Discrete Surfaces and Coordinate Systems: Approximation Theorems and Computation

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Chapter 1

Introduction

This doctoral thesis is concerned with the approximation of selected models in classical differential geometry by models from discrete differential geometry.
Numerical schemes are developed for solution of the Goursat and Cauchy problem for a class of nonlinear hyperbolic and elliptic equations, respectively. Convergence results are proven and applied to the equations associated to the geometric models.

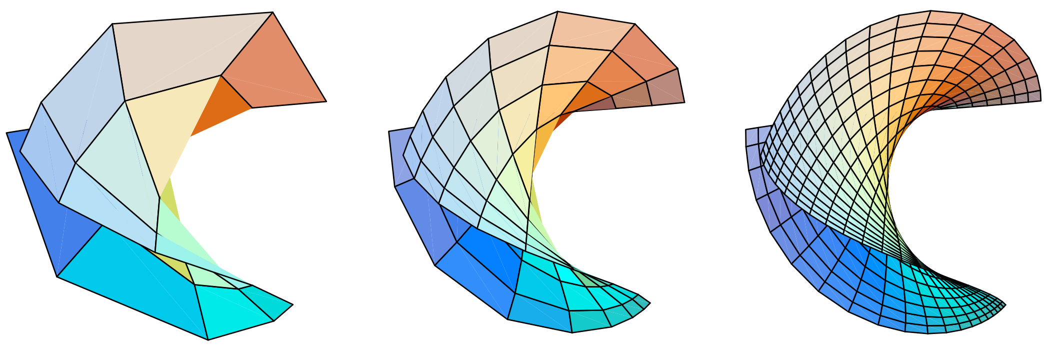

Figure 1.1: A K-surface in different discretizations

Motivation

The development of classical differential geometry led to the introduction and investigation of various classes of surfaces, which are of interest both for geometric reasons and for applications. Examples are minimal surfaces, surfaces of constant mean or Gaussian curvature and surfaces in special parameterizations, like immersions carrying curvature-line-coordinates. Already Euler, Gauß and Weierstraß studied such surfaces. Also, special coordinate systems have attracted the attention of mathematicians and physicists for a long time. Triply orthogonal curvilinear coordinates were used by Leibniz and Euler to evaluate multiple integrals, and later by Lamé and Jacobi to carry out calculations in analytical mechanics. Among many others, the aforementioned surfaces and coordinates played a prominent role in the classical works of Bianchi [Bi], Darboux [Da] and others, dating back to the end of the 19th century. Hence, their rich theory is – to a large extent – a classical heritage.

Recently, numerical methods began to play an important role in the investigation of such surfaces and maps. They are stimulated by applications in science and scientific computation, in particular for the purpose of visualization, see for instance [MPS]. The question of proper discretization of the classical models – preserving as many of the characteristic features as possible – is a subject of intensive study in the flourishing field of discrete differential geometry, see, for example, [BSe] and in particular [BP3].

A very distinct property of various special classical surfaces turns out to be their *integrability*, meaning that the partial differential equations underlying the geometry are integrable in the sense of the modern theory of solitons. One of the manifestations of integrability is the existence of (Bäcklund-Darboux) transformations, which enable one to construct a variety of new surfaces from a given one. Another is that some surfaces come in (associated) families, whose members share many characteristic properties.

The underlying integrable equations are of high relevance in pure and applied mathematics, often at places without apparent connection to geometry. For instance, the sine-Gordon equation belonging to surfaces of constant negative curvature plays an important role in (quantum) field theory, while the Lamé system, describing orthogonal systems, has deep relations to associativity equations; see [Dub], [Kr]. All systems have been intensively studied in soliton theory. The associated transformations were investigated, and explicit solutions in terms of elementary and transcendential functions constructed. Recent results about n-dimensional orthogonal systems as integrable systems are found in the publications of Krichever [Kr] and Zakharov [Z].

For a great variety of geometric problems described by integrable equations, it was found that *integrable discretizations* are best suited for the purpose of finding an appropriate discrete analogue to the continuous model. Here integrable discretization means that the feature of integrability passes from the partial differential equations to the difference equations which describe the discrete geometric model. Special examples for discrete surfaces of constant negative curvature are constructed in [BP1], and for quadrilateral lattices in [BoK].

It turns out that integrable discretizations can be described in pure geometric terms. It is interesting to note that the basic equation for discrete surfaces of constant negative curvature, that is, an integrable discretization of the sine-Gordon-equation, has been introduced by Hirota [Hi], without any reference to geometry; the link to discrete surfaces was established in [BP1]. Similarly, a set of equations describing quadrilateral lattices was known [BoK] before the geometrical meaning became clear [DS1].

It is a common belief that the classical theory of surfaces can be obtained as a limit of the corresponding discrete one. Explicit examples and computer experiments both suggest a good qualitative and quantitative approximation of the continuous objects by their discrete counterparts. There have been, however, only few rigorous convergence results so far. This thesis is intended as one more step to close this gap. A part of our results has already been published [BMS].

To further justify the interest in the discrete models, and underline the relevance of rigorous convergence results, let us mention two applications. Firstly, the discrete objects are natural for calculations on computers. In combination with the convergence results of this thesis, the discretizations thus provide a practical numerical scheme for approximative computation of the respective smooth objects. Secondly, discrete surfaces and their transformations appear as sublattices of higher-dimensional lattices. Hence, surfaces and their transformations, which are treated seperately in the classical context, are naturally unified on the discrete level. Our convergence results hold simultaneously for surfaces and transformations. They are obtained by refining the mesh for the individual sublattices, keeping the lattice step in the other directions constant. This shows that the classical transformations are indeed the limit of the very natural discrete constructions.

Classical Models

To be specific, let us informally introduce the models which are investigated in this thesis, along with their transformations. They are smooth immersions or mappings $f : \Omega \subset \mathbb{R}^M \to \mathbb{R}^N$ with certain properties.

For K-surfaces, the dimensions are $M = 2$ and $N = 3$, and f parameterizes an immersed surface of constant negative Gaussian curvature by asymptotic lines. K-surfaces come in *associated families*, whose members have the same angles between asymptotic lines and the same second fundamental form. A K-surface is a Bäcklund transform of another one, if the line connecting corresponding points is tangent to both surfaces.

For conjugate nets, M and N are arbitrary. It is required that the mixed second derivatives $\partial_i\partial_j f$ lie in the span of the respective first derivatives $\partial_i f$ and $\partial_j f$ at each point. Jonas transformations are used to produce new conjugate nets from a given one. The line connecting corresponding points of a conjugate net and its Jonas transform lies in the intersection of the M affine planes spanned by the pairs of respective tangential vectors.

Orthogonal systems form a subclass of conjugate nets. They are subject to the orthogonality conditions $\partial_i f \cdot \partial_j f = 0$ for $i \neq j$. Two orthogonal systems are related by a Ribaucour transformation, if any two corresponding coordinate curves envelope a circle congruence.

Two-dimensional conformal maps, $M = N = 2$, are a subclass of orthogonal systems, singled out by $|\partial_1 f| = |\partial_2 f|$. Their transformations are not considered here.

Discrete Models

The discrete models under consideration are described by point lattices $f^\epsilon : \Omega^\epsilon \subset (\epsilon_1\mathbb{Z}) \times \cdots \times (\epsilon_M\mathbb{Z}) \to \mathbb{R}^N$ defined on a subset Ω^ϵ of the M-dimensional grid with mesh sizes $\epsilon_i > 0$. While the definitions for the smooth models were classical, the definition of the discrete models are taken from recent publications.

Discrete K-surfaces, the analogues of surfaces of constant negative curvature, were first introduced by Wunderlich [Wu], then studied by Bobenko and Pinkall [BP1]. f^ϵ is a discrete K-surface if all edges of the lattice are of the same length, and each lattice point lies in a common plane with its four neighbors. Examples of discrete K-surfaces are displayed in Fig 1.1.

Quadrilateral lattices are the discrete analogue of conjugate nets. They were first considered in dimensions $M = 2$ and $N \geq 2$ by Sauer [Sa], then generalized to arbitrary dimensions by Doliwa and Santini, see the overview article in [DS2]. Their defining property is that the images of elementary M-dimensional cubes have planar faces.

The definition of discrete triply orthogonal systems ($M = N = 3$) is due to Bobenko [B2] and was generalized to arbitrary dimensions by Doliwa and Santini [DS2]. Orthogonal systems are special conjugate nets, in which the vertices of the elementary cubes lie on $M-1$-dimensional spheres.

Orthogonal circle patterns as analogous of complex-analytic functions were introduced and studied by Schramm in [Sc]. Roughly, these are planar lattices with $M = N = 2$, where elementray quadrilaterals are concircular, and neighboring circles intersect orthogonally. For an alternative approach to discrete conformal maps, we also investigate CR-1–mappings as proposed in [BP2]. For those, elementary quadrilaterals have cross-ratio minus one.

Let us make the idea of how transformations appear as sublattices more precise. Consider an M-dimensional discrete orthogonal system f^ϵ in $\mathbb{R}^N$, which for simplicity is defined on $\Omega^\epsilon = (\epsilon_1\mathbb{Z}) \times \cdots \times (\epsilon_M\mathbb{Z})$. Let $M = m + m'$ and split the domain accordingly, $\Omega^\epsilon = \Omega_0^\epsilon \times \Omega_+^\epsilon$, so that $\Omega_0^\epsilon = (\epsilon_1\mathbb{Z}) \times \cdots \times (\epsilon_m\mathbb{Z})$ contains the first m directions, and $\Omega_+^\epsilon = (\epsilon_{m+1}\mathbb{Z}) \times \cdots \times (\epsilon_M\mathbb{Z})$ the remaining m' directions. Then, for each $\xi' \in \Omega_+^\epsilon$, the function $f^\epsilon(\cdot, \xi') : \Omega_0^\epsilon \to \mathbb{R}^N$ is an m-dimensional discrete orthogonal system. Given two values $\xi', \xi'' \in \Omega_+^\epsilon$, the respective systems $f^\epsilon(\cdot, \xi')$ and $f^\epsilon(\cdot, \xi'')$ are related by a Ribaucour transformation (or a sequence thereof).

The idea for simultaneous approximation of the immersion and its transformations is evident: Refine the lattice in the directions which correspond to the discrete object ($\epsilon_i \equiv \epsilon \to 0$ for indices $i \leq m$), and keep the other directions, corresponding to transformations, discrete ($\epsilon_j \equiv 1$ for $j > m$). In the limit, smooth surfaces are obtained, which are interrelated by the corresponding continuous transformations. Once more this shows how natural the transformation theory fits into the picture.

As the existence of transformations is one of the crucial features of integrable systems, integrability is probably best understood on the discrete level. It is natural to call those classes of lattices *discrete integrable*, which can be *consistently* defined for arbitrary dimension $M \geq M_0$. The notion of consistency has been intensively investigated in the recent works of Adler, Bobenko and Suris, [BSu], [ABS]. In this thesis, consistency appears as a seemingly technical hypothesis for the well-posedness of a certain initial-value problem. Consistency has been proven for (the transformations of) discrete K-surfaces by Wunderlich [Wu], and for conjugate nets and orthogonal systems by Doliwa and Santini, see [DS1].

Summary of Results

As the central results, theorems about C^∞-approximation of K-surfaces, conjugate and orthogonal systems, and two-dimensional conformal maps by their appropriate discrete counterparts are proven. Approximation holds simultaneously for finitely many transforms, and also for associated families of K-surfaces. The results are formulated in geometrical terms.

In order to prove the approximation results, convergence in certain numerical schemes, discretizing geometrically relevant hyperbolic and elliptic PDEs, is investigated. The results are proven in an abstract analytic setting. They are of relevance on their own.

The following theorem about approximation of triply orthogonal systems is a consequence of our findings, and is intended to give a flavor of the nature of our results.

Theorem 1.1 *Assume three umbilic free*[1] *immersed surfaces* $\mathcal{F}_1$*,* $\mathcal{F}_2$ *and* $\mathcal{F}_3$ *in* $\mathbb{R}^3$ *intersect pairwise along their curvature lines,* $\mathrm{X}_1 = \mathcal{F}_2 \cap \mathcal{F}_3$ *etc. Then:*

1. *On a sufficiently small box* $\mathcal{B}(r) = [0, r]^3 \subset \mathbb{R}^3$*, there is an orthogonal coordinate system* $\mathrm{x} : \mathcal{B}(r) \to \mathbb{R}^3$*, which has* $\mathcal{F}_1$*,* $\mathcal{F}_2$ *and* $\mathcal{F}_3$ *locally as images of the coordinate planes, with each curve* X_i *being parameterized locally over the* i*-th axis by arc length. The orthogonal coordinate system* x *is uniquely determined by these properties.*

2. *Denote by* $\mathcal{B}^\epsilon(r) = \mathcal{B}(r) \cap (\epsilon\mathbb{Z})^3$ *the grid of mesh size* ϵ *inside* $\mathcal{B}(r)$*. One can define a family* $\{\mathrm{x}^\epsilon : \mathcal{B}^\epsilon(r) \to \mathbb{R}^3\}_\epsilon$ *of discrete orthogonal systems, parameterized by* $\epsilon > 0$*, such that the approximation error decays as*

$$\sup_{\xi \in \mathcal{B}^\epsilon(r)} |\mathrm{x}^\epsilon(\xi) - \mathrm{x}(\xi)| < C\epsilon.$$

   The convergence is C^∞*: all partial difference quotients of* x^ϵ *converge at the same rate to the respective partial derivatives of* x*.*

[1] This is a genericity condition, expressing that the intersection point $\mathcal{F}_1 \cap \mathcal{F}_2 \cap \mathcal{F}_3$ lies in general position on each of the surfaces.

The theorem is actually more than an abstract approximation result. A recipe to construct a convergent sequence of discrete orthogonal systems from the given surfaces is presented. If smooth data for a Ribaucour transformation is given, discrete four-dimensional orthogonal systems are calculated, in which the respective three-dimensional subsystems converge to the smooth system and its transform.

More generally, the following is performed for K-surfaces and conjugate/orthogonal systems: It is clarified which smooth geometric data must be prescribed to determine the respective continuous geometric object – and its transforms – uniquely. From the smooth data, discrete data is defined, often simply by restriction. By an explicit construction, discrete geometric objects are found. These converge to the sought smooth object.

On the analytical side, the explicit discrete construction corresponds to solving a Cauchy (or, more precisely, Goursat) problem for a system of hyperbolic *difference* equations on the grid Ω^ϵ, which approximate a system of *differential* equations in the limit $\epsilon \to 0$. By our Theorem 2.1, convergence of the discrete solutions to the unique smooth solution of the continuous Goursat problem is provided under relatively mild hypotheses on the formal C^∞-convergence of the discrete equations. It is shown that the error for the approximation of the smooth solution and all its partial derivatives is linear in ϵ. It is further shown that "C^∞-perturbations" of the discrete initial data do not harm the convergence. There are relations to various older results (see e.g. [Gi], [Ke], [Th] and the references therein) about solvability of mixed boundary problems for hyperbolic equations and their numerical approximation. The quite general type of nonlinear Goursat problem considered here has seemingly not been treated before. Also, the proof of simultaneous convergence of transformations is new.

For two-dimensional conformal maps, the equations are elliptic. A genuine Cauchy problem is studied instead of a Goursat problem. The proof of the corresponding analytical convergence Theorem 6.2 is harder, mainly because the continuous problem is well-posed in the space of analytic functions (or rather in a scale of these spaces), which has no canonical analogue on the discrete side. The approximation error decays quadratic in ϵ, and only "analytic perturbations" of the discrete data are allowed. Difference schemes discretizing ill-posed problems for elliptic equations have been proposed and analyzed by various authors, e.g. [HHR]. Also, numerical schemes for the related abstract Cauchy problems have been investigated, see [Su], [GMS]. The available convergence results are tailored to particular situations, and do not apply to the difference scheme considered here.

Some by-products of our investigations are of interest on their own. For example:

- a system of difference equations which constitutes a discrete analogue of the Lamé equations (A *general* discrete Lamé system is seemingly new. Note that equations are known for the special reduction of Darboux-Egorov-metrics, see [AKV].)

- formulation of geometric initial value problems for smooth K-surfaces and conjugate/orthogonal systems

- approximation of curvature line parameterized surfaces by circular nets
- derivation of the classical Bianchi permutability theorems for transformations from the results about multi-dimensional discrete lattices
- a new existence proof for solutions to semi-linear Cauchy problems, based on discrete methods

Simplified Scheme of the Proofs

The individual geometric models are treated in the same manner. For all the "hyperbolic models", one follows the following steps:

1. Differential and difference equations are derived from the geometric properties of the smooth and discrete objects. These are classical for the smooth models.
2. The equations derived are brought into a special form we refer to as (discrete) hyperbolic system. Also, the suitable Goursat problem is formulated in analytical terms. Formal approximation of the system of partial differential equations by its discrete counterparts is verified.
3. Local well-posedness of the Goursat problem is proven. It is equivalent to the important property of consistency of the system of difference equations.
4. The analytic approximation Theorem 2.2 now leads to a convergence result for the solutions to the discrete equations. This needs to be interpreted in geometrical terms.

Orthogonal circle patterns and CR-1–mappings are treated in a similar way. Instead of a hyperbolic system, one derives a semi-linear PDE of first order in step two.

The Thesis is Organized as Follows

In Chapter 2, the specific notion of hyperbolic systems of differential and difference equations is introduced, the canonical Goursat problem is described, and the main theorem about existence, uniqueness and discrete approximation of solutions is proven.

Chapter 3 deals with K-surfaces. These are investigated from two different points of view: Firstly, the approximation result is proven in the geometric setting where all properties of the surface are described in terms of its Gauss maps. The hyperbolic equations are equivalent to the Lorentz-harmonicity of the Gauss map. Secondly, adapted $SO(3)$-frames yield an alternative, more algebraic approach. It is suited for a simultaneous description of the whole associated family.

Chapter 4 is dedicated to conjugate nets. They provide an example where the hyperbolic equations are almost immediately read off from the definition, and the main theorem applies directly.

In contrast, the analogous considerations for orthogonal systems presented in Chapter 5 are more sophisticated. Firstly, that orthogonal systems are rather objects of Möbius than Euclidean geometry is reflected by difficulties in finding suitable discrete hyperbolic equations in the classical (euclidean) setting. Hence, we pass to a Möbius geometrical description of the lattice as suggested in [BHe]. A brief introduction to the basic concepts of Möbius geometry and Clifford algebras is included at this point. Secondly, neither the classical nor the discrete Lamé system is intrinsically hyperbolic in general dimensions. This makes it necessary to develop the two-dimensional theory (for C-surfaces) first, where the hyperbolic form and the respective approximation result is obtained. Eventually, the results for general orthogonal systems are proven by combining the two-dimensional theorems with those about conjugate nets.

Chapters 6 and 7 constitute the second part of this thesis. Continuous and discrete Cauchy problems are introduced. Their unique solvability and the convergence of discrete solutions are the issue of the main theorem in Chapter 6. Though apparently similar to the main theorem of Chapter 2, different techniques are needed for the proof.

Applications of this theorem to CR-1–mappings and orthogonal circle patterns are content of the last chapter. Again, the main difficulty is to reformulate the geometrical questions in the form of Cauchy problems.

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Chapter 2

Approximation Theorem for Hyperbolic Problems

2.1 Notations

2.1.1 Domains and Functions

For a vector $\boldsymbol{\epsilon} = (\epsilon_1, \ldots, \epsilon_M)$ of M non-negative numbers, define the grid

$$\mathcal{B}^{\boldsymbol{\epsilon}} = (\epsilon_1\mathbb{Z}) \times \ldots \times (\epsilon_M\mathbb{Z}) \subset \mathbb{R}^M. \tag{2.1}$$

We refer to ϵ_i as the *mesh size* of the grid $\mathcal{B}^{\boldsymbol{\epsilon}}$ in the i-th direction; a direction with a vanishing mesh size $\epsilon_i = 0$ is understood as a *continuous* one, i.e. in this case the corresponding factor $\epsilon_i\mathbb{Z}$ in (2.1) is replaced by $\mathbb{R}$. For instance, the choice $\boldsymbol{\epsilon} = (0, \ldots, 0)$ yields $\mathcal{B}^{\boldsymbol{\epsilon}} = \mathbb{R}^M$. We use also the following notation for sublattices of $\mathcal{B}^{\boldsymbol{\epsilon}}$

$$\mathcal{B}^{\boldsymbol{\epsilon}}_{\mathcal{S}} = \{\xi = (\xi_1, \ldots, \xi_M) \in \mathcal{B}^{\boldsymbol{\epsilon}} \mid \xi_i = 0 \text{ if } i \notin \mathcal{S}\}\,, \text{ where } \mathcal{S} \subset \{1, \ldots, M\}.$$

Considering convergence problems, the focus is on situations where

$$\epsilon_i = \epsilon \text{ for } i = 1, \ldots, m \quad \text{and} \quad \epsilon_i = 1 \text{ for } i = m+1, \ldots, M$$

with some integer $0 \le m \le M$ and some $\epsilon \ge 0$. Hence each $\xi \in \mathcal{B}^{\boldsymbol{\epsilon}}$ is of the form

$$\xi = (\hat{\xi}, \check{\xi}) \text{ with } \hat{\xi} = (\hat{\xi}_1, \ldots, \hat{\xi}_m) \in \epsilon\mathbb{Z}^m,\ \check{\xi} = (\check{\xi}_1, \ldots, \check{\xi}_{m'}) \in \mathbb{Z}^{m'},$$

where $m' = M - m$. We do allow $m = 0$ and $m = M$. We call the first m directions *quasi-continuous* and the last m' directions *purely discrete*. In our applications, $\hat{\xi}$ lies in the domain of a particular m-dimensional immersion ($\epsilon = 0$) or its lattice approximation ($\epsilon > 0$), while $\check{\xi}$ parametrizes an m'-dimensional discrete family of such maps.

The considerations about convergence are local, and the encountered continuous functions are defined on compact domains; either on an M-cube

$$\mathcal{B}(r,R) = \left\{(\hat{\xi},\check{\xi}) \in \mathcal{B}^0 \mid 0 \le \hat{\xi}_i \le r,\ 1 \le \check{\xi}_i \le R\right\} = \mathcal{B}_0(r) \times \mathcal{B}_+(R)$$
$$\mathcal{B}_0(r) = [0,r]^m, \quad \mathcal{B}_+(R) = \{0,\dots,R\}^{m'}$$

or on a subcube

$$\mathcal{B}_{\mathcal{S}}(r,R) \;=\; \mathcal{B}(r,R) \cap \mathcal{B}_{\mathcal{S}}.$$

In the lattice context, $\epsilon > 0$, we will use

$$\begin{aligned} \mathcal{B}^{\epsilon}(r,R) &= \mathcal{B}(r,R) \cap \mathcal{B}^{\epsilon} = \mathcal{B}_0^{\epsilon}(r) \times \mathcal{B}_+(R) \\ \mathcal{B}_{\mathcal{S}}^{\epsilon}(r,R) &= \mathcal{B}_{\mathcal{S}}(r,R) \cap \mathcal{B}^{\epsilon}. \end{aligned}$$

The just defined sets serve as domains for functions u with values in some Banach space $\mathcal{X}$. Denote the restriction of a continuous map $u : \Omega \to \mathcal{X}$ to the grid with mesh sizes $\boldsymbol{\epsilon}$ by

$$[u]^{\epsilon} := u\restriction_{\mathcal{B}^{\epsilon}} : \Omega \cap \mathcal{B}^{\epsilon} \to \mathcal{X}.$$

As usual, multi-index notation is used to denote higher-order partial derivatives:

$$\partial^{\alpha} = \partial_1^{\alpha_1} \cdots \partial_m^{\alpha_m}.$$

We say that a multi-index $\alpha = (\alpha_1,\dots,\alpha_M) \in \mathbb{N}^M$ is *subordinate* to a set $\mathcal{S} \subset \{1,\dots,M\}$ and write $\alpha \prec \mathcal{S}$ if $\alpha_i = 0$ for $i \notin \mathcal{S}$. In the following, any multi-index α will be subordinate to the set $\{1,\dots,m\}$, i.e., $\alpha_{m+1} = \cdots = \alpha_M = 0$.

For lattice functions $u : \mathcal{B}_{\mathcal{S}}^{\epsilon}(r,R) \to \mathcal{X}$, shift operators τ_i and partial difference operators $\delta_i = (\tau_i - \mathbf{1})/\epsilon_i$ for $i = 1,\dots,M$ replace the partial derivatives in the context of lattice functions $u : \mathcal{B}_{\mathcal{S}}^{\epsilon}(r,R) \to \mathcal{X}$, and are defined as

$$(\tau_i u)(\xi) = u(\xi + \epsilon_i \mathbf{e}_i), \text{ and } (\delta_i u)(\xi) = \frac{1}{\epsilon_i}\left(u(\xi + \epsilon_i \mathbf{e}_i) - u(\xi)\right)$$

at lattice sites $\xi \in \mathcal{B}_{\mathcal{S}}^{\epsilon}(r,R)$ for which $\xi + \epsilon_i \mathbf{e}_i \in \mathcal{B}_{\mathcal{S}}^{\epsilon}(r,R)$, too. (Here $\mathbf{e}_i$ is the ith unit vector.) Multiple shifts and higher order partial differences are also written with the help of multi-indices:

$$\tau^{\alpha} = \tau_1^{\alpha_1} \cdots \tau_M^{\alpha_M},\ \delta^{\alpha} = \delta_1^{\alpha_1} \cdots \delta_M^{\alpha_M},\ |\alpha| = \alpha_1 + \ldots + \alpha_M. \tag{2.2}$$

If the i-th direction is a continuous one, i.e., $\epsilon_i = 0$, the difference operator δ_i is understood as the respective differential operator ∂_i, and τ_i is the identity.

2.1.2 Norms and Convergence

For the problems of interest here, it is natural to consider functions of the class

$$C^{s,1}(\mathcal{B}_{\mathcal{S}}(r,R)) = \{u : \mathcal{B}_{\mathcal{S}}(r,R) \to \mathcal{X} \mid \partial^\alpha u \text{ is Lipschitz } \forall \alpha \prec \mathcal{S},\ |\alpha| \le s\},$$

which have the norm

$$\|u\|_{s+1} := \inf\left\{L > 0 \middle| |\partial^\alpha u(\hat{\xi},\check{\xi}) - \partial^\alpha u(\hat{\xi}',\check{\xi})|_{\mathcal{X}} \le L|\hat{\xi} - \hat{\xi}'|_1 \forall \alpha \prec \mathcal{S},\ |\alpha| \le s\right\}.$$

As usual, $|\hat{\xi}|_1 = |\hat{\xi}_1| + \cdots + |\hat{\xi}_m|$ is the 1-norm of $\hat{\xi}$. For $u : \mathcal{B}^\epsilon_{\mathcal{S}}(r,R) \to \mathcal{X}$, $\epsilon > 0$, define discrete $C^{s,1}$-norms

$$\|u\|_{s+1} := \max_{\substack{|\alpha| \le s+1 \\ \alpha \prec \mathcal{S}_0}} \max_{\xi \in \mathcal{B}^\epsilon_{\mathcal{S}}(r-\epsilon|\alpha|)} |\delta^\alpha u(\xi)|_{\mathcal{X}}.$$

It is easy to see that for a function $u \in C^{s,1}$

$$\|u\|_{s+1} = \sup_{\epsilon > 0} \|[u]^\epsilon\|_{s+1}.$$

In particular, the discrete norms $\|[u]^\epsilon\|_{s+1}$ in general diverges as $\epsilon \to 0$ for a function u that is not $C^{s,1}$-smooth.

Definition 2.1 *Let f be a C^∞-function defined on $\mathbb{D} \subset \mathcal{X}$, and let $\{f^\epsilon\}_{\epsilon \ge 0}$ be a family of C^∞-functions on domains $\mathbb{D}^\epsilon \subset \mathcal{X}$. We say that f^ϵ is* $\mathcal{O}(\epsilon)$-convergent *to f^0 if for every compact set $\mathcal{K} \subset \mathbb{D}$, there is an $\epsilon_0(\mathcal{K}) > 0$ so that $\mathcal{K} \subset \mathbb{D}^\epsilon$ for all $\epsilon < \epsilon_0(\mathcal{K})$, and there is a sequence of positive numbers $C_1, C_2, \ldots$, depending on $\mathcal{K}$, so that*

$$\|f^\epsilon - f^0\|_{C^s(\mathcal{K})} \le C_s \epsilon$$

for all $\epsilon < \epsilon_0$ and all numbers $s = 0, 1, 2, \ldots$.

Despite its technical definition, $\mathcal{O}(\epsilon)$-convergence in C^∞ is usually easily verified in concrete examples, for instance with the help of the following immediate observations:

- If $f^\epsilon, g^\epsilon : \mathbb{D} \to \mathcal{X}$ are $\mathcal{O}(\epsilon)$-convergent families in $C^\infty(\mathbb{D})$, then so is $f^\epsilon + g^\epsilon$, to the sum of the individual limits. If $\mathcal{X}$ has an algebra structure, and the multiplication is a smooth bilinear map, e.g., $\mathcal{X} = \mathbb{C}$, then the respective statement holds for products $f^\epsilon \cdot g^\epsilon$.
- If $f : \mathbb{D} \to \mathcal{X}$ is a C^∞ function, then the family $f^\epsilon := \epsilon f$ is $\mathcal{O}(\epsilon)$-convergent in $C^\infty(\mathbb{D})$ to zero.
- If $f : \Omega \to \mathcal{X}$ is a C^∞ function on some open neighborhood Ω of $0 \in \mathcal{X}$, then the family f^ϵ defined by
$$f^\epsilon(u) := f(\epsilon u)$$
is $\mathcal{O}(\epsilon)$-convergent in $C^\infty(\mathcal{X})$ to the constant function $\bar{f} = f(0)$.

- Suppose f above has $f(0) = Df(0) = \cdots = D^{k-1}f(0) = 0$, then the family f^ϵ with
$$f^\epsilon(u) := \epsilon^{-k} f(\epsilon u)$$
is $\mathcal{O}(\epsilon)$-convergent to the k-homogeneous function $\bar{f}(u) = D^k f(0) \cdot u^{\otimes k}$.

Definition 2.2 *Let u be a continuous function in $C^{s,1}(\mathcal{B}_{\mathcal{S}}(r,R))$, and let u^ϵ be a family of discrete functions defined on $\mathcal{B}^\epsilon_{\mathcal{S}}(r,R)$, respectively. We say that u^ϵ is* lattice-convergent to u in $C^{s,1}(\mathcal{B}_{\mathcal{S}}(r,R))$ *if there is some $C > 0$ so that,*
$$\|u^\epsilon - [u]^\epsilon\|_{s+1} \leq C\epsilon.$$
If the limit u is in C^∞ and u^ϵ is lattice convergent in every $C^{s,1}$, then we say that u is smoothly lattice convergent*, or lattice convergent in $C^\infty(\mathcal{B}_{\mathcal{S}}(r,R))$.*

A trivial, but very important example of a family u^ϵ which lattice-converges in $C^{s,1}$ to a function u of respective smoothness $C^{s,1}$, is the family of its restrictions: $u^\epsilon = [u]^\epsilon$. Analogously, $\delta^\alpha[u]^\epsilon$ is lattice convergent to $\partial^\alpha u$ in $C^{s-|\alpha|,1}$, provided that $|\alpha| \leq s$. Consequently, there is an alternative characterization of lattice-convergence:

Lemma 2.1 *If a family of discrete functions u^ϵ lattice-converges to a limit u in $C^{s,1}(\mathcal{B}_{\mathcal{S}}(r,R))$, then the first s partial derivatives of u are uniformly approximated by the respective partial difference quotients of u^ϵ:*
$$\max_{\substack{|\alpha|\leq s+1 \\ \alpha \prec \mathcal{S}}} \max_{\xi \in \mathcal{B}^\epsilon_{\mathcal{S}}(r-\epsilon|\alpha|,R)} |\partial^\alpha u(\xi) - \delta^\alpha u^\epsilon(\xi)| \leq C\epsilon.$$

The converse is not true, because Definition 2.2 also requires approximation of the $s+1$st partial difference quotients.

2.2 The Approximation Theorem

2.2.1 Hyperbolic Equations

Let $\mathcal{X} = \mathcal{X}_1 \times \cdots \times \mathcal{X}_K$ be a direct product of K Banach spaces, equipped with the norm
$$|u| = |u|_{\mathcal{X}} = \max_{1\leq k\leq K} |u_k|_{\mathcal{X}_k}. \tag{2.3}$$
(We will omit the subindices $\mathcal{X}$ and $\mathcal{X}_k$ whenever possible.)

Consider a *hyperbolic system* of first-order partial difference ($\epsilon > 0$) or partial differential ($\epsilon = 0$) equations for functions $u : \mathcal{B}^\epsilon \to \mathcal{X}$:
$$\delta_j u^\epsilon_k = f^\epsilon_{k,j}(u^\epsilon), \qquad 1 \leq k \leq K,\ j \in \mathcal{E}(k). \tag{2.4}$$
The subset $\mathcal{E}(k) \subset \{1, \ldots, M\}$ of indices is defined for each $k = 1, \ldots, K$, and called the set of *evolution directions* for the component u_k; the complementary subset

$\mathcal{S}(k) = \{1, \dots M\} \setminus \mathcal{E}(k)$ is called the set of *static directions* for the component u_k. It is convenient to partition $\mathcal{E}(k) = \mathcal{E}_0(k) \cup \mathcal{E}_+(k)$ and $\mathcal{S}(k) = \mathcal{S}_0(k) \cup \mathcal{S}_+(k)$ as follows:

$$\begin{array}{ll} \mathcal{E}_0(k) = \mathcal{E}(k) \cap \{1, \dots, m\}, & \mathcal{E}_+(k) = \mathcal{E}(k) \cap \{m+1, \dots, M\} \quad , \\ \mathcal{S}_0(k) = \mathcal{S}(k) \cap \{1, \dots, m\}, & \mathcal{S}_+(k) = \mathcal{S}(k) \cap \{m+1, \dots, M\} \quad . \end{array}$$

Recall that for $\epsilon = 0$, the partial difference operators δ_i in the directions $i = 1, \dots, m$ are partial differential operators ∂_i by definition. Explicitly, the hyperbolic system of partial difference *and* differential equations at $\epsilon = 0$ has the form:

$$\begin{aligned} \partial_j u_k &= f^0_{k,j}(u), \qquad 1 \le k \le K, \ j \in \mathcal{E}_0(k) \\ \tau_j u_k &= u_k + f^0_{k,j}(u), \qquad 1 \le k \le K, \ j \in \mathcal{E}_+(k). \end{aligned}$$

Each function $f^\epsilon_{k,j} : \mathbb{D}^\epsilon_{k,j} \to \mathcal{X}_k$ is supposed to be C^∞-smooth on an open domain $\mathbb{D}^\epsilon_{k,j} \subset \mathcal{X}$. Define $\mathbb{D}^\epsilon = \bigcap_{j \in \mathcal{E}(k)} \mathbb{D}^\epsilon_{k,j}$, the common domain of definition.

We focus on the following special type of Cauchy Problem for (2.4):

Goursat Problem 2.1 *Given functions U^ϵ_k on $\mathcal{B}^\epsilon_{\mathcal{S}(k)}(r, R)$ for each $k = 1, \dots, K$, find a function $u^\epsilon : \mathcal{B}^\epsilon(r, R) \to \mathcal{X}$ that solves the system (2.4) and continuates the initial conditions, i.e., $u^\epsilon_k(\xi) = U^\epsilon_k(\xi)$ for $\xi \in \mathcal{B}^\epsilon_{\mathcal{S}(k)}(r, R)$.*

Note that there is no formal distinction between the Goursat Problem for the difference ($\epsilon > 0$) and the differential ($\epsilon = 0$) equations. The idea is to consider a whole family of discrete Goursat Problems, parametrized by the mesh size $\epsilon > 0$, for fixed positive values of r and R. The content of the main Theorem 2.1 is that, provided the functions $f^\epsilon_{k,j}$ satisfy some natural assumptions, lattice-convergence of the discrete data U^ϵ_k to $C^{s,1}$-functions U^0_k implies lattice-convergence of the solutions u^ϵ to a $C^{s,1}$-function u, which solves the Goursat problem for $\epsilon = 0$ with data U^0_k.

The following considerations only apply to the purely discrete case $\epsilon > 0$. It is obvious that, if u^ϵ solves the Goursat problem, then it is the only solution on $\mathcal{B}^\epsilon(r, R)$. The question of existence of a solution, on the other hand, is not trivial. The problem is twofold: First, u^ϵ might leave the domain of definition of the equations in (2.4).

Definition 2.3 *We say that the solution of a Goursat problem* blows up *on $\Omega \subset \mathcal{B}^\epsilon(r, R)$ if there is a solution $\tilde{u}^\epsilon : \Omega \to \mathcal{X}$ to the Goursat problem such that $\tilde{u}^\epsilon(\xi) \notin \mathbb{D}^\epsilon$ at some site $\xi \in \Omega$.*

A second condition is more interesting: The equations (2.4) need to satisfy a compatibility condition. Discrete compatibility mimics the classical criterion

$$\partial_2 F = \partial_1 G$$

which is necessary and sufficient for the equations

$$\partial_1 u = F, \ \partial_2 u = G,$$

to admit a solution in general. It is a local criterion insofar as it is concerned with the solvability on an elementary cell of $\mathcal{B}^\epsilon(r, R)$ only.

Definition 2.4 *We say the system (2.4) at $\epsilon > 0$ is* consistent *if the Goursat Problem on the elementary M-dimensional box $\mathcal{B}^\epsilon(\epsilon, 1)$, i.e., $r = \epsilon$ and $R = 1$, is formally solvable for all initial data.*

Formally solvable means that there either exists a solution $u^\epsilon : \mathcal{B}^\epsilon(\epsilon, 1) \to \mathcal{X}$, or the solution blows up on a subset of $\mathcal{B}^\epsilon(\epsilon, 1)$. Consistency can be symbolically expressed as the condition that

$$\delta_j(\delta_i u_k^\epsilon) = \delta_i(\delta_j u_k^\epsilon), \quad i \neq j \in \mathcal{E}(k).$$

Writing this out, inserting (2.4), one obtains

$$\epsilon_j^{-1}\Big(f_{k,i}^\epsilon(\tau_j u^\epsilon) - f_{k,i}^\epsilon(u^\epsilon)\Big) = \epsilon_i^{-1}\Big(f_{k,j}^\epsilon(\tau_i u^\epsilon) - f_{k,j}^\epsilon(u^\epsilon)\Big). \tag{2.5}$$

It is implied that the function $f_{k,i}^\epsilon(u^\epsilon)$ depends only on those components u_ℓ^ϵ, for which $j \in \mathcal{E}_\ell$, so that $\tau_j u_\ell^\epsilon = f_{\ell,j}^\epsilon(u^\epsilon)$; otherwise, $\tau_j u_\ell^\epsilon$ could be prescribed arbitrarily as data, generating a contradiction in (2.5). Hence, the final algebraic form of consistency is that

- $f_{k,j}^\epsilon(u)$ only depends on the components u_ℓ, for which $\mathcal{E}(k) \subset \mathcal{E}(\ell) \cup \{j\}$
- the identity

$$\epsilon_j^{-1}\Big(f_{k,i}(U^\epsilon + \epsilon_j f_j(U^\epsilon)) - f_{k,i}(U^\epsilon)\Big) = \epsilon_i^{-1}\Big(f_{k,j}(U^\epsilon + \epsilon_i f_i(U^\epsilon)) - f_{k,j}(U^\epsilon)\Big) \tag{2.6}$$

  holds for all $U^\epsilon \in \mathcal{X}$ for which both sides are defined. ($f_i(U^\epsilon) \in \mathcal{X}$ abbreviates a vector whose $\mathcal{X}_k$-component is $f_{k,i}(U^\epsilon)$ if $i \in \mathcal{E}_k$, and arbitrarily defined otherwise.)

One could take (2.6) as definition of consistency. However, it is often more practicable to directly verify the solvability of the Goursat problem on the elementary box, using geometric theorems.

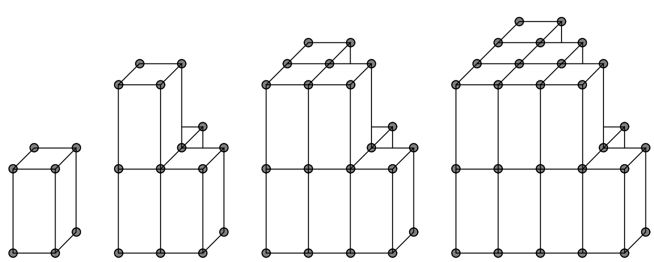

Figure 2.1: Successive construction of a discrete solution

In summary,

Proposition 2.1 *Let $\epsilon > 0$, and assume that the system (2.4) is consistent. Then either the Goursat problem 2.1 has a unique solution u^ϵ on $\mathcal{B}^\epsilon(r, R)$, or the solution blows up on a subset thereof.*

Proof: Let $d(\xi) = |\hat{\xi}|_1/\epsilon + |\check{\xi}|_1$, which is an integer function for $\hat{\xi} \in \mathcal{B}^\epsilon(r, R)$. The domain $\mathcal{B}^\epsilon(r, R)$ is a finite union of elementary boxes of the type $\mathcal{B}^\epsilon(\epsilon, 1)$. One advances inductively from box to box, solving the Goursat problem there: Start at $\xi = 0$; the solution to the Goursat problem on $\mathcal{B}^\epsilon(\epsilon, 1)$ provides data for the Goursat problems on the elementary boxes associated to the points $\xi \in \mathcal{B}^\epsilon(r, R)$ with $d(\xi) = 1$. The solution to the Goursat problem there – provided there is no blow-up – supplies data for the boxes at points ξ with $d(\xi) = 2$ etc. In figure 2.1, the first four induction steps are indicated for a grid in dimensions $m = 2$, $m' = 1$, and $R = 2$. The ξ_3-direction, which is kept discrete, is vertically oriented. The drawn boxes are those on which the Goursat Problem has been solved, and the corners are the sites $\xi \in \mathcal{B}^\epsilon(r, 2)$ at which the "new values" $u^\epsilon(\xi)$ are calculated. ■

2.2.2 Statement of the Main Theorem

We begin by stating the main Theorem in its most general form; its proof is delivered in the next subsection.

Theorem 2.1 *Consider a system of hyperbolic difference/differential equations (2.4) on the domains $\mathcal{B}^\epsilon(\bar{r}, R)$, $\bar{r} > 0$. Let respective Goursat data $U_k^\epsilon : \mathcal{B}^\epsilon_{\mathcal{S}(k)}(\bar{r}) \to \mathcal{X}_k$ be given. Further, let an open set $\mathbb{D} \in \mathcal{X}$ be given, on which all functions $f^0_{k,j}$ are defined. Provided that*

i. the hyperbolic system is consistent for every positive ϵ

ii. the functions $f^\epsilon_{k,j}$, $j \in \mathcal{E}(k)$, converge with $\mathcal{O}(\epsilon)$ in $C^\infty(\mathbb{D})$ to the limiting functions $f^0_{k,j}$

iii. the discrete initial data U_k^ϵ for $k = 1, \ldots, K$ are lattice-convergent in $C^{S,1}(\mathcal{B}_{\mathcal{S}(k)}(\bar{r}, R))$ to the continuous data U_k^0 at $\epsilon = 0$

iv. the solution u^0 for $\epsilon = 0$ does not blow up on $\mathcal{B}_+(R)$,

there is some $\epsilon_0 > 0$ and some positive $r \le \bar{r}$, so that

1. a solution $u = u^0 \in C^{S,1}$ to the Goursat problem at $\epsilon = 0$ exists on $\mathcal{B}(r, R)$ and is the unique solution there

2. discrete solutions u^ϵ to the Goursat problem exist on $\mathcal{B}^\epsilon(r, R)$ for every positive mesh size $\epsilon < \epsilon_0$

3. the solutions u^ϵ are lattice-convergent to u in $C^{S,1}(\mathcal{B}(r, R))$.

Remark: Lemma 2.1 implies that the partial derivatives $\partial^\alpha u$ of the smooth solution u are uniformly $\mathcal{O}(\epsilon)$-approximated by the respective partial difference quotients $\delta^\alpha u^\epsilon$ for $|\alpha| \le S$.

Next, another version of the Theorem is formulated, which is a bit simpler and tailored to most of the applications lying ahead: Either $m' = 0$ so that all directions are quasi-continuous, corresponding to the approximation of a single immersion; or there is exactly $m' = 1$ discrete direction, and the approximation of the immersion along with $R = 1$ of its transformations is investigated. In particular, the a priori requirement that blow-up's do not occur at $\hat{\xi} = 0$ is replaced by an assumption that is more easily verified. In the general setting of Theorem 2.1, smoothness and approximation of the transformation is proven once it is known that they exist; now, the existence of one transformation is part of the conclusions.

Theorem 2.2 *Suppose that there is exactly $m' = 1$ discrete direction, so $M = m+1$. Consider the system (2.4) on the domains $\mathcal{B}^\epsilon(\bar{r}, 1)$, with $\bar{r} > 0$, and with Goursat data $U_k^\epsilon : \mathcal{B}^\epsilon_{\mathcal{S}(k)}(\bar{r}, 1) \to \mathcal{X}_k$. Further, let an open set $\mathbb{D} \in \mathcal{X}$ be given, the role of which is clarified below. Assuming*

i. the system is consistent for all positive ϵ

ii. a) the functions $f^\epsilon_{k,j}$ with $j \in \mathcal{E}_0(k)$, i.e. $j \leq m$, are $\mathcal{O}(\epsilon)$-convergent to the limiting functions $f^0_{k,j}$ in $C^\infty(\mathcal{X})$

b) the functions $f^\epsilon_{k,M}$ converge with $\mathcal{O}(\epsilon)$ in $C^\infty(\mathbb{D})$ to $f^0_{k,M}$

iii. the initial data U_k^ϵ, $k = 1, \ldots, K$, are smoothly lattice-convergent to the limiting data U_k^0

iv. $U^0(0) = (U_1^0(0), \ldots, U_K^0(0))$ lies in $\mathbb{D}$,

then there is some $\epsilon_0 > 0$ and some positive $r < \bar{r}$, so that

1. the Goursat problem at $\epsilon = 0$ has a unique solution $u \in C^\infty(B(r, 1))$

2. the discrete Goursat problems have unique solutions u^ϵ for all $0 < \epsilon < \epsilon_0$ on $\mathcal{B}^\epsilon(r, 1)$

3. the solutions u^ϵ are smoothly lattice-convergent to u

The above statement carries over literally to the case that there are no discrete directions, $m' = 0$, upon replacing the domains $\mathcal{B}^{(\epsilon)}(r, 1)$ by $\mathcal{B}^{(\epsilon)}(r)$; the requirements ii.b) and iv. above are void in this situation.

2.2.3 Proof of the Main Theorem

Structure of the Proof

The proof of Theorem 2.1 is divided into five major steps:

1. Existence of solutions $u^\epsilon : \mathcal{B}^\epsilon(r, R) \to \mathcal{X}$ to the discrete problems for some $r > 0$ and all ϵ small enough is proven.

2. Uniform, ϵ-independent bounds on the solutions u^ϵ and their difference quotients up to $S + 1$st order are obtained.

3. One solution $u \in C^{S,1}$ to the continuous problem is constructed.

4. Uniqueness of the solution u is shown.

5. The rates of convergence are verified.

In a sixth step, proofs of some technical estimates related to the discrete chain rule are proven. After the first step, the changes needed to prove the reduced Theorem 2.2 are listed. The (small) modifications mainly concern the definition of some quantities.

The following technical lemma plays a crucial role and is repeatedly used:

Lemma 2.2 (Discrete Gronwall) *Let be given a real-valued function $W(t, T)$ for arguments $t = 0, \epsilon, 2\epsilon, \ldots, t_0$ and $T = 0, 1, \ldots, T_0$. If at any point (t, T) at least one of the* implicit *estimates*

$$\begin{aligned} \text{either } t > 0 \quad &\text{and} \quad W(t,T) \le (1 + \epsilon\mathcal{L}_1) W(t - \epsilon, T) + \epsilon\mathcal{Q}_1 && (2.7)\\ \text{or } T > 0 \quad &\text{and} \quad W(t,T) \le (1 + \mathcal{L}_2) W(t, T-1) + \mathcal{Q}_2 && (2.8)\\ &\text{or} \quad W(t,T) \le \overline{W} && (2.9) \end{aligned}$$

holds with nonnegative constants $\overline{W}$, $\mathcal{Q}_i$ and $\mathcal{L}_i$, then the following explicit *estimate is true:*

$$W(t,T) \le (\overline{W} + t\mathcal{Q}_1 + T\mathcal{Q}_2) \exp(t\mathcal{L}_1 + T\mathcal{L}_2). \tag{2.10}$$

Proof: The estimate (2.10) follows from an induction argument over the integer $N = N(t,T) = \frac{t}{\epsilon} + T$. For $N(t,T) = 0$, one has $t = T = 0$ and so (2.9) holds, confirming (2.10). Let be given $T \le T_0$ and $t \le t_0$, with $N(t,T) = N$, and assume (2.10) is proven for all T', t' with $N(t', T') = N - 1$. If (2.7) holds, then

$$\begin{aligned} W(t,T) &\le (1 + \epsilon\mathcal{L}_1)(\overline{W} + (t - \epsilon)\mathcal{Q}_1 + T\mathcal{Q}_2) \exp((t-\epsilon)\mathcal{L}_1 + T\mathcal{L}_2) + \epsilon\mathcal{Q}_1 \\ &\le (\overline{W} + t\mathcal{Q}_1 + T\mathcal{Q}_2)(1 + \epsilon\mathcal{L}_1) \exp((t-\epsilon)\mathcal{L}_1 + T\mathcal{L}_2) \\ &\le (\overline{W} + t\mathcal{Q}_1 + T\mathcal{Q}_2) \exp(t\mathcal{L}_1 + T\mathcal{L}_2) \end{aligned}$$

because $1 + x \le e^x$. If (2.8) is true instead of (2.7), the estimate is completely analogous. If both (2.7) and (2.8) fail, then (2.9) must be true, and (2.10) is trivially satisfied. ■

Existence of discrete solutions

At $\epsilon = 0$, the values of $u_k^0(0, \check{\xi})$, $\check{\xi} \in \mathcal{B}_+(R)$, can be calcuted from the data U_k^0 by a suitable finite iteration of the functions $f_{k,j}^0$, $j > m$, in the discrete evolution equations. By assumption, the solution does not blow up here, so $u^0(0, \check{\xi}) \in \mathbb{D}$ for $\check{\xi} \in \mathcal{B}_+(R)$. Choose a real number Δ so that

$$0 < \Delta \quad < \min_{\check{\xi} \in \mathcal{B}_+(R)} \operatorname{dist}(u^0(0, \check{\xi}), \partial\mathbb{D}). \tag{2.11}$$

$\Delta > 0$ is possible because $\mathbb{D}$ was assumed to be open. Further, define

$$\mathbb{D}_\Delta \quad := \bigcup_{\check{\xi} \in \mathcal{B}_+(R)} \left\{ u \in \mathcal{X} \middle| |u - u^0(0, \check{\xi})| \leq \frac{\Delta}{2} \right\}. \tag{2.12}$$

The values of $u_k^\epsilon(0, \check{\xi})$ are calculated in the same manner, iterating the respective maps $f_{k,j}^\epsilon$ for the discrete initial data U_k^ϵ. By assumption ii. about convergence of $f_{k,j}^\epsilon$ to $f_{k,j}^0$ and U_k^ϵ to U_k^0, the iteration is well-defined for ϵ small enough, and $u^\epsilon(0, \check{\xi})$ converges to $u^0(0, \check{\xi})$. Choose $\epsilon_0 > 0$ so small, that for all positive $\epsilon < \epsilon_0$

$$\mathbb{D}_\Delta \subset \mathbb{D}^\epsilon \text{ and } |u^\epsilon(0, \check{\xi}) - u^0(0, \check{\xi})| \leq \frac{\Delta}{4}. \tag{2.13}$$

Let

$$\begin{aligned} C_U &= \sup_{0 < \epsilon < \epsilon_0} \max_k \|U_k^\epsilon\|_1 \\ C_F &= \sup_{0 < \epsilon < \epsilon_0} \max_{\substack{k,j \\ j \in \mathcal{E}(k)}} \|f_{k,j}^\epsilon\|_{C^1(\mathbb{D}_\Delta)} \end{aligned}$$

These are finite positive quantities. Now choose $r > 0$ small enough to have

$$mr(C_U + C_F) \exp(m' R C_F) < \frac{\Delta}{4}. \tag{2.14}$$

Claim 1 *With these choices for ϵ_0 and r, the discrete solutions u^ϵ do not blow up on $\mathcal{B}^\epsilon(r, R)$ for any $\epsilon < \epsilon_0$.*

Actually, we show that $u^\epsilon(\xi) \in \mathbb{D}_\Delta$ for all $0 < \epsilon < \epsilon_0$ and $\xi \in \mathcal{B}^\epsilon(r, R)$. Our goal is to use Lemma 2.2 to estimate the quantity

$$W(t, T) := \max_{|\hat{\xi}|_1 \leq t} \max_{|\check{\xi}|_1 \leq T} |u^\epsilon(\hat{\xi}, \check{\xi}) - u^\epsilon(0, \check{\xi})|$$

Given t, T, let $\xi \in \mathcal{B}^\epsilon(r, R)$ be an arbitrary point with $t = |\hat{\xi}|_1$, $T = |\check{\xi}|_1$. For the k-th field u_k, at least one of the following is the case:

- *Initial data $U_k^\epsilon(\xi)$ is prescribed at ξ.*
  Then, by definition of C_U,

$$|u_k^\epsilon(\hat{\xi},\check{\xi}) - u_k^\epsilon(0,\check{\xi})| = |U_k^\epsilon(\hat{\xi},\check{\xi}) - U_k^\epsilon(0,\check{\xi})| \leq C_U|\hat{\xi}|_1 \leq C_U mr.$$

- *There is a $j \in \mathcal{E}_0(k)$ such that $\xi_j > 0$.*
  Let $\hat{\xi}' = \hat{\xi} - \epsilon\mathbf{e}_j$ and observe

$$\begin{aligned} |u_k^\epsilon(\hat{\xi},\check{\xi}) - u_k^\epsilon(0,\check{\xi})| &\leq |u_k^\epsilon(\hat{\xi}',\check{\xi}) - u_k^\epsilon(0,\check{\xi})| + |\epsilon f_{k,j}^\epsilon(u^\epsilon(\hat{\xi}',\check{\xi}))| \\ &\leq W(t-\epsilon,T) + \epsilon C_F. \end{aligned}$$

- *There is a $j \in \mathcal{E}_+(k)$ such that $\xi_j > 0$.*
  Define $\check{\xi}' = \check{\xi} - \mathbf{e}_{j-m}$, so that

$$\begin{aligned} |u_k^\epsilon(\hat{\xi},\check{\xi}) - u_k^\epsilon(0,\check{\xi})| &= |u^\epsilon(\hat{\xi},\check{\xi}') - u^\epsilon(0,\check{\xi}')| + |f_{k,j}^\epsilon(u^\epsilon(\hat{\xi},\check{\xi}')) - f_{k,j}^\epsilon(u^\epsilon(0,\check{\xi}'))| \\ &\leq (1+C_F)|u^\epsilon(\hat{\xi},\check{\xi}') - u^\epsilon(0,\check{\xi}')| \\ &\leq (1+C_F)W(t,T-1). \end{aligned}$$

Recall that $|u^\epsilon(\hat{\xi},\check{\xi}) - u^\epsilon(0,\check{\xi})|$ is the maximum of the K expressions $|u_k^\epsilon(\hat{\xi},\check{\xi}) - u_k^\epsilon(0,\check{\xi})|_{\mathcal{X}_k}$. Thus the implicit estimates (2.7)–(2.9) follow with

$$\mathcal{L}_1 = 0,\ \mathcal{L}_2 = C_F,\ \mathcal{Q}_1 = C_F,\ \mathcal{Q}_2 = 0,\ \overline{W} = C_U mr,$$

and the explicit estimate (2.10) gives

$$|u^\epsilon(\hat{\xi},\check{\xi}) - u^\epsilon(0,\check{\xi})| \leq mr(C_F + C_U)\exp(|\check{\xi}|_1 C_F), \tag{2.15}$$

which, in view of (2.13) and (2.14), yields the claim.

Modifications for the proof of Theorem 2.2

To prove the reduced Theorem, some of the above choices are made differently. Note that the value of $u^0(0,1) \in \mathcal{X}$ is well-defined because of the requirement iv., but possibly does not lie in $\mathbb{D}$. Instead of (2.11), one simply chooses a positive

$$\Delta \leq dist(u^0(0,0), \partial\mathbb{D})$$

where $u^0(0,0) = (U_1^0(0), \ldots, U_K^0(0)) \in \mathbb{D}$ by iv. In addition to the domain $\mathbb{D}_\Delta$ defined as in (2.12), let

$$\mathbb{D}'_\Delta := \{u \in \mathcal{X} \mid |u - u^0(0,0)| < \frac{\Delta}{2}\}.$$

And ϵ_0 is picked so that

$$\begin{gathered} |u^\epsilon(0,0) - u^0(0,0)|, |u^\epsilon(0,1) - u^0(0,1)| \leq \tfrac{\Delta}{4} \text{ and} \\ \mathbb{D}'_\Delta \subset \mathbb{D}^\epsilon,\ \mathbb{D}_\Delta \subset \mathbb{D}_{k,j}^\epsilon \text{ for all } j \in \mathcal{E}_0(k) \end{gathered}$$

holds for all $\epsilon < \epsilon_0$.

At sites $\xi \in \mathcal{B}^\epsilon(r,1)$ with $\check{\xi} = 0$, everything is exactly as before. However, one may no longer conclude that $u(\xi,1) \in \mathbb{D}$ for $(\xi,1) \in \mathcal{B}^\epsilon(r,1)$, i.e., formally, there might be a blow-up, but that does not hurt the arguments of the proof: The functions $f^\epsilon_{k,j}$ with $j < M$ are still defined for all values $u(\hat{\xi},1) \in \mathbb{D}_\Delta$ by the choice of ϵ_0, while the functions $f^\epsilon_{k,M}$ are *never* evaluated on $u^\epsilon(\hat{\xi},1)$ which possibly lie outside of $\mathbb{D}^\epsilon$. The arguments in the subsequent steps of the proof are changend in the obvious manner; one simply replaces explicit references to $\mathbb{D}_\Delta$ by $\mathbb{D}'_\Delta$ at the appropriate occurences. For example, constants defined as

$$C := \sup_{\epsilon<\epsilon_0} \max_{j\in\mathcal{E}(k)} \|f^\epsilon_{k,j}\|_{C^s(\mathbb{D}_\Delta)} \text{ are replaced by}$$
$$C := \max_{\epsilon<\epsilon_0} \max_{j\in\mathcal{E}_0(k)} \|f^\epsilon_{k,j}\|_{C^s(\mathbb{D}_\Delta)} + \max_{\epsilon<\epsilon_0} \max_{M\in\mathcal{E}_+(k)} \|f^\epsilon_{k,M}\|_{C^s(\mathbb{D}'_\Delta)}.$$

A priori bounds

Claim 2 *There is a constant $M > 0$ such that*

$$\|u^\epsilon\|_{S+1} \le M$$

independently of $\epsilon < \epsilon_0$.

To verify this claim, we proceed inductively from $s = 0$ to $s = S + 1$, proving

$$\|u^\epsilon\|_s \le M_s \tag{2.16}$$

for suitable constants M_s, independent of ϵ, in each step. By the previous step, all values $u^\epsilon(\xi)$ lie in $\mathbb{D}_\Delta$, so for $s = 0$, a possible choice is

$$M_0 := \max\{1, \sup_{u\in\mathbb{D}_\Delta} |u|\}.$$

For positive values $s \le S + 1$ and positive $\epsilon < \epsilon_0$, define

$$W^\epsilon_s(t,T) \quad := \quad \max_{|\alpha|=s} \max_{\substack{\xi\in\mathcal{B}^\epsilon(r-\epsilon|\alpha|,R)\\ |\hat{\xi}|_1\le t,\, |\check{\xi}|_1\le T}} |\delta^\alpha u^\epsilon(\xi)|.$$

We derive an estimate on $W^\epsilon(t,T)$, based on Lemma 2.2 and the following:

Lemma 2.3 *Given an integer $s \ge 1$, there is a constant C_s such that for every $v : \mathcal{B}^\epsilon(r,R) \to \mathbb{D}$ with arbitrary $\epsilon > 0$, every C^∞-function $f : \mathbb{D} \to \mathcal{X}$, and every multi-index α with $|\alpha| = s$*

$$|\delta^\alpha f(v(\xi))| \le \|f\|_{C^s(\mathbb{D})} \left(|\delta^\alpha v(\xi)| + C_s \max\{1, \|v\|^s_{s-1}\}\right). \tag{2.17}$$

The proof of this lemma is based on the estimates related to the discrete chain rule, and is found in the last subsection.

Define the constants

$$\begin{aligned} C_F &= \sup_{0<\epsilon<\epsilon_0} \max_{\substack{j,k \\ j\in\mathcal{E}(k)}} \|f^\epsilon_{k,j}\|_{C^{S+1}(\mathbb{D}_\Delta)} \\ C_U &= \sup_{0<\epsilon<\epsilon_0} \max_k \|U^\epsilon_k\|_{S+1} \end{aligned}$$

These quantities are finite because $f^\epsilon_{k,j} \to f^0_{k,j}$ in C^∞, and each U^ϵ_k lattice-converges to U^0_k.

We estimate $\delta^\alpha u^\epsilon_k(\xi)$ with $|\alpha| = s$ and $\xi \in \mathcal{B}^\epsilon(r - \epsilon s, R)$, setting $t := |\hat{\xi}|_1$, $T := |\check{\xi}|_1$. At least one of the following is true:

- *$\delta^\alpha u^\epsilon_k(\xi)$ can be calculated from U^ϵ_k.*

  From the definition of C_U, we conclude

  $$|\delta^\alpha u^\epsilon_k(\xi)| \le C_U.$$

- *There is a $j \in \mathcal{E}_0(k)$ such that $\alpha_j > 0$.*

  If $|\alpha| = s = 1$, i.e., $\delta^\alpha = \delta_j$, then

  $$|\delta^\alpha u^\epsilon_k(\xi)| = |f^\epsilon_{k,j}(u^\epsilon(\xi))| \le C_F.$$

  Otherwise, define $\alpha' := \alpha - \mathbf{e}_j$ and apply estimate (2.17) Lemma 2.3:

  $$|\delta^\alpha u^\epsilon_k(\xi)| = |\delta^{\alpha'} f^\epsilon_{k,j}(u^\epsilon(\xi))| \le C_F(M_{s-1} + C_{s-1}M^{s-1}_{s-2}).$$

- *There is a $j \in \mathcal{E}_0(k)$ such that $\xi_j > 0$.*

  Define $\hat{\xi}' = \hat{\xi} - \epsilon\mathbf{e}_j$, then, by the estimate (2.17),

  $$\begin{aligned} |\delta^\alpha u^\epsilon_k(\hat{\xi}, \check{\xi})| &\le |\delta^\alpha u^\epsilon_k(\hat{\xi}', \check{\xi})| + \epsilon|\delta^\alpha(f^\epsilon_{k,j}(u^\epsilon))(\hat{\xi}', \check{\xi})| \\ &\le (1 + \epsilon C_F)\underbrace{|\delta^\alpha u^\epsilon_k(\hat{\xi}', \check{\xi})|}_{\le W^\epsilon_s(t-\epsilon,T)} + \epsilon C_F C_s M^s_{s-1} \end{aligned}$$

- *There is a $j \in \mathcal{E}_+(k)$ such that $\xi_j > 0$.*

  With $\check{\xi}' = \check{\xi} - \mathbf{e}_{j-m}$ and (2.17),

  $$\begin{aligned} |\delta^\alpha u^\epsilon_k(\hat{\xi}, \check{\xi})| &= |\delta^\alpha u^\epsilon_k(\hat{\xi}, \check{\xi}')| + |(\delta^\alpha f^\epsilon_{k,j}(u^\epsilon))(\hat{\xi}, \check{\xi}')| \\ &\le (1 + C_F)\underbrace{|\delta^\alpha u^\epsilon_k(\hat{\xi}, \check{\xi}')|}_{\le W^\epsilon_s(t,T-1)} + C_F C_s M^s_{s-1} \end{aligned}$$

This confirms the implicit estimates (2.7)–(2.9) with constants independent of ϵ. So the explicit estimate (2.10) gives an ϵ-independent bound on the values $W^\epsilon_s(t, T)$ for $t \le mr - \epsilon s$, $T \le m'R$. This completes the induction since one can choose

$$M_s := M_{s-1} + W^\epsilon_s(mr - \epsilon s, m'R)$$

in formula (2.16).

Existence of a continuous solution

With the a priori estimates on the difference quotients at hand, one can basically follow a standard procedure (see, e.g. [P]) to obtain a sequence $\epsilon_n^* \to 0$ such that $u^{\epsilon_n^*}$ lattice-converges to a function $u : \mathcal{B}^\epsilon(r,R) \to \mathcal{X}$ in $C^{S,1}$.

To begin with, extend the discrete functions $u^\epsilon : \mathcal{B}^\epsilon(r,R) \to \mathcal{X}$ by piecewise linear interpolation to functions $\bar{u}^\epsilon$ on $\mathcal{B}(r,R)$: So if $(\hat{\xi},\check{\xi}) \in \mathcal{B}(r,R)$ and $(\hat{\xi}^0,\check{\xi}) \in \mathcal{B}^\epsilon(r,R)$ is so that $0 \le \hat{\xi}_j - \hat{\xi}_j^0 < \epsilon$ for $j = 1,\dots,m$, then

$$\bar{u}^\epsilon(\hat{\xi},\check{\xi}) = u^\epsilon(\hat{\xi}^0,\check{\xi}) + \sum_{j=1}^m (\hat{\xi}_j - \hat{\xi}_j^0) \cdot \delta_j u^\epsilon(\hat{\xi}^0,\check{\xi}).$$

The same is done with the functions $\delta^\alpha u^\epsilon$ for $|\alpha| \le S$, which are linearly interpolated to functions $\bar{u}^{\epsilon(\alpha)} : \mathcal{B}(r,R) \to \mathcal{X}$.

Now by the a priori estimate $\|u^\epsilon\|_{S+1} \le M$, it is clear that $\|\delta^\alpha u^\epsilon\|_1 \le M$ if $|\alpha| \le S$. Hence, M is a $\mathcal{B}(r,R)$-uniform Lipschitz constant for the piecewise linear functions $\bar{u}^{\epsilon(\alpha)}$, independent of ϵ. In particular, each ϵ-family $\{\bar{u}^{\epsilon(\alpha)}\}_\epsilon$ is *equicontinuous.*

Choose a sequence $\epsilon_n^0 \to 0$. Apply the Arzela-Ascoli Theorem: As an equicontinuous sequence of continuous functions on a compact domain, $\{\bar{u}^{\epsilon_n^0}\}_n$ posseses a subsequence which is uniformly convergent to a continuous limit function $\bar{u} : \mathcal{B}(r,R) \to \mathcal{X}$, i.e., $\bar{u}^{\epsilon_n^1} \to \bar{u}$ in $C^0(\mathcal{B}(r,R))$ for a suitable infinite subsequence $\{\epsilon_n^1\}_n \subset \{\epsilon_n^0\}_n$. Next, observe that $\{\bar{u}^{\epsilon_n^1(\alpha)}\}_n$ with, say, $\alpha = \mathbf{e}_1$, is an equicontinuous sequence, too. Choose a suitable subsubsequence $\{\epsilon_n^2\}_n \subset \{\epsilon_n^1\}_n$ such that $\bar{u}^{\epsilon_n^2(\mathbf{e}_1)} \to \bar{u}^{(\mathbf{e}_1)}$ in $C^0(\mathcal{B}(r,R))$. Repeat this procedure for all multi-indices α with $|\alpha| \le S$, obtaining in each step an infinite subsequence $\{\epsilon_n^{k+1}\}_n \subset \{\epsilon_n^k\}_n$ of the previous one by the Arzela-Ascoli theorem, so that the respective $\bar{u}^{\epsilon_n^{k+1}(\alpha)}$ converges uniformly to a continuous limit $\bar{u}^{(\alpha)} : \mathcal{B}(r,R) \to \mathcal{X}$. There are only finitely many multi-indices α to be processed. Let $\epsilon_n^* \to 0$ denote the respective sequence obtained in the last step. Obviously,

$$\bar{u}^{\epsilon_n^*(\alpha)} \to \bar{u}^{(\alpha)} \text{ in } C^0(\mathcal{B}(r,R)) \text{ as } n \to \infty \text{ for all } |\alpha| \le S.$$

It is also immediately seen that each $\bar{u}^{(\alpha)}$ is also Lipschitz-continuous on its domain with the Lipschitz-constant M from the a priori estimate.

Claim 3 *The function $u := \bar{u}$ is of class $C^{S,1}(\mathcal{B}(r,R))$ with $\partial^\alpha u = \bar{u}^{(\alpha)}$ and solves the continuous Gousart problem.*

Below, it is shown that each $\bar{u}^{(\alpha)}$ with $|\alpha| < S$ is differentiable and

$$\partial_j \bar{u}^{(\alpha)} = \bar{u}^{(\alpha+\mathbf{e}_j)}. \tag{2.18}$$

From this, the claim follows: Iteration of this argument yields that u is indeed S times differentiable, and $\partial^\alpha u = \bar{u}^{(\alpha)}$ is a Lipschitz-continous function, so $u \in C^{S,1}$. Also, from the equation $\delta_j u_k^\epsilon = f_{k,j}^\epsilon(u^\epsilon)$ of (2.4) one concludes that

$$\bar{u}_k^{\epsilon(\mathbf{e}_j)} = f_{k,j}^\epsilon(\bar{u}^\epsilon) + \mathcal{O}(\epsilon)$$

with an $\mathcal{B}(r,R)$-uniform implicit constant in $\mathcal{O}(\epsilon)$, so that

$$\partial_j u_k = \bar{u}^{(\mathbf{e}_j)} = \lim_{n\to\infty} \bar{u}^{\epsilon_n^*(\mathbf{e}_j)} = \lim_{n\to\infty} f_{k,j}^{\epsilon}(\bar{u}^{\epsilon}) = f_{k,j}^{0}(u),$$

and u solves the equations of the limiting system (2.4) at $\epsilon = 0$. Finally, the Goursat data are obviously attained, $u_k = U_k^0$, at the respective boundary subsets $\mathcal{B}_{\mathcal{S}(k)}(r,R)$ because of the lattice-convergence $U_k^\epsilon \to U_k^0$.

Thus is remains to prove (2.18). In the equivalent integral form, it reads:

$$\bar{u}^{(\alpha)}(\hat{\xi},\check{\xi}) - \bar{u}^{(\alpha)}(\hat{\xi}',\check{\xi}) = \int_0^{\hat{\xi}_j} \bar{u}^{(\alpha+\mathbf{e}_j)}(\hat{\xi}'+\mu\mathbf{e}_j,\check{\xi})d\mu \tag{2.19}$$

which is supposed to be true for all $(\hat{\xi},\check{\xi}) \in \mathcal{B}(r,R)$, where $\hat{\xi}' = \hat{\xi} - \hat{\xi}_j\mathbf{e}_j$. But from the obvious fact that for arguments $(\hat{\xi},\check{\xi})$ from the grid $\mathcal{B}^\epsilon(r,R)$

$$(\delta^\alpha u^\epsilon)(\hat{\xi},\check{\xi}) - (\delta^\alpha u^\epsilon)(\hat{\xi}',\check{\xi}) = \epsilon \sum_{m=0}^{\hat{\xi}_j/\epsilon-1} \delta_j\delta^\alpha u^\epsilon(\hat{\xi}'+m\epsilon\mathbf{e}_j,\check{\xi}).$$

Replacing the purely discrete by the respective interpolated functions, one obtains:

$$\bar{u}^{\epsilon(\alpha)}(\hat{\xi},\check{\xi}) - \bar{u}^{\epsilon(\alpha)}(\hat{\xi}',\check{\xi}) = \int_0^{\hat{\xi}_j} \bar{u}^{\epsilon(\alpha+\mathbf{e}_j)}(\hat{\xi}'+\mu\mathbf{e}_j,\check{\xi})d\mu + \mathcal{O}(\epsilon),$$

where the arguments are no longer restricted to the grid, at the price of introducing an error $\mathcal{O}(\epsilon)$ for which, however, the implicit constant is independent of $\hat{\xi},\check{\xi}$ and ϵ. Now (2.19) is obtained by passing to the limit $\epsilon = \epsilon_n^* \to 0$ on both sides: Uniform convergence of each $\bar{u}^{\epsilon_n^*(\alpha)}$ on $\mathcal{B}(r,R)$, implies convergence of the integrals to the integral of the respective limit function $\bar{u}^{(\alpha)}$.

Uniqueness of the continuous solution

Assume u and $\hat{u}$ are two solutions to the continuous Goursat problem, defined on $\mathcal{B}(r,R)$.

Claim 4 *$u(\xi) = \hat{u}(\xi)$ for all $\xi \in \mathcal{B}(r,R)$.*

The crucial technical tool is the continuous analogue of Lemma 2.2.

Lemma 2.4 (Continuous Gronwall) *Assume $W : [0,t_0] \times \{0,\ldots,T_0\} \to \mathbb{R}$ is a continuous real function. If at any point (t,T) at least one of the* implicit *estimates*

$$\begin{aligned} \text{either } t>0 \quad \text{and} \quad & W(t,T) \le \mathcal{L}\textstyle\int_0^t W(s,T)ds + \mathcal{Q} \\ \text{or } T>0 \quad \text{and} \quad & W(t,T) \le (1+\mathcal{L})W(t,T-1) + \mathcal{Q} \\ \text{or} \quad & W(t,T) \le \overline{W} \end{aligned} \tag{2.20}$$

holds with nonnegative constants $\overline{W}$, $\mathcal{Q}$ and $\mathcal{L}$, then the following explicit *estimate is true:*

$$W(t,T) \le (\overline{W} + (t+T)\mathcal{Q})\exp((t+T)\mathcal{L}). \tag{2.21}$$

To prove this lemma, combine the classical Gronwall Lemma with the estimates in the proof of Lemma 2.2. The goal is to apply this lemma to

$$W(t,T) = \sup_{|\hat{\xi}|_1 \le t} \max_{|\check{\xi}|_1 \le T} |u(\xi) - \hat{u}(\xi)|$$

Further, define

$$C_F = \max_{\substack{k,j \\ j \in \mathcal{E}(k)}} \|f^0_{k,j}\|_{C^1(\mathcal{K})}$$

where the compact set $\mathcal{K} \subset \mathbb{D}$ is chosen so that $u(\xi), \hat{u}(\xi) \in \mathcal{K}$ for all $\xi \in \mathcal{B}(r,R)$; consequently, C_F is finite.

In order to confirm the implicit estimates (2.20) for $W(t,T)$ at an arbitrary point $\xi \in \mathcal{B}(r,R)$ with $|\hat{\xi}|_1 = t$, $|\check{\xi}|_1 = T$, consider the k-th fields u_k, $\hat{u}_k$ there. One of the following is true:

- *There is a* $j \in \mathcal{E}_+(k)$ *with* $\xi_j > 0$.
  Define $\check{\xi}' = \check{\xi} - \mathbf{e}_{j-m}$ and observe
  $$\begin{aligned} |u_k(\hat{\xi},\check{\xi}) - \hat{u}(\hat{\xi},\check{\xi})| &\le |u(\hat{\xi},\check{\xi}') - \hat{u}(\hat{\xi},\check{\xi}')| + |f^0_{k,j}(u(\hat{\xi},\check{\xi}')) - f^0_{k,j}(\hat{u}(\hat{\xi},\check{\xi}'))| \\ &\le (1+C_F)\underbrace{|u(\hat{\xi},\check{\xi}') - \hat{u}(\hat{\xi},\check{\xi}')|}_{\le W(t,T-1)} \end{aligned}$$

- *There is* no $j \in \mathcal{E}_+(k)$ *with* $\xi_j > 0$.
  But then, there exists some $\hat{\xi}' \in \mathcal{B}_0(r)$ such that initial data $U^0_k(\hat{\xi}',\check{\xi})$ is prescribed and $\hat{\xi} - \hat{\xi}'$ lies in the span of the evolutionary directions, i.e.,
  $$\hat{\xi}' - \hat{\xi} = \sum_{j \in \mathcal{E}_0(k)} \lambda_j \mathbf{e}_j.$$
  Since u solves the system of differential equations, it has the integral representation
  $$\begin{aligned} u_k(\hat{\xi},\check{\xi}) &= u_k(\hat{\xi}',\check{\xi}) + \int_0^1 \sum_j \lambda_j \partial_j u_k(\hat{\xi}' + \sigma(\hat{\xi} - \hat{\xi}'),\check{\xi}) d\sigma \\ &= U^0_k(\hat{\xi}',\check{\xi}) + \sum_j \lambda_j \int_0^1 f^0_{kj}(u(\hat{\xi}' + \sigma(\hat{\xi} - \hat{\xi}'),\check{\xi})) d\sigma, \end{aligned}$$
  and the same is true with $\hat{u}$ in place of u. Substracting these equations leads to
  $$\begin{aligned} &|u_k(\hat{\xi},\check{\xi}) - \hat{u}_k(\hat{\xi},\check{\xi})| \\ &\le \sum_j \lambda_j C_F \int_0^1 |u(\hat{\xi}' + \sigma(\hat{\xi} - \hat{\xi}'),\check{\xi}) - \hat{u}(\hat{\xi}' + \sigma(\hat{\xi} - \hat{\xi}'),\check{\xi})| d\sigma \\ &\le C_F |\hat{\xi}|_1 \int_0^1 W(\sigma|\hat{\xi}|_1, T) d\sigma \\ &= C_F \int_0^t W(s,T) ds. \end{aligned}$$

Thus, Lemma 2.4 is applicable with $\mathcal{Q} = 0$, forcing $W(t,T) = 0$ for all $t \le mr$, $T \le m'R$. But this is equivalent to $\hat{u} = u$.

Rates of convergence

Claim 5 *There is a constant $R > 0$, independent of $\epsilon > 0$, such that*

$$\|u^\epsilon - [u]^\epsilon\|_S \le R\epsilon$$

for all positive $\epsilon < \epsilon_0$.

We proceed inductively from $s = 0$ to $s = S$, proving in each step that

$$\|u^\epsilon - [u]^\epsilon\|_s \le R_s\epsilon \tag{2.22}$$

for some ϵ-independent constant R_s. We use Lemma 2.2 to estimate the quantity

$$W(t,T) := \max_{|\alpha|=s} \max_{\substack{\xi\in\mathcal{B}^\epsilon(r,R)\\ |\hat{\xi}|_1\le t,\ |\check{\xi}|_1\le T}} |\delta^\alpha u^\epsilon(\xi) - \delta^\alpha[u]^\epsilon(\xi)|.$$

We also need

Lemma 2.5 *Given an integer $s \ge 1$, there is a constant C_s such that for every pair of discrete functions $v, w : \mathcal{B}^\epsilon(r,R) \to \mathbb{D}$ with arbitrary $\epsilon > 0$, for every pair of C^∞-functions $f, g : \mathbb{D} \to \mathcal{X}$, and for every multi-index α with $|\alpha| = s$*

$$\begin{aligned} |\delta^\alpha(f(v) - g(w))(\xi)| \ &\le \ C_s\|f-g\|_{C^s(\mathbb{D})}Q_s \\ +\|g\|_{C^{s+1}(\mathbb{D})}&\left(|\delta^\alpha(v-w)(\xi)| + C_s\|v-w\|_{s-1}Q_s\right) \end{aligned} \tag{2.23}$$

where

$$Q_s = \max\{1, \|v\|_s^s, \|w\|_s^s\}.$$

This lemma is again an application of the estimates on the discrete chain rule. Its proof is put in the last subsection.

And we repeatedly make use of the relation

$$\|\delta_j[u]^\epsilon - [\partial_j u]^\epsilon\|_s \le \epsilon\|u\|_{s+2}. \tag{2.24}$$

Define

$$\begin{aligned} C_F \ &:= \ \max_{\substack{j,k\\ j\in\mathcal{E}(k)}} \|f^0_{k,j}\|_{C^{S+1}(\mathbb{D}_\Delta)} \\ C_G \ &:= \ \sup_{0<\epsilon<\epsilon_0} \epsilon^{-1} \max_{\substack{j,k\\ j\in\mathcal{E}(k)}} \|f^\epsilon_{k,j} - f^0_{k,j}\|_{C^S(\mathbb{D}_\Delta)} \\ C_U \ &:= \ \sup_{0<\epsilon<\epsilon_0} \epsilon^{-1} \max_k \|U^\epsilon_k - [U^0_k]^\epsilon\|_{S+1}. \end{aligned}$$

Observe that $Q_s \le M^s$ by the a priori estimates. This specializes the estimate (2.23) in Lemma 2.5.

Consider the difference $\delta^\alpha u^\epsilon_k(\xi) - \delta^\alpha[u]^\epsilon_k(\xi)$ for $|\alpha| = s$ at $\xi \in \mathcal{B}^\epsilon(r,R)$, with $t = |\hat{\xi}|_1$, $T = |\check{\xi}|_1$. At least one of the following is true:

- *$\delta^\alpha u_k^\epsilon(\xi)$ can be calculated from the data.*
  Trivially, by the definition of C_U above,

$$|\delta^\alpha(u_k^\epsilon - [u]_k^\epsilon)(\xi)| = |\delta^\alpha(U_k^\epsilon - [U_k^0]^\epsilon)(\xi)| \leq C_U\epsilon$$

- *There is an $j \in \mathcal{E}(k)$ such that $\alpha_j > 0$.*
  If $s = 1$, i.e., $\alpha = \mathbf{e}_j$, then, observing the estimate (2.24),

$$\begin{aligned} |\delta_j(u^\epsilon - [u]^\epsilon)(\xi)| &\leq |\left(f_{k,j}^\epsilon(u^\epsilon) - f_{k,j}^0(u)\right)(\xi)| + \epsilon\|u\|_2 \\ &\leq \|f_{k,j}^0\|_{C^1(\mathbb{D}_\Delta)}|(u^\epsilon - u)(\xi)| \\ &\quad + \|f_{k,j}^\epsilon - f_{k,j}^0\|_{C^0(\mathbb{D}_\Delta)} + M\epsilon \\ &\leq C_F R_0\epsilon + C_G\epsilon + M\epsilon. \end{aligned}$$

  If $s > 1$, define $\alpha' = \alpha - \mathbf{e}_j$. Then, exploiting (2.24),

$$\begin{aligned} |\delta^\alpha(u^\epsilon - [u]^\epsilon)(\xi)| &\leq |\delta^{\alpha'}\left(f_{j,k}^\epsilon(u^\epsilon) - f_{j,k}^0([u]^\epsilon)\right)(\xi)| + \epsilon\|u\|_{s+1} \\ &\leq C_F(R_{s-1}\epsilon + C_{s-1}R_{s-2}\epsilon M^{s-1}) \\ &\quad + C_{s-1}C_G M^{s-1}\epsilon + M\epsilon. \end{aligned}$$

  where we have used estimate (2.23) of Lemma 2.5.

- *There is an $j \in \mathcal{E}(k)$ such that $\xi_j > 0$.*
  Let $\hat{\xi}' = \hat{\xi} - \epsilon\mathbf{e}_j$.

$$\begin{aligned} |\delta^\alpha(u^\epsilon - [u]^\epsilon)(\hat{\xi}, \check{\xi})| &\leq |\delta^\alpha(u^\epsilon - [u]^\epsilon)(\hat{\xi}', \check{\xi})| + \epsilon^2\|u\|_{s+1} \\ &\quad + \epsilon|\delta^\alpha(f_{k,j}^\epsilon(u^\epsilon) - f_{k,j}^0([u]^\epsilon))(\hat{\xi}', \check{\xi})| \\ &\leq (1 + \epsilon C_F)W^\epsilon(t - \epsilon, T) + M\epsilon^2 \\ &\quad + \epsilon C_s M^s(R_{s-1}\epsilon + C_G\epsilon). \end{aligned}$$

- *There is an $j \in \mathcal{E}(k)$ such that $\xi_j > 0$.*
  Define $\check{\xi}' = \check{\xi} - \mathbf{e}_{j-m}$.

$$\begin{aligned} |\delta^\alpha(u^\epsilon - [u]^\epsilon)(\hat{\xi}, \check{\xi})| &\leq |\delta^\alpha(f_{k,j}^\epsilon(u^\epsilon) - f_{k,j}^0([u]^\epsilon))(\hat{\xi}, \check{\xi}')| \\ &\leq C_F W^\epsilon(t, T - 1) + C_s M^s(R_{s-1}\epsilon + C_G\epsilon). \end{aligned}$$

These considerations confirm the implicit estimates (2.7)–(2.9) of Lemma 2.2 with constants $\mathcal{L}_1 = \mathcal{L}_2 = C_F$ independent of ϵ and source terms $\mathcal{Q}_1, \mathcal{Q}_2, \bar{W} = \mathcal{O}(\epsilon)$. But this means that, with an appropriate choice of R_s,

$$W_s^\epsilon(t, T) \leq R_s\epsilon \text{ for all } \epsilon < \epsilon_0$$

This completes the induction and proves $\mathcal{O}(\epsilon)$-convergence of the family u^ϵ to the smooth solution u in $C^{S,1}(\mathcal{B}(r, R))$.

Estimates for the Discrete Chain Rule

The Lemmata 2.3 and 2.5 are consequences of Lemma 2.6 below, which generalizes the *chain rule* for higher-order partial *difference* quotients. For a convenient formulation, introduce the following notations:

We say that an s-tupel $J = (j_1, \ldots, j_s)$ of directions $j_k \in \{1, \ldots, m\}$ is an *ordered representation* of the multi-index α with $|\alpha| = s$, if

$$\delta^\alpha = \prod_{\ell=1,\ldots,m} \delta_{j_\ell}.$$

Given an ordered representation J, define for $A \subset \{1, \ldots, s\}$

$$\delta_J^A = \prod_{\ell \in A} \delta_{j_\ell}.$$

Furthermore, let $\mathcal{Z}_k^s$ be the set of partitions of $\{1, \ldots, s\}$ into exactly k nonempty subsets,

$$\mathcal{Z}_k^s = \left\{ \mathcal{A} = (A_1, \ldots, A_k) \mid A_\ell \neq \varnothing, \ \dot{\bigcup} A_\ell = \{1, \ldots, s\} \right\}.$$

As usual, we understand $D^k f(u)$ with $f : \mathbb{D} \to \mathcal{X}$, $u \in \mathbb{D}$, as the symmetric k-linear $\mathcal{X}$-valued functional over $\mathcal{X}$ that represents the k-th derivative of f at u. To indicate the evaluation of the functional on the vectors w_1 to w_k, we write $D^k f(u)[w_1, \ldots, w_k]$.

Lemma 2.6 (discrete chain rule) *Assume $f : \mathbb{D} \subset \mathcal{X} \to \mathcal{X}$ is a function of class C^∞, and $u : \mathcal{B}^\epsilon(r, R) \to \mathbb{D}$. Let be given a multi-index α, $|\alpha| = s \geq 1$, and an ordered representation $J = (j_1, \ldots, j_s)$ for α. Then:*

$$\delta^\alpha f(u) = \sum_{k=1,\ldots,s} \sum_{\mathcal{A} \in \mathcal{Z}_k^s} \int_{[0,1]^k} d\Theta \left(D^k f(\mu^s u)[\mu^{s_1}(\delta_J^{A_1} u), \ldots, \mu^{s_k}(\delta_J^{A_k} u)] \right) \tag{2.25}$$

In this formula, $s_\ell = s - \#A_\ell$, and the operator

$$\mu^s := \sum_{|\beta| \leq s} g_\beta \tau^\beta \ \text{with } g_\beta \geq 0, \ \sum_\beta g_\beta = 1$$

yields a convex combination of shifts of the function it is applied to, with weights g_β depending on $\Theta \in [0,1]^k$, k, J and $\mathcal{A}$, but not *on u, f or the point of evaluation.*

Proof of Lemma 2.6: We prove the lemma by induction on s.
If $s = 1$, then $\alpha = \mathbf{e}_j$ and $J = \{j\}$. We write

$$\begin{aligned} \delta^\alpha f(u) &= \frac{1}{\epsilon}\Big(f(\tau_j u) - f(u)\Big) \\ &= \int_0^1 d\theta Df((1-\theta)u + \theta \tau_j u) \cdot \frac{1}{\epsilon}(\tau_j u - u) \\ &= \int_0^1 d\theta Df(\mu^1 u)[\delta_j u]. \end{aligned}$$

and (2.25) is verified in this case.

Now let $|\alpha| = s+1$ and assume the lemma is proven for all $s' \leq s$. If the given ordered representation of α is $J = (j_1, \dots, j_{s+1})$, then define $j = j_{s+1}$ and $J' = (j_1, \dots, j_s)$, which is an ordered representation of $\alpha' = \alpha - \mathbf{e}_j$. Take the discrete derivative δ_j of both sides in the formula (2.25) for α', J'. The operator δ_j obviously commutes with summation and integration, therefore it suffices to investigate the difference quotient of the expression in brackets for arbitrary, but fixed choices of $k \leq s$, a partition $\mathcal{A} = (A_1, \dots, A_k)$ and $\Theta \in [0,1]^k$. Note that these choices completely determine the weights g inside the operators μ. One finds

$$
\begin{aligned}
&\delta_j\Big(D^k f(\mu^s u)[\mu^{s_1}(\delta_{J'}^{A_1} u), \dots, \mu^{s_k}(\delta_{J'}^{A_k} u)] \\
&= \sum_{\ell=1,\dots,k} D^k f(\tau_j \mu^s u)[\tau_j \mu^{s_1}(\delta_{J'}^{A_1} u), \dots, \delta_j \mu^{s_\ell}(\delta_{J'}^{A_\ell} u), \dots, \mu^{s_k}(\delta_{J'}^{A_k} u)] \\
&\quad + \int_0^1 d\theta \Big(D^{k+1} f(\mu^{s+1} u)[\mu^{s_1}(\delta_{J'}^{A_1} u), \dots, \mu^{s_k}(\delta_{J'}^{A_k} u), \mu^s(\delta_j u)]\Big)
\end{aligned}
$$

It is easy to see that this is the sum of $k+1$ expressions of the form

$$
D^{k'}(\mu^s u)[\mu^{s_1+1}(\delta_J^{A'_1} u), \dots, \mu^{s_k+1}(\delta_J^{A'_{k'}} u)]
$$

where

1. either $k' = k$ and $\mathcal{A}' = \{A_1, \dots, A_\ell \cup \{s+1\}, \dots, A_k\} \in \mathcal{Z}_k^{s+1}$ for some ℓ between 1 and k,

2. or: $k' = k+1$ and $\mathcal{A}' = (A_1, \dots, A_k, \{s+1\}) \in \mathcal{Z}_{k+1}^{s+1}$.

From these representations, it is easily verified that: If $\mathcal{A}' \in \mathcal{Z}_{k'}^{s+1}$ for some $k' \leq s+1$, then $\mathcal{A}'$ can be obtained from an appropriate $\mathcal{A} \in \mathcal{Z}_{k''}^{s}$ either by means of 1. or 2. above (neccessarily $k' = k''$ or $k' = k''+1$). And the element $\mathcal{A}$ is uniquely determined by $\mathcal{A}'$. This concludes the induction on s. ■

Proof of Lemma 2.3 Let $\xi \in \mathcal{B}^\epsilon(r - s\epsilon, R)$ be arbitrary. First, observe that for $k = 1$, the sum in formula (2.25) contains only *one* term, corresponding to $\mathcal{A} = \{\{1, \dots, s\}\}$:

$$
Df(\mu^s u)(\xi) \cdot (\delta^\alpha u)(\xi).
$$

It now follows immediately:

$$
\begin{aligned}
|\delta^\alpha f(u)(\xi)| &\leq |Df(\mu^s u)(\xi)| \cdot |(\delta^\alpha u)(\xi)| + \\
&\quad \sum_{k=2,\dots,s} \sum_{\mathcal{A} \in \mathcal{Z}_k^s} \Big(|D^k f(\mu^k u)(\xi)| \cdot \prod_{\ell=1,\dots,k} |\mu^{s_\ell}(\delta_J^{A_\ell} u)(\xi)|\Big) \\
&\leq \|f\|_{C^1(\mathbb{D})} \cdot |(\delta^\alpha u)(\xi)| + C_s \|f\|_{C^s(\mathbb{D})} \cdot (\max\{1, \|u\|_s^{s-1}\}),
\end{aligned}
$$

C_s being the number of terms in the summation. This proves the desired formula (2.17). ■

Proof of Lemma 2.5 Substract the respective formulas (2.25) for $f(u)$ and $g(v)$ from each other. This yields a sum (over k, $\mathcal{A}$ and Θ) of terms

$$\begin{aligned}
& D^k f(\mu^s u)[\mu^{s_1}(\delta_J^{A_1} u), \ldots, \mu^{s_k}(\delta_J^{A_k} u)] - D^k g(\mu^s v)[\mu^{s_1}(\delta_J^{A_1} v), \ldots, \mu^{s_k}(\delta_J^{A_k} v)] \\
= \quad & \Big(D^k f(\mu^s u) - D^k g(\mu^s v)\Big)[\mu^{s_1}(\delta_J^{A_1} u), \ldots, \mu^{s_k}(\delta_J^{A_k} u)] \qquad (2.26) \\
+ \quad & D^k g(\mu^s v)\Big([\mu^{s_1}(\delta_J^{A_1} u), \ldots, \mu^{s_k}(\delta_J^{A_k} u)] - -[\mu^{s_1}(\delta_J^{A_1} v), \ldots, \mu^{s_k}(\delta_J^{A_k} v)]\Big) \qquad (2.27)
\end{aligned}$$

where the weights g are the same in corresponding symbols μ. To estimate (2.26), observe that for an arbitrary $\xi \in \mathcal{B}^\epsilon(r - s\epsilon, R)$:

$$\begin{aligned}
& |D^k f(\mu^s u)(\xi) - D^k g(\mu^s v)(\xi)| \\
& \quad \leq \Big|D^k f(\mu^s u)(\xi) - D^k g(\mu^s u)(\xi)\Big| + \Big|D^k g(\mu^s u)(\xi) - D^k g(\mu^s v)(\xi)\Big| \\
& \quad \leq \|f - g\|_{C^k(\mathbb{D})} + \|g\|_{C^{k+1}(\mathbb{D})}\|u - v\|_0
\end{aligned}$$

Also, for the k-linear function $Q(\xi) := D^k f(\mu^s u)(\xi) - D^k g(\mu^s v)(\xi)$,

As for the expression (2.27) there holds:

$$\begin{aligned}
& \Big|D^k g(\mu^s v)(\xi)\Big([\mu^{s_1}(\delta_J^{A_1} u)(\xi), \ldots, \mu^{s_k}(\delta_J^{A_k} u)(\xi)][\mu^{s_1}(\delta_J^{A_1} v)(\xi), \ldots, \mu^{s_k}(\delta_J^{A_k} v)(\xi)]\Big)\Big| \\
& \leq \sum_{\mathcal{L}=1,\ldots,k} \Big|D^k g(\mu^s v)(\xi)[\mu^{s_1}(\delta_J^{A_1} u)(\xi), \ldots, \mu^{s_\mathcal{L}}(\delta_J^{A_\mathcal{L}}(u - v)), \ldots, \mu^{s_k}(\delta_J^{A_k} v)(\xi)]\Big| \\
& \leq \sum_{\mathcal{L}=1,\ldots,k} \|g\|_{C^k(\mathbb{D})} \prod_{\ell<\mathcal{L}} |\mu^{s_\ell}(\delta_J^{A_\ell} u)(\xi)| \cdot |\mu^{s_\mathcal{L}}(\delta_J^{A_\mathcal{L}}(u - v))(\xi)| \cdot \prod_{\ell>\mathcal{L}} |\mu^{s_\ell}(\delta_J^{A_\ell} v)(\xi)| \\
& \leq \|g\|_{C^k(\mathbb{D})} \cdot \max\{1, \|u\|_{s-1}^{k-1}, \|v\|_{s-1}^{k-1}\} \cdot \sum_{\mathcal{L}=1,\ldots,k} |\mu^{s_\mathcal{L}}(\delta_J^{A_\mathcal{L}}(u - v))(\xi)|
\end{aligned}$$

In combination, one concludes

$$\begin{aligned}
& |\delta^\alpha(f(u) - g(v))(\xi)| \\
& \leq \sum_{k=1,\ldots,s} \hat{C}_k \Big(\|f - g\|_{C^k(\mathbb{D})} + \|g\|_{C^{k+1}(\mathbb{D})}\|u - v\|_0\Big) \max\{1, \|u\|_s^k\} \\
& \quad + \sum_{k=2,\ldots,s} \hat{C}_k \|g\|_{C^k(\mathbb{D})} k \|u - v\|_{s-1} \max\{1, \|u\|_{s-1}^{k-1}, \|v\|_{s-1}^{k-1}\} \\
& \quad + \|g\|_{C^1(\mathbb{D})} |\delta^\alpha(u - v)(\xi)|.
\end{aligned}$$

$\hat{C}_k$ is the number of elements of $\mathcal{Z}_k^s$. From here, with a proper choice of C_s, the estimate (2.23) follows immediately. ∎

Chapter 3

Surfaces with Constant Negative Gaussian Curvature

3.1 K-Surfaces by Their Gauss Maps

In this chapter, two approaches to the approximation theory of surfaces with constant negative curvature are presented. The first, presented below, is based on investigations of the surfaces' Gauss maps. The starting point for the second approach is their description by moving frames.

3.1.1 Definitions and Basic Properties

We start by considering surfaces without transformations, therefore we work on domains $\mathcal{B}(r)$ in $M = 2$ dimensions, both quasi-continuous, $m = 2$, $m' = 0$.

Definition 3.1 *A (continuous) K-surface is a smooth asymptotic line parameterization $f : \mathcal{B}(r) \to \mathbb{R}^3$ of an immersed surface with constant negative Gaussian curvature. We denote by $n : \mathcal{B}(r) \to S^2$ the corresponding Gauss map and by $\mathcal{S}_f \subset \mathbb{R}^3$ the image of f.*

Recall that for an asymptotic line parameterization, the second derivatives $\partial_1^2 f, \partial_2^2 f$ lie in the tangent space, spanned by $\partial_1 f, \partial_2 f$. It is a classical result that an immersed surface $\mathcal{S}_f$ in asymptotic line parameterization is of constant negative Gaussian curvature if and only if $|\partial_1 f(\xi_1, \xi_2)|$ is independent of ξ_2, and $|\partial_2 f(\xi_1, \xi_2)|$ is independent of ξ_1.

Definition 3.2 *A discrete K-surface is a map $f^\epsilon : \mathcal{B}^\epsilon(r) \to \mathbb{R}^3$ of two discrete variables, for which*

- *the five points $f^\epsilon(\xi_1, \xi_2)$, $f^\epsilon(\xi_1 \pm \epsilon, \xi_2)$, $f^\epsilon(\xi_1, \xi_2 \pm \epsilon)$ are coplanar*

- *$|\delta_1 f^\epsilon(\xi_1,\xi_2)|$ is independent of ξ_2, and $|\delta_2 f^\epsilon(\xi_1,\xi_2)|$ is independent of ξ_1*

We call $n^\epsilon : \mathcal{B}^\epsilon(r) \to S^2$ with

$$n^\epsilon = \frac{(\delta_1 f^\epsilon) \times (\delta_2 f^\epsilon)}{|(\delta_1 f^\epsilon) \times (\delta_2 f^\epsilon)|},$$

its Gauss map. The discrete surface $\mathcal{S}_{f^\epsilon}$ is the lattice image of f^ϵ in $\mathbb{R}^3$.

The vector $n^\epsilon(\xi)$ is normal to the plane through the five points $f^\epsilon(\xi_1,\xi_2)$, $f^\epsilon(\xi_1 \pm \epsilon, \xi_2)$, $f^\epsilon(\xi_1, \xi_2 \pm \epsilon)$. Hence the term "Gauss map" is justified.

The starting point of our considerations are two observations – from which the first one is classical and the second one was first made in [Wu], then exploited in [BP1] – which, in our notations, read:

Proposition 3.1 *A map $n : \mathcal{B}(r) \to S^2$ is the Gauss map of some K-surface $f : \mathcal{B}(r) \to \mathbb{R}^3$ iff it is Lorentz-harmonic*

$$\partial_1 \partial_2 n = \rho n, \ \rho : \mathcal{B}(r) \to \mathbb{R}. \tag{3.1}$$

f is uniquely determined by n up to translations and homoteties of the image in $\mathbb{R}^3$. In particular, given a Lorentz-harmonic map $n : \mathcal{B}(r) \to S^2$, then for an arbitrary positive number κ and any point $F \in \mathbb{R}^3$, the unique solution of

$$f(0,0) = F, \ \partial_1 f = \kappa n \times \partial_1 n, \ \partial_2 f = -\kappa n \times \partial_2 n \tag{3.2}$$

is a K-surface f for which $\mathcal{S}_f$ has constant Gaussian curvature $-\kappa^{-2}$, and whose Gauss map is n.

Remark: Lorentz-harmonicity of the Gauss map $n : \mathcal{B}(r) \to S^2$ is equivalent to the fact that n forms a two-dimensional Chebyshev-net on S^2. The latter is defined by the property, that the length $|\partial_1 n(\xi_1,\xi_2)|$ is independent of ξ_1 and $|\partial_1 n(\xi_1,\xi_2)|$ is independent of ξ_2.

Proposition 3.2 ([BP1]) *A map $n^\epsilon : \mathcal{B}^\epsilon(r) \to S^2$ is the Gauss map of some discrete K-surface $f^\epsilon : \mathcal{B}^\epsilon(r) \to \mathbb{R}^3$ iff it is discrete Lorentz-harmonic, i.e.,*

$$\delta_1 \delta_2 n^\epsilon = \frac{\rho^\epsilon}{2}(\tau_1 n^\epsilon + \tau_2 n^\epsilon), \ \rho^\epsilon : \mathcal{B}^\epsilon(r) \to \mathbb{R}. \tag{3.3}$$

f^ϵ is uniquely determined by n up to translations and homoteties in $\mathbb{R}^3$. In particular, given a discrete Lorentz-harmonic map $n^\epsilon : \mathcal{B}^\epsilon(r) \to S^2$, then for any positive number κ and any point $F \in \mathbb{R}^3$,

$$f^\epsilon(0,0) = F, \ \delta_1 f = \kappa n \times \delta_1 n, \ \delta_2 f = -\kappa n \times \delta_2 n \tag{3.4}$$

determines a discrete K-surface f^ϵ which has n as its Gauss map.

Remark: The discrete Lorentz-harmonicity of n^ϵ is equivalent to n^ϵ being a discrete Chebyshev-net in S^2, meaning that $n^\epsilon(\xi_1,\xi_2)$ and $n^\epsilon(\xi_1+\epsilon,\xi_2)$ enclose an angle which is independent of ξ_2, and the angle between $n^\epsilon(\xi_1,\xi_2)$ and $n^\epsilon(\xi_1,\xi_2+\epsilon)$ is independent of ξ_2.

3.1.2 Approximation of K-Surfaces

We first derive a hyperbolic system in the case of smooth K-surfaces.

Lemma 3.1 *Any K-surface $f : \mathcal{B}(r) \to \mathbb{R}^3$ with Gauss map $n : \mathcal{B}(r) \to S^2$ gives rise to a solution of the system*

$$\begin{array}{rclrcl} \partial_1 f &=& \kappa n \times v_1 & \partial_2 f &=& -\kappa n \times v_2 \\ \partial_1 n &=& v_1 & \partial_2 n &=& v_2 \\ \partial_1 v_2 &=& -\rho n & \partial_2 v_1 &=& -\rho n \end{array} \tag{3.5}$$

$$\textit{where } \rho = -v_1 \cdot v_2, \tag{3.6}$$

Here $v_1, v_2 : \mathcal{B}(r) \to \mathbb{R}^3$ are suitable functions, and $\kappa > 0$ is chosen so that the Gaussian curvature of the corresponding immersed surface $\mathcal{S}_f$ is $-\kappa^{-2}$.

Conversely, given a solution to the system (3.5) so that $n(\xi_1, \xi_2)$ are unit vectors, then f is a smooth K-surface and n its Gauss map.

Proof: Let some K-surface f with its Gauss map n be given. Define the auxiliary functions

$$v_1 := \partial_1 n, \quad v_2 := \partial_2 n.$$

Proposition 3.1 implies that n has the property (3.1) of Lorentz-harmonicity, and f itself satisfies (3.2) for the obvious choice of F and κ. So the system (3.5) holds. To calculate ρ, take the scalar product of (3.1) with n to find

$$\rho = n \cdot \partial_1 \partial_2 n = \partial_1 (n \cdot \partial_2 n) - \partial_1 n \cdot \partial_2 n.$$

Since the values of n lie in S^2, we have $n \cdot \partial_i n = 0$ everywhere, and (3.6) follows.

Conversely, given a solution to (3.5), the function n is obviously Lorentz-harmonic. The values of n are unit vectors by assumption, so it can be interpreted as the Gauss map of the immersion given by f, which then is a K-surface with necessity by Proposition 3.1. ∎

We turn to the theory of discrete K-surfaces.

Lemma 3.2 *Any discrete K-surface $f^\epsilon : \mathcal{B}^\epsilon(r) \to \mathbb{R}^3$ with respective discrete Gauss map $n^\epsilon : \mathcal{B}^\epsilon(r) \to S^2$ gives rise to a solution of the system*

$$\begin{array}{rclrcl} \delta_1 f^\epsilon &=& \kappa n^\epsilon \times v_1^\epsilon & \delta_2 f^\epsilon &=& -\kappa n^\epsilon \times v_2^\epsilon \\ \delta_1 n^\epsilon &=& v_1^\epsilon & \delta_2 n^\epsilon &=& v_2^\epsilon \\ \delta_1 v_2^\epsilon &=& -\rho^\epsilon (n^\epsilon + \frac{\epsilon}{2}(v_1^\epsilon + v_2^\epsilon)) & \delta_2 v_1^\epsilon &=& -\rho^\epsilon (n^\epsilon + \frac{\epsilon}{2}(v_1^\epsilon + v_2^\epsilon)) \end{array} \tag{3.7}$$

$$\textit{where } \rho^\epsilon = (v_1^\epsilon \cdot v_2^\epsilon) \cdot (1 + \tfrac{\epsilon}{2} n^\epsilon \cdot (v_1^\epsilon + v_2^\epsilon) + \tfrac{\epsilon^2}{2} v_1^\epsilon \cdot v_2^\epsilon)^{-1}, \tag{3.8}$$

κ is some positive number and $v_1^\epsilon, v_2^\epsilon$ are suitable functions.

Conversely, given a solution to (3.7) for which $n^\epsilon(\xi_1, \xi_2)$ are unit vectors, then f^ϵ is a discrete K-surface and n^ϵ its Gauss map.

Proof: Assume f^ϵ is a discrete K-surface, n^ϵ is its Gauss map, and introduce

$$v_1^\epsilon := \delta_1 n^\epsilon, \quad v_2^\epsilon := \delta_2 n^\epsilon.$$

Proposition 3.2 implies the remaining equations of the system (3.7), with a suitable choice of κ and the function ρ^ϵ from equation (3.3). To derive the formula (3.8) for ρ^ϵ, rewrite (3.3) as

$$\tau_1\tau_2 n^\epsilon = (1+\epsilon^2\frac{\rho^\epsilon}{2})(\tau_1 n^\epsilon + \tau_2 n^\epsilon) - n^\epsilon.$$

The left side is a unit vector, so the norm of the right side has to be equal to one, leading to

$$1+\epsilon^2\rho^\epsilon = \frac{2n^\epsilon \cdot (\tau_1 n^\epsilon + \tau_2 n^\epsilon)}{|\tau_1 n^\epsilon + \tau_2 n^\epsilon|^2}$$

which eventually results in (3.8).

Conversely, given a solution to (3.7), then n^ϵ is obviously discrete Lorentz-harmonic. As the $n^\epsilon(\xi)$ are unit vectors, the function n^ϵ is, by Proposition 3.2, the Gauss map of the discrete K-surface f^ϵ. ■

Recall that the eventual goal is to obtain a convergence result for discrete K-surfaces to a smooth limit. Observe that the systems (3.5) and (3.7) are already in the hyperbolic form (2.4). Hence Theorem 2.1 applies.

We choose (omitting the parameter κ)

$$\mathcal{X} = \mathbb{R}^3_f \times \mathbb{R}^3_n \times \mathbb{R}^3_{v_1} \times \mathbb{R}^3_{v_2}.$$

The quantities f and n are assigned no static directions, while the first direction is static for v_1 and the second direction is static for v_2. Since the right-hand sides of (3.5) are *always* defined, it is natural to take $\mathbb{D} = \mathcal{X}$.

Correspondingly, the Goursat problem for (3.5),(3.7) reads

Goursat Problem 3.1 (for K-surfaces) [1] *Given a point $F^\epsilon \in \mathbb{R}^3$, a unit vector $N^\epsilon \in S^2$, and two smooth functions $V_i^\epsilon : \mathcal{B}_i^\epsilon(r) \to \mathbb{R}^3$, $i = 1, 2$, find a solution to (3.5) resp.(3.7), such that $f^\epsilon(0,0) = F^\epsilon$, $n^\epsilon(0,0) = N^\epsilon$ and $v_i^\epsilon\restriction_{\mathcal{B}_i^\epsilon(r)} = V_i^\epsilon$, $i = 1, 2$.*

We conclude our considerations of the hyperbolic equations with

Lemma 3.3 *The discrete system (3.7) is compatible, and the right-hand sides of the difference equations $\mathcal{O}(\epsilon)$-converge in $C^\infty(\mathcal{X})$ to the respective right-hand sides of the differential equations in (3.5).*

[1] Throughout this thesis, Goursat problems are considered simultaneously for the smooth ($\epsilon = 0$) and the discrete ($\epsilon > 0$) equations.

Proof: Compatibility of the equations for n^ϵ is immediate since

$$\begin{aligned}\tau_2(\tau_1 n^\epsilon) &= n^\epsilon + \epsilon(v_1^\epsilon + v_2^\epsilon) + \epsilon^2 \delta_2 v_1^\epsilon, \\ \tau_1(\tau_2 n^\epsilon) &= n^\epsilon + \epsilon(v_1^\epsilon + v_2^\epsilon) + \epsilon^2 \delta_1 v_2^\epsilon,\end{aligned}$$

and the expressions for $\delta_1 v_2^\epsilon$ and $\delta_2 v_1^\epsilon$ are identical. Calculation of the two expressions $\tau_2(\tau_1 f^\epsilon)$ and $\tau_1(\tau_2 f^\epsilon)$ leads to the same result,

$$f^\epsilon + \kappa(1 - \frac{\epsilon^2}{2}\rho)\left(\epsilon n^\epsilon \times (v_1^\epsilon - v_2^\epsilon) - \epsilon^2(v_1^\epsilon \times v_2^\epsilon)\right).$$

It is elementary to verify the statement about convergence. ■

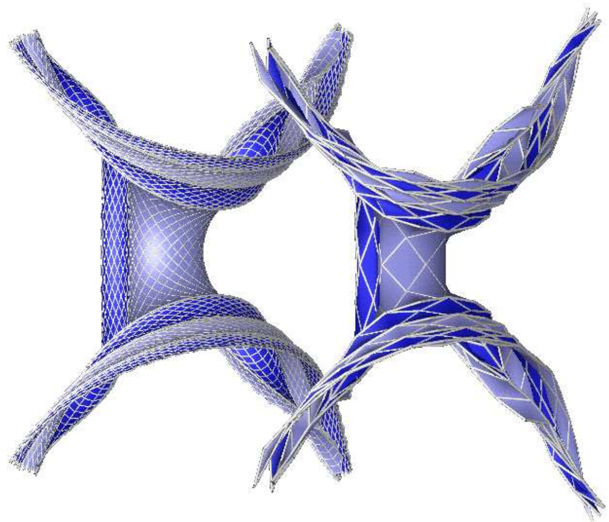

Figure 3.1: Two discrete Amsler surfaces with different mesh sizes ϵ

With the help of Theorem 2.2, we prove the following approximation result:

Theorem 3.1 *Let be given two families of unit vectors, $N_1, N_2 : [0, \bar{r}] \to S^2$ with $N_1(0) = N_2(0)$. Then*

1. *There is a smooth K-surface $f : \mathcal{B}(r) \to \mathbb{R}^3$ for which the Gauss map $n : \mathcal{B}(r) \to S^2$ satisfies $n(\xi_1, 0) = N_1(\xi_1)$ and $n(0, \xi_2) = N_2(\xi_2)$ for $0 \leq \xi_1, \xi_2 \leq r$. f is unique up to translations and homoteties.*
2. *Construct the family $\{f^\epsilon : \mathcal{B}^\epsilon(r) \to \mathbb{R}^3\}$ of discrete K-surfaces as solutions to the Goursat problem 3.1 with data $F^\epsilon = f(0,0)$, $N^\epsilon = N_1(0)$, $V_1^\epsilon = \delta_1[N_1]^\epsilon$, $V_2^\epsilon = \delta_2[N_2]^\epsilon$. Choose κ so that $-\kappa^{-2}$ is the constant negative Gaussian curvature of $\mathcal{S}_f$. Then, as $\epsilon \to 0$, the family $\{f^\epsilon\}$ is smoothly lattice-convergent to f. In particular, the images $\mathcal{S}_f^\epsilon$ lie $\mathcal{O}(\epsilon)$-close to the surface $\mathcal{S}$, and the 5-tuples of planar points approximate the respective tangent spaces.*

Remark: The approximating family f^ϵ attains the prescribed surface normals, $n^\epsilon(\xi_1, 0) = N_1(\xi_1)$ and $n^\epsilon(0, \xi_2) = N_2(\xi_2)$.

Example: The convergence result is illustrated in Figure 3.1, where two discrete approximations of an Amsler surface for different values of the mesh size ϵ are displayed. A – smooth or discrete – Amsler surface is a K-Surface for which two asymptotic curves are in fact straight lines. They are easily constructed by means of the Theorem above: If $v_1, v_2 \in \mathbb{R}^3$ are the directions of the two straight lines, one simply chooses the initial data N_1 and N_2 so that N_i is orthogonal to v_i, i.e., $N_1(\xi) \cdot v_1 = N_2(\xi) \cdot v_2 = 0$ for $\xi \in [0, \bar{r}]$.

Proof: By Theorem 2.2, there is a unique solution $(f, n, v_1, v_2) : \mathcal{B}(r) \to \mathcal{X}$ to the continuous Goursat problem 3.1 with $F \in \mathbb{R}^3$ arbitrary, $N = N_1(0)$ and $V_1(\xi_1, 0) = \partial_1 N_1(\xi_1)$, $V_2(0, \xi_2) = \partial_2 N_2(\xi_2)$. And the solutions $(f^\epsilon, n^\epsilon, v_1^\epsilon, v_2^\epsilon) : \mathcal{B}^\epsilon(r) \to \mathcal{X}$ to the discrete Goursat problem with data as prescribed in the theorem lattice-converge to the smooth solution in $C^\infty(\mathcal{B}(r))$; in particular, f^ϵ converges to f. It remains to show that f and f^ϵ are smooth and discrete K-surfaces, respectively. To apply the second part of Lemma 3.1 and 3.2, respectively, one needs to verify that the values of n and n^ϵ lie in S^2. By the above choice of initial data,

$$n^\epsilon(\xi_1, 0) = N_1(\xi_1) \text{ and } n^\epsilon(0, \xi_2) = N_2(\xi_2)$$

are unit vectors for $0 \leq \xi_1, \xi_2 \leq r$. By the calculation from the proof of Lemma 3.1 and the form (3.8) of the function ρ^ϵ, it is clear that: If $n^\epsilon(\xi)$, $\tau_1 n^\epsilon(\xi)$ and $\tau_2 n^\epsilon(\xi)$ have norm one, then so does $\tau_1\tau_2 n^\epsilon(\xi)$. Hence the functions n^ϵ are indeed S^2-valued, and so is the limiting function n. ■

3.1.3 Bäcklund Transformations

From any given K-surface, one can construct a two-parameter family of new K-surfaces by means of the following special transformation.

Definition 3.3 *For two K-surfaces $f, f^+ : \mathcal{B}(r) \to \mathbb{R}^3$ we say that f^+ is a* Bäcklund transformation *of f if their Gauss maps $n, n^+ : \mathcal{B}(r) \to S^2$ are related as follows:*

$$\partial_1 n^+ - \partial_1 n \parallel n^+ + n, \qquad \partial_2 n^+ + \partial_2 n \parallel n^+ - n. \tag{3.9}$$

Remark: Equivalently, the normals satisfy the following differential equations:

$$\partial_1 n^+ = +\partial_1 n - \frac{n^+ \cdot \partial_1 n}{1 + n \cdot n^+}(n^+ + n) \tag{3.10}$$

$$\partial_2 n^+ = -\partial_2 n + \frac{n^+ \cdot \partial_2 n}{1 - n \cdot n^+}(n^+ - n) \tag{3.11}$$

Geometric interpretation: The length of the derivatives of n are preserved:

$$|\partial_1 n^+(\xi)| = |\partial_1 n(\xi)| \text{ and } |\partial_2 n^+(\xi)| = |\partial_2 n(\xi)|,\ \xi \in \mathcal{B}(r). \tag{3.12}$$

Equivalently, corresponding asymptotic lines on f and f^+ are parameterized equivalently:

$$|\partial_1 f^+(\xi)| = |\partial_1 f(\xi)| \text{ and } |\partial_2 f^+(\xi)| = |\partial_2 f(\xi)|,\ \xi \in \mathcal{B}(r), \tag{3.13}$$

assuming that f^+ is suitably scaled to have the same Gaussian curvature as f. Furthermore, the normal vectors fields n^+ and n enclose a constant angle θ,

$$n^+(\xi) \cdot n(\xi) = \cos\theta,\ \xi \in \mathcal{B}(r). \tag{3.14}$$

The claims can be verified by direct calculations starting from (3.10) and (3.11). These two geometric properties yield one of the classical definitions of the Bäcklund transformation. Starting with these, one concludes that either the relations in (3.9) hold as stated, or they hold with the role of the first and second axis interchanged.

A comment on the uniqueness of Bäcklund transformations is in place: It is clear that the Gauss map of f^+ – provided f^+ exists – is completely determined by the Gauss map of f up to the choice of n^+ at one point, say $N^+ = n^+(0)$. Thus f^+ itself is determined by f and a free parameter N^+ in S^2, up to translation and homotety.

The corresponding definition for discrete Bäcklund transformations reads as follows:

Definition 3.4 *For two discrete K-surfaces $f^\epsilon, f^{\epsilon+} : \mathcal{B}^\epsilon(r) \to \mathbb{R}^3$ we say that $f^{\epsilon+}$ is a Bäcklund transformation of f^ϵ if their discrete Gauss maps $n^\epsilon, n^{\epsilon+} : \mathcal{B}^\epsilon(r) \to S^2$ are related by*

$$\delta_1 n^{\epsilon+} - \delta_1 n^\epsilon \parallel n^{\epsilon+} + \tau_1 n^\epsilon \qquad \delta_2 n^{\epsilon+} + \delta_2 n^\epsilon \parallel n^{\epsilon+} - \tau_2 n^\epsilon \tag{3.15}$$

Remark: Equivalently, the discrete Gauss maps satisfy

$$\delta_i n^+ = +\delta_i n - \frac{2\delta_i n \cdot n^+ + \epsilon|\delta_i n|^2}{|n^+ + \tau_i n|^2}(n^+ + \tau_i n) \tag{3.16}$$

$$\delta_j n^+ = -\delta_j n + \frac{2\delta_j n \cdot n^+ - \epsilon|\delta_j n|^2}{|n^+ - \tau_j n|^2}(n^+ - \tau_j n) \tag{3.17}$$

Geometric interpretation: The discrete Bäcklund transformation preserves the angles of the discrete Chebyshev-net

$$|\delta_1 n^{\epsilon+}(\xi)| = |\delta_1 n^\epsilon(\xi)| \text{ and } |\delta_2 n^{\epsilon+}(\xi)| = |\delta_2 n^\epsilon(\xi)|,\ \xi \in \mathcal{B}^\epsilon(r). \tag{3.18}$$

Equivalently, it preserves the lengths of the line segments

$$|\delta_1 f^{\epsilon+}(\xi)| = |\delta_1 f^\epsilon(\xi)| \text{ and } |\delta_2 f^{\epsilon+}(\xi)| = |\delta_2 f^\epsilon(\xi)|,\ \xi \in \mathcal{B}^\epsilon(r), \tag{3.19}$$

provided $f^{\epsilon+}$ is suitably scaled. Furthermore, the Gauss maps $n^{\epsilon+}$ and n^ϵ enclose a constant angle θ,

$$n^{\epsilon+}(\xi) \cdot n^\epsilon(\xi) = \cos\theta,\ \xi \in \mathcal{B}^\epsilon(r).$$

One says that n and n^+ together form a $2+1$-dimensional discrete Chebyshev-net on S^2 for obvious reasons. Also, the remarks about uniqueness carry over from after Definition 3.3.

3.1.4 Approximation of Bäcklund Transformations

We switch to three-dimensional domains $\mathcal{B}(r,1)$, i.e., $M = 3$, with $m = 2$ quasi-continuous and $m' = 1$ discrete directions, and we write $\tau_3 f = f^+$ etc. in the following. To incorporate Bäcklund transformations, the hyperbolic system (3.5) for smooth K-surfaces is enlarged by the equations

$$\begin{aligned} \delta_3 n &= 2w \\ \delta_3 v_1 &= -2\tfrac{v_1\cdot w}{1+n\cdot w}(n+w) \\ \delta_3 v_2 &= -2v_2 - 2\tfrac{v_2\cdot w}{|w|^2}w \end{aligned} \tag{3.20}$$

which represent the relations (3.10) and (3.11).

Analogously, the system (3.7) is enlarged by the following hyperbolic equations that are equivalent to (3.16) and (3.17):

$$\begin{aligned} \delta_3 n^\epsilon &= 2w^\epsilon \\ \delta_3 v_1^\epsilon &= -2\tfrac{v_1^\epsilon\cdot w^\epsilon}{1+n^\epsilon\cdot w^\epsilon+\epsilon v_1^\epsilon\cdot w^\epsilon}(n^\epsilon + w^\epsilon + \tfrac{\epsilon}{2}v_1^\epsilon) \\ \delta_3 v_2^\epsilon &= -2v_2^\epsilon - 2\tfrac{v_2^\epsilon\cdot w^\epsilon - \frac{\epsilon}{2}|v_2^\epsilon|^2}{|w^\epsilon|^2-\epsilon w^\epsilon\cdot v_2^\epsilon+\frac{\epsilon^2}{4}|v_2^\epsilon|^2}(w-\tfrac{\epsilon}{2}v_2^\epsilon) \end{aligned} \tag{3.21}$$

For the quantities f and w, the third direction is the only static direction. For v_1 and v_2, the first or second axis, respectively, is static, while n has no static directions. The Banach space is

$$\mathcal{X} = \mathbb{R}^3_f \times \mathbb{R}^3_n \times \mathbb{R}^3_{v_1} \times \mathbb{R}^3_{v_2} \times \mathbb{R}^3_w.$$

The equations in (3.20) are defined whenever neither $|w|$ nor $1 + n\cdot w$ vanishes. Consequently, we define

$$\mathbb{D} = \{(f, n, v_1, v_2, w) \in \mathcal{X} \mid w \neq 0, n\cdot w \neq -1\}.$$

Goursat Problem 3.2 (Bäcklund Transformations) *Given two functions $V_1^\epsilon : \mathcal{B}_1^\epsilon(r) \to \mathbb{R}^3$, $V_2^\epsilon : \mathcal{B}_2^\epsilon(r) \to \mathbb{R}^3$, along with two points F^ϵ and $F^{\epsilon+}$, a unit vector $N^\epsilon \in S^2$, and some $W^\epsilon \in \mathbb{R}^3$, find a solution to the system (3.5)&(3.20) or (3.7)&(3.21), respectively*[2]*, with $f^\epsilon(0,0,0) = F^\epsilon$, $f^\epsilon(0,0,1) = F^{\epsilon+}$, $n^\epsilon(0,0,0) = N^\epsilon$, and $v_1^\epsilon = V_1^\epsilon$ on $\mathcal{B}_1^\epsilon(r)$, $v_2^\epsilon = V_2^\epsilon$ on $\mathcal{B}_2^\epsilon(r)$, respectively.*

Again, note that f^ϵ corresponds to a K-surface only for those solutions, for which v_1^ϵ and v_2^ϵ attain values in S^2 only.

Proposition 3.3 *The discrete system (3.7),(3.21) is compatible. The right-hand sides of the equations in (3.21) are $\mathcal{O}(\epsilon)$-convergent to the respective expressions in (3.20) in $C^\infty(\mathbb{D})$.*

[2]Recall that the smooth and the discrete case are considered simultaneously. The system (3.5)&(3.20) corresponds to $\epsilon = 0$ while (3.7)&(3.21) corresponds to $\epsilon > 0$.

Proof: The convergence of the equations is immediately seen.

For a geometrical approach to consistency, see Wunderlich's original paper [Wu]. There, consistency is formulated as the question of existence of a geometrical figure, called "windschiefes Parallelepiped" (warped parallelepiped). The eight vertices of this figure – four of which can be prescribed freely – correspond to the image of an elementary 3-cube $\mathcal{B}^\epsilon(\epsilon, 1)$ under f^ϵ. Hence, the existence of the warped parallelepiped implies local solvability of the discrete Goursat problem. However, the existence proof seems to contain a gap, or is at least unclear at the essential point[3].

We present another approach here. Observe that the system (3.7),(3.21) of first order difference equations is equivalent to the following second-order system (cf. eqs. (3.16)&(3.17)):

$$\tau_1\tau_2 n^\epsilon = g_+(\tau_1 n^\epsilon, n^\epsilon, \tau_2 n^\epsilon) \tag{3.22}$$
$$\tau_1\tau_3 n^\epsilon = g_+(\tau_1 n^\epsilon, n^\epsilon, \tau_3 n^\epsilon) \tag{3.23}$$
$$\tau_2\tau_3 n^\epsilon = g_-(\tau_2 n^\epsilon, n^\epsilon, \tau_3 n^\epsilon) \tag{3.24}$$

where

$$\begin{aligned} g_+(P,Q,R) &= \frac{Q\cdot(P+R)}{1+P\cdot R}(P+R) - Q \\ g_-(P,Q,R) &= Q - \frac{Q\cdot(P-R)}{1-P\cdot R}(P-R). \end{aligned}$$

Solvability of this system on an elementary cube $\mathcal{B}^\epsilon(\epsilon, 1)$ is investigated: From the Goursat data prescribed, one has $n^\epsilon(0,0,0) = N^\epsilon$, $n^\epsilon(\epsilon,0,0) = N^\epsilon + \epsilon V_1^\epsilon(0)$, $n^\epsilon(0,\epsilon,0) = N^\epsilon + \epsilon V_2^\epsilon(0)$, and $n^\epsilon(0,0,1) = N^\epsilon + 2W^\epsilon$. Furthermore, the value of $n^\epsilon(\epsilon,\epsilon,0)$ is calculated from (3.22), $n^\epsilon(\epsilon,0,1)$ is calculated from (3.23), and $n^\epsilon(0,\epsilon,1)$ from (3.24). From these values, $n^\epsilon(\epsilon,\epsilon,1)$ can be defined in three different ways, using an arbitrary one of the equations (3.22)-(3.24). Consistency means that if $n^\epsilon(\epsilon,\epsilon,1)$ is calculated from any of the three equations, then also the other two equations hold as well. Hence, it is sufficient to prove that the following three expressions coincide for all choices of unit vectors N_0, N_1, N_2 and N_3:

$$g_1 := g_-(g_+(N_1,N_0,N_2), N_1, g_+(N_1,N_0,N_3)) \tag{3.25}$$
$$g_2 := g_+(g_+(N_1,N_0,N_2), N_2, g_-(N_2,N_0,N_3)) \tag{3.26}$$
$$g_3 := g_+(g_+(N_1,N_0,N_3), N_3, g_-(N_2,N_0,N_3)) \tag{3.27}$$

Coincidence of the expressions (3.25), (3.26) and (3.27) are most easily verified with a computer algebra system. Parameterize the unit vectors by

$$N_i(x_i,y_i) = (2x_i, 2y_i, 1-(x_i^2+y_i^2))/(1+x_i^2+y_i^2).$$

[3]From the given points, the four remaining points of the warped parallelepiped are constructed in an ad hoc manner. In our language, a function u^ϵ is defined on $\mathcal{B}^\epsilon(\epsilon, 1)$, which attains the Goursat data and solves a *selection* of equations from (3.7),(3.21). Certain claims are formulated about the constructed figure. They correspond to the assumption that also the remaining equations are satisfied, and u^ϵ is a genuine solution. However, these claims are not proven but are argued to be obvious from a supplementing figure. The author strongly disagrees with the obviousness.

With the help of `MATHEMATICA`, it is checked that the functions g_1, g_2 and g_3 coincide identically in x_i, y_i. ■

Theorem 3.2 *Let be given a smooth K-surface $f : \mathcal{B}(\bar{r}) \to \mathbb{R}^3$ and a unit vector N^+, which is not normal to $\mathcal{S}_f$ at $f(0)$.*

1. *Up to translations and homoteties, there is a unique Bäcklund transform $f^+ : \mathcal{B}(r) \to \mathbb{R}^3$ of f, for which N^+ is the value of the Gauss map at $\xi = 0$.*
2. *Let a family of discrete K-surfaces $\{f^\epsilon\}$ be given and assume that $\{f^\epsilon\}$ converges to f in $C^\infty(\mathcal{B}(r))$. Construct another family $\{f^{\epsilon+}\}$ of discrete K-surfaces, so that each $f^{\epsilon+}$ is a Bäcklund transformation of f^ϵ, for which N^+ is the value of its Gauss map at $\xi = 0$. Then the family $\{f^{\epsilon+}\}$ smoothly lattice-converges to f^+ (after a suitable scaling and translation of the f^ϵ if necessary).*

Remark: For the approximating sequence in 2., one may choose the solutions to the discrete Goursat problem 3.1 with data $F^\epsilon = f(0)$ and $V_i^\epsilon = \delta_i[n]^\epsilon$, where $n : \mathcal{B}(r) \to S^2$ is the Gauss map of f.

Proof: Let $n : \mathcal{B}(\bar{r}) \to S^2$ be the Gauss map of f. Solve the Goursat problem 3.2 for $\epsilon = 0$ with data $F = f(0)$, some $F^+ \in \mathbb{R}^3$, $N = n(0)$, $W = \frac{1}{2}(N^+ - N)$, $V_1(\xi_1, 0) = \partial_1 n(\xi_1, 0)$, $V_2(0, \xi_2) = \partial_2 n(0, \xi_2)$ on $\mathcal{B}(r, 1)$. The parameter $\kappa > 0$ is chosen so that $-\kappa^{-2}$ is the Gaussian curvature of $\mathcal{S}_f$. Then the K-surface $f^+ := f(\cdot, 1) : \mathcal{B}(r) \to \mathbb{R}^3$ is a Bäcklund transformation of f with the required properties. Observe that, except for the arbitrarily chosen parameters F^+ and κ, the Goursat data are determined by f and N^+. Other choices of F^+ and κ correspond to translation and scaling, respectively.

For the second part, let $N^\epsilon : \mathcal{B}^\epsilon(\bar{r}) \to S^2$ be the respective Gauss maps and solve for each $\epsilon > 0$ the Goursat problem 3.2 with data $F^\epsilon = f^\epsilon(0)$, $F^{\epsilon+} = F^+$, $N^\epsilon = n^\epsilon(0)$, $W^\epsilon = N^+ - N^\epsilon$, $V_1^\epsilon(\xi_1, 0) = \delta_1 n^\epsilon(\xi_1, 0)$, $V_2^\epsilon(0, \xi_2) = \delta_2 n^\epsilon(0, \xi_2)$ on $\mathcal{B}^\epsilon(r, 1)$. Since f^ϵ was assumed to lattice-converge to f smoothly, so do the data derived from f^ϵ, and lattice-convergence of $f^{\epsilon+}$ to f^+ now follows from Theorem 2.2. ■

3.2 K-Surfaces by Lax Matrices

The approach presented in this section is more algebraic. Smooth and discrete K-surfaces are described with the help of *adapted frames*. This leads naturally to the so-called Lax pair representations of the Sine-Gordon-Equation and of the Hirota equation in the smooth and discrete case, respectively.

3.2.1 Associated Families of Smooth K-surfaces

It is a classical observation that each K-surface f is member of a distinct one-parameter family $\{f_\lambda\}_\lambda$ of K-surfaces, called the *associated family of f*. All elements

f_λ have the same Gaussian curvature and the same second fundamental form. Furthermore, the f_λ depend continuously on the parameter λ, which varies over some finite, closed interval $\mathcal{I}$. The approach via adapted frames and Lax matrices allows to investigate all elements of the associated family at once.

For technical simplicity, assume that the interval $\mathcal{I}$ contains a neighborhood of 1 but does not contain 0.

Definition 3.5 *The* associated family $\{f_\lambda : \mathcal{B}(r) \to \mathbb{R}^3\}_{\lambda\in\mathcal{I}}$ *is a family of K-surfaces parameterized by $\lambda \in \mathcal{I}$ for which the angle between asymptotic lines at corresponding points is independent of λ,*

$$\angle(\partial_1 f_\lambda(\xi), \partial_2 f_\lambda(\xi)) = \angle(\partial_1 f_1(\xi), \partial_2 f_1(\xi)) \text{ for all } \xi \in \mathcal{B}(r),$$

and the lengths of corresponding tangential vectors scale as

$$|\partial_1 f_\lambda| = \lambda|\partial_1 f_1|,\ |\partial_2 f_\lambda| = \lambda^{-1}|\partial_2 f_1|.$$

The surface f generating the associated family $\{f_\lambda\}_\lambda$ embeds as f_1. Existence of the associated family for a prescribed f and a suitable interval $\mathcal{I}$ is a classical result.

The most convenient description of the associated families is given in terms of λ-dependent frames with values in SU(2). We use the standard identification between the Lie algebra su(2) and $\mathbb{R}^3$, see e.g. [BP3], and we will not distinguish between the two.

Definition 3.6 *A frame $\psi : \mathcal{B}(r) \to \mathrm{SU}(2)$ is called* adapted *to the K-surface $f : \mathcal{B}(r) \to \mathbb{R}^3$ if $\psi^{-1}(\xi)\mathbf{e}_1\psi(\xi)$ is parallel (with the same orientation) to $\partial_1 f(\xi)$ for all $\xi \in \mathcal{B}(r)$, and $\psi^{-1}\mathbf{e}_3\psi : \mathcal{B}(r) \to S^2$ is the Gauss map of f.*

An associated family determines its frames uniquely; the frames determine the associated family up to a λ-independent homotety and a λ-dependent translation of the immersions in $\mathbb{R}^3$.

The following results can be found in [B1]:

Proposition 3.4 *Let $\psi_\lambda : \mathcal{B}(r) \to \mathrm{SU}(2)$, with $\lambda \in \mathcal{I}$, be frames adapted to an associated family $\{f_\lambda : \mathcal{B}(r) \to \mathbb{R}^3\}$ of K-surfaces. Then the logarithmic derivatives, or* Lax matrices, *$U_\lambda, V_\lambda : \mathcal{B}(r) \to \mathrm{su}(2)$ defined by*

$$\partial_1\psi_\lambda = U_\lambda \cdot \psi_\lambda,\ \partial_2\psi_\lambda = V_\lambda \cdot \psi_\lambda \tag{3.28}$$

are of the form

$$U_\lambda = \frac{i}{2}\begin{pmatrix} \alpha & -a\lambda \\ -a\lambda & -\alpha \end{pmatrix},\ V_\lambda = \frac{i}{2}\begin{pmatrix} 0 & \frac{b}{\lambda}e^{i\beta} \\ \frac{b}{\lambda}e^{-i\beta} & 0 \end{pmatrix} \tag{3.29}$$

with suitable real functions $a, b, \alpha, \beta : \mathcal{B}(r) \to \mathbb{R}$ independent of λ. Naturally, the zero-curvature condition is satisfied

$$\partial_2 U_\lambda - \partial_1 V_\lambda + [U_\lambda, V_\lambda] = 0, \tag{3.30}$$

which is equivalent to

$$\partial_2 a = \partial_1 b = 0,\ \partial_2\alpha = ab\sin\beta,\ \partial_1\beta = \alpha. \tag{3.31}$$

Conversely, given a solution to the equations (3.31), define the matrices $U_\lambda, V_\lambda : \mathcal{B}(r) \to \mathrm{su}(2)$ *according to (3.29). Then (3.28) has a solution* $\psi_\lambda : \mathcal{B}(r) \to \mathrm{SU}(2)$ *and the* Sym formula

$$f_\lambda = 2\kappa\lambda\psi_\lambda^{-1} \cdot (\partial_\lambda\psi_\lambda) \tag{3.32}$$

produces an associated family $\{f_\lambda\}$ *of K-surfaces. The immersions* $\mathcal{S}_{f_\lambda}$ *have constant negative Gaussian curvature* $-\kappa^2$*, and each frame* ψ_λ *is adapted to the corresponding K-surface* f_λ*.*

Naturally, K-surfaces that differ only by an Euclidean motion are considered as equivalent. In the present context, it is convenient to assume that *the immersions* f_λ *are generated by the Sym formula 3.32 from the adapted frame* ψ_λ. Multiplication of the function ψ_λ by a λ-dependent matrix from the right simply results in translations and rotations of the individual f_λ.

The interpretation of the functions in (3.29) is the following:

$$\kappa\lambda a = |\partial_1 f_\lambda|,\ \kappa\lambda^{-1} b = |\partial_2 f_\lambda|,\ \alpha = \partial_1\beta,\ \beta = \angle(\partial_1 f_\lambda, \partial_2 f_\lambda), \tag{3.33}$$

i.e., $\beta(\xi)$ is the (λ-independent) angle between the two asymptotic lines crossing at $f_\lambda(\xi)$; the identity (3.31) shows that it satisfies the classical sine-Gordon-equation

$$\partial_1\partial_2\beta = ab\sin\beta. \tag{3.34}$$

As for single K-surfaces, there exists a notion of Bäcklund transformations for associated families. These transformations produce a new associated family $\{f_\lambda^+\}$ from a given one $\{f_\lambda\}$. The induced transformation for each fixed $\lambda \in \mathcal{I}$ coincides with the Bäcklund transformation from Definition 3.3.

Definition 3.7 *An associated family of K-surfaces* $\{f_\lambda^+\}$ *is called a* Bäcklund transformed *of another associated family* $\{f_\lambda\}$ *iff the corresponding adapted frames* ψ_λ^+, ψ_λ *are related by*

$$\psi_\lambda^+ = \mathcal{W}_\lambda \cdot \psi_\lambda \tag{3.35}$$

with matrix functions $\mathcal{W}_\lambda : \mathcal{B}(r) \to \mathrm{SU}(2)$ *of the form*

$$\mathcal{W}_\lambda = \frac{1}{\sqrt{1+(c\lambda)^2}} \begin{pmatrix} e^{i\gamma} & -ic\lambda \\ -ic\lambda & e^{-i\gamma} \end{pmatrix} \tag{3.36}$$

where c *is a non-zero real constant and* $\gamma : \mathcal{B}(r) \to \mathbb{R}$ *is a smooth function, both independent of* λ*.*

The motivation for the ansatz for $\mathcal{W}_\lambda$ is that it represents the simplest (namely, up to a factor affine in λ) of the associated *loop group* of the frame (see [BP3] for details).

Note that $\mathcal{W}_\lambda^{-1}$ is of the same form as $\mathcal{W}_\lambda$. Hence $\{f_\lambda\}$ is also a Bäcklund transform of $\{f_\lambda^+\}$.

Geometric interpretation: The geometric properties stated after Definition 3.3 carry over: For each fixed $\lambda \in \mathcal{I}$, the immersions f_λ and f_λ^+ are equivalently parameterized, see eq. (3.13) and (3.12), and their Gauss maps enclose the constant angle

$$\theta = \arccos \frac{1-(c\lambda)^2}{1+(c\lambda)^2}, \tag{3.37}$$

see eq. (3.14). In addition, the Sym formula (3.32) implies

$$f_\lambda^+ = f_\lambda + 2\kappa\lambda\psi_\lambda^{-1}(\mathcal{W}_\lambda^{-1}\cdot\partial_\lambda\mathcal{W}_\lambda)\psi_\lambda.$$

One calculates that $\mathcal{W}_\lambda^{-1}\cdot\partial_\lambda\mathcal{W}_\lambda$ is of length $c/(1+(c\lambda)^2)$, so the distance between the points $f_\lambda^+(\xi)$ and $f_\lambda(\xi)$ is

$$\Delta_\lambda = \frac{2\kappa c\lambda}{1+(c\lambda)^2}. \tag{3.38}$$

Also, $\mathcal{W}_\lambda^{-1}\cdot\partial_\lambda\mathcal{W}_\lambda$ is orthogonal to $\mathbf{e}_3$, so the vector $f_\lambda^+(\xi)-f_\lambda(\xi)$ is tangent to $\mathcal{S}_{f_\lambda}$ at $f_\lambda(\xi)$. Since f_λ is also a Bäcklund transformation of f_λ^+, the $f_\lambda^+(\xi)-f_\lambda(\xi)$ is tangent to $\mathcal{S}_{f_\lambda^+}$ at $f_\lambda^+(\xi)$, too.

Corollary 1 *If $\{f_\lambda^+\}$ is a Bäcklund transform $\{f_\lambda\}$, then the vectors $f_\lambda^+(\xi)-f_\lambda(\xi)$ have an ξ-independent length and are tangent to both immersions $\mathcal{S}_{f_\lambda}$ and $\mathcal{S}_{f_\lambda^+}$ at the respective points.*

This corollary actually represents the most classical definition of the Bäcklund transformation. Furthermore, it yields an geometric interpretation of the geometric parameters that determine a Bäcklund transformation: Given the original family $\{f_\lambda\}$, a Bäcklund transform $\{f_\lambda^+\}$ is fixed by a point, say $f_1^+(0)$, in the two-dimensional affine tangent space of $\mathcal{S}_{f_1}$ at $f_1(0)$.

Algebraically, a Bäcklund transform is fixed by two real parameters: the constant $c>0$, and the value of $\Gamma=\gamma(0)$. The function γ is then determined as solutions to two ordinary differential equations:

Proposition 3.5 *If $\{f_\lambda^+\}$ is a Bäcklund transform of $\{f_\lambda\}$, so that the respective adapted frames ψ_λ^+, ψ_λ are related by (3.35) with $\mathcal{W}_\lambda$ of the form (3.36), then*

$$\partial_1\gamma = -\alpha + \frac{a}{c}\sin\gamma, \quad \partial_2\gamma = bc\sin(\beta+\gamma). \tag{3.39}$$

Conversely, let $\{f_\lambda\}$ be an associated family of K-surfaces with adapted frames ψ_λ. Let $c \neq 0$ be arbitrary and assume $\gamma : \mathcal{B}(r) \to \mathbb{R}$ satisfies (3.39). Then $\psi_\lambda^+ = \mathcal{W}_\lambda \cdot \psi_\lambda$ with $\mathcal{W}_\lambda$ as in (3.36) are frames adapted to a Bäcklund transformed family $\{f_\lambda^+\}$ of K-surfaces. The corresponding quantities in the Lax matrices for f_λ^+ are

$$a^+ = a,\ b^+ = b,\ \alpha^+ = -\alpha + 2\frac{a}{c}\sin\gamma,\ \beta^+ = \beta + 2\gamma. \tag{3.40}$$

Proof: The frames ψ^+ have logarithmic derivatives

$$U_\lambda^+ = \mathcal{W}_\lambda U_\lambda \mathcal{W}_\lambda^{-1} + (\partial_1 \mathcal{W}_\lambda)\mathcal{W}_\lambda^{-1},\ V_\lambda^+ = \mathcal{W}_\lambda V_\lambda \mathcal{W}_\lambda^{-1} + (\partial_2 \mathcal{W}_\lambda)\mathcal{W}_\lambda^{-1}. \tag{3.41}$$

The requirement that these are of the form (3.29) is equivalent to the equations (3.39). Inserting (3.39) into (3.41) implies the relations (3.40). ∎

3.2.2 Associated Families of Discrete K-surfaces

Definition 3.8 *A* discrete associated family $\{f_\lambda^\epsilon : \mathcal{B}^\epsilon(r) \to \mathbb{R}^3\}_{\lambda\in\mathcal{I}}$ *is a family of discrete K-surfaces, parameterized by $\lambda \in \mathcal{I}$ for which the angles between adjacent edges at corresponding points are independent of λ*

$$\angle(\delta_1 f_\lambda^\epsilon, \delta_2 f_\lambda^\epsilon) = \angle(\delta_1 f_1^\epsilon, \delta_2 f_1^\epsilon)$$

and the lengths of the edges scale as

$$|\delta_1 f_\lambda^\epsilon| = \frac{\lambda|\delta_1 f_1^\epsilon|}{1 + (\frac{\epsilon\lambda}{2}|\delta_1 f_1^\epsilon|)^2},\ |\delta_2 f_\lambda^\epsilon| = \frac{\lambda^{-1}|\delta_2 f_1^\epsilon|}{1 + (\frac{\epsilon}{2\lambda}|\delta_1 f_1^\epsilon|)^2}.$$

We will need adapted frames for discrete K-surfaces. An algebraic definition takes the place of the geometric Definition 3.6 given in the smooth case:

Definition 3.9 *A family of maps $\{\psi_\lambda^\epsilon : \mathcal{B}^\epsilon(r) \to \mathrm{SU}(2)\}$ is called* adapted *to an associated family of discrete K-surfaces $\{f^\epsilon : \mathcal{B}^\epsilon(r) \to \mathbb{R}^3\}$, iff $(\psi_\lambda^\epsilon)^{-1}\mathbf{e}_3\psi_\lambda^\epsilon$ is the Gauss map for f_λ^ϵ, and the logarithmic derivatives, or* discrete Lax matrices, *$\mathcal{U}_\lambda^\epsilon$ and $\mathcal{V}_\lambda^\epsilon$ defined by*

$$\tau_1\psi_\lambda^\epsilon = \mathcal{U}_\lambda^\epsilon \cdot \psi_\lambda^\epsilon,\ \ \tau_2\psi_\lambda^\epsilon = \mathcal{V}_\lambda^\epsilon \cdot \psi_\lambda^\epsilon \tag{3.42}$$

are of the special form

$$\mathcal{U}_\lambda^\epsilon = \frac{1}{\sqrt{1 + (\frac{\epsilon\lambda}{2}a^\epsilon)^2}} \begin{pmatrix} e^{\frac{i\epsilon}{2}\alpha^\epsilon} & -\frac{i\epsilon\lambda}{2}a^\epsilon \\ -\frac{i\epsilon\lambda}{2}a^\epsilon & e^{-\frac{i\epsilon}{2}\alpha^\epsilon} \end{pmatrix}, \tag{3.43}$$

$$\mathcal{V}_\lambda^\epsilon = \frac{1}{\sqrt{1 + (\frac{\epsilon}{2\lambda}b^\epsilon)^2}} \begin{pmatrix} 1 & \frac{i\epsilon}{2\lambda}b^\epsilon e^{i\beta^\epsilon} \\ \frac{i\epsilon}{2\lambda}b^\epsilon e^{-i\beta^\epsilon} & 1 \end{pmatrix} \tag{3.44}$$

with suitable real functions $a^\epsilon, b^\epsilon, \alpha^\epsilon, \beta^\epsilon : \mathcal{B}^\epsilon(r) \to \mathbb{R}$.

Any given associated family of discrete K-surface possesses a family of adapted frames. The frames are not uniquely determined by the surfaces alone, see the discussion about the gauge freedom of the Hirota variables in [BP1] for more details. The interpretation of the quantities in the Lax matrices is:

$$|\delta_1 f_1^\epsilon| = a^\epsilon, \quad |\delta_2 f_\lambda^\epsilon| = b^\epsilon$$
$$\alpha^\epsilon + \tau_2\alpha^\epsilon = 2\delta_1\beta^\epsilon, \quad \angle(\delta_1 f_\lambda^\epsilon, \delta_2 f_\lambda^\epsilon) = \beta^\epsilon + \tfrac{\epsilon}{2}\alpha^\epsilon$$

Proposition 3.6 *Let $\{f_\lambda^\epsilon\}$ be an associated family of discrete K-surfaces with adapted frames ψ_λ^ϵ. The respective Lax matrices (3.42) satisfy the discrete zero curvature condition*

$$(\tau_2 \mathfrak{U}_\lambda^\epsilon) \cdot \mathcal{V}_\lambda^\epsilon = (\tau_1 \mathcal{V}_\lambda^\epsilon) \cdot \mathfrak{U}_\lambda^\epsilon. \tag{3.45}$$

Equivalently, the coefficients satisfy

$$\delta_2 a^\epsilon = \delta_1 b^\epsilon = 0, \quad \delta_2\alpha^\epsilon = q^\epsilon, \quad \delta_1\beta^\epsilon = \alpha + \tfrac{\epsilon}{2}q^\epsilon, \tag{3.46}$$
$$\text{with } q^\epsilon = -\tfrac{4}{\epsilon^2}a^\epsilon b^\epsilon \arg(1 - \tfrac{\epsilon^2}{4}e^{i(b^\epsilon + \frac{\epsilon}{2}a^\epsilon)}). \tag{3.47}$$

Conversely, given a solution to the system (3.46), define matrices $\mathfrak{U}_\lambda^\epsilon$ and $\mathcal{V}_\lambda^\epsilon$ according to (3.43) and (3.44) and let ψ_λ^ϵ satisfy (3.42). Then the Sym formula

$$f_\lambda^\epsilon = 2\kappa\lambda(\psi_\lambda^\epsilon)^{-1} \cdot (\partial_\lambda \psi_\lambda^\epsilon). \tag{3.48}$$

yields an associated family $\{f_\lambda^\epsilon\}$ of discrete K-surfaces. For each $\lambda \in \mathfrak{I}$, the frame ψ_λ^ϵ is adapted to f_λ^ϵ.

Like in the continuous case, immersions f_λ^ϵ are considered equivalent if they only differ by an Euclidean motion. For simplicity, we will henceforth assume that the immersions f_λ^ϵ are generated by the Sym formula (3.48) from an adapted frame.

Let us introduce the so-called Hirota variable $h^\epsilon : \mathcal{B}^\epsilon(r) \to \mathbb{R}$ by $\delta_1 h^\epsilon = \frac{1}{2}\alpha^\epsilon$ and $h^\epsilon + \tau_2 h^\epsilon = \beta^\epsilon$. Starting from (3.46), one can show that h^ϵ is a solution to the *Hirota equation*:

$$e^{i(h^\epsilon + \tau_1\tau_2 h^\epsilon)} - e^{i(\tau_1 h^\epsilon + \tau_2 h^\epsilon)} = a^\epsilon b^\epsilon (1 - e^{i(h^\epsilon + \tau_1 h^\epsilon + \tau_2 h^\epsilon + \tau_1\tau_2 h^\epsilon)}),$$

which was suggested by Hirota in [Hi] as discretization of the sine-Gordon-equation[4].

The definition of a Bäcklund transformations for associated families of discrete K-surfaces carries over literally from the smooth setting:

[4]Note that there is also a *geometric* discrete analogue of the sine-Gordon-equation, which relates the four angles in an elementary quadrilateral. The equation can be found in [BP1].

Definition 3.10 *An associated family of K-surfaces* $\{f_\lambda^{\epsilon+}\}$ *is called a* Bäcklund transform *of another associated family* $\{f_\lambda^\epsilon\}$ *iff there are adapted frames* $\psi_\lambda^{\epsilon+}$, ψ_λ^ϵ *for them, related by*

$$\psi_\lambda^{\epsilon+} = \mathcal{W}_\lambda^\epsilon \cdot \psi_\lambda^\epsilon \tag{3.49}$$

with matrix functions $\mathcal{W}_\lambda^\epsilon : \mathcal{B}^\epsilon(r) \to \mathrm{SU}(2)$, $\lambda \in \mathcal{I}$, *of the form*

$$\mathcal{W}_\lambda^\epsilon = \frac{1}{\sqrt{1+(c^\epsilon\lambda)^2}} \begin{pmatrix} e^{i\gamma^\epsilon} & -ic^\epsilon\lambda \\ -ic^\epsilon\lambda & e^{-i\gamma^\epsilon} \end{pmatrix}. \tag{3.50}$$

Here c^ϵ *a positive real constant and* $\gamma^\epsilon : \mathcal{B}^\epsilon(r) \to \mathbb{R}$.

Geometric interpretation: For each individual K-surface f_λ^ϵ, the definition above agrees with Definition 3.4. The lengths of line segments are equal (possibly up to a global homotety), see (3.19) and (3.18), and the Gauss maps of f_λ and f_λ^+ enclose the constant angle θ from (3.37). Moreover, if the Sym formula is used to construct the lattices from the discrete frames, then corresponding points $f_\lambda^\epsilon(\xi)$ and $f_\lambda^{\epsilon+}(\xi)$ have constant distance Δ_λ from (3.38), and their connecting line is orthogonal to the Gauss map at these points. Corollary 1 carries over literally and is not repeated here.

Proposition 3.7 *If* $\{f_\lambda^{\epsilon+}\}$ *is a Bäcklund transform of* $\{f_\lambda^\epsilon\}$*, so that the respective adapted frames* $\psi_\lambda^{\epsilon+}$, ψ_λ^ϵ *are related by (3.49) with* $\mathcal{W}_\lambda^\epsilon$ *of the form (3.50), then*

$$\begin{aligned} \delta_1\gamma^\epsilon &= -\alpha^\epsilon - \tfrac{2}{\epsilon}\arg\!\left(1 - \tfrac{\epsilon}{2}\tfrac{a^\epsilon}{c^\epsilon} e^{i(\gamma^\epsilon - \frac{\epsilon}{2}\alpha^\epsilon)}\right), \\ \delta_2\gamma^\epsilon &= -\tfrac{2}{\epsilon}\arg\!\left(1 - \tfrac{\epsilon}{2} b^\epsilon c^\epsilon e^{i(\gamma^\epsilon+\beta^\epsilon)}\right). \end{aligned} \tag{3.51}$$

Conversely, let $\{f_\lambda^\epsilon\}$ *be an associated family of discrete K-surfaces with adapted frames* ψ_λ^ϵ*. Let* $c > 0$ *be an arbitrary constant and* $\gamma^\epsilon : \mathcal{B}^\epsilon(r) \to \mathbb{R}$ *a solution to (3.51). Then* $\psi_\lambda^{\epsilon+} = \mathcal{W}_\lambda^\epsilon \cdot \psi_\lambda^\epsilon$ *with* $\mathcal{W}_\lambda^\epsilon$ *as in (3.50) are frames adapted to a Bäcklund transform* $\{f_\lambda^{\epsilon+}\}$*. The corresponding quantities in the Lax matrices (3.43) and (3.44) for* $f_\lambda^{\epsilon+}$ *are*

$$\begin{aligned} a^{\epsilon+} &= a^\epsilon,\ b^{\epsilon+} = b, \\ \alpha^{\epsilon+} &= -a^\epsilon - \tfrac{4}{\epsilon}\arg\!\left(1 - \tfrac{\epsilon}{2}\tfrac{a^\epsilon}{c^\epsilon} e^{i(\gamma^\epsilon - \frac{\epsilon}{2}\alpha^\epsilon)}\right), \\ \beta^{\epsilon+} &= \beta^\epsilon + 2\gamma^\epsilon - 2\arg\!\left(1 - \tfrac{\epsilon}{2} b^\epsilon c^\epsilon e^{i(\gamma^\epsilon+\beta^\epsilon)}\right). \end{aligned} \tag{3.52}$$

Proof: The logarithmic derivatives of the transformed frames $\psi_\lambda^{\epsilon+}$ are

$$\mathcal{U}_\lambda^{\epsilon+} = (\tau_1\mathcal{W}_\lambda^\epsilon)\cdot\mathcal{U}_\lambda^\epsilon\cdot(\mathcal{W}_\lambda^\epsilon)^{-1},\ \mathcal{V}_\lambda^{\epsilon+} = (\tau_2\mathcal{W}_\lambda^\epsilon)\cdot\mathcal{V}_\lambda^\epsilon\cdot(\mathcal{W}_\lambda^\epsilon)^{-1}.$$

We require that these are of the forms (3.43) and (3.44), respectively. A first consequence is that the expressions under the square roots in the formulas (3.43), (3.44) must be the same for $\mathcal{U}_\lambda^{\epsilon+}$ and $\mathcal{U}_\lambda^\epsilon$, and for $\mathcal{V}_\lambda^{\epsilon+}$ and $\mathcal{V}_\lambda^\epsilon$, respectively, so that $a^{\epsilon+} = a^\epsilon$ and $b^{\epsilon+} = b^\epsilon$. This simplifies the further calculations which eventually lead to the equations in (3.51). These, on the other hand, immediately imply $a^{\epsilon+} = a^\epsilon$ and $b^{\epsilon+} = b^\epsilon$, so the calculation can be reversed, and one eventually finds that the system (3.51) is truly equivalent to saying that $\mathcal{U}_\lambda^{\epsilon+}$, $\mathcal{V}_\lambda^{\epsilon+}$ are of the forms (3.43), (3.44). The remaining identities in (3.52) follow. ∎

3.2.3 Approximation of Associated Families

Note that the compatibility conditions written out in (3.31) and (3.46) are already in the form of a continuous and discrete hyperbolic system, respectively. We summarize the equations again. For the smooth case, one has

$$\begin{array}{rclrcl} \partial_1\psi_\lambda &=& U_\lambda\cdot\psi_\lambda & \qquad \partial_2\psi_\lambda &=& V_\lambda\cdot\psi_\lambda \\ \partial_1 b &=& 0 & \partial_2 a &=& 0 \\ \partial_1\beta &=& \alpha & \partial_2\alpha &=& ab\sin\beta \end{array} \tag{3.53}$$

The matrices $U_\lambda = U_\lambda(a,\alpha)$ and $V_\lambda = V_\lambda(b,\beta)$ are given by formula (3.29).

In the discrete case, the system reads:

$$\begin{array}{rclrcl} \delta_1\psi^\epsilon_\lambda &=& U^\epsilon_\lambda\cdot\psi^\epsilon_\lambda & \qquad \delta_2\psi^\epsilon_\lambda &=& V^\epsilon_\lambda\cdot\psi^\epsilon_\lambda \\ \delta_1 b^\epsilon &=& 0 & \delta^2 a^\epsilon &=& 0 \\ \delta_1\beta^\epsilon &=& \alpha^\epsilon + \frac{\epsilon}{2}q^\epsilon & \delta_2\alpha^\epsilon &=& q^\epsilon \end{array} \tag{3.54}$$

where we have used

$$U^\epsilon_\lambda := \frac{1}{\epsilon}(\mathcal{U}^\epsilon_\lambda - \mathbf{1}),\ V^\epsilon_\lambda := \frac{1}{\epsilon}(\mathcal{V}^\epsilon_\lambda - \mathbf{1})$$

with $\mathcal{U}^\epsilon_\lambda = \mathcal{U}^\epsilon_\lambda(a^\epsilon,\alpha^\epsilon)$ and $\mathcal{V}^\epsilon_\lambda = \mathcal{V}^\epsilon_\lambda(b^\epsilon,\beta^\epsilon)$ as defined in (3.43) and (3.44). Also q^ϵ is a smooth function of the involved quantities, see (3.47).

For a and α, the ξ_1-direction is static and the ξ_2-direction is evolutionary; for b and β, it is the other way around. ψ_λ has no static directions.

As Banach space $\mathcal{X}$, we choose

$$\mathcal{X} = \mathcal{M}[\lambda]\times\mathbb{R}_a\times\mathbb{R}_b\times\mathbb{R}_\alpha\times\mathbb{R}_\beta$$

where the first component

$$\mathcal{M}[\lambda] = \{\psi : \mathcal{I}\to \mathrm{Mat}(2\times 2,\mathbb{C}) | \psi \text{ continuously differentiable}\}$$

consists of C^1-matrix functions depending on λ and is equipped with the usual C^1-norm. Pointwise (matrix-)multiplication of functions is a bilinear C^∞-mapping from $\mathcal{M}[\lambda]\times\mathcal{M}[\lambda]$ to $\mathcal{M}[\lambda]$. It is obvious that the matrix functions $U^{(\epsilon)}_\lambda$ and $V^{(\epsilon)}_\lambda$ lie in $\mathcal{M}[\lambda]$, and their dependence on $a^{(\epsilon)}$, $b^{(\epsilon)}$, $\alpha^{(\epsilon)}$ and $\beta^{(\epsilon)}$ is smooth. Therefore, the expressions $U_\lambda\cdot\psi_\lambda$ etc. in (3.53) and (3.54) are indeed C^∞ from $\mathcal{X}$ to $\mathcal{M}[\lambda]$.

Goursat Problem 3.3 (for associated families) *Given a λ-dependent frame $\Psi^\epsilon\in\mathcal{M}[\lambda]$, a pair of real functions $A^\epsilon,\mathcal{A}^\epsilon$ on $\mathcal{B}^\epsilon_1(r)$ and another pair $B^\epsilon,\mathcal{B}^\epsilon$ on $\mathcal{B}^\epsilon_2(r)$, find a solution to (3.53) or (3.54), respectively, so that $\psi^\epsilon_\lambda(0) = \Psi^\epsilon_\lambda$; $a^\epsilon = A^\epsilon$ and $\alpha^\epsilon = \mathcal{A}^\epsilon$ on $\mathcal{B}^\epsilon_1(r)$; $b^\epsilon = B^\epsilon$ and $\beta^\epsilon = \mathcal{B}^\epsilon$ on $\mathcal{B}^\epsilon_2(r)$.*

Lemma 3.4 *The discrete system (3.54) is compatible, and the functions on the right-hand sides $\mathcal{O}(\epsilon)$-converge smoothly to the respective functions in (3.53).*

Proof: The only compatibility condition to check is $\tau_2(\tau_1\psi^\epsilon) = \tau_1(\tau_2\psi^\epsilon)$, or, equivalently, the discrete zero curvature condition (3.45). But by Proposition 3.6, the condition (3.45) is in fact equivalent to to the last four equations of (3.54). Convergence of the right-hand sides is verified using the elementary rules following definition 2.1. ∎

Theorem 3.3 *Assume two positive functions $A, B : [0, \bar{r}] \to \mathbb{R}$ and two general real valued functions, $\phi_1, \phi_2 : [0, \bar{r}] \to \mathbb{R}$ are given, satisfying $\phi_1(0) = \phi_2(0)$.*

1. *For any given positive number κ, there is an associated family $\{f_\lambda : \mathcal{B}(r) \to \mathbb{R}^3\}_{\lambda\in\mathfrak{I}}$ of K-surfaces with constant negative Gaussian curvature $-\kappa^{-2}$, and some positive $r \leq \bar{r}$, that satisfies*

$$\begin{aligned} |\partial_1 f_\lambda(\xi_1,\xi_2)| = \kappa\lambda A(\xi_1),\ |\partial_2 f_\lambda(\xi_1,\xi_2)| &= \kappa\lambda^{-1}B(\xi_2), \\ \angle(\partial_1 f_\lambda(\xi_1,0), \partial_2 f_\lambda(\xi_1,0)) &= \phi_1(\xi_1), \\ \angle(\partial_1 f_\lambda(0,\xi_2), \partial_2 f_\lambda(0,\xi_2)) &= \phi_2(\xi_2) \end{aligned} \tag{3.55}$$

   for all $\lambda \in \mathfrak{I}$ and all $\xi_1, \xi_2 \in [0, r]$. $\{f_\lambda\}$ is unique up to a λ-dependent Euclidean motion. Without loss of generality, they are normalized by

$$f_\lambda(0) = 0,\ \partial_1 f_\lambda(0) \parallel \mathbf{e}_1,\ n_\lambda(0) = \mathbf{e}_3.$$

2. *For $\epsilon > 0$, denote by $\psi_\lambda^\epsilon : \mathcal{B}^\epsilon(r) \to \mathcal{M}[\lambda]$ the frames associated to the solutions to Goursat problem 3.3 with $A^\epsilon = [A]^\epsilon$, $B^\epsilon = [B]^\epsilon$, $\mathcal{A}^\epsilon = [\partial_1\phi_1]^\epsilon$, $\mathcal{B}^\epsilon = [\phi_2]^\epsilon$ and $\Psi_\lambda^\epsilon \equiv \mathbf{1}$. And denote by $\{f_\lambda^\epsilon : \mathcal{B}^\epsilon(r) \to \mathbb{R}^3\}_{\lambda\in\mathfrak{I}}$ the associated families of discrete K-surfaces obtained by the Sym formula (3.48). For each $\lambda \in \mathfrak{I}$, the discrete surfaces f_λ^ϵ lattice-converge to the smooth K-surface f_λ from above in $C^\infty(\mathcal{B}(r))$.*

Proof: The Goursat Problem 3.3 with data A, B as given in the assumptions, $\mathcal{A}(\xi_1, 0) = \partial_1\phi_1(\xi_1)$, $\mathcal{B}(0, \xi_2) = \phi_2(\xi_2)$ and $\Psi_\lambda \equiv \mathbf{1}$ has a unique solution $(\psi, a, b, \alpha, \beta) : \mathcal{B}(r) \to \mathfrak{X}$ by Theorem 2.2. The Sym formula (3.32) yields an associated family $\{f_\lambda : \mathcal{B}(r) \to \mathbb{R}^3\}_\lambda$ of discrete K-surfaces. Since the matrix functions $\psi(\xi) \in \mathcal{M}[\lambda]$ are continuously differentiable in λ for any $\xi \in \mathcal{B}(r)$, the functions $f_\lambda : \mathcal{B}(r) \to \mathbb{R}^3$ are indeed well-defined. Each $f_\lambda : \mathcal{B}(r) \to \mathbb{R}^3$ is a K-surface of Gaussian curvature $-\kappa^{-2}$ by Proposition 3.4. Also, f_λ is smooth since the Sym formula smoothly combines the functions ψ_λ^{-1} and $\partial_\lambda\psi_\lambda$, which are obviously C^∞ with respect to ξ. The conditions (3.55) are a consequence from the interpretation of the quantities a, b, α and β given in (3.33). The normalizations follow from our choice $\Psi_\lambda \equiv \mathbf{1}$.

Now consider the family $\{\{f_\lambda^\epsilon\}_\lambda\}_\epsilon$ – which is parameterized both by $\epsilon > 0$ and by $\lambda \in \mathfrak{I}$ – constructed according to the second part of the Theorem. By Theorem 2.2, the frame functions $\psi^\epsilon : \mathcal{B}^\epsilon(r) \to \mathcal{M}[\lambda]$ that are component of the solutions to the discrete Goursat problem 3.3, are smoothly lattice-convergent to the respective frame $\psi : \mathcal{B}(r) \to \mathcal{M}[\lambda]$ of the continuous solution defined above. Since convergence in C^1 means uniform convergence of the function and its first derivative (w.r.t. λ), both ψ_λ^ϵ and $\partial_\lambda \psi_\lambda^\epsilon$ smoothly lattice-converge for any fixed $\lambda \in \mathcal{I}$ to the limits ψ_λ and $\partial_\lambda \psi_\lambda$. Hence also the matrix product $(\psi_\lambda^\epsilon)^{-1} \cdot (\partial_\lambda \psi_\lambda^\epsilon)$ is lattice-convergent in $C^\infty(\mathcal{B}(r))$ (w.r.t. ξ) to $(\psi_\lambda)^{-1} \cdot (\partial_\lambda \psi_\lambda)$, proving convergence of the discrete associated families to the smooth one. ■

3.2.4 Approximation of Bäcklund Transformations

Finally, the picture is completed by including both associated families and Bäcklund transformations in a convergence result.

Change to $M = 3$ dimensions with $m = 2$ quasi-continuous and $m' = 1$ discrete directions. Incorporating the transformation theory in our hyperbolic description, new equations are added to the systems (3.53) and (3.54). To interprete Bäcklund transformations as a shift in the third direction, it is natural to consider the transformation parameters c and γ as functions on the respective $\mathcal{B}(r,1)$, too. For both quantities, the third direction is static. This is in coherence with the interpretation of a sequence of Bäcklund transformations: At each step, parameters c and $\Gamma = \gamma(0)$ can be arbitrarily prescribed. Equations for γ have been presented in (3.39) and (3.51), and since c is a constant for each step of the transformation, it trivially satisfies $\partial_1 c = \partial_2 c = 0$.

We enlarge the system (3.53) by the defining equation (3.35) of continuous Bäcklund transformation and the equations (3.39) and (3.40) derived from it:

$$
\begin{array}{rclrcl}
\delta_3 \psi_\lambda & = & W_\lambda \cdot \psi_\lambda & & & \\
\delta_3 a & = & 0 & \delta_3 b & = & 0 \\
\partial_1 c & = & 0 & \partial_2 c & = & 0 \\
\delta_3 \alpha & = & -2\alpha + 2\frac{a}{c}\sin\gamma & \delta_3 \beta & = & 2\gamma \\
\partial_1 \gamma & = & -\alpha + \frac{a}{c}\sin\gamma & \partial_2 \gamma & = & bc\sin(\beta+\gamma)
\end{array}
\tag{3.56}
$$

where $W_\lambda = \mathcal{W}_\lambda - \mathbf{1}$ is the matrix function of c and γ given by (3.36). Respectively, the discrete hyperbolic system in (3.54) is enlarged by the definition given in (3.49) and the derived equations (3.51) and (3.52)

$$
\begin{array}{rclrcl}
\delta_3 \psi_\lambda^\epsilon & = & W_\lambda^\epsilon \cdot \psi_\lambda^\epsilon & & & \\
\delta_3 a^\epsilon & = & 0 & \delta_3 b^\epsilon & = & 0 \\
\delta_1 c^\epsilon & = & 0 & \delta_2 c^\epsilon & = & 0 \\
\delta_3 \alpha^\epsilon & = & -2a^\epsilon - \frac{4}{\epsilon}\arg(1 - \frac{\epsilon}{2}\frac{a^\epsilon}{c^\epsilon} e^{i(\gamma^\epsilon - \frac{\epsilon}{2}\alpha^\epsilon)}) & & & \\
\delta_3 \beta^\epsilon & = & 2\gamma^\epsilon - 2\arg(1 - \frac{\epsilon}{2} b^\epsilon c^\epsilon e^{i(\gamma^\epsilon + \beta^\epsilon)}) & & & \\
\delta_1 \gamma^\epsilon & = & -\alpha^\epsilon - \frac{2}{\epsilon}\arg(1 - \frac{\epsilon}{2}\frac{a^\epsilon}{c^\epsilon} e^{i(\gamma^\epsilon - \frac{\epsilon}{2}\alpha^\epsilon)}) & & & \\
\delta_2 \gamma^\epsilon & = & -\frac{2}{\epsilon}\arg(1 - \frac{\epsilon}{2} b^\epsilon c^\epsilon e^{i(\gamma^\epsilon + \beta^\epsilon)}). & & &
\end{array}
\tag{3.57}
$$

The canonical extension of the Banach space is

$$
\mathcal{X} = \mathcal{M}[\lambda] \times \mathbb{R}_a \times \mathbb{R}_b \times \mathbb{R}_\alpha \times \mathbb{R}_\beta \times \mathbb{R}_c \times \mathbb{R}_\gamma,
$$

and the domain of definition of the right-hand sides in (3.56) and (3.57) is $\mathbb{D} = \{c^\epsilon > 0\} \subset \mathcal{X}$ – this restriction does not cause any analytical difficulties, since c^ϵ is independent of ξ. (The absence of singularities is an advantage of Definition 3.7 compared with Definition 3.3.) As mentioned before, the new quantities c and γ have the third as their only static direction, and all other fields keep the static directions assigned before.

Goursat Problem 3.4 (BT for associated families) *Given a frame $\Psi^\epsilon_\lambda \in \mathcal{M}[\lambda]$ and real valued functions $A^\epsilon, \mathcal{A}^\epsilon$ on $\mathcal{B}^\epsilon_1(\bar{r}, R)$, $B^\epsilon, \mathcal{B}^\epsilon$ on $\mathcal{B}^\epsilon_2(\bar{r}, R)$, and C^ϵ, Γ^ϵ on $\mathcal{B}^\epsilon_3(\bar{r}, R)$, find a solution to the system (3.56) or (3.57), respectively, with $\psi^\epsilon_\lambda(0) = \Psi^\epsilon_\lambda$, $a^\epsilon = A^\epsilon, \alpha^\epsilon = \mathcal{A}^\epsilon$ on $\mathcal{B}^\epsilon_1(\bar{r}, R)$, $b^\epsilon = B^\epsilon, \beta^\epsilon = \mathcal{B}^\epsilon$ on $\mathcal{B}^\epsilon_2(\bar{r}, R)$, and $c^\epsilon = C^\epsilon$, $\gamma^\epsilon = \Gamma^\epsilon$ on $\mathcal{B}^\epsilon_3(\bar{r}, R)$.*

Proposition 3.8 *The combination of the discrete hyperbolic systems (3.54) and (3.57) is compatible and the right-hand sides of the equations in (3.57) are $\mathcal{O}(\epsilon)$-convergent to the respective right-hand sides of (3.56) in $C^\infty(\mathcal{X})$.*

Proof: Compatibility of the discrete system is a direct consequence of the existence of the Bäcklund transform for discrete K-surfaces. The latter has been proven in [BP1]. There, more general matrices $\mathcal{W}_\lambda$ with polynomial dependence on λ are considered, corresponding to iterates of the transform. Below, an abridged argument tailored to the simpler situation at hand is presented.

Consider the three-dimensional cube $\mathcal{B}^\epsilon(\epsilon, 1)$, and let data $a^\epsilon, b^\epsilon, c^\epsilon$ and $\alpha^\epsilon, \beta^\epsilon, \gamma^\epsilon$ be given at the origin, or, equivalently, matrices $\mathcal{U}^\epsilon_\lambda$, $\mathcal{V}^\epsilon_\lambda$ and $\mathcal{W}^\epsilon_\lambda$. Compatibility is proven if one can show:

Claim: *To each point $\xi \in \mathcal{B}^\epsilon(\epsilon, 1)$ a frame $\psi(\xi) \in \mathcal{M}[\lambda]$ can be assigned, so that the matrices $\tau_1\psi_\lambda \cdot \psi_\lambda^{-1}$, $\tau_2\psi_\lambda \cdot \psi_\lambda^{-1}$ and $\tau_3\psi_\lambda \cdot \psi_\lambda^{-1}$ are of the forms (3.43), (3.44) and (3.50), respectively. And in particular, $\tau_1\psi_\lambda(0)\cdot(\psi_\lambda(0))^{-1} = \mathcal{U}^\epsilon_\lambda$, $\tau_2\psi_\lambda(0)\cdot(\psi_\lambda(0))^{-1} = \mathcal{V}^\epsilon_\lambda$ and $\tau_3\psi_\lambda(0) \cdot (\psi_\lambda(0))^{-1} = \mathcal{W}^\epsilon_\lambda$.*

The claim implies that the hyperbolic equations (3.54) and (3.57) are satisfied because of their equivalence to the discrete zero-curvature conditions of the Lax matrices.

We turn to verify the claim. By Propositions 3.6 and 3.7, there is a unique possibility to consistently assign frames $\psi^\epsilon_\lambda(\xi)$ to all $\xi \in \mathcal{B}^\epsilon(\epsilon, 1)$, except for $\xi^* = (\epsilon, \epsilon, 1)$. There are three different ways to calculate $\psi^\epsilon_\lambda(\xi^*)$ from the data. We choose to use system (3.54) to define the matrices $\mathcal{U}^\epsilon_\lambda(\xi)$, $\mathcal{V}^\epsilon_\lambda(\xi)$ at points ξ with $\xi_3 = 1$, and $\psi^\epsilon_\lambda(\xi^*)$ accordingly. The claim is obviously proven if $\psi^\epsilon_\lambda(\xi^*) = \mathcal{W}^*_\lambda \cdot \psi^\epsilon_\lambda(\epsilon, \epsilon, 0)$ with a matrix family $\mathcal{W}^*_\lambda$ of the form (3.50). The latter can be shown by closer inspection of the the following two representations of $\mathcal{W}^*_\lambda$:

$$\mathcal{W}^*_\lambda = \mathcal{V}^\epsilon_\lambda(\epsilon, 0, 1) \cdot \mathcal{W}^\epsilon_\lambda(\epsilon, 0, 0) \cdot (\mathcal{V}^\epsilon_\lambda(\epsilon, 0, 0))^{-1} \tag{3.58}$$

$$\mathcal{W}^*_\lambda = \mathcal{U}^\epsilon_\lambda(0, \epsilon, 1) \cdot \mathcal{W}^\epsilon_\lambda(0, \epsilon, 0) \cdot (\mathcal{U}^\epsilon_\lambda(0, \epsilon, 0))^{-1} \tag{3.59}$$

From these two equations, one learns that

$$\mathcal{W}_\lambda^* = (1+(c^\epsilon\lambda)^2)^{-1/2}(1+(\frac{\epsilon b^\epsilon}{2\lambda})^2)^{-1}\mathbf{M}_1(\lambda) \quad (3.60)$$

$$\mathcal{W}_\lambda^* = (1+(c^\epsilon\lambda)^2)^{-1/2}(1+(\frac{\epsilon a^\epsilon}{2}\lambda)^2)^{-1}\mathbf{M}_2(\lambda) \quad (3.61)$$

where $\mathbf{M}_1(\lambda)$ and $\mathbf{M}_2(\lambda)$ are matrix Laurent polynomials in λ, with algebraic singularities at most at $\lambda = 0$ and $\lambda = \infty$. Since the expression in (3.60) is singular at a point λ only when (3.61) is, and vice versa, the factors containing a^ϵ and b^ϵ necessarily cancel with a zero of $\mathbf{M}_1$ and $\mathbf{M}_2$, respectively. Thus,

$$\mathcal{W}_\lambda^\epsilon = (1+(c^\epsilon\lambda)^2)^{-1/2}\mathbf{M}_3(\lambda),$$

where $\mathbf{M}_3$ is meromorphic with poles at most at $\lambda = 0$ and $\lambda = \infty$. Investigation of the limiting behavior of the expressions in (3.58) as $\lambda \to \infty$, and in (3.59) as $\lambda \to 0$ show that $\mathcal{W}_\lambda^\epsilon$ is indeed of the form (3.50). ■

Theorem 3.4 *Assume $\{f_\lambda : \mathcal{B}(\bar{r}) \to \mathbb{R}^3\}$ is an associated family of of K-surfaces. For simplicity, let $\{f_\lambda\}$ be normalized so that $f_\lambda(0) = 0$, $\partial_1 f_\lambda(0) \parallel \mathbf{e}_1$ and $n_\lambda(0) = \mathbf{e}_3$. Further, let a point $F^+ \in \mathbb{R}^3$ be given, $F^+ \neq 0$, which lies in the plane orthogonal to $\mathbf{e}_3$.*

1. *$\{f_\lambda\}$ possesses a unique Bäcklund transform $\{f_\lambda^+ : \mathcal{B}(r) \to \mathbb{R}^3\}_\lambda$ with $f_1^+(0) = F^+$.*

2. *Let further for each $\epsilon > 0$ an associated family $\{f_\lambda^\epsilon\}$ of discrete K-surfaces be given. Assume that each f_λ^ϵ with $\lambda \in \mathcal{I}$ is smoothly lattice convergent to the corresponding f_λ. For simplicity, also assume the normalizations $f_\lambda^\epsilon(0) = 0$, $\delta_1 f_\lambda^\epsilon(0) \parallel \mathbf{e}_1$ and $n_\lambda^\epsilon(0) = \mathbf{e}_3$.*

   Then, for each $\epsilon > 0$ small enough, there is a Bäcklund transform $\{f_\lambda^{\epsilon+} : \mathcal{B}^\epsilon(r) \to \mathbb{R}^3\}_\lambda$ of $\{f_\lambda^\epsilon\}$, such that $f_1^{\epsilon+}(0) = F^+$. And for any fixed $\lambda \in \mathcal{I}$, the discrete K-surface $f_\lambda^{\epsilon+}$ is smoothly lattice-convergent to f_λ^+.

Proof: Let ψ_λ be the family of frames adapted to $\{f_\lambda\}$. Because of the normalizations, it is obvious that $\{f_\lambda\}$ can be thought of "generated by the Sym formula" from its adapted frame ψ_λ, which obeys $\psi_\lambda(0) = \mathbf{1}$. Choose $\Gamma = \angle(F^+ - F, \partial_1 f_1(0))$ and C so that $C/(1+C^2) = |F^+ - F|$. Defining $\mathcal{W}_\lambda$ according to (3.36) with $c = C$ and $\gamma = \Gamma$, it is easy to check that

$$F^+ - F = (\psi_1(0))^{-1}\left(W_\lambda^{-1}\partial_\lambda\mathcal{W}_\lambda\right)\psi_1(0). \quad (3.62)$$

This requirement also makes the choice of C and Γ unique. To prove the statement about smooth Bäcklund transformations, solve the Goursat problem 3.4 above with the help of Theorem 2.2 on $\mathcal{B}(r,1)$ for the data $A(\xi_1,0,0) = |\partial_1 f_1(\xi_1,0,0)|$, $B(0,\xi_2,0) = |\partial_2 f_1(0,\xi_2,0)|$ and $\mathcal{A}(\xi_1,0,0) = \partial_1\phi(\xi_1,0,0)$, $\mathcal{B}(0,\xi_2,0) = \phi(0,\xi_2,0)$

where $\phi = \arccos(\partial_1 f_1 \cdot \partial_2 f_1)$ is the angle between asymptotic lines on the f_λ, and C, Γ as above. Then use the Sym formula to construct the immersion of the associated family of the Bäcklund transform:

$$f_\lambda^+ := 2\kappa\lambda(\tau_3\psi_\lambda)^{-1} \cdot (\partial_\lambda \tau_3 \psi_\lambda).$$

In combination with equation (3.62) above, it follows that $f_1^+ = F^+$.

The proof of the statement about approximation is another direct application of Theorem 2.2: Read off the discrete Goursat data from the frames ψ_λ^ϵ adapted to the f_λ^ϵ (to make the discrete frames unique, choose $\psi_\lambda(0) = \mathbf{1}$), and solve the Goursat problems with the additional data $C^\epsilon = C$ and $\Gamma^\epsilon = \Gamma$. ∎

Chapter 4

Conjugate Nets

Definitions of continuous and discrete conjugate nets and their transformations are given. A Goursat problem is formulated, and convergence of discrete conjugate nets towards continuous ones are proven.

4.1 Definitions and Basic Properties

Let M, m be positive integers, $\boldsymbol{\epsilon} = (\epsilon_1, \ldots, \epsilon_M)$ a vector with positive entries.

Definition 4.1 *A map* $\mathrm{x} : \Omega \subset \mathbb{R}^m \to \mathbb{R}^N$ *is called an* m*-dimensional* conjugate net *in* $\mathbb{R}^N$*, if* $\partial_i\partial_j\mathrm{x} \in \mathrm{span}(\partial_i\mathrm{x}, \partial_j\mathrm{x})$ *at any point* $\xi \in \Omega$ *for all pairs* $1 \le i \ne j \le m$*, i.e. if there exist functions* $c_{ij}, c_{ji} : \Omega \to \mathbb{R}$ *such that*

$$\partial_i\partial_j\mathrm{x} = c_{ji}\partial_i\mathrm{x} + c_{ij}\partial_j\mathrm{x}\,. \tag{4.1}$$

Discrete conjugate nets were introduced by Doliwa and Santini, see [DS1].

Definition 4.2 *A map* $\mathrm{x} : \Omega^{\boldsymbol{\epsilon}} \subset \mathcal{B}^{\boldsymbol{\epsilon}} \to \mathbb{R}^N$ *is called an* M*-dimensional* discrete conjugate net *in* $\mathbb{R}^N$*, if the four points* x*,* $\tau_i\mathrm{x}$*,* $\tau_j\mathrm{x}$*, and* $\tau_i\tau_j\mathrm{x}$ *are coplanar at any* $\xi \in \Omega^{\boldsymbol{\epsilon}}$ *for all pairs* $1 \le i \ne j \le M$*, i.e., there exist functions* $c_{ij} : \Omega^{\boldsymbol{\epsilon}} \to \mathbb{R}$ *such that*

$$\delta_i\delta_j\mathrm{x} = c_{ji}\delta_i\mathrm{x} + c_{ij}\delta_j\mathrm{x}. \tag{4.2}$$

To improve readability of the following formulae, the super-index $\boldsymbol{\epsilon}$ for the discrete quantities will be suppressed whenever confusion is unlikely. We return to the usual notation in the formulation of the theorems. We investigate discrete conjugate nets first. And we assume that $\Omega^{\boldsymbol{\epsilon}} = \mathcal{B}^{\epsilon}(r, R)$ for a suitable choice of m, m', r and R. Introducing $M = m + m'$ new functions $w_i : \mathcal{B}^{\epsilon}(r, R) \to \mathbb{R}^N$, we can rewrite (4.2) as system of first order:

$$\begin{aligned}
\delta_i\mathrm{x} &= w_i\,, & (4.3)\\
\delta_i w_j &= c_{ji}w_i + c_{ij}w_j\,, \quad i \ne j, & (4.4)\\
\delta_i c_{kj} &= (\tau_j c_{ik})c_{kj} + (\tau_j c_{ki})c_{ij} - (\tau_i c_{kj})c_{ij}, \quad i \ne j \ne k \ne i. & (4.5)
\end{aligned}$$

For a given discrete conjugate net $\mathrm{x} : \mathcal{B}^\epsilon(r,R) \to \mathbb{R}^N$, equation (4.3) defines the functions w_j, then (4.4) reflects the property (4.2), and (4.5) is just a transcription of the compatibility condition $\delta_i(\delta_j w_k) = \delta_j(\delta_i w_k)$. Conversely, for any solution of (4.3)–(4.5) the map $\mathrm{x} : \mathcal{B}^\epsilon(r,R) \to \mathbb{R}^N$ has the defining property (4.2) and thus is a discrete conjugate net. So, discrete conjugate nets are in a one-to-one correspondence to solutions of the system (4.3)-(4.5).

This system almost suits the framework of the hyperbolic systems (2.4); the only obstruction is the implicit nature of the equations (4.5): their right–hand sides depend on the shifted variables like $\tau_i c_{kj}$ which is not allowed in (2.4). For the moment being, we handle the system (4.3)-(4.5) as if it would belong to the class (2.4). In the context of hyperbolic systems, we have to assign to every variable a Banach space and static/evolution directions. Abbreviating $w = (w_1, \ldots, w_M)$ and $c = (c_{ij})_{i \neq j}$, we set

$$\mathcal{X} = \mathbb{R}^N_{\mathrm{x}} \times (\mathbb{R}^N_w)^M \times \mathbb{R}^{M(M-1)}_c,$$

with the norm from (2.3). We assign to x no static directions, to w_i the only static direction i, and to c_{ij} two static directions i and j.

Proposition 4.1 *The system (4.3)-(4.5) is consistent in any dimension M.*

Proof: The geometric version of this Proposition is proven in [DS1]. We only establish the link between the Goursat problem an the geometric interpretation: Recall the definition of compatibility: One needs to show that the discrete Goursat problem is formally solvable on the elementary M-cube $\mathcal{B}^\epsilon(\epsilon, 1)$, i.e., either there exists a genuine solution, or the solution blows up on a subset of $\mathcal{B}^\epsilon(\epsilon, 1)$. Goursat data on the elementary M-cube are a vertex $\mathrm{x}(0) \in \mathbb{R}^N$, M vectors $w_i(0) \in \mathbb{R}^N$, and $M(M-1)$ real numbers $c_{ij}(0)$, with $i \neq j = 1, \ldots, M$.

First, calculate $\tau_i w_j(0)$ for $i \neq j$ with the help of (4.4). Next, construct the $\binom{M}{2}$ vertices $\tau_i \mathrm{x}(0)$ and $\tau_i\tau_j \mathrm{x}(0)$ by means of (4.3); the two expressions for $\tau_j\tau_i\mathrm{x}$ and $\tau_i\tau_j\mathrm{x}$ agree since

$$\epsilon_j w_j + \epsilon_i \tau_j w_i = \epsilon_j(1 + \epsilon_i c_{ij})w_j + \epsilon_i(1 + \epsilon_j c_{ji})w_i = \epsilon_i w_i + \epsilon_j \tau_i w_j.$$

Now, by [DS1], it is possible to build an M-dimensional polyhedron which has all the previously defined vertices as corners. I.e., one can define $\mathrm{x}(\xi) \in \mathbb{R}^N$ for all $\xi \in \mathcal{B}^\epsilon(r,R)$ such that the following is true: Any four points that are corners of a 2-face in the elementary M-cube are mapped to a common plane in $\mathbb{R}^N$ by x. Thus, the map $\mathrm{x} : \mathcal{B}^\epsilon(r,R) \to \mathbb{R}^N$ has the properties of a conjugate net. From x, one can then read off the quantities $w_i(\xi)$, and also $c_{ij}(\xi)$, provided there is no blow-up (the coefficient $c_{ij}(\xi)$ is well-defined only when the vectors $w_i(\xi)$ and $w_j(\xi)$ are linearly independent for $i \neq j$; see the discussion below). These read-off quantities necessarily fulfill (4.3)-(4.5). ■

We turn to the implicit nature of the system (4.3)–(4.5): Its geometric interpretation is the following. Consider the image of an elementary 3-cube under x^ϵ; from the seven vertices $\mathrm{x}^\epsilon(\xi)$, $\mathrm{x}^\epsilon(\xi + \epsilon_i \mathbf{e}_i)$, $\mathrm{x}^\epsilon(\xi + \epsilon_i \mathbf{e}_i + \epsilon_j \mathbf{e}_j)$, it is always possible to find an eighths one, x*, so that the vertices are the corners of an hexahedron with flat surfaces. However, it can happen that the resulting hexahedron is degenerate; for instance, if the new vertex x* coincides with a prescribed one. Typical situations are illustrated on Fig.4.1. While the example on the left of Fig. 4.1 is fine, the right one does not

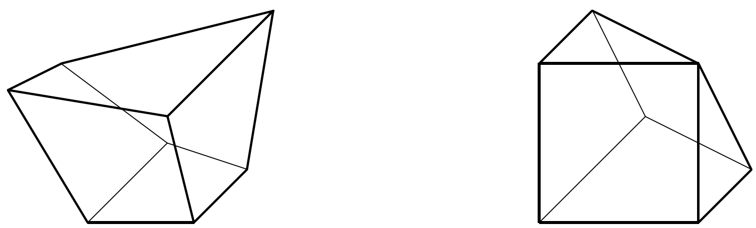

Figure 4.1: A non-degenerate and a degenerate hexahedron.

have the combinatorics of a 3-cube (the top-right edge has degenerated to a point). As a result, some of the quantities c_{kj} used to describe the geometric properties of the lattice are not well-defined; this is reflected by the fact that the equations (4.5) are implicit and there is no (or no unique) solution $\tau_i c_{kj}$ for certain initial values.

To investigate this point, rewrite (4.5) as

$$\delta_i c_{kj} = F_{(ijk)}(c) + \epsilon_j(\delta_j c_{ik})c_{kj} + \epsilon_j(\delta_j c_{ki})c_{ij} - \epsilon_i(\delta_i c_{kj})c_{ij}$$

Introducing vectors δc and $F(c)$ with $M(M-1)(M-2)$ components labeled by triples (ijk) of pairwise distinct numbers $1 \le i, j, k \le M$,

$$\begin{aligned} \delta c_{(ijk)} &= \delta_i c_{kj} \\ F_{(ijk)}(c) &= c_{ik}c_{kj} + c_{ki}c_{ij} - c_{kj}c_{ij}, \end{aligned}$$

we restate the above equations in the form

$$(\mathbf{1} - Q^\epsilon(c))\delta c = F(c), \tag{4.6}$$

with a suitable matrix $Q^\epsilon(c)$. We need to find conditions for $\mathbf{1}-Q^\epsilon(c)$ to be invertible.

4.2 Approximation of Conjugate Nets

We start with the case of m quasi-continuous and $m' = 0$ purely discrete directions. Since all directions are quasi-continuous, in equation (4.6), one learns from the form of equation (4.5) that $Q^\epsilon(c) = \epsilon A(c)$ with an ϵ-independent matrix $A(c)$. Considering Q^ϵ as a matrix-valued function on some compact subset $\mathbb{K} \subset \mathfrak{X}$, then for $\epsilon > 0$ small enough, the function $\mathbf{1} - Q^\epsilon$ is pointwise invertible and the function $(\mathbf{1} - Q^\epsilon(c))^{-1}$ of the inverse matrices is smooth on $\mathbb{K}$, and $O(\epsilon)$-convergent in $C^\infty(\mathbb{K})$ to $\mathbf{1}$. So, the system (4.6) is solvable, and

$$\delta_i c_{kj} = F_{(ijk)}(c) + O(\epsilon) = c_{ik}c_{kj} + c_{ki}c_{ij} - c_{kj}c_{ij} + O(\epsilon),$$

where the implicit constant in $O(\epsilon)$ depends on $\mathbb{K}$ only.

The limiting equations for (4.3)-(4.5) in this case are:

$$\begin{aligned} \partial_i \mathrm{x} &= w_i && (4.7)\\ \partial_i w_j &= c_{ji} w_i + c_{ij} w_j && (4.8)\\ \partial_i c_{kj} &= c_{ik} c_{kj} + c_{ki} c_{ij} - c_{kj} c_{ij}. && (4.9) \end{aligned}$$

For any solution $(\mathrm{x}, w, c) : \mathcal{B}_0(r) \to \mathfrak{X}$ of this system the function $\mathrm{x} : \mathcal{B}_0(r) \to \mathbb{R}^N$ is a conjugate net, because equations (4.7) and (4.8) yield the defining property (4.1). Conversely, if $\mathrm{x} : \mathcal{B}(r) \to \mathbb{R}^N$ is a conjugate net, then equation (4.7) defines the vectors w_i, these fulfill (4.8) since this is the defining property of a conjugate net, and (4.9) results from equating $\partial_i(\partial_j w_k) = \partial_j(\partial_i w_k)$.

Lemma 4.1 *The right-hand sides of the equations (4.3)-(4.5) are $\mathcal{O}(\epsilon)$-convergent in $C^\infty(\mathfrak{X})$ to the right-hand sides in (4.7)-(4.9), which describe continuous conjugate nets.*

We state the Goursat Problem for this situation:

Goursat Problem 4.1 (for conjugate nets) *Given a point $\mathrm{X}^\epsilon \in \mathbb{R}^N$, M functions $W_i^\epsilon : \mathcal{B}_i^\epsilon(r) \to \mathbb{R}^N$ on the coordinate axes, and $M(M-1)$ functions $C_{ij}^\epsilon : \mathcal{B}_{ij}^\epsilon(r) \to \mathbb{R}$ on the coordinate planes, find a solution $(\mathrm{x}^\epsilon, w^\epsilon, c^\epsilon) : B^\epsilon(r) \to \mathfrak{X}$ to the difference equations (4.3)-(4.5) or the differential equations (4.7)-(4.9) satisfying the initial conditions*

$$\mathrm{x}^\epsilon(0) = \mathrm{X}^\epsilon, \qquad w_i^\epsilon \restriction_{\mathcal{B}_i^\epsilon} = W_i^\epsilon, \qquad c_{ij}^\epsilon \restriction_{\mathcal{B}_{ij}^\epsilon} = C_{ij}^\epsilon. \tag{4.10}$$

We are now ready to formulate and prove the main results of this chapter.

Theorem 4.1 (approximation of a conjugate net) *Let there be given: m smooth curves $\mathrm{X}_i : \mathcal{B}_i(\bar{r}) \to \mathbb{R}^N$, $i = 1, \dots, m$, intersecting at a common point $\mathrm{X} = \mathrm{X}_1(0) = \cdots = \mathrm{X}_m(0)$; and for each pair $1 \le i < j \le m$, two smooth functions $C_{ij}, C_{ji} : \mathcal{B}_{ij}(\bar{r}) \to \mathbb{R}$. Then, for some positive $r < \bar{r}$:*

1. *There is a unique conjugate net $\mathrm{x} : \mathcal{B}(r) \to \mathbb{R}^N$ such that*

$$\mathrm{x}\restriction_{\mathcal{B}_i} = X_i\,, \quad c_{ij}\restriction_{\mathcal{B}_{ij}} = C_{ij}\,. \tag{4.11}$$

2. *The family of discrete conjugate nets $\{\mathrm{x}^\epsilon : \mathcal{B}^\epsilon(r) \to \mathbb{R}^N\}$ uniquely determined by requiring*

$$\mathrm{x}^\epsilon\restriction_{\mathcal{B}_i^\epsilon} = [X_i]^\epsilon\,, \quad c_{ij}^\epsilon\restriction_{\mathcal{B}_{ij}^\epsilon} = [C_{ij}]^\epsilon\,, \tag{4.12}$$

   where c_{ij}^ϵ are the respective coefficients of the net x^ϵ, lattice-converges in $C^\infty(B(r))$ to x.

Proof. The requirements (4.11) are immediately reformulated into data for the Goursat problem 4.1 at $\epsilon = 0$:

$$\mathrm{X} \text{ and } C_{ij} \text{ as given}, \quad W_i = \partial_i \mathrm{X}_i.$$

For (4.12), it is

$$\mathrm{X} \text{ as given}, \quad W_i^\epsilon = \delta_i[\mathrm{X}_i]^\epsilon, \quad C_{ij}^\epsilon = [C_{ij}]^\epsilon. \tag{4.13}$$

The smooth lattice-convergence of the discrete data is obvious, and the claim follows directly from Lemma 4.1 and Theorem 2.2. ∎

4.3 Jonas Transformations

In this and the following section, the talk is about *pairs* of – discrete and continuous – conjugate nets. Consequently, we switch from $m' = 0$ to $m' = 1$ purely discrete directions, and we frequently identify the two m-dimensional immersions $\mathrm{x}^{\epsilon-}, \mathrm{x}^{\epsilon+} : \mathcal{B}_0^\epsilon(r) \to \mathbb{R}^N$ with a single function x^ϵ defined on the $M = m+1$-dimensional domain $\mathcal{B}^\epsilon(r,1)$ by $\mathrm{x}^\epsilon(\cdot,0) = \mathrm{x}^{\epsilon-}(\cdot)$ and $\mathrm{x}^\epsilon(\cdot,1) = \mathrm{x}^{\epsilon+}(\cdot)$. We call x the *composite map.*

Definition 4.3 *Two m-dimensional* smooth *conjugate nets* $\mathrm{x}^-, \mathrm{x}^+ : \mathcal{B}_0(r) \to \mathbb{R}^N$ *are called a* Jonas pair*, if three vectors $\partial_i\mathrm{x}^-$, $\partial_i\mathrm{x}^+$ and $\delta_M\mathrm{x} = \mathrm{x}^+ - \mathrm{x}^-$ are coplanar at any point $\xi \in \mathcal{B}_0(r)$ for all $1 \le i \le m$, i.e. if there exist functions $c_{iM}, c_{Mi} : \mathcal{B}_0(r) \to \mathbb{R}$ such that*

$$\partial_i\mathrm{x}^+ - \partial_i\mathrm{x}^- = c_{Mi}\partial_i\mathrm{x}^- + c_{iM}(\mathrm{x}^+ - \mathrm{x}^-)\,. \tag{4.14}$$

Remarks.

- x^+ is also called a *Jonas transformation* of x^-. One can iterate these transformations and obtain sequences $(\mathrm{x}^{(0)}, \mathrm{x}^{(1)}, \ldots, \mathrm{x}^{(R)})$ of conjugate nets, i.e., each pair $\mathrm{x}^{(t)}$ and $\mathrm{x}^{(t+1)}$ of nets is in fact a Jonas pair. These sequences are canonically identified with a single composite map $\mathrm{x} : \mathcal{B}(r,R) \to \mathbb{R}^N$.

- A *Combescure transformation* x^+ of x^- is a Jonas transformation for which vectors $\partial_i\mathrm{x}^+$ and $\partial_i\mathrm{x}^-$ are parallel, $i = 1, \ldots, m$. This class of transformations is singled out by requiring $c_{iM} = 0$ in equation (4.14).

Definition 4.4 *A pair of m-dimensional* discrete *conjugate nets* $\mathrm{x}^-, \mathrm{x}^+ : \mathcal{B}_0^\epsilon(r) \to \mathbb{R}^N$ *is called a* Jonas pair*, if the composite map* $\mathrm{x} : \mathcal{B}^\epsilon(r,1) \to \mathbb{R}^N$ *is an $m+1$-dimensional discrete conjugate net according to Definition 4.1.*

By definition, the equations describing discrete Jonas pairs are identical to the system (4.3)-(4.5) describing a single conjugate net x in $m+1$ dimensions. The analytical difference is that in (4.6) the matrix $\mathbf{1} - Q^\epsilon(c)$ is no longer a small perturbation of the identity, since $\epsilon_M = 1$. More precisely, the matrix $Q^\epsilon(c)$ is block–diagonal, its $\binom{M}{3}$

diagonal blocks $Q^{\epsilon}_{\{ijk\}}(c)$ of size 6×6 correspond to non–ordered triples of pairwise distinct indices $\{i,j,k\}$, with rows and columns labeled by the six possible permutations of these indices. Blocks for which $i,j,k\neq M$ are of the form considered before – their entries are either zero or of the form ϵc_{ij}. Blocks where, say, $k=M$, admit a decomposition $Q^{\epsilon}_{\{ijM\}}(c)=A(c)+\epsilon B(c)$, where the matrices A and B do not depend on ϵ. It is not difficult to calculate that $\det(\mathbf{1}-A(c))=(1+c_{Mi})(1+c_{Mj})$. So $\mathbf{1}-Q^{\epsilon}_{\{ijM\}}(c)$ is invertible if $c_{Mi},c_{Mj}\neq -1$, provided ϵ is small enough. The inverse is smooth in a neighborhood any such point c, and $O(\epsilon)$-convergent to $(\mathbf{1}-A(c))^{-1}$. Hence, the natural choice for the domain $\mathbb{D}$ is

$$\mathbb{D}=\{(\mathrm{x},w,c)\in\mathcal{X}\mid c_{Mi}\neq -1\ \text{ for }\ 1\leq i\leq m\}.$$

As C^∞-limit of (4.3) and (4.4) on $\mathbb{D}$, we obtain ($i\neq j$):

$$\begin{aligned}
\partial_i\mathrm{x} &= w_i \qquad (1\leq i\leq m), && (4.15)\\
\delta_M\mathrm{x} &= w_M, && (4.16)\\
\partial_i w_j &= c_{ji}w_i+c_{ij}w_j \qquad (1\leq i\leq m), && (4.17)\\
\delta_M w_j &= c_{jM}w_M+c_{Mj}w_j. && (4.18)
\end{aligned}$$

There are also four different limits of equation (4.5), depending on which directions i and j are kept discrete. We shall not write them down; they are easily reconstructed as the compatibility conditions for (4.17), (4.18), like $\delta_M(\partial_i w_k)=\partial_i(\delta_M w_k)$ etc. Comparing (4.15)–(4.18) with the definition of the Jonas transformation (4.14), the following is demonstrated.

Lemma 4.2 *The equations describing a Jonas pair are $\mathcal{O}(\epsilon)$-convergent in $C^\infty(\mathbb{D})$ to a limiting system describes Jonas pairs of continuous conjugate nets.*

The Goursat Problem 4.1 formally carries over, with indices running from 1 to $m+1$, and after replacing the quasi-continuous domain $\mathcal{B}^\epsilon(r)$ by $\mathcal{B}^\epsilon(r,1)$; or even by $\mathcal{B}^\epsilon(r,R)$, when iterated Jonas transformations are considered, cf. the remark above.

4.4 Approximation of Jonas Transformations

Recall that we work in m quasi-continuous and $m'=1$ purely discrete dimensions.

Theorem 4.2 (approximation of a Jonas pair). *Let, in addition to the data listed in Theorem 4.1, be given: m smooth curves $\mathrm{X}_i^+:\mathcal{B}_i(\bar r)\to\mathbb{R}^N$, $i=1,\dots,m$, such that all X_i^+ intersect at a common point $\mathrm{X}^+=\mathrm{X}_1^+(0)=\ldots=\mathrm{X}_m^+(0)$, and such that for any $1\leq i\leq m$ and for any point $\xi\in\mathcal{B}_i(\bar r)$ the three vectors $\partial_i\mathrm{X}_i(\xi)$, $\partial_i\mathrm{X}_i^+(\xi)$ and $\mathrm{X}_i^+(\xi)-\mathrm{X}_i(\xi)$ are coplanar.*

Assume that $\mathrm{X}^+-\mathrm{X}$ is not parallel to any of the m vectors $\partial_i\mathrm{X}_i^+(0)$. Define the functions $C_{Mi},C_{iM}:\mathcal{B}_i\to\mathbb{R}$ by the formula

$$\partial_i X_i^+-\partial_i X_i=C_{Mi}\partial_i X_i+C_{iM}(X_i^+-X_i). \tag{4.19}$$

Then, for some positive $r<\bar r$:

1. *In addition to the conjugate net* $\mathrm{x}^- : \mathcal{B}_0(r) \to \mathbb{R}^N$ *which is the smooth solution* x *from Theorem 4.1, there is its unique Jonas transformation* $\mathrm{x}^+ : \mathcal{B}_0(r) \to \mathbb{R}^N$ *such that*
$$\mathrm{x}^+\restriction_{\mathcal{B}_i} = \mathrm{X}_i^+ \quad (1 \le i \le m); \tag{4.20}$$

2. *The* ϵ*-family* $\{\mathrm{x}^\epsilon : \mathcal{B}^\epsilon(r,1) \to \mathbb{R}^N\}$ *of discrete* $m+1$*-dimensional conjugate nets, uniquely determined by*
$$\mathrm{x}^\epsilon\restriction_{\mathcal{B}_i^\epsilon} = [\mathrm{X}_i]^\epsilon, \quad c_{ij}^\epsilon\restriction_{\mathcal{B}_{ij}^\epsilon} = [C_{ij}]^\epsilon, \quad (1 \le i \ne j \le m) \tag{4.21}$$
and
$$\mathrm{x}^{\epsilon+}(0) = \mathrm{X}^+, \quad c_{Mi}^\epsilon\restriction_{\mathcal{B}_i^\epsilon} = [C_{Mi}]^\epsilon, \quad c_{iM}^\epsilon\restriction_{\mathcal{B}_i^\epsilon} = [C_{iM}]^\epsilon \quad (1 \le i \le m) \tag{4.22}$$
lattice-converges in $C^\infty(\mathcal{B}(r))$ *to the continuous Jonas pair above, in the sense that* x^ϵ *converges to the composed map* x.

Proof. Consider the Goursat problem 4.1 for M-dimensional discrete conjugate nets (corresponding to the composed map) with the initial data
$$\mathrm{X}^\epsilon = \mathrm{X}\,, \quad W_i^\epsilon = \delta_i[\mathrm{X}_i]^\epsilon, \quad W_M^\epsilon(0) = \mathrm{X}^+ - \mathrm{X}, \quad (i = 1, \ldots, m) \tag{4.23}$$
and
$$C_{ij}^\epsilon = [C_{ij}]^\epsilon, \qquad (1 \le i \ne j \le M). \tag{4.24}$$
The statement of the theorem follows by applying Theorem 2.2 to this situation. This, in turn, is possible due to Lemma 4.2 and the following observation. The condition of the theorem yields that $\mathrm{X}_i^+(\xi) - \mathrm{X}_i(\xi)$ is not parallel to any of $\partial_i \mathrm{X}_i^+(\xi)$ not only at $\xi = 0$, but also in some neighborhoods of zero on the corresponding axes $\mathcal{B}_i$. Now one deduces from the definition (4.19) of the rotation coefficients C_{iM}, C_{Mi} that in the same neighborhoods $C_{Mi} \ne -1$. Hence, the data of our Goursat problem belong to the set $\mathbb{D}$. ∎

We close with the following remark: The cases $m' > 1$ with more than one parameter for the Jonas transformations are handled in an analogous manner; formally, the Goursat problem does not change, and because of Lemma 4.1, the discrete problems are locally well-posed for any choice of m and m'. Applying the more general Theorem 2.1 to the situations $m' = 2$ and $m' = 3$, one arrives at the permutability properties of the Jonas transformations:

Theorem 4.3 permutability of Jonas transformations

- $\mathbf{m' = 2}$ *Given an* m*-dimensional conjugate net* $\mathrm{x}(\cdot;0,0) : \mathcal{B}_0(r) \to \mathbb{R}^N$ *and its two Jonas transformations* $\mathrm{x}(\cdot;1,0), \mathrm{x}(\cdot;0,1) : \mathcal{B}_0(r) \to \mathbb{R}^N$*, there exists a two-parameter family of conjugate nets* $\mathrm{x}(\cdot;1,1) : \mathcal{B}_0(r) \to \mathbb{R}^N$ *that are Jonas transformations of both* $\mathrm{x}(\cdot;1,0)$ *and* $\mathrm{x}(\cdot;0,1)$*. Corresponding points of the four conjugate nets are coplanar.*

- $\mathbf{m'=3}$ *Given three Jonas transformations*

$$\mathrm{x}(\cdot;1,0,0),\mathrm{x}(\cdot;0,1,0),\mathrm{x}(\cdot;0,0,1):\mathcal{B}_0(r)\to\mathbb{R}^N$$

of a given m–dimensional conjugate net $x(\cdot;0,0,0)$, as well as three further conjugate nets

$$\mathrm{x}(\cdot;1,1,0),\mathrm{x}(\cdot;0,1,1),\mathrm{x}(\cdot;1,0,1):\mathcal{B}_0(r)\to\mathbb{R}^N$$

such that $\mathrm{x}(\cdot;1,1,0)$ *is a Jonas transformation of both* $\mathrm{x}(\cdot;1,0,0)$ *and* $\mathrm{x}(\cdot;0,1,0)$ *etc., there exists generically a unique conjugate net*

$$\mathrm{x}(\cdot;1,1,1):\mathcal{B}_0(r)\to\mathbb{R}^N$$

which is a Jonas transformation of all three $\mathrm{x}(\cdot;1,1,0),\mathrm{x}(\cdot;0,1,1)$ *and* $\mathrm{x}(\cdot;1,0,1)$.

These results are proved in the same spirit as Theorem 4.2: One translates the prescribed nets into Goursat data at $\epsilon=0$, finds the unique solution of the continuous hyperbolic system with the help of Theorem 2.1, and then interpretes the solution geometrically. The two parameters in 1. correspond to the freedom of choosing $\mathrm{x}(0,1,1)$ in the 2-plane through the points $\mathrm{x}(0,0,0)$, $\mathrm{x}(0,1,0)$ and $\mathrm{x}(0,0,1)$; the conjugate net $\mathrm{x}(\cdot,1,1)$ exists and is unique for a generic choice of this point.

Chapter 5

Orthogonal Systems

Smooth and discrete orthogonal systems form subclasses of smooth and discrete conjugate nets. Smooth orthogonal systems are subject to the additional condition that coordinate lines intersect orthogonally. The corresponding reduction to obtain discrete orthogonal systems was first formulated by Bobenko [B2] in the three-dimensional case, then generalized to arbitrary dimensions by Doliwa and Santini [DS2]. Discrete orthogonal systems are discrete conjugate nets in which elementary quadrilaterals are circular.

This reduction comes rather natural in the context of Möbius geometry, which will play a prominent role in this chapter. Since the class of smooth orthogonal systems is invariant under the Moebius group, the same is supposed to be true for a reasonable definition of discrete orthogonal systems. And as the invariant objects of Möbius geometry are spheres of any dimension, the above definition is sensible.

A Goursat problem is formulated for orthogonal systems, and the approximation of smooth orthogonal systems by discrete ones is proven. Also, Ribaucour transformations are considered, and a result about simultaneous approximation of an orthogonal system and its transform is obtained.

The crucial step in the derivation of hyperbolic equations for discrete orthogonal systems is to pass from the classical, Euclidean picture to a Möbius description. There, adapted frame can be defined, and the hyperbolic difference equations follow from the respective moving-frame equations.

5.1 Properties of Orthogonal Systems

5.1.1 Definitions and Classical Description

Let $m, M \geq 1$ be integers, and $\boldsymbol{\epsilon} = (\epsilon_1, \ldots, \epsilon_M)$ of vector with positive entries.

Definition 5.1 *A conjugate net* $x : \Omega \subset \mathbb{R}^m \to \mathbb{R}^N$ *is called an m-dimensional* orthogonal system *in* $\mathbb{R}^N$*, if*

$$\partial_i x \cdot \partial_j x = 0. \tag{5.1}$$

If $m = N$*, then* x *is also called* orthogonal coordinate system.

Definition 5.2 *A map* $x : \Omega^\epsilon \subset \mathcal{B}^\epsilon \to \mathbb{R}^N$ *is called an M-dimensional* discrete orthogonal system *in* $\mathbb{R}^N$*, if the four points* x, τ_ix, τ_jx *and* $\tau_i\tau_j$x *are concircular for all* $1 \le j \ne j \le M$*, everywhere on* Ω^ϵ.

As in the previous chapter, the super-index $\boldsymbol{\epsilon}$ will be suppressed whenever possible to increase readability of the formulae.

Remark: Two-dimensional orthogonal systems, smooth or discrete, are called *C-surfaces*. The following fact is easily verified: If a surface is parameterized by x so that x is a conjugate net, $\partial_1\partial_2 x \in span\{\partial_1 x, \partial_2 x\}$, and orthogonality $\partial_1 x \cdot \partial_2 x = 0$ holds, then the coordinate curves of x are curvature lines of the surface. Hence, smooth C-surfaces in $\mathbb{R}^3$ are surfaces in curvature-line parameterization, and the "concircular nets" coming from discrete C-surfaces are regarded as their discrete analogue.

The classical description of continuous orthogonal systems is this (in the following equations it is assumed that $i \ne j \ne k \ne i$):

$$\begin{aligned}
\partial_i x &= h_i v_i\,, && (5.2)\\
\partial_i v_j &= \beta_{ji} v_i\,, && (5.3)\\
\partial_i h_j &= h_i \beta_{ij}\,, && (5.4)\\
\partial_i \beta_{kj} &= \beta_{ki}\beta_{ij}\,, && (5.5)\\
\partial_i \beta_{ij} + \partial_j \beta_{ji} &= -\partial_i v_i \cdot \partial_j v_j\,. && (5.6)
\end{aligned}$$

Equations (5.2) defines a system $\{v_i\}_{1\le i\le m}$ of m orthonormal vectors at each point $\xi \in \mathcal{B}_0(r)$; the quantities $h_i = |\partial_i x|$ are the respective metric coefficients. Conjugacy of the net x and normalization of v_i imply that (5.3) holds with some real–valued functions β_{ij}. Equation (5.4) expresses the consistency condition $\partial_i(\partial_j x) = \partial_j(\partial_i x)$ for $i \ne j$. Analogously, (5.5) expresses the consistency condition $\partial_i(\partial_j v_k) = \partial_j(\partial_i v_k)$ for $i \ne j \ne k \ne i$. The m equations (5.4) and $\binom{m}{3}$ equations (5.5) are called *Darboux system*; they constitute a description of conjugate nets alternative to the system (4.7)–(4.9) used before. The orthogonality constraint is expressed by $\binom{m}{2}$ additional equations (5.6), derived from the identity $\partial_i\partial_j\langle v_i, v_j\rangle = 0$. In the case $m = N$ the scalar product on the right-hand side of (5.6) can be expressed in terms of rotation coefficients β_{ij} only:

$$\partial_i\beta_{ij} + \partial_j\beta_{ji} + \sum_{k\ne i,j} \beta_{ki}\beta_{kj} = 0. \tag{5.7}$$

The Darboux system (5.4), (5.5) together with equations (5.7) forms the *Lamé system*.

The classical approach as presented is based on the Euclidean geometry. However, the invariance group of orthogonal systems is the Möbius group, which acts on the compactification $\mathbb{R}^N \cup \{\infty\} \approx S^N$, rather than on $\mathbb{R}^N$. This is a motivation to consider orthogonal systems in the sphere S^N and to study their Möbius-invariant description. In particular, this enables one to give a frame description of orthogonal systems which can be generalized to the discrete context in a straightforward manner. This turns out to be the key to derivation of the discrete analogue of the Lamé system.

5.1.2 A Sketch of Möbius Geometry

Below, a brief presentation of Möbius geometry is given, tailored to the current needs. A more profound introduction may be found in [He].

The N–dimensional Möbius geometry is associated with the unit sphere $S^N \subset \mathbb{R}^{N+1}$. Fix two antipodal points on S^N, $p_0 = (0, \ldots, 0, 1)$ and $p_\infty = -p_0$. The standard stereographic projection σ from the point p_∞ is a conformal bijection from $S^N_* = S^N \setminus \{p_\infty\}$ to $\mathbb{R}^N$ that maps spheres (of any dimension) in S^N to spheres or affine subspaces in $\mathbb{R}^N$. Its inverse map

$$\sigma^{-1}(x) = \frac{2}{1+|x|^2}\, x + \frac{1-|x|^2}{1+|x|^2}\, p_0$$

lifts an orthogonal system in $\mathbb{R}^N$, continuous or discrete, to an orthogonal system in S^N.

The group $\mathcal{M}(N)$ of N-dimensional Möbius transformations consists of those bijective maps $\mu : S^N \to S^N$ that map any sphere in S^N to a sphere of the same dimension; it then follows that μ is also conformal.

Fact 1 *The notion of orthogonal system on S^N is invariant under Möbius transformations.*

Möbius transformations can be embedded into a matrix group, namely the group of pseudo-orthogonal transformations on the Minkowski space $\mathbb{R}^{N+1,1}$, i.e. the $(N+2)$-dimensional space spanned by $\{\mathbf{e}_1, \ldots, \mathbf{e}_{N+2}\}$ and equipped with the Lorentz scalar product

$$\Big\langle \sum_{i=1}^{N+2} u_i \mathbf{e}_i, \sum_{j=1}^{N+2} v_j \mathbf{e}_j \Big\rangle = \sum_{k=1}^{N+1} u_k v_k - u_{N+2} v_{N+2}.$$

To do this, a model is used where points of S^N are identified with lines on the *light cone*

$$\mathcal{L}^{N+1} = \big\{ u \in \mathbb{R}^{N+1,1} \mid \langle u, u \rangle = 0 \big\} \subset \mathbb{R}^{N+1,1}.$$

This identification is achieved via the projection map

$$\begin{aligned} \pi : \mathbb{R}^{N+1,1} \setminus (\mathbb{R}^{N+1} \times \{0\}) \quad &\to \quad \mathbb{R}^{N+1}, \\ (u_1, \ldots, u_{N+1}, u_{N+2}) \quad &\mapsto \quad (u_1/u_{N+2}\,, \ldots, u_{N+1}/u_{N+2}), \end{aligned}$$

which sends lines on $\mathcal{L}^{N+1}$ to points of S^N. In particular, the lines through

$$\mathbf{e}_0 = \frac{1}{2}(\mathbf{e}_{N+2} + \mathbf{e}_{N+1}) \quad \text{and} \quad \mathbf{e}_\infty = \frac{1}{2}(\mathbf{e}_{N+2} - \mathbf{e}_{N+1})$$

are mapped to the points p_0 and p_∞, respectively.

The group $O^+(N+1,1)$ of genuine Lorentz transformations consists of pseudo-orthogonal linear maps L which preserve "the direction of time":

$$\langle L(u), L(v)\rangle = \langle u, v\rangle \quad \forall u, v \in \mathbb{R}^{N+1,1}, \qquad \langle L(\mathbf{e}_{N+2}), \mathbf{e}_{N+2}\rangle < 0. \tag{5.8}$$

For $L \in O^+(N+1,1)$, the projection π induces a Möbius transformation of S^N via $\mu(L) = \pi \circ L \circ \pi^{-1}$. Indeed, L is linear and preserves the light cone because of (5.8), so $\mu(L)$ maps S^N to itself. Further, L maps linear planes to linear planes; under π, linear planes in $\mathbb{R}^{N+1,1}$ project to affine planes in $\mathbb{R}^{N+1}$. Since any sphere in S^N is uniquely represented as the intersection of an affine plane in $\mathbb{R}^{N+1}$ with $S^N \subset \mathbb{R}^{N+1}$, we conclude that spheres of any dimension are mapped by $\mu(L)$ to spheres of the same dimension, which is the defining property of Möbius transformations. It can be shown that the correspondence $L \mapsto \mu(L)$ between $L \in O^+(N+1,1)$ and $\mathcal{M}(N)$ is indeed one-to-one.

Recall that we are interested in orthogonal systems in $\mathbb{R}^N$; their lifts to S^N actually do not contain p_∞. Therefore, we focus on those Möbius transformations which preserve p_∞, thus corresponding to Euclidean motions and homoteties in $\mathbb{R}^N = \sigma(S^N_*)$. For these, we can restrict our attention to the section

$$\mathcal{K} = \Big\{u \in \mathbb{R}^{N+1,1} : \langle u, u\rangle = 0,\, \langle u, \mathbf{e}_\infty\rangle = -1/2\Big\} = \mathcal{L}^{N+1} \cap (\mathbf{e}_0 + \mathbf{e}_\infty^{\perp})$$

of the light cone. (The notation $u^{\perp}$ stands for the hyperplane orthogonal to u.) The canonical lift

$$\begin{aligned} \lambda : \mathbb{R}^N &\to \mathcal{K}, \\ \mathrm{x} &\mapsto \mathrm{x} + \mathbf{e}_0 + |\mathrm{x}|^2 \mathbf{e}_\infty, \end{aligned}$$

factors the (inverse) stereographic projection, $\sigma^{-1} = \pi \circ \lambda$, and is an isometry in the following sense: for any four points $\mathrm{x}_1, \ldots, \mathrm{x}_4 \in \mathbb{R}^N$ one has:

$$\langle \lambda(\mathrm{x}_1) - \lambda(\mathrm{x}_2), \lambda(\mathrm{x}_3) - \lambda(\mathrm{x}_4)\rangle = (\mathrm{x}_1 - \mathrm{x}_2) \cdot (\mathrm{x}_3 - \mathrm{x}_4). \tag{5.9}$$

We summarize these results in a diagram

$$\begin{array}{ccc} \mathcal{K} & \hookrightarrow & \mathcal{L}^{N+1} \subset \mathbb{R}^{N+1,1} \\ {\scriptstyle\lambda}\uparrow & & \downarrow{\scriptstyle\pi} \\ \mathbb{R}^N & \xrightarrow{\sigma^{-1}} & S^N \subset \mathbb{R}^{N+1} \end{array}$$

Denote by $O^+_\infty(N+1,1)$ the subgroup of Lorentz transformations fixing the vector $\mathbf{e}_\infty$ (and consequently, preserving $\mathcal{K}$). The same arguments as before allow one to identify this subgroup with the group $E(N)$ of Euclidean motions of $\mathbb{R}^N$.

Fact 2 *Möbius transformations of S^N are in a one-to-one correspondence with the genuine Lorentz transformations of $\mathbb{R}^{N+1,1}$. Euclidean motions of $\mathbb{R}^N$ are in a one-to-one correspondence with the Lorentz transformations that fix $\mathbf{e}_\infty$.*

A useful technical device in this context Clifford algebras which are very convenient to describe Lorentz transformations in $\mathbb{R}^{N+1,1}$, and hence Möbius transformations in S^N. Recall that the Clifford algebra $\mathcal{C}(N+1,1)$ is an algebra over $\mathbb{R}$ with generators $\mathbf{e}_1, \dots, \mathbf{e}_{N+2} \in \mathbb{R}^{N+1,1}$ subject to the relations

$$uv + vu = -2\langle u, v\rangle \mathbf{1} = -2\langle u, v\rangle \qquad \forall u, v \in \mathbb{R}^{N+1,1}. \tag{5.10}$$

Relation (5.10) implies that $u^2 = -\langle u, u\rangle$, so any vector $u \in \mathbb{R}^{N+1,1} \setminus \mathcal{L}^{N+1}$ has an inverse $u^{-1} = -u/\langle u, u\rangle$. The multiplicative group generated by the invertible vectors is called the *Clifford group*. It contains the subgroup $Pin(N+1,1)$ which is a universal cover of the group of Möbius transformations. We shall need the *genuine Pin group*:

$$\mathcal{H} = Pin^+(N+1,1) = \{u_1 \cdots u_n \mid u_i^2 = -1\},$$

and its subgroup generated by vectors orthogonal to $\mathbf{e}_\infty$:

$$\mathcal{H}_\infty = \{u_1 \cdots u_n \mid u_i^2 = -1,\ \langle u_i, \mathbf{e}_\infty\rangle = 0\} \subset \mathcal{H}.$$

$\mathcal{H}$ and $\mathcal{H}_\infty$ are Lie groups with Lie algebras

$$\mathfrak{h} \;=\; spin(N+1,1) \;=\; \mathrm{span}\Big\{\mathbf{e}_i\mathbf{e}_j \,:\, i,j \in \{0,1,\dots,N,\infty\},\ i \neq j\Big\}, \tag{5.11}$$

$$\mathfrak{h}_\infty \;=\; spin_\infty(N+1,1) \;=\; \mathrm{span}\Big\{\mathbf{e}_i\mathbf{e}_j \,:\, i,j \in \{1,\dots,N,\infty\},\ i \neq j\Big\}. \tag{5.12}$$

In fact, $\mathcal{H}$ and $\mathcal{H}_\infty$ are universal covers of the previously defined Lorentz subgroups $O^+(N+1,1)$ and $O^+_\infty(N+1,1)$, respectively. The covers are double, since the the lift $\psi(L)$ of a Lorentz transformation L is defined up to a sign. To show this, consider the co-adjoint action of $\psi \in \mathcal{H}$ on $v \in \mathbb{R}^{N+1,1}$:

$$A_\psi(v) = \psi^{-1} v \psi. \tag{5.13}$$

Obviously, for a vector u with $u^2 = -1$ one has:

$$A_u(v) = u^{-1} v u = 2\langle u, v\rangle u - v. \tag{5.14}$$

Thus, A_u is, up to sign, the reflection in the (Minkowski) hyperplane $u^\perp$ orthogonal to u; these reflections generate the genuine Lorentz group. The induced Möbius transformation on S^N (for which the minus sign is irrelevant) is the inversion of S^N in the hypersphere $\pi(u^\perp \cap \mathcal{L}^{N+1}) \subset S^N$; inversions in hyperspheres generate the Möbius group. In particular, if u is orthogonal to $\mathbf{e}_\infty$, then A_u fixes $\mathbf{e}_\infty$, hence leaves $\mathcal{K}$ invariant and induces a Euclidean motion on $\mathbb{R}^N$, namely the reflection in an affine hyperplane; such reflections generate the Euclidean group.

Fact 3 *Möbius transformations of S^N are in a one-to-one correspondence to elements of the group $\mathcal{H}/\{\pm 1\}$; Euclidean transformations of $\mathbb{R}^N$ are in a one-to-one correspondence to elements of $\mathcal{H}_\infty/\{\pm 1\}$.*

5.1.3 Orthogonal Systems in Möbius geometry

For an orthogonal system x, continuous or discrete, consider its lift to the conic section $\mathcal{K}$:

$$\hat{\mathrm{x}} = \lambda \circ \mathrm{x} : \Omega^{\epsilon} \to \mathcal{K}. \tag{5.15}$$

Functions $\psi : \Omega^{\epsilon} \to \mathcal{H}_\infty$ will be used as frames to describe $\hat{\mathrm{x}}$ in an Möbius geometric (or, rather: Clifford-algebraic) manner: For a frame function adapted to $\hat{\mathrm{x}}$ in a sense to be defined, the zero-curvature condition, i.e. $\delta_i(\delta_j\psi) = \delta_j(\delta_i\psi)$, yields a system of hyperbolic differential and difference equations, respectively. The solutions to this system are in one-to-one correspondence with (lifted) orthogonal systems $\hat{\mathrm{x}}$. In the continuous case, we end up with the classical equations (5.2)-(5.6). The equations presented for discrete orthogonal systems seem to be new.

Continuous orthogonal systems

In the following, we work in $m > 1$ dimensions, all of which are continuous, $m' = 0$. For simplicity, assume $\Omega = \mathcal{B}(r)$ for some $r > 0$.

Define $\hat{v}_i : \mathcal{B}(r) \to T\mathcal{K}$ for $i = 1, \dots, m$ by

$$\partial_i \hat{\mathrm{x}} = h_i \hat{v}_i, \qquad h_i = |\partial_i \mathrm{x}|. \tag{5.16}$$

The vectors $\hat{v}_i$ are pairwise orthogonal, orthogonal to $\mathbf{e}_\infty$, and can be written as

$$\hat{v}_i = v_i + 2\langle x, v_i\rangle \mathbf{e}_\infty \tag{5.17}$$

with the vector fields v_i from (5.2). As an immediate consequence, these vectors satisfy

$$\partial_i \hat{v}_j = \beta_{ji}\hat{v}_i\,, \quad 1 \le i \ne j \le m\,, \tag{5.18}$$

with the same rotation coefficients β_{ji} as in (5.3).

Definition 5.3 *Given a point $\hat{\mathrm{x}} \in \mathcal{K}$ and vectors $\hat{v}_i \in T_{\hat{\mathrm{x}}}\mathcal{K}$, $1 \le i \le m$, we call an element $\psi \in \mathcal{H}_\infty$* suited *to $(\hat{\mathrm{x}}; \hat{v}_1, \dots, \hat{v}_m)$ if*

$$A_\psi(\mathbf{e}_0) = \hat{\mathrm{x}}, \tag{5.19}$$

$$A_\psi(\mathbf{e}_k) = \hat{v}_k \quad (1 \le k \le m), \tag{5.20}$$

A frame $\psi : \mathcal{B}(r) \to \mathcal{H}_\infty$ is called adapted *to the orthogonal system $\hat{\mathrm{x}}$, if it is suited to $(\hat{\mathrm{x}}; \hat{v}_1, \dots, \hat{v}_m)$ at any point $\xi \in B_0(r)$, and*

$$\partial_i A_\psi(\mathbf{e}_k) = \beta_{ki}\hat{v}_i \quad (1 \le k \le N,\ \ 1 \le i \le m). \tag{5.21}$$

with some scalar functions β_{ki}.

It is an immediate observation that the vectors $A_\psi(\mathbf{e}_i)$ for $i = 1, \dots, N$ form an orthogonal basis in $T\mathcal{K}$ at the point $A_\psi(\mathbf{e}_0) \in \mathcal{K}$. Hence, a frame suited to $(\hat{\mathrm{x}}; \hat{v}_1, \dots, \hat{v}_m)$ exists iff the the vectors $\hat{v}_i$ are pairwise orthogonal elements of $T_{\hat{\mathrm{x}}}\mathcal{K}$.

Proposition 5.1 *For a given continuous orthogonal system $\hat{x} : \mathcal{B}(r) \to \mathcal{K}$, an adapted frame ψ always exists. It is unique, if $m = N$. If $m < N$, then an adapted frame is uniquely determined by its value at one point, say $\xi = 0$. An adapted frame ψ satisfies the following differential equations.*

- *Static frame equations:*

$$\partial_i \psi = -\mathbf{e}_i \psi s_i \quad (1 \le i \le m) \tag{5.22}$$

  with $s_i = (1/2)\partial_i \hat{v}_i$.

- *Moving frame equations*

$$\partial_i \psi = -S_i \mathbf{e}_i \psi \quad (1 \le i \le m) \tag{5.23}$$

  with

$$S_i = A^{-1}_{\mathbf{e}_i\psi}(s_i) = \frac{1}{2}\sum_{k \neq i} \beta_{ki}\mathbf{e}_k - h_i \mathbf{e}_\infty, \tag{5.24}$$

  where the functions β_{ij} and h_i constitute a solution to the Lamé system (in the following equations the indices $1 \le i, j \le m$ and $1 \le k \le N$ are pairwise distinct):

$$\begin{aligned} \partial_i h_j &= h_i \beta_{ij}\,, & (5.25)\\ \partial_i \beta_{kj} &= \beta_{ki}\beta_{ij}\,, & (5.26)\\ \partial_i \beta_{ij} + \partial_j \beta_{ji} &= -\sum_{k \neq i,j} \beta_{ki}\beta_{kj}\,. & (5.27) \end{aligned}$$

Conversely, if β_{ij} and h_i are solutions to the equations (5.25)-(5.27), then the moving frame equations (5.23) are compatible, and any solution is a frame adapted to an orthogonal system.

Proof. For an orthogonal coordinate system ($m = N$) the adapted frame is uniquely determined at any point by the requirements (5.19) and (5.20), while eqs. (5.21) follow from (5.20). If $m < N$, extend the m vectors $\hat{v}_i(0)$, $1 \le i \le m$, by $N - m$ vectors $\hat{v}_k(0)$, $m < k \le N$, to a orthonormal basis of $T_{\hat{\mathrm{x}}(0)}\mathcal{K}$. There exist unique vector fields $\hat{v}_k : \mathcal{B}(r) \to T\mathcal{K}$, $m < k \le N$ with these prescribed values at $\xi = 0$, normalized ($\hat{v}_k^2 = -1$), orthogonal to each other, to $\hat{v}_i$, $1 \le i \le m$, to $\hat{\mathrm{x}}$ and to $\mathbf{e}_\infty$, that satisfy the differential equations

$$\partial_i \hat{v}_k = \beta_{ki}\hat{v}_i\,, \quad 1 \le i \le m, \quad m < k \le N \tag{5.28}$$

with some scalar functions β_{ki}. Indeed, orthogonality conditions yield that necessarily $\beta_{ki} = -\langle \partial_i \hat{v}_i, \hat{v}_k\rangle$. The equations (5.28) with these expressions for β_{ki} form a well-defined linear system of first order partial differential equations for $\hat{v}_k$. The compatibility condition for this system, $\partial_j(\partial_i \hat{v}_k) = \partial_j(\partial_i \hat{v}_k)$ $(1 \le i \ne j \le m)$, is easily verified using (5.18). The unique frame $\psi : \mathcal{B}(r) \to \mathcal{H}_\infty$ with $A_\psi(\mathbf{e}_0) = \hat{x}$, $A_\psi(\mathbf{e}_k) = \hat{v}_k$ $(1 \leq k \leq N)$ is adapted to $\hat{x}$.

Next, we prove the statements about the moving frame equation. The equations

$$\partial_i A_\psi(\mathbf{e}_0) = h_i A_\psi(\mathbf{e}_i), \qquad \partial_i A_\psi(\mathbf{e}_k) = \beta_{ki} A_\psi(\mathbf{e}_i)$$

are equivalent to

$$[\mathbf{e}_0, (\partial_i\psi)\psi^{-1}] = h_i \mathbf{e}_i\,, \qquad [\mathbf{e}_k, (\partial_i\psi)\psi^{-1}] = \beta_{ki}\mathbf{e}_i\,.$$

Since, by (5.12), $(\partial_i\psi)^{-1}\psi$ is spanned by bi-vectors, we come to the representation of this element in the form $(\partial_i\psi)^{-1}\psi = -S_i\mathbf{e}_i$ with S_i as in (5.24). The compatibility conditions for the moving frame equations, $\partial_i(\partial_j\psi) = \partial_j(\partial_i\psi)$, are equivalent to

$$\partial_j S_i \mathbf{e}_j + \partial_i S_j \mathbf{e}_i + S_i(\mathbf{e}_i S_j \mathbf{e}_i) + S_j(\mathbf{e}_j S_i \mathbf{e}_j) = 0,$$

which, in turn, are equivalent to the system (5.25)-(5.27). Conversely, for a solution $\psi : \mathcal{B}(r) \to \mathcal{H}_\infty$ of the moving frame equations, define $\hat{x} = A_\psi(\mathbf{e}_0)$ and $\hat{v}_k = A_\psi(\mathbf{e}_k)$. Then $\partial_i\hat{x} = h_i\hat{v}_i$, and orthogonality of $\hat{v}_i$ yields that $\hat{x}$ is an orthogonal system.

We conclude by showing the static frame equation. If ψ is adapted, then (5.24) in combination with the orthogonality relation of the v_i yields:

$$\begin{aligned} s_i &= A_{\mathbf{e}_i\psi}(S_i) \;=\; \frac{1}{2}\sum_{k\neq i}\beta_{ki}A_{\mathbf{e}_i\psi}(\mathbf{e}_k) - h_i A_{\mathbf{e}_i\psi}(\mathbf{e}_\infty) \\ &= -\frac{1}{2}\sum_{k\neq i}\beta_{ki}A_\psi(\mathbf{e}_k) + h_i A_\psi(\mathbf{e}_\infty) \;=\; -\frac{1}{2}\sum_{k\neq i}\beta_{ki}\hat{v}_k + h_i\mathbf{e}_\infty \;=\; \frac{1}{2}\,\partial_i\hat{v}_i\,. \end{aligned}$$

The last equality follows from $\beta_{ki} = -\langle\partial_i\hat{v}_i, \hat{v}_k\rangle$ and the fact that the $\mathbf{e}_\infty$–component of $\partial_i\hat{v}_i$ is equal to $2h_i$ (the latter follows from (5.17)). ∎

Discrete orthogonal systems

Now consider a discrete orthogonal system $\hat{x} : \mathcal{B}^\epsilon(r) \to \mathcal{K}$. Although we are only interested in the case of $m > 1$ quasi-continuous and $m' = 0$ discrete directions, we also include the cases $m' > 0$ in our considerations here by explicitly allowing for different mesh sizes $\epsilon_i > 0$; this simplifies the treatment of Ribaucour transformations later. So if $m' > 0$, read $\mathcal{B}^\epsilon(r, 1)$ instead of $\mathcal{B}^\epsilon(r)$ in the following.

Appropriate discrete analogues of the metric coefficients h_i and vectors $\hat{v}_i$ are

$$h_i = |\delta_i\hat{x}|, \quad \hat{v}_i = h_i^{-1}\delta_i\hat{x}\,. \tag{5.29}$$

These are unit vectors, i.e. $\hat{v}_i^2 = -1$, representing the reflections taking $\hat{x}$ to $\tau_i\hat{x}$ in the Möbius picture, cf. Fig. 5.1:

$$\tau_i\hat{x} = -A_{\hat{v}_i}\hat{x} = \hat{x} + \epsilon_i h_i \hat{v}_i\,. \tag{5.30}$$

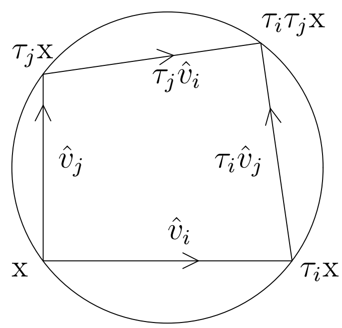

Figure 5.1: An elementary quadrilateral of a discrete orthogonal system

It is important to note that the vectors $\hat{v}_i$ are *not* mutually orthogonal.

Next, we derive equations that are discrete analogues of (5.18). Since four points $\hat{x}$, $\tau_i\hat{x}$, $\tau_j\hat{x}$ and $\tau_i\tau_j\hat{x}$ are coplanar, the vectors $\tau_j\delta_i\hat{x}$ and $\tau_i\delta_j\hat{x}$ lie in the span of $\delta_i\hat{x}$ and $\delta_j\hat{x}$. The lift λ is affine, therefore the same holds for the $\hat{v}_i$:

$$\tau_i\hat{v}_j = (n_{ji})^{-1} \cdot (\hat{v}_j + \rho_{ji}\hat{v}_i). \tag{5.31}$$

Here ρ_{ji} are discrete rotation coefficients, with expected limit behavior $\rho_{ji} \approx \epsilon_i\epsilon_j\beta_{ji}$ as $\epsilon \to 0$, and

$$n_{ji}^2 = 1 + \rho_{ji}^2 + 2\rho_{ji}\langle\hat{v}_i, \hat{v}_j\rangle \tag{5.32}$$

is the normalizing factor that is expected to converge to 1 as $\epsilon \to 0$. Circularity implies that the angle between $\delta_i\hat{x}$ and $\delta_j\hat{x}$ and the angle between $\tau_j\delta_i\hat{x}$ and $\tau_i\delta_j\hat{x}$ sum up to π:

$$\langle\tau_j\hat{v}_i, \hat{v}_j\rangle + \langle\hat{v}_i, \tau_i\hat{v}_j\rangle = 0. \tag{5.33}$$

Under the coplanarity condition (5.31) the latter condition is equivalent to

$$\hat{v}_j(\tau_j\hat{v}_i) + \hat{v}_i(\tau_i\hat{v}_j) = 0. \tag{5.34}$$

Inserting (5.31) and (5.32) into (5.33), one finds that either $\langle\hat{v}_i, \hat{v}_j\rangle = 1$, i.e., the circle degenerates to a line (we exclude this from consideration), or

$$\rho_{ij} + \rho_{ji} + 2\langle\hat{v}_i, \hat{v}_j\rangle = 0. \tag{5.35}$$

From (5.32) and (5.35), it follows:

$$n_{ij}^2 = n_{ji}^2 = 1 - \rho_{ij}\rho_{ji}. \tag{5.36}$$

So, the orthogonality constraint in the discrete case is expressed as the condition (5.35) on the rotation coefficients; recall that in the continuous case this was a condition on *the derivatives* of the rotation coefficients, see (5.6), resp. (5.7).

The following definition mimics the static frame equations of the continuous case.

Definition 5.4 *A frame $\psi : \mathcal{B}^\epsilon(r) \to \mathcal{H}_\infty$ is called adapted to a discrete orthogonal system $\hat{\mathrm{x}}$, if*

$$\begin{aligned} A_\psi(\mathbf{e}_0) &= \hat{\mathrm{x}}, && (5.37)\\ \tau_i\psi &= -\mathbf{e}_i\psi\hat{v}_i \quad (1 \le i \le M). && (5.38)\end{aligned}$$

Existence of such a frame is a consequence of the circularity condition. Indeed, the consistency $\tau_j\tau_i\psi = \tau_i\tau_j\psi$ reads

$$\mathbf{e}_i\mathbf{e}_j\psi\hat{v}_j(\tau_j\hat{v}_i) = \mathbf{e}_j\mathbf{e}_i\psi\hat{v}_i(\tau_i\hat{v}_j).$$

which is equivalent to (5.34). An adapted frame is uniquely defined by the choice of $\psi(0)$ (and thus is not unique, even if $M = N$). Indeed, (5.38) together with the definition (5.29) of $\hat{v}_i$ imply: If (5.37) holds at one point of $\mathcal{B}^\epsilon(r)$, then it holds everywhere.

Proposition 5.2 *An adapted frame ψ for a discrete orthogonal system $\hat{\mathrm{x}}$ satisfies the moving frame equations:*

$$\tau_i\psi = -\Sigma_i\mathbf{e}_i\psi \quad (1 \le i \le M) \tag{5.39}$$

with

$$\Sigma_i = A^{-1}_{\mathbf{e}_i\psi}(\hat{v}_i) = N_i\mathbf{e}_i + \frac{\epsilon_i}{2}\sum_{k\neq i}\beta_{ki}\mathbf{e}_k - \epsilon_i h_i\mathbf{e}_\infty, \tag{5.40}$$

where

$$N_i^2 = 1 - \frac{\epsilon_i^2}{4}\sum_{k\neq i}\beta_{ki}^2\,, \tag{5.41}$$

and the functions β_{ij} and h_i solve the following discrete Lamé system *(in the equations below $1 \le i, j \le M$, $1 \le k \le N$, and $i \neq j \neq k \neq i$):*

$$\begin{aligned} \tau_i h_j &= (n_{ji}\epsilon_j)^{-1}\cdot(\epsilon_j h_j + \epsilon_i h_i\rho_{ij}), && (5.42)\\ \tau_i\beta_{kj} &= (n_{ji}\epsilon_j)^{-1}\cdot(\epsilon_j\beta_{kj} + \epsilon_i\beta_{ki}\rho_{ij}), && (5.43)\\ \tau_i\beta_{ij} &= (n_{ji}\epsilon_j)^{-1}\cdot(2N_i\rho_{ij} - \epsilon_j\beta_{ij}), && (5.44)\\ \rho_{ij} + \rho_{ji} &= N_i\beta_{ij}\epsilon_j + N_j\beta_{ji}\epsilon_i - \frac{\epsilon_i\epsilon_j}{2}\sum_{k\neq i,j}\beta_{ki}\beta_{kj}. && (5.45)\end{aligned}$$

where the abbreviation $n_{ij} = (1 - \rho_{ij}\rho_{ji})^{1/2}$ is used, and ρ_{ij} are suitable real–valued functions. Conversely, given a solution β_{ij} and h_i of the equations (5.42)-(5.45), the system of moving frame equations (5.39) is consistent, and its solution ψ is an adapted frame of a discrete orthogonal system.

Proof. From the definition $\Sigma_i = A^{-1}_{\mathbf{e}_i\psi}(\hat{v}_i)$ it follows that Σ_i is a vector without $\mathbf{e}_0$–component, i.e. admits a decomposition of the form (5.40). Again, from the definition of Σ_i and the moving frame equation (5.38) we find:

$$A_{\mathbf{e}_i}(\Sigma_j) = \mathbf{e}_i\mathbf{e}_j\psi\hat{v}_j\psi^{-1}\mathbf{e}_j\mathbf{e}_i\,, \qquad \tau_i\Sigma_j = -\mathbf{e}_j\mathbf{e}_i\psi\hat{v}_i(\tau_i\hat{v}_j)\hat{v}_i\psi^{-1}\mathbf{e}_i\mathbf{e}_j\,.$$

Upon using the identity $\hat{v}_i(\hat{v}_j+\rho_{ji}\hat{v}_i)\hat{v}_i=\hat{v}_j+\rho_{ij}\hat{v}_i$, which follows easily from (5.35), we can now represent eq. (5.31) in the following equivalent form:

$$n_{ji}(\tau_i\Sigma_j)=-A_{\mathbf{e}_i}(\Sigma_j)-\rho_{ij}A_{\mathbf{e}_j}(\Sigma_i). \tag{5.46}$$

This equation is equivalent to eqs. (5.42)–(5.44). Next, the consistency $\tau_j(\tau_i\psi)=\tau_i(\tau_j\psi)$ of the moving frame equations (5.39) is equivalent to

$$(\tau_i\Sigma_j)A_{\mathbf{e}_j}(\Sigma_i)+(\tau_j\Sigma_i)A_{\mathbf{e}_i}(\Sigma_j)=0. \tag{5.47}$$

Inserting the expressions for $\tau_i\Sigma_j$ and $\tau_j\Sigma_i$ from (5.46) into the left side give:

$$\begin{aligned} A_{\mathbf{e}_i}(\Sigma_j)A_{\mathbf{e}_j}(\Sigma_i)+A_{\mathbf{e}_j}(\Sigma_i)A_{\mathbf{e}_i}(\Sigma_j)+\rho_{ji}(A_{\mathbf{e}_i}(\Sigma_j))^2+\rho_{ij}(A_{\mathbf{e}_j}(\Sigma_i))^2 \\ = N_i\beta_{ji}\epsilon_i+N_j\beta_{ij}\epsilon_j-(\epsilon_i\epsilon_j/2)\sum_{k\neq i,j}\beta_{ki}\beta_{kj}-(\rho_{ij}+\rho_{ji}) \quad = \quad 0, \end{aligned}$$

the last equality delivers (5.45). Conversely, (5.42)–(5.45) imply (5.46), as well as the compatibility of the frame equations (5.39). So for an arbitrary initial value $\psi(0)\in\mathcal{H}_\infty$, the frame equations for $\psi:\mathcal{B}^\epsilon(r)\to\mathcal{H}_\infty$ can be solved uniquely. Set $\hat{\mathrm{x}}^\epsilon=A_\psi(\mathbf{e}_0)$, $\hat{v}_i=A_{\mathbf{e}_i\psi}(\Sigma_i)$. From (5.46), (5.47) for the quantities Σ_i there follow (5.31), (5.33) for the quantities $\hat{v}_i$. Using the fact that Σ_i has no $\mathbf{e}_0$–component, one shows easily that there holds also (5.30). Therefore, $\hat{\mathrm{x}}$ is a discrete orthogonal system. ■

5.2 Approximation of Orthogonal Systems

5.2.1 Hyperbolic Equations

It would be tempting to derive an result about approximation of continuous orthogonal systems by discrete ones from the observation that the classical Lamé system (5.2)-(5.6) is the limit of the discrete system (5.42)-(5.45). However, these systems are not of the hyperbolic type (2.4). And it turns out that in dimensions $M\geq 3$ it is necessary to enlarge the set of dependent variables and equations in a cumbersome manner to obtain the hyperbolic form. The way around is based on the following fundamental lemma, which allows one to take care of two–dimensional orthogonal systems (associated to the coordinate 2-planes) only, and to use then the results of Theorem 4.1 and Theorem 4.2 for conjugate systems.

Lemma 5.1 *If for a conjugate net* x*, continuous or discrete, its restriction to each plane* $\mathcal{B}_{ij}$*,* $1\leq i\neq j\leq M$*, is a C-surface, then* x *is a (continuous or discrete) orthogonal system.*

Proof in the discrete case is based on the Miguel theorem, and can be found in [CDS]. For the continuous case it is a by-product of the proof of Theorem 5.1 below. ■

So, suppose we are in dimension $M = 2$. Then the moving frame equations (5.39) and the system (5.42)-(5.45) take the form ($1 \le i, j \le 2,\ 1 \le k \le N,\ i \ne j \ne k \ne i$):

$$\begin{aligned}
\delta_i \psi &= \left(\frac{N_i - 1}{\epsilon_i} - \frac{1}{2} \sum_{k \ne i} \beta_{ki} \mathbf{e}_k \mathbf{e}_i + h_i \mathbf{e}_\infty \mathbf{e}_i \right) \psi \,, && (5.48) \\
\delta_i h_j &= \frac{\rho_{ij}}{\epsilon_j n} h_i + \frac{1-n}{\epsilon_i n} h_j \,, && (5.49) \\
\delta_i \beta_{ij} &= \frac{2 N_i \rho_{ij}}{\epsilon_i \epsilon_j n} - \frac{1+n}{\epsilon_i n} \beta_{ij}, && (5.50) \\
\delta_i \beta_{kj} &= \frac{1-n}{\epsilon_i n} \beta_{kj} + \frac{\rho_{ij}}{\epsilon_j n} \beta_{ki}, && (5.51) \\
\rho_{12} + \rho_{21} &= \epsilon_2 N_1 \beta_{12} + \epsilon_1 N_2 \beta_{21} - \epsilon_1 \epsilon_2 \Theta. && (5.52)
\end{aligned}$$

Here we use the abbreviations

$$n = n_{12} = n_{21} = \sqrt{1 - \rho_{12}\rho_{21}}, \quad \Theta = \frac{1}{2} \sum_{k>2} \beta_{k1} \beta_{k2}, \quad N_i^2 = 1 - \frac{\epsilon_i^2}{4} \sum_{k \ne i} \beta_{ki}^2 \,. \quad (5.53)$$

We show how to re-formulate the above system in the hyperbolic form and to pose a Cauchy problem for it.

5.2.2 Approximation of C-Surfaces

In the derivation of the system above, we have worked with possibly different mesh sizes ϵ_1 and ϵ_2. Now, we turn to the case at hand, with $m = M = 2$ quasi-continuous directions, so we have $\epsilon_1 = \epsilon_2 = \epsilon$ in (5.48)-(5.52). Set

$$\begin{aligned}
\rho_{12} &= \epsilon N_1 \beta_{12} - (\epsilon^2/2)(\Theta - \gamma), && (5.54) \\
\rho_{21} &= \epsilon N_2 \beta_{21} - (\epsilon^2/2)(\Theta + \gamma), && (5.55)
\end{aligned}$$

with a suitable function $\gamma : \mathcal{B}_{12}^{\epsilon} \to \mathbb{R}$. It can be said that the auxiliary function γ splits the constraint, making the Lamé system hyperbolic. This splitting plays a crucial role for our approximation results. In the smooth limit $2\gamma = \partial_1 \beta_{12} - \partial_2 \beta_{21}$ (cf. eqs. (5.56), (5.57) below).

As a Banach space for the system take

$$\mathcal{X} = \hat{\mathcal{H}}_\psi \times \mathbb{R}_\beta^{2(N-1)} \times \mathbb{R}_h^2 \times \mathbb{R}_\gamma \,,$$

where h stands for h_1 and h_2, and β denotes the collection of β_{kj} with $k = 1, 2$, $1 \le j \le N$, $k \ne j$. $\hat{\mathcal{H}}$ is the space of an (arbitrary) matrix representation of $Spin(N+1, 1)$; note that if $\psi \in \mathcal{H}_\infty \subset \hat{\mathcal{H}}$ at some point, then eq. (5.48) guarantees that this is the case everywhere.

Both directions $i = 1, 2$ are assumed to be evolution directions for ψ; for h_i and β_{ki} there is one evolution direction $j \ne i$, while for the additional function γ both directions $i = 1, 2$ are static.

Lemma 5.2 *The hyperbolic system (5.48)–(5.51) with (5.54), (5.55) is consistent. The right-hand sides in the equations are $\mathcal{O}(\epsilon)$-convergent in $C^\infty(\mathcal{X})$ to (5.23) with (5.24), (5.25), (5.26) for pairwise distinct indices $1 \le i, j \le 2$, $1 \le k \le N$, and*

$$\partial_1 \beta_{12} = -(1/2) \sum_{k>2} \beta_{k1}\beta_{k2} + \gamma\,, \tag{5.56}$$

$$\partial_2 \beta_{21} = -(1/2) \sum_{k>2} \beta_{k1}\beta_{k2} - \gamma\,, \tag{5.57}$$

which describe continuous C-surfaces.

Recall that the C-surface x itself is calculated from the solution to the hyperbolic system by $\mathrm{x} = \lambda^{-1}(A_\psi(\mathbf{e}_0))$.

Proof. Consistency is easy to see: the only condition to be checked is $\delta_1\delta_2\psi = \delta_2\delta_1\psi$, but this has already been shown in Proposition 5.2.

The other statements are obvious from

$$N_1 = 1 + O(\epsilon^2), \qquad N_2 = 1 + O(\epsilon^2), \qquad n = 1 + O(\epsilon^2),$$

where the remainder constants in O–symbols are uniform on compact subsets of $\mathcal{X}$. Notice that hyperbolic equations (5.56), (5.57) come to replace the non–hyperbolic orthogonality constraint (5.27). ■

Goursat Problem 5.1 (for C-Surfaces) *Given $\Psi^\epsilon \in \mathcal{H}_\infty$, functions $H_i^\epsilon : \mathcal{B}_i^\epsilon(r) \to \mathbb{R}$ and $B_{ki}^\epsilon : \mathcal{B}_i^\epsilon(r) \to \mathbb{R}$ for $i = 1, 2$, $1 \le k \le N$, $k \ne i$, and $\Gamma^\epsilon : \mathcal{B}_{12}^\epsilon(r) \to \mathbb{R}$, find a solution to the equations (5.48)-(5.51) with (5.54), (5.55) if $\epsilon > 0$, or to the limiting system described in Lemma 5.2 above if $\epsilon = 0$, respectively, so that*

$$\psi(0) = \Psi^\epsilon, \quad h_i\restriction_{\mathcal{B}_i^\epsilon} = H_i^\epsilon, \quad \beta_{ki}\restriction_{\mathcal{B}_i^\epsilon} = B_{ki}^\epsilon, \quad \gamma\restriction_{\mathcal{B}_{12}^\epsilon} = \Gamma^\epsilon.$$

Now we discuss the way to prescribe the data in the Goursat problem 5.1 in order to get an approximation of a given smooth C-surface. Unlike in the case of general conjugate nets, it is not possible to prescribe the discrete curves $\mathrm{X}_i^\epsilon = \mathrm{x}^\epsilon\restriction_{\mathcal{B}_i^\epsilon}$ coinciding with their continuous counterparts $\mathrm{X}_i = \mathrm{x}\restriction_{\mathcal{B}_i}$ at the lattice points. However, it can be achieved that each X_i^ϵ is completely determined by the respective X_i.

Given a curve $\hat{X}_i : \mathcal{B}_i \to \mathcal{K}$, one has the corresponding tangential vector field $\partial_i \hat{X}_i$, and therefore the function $h_i = |\partial_i \hat{X}_i|$ and the field of unit vectors $\hat{v}_i = h_i^{-1}\partial_i \hat{X}_i$. Take $\Psi \in \mathcal{H}_\infty$ suited to $(\hat{\mathrm{X}}_i(0); \hat{v}_i(0))$. Then by Proposition 5.1, there is a unique frame $\psi : \mathcal{B}_i \to \mathcal{H}_\infty$ adapted to the curve $\hat{\mathrm{X}}_i$ with $\psi(0) = \Psi$; this frame is defined as the unique solution of the differential equation $\partial_i \psi = -S_i \mathbf{e}_i \psi$ with S_i given by eq. (5.24). According to the proof of Proposition 5.1, the latter equation is equivalent to the system of equations

$$\partial_i \hat{v}_k = \beta_{ki}\hat{v}_i, \quad \beta_{ki} = -\langle \hat{v}_k, \partial_i \hat{v}_i \rangle, \quad k \ne i.$$

So, in order to determine the rotation coefficients $\beta_{ki} : \mathcal{B}_i \to \mathbb{R}$ for $1 \le k \le N$, $k \neq i$, one has to solve the latter system of ordinary differential equations with the initial data $\hat{v}_k(0) = A_\Psi(\mathbf{e}_k)$. This produces the functions h_i and β_{ki}, or, what is equivalent, the Clifford elements

$$S_i = \frac{1}{2}\sum_{k\neq i}\beta_{ki}\mathbf{e}_k - h_i\mathbf{e}_\infty$$

for a given curve $\hat{X}_i : \mathcal{B}_i \to \mathcal{K}$. We say that h_i, β_{ki} are *read off* the curve $\hat{X}_i$.

Definition 5.5 *The* canonical discretization of the curve $\hat{\mathrm{X}}_i : \mathcal{B}_i \to \mathcal{K}$ *with respect to the initial frame* $\Psi \in \mathcal{H}_\infty$ *suited to* $(\hat{X}_i(0); \hat{v}_i(0))$ *is the function* $\hat{\mathrm{X}}_i^\epsilon = A_{\psi^\epsilon}(\mathbf{e}_0) : \mathcal{B}_i^\epsilon \to \mathcal{K}$*, where* $\psi^\epsilon : \mathcal{B}_i^\epsilon \to \mathcal{H}_\infty$ *is the solution of the discrete moving frame equation* $\tau_i\psi^\epsilon = -\Sigma_i\mathbf{e}_i\psi^\epsilon$ *with*

$$\Sigma_i = \left(1 - \frac{\epsilon^2}{4}\sum_{k\neq i}\beta_{ki}^2\right)^{1/2}\mathbf{e}_i + \frac{\epsilon}{2}\sum_{k\neq i}\beta_{ki}\mathbf{e}_k - \epsilon h_i\mathbf{e}_\infty \tag{5.58}$$

with the restricted quantities h_i *and* β_{ij} *read off from* $\hat{\mathrm{X}}_i$*, and the initial condition* $\psi^\epsilon(0) = \Psi$*.*

Obviously, ψ^ϵ is an adapted frame for the discrete curve X_i^ϵ. As $\epsilon \to 0$, the canonical discretization X_i^ϵ lattice-converges to X_i smoothly.

Proposition 5.3 (approximation for C–surfaces). *Assume* $m = M = 2$*, and let there be given: two smooth curves* $\mathrm{X}_i : \mathcal{B}_i(\bar{r}) \to \mathbb{R}^N$ *(*$i = 1, 2$*), intersecting orthogonally at* $\mathrm{X} = \mathrm{X}_1(0) = \mathrm{X}_2(0)$*, and a smooth function* $\Gamma : \mathcal{B}(\bar{r}) \to \mathbb{R}$*. Assume* $\Psi \in \mathcal{H}_\infty$ *is suited to* $(\hat{\mathrm{X}}; \hat{v}_1(0), \hat{v}_2(0))$*. Then, for some* $r > 0$*:*

1. *There exists a unique C-surface* $\mathrm{x} : \mathcal{B}(r) \to \mathbb{R}^N$ *that coincides with* X_i *on* $\mathcal{B}_i(r)$ *and satisfies*
$$\partial_1\beta_{12} - \partial_2\beta_{21} = 2\Gamma.$$
2. *Consider the family of discrete C–surfaces* $\{\mathrm{x}^\epsilon : \mathcal{B}^\epsilon(r) \to \mathbb{R}^N\}_{0<\epsilon<\epsilon_1}$ *defined as the solutions to Goursat problem 5.1 with data*
$$\Psi^\epsilon = \Psi, \quad H_i^\epsilon = [H_i]^\epsilon, \quad B_{ki}^\epsilon = [B_{ki}]^\epsilon, \quad \Gamma^\epsilon = [\Gamma]^\epsilon,$$
where the functions H_i *and* $\mathcal{B}_{ki}$ *are read off the smooth curves* X_i*, so that* $\mathrm{x}^\epsilon\lceil_{\mathcal{B}_i^\epsilon}$ *are the canonical* ϵ*-discretizations of* X_i*. This family of discrete C–surfaces is smoothly lattice-convergent to* x*.*

Proof follows from Lemma 5.2 and Theorem 2.2. ■

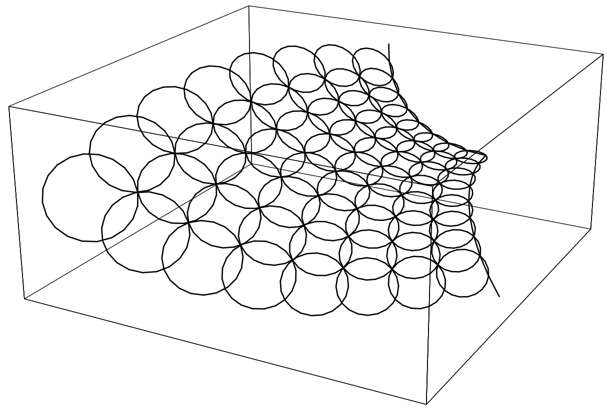

Figure 5.2: Two curves determine a discrete C–surface via Proposition 5.3.

5.2.3 Approximation of General Orthogonal Systems

Theorem 5.1 (approximation of an orthogonal system). *Let $\binom{m}{2}$ smooth C–surfaces $\mathcal{S}_{ij} : \mathcal{B}_{ij}(\bar{r}) \to \mathbb{R}^N$ be given, labeled by $1 \leq i \neq j \leq m$ with $\mathcal{S}_{ij} = \mathcal{S}_{ji}$. Assume that for given i, the surfaces $\mathcal{S}_{ij}$ intersect along a common curvature line $\mathrm{X}_i = \mathcal{S}_{ij}\restriction_{\mathcal{B}_i}$; the X_i are pairwise orthogonally intersecting at $\mathrm{X} = \mathrm{X}_1(0) = \cdots = \mathrm{X}_m(0)$.*

Set $h_i = |\partial_i \mathrm{X}_i|$, assume $h_i(0) \neq 0$, and set further $v_i = h_i^{-1}\partial_i X_i$. Let $\Psi \in \mathcal{H}_\infty(N)$ be suited to $(\hat{\mathrm{X}}, \hat{v}_1(0), \ldots, \hat{v}_m(0))$. Construct the discrete C–surfaces $\mathcal{S}^\epsilon_{ij} : \mathcal{B}^\epsilon_{ij}(\tilde{r}) \to \mathbb{R}^N$ according to Proposition 5.3, $0 < \tilde{r} < \bar{r}$, with functions $\Gamma_{ij} = (\partial_i \beta_{ij} - \partial_j \beta_{ji})/2$.

Then for some positive $r < \tilde{r}$:

1. *There exists a unique orthogonal system $\mathrm{x} : \mathcal{B}(r) \to \mathbb{R}^N$, coinciding with $\mathcal{S}_{ij}$ on $\mathcal{B}_{ij}(r)$.*

2. *There exists a unique family of discrete orthogonal systems $\{\mathrm{x}^\epsilon : \mathcal{B}^\epsilon(r) \to \mathbb{R}^N\}_{0<\epsilon<\epsilon_0}$ coinciding with $\mathcal{S}^\epsilon_{ij}$ on $\mathcal{B}^\epsilon_{ij}(r)$. The family x^ϵ lattice-converges to x smoothly.*

Proof. C–surfaces are special two-dimensional conjugate nets, therefore the surfaces $\mathcal{S}_{ij}$ can be supplied with the respective coefficients $C_{ij}, C_{ji} : \mathcal{B}_{ij}(\bar{r}) \to \mathbb{R}$. Now Theorem 4.1 applies: It yields the existence of a unique conjugate net x which has X_i as image of the i-th coordinate axes and C_{ij}, C_{ji} as coefficients on the respective $\mathcal{B}_{ij}$.

Similarly, discrete C–surfaces are special discrete two-dimensional conjugate nets, so for each $\mathcal{S}^\epsilon_{ij}$, the coefficients $C^\epsilon_{ij}, C^\epsilon_{ji}$ are defined, and they lattice-converge in C^∞ to the coefficients C_{ij}, C_{ji} of the respective $\mathcal{S}_{ij}$. Note that for any $1 \leq i \leq m$ the discrete surfaces $\mathcal{S}^\epsilon_{ij}$ intersect along the discrete curve $\mathrm{X}^\epsilon_i = \mathcal{S}^\epsilon_{ij}\restriction_{\mathcal{B}_i}$, which is the canonical discretization of the curve X_i. The conclusions of Theorem 4.1 is still valid if the Goursat data for c^ϵ_{ij} in (4.12) are taken as C^ϵ_{ij} instead of C_{ij}, i.e. read off $\mathcal{S}^\epsilon_{ij}$ instead of $\mathcal{S}_{ij}$. Thus, Theorem 4.1 delivers a family of discrete conjugate nets $\{\mathrm{x}^\epsilon\}$

that is smoothly lattice-convergent to x. Since the restrictions of x^ϵ to the coordinate planes $\mathcal{B}_{ij}^\epsilon$ are discrete C–surfaces, the discrete variant of Lemma 5.1 implies that the nets x^ϵ are actually discrete orthogonal systems. It remains to show that the limiting net x is an orthogonal one. But this follows immediately from the fact of C^∞–convergence and equation (5.34): indeed, we can conclude that for the net x everywhere holds

$$\hat{v}_j\hat{v}_i + \hat{v}_i\hat{v}_j = 0 \quad \Leftrightarrow \quad \langle \hat{v}_i, \hat{v}_j \rangle = 0,$$

that is, $\partial_i \mathrm{x} \cdot \partial_j \mathrm{x} = 0$. ∎

This result immediately implies Theorem 1.1 from the introduction: to bring its assumptions into the form of Theorem 5.1 above, one needs only to introduce a curvature line parameterization on surfaces $\mathcal{F}_i$ such that the curves X_i are parameterized by arc-length.

5.2.4 Example: Elliptic Coordinates

The simplest nontrivial example, to which the above theory can be applied, is the approximation of two-dimensional conformal maps by circular patterns. Starting with a conformal map $F : \mathbb{R}^2 \to \mathbb{R}^2$, i.e. $M = N = 2$,

$$h = |\partial_1 F| = |\partial_2 F| \quad \text{and} \quad \partial_1 F \cdot \partial_2 F = 0,$$

one calculates the metric and rotation coefficients according to equations (5.2) and (5.4), and the quantity γ according to (5.56):

$$h_1 = h_2 = h, \quad \beta_{12} = \frac{\partial_1 h}{h}, \quad \beta_{21} = \frac{\partial_2 h}{h}, \quad \gamma = \partial_1 \beta_{12} = -\partial_2 \beta_{21}$$

To construct a discrete approximation of F, solve the Goursat problem 5.1 for $\epsilon > 0$ with data H_i^ϵ, B_{ij}^ϵ and Γ^ϵ that are close to the values of the respective functions h_i, β_{ij} and γ at corresponding lattice sites. A good choice is, for example, to prescribe

$$\begin{gathered} H_1^\epsilon(\xi_1) = h(\xi_1 + \epsilon/2, 0), \quad H_2^\epsilon(\xi_2) = h(0, \xi_2 + \epsilon/2), \\ B_{21}^\epsilon(\xi_1) = \beta_{21}(\xi_1 + \epsilon/2, 0), \quad B_{12}^\epsilon(\xi_2) = \beta_{12}(0, \xi_2 + \epsilon/2), \\ \Gamma^\epsilon(\xi_1, \xi_2) = \gamma(\xi_1 + \epsilon/2, \xi_2 + \epsilon/2), \end{gathered}$$

for $\xi_1, \xi_2 = i\epsilon,\ \ 0 \le i \le R$. If $\psi^\epsilon : B^\epsilon(\epsilon R) \to \mathcal{H}_\infty$ is the frame of the respective solution to the system (5.48)-(5.51), then the function

$$F^\epsilon : B^\epsilon(R\epsilon) \to \mathbb{R}^2, \quad F^\epsilon(\xi_1, \xi_2) = \pi(\psi(\xi_1, \xi_2)^{-1}\mathbf{e}_0\psi(\xi_1, \xi_2))$$

is a two-dimensional discrete orthogonal system, i.e., the points $F^\epsilon(\xi_1, \xi_2)$, $F^\epsilon(\xi_1 + \epsilon, \xi_2)$, $F^\epsilon(\xi_1, \xi_2 + \epsilon)$ and $F^\epsilon(\xi_1 + \epsilon, \xi_2 + \epsilon)$ lie on a circle $\mathrm{C}^\epsilon(\xi_1, \xi_2)$ in $\mathbb{R}^2$, and $F^\epsilon(\xi_1, \xi_2) = F(\xi_1, \xi_2) + O(\epsilon)$.

This is illustrated with the planar elliptic coordinate system:

$$F(\xi_1, \xi_2) = (\cosh(\xi_1)\cos(\xi_2), \sinh(\xi_1)\sin(\xi_2)),$$

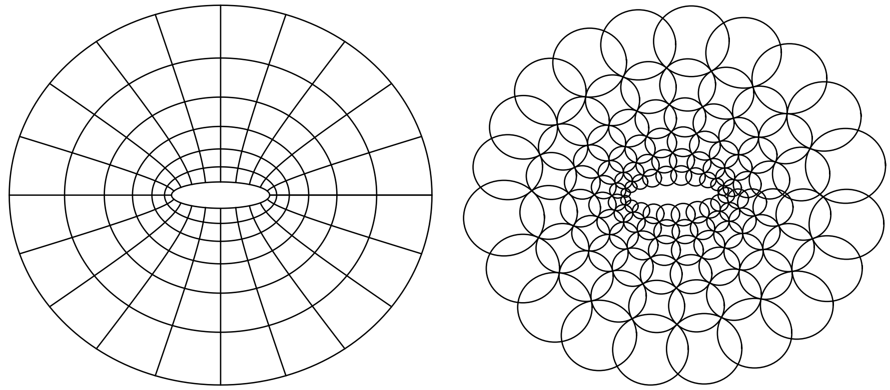

Figure 5.3: Approximation of elliptic coordinates, $\epsilon = \pi/10$.

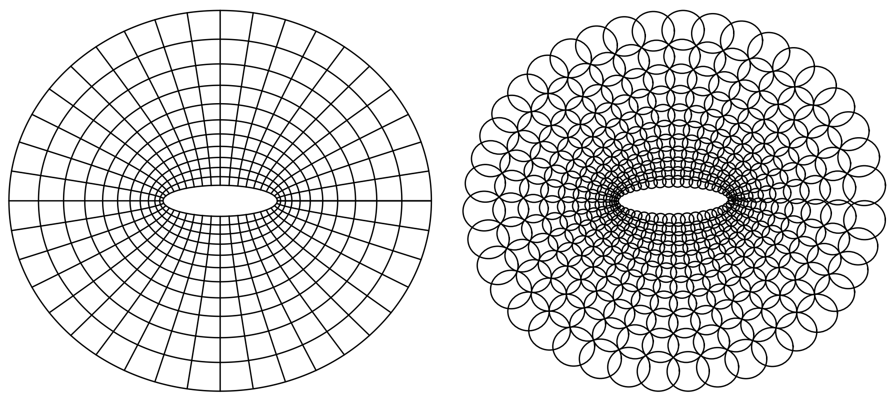

Figure 5.4: Approximation of elliptic coordinates, $\epsilon = \pi/20$.

whose coordinate lines are ellipses and hyperbolas. One finds

$$h = \left(\sinh^2(\xi_1) + \sin^2(\xi_2)\right)^{1/2},$$
$$\beta_{12} = \sinh(2\xi_1)/(2h^2), \quad \beta_{21} = \sin(2\xi_2)/(2h^2),$$
$$\gamma = \left(1 - \cosh(2\xi_1)\cos(2\xi_2)\right)/(4h^4).$$

Results are displayed on Figs. 5.3, 5.4. Their left sides show the original coordinate lines of F, and on the right sides the circles $\mathrm{C}^\epsilon(\xi_1, \xi_2)$ are drawn; each intersection point of two coordinate lines on the left corresponds to an intersection point of four circles on the right. There are small defects (the circles do not close up) on the very right of the pictures since, in contrast to F, the discrete maps F^ϵ are not periodic with respect to ξ_2.

In figure 5.5, three intersecting discrete C-surfaces are shown, which are coordinate surfaces in a discrete triply orthogonal system. This discrete system approximates

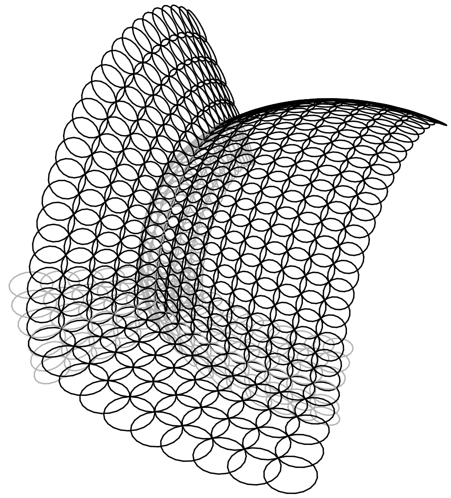

Figure 5.5: Coordinate surfaces in a discrete triply orthogonal system.

the (degenerate) elliptic coordinates

$$\mathrm{x}(\xi_1, \xi_2, \xi_3) = \begin{pmatrix} \cosh(\xi_1)\cos(\xi_2) \\ \sinh(\xi_1)\sin(\xi_2)\cos(\xi_3) \\ \sinh(\xi_1)\sin(\xi_2)\sin(\xi_3) \end{pmatrix}$$

around the point $(\xi_1, \xi_2, \xi_3) = (1, \pi/4, 0)$.

5.3 Ribaucour Transformations

5.3.1 Definitions and Möbius Description

Now transformations of m-dimensional orthogonal systems are considered. As before, we pass from an m-dimensional description of the $R+1$ individual orthogonal systems $\mathrm{x}^{\epsilon(t)} : \mathcal{B}^{\epsilon}(r) \to \mathbb{R}^N$ to the $M = m+1$-dimensional composite map $\mathrm{x}^{\epsilon} : \mathcal{B}^{\epsilon}(r, R) \to \mathbb{R}^N$, where $m' = 1$ direction is purely discrete. Of particular interest are pairs of systems $\mathrm{x}^-, \mathrm{x}^+$.

Definition 5.6 *A pair of m-dimensional orthogonal systems $\mathrm{x}^-, \mathrm{x}^+ : \mathcal{B}_0(r) \to \mathbb{R}^N$ is called a* Ribaucour pair*, if for any $1 \leq i \leq m$ the corresponding coordinate lines of x^- and x^+ envelope a one–parameter family of circles, i.e. if it is a Jonas pair of conjugate nets and*

$$\left(\frac{\partial_i \mathrm{x}^+}{|\partial_i \mathrm{x}^+|} + \frac{\partial_i \mathrm{x}^-}{|\partial_i \mathrm{x}^-|}\right) \cdot (\mathrm{x}^+ - \mathrm{x}^-) = 0. \tag{5.59}$$

Remark: There are different definitions of Ribaucour transformations in the literature (cf. [He]). We have chosen the one that is best suited for our present purposes.

Definition 5.7 *A pair of discrete m–dimensional orthogonal systems* $\mathrm{x}^-, \mathrm{x}^+ : \mathcal{B}_0^\epsilon(r) \to \mathbb{R}^N$ *is called a* Ribaucour pair*, if the composite map* $\mathrm{x} : \mathcal{B}^\epsilon(r,1) \to \mathbb{R}^N$ *is an $m+1$-dimensional discrete orthogonal system.*

The system x^+ is also called a Ribaucour transformation of x^-; Ribaucour transformations can be iterated, obtaining a sequence $(\mathrm{x}^{(0)}, \mathrm{x}^{(1)}, \ldots, \mathrm{x}^{(R)})$ of orthogonal systems, where for each $k < R$, the systems $\mathrm{x}^{(k)}$ and $\mathrm{x}^{(k+1)}$ form a Ribaucour pair. We will stick to the description of the original system and its R transformations as a single function on the $M = m+1$-dimensional domain $\mathcal{B}^\epsilon(r,R)$; in the following, the super-index $+$ denotes the shift in the M-th direction, i.e., $\mathrm{x}^+(\hat{\xi}, \check{\xi}) = \mathrm{x}(\hat{\xi}, \check{\xi}+1)$ etc. The results for Ribaucour pairs are singled out by choosing $R = 1$.

To find a Clifford-algebraic picture of Ribaucour transformations of orthogonal systems, one combines the discrete and continuous elements from the descriptions developed in subsection 5.1.3. Again, we consider the orthogonal system lifted to the section $\mathcal{K}$ of S^N:

$$\hat{\mathrm{x}} := \lambda \circ \mathrm{x} : \mathcal{B}(r,R) \to \mathcal{K}.$$

The description of *discrete* Ribaucour transformations was already part of the considerations about discrete orthogonal systems in subsection 5.1.3; one simply chooses $\epsilon_i = \epsilon$ for $i = 1, \ldots, m$ and $\epsilon_M = 1$ in the equations obtained there, and replaces the domains by $\mathcal{B}^\epsilon(r,R)$.

Thus, we turn to *continuous* Ribaucour transforms: Denote by h_i, $\hat{v}_i$, β_{ij} for $i \neq j \leq m$ the metric and rotation coefficients for $\hat{\mathrm{x}}$, as in the definitions (5.16), (5.18) of subsection 5.1.3. (So that $h_i(\cdot, k)$ is the function of metric coefficients for the k-th transformation $\hat{\mathrm{x}}^{(k)}$ etc.) Let $\hat{v}_M$ be unit vectors such that

$$\hat{\mathrm{x}}^+ = -A_{\hat{v}_M}(\hat{\mathrm{x}}), \tag{5.60}$$

cf. eq. (5.30). The defining property (5.59) of Ribaucour transformations and the normalization of $\hat{v}_M$ imply:

$$\begin{aligned} \hat{v}_i^+ &= -A_{\hat{v}_M}(\hat{v}_i), && (5.61)\\ \partial_i \hat{v}_M &= \frac{\alpha_i}{2}\,(\hat{v}_i^+ + \hat{v}_i) \;=\; \alpha_i(\hat{v}_i - \langle \hat{v}_M, \hat{v}_i\rangle \hat{v}_M), && (5.62)\end{aligned}$$

with m auxiliary functions $\alpha_i : \mathcal{B}(r,R) \to \mathbb{R}$.

Definition 5.8 *A frame* $\psi : \mathcal{B}(r,R) \to \mathcal{H}_\infty$ *is called adapted to the composite map of Ribaucour transformations* $\hat{\mathrm{x}} : \mathcal{B}(r,R)\mathbb{R}\mathcal{K}$*, if* $\psi(\cdot,0)$ *is adapted to* $\hat{\mathrm{x}}(\cdot,0)$*, and*

$$\psi^+ = -\mathbf{e}_M \psi \hat{v}_M. \tag{5.63}$$

Proposition 5.4 *Let ψ be a frame adapted to the composite function $\hat{\mathrm{x}}$ of Ribaucour transformations. Then each frame $\psi(\cdot,k)$ is adapted to the k-th transformation $\hat{\mathrm{x}}^{(k)}$, and in the moving frame equation for ψ,*

$$\begin{aligned} \partial_i \psi &= -S_i \mathbf{e}_i \psi \quad (1 \leq i \leq m), && (5.64)\\ \psi^+ &= -\Sigma_M \mathbf{e}_M \psi, && (5.65)\end{aligned}$$

one has

$$
\begin{aligned}
S_i &= \frac{1}{2}\sum_{k\neq i}\beta_{ki}\mathbf{e}_k - h_i\mathbf{e}_\infty \quad (1\le i\le m), & (5.66)\\
\Sigma_M &= A^{-1}_{\mathbf{e}_M\psi}(\hat{v}_M) \;=\; N_M\mathbf{e}_M + \frac{1}{2}\sum_{k\neq M}\beta_{kM}\mathbf{e}_k - h_M\mathbf{e}_\infty, & (5.67)\\
N_M^2 &= 1-\frac{1}{4}\sum_{k\neq M}\beta_{kM}^2. & (5.68)
\end{aligned}
$$

The functions h_i, β_{ij} are solutions to the system (5.25)-(5.27), and the following equations are satisfied (for $1\le i\neq j\le m$, $1\le k\le N$, and $i\neq k\neq M\neq i$):

$$
\begin{aligned}
h_i^+ &= h_i + h_M\alpha_i\,, & (5.69)\\
\beta_{ki}^+ &= \beta_{ki} + \beta_{kM}\alpha_i\,, & (5.70)\\
\beta_{Mi}^+ &= -\beta_{Mi} + 2N_M\alpha_i\,, & (5.71)\\
\partial_i h_M &= \frac{1}{2}\,(h_i + h_i^+)\beta_{iM}\,, & (5.72)\\
\partial_i\beta_{kM} &= \frac{1}{2}\,(\beta_{ki}+\beta_{ki}^+)\beta_{iM}\,, & (5.73)\\
\partial_i\beta_{iM} &= -\frac{1}{2}\sum_{k\neq i,M}(\beta_{ki}+\beta_{ki}^+)\beta_{kM} + N_M(\beta_{Mi}-\beta_{Mi}^+)\,, & (5.74)\\
\partial_i\alpha_j &= \frac{1}{2}\,\alpha_i(\beta_{ij}+\beta_{ij}^+)\,. & (5.75)
\end{aligned}
$$

Conversely, given solutions h_i, β_{ij} of the system above with suitable auxiliary functions α_i, the moving frame equations (5.64), (5.65) are compatible, and the solution ψ is a pair of frames adapted to a composite function of Ribaucour transformations.

Proof. First we show that $\psi^+ = \psi(\cdot, t+1)$ is an adapted frame for $\mathrm{x}^+ = \mathrm{x}^{(t+1)}$ if $\psi = \psi(\cdot, t)$ is adapted to $\mathrm{x} = \mathrm{x}(\cdot, t)$: We have for $1\le i\le m$, $1\le k\le N$:

$$
\begin{aligned}
A_{\psi^+}(\mathbf{e}_0) &= A_{\hat{v}_M}A_{\mathbf{e}_M\psi}(\mathbf{e}_0) = -A_{\hat{v}_M}(\hat{\mathrm{x}}) = \hat{\mathrm{x}}^+\,,\\
A_{\psi^+}(\mathbf{e}_i) &= A_{\hat{v}_M}A_{\mathbf{e}_M\psi}(\mathbf{e}_i) = -A_{\hat{v}_M}(\hat{v}_i) = \hat{v}_i^+\,,\\
\partial_i A_{\psi^+}(\mathbf{e}_k) &= \partial_i(A_{\hat{v}_M}A_{\mathbf{e}_M\psi}(\mathbf{e}_k)) \;=\; \pm\partial_i(A_{\hat{v}_M}A_\psi(\mathbf{e}_k))\\
&= \pm\partial_i(A_\psi(\mathbf{e}_k) - 2\langle\hat{v}_M, A_\psi(\mathbf{e}_k)\rangle\hat{v}_M)\\
&= \pm(\beta_{ki} - 2\alpha_i\langle\hat{v}_M, A_\psi(\mathbf{e}_k)\rangle)(\hat{v}_i - 2\langle\hat{v}_M,\hat{v}_i\rangle\hat{v}_M)\\
&= \pm(\beta_{ki} - 2\alpha_i\langle\hat{v}_M, A_\psi(\mathbf{e}_k)\rangle)\hat{v}_i^+ \;=\; \beta_{ki}^+\hat{v}_i^+,
\end{aligned}
$$

where the minus sign applies iff $k = M$ (and if $k = 0$, the latter calculation goes through almost literally, with replacing β_{ki}, β_{ki}^+ by h_i, h^+, respectively). Next, from $\Sigma_M = A^{-1}_{\mathbf{e}_M\psi}(\hat{v}_M)$ there follows:

$$
\begin{aligned}
\beta_{kM} &= 2\langle\Sigma_M, \mathbf{e}_k\rangle = 2\langle\hat{v}_M, A_{\mathbf{e}_M\psi}(\mathbf{e}_k)\rangle = -2\langle\hat{v}_M, A_\psi(\mathbf{e}_k)\rangle,\\
h_M &= 2\langle\Sigma_M, \mathbf{e}_0\rangle = 2\langle\hat{v}_M, A_{\mathbf{e}_M\psi}(\mathbf{e}_0)\rangle = -2\langle\hat{v}_M, \mathrm{x}\rangle,\\
N_M &= \langle\Sigma_M, \mathbf{e}_M\rangle = \langle\hat{v}_M, A_{\mathbf{e}_M\psi}(\mathbf{e}_M)\rangle = \langle\hat{v}_M, A_\psi(\mathbf{e}_M)\rangle.
\end{aligned}
$$

This proves eqs. (5.69)–(5.71). Further, eqs. (5.72)–(5.74) are readily derived by calculating the i-th partial derivative of the respective scalar product. Finally, eq. (5.75) comes from the consistency condition $\partial_i(\partial_j \hat{v}_M) = \partial_j(\partial_i \hat{v}_M)$.

Conversely, given a solution to the equations (5.25)–(5.27) and (5.69)–(5.75), the moving frame equations are consistent, thus defining the frame ψ. It follows from Proposition 5.1 that all $\hat{\mathrm{x}}^{(k)} = A_{\psi(\cdot,k)}(\mathbf{e}_0)$ are orthogonal systems. Furthermore, eq. (5.65), yields the defining relations (5.60)–(5.62) of the Ribaucour transformation, with $\hat{v}_M = A_{\mathbf{e}_M\psi}(\Sigma_M)$. ∎

5.3.2 Approximation of Ribaucour Transformations

The following is the direct analogue of Lemma 5.1, but for pairs of orthogonal systems:

Lemma 5.3 *If for a Jonas pair $\mathrm{x}^-, \mathrm{x}^+$, continuous or discrete, the net x^- is an orthogonal system, and the corresponding coordinate curves $\mathrm{x}^-\restriction_{\mathcal{B}_i^\epsilon}$ and $\mathrm{x}^+\restriction_{\mathcal{B}_i^\epsilon}$ envelope one–dimensional families of circles for all $i = 1, \ldots, m$, then $\mathrm{x}^-, \mathrm{x}^+$ form in fact a Ribaucour pair of orthogonal systems.*

As we did before, we reduce our considerations to the case of $M = 2$ dimensions, now with only $m = 1$ quasi-continuous and $m' = 1$ purely discrete direction. Recall that in this case, $\mathcal{B}^\epsilon(r, R) = \mathcal{B}_0^\epsilon(r) \times \mathcal{B}_+(R)$ consists of $R + 1$ copies of the discrete interval $\mathcal{B}_0^\epsilon(r)$. The shift τ_2 is still denoted by the superscript "+". We start again

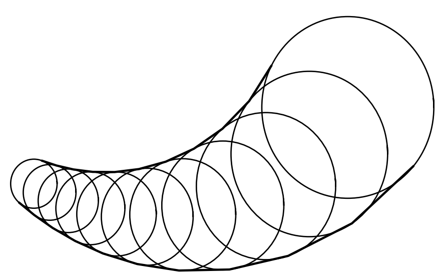

Figure 5.6: A pair of curves enveloping a circle congruence

from the system (5.48)–(5.51), but now $\epsilon_1 = \epsilon$ while $\epsilon_2 = 1$. A good resolution of the constraint (5.52), leading to two curves enveloping a circle congruence (Fig. 5.6) in the limit $\epsilon \to 0$, is to write

$$\rho_{21} = \epsilon\alpha\,, \qquad \rho_{12} = N_1\beta_{12} + \epsilon(N_2\beta_{21} - \Theta - \alpha); \tag{5.76}$$

with a suitable function $\alpha : \mathcal{B}^\epsilon(r, R) \to \mathbb{R}$.

As the Banach space where the system lives we choose, exactly as before,

$$\mathcal{X} = \hat{\mathcal{H}}_\psi \times \mathbb{R}_\beta^{2(N-1)} \times \mathbb{R}_h^2 \times \mathbb{R}_\alpha\,.$$

Also, we designate the subset

$$\mathbb{D} = \Big\{ (\psi, \beta, h_1, h_2, \alpha) : \sum_{k \neq 2} \beta_{k2}^2 < 4 \Big\} \subset \mathcal{X}. \tag{5.77}$$

Lemma 5.4 *The hyperbolic system (5.48)–(5.51) with (5.76) is consistent. The right-hand sides are $\mathcal{O}(\epsilon)$-convergent in $\mathbb{D}$; the limiting system for $\epsilon = 0$ consists of equations (5.23) with (5.24) for $i = 1$, (5.39) with (5.40) for $i = 2$, and*

$$\begin{aligned}
\partial_1 h_2 &= \beta_{12}(h_1 + \alpha h_2/2)\,, && (5.78)\\
\partial_1 \beta_{k2} &= \beta_{12}(\beta_{k1} + \alpha\beta_{k2}/2)\,, \quad k > 2\,, && (5.79)\\
\partial_1 \beta_{12} &= 2(N_2\beta_{21} - \Theta - \alpha) + \alpha\beta_{12}^2/2\,, && (5.80)\\
h_1^+ - h_1 &= \alpha h_2\,, && (5.81)\\
\beta_{k1}^+ - \beta_{k1} &= \alpha\beta_{k2}\,, \quad k > 2\,, && (5.82)\\
\beta_{21}^+ - \beta_{21} &= 2(N_2\alpha - \beta_{21})\,, && (5.83)
\end{aligned}$$

and describes Ribaucour transformations of curves.

Recall that, although the orthogonal system x itself does not appear in the hyperbolic system, is is immediately extracted from any solution as $\mathrm{x} := \lambda^{-1}(A_\psi(\mathbf{e}_0))$. In this sense it is to be understood that the solutions describe curves.

Proof. Consistency is shown exactly as in Lemma 5.2, the limiting system is calculated directly from (5.48)-(5.51) using (5.84). One sees that the system of the lemma coincides with (5.69)-(5.74) for $M = 2$.

In the right-hand sides of the equations, we have:

$$N_1^2 = 1 + O(\epsilon^2), \quad N_2^2 = 1 - \frac{1}{4}\sum_{k\neq 2}\beta_{k2}^2\,, \quad n = 1 - \frac{\epsilon}{2}\,\alpha\beta_{12} + O(\epsilon^2)\,, \tag{5.84}$$

where the constants in $\mathcal{O}$–symbols are uniform on compact subsets of $\mathcal{X}$. However, the system itself is not defined on all of $\mathcal{X}$ but only on the subset $\mathbb{D}$ where $N_2^2 > 0$. ∎

Goursat Problem 5.2 (for Ribaucour transformations of curves) *Given a frame $\Psi^\epsilon \in \mathcal{H}_\infty$, real functions H_1^ϵ and B_{k1}^ϵ on the quasi-continuous interval $\mathcal{B}_0^\epsilon(r)$, more real functions H_2^ϵ and B_{k2}^ϵ on the purely discrete interval $\mathcal{B}_+(R)$, and another real function A^ϵ on all $\mathcal{B}^\epsilon(r, R)$, find a solution to the discrete equations (5.48)-(5.51) with (5.76) if $\epsilon > 0$, or to the limiting system described in Lemma 5.4 above if $\epsilon = 0$, on $\mathcal{B}^\epsilon(r, 1)$ with the Goursat data*

$$\psi(0) = \Psi^\epsilon, \quad h_1\restriction_{\mathcal{B}_1^\epsilon} = H_1^\epsilon, \quad \beta_{k1}\restriction_{\mathcal{B}_1^\epsilon} = B_{k1}^\epsilon, \quad \alpha\restriction_{\mathcal{B}_1^\epsilon} = A^\epsilon,$$
$$h_2\restriction_{\mathcal{B}_2^\epsilon} = H_2^\epsilon, \quad \beta_{k2}\restriction_{\mathcal{B}_2^\epsilon} = B_{k2}^\epsilon.$$

We now go back to Ribaucour pairs, i.e. $R = 1$ in the following. It is not hard to see that in this case, the data H_2^ϵ and B_{k2}^ϵ need only be prescribed at one point, $\xi = 0$, and the auxiliary function A^ϵ needs only to be defined on the discrete interval $\mathcal{B}_0^\epsilon(r)$ in order to determine the geometry – the frame ψ on $\mathcal{B}^\epsilon(r, 1)$ – uniquely.

Proposition 5.5 (approximation of a circle congruence) *Let $m = 1$, $M = 2$, and let there be given: a smooth curve $\mathrm{X}^- : \mathcal{B}_1(\bar r) \to \mathbb{R}^N$, a smooth function $A : \mathcal{B}_1(\bar r) \to \mathbb{R}$, and a point $\mathrm{X}^+(0) \in \mathbb{R}^N$. Set $H_2(0) = |X^+(0) - X^-(0)|$, $\hat v_2(0) = H_2(0)^{-1}(X^+(0) - X^-(0))$. Assume that $\Psi \in \mathcal{H}_\infty$ is suited for $(\hat{\mathrm{X}}_1^-(0); \hat v_1(0))$, and set $\hat v_k(0) = A_\Psi(\mathbf{e}_k)$ for $k \neq 2$. Then, for some $r > 0$:*

1. *There exists a unique curve $\mathrm{X}^+ : \mathcal{B}_0(r) \to \mathbb{R}^N$ through the point $\mathrm{X}^+(0)$ such that the pair $(\mathrm{X}^-, \mathrm{X}^+)$ envelopes a circle congruence, and*

$$|\partial_1 \mathrm{X}^+| - |\partial_1 \mathrm{X}^-| = A \cdot |\mathrm{X}^+ - \mathrm{X}^-| \,. \tag{5.85}$$

2. *Consider the family of pairs of discrete curves $\mathrm{X}^{\epsilon-}, \mathrm{X}^{\epsilon+}$ defined as the solutions to the Goursat problem 5.2 with the data*

$$\Psi^\epsilon = \Psi, \quad H_1^\epsilon = [H_1]^\epsilon, \quad B_{k1}^\epsilon = [B_{k1}]^\epsilon, \quad A^\epsilon = [A]^\epsilon,$$
$$H_2^\epsilon = H_2(0)\,, \quad B_{k2}^\epsilon = -2\langle \hat v_2(0), \hat v_k(0)\rangle.$$

where the functions H_1 and B_{k1} are read off the smooth curve X^-, so that $\mathrm{X}^{\epsilon-}$ is the canonical ϵ-discretization of X^-. These discrete curves smoothly lattice-converge to $(\mathrm{X}^-, \mathrm{X}^+)$.

Proof follows from Lemma 5.4 and Theorem 2.2. ∎

Theorem 5.2 (approximation of a Ribaucour pair) *Let, in addition to the data of Theorem 5.1, there be given m curves $\mathrm{X}_i^+ : \mathcal{B}_i(\bar r) \to \mathbb{R}^N$ with a common intersection point $\mathrm{X}^+ = \mathrm{X}_1^+(0) = \ldots = \mathrm{X}_m^+(0)$, and such that each pair $(\mathrm{X}_i, \mathrm{X}_i^+)$ envelopes a circle congruence. In addition to the discrete surfaces $\mathcal{S}_{ij}^\epsilon$ from Theorem 5.1, construct discrete curves $\mathrm{X}_i^{\epsilon+}$ according to Proposition 5.5, with the functions $A_i = (|\partial_i \mathrm{X}_i^+| - |\partial_i \mathrm{X}_i|)/|\mathrm{X}_i^+ - \mathrm{X}_i|$. Then for some positive $r < \bar r$:*

1. *There exists a unique Ribaucour pair of orthogonal systems $\mathrm{x}^-, \mathrm{x}^+ : \mathcal{B}_0(r) \to \mathbb{R}^N$ such that x^- coincides with $\mathcal{S}_{ij}$ on $\mathcal{B}_{ij}(r)$, and x^+ coincides with X_i^+ on $\mathcal{B}_i(r)$.*

2. *There exists a unique family of Ribaucour pairs of discrete orthogonal systems $\{\mathrm{x}^\epsilon : \mathcal{B}^\epsilon(r) \to \mathbb{R}^N\}_{0<\epsilon<\epsilon_0}$ such that x coincides with $\mathcal{S}_{ij}^\epsilon$ on $\mathcal{B}_{ij}^\epsilon(r)$ and x^+ coincides with $\mathrm{X}_i^{\epsilon+}$ on $\mathcal{B}_i^\epsilon(r)$. The family x^ϵ lattice-converges to the pairs x, x^+ smoothly.*

Proof is similar to that of Theorem 5.1, with the only change in the concluding argument: for the limiting Jonas pair x, x^+ of orthogonal nets we derive from eq. (5.34):

$$\hat{v}_M \hat{v}_i^+ + \hat{v}_i \hat{v}_M = 0 \quad \Rightarrow \quad \langle \hat{v}_i + \hat{v}_i^+, \hat{v}_M \rangle = 0,$$

which is the defining property of the Ribaucour pair. ■

The above considerations are of course easily extended to the cases of more than one purely discrete directions: Two-dimensional Goursat problems are posed on all two-planes of the respective $\mathcal{B}^\epsilon(r,R)$, and the solution is extended into the interior of $\mathcal{B}^\epsilon(r,R)$ as a conjugate net (cf. the discussion there). The following theorem is an application of the Theorem 2.1 to the cases $m' = 2$ and $m' = 3$.

Theorem 5.3 (permutability of Ribaucour transformations) *1. Given an m-dimensional orthogonal system* $\mathrm{x}(\cdot,0,0) : \mathcal{B}_0^\epsilon(r) \to \mathbb{R}^N$ *and its two Ribaucour transformations* $\mathrm{x}(\cdot,1,0), \mathrm{x}(\cdot,0,1) : \mathcal{B}_0(r) \to \mathbb{R}^N$ *and, there is a one–parameter family of orthogonal systems* $\mathrm{x}(\cdot,1,1) : \mathcal{B}_0(r) \to \mathbb{R}^N$ *that are Ribaucour transformations of both* $\mathrm{x}(\cdot,1,0)$ *and* $\mathrm{x}(\cdot,0,1)$*. Corresponding points of the four conjugate nets are concircular.*

2. Given three Ribaucour transformations

$$\mathrm{x}(\cdot,1,0,0), \mathrm{x}(\cdot,0,1,0), \mathrm{x}(\cdot,0,0,1) : \mathcal{B}_0(r) \to \mathbb{R}^N$$

of a given m–dimensional orthogonal system $\mathrm{x}(\cdot,0,0,0) : PP_0(r) \to \mathbb{R}^N$*, as well as three further orthogonal systems*

$$\mathrm{x}(\cdot,1,1,0), \mathrm{x}(\cdot,0,1,1), \mathrm{x}(\cdot,1,0,1) : \mathcal{B}_0(r) \to \mathbb{R}^N$$

such that $\mathrm{x}(\cdot,1,1,0)$ *is a Ribaucour transformation of both* $\mathrm{x}(\cdot,1,0,0)$ *and* $\mathrm{x}(\cdot,0,1,0)$ *etc., there exists generically a unique orthogonal system*

$$\mathrm{x}(\cdot,1,1,1) : \mathcal{B}_0(r) \to \mathbb{R}^N$$

which is a Ribaucour transformation of all three $\mathrm{x}(\cdot,1,1,0), \mathrm{x}(\cdot,0,1,1)$ *and* $\mathrm{x}(\cdot,1,0,1)$.

Chapter 6

Approximation Theorem for Cauchy Problems

6.1 The Approximation Theorem

Discretizations of the following semi-linear Cauchy problem are investigated

$$\begin{aligned} \partial_t u(t,x) &= M\,\partial_x u(t,x) + f(u(t,x)) \\ u(0,x) &= u_0(x). \end{aligned} \tag{6.1}$$

Here $u : \Omega \subset \mathbb{R}^2 \to \mathbb{C}^d$ is a function of two arguments, x and t. The matrix M is constant and the nonlinearity f is analytic. By the classical Cauchy-Kowalevsky Theorem [Ta], problem (6.1) has a local solution about any point $(x,t) = (x_0, 0)$, provided that u_0 is analytic at x_0.

Now consider *discrete* Cauchy problems of a similar form: u is replaced by a function v^ϵ defined on a square grid of mesh size ϵ, and differential operators are replaced by suitable difference operators. So formally,

$$\delta_t v^\epsilon = M\,\delta_x v^\epsilon + f^\epsilon(v^\epsilon). \tag{6.2}$$

Initial values are calculated from the original data u_0. The solution v^ϵ to (6.2) is readily constructed by discrete time-iteration.

The main Theorem 6.2 states that – under mild hypotheses on f^ϵ – the solutions v^ϵ are convergent to the solution u of (6.1) as $\epsilon \to 0$. The convergence is uniform with an approximation error quadratic in ϵ. Furthermore, central difference quotients of v^ϵ of arbitrary order converge to the respective derivatives of u in the same manner, justifying the notion that the convergence is C^∞. This result is of interest of its own since it provides approximation of solutions to a Cauchy problem by an explicit, natural and extremely simple scheme. Moreover, the proof also provides a new approach to showing existence of solutions to (6.1).

Two situations in which Theorem 6.2 immediately applies are given below: Solutions to an ill-posed elliptic and parabolic problem are approximated by discrete functions.

The applications of main interest in this thesis, i.e., the convergence theorems for CR-1–mappings and orthogonal circle patterns, are deferred to the next chapter. It is likely that for other models from discrete differential geometry, which are related to elliptic PDEs or mixed elliptic/hyperbolic systems (e.g. isothermic and minimal surfaces [BHS]), a similar method can be applied to prove approximation of the respective smooth models.

The strategy to obtain the convergence result is closely related to the proof of an abstract version of the Cauchy-Kowalevsky Theorem [N], which is formulated in a scale of Banach spaces. (See also [Tr]. Furthermore, [Wa] provides a simple proof and an overview over the history of abstract Cauchy problems.) In the situation of problem (6.1), the Banach spaces are sets of analytic functions equipped with the maximum norm. The key step in the proof here is the definition of analogous Banach spaces and norms for functions only defined on a grid.

6.1.1 Main Theorem

A function $u : \mathrm{I} \to \mathbb{C}^d$ defined on a closed interval $\mathrm{I}_\xi = [-\xi, +\xi] \subset \mathbb{R}$ is called *analytic*, if it is C^∞-smooth and there exists a $\rho > 0$ such that the Taylor series at any point $x \in \mathrm{I}$ has radius of convergence larger than ρ. A function $\tilde{u} : \mathcal{D} \to \mathbb{C}^d$ is called a *complex extension* of $u : \mathrm{I} \to \mathbb{C}^d$, if $\tilde{u}$ is (complex) analytic on the open complex domain $\mathcal{D} \subset \mathbb{C}$, and $\tilde{u}(x) = u(x)$ for $x \in \mathcal{D} \cap \mathrm{I}$.

Solutions $u = (u_{(1)}, \ldots, u_{(d)}) : \Omega \to \mathbb{C}^d$ to problem (6.1) are considered on diamond-shaped domains

$$\Omega = \Omega(r) = \left\{ (t, x) \in \mathbb{R}^2 \middle| |x| + |t| \le r \right\}.$$

We assume that the $d \times d$-matrix M is constant and f is analytic near $u_0(0) \in \mathbb{C}^d$. The abstract version of the Cauchy-Kowalevsky-Theorem [N] implies

Theorem 6.1 *Suppose u_0 is analytic on I_ξ. Then there exists a positive $r < \xi$ for which the problem (6.1) has a solution u on $\Omega = \Omega(r)$, analytic in x.*

Replace u by a function $v^\epsilon = (v^\epsilon_{(1)}, \ldots, v^\epsilon_{(d)})$ defined on the discrete set

$$\Omega^\epsilon = \Omega^\epsilon(r) = \left\{ (t, x) \in \Omega(r) \middle| x + t \in \epsilon\mathbb{Z} \right\},$$

which is the intersection of $\Omega(r)$ with the 45-degree rotated standard lattice $(\lambda\mathbb{Z})^2$ of mesh size $\lambda = \epsilon/\sqrt{2}$. Also, its “dual” lattice

$$\Omega^\epsilon_*(r) = \left\{ (t, x) \in \Omega(r - \tfrac{\epsilon}{2}) \middle| \tfrac{\epsilon}{2} + x + t \in \epsilon\mathbb{Z} \right\}$$

will frequently be used. Define for each $\epsilon > 0$ the canonical difference quotient

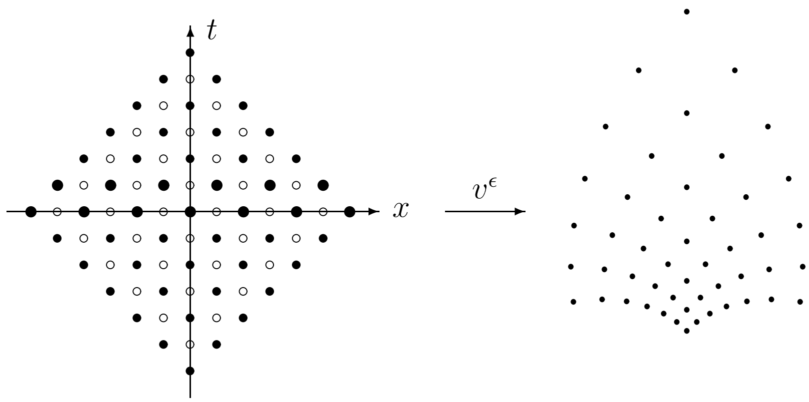

Figure 6.1: A CR-1–mapping $v^\epsilon : \Omega^\epsilon(r) \to \mathbb{R}^2$. Left points of $\Omega^\epsilon(r)$ and Ω^ϵ_* (marked $\bullet$ and $\circ$, respectively) are shown. Initial data for v^ϵ is prescribed at the bold marked sites.

operators

$$\begin{aligned}
(\delta_x v^\epsilon)(t,x) &= \tfrac{1}{\epsilon}\left(v^\epsilon(t,x+\tfrac{\epsilon}{2}) - v^\epsilon(t,x-\tfrac{\epsilon}{2})\right)\\
(\delta_t v^\epsilon)(t,x) &= \tfrac{1}{\epsilon}\left(v^\epsilon(t+\tfrac{\epsilon}{2},x) - v^\epsilon(t-\tfrac{\epsilon}{2},x)\right),
\end{aligned}$$

which map functions on Ω^ϵ to functions on Ω^ϵ_*. The problem (6.1) is replaced by

$$\begin{aligned}
\delta_t v^\epsilon(t,x) &= M\delta_x v^\epsilon(t,x) + f^\epsilon(v^\epsilon)(t,x) && ((t,x)\in\Omega^\epsilon_*)\\
v^\epsilon(0,x) &= v^\epsilon_0(x) && ((0,x)\in\Omega^\epsilon)\\
v^\epsilon(\tfrac{\epsilon}{2},x) &= v^\epsilon_+(x) && ((\tfrac{\epsilon}{2},x)\in\Omega^\epsilon)
\end{aligned}\tag{6.3}$$

The nonlinearity $F^\epsilon(v)(t,x)$ is of the form

$$f^\epsilon(v^\epsilon)(t,x) = F^\epsilon(v^\epsilon(t,x+\tfrac{\epsilon}{2}), v^\epsilon(t,x-\tfrac{\epsilon}{2}), v^\epsilon(t-\tfrac{\epsilon}{2},x)).$$

Theorem 6.2 *Assume that u_0 is analytic on some interval I_ξ, that f is analytic on a neighborhood $\mathcal{D}\subset\mathbb{C}^d$ of $u_0(0)$ and that F^ϵ is analytic on $\mathcal{D}\times\mathcal{D}\times\mathcal{D}\subset\mathbb{C}^{3d}$ for all $\epsilon>0$. Furthermore, assume that*

$$|F^\epsilon(u^+,u^-,u^*) - f(\tfrac{u^+ + u^-}{2})| \le K(\epsilon + |u^+ - u^-|)^2 \tag{6.4}$$

holds for all u^+, u^-, $u^*\in\mathcal{D}$ with $K>0$ independent of ϵ.

Then there exists a positive $r<\xi$ such that the solutions v^ϵ to problem (6.3) on $\Omega^\epsilon(r)$ with the initial data

$$\begin{aligned}
v^\epsilon_0(x) &= u_0(x) && ((0,x)\in\Omega^\epsilon(r))\\
v^\epsilon_+(x) &= u_0(x) + \tfrac{\epsilon}{2}\Big(M\partial_x u_0(x) + f(u_0(x))\Big) && ((\tfrac{\epsilon}{2},x)\in\Omega^\epsilon_*(r))
\end{aligned}\tag{6.5}$$

converge to a continuous function u on $\Omega(r)$. In particular, the following estimate holds

$$\sup_{(t,x)\in\Omega^\epsilon(r)} |u(t,x) - v^\epsilon(t,x)| \leq C\epsilon^2 \tag{6.6}$$

with some $C > 0$. The function u is analytic in x and differentiable in t, and constitutes a classical solution to Cauchy problem (6.1). Moreover, u is the only classical solution to (6.1) on $\Omega(r)$ in the class of x-analytic functions.

Note that Theorem 6.1 is a corollary of Theorem 6.2. In other words, the proof of Theorem 6.2 also provides an alternative proof for the local solvability of the Cauchy problem 6.1.

Certain "soft" perturbations of the initial conditions v_0^ϵ and v_+^ϵ are allowed.

Theorem 6.3 *Assume $h_0^\epsilon, h_+^\epsilon : \mathcal{D}' \to \mathbb{C}^d$ are families of analytic functions on a common complex neighborhood $\mathcal{D}'$ of $x = 0$, bounded independently of ϵ there. Under the hypotheses of Theorem 6.2, the estimate (6.6) still holds if the initial conditions v_0^ϵ, v_+^ϵ are replaced by $v_0^\epsilon + \epsilon^2 h_0^\epsilon$, $v_+^\epsilon + \epsilon^2 h_+^\epsilon$.*

The convergence of v^ϵ to u turns out to be C^∞. To make this precise, introduce in addition to $\Omega^\epsilon(r)$ and $\Omega_*^\epsilon(r)$ also the "shrunk" domains $\Omega_k^\epsilon(r)$ by

$$\Omega_k^\epsilon(r) = \begin{cases} \Omega^\epsilon(r - k\epsilon) \text{ if } k \text{ is even} \\ \Omega_*^\epsilon(r - (k-1)\epsilon) \text{ if } k \text{ is odd} \end{cases}$$

The central difference quotients $\delta_x^m \delta_t^n \psi^\epsilon$ of a function ψ^ϵ on $\Omega^\epsilon(r)$ are naturally defined at the points of $\Omega_{m+n}^\epsilon(r)$.

Theorem 6.4 *Under the hypothesis of Theorem 6.2, there are real numbers C_{mn} such that*

$$\sup_{(x,t)\in\Omega_{m+n}^\epsilon(r)} |\delta_x^m \delta_t^n v^\epsilon(x,t) - \partial_x^m \partial_t^n u(x,t)| \leq C_{mn}\epsilon^2. \tag{6.7}$$

In particular, u is infinitely often differentiable.

A remark is in order here: The error estimate of order $\mathcal{O}(\epsilon^2)$ in (6.7) is only possible because we consider *central* difference quotients here. For the usual one-sided difference quotients used in the previous chapters, one would face an error of order $\mathcal{O}(\epsilon)$ even if v^ϵ is *identical* to u on the grid. For central difference quotients, it is an exercise in elementary calculus to prove that

$$\begin{aligned} &\sup_{(x,t)\in\Omega_{m+n}^\epsilon(r)} |\delta_x^m \delta_t^n u^\epsilon(x,t) - \partial_x^m \partial_t^n u(x,t)| \\ &\leq (m+n) \sup_{(x,t)\in\Omega(r)} (|\partial_x^m \partial_t^{n+2} u(x,t)| + |\partial_x^{m+2} \partial_t^n u(x,t)|)\epsilon^2 \end{aligned}$$

for an arbitrary functions $u : \Omega(r) \to \mathbb{C}^d$ of class C^{m+n+2}. Here u^ϵ denotes the restriction of u to $\Omega^\epsilon(r)$.

Before the proofs of Theorems 6.2 - 6.4 are given in section 6.2, we shortly present two canonical applications.

6.1.2 Examples: Two Ill-Posed Problems

As pointed out before, our main applications of Theorem 6.2 is to derive convergence results for orthogonal circle patterns and CR-1–mappings towards conformal mappings. However, the following two basic examples are intended to show that the Theorem is useful in many more situations (not necessarily geometrically motivated).

A Nonlinear Elliptic Problem

Rewrite the ill-posed elliptic problem

$$\begin{aligned} \partial_t^2\phi(t,x) + \partial_x^2\phi(t,x) &= g(\phi(x,t)) && (6.8)\\ \phi(0,x) &= \phi_0(x) && (6.9)\\ \partial_t\phi(0,x) &= \phi_+(x) && (6.10)\end{aligned}$$

for the scalar function ϕ in the form (6.1):

$$\partial_t \begin{pmatrix} u_{(1)} \\ u_{(2)} \\ u_{(3)} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \partial_x \begin{pmatrix} u_{(1)} \\ u_{(2)} \\ u_{(3)} \end{pmatrix} + \begin{pmatrix} u_{(3)} \\ 0 \\ g(u_{(1)}) \end{pmatrix}. \qquad (6.11)$$

Here $u_{(1)} = \phi$, $u_{(2)} = \partial_x\phi$ and $u_{(3)} = \partial_t\phi$. For functions ϕ_0, ϕ_+ analytic on $[-\xi, +\xi]$, the respective initial data $u_{0(1)}(x) = \phi_0(x)$, $u_{0(2)} = \partial_x\phi_0(x)$, $u_{0(3)} = \phi_+(x)$ are analytic there, too. Assuming further that the nonlinearity g is analytic at $\phi_0(0)$, a solution to (6.11) exists on some $\Omega = \Omega(r)$ and is unique.

To approximate (6.11) by a discrete problem (6.3), pick

$$F^\epsilon(v^+, v^-, v^*) = \tfrac{1}{2}(v^+_{(3)} + v^-_{(3)}, 0, g(v^+_{(1)}) + g(v^-_{(1)}))$$

as nonlinearity, independent of ϵ (and of v^*). The assumptions of Theorem 6.2 concerning analyticity and boundedness of F^ϵ are obviously fulfilled. A Taylor expansion of g around $\bar V = (v^+_{(1)} + v^-_{(1)})/2$

$$\tfrac{1}{2}(g(v^+_{(1)}) + g(v^-_{(1)})) = g(\bar V) + \tfrac{1}{2}(v^+_{(1)} - v^-_{(1)})g'(\bar V) + \mathcal{O}(|v^+ - v^-|^2)$$

implies the estimate (6.4). Due to Theorem 6.2, the component $v^\epsilon_{(1)}$ of the discrete solutions to problem (6.3) converges to the smooth solution ϕ on $\Omega(r')$ with a suitable positive $r' < r$ in the sense of (6.6).

Sidewards heat equation

The 1+1-dimensional sidewards heat equation

$$\partial_t^2\phi(t,x) = \partial_x\phi(t,x) + g(\phi(t,x)) \qquad (6.12)$$

is obtained from the standard heat equation after exchanging roles of time and space. Posing initial data $\phi(0,x) = \phi_0(x)$ and $\partial_t\phi(0,x) = \phi_+(x)$ corresponds to prescribing the temperature and its spatial derivative at a fixed point over some time interval in the physical world. Introducing $u_{(1)} = \phi$ and $u_{(2)} = \partial_t\phi$, equation (6.12) is equivalent to

$$\partial_t \begin{pmatrix} u_{(1)} \\ u_{(2)} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \partial_x \begin{pmatrix} u_{(1)} \\ u_{(2)} \end{pmatrix} + \begin{pmatrix} u_{(2)} \\ g(u_{(1)}) \end{pmatrix}.$$

Local solvability is guaranteed by Theorem 6.2 for analytic nonlinearity g and analytic initial data ϕ^0, ϕ^+. A discrete scheme of type (6.3) with convergent solutions is obtained in the same fashion as for elliptic problems.

6.2 Proof of the Main Theorem

Let us – informally – recall the classical existence theorem by Nirenberg first: In [N], abstract Cauchy problems

$$\partial_t u = F(u), \quad u(0) = u_0 \tag{6.13}$$

are considered. It is assumed that the action of the non-linear operator F is "not worse" than that of ∂_z on an analytic function, i.e., F satisfies a certain Cauchy-type estimate. To show existence of a solution to problem (6.13) locally in time, it is rewritten as an integral equation

$$u(t) = J(u)(t) := u_0 + \int_0^t F(u(s))ds.$$

A scale of Banach spaces $X_1 \subset X_2 \subset \cdots \subset X_\infty$ is defined so that J maps X_k into X_{k+1} and is a contraction. By adaption of the Banach contraction principle to scales of spaces, it is shown that the iteration $u^{(k+1)} := J(u^{(k)})$ converges to a solution $u \in X_\infty$. The procedure is a simple example of the so-called Nash-Moser-iteration.

Our general strategy to prove Theorem 6.2 is to translate the idea to consider *scales of Banach spaces* and the essential estimates from the proof of the contraction property

$$\|u^{(k+1)} - u^{(k)}\|_{(k+1)} \leq L\|u^{(k)} - u^{(k-1)}\|_{(k)}$$

to the discrete situation. Below, we define a scale of norms on (finite-dimensional) spaces of discrete functions. In these norms, we are able to estimate the difference $u - v^\epsilon$ between the smooth and the discrete solution at some instant of time in terms of their difference at the previous time step; roughly, we obtain

$$\|v^\epsilon(t + \tfrac{\epsilon}{2}) - u(t + \tfrac{\epsilon}{2})\|_{(t+\frac{\epsilon}{2})} \leq L\|v^\epsilon(t) - u(t)\|_{(t)}.$$

The norms for discrete functions are by no means immediate generalizations of the ones used in the continuous setting. For complex-analytic functions u, one usually considers the supremum of $|u|$ on a complex disc of prescribed diameter; there exists no obvious analogue for discrete functions v^ϵ, which are only defined on finitely many points on the real axis. It is a nontrivial fact (see Lemma 6.1) that our discrete norms posses the fundamental properties of the supremum-norm, which are essential to derive the contraction property.

6.2.1 Discrete norms and their properties

Without loss of generality, we assume that $|u|^2 = u_{(1)}^2 + \cdots + u_{(d)}^2$ and $|Mu| \leq u$ for all $u \in \mathbb{C}^d$. Let $D(\rho) \subset \mathbb{C}$ denote the complex disc of radius ρ centered at 0, and for $p > 1$, $D^p(\rho) = D(\rho) \times \cdots \times D(\rho) \subset \mathbb{C}^p$ is a p-dimensional poly-disc. Given an interval $\mathrm{I}_\xi = [-\xi, +\xi]$ and $\rho \geq 0$, then

$$B_\rho(\mathrm{I}_\xi) = \{z \in \mathbb{C} \mid \mathrm{dist}(z, \mathrm{I}_\xi) \leq \rho\}$$

is its complex extension of width ρ. For an integer $m \geq 0$, define discrete intervals

$$\mathcal{I}_m^\epsilon = \begin{cases} \{-\frac{m}{2}\epsilon, \ldots, -\epsilon, 0, \epsilon, \ldots, \frac{m}{2}\epsilon\} & \text{if } m \text{ is even} \\ \{-\frac{m}{2}\epsilon, \ldots, -\frac{\epsilon}{2}, \frac{\epsilon}{2}, \ldots, \frac{m}{2}\epsilon\} & \text{if } m \text{ is odd} \end{cases}$$

containing $m+1$ points. Given a function $v : \mathcal{I}_m^\epsilon \to \mathbb{C}^d$, its difference quotient with respect to x is

$$\delta_x v : \mathcal{I}_{m-1}^\epsilon \to \mathbb{C}^d, \quad (\delta_x v)(x) = \tfrac{1}{\epsilon}\left(v(x + \tfrac{\epsilon}{2}) - v(x - \tfrac{\epsilon}{2})\right).$$

Define also the half-shifts and the restriction

$$\begin{aligned} \tau_\pm v : \mathcal{I}_{m-1}^\epsilon \to \mathbb{C}^d, &\qquad (\tau_\pm v)(x) = v(x \pm \tfrac{\epsilon}{2}) \\ \pi v : \mathcal{I}_{m-2}^\epsilon \to \mathbb{C}^d, &\qquad (\pi v)(x) = v(x). \end{aligned}$$

as well as the linear interpolation

$$Ev : [-\tfrac{\epsilon}{2}m, +\tfrac{\epsilon}{2}m] \to \mathbb{C}^d,\ (Ev)(x) = (1 - \frac{x - x^\epsilon}{\epsilon})v(x^\epsilon) + \frac{x - x^\epsilon}{\epsilon}v(x^\epsilon + \epsilon),$$

where $x^\epsilon \in \mathcal{I}_m^\epsilon$ and $0 \leq x - x^\epsilon < \epsilon$. For $\rho > 0$, define a functional on discrete functions $v : \mathcal{I}_n^\epsilon \to \mathbb{C}^d$ by

$$\|v\|_\rho = \sum_{k=0}^{n} \frac{\rho^k}{k!} \max_{x \in \mathcal{I}_{n-k}^\epsilon} |\delta^k v(x)|.$$

The following lemma summarizes the essential properties of the $\|\cdot\|_\rho$. These norms behave in many ways similar to the maximum-norm for analytic functions. In particular, the estimate on difference quotients (item 2) comes as a substitute for the classical Cauchy estimate

$$\sup_{|z|<\rho} |\partial_z u(z)| \leq \frac{1}{\rho' - \rho} \sup_{|z|<\rho'} |u(z)|,$$

which holds for analytic functions $u : D(\rho') \to \mathbb{C}^d$.

Lemma 6.1 *The functionals $\|\cdot\|_\rho$ provide a scale of norms on discrete functions: For $u : \mathfrak{I}^\epsilon_n \to \mathbb{C}^d$ and $0 \le \rho \le \rho'$, one has $\|u\|_\rho \le \|u\|_{\rho'}$. In addition, each $\|\cdot\|_\rho$ has the following properties:*

1. Submultiplicativity: $\|u_i\, v_i\|_\rho \le \|u_i\|_\rho \|v_i\|_\rho$.
2. Estimate on difference quotients: $\|u\|_\rho + \epsilon\|\delta u\|_\rho \le \|u\|_{\rho+\epsilon}$
3. Estimate on compositions: *If f is analytic on $D^d(\gamma U)$, $\gamma > 1$, and $\|u\|_\rho \le U$, then $f(u)$ is defined on $\mathfrak{I}^\epsilon_n$ and*
$$\|f(u)\|_\rho \le (1 - \tfrac{1}{\gamma})^{-d} \sup_{D^d(\gamma U)} |f|. \tag{6.14}$$
4. Estimate on restrictions: *If $u : \mathrm{I} \to \mathbb{C}^d$ extends analytically to $B_{\rho'}(I)$ and u^ϵ is its restriction to $\mathfrak{I}^\epsilon_n \subset \mathrm{I}$, then, provided $\rho < \rho'$,*
$$\|u^\epsilon\|_\rho \le (1 - \rho/\rho')^{-1} \sup_{B_{\rho'}(I)} |u|. \tag{6.15}$$
5. Completeness: *For a sequence $\epsilon \to 0$, let functions $v^\epsilon : \mathfrak{I}^\epsilon_{n^\epsilon} \to \mathbb{C}^d$ with $n^\epsilon \epsilon \ge \xi \ge 0$ and $\|v^\epsilon\|_\rho \le C$ be given. Then there exists an analytic $u : \mathrm{I}_\xi \to \mathbb{C}^d$, possessing a complex extension to $B_\rho(\mathrm{I}_\xi)$ which is bounded by C there, so that: $E\delta^k_x v^{\epsilon(k)}$ converges to $\partial^k_x u$ uniformly for each $k \ge 0$ and a suitable subsequence $\epsilon(k) \to 0$ of ϵ.*

This lemma is proven in subsection 6.2.7.

In the following, it will be shown that for an appropriate $r > 0$ and ϵ small enough, discrete solutions v^ϵ exist on $\Omega^\epsilon(r)$, and a limiting function u can be defined on $\Omega(r)$. Let $N = [\frac{2r}{\epsilon}]$ be the largest integer such that $\frac{\epsilon}{2}N \le r$. The integer variable $n \le N$ denotes the discrete time step, i.e., $t = \epsilon n$. Let $n' = N - n$.

For integers n with $|n| \le N$, the functions

$$\begin{aligned} u^\epsilon_n(\cdot) &:= u(\tfrac{\epsilon}{2}n, \cdot) \\ v^\epsilon_n(\cdot) &:= v(\tfrac{\epsilon}{2}n, \cdot) \\ w^\epsilon_n &:= v^\epsilon - u^\epsilon \end{aligned}$$

are naturally defined on $\mathfrak{I}^\epsilon_{n'}$. In our notations, the problem (6.3) takes the convenient form

$$v^\epsilon_{n+1} = \pi v^\epsilon_{n-1} + \epsilon\delta_x v^\epsilon_n + \epsilon f^\epsilon_n(v^\epsilon) \tag{6.16}$$
$$f^\epsilon_n(v^\epsilon) = F^\epsilon(\tau_+ v^\epsilon_n, \tau_- v^\epsilon_n, \pi v^\epsilon_{n-1}) \tag{6.17}$$

where pointwise evaluation of the arguments is understood.

6.2.2 Existence of a Continuous Solution

Without loss of generality, it can be assumed that $u_0(0) = 0$. (If $\bar{u} := u_0(0) \neq 0$ in the beginning, replace $u_0(x) \to u_0(x) - \bar{u}$, $f(u) \to f(u+\bar{u})$ etc.) For an appropriate choice of $U > 0$, f is defined on $D^d(2U)$ and F^ϵ on $D^{3d}(2U)$, respectively. Let $\mathcal{F} = 2^{3d} \sup_{\epsilon>0} \sup_{v^+,v^-,v^* \in D^d(U)} |F^\epsilon(v^+, v^-, v^*)|$.

Choose $\xi' \in (0, \xi)$ and $\epsilon_0 > 0$ so that

$$|u_0(x)|,\ |u_0(x) + \tfrac{\epsilon}{2}(M\partial_x u_0(x) + f(u_0(x)))| \leq \tfrac{U}{8} \tag{6.18}$$

for all $x \in \mathcal{I}^\epsilon_{\xi'}$ and all positive $\epsilon < \epsilon_0$. Since u_0 is analytic, it has a complex extension to a neighborhood of $\mathrm{I}_{\xi'}$. By the property 4 of Lemma 6.1, there is some $\rho > 0$ such that

$$\|v_0^\epsilon\|_\rho,\ \|v_+^\epsilon\|_\rho \leq \tfrac{U}{4} \tag{6.19}$$

Choose $r > 0$ such that the following inequalities are fulfilled:

$$\text{I. } 4r \leq \rho,\ \text{II. } r \leq \xi',\ \text{III. } 2r\mathcal{F} \leq U. \tag{6.20}$$

The following estimates are carried out for $n \geq 0$; the case of negative times is treated analogously. It will now be shown that for $\rho_n = 4r - n\epsilon$

$$\mathcal{M}_n \ := \ \|v_n\|_{\rho_n} + \|v_{n-1}\|_{\rho_n} \leq U(\tfrac{1}{2} + \tfrac{n}{2N}). \tag{6.21}$$

Let $\epsilon < \epsilon_0$ be arbitrary. Then $\mathcal{M}_1 \leq U/2$ by I. and II. of (6.20). Proceed inductively: Suppose (6.21) holds for n. Then $\mathcal{M}_n \leq U$ and $f_n^\epsilon(v^\epsilon)$ is defined, so

$$\mathcal{M}_{n+1} = \|v^\epsilon_{n+1}\|_{\rho_n} + \|v^\epsilon_n\|_{\rho_n} \leq \|v^\epsilon_n\|_{\rho_n} + \epsilon\|\delta_x v^\epsilon_n\|_{\rho_n} + \|v^\epsilon_{n-1}\|_{\rho_n} + \epsilon\|f^\epsilon_n(v)\|_{\rho_n}$$

by (6.16). The sum can be estimated using properties 2 and 3 ($\alpha = 2$, $p = 3d$) of Lemma 6.1 as follows

$$\leq \|v^\epsilon_n\|_{\rho_n+\epsilon} + \|v^\epsilon_{n-1}\|_{\rho_n} + \epsilon\|F^\epsilon(\tau_+ v^\epsilon_n, \tau_- v^\epsilon_n, \pi v^\epsilon_{n-1})\|_{\rho_n} \leq \mathcal{M}_n + \epsilon\mathcal{F}.$$

Estimate (6.21) now follows from III.

Observe that, in particular, for $x \in \mathcal{I}^\epsilon_{n'}$ and $x' \in \mathcal{I}^\epsilon_{n'-1}$

$$\begin{aligned} |v^\epsilon_n(x)| &\leq \|v^\epsilon_n\|_{\rho_n} \leq U \\ |\delta_x v^\epsilon_n(x')| &\leq \tfrac{1}{\rho_n}\|v^\epsilon_n\|_{\rho_n} \leq U/(3r) \\ |\delta_t v^\epsilon_n(x')| &\leq |\delta_x v^\epsilon_n(x')| + |f_n(v^\epsilon)(x')| \leq U/(3r) + \mathcal{F}. \end{aligned}$$

From each discrete solution v^ϵ, construct two continuous functions v^ϵ_I, v^ϵ_{II} as follows:

$$v^\epsilon_I(t,x) = (1 - \frac{t - \frac{\epsilon}{2}n_\epsilon}{\epsilon})Ev^\epsilon_{n_\epsilon+2}(x) + \frac{t - \frac{\epsilon}{2}n_\epsilon}{\epsilon}Ev^\epsilon_{n_\epsilon}(x)$$

where n_ϵ is the largest *odd* integer smaller than t; for v^ϵ_{II}, take n_ϵ to be the largest *even* integer, respectively. The two families of functions $\{v^\epsilon_I\}_\epsilon$, $\{v^\epsilon_{II}\}_\epsilon$ are equicontinuous as can be seen from the estimates on v^ϵ and it difference quotients above. By the Arzela-Ascoli theorem, there is a sequence $\epsilon' \to 0$ such that $v_I^{\epsilon'}$ and $v_{II}^{\epsilon'}$ converge uniformly to a continuous functions u_I and u_{II} on $\Omega(r)$, respectively. Moreover, it is not hard to see (cf. the proof of property 5 in Lemma 6.1) that there is a subsequence $\epsilon'' \to 0$ of ϵ' such that $\delta_x v_{I/II}^{\epsilon''}$ are uniformly convergent to $\partial_x u_{I/II}$. Also, $f_n^\epsilon(v_{I/II}^\epsilon)$ converge to $f(u_{I/II})$ uniformly as can be deduced from the estimate (6.4). Since v^ϵ solves (6.16), it is easily verified that for (t,x), $(t',x) \in \Omega(r)$

$$v_{I/II}^{\epsilon''}(t',x) = v_{I/II}^{\epsilon''}(t,x) + \int_t^{t'} (\delta_x v_{II/I}^{\epsilon''}(s,x) + f^{\epsilon''}(v_{II/I}^{\epsilon''})(s,x))\, ds + \mathcal{O}(\epsilon'')$$

Passing to the limit $\epsilon'' \to 0$ (observe that the term under the integral sign converges uniformly)

$$u_{I/II}(t',x) = u_{I/II}(t,x) + \int_t^{t'} (\partial_x u_{II/I}(s,x) + f(u_{II/I})(s,x))\, ds$$

from where it is obvious that u_I and u_{II} are differentiable in time and satisfy

$$\partial_t \begin{pmatrix} u_I \\ u_{II} \end{pmatrix} = \begin{pmatrix} 0 & M \\ M & 0 \end{pmatrix} \partial_x \begin{pmatrix} u_I \\ u_{II} \end{pmatrix} + \begin{pmatrix} f(u_{II}) \\ f(u_I) \end{pmatrix} \tag{6.22}$$

By uniqueness of x-analytic solutions to the problems of type (6.1) – which is proven in subsection 6.2.4 – a solution to (6.22) with initial conditions $u_I(0,x) = u_{II}(0,x) = u_0(x)$ must satisfy $u_I = u_{II} =: u$ on $\Omega(r)$ and hence u solves (6.1) with $u(0,x) = u_0(x)$ for $x \in \mathrm{I}_r$.

Regularity

By property 5 of Lemma 6.1, the estimates $\|v_n^\epsilon\|_{\rho_n} \leq U$ imply for fixed $t \in [-r,r]$ that $u(t,x)$ extends x-analytically to $B_{4r-2|t|}(\mathrm{I}_{r-|t|})$, where it is bounded by U. It is easy to see, using the Cauchy estimate (6.36), that for integers $k \geq 0$,

$$|\partial_x^k u(t,x)| \leq Uk!\, r^{-k}, \quad |t| < r,\ x \in B_{3r-2|t|}(\mathrm{I}_{r-|t|}).$$

It is also clear that $\partial_t^k u(t,x)$ is analytic in $x \in B_{3r-2|t|}(\mathrm{I}_{r-|t|})$, and bounded by a expression depending on k, U, r and the function f, but independent of x and t. Hence

$$\mathcal{Q} = \max_{k=0,1,2,3} \sup_{|t|<r} \sup_{x \in B_{3r-2|t|}(\mathrm{I}_{r-t})} (|\partial_x^k u(t,x)| + |\partial_t^k u(t,x)|) \tag{6.23}$$

is a finite quantity.

6.2.3 Approximation

The idea of the proof is to calculate bounds on the functions

$$\mathcal{L}_n \quad := \quad \|w_n\|_{n'\epsilon} + \|w_{n-1}\|_{n'\epsilon} \tag{6.24}$$

which is defined for $|n| \leq N$. Again, only $n \geq 0$ is considered here, and the analogous case $n \leq 0$ is left to the reader. It will be shown that

$$\mathcal{L}_n \leq (1 + \epsilon B^*)\mathcal{L}_{n-1} + C^*\epsilon^3 \text{ and } \mathcal{L}_1 \leq D^*\epsilon^2, \tag{6.25}$$

leading by the standard Gronwall estimate to

$$\mathcal{L}_n \leq (C^* + D^*)e^{rB^*}\epsilon^2 \tag{6.26}$$

which implies the assertion of the Theorem with $K = (C^* + D^*)\exp(rB^*)$.

To prove the estimate (6.25) at the instant of time $t = \epsilon n$ with $n \geq 2$, express w_n in the definition (6.24) in terms of u and v at previous time steps:

$$\begin{aligned}
\mathcal{L}_n \quad \leq \quad & \|\pi w^\epsilon_{n-2} + \delta_x w^\epsilon_{n-1}\|_{n'\epsilon} + \|w^\epsilon_{n-1}\|_{n'\epsilon} & (A) \\
& + \epsilon\|F^\epsilon_n(v^\epsilon) - F^\epsilon_n(u^\epsilon)\|_{n'\epsilon} & (B) \\
& + \|\epsilon(M\delta_x u^\epsilon_{n-1} + F^\epsilon_n(u^\epsilon)) - (u^\epsilon_n - \pi u^\epsilon_{n-2})\|_{n'\epsilon} & (C)
\end{aligned}$$

The three resulting expressions (A)-(C) as well as $\mathcal{L}_1$ will be estimated separately in the following.

Estimates for expression (A)

By the properties of the discrete norms, especially estimate 2 of Lemma 6.1, one has

$$\begin{aligned}
(A) \quad & \leq \quad \|w^\epsilon_{n-1}\|_{n'\epsilon} + \epsilon\|\delta_x w^\epsilon_{n-1}\|_{n'\epsilon} + \|w^\epsilon_{n-2}\|_{n'\epsilon} \\
& \leq \quad \|w^\epsilon_{n-1}\|_{n'\epsilon+\epsilon} + \|w^\epsilon_{n-2}\|_{n'\epsilon} \ \leq \mathcal{L}_{n-1}
\end{aligned}$$

since $(n-1)' = n' + 1$.

Estimates for expression (B)

Recall an elementary result from calculus of several complex variables: If $g : \mathcal{D} \subset \mathbb{C}^p \rightarrow \mathbb{C}^d$ is an analytic function, then there exist p functions $g^*_{(j)} : \mathcal{D} \times \mathcal{D} \rightarrow \mathbb{C}^d$ (not uniquely determined for $p > 1$) which are analytic and satisfy for $\xi = (\xi_{(1)}, \ldots, \xi_{(p)})$, $\eta = (\eta_{(1)}, \ldots, \eta_{(p)}) \in \mathcal{D}$

$$g(\xi) - g(\eta) = \sum_{j \leq p} g^*_{(j)}(\xi, \eta)\,(\xi_{(j)} - \eta_{(j)})$$

Apply this to the case $g = F^\epsilon$ and $p = 3d$, $\mathcal{D} = D^{3d}(2U)$ with $\xi = (\tau_+ u^\epsilon, \tau_- u^\epsilon, u^\epsilon)$, $\eta = (\tau_+ v^\epsilon, \tau_- v^\epsilon, v^\epsilon)$. From the submultiplicativity property 1 stated in Lemma 6.1, it follows

$$\begin{aligned}
\|F^\epsilon(\xi) - F^\epsilon(\eta)\|_{n'\epsilon} &\leq \sum_{j\leq 3d} \|(F^\epsilon)^*_{(j)}(\xi,\eta)\,(\xi_{(j)} - \eta_{(j)})\|_{n'\epsilon} \\
&\leq \sum_{j\leq d} \Big(\|(F^\epsilon)^*_{(j)}(\xi,\eta)\|_{n'\epsilon}\,\|\tau_+ u^\epsilon_{(j)} - \tau_+ v^\epsilon_{(j)}\|_{n'\epsilon} \\
&\qquad + \|(F^\epsilon)^*_{(j+d)}(\xi,\eta)\|_{n'\epsilon}\,\|\tau_- u^\epsilon_{(j)} - \tau_- v^\epsilon_{(j)}\|_{n'\epsilon} \\
&\qquad + \|(F^\epsilon)^*_{(j+2d)}(\xi,\eta)\|_{n'\epsilon}\,\|\pi u^\epsilon_{(j)} - \pi v^\epsilon_{(j)}\|_{n'\epsilon} \Big).
\end{aligned}$$

Now apply the estimate (6.14) of Lemma 6.1 with $\gamma = 2$ to bound all expressions $\|(F^\epsilon)^*_{(j)}\|_{n'\epsilon}$ on $D^{3d}(2U) \times D^{3d}(2U)$ by a suitable constant. In summary,

$$(B) \;\leq\; \epsilon B^* \mathcal{L}_{n-1}.$$

Estimates for expression (C)

Recall that $u(t,x)$ allowed an analytic continuation for $x \in B_{3r-2|t|}(\mathrm{I}_{r-|t|})$, which will also be denoted by u. The functions

$$\begin{aligned}
\mu(t,x) &= \tfrac{1}{2}(u(t,x+\tfrac{\epsilon}{2}) + u(t,x-\tfrac{\epsilon}{2})) \\
\Delta_x(t,x) &= \tfrac{1}{\epsilon}(u(t,x+\tfrac{\epsilon}{2}) - u(t,x-\tfrac{\epsilon}{2})) \\
\Delta_t(t,x) &= \tfrac{1}{\epsilon}(u(t+\tfrac{\epsilon}{2},x) - u(t-\tfrac{\epsilon}{2},x))
\end{aligned}$$

are defined for $|t| \leq r - \frac{\epsilon}{2}$ and analytic in $x \in B_{3r-2|t|-\frac{\epsilon}{2}}(\mathrm{I}_{3r-2|t|})$. The restrictions of Δ_x, Δ_t to $\Omega^\epsilon_*(r)$ coincide with $\delta_x u^\epsilon$ and $\delta_t u^\epsilon$, respectively. Moreover,

$$|\Delta_x(t,x) - \partial_x u(t,x)|,\; |\Delta_t(t,x) - \partial_t u(t,x)|,\; |\mu(t,x) - u(t,x)| \leq \mathcal{Q}\epsilon^2$$

with $\mathcal{Q}$ as defined in (6.23). Estimate (6.15) of Lemma 6.1 implies that at $t = n\epsilon$

$$\|\delta_x u^\epsilon_n - \partial_x u(t)\|_{n'\epsilon},\; \|\delta_t u^\epsilon_n - \partial_t u(t)\|_{n'\epsilon} \leq (1 - \frac{n'\epsilon}{3r - 2|t|})^{-1} C\epsilon^2 \leq 3C\epsilon^2.$$

The expression $F^\epsilon(u)(t,x) = F^\epsilon(u(t,x+\frac{\epsilon}{2}), u(t,x-\frac{\epsilon}{2}), u(t-\frac{\epsilon}{2},x))$ is analytic in $x \in B_{3r-2|t|}(\mathrm{I}_{r-|t|})$ for all $|t| \leq r - \frac{\epsilon}{2}$. By the estimate (6.4) from the hypothesis of the Theorem:

$$\begin{aligned}
|F^\epsilon(u) - f(u)| &\leq |F^\epsilon(\mu + \tfrac{\epsilon}{2}\Delta_x, \mu - \tfrac{\epsilon}{2}\Delta_x, \tau u) - f(\mu)| + |f(\mu) - f(u)| \\
&\leq K(\epsilon + \epsilon|\Delta_x|)^2 + K'|\mu - u| \\
&\leq (K(1+Q)^2 + K'Q^2)\epsilon^2
\end{aligned}$$

with K' a Lipschitz constant for f and $(\tau u)(t,x) = u(t-\frac{\epsilon}{2}, x)$. Combining the above estimates, one obtains at $t = n\frac{\epsilon}{2}$

$$\begin{aligned}
(C) &\leq \epsilon \|(M\delta_x u_n^\epsilon + F_n^\epsilon(u) - \delta_t u^\epsilon) - (M\partial_x u(t) + f(u(t)) - \partial_t u(t))\|_{n'\epsilon} \\
&\leq \epsilon \left(\|\delta_x u_n^\epsilon - \partial_x u(t)\|_{n'\epsilon} + \|\delta_t u_n^\epsilon - \partial_t u(t)\|_{n'\epsilon} + \|F_n^\epsilon(u) - f(u(t))\|_{n'\epsilon} \right) \\
&\leq C^* \epsilon^3
\end{aligned}$$

Estimates on the initial conditions

Obviously, $w_0^\epsilon = 0$ by (6.5). $v_1^\epsilon = v_+^\epsilon$ is the restriction of $u_+^\epsilon := u_0 + \frac{\epsilon}{2}\partial_t u_0$, which is an x-analytic function on $B_{3r-\epsilon}(\mathrm{I}_{r-\frac{\epsilon}{2}})$ and satisfies

$$|u_+^\epsilon(x) - u(\tfrac{\epsilon}{2}, x)| \leq \tfrac{\epsilon^2}{8} \sup_{0<s<\epsilon/2} |\partial_t^2 u(s,x)| \leq \mathfrak{Q}\epsilon^2.$$

Exploiting the estimate (6.15) once again yields

$$\mathfrak{L}_1 \leq \|w_1^\epsilon\|_{r-\epsilon} \leq D^* \epsilon^2.$$

■

6.2.4 Uniqueness of the Continuous Solution

In the approximation part of the proof, no essential reference to the particular solution u constructed in the existence part was made. The only property of u that has been used (to derive estimates on (C)) was that at each time $t \in [-r, +r]$, $u(t)$ possesses an appropriate analytic extension, bounded by U. Hence, *any* solution to (6.1) of this regularity is approximated by v^ϵ. The v^ϵ,on the other hand, are uniquely determined from u_0. This implies uniqueness of solutions to (6.1). ■

6.2.5 Proof of Theorem 6.3 (Perturbation)

Only a small change of the proof is necessary to incorporate perturbations of the initial conditions. If v_0^ϵ, v_1^ϵ are perturbed by $\epsilon^2 h_0^\epsilon$, $\epsilon^2 h_1^\epsilon$, then the initial approximation error $\mathfrak{L}_1$ is simply replaced by

$$\mathfrak{L}_1 \to \mathfrak{L}_1 + C\epsilon^2 \sup_{\epsilon>0} \sup_{x\in\mathfrak{D}} (|h_0^\epsilon(x)| + |h_1^\epsilon(x)|).$$

The constant C is due to estimate (6.15), and r is assumed to be chosen suitable small. The rest of the proof remains unchanged. ■

6.2.6 Proof of Theorem 6.4 (Smooth Convergence)

Before we present the proof of Theorem 6.4, we need to show another technical lemma.

Lemma 6.2 *Assume $h : \mathcal{D} \subset \mathbb{C}^p \to \mathbb{C}^d$ is analytic on $B^p(\tilde{U})$. Let further a sub-multiplicative semi-norm $\|\cdot\|$ for functions on $\Omega^\epsilon(r)$ be given, i.e., for arbitrary $u, v : \Omega^\epsilon(r) \to \mathbb{C}^p$, and $\lambda \in \mathbb{C}$,*

$$\|\lambda u\| = |\lambda|\,\|u\|, \quad \|u+v\| \leq \|u\| + \|v\|, \quad \|u_i\, v_i\| \leq \|u\|\,\|v\|.$$

If $\|u\|, \|v\| \leq U < \tilde{U}$, then

$$\|h(u) - h(v)\| \leq C\|u - v\|, \tag{6.27}$$

where C depends on the dimensions d and p, on the ratio $U/\tilde{U}$, and on $\sup_{|u|<\tilde{U}} |h(u)|$, but not *on the particular norm $\|\cdot\|$ or the domain $\Omega^\epsilon(r)$.*

Proof of Lemma 6.2: The listed properties of $\|\cdot\|$ are sufficient to prove the composition estimate of Lemma 6.1. Indeed, in the proof of the composition estimate, no reference to the particular form of the semi-norms $\|\cdot\|_\rho$ are made. Only submultiplicativity and the semi-norm properties have been used.

Furthermore, also the estimates for expression (B) carry over – word by word – from the proof of Theorem 6.2. Again, only the basic properties of the semi-norm and the composition estimate play a role in the argument. ■

Proof of Theorem 6.4: First of all, note that the continuous solution u is actually infinitely often differentiable in x and t on its domain. This follows because u is analytic in x and derivatives with respect to t are expressible with the help of x-derivatives and superpositions with f, using the PDE (6.1).

For each $\epsilon > 0$, define the functions u^ϵ, v^ϵ and w^ϵ as before. As a first step, estimate the modulus of the expressions $\delta_x^m \delta_t^n u^\epsilon - \partial_x^m \partial_t^n u$. Using the C^∞-smoothness of u, each difference is bounded by $A_{mn}\epsilon^2$ uniformly on $\Omega^\epsilon_{m+n}(r)$, for suitable constants A_{mn}. Hence, it remains to show that

$$\sup_{(x,t)\in\Omega^\epsilon_{m+n}(r)} |\delta_x^m \delta_t^n w^\epsilon| \leq B_{mn}\epsilon^2 \tag{6.28}$$

with some finite constants B_{mn}.

Define the positive numbers $\rho_N = 2^{-N}\rho_0$ (where $\rho_0 > 0$ is smaller than r), and for $w^\epsilon : \Omega^\epsilon(r) \to \mathbb{R}^d$

$$\|w^\epsilon\|_{(N)} = \sum_{\substack{n\leq N\\ m=1,2,\ldots}} \frac{\rho_N^{m+n}}{m!n!} \sup_{(x,t)\in\Omega^\epsilon_{m+n}(r)} |\delta_x^m \delta_t^n w^\epsilon(x,t)|.$$

The functionals $\|\cdot\|_{(N)}$ are submultiplicative semi-norms, i.e., they satisfy the hypotheses of Lemma 6.2 above.

We show inductively that if

$$\|w^\epsilon\|_{(N)} \le Q_N \epsilon^2 \tag{6.29}$$

is true for *some* $N \ge 0$, then (6.29) is also true for $N+1$ with an appropriate choice of Q_{N+1}. Consequently, the next step is to give an estimate of $\|w^\epsilon\|_{(N+1)}$ in terms of $\|w^\epsilon\|_{(N)}$. By definition,

$$\|w^\epsilon\|_{(N+1)} \;\le\; \|w^\epsilon\|_{(N)} + \sum_{m=1,2,\ldots} \frac{\rho_{N+1}^{m+N+1}}{m!(N+1)!} \sup_{(x,t)} |\delta_x^m \delta_t^{N+1} w^\epsilon(x,t)|. \tag{6.30}$$

Since v^ϵ solves the discrete equation (6.3),

$$\begin{aligned} |\delta_x^m \delta_t^{N+1} w^\epsilon(x,t)| \;\le\; & |\delta_x^m \delta_t^N (M\delta_x w^\epsilon + f^\epsilon(v^\epsilon) - f^\epsilon(u^\epsilon))| \\ & + |\delta_x^m \delta_t^N (M\delta_x u^\epsilon + f^\epsilon(u^\epsilon) - \delta_t u^\epsilon)|. \end{aligned}$$

So for the sum in (6.30), one obtains

$$\sum_{m=1,2,\ldots} \frac{\rho_{N+1}^{m+N+1}}{m!(N+1)!} \sup_{(x,t)} |\delta_x^m \delta_t^{N+1} w^\epsilon(x,t)|$$

$$\le \; \sum \tfrac{m+1}{N+1} 2^{-(m+N+1)} \frac{\rho_N^{m+N+1}}{(m+1)!N!} \sup_{(x,t)} |\delta_x^{m+1} \delta_t^N w^\epsilon(x,t)| \tag{6.31}$$

$$+ \; \sum \tfrac{\rho_{N+1}}{N+1} 2^{-(m+N)} \frac{\rho_N^{m+N}}{m!N!} \sup_{(x,t)} |\delta_x^m \delta_t^N (f^\epsilon(v^\epsilon) - f^\epsilon(u^\epsilon))(x,t)| \tag{6.32}$$

$$+ \; \sum \frac{\rho_{N+1}^{m+N+1}}{m!(N+1)!} \sup_{(x,t)} |\delta_x^m \delta_t^N (M\delta_x u^\epsilon + f^\epsilon(u^\epsilon) - \delta_t u^\epsilon)(x,t)|. \tag{6.33}$$

The three expressions (6.31)–(6.33) are now estimated separately. First, observe that $m2^{-m} < 1$ for all numbers m. Thus,

$$(6.31) \;\le\; 2^{-N} \|w^\epsilon\|_{(N)}.$$

To estimate (6.32), we use Lemma 6.2. For the moment, assume that for all $\epsilon > 0$ and all positive integers N one has

$$\|v^\epsilon\|_{(N)} \le \tilde{U}/3, \tag{6.34}$$

where $\tilde{U} > 0$ is such that each f^ϵ is analytic on $B^{3d}(\tilde{U})$. This assumption will be justified at the end of the proof. Assume further that ϵ be small enough so that $Q_N \epsilon^2 < \tilde{U}/3$ and hence

$$\|u^\epsilon\|_{(N)} \le \|v^\epsilon\|_{(N)} + \|w^\epsilon\|_{(N)} \le 2\tilde{U}/3.$$

Then, for a suitable constant $\tilde{C}$ which is independent of ϵ and N,

$$(6.32) \;\le\; \tilde{C} 2^{-N} \|w^\epsilon\|_{(N)}.$$

Eventually, exploiting that u and all of its t-derivatives are x-analytic functions, one easily shows (cf. the estimate for expression (C) in the proof of Theorem 6.2) that for each N, there exists a constant $\hat{C}_N$, so that

$$(6.33) \quad \leq \quad \hat{C}_N \epsilon^2.$$

In combination of these estimates, one obtains in (6.30)

$$\|w^\epsilon\|_{(N+1)} \quad \leq \quad (1 + \frac{\tilde{C}}{2^N})\|w^\epsilon\|_{(N)} + \hat{C}_N \epsilon^2, \tag{6.35}$$

at least for $\epsilon > 0$ small enough. With a suitable choice of Q_{N+1}, the estimate (6.29) holds for all $\epsilon > 0$.

Finally, we justify the assumption (6.34). Formally estimating $\|v^\epsilon\|_{(N+1)}$ in terms of $\|v^\epsilon\|_{(N)}$ along the same lines as above, one finds

$$\|v^\epsilon\|_{(N+1)} \quad \leq \quad (1 + \frac{\tilde{C}}{2^N})\|v^\epsilon\|_{(N)}.$$

Note that, in contrast to (6.35), there is no perturbation $\mathcal{O}(\epsilon^2)$. Consequently,

$$\|v^\epsilon\|_{(N)} \leq \prod_{k=0}^{N}(1 + \frac{\tilde{C}}{2^N}) \cdot \|v^\epsilon\|_{(0)} \leq e^{2\tilde{C}}\|v^\epsilon\|_{(0)}.$$

This expression can indeed be made smaller than $\tilde{U}/3$ because of the following reasoning: By possibly diminishing r and ρ in the proof of Theorem 6.2, it can be achieved that $\mathcal{L}_n^\epsilon < L$ for all ϵ which are small enough, where $L > 0$ is arbitrary. In particular, also $\|v^\epsilon\|_{(0)} < L$ (after adjusting ρ_0 such that $0 < \rho_0 < \rho$).

It is obvious that (6.29) for arbitrary N implies (6.28). ■

6.2.7 Proof of Lemma 6.1

As a finite sum of norms, $\|\cdot\|_\rho$ is seen to constitute a norm by itself.

Submultiplicativity: For two functions $u, v : \mathcal{I}_n^\epsilon \to \mathbb{C}$,

$$\begin{aligned}
\|uv\|_\rho &= \sum_{k=0}^{n} \frac{\rho^k}{k!} \sup_{x \in \mathcal{I}_{n-k}^\epsilon} |\delta_x^k (uv)(x)| \\
&\leq \sum_{k \leq n} \frac{\rho^k}{k!} \sum_{\ell=0}^{k} \binom{k}{\ell} \sup_{x \in \mathcal{I}_{n-\ell}^\epsilon} |\delta_x^\ell u(x)| \sup_{y \in \mathcal{I}_{n-k+\ell}^\epsilon} |\delta_x^{k-\ell} v(y)| \\
&\leq \sum_{\ell=0}^{n} \sum_{m=0}^{n} \frac{\rho^{\ell+m}}{\ell!\, m!} \sup_{x \in \mathcal{I}_{n-\ell}^\epsilon} |\delta_x^\ell u(x)| \sup_{y \in \mathcal{I}_{n-m}^\epsilon} |\delta_x^m v(y)| \\
&\leq \|u\|_\rho \|v\|_\rho.
\end{aligned}$$

Difference estimate: For $u : \mathfrak{I}_n^\epsilon \to \mathbb{C}^d$

$$\begin{aligned}
\|u\|_\rho + \epsilon \|\delta_x u\|_\rho &= \sum_{k=0}^{n} \frac{\rho^k}{k!} \max_{x \in \mathfrak{I}_{n-k}^\epsilon} |\delta_x^k u(x)| + \epsilon \sum_{k=0}^{n-1} \frac{\rho^k}{k!} \max_{x \in \mathfrak{I}_{n-k-1}^\epsilon} |\delta_x^{k+1} u(x)| \\
&\leq \sum_{k=0}^{n} \frac{\rho^k}{k!} (1 + \epsilon \frac{k}{\rho}) \max_{x \in \mathfrak{I}_{n-k}^\epsilon} |\delta_x^k u(x)| \\
&= \sum_{k=0}^{n'+1} \omega(\rho; k) \frac{(\rho+\epsilon)^k}{k!} \max_{x \in \mathfrak{I}_{n'-k}^\epsilon} |\delta_x^k u(x)| \ \leq \|u\|_{\rho+\epsilon}
\end{aligned}$$

because for $k = 0, 1, \ldots$, one has

$$\omega(\rho; k) = (1 + k\frac{\epsilon}{\rho})(1 + \frac{\epsilon}{\rho})^{-k} \leq 1$$

Composition estimate: If $g : D^p(U) \to \mathbb{C}^d$ is analytic, $\alpha = (\alpha_1, \ldots, \alpha_p)$ is a multi-index, then one has the classical *Cauchy estimate:*

$$|\partial^\alpha g(0)| \leq \frac{\alpha!}{U^{|\alpha|}} \sup_{u \in D^p(U)} |g(u)| \tag{6.36}$$

by the Cauchy integral representation of partial derivatives. From the power series representation of $f : D^d(\gamma U) \to \mathbb{C}^d$ around $u = 0$

$$f(u) = \sum_{\alpha \in \mathbb{Z}_+^p} \frac{\partial^\alpha f(0)}{\alpha!} u^\alpha, \qquad u^\alpha = \prod_{j \leq p} u_j^{\alpha_j}$$

one learns that for $u : \mathfrak{I}_n^\epsilon \to \mathbb{C}^d$ with $\|u\|_\rho < U$

$$\begin{aligned}
\|f(u)\|_\rho &\leq \sum_{\alpha \in \mathbb{N}_0^p} \left| \frac{\partial^\alpha f(0)}{\alpha!} \right| \|u^\alpha\|_\rho \leq \sup_{u \in D^p(\gamma U)} |f(u)| \sum_{\alpha \in \mathbb{N}_0^p} \left(\frac{\|u\|}{\gamma U} \right)^{|\alpha|} \\
&\leq \sup_{u \in D^p(\gamma U)} |f(u)| \left(\sum_{m=0}^{\infty} \gamma^{-m} \right)^p \leq (1 - \tfrac{1}{\gamma})^{-p} \sup_{u \in D^p(\gamma U)} |f(u)|
\end{aligned}$$

where submultiplicativity and the Cauchy estimate (6.36) have been used in the first line. The last estimate follows from evaluating the geometric series, which is convergent for $\gamma > 1$.

Discrete norm estimate: For any fixed positive ϵ, the respective operator δ_x also makes sense for any analytic function $u : B_r(\mathrm{I}_\xi) \to \mathbb{C}^d$:

$$\delta_x u : B_r(\mathrm{I}_{\xi - \epsilon/2}) \to \mathbb{C}^d, \quad (\delta_x u)(x) = \tfrac{1}{\epsilon} \left(u(x + \tfrac{\epsilon}{2}) - u(x - \tfrac{\epsilon}{2}) \right).$$

So $\delta_x^k u : B_r(\mathrm{I}_{\xi - k\epsilon/2}) \to \mathbb{C}^d$ is analytic if $k\frac{\epsilon}{2} < \xi$ and from the mean-value theorem, it follows that

$$\sup_{x \in \mathrm{I}_{\xi - k\epsilon/2}} |\delta_x^k u(x)| \leq \sup_{x \in \mathrm{I}_\xi} |\partial_x^k u(x)| \tag{6.37}$$

Now let u^ϵ the restriction of u to the points of $\mathcal{I}_n^\epsilon \subset \mathrm{I}_\xi$. Combining (6.37) with the Cauchy estimate (6.36), one arrives at

$$\|u^\epsilon\|_\rho \;\le\; \sum_{k=0}^{n} \frac{\rho^k}{k!} \sup_{x\in \mathrm{I}_\xi} |\partial_x^k u(x)| \;\le\; \Big(1-\frac{\rho}{r}\Big)^{-1} \sup_{x\in B_r(\mathrm{I}_\xi)} |u(x)|.$$

Completeness: For simplicity, assume that all n^ϵ are odd. $\|v^\epsilon\|_\rho \le C$ implies $|\delta^s v^\epsilon(x)| \le Cs!\rho^{-s}$ for all $x \in \mathcal{I}^\epsilon_{n^\epsilon}$ and all $s \le n^\epsilon$. Hence, for fixed $s \ge 0$ and ϵ small enough, the sequence of interpolated functions $E\delta_x^s v^\epsilon$ is equicontinuous.

At $s = 0$, the Arzela-Ascoli theorem yields a subsequence $\epsilon(0) \to 0$ of ϵ so that $Ev^{\epsilon(0)}$ uniformly converges to a continuous function u. From here, proceed inductively: Assume that $E\delta_x^s v^{\epsilon(s)}$ converges uniformly to $\partial_x^s u$. Apply the Arzela-Ascoli theorem at $s+1$ to obtain an infinite subsequence $\epsilon(s+1)$ of $\epsilon(s)$ for which $E\delta_x^{s+1} v^{\epsilon(s+1)}$ converges uniformly to some $u^{(s+1)}$. To show that $u^{(s+1)}$ is indeed the $s+1$-st derivative of u, consider the identity

$$(\delta_x^s u^{\epsilon(s+1)})(x) = (\delta_x^s u^{\epsilon(s+1)})(0) + \epsilon \sum_{0\le j<J} (\delta_x^{s+1} u^{\epsilon(s+1)})(\tfrac{\epsilon}{2} + \epsilon j)$$

with arbitrary $x = \epsilon J \in \mathcal{I}^\epsilon_{n^\epsilon - s}$. This implies for the interpolated functions

$$(E\delta_x^s u^{\epsilon(s+1)})(x) = (E\delta_x^s u^{\epsilon(s+1)})(0) + \int_0^x (E\delta_x^{s+1} u^{\epsilon(s+1)})(z)\,dz + \mathcal{O}(\epsilon')$$

$x \in \mathrm{I}$. Pass to the limit $\epsilon' \to 0$ on both sides: Obviously, $u^{(s+1)}$ is the x-derivative of $\partial^s u$, so $u \in C^s(\mathrm{I}')$. From Fatou's lemma, it is easily seen that for $x \in \mathrm{I}$

$$\sum_{s=0}^{\infty} \left|\frac{\partial^s u(x)}{s!}\right| \rho^s \le C.$$

This means that u possesses a convergent Taylor expansion around any point of I, with convergence radius ρ, and the respective analytic extensions are bounded by C.

6.3 A Remark About Numerical Application

The iteration solving the discrete problem (6.3) is tailored to straight-forward numerical implementation. Assume, the values of v^ϵ_{n-1} and v^ϵ_n are known. Then:

$$v^\epsilon_{n+1}(x) = v^\epsilon_{n-1}(x) + M(v^\epsilon_n(x+\tfrac{\epsilon}{2}) - v^\epsilon_n(x-\tfrac{\epsilon}{2})) + \epsilon F^\epsilon_n(v)(x).$$

To investigate the effect of round-off errors, consider the perturbed quantity $\hat{v}^\epsilon$ solving

$$\hat{v}^\epsilon_n(x) = \hat{v}^\epsilon_{n+2}(x) + M(\hat{v}^\epsilon_{n+1}(x+\tfrac{\epsilon}{2}) - \hat{v}^\epsilon_{n+1}(x-\tfrac{\epsilon}{2})) + \epsilon F^\epsilon_n(\hat{v})(x) + \mu_n(x).$$

The only property of the perturbation μ, which will be used, is its boundedness, $|\mu_n(x)| \leq 10^{-\theta}$. One should think of θ as the number of digits used to perform the calculation. To get estimates on the deviation of $\hat{v}^\epsilon$ from u, modify inequality (6.25) in the obvious way:

$$\hat{\mathfrak{L}}_n \leq (1 + \epsilon A^*)\hat{\mathfrak{L}}_{n+1} + C^*\epsilon^3 + E^*,$$

where E^* bounds the error introduced by μ. In the worst case, some component of μ takes values $+\theta$ and $-\theta$ interchangingly at consecutive lattice points, so that

$$E^* \approx 10^{-\theta} \exp\left(\frac{R}{\epsilon}\right).$$

Here R is of the magnitude of the diameter of the domain Ω, so that $T = R/\epsilon$ approximates the number of time steps. The iteration produces reasonable results only if E^* is of the order ϵ^3. Hence, the number of digits θ needs to grow linearly with T as ϵ becomes smaller. Assuming that the values of u are of the order of one, and that the precision $10^{-\theta}$ of the calculation is known a priori, it is reasonable to choose ϵ so that $T = R/\epsilon$ is of the order of θ. Then the effect of E^* is neglegible, and the fast convergence rate $\mathcal{O}(\epsilon^2)$ pays off. For instance, the pictures in the next chapter were calculated with $R \approx 1$, $\epsilon \approx 0.1$ and $\theta \approx 10$.

In figure 6.2, an example illustrates the distortion of the solution v^ϵ due to insufficient precision. The two circle patterns represent CR-1–mappings (see Definition 7.1), which are supposed to approximate the holomorphic function $u(z) = \sinh z$ on the square $[-1, +1] \times [-1, +1] \subset \mathbb{C}$. Values for ten time steps in both directions were calculated, corresponding to $\epsilon = 0.1$. While the figure on the left was produced with $\theta = 10$, only $\theta = 7$ was used for the right one.

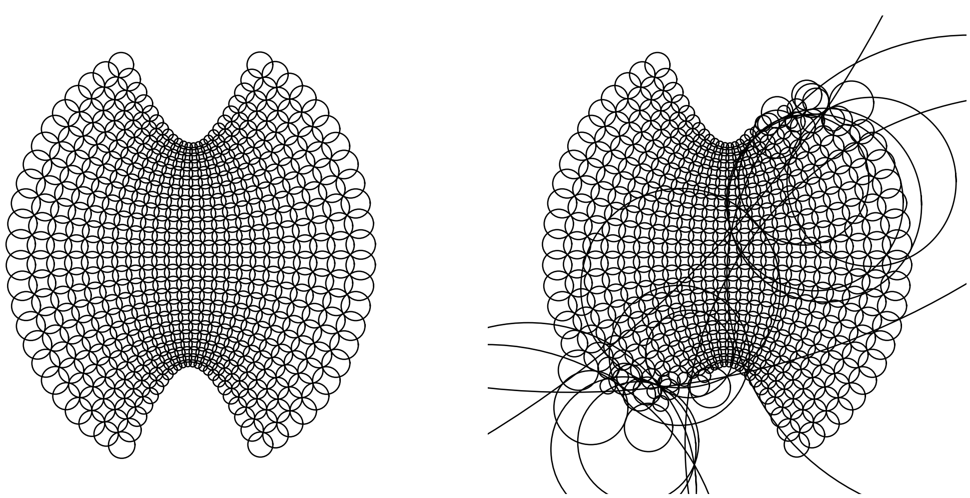

Figure 6.2: Effect of round-off errors.

Chapter 7

Two-Dimensional Conformal Mappings

In application of Theorem 6.2 from the last chapter, results about the approximation properties of two models for discrete conformal maps are derived: CR-1–mappings and orthogonal circle packings. The results hold in two dimensions.

7.1 Cross-Ratio-Equation

The *cross-ratio* of four mutually distinct complex numbers $q_1, \dots, q_4 \in \mathbb{C}$ is

$$CR(q_1, q_2, q_3, q_4) = \frac{(q_1 - q_2)(q_3 - q_4)}{(q_2 - q_3)(q_4 - q_1)}.$$

Consider $\Omega(R)$ as a subset of $\mathbb{C}$, i.e, identify each point $(t, x) \in \Omega(R)$ with the complex number $z = x + it \in \mathbb{C}$.

Definition 7.1 ([BP2, BP3]) *A* CR-1–mapping *is a discrete function $\psi^\epsilon : \Omega^\epsilon(R) \to \mathbb{C}$ such that for all $z_* \in \Omega^\epsilon_*(R) \subset \mathbb{C}$*

$$CR\Big(\psi^\epsilon(z_* + \tfrac{\epsilon}{2}), \psi^\epsilon(z_* + i\tfrac{\epsilon}{2}), \psi^\epsilon(z_* - \tfrac{\epsilon}{2}), \psi^\epsilon(z_* - i\tfrac{\epsilon}{2})\Big) = -1. \tag{7.1}$$

CR-1–mappings are suitable discrete analogues of holomorphic functions in the sense that any holomorphic function can be locally approximated.

Theorem 7.1 *Assume $\phi : \Omega(R) \to \mathbb{C}$ is a holomorphic function with $\phi'(0) \neq 0$. Then, there are positive constants $r < R$ and C, so that for each $\epsilon > 0$, a CR-1–mapping ψ^ϵ is defined on $\Omega^\epsilon(r)$ and satisfies*

$$\sup_{z \in \Omega^\epsilon(r)} |\phi(z) - \psi^\epsilon(z)| \leq C\epsilon^2. \tag{7.2}$$

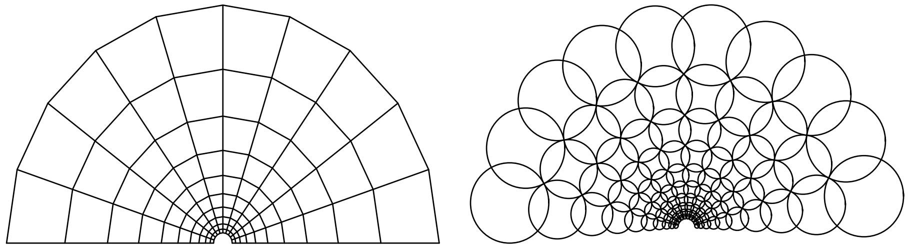

Figure 7.1: A CR-1–mapping approximating the exponential function

Proof: For each mesh size $\epsilon > 0$, denote by $\tau_\pm$ the shift operators

$$(\tau_+ h)(z) = h(z + \tfrac{\epsilon}{2}(i+1)), \quad (\tau_- h)(z) = h(z + \tfrac{\epsilon}{2}(i-1)).$$

From a given CR-1–mapping $\psi^\epsilon : \Omega^\epsilon(R) \to \mathbb{C}$, derive the quantities

$$\alpha^\epsilon = (\tau_+\psi^\epsilon - \psi^\epsilon)/\epsilon \qquad \beta^\epsilon = (\tau_-\psi^\epsilon - \psi^\epsilon)/\epsilon. \tag{7.3}$$

It is clear from definitions (7.3) that

$$\alpha^\epsilon + \tau_+\beta^\epsilon = \beta^\epsilon + \tau_-\alpha^\epsilon. \tag{7.4}$$

Let further the function Q^ϵ be given as the quotient

$$Q^\epsilon = \beta^\epsilon/\alpha^\epsilon. \tag{7.5}$$

From the defining property (7.1), one deduces $\tau_+\beta = -(Q^\epsilon)^{-1}\tau_-\alpha$, and the following two formulas are deduced:

$$\tau_-\alpha^\epsilon = \frac{1-Q^\epsilon}{1+Q^\epsilon}\, Q^\epsilon\, \alpha^\epsilon, \qquad \tau_+\alpha^\epsilon = -\frac{1-Q^\epsilon}{1+Q^\epsilon}\, \frac{\alpha^\epsilon}{\tau_+ Q^\epsilon}. \tag{7.6}$$

Their compatibility condition $\tau_+(\tau_-\alpha^\epsilon) = \tau_-(\tau_+\alpha^\epsilon)$ provides one with a discrete analogue of the Cauchy-Riemann equation (below, $z_* \in \Omega^\epsilon_*(R)$)

$$\frac{Q^\epsilon(z_* + i\frac{\epsilon}{2})}{Q^\epsilon(z_* - i\frac{\epsilon}{2})} = \frac{1 + Q^\epsilon(z_* + \frac{\epsilon}{2})}{1 - Q^\epsilon(z_* + \frac{\epsilon}{2})} \cdot \frac{1 - Q^\epsilon(z_* - \frac{\epsilon}{2})}{1 + Q^\epsilon(z_* - \frac{\epsilon}{2})}. \tag{7.7}$$

Now introduce $v^\epsilon : \Omega^\epsilon_*(R) \to \mathbb{C}$ by

$$i\exp(\epsilon v^\epsilon(z_*)) = Q^\epsilon(z_* - i\tfrac{\epsilon}{2}) \tag{7.8}$$

Expressing the relation (7.7) with the help of this quantity, one obtains

$$\delta_t v^\epsilon = M\delta_x v^\epsilon + F^\epsilon(v^\epsilon) \tag{7.9}$$

which is in the form of the discrete problem (6.3) with $M = i$ and

$$\begin{aligned} F^\epsilon(v^+, v^-, v^*) &= \frac{1}{\epsilon^2}(g(\epsilon v^+) - g(\epsilon v^-)) \\ g(V) &= \log\left(-i\exp(V)\frac{1 + i\exp(V)}{1 - i\exp(V)}\right). \end{aligned}$$

Note that v^ϵ is defined on $\Omega^\epsilon_*(R)$ instead of $\Omega^\epsilon(R)$, but this does obviously not affect any of the results, except for a minor mix-up of notations.

The process of getting a solution v^ϵ to (7.9) from a CR-1–mapping ψ^ϵ is reversible:

Lemma 7.1 *For any solution $v^\epsilon : \Omega^\epsilon_*(r) \to \mathbb{C}$ to (7.9), there exists a corresponding CR-1–mapping $\psi^\epsilon : \Omega^\epsilon(r) \to \mathbb{C}$. It is uniquely determined by v^ϵ up to Euclidean motions and a homothety.*

Proof of Lemma 7.1: Let $\hat{z} = -iN\frac{\epsilon}{2}$ be the "bottom" of $\Omega^\epsilon(r)$. Assign $\alpha^\epsilon(\hat{z})$ an arbitrary, non-zero number $A^\epsilon \in \mathbb{C}$. From v^ϵ, define Q^ϵ according to (7.5), and use (7.6) to define α^ϵ on $\Omega^\epsilon(r)$. Consistency is provided, i.e. $\tau_+(\tau_-\alpha^\epsilon) = \tau_-(\tau_+\alpha^\epsilon)$, because Q^ϵ solves (7.7). Next, define $\beta^\epsilon = Q^\epsilon\alpha^\epsilon$. Assign an arbitrary number $\Psi^\epsilon \in \mathbb{C}$ to $\psi^\epsilon(\hat{z})$, then use (7.3) to define ψ^ϵ on $\Omega^\epsilon(r)$. Compatibility (7.4) is guaranteed because α^ϵ solves (7.6). To verify ψ^ϵ is indeed a CR-1–mapping, combine equations (7.6) with (7.4). Variations of A^ϵ correspond to rigid rotations and dilations of the whole lattice ψ^ϵ, variations of Ψ^ϵ to translations. ■

Proof of Theorem 7.1, continued: In the continuous case, the role of the function v^ϵ is played by

$$u(z) = -\frac{1}{2}\frac{\phi''(z)}{\phi'(z)}, \tag{7.10}$$

which is defined on some $\Omega(R)$ with $R > 0$ small enough (since $\phi'(0) \neq 0$ by hypothesis). To convince the reader that this is indeed a reasonable ansatz for u, we make an asymptotic expansion of the quantity v^ϵ, supposing it was defined from the holomorphic function ϕ instead of the lattice ψ^ϵ. One finds:

$$\begin{aligned}
v^\epsilon(z + i\tfrac{\epsilon}{2}) &= \frac{1}{\epsilon}\log\left(-i\frac{\tau_-\phi - \phi}{\tau_+\phi - \phi}(z)\right) \\
&= \tfrac{1}{\epsilon}\log\left(-i\frac{\phi'(z)\frac{\epsilon}{2}(i-1) + \phi''(z)\frac{\epsilon^2}{4}(-2i) + \mathcal{O}(\epsilon^3)}{\phi'(z)\frac{\epsilon}{2}(i+1) + \phi''(z)\frac{\epsilon^2}{4}(2i) + \mathcal{O}(\epsilon^3)}\right) \\
&= \tfrac{1}{\epsilon}\log\left(1 + \frac{\phi''(z)}{\phi'(z)}(-\tfrac{\epsilon}{2}) + \mathcal{O}(\epsilon^2)\right) \\
&= -\frac{1}{2}\frac{\phi''(z)}{\phi'(z)} + \mathcal{O}(\epsilon).
\end{aligned}$$

The Cauchy-Riemann equation for u,

$$\partial_t u = i\partial_x u, \tag{7.11}$$

corresponds to $M = i$ and $f \equiv 0$ in problem (6.1). Since $g(0) = g'(0) = g''(0) = 0$, the function F^ϵ satisfies the estimate (6.4). In particular, one easily verifies $|F^\epsilon(v^+, v^-, v^*)| \leq K\epsilon|v^+ - v^-|$. Let v^ϵ be the solution to (7.9) with initial data (6.5) derived from the function u in (7.10). Theorem 6.2 gives

$$|u(t,x) - v^\epsilon(t,x)| \leq C\epsilon^2 \tag{7.12}$$

for all $(t,x) \in \Omega^\epsilon(r)$ (or rather $\Omega^\epsilon_*(r)$), with suitable $r>0$, $C>0$ independent of ϵ.

By Lemma 7.1, v^ϵ corresponds to a CR-1–mapping ψ^ϵ. Make ψ^ϵ unique by choosing $\psi^\epsilon(\hat z) = \phi(\hat z)$ and $\alpha^\epsilon(\hat z) = (\phi(\hat z + (i+1)\frac{\epsilon}{2}) - \phi(\hat z))/\epsilon$. It remains to be shown that approximation of u by v^ϵ implies approximation of ϕ by ψ^ϵ. Define quantities analogous to α^ϵ and β^ϵ,

$$a^\epsilon = (\tau_+\phi - \phi)/\epsilon \qquad b^\epsilon = (\tau_-\phi - \phi)/\epsilon \tag{7.13}$$

Further define $q^\epsilon(z) = i\exp(\epsilon u(z + i\frac{\epsilon}{2}))$ for $z \in \Omega(r)$, which corresponds to Q^ϵ. By a straightforward calculation[1], one verifies that (7.6) holds with a^ϵ, b^ϵ and q^ϵ in place of α^ϵ, β^ϵ and Q^ϵ, up to an error of order $\mathcal{O}(\epsilon^3)$:

$$\tau_- a^\epsilon = \frac{1-q^\epsilon}{1+q^\epsilon}\, q^\epsilon\, a^\epsilon + \mathcal{O}(\epsilon^3), \qquad \tau_+ a^\epsilon = -\frac{1-q^\epsilon}{1+q^\epsilon}\,\frac{a^\epsilon}{\tau_+ q^\epsilon} + \mathcal{O}(\epsilon^3) \tag{7.14}$$

Subtracting these relations from (7.6), one obtains

$$\begin{aligned}
|\tau_-(a^\epsilon - \alpha^\epsilon)| &= \left|\frac{1-q^\epsilon}{1+q^\epsilon}\, q^\epsilon\, a^\epsilon - \frac{1-Q^\epsilon}{1+Q^\epsilon}\, Q^\epsilon\, \alpha^\epsilon\right| + \mathcal{O}(\epsilon^3) \\
&\le |a^\epsilon - \alpha^\epsilon| + \left|\frac{1-q^\epsilon}{1+q^\epsilon}\, q^\epsilon - \frac{1-Q^\epsilon}{1+Q^\epsilon}\, Q^\epsilon\right| |a^\epsilon| + \mathcal{O}(\epsilon^3) \\
&\le (1+\mathcal{O}(\epsilon))|a^\epsilon - \alpha^\epsilon| + \mathcal{O}(\epsilon^3),
\end{aligned}$$

where (7.12) has been used to estimate the difference of the expressions in q^ϵ and Q^ϵ in the second line. An analogous estimate holds for the difference $\tau_+(a^\epsilon - \alpha^\epsilon)$.

Since there is a path from $\hat z$ to any other point $z \in \Omega^\epsilon(r)$ of length not more than $2N = \mathcal{O}(\epsilon^{-1})$, if follows (cf. the Gronwall estimate in Lemma 2.2) that $\alpha^\epsilon(z) = a^\epsilon(z) + \mathcal{O}(\epsilon^2)$. From the definition $\beta^\epsilon = Q^\epsilon\alpha^\epsilon$, it eventually follows that also $\beta^\epsilon(z) = b^\epsilon(z) + \mathcal{O}(\epsilon^2)$. Subtracting equations (7.3) and (7.13) leads to

$$\begin{aligned}
|\tau_+(\phi - \psi^\epsilon)| &\le |\phi - \psi^\epsilon| + \epsilon|a^\epsilon - \alpha^\epsilon| \\
|\tau_-(\phi - \psi^\epsilon)| &\le |\phi - \psi^\epsilon| + \epsilon|b^\epsilon - \beta^\epsilon|
\end{aligned}$$

from where the result (7.2) follows immediately by the choice of $\psi^\epsilon(\hat z)$. ■

Theorem 7.2 *Under the hypothesis of Theorem 7.1, the CR-1–mapping ψ^ϵ converges to ϕ in C^∞:*

$$\sup_{z\in\Omega^\epsilon_{m+n}(r)} |\delta_x^m \delta_t^n \psi^\epsilon(z) - \partial_x^m \partial_t^n \phi(z)| \le C_{mn}\epsilon^2. \tag{7.15}$$

[1] One needs to make a Taylor expansions up to $\mathcal{O}(\epsilon^3)$. Here and at similar occasions later on, we have used the computer algebra system `MATHEMATICA` for the explicit calculation of the Taylor coefficients.

Proof: Let everything be defined as in the proof of Theorem 7.1. By Theorem 6.4, each $\delta_x^m \delta_t^n v^\epsilon$ approximates the respective $\partial_x^m \partial_t^n u$ with an error $\mathcal{O}(\epsilon^2)$. In a first step, we deduce that for all m, n, there are constants A_{mn} such that

$$|\delta_x^m \delta_t^n (\alpha^\epsilon - a^\epsilon)| \leq A_{mn}\epsilon^2 \tag{7.16}$$

on $\Omega^\epsilon_{m+n}(r)$. Introduce the functions g^ϵ, G^ϵ by

$$\alpha(z) = \exp(g^\epsilon(z))\alpha(\hat{z}), \quad a(z) = \exp(G^\epsilon(z))a(\hat{z}). \tag{7.17}$$

Combining the formulas in (7.6) yields the simple relation

$$\tau_x \alpha^\epsilon = \tau_-^{-1}(\tau_+ \alpha^\epsilon) = \frac{-1}{(\tau_x Q^\epsilon)(\tau_-^{-1} Q^\epsilon)} \alpha^\epsilon,$$

which implies for g^ϵ that

$$\delta_x g^\epsilon(z) = v^\epsilon(z + i\tfrac{\epsilon}{2}) + v^\epsilon(z - i\tfrac{\epsilon}{2}).$$

By Taylor expansion, one verifies that

$$\delta_x G^\epsilon(z) = u(z + i\tfrac{\epsilon}{2}) + u(z - i\tfrac{\epsilon}{2}) + \mathcal{O}(\epsilon^2).$$

In combination with the convergence of v^ϵ to u, it follows

$$|\delta_x^m \delta_t^n (g^\epsilon - G^\epsilon)| \leq 2|\delta_x^{m-1}\delta_t^n v^\epsilon - \partial_x^{m-1}\partial_t^n u| + \mathcal{O}(\epsilon^2) \leq B_{mn}\epsilon^2. \tag{7.18}$$

If $m = 0$, $n > 0$, the argument is modified in the obvious way, exchanging the roles of x and t. The functions α^ϵ, a^ϵ are now reconstructed from g^ϵ, G^ϵ, using (7.17). We derive the estimate

$$\sup_{z \in \Omega^\epsilon_{m+n}(r)} |\exp(g^\epsilon(z)) - \exp(G^\epsilon(z))| \leq B'_{mn}\epsilon^2 \tag{7.19}$$

in application of Lemma 6.2 as follows: Introduce the submultiplicative semi-norms

$$\|w^\epsilon\|_{(N)} := \sum_{m,n \leq N} \frac{1}{m!n!} \sup_{(x,t)\in\Omega^\epsilon_{m+n}(r)} |\delta_x^m \delta_t^n w^\epsilon(x,t)|.$$

For any fixed $N > 0$, estimate (7.18) implies

$$\|g^\epsilon - G^\epsilon\|_{(N)} \leq \hat{B}_N \epsilon^2.$$

Furthermore, the expression $\|G^\epsilon\|_{(N)}$ is obviously bounded independent of $\epsilon > 0$, and so is $\|g^\epsilon\|_{(N)} \leq \|G^\epsilon\|_{(N)} + \hat{B}_N\epsilon^2$. The exponential function is entire, so the hypotheses of Lemma 6.2 are satisfied for $u = g^\epsilon$, $v = G^\epsilon$ and $h = \exp$. Choosing $N = m+n$, the lemma yields the estimate (7.19), indeed. Since $\alpha^\epsilon(\hat{z}) = a^\epsilon(\hat{z})$ by our construction, (7.19) also implies (7.16).

Approximation of $b^\epsilon = q^\epsilon a^\epsilon$ by $\beta^\epsilon = Q^\epsilon \alpha^\epsilon$ similar to (7.16) is easily concluded from

$$\|\beta^\epsilon - b^\epsilon\|_{(N)} \leq \|Q^\epsilon - q^\epsilon\|_{(N)}\|a^\epsilon\|_{(N)} + \|q^\epsilon\|_{(N)}\|\alpha^\epsilon - a^\epsilon\|_{(N)},$$

which follows by submultiplicativity of $\|\cdot\|_{(N)}$. Apply Lemma 6.27 again to estimate $\|Q^\epsilon - q^\epsilon\|_{(N)} = \mathcal{O}(\epsilon^3)$. Hence, $\|\beta^\epsilon - b^\epsilon\|_{(N)} = \mathcal{O}(\epsilon^2)$.

In the second step, ψ^ϵ and ϕ are reconstructed from α^ϵ, β^ϵ and a^ϵ, b^ϵ, respectively. By definition of α^ϵ and a^ϵ,

$$\delta_x\psi^\epsilon(z) = \alpha^\epsilon(z - \tfrac{\epsilon}{2}) - \beta^\epsilon(z + \tfrac{\epsilon}{2}), \quad \partial_x\phi(z) = a^\epsilon(z - \tfrac{\epsilon}{2}) - b^\epsilon(z + \tfrac{\epsilon}{2}) + \mathcal{O}(\epsilon^2),$$

and similar for $\delta_t\psi^\epsilon$ and $\partial_t\phi$. With the estimate (7.16) at hand, we conclude that

$$\sup|\delta_x^m\delta_t^n\psi^\epsilon - \partial_x^m\partial_t^n\phi| \;\leq\; \sup|\delta_x^{m-1}\delta_t^n(\alpha^\epsilon - a^\epsilon)| + \sup|\delta_x^{m-1}\delta_t^n(\beta^\epsilon - b^\epsilon)| + \mathcal{O}(\epsilon^2).$$

Interchange the roles of m and n if $m = 0$.

In conclusion, the relations (7.15) are proven. ■

7.2 Schramm's Circle Patterns

In [Sc], orthogonal circle patterns are proposed as discrete analogues of conformal maps. Roughly, an orthogonal circle pattern $\mathcal{C}^\epsilon$ assigns to each vertex $(x,t) \in \tilde{\Omega}^\epsilon \subset (\epsilon\mathbb{Z})^2$ an embedded circle $\mathcal{C}^\epsilon(x,t)$ in $\mathbb{R}^2 \equiv \mathbb{C}$ in such a way, that circles assigned to neighboring vertices intersect orthogonally, and circles assigned to opposite corners of an elementary square in $(\epsilon\mathbb{Z})^2$ are tangent. For the formal definition, refer to the original article.

The theory developed here allows for an easy proof of a local approximation property of the patterns. Our result differs in nature from the convergence theorem presented in [Sc], which deals with global boundary problems.

For notational simplicity, we assume that the domain for orthogonal circle pattern $\mathcal{C}^\epsilon$ is some

$$\tilde{\Omega}^\epsilon(r) = \Omega^\epsilon(r) \cap (\epsilon\mathbb{Z})^2,$$

which contains half of the grid points of $\Omega^\epsilon(r)$. In obvious analogy to $\Omega_k^\epsilon(r)$, the sets $\tilde{\Omega}_{mn}^\epsilon(r) \subset \Omega_{m+n}^\epsilon(r)$ are introduced so that the central difference quotient $\delta_x^m\delta_t^n\psi^\epsilon$ of a function ψ^ϵ on $\tilde{\Omega}^\epsilon(r)$ is naturally evaluated on $\tilde{\Omega}_{mn}^\epsilon(r)$. Note that $\tilde{\Omega}_{mn}^\epsilon = \tilde{\Omega}_{nm}^\epsilon$ is not true in general.

A pattern $\mathcal{C}^\epsilon$ is essentially (up to rigid motions) determined by its *radius function* ρ^ϵ which assigns to each point $(x,t) \in (\epsilon\mathbb{Z})^2$ the radius of the circle $\mathcal{C}^\epsilon(x,t)$.

Theorem 7.3 *Given a conformal map $\phi : \Omega(R) \to \mathbb{C}$ with $\partial_x\phi(0) \neq 0$, there is a positive $r < R$, and there exists a family of orthogonal circle patterns $\mathcal{C}^\epsilon$ defined on $\tilde{\Omega}^\epsilon(r)$ for all $\epsilon > 0$ small enough, whose radii functions ρ^ϵ are convergent to the metric factor $\rho = |\partial_x\phi|$ of ϕ in C^∞,*

$$\sup_{(x,t)\in\tilde{\Omega}_{mn}^\epsilon(r)} |\delta_x^m\delta_t^n\rho(x,t) - \partial_x^m\partial_t^n\rho^\epsilon(x,t)| \leq C_{mn}\epsilon^2. \tag{7.20}$$

Proof: The function $\log\rho$ is harmonic, i.e., satisfies Laplace's equation

$$\partial_x^2(\log\rho)+\partial_t^2(\log\rho)=0. \tag{7.21}$$

In terms of $u_{(1)}=\partial_x(\log\rho)$ and $u_{(2)}=\partial_t(\log\rho)$, harmonicity reads as

$$\partial_t\begin{pmatrix}u_{(1)}\\u_{(2)}\end{pmatrix}=\begin{pmatrix}0&1\\-1&0\end{pmatrix}\partial_x\begin{pmatrix}u_{(1)}\\u_{(2)}\end{pmatrix}. \tag{7.22}$$

For the radius function ρ^ϵ of an orthogonal circle pattern, an "exponential Laplace equation" has been derived in [Sc]. In our notations,

$$(\tau_x\rho^\epsilon)(\tau_t\rho^\epsilon)(\tau_x^{-1}\rho^\epsilon)(\tau_t^{-1}\rho^\epsilon)=(\rho^\epsilon)^2\left(\frac{(\tau_x\rho^\epsilon)+(\tau_t\rho^\epsilon)+(\tau_x^{-1}\rho^\epsilon)+(\tau_t^{-1}\rho^\epsilon)}{(\tau_x\rho^\epsilon)^{-1}+(\tau_t\rho^\epsilon)^{-1}+(\tau_x^{-1}\rho^\epsilon)^{-1}+(\tau_t^{-1}\rho^\epsilon)^{-1}}\right). \tag{7.23}$$

Equation (7.23) is satisfied by a positive function $\rho^\epsilon:\tilde\Omega^\epsilon(r)\to\mathbb{R}_+$ if and only if it is the radius function of an orthogonal circle pattern. Introduce a function[2] $v^\epsilon:\Omega_*^\epsilon(r)\to\mathbb{R}^2$ by

$$\rho^\epsilon(x+\tfrac{\epsilon}{2},t) = \rho^\epsilon(x-\tfrac{\epsilon}{2},t)\exp(\epsilon v_{(1)}^\epsilon(x,t)) \tag{7.24}$$
$$\rho^\epsilon(x,t+\tfrac{\epsilon}{2}) = \rho^\epsilon(x,t-\tfrac{\epsilon}{2})\exp(\epsilon v_{(2)}^\epsilon(x,t)). \tag{7.25}$$

In these variables, (7.23) and the compatibility condition $\delta_x v_{(2)}^\epsilon=\delta_t v_{(1)}^\epsilon$ imply

$$\delta_t\begin{pmatrix}v_{(1)}^\epsilon\\v_{(2)}^\epsilon\end{pmatrix}=\begin{pmatrix}0&1\\-1&0\end{pmatrix}\delta_x\begin{pmatrix}v_{(1)}^\epsilon\\v_{(2)}^\epsilon\end{pmatrix}+\begin{pmatrix}0\\G^\epsilon\end{pmatrix}, \tag{7.26}$$
$$G^\epsilon(v^+,v^-,v^*)=\frac{1}{\epsilon^2}\log\left(\frac{\exp(\epsilon v_{(1)}^+)+\exp(-\epsilon v_{(1)}^-)+\exp(-\epsilon v_{(2)}^*)-\exp(\epsilon(v_{(1)}^+-v_{(1)}^--v_{(2)}^*))}{\exp(-\epsilon v_{(1)}^+)+\exp(\epsilon v_{(1)}^-)+\exp(\epsilon v_{(2)}^*)-\exp(\epsilon(-v_{(1)}^++v_{(1)}^-+v_{(2)}^*))}\right).$$

On the other hand, if $v^\epsilon:\Omega_*^\epsilon(r)\to\mathbb{R}^2$ is a solution to (7.26), then the equations (7.24) and (7.25) are compatible. The respective solution[3] $\rho^\epsilon:\tilde\Omega^\epsilon(r)\to\mathbb{R}_+$ – which is uniquely determined by v^ϵ up to a global scalar factor – satisfies the exponential Laplace equation (7.23), hence is the radius function of some orthogonal circle pattern $\mathcal{C}^\epsilon$ on $\tilde\Omega^\epsilon(r)$.

Now let v^ϵ be the solution to (7.26) with the restrictions of u as initial data in (6.5). By Taylor expansion, one verifies that the nonlinearity $F^\epsilon=(0,G^\epsilon)$ satisfies the estimate (6.4); recall that $f\equiv 0$. Hence Theorem 6.2 applies and yields convergence of v^ϵ to u on a suitable $\Omega(r)$.

Make the respective solution ρ^ϵ of (7.24)–(7.25) unique by choosing $\rho^\epsilon(0,0)=\rho(0,0)$. By estimates completely analogous to those used in the proofs of Theorems 7.1 and 7.2 (reconstruction of α^ϵ from v^ϵ), one obtains C^∞-convergence of ρ^ϵ to ρ. ∎

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[2] Since ρ^ϵ is only defined on $\tilde\Omega^\epsilon(r)$, v^ϵ cannot be calculated everywhere on $\Omega_*^\epsilon(r)$. In fact, $v_{(1)}^\epsilon$ is defined on one half of the grid sites, $v_{(2)}^\epsilon$ on the other half. This does not hurt the argument, since we only intend to motivate formula (7.26) at this point.

[3] From a solution v^ϵ to 7.26 on $\Omega_*^\epsilon(r)$, the function ρ^ϵ could be defined everywhere on $\Omega^\epsilon(r)$, using (7.24) and (7.25). The restrictions of ρ^ϵ to $\tilde\Omega^\epsilon(r)$ and $\Omega^\epsilon(r)\setminus\tilde\Omega^\epsilon(r)$ are radius functions for two independent orthogonal circle patterns.

The radius function $\rho^\epsilon : \tilde{\Omega}^\epsilon(r) \to \mathbb{R}_+$ is accompanied by a function $\psi^\epsilon : \tilde{\Omega}^\epsilon(r) \to \mathbb{C}$, which determines the positions of the circles of the pattern $\mathcal{C}^\epsilon$. More precisely, we assume that $\mathcal{C}^\epsilon(x,t)$ is a circle of radius $\rho^\epsilon(x,t) > 0$ around the center $\psi^\epsilon(x,t) \in \mathbb{C}$. As pointed out before, the radius function alone determines the pattern $\mathcal{C}^\epsilon$ – and hence ψ^ϵ – up to a rigid motion.

Theorem 7.4 *Under the hypotheses of Theorem 7.3 and for every $\epsilon > 0$ small enough, there exists an orthogonal circle pattern $\mathcal{C}^\epsilon$ on $\tilde{\Omega}^\epsilon(r)$, such that the circle centers ψ^ϵ converge to the conformal map ϕ in C^∞:*

$$\sup_{(x,t)\in\tilde{\Omega}^\epsilon_{mn}(r)} |\delta_x^m \delta_t^n \psi^\epsilon(x,t) - \partial_x^m \partial_t^n \phi(x,t)| \le C_{mn}\epsilon^2. \tag{7.27}$$

Proof: We show how to construct the map ϕ from its conformal factor ρ and the map ψ^ϵ from the radius function ρ^ϵ, respectively. The estimate (7.27) turns out to be a consequence of (7.20).

Introduce the real-valued function w on $\Omega(R)$ by

$$\phi' = \rho \exp(iw), \tag{7.28}$$

where $\phi'(x,t) := \partial_x \phi(x,t)$ is holomorphic with respect to the complex variable $z = x + it$. From the Cauchy-Riemann equation for ϕ', the following relations are deduced for w:

$$\begin{aligned} \partial_x w &= -\partial_t \log \rho = -u_{(2)} \\ \partial_t w &= \partial_x \log \rho = u_{(1)}, \end{aligned}$$

with u defined as before, $u_{(1)} = \partial_x(\log \rho)$, $u_{(2)} = \partial_t(\log \rho)$.

Analogous quantities and relations are now given for an arbitrary orthogonal circle pattern $\mathcal{C}^\epsilon$. Define the real functions ω^ϵ and d^ϵ on $\tilde{\Omega}^\epsilon_{1,0}(r)$ by

$$\delta_x \psi^\epsilon = d^\epsilon \exp(i\omega^\epsilon). \tag{7.29}$$

Here $d^\epsilon(x,t)$ denotes the euclidian distance between the circle centers $\psi^\epsilon(x + \frac{\epsilon}{2}, t)$ and $\psi^\epsilon(x + \frac{\epsilon}{2}, t)$, and $\omega^\epsilon(x,t)$ describes the slope of their connecting line with respect to the x-axis.

In Fig. 7.2, two pieces of an orthogonal circle pattern are displayed. From the left sketch, one learns that

$$\begin{aligned} \angle(\psi_{++} - \psi_{0+}, \psi_{+0} - \psi_{00}) &= \angle(\mu, \psi_{0+}, \psi_{++}) - \angle(\psi_{00}, \psi_{+0}, \mu) \\ &= \arctan\left(\frac{\rho_{++}}{\rho_{0+}}\right) - \arctan\left(\frac{\rho_{00}}{\rho_{+0}}\right). \end{aligned}$$

Introducing v^ϵ by the formulas (7.24), (7.25),

$$\delta_t \omega^\epsilon(x,t) = \tfrac{1}{\epsilon}\left(\arctan\exp(\epsilon v^\epsilon_{(1)}(x, t + \tfrac{\epsilon}{2})) - \arctan\exp(-\epsilon v^\epsilon_{(1)}(x, t - \tfrac{\epsilon}{2}))\right). \tag{7.30}$$

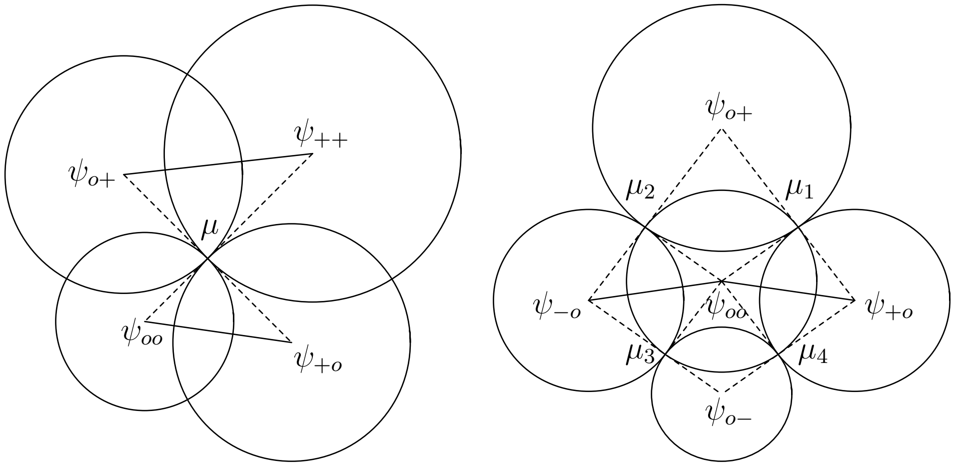

Figure 7.2: Relations between centers of adjacent circles.

From the sketch on the right, it follows that

$$\begin{aligned} \angle(\psi_{+0}-\psi_{00},\psi_{00}-\psi_{-0}) &= +\pi-\angle(\mu_2,\psi_{00},\psi_{-0})- \\ &\quad -\angle(\mu_1,\psi_{00},\mu_2)-\angle(\psi_{+0},\psi_{00},\mu_1), \end{aligned} \tag{7.31}$$

and as well it follows that

$$\begin{aligned} \angle(\psi_{+0}-\psi_{00},\psi_{00}-\psi_{-0}) &= -\pi+\angle(\mu_3,\psi_{00},\psi_{-0})+ \\ &\quad +\angle(\mu_4,\psi_{00},\mu_3)+\angle(\psi_{+0},\psi_{00},\mu_4). \end{aligned} \tag{7.32}$$

The sum of the equations (7.31) and (7.32) yields the relation

$$\begin{aligned} \angle(\psi_{+0}-\psi_{00},\psi_{00}-\psi_{-0}) &= \angle(\mu_4,\psi_{00},\mu_3)-\angle(\mu_1,\psi_{00},\mu_2) \\ &= \arctan\left(\frac{\rho_{0-}}{\rho_{00}}\right)-\arctan\left(\frac{\rho_{0+}}{\rho_{00}}\right). \end{aligned}$$

By definition of ω^ϵ and v^ϵ,

$$\delta_x\omega^\epsilon(x,t)=\tfrac{1}{\epsilon}\left(\arctan\exp(-\epsilon v^\epsilon_{(2)}(x,t-\tfrac{\epsilon}{2}))-\arctan\exp(\epsilon v^\epsilon_{(2)}(x,t+\tfrac{\epsilon}{2}))\right). \tag{7.33}$$

Since $\arctan\exp(\epsilon v)=\pi/4+\frac{\epsilon}{2}v+\mathcal{O}(\epsilon^3)$, the equations for ω^ϵ have the asymptotic behavior

$$\begin{aligned} \delta_x\omega^\epsilon(x,t) &= -\left(v^\epsilon_{(2)}(x,t+\tfrac{\epsilon}{2})+v^\epsilon_{(2)}(x,t-\tfrac{\epsilon}{2})\right)/2+\mathcal{O}(\epsilon^2) \\ \delta_t\omega^\epsilon(x,t) &= \left(v^\epsilon_{(1)}(x,t+\tfrac{\epsilon}{2})+v^\epsilon_{(1)}(x,t-\tfrac{\epsilon}{2})\right)/2+\mathcal{O}(\epsilon^2), \end{aligned}$$

which, in view of (7.28), justifies to consider ω^ϵ as a discrete analogue of w. In fact, from a Taylor expansion one learns that equations (7.30), (7.33) hold with an error of order $\mathcal{O}(\epsilon^2)$ upon replacing v^ϵ and ω^ϵ by the restrictions of u and w to $\Omega^\epsilon_*(r)$.

With formulas (7.30), (7.33) at hand, we continue the argument of the proof for Theorem 7.3. Let $\rho^\epsilon:\tilde{\Omega}^\epsilon(r)\to\mathbb{R}_+$ be the radius function approximating the conformal factor ρ of ϕ, and v^ϵ the corresponding quantity approximating u.

Define ω^ϵ on $\tilde{\Omega}^\epsilon_{1,0}$ using the formulas (7.30), (7.33); the choice $\omega^\epsilon(\frac{\epsilon}{2},0) = w(\frac{\epsilon}{2},0)$ makes ω^ϵ unique. To prove C^∞-convergence of ω^ϵ to w from the convergence of v^ϵ to u, apply Lemma 6.2 with the natural choice of the semi-norms

$$\|w^\epsilon\|_{(N)} = \sum_{m,n\le N} \frac{1}{m!n!} \sup_{(x,t)\in\tilde{\Omega}^\epsilon_{m+1,n}(r)} |\delta_x^m \delta_t^n w^\epsilon(x,t)|$$

to the family of functions

$$h^\epsilon(V) := \tfrac{1}{\epsilon}(\arctan\exp(\epsilon V)).$$

The h^ϵ do not have infinite radius of convergence, but for $\epsilon < \epsilon(U)$, they are analytic on the disc $D(U)$ for arbitrary $U > 0$, and they are bounded independently of ϵ there. In summary, one finds

$$\begin{aligned}&\sup_{(x,t)\in\tilde{\Omega}^\epsilon_{m+1,n}} |\delta_x^m\delta_t^n\omega^\epsilon(x,t) - \partial_x^m\partial_t^n w(x,t)| \\ &\le L_{mn} \sup_{(x,t)\in\Omega^\epsilon_{m+n}} |\delta_x^{m-1}\delta_t^n v^\epsilon(x,t) - \partial_x^{m-1}\partial_t^n u(x,t)| + \mathcal{O}(\epsilon^2) \le C_{mn}\epsilon^2.\end{aligned}$$

Completely analogous estimates hold for the approximation of $\tilde{w}$ and $\tilde{\omega}^\epsilon$ defined by

$$\partial_t\phi = \rho\exp(i\tilde{w}) \tag{7.34}$$
$$\delta_t\psi^\epsilon = \tilde{d}^\epsilon\exp(i\tilde{\omega}^\epsilon). \tag{7.35}$$

Equations (7.30),(7.33) change in the obvious way, and the choice $\omega^\epsilon(\frac{\epsilon}{2},0) = w(\frac{\epsilon}{2},0)$ implies that $\tilde{\omega}^\epsilon(0,\frac{\epsilon}{2}) = \pi/2 + w(0,\frac{\epsilon}{2}) + \mathcal{O}(\epsilon^2)$. (Unfortunately, there is no such simple relation as $\tilde{\omega}^\epsilon = \omega^\epsilon + \pi/2$.)

With the functions ω^ϵ and $\tilde{\omega}^\epsilon$ being defined, we construct the lattice ψ^ϵ from the relations (7.29) and (7.35). To do so, the quantities d^ϵ and $\tilde{d}^\epsilon$ need to be specified. Observe that the distance between the centers of two orthogonally intersecting circles with radii ρ_1 and ρ_2 is

$$d = \sqrt{\rho_1^2+\rho_2^2} = \rho_1\sqrt{1+(\rho_2/\rho_1)^2}.$$

Having the convergence $\rho^\epsilon \to \rho$ and the limiting equations (7.28) and (7.34) in mind, it is natural to choose[4]

$$d^\epsilon(x,t) = \sqrt{\frac{1+\exp(2\epsilon v^\epsilon_{(1)}(x,t))}{2}}\rho^\epsilon(x-\tfrac{\epsilon}{2},t) \tag{7.36}$$
$$\tilde{d}^\epsilon(x,t) = \sqrt{\frac{1+\exp(2\epsilon v^\epsilon_{(2)}(x,t))}{2}}\rho^\epsilon(x,t-\tfrac{\epsilon}{2}). \tag{7.37}$$

[4]The reason we employ the particular form (7.36) instead of the symmetric representation $d = \sqrt{\rho_1^2+\rho_2^2}$ is that the former is an analytic function with arbitrarily large radius of convergence as $\epsilon \to 0$, whereas the latter has a singularity at $\rho = 0$.

Indeed, with these choices for d^ϵ, $\tilde{d}^\epsilon$, equations (7.29) and (7.35) are satisfied by ϕ, ρ, w and $\tilde{w}$ in place of ψ^ϵ, ρ^ϵ, ω^ϵ and $\tilde{\omega}^\epsilon$, up to an error $\mathcal{O}(\epsilon^2)$.

Construct ψ^ϵ on $\tilde{\Omega}(r)$, using the canonical choice $\psi^\epsilon(0,0) = \phi(0,0)$. Note that (7.29) and (7.35) are necessarily compatible. In fact, the function ψ^ϵ to be constructed describes the circle centers in the orthogonal circle pattern $\mathcal{C}^\epsilon$ with radius function $\hat{\rho}^\epsilon = \epsilon/\sqrt{2}\rho^\epsilon$, satisfying the side conditions

$$\psi^\epsilon(0,0) = \phi(0,0), \quad \arg(\psi^\epsilon(\epsilon,0) - \psi^\epsilon(0,0)) = w(\tfrac{\epsilon}{2}, 0).$$

And $\mathcal{C}^\epsilon$ exists *a priori* by the result in [Sc] that there is an orthogonal circle pattern, unique up to rigid motions, for every radius function $\hat{\rho}^\epsilon$ satisfying the exponential Laplace equation (7.23).

Finally, one passes from C^∞-approximation of ρ, u, w and $\tilde{w}$ by ρ^ϵ, v^ϵ, ω^ϵ and $\tilde{\omega}^\epsilon$ to C^∞-approximation of ϕ by ψ^ϵ, using Lemma 6.2 once again. The semi-norms $\|\cdot\|_{(N)}$ and functions h^ϵ are defined in the obvious way. ■

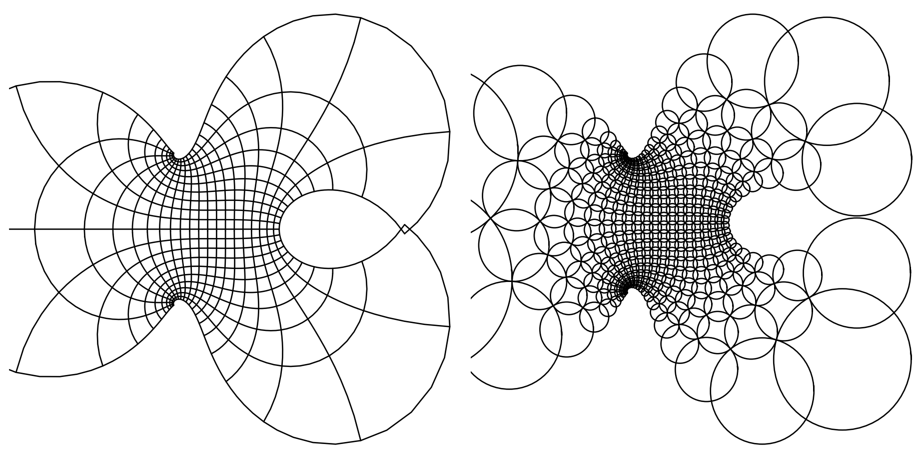

Figure 7.3: The quotient of two Airy functions – exact coordinate curves are shown on the left, and an approximating CR-1–mapping is shown on the right.

In figures 7.3 and 7.4 a CR-1–mapping and an orthogonal circle pattern are compared with the respective exact holomorphic map. The quotient of two Airy functions

$$\phi(z) = \frac{\mathrm{Bi}(az+b) - \sqrt{3}\mathrm{Ai}(az+b)}{\mathrm{Bi}(az+b) + \sqrt{3}\mathrm{Ai}(az+b)}$$

has been chosen for ϕ, with different values for $a, b \in \mathbb{C}$ in both examples. The orthogonal circle pattern obtained by solving the Cauchy problem (7.26) with the restriction of u as initial data is different than the one constructed in [BHo]. As pointed out in section 6.3, the numerical calculations for these pictures are very unstable. A precision of 10^{-10} has been used to produced them. A noticeable reduction of this precision, or an extension of the pattern across its current boundary would result in a chaotic arrangement of circles that does not resemble the original smooth function anymore.

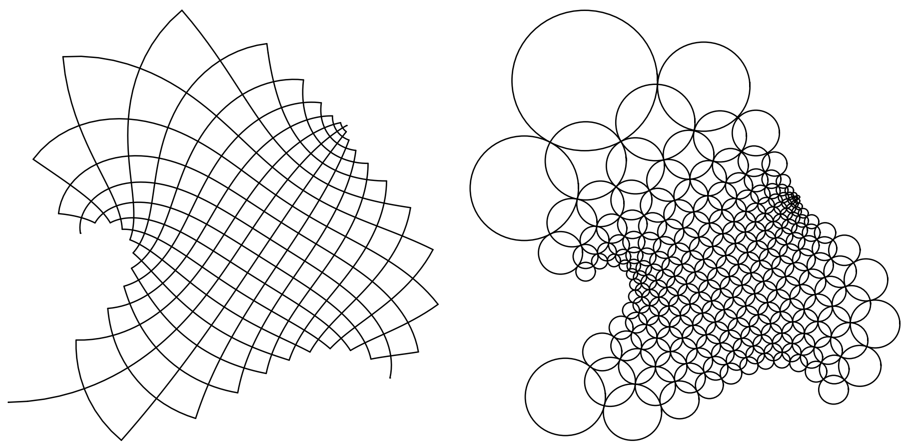

Figure 7.4: The quotient of two Airy functions – exact coordinate curves are shown on the left, and an approximating orthogonal circle pattern is shown on the right.

Chapter 8

Concluding Remarks

This concluding chapter is intended to provide a short collection of *some* ideas for possible continuation of the presented work.

New Examples

The most straightforward continuation of this work is to further complete the list of convergence proofs for the known discrete geometric models. Possible classes to investigate are, for instance, surfaces of constant mean curvature or isothermic surfaces (see below). Also, reductions of orthogonal systems could be focused on, in particular the Darboux-Egorov-metrics which have already been studied in [AKV].

General Combinatorics

One of the limitations of the theory presented here is that it only applies to lattices with the combinatorics of the standard-grid $\mathbb{Z}^M$. It is a natural desire also to include certain singularities that are reflected by a change of this combinatorics. For instance, it would be nice to have convergence results for geometries including umbilic points.

Rates of Convergence

A quite remarkable feature of the approximation theorem for general Cauchy problems is that the rate of convergence is quadratic in ϵ. It is likely that the same convergence speed can be obtained for hyperbolic equations if the initial value problem is posed in a different way. Numerical experiments support this conjecture in cases where discrete Cauchy problems are solved instead of Goursat problems.

Mixed Problems

The approximation Theorem 6.2 allows for many obvious generalizations. For instance, it is possible also to treat Cauchy problems for PDEs of "mixed type". A geometric application is the class of isothermic surfaces, which correspond to solutions of

$$\begin{aligned}
\partial_\xi^2 h + \partial_\eta^2 h + 2k_1k_2e^h &= 0\\
2\partial_\xi k_2 - (k_1 - k_2)\partial_\xi h &= 0\\
2\partial_\eta k_1 + (k_1 - k_2)\partial_\eta h &= 0
\end{aligned}$$

Here h is the metric factor, and k_1, k_2 are the sectional curvatures. This system of PDEs is neither elliptic nor hyperbolic. A Cauchy problem, however, is posed in a straightforward manner, upon introducing new coordinates $x = \xi + \eta$ and $t = \xi - \eta$. Discrete isothermic surfaces have been defined in [BP2]. They are described by difference equations for which a discrete Cauchy problem, analogous to the continuous one, can be posed.

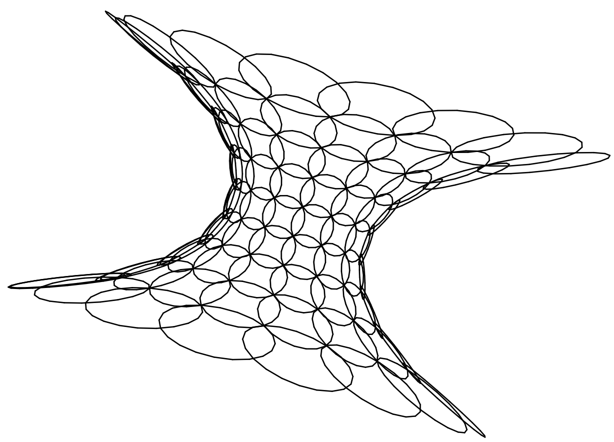

Figure 8.1: A discrete isothermic surface approximating the catenoid

Figure 8.1 represents a discrete isothermic surface that was produced as solution of such a Cauchy problem; it is supposed to approximative the smooth catenoid. A precise formulation and proof of the corresponding convergence theorem, however, seems to be technically very involved.

Note that the catenoid is not only an isothermic, but also a minimal surface. Approximation of minimal surfaces by *discrete minimal surfaces* has been proven in [BHS].

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