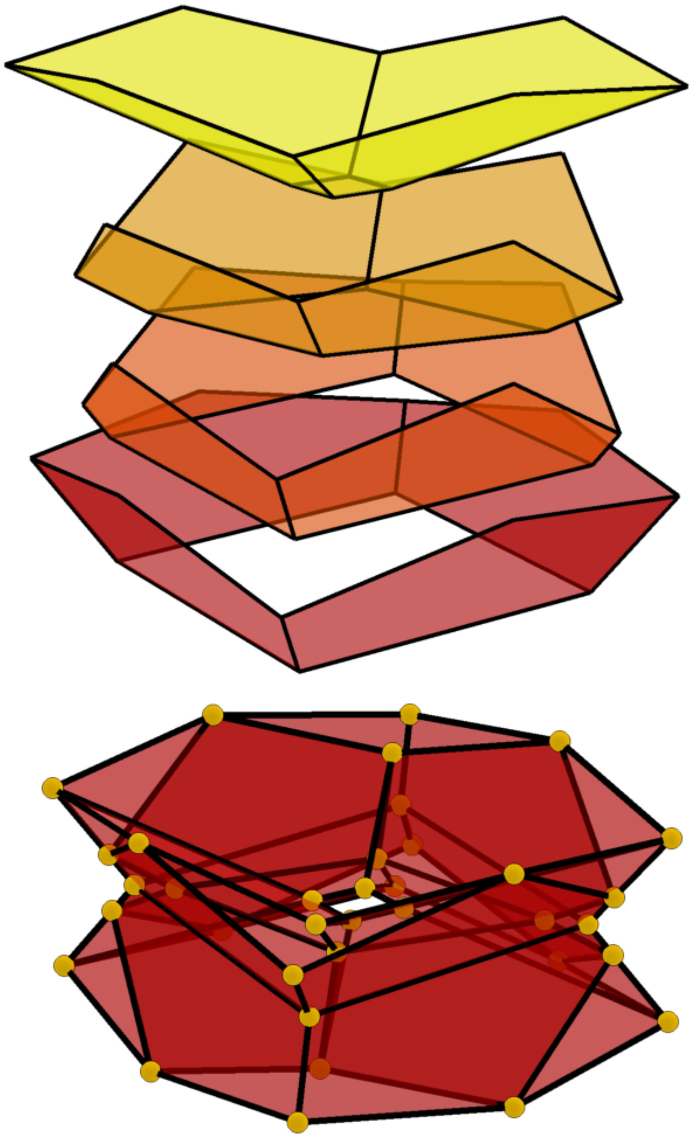
Polyhedral Surfaces, Polytopes, and Projections



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Polyhedral Surfaces, Polytopes, and Projections

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Contents

Introduction

The construction of "interesting" combinatorial and geometric polyhedral surfaces and polytopes is the main subject of this thesis. On the way we develop some new methods for the study of projection problems of polytopes and for the analysis of the realization spaces of polytopes as well as polyhedral surfaces obtained via projections.

The starting point of our investigation of polyhedral surfaces were the two articles by McMullen, Schulz & Wills [36, 37] on equivelar surfaces and polyhedral surfaces of "unusually large genus" from the 1980ies. In their first article they propose different techniques to realize surfaces via 4-dimensional polytopes and give explicit constructions for certain parameters $\{p, q\}$, where p denotes the size of the polygons and q the degree of the vertices. In the second article they construct three families of equivelar polyhedral surfaces of types $\{3, q\}$, $\{4, q\}$, and $\{p, 4\}$ in $\mathbb{R}^3$. These surfaces have an astonishingly large genus. More precisely, the surfaces have a genus of order $\mathcal{O}(n \log n)$ on only n vertices. Further, they are constructed explicitly in $\mathbb{R}^3$ with planar polygon faces and without self-intersection.

From the combinatorial point of view, the maximal genus of a surface on n vertices is of order $\mathcal{O}(n^2)$. All the polyhedral surfaces of McMullen, Schulz & Wills in $\mathbb{R}^3$ have either vertex degree or polygon size of less than 5. This leads to a question of Brehm and Wills [13, Sect. 4.2] whether polyhedral surfaces of vertex degree *and* polygon size of at least 5 may be embedded in $\mathbb{R}^3$. Furthermore, little is known about the realization spaces of polyhedral surfaces. Brehm [10] announced a universality theorem for polyhedral surfaces but even for special families of polyhedral surfaces we only have very naïve approaches.

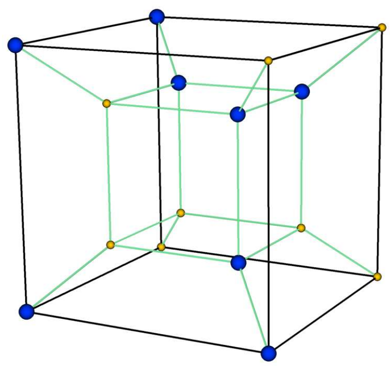
At that time, McMullen, Schulz & Wills did not know the neighborly cubical polytopes (NCP) of Joswig & Ziegler [29] or the projected deformed products of polygons (PPP) of Sanyal & Ziegler [44] which we use to construct polyhedral surfaces. These polytopes have many very interesting properties:

- First of all, they are neighborly, that is, the d-dimensional NCP resp. PPP has the $\lfloor \frac{d-1}{2} \rfloor$-skeleton of the corresponding high dimensional cube resp. product of polygons. Both polytopes are obtained by projections of particular deformed realizations of the cubes resp. products that retain the required skeleta. So they are a manifestation of *dimensional ambiguity*, which is studied since the 1960ies with the cyclic polytopes as its most prominent example.

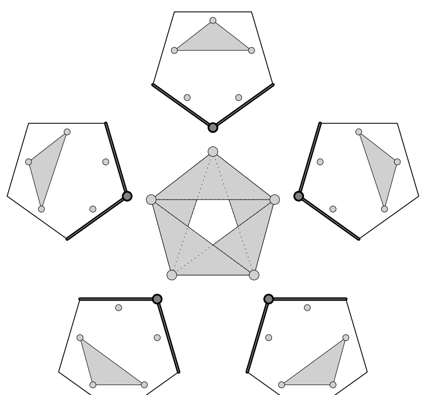
- Secondly, the 4-dimensional NCPs and the PPPs both are relevant to the study of the space of f-vectors of 4-dimensional polytopes. A major open question in this line of research is whether the *fatness* of 4-polytopes is bounded or not, and both the NCPs (2000) and the PPPs (2004) are polytopes with large fatness $5 - \varepsilon$ resp. $9 - \varepsilon$.

- Thirdly, the NCPs and the PPPs both contain interesting families of polyhedral surfaces. The surfaces in the 4-dimensional neighborly cubical polytopes are the Hamiltonian surfaces described by Coxeter [15] and later by Ringel [41], as observed in [28]. The polyhedral surfaces contained in the 4-dimensional PPPs are analyzed in Chapter 5.

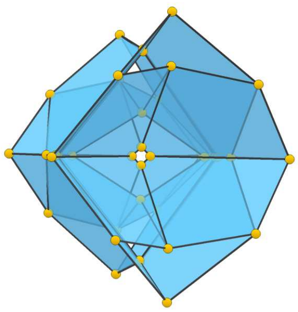
- Finally, the polytopes used in the constructions of both NCPs and PPPs involve projections of deformed products of intervals to obtain the NCPs and of polygons to obtain the PPPs. These deformed polytopes are also worst case examples for linear programming.



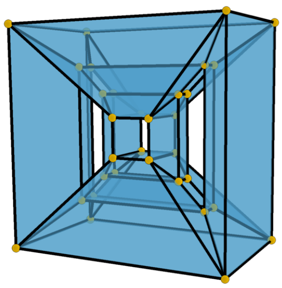
After having introduced the basic objects and methods in **Chapter 1** we present two new methods concerning projections of polytopes and their realization spaces. In the first part of **Chapter 2** we build a bridge between the projection problems and embedding problems via Gale duality. A projection of a polytope preserving a certain subcomplex gives rise to a simplicial complex embedded in the boundary of an associated polytope. We will use the contraposition, that is, if the associated subcomplex cannot be embedded into a sphere of a certain dimension (the boundary of the associated polytope) then there exists no realization of the polytope such that a projection preserves a given subcomplex. As a first application of this scheme we use the non-planarity of the complete bipartite graph on $3 + 3$ vertices to show that the product of two triangles may not be projected to the plane preserving all 9 vertices. Our main tool to show the non-embeddability of the associated simplicial complexes into spheres is the Sarkaria coloring/embedding theorem, which is a combinatorial criterion based on colorings of Kneser graphs. In view of the constructions known for dimensionally ambiguous polytopes, projections preserving entire skeleta are of particular interest. This special case is treated at the end of Section 2.1 where we derive a combinatorial upper bound on the target dimension of a projection of the form: If the target dimension is smaller than the given bound then the projection cannot preserve the desired skeleton. In the second section of **Chapter 2** we describe a new way to parametrize the realization spaces of simple polytopes. Many polytopes studied in this thesis are projections of high dimensional simple polytopes. Unfortunately, it is not clear how the usual parametrization of their realizations via facet normals behaves under projections, since every modification of a single facet normal changes the coordinates of many vertices. So our approach is to give a new parametrization of the realization space of a simple polytope via certain subsets of the vertices. These affine support sets are affinely independent restricted to every facet in *every* realization of the polytope. Thus every small perturbation of the vertices of the affine support set yields a new realization of the simple polytope. So the coordinates of the vertices of the affine support set yield parameters of a part of the realization space of the simple polytope. This parametrization is suited to understanding the dimension of the realization space (moduli) of the projected polytope: Modifications of the vertices of the affine support set orthogonal to the direction of projection become modifications of the projected vertices. Further, this parametrization also yields moduli for arbitrary subcomplexes that are preserved by a projection. In Section 2.2.3 we apply this new technique to the surfaces of Ringel, which may be obtained by a projection of a deformed high-dimensional cube.



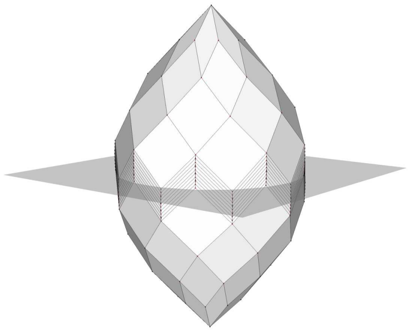
As mentioned earlier, the projected deformed products of polygons (PPPs) are interesting polytopes with respect to various problems: dimensional ambiguity, fatness, polyhedral surfaces, and complexity of linear programming. One curiosity about the construction of the PPPs is that it only works for products of *even* polygons. In the first section of **Chapter 3** we provide a systematic approach to non-projectability of skeleta of products of polytopes that leads to a "knapsack-type" integer program. The optimal value of this integer program serves as a lower bound on the embeddability dimension of the skeleton complex, which is a subcomplex of the associated complex introduced in Section 2.1.1. We use this method to prove the non-projectability of skeleta of products of odd polygons and of products of odd and even polygons. These results complement the previous projectability results for products of even polygons of Ziegler [48] and Sanyal & Ziegler [44]. They also generalize the well known fact that the projection of the product of two odd polygons to the plane cannot preserve all the vertices. Another generalization of the projection of two triangles to the plane preserving all vertices is the projection of products of simplices preserving entire skeleta. Again using the methods developed in Section 2.1, we are able to give upper bounds on the target dimension of the projection to make a projection preserving the skeleton impossible. This result may also be interpreted as a relative of the Flores–Van Kampen Theorem which deals with the (topological) embeddability of skeleta of simplices. As we will see, we are able to reuse the projectability results on products of simplices for projections of polyhedral surfaces in wedge products in Section 4.4.



The starting point of **Chapter 4** were the equivelar polyhedral surfaces of type $\{p, 4\}$ of McMullen, Schulz & Wills [37] with p-gon faces and vertex degree 4. These surfaces are contained in a new kind of polytope constructed from two polytopes—the wedge product. This class of polytopes is dual to the wreath products of Joswig & Lutz [27] and may be obtained by iterating the subdirect sum construction of McMullen [35]. We provide a definition by an inequality system, which combines the inequality systems of the two constituents. Later we observe that there is also a purely combinatorial description, which may serve as a definition as well. The wedge product of a p-gon and a $(q-1)$-simplex contains a family of regular surfaces with p-gon faces and vertex degree $2q$. These surfaces generalize the surfaces of McMullen, Schulz & Wills of type $\{p, 4\}$ and the surfaces of type $\{3, 2q\}$ of Coxeter [15]. Since they are subcomplexes of the wedge products we are able to realize the surfaces in $\mathbb{R}^5$ by a lemma about projections of 2-dimensional complexes by Perles. Moreover, we obtain another way of realizing the surfaces of type $\{p, 4\}$ in $\mathbb{R}^3$ by projecting a deformed wedge product of a p-gon and an interval. As a benefit of our construction we are able to give a lower bound on the number of moduli of the projected surfaces by using the new methods developed in Section 2.2. In Section 4.3.3 we construct 4-polytopes with prisms over the surfaces in their boundary. This allows us to use polytope duality to obtain realizations of the dual surfaces in the boundary of the dual 4-polytope. Consequently, we may realize the dual surfaces in $\mathbb{R}^3$ as well via Schlegel projection. (Un)fortunately, we are able to use our results about the projectability of skeleta of products of simplices of Section 3.3 to deduce the following for the wedge product surfaces: For $p \geq 4$ and $q \geq 3$ there exists no realization of the corresponding wedge product such that the surface survives the projection to the boundary of a 4-polytope. For Coxeter's surfaces of type $\{3, 2q\}$ our methods did not yield any new results.



The quadrilateral surfaces constructed by McMullen, Schulz & Wills [37] (MSW surfaces) are equipped with three parameters: the vertex degree, and two parameters arising from the torus symmetry. The construction is performed entirely in $\mathbb{R}^3$ using great geometric intuition and the symmetry of the standard $(m \times n)$-torus. In **Chapter 5** we identify these surfaces as subcomplexes of products of polygons. This leads to the generalized MSW surfaces contained in the 2-skeleton of the product of polygons described in Section 5.1.2. These surfaces provide a large number of parameters and include the original MSW surfaces as well as the surfaces described in terms of their symmetry group by Coxeter [15]. Another family of surfaces containing all "polygon" faces and some quadrilateral faces of the product is constructed in the subsequent section. Both families of surfaces may be realized in the boundary of a 4-polytope and in $\mathbb{R}^3$ via an orthogonal projection of a suitable deformed realization of the product of polygons. Hence we are able to apply the methods of Section 2.2 to obtain a large number of moduli despite the fact that the surfaces neither have vertices of degree three nor triangle faces. Further, the surfaces of the second family are the first to achieve an average vertex degree and an average polygon size larger than 5 embedded in $\mathbb{R}^3$ with planar convex faces. This gives an affirmative answer to a variation of the question of Brehm & Wills [13, Sect. 4.2] asking for the existence of such polyhedral surfaces with large vertex degree *and* large polygon size in $\mathbb{R}^3$. As for the surfaces contained in the projected wedge products, we are again able to construct 4-polytopes with the prisms over the surfaces in their boundaries. Hence we obtain realizations of the dual surfaces in $\mathbb{R}^3$ as well.



A particularly interesting family of zonotopes are the "Ukrainian Easter eggs" from Eppstein's online "Geometry Junkyard" [18]. These zonotopes have a large 2-dimensional section: The Ukrainian easter egg generated by n vectors, that is, with n zones, has a central 2-dimensional cut with $\Omega(n^2)$ sides. This is optimal in dimension 3 since a zonotope on n zones has at most $\mathcal{O}(n^2)$ facets. Dually, the duals of the Ukrainian Easter eggs have a projection to the plane that has $\Omega(n^2)$ vertices. In **Chapter 6** we provide two constructions of high-dimensional dual zonotopes that have 2-dimensional projections with many vertices: The first construction is based on the projected deformed products of polygons. It yields a d-dimensional dual zonotope with n zones whose projection to the plane has $\Theta(n^{(d-1)/2})$ vertices. In Section 6.3 we construct another d-dimensional zonotope on n zones such that the projection to the plane has $\Theta(n^{d-1})$ vertices. This is the maximal asymptotic bound for fixed dimension, since the number of vertices of a dual zonotope is also $\mathcal{O}(n^{d-1})$. The construction is also relevant to the arrangement method for linear programming proposed by Koltun [30]. This method transforms a linear program into a linear program on an arrangement polytope, that is, a linear program on a dual zonotope.

Co-authors

The new methods of Chapter 2 were developed together with Raman Sanyal (Section 2.1) and Günter M. Ziegler (Section 2.2). Chapter 3 on topological obstructions to projections of polytope skeleta is joint work with Raman Sanyal. The polyhedral surfaces and wedge products of Chapter 4 have been worked out together with Günter M. Ziegler. The final Chapter 6 was written in collaboration with Nikolaus Witte and Günter M. Ziegler and is published in *Discrete & Computational Geometry (Online first)* [42].

Chapter 1

Basics

In this chapter we introduce the basic concepts of this thesis and give references to the relevant literature for further reading. All the important definitions are marked by **Definition**, whereas other basic concepts are only *emphasized* in the text. For $n \in \mathbb{N}$ we define $[n] := \{0, 1, \cdots, n-1\}$. This is most suitable for calculating modulo n.

1.1 Polytopes

In this section we provide some basic notions about convex polytopes used throughout this thesis. We will introduce some notation and ways of speaking without giving an exhaustive list of basic definitions. For a thorough treatment of polytopes we refer the reader to the books of Grünbaum [25] and Ziegler [47].

There are an interior and an exterior definition of a polytope. They are equivalent by the Minkowski–Weyl Theorem [47, Thm. 1.1, p. 29]:

Interior/$\mathcal{V}$-description: A *polytope* $P \subseteq \mathbb{R}^d$ is the convex hull of a finite set of points $V = \{v_0, \ldots, v_{n-1}\} \subseteq \mathbb{R}^d$:

$$P = \operatorname{conv}(V) = \operatorname{conv}(\{v_0, \ldots, v_{n-1}\}).$$

Exterior/$\mathcal{H}$-description: A *polytope* $P \subseteq \mathbb{R}^d$ is the bounded intersection of a finite number of halfspaces $h_i^+ = \{x \in \mathbb{R}^d \mid a_i x \leq b_i\}$:

$$P = \bigcap_{i=0}^{m-1} h_i^+ = \{x \in \mathbb{R}^d \mid a_i x \leq b_i \text{ for } i = 0, \ldots, m-1\}.$$

The *dimension* of a polytope is the dimension of its affine hull. If not stated otherwise, we always consider full-dimensional polytopes $P \subset \mathbb{R}^d$, that is, the dimension of the polytope coincides with the dimension of the ambient space. A d-dimensional polytope is called a *d-polytope* for short.

A *face* G of a d-polytope $P \subset \mathbb{R}^d$ is the intersection $G = P \cap h_c$ of the polytope P with a (valid/admissible) hyperplane $h_c = \{x \in \mathbb{R}^d \mid cx = c_0\}$ with $cx \leq c_0$ for all $x \in P$. The dimension of the face G is the dimension of its affine hull.

The 0-dimensional faces of P are the *vertices* denoted by $V = \operatorname{vert}(P)$. Sometimes we identity the set of vertices $V = \{v_0, \ldots, v_{n-1}\}$ with the index set $[n] = \{0, \ldots, n-1\}$ or the coordinate matrix $V \in \mathbb{R}^{d \times n}$ whose columns contain the coordinates of the corresponding vertices. The *facets* of P are the $(d-1)$-dimensional faces. A facet F will often be identified with its defining hyperplane $h_i = \{x \in \mathbb{R}^d \mid a_i x \leq b_i\}$ as given in the exterior $\mathcal{H}$-description or just with an index $i \in [m] = \{0, \ldots, m-1\}$ such that $F = P \cap h_i$. The empty set $\emptyset$ and the entire polytope P are also faces of the polytope. The number of k-dimensional faces (k-faces) of a polytope is denoted by $f_k(P)$. The vector $f(P) := (f_k(P))_{k=0}^{d-1} \in \mathbb{Z}^d$ is the *f-vector* of the polytope.

Each face G of P is uniquely determined by the subset $V_G := \operatorname{vert}(P) \cap G$ of the vertices of P contained in G, such that $G = \operatorname{conv}(V_G)$. The set of facets containing a face G is denoted by $H_G := \{F \mid F \text{ facet}, G \subset F\}$. This set also determines the face G uniquely as intersection of facets, that is, $G = \bigcap\{F \mid F \in H_G\}$. If we identify the vertices with $[n]$ and the facets with $[m]$ then $V_G \subseteq [n]$ and $H_G \subseteq [m]$.

The combinatorial structure of a d-polytope P with n vertices and m facets is captured in its *face lattice* $\mathcal{L}(P)$. Its elements are the faces of P partially ordered by inclusion. We may also construct the face lattice by ordering the vertex sets of the faces $V_G \subseteq [n]$ by inclusion or the sets of facets $H_G \subseteq [m]$ containing a given face by reverse inclusion. A d-polytope is *simple* if every vertex is contained in exactly d facets, that is, if $|H_v| = d$ for every vertex v. A *flag* of a d-polytope is a sequence of faces $(G_k)_{k=0}^{d-1}$ such that G_k is a k-face and $G_{k-1} \subset G_k$ for $k = 1, \ldots, n-1$.

Two polytopes P and Q are *combinatorially isomorphic* if their face lattices $\mathcal{L}(P)$ and $\mathcal{L}(Q)$ are isomorphic lattices. This leads to our first new definition.

Definition 1.1 (Combinatorial type/d-type)**.** The *combinatorial type* or *d-type* $\mathcal{P}$ of a d-polytope P consists of all polytopes combinatorially equivalent to P. We call every such polytope P a *realization* of the combinatorial type (d-type) $\mathcal{P}$.

We will denote combinatorial types by script letters and their corresponding realizations by normal letters, e.g. P is a realization of $\mathcal{P}$. All combinatorial properties of the realizations will also be assigned to the corresponding combinatorial types.

The space of all realizations of a given d-dimensional combinatorial type $\mathcal{P}$ with n vertices in $\mathbb{R}^d$ is the *realization space* $\mathfrak{R}(\mathcal{P})$ of $\mathcal{P}$. The realization space $\mathfrak{R}(\mathcal{P})$ may be interpreted as a subspace of $\mathbb{R}^{nd}$ by identifying the $n \cdot d$ coordinates of the vertices of a realization P of $\mathcal{P}$ with a point in $\mathbb{R}^{nd}$. It may easily be described by a set of polynomial equations (vanishing determinants for vertices on one facet) and strict inequalities for convexity. Such sets given by polynomial equations and strict polynomial inequalities are called primary basic semialgebraic sets. The universality theorems of Mnëv [38] and Richter-Gebert [40] state that every primary basic semialgebraic set is stably equivalent to the realization space of a d-polytope on $d + 4$ vertices (Mnëv) or the realization space of a 4-polytope (Richter-Gebert).

The local dimension of the realization space of a combinatorial type $\mathcal{P}$ is the number of *moduli*. For example, every realization P of a simple d-type $\mathcal{P}$ has $d \cdot f_{d-1}(\mathcal{P})$ moduli since every small perturbation of the facet normals yields another realization of the same combinatorial type. So the realization space of a simple d-type has dimension $d \cdot f_{d-1}(\mathcal{P})$ (everywhere).

1.2 Gale transform

Gale transformation is a simple linear algebra method to transform one sequence of vectors into another. (The linear algebra behind Gale transformation is described in Matoušek [33, Ch. 5.6]. Another introduction to Gale transforms with a special focus on the link to matroid duality is given in Ziegler [47, Ch. 6].) In polytope theory it is used to study high dimensional polytopes with few vertices. We use it to construct vector configurations needed for our projection problems.

Definition 1.2 (Gale transform)**.** Let $V = (v_0, \dots, v_{n-1})$ be a sequence of points in $\mathbb{R}^d$ forming the columns of a matrix V with $\operatorname{rank}(\binom{\mathbf{1}}{V}) = d+1$. The *Gale transform* G of V is a sequence of vectors $G = (g_0, \dots, g_{n-1})$ in $\mathbb{R}^{n-d-1}$ such that the rows of G span the orthogonal complement of the rows of V and are orthogonal to the vector $\mathbf{1} = (1, \dots, 1)$, that is, $\binom{\mathbf{1}}{V} G^t = \mathbb{0}$.

Since the choice of basis for the orthogonal complement of the row space of V is arbitrary, the Gale transform of a sequence of points is only determined up to linear isomorphism.

Since the sequences of points V and G are linked by orthogonal complement, the affine values of V correspond to the linear dependencies of G [33, Lemma 5.6.2, p. 110] and the linear values of G are the affine dependencies of V. This allows us to read off various properties of the configuration V

from the configuration G and vice versa. We may, for example, determine whether the points of V are in convex position.

Proposition 1.3 (Polytopal Gale transform [33]). Let $V = (v_0, \dots, v_{n-1})$ be a set of points in $\mathbb{R}^d$ with $\operatorname{rank}(\binom{\mathbf{1}}{V}) = d+1$ and $G = (g_0, \dots, g_{n-1})$ its Gale transform. Then the v_i are the vertices of a convex d-polytope if and only if there exists *no* oriented hyperplane containing at most one of the vectors g_i on its positive side.

The Gale transform of a polytope allows us to recover the entire facial structure of the polytope via the following dictionary. There are of course more properties to read off the Gale transform but we only state the ones important for our further investigations.

Proposition 1.4 (Gale transform dictionary [33]). Let P be a d-polytope with vertices $V = (v_0, \dots, v_{n-1})$ and $G = (g_0, \dots, g_{n-1})$ its Gale transform in $\mathbb{R}^{n-d-1}$. Then there is the following correspondence between the vertices v_i and the corresponding g_i:

1. The set of vertices $\{v_i \mid i \in I \subseteq [n]\}$ with $|I| = d+1$ is affinely independent if and only if $\{g_i \mid i \notin I\}$ is linearly independent.
2. $F = \operatorname{conv}(\{v_i \mid i \in I_F \subseteq [n]\})$ is a face of P if and only if the complement of the corresponding vectors $\{g_i \mid i \notin I_F\}$ is positively dependent.
3. $F = \operatorname{conv}(\{v_{i_0}, \dots, v_{i_k}\})$ is a simplicial face of P if and only if the complement of the corresponding vectors $G \setminus \{g_{i_0}, \dots, g_{i_k}\}$ is positively dependent and spanning.

Example 1.5 (Gale transform of a pyramid). Let $V = (v_1, \dots, v_n)$ be the vertices of a d-polytope P in $\mathbb{R}^d$ and $G = (g_1, \dots, g_n)$ its Gale transform in $\mathbb{R}^{n-d-1}$. Then the vertices $(\hat{v}_0, \hat{v}_1, \dots, \hat{v}_n) = (\binom{1}{\mathbb{0}}, \binom{0}{v_1}, \dots, \binom{0}{v_n})$ are the vertices of a pyramid $\operatorname{pyr}(P)$. The Gale transform $\hat{G} = (\hat{g}_0, \hat{g}_1, \dots, \hat{g}_n)$ of the pyramid consists of $n+1$ vectors in $\mathbb{R}^{n+1-(d+1)-1}$, that is, the same ambient dimension as the Gale transform of the polytope P. Since the polytope P itself is a face of the pyramid $\operatorname{pyr}(P)$ we obtain $\hat{g}_0 = \mathbb{0}$. Further for every face G of P, G and $G \cup \{\hat{v}_0\}$ are faces of $\operatorname{pyr}(P)$ and thus $\hat{G} = (\mathbb{0}, g_1, \dots, g_n)$ is a Gale transform of $\operatorname{pyr}(P)$. A pyramid over a 5-gon and its Gale diagram are shown in Figure 1.1.

1.3 Polyhedral surfaces

There are many different ways used to describe surfaces, for example polyhedral maps, which may be given by a graph on a surface, or cell complexes

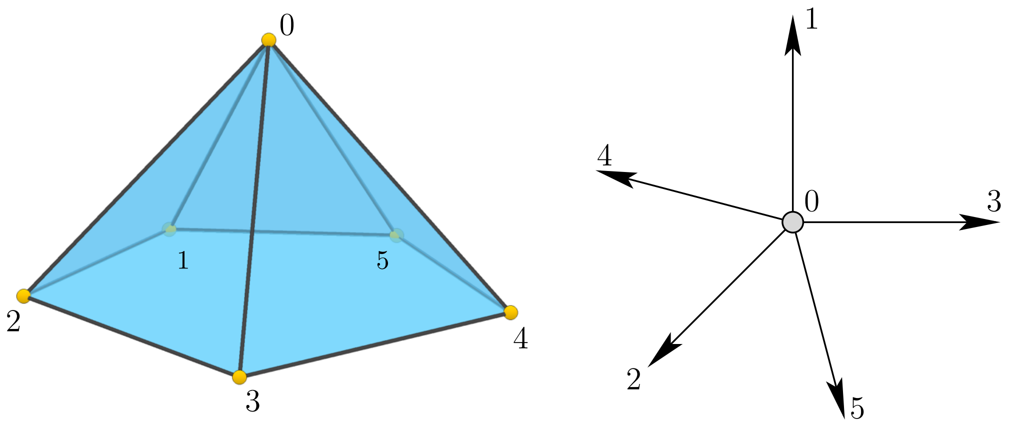


Figure 1.1: A pyramid over a 5-gon and its Gale diagram. The 0-vector in the Gale diagram is indicated by the point in the center.

with intersection property (see the handbook articles of Brehm & Wills [13] and Brehm & Schulte [12] for further discussion and references). Since we are heading for realizations of certain combinatorial types of surfaces we consider the following complexes whose topology is determined by its face poset as described by Björner in [5]. So a polytopal complex is given by a poset whose intervals are isomorphic to posets of polytopes. We do not require that a polytopal complex has any realization as in the definition of a polyhedral complex given by Ziegler [47, p. 127].

Definition 1.6 (Polyhedral surface)**.** A *polyhedral surface* $\mathcal{S}$ is a 2-dimensional polytopal complex homeomorphic to a connected 2-manifold without boundary, that is, $\mathcal{S}$ is a polytopal complex satisfying the following conditions:

1. $\mathcal{S}$ is connected,
2. the star of every vertex is homeomorphic to a 2-ball, and
3. every edge of $\mathcal{S}$ is contained in exactly two polygons.

The *face lattice* $\mathcal{L}(\mathcal{S})$ of a polyhedral surface $\mathcal{S}$ is obtained by ordering the faces of the vertices, edges, and polygons by inclusion and attaching a unique minimal and maximal element. The *dual surface* $\mathcal{S}^*$ is given by the dual face lattice, that is, by reversing the order of the face lattice $\mathcal{L}(\mathcal{S})$. The polygons of the surface $\mathcal{S}$ correspond to vertex figures of the dual surface $\mathcal{S}^*$, and vice versa. So the dual of a polyhedral surface is again a polyhedral surface. If the polygons may be oriented such that each edge of the polyhedral surface is oriented in different direction in the two adjacent polygons, then the polyhedral surface is *orientable*. The *Euler characteristic* of a surface is $\chi(\mathcal{S}) = f_0 - f_1 + f_2$, where (f_0, f_1, f_2) is the f-vector of the surface, that is, $\mathcal{S}$ has f_0 vertices, f_1 edges, and f_2 polygons. The *genus* $g(\mathcal{S})$ of an orientable polyhedral surface may easily be calculated from its Euler characteristic: $g(\mathcal{S}) = 2 - 2\chi(\mathcal{S})$.

Most of the surfaces described in this thesis are of a particular type of polyhedral surfaces given by the next definition.

Definition 1.7 (Equivelar surface of type $\{p, q\}$)**.** A polyhedral surface is *equivelar* of type $\{p, q\}$ if all its faces are p-gons and every vertex is incident to exactly q of these p-gons.

We consider equivelar surfaces as abstract surfaces that need not have any realization. The *automorphism group* $\mathsf{Aut}(\mathcal{S})$ of a polyhedral surface is the subgroup of the vertex permutations inducing an automorphism on the face lattice $\mathcal{L}(\mathcal{S})$. This leads to the following definition.

Definition 1.8 (Regular polyhedral surfaces)**.** A polyhedral surface $\mathcal{S}$ is *regular* if its automorphism group acts transitively on the flags of $\mathcal{S}$.

A *realization* S of a polyhedral surface $\mathcal{S}$ is an embedding of the surface in some $\mathbb{R}^d$ with planar convex polygons. The space of all realizations of a polyhedral surface $\mathcal{S}$ in some $\mathbb{R}^d$ is the realization space $\mathfrak{R}(\mathcal{S}, d)$ of the surface. In contrast to polytopes the polyhedral surfaces do not have a natural ambient space, so we have to introduce an additional parameter d. Topologically, every orientable surface may be embedded in $\mathbb{R}^3$ and every non-orientable surface in $\mathbb{R}^4$. But for orientable polyhedral surfaces the realization space $\mathfrak{R}(\mathcal{S}, 3)$ may be empty: Bokowski & Guedes de Oliveira [8] proved that one particular triangulated surface of genus 6 with the complete graph on 12 vertices cannot be realized in $\mathbb{R}^3$. Using a similar approach Schewe [45] was able to show that none of the triangulations of a surface of genus 6 with 12 vertices may be realized in $\mathbb{R}^3$ with a more sophisticated algorithm in combination with faster computers. On the other hand, Brehm [10] announced a theorem on the universality of polyhedral surfaces (see Ziegler [49]), that is, every primary basic semialgebraic set is stably equivalent to the realization space of a polyhedral surface. If there exists a realization $S \subset \mathbb{R}^d$ of a polyhedral surface $\mathcal{S}$, then we may ask for the local dimension of the realization space in the vicinity of S. This dimension of the realization space close to a given realization S is the *number of moduli* $\mathfrak{M}(S, d)$ of the realization S.

1.3.1 Naïve estimate on the number of moduli

The naïve way to estimate the number of moduli for a realization S of a polyhedral surface $\mathcal{S}$ is to count the "degrees of freedom" and subtract the number of "constraints". The idea is the following: Every vertex embedded in some $\mathbb{R}^d$ has d degrees of freedom. Now the generic constraints mimic the behavior of a generic system of linear equations. They are given by generic equations, that is, the equations define co-dimension 1 hypersurfaces that are stable under perturbations of the coefficients of the defining equations. Hence every k generic constraints should intersect in a set of co-dimension k.

We adapt this naïve approach to our problem of embedding surfaces. If the surface is embedded in $\mathbb{R}^d$ then every vertex has d degrees of freedom making a total of $d \cdot f_0(\mathcal{S})$ degrees of freedom. The constraints originate from the planarity of the faces. These constraints are generic since we start with an embedding of the surface and consider only small perturbations of the vertices. To guarantee the planarity of the polygons we consider a triangulation of the surface. This triangulation induces a triangulation on every p-gon face with $p-3$ diagonals (non-polygon edges). If the two triangles adjacent to each of the diagonals in the triangulation of the p-gon are coplanar then the entire p-gon is also planar. Since a 2-dimensional subspace of d-space is the intersection of $d-2$ hyperplanes, we need $d-2$ constraints to assure that two neighboring triangles are coplanar. Summing up over all diagonals of all p-gons we obtain $(d-2)(f_{02}(\mathcal{S}) - 3f_2(\mathcal{S}))$ constraints, where $f_{02}(\mathcal{S})$ counts the number of vertex-polygon incidences. Thus the naïve "degrees of freedom minus number of constraints" count yields the following estimate for the number of moduli of a realization S of the polyhedral surface $\mathcal{S}$:

$$\begin{aligned}\mathfrak{M}(S,d) &\sim df_0(\mathcal{S}) - (d-2)(f_{02}(\mathcal{S}) - 3f_2(\mathcal{S})) \\ &= df_0(\mathcal{S}) - 2(d-2)f_1(\mathcal{S}) + 3(d-2)f_2(\mathcal{S}),\end{aligned}$$

since $f_{02}(\mathcal{S}) = f_{12}(\mathcal{S}) = 2f_1(\mathcal{S})$. For orientable surfaces we can express the above estimate in terms of the genus:

$$\mathfrak{M}(S,d) \sim 4(d-2) + (d-2)f_2(\mathcal{S}) - (d-4)f_0(\mathcal{S}) - 4(d-2)g(\mathcal{S})$$

since $\chi(\mathcal{S}) = 2 - 2g(\mathcal{S}) = f_0(\mathcal{S}) - f_1(\mathcal{S}) + f_2(\mathcal{S})$. For surfaces in $\mathbb{R}^3$ this amounts to:

$$\begin{aligned}\mathfrak{M}(S,3) &\sim 3f_0(\mathcal{S}) - 2f_1(\mathcal{S}) + 3f_2(\mathcal{S}) \\ &= 4 + f_2(\mathcal{S}) + f_0(\mathcal{S}) - 4g(\mathcal{S}).\end{aligned} \tag{1.1}$$

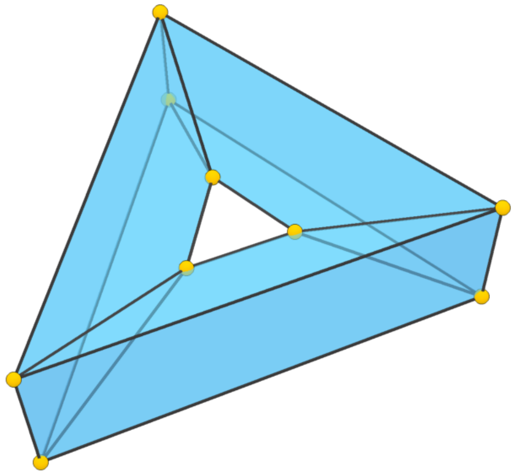


Figure 1.2: The 3×3-torus.

Thus if the genus of the surface is asymptotically larger than the number of vertices and facets then we expect the surface to have few moduli or no moduli at all. The f-vector of an equivelar surface $\mathcal{S}_{p,q}$ of type $\{p, q\}$ with $2np$ vertices is $n(2p, pq, 2q)$. If realized in $\mathbb{R}^3$ then the estimate for the number of moduli of a realization $S_{p,q}$ is:

$$\mathfrak{M}(S_{p,q}, 3) \;\sim\; n(3 \cdot 2p - 2 \cdot pq + 3 \cdot 2q) = 2n(3p + 3q - pq).$$

For triangulated surfaces (i.e. equivelar surfaces of type $\{3, q\}$) in $\mathbb{R}^3$ we know that the number of moduli is $3 \cdot f_0(\mathcal{S}_{3,q}) - 15$. This coincides with the above estimate for equivelar surfaces of type $\{3, q\}$. We also observe that if p and q are large then the estimated number of moduli becomes negative. Hence one could expect that those surfaces do not have any non-trivial moduli (i.e. moduli which do not come from projective transformations).

Remark 1.9. Every embedded surface always allows for small projective transformations. As we are interested in asymptotic results for families of surfaces we omit this additive constant obtained from fixing a projective basis chosen from the vertices.

As we will see in Chapter 4 and Chapter 5, particular realizations of surfaces with large polygon size or large vertex degree may nonetheless have many moduli.

This is captured in a meta-theorem of Crapo [17] saying that the number of moduli of a configuration is “the number of degrees of freedom of the

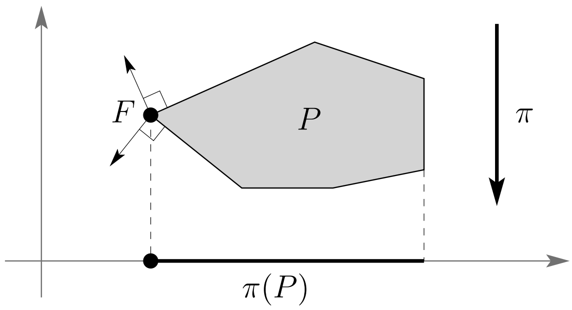


Figure 1.3: Survival of a face F in the projection π to the first coordinate. The right vertex of the interval is not strictly preserved.

vertices minus the number of generic constraints *plus the number of hidden (incidence) theorems*." For the (3×3)-torus shown in Figure 1.2 we know that if 8 of the 9 quadrilaterals are planar, then the 9th quadrilateral is also planar. So one of the planarity conditions is superfluous (see Bokowski & Sturmfels [9, p. 72] for a proof and references). In the case of the (3×3)-torus the Desargues Theorem is the incidence theorem that implies that the ninth quadrilateral is planar.

1.4 Projections and linear algebra

A linear projection π may map the k-faces of a polytope P to k-faces of $\pi(P)$, to lower dimensional faces of $\pi(P)$, to subsets of faces of $\pi(P)$, or into the interior of $\pi(P)$. We restrict ourselves to the nicest case of *strictly* preserved faces as defined by Ziegler [48].

Definition 1.10 (Strictly preserved faces)**.** Let $P \subset \mathbb{R}^d$ be a polytope and $Q = \pi(P)$ be the image of P under the affine projection map $\pi : \mathbb{R}^d \to \mathbb{R}^e$. A proper face F of P is *strictly preserved* by π if

(D1) $\pi(F)$ is a face of Q,
(D2) $\pi(F)$ is combinatorially equivalent to F, and
(D3) the preimage $\pi^{-1}(\pi(F)) \cap P$ is F.

We also say that a face *survives* the projection or is *retained* by a projection, if it is strictly preserved by the projection. The projection π is *generic with respect to a subcomplex* of the polytope, if the faces of the subcomplex are strictly preserved by the projection. (Note that such projections do not exist for arbitrary subcomplexes.) A projection is *generic* if it is generic with respect to the entire projected polytope, that is, all the proper faces of the projected polytope are strictly preserved. This is motivated by the fact, that

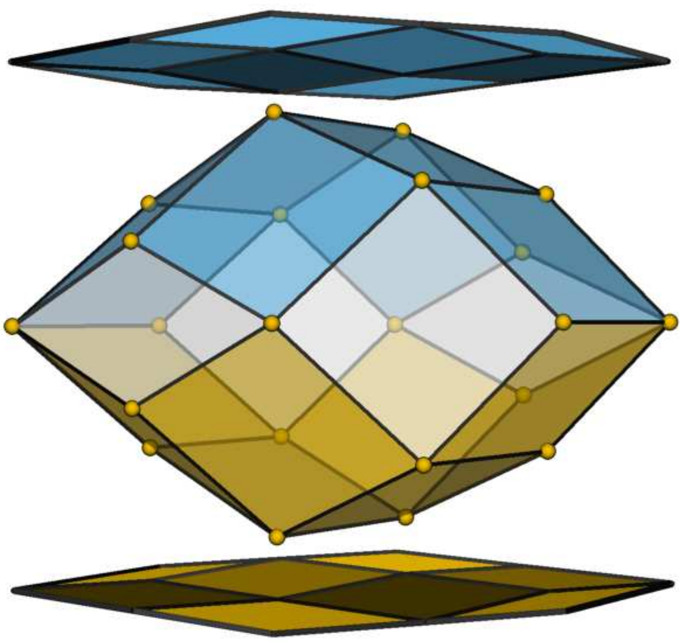


Figure 1.4: The lower and the upper hull of a zonotope. Orthogonal projections of the lower resp. upper hull yield a polytopal complex (without overlap).

preserved faces are stable under perturbation: If a face F of a polytope P is strictly preserved under projection, then it is also strictly preserved by any small perturbation of the projection. Moreover, in the case of a simple polytope P we may consider small perturbations of (the facet-defining inequalities/hyperplanes of) the polytope P that do not change the combinatorial type, and for sufficiently-small such perturbations the corresponding face $\tilde{F}$ of the slightly perturbed polytope $\tilde{P}$ is also preserved under the same projection. To be able to apply this definition to our polytopes we use the following lemma which connects the strictly preserved faces to the inequality description of a polytope.

Lemma 1.11 (Preserved faces: linear algebra version [48])**.** Let $P \subset \mathbb{R}^d$ be a d-polytope with facets $a_i x \leq 1$ for $i \in [m]$, F a non-empty face of P, and H_F the index set of the inequalities that are tight at F. Then F is strictly preserved by the projection to the first e coordinates if and only if the facet normals truncated to the last $d - e$ coordinates $\{a_i^{(d-e)} : i \in H_F\}$ positively span $\mathbb{R}^{d-e}$.

The previous lemma is illustrated in Figure 1.3. Every affine projection of a polytope to some e-dimensional subspace may be transformed into a projection to the first e-coordinates by a suitable change of coordinates. Thus we will only consider such projections to the first e coordinates in the following.

The above Lemma 1.11 suffices to obtain realizations of interesting polyhedral surfaces in the boundaries of 4-polytopes via projections of high-dimensional polytopes (see Chapters 4 and 5). To get a realizations of these surfaces in $\mathbb{R}^3$ we might construct the Schlegel diagram of the projected polytope via a central projection. To avoid this central projection we will project the surface onto the lower or upper hull of the 4-polytope.

Definition 1.12. The *lower/upper hull* of a polytope with respect to some coordinate direction x_ℓ is the polytopal complex consisting of all faces that have a normal vector with negative/positive x_ℓ-coordinate (see Figure 1.4).

We will use the following lemma to prove the existence of realizations of some of the surfaces in the lower resp. upper hull of projected (4-)polytopes.

Lemma 1.13 (Preserved faces on the lower/upper hull)**.** Let $P \subset \mathbb{R}^d$ be a polytope given by $P = \{x \in \mathbb{R}^d \mid a_i x \leq 1, i \in [m]\}$, G a non-empty face of P, and $H_G \subset [m]$ the index set of the inequalities that are tight at G. Then $\pi(G)$ is on the lower/upper hull with respect to x_{e-1} of the projection to the first e coordinates if and only if

(L1) the facet normals $a_i^{(d-e)}$ with $i \in H_G$ truncated to the last $d-e$ coordinates positively span $\mathbb{R}^{d-e}$, and

(L2) there exist $\lambda_i \geq 0$ such that $\nu = \sum_{i \in H_G} \lambda_i a_i$ with $(\nu_e, \ldots, \nu_{d-1}) = \mathbf{0}$ and $\nu_{e-1} < 0$ (lower hull) resp. $\nu_{e-1} > 0$ (upper hull).

Proof. The first part (L1) of this lemma is exactly the Projection Lemma 1.11 and the second part (L2) corresponds to Definition 1.12. $\square$

Chapter 2

New methods

In this chapter we introduce two new methods used in the following chapters. The first section describes a new approach to attack problems concerning polytope projections. By Gale duality we are able to associate an embedding problem to a projection problem. To solve the associated embedding problems we use methods from combinatorial topology. We apply the method to products of polytopes in Chapter 3 and to polyhedral surfaces in wedge products in Chapter 4. The second section introduces a new way to parametrize the realization spaces of simple polytopes via affine support sets, which are subsets of the vertices. The advantage of this parametrization is that the moduli of the vertices are preserved by generic projections and thus yield lower bounds on the moduli of surfaces obtained via projections of high-dimensional polytopes. With this new method we estimate the dimension of the realization spaces for the surfaces in products of polygons in Chapter 5 and for the surfaces contained in wedge products in Chapter 4.

2.1 Topological obstructions and polytope projections
joint with Raman Sanyal

We devise a criterion for projections of polytopes that allows us to state when a certain subcomplex may be strictly preserved by a projection. First we associate an embedding problem to the projection problem in Section 2.1.1. Then we describe methods from combinatorial topology, which may yield obstructions to the associated embeddability problem in Section 2.1.2. Finally, in Section 2.1.4 we specialize the obstructions to the problem of preserving certain skeleta of polytopes by projections.

2.1.1 Associated polytope and subcomplex

We build a bridge between projection problems and embeddability problems as follows: We associate a polytope with certain simplex faces to a projection of a polytope with certain strictly preserved faces via Gale duality. The simplex faces of this associated polytope form a simplicial complex. If we can show that this simplicial complex cannot be embedded into the boundary of the associated polytope, then there is no realization of the polytope that allows for a projection preserving the given subcomplex.

Sanyal [43] uses the same approach to analyze the number of vertices of Minkowski sums of polytopes, since Minkowski sums are projections of products of polytopes. The vertices of a (simple) polytope P give rise to a simplicial complex Σ_0. If $\pi : P \to \pi(P)$ is a projection preserving the vertices, then Σ_0 is realized in a (simplicial) sphere whose dimension depends on $\mathsf{dim}\, \pi(P)$. So if the simplicial complex Σ_0 cannot be embedded into that sphere then there exists no realization of the polytope such that all vertices survive the projection.

Theorem 2.4 below is a generalization of this result from vertices to arbitrary subcomplexes, which should be preserved. The next proposition links strictly preserved faces to the associated polytope via Gale duality.

Proposition 2.1. Let $\pi : \mathbb{R}^d \to \mathbb{R}^e$ be the projection to the first e coordinates of a d-polytope P given by its facet inequalities $(A^{(e)}, A^{(d-e)})\binom{x}{x'} \leq \mathbb{1}$ with $A^{(e)} \in \mathbb{R}^{m\times e}$, $A^{(d-e)} \in \mathbb{R}^{m\times(d-e)}$, $x \in \mathbb{R}^e$, and $x' \in \mathbb{R}^{d-e}$. If for each facet F of P at least one vertex $v \notin F$ survives the projection then the rows of $A^{(d-e)}$ are the Gale transform of a polytope.

Proof. The rows of the matrix $A^{(d-e)}$ are the Gale transform of a polytope if for every row $a_i^{(d-e)}$ ($i \in [m]$) the remaining rows of $A^{(d-e)} \setminus a_i^{(d-e)}$ are positively spanning. But for every facet F there exists a vertex $v \notin F$ that survives the projection. Hence by Lemma 1.11 the truncated normals corresponding to the facets containing this vertex positively span $\mathbb{R}^{d-e}$. Thus $A^{(d-e)}$ is the Gale transform of a polytope. □

So if we project a d-polytope to $\mathbb{R}^e$ such that some of the vertices survive the projection as described in the above proposition we obtain a polytope by Gale duality.

Definition 2.2 (Associated polytope)**.** Let π be a projection of a d-dimensional polytope P on m facets to $\mathbb{R}^e$ that preserves one vertex $v \notin F$ for every facet F of P. Then the $(m-(d-e)-1)$-dimensional polytope on m vertices obtained via Gale transformation as described in Proposition 2.1 is the *associated polytope* $\mathcal{A}(P, \pi)$.

Further every face G that is preserved by the projection in the sense of Lemma 1.11 yields an associated face $\mathcal{A}_G = [m] \setminus H_G$ of the associated polytope $\mathcal{A}(P, \pi)$ since Gale duality transforms positively spanning vectors into faces of the polytope. By Proposition 1.4 all these associated faces $\mathcal{A}_G$ are simplices. This yields the following subcomplex of the associated polytope.

Definition 2.3 (Associated subcomplex)**.** Let π be a projection of a d-dimensional polytope P on m facets to $\mathbb{R}^e$ that preserves one vertex $v \notin F$ for every facet F of P, and let $\mathcal{S}$ be the subcomplex of P that is preserved under projection in the sense of Lemma 1.11. Then the *associated subcomplex* $\mathsf{K}(P, \pi)$ is the simplicial complex:

$$\mathsf{K}(P, \pi) \ := \ \overline{\{[m] \setminus H_G \mid G \in \mathcal{S}\}}.$$

The subcomplex consists of all the facets and their faces.

Now we obtain the following theorem which links the projection of a polytope preserving certain faces with the embedding of the associated subcomplex into the associated polytope.

Theorem 2.4. Let π be a projection of a d-dimensional polytope P on m facets to $\mathbb{R}^e$ that preserves one vertex $v \notin F$ for every facet F of P. Then the associated subcomplex $\mathsf{K}(P, \pi)$ is embedded in the boundary of the associated polytope $\mathcal{A}(P, \pi)$ of dimension $m + e - d - 1$. $\square$

Example 2.5 (Projection of the product of triangles preserving all vertices)**.** We will use the technique developed in this section to show that there exists no realization of the product $(\Delta_2)^2 \subset \mathbb{R}^4$ of two triangles Δ_2 such that the projection $\pi : \mathbb{R}^4 \to \mathbb{R}^2$ to the plane preserves all 9 vertices.

The product of two triangles is a 4-polytope on 6 facets. Since the projection is to $\mathbb{R}^2$, the associated polytope $\mathcal{A}((\Delta_2)^2, \pi)$ is a 3-dimensional polytope. Let us label the facets of the two triangles by a_0, a_1, a_2 and a'_0, a'_1, a'_2. These are also the labels of the vertices of the associated polytope $\mathcal{A}((\Delta_2)^2, \pi)$. Each vertex of the product lies on two facets corresponding to two edges of each of the factors. Thus the associated complex $\mathsf{K}((\Delta_2)^2, \pi)$ has an edge for every pair (a_i, a'_j) with $i, j \in [3]$. So if there exists a projection of the product of two triangles to the plane preserving all its vertices, then this yields an embedding of the complete bipartite graph on $3 + 3$ vertices $K_{3,3}$ into the boundary of a 3-polytope. But since $K_{3,3}$ is not planar there exists no 3-polytope with $K_{3,3}$ in its boundary. This implies that there exists no realization of $(\Delta_2)^2$ such that all vertices survive the projection to the plane.

In the above example we used the non-planarity of the graph $K_{3,3}$ as a topological obstruction to show that the projection of a product of two triangles to the plane cannot preserve all the vertices. More sophisticated obstructions are subject of the next section.

2.1.2 Embeddability dimension and Sarkaria index

In general it is hard to decide the embeddability of a simplicial complex K into some sphere $\mathbb{S}^d$. The following notions, taken and adapted from Matoušek's book [34], show that in fortunate cases we can determine non trivial lower bounds on the dimension of a sphere that the complex K can be embedded into.

Let $\mathsf{K} \subseteq 2^{[m]}$ be a (finite) simplicial complex. Simplicial complexes on m vertices have the nice property that they may always be embedded as a subcomplex of the $(m-1)$-simplex in $\mathbb{R}^{m-1}$. Topologically, this yields an embedding of the underlying topological space $\|\mathsf{K}\|$ of K into $\mathbb{R}^{m-1}$. In the following we use K for the simplicial complex and the corresponding topological space (usually denoted by $\|\mathsf{K}\|$). We will look for the smallest dimension of a sphere in which K can be embedded.

Definition 2.6 (Embeddability dimension)**.** Let $\mathsf{K} \subseteq 2^{[m]}$ be a simplicial complex. The *embeddability dimension* $\mathsf{e\text{-}dim}(\mathsf{K})$ is the smallest integer d such that K may be embedded into the d-sphere, i.e. K is homeomorphic to a closed subset the d-sphere.

It is a well-known result [25, Ex. 4.8.25, Thm. 11.1.8] that the embeddability dimension can be bounded in terms of the dimension of K.

Proposition 2.7. Let K be a simplicial complex of dimension $\mathsf{dim}\,\mathsf{K} = \ell$. Then

$$\ell \;\leq\; \mathsf{e\text{-}dim}(\mathsf{K}) \;\leq\; 2\ell+1.$$

To find lower bounds other then the dimension of the complex is not obvious. The methods derived in the book of Matoušek [34, Sect. 5] establish a combinatorial bound for the embeddability dimension of a simplicial complex. This approach is summarized in Section 2.1.3. We take a shortcut and rephrase the relevant theorem in terms of the embeddability dimension.

Before we are able to state the theorem we need some more definitions. For a simplicial complex $\mathsf{K} \subseteq 2^{[m]}$ we denote by $\mathcal{F}(\mathsf{K})$ the set of *minimal non-faces*, i.e. the inclusion-minimal sets in $2^{[m]} \setminus \mathsf{K}$. The *Kneser graph* $\mathsf{KG}(\mathcal{F})$ on a set system $\mathcal{F} \subseteq 2^{[m]}$ has the elements of $\mathcal{F}$ as vertices and two vertices $S, S' \in \mathcal{F}$ share an edge if and only if S and S' are disjoint. Furthermore, for a graph G we denote by $\chi(G)$ the *chromatic number* of G.

Definition 2.8 (Sarkaria index)**.** Let K be a simplicial complex on m vertices and $\mathcal{F} = \mathcal{F}(\mathsf{K})$ the collection of minimal non-faces. The *Sarkaria index* is

$$\mathsf{ind}_{\mathsf{SK}}\,\mathsf{K} := m - \chi(\mathsf{KG}(\mathcal{F})) - 1.$$

Using the methods developed in [34] we are able to state a very compact criterion for the embeddability dimension of a simplicial complex. (This is a reformulation of Sarkaria's Coloring/Embedding Theorem [34, Sect. 5.8] without deleted joins.)

Theorem 2.9 (Sarkaria's index theorem)**.** Let K be a simplicial complex. Then

$$\mathsf{e\text{-}dim}(\mathsf{K}) \geq \mathsf{ind}_{\mathsf{SK}}\,\mathsf{K}.$$

We use the above theorem to show the non-embeddability of a simplicial complex via the following corollary.

Corollary 2.10 (Non-embeddability and Sarkaria index)**.** Let K be a simplicial complex. If

$$\mathsf{ind}_{\mathsf{SK}}\,\mathsf{K} > d$$

then there exists no embedding of the complex K into the d-sphere.

In its original form in [34] the above theorem bounds from below the $\mathbb{Z}_2$-index of the deleted join of the complex K. The relation between the embeddability dimension of a complex and the $\mathbb{Z}_2$-index of the deleted join of the complex K is given in the next section.

Example 2.11. With the technique described in this section we are able to show that $K_{3,3}$ is not planar. The graph $K_{3,3}$ is the join $\mathsf{D}_3 * \mathsf{D}_3$ of the simplicial complex with three isolated vertices D_3. We label the vertices of the two "coasts" of $K_{3,3}$ by $1, 2, 3$ and $1', 2', 3'$ respectively. The minimal non-faces of $K_{3,3}$ are the edge 12,23,13 and $1'2'$,$2'3'$,$1'3'$. Since the edges of the one side are disjoint from the other side, the Kneser graph on the minimal non-faces is again a $K_{3,3}$ but with vertices labeled by the minimal non-faces as shown in Figure 2.1. This graph has chromatic number 2 and hence the Sarkaria index of $K_{3,3}$ is:

$$\mathsf{ind}_{\mathsf{SK}}\,K_{3,3} = 6 - 2 - 1 = 3.$$

Thus there exists no embedding of $K_{3,3}$ into a 2-sphere, i.e. the graph $K_{3,3}$ is not planar.

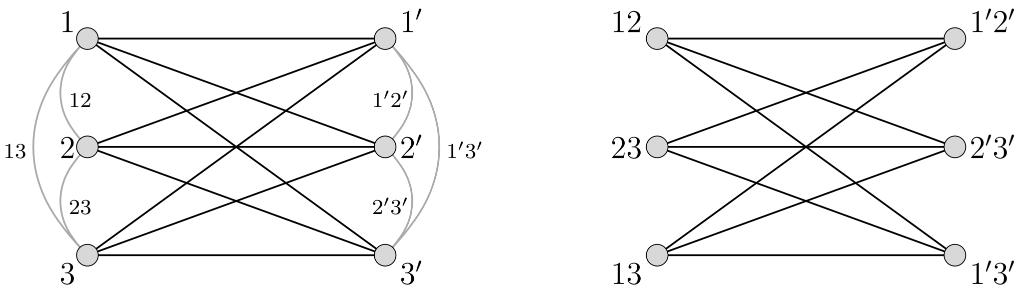


Figure 2.1: The Kneser graph (right) on the minimal non-faces of the graph $K_{3,3}$ is isomorphic to $K_{3,3}$. The light edges are the minimal non-faces of the graph (left).

2.1.3 $\mathbb{Z}_2$-maps and non-embeddability

In this section we take a look at the methods developed in Matoušek [34] to show non-embeddability for certain simplicial complexes. They will lead to a combinatorial criterion for the embeddability of a simplicial complex into a sphere called Sarkaria's Coloring/Embedding Theorem (see Theorem 2.15).

The idea is the following: Every embedding f of a simplicial complex K into a d-sphere gives rise to a map f^{*2} of another space K^{*2}_{Δ} to the d-sphere satisfying certain properties. So if we are able to show, that such a map f^{*2} cannot exist, then this contradicts the embeddability of the original complex K into the d-sphere via the map f. The tool used to show the non-existence of such a map is the $\mathbb{Z}_2$-index of K^{*2}_{Δ} described in the following.

First we need to introduce some basic notions: A pair (X, ν) of a topological space X and a homeomorphism $\nu : X \to X$ with $\nu(x) \neq x$ for all $x \in X$ and $\nu^2 = \mathsf{id}_X$ is called a *(free)* $\mathbb{Z}_2$*-space*. (Since we will only consider free $\mathbb{Z}_2$-spaces, we will omit the attribute free.) The most prominent example of a $\mathbb{Z}_2$-space is the d-sphere together with the antipodal map $x \mapsto -x$. Continuous maps $f : (X, \nu) \to (Y, \mu)$ between two $\mathbb{Z}_2$-spaces with $f \circ \nu = \mu \circ f$ are called *equivariant* $\mathbb{Z}_2$*-maps* denoted by $f : X \xrightarrow{\mathbb{Z}_2} Y$.

Definition 2.12 ($\mathbb{Z}_2$-index)**.** The $\mathbb{Z}_2$-index of a $\mathbb{Z}_2$-space (X, ν) is

$$\mathsf{ind}_{\mathbb{Z}_2} X \ := \ \min\{d \in \mathbb{N} \mid X \xrightarrow{\mathbb{Z}_2} \mathbb{S}^d\},$$

where $\mathbb{S}^d$ is the d-sphere with the usual antipodal action.

The Borsuk–Ulam Theorem tells us that there exists no $\mathbb{Z}_2$-map $\mathbb{S}^d \xrightarrow{\mathbb{Z}_2} \mathbb{S}^{d-1}$. Hence the $\mathbb{Z}_2$-index of the d-sphere is d.

We will work with combinatorial $\mathbb{Z}_2$-actions ν on simplicial complexes K, i.e. maps that act on the vertex set $[n]$ of the complexes. These actions are free if no simplex is fixed under the action, that is, $\nu(F) \neq F$ for all faces $F \in \mathsf{K}$. They easily yield $\mathbb{Z}_2$-actions on all the faces of the complex as well as on the topological space $\|\mathsf{K}\|$. The canonical triangulation of the d-sphere with a free $\mathbb{Z}_2$-action is the boundary of the cross polytope $\mathrm{cr}_{d+1} = \mathrm{conv}(\{\pm e_i \mid i \in [d+1]\})$ where e_i is the ith unit vector in $\mathbb{R}^{d+1}$. The antipodality maps the vertices of the crosspolytope onto their antipodes, i.e. $e_i \mapsto -e_i$.

Since the simplicial complexes we would like to embed do not have a natural free $\mathbb{Z}_2$-action we need the following construction.

Definition 2.13 (Deleted join)**.** Let K be a simplicial complex. The *deleted join* K^{*2}_{Δ} of K is a subcomplex of the join $\mathsf{K} * \mathsf{K}$ with the following faces:

$$\mathsf{K}^{*2}_{\Delta} \ := \ \{F_1 \uplus F_2 \mid F_1, F_2 \in \mathsf{K}, F_1 \cap F_2 = \emptyset\},$$

where $F_1 \uplus F_2 = (F_1 \times \{0\}) \cup (F_2 \times \{1\})$. The $\mathbb{Z}_2$-action on the deleted join is $F_1 \uplus F_2 \mapsto F_2 \uplus F_1$. This is a free $\mathbb{Z}_2$-action since $F_1 \cap F_2 = \emptyset$.

The $\mathbb{Z}_2$-index of the deleted join of the d-sphere is the same as the $\mathbb{Z}_2$-index of the d-sphere. We obtain our first result about non-embeddability of a simplicial complex.

Theorem 2.14 (Non-embeddability [34, Thm. 5.5.5])**.** Let K be a simplicial complex. If

$$\mathsf{ind}_{\mathbb{Z}_2} \mathsf{K}^{*2}_{\Delta} > d$$

then for every continuous mapping $f : \mathsf{K} \to \mathbb{S}^d$, the images of some two disjoint faces of K intersect.

So the task to show non-embeddability for a complex K amounts to finding lower bounds on the $\mathbb{Z}_2$-index of the deleted join K^{*2}_{Δ}. A combinatorial way to find lower bounds on the $\mathbb{Z}_2$-index is via coloring the Kneser graph of the minimal non-faces of the complex K.

Theorem 2.15 (Sarkaria's coloring/embedding theorem)**.** Let K be a simplicial complex on n vertices, $\mathsf{KG}(\mathcal{F})$ the Kneser graph on the minimal non-faces $\mathcal{F}$ of K. Then

$$\mathsf{ind}_{\mathbb{Z}_2} \mathsf{K}^{*2}_{\Delta} \geq \mathsf{ind}_{\mathsf{SK}} \mathsf{K} = n - \chi(\mathsf{KG}(\mathcal{F})) - 1.$$

Consequently, if $n - \chi(\mathsf{KG}(\mathcal{F})) - 1 > d$ then for every continuous mapping $f : \mathsf{K} \to \mathbb{S}^d$, the images of some two disjoint faces of K intersect.

Every embedding of a simplicial complex K into a d-sphere gives rise to a $\mathbb{Z}_2$-equivariant map of the *deleted join* K^{*2}_{Δ} to a d-sphere. Thus the embeddability dimension of K is at least as big as the $\mathbb{Z}_2$-index of its deleted join K^{*2}_{Δ}, which yields Theorem 2.9 from Theorem 2.15.

2.1.4 Projections of skeleta and skeleton complexes

We take a closer look at projections of polytopes retaining entire skeleta of polytopes. If a projection preserves an entire skeleton then this leads to a subcomplex of the associated subcomplex $\mathsf{K}(P,\pi)$, which is embedded in the associated polytope. So we need to show that the embeddability dimension of this subcomplex is large in order to prove the non-projectablity of entire skeleta. The subcomplex may be defined in a purely combintorial way.

Definition 2.16 (Skeleton complex)**.** Let $\mathcal{P}$ be a combinatorial d-type with m facets. For $0 \leq k \leq d$, the *kth skeleton complex* is the simplicial complex $\Sigma_k(\mathcal{P}) \subseteq 2^{[m]}$ with facets $[m] \setminus H_G$ for all k-faces $G \subseteq P$.

Since every k-face of a d-type $\mathcal{P}$ is contained in at least $d-k$ facets, the dimension of the kth skeleton complex is at most $m-d+k-1$. If $\mathcal{P}$ is simple then $\Sigma_k(\mathcal{P})$ is a pure simplicial complex of dimension $m+k-d-1$. In [43], $\Sigma_0(\mathcal{P})$ was defined in terms of the complement complex of the boundary complex of the dual to $\mathcal{P}$. Here, we abandon the restriction to simple polytopes. The connection to $\mathsf{K}(P,\pi)$ is given by the following observation.

Observation. If $\pi : P \to \pi(P)$ is a projection retaining the k-skeleton then

$$\Sigma_0(P) \subset \Sigma_1(P) \subset \cdots \subset \Sigma_k(P) \subset \mathsf{K}(P,\pi)$$

is an increasing sequence of subcomplexes.

Putting it all together we obtain a criterion for the non-projectability of the k-skeleton of a combinatorial type of polytope.

Corollary 2.17. Let $\mathcal{P}$ be a d-type with m facets and for $0 \leq k < d$ let $\Sigma_k(\mathcal{P})$ be the kth skeleton complex of $\mathcal{P}$. If

$$e \ < \ \mathsf{e\text{-}dim}(\Sigma_k(\mathcal{P})) + d - m + 2$$

then there is no realization of $\mathcal{P}$ such that a projection to $\mathbb{R}^e$ retains the k-skeleton.

Proof. We prove the result by contradiction. Assume that P is a realization of $\mathcal{P}$ and $\pi : P \to \pi(P)$ is a projection retaining the k-skeleton with

$$\begin{aligned} & e < \mathsf{e\text{-}dim}(\Sigma_k(\mathcal{P})) + d - m + 2 \\ \iff \quad & m + e - d - 2 < \mathsf{e\text{-}dim}(\Sigma_k(\mathcal{P})). \end{aligned}$$

Since $\Sigma_k(\mathcal{P})$ is a subcomplex of $\mathsf{K}(P, \pi)$ it is a subcomplex of the boundary of the associated polytope $\mathcal{A}(P, \pi)$ by Theorem 2.4. Hence the embeddability dimension of $\Sigma_k(\mathcal{P})$ is:

$$\mathsf{e\text{-}dim}(\Sigma_k(\mathcal{P})) \leq m + e - d - 2 < \mathsf{e\text{-}dim}(\Sigma_k(\mathcal{P}))$$

which is a contradiction. □

Theorem 2.9 gives a combintorial lower bound on the embeddability dimension via the Sarkaria index. Hence we are able to state a purely combinatorial criterion for the projectability of polytope skeleta by rephrasing the above corollary

Corollary 2.18. Let $\mathcal{P}$ be a d-type with m facets and for $0 \leq k < d$ let $\Sigma_k(\mathcal{P})$ be the kth skeleton complex of $\mathcal{P}$. If

$$e \ < \ \mathsf{ind}_{\mathsf{SK}}\, \Sigma_k(\mathcal{P}) + d - m + 2$$

then there is no realization of $\mathcal{P}$ such that a projection to $\mathbb{R}^e$ retains the k-skeleton. □

As a plausibility check, consider the statement of Corollary 2.17 with the bounds given in Proposition 2.7. If $\mathsf{e\text{-}dim}(\Sigma_k)$ attains the lower bound in Proposition 2.7 then Corollary 2.17 implies that the dimension of the target space has to be at least $e \geq k + 1$. This sounds reasonable as the projection embeds the k-skeleton into a sphere of dimension $e - 1$. If $\mathsf{e\text{-}dim}(\Sigma_k(\mathcal{P}))$ attains the upper bound then the k-skeleton is not projectable to e-space if $e < m - d + 2k + 1$. This implies the polyhedral counterpart of the classical Van Kampen–Flores result:

Theorem 2.19. Let $\mathcal{P}$ be a d-type and let $0 \leq k \leq \lfloor \frac{d-2}{2} \rfloor$. If

$$e \leq 2k + 1$$

then there is no realization of $\mathcal{P}$ such that a projection to e-space retains the k-skeleton.

Proof. By a result of Grünbaum [24] the boundary complex of a d-polytope is a refinement of the boundary complex of a d-simplex Δ_d. This implies that the k-skeleton of $\mathcal{P}$ contains a refinement of the k-skeleton of Δ_d. The Van Kampen–Flores theorem (see [34]) states that for $k \leq \lfloor \frac{d-2}{2} \rfloor$ the k-skeleton of a d-simplex is not homeomorphic to a subset of a $2k$-sphere. □

The embeddability dimensions of the skeleton complexes depend heavily on the combintorial structure of the polytope. But for certain dimensions of skeleta we can determine the embeddability dimension and the Sarkaria index of the skeleton complexes exactly.

Proposition 2.20. Let $\mathcal{P}$ be a d-type on m facets. Then $\Sigma_d(\mathcal{P}) = \Delta_{m-1}$ and $\Sigma_{d-1}(\mathcal{P}) = \partial\Delta_{m-1}$. In particular,

- $\mathsf{e\text{-}dim}(\Sigma_d(\mathcal{P})) = \mathsf{ind}_{\mathsf{SK}}\, \Sigma_d(\mathcal{P}) = m - 1$ and
- $\mathsf{e\text{-}dim}(\Sigma_{d-1}(\mathcal{P})) = \mathsf{ind}_{\mathsf{SK}}\, \Sigma_{d-1}(\mathcal{P}) = m - 2$.

Proof. The first claim follows from the definition of the skeleton complex. Thus the embeddability dimensions are $m - 1$ and $m - 2$, respectively. For the Sarkaria index we get in the former case that the Kneser graph of the minimal nonfaces of $\Sigma_d(\mathcal{P})$ has no vertices, whereas in the latter case the graph has no edges. □

2.2 Realization spaces of projected polytopes
joint with Günter M. Ziegler

The usual approach to parametrize the realization space of a simple polytope via its facet normals is not very useful when we try to understand the space of realizations of its projections. We define a certain subset of the vertices that will yield parameters for the realization space of the high-dimensional polytope as well as for the realization space of its projection. These subsets are the subject of the next section.

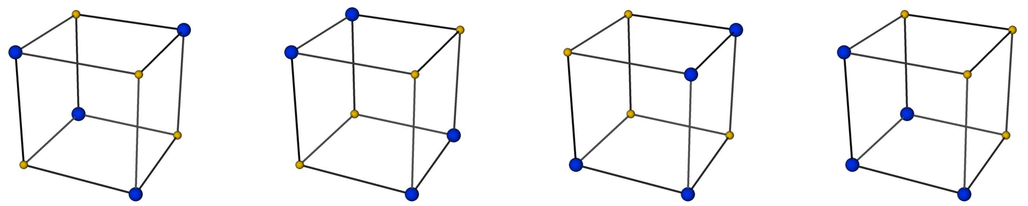
2.2.1 Affine support sets

We will consider the following kind of subsets of the vertices of a combintorial type of polytope.

Definition 2.21 (Affine support set)**.** A subset $\mathsf{A} \subseteq V$ of the vertices V of a simple d-type $\mathcal{P}$ is an *affine support set* if for every realization P and every facet F of $\mathcal{P}$ the vertices $\mathsf{A} \cap F$ are affinely independent.

An affine basis of d-space has $d + 1$ points. Hence there exists an affinely independent subset of the vertices of cardinality $d+1$ for each realization P of a combinatorial d-type. But a set of vertices may be affinely independent for one realization, but affinely dependent for another, as shown in the following example.

Example 2.22. The facets of the 4-cube are 3-cubes and every realization of the 3-cube may occur as a facet of a 4-cube. So let us have a look at different subsets of the vertices of the 3-cube, some of which are affinely independent and others that are not affinely independent for all possible realizations of the 3-cube. The subsets that are affinely independent for all realizations are candidates for affine support sets of the 4-cubes. We label the vertices of the 3-cube by $\{\pm\}$-vectors of length 3. See Section 2.2.3.



A. B. C. D.

A. The first set of vertices consists of the three neighbors of the $(---)$-vertex and the vertex $(+++)$. These vertices are affinely independent for all realizations of the cube: The neighbors of the vertex $(---)$ are affinely independent and form a triangle. The three halfspaces defining the facets adjacent to the vertex $(---)$ form a cone that contains the vertex $(+++)$ in its interior. So if the neighbors of $(---)$ and the vertex $(+++)$ were affinely dependent then the vertex $(+++)$ would lie in the simplex spanned by the neighbors of $(---)$, which cannot happen in any realization of a cube.

B. The second set of vertices is affinely independent for a "generic" realization of the cube, but for the regular cube, the chosen four points are affinely dependent and lie on a planar quadrilateral. A perturbation of the facets of the regular cube breaks the affine dependence of the chosen vertices but does not change the combinatorial type of the polytope.

C. The third set of vertices is also affinely independent for all realizations by Lemma 2.23 since there exists a flag of faces, such that each k-face contains exactly $k + 1$ of the vertices: The vertex subset consists of the vertices $(---)$, $(--+)$, $(-++)$, and $(+++)$. So a suitable flag which assures the affine independence is

$$(---) \subset (--\emptyset) \subset (-\emptyset\emptyset) \subset (\emptyset\emptyset\emptyset).$$

D. The fourth set consists of a vertex and its three neighbors. These points are also affinely independent in every realization of the cube.

As the example shows there are combinatorial criteria for a subset of vertices to be affinely independent in every realization of a polytope. The following lemma captures the combinatorial criterion used in Example 2.22/C.

Lemma 2.23. Let $\{G_k\}_{k=0}^{d-1}$ be a flag of the simple d-type $\mathcal{P}$ and let $\{v_k\}_{k=0}^{d}$ be a set of vertices such that $v_0 = G_0$, $v_d \in \mathcal{P} \setminus G_{d-1}$ and $v_k \in G_k \setminus G_{k-1}$ for $k = 1, \dots, d-1$. Then $\{v_k\}_{k=0}^{d}$ is affinely independent for every realization of $\mathcal{P}$. □

Since every (simple) polytope has a flag we obtain the following immediate corollary about the existence of an affine support set for a simple polytope.

Corollary 2.24. For every simple d-type $\mathcal{P}$ there exists an affine support set of cardinality at least $d + 1$. □

The size of an affine support set is bounded from above by the following lemma.

Lemma 2.25. The cardinality of an affine support set A of a d-type $\mathcal{P}$ is bounded from above by the number of facets of $\mathcal{P}$, that is, $|\mathsf{A}| \leq f_{d-1}(\mathcal{P})$.

Proof. Since the dimension of a facet F of a d-polytope is $d-1$, the maximal number of affinely independent vertices in $\mathsf{A} \cap F$ is d. But each vertex of A is contained in at least d facets, so $d \cdot |\mathsf{A}| \leq d \cdot f_{d-1}(P)$. □

In some fortunate cases we find an affine support set of maximal size, but in other cases, the upper bound becomes arbitrarily bad as shown in the following example.

Example 2.26. Let us try to construct affine support sets for 3-dimensional prisms Π_p over p-gons. The number of facets of the prism is $p + 2$. For the triangular prism Π_3 (with 5 facets) there exist only affine support sets of size at most 4, since every set of 5 points contains an entire quadrilateral and is hence affinely dependent. For the cube Π_4 (the prism over a quadrilateral) there exist affine support sets of cardinality 6 which are generalized in Section 2.2.3 to higher dimensional cubes. For $p \geq 5$ the maximal cardinality of an affine support set for the prism Π_p is 6 because both bottom and top polygon may only contain at most 3 vertices. So the gap between the maximal size of an affine support set and the upper bound of Lemma 2.25 becomes arbitrarily bad.

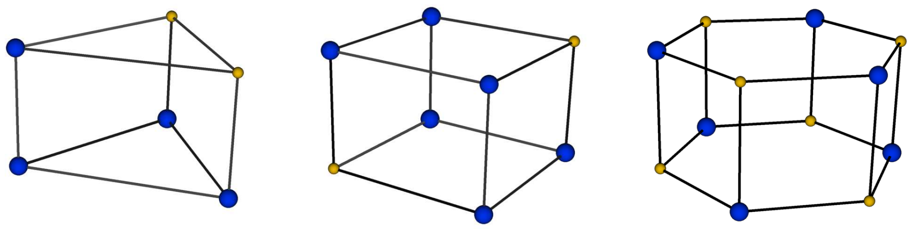


Figure 2.2: Maximal affine support sets for prisms over polygons.

We recall that the vertices of an affine support set must only be affinely independent restricted to every facet and not in the entire polytope. (Requiring the affine independence of the affine support set in the entire polytope would upper bound the size of the set to the dimension of the polytope plus 1.)

2.2.2 Moduli of projected polytopes

In this section we use the affine support sets of the previous section to find a lower bound on the number of moduli of projected polytopes and their subcomplexes. The moduli of simple polytopes are easily obtained from their facets. But how many of these moduli are preserved under projection is not clear. With the affine support sets introduced in the previous section the modifications of the vertices not in direction of the projection are preserved. So the parametrization of parts of the realization space via an affine support set provides moduli for the projected polytope as well.

The following lemma shows that small modifications of the vertices in an affine support set of a simple polytope may be completed to a realization of the polytope.

Lemma 2.27 (Realizations via affine support sets)**.** Let $\mathcal{P}$ be a simple d-type and A an affine support set. Then every small modification of the coordinates of the vertices contained in A of an arbitrary realization P yields a realization $\tilde{P}$ of the same d-type $\mathcal{P}$.

Proof. Let $F_1, \dots, F_m$ be an arbitrary ordering of the facets. For $i = 0, \dots, m$ let $\mathsf{A}_i := \mathsf{A} \cap (\bigcup_{j=1}^{i} F_i)$ be the subset of the vertices contained in the first i facets and let $\tilde{V}_i$ be the vertices with modified coordinates in A_i. We start with an arbitrary realization $P = \tilde{P}_0$ of $\mathcal{P}$. Then for $i = 1, \dots, m$ we construct a polytope $\tilde{P}_i$ from $\tilde{P}_{i-1}$ such that all the modified vertices $\tilde{V}_i$ are vertices of $\tilde{P}_i$. In the last step we obtain a realization $\tilde{P} = \tilde{P}_m$ containing all modified vertices.

For $i = 0$ the set A_0 is empty and $\tilde{P}_0$ obviously contains all modified vertices of A_0. So assume that all the modified vertices $\tilde{V}_{i-1}$ are vertices of $\tilde{P}_{i-1}$. Then we construct $\tilde{P}_i$ in the following way depending on the size of $F_i \cap \mathsf{A}$:

- If the facet F_i of $\tilde{P}_{i-1}$ contains d vertices of the affine support set A then the hyperplane supporting the new facet $\tilde{F}_i$ is uniquely determined by these vertices.
- If a facet F_i contains $k < d$ vertices of A then we extend the set $\mathsf{A} \cap F_i$ to an affine basis of the facet by vertices of $\tilde{P}_{i-1}$ in $F_i \setminus \mathsf{A}$. This affine basis defines the new facet $\tilde{F}_i$.

In both cases we replace the facet F_i with the new (perturbed) facet $\tilde{F}_i$ to obtain $\tilde{P}_i$ from $\tilde{P}_{i-1}$. Since $\mathcal{P}$ is simple this procedure yields a sequence of realizations $\tilde{P}_0, \dots, \tilde{P}_m$ of the same combinatorial type $\mathcal{P}$. Finally, $\tilde{P} = \tilde{P}_m$ is a realization of $\mathcal{P}$ that contains all the modified vertices of the affine support set. □

The realizations of the surfaces in Chapters 5 and 4 are obtained from projections of high dimensional simple polytopes to $\mathbb{R}^4$ resp. $\mathbb{R}^3$. The following theorem allows us to establish a lower bound for the dimension of the realization space of a generic projection of a simple polytope.

Theorem 2.28 (Moduli of projected polytopes)**.** Let P be a realization of the simple d-type $\mathcal{P}$, A an affine support set and $\pi : \mathbb{R}^d \to \mathbb{R}^e$ a generic projection. Then the dimension of the realization space of the projected polytope $\pi(P)$ is at least $e \cdot |\mathsf{A}|$.

Proof. Let A_π denote the vertices of the projected polytope $\pi(P)$ corresponding to the affine support set A. If we perturb the vertices of A_π in $\mathbb{R}^e$ then this induces small modifications of the vertices of A in the realization $P \subset \mathbb{R}^d$. By Lemma 2.27 small modifications of the vertices in A yield a new realization $\tilde{P}$ of the same combinatorial type $\mathcal{P}$. Since the projection is generic, the projected polytopes $\pi(P)$ and $\pi(\tilde{P})$ have the same combinatorial type.

Hence the e degrees of freedom at each vertex of A_π in $\mathbb{R}^e$ imply the lower bound of $e \cdot |\mathsf{A}|$ on the moduli of the projected polytope. $\square$

It is very important that we are working with generic projections of simple polytopes since generic projections are "stable under perturbation" and vertices of simple polytopes are the intersection of exactly d facets.

Remark 2.29 (Moduli of simplicial polytopes)**.** The realization space of a simplicial polytope is easily parametrized via its vertices, that is, a d-dimensional simplicial polytope P on n vertices has dn moduli. This may also obtained from Theorem 2.28: A simplicial polytope on n vertices is the generic projection of an $(n-1)$-simplex. An affine support set of the $(n-1)$-simplex may obviously contain all n vertices, so by Theorem 2.28 the projected simplicial polytope P has the required dn moduli.

Another advantage of the parametrization of the realization space of a high dimensional polytope via affine support sets is that it can easily be restricted to subcomplexes.

Theorem 2.30 (Moduli of projected subcomplexes)**.** Let $\mathcal{S}$ be a subcomplex of a simple d-type $\mathcal{P}$, A an affine support set of $\mathcal{P}$ and $\pi : \mathbb{R}^d \to \mathbb{R}^e$ a generic projection preserving the subcomplex $\mathcal{S}$. Then the realization space of the projected subcomplex $\pi(\mathcal{S})$ has dimension at least $e \cdot |\mathsf{A} \cap \mathcal{S}|$. $\square$

This result is the key to show the existence of non-trivial moduli for realizations of surfaces that have neither triangle faces, nor vertices of degree three. The naïve estimate of Section 1.3.1 will suggest that those surfaces will have a decreasing number of moduli if the vertex degree and polygon size increase. Finally we would not expect to have any non-trivial moduli for large vertex degree and large polygon size. But our approach will show that the realizations of the surfaces obtained in Sections 2.2.3, Section 4.3.1, and Section 5.2 have and increasing number of moduli even though the vertex degrees and the polygon sizes increase.

2.2.3 Affine support sets of cubes and Ringel surfaces

In this section we construct two different affine support sets for the d-dimensional cube. These sets are used to derive lower bounds on the number of moduli of the surfaces studied by Ringel [41]. These surfaces are contained in the 2-skeleton of the neighborly cubical polytopes of Joswig & Ziegler [29] as shown in [28]. Since for even dimensions the cube is just the product of quadrilaterals, the surfaces contained in the even dimensional cubes are special cases of the surfaces constructed in Section 5.1.2.

The $2d$ facets of the d-dimensional cube consists of d pairs of opposite facets denoted by $F_i^{\pm}$ for $i \in [d]$. Since the opposing facets F_i^+ and F_i^- do not intersect, we may identify each non-empty face G of the d-cube with a $\{+,-,\emptyset\}$-vector H_G of length d in the following way:

$$(H_G)_i \;=\; \begin{cases} + & \text{if } G \subset F_i^+, \\ - & \text{if } G \subset F_i^-, \\ \emptyset & \text{otherwise.} \end{cases}$$

The dimension of the face G is exactly the number of $\emptyset$-entries in the corresponding vector H_G. The vertices of the cube are identified with $\{+,-\}$-vectors without $\emptyset$-entries.

We consider two different subsets of the vertices of the cube which will be proved to be affine support sets for the d-cube for $d \geq 3$. The first set consists of the union of the neighbors of the vertex $(-\cdots-)$ with the neighbors of the vertex $(+\cdots+)$:

$$\mathsf{A}_{\mathsf{neigh}}(d) \;:=\; \left\{ v \in \{\pm\}^d \;\middle|\; \#\{i : v_i = +\} = 1 \text{ or } d-1 \right\} \tag{2.1}$$

The second set zigzags through the cube and has the following combinatorial description:

$$\mathsf{A}_{\mathsf{zigzag}}(d) := \left\{ v \in \{\pm\}^d \;\middle|\; v = \pm(\underbrace{+\ldots+}_{k \text{ times}}\;\underbrace{-\ldots-}_{d-k \text{ times}}) \text{ for } k \in [d] \right\} \tag{2.2}$$

Example 2.31. The subsets $\mathsf{A}_{\mathsf{neigh}}(3)$ and $\mathsf{A}_{\mathsf{zigzag}}(3)$ of the vertices of the 3-cube are:

$$\mathsf{A}_{\mathsf{neigh}}(3) = \left\{ \begin{array}{c} (+--),(-+-),(--+), \\ (-++),(+-+),(++-) \end{array} \right\},$$

$$\mathsf{A}_{\mathsf{zigzag}}(3) = \left\{ \begin{array}{c} (---),(--+),(-++), \\ (+++),(++-),(+--) \end{array} \right\}.$$

These two sets are obviously affine support sets, since every facet, i.e. quadrilateral face, of the 3-cube contains exactly 3 selected vertices. But they are equivalent by flipping the sign in the middle which is a combinatorial symmetry of the cube. This is illustrated in Figure 2.3.

In the 4-dimensional cube the two sets are no longer equivalent since the vertices of $\mathsf{A}_{\mathsf{zigzag}}(4)$ form a cycle of length 8, whereas the vertices of $\mathsf{A}_{\mathsf{neigh}}(4)$ do not share any edge (see Figure 2.4).

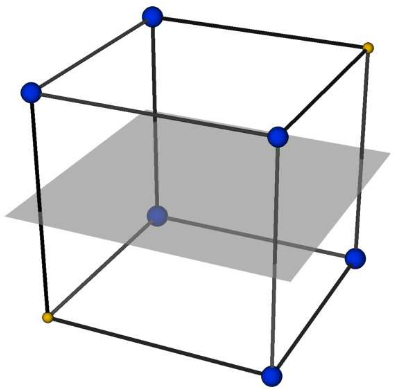


Figure 2.3: The two sets $\mathsf{A}_{\mathsf{neigh}}(3)$ (left) and $\mathsf{A}_{\mathsf{zigzag}}(3)$ (right) are isomorphic by flipping the vertical coordinate, that is, the second sign in the vector corresponding to a vertex.

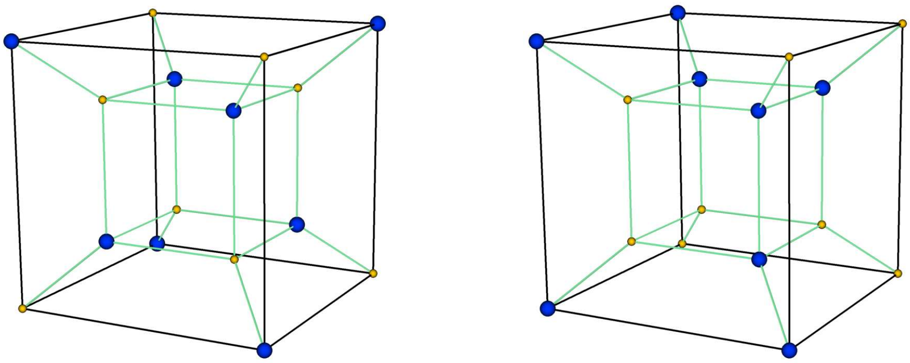


Figure 2.4: The subsets $\mathsf{A}_{\mathsf{neigh}}(4)$ (left) and $\mathsf{A}_{\mathsf{zigzag}}(4)$ (right) of the vertices of the 4-cube are two different affine support sets.

For the vector descriptions this implies that there exist no two vertices in $\mathsf{A}_{\mathsf{neigh}}(4)$ that differ at exactly one position:

$$\mathsf{A}_{\mathsf{neigh}}(4) = \left\{ \begin{array}{c} (+---),(-+--),(--+-),(---+) \\ (-+++),(+-++),(++-+),(+++-) \end{array} \right\},$$

$$\mathsf{A}_{\mathsf{zigzag}}(4) = \left\{ \begin{array}{c} (----),(---+),(--++),(-+++) \\ (++++),(+++-),(++--),(+---) \end{array} \right\}.$$

Both sets are affine support sets of the 4-cube. The facet $F_0^- = (-\emptyset\emptyset\emptyset)$ contains the following subsets of the vertices:

$$\begin{aligned}\mathsf{A}_{\mathsf{neigh}}(4) \cap F_0^- &= \{(-+--), (--+-), (---+), (-+++)\},\\ \mathsf{A}_{\mathsf{zigzag}}(4) \cap F_0^- &= \{(----), (---+), (--++), (-+++)\}.\end{aligned}$$

These are exactly the subsets of the vertices considered in Example 2.22/A ($\mathsf{A}_{\mathsf{neigh}}(4) \cap F_0^-$) and Example 2.22/C ($\mathsf{A}_{\mathsf{zigzag}}(4) \cap F_0^-$) and hence affinely independent for every realization of the 4-cube. The same is true for all the other facets by symmetry.

Theorem 2.32 (Maximal affine support sets for cubes)**.** The sets $\mathsf{A}_{\mathsf{neigh}}(d)$ and $\mathsf{A}_{\mathsf{zigzag}}(d)$ of vertices of the d-cube defined in Equation (2.1) and (2.2) are affine support sets of size $2d$.

Proof. The two subsets $\mathsf{A}_{\mathsf{neigh}}(d)$ and $\mathsf{A}_{\mathsf{zigzag}}(d)$ are invariant under the following automorphisms:

- flipping all the sign entries,

$$(\sigma_0, \dots, \sigma_{d-1}) \mapsto -(\sigma_0, \dots, \sigma_{d-1})$$

- and cyclic rotation of the vectors with a flip in the zigzag case,

$$(\sigma_0, \dots, \sigma_{d-1}) \mapsto \begin{cases} (\sigma_1, \dots, \sigma_{d-1}, \sigma_0) & \text{for } \mathsf{A}_{\mathsf{neigh}}(d) \\ (\sigma_1, \dots, \sigma_{d-1}, -\sigma_0) & \text{for } \mathsf{A}_{\mathsf{zigzag}}(d). \end{cases}$$

A suitable sequence of these automorphisms maps an arbitrary facet to the facet $F_0^- = (-\emptyset \dots \emptyset)$ and leaves the chosen vertex subsets invariant. So we only need to verify that the sets $\mathsf{A}_{\mathsf{neigh}}(d) \cap F_0^-$ and $\mathsf{A}_{\mathsf{zigzag}}(d) \cap F_0^-$ are affinely independent for every realization of the cube.

$\boldsymbol{\mathsf{A}_{\mathsf{neigh}}(d)}$: The vertices of $\mathsf{A}_{\mathsf{neigh}}(d) \cap F_0^-$ are the $d-1$ neighbors of the vertex $(-- \dots -)$ in F_0^- and the vertex $(-+ \dots +)$. The neighbors of $(-- \dots -)$ span a $(d-2)$-simplex. The line segment connecting the vertex $(-- \dots -)$ with the vertex $(-+ \dots +)$ intersects this $(d-2)$-simplex in its relative interior. So if $(-+ \dots +)$ would lie in the affine hull of the simplex then it would be in its relative interior which is impossible. Hence $\mathsf{A}_{\mathsf{neigh}}(d) \cap F_0^-$ is affinely independent for every realization of the cube. By combinatorial symmetry the same holds for all other facets.

$\mathsf{A}_{\mathsf{zigzag}}(d)$: Consider the flag $\{F_k\}_{k=0}^{d-1}$ with $F_{d-1} = F_0^-$ and

$$F_k = (\underbrace{- \dots -}_{d-k \text{ times}} \underbrace{\emptyset \dots \emptyset}_{k \text{ times}}).$$

Then $F_0 = (- \cdots -)$ is a vertex and for $k = 1, \dots, d-1$ the following intersection of subsequent elements of the flag with the set $\mathsf{A}_{\mathsf{zigzag}}(d)$ contains exactly one vertex:

$$\mathsf{A}_{\mathsf{zigzag}}(d) \cap (F_k \setminus F_{k-1}) \;=\; \{(\underbrace{- \cdots -}_{d-k \text{ times}} \underbrace{+ \cdots +}_{k \text{ times}})\}.$$

It follows from Lemma 2.23 that $\mathsf{A}_{\mathsf{zigzag}}(d) \cap F_0^-$ is affinely independent for every realization of the cube. □

The surfaces studied by Ringel [41] consist of a subset of the 2-faces of the d-cube given by the following vectors:

$$\mathcal{R}_d \;:=\; \left\{ \begin{array}{cccccccc} \emptyset & \emptyset & \pm & \pm & \dots & \pm & \pm & \pm \\ \pm & \emptyset & \emptyset & \pm & \dots & \pm & \pm & \pm \\ . & . & . & . & \cdots & . & . & . \\ . & . & . & . & \cdots & . & . & . \\ \pm & \pm & \pm & \pm & \dots & \pm & \emptyset & \emptyset \\ \emptyset & \pm & \pm & \pm & \dots & \pm & \pm & \emptyset \end{array} \right.$$

Each of the rows corresponds to a family of quadrilaterals containing 2^{d-2} faces. Further the surface contains all vertices and edges of the cube. So the f-vector of the surface $\mathcal{R}_d$ is $2^{d-2}(4, 2d, d)$. The surfaces for $d = 4$ and 5 are shown in Figure 2.5. The realizations of these surfaces obtained via projections of d-cubes inherit the moduli of the affine support sets of the cubes via Theorem 2.30 since they contain all the vertices of the cube.

Theorem 2.33 (Moduli of Ringel surfaces)**.** The realizations of the Ringel surfaces $\mathcal{R}_d$ obtained via projections of the deformed d-cubes to $\mathbb{R}^3$ have at least $6d$ moduli. □

For $d = 3$ the surface is just the boundary of the 3-cube. It has 18 moduli corresponding to the degrees of freedom of the facets or the affine support set of size 6 shown in Figure 2.3. Starting from $d = 4$ the surface does not have any degree 3 vertices any more. So the perturbation of a single normal of a quadrilateral will immediately destroy the combinatorial structure. A small modifications of a single vertex will not yield any moduli either, since all the faces of the surfaces are quadrilaterals. This is captured in the rule of

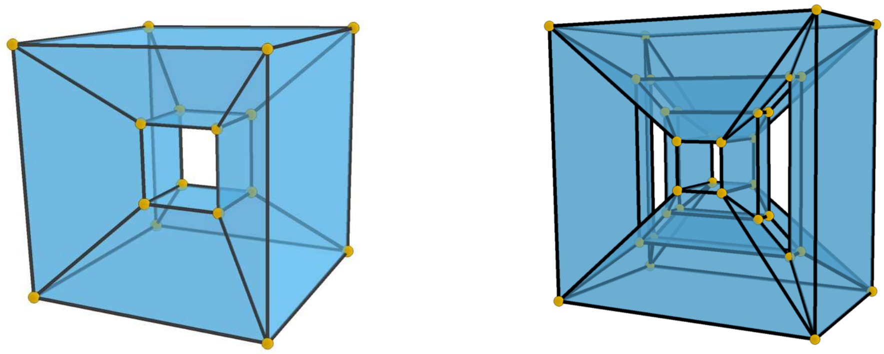


Figure 2.5: The Ringel surfaces for $d = 4$ and 5. The surfaces are embedded in the Schlegel diagram of a 4-cube and in the neighborly cubical polytope of dimension 4 with the graph of a 5-cube.

thumb explained in Section 1.3.1. For realizations $S_{\mathcal{R}_d}$ of the Ringel surfaces in $\mathbb{R}^3$, Equation (1.1) yields the following estimate for the moduli:

$$\mathfrak{M}(S_{\mathcal{R}_d}, 3) \;\sim\; 2^{d-2}(12 - 4d + 3d) = 2^{d-2}(12 - d).$$

Hence the number of $6d$ moduli achieved by Theorem 2.33 is quite astonishing, since the naïve count would yield no non-trivial moduli at all for large d.

Chapter 3

Non-projectability of polytope skeleta

joint with Raman Sanyal

In this chapter we prove some non-projectibility results for products of polytopes. We first adapt the techniques presented in Section 2.1 to the special case of products. Then in Section 3.2 we show that certain skeleta of products of odd and even polygons cannot be preserved by a projection. This complements the results of Ziegler [48] and Sanyal & Ziegler [44] about projections of products of *even* polygons. In Section 3.3 we end this chapter with a result about projections of products of simplices. The non-projectability of skeleta of these products will be used in Chapter 4 to show that certain families of polyhedral surfaces contained in the wedge product cannot be realized in $\mathbb{R}^3$ via projection.

3.1 Embeddability dimension of skeleton complexes of products

In this section we have a closer look at the combinatorial structure of products and derive bounds on the Sarkaria index using a knapsack type integer program.

The faces of the product of polytopes are products of the faces of its factors and the dimensions of the product faces are the sums of the dimensions of its constituents. The following definition distinguishes the faces of the product by their "type".

Definition 3.1 (Face type/face complex)**.** For $i = 1, \dots, r$ let $\mathcal{P}_i$ be combinatorial d_i-types on m_i facets and let $\mathcal{P} = \mathcal{P}_1 \times \mathcal{P}_2 \times \cdots \times \mathcal{P}_r$ be their product of dimension $d = d_1 + \cdots + d_r$. For a fixed $0 \le k < d$ the *face type* $\Lambda_k(\mathcal{P})$ of dimension k is the set of the following vectors:

$$\Lambda_k(\mathcal{P}) := \{\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \dots, \lambda_r) \in \mathbb{Z}^n : |\boldsymbol{\lambda}| = k,\ 0 \le \lambda_i \le d_i \text{ for all } i \in [r]\}.$$

Further for $\boldsymbol{\lambda} \in \Lambda_k(\mathcal{P})$ the *face complex* $\Sigma_{\boldsymbol{\lambda}}(\mathcal{P})$ of type $\boldsymbol{\lambda}$ is

$$\Sigma_{\boldsymbol{\lambda}}(\mathcal{P}) := \Sigma_{\lambda_1}(\mathcal{P}_1) * \Sigma_{\lambda_2}(\mathcal{P}_2) * \cdots * \Sigma_{\lambda_r}(\mathcal{P}_r).$$

It is clear from the definition of the product that every face of $\mathcal{P}$ belongs to some face type and the next observation states that this partition yields a cover of the skeleton complex.

Proposition 3.2. Let $\mathcal{P} = \mathcal{P}_1 \times \mathcal{P}_2 \times \cdots \times \mathcal{P}_r$ and $0 \leq k < \mathsf{dim}\,\mathcal{P}$. Then

$$\Sigma_k(\mathcal{P}) = \bigcup_{\boldsymbol{\lambda} \in \Lambda_k(\mathcal{P})} \Sigma_{\boldsymbol{\lambda}}(\mathcal{P}).$$

□

Obviously, the embeddability dimension of a complex is bounded from below by the embeddability dimensions of arbitrary subcomplexes. This monotonicity of the embeddability dimension yields our first bound on the embeddability dimension of skeleton complexes of products, since for every face type $\boldsymbol{\lambda} \in \Lambda_k(\mathcal{P})$ of a product $\mathcal{P}$ we have

$$\Sigma_{\boldsymbol{\lambda}}(\mathcal{P}) \subset \Sigma_k(\mathcal{P}) \qquad \Longrightarrow \qquad \mathsf{e\text{-}dim}(\Sigma_{\boldsymbol{\lambda}}(\mathcal{P})) \leq \mathsf{e\text{-}dim}(\Sigma_k(\mathcal{P})).$$

This observation yields another simple corollary of Theorem 2.4. It may also be derived from Corollary 2.17.

Corollary 3.3. Let $\mathcal{P}$ be a product and $0 \leq k < \mathsf{dim}\,\mathcal{P}$. If there is a face type $\boldsymbol{\lambda} \in \Lambda_k(\mathcal{P})$ such that

$$e < \mathsf{e\text{-}dim}(\Sigma_{\boldsymbol{\lambda}}(\mathcal{P})) + d - m + 2$$

then there is no realization of $\mathcal{P}$ such that a projection to $\mathbb{R}^e$ retains the k-skeleton. □

Remark 3.4. The definition of the face complex relies on properties of the product that are shared by other polytope constructions such as joins and direct sums. Therefore, the methods developed in this section can be suitably adapted. For lack of interesting applications we refrain from developing the methods in full generality.

The Sarkaria index will help us determine bounds on the embeddability dimension of the skeleton complexes of products. We use the following observation to simplify the calculation.

Proposition 3.5 (Sanyal [43]). Let K and L be simplicial complexes. Then

$$\mathsf{ind}_{\mathsf{SK}}\,\mathsf{K} * \mathsf{L} = \mathsf{ind}_{\mathsf{SK}}\,\mathsf{K} + \mathsf{ind}_{\mathsf{SK}}\,\mathsf{L} + 1.$$

Thus the Sarkaria index of a given face type $\boldsymbol{\lambda} \in \Lambda_k(\mathcal{P})$ of a product $\mathcal{P}$ is readily calculated from the Sarkaria indices of the factors.

Corollary 3.6. Let $\mathcal{P} = \mathcal{P}_1 \times \mathcal{P}_2 \times \cdots \times \mathcal{P}_r$ and let $\boldsymbol{\lambda} \in \Lambda_k(\mathcal{P})$. Then

$$\mathsf{ind}_{\mathsf{SK}}\, \Sigma_{\boldsymbol{\lambda}}(\mathcal{P}) = \sum_{i=1}^{r} \mathsf{ind}_{\mathsf{SK}}\, \Sigma_{\lambda_i}(\mathcal{P}_i) + r - 1.$$

□

In the special case that we have an r-fold product $\mathcal{P}^r$ of a combinatorial type $\mathcal{P}$, bounds on the embeddability dimension of $\Sigma_k(\mathcal{P}^r)$ can be obtained by solving a *knapsack-type* problem.

Proposition 3.7. Let $\mathcal{P}$ be a d-type and let $r \geq 1$ and $0 \leq k \leq rd - 1$. For $i = 0, \ldots, d$ set $s_i = \mathsf{ind}_{\mathsf{SK}}\, \Sigma_i(\mathcal{P})$ and let s^* be the optimal value of the following integer program

$$\begin{array}{rrcrcccrcl}
\max & s_0\,\mu_0 & + & s_1\,\mu_1 & + & \cdots & + & s_d\,\mu_d & & \\
\mathsf{s.t.} & 0\,\mu_0 & + & 1\,\mu_1 & + & \cdots & + & d\,\mu_d & = & k \\
 & \mu_0 & + & \mu_1 & + & \cdots & + & \mu_d & = & r \\
 & & & & & & & \mu_i & \geq & 0
\end{array}$$

with $\mu_0, \ldots, \mu_d \in \mathbb{Z}$. Then $\mathsf{e\text{-}dim}(\Sigma_k(\mathcal{P}^r)) \geq s^* + r - 1$.

Proof. Let $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_r) \in \Lambda_k(\mathcal{P}^r)$ be a face type with $|\boldsymbol{\lambda}| = k$. For $i = 0, \ldots, d$ we associate non-negative numbers μ_i to the face type $\boldsymbol{\lambda}$ with

$$\mu_i \;=\; \#\{j \in [r] : \lambda_j = i\}.$$

So μ_i counts the number of factors of dimension i composing a k-face of face type $\boldsymbol{\lambda}$ of the product $\mathcal{P}^r$. Hence the μ_i satisfy

$$\begin{array}{rcrcccrcl}
0\,\mu_0 & + & 1\,\mu_1 & + & \cdots & + & d\,\mu_d & = & k \\
\mu_0 & + & \mu_1 & + & \cdots & + & \mu_d & = & r \\
 & & & & & & \mu_i & \geq & 0\,.
\end{array}$$

Conversely, every such non-negative collection of numbers μ_i that satisfies the conditions of the integer program gives rise to a valid face type. The Sarkaria index of the face complex $\Sigma_{\boldsymbol{\lambda}}(\mathcal{P}^r)$ is obtained from Corollary 3.6

$$\mathsf{ind}_{\mathsf{SK}}\, \Sigma_{\boldsymbol{\lambda}}(\mathcal{P}^r) = \sum_{j=0}^{r} s_{\lambda_j} + r - 1 = \sum_{i=0}^{d} s_i\,\mu_i + r - 1.$$

Hence the integer program calculates the distribution of dimensions μ_i such that the Sarkaria index of a corresponding face type is maximal. Since every face complex $\Sigma_{\boldsymbol{\lambda}}(\mathcal{P}^r)$ with $|\boldsymbol{\lambda}| = k$ is a subcomplex of the respective skeleton complex $\Sigma_k(\mathcal{P}^r)$ this yields the stated lower bound for the embeddability dimension. □

3.2 Products of polygons

Let $\mathcal{P} = \mathcal{D}_{m_1} \times \mathcal{D}_{m_2} \times \cdots \times \mathcal{D}_{m_r}$ be a product of m_i-gons $\mathcal{D}_{m_i}$. In this section we investigate the embeddability dimension of the skeleton complex $\Sigma_k(\mathcal{P})$ for $0 \leq k < 2r = \operatorname{dim} \mathcal{P}$. An interesting feature of the results to come is that (bounds on) the embeddability will only depend on the parity of the m_i. For this reason, we fix the following notation for the product of r_e even polygons and r_o odd polygons:

$$\mathcal{P} \;=\; \mathcal{D}_{\mathsf{even}}^{r_e} \times \mathcal{D}_{\mathsf{odd}}^{r_o}.$$

Furthermore, we denote by $r = r_e + r_o$ the total number of factors and by m the total number of facets.

3.2.1 Skeleton complexes of polygons

The embeddability dimension of the skeleton complex $\Sigma_k(\mathcal{D}_m)$ of an m-gon for $k = 1, 2$ is already given by Proposition 2.20. So we are left to determine the Sarkaria index for the 0th skeleton complex.

Lemma 3.8. Let $m \geq 3$ and $\mathcal{D}_m$ the combinatorial type of an m-gon. The Sarkaria bound for the 0th skeleton complex is

$$\mathsf{ind}_{\mathsf{SK}}\, \Sigma_0(\mathcal{D}_m) = \begin{cases} m-3, & \text{if } m \text{ is even, and} \\ m-2, & \text{if } m \text{ is odd.} \end{cases}$$

Proof. We show that the Kneser graph of minimal non-faces of $\Sigma_0(\mathcal{D}_m)$ has chromatic number 2 and 1, respectively, depending on the parity of m. For that let us determine the minimal non-faces of $\Sigma_0(\mathcal{D}_m)$: A subset $\sigma \subseteq [m]$ of the edges of $\mathcal{D}_m$ is a non-face of $\Sigma_0(\mathcal{D}_m)$ if and only if every vertex of $\mathcal{D}_m$ is incident to at least one edge F_i of D_m with $i \in \sigma$.

If a vertex of $\mathcal{D}_m$ is covered twice by σ then every other minimal non-face intersects σ and thus σ is an isolated vertex in the Kneser graph. If σ covers every vertex exactly once, then $[m] \setminus \sigma$ is again a minimal non-face.

It follows that for odd m the Kneser graph consists of isolated vertices alone while for even m there is exactly one edge. Hence the chromatic number of the Kneser graph for even m is 2, whereas for odd m it is 1. This yields the stated result for the Sarkaria index. □

Example 3.9. As an illustration, let us consider $\Sigma_0(\mathcal{D}_5)$ – the 0th skeleton complex of a pentagon $\mathcal{D}_5$. By the above Lemma 3.8 the Sarkaria index of the 0th skeleton complex of the pentagon is 3. Hence the complex should

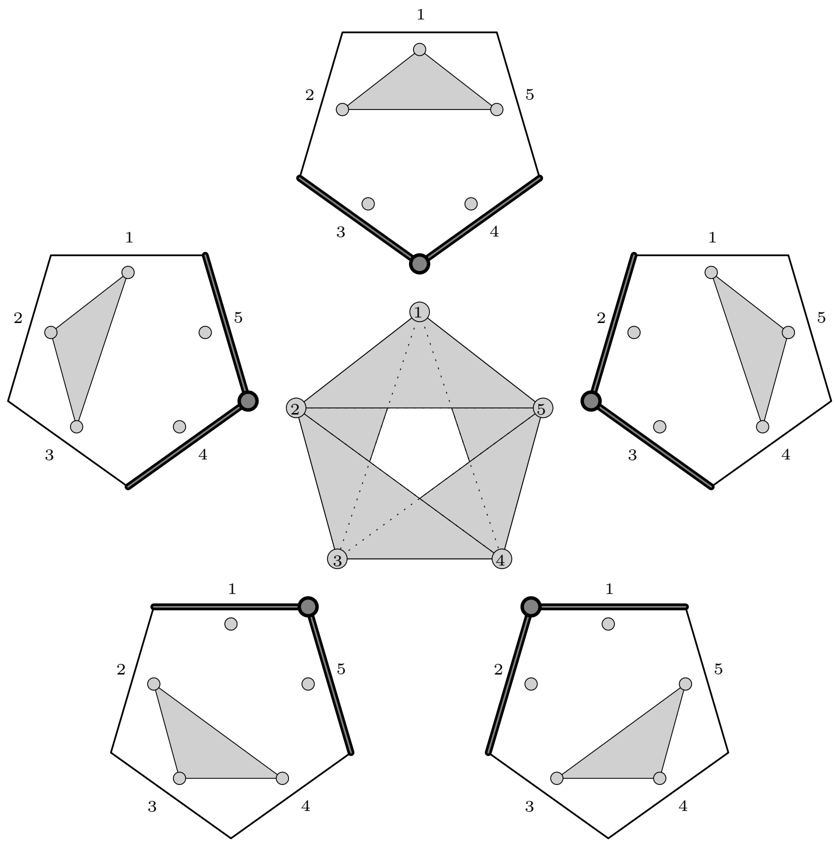


Figure 3.1: The five triangles of the 0th skeleton complex of a pentagon fit together to form a Möbius strip.

not be embeddable into a 2-sphere. If the facets (edges) of the pentagon are labeled by $1, 2, 3, 4, 5$ in cyclic order then the faces of $\Sigma_0(\mathcal{D}_5)$ are the triangles with three cyclically adjacent vertices shown in Figure 3.1. These triangles fit together to form a Möbius strip which is not embeddable in a 2-sphere. Thus $\Sigma_0(\mathcal{D}_5)$ not embeddable in the 2-sphere as stated by the lemma.

The example shows that the 0th skeleton complex of an *odd* polygon has a certain twist to it that obstructs the embeddability into $m-2$ dimensional space.

3.2.2 Skeleton complexes of products of polygons

We are now ready to deal with the skeleton complexes of products of polygons using the *knapsack-type* integer program introduced in Proposition 3.7.

Theorem 3.10. Let $\mathcal{P} = \mathcal{D}_{m_1} \times \cdots \times \mathcal{D}_{m_r}$ be a product of r_e even and r_o odd polygons with a total of m facets and $r = r_o + r_e$ factors. Then the embeddability dimension of the kth skeleton complex for $0 \leq k \leq 2r$ bounded from below by:

$$\mathsf{e\text{-}dim}(\Sigma_k(\mathcal{P})) \geq m - 1 - r + \left\lfloor \frac{k}{2} \right\rfloor + \min\left\{0, \left\lceil \frac{k}{2} \right\rceil - r_e\right\}.$$

Proof. The Sarkaria indices of the kth skeleton complexes of even and odd polygons coincide for $k = 1, 2$ and differ by one for $k = 0$. Similar to Proposition 3.7 we sort the face type by dimension but this time we distinguish two different kinds of vertices in the following way: To every face type $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_r) \in \Lambda_k(\mathcal{P})$ we associate a vector $\boldsymbol{\mu} = (\mu_0^{\mathsf{odd}}, \mu_0^{\mathsf{even}}, \mu_1, \mu_2)$ with

$$\begin{array}{lcll}
\mu_2 & := & \#\{i : \lambda_i = 2\} & \text{(polygons)} \\
\mu_1 & := & \#\{i : \lambda_i = 1\} & \text{(edges)} \\
\mu_0^{\mathsf{odd}} & := & \#\{i : \lambda_i = 0, m_i \text{ odd}\} & \text{(odd vertices)} \\
\mu_0^{\mathsf{even}} & := & \#\{i : \lambda_i = 0, m_i \text{ even}\} & \text{(even vertices)}
\end{array}$$

Using Corollary 3.6 and the fact that $\mu_0^{\mathsf{odd}} + \mu_0^{\mathsf{even}} + \mu_1 + \mu_2 = r$, the Sarkaria index of a face type $\boldsymbol{\lambda}$ may be expressed in terms $\boldsymbol{\mu}$ in the following way:

$$\begin{aligned}
\mathsf{ind_{SK}}\, \Sigma_{\boldsymbol{\lambda}}(\mathcal{P}) &= \sum_{j=0}^{r} \mathsf{ind_{SK}}\, \Sigma_{\lambda_i}(\mathcal{D}_{m_i}) + r - 1 \\
&= m - 3\ \mu_0^{\mathsf{even}} - 2\ \mu_0^{\mathsf{odd}} - 2\ \mu_1 - \mu_2 + r - 1 \\
&= m - r - 1 + \mu_2 - \mu_0^{\mathsf{even}}.
\end{aligned}$$

Now the knapsack-type integer program similar to Proposition 3.7 is

$$\begin{array}{llcccccccl}
\max & -\mu_0^{\mathsf{even}} & & & & & + & \mu_2 & & \\
\mathsf{s.t.} & & & & & \mu_1 & + & 2\,\mu_2 & = & k \\
& \mu_0^{\mathsf{even}} & + & \mu_0^{\mathsf{odd}} & + & \mu_1 & + & \mu_2 & = & r \\
& \mu_0^{\mathsf{even}} & & & & & & & \leq & r_e \\
& & & \mu_0^{\mathsf{odd}} & & & & & \leq & r_o \\
& & & & & & \multicolumn{2}{r}{\mu_0^{\mathsf{even}}, \mu_0^{\mathsf{odd}}, \mu_1, \mu_2} & \geq & 0\ .
\end{array}$$

Every face type $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_r) \in \Lambda_k(\mathcal{P})$ gives rise to a feasible solution and, conversely, every feasible solution yields a face type. We eliminate the variables μ_0^{odd} and μ_1 from the program using the two equalities and obtain an integer program in the variables μ_0^{even} and μ_2 only:

$$\begin{array}{lrcl} \max & & -\mu_0^{\mathsf{even}} + \mu_2 & \\ \mathsf{s.t.} & 0 \leq r - k - \mu_0^{\mathsf{even}} + \mu_2 & \leq & r_o \\ & 0 \leq \quad k - 2\mu_2 & \leq & r_e \\ & \mu_0^{\mathsf{even}}, \mu_2 & \geq & 0 \end{array}$$

The optimal value μ^* of this program is

$$\mu^* = \min\left\{ \left\lfloor \frac{k}{2} \right\rfloor, k - r_e \right\} = \left\lfloor \frac{k}{2} \right\rfloor + \min\left\{0, \left\lceil \frac{k}{2} \right\rceil - r_e \right\}.$$

The result then follows from the fact that $\mathsf{e\text{-}dim}(\Sigma_k(\mathcal{P})) \geq m - r - 1 - \mu^*$. $\square$

In order to put the above result in perspective, let us calculate upper bounds on the embeddability dimension.

Proposition 3.11. Let $\mathcal{P} = \mathcal{D}_{\mathsf{even}}^{r_e} \times \mathcal{D}_{\mathsf{odd}}^{r_o}$ be a product of r_e even and r_o odd polygons with a total of r factors and m facets. For $0 \leq k < 2r$ the embeddability dimension of the kth skeleton complex satisfies the following bounds:

$$\mathsf{e\text{-}dim}(\Sigma_k(\mathcal{P})) \leq \begin{cases} m - r - r_e - 1, & \text{if } k = 0 \\ m - r - 1, & \text{if } k = 1 \\ m - 2, & \text{otherwise.} \end{cases}$$

Proof. Let m_e and m_o be the number of vertices of even and odd polygons, respectively. The 0th skeleton complex of an even m_e-gon consists of a subset of the facets of the cyclic polytope $\mathrm{cyc}_{m_e-2}(m_e)$ of dimension $m_e - 2$ with m_e vertices: Let $i, i' \in [m_e]$ with $i < i'$. Then Gale's Evenness Condition (e.g. found in [47, Ch. 0]) tells us that the simplex σ with vertices $[m_e] \setminus \{i, i'\}$ is a facet of $\mathrm{cyc}_{m_e-2}(m_e)$ if and only if the cardinality of $\{j \in \sigma \mid i < j < i'\}$ is even. Hence the 0th skeleton complex with facets $[m_e] \setminus \{i, i+1 \text{ mod } m_e\}$ for $i \in [m_e]$ is a subcomplex of the boundary of the cyclic polytope. The 0th skeleton complex of an odd m_o-gon may be embedded in the boundary of an $(m_o - 1)$-dimensional simplex.

The 0th skeleton complex $\Sigma_0(\mathcal{P})$ of the product $\mathcal{P}$ is just the join of the 0th skeleton complexes of the factors. Hence it embeds into $(\partial\, \mathrm{cyc}_{m_e-2}(m_e))^{*r_e} * (\partial\, \Delta_{m_o-1})^{*r_o}$ yielding the bound on the embeddability dimension for $k = 0$.

For $k = 1$ the skeleton complexes $\Sigma_1(\mathcal{D}_{m_e})$ and $\Sigma_1(\mathcal{D}_{m_o})$ embed into the boundaries $\partial\, \Delta_{m_e-1}$ and $\partial\, \Delta_{m_o-1}$ of suitable simplices. Hence the skeleton complex $\Sigma_1(\mathcal{P})$ of the product embeds into the join $(\partial\Delta_{m_e-1})^{*r_e} * (\partial\Delta_{m_o-1})^{*r_o}$ which is homeomorphic to a $(m - r - 1)$-sphere.

For $k \geq 2$ the skeleton complex only embeds into $\partial((\Delta_{m_e-1})^{*r_e} * (\Delta_{m_o-1})^{*r_o})$, which is the boundary of an $(m-1)$-polytope, i.e. homeomorphic to a $(m-2)$-sphere. $\square$

3.2.3 Projections of products of polygons

Combining the bounds on the embeddability dimensions of the skeleton complexes of Theorem 3.10 with Corollary 2.17 we obtain the following obstructions to projectability of products of polygons.

Theorem 3.12. Let $\mathcal{P} = \mathcal{D}_{\mathsf{even}}^{r_e} \times \mathcal{D}_{\mathsf{odd}}^{r_o}$ be a product of $r = r_e + r_o$ polygons with m facets. Then for $0 \leq k < 2r$ there exists no realization of the product $\mathcal{P}$ in $\mathbb{R}^{2r}$ such that the projection $\pi : \mathbb{R}^{2r} \to \mathbb{R}^e$ preserves the k-skeleton if

$$e < r + 1 + \left\lfloor \frac{k}{2} \right\rfloor + \min\left\{0, \left\lceil \frac{k}{2} \right\rceil - r_e\right\}. \qquad \square$$

In Sanyal & Ziegler [44] it was shown that there exist e-dimensional polytopes with the $\lfloor \frac{e-2}{2} \rfloor$-skeleton of the r-fold product of even polygons. For the product of odd polygons we obtain the following obstruction to the projectability.

Corollary 3.13. Let $\mathcal{P} = \mathcal{D}_{\mathsf{odd}}^{r_o}$ be a product of *odd* polygons, $0 \leq k < 2r_o$ with m facets. If

$$e < r_o + 1 + \left\lfloor \frac{k}{2} \right\rfloor$$

then there is no realization of $\mathcal{P}$ such that the projection to $\mathbb{R}^e$ preserves the k-skeleton. $\square$

This corollary generalizes the classical result that the product of two triangles ($r_o = 2$), or more generally, the product of two odd polygons may not be projected into the plane ($e = 2$) such that all vertices survive the projection. For the product of even polygons ($r = r_e$) Theorem 3.12 yields a bound of $e < k + 1$, which is just the trivial dimension bound for the k-skeleton of the product. Another interesting case studied for the product of even polygons in [44] is when $k = \lfloor \frac{e-2}{2} \rfloor$, i.e., the "neighborly" case.

Corollary 3.14. Let $\mathcal{P} = \mathcal{D}_{\mathsf{even}}^{r_e} \times \mathcal{D}_{\mathsf{odd}}^{r_o}$ be a product of $r = r_e + r_o$ polygons and let $e \geq 1$. If

$$\begin{cases} \left\lceil \frac{3e-2}{4} \right\rceil < r & \text{for } r_e < \left\lfloor \frac{e}{4} \right\rfloor, \\ \left\lfloor \frac{e}{2} \right\rfloor < r_o & \text{for } r_e \geq \left\lfloor \frac{e}{4} \right\rfloor \end{cases}$$

then there is no realization of $\mathcal{P}$ such that the image under projection to e-space is neighborly, i.e. the image and $\mathcal{P}$ have isomorphic $\lfloor \frac{e-2}{2} \rfloor$-skeleta.

This corollary implies that one cannot project any realization of the product of two odd polygons with an arbitrary number of even polygons into the plane or $\mathbb{R}^3$ such that all the vertices survive. Further we deduce that the number of factors r_o must not exceed $\lceil \frac{3e-2}{4} \rceil$ if the projection of a product of odd polygons to $\mathbb{R}^e$ with $e \geq 4$ should be neighborly.

3.3 Products of simplices

In this section we will establish bounds on the embeddability dimension of the skeleton complexes of the product of simplices. In the spirit of the previous section, we determine the embeddability dimension as well as the Sarkaria index of the skeleton complex of a single simplex first. Then we use this knowledge to prove tight bounds on the embeddability dimensions of face complexes. Appealing to results from Section 2.1, this yields bounds on the projectability of products of simplices.

3.3.1 Skeleton complexes of simplices

The key to determining the embeddability dimension and the Sarkaria index of $\Sigma_k(\Delta_{n-1})$ will be the following observation: The k-faces of the simplex are intersections of exactly $n-1-k$ facets. Hence the complements of the facets defining a k-face consists of $k+1$ facets.

Observation. For $n \geq 1$ and $0 \leq k \leq n-1$ the kth skeleton complex $\Sigma_k(\Delta_{n-1})$ of the $(n-1)$-simplex is isomorphic to the k-skeleton of Δ_{n-1}.

Thus the skeleton complexes $\Sigma_k(\Delta_{n-1})$ are well known complexes and the calculation of the Sarkaria indices involves the *classical* Kneser graphs, which we now recall.

Theorem 3.15 (Lovász [31])**.** For $n \geq 1$ and $1 \leq k \leq n$ denote by $\mathsf{KG}_{n,k} = \mathsf{KG}(\binom{[n]}{k})$ the Kneser graph on the collection of all k-sets of $[n]$. Then

$$\chi(\mathsf{KG}_{n,k}) = \begin{cases} n-2k+2 & \text{if } k \leq \frac{n+1}{2} \\ 1 & \text{otherwise.} \end{cases}$$

The Sarkaria index of the kth skeleton complex may easily be calculated as follows.

Lemma 3.16. For $n \geq 2$ let Δ_{n-1} be an $(n-1)$-simplex and let $\Sigma_k(\Delta_{n-1})$ be the kth skeleton complex for $0 \leq k \leq n-1$. Then the Sarkaria index is the following:

$$\mathsf{ind}_{\mathsf{SK}}\,\Sigma_k(\Delta_{n-1}) = \begin{cases} 2k+1, & \text{if } \quad 0 \leq k \leq \frac{n-3}{2}, \\ n-2, & \text{if } \quad \frac{n-3}{2} < k \leq n-2, \\ n-1, & \text{if } \quad k = n-1. \end{cases}$$

Proof. To calculate the Sarkaria index we need to determine the chromatic number of the Kneser graph on the minimal non-faces of the kth skeleton complex of the $(n-1)$-simplex. By the observation above, the minimal non-faces are all $(k+2)$-subsets of $[n]$. Note that for $k = n-1$ there are no minimal non-faces. For $k \leq n-2$ the Kneser graph to be investigated is $\mathsf{KG}_{n,k+2}$. This yields the stated bound for $0 \leq k \leq n-2$. If $k = n-1$ the Kneser graph is empty and its chromatic number is 0. So the Sarkaria index for $k = n-1$ is $n-1$. □

In combination with Proposition 2.7 we obtain the following corollary.

Corollary 3.17. Let $\Sigma_k = \Sigma_k(\Delta_{n-1})$ be the kth skeleton complex of an $(n-1)$-simplex for $n \geq 2$. Then the embeddability dimension satisfies

$$\mathsf{e\text{-}dim}(\Sigma_k) = \begin{cases} 2k+1 & \text{if } \quad 0 \leq k \leq \frac{n-4}{2}, \\ n-2 & \text{if } \quad \frac{n-4}{2} < k \leq n-2, \\ n-1 & \text{otherwise.} \end{cases}$$

3.3.2 Skeleton complexes of products of simplices

We follow the same path to determine the embeddability dimension as in Section 3.2 about products of polygons: First we prove upper bounds using Proposition 2.7, then we find face types of the product of simplices maximizing the Sarkaria index among all face types. Finally we combine these results to obtain obstructions to the projectability of products of simplices.

In the following we denote by

$$\Delta_{n-1}^r = \underbrace{\Delta_{n-1} \times \Delta_{n-1} \times \cdots \times \Delta_{n-1}}_{r}$$

an r-fold product of $(n-1)$-simplices. The next lemma establishes an upper bound on the embeddability dimension of $\Sigma_k(\Delta_{n-1}^r)$.

Lemma 3.18. Let $\Sigma_k = \Sigma_k(\Delta^r_{n-1})$ be the kth skeleton complex of the r-fold product of $(n-1)$-simplices with $n \geq 2$ and $0 \leq k \leq r(n-1)$. Then

$$\begin{aligned}\mathsf{e\text{-}dim}(\Sigma_k) &\leq \begin{cases} 2k+2r-1 & \text{if} \quad 0 \leq k \leq \frac{1}{2}r(n-2), \\ rn-1 & \text{if} \quad \frac{1}{2}r(n-2) < k \leq r(n-1) \end{cases} \\ &\leq \min\{2k+2r-1, rn-1\}.\end{aligned}$$

Proof. For $\boldsymbol{\lambda} \in \Lambda_k(\Delta^r_{n-1})$ we have that

$$\mathsf{dim}\, \Sigma_{\boldsymbol{\lambda}}(\Delta^r_{n-1}) = \sum_{i=1}^{r} \mathsf{dim}\, \Sigma_{\lambda_i}(\Delta_{n-1}) + r - 1 = \sum_{i=1}^{r} \lambda_i + r - 1 = k + r - 1.$$

By using the covering of Proposition 3.2 we obtain that $\mathsf{dim}\, \Sigma_k = k+r-1$ and by Proposition 2.7 we have $\mathsf{e\text{-}dim}(\Sigma_k) \leq 2k+2r-1$. On the other hand, the kth skeleton complex naturally embeds into the r-fold join of $(n-1)$-simplices and therefore $\mathsf{e\text{-}dim}(\Sigma_k) \leq r(n-1)+r-1 = rn-1$. □

As in Section 3.2 we use the Sarkaria index to get lower bounds on the embeddability dimension. In the following technical lemma we determine face types of the product of simplices that maximize the Sarkaria index and thus give the best possible lower bounds on the embeddability dimension via face types.

Lemma 3.19. Let $n \geq 2$ and $0 \leq k \leq r(n-1)$. Let $\Sigma_{\boldsymbol{\lambda}} = \Sigma_{\boldsymbol{\lambda}}(\Delta^r_{n-1})$ be the face complex of Δ^r_{n-1} of type $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_r) \in \Lambda_k(\Delta^r_{n-1})$. Then for n odd:

$$\mathsf{ind}_{\mathsf{SK}}\, \Sigma_{\boldsymbol{\lambda}} \leq \begin{cases} 2k+2r-1, & \text{if} \quad 0 \leq k \leq \frac{1}{2}r(n-3), \\ r(n-1)+j-1, & \text{if} \quad j\frac{n+1}{2} \leq k - \frac{1}{2}r(n-3) < (j+1)\frac{n+1}{2} \\ rn-1, & \text{if} \quad r(n-1) = k \end{cases}$$

And for n even:

$$\mathsf{ind}_{\mathsf{SK}}\, \Sigma_{\boldsymbol{\lambda}} \leq \begin{cases} 2k+2r-1, & \text{if} \quad 0 \leq k \leq \frac{1}{2}r(n-4), \\ k+\frac{1}{2}rn-1, & \text{if} \quad 0 < k - \frac{1}{2}r(n-4) \leq r \\ r(n-1)+j-1, & \text{if} \quad j\frac{n}{2} \leq k - \frac{1}{2}r(n-2) < (j+1)\frac{n}{2} \\ rn-1, & \text{if} \quad r(n-1) = k \end{cases}$$

where $j = 0, \ldots, r-1$. There are face types for which the bounds are sharp.

Proof. Let $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_r)$ with $|\boldsymbol{\lambda}| = k$. From Proposition 3.5 we obtain that the Sarkaria index is additive for joins of complexes, hence:

$$\mathsf{ind}_{\mathsf{SK}}\, \Sigma_{\boldsymbol{\lambda}}(\Delta^r_{n-1}) = \sum_{i=1}^{r} \mathsf{ind}_{\mathsf{SK}}\, \Sigma_{\lambda_i}(\Delta_{n-1}) + r - 1.$$

In the following we will describe the optimal distribution of k onto the r factors of the skeleton complex. According to Lemma 3.16 the Sarkaria index of the skeleton complex of a single $(n-1)$-simplex is monotone in λ_i. But note that for $\frac{1}{2}(n-4) \leq \lambda_i \leq n-2$ it is constant. We will treat two cases depending on the parity of n.

For n odd: The Sarkaria index is $n-2$ for $\frac{1}{2}(n-3) \leq \lambda_i \leq n-2$. Hence for $0 \leq k \leq \frac{1}{2}r(n-3)$ all distributions $\boldsymbol{\lambda}$ with $\lambda_i \leq \frac{1}{2}(n-3)$ for $i = 0, \dots, r$ yield the same bound on the Sarkaria index of $\Sigma_{\boldsymbol{\lambda}}(\Delta^r_{n-1})$. Suppose $\boldsymbol{\lambda}'$ is a distribution with $\lambda'_0 > \frac{1}{2}(n-3)$. Then $\mathsf{ind}_{\mathsf{SK}}\, \Sigma_{\boldsymbol{\lambda}}(\Delta^r_{n-1}) > \mathsf{ind}_{\mathsf{SK}}\, \Sigma_{\boldsymbol{\lambda}'}(\Delta^r_{n-1})$ since redistributing the surplus $\lambda'_0 - \frac{1}{2}(n-3)$ onto some $\lambda_i < \frac{1}{2}(n-3)$ increases the Sarkaria index.

If $k = \frac{1}{2}r(n-3) + j\frac{n+1}{2} + k'$ then the best distribution is obtained by the following:

$$\lambda_i = \begin{cases} n-1 & \text{if } i = 1, \dots, j\ , \\ \frac{1}{2}(n-3) & \text{if } i = j+1, \dots, r-1\ , \\ \frac{1}{2}(n-3) + k' & \text{if } i = r. \end{cases}$$

For n even: In the even dimensional case Lemma 3.16 yields the following Sarkaria index for the simplex:

$$\mathsf{ind}_{\mathsf{SK}}\, \Sigma_k(\Delta_{n-1}) = \begin{cases} 2k+1 & \text{if } 0 \leq k \leq \frac{1}{2}(n-4), \\ n-2 & \text{if } \frac{1}{2}(n-2) \leq k \leq n-2 \\ n-1 & \text{if } k = n-1. \end{cases}$$

Note that, in contrast to the odd dimensional case, the index increases only by 1 between the first $(0 \leq k \leq \frac{1}{2}(n-4))$ and the second case $(\frac{1}{2}(n-2) \leq k \leq n-2)$. This leads to the extra case for the index $\mathsf{ind}_{\mathsf{SK}}\, \Sigma_{\boldsymbol{\lambda}}(\Delta^r_{n-1})$ if n is even. The optimal distributions are obtained as follows:

- If $0 \leq k \leq \frac{1}{2}r(n-4)$ then any distribution with $\lambda_i \leq \frac{1}{2}(n-4)$ yields a Sarkaria index of $2k + 2r - 1$.
- If $k = \frac{1}{2}r(n-4) + k'$ with $1 \leq k' \leq r$ then the optimal distribution is:

$$\lambda_i = \begin{cases} \frac{1}{2}(n-2) & \text{if } i = 1, \dots, k' \\ \frac{1}{2}(n-4) & \text{otherwise.} \end{cases}$$

- If $k = \frac{1}{2}r(n-2) + j(\frac{n}{2}) + k'$ with $j = 0, \dots, r-1$ and $k' \in [\frac{n}{2} - 1]$ then $\boldsymbol{\lambda}$ with

$$\lambda_i = \begin{cases} n-1 & \text{if } i = 1, \dots, j\ , \\ \frac{1}{2}(n-2) & \text{if } i = j+1, \dots, r-1\ , \\ \frac{1}{2}(n-2) + k' & \text{if } i = r. \end{cases}$$

To see that these distribution are really optimal, one can check that any redistribution does not increase the Sarkaria bound. □

In the above lemma we investigated the Sarkaria index for different face types of the product of simplices. Since any decomposition $\boldsymbol{\lambda}$ of k yields a subcomplex $\Sigma_{\boldsymbol{\lambda}}(\Delta_{n-1}^r)$ of the kth skeleton complex $\Sigma_k(\Delta_{n-1}^r)$, Lemma 3.19 yields a lower bound on the embeddability dimension of the kth skeleton complex. After a little calculation and appropriate rounding we unify the odd and even cases to obtain the following theorem.

Theorem 3.20. Let $n \geq 2$, $r \geq 1$ and $0 \leq k \leq r(n-1)$. The embeddability dimension of the kth skeleton complex $\Sigma_k(\Delta_{n-1}^r)$ of the r-fold product of simplices satisfies the following inequalities:

$$\mathsf{e\text{-}dim}(\Sigma_k(\Delta_{n-1}^r)) \geq \begin{cases} 2r+2k-1 & \text{if } \quad 0 \leq k \leq r\lfloor\frac{n-3}{2}\rfloor \\ \frac{1}{2}rn+k-1 & \text{if}\, r\lfloor\frac{n-3}{2}\rfloor < k \leq r\lfloor\frac{n-2}{2}\rfloor \\ r(n-1)+j-1 & \text{if}\, j\lfloor\frac{n+1}{2}\rfloor \leq k - r\lfloor\frac{n-2}{2}\rfloor < (j+1)\lfloor\frac{n+1}{2}\rfloor \\ rn-1 & \text{if} \quad k = r(n-1), \end{cases}$$

with $j = 0, \ldots, r-1$. □

The second case in the above theorem is empty if n is odd. We combine Lemma 3.18 and Theorem 3.20 to obtain the embeddability dimension of the skeleton complex of the product of simplices for certain parameters.

Corollary 3.21. Let $n \geq 2$, $r \geq 1$ and $0 \leq k \leq r\lfloor\frac{n-3}{2}\rfloor$. The embeddability dimension of the kth skeleton complex $\Sigma_k(\Delta_{n-1}^r)$ of the r-fold product of simplices is

$$\mathsf{e\text{-}dim}(\Sigma_k(\Delta_{n-1}^r)) = 2r+2k-1.$$

3.3.3 Projections of products of simplices

We use the bounds on the embeddability dimension established in the previous section to determine whether or not a certain skeleton may be preserved under projection.

Theorem 3.22 (Non-projectability of skeleta of products of simplices)**.** For $n \geq 2$ and $r \geq 1$ there exists no realization of the r-fold product Δ_{n-1}^r of $(n-1)$-simplices such that the projection from $\mathbb{R}^{r(n-1)}$ to $\mathbb{R}^e$ preserves the k-skeleton if

$$e < \begin{cases} r+2k+1 & \text{if} \quad 0 \leq k \leq r\lfloor\frac{n-3}{2}\rfloor \\ \frac{1}{2}r(n-2)+k+1 & \text{if } r\lfloor\frac{n-3}{2}\rfloor < k \leq r\lfloor\frac{n-2}{2}\rfloor \\ r(n-2)+j+1 & \text{if } j\lfloor\frac{n+1}{2}\rfloor \leq k-r\lfloor\frac{n-2}{2}\rfloor < (j+1)\lfloor\frac{n+1}{2}\rfloor \\ r(n-1)+1 & \text{if} \quad k = r(n-1). \end{cases}$$

Proof. By Corollary 3.3 we get the following bound on the dimension e projected onto:

$$e < \mathsf{e\text{-}dim}(\Sigma_k(\Delta_{n-1}^r)) + r(n-1) - rn + 2 = \mathsf{e\text{-}dim}(\Sigma_k(\Delta_{n-1}^r)) - r + 2.$$

We obtain the stated result by inserting the bounds of Theorem 3.20 into this inequality. □

For $r = 1$ Theorem 3.22 yields bounds on the realizability of the k-skeleton of the $(n-1)$-simplex, i.e. there exists no projection of the $(2k+2)$-simplex to $\mathbb{R}^{(2k+1)}$ which preserves the k-skeleton. This is exactly the Van Kampen–Flores Theorem. In this sense, Theorem 3.22 is a generalization of the classical polyhedral Van Kampen–Flores Theorem from simplices to products of simplices. Furthermore, the above theorem gives yet another proof of Corollary 3.13 concerning the projection of products of 2-simplices (triangles).

Chapter 4

Polyhedral surfaces in wedge products

joint with Günter M. Ziegler

In this chapter we discuss a family of surfaces that is contained in a new family of polytopes. The new polytopes are "wedge products." They are dual to the wreath products of Joswig & Lutz [27]. They may be obtained by iterating the generalized wedge construction described in Section 4.1.1, which is a special kind of subdirect product, as introduced by McMullen [35]. The new surfaces are constructed as subcomplexes of the 2-skeleta of wedge products. We may deform the wedge products containing the surfaces in a way that the surfaces survive the projection to $\mathbb{R}^4$ and $\mathbb{R}^3$ for certain parameters. Using the techniques introduced in Section 2.2, we obtain lower bounds on the number of moduli for these realizations. Furthermore, we observe that the dual surface is contained in the 2-skeleton of the dual 4-polytope, if the prism over the primal surface is contained in the primal 4-polytope. So by projecting the prism over the surface to the boundary of a 4-polytope we obtain realizations of the dual surfaces in $\mathbb{R}^3$ as well. For other parameters we use the techniques of Section 2.1 to show that we are not able to obtain realizations of these surfaces via projections of a wedge products.

4.1 Wedge products

We begin this section with the construction of the generalized wedge. As the name suggests, this construction generalizes the wedge construction for polytopes which was used, e.g. by Fritzsche & Holt [21], to study the Hirsch conjecture. We define the generalized wedge of two polytopes in terms of an inequality system, which merges the inequality systems of the two constituents. We also interpret the generalized wedge as a degenerate deformed product and determine faces that are affinely equivalent to the two polytopes involved in the construction. The wedge product is also defined by its inequality description but may also be obtained as an iterated generalized wedge.

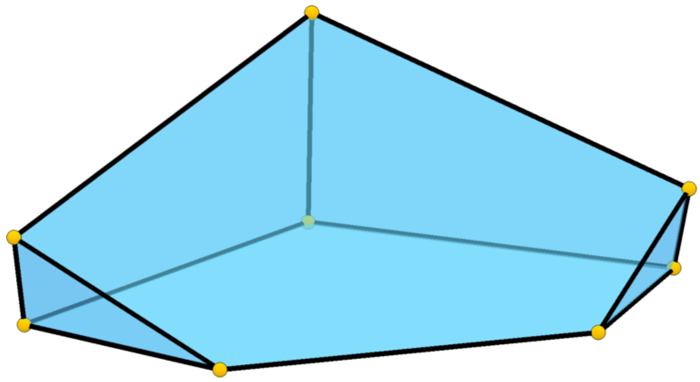


Figure 4.1: The classical wedge over a pentagon.

4.1.1 Generalized wedges

Let P be a d-dimensional polytope in $\mathbb{R}^d$ with m facets, given by its facet description $Ax \leq \mathbb{1}$, with $A \in \mathbb{R}^{m \times d}$, $x = (x_0, \dots, x_{d-1})^t$. Let F be the facet of P defined by the hyperplane $a_0 x = 1$. The classical *wedge* over the polytope P at F is constructed as follows: Embed $P \times \{0\}$ in $\mathbb{R}^{d+1}$ and construct the cylinder $P \times \mathbb{R} \subset \mathbb{R}^{d+1}$. Then cut the cylinder with two distinct hyperplanes through $F \times \{0\}$ such that both cuts are bounded. These hyperplanes divide the cylinder into one bounded and two unbounded components. The bounded part is the *wedge*. This construction can be performed in terms of the inequality system

$$\operatorname{wedge}_F(P) \;:=\; \left\{ \begin{pmatrix} x \\ x_d \end{pmatrix} \in \mathbb{R}^{d+1} \;\middle|\; \begin{pmatrix} A' & \\ a_0 & \pm 1 \end{pmatrix} \begin{pmatrix} x \\ x_d \end{pmatrix} \leq \begin{pmatrix} \mathbb{1} \\ 1 \end{pmatrix} \right\},$$

where A' is the matrix A with the row a_0 removed. The two hyperplanes that cut the cylinder are $a_0 x + x_d = 1$ and $a_0 x - x_d = 1$. They may be constructed by combining the equation $a_0 x = 1$ that defines the facet F with the inequality description $\pm x_d \leq 1$ of the interval $[-1, +1]$ in x_d-direction.

Deletion of the last coordinates yields a projection $\operatorname{wedge}_F(P) \to P$: Fourier-Motzkin elimination of x_d (that is, addition of the two inequalities involving x_d) recovers $a_0 x \leq 1$ as an inequality that is valid, but not facet-defining for $\operatorname{wedge}_F(P)$.

For the projection $\mathrm{wedge}_F(P) \to P$ the fiber above every point of P is an interval I, except that it is a single point $\{*\}$ above every point of F. This might be indicated by

$$(I, \{*\}) \quad \longrightarrow \quad \mathrm{wedge}_F(P) \quad \longrightarrow \quad (P, F).$$

For our purposes we need the following more general construction.

Definition 4.1 (Generalized Wedge $P \lhd_F Q$)**.** Let P be a d-polytope in $\mathbb{R}^d$ with m facets given by the inequality system $Ax \leq \mathbb{1}$, and let Q be a d'-polytope in $\mathbb{R}^{d'}$ with m' facets given by $By \leq \mathbb{1}$. Let G be the face of P defined by the hyperplane $cx = 1$.

The *generalized wedge* $P \lhd_G Q$ of P and Q at G is the $(d+d')$-dimensional polytope defined by

$$P \lhd_G Q \;:=\; \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^{d+d'} \;\middle|\; \begin{pmatrix} A' & \\ C & B \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \leq \begin{pmatrix} \mathbb{1} \\ \mathbb{1} \end{pmatrix} \right\}, \tag{4.1}$$

where $C = \mathbb{1}c$ is the $m' \times d$ matrix all of whose rows are equal to c, and A' is the matrix A if G is not a facets and A' is the matrix A without the row a_0 if G is a facet defined by $a_0 x = 1$.

The generalized wedge $P \lhd_G Q$ is a $(d+d')$-dimensional polytope with $m+m'$ or $m+m'-1$ facets depending on the dimension of the face G. The classical wedge $\mathrm{wedge}_F(P)$ may be viewed as the generalized wedge $P \lhd_F [-1,1]$, where F is a facet of P. The generalized wedge $P \lhd_G Q$ comes with a projection to P similar to the projection of the classical wedge described above.

Proposition 4.2. If P and Q are polytopes of dimension d resp. d', then the generalized wedge is a $(d+d')$-polytope $P \lhd_G Q$. It comes with a projection to P (to the first d coordinates) such that the fiber above every point of P is an affine copy of Q, except that it is a single point $\{*\}$ above every point of G. That is,

$$(Q, \{*\}) \quad \longrightarrow \quad P \lhd_G Q \quad \longrightarrow \quad (P, G).$$

Proof. First we show that the projection maps to P. If G is not a facet of P, then this is obvious, since all facet inequalities of P also define facets of the generalized wedge.

If G is a facet of P defined by the inequality $a_0 x \leq 1$ then $C = \mathbb{1}a_0$ in the inequality system (4.1) of the generalized wedge. We need to show that

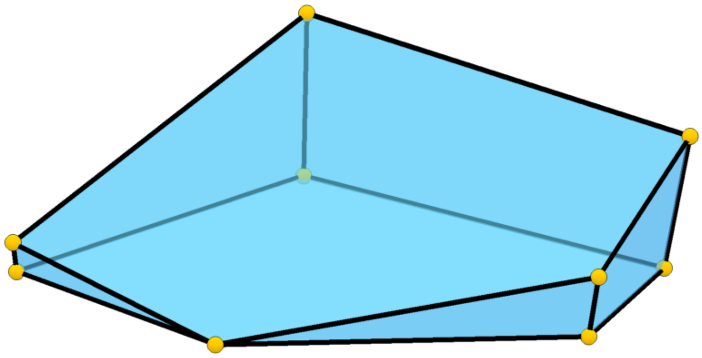


Figure 4.2: The generalized wedge of a pentagon and an interval at a vertex.

this inequality is valid (but not facet-defining) for the generalized wedge as well: Since Q is bounded, its facet-normals (the rows of B) are positively-dependent, so there is a positive row-vector λ satisfying $\lambda B = \mathbf{0}$ and $\lambda \mathbb{1} = 1$; thus summing the inequalities in the system $Cx + By \leq \mathbb{1}$ with coefficients given by λ yields

$$a_0 x = \lambda C x + \lambda B y \leq \lambda \mathbb{1} = 1$$

since $\lambda C = \lambda(\mathbb{1} a_0) = (\lambda \mathbb{1}) a_0 = a_0$.

Now given any point $x \in P$, the fiber above x is given by the inequality system $By \leq \mathbb{1} - Cx$. For $x \in G$ we have $Cx = \mathbb{1}$, and $By \leq \mathbb{0}$ describes a point. For $x \in P \setminus G$ we have $Cx < \mathbb{1}$, and $By \leq \mathbb{1} - Cx$ describes a copy of Q that has been scaled by a factor of $1 - cx$. This is schematically shown in Figure 4.3. □

Remark 4.3. The subdirect product construction introduced when studying projectively unique polytopes by McMullen [35] subsumes the generalized wedge $P \lhd_G Q$ as the special case $(P, G) \otimes (Q, \emptyset)$.

Remark 4.4. The generalized wedge may be interpreted as a limit case (degeneration) of a deformed product in the sense of Amenta and Ziegler [2]: If we consider an inequality $cx \leq 1 + \varepsilon$ for small $\varepsilon > 0$ instead of the inequality $cx \leq 1$ defining the face, then this inequality is strictly satisfied by all $x \in P$. Further an inequality system similar to Equation (4.1) in

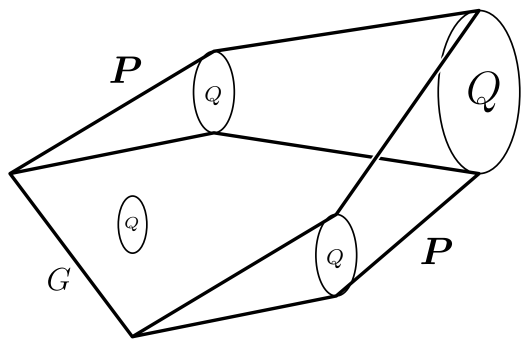


Figure 4.3: A schematic drawing of the generalized wedge $P \lhd_G Q$. It is a degeneration of the product $P \times Q$ of two polytopes and contains many copies of both constituents.

Definition 4.1 defines a deformed product:

$$\left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^{d+d'} \;\middle|\; \begin{pmatrix} A & \\ \frac{1}{1+\varepsilon} C & B \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \le \begin{pmatrix} \mathbb{1} \\ \mathbb{1} \end{pmatrix} \right\},$$

where C is the $m' \times d$ matrix with all rows equal to c. If $\varepsilon \to 0$ then the face $G \times Q$ of the product $P \times Q$ degenerates to a lower dimensional face $G \times \{0\}$, and we obtain the generalized wedge (see Figure 4.4).

The following example illustrates the relation between the product, the deformed product and the generalized wedge.

Example 4.5 (From product to generalized wedge)**.** To construct the generalized wedge of a pentagon D_5 and a triangle Δ_2 we need an inequality description of the two polytopes:

$$\mathrm{D}_5 = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 \;\middle|\; \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \le \mathbb{1} \right\},$$

$$\Delta_2 = \left\{ \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in \mathbb{R}^2 \;\middle|\; \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \le \mathbb{1} \right\}.$$

We now choose the edge e_5 given by the inequality $-x_1 - x_2 \le 1$ as the base facet of the generalized wedge. So the generalized wedge $\mathrm{D}_5 \lhd_{e_5} \Delta_2$ has the following inequality description:

$$\mathrm{D}_5 \lhd_{e_5} \Delta_2 = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \end{pmatrix} \in \mathbb{R}^4 \;\middle|\; \left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ \hline -1 & -1 & 1 & 0 \\ -1 & -1 & 0 & 1 \\ -1 & -1 & -1 & -1 \end{array}\right) \begin{pmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \end{pmatrix} \le \mathbb{1} \right\}.$$

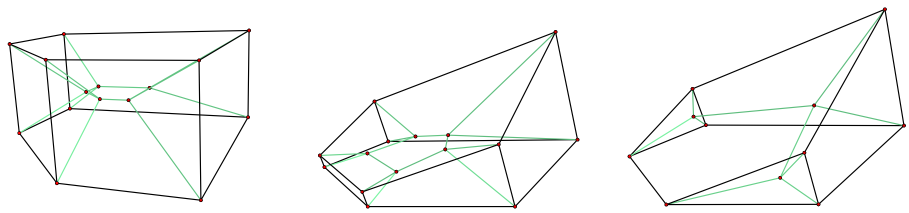


Figure 4.4: Schlegel diagrams showing the degeneration of a product to a generalized wedge: The orthogonal product of pentagon and triangle (left), a deformed product of pentagon and triangle (middle), and the generalized wedge of pentagon and triangle (right).

The first two columns of the matrix contain the normals of the pentagon, where the normal of the edge e_5 is tripled. The lower part of the second two columns contains exactly the normals of the triangle. In Figure 4.4 we show the process of how an orthogonal product degenerates via a deformed product to a generalized wedge.

The facet normals of the three steps in the degeneration for $0 < \lambda < 1$ from the product $\mathrm{D}_5 \times \Delta_2$ via the deformed product $\mathrm{D}_5 \mathbin{\widetilde{\times}} \Delta_2$ to the generalized wedge $\mathrm{D}_5 \triangleleft \Delta_2$ are the following:

$$\begin{array}{ccc}
\mathrm{D}_5 \times \Delta_2 & \mathrm{D}_5 \mathbin{\widetilde{\times}} \Delta_2 & \mathrm{D}_5 \triangleleft \Delta_2 \\
\left(\begin{array}{rr|rr}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
-1 & -1 & 0 & 0 \\
\hline
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & -1 & -1
\end{array}\right)
&
\left(\begin{array}{rr|rr}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
-1 & -1 & 0 & 0 \\
\hline
-\lambda & -\lambda & 1 & 0 \\
-\lambda & -\lambda & 0 & 1 \\
-\lambda & -\lambda & -1 & -1
\end{array}\right)
&
\left(\begin{array}{rr|rr}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
\hline
-1 & -1 & 1 & 0 \\
-1 & -1 & 0 & 1 \\
-1 & -1 & -1 & -1
\end{array}\right)
\end{array}$$

For the boundary cases of λ we obtain the orthogonal product for $\lambda = 0$ and the generalized wedge for $\lambda = 1$. Note that the inequality defining the edge e_5 is redundant and no longer defines a facet of the generalized wedge, but only an edge.

Using the degeneration of the deformed product to a generalized wedge or a little linear algebra we obtain the vertices of the generalized wedge.

Lemma 4.6 (Vertices of the generalized wedge $P \triangleleft_G Q$)**.** Let $P \triangleleft_G Q$ be the generalized wedge of P and Q at G, where P has n vertices and Q has n' vertices, and let $H = \{x \in \mathbb{R}^d : cx = 1\}$ be a hyperplane defining the face G with $\overline{n}$ vertices. Then $P \triangleleft_G Q$ has $(n - \overline{n})n' + \overline{n}$ vertices. These belong to two families

$$u_{k\ell} = \begin{cases} \binom{v_k}{0} & \text{for } v_k \in G, \quad 0 \le k < n, \\ \binom{v_k}{(1-cv_k)w_\ell} & \text{for } v_k \notin G, \quad 0 \le k < n, \ 0 \le \ell < n', \end{cases}$$

where v_k is a vertex of P and w_ℓ is a vertex of Q. □

With the above lemma or again using the degeneration of the deformed product we can determine the combinatorial types of the facets of the generalized wedge.

Lemma 4.7 (Facets of the generalized wedge $P \lhd_G Q$)**.** The inequalities defining the generalized wedge $P \lhd_G Q$ as given by Definition 4.1 are of two different kinds: (i) $a_i x = 1$ for $i \in [m]$ resp. $i \in [m] \setminus \{0\}$ if G is a facet defined by $a_0 x \le 1$ and (ii) $cx + b_j y = 1$ for $j \in [m']$.

(i) Let $a_i x = 1$ define the facet $F_i \neq G$ of P. Then $a_i x = 1$ defines a facet of the generalized wedge combinatorially equivalent to

  (a) the product $F_i \times Q$ if $F_i \cap G = \emptyset$, and

  (b) the generalized wedge $F_i \lhd_{(F_i \cap G)} Q$ if $F_i \cap G \neq \emptyset$.

(ii) Let $b_j y = 1$ define the facet F_j of Q. Then $a_0 x + b_j y = 1$ defines a facet combinatorially equivalent to the generalized wedge $P \lhd_G F_j$. □

The generalized wedge $P \lhd_G Q$ contains many faces that are affinely equivalent to the "base" P. These are characterized in the following proposition.

Proposition 4.8 (P-faces of $P \lhd_G Q$)**.** Let $P \lhd_G Q$ be the generalized wedge of P and Q at G defined by the inequality $G = P \cap \{x \in \mathbb{R}^d \mid cx = 1\}$. For an arbitrary vertex w of Q the convex hull of the vertices $\binom{v_k}{(1-cv_k)w}$ for $k = 0, \ldots, n-1$ is a face that is affinely equivalent to P.

Proof. The vertex $w \in Q$ is described by $\overline{B}y = \mathbb{1}$, where $\overline{B}$ is an invertible square matrix, and $\overline{B}y \le \mathbb{1}$ is a subsystem of $By \le \mathbb{1}$. The corresponding subsystem $\overline{C}x + \overline{B}y \le \mathbb{1}$ defines a face G_w of $P \lhd_G Q$, since it is a valid subsystem. This system is tight for the point $\binom{\mathbb{0}}{w}$ that lies on the boundary of $P \lhd_G Q$. For any $x \in P$ we get a unique solution y for $\overline{C}x + \overline{B}y = \mathbb{1}$, which depends affinely on x. Hence $x \mapsto (x, y)$ yields an affine equivalence between P and the face G_w of $P \lhd_G Q$ that maps the vertices v_k of P to $\binom{v_k}{(1-cv_k)w}$ of G_w. □

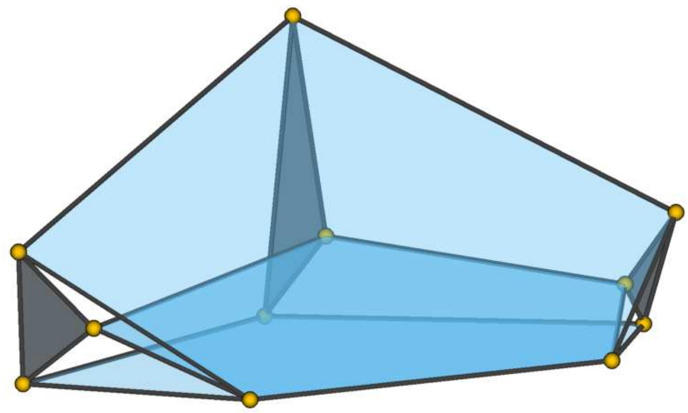


Figure 4.5: The Schlegel diagram of the generalized wedge of a 5-gon and a triangle displays the properties proved in this section: (1) It has a triangle face for every vertex of the pentagon not on the base edge (Proposition 4.2). (2) It has a pentagon face for every vertex of the triangle (Proposition 4.8). (3) It is a simple polytope (Corollary 4.9).

In rare cases the generalized wedge of two polytopes is simple. To characterize the simple wedge products we simply count the facets at each vertex.

Corollary 4.9 (Simple generalized wedges)**.** The generalized wedge $P \lhd_G Q$ of P and Q at G is simple if and only if

- P is a point and Q is simple (trivial case) or
- P is simple and Q is a simplex.

Proof. If P is a point then the inequality system (4.1) of Definition 4.1 is exactly the inequality description of Q, that is, $P \lhd_G Q = Q$. Hence the generalized wedge of a point with a simple polytope is simple if and only if Q is simple.

If the dimension of P is at least 1 we proceed as follows. The generalized wedge is a simple polytope if and only if the number of facets incident to every vertex is $d + d'$. Every vertex of P satisfies at least $d \geq 1$ of the inequalities $a_i x \leq 1$ with equality and every vertex w of Q satisfies at least d' of the inequalities $b_j x \leq 1$ with equality. We will distinguish two kinds of vertices:

(1) Let $\binom{v}{0}$ be a vertex of $P \lhd_G Q$ where v is a vertex of P on the face G defined by $cx = 1$. Then v satisfies at least d inequalities of type $a_i x \leq 1$

with equality. But maybe one of the inequalities $a_0x = 1$ is redundant if $cx = 1$ defines a facet. The vertex $\binom{v}{0}$ also satisfies all m' inequalities of type $cx+b_jy \le 1$ with equality. Hence $\binom{v}{0}$ satisfies at least $d-1+m'$ of the inequalities defining the generalized wedge with equality. Since m' is at least $d'+1$, the vertex $\binom{v}{0}$ lies in exactly $d+d'$ facets of the generalized wedge if and only if v lies in exactly d of the facets of P (one of which is G) and Q has exactly $d'+1$ facets. In other words, the vertex $\binom{v}{0}$ is simple if and only if v is simple in P, Q is a simplex and G is a facet of P.

(2) Let $\binom{v}{(1-cv)w}$ be a vertex of $P \lhd_G Q$ where v is a vertex of P not contained in the face G and w a vertex of Q. Since v is a vertex of P it satisfies at least d inequalities of type $a_ix \le 1$ for $i = 0, \dots, m-1$ with equality, non of which is redundant. Since v is not on G, $cv < 1$, and hence $\binom{v}{(1-cv)w}$ satisfies at least d' of the inequalities of type $cx + b_jy \le 1$ with equality. Thus the vertices of this type are simple if and only if v is a simple vertex of P and Q is simple.

Taking into account both types of vertices yields the lemma. $\square$

4.1.2 Wedge products

The wedge product of two polytopes P and Q may be obtained by iterating the generalized wedge construction for all facet defining inequalities $a_ix = 1$ of P. This is made explicit in the following definition.

Definition 4.10 (Wedge product $P \lhd Q$)**.** Let P be a d-polytope in $\mathbb{R}^d$ given by $Ax \le \mathbb{1}$ with m facets defined by $a_ix \le 1$ for $i \in [m]$ and let Q be a d'-polytope in $\mathbb{R}^{d'}$ given by $By \le \mathbb{1}$ with m' facets that are given by $b_jy \le 1$ for $j \in [m']$. For $i \in [m]$ denote by A_i the $(m' \times d)$-matrix $\mathbb{1}a_i$ with rows equal to a_i. The *wedge product* $P \lhd Q$ is defined by the following system of inequalities:

$$\left\{ \begin{pmatrix} x \\ y_0 \\ y_1 \\ \vdots \\ y_{m-1} \end{pmatrix} \in \mathbb{R}^{d+md'} \middle| \begin{pmatrix} A_0 & B & & & \\ A_1 & & B & & \\ \vdots & & & \ddots & \\ A_{m-1} & & & & B \end{pmatrix} \begin{pmatrix} x \\ y_0 \\ y_1 \\ \vdots \\ y_{m-1} \end{pmatrix} \le \begin{pmatrix} \mathbb{1} \\ \mathbb{1} \\ \vdots \\ \mathbb{1} \end{pmatrix} \right\}. \tag{4.2}$$

We denote the hyperplanes $a_ix + b_jy_i = 1$ defining the facets of the wedge product by $h_{i,j}$ with $(i,j) \in [m] \times [m']$.

Remark 4.11. Comparing the inequality description of the wedge product to the vertex description of the wreath products of Joswig & Lutz [27], we observe that wedge product and wreath product are dual constructions. In other words, if P and Q are polytopes and P^* and Q^* their duals, then the wedge product $P \lhd Q$ is the dual of the wreath product $Q^* \wr P^*$.

According to Proposition 4.2 the generalized wedge $P \lhd_G Q$ comes with a natural projection onto P whose fibers are affine copies of the polytope Q. In the wedge product we have a similar structure proved by analogous techniques.

Proposition 4.12 (Q^m-faces of $P \lhd Q$)**.** The wedge product $P \lhd Q$ of P and Q is a $(d + md')$-dimensional polytope with mm' facets $h_{i,j}$ indexed by $i \in [m]$ and $j \in [m']$. It comes with a linear projection $P \lhd Q \to P$ (to the first d coordinates). The fiber above every interior point of P is a product Q^m, while the ith factor in the fiber degenerates to a point above every point of P that is contained in the ith facet of P. □

In a "fiber bundle" interpretation, the situation might be denoted as

$$(Q, \{*\})^m \quad \longrightarrow \quad P \lhd Q \quad \longrightarrow \quad (P, \{F_i\}_i).$$

This picture has an analogy to MacPherson's topological description of the moment map $\mathbf{T}(P) \to P$ for a toric variety, as presented in [20] and in [32, Sect. 2.8].

We now give a purely combinatorial description of the faces of the wedge product. Each face G of the wedge product $P \lhd Q$ is determined by a subset of the facets $H_G = \{(i, j) \in [m] \times [m'] \mid G \subset h_{i,j}\}$. Ordering this subset by the first index i, the faces may be identified with a vector $(H_0, \dots, H_{m-1})$ with $H_i \subseteq [m']$ in the following way:

$$j \in H_i \iff F \text{ lies on } h_{i,j}. \tag{4.3}$$

In this correspondence, the facets of the wedge correspond to the mm' "unit coordinate vectors" with one entry 1 and all other coordinates equal to 0. The vertices of a simple polytope P correspond to vectors with $\mathsf{dim}\,(P)$ ones and zeroes otherwise. With this notation we now describe the vertex facet incidences of the wedge product. It follows from iterating the generalized wedge construction or by duality from the description of the wreath products of Joswig & Lutz [27].

Theorem 4.13. Let $P \triangleleft Q$ be the wedge product of polytopes P and Q with m resp. m' facets. Then $(H_0, \dots, H_{m-1})$ with $H_i \subseteq [m']$ corresponds to a vertex of $P \triangleleft Q$ if and only if:

- $\{i \in [m] \mid H_i = [m']\} \subseteq [m]$ corresponds to a vertex of P, and
- $H_i \neq [m']$ corresponds to a vertex of Q. □

The previous theorem is a purely combinatorial result and allows us to define the wedge product of combinatorial types: Let $\mathcal{P}$ and $\mathcal{Q}$ be combinatorial types of polytopes. The *wedge product* $\mathcal{P} \triangleleft \mathcal{Q}$ is the polytope obtained from the vertex-facet-incidences of Theorem 4.13.

Certain faces in a wedge product $P \triangleleft Q$ that are affinely equivalent to P will be particularly interesting to us.

Proposition 4.14 (P-faces of $P \triangleleft Q$)**.** Let $(H_0, \dots, H_{m-1})$ with $H_i \subseteq [m']$ correspond to a face F of the wedge product $P \triangleleft Q$. If the intersection of the facets $b_j y \leq 1$ $(j \in H_i)$ is a vertex w_i of Q for all $i \in [m]$, then F is affinely equivalent to P.

Proof. Every H_i gives rise to a submatrix $\overline{B}_i = (b_j)_{j \in H_i}$ of B, such that the vertex w_i is the unique solution of $\overline{B}_i y_i = \mathbb{1}$. The corresponding subsystem of the inequality system of $P \triangleleft Q$ is:

$$\begin{pmatrix} \overline{A}_0 & \overline{B}_0 & & & \\ \overline{A}_1 & & \overline{B}_1 & & \\ \vdots & & & \ddots & \\ \overline{A}_{m-1} & & & & \overline{B}_{m-1} \end{pmatrix} \begin{pmatrix} x \\ y_0 \\ y_1 \\ \vdots \\ y_{m-1} \end{pmatrix} \leq \begin{pmatrix} \mathbb{1} \\ \mathbb{1} \\ \vdots \\ \mathbb{1} \end{pmatrix}.$$

It defines a face of the wedge product, since it is satisfied with equality at the point $(\mathbb{0}, w_0, \dots, w_{m-1})^t$ on the boundary of $P \triangleleft Q$. Analogous to the proof of Proposition 4.8, every point $x \in P$ corresponds to a unique point $(x, y_0, \dots, y_{m-1})$ on F where each y_i is the unique solution of the subsystem $\overline{B}_i y_i = 1 - \overline{A}_i x$. So F is affinely equivalent to P. □

Using either duality and [27, Cor. 2.4], or Corollary 4.9, we obtain the following characterization of simple wedge products.

Corollary 4.15. The wedge product $P \triangleleft Q$ of two polytopes P and Q is simple if and only if

- P is a point and Q is simple (trivial case) or
- P is simple and Q is a simplex. □

4.2 The polyhedral surfaces $\mathcal{S}_{p,2q}$

In this section we study a particularly interesting polytope, the wedge product $\mathcal{W}_{p,q-1}$ of a p-gon with a $(q-1)$-simplex. We use the general results of the previous section and observe that this wedge product is a simple polytope with many p-gon faces. These p-gon faces will be used to construct a regular polyhedral surface of type $\{p, 2q\}$ in the 2-skeleton of $\mathcal{W}_{p,q-1}$.

4.2.1 Wedge product of p-gon and $(q-1)$-simplex

The wedge product of a p-gon $\mathcal{D}_p$ and a $(q-1)$-simplex Δ_{q-1} will be denoted by

$$\mathcal{W}_{p,q-1} \;:=\; \mathcal{D}_p \lhd \Delta_{q-1}\,.$$

This is a $(2+p(q-1))$-dimensional polytope with pq facets. By Corollary 4.15 it is simple.

Let us first fix some notation. We assume that the facets (edges) of $\mathcal{D}_p$ are labeled in cyclic order, that is, if $i, i' \in [p]$ are indices of edges of $\mathcal{D}_p$, then they intersect in a vertex of the p-gon if and only if $i' \equiv i \pm 1 \bmod p$. For $j \in [q]$ we denote by $\overline{j}$ the set complement $[q] \setminus \{j\}$ of j in $[q]$. A vertex of the $(q-1)$-simplex Δ_{q-1} is the intersection of any $q-1$ facets of the simplex, hence for $j \in [q]$ the intersection $\bigcap_{j' \in \overline{j}} F_{j'}$ of the facets $F_{j'}$ $(j' \in \overline{j})$ of the simplex is a vertex. So Theorem 4.13 specializes as follows.

Corollary 4.16 (Vertices of wedge product $\mathcal{W}_{p,q-1}$)**.** Let $\mathcal{W}_{p,q-1}$ be the wedge product of p-gon and $(q-1)$-simplex. Then the vertices of the wedge product $\mathcal{W}_{p,q-1}$ correspond to the vectors $(H_0, \ldots, H_{p-1})$ with

$$(H_0, \ldots, H_{p-1}) = \begin{cases} (\overline{j_0}, \ldots, \overline{j_{i-1}}, [q], [q], \overline{j_{i+2}}, \ldots, \overline{j_{p-1}}), & \text{or} \\ ([q], \overline{j_1}, \overline{j_2}, \ldots\ldots\ldots, \overline{j_{p-3}}, \overline{j_{p-2}}, [q]) \end{cases} \tag{4.4}$$

with $j_i \in [q]$ and $\overline{j_i} = [q] \setminus \{j_i\}$. In other words, each vector that corresponds to a vertex has two cyclically-adjacent $[q]$ entries while all other entries are subsets of $[q]$ with $q-1$ elements. The number of vertices is pq^{p-2}.

For the construction of the surface in the next section we are interested in the p-gon faces of $\mathcal{W}_{p,q-1}$ that we obtain from Proposition 4.14.

Corollary 4.17 (p-gon faces of the wedge product $\mathcal{W}_{p,q-1}$)**.** The faces of the wedge product $\mathcal{W}_{p,q-1}$ of p-gon and $(q-1)$-simplex corresponding to the vectors

$$(H_0, \ldots, H_{p-1}) \;=\; (\overline{j_0}, \ldots, \overline{j_{p-1}}),$$

where $j_i \in [q]$ and $\overline{j_i} = [q] \setminus \{j_i\}$ are p-gons. The number of such p-gons in $\mathcal{W}_{p,q-1}$ is q^p. $\square$

The following example illustrates the incidences of the p-gons at a vertex in the above notation.

Example 4.18 (Vertices and 5-gons of $\mathcal{W}_{5,2}$)**.** Let us consider the wedge product $\mathcal{W}_{5,2}$ of a pentagon $\mathcal{D}_5$ and a triangle Δ_2 to get used to the vector notation for the faces. The wedge product $\mathcal{W}_{5,2}$ has dimension 12 and 15 facets. As described in Corollary 4.16, the vertices of $\mathcal{W}_{5,2}$ correspond to the vectors $(H_1, H_2, H_3, H_4, H_5)$ with two cyclically adjacent $H_{i_0} = H_{i_0+1} = [3]$ and $H_i = \overline{j_i}$ for $i \notin \{i_0, i_0+1\}$:

$$(H_1, H_2, H_3, H_4, H_5) = \begin{cases} ([3],[3],\overline{j_3},\overline{j_4},\overline{j_5}) \\ (\overline{j_1},[3],[3],\overline{j_4},\overline{j_5}) \\ (\overline{j_1},\overline{j_2},[3],[3],\overline{j_5}) \\ (\overline{j_1},\overline{j_2},\overline{j_3},[3],[3]) \\ ([3],\overline{j_2},\overline{j_3},\overline{j_4},[3]) \end{cases}.$$

Each of the five families of vertices of the wedge product $\mathcal{W}_{5,2}$ contains 3^3 vertices which makes a total of $5 \cdot 3^3 = 135$ vertices obtained from the 5 families of vertices by choosing the j_i's. Further each of the vertices is the intersection of $2 \cdot 3 + 3 \cdot 2 = 12$ facets, that is, $\mathcal{W}_{5,2}$ is a simple polytope.

There are $3^5 = 243$ pentagons in the 2-skeleton each corresponding to a vector $(\overline{j_1}, \overline{j_2}, \overline{j_3}, \overline{j_4}, \overline{j_5})$. Each vertex is adjacent to $3^2 = 9$ of the pentagons. If we look at the pentagon vertex figure of the vertex $([3], [3], \overline{3}, \overline{3}, \overline{3})$, that is, we intersect the pentagons adjacent to a vertex with a little sphere, then the 9 pentagons form a complete bipartite graph $K_{3,3}$ on $(3+3)$ vertices. Each vertex of the $K_{3,3}$ corresponds to an edge of the form $([3] \setminus \{j_1\}, [3], \overline{3}, \overline{3}, \overline{3})$ resp. $([3], [3] \setminus \{j_2\}, \overline{3}, \overline{3}, \overline{3})$, with $j_1 \in [3]$ resp. $j_2 \in [3]$. The 9 pentagons with vector representation $([3] \setminus \{j_1\}, [3] \setminus \{j_2\}, \overline{3}, \overline{3}, \overline{3})$ with $j_1, j_2 \in [3]$ are the edges of $K_{3,3}$ (see Figure 4.6).

The task of the next section is to choose an appropriate subset of the p-gons of the wedge product $\mathcal{W}_{p,q-1}$ that forms a surface. It is not difficult to find a set of p-gons that forms a 2-ball at each vertex, but the problem is to select the p-gons such that they fit together globally to form a polyhedral surface.

4.2.2 Combinatorial construction

In this section we describe a combinatorially regular surface of type $\{p, 2q\}$, that is, a surface composed of p-gon faces, whose vertices have uniform degree $2q$, and with a combinatorial automorphism group that acts transitively on its flags. It will be a subcomplex formed by some p-gons of the wedge product $\mathcal{W}_{p,q-1} = \mathcal{D}_p \triangleleft \Delta_{q-1}$ of p-gon and $(q-1)$-simplex defined in the previous Section 4.2.1.

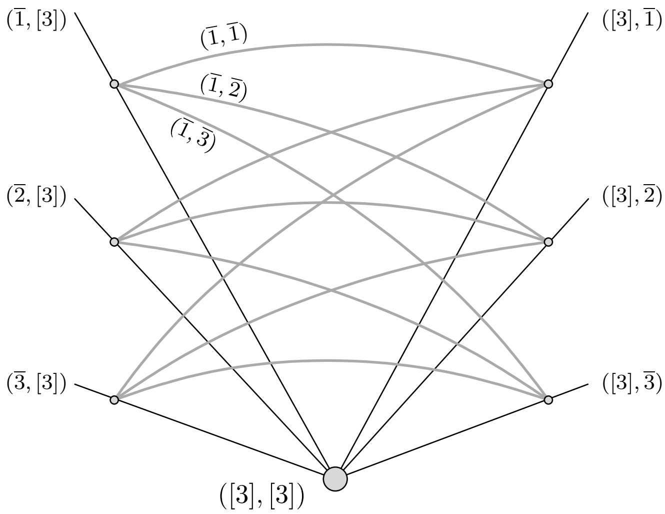


Figure 4.6: The pentagon vertex figure of the wedge product $\mathcal{W}_{5,2}$ of pentagon and 2-simplex at the vertex $([3],[3],\overline{3},\overline{3},\overline{3})$. The labels only contain the first two entries of the vector since the last three are always the same. The vertices of the graph correspond to edges and the edges to 5-gons of the polytope.

To construct the surface we have to select certain p-gon faces of the wedge product. By Corollary 4.17 we know that the p-gon faces of $\mathcal{W}_{p,q-1}$ correspond to vectors $(\overline{j_0},\ldots,\overline{j_{p-1}})$ with $\overline{j_i} = [q] \setminus \{j_i\}$.

Definition 4.19 (Polytopal subcomplex $\mathcal{S}_{p,2q}$ of $\mathcal{W}_{p,q-1}$)**.** For $p \geq 3$ and $q \geq 2$, the subcomplex $\mathcal{S}_{p,2q}$ is defined by the p-gon faces of the wedge product $\mathcal{W}_{p,q-1}$ that correspond to the following set of vectors:

$$\mathcal{S}_{p,2q} \;=\; \Big\{(\,\overline{j_0},\ldots,\overline{j_{p-1}}\,) \;\Big|\; \sum_{i=0}^{p-1} j_i \equiv 0 \text{ or } 1 \ \mathsf{mod}\ q\Big\}.$$

The subcomplex consists of all these p-gons, their edges and vertices.

Let us start with an easy observation on the faces contained in the subcomplex $\mathcal{S}_{p,2q}$.

Lemma 4.20 (Vertices and edges of $\mathcal{S}_{p,2q}$)**.** The subcomplex $\mathcal{S}_{p,2q}$ contains *all* the vertices of $\mathcal{W}_{p,q-1}$. It contains all edges corresponding to vectors with exactly one $[q]$ entry, that is, all edges contained in at least one p-gon. Thus the f-vector of $\mathcal{S}_{p,2q}$ is given by $(f_0,f_1,f_2) \;=\; (p,pq,2q)q^{p-2}$. □

In the following we will prove that the polytopal complex $\mathcal{S}_{p,2q}$ is a regular surface. We start by proving the regularity.

Proposition 4.21 (Regularity of the polytopal complex $\mathcal{S}_{p,2q}$)**.** The polytopal complex $\mathcal{S}_{p,2q}$ is regular, that is, the combinatorial automorphism group acts transitively on its flags.

Proof. We use four special combinatorial automorphisms of the subcomplex to show that the flag $\mathcal{F}_0 : ([q],[q],\overline{0},\ldots,\overline{0}) \subseteq ([q],\overline{0},\overline{0},\ldots,\overline{0}) \subseteq (\overline{0},\overline{0},\overline{0},\ldots,\overline{0})$ may be mapped onto any other flag. Acting on (index vectors of) vertices, they may be described as follows:

$$\begin{aligned}
&\mathsf{F} : (\,\overline{i_0},\overline{i_1},\overline{i_2},\ldots,\overline{i_{p-1}}\,) \longmapsto (\,\overline{i_{p-1}},\ldots,\overline{i_2},\overline{i_1},\overline{i_0}) && \text{(Flip)}\\
&\mathsf{P} : (\,\overline{i_0},\overline{i_1},\overline{i_2},\ldots,\overline{i_{p-1}}\,) \longmapsto (\,\overline{q-i_0+1},\overline{q-i_1},\ldots,\overline{q-i_{p-1}}\,) && \text{(Parity)}\\
&\mathsf{R} : (\,\overline{i_0},\overline{i_1},\overline{i_2},\ldots,\overline{i_{p-1}}\,) \longmapsto (\,\overline{i_0+1},\overline{i_1-1},\overline{i_2},\ldots,\overline{i_{p-1}}\,) && \text{(Rotate)}\\
&\mathsf{S} : (\,\overline{i_0},\overline{i_1},\overline{i_2},\ldots,\overline{i_{p-1}}\,) \longmapsto (\,\overline{i_1},\overline{i_2},\ldots,\overline{i_{p-1}},\overline{i_0}\,) && \text{(Shift)}
\end{aligned}$$

All four maps act on the vectors representing the faces of the subcomplex $\mathcal{S}_{p,2q}$. The map P changes the parity of the p-gon, S shifts the vector cyclically, F reverses the order of the vector, and R rotates around a vertex preserving parity. Hence by applying an appropriate combination of F, S, and P we may map an arbitrary flag to a flag of the type

$$\mathcal{F} = (\,[q],[q],\overline{j_2},\ldots,\overline{j_{p-1}}\,) \subseteq (\,[q],\overline{j_1},\overline{j_2},\ldots,\overline{j_{p-1}}\,) \subseteq (\,\overline{j_0},\overline{j_1},\overline{j_2},\ldots,\overline{j_{p-1}}\,).$$

with $\sum j_i \equiv 0 \bmod q$. The two maps S and R do not change the parity of the p-gon. If we now apply the following sequence of S and R to the flag $\mathcal{F}_0$ we obtain the flag $\mathcal{F}$:

$$\mathcal{F} = (\mathsf{S}(\mathsf{SR}^{s_{p-2}})(\mathsf{SR}^{s_{p-3}})\cdots(\mathsf{SR}^{s_1})(\mathsf{SR}^{s_0}))(\mathcal{F}_0)$$

where $s_\ell = \sum_{i=0}^{\ell} j_i$. Each of the SR pairs adjusts one of the entries of the flag, and the entire sequence maps $\mathcal{F}_0$ to $\mathcal{F}$. $\square$

Remark 4.22. The symmetry group $\mathsf{Aut}(\mathcal{S}_{p,2q})$ of the surface $\mathcal{S}_{p,2q}$ is a subgroup of the symmetry group $[p,2q]$ of the regular tiling of type $\{p,2q\}$, that is, the regular tiling with p-gon faces and uniform vertex degree $2q$. (Depending on the parameters p and q these tilings are Euclidean, spherical, or hyperbolic.) The group $\mathsf{Aut}(\mathcal{S}_{p,2q})$ is generated by "combinatorial reflections" at the lines bounding a fundamental triangle of the barycentric subdivision of the surface. The subgroup $G^{p,2q,r} \le [p,2q]$ studied by Coxeter [16] also contains the group $\mathsf{Aut}(\mathcal{S}_{p,2q})$ for suitable parameter r.

We are now able to prove the following theorem on the structure of our selected subcomplex.

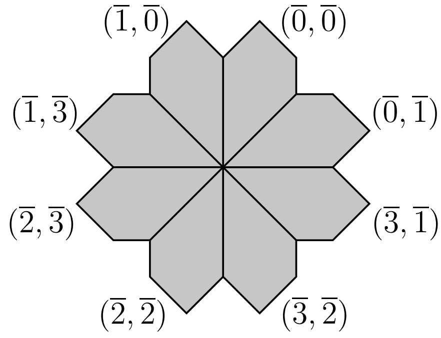


Figure 4.7: The p-gons incident to the vertex $([q],[q],\overline{0},\overline{0},\overline{0})$ of $\mathcal{S}_{p,2q}$ form a 2-ball. In this case $p=5$ and $q=4$ and the pentagons are labeled by the first two entries of the vector representation.

Theorem 4.23 (Properties of $\mathcal{S}_{p,2q}$)**.** The subcomplex $\mathcal{S}_{p,2q}$ of the wedge product $\mathcal{W}_{p,q-1} = \mathcal{D}_p \lhd \Delta_{q-1}$ of a p-gon and a $(q-1)$-simplex is a closed connected orientable regular 2-manifold of type $\{p,2q\}$ with f-vector

$$f(\mathcal{S}_{p,2q}) = (p, pq, 2q)q^{p-2}$$

and genus $1 + \frac{1}{2}q^{p-2}(pq-p-2q)$.

Proof. We start by proving that $\mathcal{S}_{p,2q}$ is a manifold, that is, that the p-gons form a 2-ball at every vertex. By Proposition 4.21 all the vertices are equivalent, so it suffices to consider the vertex $v = ([q],[q],\overline{0},\dots,\overline{0})$. The p-gons adjacent to the vertex v correspond to the vectors

$$(\overline{j_0},\overline{j_1},\overline{0},\dots,\overline{0}) \qquad \text{with} \qquad j_0+j_1 \equiv 0,1 \bmod q.$$

Starting from the p-gon $(\overline{0},\overline{0},\overline{0},\dots,\overline{0})$ we obtain all the other p-gons adjacent to v if we alternately increase the first component j_0 or decrease the second component j_1 as shown in Figure 4.7. The $2q$ edges joining the p-gons correspond to the vectors $([q],\overline{j_1},\overline{0},\dots,\overline{0})$ or $(\overline{j_0},[q],\overline{0},\dots,\overline{0})$ with $j_0,j_1 \in [q]$. Thus the p-gons around each vertex form a 2-ball and $\mathcal{S}_{p,2q}$ is a manifold with uniform vertex degree $2q$.

We proceed by showing that the manifold is connected by constructing a sequence of p-gons connecting two arbitrary p-gons. Consider two arbitrary p-gons $F = (\overline{j_0},\overline{j_1},\overline{j_2},\dots,\overline{j_{p-1}})$ and $G = (\overline{j'_0},\overline{j'_1},\overline{j'_2},\dots,\overline{j'_{p-1}})$. Then there exists a sequence of p-gons in the star of the vertex $v_0 = ([q],[q],\overline{j_2},\dots,\overline{j_{p-1}})$ connecting F to a p-gon $F_1 = (\overline{j'_0},\overline{j''_1},\overline{j_2},\dots,\overline{j_{p-1}})$. For $k=1,\dots,p-2$ we continue around the vertices $v_k = (\overline{j'_1},\dots,\overline{j'_{k-1}},[q],[q],\overline{j_{k+2}},\dots,\overline{j_{p-1}})$ to obtain $F_k = (\overline{j'_1},\dots,\overline{j'_k},\overline{j''_{k+1}},\overline{j_{k+2}},\dots,\overline{j_{p-1}})$ with the first k components equal to those of G. Then either $F_{p-1} = G$ or they share the common edge $(\overline{j_0'}, \ldots, \overline{j_{p-2}'}, [q])$. Hence we have shown so far that $\mathcal{S}_{p,2q}$ is a closed connected equivelar 2-manifold of type $\{p, 2q\}$ without boundary.

The surface $\mathcal{S}_{p,2q}$ consists of two families of p-gons $(\overline{j_0}, \ldots, \overline{j_{p-1}})$, distinguished by $j_0 + \cdots + j_{p-1} \bmod q$. We assign an orientation to the edges of the p-gons as follows

$$([q], \overline{j_1}, \overline{j_2}, \ldots, \overline{j_{p-1}}) \;\rightarrow\; (\overline{j_0}, [q], \overline{j_2}, \ldots, \overline{j_{p-1}}) \;\rightarrow\; (\overline{j_0}, \overline{j_1}, [q], \ldots, \overline{j_{p-1}}) \;\rightarrow\; \ldots$$

if $j_0 + \cdots + j_{p-1} \equiv 0 \bmod q$, and

$$([q], \overline{j_1}, \overline{j_2}, \ldots, \overline{j_{p-1}}) \;\leftarrow\; (\overline{j_0}, [q], \overline{j_2}, \ldots, \overline{j_{p-1}}) \;\leftarrow\; (\overline{j_0}, \overline{j_1}, [q], \ldots, \overline{j_{p-1}}) \;\leftarrow\; \ldots$$

if $j_0 + \cdots + j_{p-1} \equiv 1 \bmod q$. In Figure 4.8 this is illustrated for $p = 5$. Since every edge is contained in one p-gon with sum $\equiv 0$ and one with sum $\equiv 1$ this yields a consistent orientation for the surface.

As $\mathcal{S}_{p,2q}$ is an orientable manifold we calculate the genus of the surface from the f-vector given in Lemma 4.20 via the Euler characteristic:

$$g = 1 - \frac{1}{2}\chi(\mathcal{S}_{p,2q}) = 1 + \frac{1}{2}((q-1)p - 2q)q^{p-2}.$$

□

For $p = 3$ we obtain a regular triangulated surface $\mathcal{S}_{3,2q}$ of type $\{3, 2q\}$ with f-vector $(3q, 3q^2, 2q^2)$ in the wedge product $\mathcal{W}_{3,q-1}$. The genus of the surface is $1 + \frac{1}{2}q(q-3)$ and thus quadratic in the number of vertices. Unfortunately, the wedge product of a triangle and a $(q-1)$-simplex is a polytope of dimension $3q-1$ with $3q$ facets, hence a $(3q-1)$-simplex. So our construction does not provide an "interesting" realization of the surface. The surface $\mathcal{S}_{3,2q}$ is well known and occurs already in Coxeter [15] called $\{3, 2q|, 3\}$. For $q = 2$ the surface is the octahedron and for $q = 3$ Dyck's Regular Map. For Dyck's regular map there exist two realizations in $\mathbb{R}^3$, one by Bokowski [7] and a more symmetric one by Brehm [11].

For $q = 2$ the surface $\mathcal{S}_{p,4}$ is the surface of type $\{p, 4\}$ constructed by McMullen, Schulz and Wills [37, Sect. 4]. In their paper they construct a realization of the surface directly in $\mathbb{R}^3$. Their construction also provides two additional parameters m and n arising from the reflections around an $m \times n$-torus (see Section 5.1.1 for the dual construction). Our surfaces coincide with the surfaces of McMullen, Schulz & Wills for $m = 2$ and $n = 2$.

So our surface generalizes two interesting families of surfaces. As we will see, for some parameters it also provides a new way of realizing the surface in the boundary complex of a 4-polytope and by orthogonal projection in $\mathbb{R}^3$.

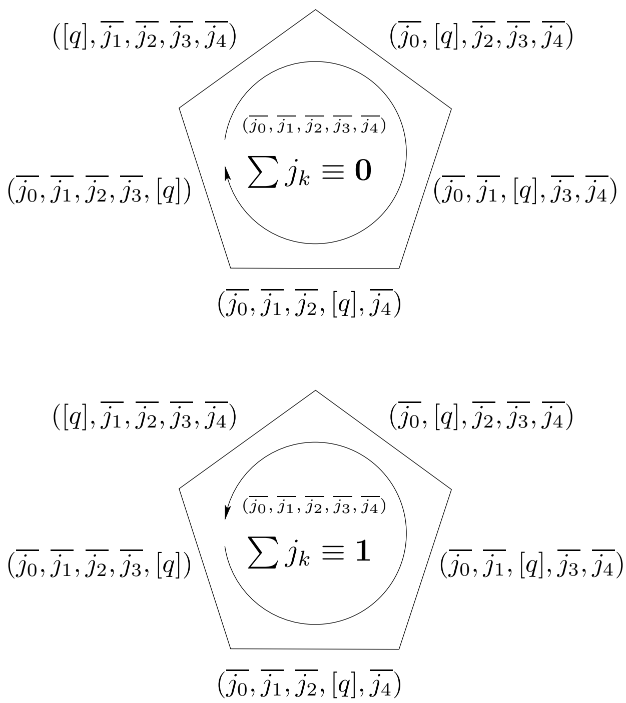


Figure 4.8: The orientation of the p-gons in the surface $\mathcal{S}_{p,2q}$, for $p = 5$.

In contrast to simplicial complexes, which may always be realized in a high-dimensional simplex, there is no fool-proof strategy for realizing general polytopal complexes. For abstract non-simplicial polyhedral 2-manifolds in general not even a realization in $\mathbb{R}^N$ for large N is possible. For example, equivelar surfaces of type $\{p, 2q+1\}$ are not realizable in general:

Proposition 4.24 (Betke and Gritzmann [4])**.** Let $S \subset \mathbb{R}^N$ be an equivelar polyhedral 2-manifold of type $\{p, 2q+1\}$ with $q \geq 1$ in $\mathbb{R}^d$. Then

$$2(2q+1) \geq p+1.$$

If realized in some $\mathbb{R}^N$, a polyhedral surface can be embedded into $\mathbb{R}^5$ via an arbitrary general-position projection. Combining this observation with Theorem 4.23 we get the following corollary.

Corollary 4.25. The regular surfaces $\mathcal{S}_{p,2q}$ of Theorem 4.23 ($p \geq 3$, $q \geq 2$) can be realized in $\mathbb{R}^5$.

4.3 Realizing the surfaces $\mathcal{S}_{p,4}$ and $\mathcal{S}^*_{p,4}$.

In the following we provide a construction for the surfaces $\mathcal{S}_{p,4}$ and $\mathcal{S}^*_{p,4}$ in $\mathbb{R}^3$ via projection. We construct a realization of $\mathcal{W}_{p,1}$ that allows for a projection of the surface into the boundary of a 4-dimensional polytope. A particular property of the embedding will be that all the faces of the surface lie on the "lower hull" of the polytope. In this way, we obtain the surface by an orthogonal projection to $\mathbb{R}^3$ and we do not need to take the Schlegel diagram. Using the new method introduced in Section 2.2, we prove a non-trivial lower bound for the number of moduli of the surfaces $\mathcal{S}_{p,4}$. The dual surface $\mathcal{S}^*_{p,4}$ is constructed via a projection of the product $\mathcal{W}_{p,1} \times I$ of the wedge product with an interval I.

4.3.1 Projection of the surface to $\mathbb{R}^4$ and to $\mathbb{R}^3$

We are now ready to state our main result about the projections of the surfaces contained in the wedge products of p-gons and intervals.

Theorem 4.26. The wedge product $\mathcal{W}_{p,1} = \mathcal{D}_p \lhd \Delta_1$ of a p-gon and a 1-simplex of dimension $2+p$ has a realization in $\mathbb{R}^{2+p}$ such that all the faces corresponding to the surface $\mathcal{S}_{p,4} \subset \mathcal{W}_{p,1}$ are preserved by the projection to the first four resp. three coordinates.

This realizes $\mathcal{S}_{p,4}$ as a subcomplex of a polytope boundary in $\mathbb{R}^4$, and as an embedded polyhedral surface in $\mathbb{R}^3$.

Proof. We proceed in two steps. In the first step we construct a wedge product of a p-gon with a 1-simplex and describe a suitable deformation. In the second step we use the Projection Lemma 1.13 to show that the projection of the deformed wedge product to the first four coordinates preserves all the p-gons of the surface $\mathcal{S}_{p,4}$. Furthermore, all the faces of the projected surface lie on the lower hull of the projected polytope and hence the surface may be realized by an orthogonal projection to the first four/three coordinates.

Let the p-gon D_p be given by $\mathrm{D}_p = \{x \in \mathbb{R}^2 \mid a_i x \le 1, i \in [p]\}$ with facets in cyclic order. Let $\Delta_1 = \{y \in \mathbb{R} \mid \pm\varepsilon y \le 1\}$ be a 1-simplex for a small $\varepsilon > 0$.

Then by Definition 4.10 the inequality description of the wedge product $\mathcal{W}_{p,1}$ is:

$$\left(\begin{array}{ccc|cccc} a_0 & \pm\varepsilon & & & & & \\ a_1 & & \pm\varepsilon & & & & \\ a_2 & & & \pm\varepsilon & & & \\ \vdots & & & & \ddots & & \\ a_{p-2} & & & & & \pm\varepsilon & \\ a_{p-1} & & & & & & \pm\varepsilon \end{array}\right) \begin{pmatrix} x \\ y_0 \\ y_1 \\ \vdots \\ y_{p-1} \end{pmatrix} \leq \begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}.$$

Each of the rows in the matrix corresponds to two facets – one for each sign. Since $\mathcal{W}_{p,1}$ is a simple polytope we may perturb the facet normals of the wedge product without changing the combinatorial structure. So for $M > 0$ large enough we obtain a realization of $\mathcal{W}_{p,1}$ of the form

$$\left(\begin{array}{ccc|cccc} a_0 & \pm\varepsilon & -\frac{1}{M} & -\frac{1}{M^2} & \cdots & -\frac{1}{M^{p-2}} & -\frac{1}{M^{p-1}} \\ a_1 & & \pm\varepsilon & \frac{1}{M} & & & \\ a_2 & & & \pm\varepsilon & \frac{1}{M} & & \\ \vdots & & & & \ddots & \ddots & \\ a_{p-2} & & & & & \pm\varepsilon & \frac{1}{M} \\ a_{p-1} & & & & & & \pm\varepsilon \end{array}\right) \begin{pmatrix} x \\ y_0 \\ y_1 \\ \vdots \\ y_{p-1} \end{pmatrix} \leq \begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}.$$

We rescale the inequalities of the wedge product and replace the variables by multiplying the ith pair of rows for $i = 0, \ldots, p-1$ with M^{p-1-i} and setting $y'_i = M^{p-1-i} y_i$ to get

$$\left(\begin{array}{ccc|cccc} M^{p-1}a_0 & \pm\varepsilon & -1 & -1 & \cdots & -1 & -1 \\ M^{p-2}a_1 & & \pm\varepsilon & 1 & & & \\ M^{p-3}a_2 & & & \pm\varepsilon & 1 & & \\ \vdots & & & & \ddots & \ddots & \\ Ma_{p-2} & & & & & \pm\varepsilon & 1 \\ a_{p-1} & & & & & & \pm\varepsilon \end{array}\right) \begin{pmatrix} x \\ y'_0 \\ y'_1 \\ \vdots \\ y'_{p-1} \end{pmatrix} \leq \begin{pmatrix} M^{p-1} \\ M^{p-2} \\ M^{p-3} \\ \vdots \\ 1 \end{pmatrix}.$$

The above modifications do not change the combinatorial structure: scaling the inequalities does not change the realization, and the change of variables is just a scaling of the coordinate axes. According to Definition 4.19, the surface $\mathcal{S}_{p,4}$ contains the following p-gon faces of $\mathcal{W}_{p,1}$:

$$\mathcal{S}_{p,4} = \Big\{ (\overline{j_0}, \ldots, \overline{j_{p-1}}) \in [2]^p \;\Big|\; \sum_{i=0}^{p-1} j_i \equiv 0, 1 \ \mathsf{mod}\ 2 \Big\}.$$

Since $[2] = \{0, 1\}$, the surface $\mathcal{S}_{p,4}$ contains *all* the "special" p-gons of $\mathcal{W}_{p,1}$ specified by Corollary 4.17. Each of the p-gons is obtained by intersecting p facets with one facet chosen from each pair of rows, that is, the p-gon $(\overline{j_0}, \dots, \overline{j_{p-1}})$ corresponds to a choice of signs $((-1)^{j_0}, \dots, (-1)^{j_{p-1}})$ in the above matrix. So the normals to the facets containing the p-gon $(\overline{j_0}, \dots, \overline{j_{p-1}})$ are:

$$\left(\begin{array}{ccc|cccc}
M^{p-1}a_0 & (-1)^{j_0}\varepsilon & -1 & -1 & \cdots & -1 & -1 \\
M^{p-2}a_1 & & (-1)^{j_1}\varepsilon & 1 & & & \\
M^{p-3}a_2 & & & (-1)^{j_3}\varepsilon & 1 & & \\
\vdots & & & & \ddots & \ddots & \\
Ma_{p-2} & & & & & (-1)^{j_{p-2}}\varepsilon & 1 \\
a_{p-1} & & & & & & (-1)^{j_{p-1}}\varepsilon
\end{array}\right) \tag{4.5}$$

By Lemma 1.11 a p-gon survives the projection to the first four coordinates if the last $p - 2$ columns of the matrix are positively spanning. Since ε is very small and the conditions of the projection lemma are stable under perturbation, the last $p - 2$ columns are positively spanning independent of the choice of signs $(-1)^{j_i}$. Consequently all the p-gons survive the projection to the first four coordinates.

This deformed realization of the wedge product has the additional property that all the p-gon faces of the surface have a face normal that has a negative fourth (y_1) coordinate as required in Lemma 1.13: The normal cone of a p-gon face is spanned by the normals of the facets containing the p-gon given by the matrix in Equation (4.5). Since the -1 in the y_1 coordinate of the first row dominates the y_1 coordinates of the other normals, the normal cone of the projected p-gon contains a vector $\nu = (\nu_x, \nu_y) \in \mathbb{R}^{2+p}$ with $\nu_{y_j} = 0$ for $j = 2, \dots, p - 1$ and negative $\nu_{y_1} < 0$. Hence the p-gons of the surface lie on the lower hull of the projected polytope.

Thus we get a coordinatization of the surface by orthogonal projection to the first three coordinates without the need of a Schlegel projection. $\square$

The construction of the surfaces via the projections of the deformed wedge products yields very "skinny" realizations for large p. A realization of the surface $\mathcal{S}_{5,4}$ of genus 5 is shown in Figure 4.9: Its 32 pentagon faces are arranged symmetrically to a horizontal plane. They come in 8 families of 4 pentagons. The 4 families above the horizontal plane are arranged as indicated by Figure 4.9 (right).

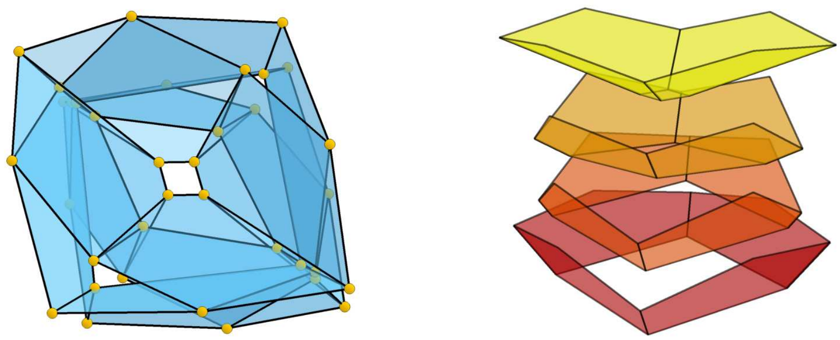


Figure 4.9: The surface $\mathcal{S}_{5,4}$ consists of 32 pentagons on 40 vertices.

4.3.2 Moduli of the projected surfaces

For $p \geq 4$, the surfaces $\mathcal{S}_{p,4}$ do neither contain vertices of degree three, nor triangle faces. So we cannot perform any local perturbations of the vertices or the face normals without changing the combinatorial structure. Following the approach of Section 1.3.1, the naïve estimate for the number of moduli of a realization S of the surface $\mathcal{S}_{p,2q}$ in $\mathbb{R}^3$ is:

$$\mathfrak{M}(S,3) \;\sim\; 2^{p-2}(3p-4p+12) = 2^{p-2}(12-p).$$

Hence for large p we would not expect that realizations of the surfaces $\mathcal{S}_{p,2q}$ have any non-trivial moduli. But as the surfaces are the projections of high-dimensional simple polytopes, we may use the moduli of the simple wedge product $\mathcal{W}_{p,1}$ to obtain moduli for the surfaces according to Theorem 2.30. So to obtain a lower bound on the dimension of the realization space of the surfaces $\mathcal{S}_{p,4}$ in $\mathbb{R}^3$, we need to determine an affine support set in the corresponding wedge product $\mathcal{W}_{p,1}$. The maximal size of an affine support A is:

$$\dim \mathcal{W}_{p,1} + 1 = p + 3 \;\leq\; |\mathsf{A}| \;\leq\; 2p = f_{p+1}(\mathcal{W}_{p,1})$$

by Corollary 2.24 and Lemma 2.25. Unfortunately, we have not been able to prove the existence of an affine support set of size $2p$ for this wedge product yet. But we may use the general approach as described in Corollary 2.24. This yields the following lower bound on the number of moduli for the realizations of the surfaces $\mathcal{S}_{p,4}$ in $\mathbb{R}^3$ obtained from Theorem 4.26.

Theorem 4.27 (Moduli of $\mathcal{S}_{p,4}$)**.** The realizations of the surfaces $\mathcal{S}_{p,4}$ in $\mathbb{R}^3$ obtained via projections of the wedge products $\mathcal{W}_{p,1}$ have at least $3(p+3)$ moduli. $\square$

The above theorem states that the number of moduli grows linearly with the size of the polygons p, whereas the naïve count suggests that the number of moduli decreases with the size of the polygons. This shows either that our realizations of the surfaces are very specific compared to arbitrary realizations or that there must be many incidence theorems hidden in the combinatorial structure of the surface.

The maximal number of moduli for our realizations of the wedge product surfaces obtainable from Theorem 2.30 is $6p$ since the wedge product $\mathcal{W}_{p,1}$ has $2p$ facets. One candidate for an affine support set which would yield the desired number of moduli is described in the following question. The set is motivated by the affine support set $\mathsf{A}_{\mathsf{neigh}}(p)$ of the p-cube discussed in Section 2.2.3. The p-cube is the fiber of an interior point of the polygon with respect to the canonical projection of the wedge product $\mathcal{W}_{p,1}$ onto the polygon (see Proposition 4.2). Maybe one can also use the fact that the wedge product is a degenerate deformed product to prove the existence or even the non-existence of an affine support set of size $2p$ for the wedge product $\mathcal{W}_{p,1}$.

Question 4.28. Let $\mathcal{W}_{p,1}$ be the wedge product of p-gon and 1-simplex for $p \geq 3$. Consider the subset A consisting of the vertices corresponding to the following vectors:

$$\mathsf{A} := \left\{ \begin{array}{l} ([2],[2],1,\ldots,0,0,0) \\ ([2],[2],0,\ldots,1,1,1) \\ (0,[2],[2],\ldots,0,0,0) \\ (1,[2],[2],\ldots,1,1,1) \\ (\,.\,,\,.\,,\,.\,,\ldots,\,.\,,\,.\,,\,.\,) \\ (0,0,0,\ldots,[2],[2],1) \\ (1,1,1,\ldots,[2],[2],0) \\ (1,0,0,\ldots,0,[2],[2]) \\ (0,1,1,\ldots,1,[2],[2]) \\ ([2],1,0,\ldots,0,0,[2]) \\ ([2],0,1,\ldots,1,1,[2]) \end{array} \right\}.$$

Is A an affine support for the wedge product $\mathcal{W}_{p,1}$?

4.3.3 Surface duality and polytope duality

In Section 4.3.1 we obtained a realization of the surface $\mathcal{S}_{p,4}$ as a subcomplex of a polytope boundary in $\mathbb{R}^4$, and thus as an embedded polyhedral surface in $\mathbb{R}^3$. Now our ambition is to derive a realization of the dual surface $\mathcal{S}^*_{p,4}$ from the primal one. This is not automatic: For this the dimension 4 is

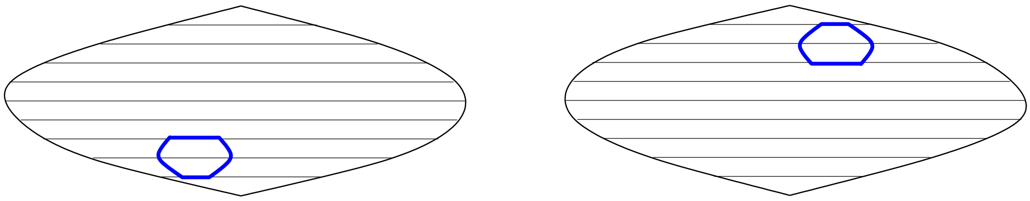


Figure 4.10: The face lattice of a polytope (left) and its dual (right). The poset of the surface is a subposet of the face lattice of the polytope (left) containing elements of dimensions 0, 1 and 2. The face lattice of the dual polytope contains a subposet corresponding to the dual surface, but the faces are of co-dimensions 0, 1, and 2.

crucial, and we also need that not only the primal surface $\mathcal{S}_{p,4}$, but also the "prism" $\mathcal{S}_{p,4} \times I$ over the primal surface embeds into a 4-polytope as a subcomplex.

(Indeed, a surface embedded as a subcomplex in the boundary of a d-polytope P exhibits a collection of faces of dimensions 0, 1, and 2. This corresponds to faces of dimensions $d-1$, $d-2$ and $d-3$ in the boundary of the dual polytope P^* (see Figure 4.10). These do not form a subcomplex unless the dimension is $d = 3$; for larger d this is a collection of high-dimensional faces that just have the inclusion relations dictated by the face poset of $\mathcal{S}^*_{p,4}$.)

For the following, the *prism* over a cell complex or polyhedral complex Σ refers to the product $\Sigma \times I$ with an interval $I = [0, 1]$, equipped with the obvious cellular structure that comes from the cell decompositions of Σ (as given) and of I (with two vertices and one edge). In particular, if Σ is a polytope (with the canonical face structure), then $\Sigma \times I$ is the prism over Σ in the classical sense of polytope theory.

Theorem 4.29. The prism $\mathcal{W}_{p,1} \times I$ over the wedge product $\mathcal{W}_{p,1}$ has a realization such that the prism over the surface $\mathcal{S}_{p,4} \times I$ survives the projection to $\mathbb{R}^4$ resp. $\mathbb{R}^3$.

Furthermore, the dual of the projected 4-polytope contains the dual surface $\mathcal{S}^*_{p,4}$ as a subcomplex, thus by constructing a Schlegel diagram we obtain a realization of the surface $\mathcal{S}^*_{p,4}$ in $\mathbb{R}^3$.

Proof. The proof follows the same line as the proof of Theorem 4.26. For small positive $0 < \delta \ll 1$ we construct the product $\mathcal{W}_{p,1} \times I$ of an orthogonal wedge product with an interval $\{z \in \mathbb{R}\colon \pm\delta z \leq 1\}$ which has the following inequality description:

$$\left(\begin{array}{ccccccc|c}
a_0 & \pm\varepsilon & & & & & & \\
a_1 & & \pm\varepsilon & & & & & \\
a_2 & & & \pm\varepsilon & & & & \\
\vdots & & & & \ddots & & & \\
a_{p-2} & & & & & \pm\varepsilon & & \\
a_{p-1} & & & & & & \pm\varepsilon & \\
\hline
 & & & & & & & \pm\delta
\end{array}\right)
\begin{pmatrix} x \\ y_0 \\ y_1 \\ \vdots \\ y_{p-1} \\ z \end{pmatrix}
\leq
\begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \\ 1 \\ 1 \end{pmatrix}.$$

As in Theorem 4.26 we will project onto the first 4 coordinates indicated by the vertical line in the next matrix. We perform a suitable deformation and obtain a deformed polytope combinatorially equivalent to $\mathcal{W}_{p,1} \times I$:

$$\left(\begin{array}{ccc|ccccc}
M^p a_0 & \pm\varepsilon & -1 & -1 & -1 & -1 & -1 & -1 \\
M^{p-1} a_1 & & \pm\varepsilon & 1 & & & & \\
M^{p-2} a_2 & & & \pm\varepsilon & 1 & & & \\
\vdots & & & & \ddots & \ddots & & \\
M^2 a_{p-2} & & & & & \pm\varepsilon & 1 & \\
M^1 a_{p-1} & & & & & & \pm\varepsilon & 1 \\
\mathbf{0} & & & & & & & \pm\delta
\end{array}\right)
\begin{pmatrix} x \\ y_0' \\ y_1' \\ \vdots \\ y_{p-1}' \\ z \end{pmatrix}
\leq
\begin{pmatrix} M^p \\ M^{p-1} \\ M^{p-2} \\ \vdots \\ M^2 \\ M^1 \\ 1 \end{pmatrix}.$$

The matrix has the same structure as the one used in Theorem 4.26 except for the $\mathbf{0}$ in the last row of the first column. The prism $\mathcal{S}_{p,4} \times I$ over the surface is a union of prisms over p-gons, where each prism is identified with the corresponding vector $(\overline{j_0}, \ldots, \overline{j_{p-1}})$ of the p-gon face in the surface $\mathcal{S}_{p,4}$. The normals of the facets containing a prescribed p-gon prism are:

$$\left(\begin{array}{ccc|ccccc}
M^p a_0 & (-1)^{j_0}\varepsilon & -1 & -1 & -1 & -1 & -1 & -1 \\
M^{p-1} a_1 & & (-1)^{j_1}\varepsilon & 1 & & & & \\
M^{p-2} a_2 & & & (-1)^{j_2}\varepsilon & 1 & & & \\
\vdots & & & & \ddots & \ddots & & \\
M^2 a_{p-2} & & & & & (-1)^{j_{p-2}}\varepsilon & 1 & \\
M^1 a_{p-1} & & & & & & (-1)^{j_{p-1}}\varepsilon & 1
\end{array}\right)$$

As in the previous proof, the last $p-2$ columns of these vectors are positively spanning because they are positively spanning for $\varepsilon = 0$ and the given configuration is only a perturbation since ε is very small. Further the -1 in the y_1 coordinate of the first row dominates and yields a normal with negative y_1 coordinate for the prisms over the p-gons. So the prism over the surface survives the projection to a 4-dimensional polytope and lies on its lower hull using Lemma 1.13. This way we obtain a realization of the prism over the surface in $\mathbb{R}^3$ by orthogonal projection.

Looking at the face lattice of the projected polytope we observe that it contains three copies of the face lattice of the surface $\mathcal{S}_{p,4}$ – the top and the bottom copy and another copy raised by one dimension corresponding to the prism faces connecting top and bottom copy shown in Figure 4.11 (left). The face lattice of the dual polytope contains three copies of the face lattice of the dual surface. One of those copies based at the vertices corresponds to the dual surface contained in the 2-skeleton of the dual polytope. $\square$

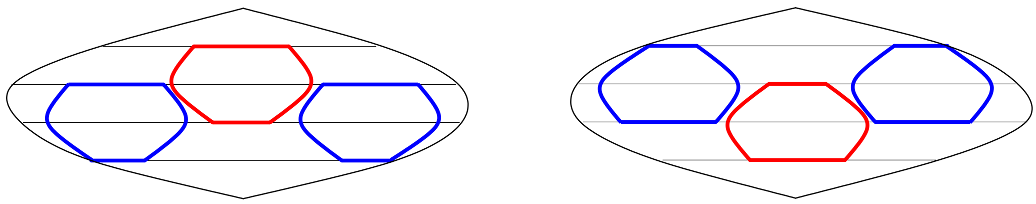


Figure 4.11: The face lattice of a 4-polytope containing the face poset of the prism over the surface $\mathcal{S}_{p,4}$ (left). The face lattice of the dual polytope contains the face poset of dual surface (right).

4.4 Topological obstructions

In this section we show that for other parameters p and q the technique used in this chapter to obtain realizations of the surfaces $\mathcal{S}_{p,2q}$ does not work. Using the combinatorial structure of the surface we may use a result about the non-projectability of the product of simplices from Section 3.3.3 to show the non-projectability of some of the surfaces contained in the wedge products.

Theorem 4.30 (Non-projectability of wedge product surfaces)**.** There is no realization of the wedge product $\mathcal{W}_{p,q-1}$, with $p \geq 3$ and $q \geq 4$, such that all the faces corresponding to the surface $\mathcal{S}_{p,2q}$ are strictly preserved by the projection $\pi : \mathbb{R}^{2+p(q-1)} \longrightarrow \mathbb{R}^e$ for $e < p+1$.

Proof. The regular surface contained in the wedge product consists of the following p-gons:

$$\mathcal{S}_{p,2q} = \Big\{ (\overline{j_0}, \dots, \overline{j_{p-1}}) \;\Big|\; \sum_{i=0}^{p-1} j_i \equiv 0, 1 \ \mathsf{mod}\ q \Big\}.$$

We prove the theorem by contradiction. So assume that there exists a realization of $\mathcal{W}_{p,q-1}$ such that the surface $\mathcal{S}_{p,2q}$ is strictly preserved by the projection. According to Theorem 2.4 the associated simplicial complex $\mathsf{K}(\mathcal{W}_{p,q-1},\pi)$ of the strictly preserved faces is embedded in a sphere of dimension:

$$pq - (p(q-1)+2) + e - 2 = p + e - 4.$$

Since the polygons of the wedge product surface $\mathcal{S}_{p,2q}$ are strictly preserved by the projection π the simplicial complex $\mathsf{K}(\mathcal{W}_{p,q-1},\pi)$ contains a subcomplex Σ corresponding to the polygons of $\mathcal{S}_{p,2q}$. This subcomplex Σ consists of all $(p-1)$-simplices $(j_0,\ldots,j_{p-1})$ in the join $(\mathsf{D}_q)^{*p}$ with $\sum_{i=0}^{p-1} j_i \equiv 0,1 \bmod q$, where D_q is the simplicial complex on q disjoint vertices. To analyze the projectability of the wedge product surface we first remove the asymmetry from Σ by only considering the first $p-1$ simplices in the wedge product:

$$\Sigma' = \{(j_0,\ldots,j_{p-2}) : j_i \in [q]\}.$$

This is exactly the 0th skeleton complex $\Sigma_0(\Delta_{q-1}^{p-1})$ of the $(p-1)$-fold product of $(q-1)$-simplices. By Corollary 3.21 the embeddability dimension of $\Sigma_0(\Delta_{q-1}^{p-1})$ is $2p-3$. Hence the embeddability dimension of $\mathsf{K}(\mathcal{W}_{p,q-1},\pi)$ is at least $2p-3$ because $\Sigma_0(\Delta_{q-1}^{p-1}) = \Sigma' \subseteq \Sigma \subseteq \mathsf{K}(\mathcal{W}_{p,q-1},\pi)$. Thus if $e < p+1$ we obtain:

$$p + e - 4 < 2p - 3.$$

This is a contradiction to the embeddability of $\mathsf{K}(\mathcal{W}_{p,q-1},\pi)$ into an $(p+e-4)$-dimensional sphere. So there exists no realization of $\mathcal{W}_{p,q-1}$ such that the surface $\mathcal{S}_{p,2q}$ is strictly preserved by the projection to $\mathbb{R}^e$. $\square$

The above theorem does not claim that there is no realization of the surfaces $\mathcal{S}_{p,2q}$ for $p \geq 4$ and $q \geq 3$ in $\mathbb{R}^3$ at all. It only proves that our technique of embedding the surface in the wedge product and then projecting it to the lower hull of a 4-polytope will not yield a proper realization. It would hence be interesting to find other simple polytopes containing the surfaces or to find realizations directly in $\mathbb{R}^3$. For $q = 3$ we obtain the triangulated surfaces $\mathcal{S}_{3,2q}$ studied by Coxeter [15]. Unfortunately we can neither construct wedge products $\mathcal{W}_{3,q-1}$ such that the surfaces survive the projection to $\mathbb{R}^4$ nor prove the non-projectability.

Chapter 5

Polyhedral surfaces in products of polygons

The surfaces of McMullen, Schulz & Wills [37] are of particular interest, because they have an "unusually large genus" compared with the number of vertices. Further they can be realized in $\mathbb{R}^3$ with planar convex quadrilaterals without self-intersection. The construction of McMullen, Schulz & Wills is done entirely in $\mathbb{R}^3$ by building a "corner" of the surface and then reflecting it around a torus as sketched in Section 5.1.1.

We take a different approach and describe two families of surfaces primarily as subcomplexes of high-dimensional polytopes. They appear in the 2-skeleta of the (deformed) products of polygons. For a suitable deformed realization of the product we show that these subcomplexes are strictly preserved under a projection to $\mathbb{R}^4$, resp. $\mathbb{R}^3$. One family of surfaces, given in Section 5.1.2, is a generalization of the surfaces constructed by McMullen, Schulz & Wills. The other family, constructed in Section 5.1.3, contains all the "polygon" faces of the product of polygons, that is, all the faces of the product that are products of one polygon with vertices of the other factors. These surfaces have arbitrarily large vertex degree and an average polygon size arbitrarily close to 8. A natural variation of Wills' question concerning the existence of equivelar surfaces of type $\{p, q\}$ for $p > 4$ and $q > 4$ is: Can polyhedral surfaces with average vertex degree and average polygon size larger than 4 be embedded in $\mathbb{R}^3$? Our quad-polygon surfaces of Section 5.1.3 provide such polyhedral surfaces. Since we are able to construct a large affine support set for the product of polygons, we obtain a large lower bound on the number of moduli for our realizations of the surfaces in Section 5.2.2. Moreover, in Section 5.2.3, we use polytope duality to obtain realizations of the duals of the surfaces.

Before we start with the combinatorial description of the surfaces, let us fix some notation for the facets of the product of polygons. Let D_{2p} be a $2p$-gon given by the inequality system $Ax \leq \mathbb{1}$ where $A \in \mathbb{R}^{2p\times 2}$ and $x \in \mathbb{R}^2$. Then the inequality description of the product $(\mathrm{D}_{2p})^r \subset \mathbb{R}^{2r}$ is

$$\begin{pmatrix} A & & & \\ & A & & \\ & & \ddots & \\ & & & A \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_{r-1} \end{pmatrix} \leq \begin{pmatrix} \mathbb{1} \\ \mathbb{1} \\ \vdots \\ \mathbb{1} \end{pmatrix}.$$

with $x_i \in \mathbb{R}^2$ and rows $a_j x_i \leq 1$ for $i \in [r]$ and $j \in [2p]$. We denote the facet defined by the inequality $a_j x_i \leq 1$ by $h_{i,j}$.

5.1 Hamiltonian surfaces

Hamiltonian surfaces are polyhedral surfaces in the 2-skeleton of a polytope which contain the entire graph. They are higher dimensional analogs of Hamiltonian cycles in the graph of a polytope, which are connected 1-manifolds without boundary containing all the vertices. (A general definition of k-Hamiltonian m-manifold is given in Ewald et al. [19].) The products of polygons contain two such families of polyhedral surfaces, which will be described in the following. The family described in Section 5.1.2 includes the equivelar surfaces of type $\{4, q\}$ of McMullen, Schulz & Wills [37], reviewed in Section 5.1.1. The surfaces given in Section 5.1.3 are not equivelar any more but of high “complexity” since they use all the “polygon” faces of the product. Both surfaces are defined in a purely combinatorial way as subcomplexes of the 2-skeleton of the product of polygons.

To construct the surfaces we introduce some notation for the 2-skeleton of the product of even polygons. All index calculations are performed modulo $2p$ resp. the size of the polygon in the following. We start with a single $2p$-gon $\mathcal{D}_{2p}$, whose vertices are labeled with $(j, j+1)$ or $(j, j-1)$ where the first component $j \in [2p]$ is even (and hence $j+1$ resp. $j-1$ is odd). The edge from $(j, j-1)$ to $(j, j+1)$ is labeled by $(j, \emptyset)$ and the edge from $(j, j+1)$ to $(j+2, j+1)$ is labeled by $(\emptyset, j+1)$. The polygon itself is denoted by $(\emptyset, \emptyset)$. See Figure 5.1.

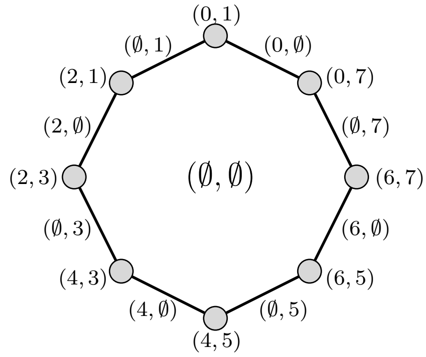


Figure 5.1: The non-empty faces of an 8-gon are denoted by vectors (j, j') with $j \in \{0, 2, 4, 6\} \cup \{\emptyset\}$ and $j' \in \{1, 3, 5, 7\} \cup \{\emptyset\}$.

Summarizing, we identify the non-empty faces of an even $2p$-gon with the following vectors:

$$\begin{aligned}
\mathcal{D}_{2p} = &\{(j, \emptyset) : j \in [2p] \text{ even }\} && \text{(even edges)} \\
\cup &\{(\emptyset, j) : j \in [2p] \text{ odd }\} && \text{(odd edges)} \\
\cup &\{(j, j') : j, j' \in [2p],\ j \text{ even, } j' \text{ odd, } j' = j \pm 1\ \} && \text{(vertices)} \\
\cup &\{(\emptyset, \emptyset)\} && \text{(polygon)}
\end{aligned} \tag{5.1}$$

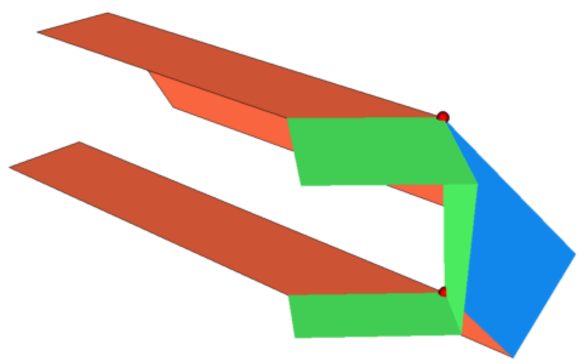
All the faces of the r-fold product $(\mathcal{D}_{2p})^r$ of even polygons are products of non-empty faces of its factors. Hence they may be identified with vectors $(j_0, j_0'; j_1, j_1'; \ldots; j_{r-1}, j_{r-1}')$ where each (j_i, j_i') is a non-empty face of the $2p$-gon in the ith factor. This is exactly the representation of the faces of the product as intersection of facets: The facets of the product of polygons $(\mathcal{D}_{2p})^r$ are denoted by $h_{i,j}$ with $i \in [r]$ and $j \in [2p]$. Then the vector $(j_0, j_0'; j_1, j_1'; \ldots; j_{r-1}, j_{r-1}')$ represents the face $(\mathcal{D}_{2p})^r \cap \left(\bigcap_{i\in[r]} h_{i,j_i} \cap h_{i,j_i'}\right)$ (where $h_{i,\emptyset} = \mathbb{R}^{2r}$).

The vertices of the product correspond to vectors with no $\emptyset$-entry, that is, vectors $(j_0, j_0'; j_1, j_1'; \ldots; j_{r-1}, j_{r-1}')$ with $j_i \in [2p]$ even and $j_i' = j_i \pm 1 \in [2p]$ odd. The edges are products of one edge and $r-1$ vertices and are identified with vectors with one $\emptyset$-entry and again $j_i \in [2p]$ even and $j_i' = j_i \pm 1 \in [2p]$ odd. A 2-face of the product $(\mathcal{D}_{2p})^r$ is either a $2p$-gon or a quadrilateral. Both 2-faces correspond to vectors with two $\emptyset$-entries: If the $\emptyset$-entries belong to one factor then the face is a $2p$-gon, if the $\emptyset$-entries belong to distinct factors the face is a quadrilateral.

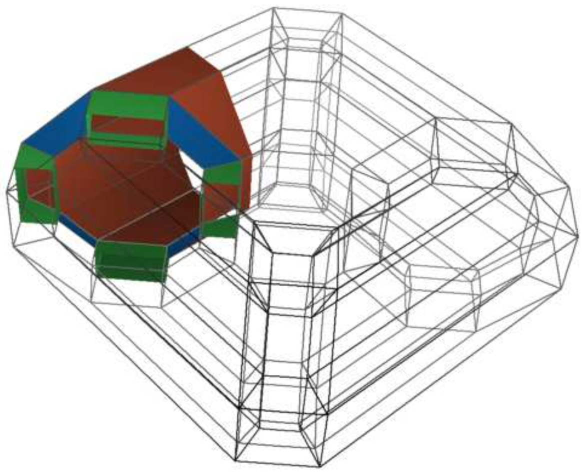
5.1.1 MSW surfaces

In this section we review the construction of McMullen, Schulz & Wills [37] for equivelar surfaces of type $\{4, q\}$, which we call *MSW surfaces* for short. These surfaces are included in the family of surfaces considered by Coxeter [15] in terms of their symmetry groups—the group $\{4, 2m|4^{m-1}\}$ is the symmetry group of the generalized MSW surfaces constructed in Section 5.1.2. Ringel [41] considered three problems concerning cubes and identified the Hamiltonian surface with symmetry group $\{4, m\}$ in the 2-skeleton of the cube. Since the even-dimensional cube is just the product of quadrilaterals, that is, even polygons, these surfaces are a subclass of the surfaces constructed in Section 5.1.2 as well. Using geometric intuition McMullen, Schulz & Wills [37] were able to construct realizations of the MSW surfaces in $\mathbb{R}^3$ that include the surfaces of Ringel, but are not as general as Coxeter's. Their construction starts with a "corner" of the surface. This corner is sheared such that it may be reflected around a $2p_1 \times 2p_2$ torus building a two parameter family of surfaces. For $p_1 = p_2 = 2$ the MSW surfaces are exactly the surfaces considered by Ringel. An example for $p_1 = p_2 = 4$ of the construction of McMullen, Schulz & Wills is described in the following example.

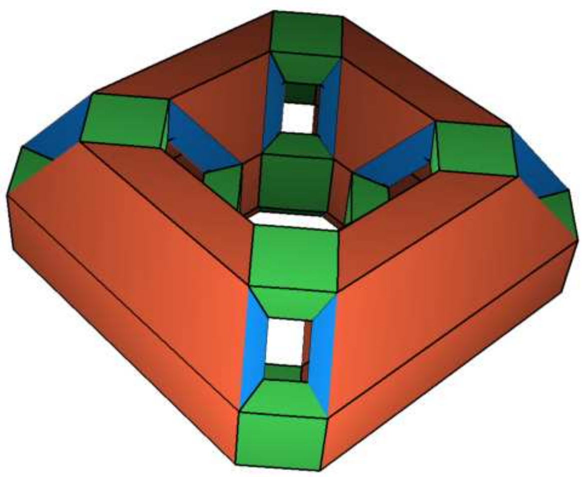
Example 5.1. We start with a corner consisting of two vertices, one edge, and 5 rays at each vertex. The rays and edges are connected by half- and



In this case, the constructed corner fits nicely into an (8×8)-torus. After a reflection around one of the meridian curves the blue half-planes close to form quadrilaterals, and the green and red quarter-planes fit together to build half-planes.



By reflecting the above part of the surface around the other meridian of the torus, all the remaining green and red half-planes form quadrilaterals. So the this MSW surface consists of 64 corners which are reflected around an (8×8)-torus.



This surface is exactly the generalized MSW surface in the product of two 8-gons, as constructed in the following section.

5.1.2 Generalized MSW surfaces

In this section we generalize the MSW surfaces by defining a certain subcomplex of the product of even polygons and showing that the subcomplex is an orientable surface. The projection is postponed to Section 5.2. We start by describing a certain subcomplex of the 2-skeleton of the product $(\mathcal{D}_{2p})^r$ of even polygons. Since we want to construct a surface of type $\{4, 2r\}$ the subcomplex consists of quadrilateral faces only. Each quadrilateral corresponds to a vector $(j_0, j_0'; j_1, j_1'; \ldots; j_{r-1}, j_{r-1}')$ with

- $j_i, j_i' \in [2p] \cup \{\emptyset\}$,
- j_i even and $j_i' = j_i \pm 1$ odd, and
- exactly two $\emptyset$-entries which must not be in the same factor.

Each vertex of the product corresponds to a vector without $\emptyset$-entries and each edge is identified with a vector with exactly one $\emptyset$-entry. To illustrate the notation consider the following example.

Example 5.2. The faces of the product of three octagons $(\mathcal{D}_8)^3$ correspond to vectors $(j_0, j_0'; j_1, j_1'; j_2, j_2')$ with $j_0, j_1, j_2 \in \{0, 2, 4, 6, \emptyset\}$ and $j_0', j_1', j_2' \in \{1, 3, 5, 7, \emptyset\}$. In preparation of Theorem 5.5 we have a look at the following set of faces:

$$\mathcal{Q}_{4(8),6} = \left\{ \begin{array}{l} (\,\emptyset\,,\ j_0'\ ;\ \emptyset\,,\ j_1'\ ;\ j_2,\ j_2') \\ (j_0,\ j_0'\ ;\ \emptyset\,,\ j_1'\ ;\ \emptyset\,,\ j_2') \\ (j_0,\ \emptyset\ ;\ j_1,\ j_1'\ ;\ \emptyset\,,\ j_2') \\ (j_0,\ \emptyset\ ;\ j_1,\ \emptyset\ ;\ j_2,\ j_2') \\ (j_0,\ j_0'\ ;\ j_1,\ \emptyset\ ;\ j_2,\ \emptyset\,) \\ (\,\emptyset\,,\ j_0'\ ;\ j_1,\ j_1'\ ;\ j_2,\ \emptyset\,) \end{array} \right\}$$

Every vector of this set corresponds to 128 quadrilateral faces: In each factor of the product we choose either an even edge $(j_i, \emptyset)$, an odd edge $(\emptyset, j_i')$, or a vertex (j_i, j_i') to obtain a quadrilateral of each family. Since for each even/odd edge we have four possibilities and there are eight vertices in each factor we get 128 quadrilaterals per family.

Every edge of $(\mathcal{D}_8)^3$ corresponds to a vector with exactly one $\emptyset$-entry and is contained in exactly two quadrilaterals belonging to two different families. This makes the subcomplex $\mathcal{Q}_{4(8),6}$ a closed pseudo-manifold with 768 quadrilaterals, 1536 edges, and 512 vertices.

The quadrilaterals incident to the vertex $(0, 1; 4, 3; 0, 7)$ are given in Figure 5.2. These quadrilaterals fit together to form a two dimensional ball.

$$\left\{\begin{array}{l}(\,\emptyset,\,1\;;\,\emptyset,\,3\;;\,0,\,7\,)\\(\,0,\,1\;;\,\emptyset,\,3\;;\,\emptyset,\,7\,)\\(\,0,\,\emptyset\;;\,4,\,3\;;\,\emptyset,\,7\,)\\(\,0,\,\emptyset\;;\,4,\,\emptyset\;;\,0,\,7\,)\\(\,0,\,1\;;\,4,\,\emptyset\;;\,0,\,\emptyset\,)\\(\,\emptyset,\,1\;;\,4,\,3\;;\,0,\,\emptyset\,)\end{array}\right\}$$

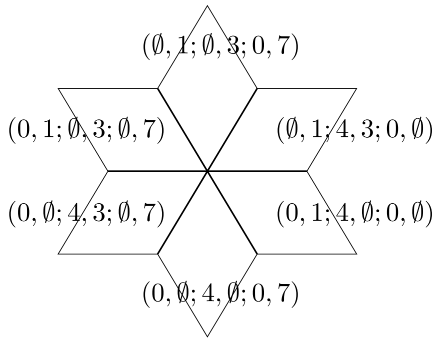


Figure 5.2: The vertex star of the vertex $(0,1;4,3;0,7)$ in the subcomplex $\mathcal{Q}_{4(8),6} \subset (\mathrm{D}_8)^3$. The combinatorial representation of the quadrilaterals adjacent to the vertex (left). The six quadrilaterals form a 2-dimensional ball (right).

This is true for any vertex of the product and hence $\mathcal{Q}_{4(8),6}$ is a Hamiltonian 2-manifold.

Based on the previous example we define a subcomplex of the quadrilaterals of the product of even polygons.

Definition 5.3 (Generalized MSW surface)**.** The *generalized MSW surface* is the subcomplex $\mathcal{Q}_{4(2p),2r}$ of the product $(\mathcal{D}_{2p})^r$ of r even $2p$-gons generated by the quadrilaterals that correspond to the vectors $(j_0, j_0'; j_1, j_1'; \ldots; j_{r-1}, j_{r-1}')$ with exactly two $\emptyset$-entries at the following positions:

1. two consecutive even edges, i.e. $j_i = j_{i+1} = \emptyset$ with $i = 0, \ldots, r-2$, or
2. an odd edge of the first factor and an even edge of the last factor, i.e. $j_0' = \emptyset$ and $j_{r-1} = \emptyset$, or
3. two consecutive odd edges, i.e. $j_i' = j_{i+1}' = \emptyset$ with $i = 0, \ldots, r-2$, or
4. an even edge of the first factor and an odd edge of the last factor, i.e. $j_0 = \emptyset$ and $j_{r-1}' = \emptyset$.

We collect some easy facts of the subcomplex $\mathcal{Q}_{4(2p),2r}$ in the next lemma.

Lemma 5.4. The subcomplex $\mathcal{Q}_{4(2p),2r}$ of the product $(\mathcal{D}_{2p})^r$ of even $2p$-gons $\mathcal{D}_{2p}$ is a closed connected 2-dimensional pseudomanifold containing all the vertices and edges of the product. Its f-vector is $((2p)^r, r(2p)^r, \frac{1}{2}r(2p)^r)$.

Proof. Each edge of the product $(\mathcal{D}_{2p})^r$ corresponds to a vector with exactly one $\emptyset$-entry. From the description of $\mathcal{Q}_{4(2p),2r}$ we easily see that every edge of the product is contained in exactly two quadrilaterals. So $\mathcal{Q}_{4(2p),2r}$ is a closed 2-dimensional pseudomanifold containing all $r(2p)^r$ edges of the product. It is connected because the graph of the polytope $(\mathcal{D}_{2p})^r$ is connected.

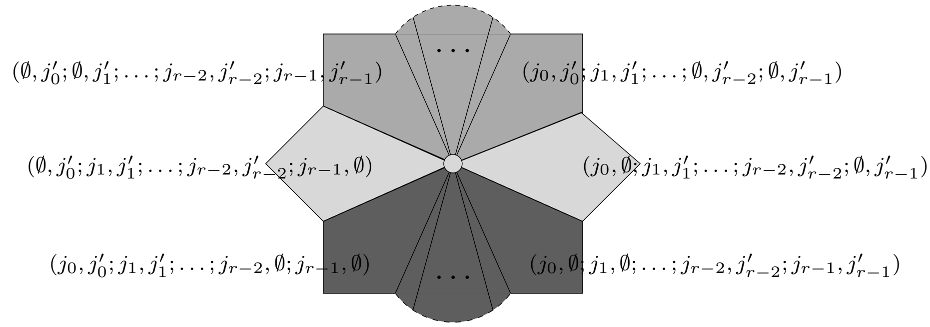


Figure 5.3: The star of a vertex $(j_0, j_0'; j_1, j_1'; \ldots; j_{r-1}, j_{r-1}')$ of the complex $\mathcal{Q}_{4(2p),2r}$. The faces at the top are the quadrilaterals that are products of even edges. The faces at the bottom are products of odd edges. The left and the right quadrilateral correspond to the products of an even and an odd edge.

Further, $\mathcal{Q}_{4(2p),2r}$ contains all $(2p)^r$ vertices of the product, since every vertex is contained in some edge. Finally, each of the $2r$ families of quadrilaterals given in Definition 5.3 contains $p^2(2p)^{r-2}$ quadrilaterals yielding a total number of $\frac{1}{2}r(2p)^r$ quadrilaterals in $\mathcal{Q}_{4(2p),2r}$. $\square$

Theorem 5.5. The subcomplex $\mathcal{Q}_{4(2p),2r}$ of the product $(\mathcal{D}_{2p})^r$ of $r \geq 2$ even $2p$-gons is an equivelar surface of type $\{4, 2r\}$ with f-vector

$$f(\mathcal{Q}_{4(2p),2r}) \;=\; \tfrac{1}{2}(2p)^r(2, 2r, r)$$

and genus $1 + \frac{1}{4}(r-2)(2p)^r$.

Proof. By Lemma 5.4 we obtain that $\mathcal{Q}_{4(2p),2r}$ is a closed connected 2-dimensional pseudomanifold. All the faces of $\mathcal{Q}_{4(2p),2r}$ are quadrilaterals because they are products of two edges. Since a vertex of the product is the product of vertices and each vertex is contained in exactly two edges in each factor, every vertex has degree $2r$. So the complex is equivelar of type $\{4, 2r\}$.

To prove that the given complex is a 2-manifold we show that the star of every vertex is a 2-dimensional ball. Consider an arbitrary vertex v corresponding to the vector $(j_0, j_0'; j_1, j_1'; \ldots; j_{r-1}, j_{r-1}')$ with $j_i \in [2p]$ even and $j_i' \in [2p]$ odd. The quadrilaterals adjacent to the vertex v are arranged as in Figure 5.3, so $\mathcal{Q}_{4(2p),2r}$ is a 2-manifold.

It remains to prove that $\mathcal{Q}_{4(2p),2r}$ is orientable, which is not difficult but a little technical. We assign an orientation to each quadrilateral and show that it is consistent, that is, every edge is oriented in opposite directions by the two adjacent quadrilaterals.

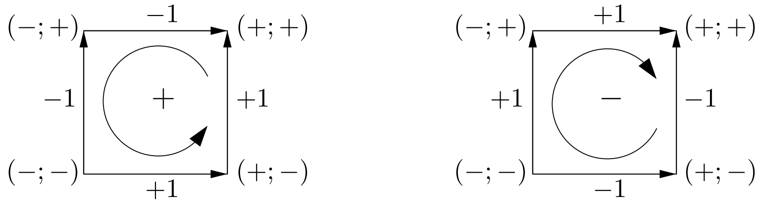
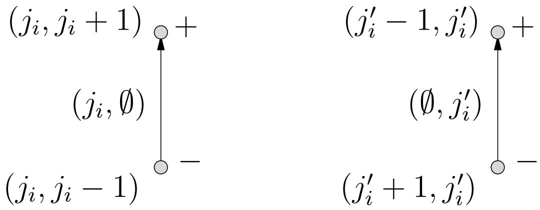


Figure 5.4: The orientation of the quadrilaterals of the surface $\mathcal{Q}_{4(2p),2r}$. The sign of the quadrilateral determines its orientation. Each vertex of the quadrilateral is the product of two vertices $v = (j_i, j'_i)$ and $w = (j_k, j'_k)$ where i and k are cyclically adjacent. They are labeled with their sign $(\sigma(v), \sigma(w))$. The edges are labeled with $+1$ if the orientation coincides with the canonical orientation and -1 if not.

We consider the following map on the vertices (j_i, j'_i) of the polygons:

$$\sigma(j_i, j'_i) = \begin{cases} +1 & \text{if } j'_i \equiv j_i + 1 \;\mathsf{mod}\; p \\ -1 & \text{if } j'_i \equiv j_i - 1 \;\mathsf{mod}\; p. \end{cases}$$

Every edge $(j_i, \emptyset)$ resp. $(\emptyset, j'_i)$ is adjacent to a positive vertex $(j_i, j_i + 1)$ resp. $(j'_i - 1, j'_i)$ and a negative vertex $(j_i, j_i - 1)$ resp. $(j'_i + 1, j'_i)$ and thus has a canonical orientation from the negative to the positive vertex as follows:



Each quadrilateral is the product of two edges of cyclically adjacent factors and $r - 2$ vertices. We extend the map σ to the quadrilaterals by taking the product of the signs of the $r - 2$ vertices. In Figure 5.4 we define the orientation of the quadrilaterals according to their sign. The vertices of the quadrilaterals are denoted only by their signs. The edges are labeled with a $+1$ if the orientation of the edge induced by the orientation of the quadrilateral coincides with the canonical orientation of the edge, and -1 otherwise. We are now ready to prove the orientability of the surface. Consider the following edge:

$$e = (j_0, j'_0; \ldots; j_{i-1}, j'_{i-1}; j_i, \emptyset; j_{i+1}, j'_{i+1}; \ldots; j_{r-1}, j'_{r-1}).$$

The two adjacent quadrilaterals are:

$$\begin{aligned} Q &= (j_0, j'_0; \ldots; j_{i-1}, \emptyset; j_i, \emptyset; j_{i+1}, j'_{i+1}; \ldots; j_{r-1}, j'_{r-1}) \qquad \text{and} \\ Q' &= (j_0, j'_0; \ldots; j_{i-1}, j'_{i-1}; j_i, \emptyset; j_{i+1}, \emptyset; \ldots; j_{r-1}, j'_{r-1}) \end{aligned}$$

In the case where $i = 0$ or $i = r - 1$ we have to take the cyclically adjacent factors, but the calculations remain the same. Define $\sigma_{i-1} = \sigma(j_{i-1}, j'_{i-1})$ and $\sigma_{i+1} = \sigma(j_{i+1}, j'_{i+1})$. Now we read the orientation of e off Figure 5.4 and obtain:

1. the edge e is $\sigma(Q) \cdot \sigma_{i-1}$ oriented in Q, and
2. the edge e is $-\sigma(Q') \cdot \sigma_{i+1}$ oriented in Q'.

If $\sigma_{i-1} = \sigma_{i+1}$ then the quadrilaterals Q and Q' adjacent to e have the same orientation and thus $\sigma(Q) \cdot \sigma_{i-1} = -(-\sigma(Q') \cdot \sigma_{i+1})$. So the edge has opposite direction in Q and Q'. If $\sigma_{i-1} \neq \sigma_{i+1}$ then Q and Q' have different orientations and thus $\sigma(Q) \cdot \sigma_{i-1} = -(-\sigma(Q') \cdot \sigma_{i+1})$ as well. This proves that the orientation defined via the extension of the vertex signs onto the quadrilaterals is an orientation of the surface $\mathcal{Q}_{4(2p),2r}$.

For orientable surfaces we have the equality $\chi(\mathcal{Q}_{4(2p),2r}) = 2 - 2g(\mathcal{Q}_{4(2p),2r})$ relating the Euler characteristic χ and the genus g. Taking the f-vector from Lemma 5.4 we obtain $g(\mathcal{Q}_{4(2p),2r}) = 1 + \frac{1}{4}(2p)^r(r-2)$. $\square$

Remark 5.6. Our construction can easily be generalized to obtain surfaces contained in the product $\prod \mathcal{D}_{2p_i} \times I^k$ of arbitrary even polygons $\mathcal{D}_{2p_i}$ and intervals I. The intervals allow us to construct surfaces with odd vertex degree. The MSW surfaces correspond to the surfaces in the 2-skeleton of the product $\mathcal{D}_{2p_1} \times \mathcal{D}_{2p_2} \times I^k$ of a $2p_1$-gon $\mathcal{D}_{2p_1}$, a $2p_2$-gon $\mathcal{D}_{2p_2}$ and k intervals.

Proposition 5.7. The automorphism group of the subcomplex $\mathcal{Q}_{4(2p),2r}$ of the product of r even $2p$-gons $(\mathcal{D}_{2p})^r$ acts transitively on the flags, that is, $\mathcal{Q}_{4(2p),2r}$ is a regular polyhedral surface.

Proof. We will show that every flag of the surface may be mapped to one particularly simple flag by a certain sequence of a set of combinatorial automorphisms. Our basis flag is the following.

$$\mathcal{F}_0 : (0,1;0,1;\ldots;0,1) \subset (0,\emptyset;0,1;\ldots;0,1) \subset (0,\emptyset;0,\emptyset;0,1;\ldots;0,1).$$

The automorphisms of the surface act on the vectors representing the vertices, edges and quadrilaterals simultaneously. Let us consider the following two maps:

$$\mathsf{F} : (j_0, j'_0; \ldots; j_{r-1}, j'_{r-1}) \mapsto (j_{r-1}, j'_{r-1}; \ldots; j_1, j'_1; j_0, j'_0) \qquad \text{(Flip)}$$
$$\mathsf{S} : (j_0, j'_0; \ldots; j_{r-1}, j'_{r-1}) \mapsto (1 - j'_{r-1}, 1 - j_{r-1}; j_0, j'_0; \ldots; j_{r-2}, j'_{r-2}) \qquad \text{(Shift)}$$

The two maps are obviously automorphisms of the vertices, edges and quadrilaterals of the product $(\mathcal{D}_{2p})^r$. But they are also automorphisms of the surface $\mathcal{Q}_{4(2p),2r}$: Since the surface contains all vertices and edges we only need to verify that the quadrilaterals of the surface are invariant under F and S. But the two maps are exactly constructed in such a way that they map the four families of quadrilaterals contained in the surface (see Definition 5.3) onto each other.

By applying a suitable number of shifts S and a flip F if needed to an arbitrary flag we obtain a flag $\mathcal{F}$ of the form:

$$\begin{aligned}\mathcal{F}:\ (j_0, j_0'; j_1, j_1'; \ldots ; j_{r-1}, j_{r-1}') & \\ \subset (j_0, \emptyset; j_1, j_1'; \ldots ; j_{r-1}, j_{r-1}') & \\ \subset (j_0, \emptyset; j_1, \emptyset; \ldots ; j_{r-1}, j_{r-1}').\end{aligned}$$

The automorphisms of each of the factors $\mathcal{D}_{2p}$ of the product induce automorphisms on the surface as well given by the following maps:

$$\begin{aligned}\mathsf{E}_i : (\ldots ; j_i, j_i'; \ldots) &\mapsto (\ldots ; -j_i, -j_i'; \ldots) && \text{(Exchange)}\\ \mathsf{R}_i : (\ldots ; j_i, j_i'; \ldots) &\mapsto (\ldots ; j_i + 2, j_i' + 2; \ldots) && \text{(Rotate)}\end{aligned}$$

All other entries of the vectors are constant. Since each of the E_i and R_i allows to map an arbitrary pair (j_i, j_i') onto the pair $(0, 1)$ we are able to map the flag $\mathcal{F}$ to the flag $\mathcal{F}_0$. Thus the group of automorphisms acts transitively on the flags of the surface. □

5.1.3 Quad-polygon surfaces

In this section we describe another surface contained in the r-fold product $(\mathcal{D}_{2p})^r$ of even $2p$-gons. It is no longer equivelar but contains all $2p$-gon faces of the product and thus all the vertices and the entire graph.

Definition 5.8 (Quad-polygon surface $\mathcal{QP}_{\{4,2p\},2r}$)**.** The facets of the polyhedral surface $\mathcal{QP}_{\{4,2p\},2r}$ contained in the 2-skeleton of the r-fold product $(\mathcal{D}_{2p})^r$ of even $2p$-gons are given by the vectors $(j_0, j_0'; j_1, j_1'; \ldots ; j_{r-1}, j_{r-1}')$ with exactly two $\emptyset$-entries at cyclically adjacent positions.

Let us get used to the structure of the surface by the following simple example.

Example 5.9. Consider the product of two 8-gons $(\mathcal{D}_8)^2$. Then the polyhedral surface $\mathcal{QP}_{\{4,8\},4}$ contains the following polygons:

$$\mathcal{QP}_{\{4,8\},2} = \left\{\begin{matrix}(\ \emptyset,\ \emptyset\ ;\ j_1,\ j_1')\\ (j_0,\ \emptyset\ ;\ \emptyset,\ j_1')\\ (j_0,\ j_0'\ ;\ \emptyset,\ \emptyset\ )\\ (\ \emptyset,\ j_0'\ ;\ j_1,\ \emptyset\ )\end{matrix}\right\}.$$

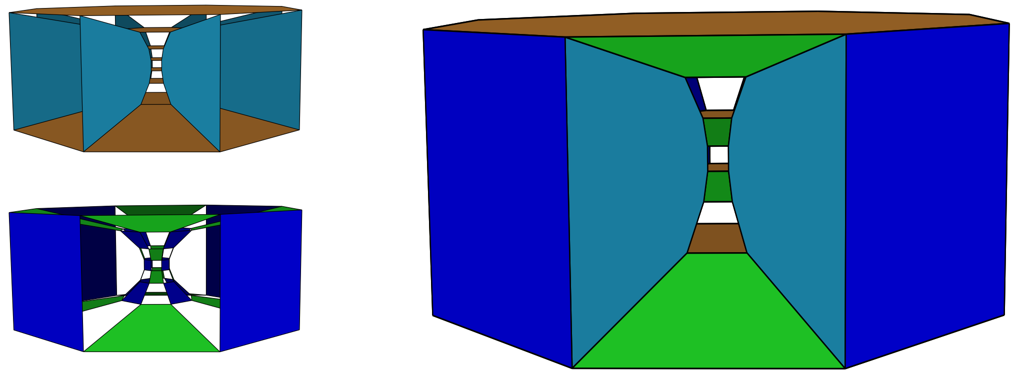


Figure 5.5: The Hamiltonian surface $\mathcal{QP}_{\{4,8\},4}$ contained in the Schlegel diagram of the product of two 8-gons. The three picture show the different components of the surface: the 8-gons and the quadrilaterals (left), and the surface (right). The different families of 8-gons and quadrilaterals are displayed in different colors.

The first and the third row each correspond to a family of eight 8-gons, that is, the product of a vertex of the one factor with the entire 8-gon of the other factor. The second and the fourth row each correspond to 16 quadrilateral faces, that is, the product of two edges, one from each factor. Each vector with exactly one $\emptyset$-entry is an edge of the product and is contained in exactly two quadrilaterals belonging to two different families. As we will show in Theorem 5.11, the 8-gons and the quadrilaterals around one vertex form a 2-dimensional ball. Thus $\mathcal{QP}_{\{4,8\},4}$ is a two dimensional manifold contained in the 2-skeleton of $(\mathcal{D}_8)^2$. A picture of the surface in the Schlegel diagram of the product is shown in Figure 5.5.

We collect some simple facts about the subcomplex $\mathcal{QP}_{\{4,2p\},2r}$ in the following lemma.

Lemma 5.10. The subcomplex $\mathcal{QP}_{\{4,2p\},2r}$ of the product $(\mathcal{D}_{2p})^r$ is a closed connected 2-dimensional pseudomanifold containing all vertices and edges of the product.

Proof. An edge of the product $(\mathcal{D}_{2p})^r$ corresponds to a vector with exactly one $\emptyset$-entry. Since the 2-faces of the complex are given by vectors with exactly two cyclically adjacent $\emptyset$-entries, every edge of the polytope is contained in exactly two faces of $\mathcal{QP}_{\{4,2p\},2r}$. As the graph of the product $(\mathcal{D}_{2p})^r$ is connected, we obtain that $\mathcal{QP}_{\{4,2p\},2r}$ is a closed connected 2-dimensional pseudomanifold. The complex $\mathcal{QP}_{\{4,2p\},2r}$ contains all the vertices of $(\mathcal{D}_{2p})^r$ because every vertex is contained in an edge and the complex contains all the edges. $\square$

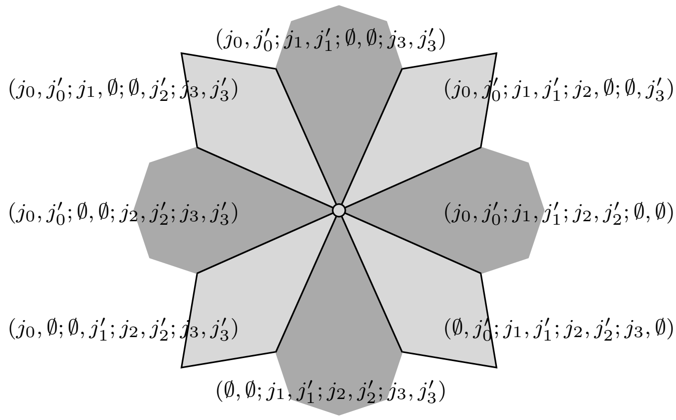


Figure 5.6: The vertex star of a vertex $(j_0, j_0'; \ldots; j_{r-1}, j_{r-1}')$ in the complex $\mathcal{QP}_{\{4,2p\},r}$ is a 2-ball consisting of r quadrilaterals and r $2p$-gons. The quadrilaterals and $2p$-gons alternate around the vertex. The figure shows the case $r = 4$.

Theorem 5.11. The subcomplex $\mathcal{QP}_{\{4,2p\},2r}$ of the r-fold product $(\mathcal{D}_{2p})^r$ of even $2p$-gons is an orientable polyhedral surface in $\mathbb{R}^{2r}$ with f-vector:

$$f(\mathcal{QP}_{\{4,2p\},2r}) \;=\; \tfrac{1}{2}(2p)^{r-1}(4p, 4pr, pr + 2r)$$

and genus $1 + \frac{1}{4}(2p)^{r-1}(3pr - 4p - 2r)$. This surface is *Hamiltonian* in the sense that it contains all vertices and all edges (and all the $2p$-gon faces) of $(\mathcal{D}_{2p})^r$.

Proof. As the combinatorial description of the complex $\mathcal{QP}_{\{4,2p\},2r}$ is similar to the description of the surface $\mathcal{Q}_{4(2p),2r}$, the proof is also similar to the proof of Theorem 5.5. By Lemma 5.10 we obtain that $\mathcal{QP}_{\{4,2p\},2r}$ is a closed connected 2-dimensional pseudomanifold. Further, Figure 5.6 shows that the star of every vertex is a 2-dimensional ball, hence $\mathcal{QP}_{\{4,2p\},2r}$ is a 2-dimensional manifold. The orientation is assigned to the polygons as in the proof of Theorem 5.5 by taking the product of the signs of the vertices involved, that is, for the $2p$-gons we take the product of $r - 1$ vertex signs, for the quadrilaterals of $r - 2$ vertex signs. An easy calculation then yields the orientability of the surface $\mathcal{QP}_{\{4,2p\},r}$.

To complete the theorem we need to calculate the f-vector (f_0, f_1, f_2) of the surface. The surface contains all vertices and edges of the product and thus $f_0 = (2p)^r$ and $f_1 = r(2p)^r$. Further, the combinatorial description implies that the surface contains all $r(2p)^{r-1}$ $2p$-gon faces and $rp^2(2p)^{r-2}$ quadrilaterals of the product. So $f_2 = r(2p)^{r-1} + \frac{1}{4}r(2p)^r$. ☐

5.2 Realizing the surfaces in $\mathbb{R}^3$

In this section we prove that there exist realizations of the products of even polygons such that the two surfaces described in Sections 5.1.2 and 5.1.3 survive the projection to the upper hull of a 4-polytope and hence by orthogonal projection to $\mathbb{R}^3$. We construct particular deformed realizations of the products with the prescribed projection properties similar to the constructions of Ziegler [48] and Sanyal & Ziegler [44]. In Section 5.2.2 we construct an affine support set for the products of polygons. This affine support set yields a lower bound on the number of moduli for the projected surfaces. Finally, we follow the same path as in Section 4.3.3 to realize the duals of all the surfaces in $\mathbb{R}^3$ as well.

5.2.1 Projection of the surface to $\mathbb{R}^4$ and to $\mathbb{R}^3$

To be able to prove our projection results, we work with a very special geometric realization of the $2p$-gon: Let $\varepsilon \ll 1$, $a_j = (\alpha_j, 1-\alpha_j^2)$ on the parabola $y = 1 - x^2$ with $\alpha_j = \frac{\varepsilon}{2}(\frac{p-1-j}{p-1})$ for $j = 0, \dots, 2p-2$ and $a_{2p-1} = (0, -1)$. Since the a_i are in convex position and their convex hull contains $\mathbf{0}$ in its interior, the inequality system $a_j\binom{x}{y} \leq 1$ for $j \in [2p]$ describes a convex $2p$-gon. By scaling the inequalities for odd j by ε we obtain the following inequality description for our $2p$-gon:

$$\mathrm{D}_{2p}^{\frown} := \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \;\middle|\; \begin{pmatrix} \frac{\varepsilon}{2} & 1-\frac{\varepsilon^2}{4} \\ \frac{\varepsilon^2}{2}(\frac{p-2}{p-1}) & \varepsilon - \frac{\varepsilon^3}{4}(\frac{p-2}{p-1})^2 \\ \vdots & \vdots \\ \frac{\varepsilon}{2}(\frac{p-1-j}{p-1}) & 1-\frac{\varepsilon^2}{4}(\frac{p-1-j}{p-1})^2 \\ \frac{\varepsilon^2}{2}(\frac{p-j-2}{p-1}) & \varepsilon - \frac{\varepsilon^3}{4}(\frac{p-j-2}{p-1})^2 \\ \vdots & \vdots \\ -\frac{\varepsilon}{2} & 1-\frac{\varepsilon^2}{4} \\ 0 & -\varepsilon \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \leq \begin{pmatrix} 1 \\ \varepsilon \\ \vdots \\ 1 \\ \varepsilon \\ \vdots \\ 1 \\ \varepsilon \end{pmatrix} \right\} \tag{5.2}$$

As ε is small, the normals of the odd edges are perturbed $(0,0)$ vectors with entries of orders $(\varepsilon^2, \varepsilon)$ and the normals to the even edges are perturbed $(0,1)$ vectors with entries of orders $(\varepsilon, 1-\varepsilon^2)$ as shown in Figure 5.7. The advantage of this realization of an even polygon is that every vertex lies on the intersection of two edges whose normals are a perturbation of the $(0,0)$ and the $(0,1)$ vector, respectively. In the following we will disregard the exact

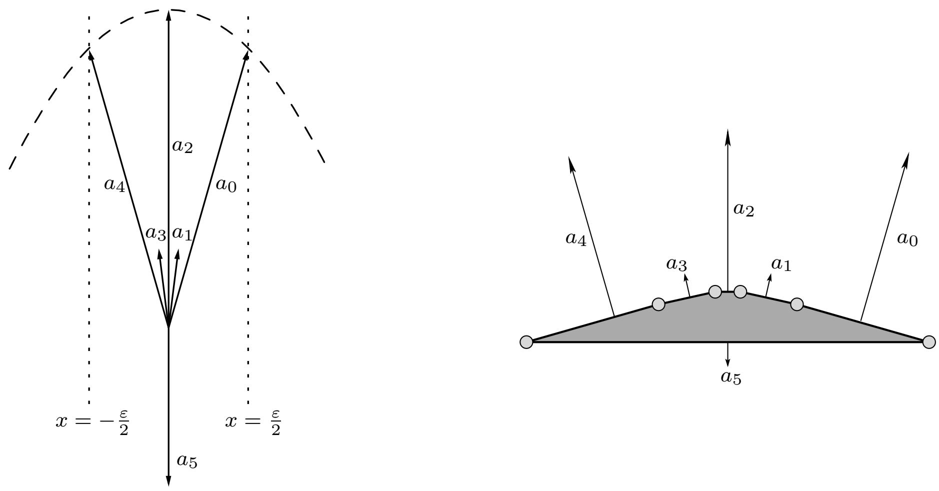


Figure 5.7: The normals of the special hexagon $\mathrm{D}_{\overset{\frown}{6}}$ are perturbations of the vector $(0,0)$ and $(0,1)$ (left). The resulting hexagon is very skinny, depending on the size of ε (right).

coordinates and denote the normal of odd and even edges by $(*,*)$ and $(*,1)$, respectively. This is very useful in the proofs of the next two theorems.

Theorem 5.12 (Realization of $\mathcal{Q}_{4(2p),2r}$)**.** There exists a realization of the r-fold product of even polygons $(\mathcal{D}_{2p})^r$ such that the surface $\mathcal{Q}_{4(2p),2r}$ survives the projection to the upper hull of a 4-polytope. Hence the surface may be projected orthogonally to $\mathbb{R}^3$.

Proof. The proof proceeds as follows: We start with a short discussion of the Gale transform of the pyramid over a properly labeled $(2r-1)$-gon. Then we merge this Gale transform with the r-fold orthogonal product of our special $2p$-gon $\mathrm{D}_{\overset{\frown}{2p}}$ to obtain a deformed product of polygons. Finally, the positive dependences of the Gale transform of the pyramid (that is, the faces of the pyramid) yield strictly preserved faces of the product under the projection to $\mathbb{R}^4$. These faces in particular contain the quadrilaterals that form the surface $\mathcal{Q}_{4(2p),2r}$.

Let us start with the Gale transform of $\mathrm{pyr}(\mathcal{D}_{2r-1})$ with the apex labeled by $2r-1$ and the vertices of the base labeled cyclically by increasing odd numbers $1, 3, \ldots, 2r-3$ followed by increasing even numbers $0, 2, \ldots, 2r-2$ as shown in Figure 5.8. Since the vertices 0, 1, 2, and $2r-1$ are affinely independent, the vectors $g_3, \ldots, g_{2r-2}$ of the Gale transform are linearly independent by Proposition 1.4. Hence we may assume, that the Gale transform

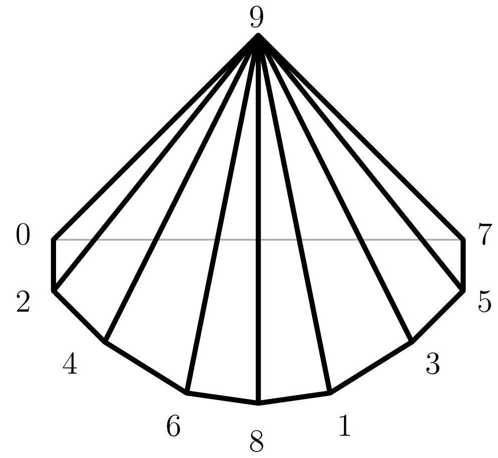


Figure 5.8: Labeling of the pyramid over a $(2r-1)$-gon need for the realization of the surface $\mathcal{Q}_{4(2p),2r}$ via projection of deformed product (example $r=5$).

of $\operatorname{pyr}(\mathcal{D}_{2r-1})$, which contains $2r$ vectors of dimension $2r-4$, looks as follows:

$$G^t = \begin{pmatrix} \text{———} & & g_0 & \text{———} & & \\ \text{———} & & g_1 & \text{———} & & \\ \text{———} & & g_2 & \text{———} & & \\ 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

The last 0-row corresponds to the apex of the pyramid. Since this is the Gale transform of the pyramid it comes with certain positive dependences. In particular, the edges of the polytope yield positive spanning subsets which will correspond to the quadrilaterals of the surface $\mathcal{Q}_{4(2p),2r}$.

The r-fold product of our particular realization of the $2p$-gon $\mathrm{D}_{\widehat{2p}}$ has the following inequality description:

$$(\mathrm{D}_{\widehat{2p}})^r = \left\{ (x_j)_{j=0}^{r-1} \in \mathbb{R}^{2r} \;\middle|\; \left(\begin{array}{cc|cccc} \boxed{A} & & & & & \\ & \boxed{A} & & & & \\ & & \boxed{A} & & & \\ & & & \ddots & & \\ & & & & \boxed{A} & \\ & & & & & \boxed{A} \end{array}\right) \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \\ x_{r-2} \\ x_{r-1} \end{pmatrix} \leq \begin{pmatrix} b \\ b \\ \vdots \\ b \\ b \\ b \end{pmatrix} \right\}$$

where the matrices $A \in \mathbb{R}^{2p \times 2}$ contain the normals of our $2p$-gon realization and $b \in \mathbb{R}^{2p}$ the corresponding right-hand sides as given in Equation (5.2). The vertical line indicates that we are going to project onto the first 4 coordinates.

Since the even resp. odd edges of our special realization of the $2p$-gon are almost equal, that is, the normals to the even resp. odd edges are perturbations of the $(0,0)$ resp. $(0,1)$ vector, we use the following notation for the left-hand side of the inequality description:

$$\left(\begin{array}{cc|ccc} \boxed{\begin{matrix} * & 1 \\ * & * \end{matrix}} & & & & \\ & \boxed{\begin{matrix} * & 1 \\ * & * \end{matrix}} & & & \\ & & \boxed{\begin{matrix} * & 1 \\ * & * \end{matrix}} & & \\ & & & \ddots & \\ & & & & \boxed{\begin{matrix} * & 1 \\ * & * \end{matrix}} \end{array}\right) \quad \text{with} \quad \boxed{\begin{matrix} * & 1 \\ * & * \end{matrix}} \sim \begin{array}{|cc|} \hline \varepsilon & 1 \\ \varepsilon^2 & \varepsilon \\ \hline \varepsilon & 1 \\ \varepsilon^2 & \varepsilon \\ \hline \vdots & \vdots \\ \hline \varepsilon & 1 \\ \varepsilon^2 & \varepsilon \\ \hline \end{array}.$$

Since the product of polygons is a simple polytope, a small perturbation of its facet normals will not change the combinatorial type. So if M is a very large number such that $\frac{1}{M} \ll \varepsilon$, then the following is still an inequality description of a product of polygons:

$$
\left(\begin{array}{cccc|cccccccc}
* & 1 & & \frac{1}{M} & \multicolumn{8}{c}{\text{———}\ \overline{g}_0\ \text{———}} \\
* & * & & \frac{1}{M} & \multicolumn{8}{c}{\text{———}\ \overline{g}_1\ \text{———}} \\
 & & * & 1 & \multicolumn{8}{c}{\text{———}\ \overline{g}_2\ \text{———}} \\
 & & * & * & \frac{1}{M} & & & & & & & \\
 & & & & * & 1 & & & & & & \\
 & & & & * & * & \frac{1}{M} & & & & & \\
 & & & & & & \ddots & \ddots & & & & \\
 & & & & & & & & * & 1 & & \\
 & & & & & & & & * & * & \frac{1}{M} & \\
 & & & & & & & & & & * & 1 \\
 & & & & & & & & & & * & *
\end{array}\right)
\begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_{r-3} \\ x_{r-2} \\ x_{r-1} \end{pmatrix}
\leq
\begin{pmatrix} b \\ b \\ \vdots \\ b \\ b \\ b \end{pmatrix}
$$

where the $\overline{g}_i$ are componentwise scaled vectors of the Gale transform G of the pyramid $\operatorname{pyr}(\mathcal{D}_{2p-1})$ in the following way:

$$
\begin{aligned}
\overline{g}_0 &:= \left(\tfrac{1}{M^2}, \tfrac{1}{M^2}, \tfrac{1}{M^3}, \tfrac{1}{M^3}, \ldots, \tfrac{1}{M^{r-1}}, \tfrac{1}{M^{r-1}}\right) \odot g_0 \\
\overline{g}_1 &:= \left(\tfrac{1}{M^2}, \tfrac{1}{M^2}, \tfrac{1}{M^3}, \tfrac{1}{M^3}, \ldots, \tfrac{1}{M^{r-1}}, \tfrac{1}{M^{r-1}}\right) \odot g_1 \\
\overline{g}_2 &:= \left(\tfrac{1}{M}, \tfrac{1}{M}, \tfrac{1}{M^2}, \tfrac{1}{M^2}, \ldots, \tfrac{1}{M^{r-2}}, \tfrac{1}{M^{r-2}}\right) \odot g_2
\end{aligned}
$$

where $\odot$ denotes the componentswise multiplication of two vectors, that is, $(v \odot w)_i = v_i \cdot w_i$. All the following modifications are applied to the normals of the facets of the product corresponding to the even and the odd edges of the factors $\mathrm{D}_{\widehat{2p}}$ in the same way. We scale the inequalities of the jth factor with M^{r-1-j} for $j = 0, \ldots, r-1$ and replace the coordinates $x_j \in \mathbb{R}^2$ with $x'_j = M^{r-1-j} x_j$. This way we obtain the following inequality system for a deformed product of polygons:

$$
\left(\begin{array}{cccc|ccccccc}
* & 1 & & 1 & \multicolumn{7}{c}{\text{———} \; g_0 \; \text{———}} \\
* & * & & 1 & \multicolumn{7}{c}{\text{———} \; g_1 \; \text{———}} \\
 & & * & 1 & \multicolumn{7}{c}{\text{———} \; g_2 \; \text{———}} \\
 & & * & * & 1 & & & & & & \\
 & & & & * & 1 & & & & & \\
 & & & & * & * & 1 & & & & \\
 & & & & & \ddots & \ddots & & & & \\
 & & & & & & * & 1 & & & \\
 & & & & & & * & * & 1 & & \\
 & & & & & & & & * & 1 & \\
 & & & & & & & & * & * &
\end{array}\right)
\begin{pmatrix} x'_0 \\ x'_1 \\ x'_2 \\ \vdots \\ x'_{r-2} \\ x'_{r-1} \end{pmatrix}
\leq
\begin{pmatrix} M^{r-1} \\ M^{r-1}\varepsilon \\ M^{r-2} \\ M^{r-2}\varepsilon \\ M^{r-3} \\ M^{r-3}\varepsilon \\ \vdots \\ M \\ M\varepsilon \\ 1 \\ \varepsilon \end{pmatrix}.
$$

Each row of the above system corresponds to p odd resp. even edges of the factors. Let us denote the last $2r-4$ columns of the "condensed" matrix with $*$-entries by $\tilde{G}$, that is, $\tilde{G}$ has one row for the even and one row for the odd edges of each factor and is a $2r \times (2r-4)$ matrix.

We need to verify that the quadrilaterals contained in the surface $\mathcal{Q}_{4(2p),2r}$ are strictly preserved by the projection to the first 4 coordinates. By Lemma 1.11 a quadrilateral Q is strictly preserved if the corresponding truncated facet normals are positively spanning. The quadrilaterals of $\mathcal{Q}_{4(2p),2r}$ come in four families corresponding to the following vectors:

1. $(j_0, j'_0; \ldots; j_{i-1}, j'_{i-1}; \emptyset, j'_i; \emptyset, j'_{i+1}; j_{i+2}, j'_{i+2}; \ldots; j_{r-1}, j'_{r-1})$ for $i \in [r]$
2. $(j_0, \emptyset; j_1, j'_1; \ldots; j_{r-2}, j'_{r-2}; \emptyset, j'_{r-1})$
3. $(j_0, j'_0; \ldots; j_{i-1}, j'_{i-1}; j_i, \emptyset; j_{i+1}, \emptyset; j_{i+2}, j'_{i+2}; \ldots; j_{r-1}, j'_{r-1})$ for $i \in [r]$
4. $(\emptyset, j'_0; j_1, j'_1; \ldots; j_{r-2}, j'_{r-2}; j_{r-1}, \emptyset)$

Each of the j_i resp. j'_i corresponds to an even resp. odd facet of the ith factor. Hence the truncated normals of the facets containing a quadrilateral are the rows of $\tilde{G}$ with the rows corresponding to the $\emptyset$-entries removed. The corresponding rows of the matrix G^t are positively spanning since G is the Gale transform of the pyramid $\mathrm{pyr}(\mathcal{D}_{2r-1})$ with our carefully chosen labeling of the vertices. But $\tilde{G}$ is just a perturbation of the matrix G^t and positively spanning vector configurations are stable under perturbation. Hence the chosen rows of $\tilde{G}$ are also positively spanning and all the quadrilaterals of the surface survive the projection.

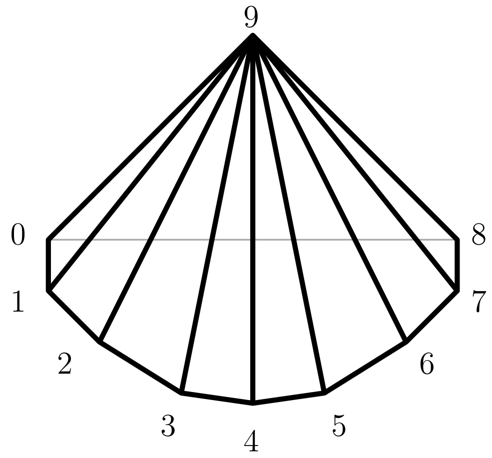


Figure 5.9: Labeling of the pyramid over a $(2r-1)$-gon $(r=5)$ needed for the realization of the surface $\mathcal{QP}_{\{4,2p\},2r}$ via projection of a deformed product.

Further the quadrilaterals all project to the upper hull of the projected polytope with respect to the fourth coordinate since the 1 of the first three rows dominate the direction of normals of the projected quadrilaterals. Thus an orthogonal projection to the first three coordinates yields a realization of the surface in $\mathbb{R}^3$. $\square$

Theorem 5.13 (Realization of $\mathcal{QP}_{\{4,2p\},2r}$)**.** There exists a realization of the r-fold product of even polygons $(\mathcal{D}_{2p})^r$ such that the surface $\mathcal{QP}_{\{4,2p\},2r}$ survives the projection to the upper hull of a 4-polytope. Hence the surface may be projected orthogonally to $\mathbb{R}^3$.

Proof. The proof is very similar to the proof of the previous theorem. We only have to label the vertices of the pyramid over an $(2r-1)$-gon differently as shown in Figure 5.9. This induces different positive dependences in the Gale transform and implies that all the quadrilaterals and all the $2p$-gons of the surface $\mathcal{QP}_{\{4,2p\},2r}$ survive the projection to $\mathbb{R}^4$, resp. $\mathbb{R}^3$. $\square$

The above theorem provides realizations of the surfaces $\mathcal{QP}_{\{4,2p\},2r}$ in $\mathbb{R}^3$. These surfaces have constant vertex degree $2r$. The 2-faces of these surfaces consist of $r(2p)^{r-1}$ $2p$-gons and $rp^2(2p)^{r-2}$ quadrilaterals. So the average polygon size is:

$$\frac{2p \cdot r(2p)^{r-1} + 4 \cdot rp^2(2p)^{r-2}}{r(2p)^{r-1} + rp^2(2p)^{r-2}} = \frac{2r(2p)^r}{\frac{1}{2}r(2p)^{r-1}(2+p)} = \frac{8p}{p+2}.$$

Brehm & Wills [13, Sect. 4.2] raised the question whether equivelar polyhedral surfaces of type $\{p, q\}$ with $p \geq 5$ and $q \geq 5$ existed in $\mathbb{R}^3$. This question relates to the following corollary since we obtain polyhedral surfaces in $\mathbb{R}^3$ with vertex degree and average polygon size at least 5.

Corollary 5.14 ($\{p, q\}$-surface in $\mathbb{R}^3$)**.** The surfaces $\mathcal{QP}_{\{4,2p\},2r}$ have average polygon size $\delta_{poly} = \frac{8p}{p+2}$ and obviously (average) vertex degree $\delta_{vert} = 2r$.

In particular, for $p \geq 4$ and $r \geq 3$, both δ_{vert} and δ_{poly} are at least 5 and Theorem 5.13 shows the existence of surfaces with $\delta_{vert} \geq 5$ and $\delta_{poly} \geq 5$ in $\mathbb{R}^3$. □

5.2.2 Moduli of the projected surfaces

We use the methods developed in Section 2.2 to obtain a lower bound on the moduli of the surfaces constructed in Sections 5.1.2 and 5.1.3. As all the surfaces described in this chapter contain all the vertices of the product of polygons, we need to find a large affine support set of the product of polygons $(\mathcal{D}_{2p})^r$.

The vertices of the product $(\mathcal{D}_{2p})^r$ are products of vertices of the factors and are hence identified with the vectors $(j_0, j'_0; \ldots; j_{r-1}, j'_{r-1})$. For $i \in [r]$ consider the following vertices:

$$v_{i,j} \;:=\; \begin{cases} (\underbrace{j, j+1; \ldots; j, j+1}_{i \text{ factors}}; \underbrace{j, j-1; \ldots; j, j-1}_{r-i \text{ factors}}) & \text{if } j \in [2p] \text{ even,} \\ (\underbrace{j+1, j; \ldots; j+1, j}_{i \text{ factors}}; \underbrace{j-1, j; \ldots; j-1, j}_{r-i \text{ factors}}) & \text{if } j \in [2p] \text{ odd,} \end{cases}$$

where the calculations are made modulo $2p$. These vertices form a subset of the vertices of size $2pr$ of the product, denoted by:

$$\mathsf{A}^r_{2p} \;:=\; \{v_{i,j} \mid i \in [r], j \in [2p]\}. \tag{5.3}$$

For $r = 1$ the set A^1_{2p} contains all the vertices of the polygon. An example of the case $r = 2$ is illustrated in the following example.

Example 5.15. In this example we investigate the set A^2_6 which is a subset of the vertices of the product of two 6-gons. The vertices $v_{i,j}$ contained in A^2_6 are the following:

$i \backslash j$	0	1	2	3	4	5
0	$(0,5;0,5)$	$(0,1;0,1)$	$(2,1;2,1)$	$(2,3;2,3)$	$(4,3;4,3)$	$(4,5;4,5)$
1	$(0,1;0,5)$	$(2,1;0,1)$	$(2,3;2,1)$	$(4,3;2,3)$	$(4,5;4,3)$	$(0,5;4,5)$

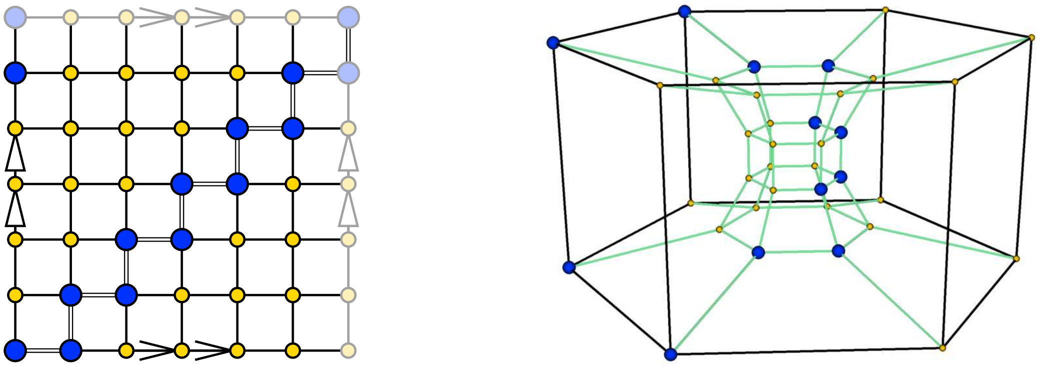


Figure 5.10: The vertices of the subset A_6^2 in the product of two hexagons. The (6×6)-grid on the left is a schematic picture of the torus (identify top & bottom, and left & right) contained in the boundary of the product. The Schlegel diagram of the product is shown on the right. Both images show the zigzag of the selected vertices around the torus.

These vertices form a zigzag on the (6×6)-torus contained in the boundary of the product $\mathcal{D}_6 \times \mathcal{D}_6$ shown in Figure 5.10. The vertices contained in the facet $F = (2, \emptyset; \emptyset, \emptyset)$ are:

$$F \cap \mathsf{A}_6^2 \;=\; \{(2,1;0,1), (2,1;2,1), (2,3;2,1), (2,3;2,3)\}.$$

Now the flag $(2,3;2,3) \subset (2,3;2,\emptyset) \subset (2,\emptyset;2,\emptyset) \subset (2,\emptyset;\emptyset,\emptyset)$ "cuts off" one vertex after the other and thus the vertices in $F \cap \mathsf{A}_6^2$ are affinely independent for every realization. It follows from the symmetry that the selected vertices in every facet are affinely independent for every realization of the product. Hence A_6^2 is an affine support set of the product $(\mathcal{D}_6)^2$.

Theorem 5.16 (Affine support set for products of polygons)**.** The subset A_{2p}^r of the vertices of the r-fold product $(\mathcal{D}_{2p})^r$ of $2p$-gons is an affine support set of size $2pr$.

Proof. We use a flag to show that the set A_{2p}^r is an affine support set. By the combinatorial symmetry of the product and the set A_{2p}^r we only need to consider the vertices contained in the facet $F := h_{0,2} = (2,\emptyset;\emptyset,\emptyset;\ldots;\emptyset,\emptyset)$.

The vertices of A^r_{2p} in this facet are:

$$\begin{aligned}\mathsf{A}^r_{2p} \cap F &= \left\{(j_0, j_0'; \ldots; j_{r-1}, j_{r-1}') \in \mathsf{A}^r_{2p} \;\middle|\; j_0 = 2\right\} \\ &= \left\{(\underbrace{2,1;2,1;\ldots;2,1}_{i};\underbrace{0,1;\ldots;0,1}_{r-i}) \text{ for } i = 1,\ldots,r\right\} \\ &\quad \cup \left\{(\underbrace{2,3;2,3;\ldots;2,3}_{i};\underbrace{2,1;\ldots;2,1}_{r-i}) \text{ for } i = 1,\ldots,r\right\}.\end{aligned}$$

The flag which shows that A^r_{2p} is an affine support set is constructed from the vertex $(2,3;\ldots;2,3)$ by intersecting the facets defining this vertex in the proper order: We start with the facet $G_{2r-1} := (2,\emptyset;\emptyset,\emptyset;\ldots;\emptyset,\emptyset)$. Then we add the even entries of the vector to the intersection such that the face $G_{2r-1-\ell}$ is the intersection of the face $G_{2r-\ell}\cap h_{\ell,2}$, that is, for $\ell = 1,\ldots,r$:

$$G_{2r-\ell} \;:=\; (\underbrace{2,\emptyset;\ldots;2,\emptyset}_{\ell \text{ factors}};\underbrace{\emptyset,\emptyset;\ldots;\emptyset,\emptyset}_{r-\ell \text{ factors}}).$$

Then we start to intersect with the odd faces of the factors and obtain the following faces for $\ell = r+1,\ldots,2r$:

$$G_{2r-\ell} \;:=\; (\underbrace{2,3;\ldots;2,3}_{\ell - r \text{ factors}};\underbrace{2,\emptyset;\ldots;2,\emptyset}_{2r-\ell \text{ factors}}).$$

The sequence $(G_k)_{k=0}^{2r-1}$ obviously forms a flag. Finally, each G_k contains exactly $k+1$ of the vertices in $\mathsf{A}^r_{2p}\cap F$ and hence $\mathsf{A}^r_{2p}\cap F$ is affinely independent for every realization of the product. □

The size of an affine support set is bounded by the number of facets, as shown in Lemma 2.25. The number of facets of the r-fold product $(\mathcal{D}_{2p})^r$ of $2p$-gons is $2pr$. Hence the affine support set A^r_{2p} defined in Equation (5.3) is maximal and parametrizes the entire realization space of the product.

Remark 5.17. We can construct a similar affine support set A^r_{2p+1} for products of odd polygons as well. But as this chapter deals with the construction of surfaces in the boundary of the projection of products of even polygons, we omit this simple generalization.

We are now ready to apply Theorem 2.30 to the quadrilateral surface $\mathcal{Q}_{4(2p),2r}$ of Section 5.1.2 and the surface $\mathcal{QP}_{\{4,2p\},2r}$ of Section 5.1.3. Since both surfaces contain all the vertices of the product of polygons we obtain the following corollary.

Corollary 5.18 (Moduli of $\mathcal{Q}_{4(2p),2r}$ and $\mathcal{QP}_{\{4,2p\},2r}$)**.** The realizations of the surfaces $\mathcal{Q}_{4(2p),2r}$ resp. $\mathcal{QP}_{\{4,2p\},2r}$ in $\mathbb{R}^3$ obtained from the projections of r-fold products $(\mathcal{D}_{2p})^r$ of $2p$-gons have at least $6pr$ moduli. $\square$

For $r = 2$ the surface $\mathcal{Q}_{4(2p),4}$ is the standard quadrangulation of a $2p \times 2p$-torus. This torus has no triangular faces or faces with only degree three vertices, hence there are no trivial moduli. For larger parameters p and r all the surfaces of the families $\mathcal{Q}_{4(2p),2r}$ and $\mathcal{QP}_{\{4,2p\},2r}$ have no triangular faces and only vertices of degree $2r$. This is reflected in the naïve estimates obtained from Equation (1.1) in Section 1.3.1. For arbitrary realizations $S_{\mathcal{Q}}$ resp. $S_{\mathcal{QP}}$ of the surfaces $\mathcal{Q}_{4(2p),2r}$ resp. $\mathcal{QP}_{\{4,2p\},2r}$ in $\mathbb{R}^3$ we obtain the following naïve estimates on the number of moduli:

$$\begin{aligned}
\mathfrak{M}(S_{\mathcal{Q}}, 3) &\sim \frac{1}{2}(2p)^r(6 - 4r + 3r) \\
&= \frac{1}{2}(2p)^r(6 - r), \\
\mathfrak{M}(S_{\mathcal{QP}}, 3) &\sim \frac{1}{2}(2p)^{r-1}(12p - 8pr + 3(rp + 2r)) \\
&= \frac{1}{2}(2p)^{r-1}(12p + 6r - 5pr)
\end{aligned}$$

from their f-vectors given in Lemma 5.4 resp. Theorem 5.11. So similar to the wedge product surfaces in Section 4.3.2 this reveals a huge discrepancy between the lower bounds on the moduli of the above Corollary 5.18 and the estimate by rule of thumb.

5.2.3 Realization of the dual surfaces

In Section 4.3.3 we established a relation between the duality of surfaces and the duality of 4-polytopes: If a 4-polytope P contains the prism over a surface S in its boundary, then the dual polytope P^* contains the dual surface S^* as a subcomplex of its 2-skeleton.

So to obtain realizations of the polyhedral surfaces $\mathcal{Q}^*_{4(2p),2r}$ and $\mathcal{QP}^*_{\{4,2p\},2r}$, we need to construct 4-polytopes containing the prisms over the respective primal surfaces. In the following we construct these 4-polytopes as projections of suitable deformed products of $2p$-gons with an interval I.

Theorem 5.19. There exists a realization of the product $(\mathcal{D}_{2p})^r \times I$ of $2p$-gons with an interval I in $\mathbb{R}^{2r+1}$ such that the prism $\mathcal{Q}_{4(2p),2r} \times I$ over the surface survives the projection to the boundary of a 4-polytope P.

Furthermore, the dual surface $\mathcal{Q}^*_{4(2p),2r}$ is a subcomplex of the 2-skeleton of the dual polytope P^* and may be realized in $\mathbb{R}^3$ by Schlegel projection.

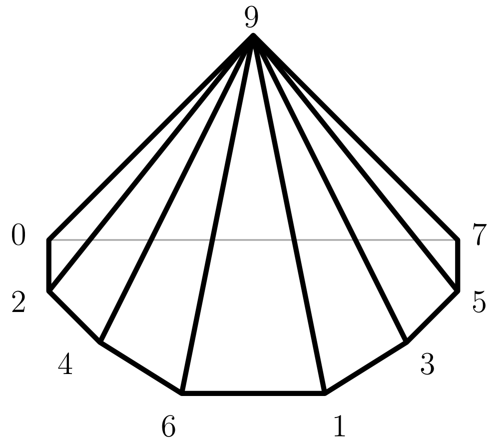


Figure 5.11: Labeling of the pyramid over a $2r$-gon needed for the realization of the prism over the surface $\mathcal{Q}_{4(2p),2r} \times I$ via a projection of a deformed product (example $r = 4$).

Proof. We mimic the proof of Theorem 5.12. We start with the orthogonal product of the special $2p$-gons. We attach the interval $I = \{x \in \mathbb{R} \mid \pm\delta x \leq 1\}$ for small positive δ to obtain a polytope of combinatorial type $(\mathcal{D}_{2p})^r \times I$. We deform this product using the Gale transform of the pyramid over a $2r$-gon labeled as in Figure 5.11 to obtain the following facet normals:

$$\left(\begin{array}{cccc|ccccccccc}
* & 1 & & 1 & \multicolumn{9}{c}{\rule[0.5ex]{5em}{0.4pt}\ g_0\ \rule[0.5ex]{5em}{0.4pt}} \\
* & * & & 1 & \multicolumn{9}{c}{\rule[0.5ex]{5em}{0.4pt}\ g_1\ \rule[0.5ex]{5em}{0.4pt}} \\
 & & * & 1 & \multicolumn{9}{c}{\rule[0.5ex]{5em}{0.4pt}\ g_2\ \rule[0.5ex]{5em}{0.4pt}} \\
 & & * & * & 1 & & & & & & & & \\
 & & & & * & 1 & & & & & & & \\
 & & & & * & * & 1 & & & & & & \\
 & & & & & & \ddots & \ddots & & & & & \\
 & & & & & & & & * & 1 & & & \\
 & & & & & & & & * & * & 1 & & \\
 & & & & & & & & & & * & 1 & \\
 & & & & & & & & & & * & * & 1 \\
 & & & & & & & & & & & & \pm\delta
\end{array}\right)$$

Then the triangles of the pyramid provide the positive dependences that make sure that the prisms of the product of the surface with an interval survive the projection to $\mathbb{R}^4$. $\square$

Remark 5.20. The construction of Theorem 5.19 can easily be generalized to products $\prod \mathcal{D}_{2p_i} \times I$ of arbitrary even polygons with an interval. So the above result finally allows us to obtain realizations for polyhedral surfaces including *all* the combinatorial types of surfaces of type $\{4, p\}$ and their duals of type $\{q, 4\}$ constructed in McMullen, Schulz & Wills [37].

As in the previous section, we only need to relabel the vertices of the pyramid used in the proof to obtain the analogous result for the duals of the quad-polygon surface $\mathcal{QP}_{\{4,2p\},2r}$.

Theorem 5.21. There exists a realization of the product $(\mathcal{D}_{2p})^r \times I$ of $2p$-gons with an interval I in $\mathbb{R}^{2r+1}$ such that the prism $\mathcal{QP}_{\{4,2p\},2r} \times I$ over the surface survives the projection to the boundary of a 4-polytope P.

Furthermore, the dual surface $\mathcal{QP}^*_{\{4,2p\},2r}$ is a subcomplex of the 2-skeleton of the dual polytope P^* and can be realized in $\mathbb{R}^3$ by Schlegel projection. $\square$

Chapter 6

Zonotopes with large 2D-cuts

joint with Nikolaus Witte and Günter M. Ziegler
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Zonotopes, the Minkowski sums of finitely many line segments, may also be defined as the images of cubes under affine maps, while their duals can be described as the central sections of cross polytopes. So, asking for images of zonotopes under projections, or for central sections of their duals doesn't give anything new: We get again zonotopes, resp. duals of zonotopes. The combinatorics of zonotopes and their duals is well understood (see e.g. Ziegler [47]): The face lattice of a dual zonotope may be identified with that of a real hyperplane arrangement.

However, surprising effects arise as soon as one asks for *sections* of zonotopes, resp. *projections* of their duals. Such questions arise in a variety of contexts.

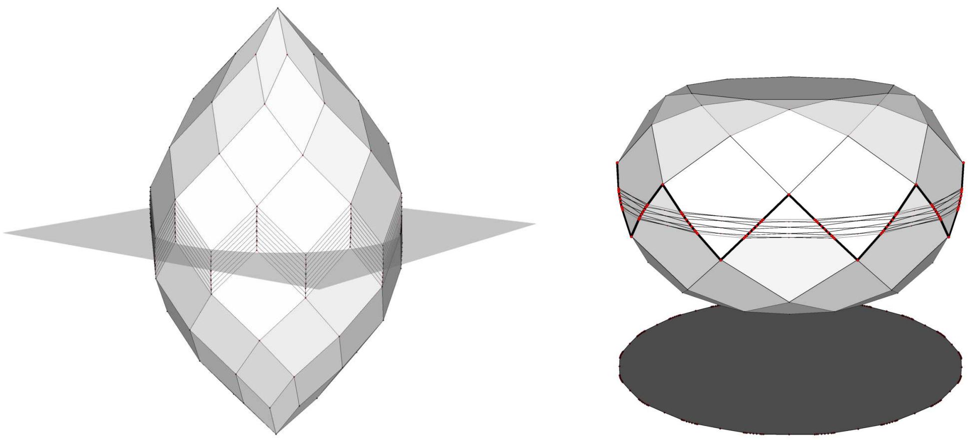

Figure 6.1: Eppstein's Ukrainian easter egg, and its dual. The 2D-cut, resp. shadow boundary, of size $\Omega(n^2)$ are marked.

For example, the "Ukrainian Easter eggs" as displayed by Eppstein in his wonderful "Geometry Junkyard" [18] are 3-dimensional zonotopes with n zones that have a 2-dimensional section with $\Omega(n^2)$ vertices; see also Figure 6.1. For "typical" 3-dimensional zonotopes with n zones one expects only a linear number of vertices in any section, so the Ukrainian Easter eggs are surprising objects. Moreover, such a zonotope has at most $2\binom{n}{2} = O(n^2)$ faces, so any 2-dimensional section is a polygon with at most $O(n^2)$ edges/vertices, which shows that for dimension $d = 3$ the quadratic behavior is optimal.

Eppstein's presentation of his model draws on work by Bern, Eppstein et al. [3], where also complexity questions are asked. (Let us note that it takes a closer look to interpret the picture given by Eppstein correctly: It is "clipped", and a close-up view shows that the vertical "chains of vertices" hide lines of diamonds; see Figure 6.2.)

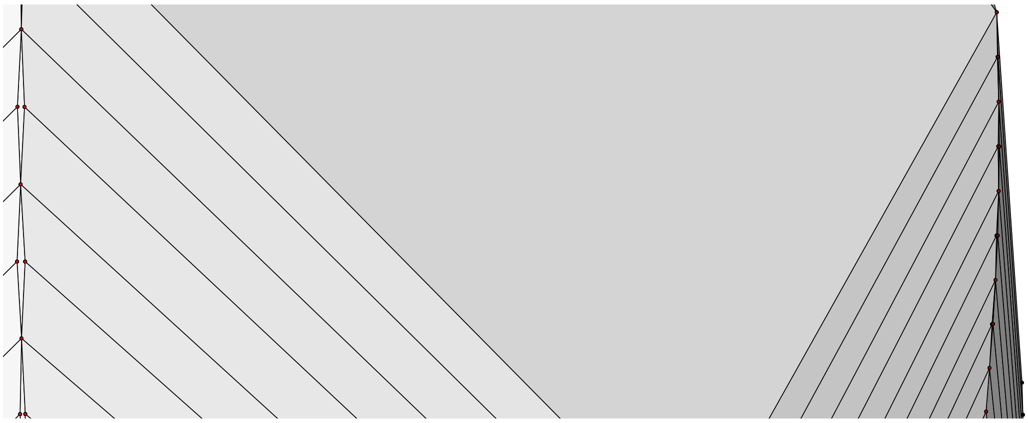

Figure 6.2: Close-up view of an Ukrainian Easter egg.

It is natural to ask for high-dimensional versions of the Easter eggs.

Problem 6.1. What is the maximal number of vertices for a 2-dimensional central section of a d-dimensional zonotope with n zones?

For $d = 2$ the answer is trivially $2n = \Theta(n)$, while for $d = 3$ it is of order $\Theta(n^2)$, as seen above. We answer this question optimally for all fixed $d \geq 2$.

Theorem 6.2. For every $d \geq 2$ the maximal complexity (number of vertices) for a central 2D-cut of a d-dimensional zonotope Z with n zones is $\Theta(n^{d-1})$.

The upper bound for this theorem is quite obvious: A d-dimensional zonotope with n zones has at most $2\binom{n}{d-1}$ facets, thus any central 2D-section has at most $2\binom{n}{d-1} = O(n^{d-1})$ edges.

To obtain lower bound constructions, it is advisable to look at the dual version of the problem.

Problem 6.3 (Koltun [46, Problem 3])**.** What is the maximal number of vertices for a 2-dimensional affine image (a "2D-shadow") of a d-dimensional dual zonotope with n zones?

Indeed, this question arose independently: It was posed by Vladlen Koltun based on the investigation of his "arrangement method" for linear programming (*see* [30]), which turned out to be equivalent to a Phase I procedure for the "usual" simplex algorithm (Hazan & Megiddo [26]). Our construction in Section 6.2 shows that the "shadow vertex" pivot rule is exponential in worst-case for the arrangement method.

Indeed, a quick approach to Problem 6.3 is to use known results about large projections of polytopes. Indeed, if Z is a d-zonotope with n zones, then the polar dual Z^* of the zonotope Z has the combinatorics of an arrangement of n hyperplanes in $\mathbb{R}^d$. The facets of Z^* are $(d-1)$-dimensional polytopes with at most n facets — and indeed *every* $(d-1)$-dimensional polytope with at most n facets arises this way. It is known that such polytopes have exponentially large 2D-shadows, which in the old days was bad news for the "shadow vertex" version of the simplex algorithm [23] [39]. Lifted to the dual d-zonotope Z^*, this also becomes relevant for Koltun's arrangement method; in Section 6.2 we briefly present this, and derive the $\Omega(n^{(d-1)/2})$ lower bound.

However, what we are really heading for is an optimal result, dual to Theorem 6.2. It will be proved in Section 6.3, the main part of this paper.

Theorem 6.2*. *For every $d \geq 2$ the maximal complexity (number of vertices) for a 2D-shadow of the dual of a d-dimensional zonotope Z^* with n zones is $\Theta(n^{d-1})$.*

Acknowledgments. We are grateful to Vladlen Koltun for his inspiration for this paper. Our investigations were greatly helped by use of the `polymake` system by Gawrilow & Joswig [22]. In particular, we have built `polymake` models that were also used to produce the main figures in this paper.

6.1 Arrangements and zonotopes

Let $A \in \mathbb{R}^{m \times d}$ be a matrix. We assume that A has full (column) rank d, that no row is a multiple of another one, and none is a multiple of the first unit-vector $(1, 0, \ldots, 0)$. We refer to [6, Chap. 2] or [47, Lect. 7] for more detailed expositions of real hyperplane arrangements, the associated zonotopes, and their duals.

6.1.1 Hyperplane arrangements

The matrix A determines an essential *linear hyperplane arrangement* $\widehat{\mathcal{A}} = \widehat{\mathcal{A}}_A$ in $\mathbb{R}^d$, whose m hyperplanes are

$$\widehat{H}_j = \{x \in \mathbb{R}^d : a_j x = 0\} \qquad \text{for } j = 1, \dots, m$$

corresponding to the rows a_j of A, and an *affine hyperplane arrangement* $\mathcal{A} = \mathcal{A}_A$ in $\mathbb{R}^{d-1}$, whose hyperplanes are

$$H_j = \{x \in \mathbb{R}^{d-1} : a_j \tbinom{1}{x} = 0\} \qquad \text{for } j = 1, \dots, m.$$

Given A, we obtain $\mathcal{A}$ from $\widehat{\mathcal{A}}$ by intersection with the hyperplane $x_0 = 1$ in $\mathbb{R}^d$, a step known as *dehomogenization*; similarly, we obtain $\widehat{\mathcal{A}}$ from $\mathcal{A}$ by *homogenization.*

The points $x \in \mathbb{R}^d$ and hence the faces of $\widehat{\mathcal{A}}$ (and by intersection also the faces of $\mathcal{A}$) have a canonical encoding by *sign vectors* $\sigma(x) \in \{+1, 0, -1\}^m$, via the map $s_A : x \mapsto (\operatorname{sign} a_1 x, \dots, \operatorname{sign} a_m x)$. In the following we use the shorthand notation $\{+, 0, -\}$ for the set of signs. The sign vector system $s_A(\mathbb{R}^d) \subseteq \{+, 0, -\}^m$ generated this way is the *oriented matroid* [6] of $\widehat{\mathcal{A}}$.

The sign vectors $\sigma \in s_A(\mathbb{R}^d) \cap \{+, -\}^m$ in this system (i.e., without zeroes) correspond to the *regions* (d-dimensional cells) of the arrangement $\widehat{\mathcal{A}}$. For a non-empty low-dimensional cell F the sign vectors of the regions containing F are precisely those sign vectors in $s_A(\mathbb{R}^d)$ which may be obtained from $\sigma(F)$ by replacing each "0" by either "+" or "−".

6.1.2 Zonotopes and their duals

The matrix A also yields a zonotope $Z = Z_A$, as

$$Z = \Big\{ \sum_{i=1}^{m} \lambda_i a_i : \lambda_i \in [-1, +1] \text{ for } i = 1, \dots, m \Big\}.$$

(In this set-up, Z lives in the vector space $(\mathbb{R}^d)^*$ of row vectors, while the dual zonotope Z^* considered below consists of column vectors.)

The dual zonotope $Z^* = Z_A^*$ may be described as

$$Z^* = \Big\{ x \in \mathbb{R}^d : \sum_{i=1}^{m} |a_i x| \le 1 \Big\}. \tag{6.1}$$

The domains of linearity of the function $f_A : \mathbb{R}^d \to \mathbb{R}$, $x \mapsto \sum_{i=1}^m |a_i x|$ are the regions of the hyperplane arrangement $\widehat{\mathcal{A}}$. Their intersections yield the faces of $\widehat{\mathcal{A}}$, and these may be identified with the cones spanned by the proper faces of Z^*. Thus the proper faces of Z^* (and, by duality, the non-empty faces of Z) are identified with sign vectors in $\{+, 0, -\}^m$: These are the same sign vectors as we got for the arrangement $\widehat{\mathcal{A}}$. (See Figure 6.4 on the left for a 3-zonotope and the corresponding linear arrangement.)

Expanding the absolute values in Equation (6.1) yields a system of 2^m inequalities describing Z^*. However, a non-redundant facet description of Z^* can be obtained from A and the combinatorics of $\widehat{\mathcal{A}}$ by considering the inequalities $\sigma(F)Ax \leq 1$ for all sign vectors $\sigma(F)$ of maximal cells F of $\widehat{\mathcal{A}}$:

$$Z^* = \left\{x \in \mathbb{R}^d : \sigma A x \leq 1 \text{ for all } \sigma \in s_A(\mathbb{R}^d) \cap \{+, -\}^m\right\}.$$

To determine the faces that survive the projection to the first k coordinates we need to find the normals of the facets adjacent to a given face. But since the faces of the dual zonotope correspond to faces of the arrangement $\widehat{\mathcal{A}}$ they are easily obtained from the following lemma.

Lemma 6.4. Let Z^* be a d-dimensional dual zonotope corresponding to the linear arrangement $\widehat{\mathcal{A}}$ given by the matrix A, and let $F \subset Z^*$ a be non-empty face. Then the normals of the facets containing F are the linear combinations σA of the rows of A for all sign vectors $\sigma \in s_A(\mathbb{R}^d)$ obtained from $\sigma(F)$ by replacing each "0" by either "+" or "−". □

So the information needed to verify that a face of a dual zonotope is preserved by a projection to the first k coordinates may be obtained from the combinatorial structure of the arrangement $\widehat{\mathcal{A}}$ and the matrix A.

6.2 Dual zonotopes with 2D-shadows of size $\Omega\left(n^{(d-1)/2}\right)$

In this section we present an exponential (yet not optimal) lower bound for the maximal size of 2D-shadows of dual zonotopes. It is merely a combination of known results about polytopes and their projections. For simplicity, we restrict to the case of odd dimension d.

Theorem 6.5. Let $d \geq 3$ be odd and n an even multiple of $\frac{d-1}{2}$. Then there is a d-dimensional dual zonotope $Z^* \subset \mathbb{R}^d$ with n zones and a projection $\pi : \mathbb{R}^d \to \mathbb{R}^2$ such that the image $\pi(Z^*)$ has at least $\left(\frac{2n}{d-1}\right)^{\frac{d-1}{2}}$ vertices.

Here is a rough sketch of the construction.

(1) According to Amenta & Ziegler [2, Theorem 5.2] there are $(d-1)$-polytopes with n facets and exponentially many vertices such that the projection $\pi_2 : \mathbb{R}^{d-1} \to \mathbb{R}^2$ to the first two coordinates preserves all the vertices and thus yields a "large" polygon.

(2) We construct a d-dimensional dual zonotope Z^* with n zones that has such a $(d-1)$-polytope as a facet F.

(3) The extension of π_2 to a projection

$$\pi_3 : \mathbb{R} \times \mathbb{R}^{d-1} \to \mathbb{R}^3, \ (x_0, x) \mapsto (x_0, \pi_2(x))$$

maps Z^* to a centrally symmetric 3-polytope P with a *large* polygon as a facet. P has a projection to $\mathbb{R}^2$ that preserves many vertices.

In the following we give a few details to enhance this sketch.

Some details for (1): Here is the exact result by Amenta & Ziegler, which sums up previous constructions by Goldfarb [23] and Murty [39]

Theorem 6.6 (Amenta & Ziegler [2])**.** Let d be odd and n an even multiple of $\frac{d-1}{2}$. Then there is a $(d-1)$-polytope $F \subset \mathbb{R}^{d-1}$ with n facets and $\left(\frac{2n}{d-1}\right)^{\frac{d-1}{2}}$ vertices such that the projection $\pi_2 : \mathbb{R}^{d-1} \to \mathbb{R}^2$ to the first two coordinates preserves all vertices of F. The polytope F is combinatorially equivalent to a $\left(\frac{d-1}{2}\right)$-fold product of $\left(\frac{2n}{d-1}\right)$-gons.

Explicit matrix descriptions of deformed products of n-gons with "large" 4-dimensional projections are given in Ziegler [48] and Sanyal & Ziegler [44] These can easily be adapted (indeed, simplified) to yield explicit coordinates for the polytopes of Theorem 6.6.

Some details for (2): We have to construct a dual zonotope Z^* with F as a facet.

Lemma 6.7. Given a $(d-1)$-polytope F with n facets, there is a d-dimensional dual zonotope Z^* with n zones that has a facet affinely equivalent to F.

Proof. Let $\{x \in \mathbb{R}^{d-1} : Ax \leq b\}$ be an inequality description of F, and let $(-b_i, a_i)$ denote the ith row of the matrix $(-b, A) \in \mathbb{R}^{n \times d}$.

The n hyperplanes $H_i = \{x \in \mathbb{R}^d : (-b_i, a_i)x = 0\}$ yield a linear arrangement of n hyperplanes in $\mathbb{R}^d$, which may also be viewed as a fan (polyhedral

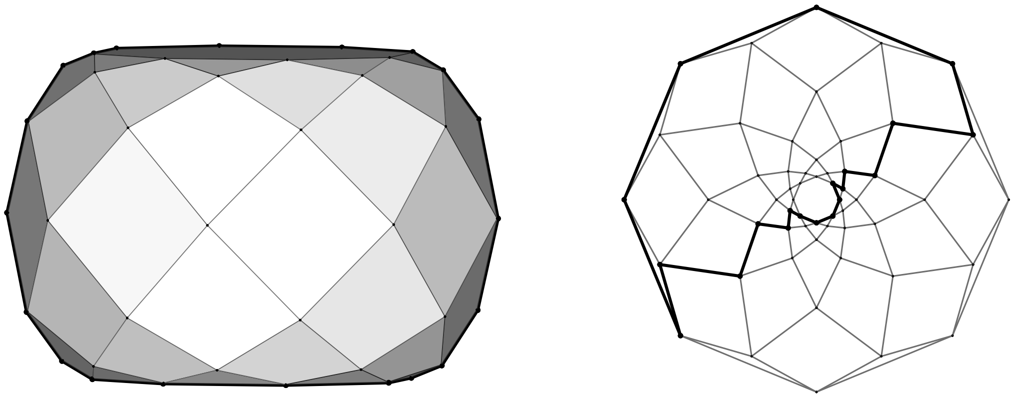

Figure 6.3: Shadow boundary of a centrally symmetric 3-polytope, on the right displayed as its Schlegel diagram.

complex of cones). According to [47, Cor. 7.18] the fan is polytopal, and the dual Z^* of the zonotope Z generated by the vectors $(-b_i, a_i)$ spans the fan.

The resulting dual zonotope Z^* has a facet that is *projectively equivalent* to F; however, the construction does not yet yield a facet that is affinely equivalent to F. In order to get this, we construct Z^* such that the hyperplane spanned by F is $x_0 = 1$. This is equivalent to constructing Z such that the vertex v_F corresponding to F is e_0. Therefore we have to normalize the inequality description of F such that

$$\sum_{i=1}^{n}(-b_i, a_i) = (1, 0, \ldots, 0).$$

The row vectors of A positively span $\mathbb{R}^{d-1}$ and are linearly dependent, hence there is a linear combination of the row vectors of A with coefficients $\lambda_i > 0$, $i = 1, \ldots, n$, which sums to 0. Thus if we multiply the ith facet-defining inequality for F, corresponding to the row vector $(-b_i, a_i)$, by

$$\frac{-\lambda_i}{\sum\limits_{j=1}^{n} \lambda_j \, b_j},$$

then we obtain the desired normalization of A and b. □

Some details for (3): The following simple lemma provides the last part of our proof; it is illustrated in Figure 6.3.

Lemma 6.8. Let P be a centrally symmetric 3-dimensional polytope and let $G \subset P$ be a k-gon facet. Then there exists a projection $\pi_G : \mathbb{R}^3 \to \mathbb{R}^2$ such that $\pi_G(P)$ is a polygon with at least k vertices.

Proof. Since P is centrally symmetric, there exists a copy G' of G as a facet of P opposite and parallel to G. Consider a projection π parallel to G (and to G') but otherwise generic and let n_G be the normal vector of the plane defining G. If we perturb π by adding $\pm\varepsilon n_G$, $\varepsilon > 0$, to the projection direction of π, parts of ∂G and $\partial G'$ appear on the shadow boundary. Since P is centrally symmetric, the parts of ∂G and $\partial G'$ appearing on the shadow boundary are the same. Therefore perturbing π either by $+\varepsilon n_G$ or by $-\varepsilon n_G$ yields a projection π_G such that $\pi_G(P)$ is a polygon with at least k vertices. □

6.3 Dual zonotopes with 2D-shadows of size $\Omega\left(n^{d-1}\right)$

In this section we prove our main result, Theorem 6.2*, in the following version.

Theorem 6.9. For any $d \geq 2$ there is a d-dimensional dual zonotope Z^* on $n(d-1)$ zones which has a 2D-shadow with $\Omega(n^{d-1})$ vertices.

We define a dual zonotope Z^* and examine its crucial properties. These are then summarized in Theorem 6.12, which in particular implies Theorem 6.9. Figure 6.4 displays a 3-dimensional example, Figure 6.7 a 4-dimensional example of our construction.

6.3.1 Geometric intuition

Before starting with the formalism for the proof, which will be rather algebraic, here is a geometric intuition for an inductive construction of $Z^* = Z_d^* \subset \mathbb{R}^d$, a d-dimensional zonotope on $n(d-1)$ zones with a 2D-shadow of size $\Omega(n^{d-1})$ when projected to the first two coordinates. For $d = 2$ any centrally-symmetric $2n$-gon (i.e., a 2-dimensional zonotope with n zones) provides such a dual zonotope Z_2^*. The corresponding affine hyperplane arrangement $\mathcal{A}_2 \subset \mathbb{R}^1$ consists of n distinct points.

We derive a hyperplane arrangement $\mathcal{A}_3' \subset \mathbb{R}^2$ from $\mathcal{A}_2$ by first considering $\mathcal{A}_2 \times \mathbb{R}$, and then "tilting" the hyperplanes in $\mathcal{A}_2 \times \mathbb{R}$. The hyperplanes in $\mathcal{A}_2 \times \mathbb{R}$ are ordered with respect to their intersections with the x_1-axis. The hyperplanes in $\mathcal{A}_2 \times \mathbb{R}$ are tilted alternatingly in x_2-direction as in Figure 6.5 (left): Each black vertex of $\mathcal{A}_2$ corresponds to a north-east line and each white vertex becomes a north-west line of the arrangement $\mathcal{A}_3'$. For each vertex in the 2D-shadow of Z_2^* we obtain an edge in the 2D-shadow of

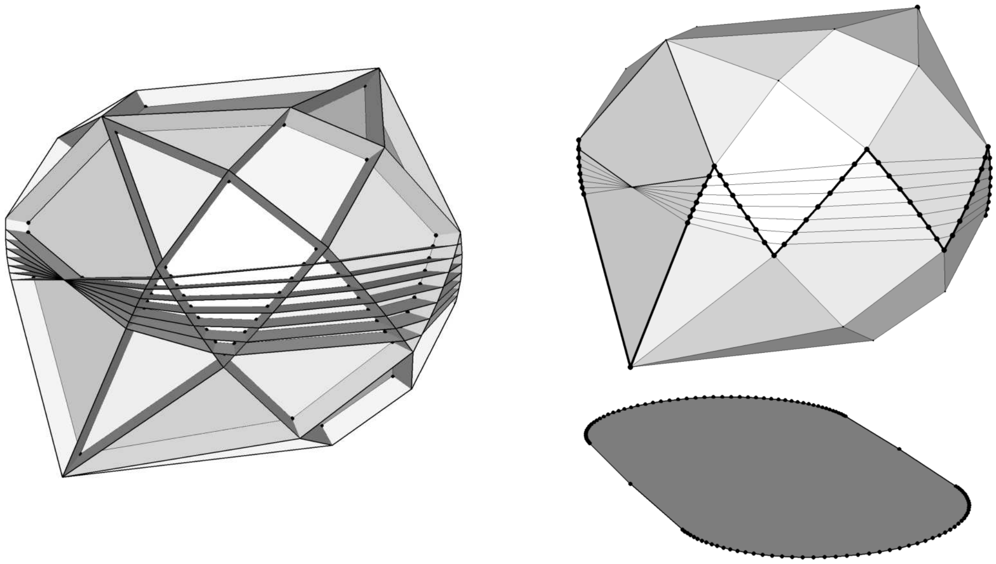

Figure 6.4: A dual 3-zonotope with quadratic 2D-shadow, on the left with the corresponding linear arrangement and on the right with its 2D-shadow.

the dual 3-zonotope $Z_3^{*\prime}$ corresponding to $\mathcal{A}_3'$. Now $\mathcal{A}_3 \subset \mathbb{R}^2$ is constructed from $\mathcal{A}_3'$ by adding a set of n parallel hyperplanes to $\mathcal{A}_3'$, all of them close to the x_1-axis, and each intersecting each edge of the 2D-shadow of $Z_3^{*\prime}$; see Figure 6.5 (right).

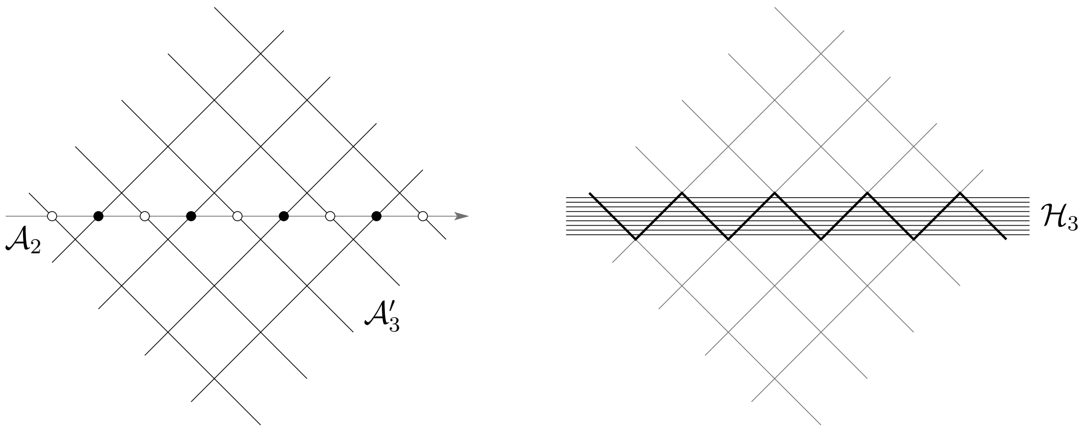

Figure 6.5: Constructing the arrangement $\mathcal{A}_3'$ from $\mathcal{A}_2$ (left) and $\mathcal{A}_3$ from $\mathcal{A}_3'$ (right).

For general d, let $\mathcal{H}_d \subset \mathcal{A}_d$ be the subarrangement of the n parallel hyperplanes added to $\mathcal{A}_d'$ in order to obtain $\mathcal{A}_d$. Then $\mathcal{A}_d' \subset \mathbb{R}^{d-1}$ is constructed from $\mathcal{A}_{d-1} \times \mathbb{R}$ by tilting the hyperplanes $\mathcal{H}_{d-1} \times \mathbb{R}$, this time with respect to their intersections with the x_{d-2}-axis. The corresponding d-dimensional dual zonotope $Z_d^{*\prime}$ has $\Omega(n^{d-2})$ edges in its 2D-shadow and each of these $\Omega(n^{d-2})$ edges is subdivided n times by the hyperplanes in $\mathcal{H}_d$ when constructing $\mathcal{A}_d$, respectively Z_d^*. See Figure 6.6 for an illustration of the arrangement $\mathcal{A}_4'$.

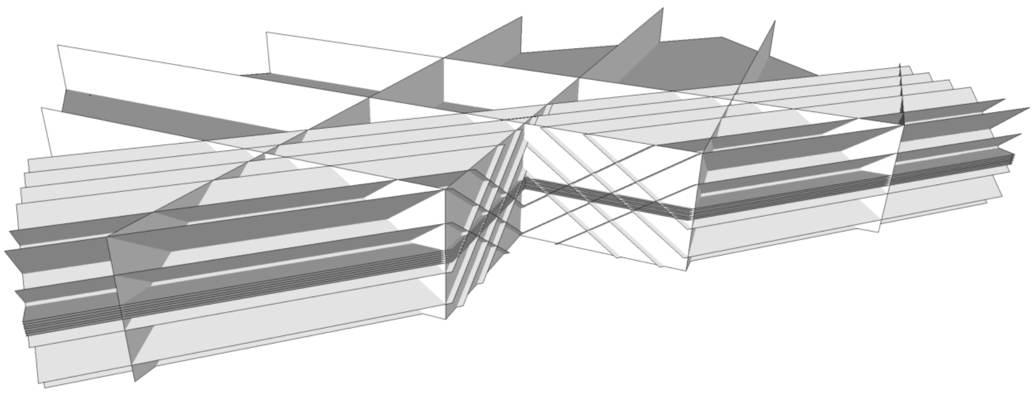

Figure 6.6: The affine arrangement $\mathcal{A}_4$ consists of three families of planes: The first family $\mathcal{A}_3' \times \mathbb{R}$ forms a coarse vertical grid; the second family (derived from $\mathcal{H}_3 \times \mathbb{R}$ by tilting) forms a finer grid running from left to right; the last family $\mathcal{H}_4$ contains the parallel horizontal planes.

6.3.2 The algebraic construction.

For $k \geq 1$, $n = 4k+1$, and $d \geq 2$ we define

$$\mathbf{b} = (k-i)_{0 \leq i \leq 2k} = \begin{pmatrix} k \\ \vdots \\ -k \end{pmatrix} \in \mathbb{R}^{2k+1} \qquad \text{and}$$

$$\mathbf{b}' = \left(i - k + \frac{1}{2}\right)_{0 \leq i \leq 2k-1} = \begin{pmatrix} -k + \frac{1}{2} \\ \vdots \\ k - \frac{1}{2} \end{pmatrix} \in \mathbb{R}^{2k}.$$

Let $\mathbb{0}, \mathbb{1} \in \mathbb{R}^\ell$ denote vectors with all entries equal to 0, respectively 1, of suitable size. For convenience we index the columns of matrices from 0 to $d-1$ and the coordinates accordingly by $x_0, \ldots, x_{d-1}$. Let $\varepsilon_i > 0$, and for $1 \leq i \leq d-1$ let $A_i \in \mathbb{R}^{n \times d}$ be the matrix with $\varepsilon_i \binom{\mathbf{b}}{\mathbf{b}'}$ as its 0-th column vector, $\binom{\mathbb{1}}{-\mathbb{1}}$ as its ith column vector, $\binom{\mathbb{1}}{\mathbb{1}}$ as its $(i+1)$-st column vector, and zeroes otherwise. In the case $i = d-1$ there is no $(i+1)$-st column of A_d and the final $\binom{\mathbb{1}}{\mathbb{1}}$-column is omitted:

$$A_i = \begin{array}{c} \begin{array}{ccccccccc} \scriptstyle 0 & \scriptstyle 1 & \scriptstyle \cdots & \scriptstyle i-1 & \scriptstyle i & \scriptstyle i+1 & \scriptstyle i+2 & \scriptstyle \cdots & \scriptstyle d-1 \end{array} \\ \begin{pmatrix} \varepsilon_i \mathbf{b} & \mathbb{0} & \cdots & \mathbb{0} & \mathbb{1} & \mathbb{1} & \mathbb{0} & \cdots & \mathbb{0} \\ \varepsilon_i \mathbf{b}' & \mathbb{0} & \cdots & \mathbb{0} & -\mathbb{1} & \mathbb{1} & \mathbb{0} & \cdots & \mathbb{0} \end{pmatrix} \end{array} \in \mathbb{R}^{(4k+1)\times d}.$$

The linear arrangement $\widehat{\mathcal{A}}$ given by the $((d-1)n \times d)$-matrix A whose horizontal blocks are the (scaled) matrices $\delta_1 A_1, \ldots, \delta_{d-1} A_{d-1}$ for $\delta_i > 0$ defines a dual zonotope by the construction of Section 6.1.2. Since the parameters δ_i do not change the arrangement $\widehat{\mathcal{A}}$, any choice of the δ_i yields the same combinatorial type of dual zonotope, but possibly different realizations. The choice of the ε_i however may (and for sufficiently large values will) change the combinatorics of $\widehat{\mathcal{A}}$ and hence the combinatorics of the corresponding dual zonotope. For the purpose of constructing Z^* we set $\alpha = \frac{1}{n+1}$, and $\varepsilon_i = \delta_i = \alpha^{i-1}$. This choice for ε_i ensures that the "interesting" part of the next family of hyperplanes nicely fits into the previous family. Compare Figure 6.5 (right): The interesting zig-zag part of family A_i is contained by the interval $\varepsilon_i[-k-\frac{1}{4}, k+\frac{1}{4}]$ in x_i-direction and by $\varepsilon_i[-\frac{1}{4}, \frac{1}{4}]$ in x_{i+1}-direction; since $\varepsilon_{i+1} = \frac{1}{n+1}\varepsilon_i$ we obtain $\varepsilon_{i+1}(k+\frac{1}{4}) < \varepsilon_i \frac{1}{4}$ and the zig-zags nicely fit into each other. For these parameters we obtain

$$A = \begin{pmatrix} A_1 \\ \alpha A_2 \\ \vdots \\ \alpha^{d-2} A_{d-1} \end{pmatrix} = \begin{pmatrix} \mathbf{b} & \mathbb{1} & \mathbb{1} & & \\ \mathbf{b}' & -\mathbb{1} & \mathbb{1} & & \\ \alpha^2 \mathbf{b} & & \alpha\mathbb{1} & \alpha\mathbb{1} & \\ \alpha^2 \mathbf{b}' & & -\alpha\mathbb{1} & \alpha\mathbb{1} & \\ \vdots & & & \ddots & \\ \alpha^{2(d-2)}\mathbf{b} & & & & \alpha^{d-2}\mathbb{1} \\ \alpha^{2(d-2)}\mathbf{b}' & & & & -\alpha^{d-2}\mathbb{1} \end{pmatrix}. \tag{6.2}$$

This matrix has size $(d-1)(4k+1)\times d = n(d-1)\times d$. The dual zonotope $Z^* = Z^*_A$ has $(d-1)n$ zones and is d-dimensional since A has rank d. According to Section 6.1.1, any point $x \in \mathbb{R}^d$ is labeled in $\widehat{\mathcal{A}}$ by a sign vector $\sigma(x) = (\sigma_1, {\sigma_1}'; \sigma_2, {\sigma_2}'; \ldots; \sigma_{d-1}, {\sigma_{d-1}}')$ with $\sigma_i \in \{+,0,-\}^{2k+1}$ and ${\sigma_i}' \in \{+,0,-\}^{2k}$. The following Lemma 6.10 selects n^{d-1} vertices of the corresponding affine arrangement $\mathcal{A}$.

Lemma 6.10. Let H_{j_1}, H_{j_2}, …, $H_{j_{d-1}}$ be hyperplanes in $\mathcal{A}$, where each H_{j_i} is given by some row a_{j_i} of A_i, which is indexed by $j_i \in \{1, \dots, n\}$. Then the $d-1$ hyperplanes H_{j_1}, H_{j_2}, …, $H_{j_{d-1}}$ intersect in a vertex of $\mathcal{A}$ with sign vector $(\sigma_1, {\sigma_1}'; \sigma_2, {\sigma_2}'; \dots; \sigma_{d-1}, {\sigma_{d-1}}') \in \{+, 0, -\}^{n(d-1)}$ with 0 at position j_i of the form

$$({\sigma_i}, {\sigma_i}') = \begin{cases} (+\cdots+0-\cdots-, -\cdots-+\cdots+) & \text{with sum 0 or} \\ (+\cdots+-\cdots-, -\cdots-0+\cdots+) & \text{with sum 0} \end{cases} \tag{6.3}$$

for each $i = 1, 2, \dots, d-1$. Conversely, each of these sign vectors corresponds to a vertex v of the arrangement. In particular, v is a generic vertex, i.e., v lies on exactly $d-1$ hyperplanes.

Proof. The intersection $v = H_{j_1} \cap H_{j_2} \cap \cdots \cap H_{j_{d-1}}$ is indeed a vertex since the matrix minor $(a_{j_i,\ell})_{i,\ell=1,\dots,d-1}$ has full rank. We solve the system $A'\binom{1}{v} = 0$ to obtain v, where $A' = (a_{j_i})_{i=1,\dots,d-1}$. As we will see, the entire sign vector of the vertex v is determined by its "0" entries whose positions are given by the j_i. Hence every sign vector agreeing with Equation (6.3) determines a set of hyperplanes H_{j_i} and thus a vertex v of the arrangement.

To compute the position of v with respect to the other hyperplanes we take a closer look at a block A_i of the matrix that describes our arrangement. For an arbitrary point $x \in \mathbb{R}^d$ with $x_0 = 1$ we obtain

$$A_i x = \begin{pmatrix} \alpha^{i-1}\mathbf{b} & \mathbb{1} & \mathbb{1} \\ \alpha^{i-1}\mathbf{b}' & -\mathbb{1} & \mathbb{1} \end{pmatrix} \begin{pmatrix} 1 \\ x_i \\ x_{i+1} \end{pmatrix}.$$

This is equivalent to the 2-dimensional(!) affine arrangement shown in Figure 6.5 on the left. We will show that if x lies on one of the hyperplanes and if $|x_{i+1}| < \frac{1}{4}\alpha^{i-1}$, then x satisfies the required sign pattern (6.3).

We start with an even simpler observation: If x' lies on one of the hyperplanes and has $x'_{i+1} = 0$ (so in effect we are looking at a 1-dimensional affine hyperplane arrangement), then there are:

- $2k$ "positive" row vectors a_j of A_i with $a_j x' > 0$,
- $2k$ "negative" row vectors a_j of A_i with $a_j x' < 0$, and
- one "zero" row vector corresponding to the hyperplane x' lies on.

The order of the rows of A_i is such that the signs match the sign pattern of $(\sigma_i, {\sigma_i}')$ in (6.3). Since the values in $\alpha^{i-1}\mathbf{b}$ and $\alpha^{i-1}\mathbf{b}'$ differ by at least $\frac{1}{2}\alpha^{i-1}$ we have in fact $a_j x' \geq \frac{1}{2}\alpha^{i-1}$ for "positive" row vectors and $a_j x' \leq -\frac{1}{2}\alpha^{i-1}$ for the "negative" row vectors of A_i. Hence we have

$$|a_j x'| \geq \tfrac{1}{2}\alpha^{i-1}.$$

If we now consider a point x with $|x_{i+1}| < \frac{1}{4}\alpha^{i-1}$ on the same hyperplane as x', then $|x_i{}' - x_i| = |x_{i+1}| < \frac{1}{4}\alpha^{i-1}$. For the row vectors a_j with $a_j x' \neq 0$ we obtain:

$$\begin{aligned} |a_j x| &\geq |a_j x'| - |a_j(x - x')| \\ &\geq \tfrac{1}{2}\alpha^{i-1} - (|x_i - x_i{}'| + |x_{i+1} - x_{i+1}{}'|) \\ &> \tfrac{1}{2}\alpha^{i-1} - \tfrac{1}{4}\alpha^{i-1} - \tfrac{1}{4}\alpha^{i-1} = 0. \end{aligned}$$

Hence the sign pattern of x is the same as the sign pattern of x'.

To conclude the proof we show that the required upper bound $|v_{i+1}| < \frac{1}{4}\alpha^{i-1}$ holds for the coordinates of the selected vertex v. For all $i' = 1, 2, \ldots, d-2$ the inequality $a_{j_{i'}}\binom{1}{v} = 0$ directly yields the bound $|v_{i'}| \leq k\alpha^{i'-1} + |v_{i'+1}|$. Further $a_{j_{d-1}}\binom{1}{v} = 0$ implies $|v_{d-1}| \leq k\alpha^{d-2}$ and thus recursively

$$\begin{aligned} |v_{i+1}| &\leq k\alpha^i + |v_{i+2}| \leq k\alpha^i + k\alpha^{i+1} + |v_{i+3}| \\ &\leq \cdots \leq k\alpha^i + k\alpha^{i+1} + \cdots + |v_{d-1}| \leq k\sum_{l=i}^{d-2}\alpha^l < k\alpha^i\sum_{l=0}^{\infty}\alpha^l \\ &= \frac{k\alpha^i}{1-\alpha} = \frac{k}{4k+1}\alpha^{i-1} < \frac{1}{4}\alpha^{i-1}. \end{aligned}$$ □

The selected vertices of Lemma 6.10 correspond to certain vertices of the dual zonotope Z^* associated to the arrangement $\mathcal{A}$. Rather than proving that these vertices of Z^* survive the projection to the first two coordinates, we consider the edges corresponding to the sign vectors obtained from Equation (6.3) by replacing the "0" in $(\sigma_{d-1}, \sigma'_{d-1})$ by either a "+" or a "−", and their negatives, which correspond to the antipodal edges.

Lemma 6.11. Let S be the set of sign vectors $\pm(\sigma_1, \sigma_1{}'; \ldots; \sigma_{d-1}, \sigma_{d-1}{}')$ of the form

$$(\sigma_i, \sigma_i{}') = \begin{cases} (+\cdots+0-\cdots-, -\cdots-+\cdots+) & \text{with sum 0 or} \\ (+\cdots+-\cdots-, -\cdots-0+\cdots+) & \text{with sum 0} \end{cases}$$

for $1 \leq i \leq d-2$ and

$$(\sigma_{d-1}, \sigma_{d-1}{}') = (+\cdots+-\cdots-, -\cdots-+\cdots+) \text{ with sum } \pm 1.$$

Then the sign vectors in S correspond to $2n^{d-2}(n+1)$ edges of Z^*, all of which survive the projection to the first two coordinates.

Proof. The sign vectors of S indeed correspond to edges of Z^* since they are obtained from sign vectors of non-degenerate(!) vertices by substituting one "0" by a "+" or a "−".

Further there are $2n^{d-2}(n+1)$ edges of the specified type: Firstly there are n choices where to place the "0" in $(\sigma_i, {\sigma_i}')$ for each $i = 1, \dots, d-2$, which accounts for the factor n^{d-2}. Let p be the number of "+"-signs in σ_{d-1}. Thus there are $2k+2$ choices for p, and for each choice of p there are two choices for ${\sigma_{d-1}}'$, except for $p = 0$ and $p = 2k+1$ with just one choice for ${\sigma_{d-1}}'$. This amounts to $2(2k+2) - 2 = n+1$ choices for $(\sigma_{d-1}, {\sigma_{d-1}}')$. The factor of 2 is due to the central symmetry.

Let e be an edge with sign vector $\sigma(e) \in S$. In order to apply Lemma 1.11 we need to determine the normals to the facets containing e. So let F be a facet containing e. The sign vector $\sigma(F)$ is obtained from $\sigma(e)$ by replacing each "0" in $\sigma(e)$ by either "+" or "−"; see Lemma 6.4. For brevity we encode F by a vector $\tau(F) \in \{+,-\}^{d-2}$ corresponding to the choices for "+" or "−" made. Conversely, there is a facet F_τ containing e for each vector $\tau \in \{+,-\}^{d-2}$, since e is non-degenerate.

The supporting hyperplane for F is $a(F)x = 1$ with $a(F) = \sigma(F)A$ being a linear combination of the rows of A. We compute the ith component of $a(F)$ for $i = 2, 3, \dots, d-1$:

$$\begin{aligned} a(F)_i = (\sigma(F)A)_i &= ((\sigma_{i-1}, \sigma'_{i-1})A_{i-1})_i + ((\sigma_i, \sigma'_i)A_i)_i \\ &= \alpha^{i-2}(\sigma_{i-1}, \sigma'_{i-1}) \begin{pmatrix} \mathbb{1} \\ \mathbb{1} \end{pmatrix} + \alpha^{i-1}(\sigma_i, \sigma'_i) \begin{pmatrix} \mathbb{1} \\ -\mathbb{1} \end{pmatrix} \end{aligned}$$

Since we replace the zero of $(\sigma_{i-1}, \sigma'_{i-1})$ by $\tau(F)_{i-1}$ in order to obtain $\sigma(F)$ from $\sigma(e)$ we have $(\sigma_{i-1}, \sigma'_{i-1})\binom{\mathbb{1}}{\mathbb{1}} = \tau(F)_{i-1}$. Since $\left|(\sigma_i, \sigma'_i)\binom{\mathbb{1}}{-\mathbb{1}}\right|$ is at most n it follows that

- $a(F)_i \geq \alpha^{i-2} - n\alpha^{i-1} = \alpha^{i-1} > 0$ holds for $\tau(F)_{i-1} = +$ and
- $a(F)_i \leq -\alpha^{i-2} + n\alpha^{i-1} = -\alpha^{i-1} < 0$ holds for $\tau(F)_{i-1} = -$.

In other words, we have for $i = 2, 3, \dots, d-1$:

$$\operatorname{sign} a(F)_i = \tau(F)_i \tag{6.4}$$

It remains to show that the last $d-2$ coordinates of the 2^{d-2} normals of the facets containing e, that is, the facets F_τ for all $\tau \in \{+,-\}^{d-2}$, span $\mathbb{R}^{d-2}$. But Equation (6.4) implies that each of the orthants of $\mathbb{R}^{d-2}$ contains one of the (truncated) normal vectors $(a(F_\tau)_i)_{i=2,\dots,d-1}$. Hence the (truncated) normals of all facets containing e positively span $\mathbb{R}^{d-2}$ and e survives the projection to the first two coordinates by Lemma 1.11. □

This completes the construction and analysis of Z^*. Scrutinizing the sign vectors of the edges specified in Lemma 6.11 one can further show that these edges actually form a closed polygon in Z^*. Thus this closed polygon is the shadow boundary of Z^* (under projection to the first two coordinates) and its projection is a $2n^{d-2}(n+1)$-gon. This yields the precise size of the projection of Z^*. The reader is invited to localize the edges corresponding to the closed polygon from Lemma 6.11 and the vertices from Lemma 6.10 in Figures 6.5 and 6.6.

The following Theorem 6.12 summarizes the construction of Z^* and its properties. Our main result as stated in Theorem 6.9 follows. Figure 6.4 displays a 3-dimensional example, Figure 6.7 a 4-dimensional example.

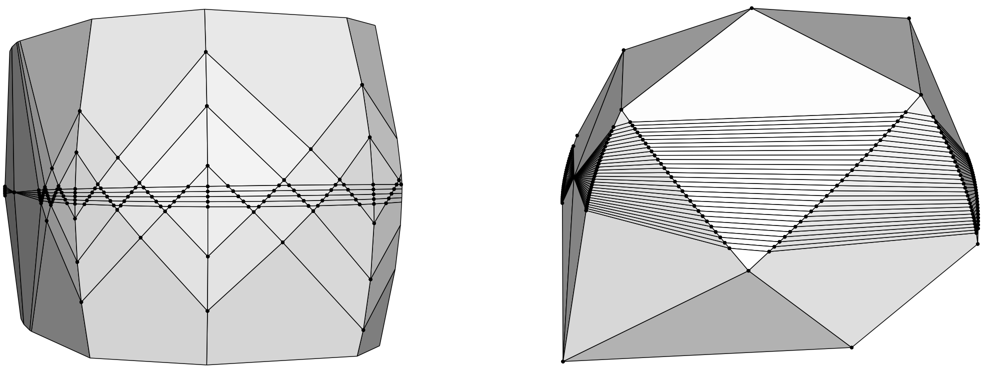

Figure 6.7: Two different projections of a dual 4-zonotope with cubic 2D-shadow. On the left the projection to the first two and last coordinate (clipped in vertical direction) and on the right the projection to the first three coordinates.

Theorem 6.12. Let k and $d \geq 2$ be positive integers, and let $n = 4k+1$. The dual d-zonotope $Z^* = Z_A^*$ corresponding to the matrix A from Equation (6.2) has $(d-1)n$ zones and its projection to the first two coordinates has (at least) $2n^{d-1} + 2n^{d-2}$ vertices. $\square$

Remark 6.13. As observed in Amenta & Ziegler [1, Sect. 5.2] any result about the complexity lower bound for projections to the plane (2D-shadows) also yields lower bounds for the projection to dimension k, a question which interpolates between the upper bound problems for polytopes/zonotopes ($k = d-1$) and the complexity of parametric linear programming ($k = 2$), the task to compute the LP optima for all linear combinations of two objective functions (see [14, pp. 162-166]).

In this vein, from Theorem 6.9 and the fact that in a dual of a cubical zonotope every vertex lies in exactly $f_k((d-1)\text{-cube}) = \binom{d-1}{k}2^k$ different k-faces (for $k < d$), and every such polytope contains at most n^{d-1} faces of dimension k, one derives that in the worst case $\Theta(n^{d-1})$ faces of dimension $k-1$ survive in a kD-shadow of the dual of a d-zonotope with n zones.

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