi = \chi(a) \leq 1$

Add formally all of these predicates as **objects/subsets** to *theory* **PR**
– via a schema of *predicate-into-subset abstraction* –
and get an embedding extension of theory **PR** into
basic p. r. set theory S, subsystem of **T**.

2.1 Boolean truth object

Introduce *2-valued boolean truth object* of **S**
as **S** *set* $\mathbb{2} = \{\alpha \in \mathbb{N} : \alpha \leq 1\} \equiv \{0, 1\} \subset \mathbb{N}$
and **define S** *predicates* as **S** *maps* of form
$\chi = \chi(a) : A \to \mathbb{2}$, A **object** of **S**.

S is the *basic primitive recursive set theory.*

2.2 FERMAT free-variables p. r. predicates

Small fermat theorems

$$\begin{aligned}
& a > 0 \wedge b > 0 \wedge c > 0 \\
& \;\Longrightarrow \neg[a^4 + b^4 = c^4] : \\
& \quad ((\mathbb{N} \times \mathbb{N}) \times \mathbb{N}) \times \mathbb{1} \to \mathbb{2} \\
& \quad \textbf{as well as} \\
& (a+1)^3 + (b+1)^3 \neq (c+1)^3 : \\
& \quad ((\mathbb{N} \times \mathbb{N}) \times \mathbb{N}) \times \mathbb{1} \to \mathbb{N} \times \mathbb{N} \xrightarrow{\neq} \mathbb{2}
\end{aligned}$$

Last fermat theorem[7]

$$\begin{aligned}
& (a+1)^{n+3} + (b+1)^{n+3} \neq (c+1)^{n+3} : \\
& \quad (((\mathbb{N} \times \mathbb{N}) \times \mathbb{N}) \times \mathbb{N}) \times \mathbb{1} \to \mathbb{N} \times \mathbb{N} \xrightarrow{\neq} \mathbb{2} \\
& a, b, c, \ \mathit{free} \ \textbf{and} \ n \in \mathbb{N} \ \mathit{free}
\end{aligned}$$

[$\neq$ is a p. r. predicate $[m \neq n] = \neg[m = n] : \mathbb{N} \times \mathbb{N} \to \mathbb{2} \to \mathbb{2}$]

3 Strings and polynomials

This section introduces ordinal semiring $\mathbb{N}[\omega]$ needed as domain of complexity values for Complexity Controlled Iteration; evaluation is defined as such a CCI.

[7] cf. Singh 1997/1998

Strings $a = a_0\, a_1\, \ldots\, a_n$ of natural numbers are coded as *prime power products*

$2^{a_0} \cdot 3^{a_1} \cdot \ldots \cdot p_n^{a_n} \in \mathbb{N}_{>0} \subset \mathbb{N}$

iteratively defined as

$((2^{a_0} \cdot 3^{a_1}) \cdot \ldots) \cdot p_n^{a_n} \in \mathbb{N}_{>0}$

Strings are identified with/interpreted as
"their" *polynomials*:

$p(\omega) \equiv 0$ or

$$p(\omega) = \sum_{j=0}^{n} a_j \omega^j = a_0 + a_1\omega^1 + \ldots + a_n\omega^n,\ a_n > 0,$$

ω an indeterminate for (arbitrarily) big natural numbers.

Order of polynomials is first by *degree,* second by *pivot coefficient,* and then – if these are equal – by comparison of the two polynomials with their equal pivot monomes removed, recursively, down to the zero polynomial (which has no degree).

Call $\mathbb{N}[\omega]$ the linearily ordered semiring of (coefficient strings) of these polynomials.

The linear order has – intuitively and *formally* within **set** theory – only *finite descending chains,* $\mathbb{N}[\omega]$ is (there) an *ordinal*:

There is no infinitely descending chain $c = c(n) : \mathbb{N} \to \mathbb{N}[\omega]$.

4 Complexity controlled iteration

Here we introduce terminating Complexity Controlled Iteration CCI in general, *terminating* – within **set** theory **T** – by availability of Hilbert's iota operator[8] (Bourbaki's τ operator[9]) given as a weak choice operator.

This has as **consequence** the following **T schema** CCI_τ of *termination witnessed, complexity controlled iteration*:

$$\begin{array}{ll}
& c = c(a) : A \to \mathbb{N}[\omega] \ (\mathit{complexity}) \\
& p = p(a) : A \to A \ (\mathit{predecessor}) \\
& c(a) = 0 \implies p(a) = a \ (\mathit{stationarity}) \\
& \mathbb{N}[\omega] \ni c(a) > 0 \implies c\,p(a) < c(a) \\
& \qquad \mathit{complexity\ descent} \\
\mathrm{CCI}_\tau & \overline{\hspace{12em}} \\
& \exists \tau = \tau_{c:p} \in \mathbb{N})[(\forall n \geq \tau)\, c\,p^n(a) = 0], \\
& \text{hence } (\forall n \geq \tau)[p^n(a) = p^\tau(a)], \\
& \text{and } \textbf{defines} \text{ p. r. map} \\
& \mathrm{cci} = \mathrm{cci}_{c:p} = p^\S(a, \tau_{c:p}) : A \to A
\end{array}$$

5 Evaluation of PR within set theory T

In this section we code, "gödelise", *arithmetise* theory **PR** – just its maps – in a very simple, litteral way into an *internal* theory PR, construct – central part of the paper – evaluation transformation ev as terminating Complexity Controlled Iteration CCC_τ, and show a characterisation theorem for the

[8] see Hilbert 1900/1970
[9] see Bourbaki 1966

evaluation as well as evaluation **objectivity**.

5.1 Coding, "gödelisation"

Coding[10] of theory **PR** : ***internal, arithmetised*** theory
$\mathrm{PR} = \langle \mathrm{Obj}_{\mathbf{PR}} = \mathrm{Obj}_{\mathbf{CA}}, \mathrm{Map}_{\mathrm{PR}}, \dot{\mathrm{bas}}, \odot, \langle \_ ; \_ \rangle, {}^{\$} \rangle$

- $\mathrm{Map}_{\mathrm{PR}} \subset \mathbb{N}$ an (algebraic carrier-)set of natural numbers within **set** theory **T**, set of all arithmetised **PR** maps
- $\dot{\mathrm{bas}} = \{ \ulcorner 0 \urcorner, \ulcorner \mathtt{s} \urcorner, \ulcorner \mathrm{id} \urcorner, \ulcorner \Pi \urcorner, \ulcorner \ell \urcorner, \ulcorner r \urcorner \} \subset \mathrm{Map}_{\mathrm{PR}} \subset \mathbb{N}$
  with *basic* (LaTeX) *map codes*
  $\ulcorner 0 \urcorner = \mathrm{utf8}[\mathtt{0}]$, $\ulcorner \mathtt{s} \urcorner = \mathrm{utf8}[\backslash\mathtt{texttt}\{\mathtt{s}\}]$ etc.
- ***composition operator*** $\odot = \ulcorner \circ \urcorner = \mathrm{utf8}[\backslash\mathtt{circ}]$
- ***map inducing operator*** $\langle\ ;\ \rangle = \ulcorner (\ ,\ ) \urcorner = \mathrm{utf8}[\mathtt{(\ \ ,\ \ )}]$
  will say
  $\langle = \ulcorner (\urcorner \quad ; = \ulcorner , \urcorner \quad \rangle = \ulcorner) \urcorner$
- ***iteration operator*** ${}^{\$} = \ulcorner \S \urcorner$

5.2 Iterative evaluation transformation

evaluation step e merged with **complexity c**
and **evaluation** ev in recursive construction

[10] cf. Gödel 1931 and Smorynski 1977

$$\mathbf{e} = \mathbf{e}\,(f, a) = (\mathbf{e}_{\mathrm{map}}(f, a), \mathbf{e}_{\mathrm{arg}}(f, a)) :$$
$$\mathrm{PR} \times \mathbf{X} \to \ \mathrm{PR} \times \mathbf{X}$$
$$f \in \mathrm{PR},\ \ a \in \mathbf{X}\ \ \textit{free}$$
$$\mathbf{X} = \bigcup_{A\,\mathrm{in}\,\mathbf{PR}} A \quad \mathbf{PR} - \text{universal } \textit{set},$$
union of $\mathbf{T}$ sets, defined recursively

$\mathbf{e}_{\mathrm{arg}}(f, a)$ *is the intermediate argument obtained by one evaluation step applied to the pair* (f, a), *and* $\mathbf{e}_{\mathrm{map}}(f, a)$ *is the remaining map code still to be evaluated on intermediate argument* $\mathbf{e}_{\mathrm{arg}}(f, a)$, *same then applies* ***iteratively*** *to resp. obtained* $(f', a') = \mathbf{e}\,(f, a) = (\mathbf{e}_{\mathrm{map}}(f, a), \mathbf{e}_{\mathrm{arg}}(f, a))$.

This evaluation step $\mathbf{e}$ is **defined** by recursive case distinction, *controlled* by $\mathbb{N}[\omega]$-valued descending **complexity c**, and using **evaluation** ev in recursive construction.

evaluation step

$$\mathbf{e} = \mathbf{e}\,(h, a) = (\mathbf{e}_{\mathrm{map}}(h), \mathbf{e}_{\mathrm{map}}(h, a)) : \mathrm{PR} \times \mathbf{X} \to \mathbf{X}$$
$$\mathbf{e}_{\mathrm{map}} = \mathbf{e}_{\mathrm{map}}(h) : \mathrm{PR} \times \mathbf{X} \to \mathrm{PR}$$
$$\mathbf{e}_{\mathrm{arg}} = \mathbf{e}_{\mathrm{arg}}(h) : \mathrm{PR} \times \mathbf{X} \to \mathbf{X}$$

is p. r. **defined**, and is iteration complexity-controlled as follows:

- **Basic map cases:**

– case of an identity:

$$\mathbf{c}\,(\ulcorner\mathrm{id}\urcorner, a) = 0$$
$$\mathbf{e}\,(\ulcorner\mathrm{id}\urcorner, a) := (\ulcorner\mathrm{id}\urcorner, a)\ \textit{stationary}$$
$$\mathrm{ev}\,(\ulcorner\mathrm{id}\urcorner, a) = a$$

– remaining basic map cases $\mathrm{ba} \in \mathrm{bas} \setminus \{\mathrm{id}\}$:

$$\mathbf{c}\,(\ulcorner\mathrm{ba}\urcorner, a) := 1$$
$$\mathbf{e}\,(\ulcorner\mathrm{ba}\urcorner, a) := (\ulcorner\mathrm{id}\urcorner, \mathrm{ba}(a))$$
$$\mathrm{ev}\,(\ulcorner\mathrm{ba}\urcorner, a) = \mathrm{ba}(a)$$
$$\mathbf{c}\,(\ulcorner\mathrm{id}\urcorner, a) = 0$$
$$< 1$$
$$= \mathbf{c}\,(\ulcorner\mathrm{ba}\urcorner, a)$$

– in particular case of successor map $\mathtt{s}$:

$$\mathbf{c}\,(\ulcorner\mathtt{s}\urcorner, n) := 1$$
$$\mathbf{e}\,(\ulcorner\mathtt{s}\urcorner, n) := (\ulcorner\mathrm{id}\urcorner, \mathtt{s}\,n)$$
$$\mathrm{ev}\,(\ulcorner\mathtt{s}\urcorner, n) = \mathtt{s}\,n$$

- **composed map case** $g \odot f$

$$\mathbf{c}\,(g \odot f, a) := (\mathbf{c}\,(g, \mathrm{ev}\,(f, a)) + \mathbf{c}\,(f, a)) + 1$$
$$\in \mathbb{N}[\omega]\ \text{recursively}$$
$$\mathbf{e}\,(g \odot f, a)$$
$$= (\mathbf{e}_{\mathrm{map}}(g \odot f, a), \mathbf{e}_{\mathrm{arg}}(g \odot f, a))$$
$$:= (\mathbf{e}_{\mathrm{map}}(g, \mathrm{ev}\,(f, a)), \mathbf{e}_{\mathrm{arg}}(g, \mathrm{ev}\,(f, a)))$$
$$\mathrm{ev}\,(g \odot f, a) := \mathrm{ev}\,(g, \mathrm{ev}\,(f, a))$$

complexity descent, local proof:

$$\begin{aligned}
&\mathbf{c}\,\mathbf{e}\,(g \odot f, a)\\
&= \mathbf{c}\,(\mathbf{e}_{\mathrm{map}}(g, \mathrm{ev}\,(f, a)), \mathbf{e}_{\mathrm{arg}}(g, \mathrm{ev}\,(f, a)))\\
&\coloneqq \mathbf{c}\,(g, \mathrm{ev}\,(f, a))\\
&< (\mathbf{c}\,(g, \mathrm{ev}\,(f, a)) + \mathbf{c}\,(f, a)) + 1\\
&= \mathbf{c}\,(g \odot f, a)
\end{aligned}$$

- **case of an induced map**:
  - identities subcase:

$$\begin{aligned}
&\mathbf{c}\,(\langle \ulcorner \mathrm{id} \urcorner ; \ulcorner \mathrm{id} \urcorner \rangle, a) \coloneqq 1 = (0 + 0) + 1\\
&\mathbf{e}\,(\langle \ulcorner \mathrm{id} \urcorner ; \ulcorner \mathrm{id} \urcorner \rangle, a) \coloneqq (\ulcorner \mathrm{id} \urcorner, (a; a))\\
&\mathrm{ev}\,(\langle \ulcorner \mathrm{id} \urcorner ; \ulcorner \mathrm{id} \urcorner \rangle, a) \coloneqq (a, a)
\end{aligned}$$

complexity descent, local proof:

$$\begin{aligned}
&\mathbf{c}\,\mathbf{e}\,(\langle \ulcorner \mathrm{id} \urcorner ; \ulcorner \mathrm{id} \urcorner \rangle, a)\\
&= \mathbf{c}\,(\ulcorner \mathrm{id} \urcorner, (a, a)) = 0\\
&< 1\\
&= \mathbf{c}\,(\langle \ulcorner \mathrm{id} \urcorner ; \ulcorner \mathrm{id} \urcorner \rangle, a)
\end{aligned}$$

  - subcase f, g not both equal to $\ulcorner \mathrm{id} \urcorner$:

$$\begin{aligned}
&\mathbf{c}\,(\langle f; g \rangle, c) \coloneqq (\mathbf{c}\,(f, c) + \mathbf{c}\,(g, c)) + 1\\
&\mathbf{e}\,(\langle f; g \rangle, c)\\
&= (\mathbf{e}_{\mathrm{map}}(\langle f; g \rangle, c), \mathbf{e}_{\mathrm{arg}}(\langle f; g \rangle, c))\\
&\coloneqq ((\mathbf{e}_{\mathrm{map}}(f, c), \mathbf{e}_{\mathrm{map}}(g, c)), (\mathbf{e}_{\mathrm{arg}}(f, c), \mathbf{e}_{\mathrm{arg}}(g, c))\\
&\mathrm{ev}\,((f, g), c) \coloneqq (\mathrm{ev}\,(f, c), \mathrm{ev}\,(g, c))
\end{aligned}$$

complexity descent, local proof:

$$
\begin{aligned}
&\mathbf{c}\,\mathbf{e}\,(\langle f; g\rangle, c) \\
&= \mathbf{c}\,(\langle \mathbf{e}_{\mathrm{map}}(f, c); \mathbf{e}_{\mathrm{map}}(g, c)\rangle, \langle \mathbf{e}_{\mathrm{arg}}(f, c); \mathbf{e}_{\mathrm{arg}}(g), c\rangle) \\
&:= (\mathbf{c}\,(f, c) + \mathbf{c}\,(g, c))\ [> 0] \\
&< (\mathbf{c}\,(f, c) + \mathbf{c}\,(g, c)) + 1 \\
&= \mathbf{c}\,(\langle f; g\rangle, c)
\end{aligned}
$$

- **cartesian-product map case** $g\#f$ (redundant)

$$
\begin{aligned}
&\mathbf{c}\,(g\#f, (a, b)) := (\mathbf{c}\,(f, a) + \mathbf{c}\,(g, b)) + 1 \\
&\mathbf{e}\,(g\#f, (a, b) \\
&= (\mathbf{e}_{\mathrm{map}}(g\#f, (a, b)), \mathbf{e}_{\mathrm{arg}}(g\#f, (a, b))) \\
&:= ((\mathbf{e}_{\mathrm{map}}(f, a), \mathbf{e}_{\mathrm{map}}(g, b)), (\mathbf{e}_{\mathrm{arg}}(f, a), \mathbf{e}_{\mathrm{arg}}(g, b)) \\
&\mathrm{ev}\,(g\#f, (a, b)) := (\mathrm{ev}\,(f, a), \mathrm{ev}\,(g, b))
\end{aligned}
$$

complexity descent, local proof:

$$
\begin{aligned}
&\mathbf{c}\,\mathbf{e}\,(g\#f, (a, b)) \\
&= \mathbf{c}\,((\mathbf{e}_{\mathrm{map}}(f, a), \mathbf{e}_{\mathrm{map}}(g, b)), (\mathbf{e}_{\mathrm{arg}}(f, a), \mathbf{e}_{\mathrm{arg}}(g, b))) \\
&:= (\mathbf{c}\,(f, a) + \mathbf{c}\,(g, b)) \\
&< (\mathbf{c}\,(f, a) + \mathbf{c}\,(g, b)) + 1 \\
&= \mathbf{c}\,(g\#f, (a, b))
\end{aligned}
$$

- **iterated map case**
  - identity subcase $f := \ulcorner \mathrm{id} \urcorner^{\$} \in \mathbf{X}^{\mathbf{X} \times \mathbb{N}}$

$$\begin{aligned}
&\mathbf{c}\left(\ulcorner\mathrm{id}\urcorner^{\$},(a,n)\right)\\
&:=\mathbf{c}\left(\ulcorner\mathrm{id}\urcorner,a\right)\omega\\
&=0\,\omega=0
\end{aligned}$$

$$\begin{aligned}
&\mathbf{e}\left(\ulcorner\mathrm{id}\urcorner^{\$},(a,n)\right)\\
&=\left(\mathbf{e}_{\mathrm{map}}\left(\ulcorner\mathrm{id}\urcorner^{\$},(a,n)\right),\mathbf{e}_{\mathrm{arg}}\left(\ulcorner\mathrm{id}\urcorner^{\$},(a,n)\right)\right)\\
&:=\left(\ulcorner\mathrm{id}\urcorner,a\right)\\
&\mathrm{ev}\left(\ulcorner\mathrm{id}\urcorner^{\$},(a,n)\right)\\
&:=\mathrm{ev}\left(\ulcorner\mathrm{id}\urcorner,a\right)\\
&=a
\end{aligned}$$

complexity stationarity, local proof:

$$\begin{aligned}
&\mathbf{c}\,\mathbf{e}\left(\ulcorner\mathrm{id}\urcorner^{\$},(a,n)\right)\\
&=\mathbf{c}\left(\mathbf{e}_{\mathrm{map}}\left(\ulcorner\mathrm{id}\urcorner^{\$},(a,n)\right),\mathbf{e}_{\mathrm{arg}}\left(\ulcorner\mathrm{id}\urcorner^{\$},(a,n)\right)\right)\\
&=\mathbf{c}\left(\ulcorner\mathrm{id}\urcorner,a\right)\\
&=0=0\,\omega\\
&=\mathbf{c}\left(\ulcorner\mathrm{id}\urcorner^{\$},(a,n)\right)
\end{aligned}$$

– **iterated map case, step**
  $f^{\$}\in\mathbf{X}^{\mathbf{X}\times\mathbb{N}}$
  $\mathbf{c}\left(f,a\right)>0$

$\mathbf{s}\, n$

$$\mathbf{c}\,(f^{\$}, (a, \mathbf{s}\, n)) := (\mathbf{s}\, n) \cdot \mathbf{c}\,(f, a)\,\omega + n$$
$$\in \mathbb{N}[\omega] \textit{ recursively}$$
$$\mathbf{e}\,(f^{\$}, (a, \mathbf{s}\, n))$$
$$:= (f^{\$}, (a, n)))$$
$$\mathrm{ev}\,(f^{\$}, (a, \mathbf{s}\, n))$$
$$:= \mathrm{ev}\,(f, \mathrm{ev}\,(f^{\$}, (a, n)))$$

complexity descent, local proof:

$$\mathbf{c}\,\mathbf{e}\,(f^{\$}, (a, \mathbf{s}\, n))$$
$$= \mathbf{c}\,(f^{\$}, (a, n))$$
$$:= n \cdot \mathbf{c}\,(f, a)\,\omega + \mathbf{p}\, n$$
$$< (\mathbf{s}\, n) \cdot \mathbf{c}\,(f, a)\,\omega + n$$
$$\text{since here } \mathbf{c}\,(f, a) > 0$$
$$= \mathbf{c}\,(f^{\$}, (a, \mathbf{s}\, n))$$

end evaluation step

5.3 PR-evaluation within T

$\mathrm{ev} \ =_{\mathrm{def}} \ \mathrm{cci}_{\mathbf{c}:\mathbf{e}} = \mathrm{cci}_{\mathbf{c}:\mathbf{e}}(f, a)$ i. e.

$\mathrm{ev} = \mathrm{ev}(f, a)) = \mathbf{e}^{\S}(f, \tau_{\mathbf{c}:\mathbf{e}}(f, a)) : \mathrm{PR} \times \mathbf{X} \xrightarrow{r} \mathbf{X}.$

this gives – by domain/codomain restriction –

(*contra/covariant natural transformation*) *family*

$\mathrm{ev} = \mathrm{ev}_{A,B}(f, a) : B^A \times A \to B$

5.4 Recursive characterisation of evaluation

evaluation

$$\mathrm{ev} = \mathrm{ev}(h, x) = \mathrm{cci}_{\mathbf{c:e}}$$
$$=_{\text{by def}} \; r \circ \mathbf{e}^{\S}(h, \tau_{\mathbf{c:e}}(h, x)) :$$
$$\mathrm{PR} \times \mathbf{X} \to \mathrm{PR} \times \mathbf{X} \xrightarrow{r} \mathbf{X} \; = \bigcup_{A \in \mathbf{PR}} A$$

is **characterised** within theory **T** by

- $\mathrm{ev}(\ulcorner \mathrm{ba} \urcorner, a) = \mathrm{ba}(a)$
- $\mathrm{ev}(g \odot f, a) = \mathrm{ev}(g, \mathrm{ev}(f, a))$
- $\mathrm{ev}(\langle f; g \rangle, c) = (\mathrm{ev}(f, c), \mathrm{ev}(g, c))$
- $\mathrm{ev}(f \# g, (a, b)) = (\mathrm{ev}(f, a), \mathrm{ev}(g, b))$
- Iteration cases
  - $\mathrm{ev}(f^{\$}, (a, 0)) = a$
  - $\mathrm{ev}(f^{\$}, (a, \mathtt{s}\, n))) = \mathrm{ev}(f, \mathrm{ev}(f^{\$}, (a, n))$

Proof by recursive case distinction, recursively
on $\tau = \tau_{\mathbf{c:e}}(h, x) : \mathrm{PR} \times \mathbf{X} \to \mathbb{N}$

- Case $h = \mathrm{ba} \in \mathrm{bas}$ *(basic)*, $\tau \le 1$

$$\mathrm{ev}(\ulcorner \mathrm{ba} \urcorner, a) = \mathbf{e}(\ulcorner \mathrm{ba} \urcorner, a) = \mathrm{ba}(a) \text{ by definition,}$$

in particular

$$\mathrm{ev}(\ulcorner \mathrm{id} \urcorner, a) = \mathrm{id}(a) = a \text{ and}$$
$$\mathrm{ev}(\ulcorner \mathtt{s} \urcorner, n) = \mathtt{s}\, n$$

- Composition case $h = g \odot f$, $\tau(g \odot f, a) \geq 1$

$$\begin{aligned}
&\mathrm{ev}(g \odot f, a) = r \circ \mathbf{e}^{\S}((g \odot f, a), \tau(g \odot f, a)) \\
&= r \circ \mathbf{e}^{\S}((g \odot \mathbf{e}_{\mathrm{map}}(f, a), \mathbf{e}_{\mathrm{arg}}(f, a)), \tau(g \odot f, a) - 1) \\
&= \mathrm{ev}(g \odot \mathbf{e}_{\mathrm{map}}(f, a), \mathbf{e}_{\mathrm{arg}}(f, a)) \\
&\quad \text{by induction hypothesis on } \tau \\
&= \mathrm{ev}(g, \mathrm{ev}(f, a)) \\
&\quad \text{by iterative definition of ev}
\end{aligned}$$

- Induced map case $h = \langle f; g \rangle$, $\tau(\langle f; g \rangle, c) \geq 1$

$$\begin{aligned}
&\mathrm{ev}(\langle f; g \rangle, c) = r \circ \mathbf{e}^{\S}((\langle f; g \rangle, c), \tau((\langle f; g \rangle, c)) \\
&= r \circ \mathbf{e}^{\S}((\langle f; g \rangle, c), \max(\tau(f, c), \tau(g, c)) + 1) \\
&= r \circ \mathbf{e}^{\S}(\mathbf{e}\,(\langle f; g \rangle, c), \max(\tau(f, c), \tau(g, c)) \\
&= \mathrm{ev}(\langle f; g \rangle, c)) \\
&\quad \text{by induction hypothesis on } \tau \\
&= \mathrm{ev}(\langle f, c); \mathrm{ev}(g \rangle, c) \\
&\quad \text{by iterative definition of ev}
\end{aligned}$$

- Iteration cases
  - Case $(h, a) = (f^{\$}, (a, 0))$, $\tau(f^{\$}, (a, 0)) = 1$

$$\begin{aligned}
&\mathrm{ev}(f^{\$}, (a, 0)) \\
&= r \circ \mathbf{e}^{\S}(f^{\$}, (a, 0)), \tau(f^{\$}, (a, 0)) \\
&= r \circ \mathbf{e}^{\S}(f^{\$}, (a, 0)), 1) \\
&= \mathbf{e}\,(f^{\$}, (a, 0)) \\
&= \mathrm{ev}(\ulcorner \mathrm{id} \urcorner, a) = \mathrm{id}(a) = a
\end{aligned}$$

– Case $(h, (a, \mathtt{s}\, n)) = (f^{\$}, (a, \mathtt{s}\, n))$,
$\tau(f^{\$}, (a, \mathtt{s}\, n)) = (\mathtt{s}\, n) \cdot \tau(f) + n$

$$\begin{aligned}
&\mathrm{ev}(f^{\$}, (a, \mathtt{s}\, n)) \\
&= r \circ \mathbf{e}^{\S}(f^{\$}, (a, \mathtt{s}\, n)), \tau(f^{\$}, (a, \mathtt{s}\, n)) \\
&= r \circ \mathbf{e}^{\S}(f^{\$}, (a, \mathtt{s}\, n)), (\mathtt{s}\, n) \cdot \tau(f) + n) \\
&= r \circ \mathbf{e}^{\S}(\mathbf{e}\,(f^{\$}, (a, \mathtt{s}\, n)), (\mathtt{s}\, n) \cdot \tau(f) + (n - 1)) \\
&= r \circ \mathbf{e}^{\S}(f \odot (f^{\$}, (a, n)), n \cdot \tau(f) + \mathtt{p}\, n) \\
&= \mathrm{ev}(f \odot (f^{\$}, (a, n)) \\
&\quad \text{by induction hypothesis on } \tau
\end{aligned}$$

q. e. d.

5.5 Evaluation objectivity

$$\mathbf{T} \vdash \mathrm{ev}(\ulcorner f \urcorner, a) = f(a)$$

Proof by evaluation characterisation theorem:

- case of basic map $\mathrm{ba} \in \mathrm{bas}$

$$\begin{aligned}
&\mathrm{ev}(\ulcorner \mathrm{ba} \urcorner, a) = \mathbf{e}\,(\ulcorner \mathrm{ba} \urcorner, a) \\
&= \mathrm{ba}(a)
\end{aligned}$$

- Case of a composed map $(g \circ f) : A \to B \to C$

$$\begin{aligned}
&\mathrm{ev}(\ulcorner g \circ f \urcorner, a) = \mathrm{ev}(\ulcorner g \urcorner \odot \ulcorner f \urcorner, a) \\
&= \mathrm{ev}(\ulcorner g \urcorner, \mathrm{ev}(\ulcorner f \urcorner, a)) = \mathrm{ev}(\ulcorner g \urcorner, f(a)) \\
&= g(f(a)) = g \circ f(a)
\end{aligned}$$

- case of an induced map $(f,g): C \to A \times B$

$$\begin{aligned}
&\mathrm{ev}(\ulcorner (f,g) \urcorner, c) = \mathrm{ev}(\langle \ulcorner f \urcorner ; \ulcorner g \urcorner \rangle, c) \\
&= (\mathrm{ev}(\ulcorner f \urcorner, c), \mathrm{ev}(\ulcorner g \urcorner, c)) \\
&= (f(c), g(c)) = (f,g)(c)
\end{aligned}$$

- cases of an iterated map $f^{\S}: A \times \mathbb{N} \to A$

  – anchor:

$$\begin{aligned}
&\mathrm{ev}(\ulcorner f^{\S} \urcorner, (a,0)) = \mathrm{ev}(\ulcorner f \urcorner^{\$}, (a,0)) \\
&= a = f^{\S}(a,0)
\end{aligned}$$

  – step:

$$\begin{aligned}
&\mathrm{ev}(\ulcorner f^{\S} \urcorner, (a, \mathsf{s}\, n)) = \mathrm{ev}(\ulcorner f \urcorner^{\$}, (a, \mathsf{s}\, n)) \\
&= \mathrm{ev}(\ulcorner f \urcorner, \mathrm{ev}(\ulcorner f \urcorner^{\$}, (a,n)) \\
&= \mathrm{ev}(\ulcorner f \urcorner, f^{\S}(a,n)) \\
&= (f \circ f^{\S})(a,n) = f^{\S}(a, \mathsf{s}\, n) \quad \textbf{q. e. d.}
\end{aligned}$$

6 Inconsistency proof for set theory

This section is to prove – based on evaluation objectivity[11] – inconsistency of (typical) **set** theory **T** and hence of all classical, quantified **set** theories. The argument is an (anti)diagonal construction of a "liar" truth value map, equal to its own negation.

[11] objectivity in the sense of SMORYNSKI 1977

Define a "Cretian" map, *truth value* liar $: \mathbb{1} \to \mathbb{2}$
– called '*liar*' because it equals its own negation –
as follows:

Let $\mathrm{ct} : \mathbb{N} \to \mathbb{2}^{\mathbb{N}}$ be the – primitive recursive – *count* of all predicate codes on $\mathbb{N}$; it comes with a (primitive recursive) inverse isomorphism $\mathrm{ct}^{-1} : \mathbb{2}^{\mathbb{N}} \to \mathbb{N}$:

With *negated* PR-*evaluation* – within **set** theory **T** –

$$\delta \ =_{\mathrm{def}} \ \neg \circ \mathrm{ev} \circ (\mathrm{ct}, \mathrm{id}_{\mathbb{N}}) : \mathbb{N} \xrightarrow{(\mathrm{ct},\mathrm{id})} \mathbb{2}^{\mathbb{N}} \times \mathbb{N} \xrightarrow{\mathrm{ev}} \mathbb{2} \xrightarrow{\neg} \mathbb{2}$$

Consider p. r. map (truth value) liar $: \mathbb{1} \to \mathbb{2}$,

$$\begin{aligned}
&\mathrm{liar} \ =_{\mathrm{def}} \ \delta \circ \mathrm{ct}^{-1} \circ \ulcorner \delta \urcorner \\
&\ =_{\mathrm{by\,def}} \ (\neg \circ \mathrm{ev} \circ (\mathrm{ct}, \mathrm{id}_{\mathbb{N}})) \circ \mathrm{ct}^{-1} \circ \ulcorner \delta \urcorner \\
&= \neg \circ \mathrm{ev} \circ ((\mathrm{ct}, \mathrm{id}_{\mathbb{N}}) \circ \mathrm{ct}^{-1} \circ \ulcorner \delta \urcorner) \qquad (\textit{associativity of} \circ) \\
&= \neg \circ \mathrm{ev} \circ (\mathrm{ct} \circ \mathrm{ct}^{-1} \circ \ulcorner \delta \urcorner, \mathrm{ct}^{-1} \circ \ulcorner \delta \urcorner) \qquad (\textit{distributivity}) \\
&= \neg \circ \mathrm{ev}(\ulcorner \delta \urcorner, \mathrm{ct}^{-1} \circ \ulcorner \delta \urcorner) \\
&= \neg \circ \delta \circ (\mathrm{ct}^{-1} \circ \ulcorner \delta \urcorner) \qquad (\textbf{objectivity of } \mathrm{ev}) \\
&\ =_{\mathrm{by\,def}} \ \neg\, \mathrm{liar} : \mathbb{1} \to \mathbb{2} \to \mathbb{2}
\end{aligned}$$

q. e. d. contradiction within theory **T**.

Consequence

The following theories are all **inconsistent:**

Set theories as in particular **PM**, **ZF**, and **NGB**,[12] all of these

[12] see BARWISE ed. 1977

taken with Hilbert's weak (first-order) choice operator[13] iota.[14]

7 WAY OUT

In forthcoming book *Arithmetical Foundations* we replace schema CCC_τ of terminating descending iteration by schema

$$\text{(CCI)}\quad \frac{\begin{array}{l} c = c(a) : A \to \mathbb{N}[\omega] \ \textit{complexity} \\ p = p(a) : A \to A \ \textit{predecessor} \text{ endo} \\ [c(a) = 0 \implies p(a) = a] \ (\textit{stationarity}) \\ \wedge\ [c(a) > 0 \implies c\,p(a) < c(a)] \ (\textit{descent}) \\ \text{put together: } \mathrm{CCI}[c : f] \end{array}}{\begin{array}{l} \mathrm{wh}[c > 0 : p] : A \rightharpoonup A \\ = \texttt{while}[c(a) > 0] \ \texttt{do} \ a := p(a) \ \texttt{od}, \text{ formally:} \\ D_{\mathrm{wh}} = \{(a, m) \in A \times \mathbb{N} : c\,p^m(a) = 0\}, \\ d_{\mathrm{wh}}(a, m) = a : D_{\mathrm{wh}} \to A \\ \widehat{\mathrm{wh}}(a, m) = p^m(a) : D_{\mathrm{wh}} \to A \end{array}}$$

Iterative non-infinite-complexity-descent theory $\pi\mathbf{R} = \mathbf{S} + (\pi)$ is **defined** as *strengthening* of theory $\mathbf{S}$ of primitive recursion with predicate-into-subject abstraction, by the following additional **axiom schema:**

[13] cf. Gödel 1940

[14] see Hilbert 1900/1970

$c : A \to \mathbb{N}[\omega], \ p : A \to A$

data of a complexity controlled iteration – CCI –

with complexity values

in ordered polynomial semiring $\mathbb{N}[\omega]$:

$[c(a) = 0 \Rightarrow p(a) = a] \wedge [c(a) > 0 \Rightarrow cp(a) < c(a)]$;

$\psi = \psi(a) : A \to \mathbb{2}$ a *"negative" test* predicate:

$\psi(a) \implies cp^n(a) > 0$, $a \in A$, $n \in \mathbb{N}$ free

(**non**-termination for all a)

$$(\pi) \quad \frac{}{\psi = \mathrm{false}_A = 0_A : A \to \mathbb{2}}$$

Non-infinite iterative descent: "**Only** the overall **false** predicate implies overall **non**-termination of CCI : *quasi-termination.*"

Recursive theory $\pi\mathbf{R} = \mathbf{S} + (\pi)$ of *non-infinitely descending complexity-controlled iteration* turns out to be *self-consistent*:

$\pi\mathbf{R} \vdash \mathrm{Con}_{\pi\mathbf{R}}$ i. e.

$\pi\mathbf{R} \vdash \neg\mathrm{Prov}_{\pi\mathbf{R}}(n, \mathrm{false})$, $n \in \mathbb{N}$ *free* :

no $n \in \mathbb{N}$ is an arithmetised *proof* of falsity.

This means that theory $\pi\mathbf{R}$ taken as foundation is good for Hilbert's **consistency program** – presumably it is not a *conservative* extension of (categorical primitive recursive Arithmetics) **PR** and **S**, so Hilbert's **conservation program** seems not to work this way.

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