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# Close-to-Fourier heat conduction equation of solids of constant mass density

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## Abstract

Heat conduction close-to-Fourier means that we look for a minimal extension of heat conduction theory using the usual Fourier expression of the heat flux density and modifying that of the internal energy as minimally as possible by choosing the minimal state space. Applying Liu's procedure results in the class of materials and a differential equation both belonging to the close-to-Fourier case of heat conduction.

## 1. Introduction

As is well-known, one of other shortcomings of the classical Fourier heat conduction theory is caused by a parabolic differential equation which allows indefinite propagation of energy. This unphysical fact can be avoided if the parabolic heat conduction equation is replaced by an hyperbolic one [1–4]. Here the question is investigated whether a minimal change in Fourier heat conduction results in an hyperbolic differential equation of first order in time.

By use of Fourier's expression for the heat flux density and the minimal state space spanned by the temperature and its gradient, Liu's procedure is applied for exploiting the second law systematically. The Liu procedure results in coupled differential equations for the specific internal energy, the specific entropy, the heat flux density, and the entropy flux density. One result is that the internal energy and the entropy depend on only one variable, which is a state function and which transforms to the thermostatic temperature in the


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thermostatic limit. Another result is that a hyperbolic heat conduction equation is compatible with the Fourier expression for the heat flux density.

## 2. The balances

In solids of constant mass density $\varrho$, an observer exists for which the field of velocity $\boldsymbol{v}$ vanishes:

$$\varrho = \text{const}, \quad \boldsymbol{v} \equiv \boldsymbol{0}. \tag{1}$$

Consequently, the balance equations of the specific internal energy $\varepsilon$ and the specific entropy $s$ result in

$$\varrho\partial_t\varepsilon + \nabla \cdot \boldsymbol{q} = 0, \quad \varrho\partial_t s + \nabla \cdot \boldsymbol{\Phi} = \sigma \geq 0. \tag{2}$$

Here $\boldsymbol{q}$ is the heat flux density, $\boldsymbol{\Phi}$ the entropy flux density, and $\sigma$ is the entropy production density. The inequality in $(2)_2$ characterizes the second law of thermodynamics. As minimal possible state space, we choose that which is spanned by the temperature $T$ and its gradient $\nabla T$:

$$\boldsymbol{z} = (T, \nabla T). \tag{3}$$

Using the chain rule for calculating the derivatives in the balance equations (2), we obtain the so-called balances on the state space [5]:

$$\varrho\left[\frac{\partial\varepsilon}{\partial T}\partial_t T + \frac{\partial\varepsilon}{\partial\nabla T}\cdot\partial_t\nabla T\right] + \frac{\partial\boldsymbol{q}}{\partial T}\cdot\nabla T + \frac{\partial\boldsymbol{q}}{\partial\nabla T} : \nabla\nabla T = 0, \tag{4}$$

$$\varrho\left[\frac{\partial s}{\partial T}\partial_t T + \frac{\partial s}{\partial\nabla T}\cdot\partial_t\nabla T\right] + \frac{\partial\boldsymbol{\Phi}}{\partial T}\cdot\nabla T + \frac{\partial\boldsymbol{\Phi}}{\partial\nabla T} : \nabla\nabla T = \sigma. \tag{5}$$

The higher derivatives belonging to the state space (3) are

$$\boldsymbol{y} = (\partial_t T, \nabla\partial_t T, \nabla\nabla T). \tag{6}$$

Using them, the balance equations on the state space (4) und (5) can be written down in matrix formulation:

$$\begin{pmatrix} \varrho\dfrac{\partial\varepsilon}{\partial T} & \varrho\dfrac{\partial\varepsilon}{\partial\nabla T} & \dfrac{\partial\boldsymbol{q}}{\partial\nabla T} \end{pmatrix} \cdot \boldsymbol{y} = -\frac{\partial\boldsymbol{q}}{\partial T}\cdot\nabla T, \tag{7}$$

$$\left(\varrho\frac{\partial s}{\partial T} \quad \varrho\frac{\partial s}{\partial \nabla T} \quad \frac{\partial \boldsymbol{\Phi}}{\partial \nabla T}\right)\cdot \boldsymbol{y} \geq -\frac{\partial \boldsymbol{\Phi}}{\partial T}\cdot \nabla T. \tag{8}$$

This inequality, named dissipation inequality, has to be exploited by taking into account the energy balance (7). This will be done in the next section.

## 3. The Liu procedure

For exploiting the balance equations with respect to the second law, we apply Liu's well-known procedure [6–8]. This procedure is based on the following theorem:

If

$$\mathbf{A}\cdot \boldsymbol{y} = \boldsymbol{C}, \quad \boldsymbol{B}\cdot \boldsymbol{y} \geq D, \quad \mathbf{A}(\boldsymbol{z}), \boldsymbol{B}(\boldsymbol{z}), \boldsymbol{C}(\boldsymbol{z}), D(\boldsymbol{z}) \tag{9}$$

are the balance equations and the dissipation inequality in matrix formulation, a state space function $\boldsymbol{\Lambda}(\boldsymbol{z})$ exists which satisfies the so-called Liu equations and the residual inequality

$$\boldsymbol{\Lambda}\cdot \mathbf{A} = \boldsymbol{B}, \quad \boldsymbol{\Lambda}\cdot \boldsymbol{C} \geq D. \tag{10}$$

The higher derivatives $\boldsymbol{y}$ (6) are eliminated by the Liu procedure.

According to (7) and (8), in our case a comparison with (9) results in

$$\left(\varrho\frac{\partial \varepsilon}{\partial T} \quad \varrho\frac{\partial \varepsilon}{\partial \nabla T} \quad \frac{\partial \boldsymbol{q}}{\partial \nabla T}\right) \mathrel{\widehat{=}} \mathbf{A}, \quad -\frac{\partial \boldsymbol{q}}{\partial T}\cdot \nabla T \mathrel{\widehat{=}} \boldsymbol{C}, \tag{11}$$

$$\left(\varrho\frac{\partial s}{\partial T} \quad \varrho\frac{\partial s}{\partial \nabla T} \quad \frac{\partial \boldsymbol{\Phi}}{\partial \nabla T}\right) \mathrel{\widehat{=}} \boldsymbol{B}, \quad -\frac{\partial \boldsymbol{\Phi}}{\partial T}\cdot \nabla T \mathrel{\widehat{=}} D. \tag{12}$$

Using (11) and (12), we obtain the Liu equations and the residual inequality (10).

In our case, the Liu equations run as follows:

$$\lambda\varrho\frac{\partial \varepsilon}{\partial T} = \varrho\frac{\partial s}{\partial T} \rightarrow \lambda = \frac{\partial s/\partial T}{\partial \varepsilon/\partial T} = \left(\frac{ds}{d\varepsilon}\right)_{\nabla T}, \tag{13}$$

$$\lambda\varrho\frac{\partial \varepsilon}{\partial \nabla T} = \varrho\frac{\partial s}{\partial \nabla T}, \tag{14}$$

$$\lambda \frac{\partial \boldsymbol{q}}{\partial \nabla T} = \frac{\partial \boldsymbol{\Phi}}{\partial \nabla T}. \tag{15}$$

The residual inequality is

$$-\lambda \frac{\partial \boldsymbol{q}}{\partial T} \cdot \nabla T \geq -\frac{\partial \boldsymbol{\Phi}}{\partial T} \cdot \nabla T \rightarrow \frac{\partial}{\partial T}(\lambda \boldsymbol{q} - \boldsymbol{\Phi}) \cdot \nabla T \leq \frac{\partial \lambda}{\partial T} \boldsymbol{q} \cdot \nabla T. \tag{16}$$

Liu's equations and the residual inequality represent constraints for the partial derivatives of the constitutive mappings $\varepsilon$, $s$, $\boldsymbol{q}$, and $\boldsymbol{\Phi}$.

From (13) and (14) we obtain

$$\lambda \partial_t \varepsilon = \partial_t s, \tag{17}$$

and from (15) and (16),

$$\lambda \nabla \cdot \boldsymbol{q} \leq \nabla \cdot \boldsymbol{\Phi} \rightarrow -\boldsymbol{q} \cdot \nabla \lambda \leq \nabla \cdot (\boldsymbol{\Phi} - \lambda \boldsymbol{q}). \tag{18}$$

From (13) and (14), the second derivatives result in

$$\frac{\partial^2 s}{\partial \nabla T \partial T} = \frac{\partial}{\partial \nabla T}\left(\lambda \frac{\partial \varepsilon}{\partial T}\right) = \frac{\partial}{\partial T}\left(\lambda \frac{\partial \varepsilon}{\partial \nabla T}\right), \tag{19}$$

$$\frac{\partial^2 \varepsilon}{\partial \nabla T \partial T} = \frac{\partial}{\partial \nabla T}\left(\frac{1}{\lambda} \frac{\partial s}{\partial T}\right) = \frac{\partial}{\partial T}\left(\frac{1}{\lambda} \frac{\partial s}{\partial \nabla T}\right). \tag{20}$$

Consequently, we obtain from the second equations by straightforward calculation

$$\frac{\partial \lambda}{\partial \nabla T} \frac{\partial \varepsilon}{\partial T} = \frac{\partial \lambda}{\partial T} \frac{\partial \varepsilon}{\partial \nabla T}, \tag{21}$$

$$\frac{\partial (1/\lambda)}{\partial \nabla T} \frac{\partial s}{\partial T} = \frac{\partial (1/\lambda)}{\partial T} \frac{\partial s}{\partial \nabla T}. \tag{22}$$

The differential of the internal energy follows by the choice of the state space (3) as:

$$d\varepsilon = \frac{\partial \varepsilon}{\partial T} dT + \frac{\partial \varepsilon}{\partial \nabla T} \cdot \nabla T = \frac{\partial \varepsilon}{\partial T} dT + \frac{\partial \varepsilon}{\partial T} \frac{\partial \lambda / \partial \nabla T}{\partial \lambda / \partial T} \cdot \nabla T. \tag{23}$$

Here, the second equation comes out by inserting (21). From (23) follows

$$\frac{\partial\lambda}{\partial T}d\varepsilon = \frac{\partial\varepsilon}{\partial T}\left[\frac{\partial\lambda}{\partial T}dT + \frac{\partial\lambda}{\partial\nabla T}\cdot\nabla T\right] = \frac{\partial\varepsilon}{\partial T}d\lambda, \tag{24}$$

and therefore

$$d\varepsilon = \frac{\partial\varepsilon/\partial T}{\partial\lambda/\partial T}d\lambda. \tag{25}$$

Totally analogous, (22) results in

$$ds = \frac{\partial s/\partial T}{\partial(1/\lambda)/\partial T}d(1/\lambda) = \frac{\partial s/\partial T}{\partial\lambda/\partial T}d\lambda. \tag{26}$$

By (25) and (26), we obtain that the internal energy and the entropy depend on only one variable, which is a state function:

$$\varepsilon = \varepsilon(\lambda(T, \nabla T)), \quad s = s(\lambda(T, \nabla T)). \tag{27}$$

This result cannot be derived for the heat flux density and the entropy flux density, because the equation analogous to (13) does not exist and is replaced by the inequality (16). Thus we obtain from (13) and (14) the relation

$$\lambda\frac{\partial\varepsilon}{\partial\lambda} = \frac{\partial s}{\partial\lambda} \to \varepsilon = \frac{\partial}{\partial\lambda}(\lambda\varepsilon - s). \tag{28}$$

The influence of the constraints on the constitutive mappings by the Liu equations and the residual inequality is investigated in the next section.

## 4. Close-to-Fourier constitutive equations

In the frame of the chosen state space (3), we assume for the heat flux density and for the rate of the specific internal energy

$$\boldsymbol{q} = -\kappa(\boldsymbol{z})\nabla T, \tag{29}$$

$$\partial_t\varepsilon = C(\boldsymbol{z})\partial_t T + B(\boldsymbol{z})\nabla T\cdot\partial_t\nabla T. \tag{30}$$

From the last equation we get

$$C(\boldsymbol{z}) = \frac{\partial \varepsilon(\boldsymbol{z})}{\partial T} = \frac{\partial \varepsilon}{\partial \lambda}\frac{\partial \lambda}{\partial T}, \tag{31}$$

$$B(\boldsymbol{z})\nabla T = \frac{\partial \varepsilon(\boldsymbol{z})}{\partial \nabla T} = \frac{\partial \varepsilon}{\partial \lambda}\frac{\partial \lambda}{\partial \nabla T}. \tag{32}$$

A relation follows from both the last equations for $B$ and $C$, namely

$$B\nabla T = C\frac{\partial\lambda/\partial\nabla T}{\partial\lambda/\partial T}. \tag{33}$$

Inserting this into (30), we obtain

$$\partial_t \varepsilon = C\left(\partial_t T + \frac{\partial\lambda/\partial\nabla T}{\partial\lambda/\partial T}\cdot\partial_t\nabla T\right). \tag{34}$$

Using the Liu equations (13) to (15) and the dissipation inequality (16), the partial derivatives of $s$ and $\boldsymbol{\Phi}$ satisfy

$$\frac{\partial s}{\partial T} = \frac{\partial s}{\partial \lambda}\frac{\partial \lambda}{\partial t} = \lambda C, \quad \frac{\partial s}{\partial \nabla T} = \frac{\partial s}{\partial \lambda}\frac{\partial \lambda}{\partial \nabla T} = \lambda B\nabla T, \tag{35}$$

$$\frac{\partial \boldsymbol{\Phi}}{\partial T}\cdot\nabla T \geq -\lambda\frac{\partial \kappa}{\partial T}\nabla T\cdot\nabla T, \quad \frac{\partial \boldsymbol{\Phi}}{\partial \nabla T} = -\lambda\left(\frac{\partial \kappa}{\partial \nabla T}\nabla T + \kappa\mathbf{1}\right). \tag{36}$$

## 5. Close-to-Fourier heat conduction equation

Inserting the constitutive equations (29) and (30) into the balance of energy $(2)_1$, we obtain a close-to-Fourier heat conduction equation as follows:

$$\varrho(C\partial_t T + B\nabla T\cdot\partial_t\nabla T) = \frac{\partial\kappa}{\partial T}\nabla T\cdot\nabla T + \frac{\partial\kappa}{\partial\nabla T}\nabla T : \nabla\nabla T + \kappa\nabla\cdot\nabla T \tag{37}$$

This heat conduction equation is of first order in time, a fact which is essential for the initial conditions. Dependent on the values of its coefficients, this differential equation is parabolic or hyperbolic. This demonstrates that the ansatz (29) allows hyperbolic heat conduction if the expression for the internal energy, modified by (30), is chosen according to the minimal state space (3). If $B \equiv 0$, (37) becomes parabolic according to usual Fourier heat conduction.

## 6. Thermostatic limit

The state space function $\lambda$, which is the only variable that the internal energy $\varepsilon$ and the entropy $s$ depend on, is defined in $(13)_2$. If we introduce the thermostatic definition of the temperature by

$$\frac{\partial s/\partial T}{\partial \varepsilon/\partial T} = \left(\frac{ds}{d\varepsilon}\right)_{\nabla T} \doteq \frac{1}{T} = \lambda, \tag{38}$$

we obtain the thermostatic limit, which is discussed in the sequel.

In this case,

$$\frac{\partial \lambda}{\partial \nabla T} \equiv \mathbf{0} \tag{39}$$

is valid. Consequently, from (21) and (22) follows

$$\frac{\partial \varepsilon}{\partial \nabla T} \equiv \mathbf{0}, \quad \frac{\partial s}{\partial \nabla T} \equiv \mathbf{0}. \tag{40}$$

Using (31) and (32), we obtain

$$C = C(T), \quad B \equiv 0, \tag{41}$$

and (38) becomes parabolic, and because of (17)

$$\partial_t \varepsilon = T \partial_t s \tag{42}$$

is valid. From (15) follows

$$\mathbf{0} = \frac{\partial}{\partial \nabla T}\left(\boldsymbol{\Phi} - \frac{\boldsymbol{q}}{T}\right) \rightarrow \boldsymbol{\Phi} = \frac{\boldsymbol{q}}{T} + \boldsymbol{k}(T). \tag{43}$$

The inequalities $(16)_2$ and $(18)_2$ become in the thermostatic limit

$$\frac{\partial}{\partial T}\left(\boldsymbol{\Phi} - \frac{\boldsymbol{q}}{T}\right) \cdot \nabla T \geq \frac{1}{T^2} \boldsymbol{q} \cdot \nabla T, \tag{44}$$

$$\nabla \cdot \left(\boldsymbol{\Phi} - \frac{\boldsymbol{q}}{T}\right) \geq \frac{1}{T^2} \boldsymbol{q} \cdot \nabla T. \tag{45}$$

Hence, from (44) and (43) follows

$$\left(\frac{\partial \boldsymbol{k}}{\partial T} - \frac{1}{T^2}\boldsymbol{q}\right) \cdot \nabla T \geq 0. \tag{46}$$

Because of the inequality, we have a common change of the sign of both the factors:

$$\nabla T = \mathbf{0} \rightarrow \frac{\partial \boldsymbol{k}}{\partial T} = \frac{1}{T^2}\boldsymbol{q} = \mathbf{0} \rightarrow \boldsymbol{k} = \boldsymbol{const}. \tag{47}$$

Here (29) was considered. Therefore, we obtain from (46)

$$-\boldsymbol{q} \cdot \nabla T \geq 0 \rightarrow \kappa(\boldsymbol{z}) \geq 0. \tag{48}$$

The constant in $(47)_3$ is zero if the entropy flux density vanishes together with the heat flux density, an assumption which cannot be derived by Liu's procedure.

## 7. Summary

Close-to-Fourier heat conduction is characterized by two items: choice of a minimal state space and choice of Fourier's ansatz for the heat flux density. By Liu's procedure, we obtain a close-to-Fourier heat conduction equation which is hyperbolic and parabolic in the thermostatic limit. This result demonstrates that close-to-Fourier heat conduction is hyperbolic beyond the thermostatic limit, thus avoiding one of the shortcomings of Fourier heat conduction.

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