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THE ZOO OF TREE SPANNER PROBLEMS

by

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Abstract

Tree spanner problems have important applications in network design, e.g. in the telecommunications industry. Mathematically, there have been considered quite a number of max-stretch tree spanner problems and of average stretch tree spanner problems.

We propose a unified notation for 20 tree spanner problems, which we investigate for graphs with general positive weights, with metric weights, and with unit weights. This covers several prominent problems of combinatorial optimization. Having this notation at hand, we can clearly identify which problems coincide. In the case of unweighted graphs, the formally 20 problems collapse to only five different problems.

Moreover, our systematic notation for tree spanner problems enables us to identify a tree spanner problem whose complexity status has not been solved so far. We are able to provide an NP-hardness proof. Furthermore, due to our new notation of tree spanner problems, we are able to detect that an inapproximability result that is due to Galbiati (2001, 2003) in fact applies to the classical max-stretch tree spanner problem. We conclude that the inapproximability factor for this problem thus is $2-\varepsilon$, instead of only $\frac{1+\sqrt{5}}{2} \approx 1.618$ according to Peleg and Reshef (1999).

1 Introduction

We consider a weighted connected undirected graph (G, w), where $G = (V, E)$. We assume the edge weights to be positive integers, occasionally after scaling. Let T be a spanning tree of G. Depending on the context, we think of T either as a subset of the edges of G, or as a subgraph of G. For a spanning subgraph H of G and $u, v \in V$ we denote by

$$d_H(u, v)$$

the length of a shortest (u, v)-path in H.

In [5] the *t-tree spanner problem* has been introduced as follows: Decide whether there exists a t-tree spanner, i.e. a spanning tree T of G such that

$$\frac{d_T(u,v)}{d_G(u,v)} \leq t, \quad \forall (u,v) \in V \otimes V := V \times V \setminus \{(v,v) \mid v \in V\}. \tag{1}$$

The corresponding optimization problem of constructing a spanning tree that realizes the minimum value t among all spanning trees is called the MINIMUM MAX-STRETCH SPANNING TREE (MMST) problem ([10]). Applications of the MMST problem arise in the area of network design, e.g. in the telecommunications industry. There, trees are of particular interest, because they allow to "keep the routing protocols simple" ([16]).

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In [24] the related problem of finding a MINIMUM AVERAGE-STRETCH SPANNING TREE (MAST) has been considered: Let $w = \mathbf{1}$, i.e. G is an unweighted graph, find a spanning tree T that minimizes

$$\sum_{\{u,v\} \in E \setminus T} \frac{d_T(u,v)}{d_G(u,v)}. \tag{2}$$

Since for unweighted graphs there holds $d_G(u,v) = 1$ for all $\{u,v\} \in E$, it is a simple observation ([1]) that this MAST problem turns out to be nothing but a special case of the MINIMUM STRICTLY FUNDAMENTAL CYCLE BASIS (MSFCB) problem as it has been considered for instance in [8]: Find a spanning tree T that minimizes

$$\sum_{e=\{u,v\} \in E \setminus T} d_T(u,v) + w(e). \tag{3}$$

The MAST finds increasing attention in preconditioning, in particular for solving symmetric diagonally dominant linear systems ([9]).

There is another related problem for which one can detect an even larger variety in notation. In the SHORTEST TOTAL PATH LENGTH SPANNING TREE (STPLST, [8, 25]) problem we seek for a spanning tree that minimizes

$$\sum_{(u,v) \in V \otimes V} d_T(u,v). \tag{4}$$

The very same problem has also been referred to as the MINIMUM ROUTING COST SPANNING TREE (MRCST) problem ([26, 14]). In the special case of an unweighted graph, Johnson et al. ([20]) call it the SIMPLE NETWORK DESIGN problem. Alternatively, when considering complete graphs, Hu ([19]) introduced it as the OPTIMUM DISTANCE SPANNING TREE problem. In [7], the MINIMUM AVERAGE DISTANCE (MAD) spanning tree problem is considered—but setting the vertex weights in that model to one, this is another variant of the STPLST problem.

Notice that also additive tree spanner problems attracted quite a number of researchers (e.g. [22]). However, to keep the presentation focused we restrict ourselves to max-stretch and average stretch tree spanners.

Outline. We propose a unified notation for the large variety of tree spanner problems in Section 2. Subject to this notation we identify which problems coincide. More specifically, we consider two problems P and Q to coincide, if every spanning tree T_P that is optimum for P constitutes an optimum solution for Q, and vice-versa. We use the notation $P \sim Q$ as a shorthand. Notice that we have to choose this very discriminative equivalence relation. Otherwise, if we allowed for general polynomial transformations, one could no more distinguish between any two NP-complete problems. We provide coincidences for both maximum stretch tree spanners (Section 3.1) and average stretch tree spanners (Section 4.1), for the cases of graphs with general weights, with metric weights, and with unit weights. We complement our analysis by providing example graphs showing that there are no further coincidences. All the examples consist of fairly small simple 2-vertex connected planar graphs.

Consider the very rich world of (in-) approximability results for tree spanner problems, occasionally for special classes of graphs. We expect that having at hand a clear map of the relationships between the various tree spanner problems, a certain cross-fertilization between the different perspectives on much similar structures will occur. In Section 6.2 we make the first step into this direction. In the context of tree spanners, as recently as 2004 the best known inapproximability factor of the MMST problem has been cited as $\frac{1+\sqrt{5}}{2}$ ([10]), being due to [23].

In this paper we observe that in the case of unweighted graphs the MMST problem coincides with the MIN-MAX STRICTLY FUNDAMENTAL CYCLE BASIS (MMSFCB) problem, as it has been stated in [12, 13]. There, an inapproximability factor of $2-\varepsilon$ has been achieved already in 2001 ([12]). Hence, this applies immediately to the MMST problem as well. Moreover, in the family of tree spanner problems we identify the only problem whose complexity status has not been identified before. We provide an NP-hardness result for it.

2 A Unified Notation for Tree Spanners (UNTS)

There are three major criteria in which tree spanner problems may differ: First, either the maximum stretch or the average stretch is to be determined. Second, this objective may be computed with respect to different sets of pairs of vertices, e.g. for $(u,v) \in V \otimes V$ or only for $\{u,v\} \in E \setminus T$. Third, there have been considered various terms for the objective, e.g. $\frac{d_T(u,v)}{d_G(u,v)}$ or $d_T(u,v) + w(e)$.

In the remainder, we refer to a tree spanner problem P through a triple

$$(\texttt{goal}, \texttt{domain}, \texttt{term})\,.$$

We consider the following family of tree spanner problems:

- `goal`
  The goal is either the maximum stretch or the average stretch.
- `domain`
  The domain is either $\{u,v\} \in E \setminus T$, $\{u,v\} \in E$, or $(u,v) \in V \otimes V$.
- `term`
  The term may be one of $d_T(u,v) + w(e)$, $d_T(u,v)$, $\frac{d_T(u,v)}{w(e)}$, or $\frac{d_T(u,v)}{d_G(u,v)}$.

Notice first that we do not consider $(*,\, V \otimes V,\, d_T(u,v) + w(e))$ and $\left(*,\, V \otimes V,\, \frac{d_T(u,v)}{w(e)}\right)$, because $w(e)$ is not properly defined for $(u,v) \in V \otimes V \setminus E$. Second, it could appear somehow strange to count the weight of tree edges twice in the two tree spanner problems $(*,\, E,\, d_T(u,v)+w(e))$. However, this is consistent with the UNTS. Moreover, this does not cause any degeneracies, because in the next two sections we exhibit that there is always some other tree spanner problem, which coincides with $(*,\, E,\, d_T(u,v) + w(e))$. Third, observe that for a given graph, $|E|$ and $|V \otimes V|$ are constant, and $|E \setminus T|$ is independent of the tree T. Hence, we prefer to represent the goal "average" with the $\sum$ symbol.

We provide a first idea of the wide range of these tree spanner problems by locating several well-known problems of combinatorial optimization within the UNTS:

- $\left(\max,\, V \otimes V,\, \frac{d_T(u,v)}{d_G(u,v)}\right)$ is the MMST problem ([5]),
- $(\max,\, V \otimes V,\, d_T(u,v))$ is the MINIMUM DIAMETER SPANNING TREE (MDST) problem ([16]),
- $(\sum,\, V \otimes V,\, d_T(u,v))$ is the STPLST problem (or MRCST problem, [20]), and
- $(\sum,\, E \setminus T,\, d_T(u,v) + w(e))$ is the MSFCB problem ([8]).

We will establish that among the remaining 16 problems there is only one single problem which does not coincide with one of these four prominent problems in the case of unweighted graphs. Since its complexity status has not been identified before, we provide an NP-hardness proof for it. However, in the case of weighted graphs there is a much larger variety of problems, in particular in the context of average stretch tree spanners. For instance, in [9] the same techniques are applied to both $\left(\sum,\ E,\ \frac{d_T(u,v)}{w(e)}\right)$ and $\left(\sum,\ E,\ \frac{d_T(u,v)}{d_G(u,v)}\right)$. Nevertheless, in general these problems do not coincide.

In Figure 1 we summarize all the coincidences that exist between tree spanner problems, and which we are going to develop in the remainder of this paper.

Namely, in Sect. 3 we deal with max-stretch problems whereas in Section 4 average-stretch problems are considered. We organize the sections by subdividing them into three parts where we distinguish general, metric, and unit weights. However, we show that there is a bridge between unweighted and integer-weighted tree spanner problems. Here, we aim at identifying a weighted instance (G, w) immediately with the unweighted instance G' that results from replacing every edge $e = \{u, v\}$ with weight $w(e)$ with a uv-path P'_e having $w(e)$ edges. Observe that every spanning tree of G' has to contain at least $w(e) - 1$ edges of P'_e. Now, consider the term $d_T(u, v) + w(e)$. Let T be some spanning tree of G. We construct a spanning tree T' of G' such that if $e \in T$ then $P'_e \subseteq T'$. This yields

$$d_T(u,v) + w(e) = d_{T'}(u',v') + 1, \quad \forall e = \{u,v\} \in E \setminus T, \{u',v'\} = P'_e \setminus T'. \tag{5}$$

Hence, for the domain $E \setminus T$ in conjunction with the term $d_T(u, v) + w(e)$ an optimum solution to a weighted tree spanner problem is obtained by a kind of projection from an optimum solution to the corresponding unweighted problem, and vice-versa.

Proposition 1. Let `goal` be a fixed optimization goal. Then, the weighted version and the unweighted version of ($\texttt{goal},\ E \setminus T,\ d_T(u,v) + w(e)$) coincide.

3 Maximum Stretch Problems

We start our tour through the zoo of tree spanner problems with maximum stretch tree spanner problems. We first collect the pairs of problems which coincide, where we distinguish between general weights, metric weights, and unit weights. Then, we examine example graphs showing that there are no further coincidences.

3.1 Coincidences

It is an elementary observation that if two tree spanner problems coincide even for general weights, in particular they also coincide for metric weights. Moreover, if two problems coincide for metric weights, they immediately coincide for unweighted graphs, too. Hence, to present the coincidences between maximum stretch tree spanner problems, we proceed from the most general weight functions to the most specialized weight function.

General Weights. In the case of general weights, there are five families of coincident maximum stretch tree spanner problems.

Proposition 2. The following two maximum stretch problems coincide: $(\max,\ E \setminus T,\ d_T(u,v) + w(e))$ and $(\max,\ E,\ d_T(u,v) + w(e))$.

Proof. Assume for contradiction there was a weighted graph (G, w) such that a spanning tree T that is optimum with respect to $(\max,\ E,\ d_T(u,v) + w(e))$ attains its maximum exclusively on a tree edge $e \in T$. Then,

$$\forall f = \{u,v\} \in E \setminus T :\ d_T(u,v) + w(f) < 2w(e). \tag{6}$$

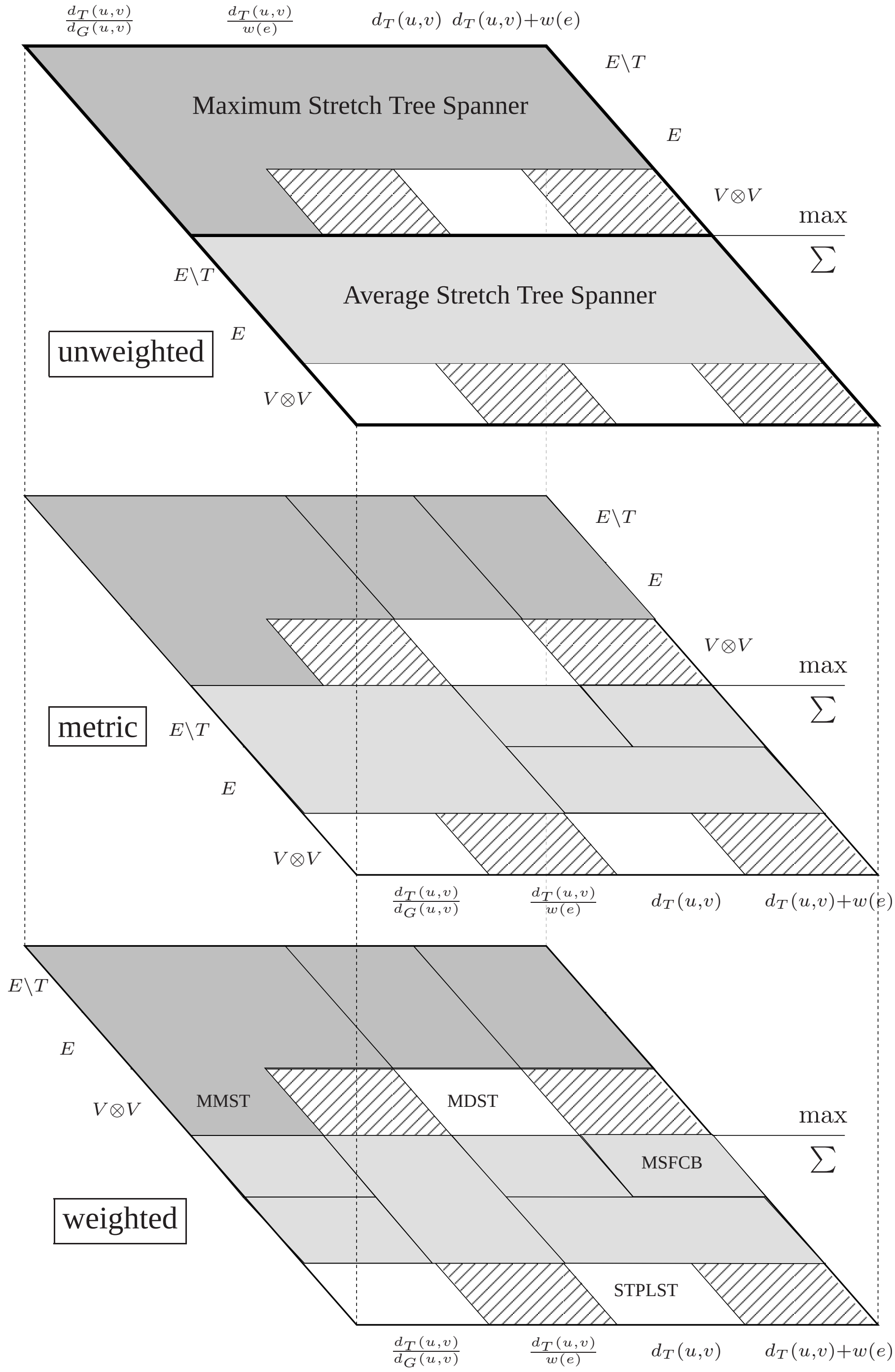


Figure 1: A guide to the zoo of tree spanner problems

Consider any edge $f \in E \setminus T$ whose fundamental circuit C contains the edge e. Such an edge exists because G is 2-vertex connected. The total weight of the circuit C is precisely $w(C) = d_T(u,v) + w(f)$, and the weight of $C \setminus \{e\}$ is $d_T(u,v) + w(f) - w(e)$. By (6) there holds

$$d_T(u,v) + w(f) - w(e) < w(e). \tag{7}$$

Now, consider the spanning tree $T' = T \cup \{f\} \setminus \{e\}$. Any fundamental circuit (different from C) with respect to T that contained the edge e is replaced with a subpath of $C \setminus \{e\}$. As we only consider positive edge weights, by (7) the new fundamental circuit is strictly shorter than the initial one. Hence, the spanning tree T has not been optimum with respect to $(\max,\ E,\ d_T(u,v) + w(e))$. □

Proposition 3. The two problems $(\max,\ E \setminus T,\ d_T(u,v))$ and $(\max,\ E,\ d_T(u,v))$ coincide.

Proof. Let T be an arbitrary spanning tree of (G,w). These two problems could only differ, if the maximum in $(\max,\ E,\ d_T(u,v))$ is attained exclusively by a tree-edge $e = \{u,v\} \in T$. But in this case, $d_T(u,v) = w(e)$. As we only consider 2-vertex connected graphs, there exists a circuit C through e. The tree T cannot contain all the edges of C. Hence, as we assume the weight function w to be positive, there exists a non-tree edge $e' = \{u',v'\} \in C \setminus T$ such that $d_T(u,v) \leq d_T(u',v')$. □

In the sequel we establish that the following five—recall that $\left(\max, V \otimes V, \frac{d_T(u,v)}{w(e)}\right)$ is not properly defined—tree spanner problems coincide: $\left(\max, *, \frac{d_T(u,v)}{w(e)}\right)$ and $\left(\max, *, \frac{d_T(u,v)}{d_G(u,v)}\right)$. In fact, most of the work has been done by Cai and Corneil ([5]):

Theorem 4 ([5])**.** Consider the following five tree spanner problems: $\left(\max, *, \frac{d_T(u,v)}{w(e)}\right)$ and $\left(\max, *, \frac{d_T(u,v)}{d_G(u,v)}\right)$. If for a given weighted graph (G,w) all of them attain an optimum stretch value of $t \geq 1$, then these five problems coincide on (G,w).

However, subject to our definition of coincidence, we are even able to relax the assertion of t being greater or equal than one. To that end, we start with an easy observation.

Lemma 5. Consider one of the four problems $\left(\max,\ E,\ \frac{d_T(u,v)}{w(e)}\right)$ and $\left(\max, *, \frac{d_T(u,v)}{d_G(u,v)}\right)$ for some weighted graph (G,w). For the optimum stretch factor t that can be obtained with respect to this problem, there holds $t \geq 1$.

Proof. In the definition of the `term` $\frac{d_T(u,v)}{d_G(u,v)}$, $d_G(u,v)$ is the length of a shortest uv-path in G. Thus $\frac{d_T(u,v)}{d_G(u,v)} \geq 1$ for all $(u,v) \in V \otimes V$. When considering the `term` $\frac{d_T(u,v)}{w(e)}$ over the `domain` E, for every tree edge $e \in T$ this edge constitutes the unique uv-path in T. In particular, $\frac{d_T(u,v)}{w(e)} = 1$ for all $e = \{u,v\} \in T \subset E$. □

Proposition 6. If a weighted graph (G,w) admits a tree spanner T such that $t < 1$ subject to $\left(\max,\ E \setminus T,\ \frac{d_T(u,v)}{w(e)}\right)$, then T is unique optimum for all five problems $\left(\max,\ *,\ \frac{d_T(u,v)}{w(e)}\right)$ and $\left(\max, *, \frac{d_T(u,v)}{d_G(u,v)}\right)$.

Proof. So, let (G,w) be a weighted graph that admits a tree spanner T such that $t < 1$ subject to $\left(\max,\ E \setminus T,\ \frac{d_T(u,v)}{w(e)}\right)$. Then, in order to prove the proposition it suffices to show that

1. for the four problems $\left(\max,\ E,\ \frac{d_T(u,v)}{w(e)}\right)$ and $\left(\max, *, \frac{d_T(u,v)}{d_G(u,v)}\right)$ there holds $t = 1$; further,

2. for every spanning tree $T' \neq T$ of G the five problems $\left(\max,\, *,\, \frac{d_T(u,v)}{w(e)}\right)$ and $\left(\max, *, \frac{d_T(u,v)}{d_G(u,v)}\right)$ have stretch factor $t' > 1$.

First, we prove 1. Therefore notice that for each $e = \{u, v\} \in T$ it holds $d_T(u, v) = w(e)$. Hence, for $\left(\max,\, E,\, \frac{d_T(u,v)}{w(e)}\right)$ one immediately observes $t = 1$. Now, consider the problem $\left(\max,\, V \otimes V, \frac{d_T(u,v)}{d_G(u,v)}\right)$. Let u and v be two vertices of G and let P_{uv} be the unique uv-path in T. Assume for contradiction that P_{uv} is not a shortest uv-path. So, let P' be a shortest uv-path in G. Then, P' contains at least one edge $f = \{u', v'\}$ that is not contained in T, since otherwise P_{uv} and P' contain a cycle in T. However, because of the proposition's assumption we know that $d_T(u', v') < w(f)$. Let $\tilde{P}$ be the $u'v'$-path in T. Then this path $\tilde{P}$ can be used to construct an uv-path with length strictly smaller than the length of P'. From this contradiction we conclude that for an arbitrary pair of vertices u and v the uv-path in T is a shortest uv-path. Hence, $d_T(u, v) = d_G(u, v)$ and the claim follows for $\left(\max,\, V \otimes V, \frac{d_T(u,v)}{d_G(u,v)}\right)$ and hereby for $\left(\max, *, \frac{d_T(u,v)}{d_G(u,v)}\right)$ with the two remaining domains as well.

Now, we prove 2. Therefore, let T and T' be defined as in 2. From $T' \neq T$ we conclude that there exists some edge $e = \{u, v\} \in T' \setminus T$. In particular, $t < 1$ provides us with $\sum_{f \in P_{uv}} w(f) < w(e)$, where $P_{uv} \subseteq T$ is the unique uv-path in T.

Consider the fundamental circuit $C_T(e) = \{e\} \cup P_{uv}$ that the edge e induces with respect to T. As T' is a tree, the set of edges $F = C_T(e) \setminus T'$ is nonempty, and in particular $e \notin F$, because $e \in T'$.

Because of $w(e) > w(f)$ for all edges $f \in P_{uv}$ and since we are only considering positive weight functions w, it remains to detect some edge $f \in F$, such that $e \in C_{T'}(f)$. Since the fundamental circuits with respect to T' form a basis of the cycle space $\mathcal{C}(G)$, and $C_T(e) \in \mathcal{C}(G)$, there exists a set $F' \subseteq E \setminus T'$ such that

$$C_T(e) = \sum_{f \in F'} C_{T'}(f),$$

where we consider the symmetric difference. Due to the special structure of cycle bases that are associated with spanning trees, we know that $F' \subset C_T(e) \setminus T'$, in fact $F' = F$ ([2]). In particular, as by definition the edge e is contained in $C_T(e)$, e has to appear in at least one fundamental circuit $C_{T'}(f)$ that is induced by an edge $f = \{u', v'\} \in F$. $\square$

Corollary 7. The following five problems coincide: $\left(\max, *, \frac{d_T(u,v)}{w(e)}\right)$ and $\left(\max, *, \frac{d_T(u,v)}{d_G(u,v)}\right)$.

Proof. Consider an optimum solution T with respect to $\left(\max,\, E \setminus T,\, \frac{d_T(u,v)}{w(e)}\right)$. In the case of a stretch factor $t \geq 1$ we are done immediately by applying Theorem 4. Otherwise, i.e., if $t < 1$, Proposition 6 ensures optimality and uniqueness of T subject to *all* five optimization problems that we consider here. $\square$

In particular, all the maximum stretch tree spanner problems that involve fractions coincide.

Metric Weights. For maximum stretch tree spanner problems, there are no coincidences in the case of metric weights, which do not apply already to the general case.

Unit Weights. For an arbitrary tree T of an unweighted connected graph G (or having weights

$w = \mathbf{1}$) with n vertices and m edges, there holds

$$\max_{e=\{u,v\}\in E} \{d_T(u,v) + w(e)\} = \max_{e=\{u,v\}\in E\setminus T} \{d_T(u,v) + w(e), 2\} \tag{8}$$
$$= \max_{e=\{u,v\}\in E} \{d_T(u,v) + 1\} \tag{9}$$
$$= \max_{e=\{u,v\}\in E\setminus T} \{d_T(u,v) + 1, 2\}\,. \tag{10}$$

Moreover, by $w(e) = 1$ we obtain immediately $d_T(u,v) = \frac{d_T(u,v)}{w(e)}$. Finally, in the case of an unweighted graph, for every edge $e = \{u,v\}$ there holds $d_T(u,v) = \frac{d_T(u,v)}{d_G(u,v)}$. Together with (8)–(10) and Corollary 7 we conclude

Proposition 8. Let G be an unweighted graph. Except for $(\max, V\otimes V, d_T(u,v))$, all max-stretch tree spanner problems coincide.

3.2 Anticoincidences

In order to prove for two problems that they do not coincide, we profit from the following transitive relation: If the problems do not coincide for unweighted graphs, then they do not coincide for graphs with metric weights. Furthermore, if there is a graph with metric weights for which the sets of optimum solutions for two tree spanner problems have empty intersection, then these problems cannot coincide for general weights either. Thus, we provide the relevant anticoincidences by moving from the most specialized weight function to general weight functions.

Unit Weights. As by Proposition 8 there are only two different maximum stretch tree spanner problems in the case of unweighted graphs, we only have to establish one single anticoincidence.

Example 9 (MMST vs. MDST)**.** Consider the unweighted simple graph G in Figure 2(a). Recall from Proposition 8 and from Theorem 4 that in the unweighted case we may think of the MMST problem as $(\max, E\setminus T, d_T(u,v))$. Hence, we are looking for a spanning tree whose non-tree edges are linked by paths in T whose maximum length is minimal. The spanning tree that we highlight in Figure 2(b) attains an objective value of two. Moreover, every spanning tree that attains an objective value of two has to induce all five triangles of G as its fundamental circuits. Thus, such a spanning tree must contain the four edges that are not incident with the infinite face. So it must not contain the edge e.

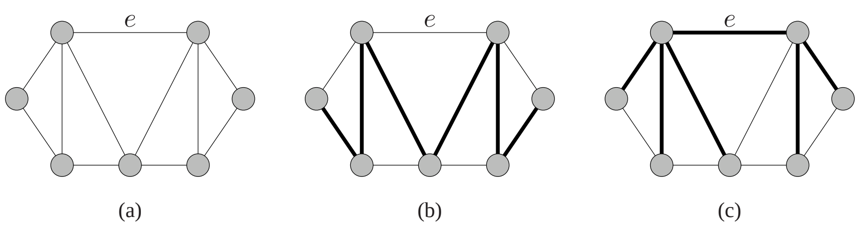


Figure 2: An unweighted graph and example trees which show that MMST and MDST do not coincide

In contrast, for that in the MDST problem a diameter of three can be achieved, the leftmost vertex and the rightmost vertex have to be connected via a path of three edges. Observe that there is only one such path. But this includes the edge e, see Figure 2(c) for one of the two optimum trees. □

Metric Weights. In order to complement the results of Section 3.1, we have to show that the following three problems do not coincide:

- $(\max,\ E \setminus T,\ d_T(u,v) + w(e))$,
- $(\max,\ E \setminus T,\ d_T(u,v))$, and
- $\left(\max,\ E \setminus T,\ \frac{d_T(u,v)}{w(e)}\right)$.

Fortunately, there exists a fairly small graph with metric weights such that the unique optimal solutions for these three problems are disjoint.

Example 10 **(**$(\max,\ E \setminus T,\ d_T(u,v) + w(e))$ **vs.** $(\max,\ E \setminus T,\ d_T(u,v))$ **vs.** $\left(\max,\ E \setminus T,\ \frac{d_T(u,v)}{w(e)}\right)$**).** Consider the graph in Figure 3(a). In Table 1 the objective values of the three spanning trees in Figures 3(b)–3(d) with respect to the three objective functions are collected.

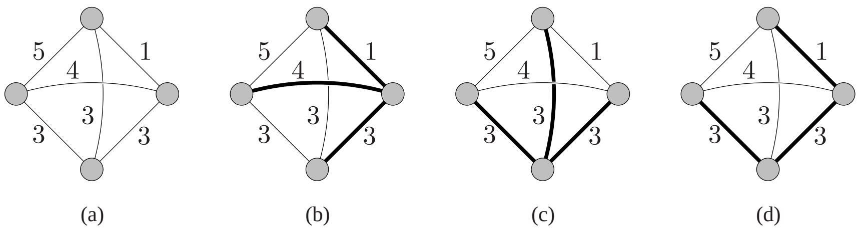


Figure 3: A graph with metric weights and its different optima with respect to the objective functions $(\max,\ E \setminus T,\ \texttt{goal})$, where $\texttt{goal} \in \{d_T(u,v) + w(e), d_T(u,v), \frac{d_T(u,v)}{w(e)}\}$

Table 1: The values with respect to the different objective functions for the spanning trees in Figures 3(b) to 3(d)

Tree	$d_T(u,v) + w(e)$	$d_T(u,v)$	$\frac{d_T(u,v)}{w(e)}$
Figure 3(b)	**10**	7	$\frac{7}{3}$
Figure 3(c)	11	**6**	6
Figure 3(d)	12	7	$\boldsymbol{\frac{3}{2}}$

There are precisely seven circuits in G. One can easily check that the values 10 and 6 are the best values with respect to the objective functions $d_T(u,v) + w(e)$ and $d_T(u,v)$, respectively, even when considering arbitrary sets of three circuits. Finally, performing a simple inspection of the few relevant cases one can further check that no other spanning tree achieves better values with respect to the three objective functions. $\square$

General Weights. As there are no coincidences between maximum stretch tree spanner problems which do only apply to metric weights but not to general weights, this paragraph has to remain void.

4 Average Stretch Problems

Our tour through the average stretch tree spanner problems follows the trace of our expedition through the maximum stretch tree spanner problems. But we will find many more different problems in the average stretch case.

4.1 Coincidences

Comparing the maximum stretch case to the average stretch case on general weights, metric weights, or unit weights, the number of different problems is by up to four larger for average stretch tree spanners.

General Weights. There are only two pairs of average stretch tree spanner problems that coincide for general weights.

Proposition 11. It holds that the two average stretch problems $(\sum,\ E,\ d_T(u,v)+w(e))$ and $(\sum,\ E,\ d_T(u,v))$ coincide.

Proof. For every spanning tree T, the objective values of these two problems differ precisely by $\sum_{e\in E} w(e)$, being independent of T. $\square$

Proposition 12. The two average stretch problems $\left(\sum,\ E\setminus T,\ \frac{d_T(u,v)}{w(e)}\right)$ and $\left(\sum,\ E,\ \frac{d_T(u,v)}{w(e)}\right)$ coincide.

Proof. For every spanning tree T, the objective values of these two problems differ precisely by $\sum_{e\in T} \frac{d_T(u,v)}{w(e)}$. As for every edge $e=\{u,v\}\in T$ the unique path in T between its endpoints is just the edge e, there holds $d_T(u,v)=w(e)$. Thus, $\sum_{e\in T}\frac{d_T(u,v)}{w(e)}=n-1$, which again is independent of T. $\square$

Metric Weights. Much similar to the case of maximum stretch tree spanners, for metric weight functions four problems whose objective functions involve fractions coincide.

Proposition 13. Let (G,w) be an undirected graph with a metric weight function w on the edges. Let *`domain`* be either $E\setminus T$ or E, and let *`term`* be one of $\frac{d_T(u,v)}{w(e)}$ and $\frac{d_T(u,v)}{d_G(u,v)}$. Then, the four problems ($\sum$, *`domain`*, *`term`*) coincide.

Proof. In the case of a metric weight function w on the edges, for every edge $e=\{u,v\}\in E$ there holds $d_G(u,v)=w(e)$. Hence, for each of the two domains that we consider here, the two problems ($\sum$, `domain`, $*$) coincide.

Moreover, for every tree edge $e=\{u,v\}\in T$ there holds $\frac{d_T(u,v)}{w(e)}=1$. Thus, for every spanning tree its objective value with respect to the domain E is precisely $n-1$ greater than the objective value with respect to the domain $E\setminus T$. $\square$

Unit Weights. With the exception of the average stretch tree spanner problems that are defined for $V\otimes V$, all other average stretch tree spanner problems coincide on unweighted graphs. Similarly to (8)–(10) we find,

$$\sum_{e=\{u,v\}\in E} d_T(u,v)+w(e) = \left(\sum_{e=\{u,v\}\in E\setminus T} d_T(u,v)+w(e)\right)+2(n-1) \quad (11)$$

$$= \left(\sum_{e=\{u,v\}\in E} d_T(u,v)\right)+m \quad (12)$$

$$= \left(\sum_{e=\{u,v\}\in E\setminus T} d_T(u,v)\right)+m+n-1. \quad (13)$$

Again, we profit from the fact that for every edge $e=\{u,v\}$ there holds $d_T(u,v)=\frac{d_T(u,v)}{w(e)}=\frac{d_T(u,v)}{d_G(u,v)}$. Together with (11)–(13) we conclude

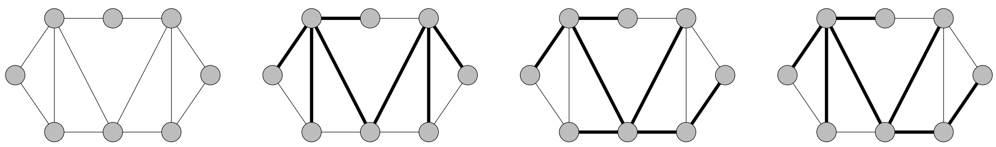
Proposition 14. Let G be an unweighted graph. Then the following eight unweighted tree spanner problems coincide: $(\sum, E \setminus T, *)$ and $(\sum, E, *)$.

4.2 Anticoincidences

In the case of average stretch tree spanner problems, it will turn out that even for the unweighted case, both problems with `domain` $V \otimes V$ do not coincide with any other problem.

Unit Weights. In the case of the most special weights, the following Example 15 shows that we remain with 3 problems.

Example 15 (MSFCB vs. STPLST vs. $\left(\sum,\ V \otimes V,\ \frac{d_T(u,v)}{d_G(u,v)}\right)$**).** Consider the unweighted planar graph G in Figure 4. Observe that the graph from Figure 2 can be obtained from G simply by contracting one single edge. Again, the unique minimum cycle basis of G consists of the five circuits which are the boundary of the finite faces of G. Hence, the optimum solution value of the MSFCB problem on G is 16 and it can be obtained by the fundamental circuits that are induced by eight spanning trees, one of which we display in Figure 4(b). These eight spanning trees all contain the four edges of G which are not incident with the infinite face of G, and yield objective values of at least 66 and $\frac{230}{6}$ for the STPLST problem and for $\left(\sum,\ V \otimes V,\ \frac{d_T(u,v)}{d_G(u,v)}\right)$, respectively.



(a) (b) (c) (d)

Figure 4: An unweighted graph and example trees which show that none of MSFCB, STPLST, and $\left(\sum,\ V \otimes V,\ \frac{d_T(u,v)}{d_G(u,v)}\right)$ coincide

In contrast, the spanning tree in Figure 4(c) is one of the four optimum solutions for the STPLST problem. Their objective value is 62. On the contrary, for the MSFCB problem and for $\left(\sum,\ V \otimes V,\ \frac{d_T(u,v)}{d_G(u,v)}\right)$ they only achieve objective values of 18 and $\frac{228}{6}$, respectively.

Finally, the four spanning trees which are optimum with respect to $\left(\sum,\ V \otimes V,\ \frac{d_T(u,v)}{d_G(u,v)}\right)$, see Figure 4(d) for an example, achieve an objective function value of $\frac{227}{6}$. But for the objective functions of MSFCB and STPLST these trees are suboptimal because of objective values of only 17 and 63, respectively. □

Metric Weights. To discover more anticoincidences we take a look at graphs with a metric weight function.

Example 16 (MSFCB vs. $(\sum, E, d_T(u,v))$ vs. $(\sum, E \setminus T, d_T(u,v))$**).** We investigate the graph G with a metric weight function w that is displayed in Figure 5(a). There are precisely two circuits in (G, w) which have weight 18, and another two circuits which have weight 19. There are indeed four spanning trees which achieve an objective function value of $18 + 18 + 19 = 55$ with respect to MSFCB (see e.g. Figure 5(b)). But since all of them include two edges of weight seven, they only achieve objective values of 62 and 40 with respect to $(\sum, E, d_T(u,v))$ and $(\sum, E \setminus T, d_T(u,v))$, respectively.

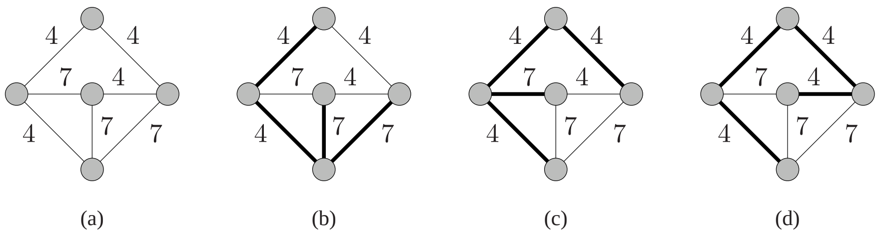


Figure 5: A graph with metric weights and example trees which show that none of MSFCB, $(\sum, E \setminus T, d_T(u,v))$, and $(\sum, E, d_T(u,v))$ coincide

Table 2: The objective values to the four considered problems for the trees of Figures 6(b) and 6(c).

Tree	$(\sum,\ E \setminus T,\ \frac{d_T(u,v)}{w(e)})$	$(\sum,\ E \setminus T,\ d_T(u,v)+w(e))$	$(\sum,\ E \setminus T,\ d_T(u,v))$	$(\sum,\ E,\ d_T(u,v))$
Figure 6(b)	$4+\frac{4}{7} \approx 4.57$	$\mathbf{46}$	$\mathbf{32}$	$\mathbf{55}$
Figure 6(c)	$4+\frac{5}{9} \approx \mathbf{4.55}$	51	35	56

In contrast to the optima with respect to MSFCB, there exist two spanning trees which only contain one single edge of weight seven each, but admit the second smallest set of fundamental circuits: $18+19+19=56$. Hence, these are precisely the trees which admit an objective function value of 38, being optimum with respect to $(\sum, E \setminus T, d_T(u,v))$. One of them is depicted in Figure 5(c). Their objective value with respect to $(\sum, E, d_T(u,v))$ is 57.

Since we identified all the optima with respect to MSFCB and $(\sum, E \setminus T, d_T(u,v))$, it suffices to provide some spanning tree T that attains a smaller objective function value with respect to $(\sum, E, d_T(u,v))$ than the former trees did. Indeed, the spanning tree T that we display in Figure 5(d) yields an objective function value of only 56. One can easily observe that T is the unique minimum spanning tree of (G,w). Actually, it is even the unique optimum solution with respect to $(\sum, E, d_T(u,v))$. $\square$

Example 17 **(**$\left(\sum,\ E \setminus T,\ \frac{d_T(u,v)}{w(e)}\right)$ **vs.** $\{(\sum,\ E \setminus T,\ d_T(u,v)+w(e)), (\sum,\ E \setminus T,\ d_T(u,v))$ **,** $(\sum,\ E,\ d_T(u,v))\}$**).** Consider the graph G of Figure 6(a). Because of the very regularly structured weights we need only to consider two families of spanning trees: those that include the edge of weight 9, and those which do not. Within both families then all trees constitute indistinguishable solutions for all the considered problems. Representatives for the families are depicted in Figures 6(b) and 6(c), respectively. The following table now proves the desired claim: whereas for the fractional problem, $\left(\sum,\ E \setminus T,\ \frac{d_T(u,v)}{w(e)}\right)$ it does not pay off to include the "expensive" edge of weight 9, it does for the other three problems. It rather turns out to be good to include this particular edge such that it can be used as a shortcut when considering $d_T(u,v)$ for $(u,v)=e \in E \setminus T$.

$\square$

General Weights. At last we need to consider graphs with non-metric weight functions to prove the remaining anticoincidences.

Example 18 **(**$\left(\sum,\ E \setminus T,\ \frac{d_T(u,v)}{w(e)}\right)$ **vs.** $\{\left(\sum,\ E \setminus T,\ \frac{d_T(u,v)}{d_G(u,v)}\right), \left(\sum,\ E,\ \frac{d_T(u,v)}{d_G(u,v)}\right)\}$**).** Consider the graph with non-metric weights in Figure 7. A first observation is that the edge with weight 8 is not metric. More important, the edge with weight 1 is included in every optimal tree for all

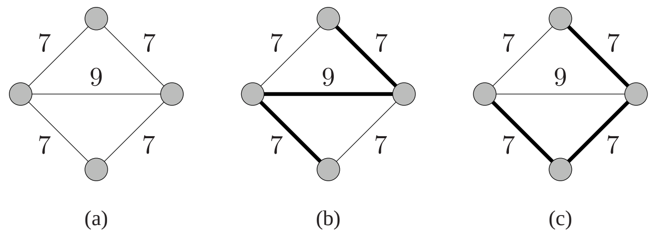


Figure 6: A graph with metric weights and example trees which show that none of MSFCB, $(\sum, E \setminus T, d_T(u,v))$, and $(\sum, E, d_T(u,v))$ coincide with $\left(\sum, E \setminus T, \frac{d_T(u,v)}{w(e)}\right)$

the three problems. Otherwise we immediately have a contribution of 14—which is the length of a shortest circuit through this edge—whereas *any* other tree induces shorter circuits w.r.t. both $\frac{d_T(u,v)}{d_G(u,v)}$ and $\frac{d_T(u,v)}{w(e)}$ even when considering the sum over all edges.

So, we remain with 5 different trees. Among these, due to symmetry reasons it suffices to consider only three trees, cf. 7(b)-7(d).

The following table provides the values for the three trees with respect to the different problems showing that the tree in 7(b) is the unique optimal solution to $\left(\sum, E \setminus T, \frac{d_T(u,v)}{w(e)}\right)$ whereas the tree indicated in Figure 7(c) is optimum for the other two problems, $\left(\sum, E \setminus T, \frac{d_T(u,v)}{d_G(u,v)}\right)$ and $\left(\sum, E, \frac{d_T(u,v)}{d_G(u,v)}\right)$.

Table 3: The objective values of the spanning trees in Figure 7 w.r.t. the three problems of Example 18. A term of $\frac{1}{840}$ is factored out for clarity reasons.

Tree	$\left(\sum, E \setminus T, \frac{d_T(u,v)}{w(e)}\right)$	$\left(\sum, E \setminus T, \frac{d_T(u,v)}{d_G(u,v)}\right)$	$\left(\sum, E, \frac{d_T(u,v)}{d_G(u,v)}\right)$
Figure 7(b)	$\mathbf{2555}$	2720	5240
Figure 7(c)	2583	$\mathbf{2688}$	$\mathbf{5208}$
Figure 7(d)	3612	3612	6252

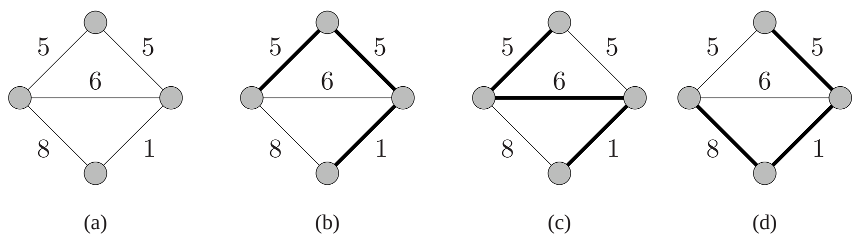


Figure 7: A weighted graph and example trees. Figures (b) and (c) show that $\left(\sum, E \setminus T, \frac{d_T(u,v)}{w(e)}\right)$ does neither coincide with $\left(\sum, E \setminus T, \frac{d_T(u,v)}{d_G(u,v)}\right)$ nor with $\left(\sum, E, \frac{d_T(u,v)}{d_G(u,v)}\right)$.

□

Example 19 ($\left(\sum,\ E \setminus T,\ \frac{d_T(u,v)}{d_G(u,v)}\right)$ vs. $\left(\sum,\ E,\ \frac{d_T(u,v)}{d_G(u,v)}\right)$)**.** We will show the anticoincidence of the two tree-spanner problems with the help of the weighted graph (G, w) in Figure 8(a). The dots in the figure shall indicate that we assume a sufficiently large number of *clips*, i.e. 4-circuits that share one common edge e or f, respectively. Let M denote this number. Further, we will refer to the edges with weight equal to 101 as *clip-edges*.

Recall from Proposition 13 that the two problems coincide in the case of metric weights. Hence, we chose the weight function w such that precisely one edge is not metric: the edge g. To show the anticoincidence we will argue as follows: in the beginning we show that each spanning tree T that is optimal for any of the two problems must have a certain structure. First, all edges having weight one are included in T, second, T does not contain any clip-edge, and third, the edges e and f are in T. See Figure 8(b), where we highlight edges that have to be in T. Edges that are not in T are depicted by dotted lines in this figure.

Observe that as we obtain this structure for parts of the graph where all edges are metric, the structural properties apply to the optimum solutions subject to both objective functions that we are investigating in this example. Thereafter, having this common structure, optimal trees to $\left(\sum,\ E \setminus T,\ \frac{d_T(u,v)}{d_G(u,v)}\right)$ and $\left(\sum,\ E,\ \frac{d_T(u,v)}{d_G(u,v)}\right)$ become distinguishable on the remaining part of the graph, where the only non-metric edge g is going to play a key role.

So, we start motivating the mentioned structure of an optimal tree T. We first show that $w(a) = 1$ implies $a \in T$. Notice first that for any of the $2M$ clips at least one edge of the clip with weight one has to be in T because otherwise the tree T would not be connected. Hence, assume that for a clip exactly one edge of weight one and its clip-edge are contained in T. In that case, however, we get immediately a contradiction to the optimality of T: A simple exchange of the clip-edge by the non-tree edge with weight one within this clip instantly effects a better tree. To see this, observe that for no other pair of vertices in G the corresponding path in T can traverse one of these two edges and compare the according values $\frac{d_T(x,y)}{d_G(x,y)}$.

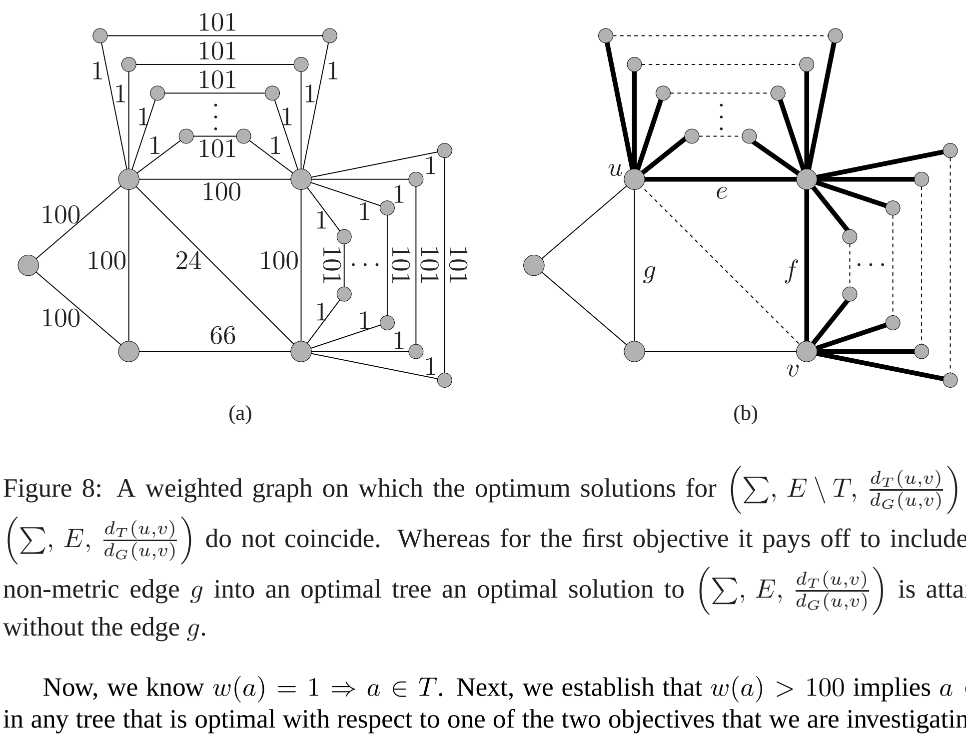


Figure 8: A weighted graph on which the optimum solutions for $\left(\sum,\ E \setminus T,\ \frac{d_T(u,v)}{d_G(u,v)}\right)$ and $\left(\sum,\ E,\ \frac{d_T(u,v)}{d_G(u,v)}\right)$ do not coincide. Whereas for the first objective it pays off to include the non-metric edge g into an optimal tree an optimal solution to $\left(\sum,\ E,\ \frac{d_T(u,v)}{d_G(u,v)}\right)$ is attained without the edge g.

Now, we know $w(a) = 1 \Rightarrow a \in T$. Next, we establish that $w(a) > 100$ implies $a \notin T$ in any tree that is optimal with respect to one of the two objectives that we are investigating in this example. To this end, consider one bundle of clips, say the one that contains the edge e, and assume an optimal tree T contains a clip-edge. Since we already know that the edges having weight one are in T, the tree can contain at most one clip-edge, because otherwise the tree T would include a cycle. Similarly, we conclude that $e \notin T$. Again, such a structure contradicts the optimality of T, because another local change on T improves the tree: This time we exchange the clip-edge c that we assume to be contained in T with the edge e and obtain a different tree $T' = T \cup \{e\} \setminus \{c\}$. This exchange shortens the length of the path in T between the vertices incident to e, hereby shortening the distances of all paths within T that contain these two vertices. In addition, even when comparing the value $\frac{d_T(x,y)}{d_G(x,y)}$ of the non-tree edge e w.r.t. T with the corresponding value of the non-tree edge c w.r.t. T' an improvement is obtained. Notice that here we only define the particular tree T' to contain the edge e. But so far nothing is said whether e is contained in any optimum tree.

The last structural property, that we are about to develop for an optimal tree T with respect to any of the two objective functions, is $\{e, f\} \subset T$. We already know that T contains all edges of weight one and no clip-edge. Hence, at least one of the edges e and f is in T, because otherwise the tree T was disconnected. So we assume for contradiction without loss of generality that $e \in T$ but $f \notin T$. Then, consider the unique path P between the vertices u and v in T where obviously $f \notin T$ implies $e \notin P$. Now, an exchange of any of the edges of P by the edge f will lead to a contradiction to the choice of T, which was an optimal tree. One can see this as follows: before the exchange, each f-clip-edge induces a path in T of length d_T at least 126. Therefore, in both objectives each f-clip-edges contributes at least $M \cdot \frac{126}{101}$. After the exchange—i.e. now with $f \in T$—this amount decreases to $M \cdot \frac{102}{101}$. It is clear that we may choose the parameter M so large that this gain compensates the possibly appearing increases of contributions of the remaining part of the graph that is independent of M.

This way we force $\{e, f\} \subset T$, and in a sense decouple the clip-edges from the remainder of the tree.

At this point we developed all of the structural properties of an optimal tree T. Let us emphasize that the properties hold for optima of *both* objectives, because up to this point we only argued for parts of the graph on which the two objective values differ by a constant term, because the edges within the clips are all metric.

For the remainder of the graph we discuss the effect of adding two more edges to our tree T. Observe that there are exactly two spanning trees that contain the edge g and three which do not.

We start by computing the objective value K that the three non-tree edges that are distinct from clip-edges contribute. If $g \in T$, then $K = \frac{491}{33} \in [14.87, 14.88]$. Otherwise, if $g \notin T$, there are two trees for which $K = \frac{6727}{450} \in [14.94, 14.95]$, and one for which $K > 16$. Hence, for the domain $E \setminus T$, the two trees that contain the expensive non-metric edge g turn out to be optimal. In contrast, for the domain E it does not pay off to include the non-metric edge g into the tree: it costs $\frac{10}{9}$ instead of only one for any other tree edge, which is in particular metric, and this extra cost of more than 0.1 gets not compensated by a reduction of less than 0.08 in K. Hence, on (G, w) the average stretch tree spanner problem $\left(\sum, E, \frac{d_T(u,v)}{d_G(u,v)}\right)$ is optimized by two of the three trees with $g \notin T$. $\square$

5 MAX-STRETCH And AVERAGE-STRETCH Problems Never Coincide

In this section we aim at detecting anticoincidences between max-stretch and average-stretch problems. Therefore we consider unweighted versions of the problems.

Example 20 (MMST vs. MSFCB)**.** Consider the unweighted simple 2-vertex connected undirected planar graph G in Figure 9. We will argue that an optimum solution for the MSFCB problem contains the edge e, whereas an optimum solution for the MMST does not.

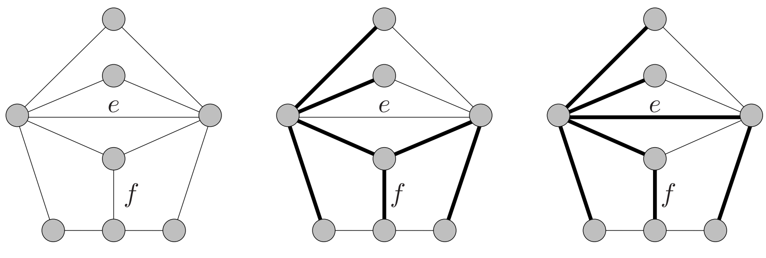


Figure 9: An unweighted graph with representatives of optimal solutions to MMST and MSFCB

Consider the MMST problem. We construct a spanning tree T all of whose fundamental circuits have at most four edges, cf. Figure 9(b). First, observe that there has to hold $f \in T$, because the only 4-circuits through the south-east most and through the south-west most vertices share the edge f. But then, in order to prevent a 5-circuit, the two edges that are incident with f must be contained in T, too. In turn, $e \notin T$. The five fundamental circuits of such a spanning tree T thus have lengths $(3, 4, 4, 4, 4)$.

In contrast, every optimum spanning tree for the MSFCB problem induces fundamental circuits of lengths $(3, 3, 3, 4, 5)$, see Figure 9(c) for an example. But this can only be achieved by including the edge e in the spanning tree, because the only three triangles in G all share this edge. $\square$

Example 21 (MDST vs. $\{\text{STPLST}, (\sum,\ V \otimes V,\ \frac{d_T(u,v)}{d_G(u,v)})\}$)**.** Consider the unweighted undirected graph G in Figure 10. We will argue that the set of spanning trees which are optimum for MDST is disjoint from the set of spanning trees optimal for STPLST or $\left(\sum,\ V \otimes V,\ \frac{d_T(u,v)}{d_G(u,v)}\right)$.

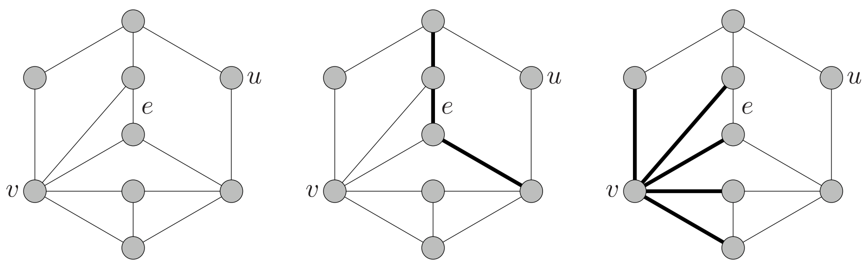


Figure 10: An unweighted graph and parts of example trees which show that MDST does neither coincide with STPLST nor with $\left(\sum,\ V \otimes V,\ \frac{d_T(u,v)}{d_G(u,v)}\right)$

We start by detecting a necessary condition for a spanning tree to be optimum for MDST. To that end, first observe that an optimum spanning tree T with respect to MDST achieves a diameter of four. Consider the vertex u. The unique shortest circuit C through u has five edges, whereas the second-shortest circuit through u has six edges. Hence, in a minimum diameter spanning tree T in G, the circuit C is the only fundamental circuit with respect to T

that contains the vertex u. But since $G \setminus \{u\}$ is 2-vertex connected, this implies the three bold edges in Figure 10(b) to be contained in T, in particular $e \in T$.

In contrast, one can check that each of the twelve spanning trees which are optimum for STPLST (having objective value 86) is also optimum for $\left(\sum,\ V \otimes V,\ \frac{d_T(u,v)}{d_G(u,v)}\right)$ (having objective value 54), and vice-versa. Moreover, in each of these trees there holds $\delta(\{v\}) \subset T$, where $\delta(\{v\})$ is the cut induced by v. In particular, $e \notin T$. □

Finally, the next example covers the remaining anticoincidences.

Example 22 ($\{$MMST, MSFCB$\}$ vs. $\{$MDST, STPLST, $\left(\sum,\ V \otimes V,\ \frac{d_T(u,v)}{d_G(u,v)}\right)\}$)**.** A key difference between MMST and MSFCB on the one side, and MDST, STPLST, and $\left(\sum,\ V \otimes V,\ \frac{d_T(u,v)}{d_G(u,v)}\right)$ on the other side, is that the former problems may be regarded as to have only the set of non-tree edges $E \setminus T$ as `domain`, whereas the latter have $V \otimes V$ as `domain`. But only the latter ensures a kind of global perspective for every spanning tree. With $E \setminus T$ as `domain`, an accordion-like tree as the one that we already displayed in Figure 2(b) admits a much more local way of counting.

Consider again the unweighted graph in Figure 2. By the very same arguments which showed that $e \in T$ for every spanning tree T which is optimum for MMST, this edge is also contained in every spanning tree which is optimum for MSFCB. More precisely, the four spanning trees which are optimum for MMST are precisely the optimum solutions for MSFCB—these four spanning trees only differ in how the left- and the rightmost vertex is connected to the tree.

Similarly, the two spanning trees of minimum diameter are precisely the optimum solutions for STPLST (having objective value 42) and even for $\left(\sum,\ V \otimes V,\ \frac{d_T(u,v)}{d_G(u,v)}\right)$ (having objective value 28). These two trees only differ in which endpoint of the edge e becomes the vertex of degree four in T. □

To summarize this section, there is not one single bridge between MAX-STRETCH and AVERAGE STRETCH tree spanner problems. In contrast, there exists a related pair of problems for which such a bridge between the maximum objective and the sum objective was established. In [6] it has been shown that any solution to the general MINIMUM CYCLE BASIS (MCB) problem is also a solution to the problem of computing a cycle basis whose longest circuit is minimum. Here, it is well-known that the MCB problem can be solved in polynomial time ([18]), more precisely in $O(m^2 n + mn^2 \log n)$ ([21]).

6 First Benefit of the UNTS

Recall that whenever a spanner problem (`goal`, `domain`, `term`) is NP-hard in the unweighted case, it is in particular NP-hard in its weighted versions, too. We are aware of three such negative complexity results for unweighted tree spanner problems.

Theorem 23 ([20])**.** The STPLST problem is NP-hard.

Theorem 24 ([8])**.** The MSFCB problem is NP-hard.

Theorem 25 ([5])**.** The MMST problem is NP-hard.

There have even been identified special classes of graphs on which these problems are still NP-hard. For instance, think of the MMST problem on planar graphs ([11]), on chordal graphs ([3]), and on chordal bipartite graphs ([4]).

But there is also one positive complexity result that has been obtained for a tree spanner problem.

Theorem 26 ([17])**.** The weighted MDST problem can be solved in $O(mn + n^2 \log n)$ time.

Now, the UNTS provides us with a clear perspective on 20 tree spanner problems. In particular, Examples 15, 22, and 21 establish that none of the above complexity results apply to the problem $\left(\sum,\, V \otimes V,\, \frac{d_T(u,v)}{d_G(u,v)}\right)$. Fortunately, we are able to settle its complexity status in Section 6.1 by establishing NP-hardness.

Moreover, by Proposition 8 we know that in the case of unweighted graphs the classical maximum stretch tree spanner problem $\left(\max,\, V \otimes V,\, \frac{d_T(u,v)}{d_G(u,v)}\right)$ coincides with $(\max,\, E \setminus T,\, d_T(u,v) + w(e))$. In Section 6.2 we compare two inapproximability results that have been obtained for these problems. Interestingly enough, the result which had never been stated before in the language of tree spanners turns out to be stronger.

6.1 NP-hardness of $\left(\sum,\, V \otimes V,\, \frac{d_T(u,v)}{d_G(u,v)}\right)$

Theorem 27. The average stretch tree spanner problem $\left(\sum,\, V \otimes V,\, \frac{d_T(u,v)}{d_G(u,v)}\right)$ is NP-hard.

Proof. As our proof is similar to the proof for Theorem 23 as it has been given by Johnson et al. ([20]), we adopt the notation of [20]. We consider the following problem, which is known to be NP-hard (Problem SP2 in [15]):

> EXACT COVER BY 3-SETS (X3C): Given a family $S = (\sigma_1, \ldots, \sigma_s)$ of 3-element subsets of a set $T = (\tau_1, \ldots, \tau_{3t})$. Does there exist a subfamily $S' \subseteq S$ of sets with pairwise empty intersection, such that $\dot{\bigcup}_{\sigma \in S'} \sigma = T$?

Given an instance $\mathcal{I}$ of X3C, we define an unweighted simple undirected graph $G = (V, E)$ as follows, see Figure 11 for an example:

- $R = \{\rho_0, \rho_1, \ldots, \rho_r\}$, where $r := |T|^2 + 2 \cdot |S| \cdot |T| + 1$, $R_0 = \{\rho_0\}$, $R^* = R \setminus R_0$,
- $V = R \cup S \cup T$,
- $E = \{\{\rho_i, \rho_0\} : i = 1, \ldots, r\} \cup \{\{\rho_0, \sigma\} : \sigma \in S\} \cup \{\{\sigma, \tau\} : \tau \in \sigma \in S\}$,

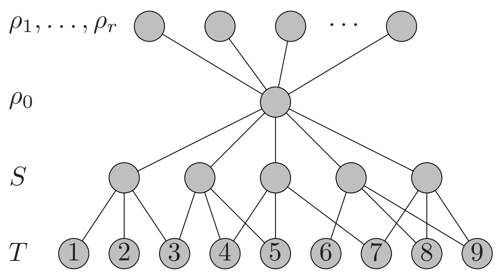


Figure 11: The instance of $\left(\sum,\, V \otimes V,\, \frac{d_T(u,v)}{d_G(u,v)}\right)$ that we associate with the instance $\{(1,2,3), (3,4,5), (4,5,7), (6,8,9), (7,8,9)\}$ of X3C

Denote by X the number of pairs of elements of T that are *not* contained together in any of the sets of S, i.e.

$$X := |\{(\tau_1, \tau_2) \in V \otimes V : \forall \sigma \in S : \tau_1 \notin \sigma \text{ or } \tau_2 \notin \sigma\}|. \tag{14}$$

We will prove that $\left(\sum, V \otimes V, \frac{d_T(u,v)}{d_G(u,v)}\right)$ has a solution of value at most

$$|V|^2 - |V| \quad + \quad |T|^2 + 6 \cdot |S| - 5 \cdot |T| - X, \tag{15}$$

if and only if the answer to the instance $\mathcal{I}$ is YES.

We denote a spanning tree of G that contains the edge $\{\rho_0, \sigma\}$ for all $\sigma \in S$ a *star tree*.

Claim. Every optimum solution of $\left(\sum, V \otimes V, \frac{d_T(u,v)}{d_G(u,v)}\right)$ is a star tree.

Claim. In Table 4, we investigate the distances between any pair of nodes in G, in an arbitrary star tree F^*, and in an arbitrary non-star tree F.

In the case of a non-star tree F, there exists one vertex $s' \in S$ such that $d_F(\rho_i, s') \geq 4$ for all $i = 1, \ldots, r$. In particular, $\frac{d_F(\rho_i,s')}{d_G(\rho_i,s')} \geq 2$, whereas $\frac{d_{F^*}(\rho_i,s')}{d_G(\rho_i,s')} = 1$, cf. the $(*)$-entry in Table 4. Hence, when considering $R^* \otimes S$, by the choice of r the objective value of F is by at least $|T|^2 + 2 \cdot |S| \cdot |T| + 1$ larger than the one of F^*.

Table 4: Distances between pairs (u, v) of nodes within the graph. The first column denotes the cardinality of the considered subset of $V \otimes V$.

set of u	set of v	number of node pairs	$d_G(u,v)$	$d_{F^*}(u,v)$	$d_F(u,v)$
R_0	R^*	$\lvert T\rvert^2 + 2\lvert S\rvert \cdot \lvert T\rvert + 1$	1	1	1
R_0	S	$\lvert S\rvert$	1	1	≥ 1
R_0	T	$\lvert T\rvert$	2	2	≥ 2
R^*	R^*	$(\lvert T\rvert^2 + 2\lvert S\rvert \cdot \lvert T\rvert + 1)\cdot$ $(\lvert T\rvert^2 + 2\lvert S\rvert \cdot \lvert T\rvert)$	2	2	2
R^*	S	$(\lvert T\rvert^2 + 2\lvert S\rvert \cdot \lvert T\rvert + 1) \cdot \lvert S\rvert$	2	2	$(*)$
R^*	T	$(\lvert T\rvert^2 + 2\lvert S\rvert \cdot \lvert T\rvert + 1) \cdot \lvert T\rvert$	3	3	≥ 3
S	S	$\lvert S\rvert \cdot (\lvert S\rvert - 1)$	2	2	≥ 2
S	T	$\lvert S\rvert \cdot \lvert T\rvert$	$\{1, 3\}$	≤ 3	≥ 1
T	T	$\lvert T\rvert \cdot (\lvert T\rvert - 1) - X$	2	≤ 4	≥ 2
T	T	X	4	4	≥ 4

According to Table 4 this can only be compensated on $S \otimes T$ and on $T \otimes T$. But there, for $(u, v) \in S \otimes T$ there holds

$$\frac{d_{F^*}(u,v)}{d_G(u,v)} \leq \frac{d_F(u,v)}{d_G(u,v)} + 2,$$

and for $(u, v) \in T \otimes T$ there holds

$$\frac{d_{F^*}(u,v)}{d_G(u,v)} \leq \frac{d_F(u,v)}{d_G(u,v)} + 1.$$

Hence, any non-star tree F can only gain $|T|^2 + 2 \cdot |S| \cdot |T|$ on $S \otimes T$ and $T \otimes T$, which is strictly smaller than its loss on $R^* \otimes S$. $\square$

Now that we know that an optimum solution to $\left(\sum, V \otimes V, \frac{d_T(u,v)}{d_G(u,v)}\right)$ is always a star tree, we will compute the objective value of an arbitrary star tree F^*. According to Table 4, it remains to investigate in detail pairs of vertices from the sets $S \otimes T$ and $T \otimes T$. We first examine the set $S \otimes T$ and compute for an arbitrary star tree F^*

$$\begin{aligned}
\sum_{(u,v)\in S\otimes T} \frac{d_{F^*}(u,v)}{d_G(u,v)} &= \sum_{\substack{(u,v)\in S\otimes T\\ d_{F^*}(u,v)=1}} \frac{d_{F^*}(u,v)}{d_G(u,v)} + \sum_{\substack{(u,v)\in S\otimes T\\ d_G(u,v)=3}} \frac{d_{F^*}(u,v)}{d_G(u,v)} + \\
&\qquad \sum_{\substack{(u,v)\in S\otimes T\\ d_G(u,v)=1,\, d_{F^*}(u,v)=3}} \frac{d_{F^*}(u,v)}{d_G(u,v)} \\
&= |T| + |S|\cdot(|T|-3) + \sum_{\substack{(u,v)\in S\otimes T\\ d_G(u,v)=1,\, d_{F^*}(u,v)=3}} \frac{d_{F^*}(u,v)}{d_G(u,v)} \\
&= |T| + |S|\cdot(|T|-3) + 3\cdot(3|S|-|T|)) \\
&= |S|\cdot|T| - 2\cdot|T| + 6\cdot|S|.
\end{aligned}$$

As this value is independent of F^*, we conclude that any two star trees differ in their objective value only for pairs $(u,v) \in T \otimes T$.

Recall the definition of X in (14). Among the pairs $(u,v) \in T \otimes T$, there are precisely X for which $d_G(u,v) = 4$—and thus $d_{F^*}(u,v) = 4$—and precisely $|T|^2 - |T| - X$ for which $d_G(u,v) = 2$. As F^* is a star tree, we know that $d_{F^*}(u,v) \in \{2,4\}$ for every $(u,v) \in T \otimes T$. Recall that the quantity X only depends on the instance $\mathcal{I}$ of X3C. Hence, a spanning tree F^* is optimum for $\left(\sum, V\otimes V, \frac{d_T(u,v)}{d_G(u,v)}\right)$, if and only if it is a star tree that maximizes the number Y of pairs $(u,v) \in T \otimes T$ for which $d_{F^*}(u,v) = 2$. How large can Y get?

We prefer to account for the quantity Y from an alternative perspective. To that end, consider the edges $F^*_{ST} := F^* \cap (S \times T)$. Note that $|F^*_{ST}| = |T|$, because F^* is a star tree. Now, we define a function $p(e)$ for the edges $e = (\sigma,\tau) \in F^*_{ST}$,

$$p(e) = \begin{cases} 2, & \text{if } |\delta_{F^*}(\sigma)| = 4, \\ 1, & \text{if } |\delta_{F^*}(\sigma)| = 3, \text{ and} \\ 0, & \text{otherwise.} \end{cases}$$

Hereby, $Y = \sum_{e\in F^*_{ST}} p(e)$. Then the following statements are equivalent:

- $Y = 2|T|$.
- For all $e \in F^*_{ST}$, $p(e) = 2$.
- For all $\sigma \in S$, $|\delta_{F^*}(\sigma)| \in \{1,4\}$.

Finally, we provide a bijection between star trees F^* of G with $|\delta_{F^*}(\sigma)| \in \{1,4\}$, for all $\sigma \in S$, and EXACT 3-COVERS S' as follows:

$$S'(F^*) := \{\sigma \in S : |\delta_{F^*}(\sigma)| = 4\} \quad \text{and} \quad F^*_{ST}(S') := S' \times T.$$

A direct computation reveals that the total objective value of the optimum solution for a graph corresponding to a YES-instances of X3C is right as given in Equation (15). □

6.2 Inapproximability of the MMST problem

Peleg and Reshef (1999, [23]) prove that the MMST problem

> "cannot be approximated within a factor better than $(1+\sqrt{5})/2$, unless $\mathcal{P} = \mathcal{NP}$."

Even recently, this result is usually cited when illustrating the complexity of the MMST problem ([9]).

In Section 3, using the UNTS to classify the large variety of similar tree spanner problems, we were able to establish that in the case of unweighted graphs the following four problems coincide:

- the MMST problem $\left(\max,\ V \otimes V,\ \frac{d_T(u,v)}{d_G(u,v)}\right)$,
- the problem $\left(\max,\ E \setminus T,\ \frac{d_T(u,v)}{d_G(u,v)}\right)$,
- the problem $(\max,\ E \setminus T,\ d_T(u,v))$, and
- the problem $(\max,\ E \setminus T,\ d_T(u,v) + w(e))$.

It is a simple observation that in the case of unweighted graphs, for every fixed tree the objective values of the first three tree spanner problems coincide and differ by one from the fourth one. In particular, any constant inapproximability factor that is obtained for one of these problems carries over to the other problems.

Now, Galbiati (2001, 2003 [12, 13]) investigated the problem $(\max,\ E \setminus T,\ d_T(u,v) + w(e))$, which she denotes the MIN-MAX STRICTLY FUNDAMENTAL CYCLE BASIS (MMSFCB) problem. This culminates in the following theorem for unweighted graphs.

Theorem 28 ([12, 13])**.** The problem of finding in a uniform graph G a spanning tree that is optimal for $(\max,\ E \setminus T,\ d_T(u,v) + w(e))$ cannot be approximated within $2 - \epsilon$, $\forall \epsilon > 0$, unless $P = NP$.

The above considerations enable us to identify the constant inapproximability factor of Theorem 28 as a stronger inapproximability factor for the MINIMUM MAX-STRETCH SPANNING TREE (MMST) problem, or $\left(\max,\ V \otimes V,\ \frac{d_T(u,v)}{d_G(u,v)}\right)$ in UNTS.

Corollary 29. The MINIMUM MAX-STRETCH SPANNING TREE problem cannot be approximated within a factor better than $2 - \varepsilon$, $\forall \epsilon > 0$, unless $P = NP$.

Note that another factor-$2 - \varepsilon$ inapproximability result has been obtained for a problem which appears much similar to $(\max,\ E,\ d_T(u,v))$. Although one could be tempted to use it immediately for the MMST/MMSFCB problem, there is still a slight difference. In [16], there are two different sets of edges involved: one are the candidate edges for choosing the spanning tree, the others indicate for which sets of pairs (u,v) of vertices the term $d_T(u,v)$ shall be evaluated for the objective function—a requirement graph. On the one hand, the proof in [16] exploits this fact. On the other hand, such additional structures are beyond the scope of the tree spanner problems that we consider in this paper.

7 Conclusions

We presented a unified notation for tree spanner (UNTS) problems. This allowed us to detect that several tree spanner problems coincide. This is complemented by a number of example graphs showing that no further coincidences exist. We even identified a tree spanner problem, whose complexity status has been open before: $\left(\sum,\ V \otimes V,\ \frac{d_T(u,v)}{d_G(u,v)}\right)$. For this problem, we present an NP-hardness proof.

Moreover, the UNTS enabled us to build the bridge between the cycle bases perspective and the tree spanner perspective on the very same problems. In particular, we establish that the inapproximability result due to Galbiati ([12, 13])—initially obtained for the MIN-MAX STRICTLY FUNDAMENTAL CYCLE BASIS (MMSFCB) problem—applies to the MINIMUM MAX-STRETCH SPANNING TREE (MMST) problem, too, and outperforms the best inapproximability result that was known in this context: $2 - \varepsilon$ compared to $\frac{1+\sqrt{5}}{2} \approx 1.618$.

Yet, it is by far beyond the scope of this paper to draw the complete map of (in-) approximability results for tree spanner problems—occasionally even for several classes of graphs. Nevertheless, when exploring this wide area of discrete mathematics, we hope the UNTS to provide an accurate common language in order to prevent double work.

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