Soliton Spheres

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Introduction

Explicit parametrizations of surfaces with special properties have always been intensively studied in differential geometry. Local explicit parametrizations were found for large classes of surfaces in the 19th and at the beginning of the 20th century. At the end of the 20th century the focus shifted to global parametrizations, e.g., conformal immersions of compact Riemann surfaces into space. A prominent example is the explicit description of all constant mean curvature tori in $\mathbb{R}^3$, S^3, and H^3 using methods from the theory of integrable systems [**PS89, Bob91**].

The main tool for the investigations of this thesis is the quaternionic approach to surface theory, as introduced in [**PP98, FLPP01, BFLPP02**]. In this theory, branched conformal immersions of Riemann surfaces into $\mathbb{R}^4$ are represented as quotients of holomorphic sections in quaternionic holomorphic line bundles. The action of $\mathrm{GL}(2, \mathbb{H})$ on the two dimensional linear system spanned by the two holomorphic sections amounts to Möbius transformations of the quotients. The appropriate target space of the branched conformal immersions is therefore $S^4 \cong \mathbb{HP}^1$. This is reminiscent of the meromorphic functions on Riemann surfaces, which are holomorphic maps into $S^2 \cong \mathbb{CP}^1$.

The well known Plücker formula for complex holomorphic curves carries over to the quaternionic setup, involving a new invariant of quaternionic holomorphic geometry: the Willmore energy. The quaternionic Plücker formula implies for a quaternionic holomorphic line bundle L over a compact Riemann surface of genus g and an $(n+1)$–dimensional linear system H of holomorphic sections of L the Plücker estimate

$$\frac{1}{4\pi} W \geq (n+1)(n(1-g)-d) + \operatorname{ord} H,$$

where W is the Willmore energy and d is the degree of L. The integer $\operatorname{ord} H$ counts the total branching of the osculating curves of the holomorphic curve in $\mathbb{HP}^n$ that corresponds to the linear system H via the quaternionic analog of the Kodaira correspondence.

The central theme of this thesis is the investigation of linear systems with equality in the Plücker estimate. It is proven that all linear systems with equality can be described by complex holomorphic data. More precisely, to every $(n+1)$–dimensional linear system with equality corresponds a complex holomorphic curve in $\mathbb{CP}^{2n+1}$, and, conversely, every complex holomorphic curve in $\mathbb{CP}^{2n+1}$ that satisfies some nondegeneracy condition yields a linear system with equality via twistor projection and dualization. The quotient of two quaternionic holomorphic sections of a linear system with equality can be obtained by algebraic operations and differentiation from a parametrization of the corresponding complex holomorphic curve in $\mathbb{C}\mathrm{P}^{2n+1}$.

The focus of this thesis is on soliton spheres, i.e., branched conformal immersions of $\mathbb{C}\mathrm{P}^1$ into $\mathbb{H}$ that are quotients of two holomorphic sections contained in a linear system with equality in the Plücker estimate. Because the corresponding complex holomorphic curve is a curve of genus zero, it can be parametrized by polynomials in a rational parameter of $\mathbb{C}\mathrm{P}^1$. This implies that soliton spheres are parametrized by rational branched conformal immersions. It is shown that the soliton spheres, investigated by Iskander Taimanov [**Ta99**], are soliton spheres in the sense of this thesis. This justifies the name and provides a direct link to the theory of integrable systems, since Taimanov's soliton spheres correspond to solutions of the Zakharov–Shabat linear problem with reflectionless potential.

Furthermore, it is shown in this thesis that all Willmore spheres in S^4 are soliton spheres. This fact is used to derive an algebraic construction for all Willmore spheres in S^4 from rational curves in $\mathbb{C}\mathrm{P}^3$. The condition on the rational curve in $\mathbb{C}\mathrm{P}^3$ which ensures that the constructed Willmore sphere lies in S^3 is given. Robert Bryant shows in [**Br88**] that there exists an immersed Willmore sphere with Willmore energy W in S^3 if and only if $W = 4\pi n$ and $n \in \mathbb{N}\setminus\{2,3,5,7\}$. In this thesis it is shown that the Willmore energy of an immersed soliton sphere in S^3 is $4\pi n$ for some $n \in \mathbb{N}\setminus\{2,3,5\}$.

The construction of the Willmore spheres from rational curves in $\mathbb{C}\mathrm{P}^3$ involves the Willmore–Bäcklund transformation introduced in [**BFLPP02**]. A generalization of this transformation to arbitrary holomorphic curves in $\mathbb{H}\mathrm{P}^n$ is proposed. It is shown that this generalized transformation includes the Willmore–Bäcklund transformation of Willmore surfaces as well as the Christoffel transformation of isothermic surfaces.

In the theory of integrable systems there often exists a method to construct all solitons of a given equation from the vacuum, i.e., the most trivial solution. Attempting to find such a construction for soliton spheres, different transformations of the round sphere were investigated. The most promising candidate seemed to be successive applications of the isothermic Darboux transformation to the round sphere. The first step, i.e., Darboux transformation of the round sphere, is nothing but Robert Bryant's representation [**Br87**] of constant mean curvature one surfaces (CMC–1) in hyperbolic 3–space (cf., [**JMN01**]). In this thesis it is shown that a countable subset of the catenoid cousins, the most famous example of CMC–1 surfaces, are smooth at their ends. Moreover, they are soliton spheres. To the authors surprise it seems that CMC–1 surfaces that smoothly extend through their ends to the ideal boundary of hyperbolic 3–space were not investigated previously.

This thesis is organized as follows. In Chapter I basic definitions and facts concerning quaternionic holomorphic vector bundles over Riemann surfaces are collected. In addition it is shown that the isomorphism class of a quaternionic holomorphic line bundle with a nontrivial holomorphic section is uniquely determined by the zero divisor of this section and its normal vector. Furthermore, it is shown that quaternionic holomorphic line bundles, even locally, do not allow nontrivial automorphisms.

Chapter II introduces the quaternionic holomorphic geometry associated to a (branched) conformal immersion into the conformal 4–sphere. In particular, the Kodaira correspondence between 2–dimensional linear systems of holomorphic line bundles and (branched) conformal immersion into S^4, and the generalized Weierstrass representation of (branched) conformal immersions is described. The behavior of the branched conformal immersions obtained via Kodaira correspondence from 2–dimensional base point free linear systems with Weierstrass points, or via Weierstrass representation from holomorphic sections with zeros at their branch points is discussed. Moreover, the relation of the four holomorphic line bundles that enter into the Kodaira correspondence and Weierstrass representation of a (branched) conformal immersion into S^4 are described in detail. This leads to the ladder of holomorphic line bundles and relations of holomorphic structures on them. Finally, the special properties of the four holomorphic line bundles associated to a conformal immersion into S^3 are described.

In Chapter III the relation between the osculating k–planes of a holomorphic curve in $\mathbb{HP}^n$ and the Weierstrass flag of its canonical linear system is explained. The Plücker formula and Plücker estimate is formulated. The Plücker formula is applied to 1– and 2–dimensional linear systems, which leads to well known formulas from classical surface theory and extends them to branched conformal immersions that have at least one smooth normal vector. It is shown that all linear systems for which equality holds in the quaternionic Plücker estimate can be obtained from complex holomorphic curves in complex projective space via twistor projection and dualization. Three equality preserving operations, two of them along the ladder of holomorphic line bundles, are described. Soliton spheres are defined in Möbius invariant terms, and it is shown that this definition is equivalent to a possible definition in Euclidean terms. The rational functions that describe the differential of all soliton spheres in $\mathbb{R}^3$ with rotationally symmetric potential, due to Iskander Taimanov [**Ta99**], are presented in our setup. Finally, it is shown that the Willmore energy of an immersed soliton sphere in $\mathbb{R}^3$ is $4\pi n$ for some $n \in \mathbb{N} \setminus \{2, 3, 5\}$.

In Chapter IV it is shown that all Willmore spheres in $\mathbb{HP}^1$ are soliton spheres. The 1–step Willmore–Bäcklund transformation, introduced in [**BFLPP02**], is used to describe a construction of all Willmore spheres in $\mathbb{HP}^1$ from complex holomorphic curves in $\mathbb{CP}^3$ that only uses algebraic operations. This construction restricted to holomorphic curves whose tangent curve in the Plücker quadric $Q^4 \subset \mathbb{CP}^5$ is contained in a space like projective hyperplane yields all Willmore spheres in $\mathbb{R}^3$. The hyperplane condition implies that the twistor projection of the curve in $\mathbb{CP}^3$ is hyperbolic superminimal. A unified description of the superminimal surfaces in $\mathbb{R}^4$, S^4, and H^4 using $\mathbb{HP}^1$ models of these spaces is given (cf., [**Br82, Fr84, Fr97**]). Furthermore, a generalized Bäcklund transformation for holomorphic curves in $\mathbb{HP}^n$ is proposed. It is shown that it includes the Willmore–Bäcklund transformation of Willmore holomorphic curves in $\mathbb{HP}^1$ as well as the Christoffel transformation of isothermic surfaces.

In the last chapter of this thesis it is shown that a countable subset of the famous catenoid cousins (cf., [**Br87**]) extends to immersed soliton spheres. The proof uses the fact that Robert Bryant's representation of mean curvature one surfaces in hyperbolic space of curvature minus one can be interpreted as Darboux transformation of branched holomorphic coverings of the round 2–sphere (cf., [**JMN01**]). A quaternionic proof of this fact is given.

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CHAPTER I

Quaternionic Holomorphic Vector Bundles

The present chapter provides basic definitions, facts, and useful formulas concerning quaternionic holomorphic vector bundles. The aim is to provide a quick reference for the following investigations.

1. Quaternions

This section collects some facts about quaternions. For proofs and more about quaternions the reader may consult [**Numbers**].

1.1. Quaternions. The skew field of quaternions $\mathbb{H}$ can be obtained defining a multiplication on $\mathbb{R}^4$. For this purpose the standard basis of $\mathbb{R}^4$ is denoted $1, \mathbb{i}, \mathbb{j}, \mathbb{k}$ and the multiplication is defined by

$$\mathbb{i}^2 = \mathbb{j}^2 = -1 \quad \text{and} \quad \mathbb{i}\mathbb{j} = -\mathbb{j}\mathbb{i} = \mathbb{k}.$$

A quaternion $\lambda = a + \mathbb{i}b + \mathbb{j}c + \mathbb{k}d \in \mathbb{H}$ is the sum of its real part, $\operatorname{Re}\lambda = a$, and its imaginary part, $\operatorname{Im}\lambda = \mathbb{i}b + \mathbb{j}c + \mathbb{k}d$. The imaginary quaternions $\operatorname{Im}\mathbb{H}$ will be identified with $\mathbb{R}^3$, where $\mathbb{i}, \mathbb{j}, \mathbb{k}$ is identified with the standard basis of $\mathbb{R}^3$. The *quaternionic conjugation* is defined by $\bar\lambda = \operatorname{Re}\lambda - \operatorname{Im}\lambda$. It satisfies $\overline{\lambda\mu} = \bar\mu\bar\lambda$ and $\lambda\bar\lambda = \bar\lambda\lambda \in \mathbb{R}_{\geq 0}$.

1.2. Euclidean Geometry of $\mathbb{R}^3$ and $\mathbb{R}^4$ with Quaternions. For the standard Euclidean inner product of $\mathbb{R}^4$ one gets $\langle\lambda,\mu\rangle = \operatorname{Re}(\lambda\bar\mu) = \operatorname{Re}(\bar\lambda\mu)$. The absolute value of a quaternion $|\lambda| = \sqrt{\lambda\bar\lambda}$ is the length of λ as a vector in $\mathbb{R}^4$. The real part of the quaternionic multiplication is commutative, because $\operatorname{Re}(\lambda\mu)$ is the standard Minkowski product of $\mathbb{R}^4$. The product of imaginary quaternions $\lambda, \mu \in \operatorname{Im}\mathbb{H}$ gets $\lambda\mu = \lambda\times\mu - \langle\lambda,\mu\rangle$, where $\times$ denotes the vector product of $\mathbb{R}^3$.

The multiplicative inverse of a quaternion λ satisfies $\lambda^{-1} = \bar\lambda|\lambda|^{-2}$. The unit 2–sphere $S^2 \subset \mathbb{R}^3$ is the set of solutions of the equation $\lambda^2 = -1$. The orientation preserving similarities of $\mathbb{R}^4$ are given by $x \mapsto \lambda x\mu + c$, and the orientation preserving similarities of $\mathbb{R}^3 = \operatorname{Im}\mathbb{H}$ by $x \mapsto \bar\mu x\mu + c$, where $\lambda, \mu \in \mathbb{H}\setminus\{0\}$ and $c \in \mathbb{H}$ or $\operatorname{Im}\mathbb{H}$.

1.3. Quaternionic Vector Spaces. In this text quaternionic vector spaces V are always right vector spaces. The *dual* V^* of a quaternionic right vector space V is naturally a left vector space. Defining the scalar multiplication of $\beta \in V^*$ with $\lambda \in \mathbb{H}$ by $\beta\lambda(x) = \bar\lambda\beta(x)$, for all $x \in V$, it is also a right vector space.

A *complex structure* on a quaternionic vector space is a quaternionic linear endomorphism $J \in \operatorname{End}(V)$ such that $J^2 = -1$. A quaternionic vector space with a complex structure is a *complex quaternionic vector space*. The complex and the quaternionic scalar multiplication of such a vector space commute.

2. Quaternionic Holomorphic Vector Bundles

Quaternionic holomorphic vector bundles on Riemann surfaces are the central object of this thesis. This section is a collection of definitions and formulas related to quaternionic holomorphic vector bundles from [**FLPP01**].

2.1. Quaternionic vector bundles. Let M be a Riemann surface. A *quaternionic vector bundle* V over M is a fiber bundle with quaternionic right vector spaces as fibers and trivializations that are quaternionic linear on the fibers. Throughout this text all vector bundles are defined over a connected Riemann surface M, and if different vector bundles occur in the same context, they are assumed to be defined over the *same* Riemann surface.

2.2. Complex Quaternionic Vector Bundles. A *complex structure* on a quaternionic vector bundle V is a quaternionic endomorphism field $J \in \Gamma(\mathrm{End}(V))$ such that $J^2 = -\mathrm{Id}$. A quaternionic vector bundle V is called a *complex quaternionic vector bundle*, if it is endowed with a complex structure. If V is a complex quaternionic vector bundle with complex structure J, then V^* denotes the dual vector bundle of V and $\bar{V}$ denotes the complex vector bundle with complex structure $-J$. As for complex line bundles, the dual of a complex quaternionic line bundle L is denoted L^{-1}.

The tensor product of a complex vector bundle E and a complex quaternionic vector bundle V is again a complex quaternionic vector bundle $EV := E \otimes_{\mathbb{C}} V$. The most important examples are the tensor products of a complex quaternionic vector bundle V with the *canonical bundle* K and the *anticanonical bundle* $\bar{K}$ of M:

$$\begin{aligned} KV &= \{\, \alpha \in T^*M \otimes V \mid *\alpha = J\alpha \,\} \\ \bar{K}V &= \{\, \alpha \in T^*M \otimes V \mid *\alpha = -J\alpha \,\} \end{aligned}$$

where $*$ denotes precomposition with the complex structure of TM. This is minus the usual Hodge $*$–operator.

The quaternionic homomorphism bundle $\mathrm{Hom}(V, \tilde{V})$ of two complex quaternionic vector bundles has no canonical quaternionic structure, but it has two complex structures induced by composition with the complex structures of V or $\tilde{V}$. Thus there are two complex tensor products of $\mathrm{Hom}(V, \tilde{V})$ with the canonical complex line bundle K of M:

$$\begin{aligned} \mathrm{KHom}(V, \tilde{V}) &= \{\, \alpha \in T^*M \otimes \mathrm{Hom}(V, \tilde{V}) \mid *\alpha = J^{\tilde{V}}\alpha \,\} \\ \mathrm{Hom}(V, \tilde{V})\mathrm{K} &= \{\, \alpha \in T^*M \otimes \mathrm{Hom}(V, \tilde{V}) \mid *\alpha = \alpha J^{V} \,\}. \end{aligned}$$

The homomorphism bundle $\mathrm{Hom}(V, \tilde{V})$ splits into the complex linear and complex antilinear homomorphisms

$$\mathrm{Hom}_{\pm}(V, \tilde{V}) := \{\, B \in \mathrm{Hom}(V, \tilde{V}) \mid \tilde{J}B = \pm BJ \,\}.$$

2.3. Quaternionic Holomorphic Structures. Let V be a complex vector bundle. A V–valued 1–form $\alpha \in \Omega(V)$ can then be decomposed into its $(1,0)$ part $\alpha' := \frac{1}{2}(\alpha - J * \alpha)$ and its $(0,1)$ part $\alpha'' := \frac{1}{2}(\alpha + J * \alpha)$. The decomposition $\alpha = \alpha' + \alpha''$ is called the *type decomposition* of α.

A *quaternionic holomorphic vector bundle* V is a complex quaternionic vector bundle with a *quaternionic holomorphic structure* D, i.e., a quaternionic linear map

$$D\colon \Gamma(V) \to \Gamma(\bar{K}V)$$

that satisfies the *Leibniz rule*

$$D(\psi\lambda) = (D\psi)\lambda + (\psi d\lambda)'',$$

for all $\psi \in \Gamma(V)$ and $\lambda\colon M \to \mathbb{H}$.

2.4. Zeros of Holomorphic Sections. Sections $\psi \in \Gamma(V)$ that lie in $\ker D =: H^0(V) \subset \Gamma(V)$ are called *holomorphic*. The zeros of holomorphic sections $\psi \in H^0(V)$ are isolated, and there is a unique integer $\operatorname{ord}_p \psi \in \mathbb{N}$ such that on an open neighborhood $U \subset M$ of p

$$\psi = z^{\operatorname{ord}_p \psi}\varphi + O(\operatorname{ord}_p \psi + 1),$$

for a non vanishing $\varphi \in \Gamma(V_{|U})$ and a holomorphic coordinate $z\colon U \to \mathbb{C}$ centered at p (cf., [**FLPP01**, Lemma 3.9]). Note that, in contrast to the complex case, it is in general not possible to choose a holomorphic φ. Moreover, $z^{-\operatorname{ord}_p \psi}\psi$ is continuous, but in general not differentiable at p.

There always exists a covering of M with local holomorphic frames, [**BP**]. From the Leibniz rule follows that any covering with local holomorphic frames completely determines the quaternionic holomorphic structure D. If M is compact, then $H^0(V)$ is finite dimensional (cf., [**FLPP01**, Theorem 2.2])

2.5. The Underlying Complex Vector Bundle. If D commutes with J then D is a usual *complex holomorphic structure* or *$\bar\partial$–operator* on the complex vector bundle (V, J). Any quaternionic holomorphic structure D splits into a $\bar\partial$–operator, the J commuting part of D, and a quaternionic J–anticommuting endomorphism field $Q \in \Gamma(\bar{\mathrm{K}}\mathrm{End}_-(V))$, called the *Hopf field* of D. For D, $\bar\partial$, and Q one has

$$D = \bar\partial + Q, \quad \bar\partial\psi = \frac{1}{2}(D\psi - JDJ\psi), \quad Q\psi = \frac{1}{2}(D\psi + JDJ\psi),$$

for all $\psi \in \Gamma(V)$.

Let

$$\hat{V} = \{\, \psi \in V \mid J\psi = \psi\mathbb{i} \,\}$$

then $V = \hat{V} \oplus \hat{V}\mathbb{j}$ and $\bar\partial$ is a complex holomorphic structure on $\hat{V}$ and $\hat{V}\mathbb{j}$, which are isomorphic as complex holomorphic vector bundles via multiplication by $\mathbb{j}$. Every quaternionic holomorphic vector bundle V is, consequently, the double of a complex holomorphic vector bundle $V = \hat{V} \oplus \hat{V}\mathbb{j}$ plus a Hopf field $Q \in \Gamma(\bar{\mathrm{K}}\mathrm{End}_-(V))$, and $Q \equiv 0$ means $D = \bar\partial$ and V is (the double of) a complex holomorphic vector bundle. If the basis M of V is compact, one defines the *degree of* V to be the degree of $\hat{V}$:

$$\deg V := \deg \hat{V}.$$

2.6. The Willmore Energy. The real part of the trace of a quaternionic endomorphism B, i.e., a quarter of the trace of B as a real endomorphism, is well defined. Hence one can define

$$\langle B\rangle := \frac{1}{4}\operatorname{tr}_{\mathbb{R}}(B).$$

The *Willmore energy* $W(V)$ of a quaternionic holomorphic vector bundle (or rather its holomorphic structure $D = \bar\partial + Q$) is then by definition

$$W(V) = 2\int_M \langle Q\wedge *Q\rangle.$$

Note that if M is not compact, then $W(V)$ can well be infinity.

2.7. Connections and Holomorphic Structures. Let ∇ be a quaternionic connection on a complex quaternionic vector bundle V and $\nabla = \nabla' + \nabla''$ its decomposition into its (1,0) part $\nabla' = \frac{1}{2}(\nabla - J*\nabla)$ and (0,1) part $\nabla'' = \frac{1}{2}(\nabla + J*\nabla)$. ∇' is a quaternionic holomorphic structure of $\bar V$ and ∇'' one of V. Further decomposition into J commuting and anticommuting parts yields the *type decomposition*:

$$\nabla = \nabla' + \nabla'' = (\partial + A) + (\bar\partial + Q).$$

∂ and $\bar\partial$ are complex holomorphic structures on $\bar V$ and V. $A\in\Gamma(K\operatorname{End}_-(V))$ and $Q\in\Gamma(\bar K\operatorname{End}_-(V))$, and A and Q are the Hopf fields of $(\bar V,\nabla')$ and (V,∇''), respectively. Because $\hat\nabla = \partial + \bar\partial$ is a complex connection, one has

$$\nabla J = 2(*Q - *A).$$

This means that $2{*Q}$ and $2{*A}$ are the $(0,1)$ and $(1,0)$ parts of ∇J, respectively. The curvature tensor $\mathcal{R}^\nabla$ of M, its J–commuting part $\mathcal{R}^\nabla_+$, and its J–anticommuting part $\mathcal{R}^\nabla_-$ satisfy

$$\mathcal{R}^\nabla = \mathcal{R}^{\hat\nabla} + Q\wedge Q + A\wedge A + d^{\hat\nabla}(A+Q),$$
$$\mathcal{R}^\nabla_+ = \mathcal{R}^{\hat\nabla} + Q\wedge Q + A\wedge A, \qquad \mathcal{R}^\nabla_- = d^{\hat\nabla}(A+Q).$$

In the calculation of the first line $Q\wedge A = A\wedge Q = 0$ is used. This is an instance of the type argument.

2.8. The Type Argument. The *type argument* is often used to show that certain wedge products of one forms on a Riemann surface vanish. It is a generalization of the fact that there are no nontrivial 2–forms of type $(2,0)$ and $(0,2)$ on a Riemann surface.

For example, the vanishing of $Q\wedge A$ follows because $*Q = QJ$ and $*A = JA$ (Q is right and A is left K) implies for all $X\in TM$: $Q\wedge A(X, J_M X) = Q_X{*A_X} - {*Q_X}A_X = Q_X J A_X - Q_X J A_X = 0$. The vanishing of $A\wedge Q$ follows because A is right and Q is left $\bar K$.

Another example is two 1–forms $\omega,\eta\in\Omega^1(\mathbb{H})$ with values in $\mathbb{H}$ such that $*\omega = \omega N$ for some map $N: M\to\mathbb{H}$ and ω does not vanish on a dense subset of M, then $\omega\wedge\eta = 0$ is equivalent to $*\eta = N\eta$.

3. Quaternionic Holomorphic Line Bundles

For the following investigations the most important quaternionic holomorphic vector bundles are line bundles, i.e., quaternionic holomorphic vector bundles with 1–dimensional fibers. For nowhere vanishing sections of such line bundles one can define the normal vector. It is shown in the present section that one can calculate the degree and the Willmore energy of the line bundle from every normal vector. In the second part of this section the definition of the Weierstrass flag, points, and order from [**FLPP01**] is given. Finally, it is shown that, in contrast to the complex case, the generic holomorphic section realizes the base locus of a linear system.

3.1. Normal Vector. Let L be a complex quaternionic line bundle and $\psi \in \Gamma(L)$ a nowhere vanishing section. Such a section always exists, by transversality. There is a unique smooth map $N\colon M \to \mathbb{H}$ such that $J\psi = \psi N$. From $J^2 = -\operatorname{Id}$ follows $N^2 = -1$. Thus N is a map into $S^2 \subset \operatorname{Im}\mathbb{H} = \mathbb{R}^3$. N is called the *normal vector* of ψ.

3.2. The Degree of Normal Vectors. On compact M a normal vector $N\colon M \to S^2$ has a mapping degree. The mapping degrees of the normal vectors of sections of a complex quaternionic line bundle are all equal.

Theorem. *Let L be a complex quaternionic line bundle on a compact Riemann surface and $\psi \in \Gamma(L)$ a nowhere vanishing section with normal vector N, then*

$$\deg L = \deg N.$$

Proof. Let ∇ be the complex quaternionic connection on L defined by the equation $\nabla\psi = -\psi\frac{1}{2}NdN$. Then the curvature tensor $\mathcal{R}^\nabla$ of ∇ satisfies $J\,\mathcal{R}^\nabla\,\psi = -\psi\frac{1}{4}NdN \wedge dN = \psi\frac{1}{2}\langle NdN, dN\rangle$. Hence

$$\deg L = \frac{1}{2\pi}\int_M J\,\mathcal{R}^\nabla = \frac{1}{4\pi}\int_M \langle NdN, dN\rangle = \deg N.$$

□

3.3. Type Decomposition by Normal Vectors. The left multiplication with a normal vector induces a complex structure on $\mathbb{H}$ and consequently a *type decomposition* of $\mathbb{H}$–valued 1–forms $\alpha \in \Omega^1(\mathbb{H})$: $\alpha = \alpha' + \alpha''$, $\alpha' = \frac{1}{2}(\alpha - N*\alpha)$ and $\alpha'' = \frac{1}{2}(\alpha + N*\alpha)$. In most situations it is clear from the context, which normal vector has to be taken, for example in

$$(\psi d\lambda)'' = \psi(d\lambda)''$$

$(d\lambda)''$ is the $(0,1)$–part of $d\lambda$ with respect to left multiplication with the normal vector of ψ. Let $\psi \in H^0(L)$ without zeros, N its normal vector and Q the Hopf field of the holomorphic structure of L, then one gets

$$Q\psi = \tfrac{1}{2}(D\psi + JDJ\psi) = \tfrac{1}{2}JD(\psi N) = \tfrac{1}{2}J(\psi dN)'' = \psi\tfrac{1}{2}NdN''.$$

3.4. The Normal Vector Determines the Holomorphic Section. In contrast to the complex case—where all sections have the normal vector $\mathbb{i}$—different holomorphic sections of quaternionic holomorphic line bundles with nontrivial Hopf field have different normal vectors.

Theorem. *If a quaternionic holomorphic line bundle L has two nontrivial holomorphic sections with the same normal vector besides their zeros, then the two sections are linearly dependent over the reals or L is a doubled complex holomorphic line bundle.*

PROOF. Let ψ, $\tilde{\psi} \in H^0(L)$ be two nontrivial holomorphic sections with the same normal vector N besides their zeros. The set $M_0 = \{ p \in M \mid \psi_p \neq 0 \}$ is dense in M and there is a map $\lambda\colon M_0 \to \mathbb{H}$ such that $\tilde{\psi} = \psi\lambda$. The quaternionic linearity of J implies $N\lambda = \lambda N$. Thus there exists λ_1 and $\lambda_2\colon M_0 \to \mathbb{R}$, such that

$$\lambda = \lambda_1 + \lambda_2 N.$$

The Leibniz rule and holomorphicity of ψ and $\tilde{\psi}$ imply

$$0 = D(\psi\lambda) = D(\psi)\lambda + (\psi d\lambda)'' = \psi(d\lambda)''.$$

Decomposition of $0 = d\lambda'' = \frac{1}{2}(d\lambda + N * d\lambda)$ into N–commuting and N–anticommuting terms yields two equations:

$$\begin{aligned} 0 &= (d\lambda_1 + d\lambda_2 N)'' = \tfrac{1}{2}(d\lambda_1 + d\lambda_2 N + *d\lambda_1 N - *d\lambda_2) \\ 0 &= \lambda_2 dN''. \end{aligned}$$

The first equation means that $\tilde{\lambda} = \lambda_1 + \lambda_2 \mathbb{i}\colon M_0 \to \mathbb{C}$ is a complex holomorphic function. This implies that $\lambda_2 \equiv 0$ or λ_2 does not vanish on a dense open subset of M_0. In the first case $\lambda = \lambda_1$ is a real constant, and $\tilde{\psi} = \psi\lambda_1$ on M. In the second case, the equation $0 = \lambda_2 dN''$ implies that $dN'' = 0$. The Hopf field Q of D satisfies $2Q\psi = \psi N dN''$ for all $\psi \in L$, see 3.3 below. Hence $dN'' = 0$ implies that Q vanishes identically on M. □

3.5. Willmore Energy. Let L be a quaternionic holomorphic line bundle and $\psi \in H^0(L)$ nontrivial. Then ψ has a normal vector N away from the zeros of ψ. The formula $Q\psi = \psi\frac{1}{2}NdN''$, from 3.3, then holds besides the zeros of ψ, which is a set of measure zero. Thus

$$\langle Q \wedge *Q \rangle = \frac{1}{4}\operatorname{Re}(NdN'' \wedge N*dN'') = \frac{1}{2}\langle dN''N, dN'' \rangle$$

away from the zeros of ψ and

$$W(L) = \int_M \langle dN''N, dN'' \rangle.$$

Let $X \in TM$ and write J_M for the complex structure of TM, then $X, J_M X$ is positively oriented and $\langle dN''N, dN'' \rangle(X, J_M X) = \langle dN''(X)N, dN''(X)N \rangle = |dN''(X)|^2$, because $*dN'' = -NdN'' = dN''N$. Thus $\langle dN''N, dN'' \rangle$ is a positive area form on M. Hence

$$W(L) = 0 \iff Q = 0,$$

which implies that every quaternionic holomorphic line bundle with vanishing Willmore energy is a (doubled) complex holomorphic line bundle.

3.6. Remark. The equations $\langle Q \wedge *Q\rangle(X, JX) = \frac{1}{2}|dN''(X)|^2$ and $Q\psi = \psi\frac{1}{2}NdN''$ imply that $\langle Q \wedge *Q\rangle$ vanishes at $p \in M$ if and only if Q vanishes at p.

3.7. Weierstrass Numbers, Flag, Gaps, Points, and Order. Let L be a quaternionic holomorphic line bundle and $H \subset H^0(L)$ an $(n+1)$–dimensional *linear system*, i.e., an $(n+1)$–dimensional quaternionic vector space of holomorphic sections of L. The *Weierstrass numbers* at $p \in M$ are defined recursively as follows

$$n_0(p) := \min\{\operatorname{ord}_p \psi \mid \psi \in H\},$$
$$n_{k+1}(p) := \min\{\operatorname{ord}_p \psi \mid \psi \in H, \operatorname{ord}_p \psi > n_k(p)\},$$

as long as there are elements ψ in H such that $\operatorname{ord}_p \psi > n_k(p)$. Let

$$H_k(p) := \{\psi \in H \mid \operatorname{ord}_p \psi \geq n_{n-k}(p)\}.$$

Lemma. $\dim(H_k/H_{k-1}) = 1.$

Proof. The definition of the H_k implies $\dim(H_k/H_{k-1}) \geq 1$. Let $\psi_1, \psi_2 \in H_k \setminus H_{k-1}$, then $\operatorname{ord}_p \psi_1 = \operatorname{ord}_p \psi_2 = n_{n-k}$. Consequently, there is a centered coordinate $z\colon M \supset U \to \mathbb{C}$, $z(p) = 0$, and nowhere vanishing sections $\varphi_1, \varphi_2 \in \Gamma(L_{|U})$ such that

$$\psi_{1,2} = z^{n_{n-k}}\varphi_{1,2} + O(n_{n-k} + 1).$$

The holomorphic section $\psi = \psi_1\varphi_1(0)^{-1} - \psi_1\varphi_2(0)^{-1}$ of L is contained in H_{k-1}, because $\lim_{z\to 0} z^{n_{n-k}}\psi = 0$. Thus ψ_1 and ψ_2 are linearly dependent modulo H_{k-1}. □

The lemma implies,

$$\dim H_k = k + 1,$$

because $H_n = H$. The flag

$$\{0\} \subset H_0 \subset \ldots \subset H_{n-1} \subset H_n = H$$

is called the *Weierstrass flag* of H. The sequence $0 \leq n_0 < n_1 < \ldots < n_n$ is called the *Weierstrass gap sequence*.

A point $p \in M$ at which the sequence differs from $0, 1, \ldots, n$ is called a *Weierstrass point*. The *Weierstrass order* $\operatorname{ord}_p H$ of H at $p \in M$ is defined by

$$\operatorname{ord}_p H = \sum_{k=0}^{n} n_k(p) - k.$$

Thus $p \in M$ is a Weierstrass point if and only if $\operatorname{ord}_p H \neq 0$. The Weierstrass points are isolated, by [**FLPP01**, Lemma 4.1 & 4.9]. Hence if M is compact, then there are only finitely many points with $\operatorname{ord}_p \psi \neq 0$. In this case one can define the *Weierstrass order* $\operatorname{ord} H$ of H by

$$\operatorname{ord} H = \sum_{p\in M} \operatorname{ord}_p H.$$

3.8. Base Points. Let $H \subset H^0(L)$ be a linear system of a quaternionic holomorphic line bundle. A point $p \in M$ at which all elements of H have a common zero is called a *base point* of H.

If H has dimension 1, then the base points of H are obviously the zeros of every element in H and $n_0(p) = \operatorname{ord}_p \psi$, for all $\psi \in H$ and $p \in M$. In contrast to the complex case, generic sections of linear systems of higher dimension also realize the base locus of the linear system.

Theorem. *If $H \subset H^0(L)$ is a linear system of a quaternionic holomorphic line bundle, then generic sections $\psi \in H$ realize the base locus of H, i.e., $\operatorname{ord}_p \psi = n_0(p)$ holds for all $p \in M$.*

Proof. The theorem follows if one shows that the set $\tilde{H} := \{\psi \in H \mid \operatorname{ord} \psi \neq n_0\}$ is a set of measure zero in H. Let $n + 1 = \dim H$. Because the n–th member of the Weierstrass flag $H_{n-1}(p)$ contains all $\psi \in H$ for which $\operatorname{ord}_p \psi \neq n_0(p)$, one has $\tilde{H} = \bigcup_{p \in M} H_{n-1}(p)$. Let $M_0 = M \setminus \{\text{base points of H}\}$. Then $\bigcup_{p \in M_0} H_{n-1}(p)$ is the union of the kernels of the evaluation maps

$$\operatorname{ev}_p \colon H \to L_p, \ \psi \mapsto \psi_p.$$

As $\operatorname{ev} \colon M_0 \times H \to L_{|M_0}$ is a surjective bundle homomorphism, its kernels form a subbundle of hyperplanes in the trivial bundle H over M_0. Because the real dimension of M is 2 and the real codimension of quaternionic hyperplanes is 4, one concludes from Sard's theorem, that $\bigcup_{p \in M_0} H_{n-1}(p)$ has measure zero in H. The base points of a linear system are isolated, because zeros of holomorphic sections are isolated. Hence $M \setminus M_0$ is countable. Thus $\tilde{H}$ has measure zero. □

4. Holomorphic Bundle Homomorphisms

In the following the questions whether a given complex quaternionic bundle homomorphism is holomorphic and whether two given quaternionic holomorphic line bundles are isomorphic occur frequently. The first question can often be answered by Lemma 4.1, whereas Theorem 4.2 is useful for answering the second one. Finally, in Theorem 4.3 an important difference between the complex and the quaternionic case is exposed: There are, even locally, no nontrivial automorphisms of quaternionic holomorphic line bundles.

4.1. Holomorphic Bundle Homomorphisms. A quaternionic bundle homomorphism $B \colon V \to \tilde{V}$ is called a *quaternionic holomorphic bundle homomorphism*, if it is *complex linear*, i.e., $BJ = \tilde{J}B$, and *holomorphic*, i.e., $BD = \tilde{D}B$. If $D = \bar\partial + Q$ and $\tilde{D} = \tilde{\bar\partial} + \tilde{Q}$, as in 2.5, then $B\bar\partial = \tilde{\bar\partial}B$, i.e., B is complex holomorphic, and $BQ = \tilde{Q}B$.

If $B \colon L \to \hat{L}$ is a holomorphic bundle homomorphism between line bundles, then one can define the *vanishing order* of B at $p \in M$ by $\operatorname{ord}_p B := \operatorname{ord}_p \hat{B}$, where $\hat{B} \colon \hat{L} \to \hat{\hat{L}}$ is the induced complex holomorphic homomorphism between the underlying complex line bundles (cf., 2.5). If M is compact, then B has finite *total vanishing order*, $\operatorname{ord} B := \sum_{p \in M} \operatorname{ord}_p B$.

Lemma. *Let V and $\tilde{V}$ be quaternionic holomorphic vector bundles and $B\colon V \to \tilde{V}$ a complex quaternionic bundle homomorphism. If U_i is an open cover of M such that there exist holomorphic frames $(\psi_{ij})_{1\leq j\leq \operatorname{rank} V}$ of $V_{|U_i}$, then B is holomorphic if and only if the sections $B\psi_{ij}$ are holomorphic sections of $\tilde{V}$.*

Proof. If B is holomorphic, then all $B\psi_{ij}$ are holomorphic. If, on the other hand, all $B\psi_{ij}$ are holomorphic, then $BD\psi = \tilde{D}B\psi$ for all $\psi \in \Gamma(L)$, by the Leibniz rule and because $BJ = \tilde{J}B$ implies $B(\alpha'') = (B\alpha)''$ for all $\alpha \in \Omega^1(L)$. □

Note that the lemma remains true if the $(\psi_{ij})_{1\leq j\leq \operatorname{rank} V}$ fail to be a basis of the fiber of V on a discrete subset $M_0 \subset M$.

4.2. Isomorphic Quaternionic Holomorphic Line Bundles. Complex holomorphic line bundles with nontrivial holomorphic sections are isomorphic if and only if they have holomorphic sections with the same zero divisor. The same holds for quaternionic holomorphic line bundles if one also requires that the normal vectors of these sections are equal.

Theorem. *Let L and $\tilde{L}$ be two quaternionic holomorphic line bundles. Suppose that there exists a nontrivial holomorphic section $\psi \in H^0(L)$. Let N be its normal vector on $M_0 = \{\, p \in M \mid \psi_p \neq 0 \,\}$. Then L and $\tilde{L}$ are isomorphic if and only if $\tilde{L}$ has a holomorphic section $\tilde{\psi}$ with normal vector N on M_0 such that $\operatorname{ord}_p \psi = \operatorname{ord}_p \tilde{\psi}$ for all $p \in M$.*

Proof. If $B \in \Gamma(\operatorname{Hom}_+(L, \tilde{L}))$ is a holomorphic isomorphism, then $\tilde{\psi} := B\psi$ is a holomorphic section of $\tilde{L}$ with the same vanishing order as ψ. Furthermore, on M_0 one gets $J\tilde{\psi} = \tilde{J}B\psi = BJ\psi = B\psi N = \tilde{\psi}N$.

Conversely, if there exists $\tilde{\psi} \in H^0(\tilde{L})$ with normal vector N on M_0 such that $\operatorname{ord}_p \psi = \operatorname{ord}_p \tilde{\psi}$, then $B\psi := \tilde{\psi}$ defines a quaternionic isomorphism from $L_{|M_0}$ to $\tilde{L}_{|M_0}$. The equality of the normal vectors of ψ and $\tilde{\psi}$ implies $B \in \Gamma(\operatorname{Hom}_+(L, \tilde{L})_{|M_0})$. Furthermore, B is holomorphic, by Lemma 4.1, and bounded near the zeros of ψ and $\tilde{\psi}$. The boundedness follows from 2.4 and the assumption that the vanishing orders of ψ and $\tilde{\psi}$ coincide. The induced bundle homomorphism $\hat{B} \in \Gamma(\operatorname{Hom}(\hat{L}, \hat{\tilde{L}})_{|M_0})$ is complex holomorphic on the underlying complex holomorphic line bundles $\hat{L}$ and $\hat{\tilde{L}}$. Thus $\hat{B}$, and consequently B, can be holomorphically extended into the zeros of ψ. B does not vanish at the zeros of ψ, once again because the vanishing orders of ψ and $\tilde{\psi}$ coincide. Hence the extension of B to M is a quaternionic holomorphic bundle isomorphism between L and $\tilde{L}$. □

4.3. Automorphisms of Holomorphic Line Bundles. In the complex case holomorphic line bundles on noncompact Riemann surfaces have many holomorphic automorphisms. This does not hold for quaternionic holomorphic line bundles with nontrivial Hopf field.

Theorem. *If a quaternionic holomorphic line bundle L has a holomorphic automorphism that is not multiplication with a real constant, then the*

Hopf field of L vanishes identically, i.e., L is a doubled complex holomorphic line bundle.

A consequence of this theorem is that holomorphic bundle homomorphisms of quaternionic holomorphic line bundles with nontrivial Hopf field, even locally, only differ by real multiplicative constants.

Proof. Let B be a holomorphic automorphism of L. Let $\psi \in H^0(L_{|U})$ be a local holomorphic section on $U \subset M$. Then $\tilde{\psi} := B\psi$ is also a holomorphic section of $L_{|U}$. Because $BJ = JB$ and B is quaternionic linear $\tilde{\psi}$ and ψ have the same normal vector N. Then Theorem 3.4 implies that $Q_{|U} \equiv 0$ or there exist $c \in \mathbb{R}$ such that $B\psi = \tilde{\psi} = \psi c$.

To conclude the proof, it is sufficient to show that if $B\psi = \psi c$ for some $c \in \mathbb{R}$ and $\psi \in \Gamma(L_{|U})$, then B is multiplication with c on all of M. Let $\psi_1 \in H^0(L_{|U_1})$ and $\psi_2 \in H^0(L_{|U_2})$ be local holomorphic sections on overlapping open subsets $U_1, U_2 \subset M$ and suppose that B is multiplication by c on U_1. Then $B\psi_2 - \psi_2 c$ is holomorphic on U_2 and vanishes on the nonempty open set $U_1 \cap U_2$, and, therefore, vanishes on all of U_2, by 2.4. Hence B is multiplication by c on U_2. Because M can be covered by local holomorphic sections of L, as mentioned in 2.4, and M is connected, it follows that B is multiplication by c on all of M. $\square$

4.4. Notation. In the rest of this text the adjective "quaternionic" is usually omitted. If it is necessary the adjective "complex" is used to emphasize that something is only assumed to be complex linear, complex holomorphic, etc. For example "$\psi \in \Gamma(V)$ is holomorphic" means $D\psi = 0$, and "$\psi \in \Gamma(V)$ is complex holomorphic" means $\bar{\partial}\psi = 0$.

CHAPTER II

Surfaces in S^4

Identifying S^4 with $\mathbb{H}\mathrm{P}^1$, a conformal immersion of a Riemann surface into S^4 can be interpreted as an immersed quaternionic holomorphic curve in $\mathbb{H}\mathrm{P}^1$. There are four quaternionic holomorphic line bundles associated to such an immersion.

Two of these quaternionic holomorphic line bundles are Möbius invariant. They are determined by the property that all stereographic projections of the immersion or its antipodal reflection are quotients of holomorphic sections of these bundles.

The quaternionic holomorphic structures of the other two, the Euclidean holomorphic line bundles, depend on the choice of a point $\infty \in S^4$. These bundles are determined by the property that they are paired and that the differential of the stereographic projection of the immersion with pole ∞ is the product of two holomorphic sections of these bundles.

For nonimmersed holomorphic curves in $\mathbb{H}\mathrm{P}^1$ one can at least define one Möbius invariant and one Euclidean holomorphic line bundle. This and the behavior of these curves at the zeros of their differentials is discussed in the section on branched conformal immersions.

The relations of the holomorphic structures of the four quaternionic holomorphic line bundles associated to the immersion can be visualized in the quadrilateral of holomorphic line bundles. This quadrilateral can then be extended to the ladder of holomorphic line bundles. Many results in the remaining text rely on a detailed investigation of the relations between the holomorphic line bundles that occur in this ladder.

If the immersed surface takes values in some 3–sphere in S^4, then the quaternionic holomorphic line bundles associated to this immersion have special properties. This is the subject of the last section of this chapter.

5. The Conformal 4–Sphere and $\mathbb{H}\mathrm{P}^1$

The *standard metric* of the 4–sphere is the induced metric on the embedding of the 4–sphere as the unit sphere $S^4 \subset \mathbb{R}^5$. The 4–sphere together with the conformal structure of the standard metric is called the *conformal 4–sphere*. In the present text the quaternionic projective line $\mathbb{H}\mathrm{P}^1$ is used as a model for the conformal 4–sphere. This model is very similar to the $\mathbb{C}\mathrm{P}^1$ model of the conformal 2–sphere. The Möbius transformations of S^4, for example, can be represented by linear transformations of $\mathbb{H}^2$. But, in contrast to the case of the conformal 2–sphere, every conformal diffeomorphism between open subsets of S^4 extends to a Möbius transformation, by Liouville's theorem, see [**Jeromin**, 1.5.4].

Another model, the light cone model, of the conformal 4–sphere is described in 26.8. Neither the light cone nor the quaternionic model for the Möbius geometry can be discussed in much detail in the present text. A comprehensive introduction to the Möbius geometry of the conformal n–sphere, the relation of the different models, applications as well as an extensive bibliography can be found in the excellent lecture note "Introduction to Möbius differential geometry" by Udo Hertrich–Jeromin [**Jeromin**].

5.1. $\mathbb{H}\mathrm{P}^1$, Affine and Euclidean coordinates. The *quaternionic projective line* $\mathbb{H}\mathrm{P}^1$ is the set of quaternionic lines in $\mathbb{H}^2$. The line through $x \in \mathbb{H}^2\backslash\{0\}$ is denoted $[x] \in \mathbb{H}\mathrm{P}^1$. Every quaternionic linear form $\beta \in (\mathbb{H}^2)^*\backslash\{0\}$ induces an *affine coordinate* A_β of $\mathbb{H}\mathrm{P}^1$

$$\begin{aligned} A_\beta \colon \mathbb{H}\mathrm{P}^1\backslash\{\infty\} &\longrightarrow \{\, x \in \mathbb{H}^2 \mid \beta(x) = 1\,\} \\ [x] &\longmapsto x(\beta(x))^{-1}, \end{aligned}$$

where $\infty = \ker\beta$ is called the *pole* of the affine coordinate. If f is a map into $\mathbb{H}\mathrm{P}^1\backslash\{\infty\}$, then $A_\beta f$ is a section of the trivial quaternionic vector bundle $\mathbb{H}^2$ over the domain of f. This section is called the *affine lift* of f. If α, β is a basis of $(\mathbb{H}^2)^*$, then

$$\begin{aligned} \sigma_{\alpha,\beta} := \alpha A_\beta \colon \mathbb{H}\mathrm{P}^1\backslash\{\infty\} &\longrightarrow \mathbb{H} \\ [x] &\longmapsto \alpha(x)(\beta(x))^{-1}. \end{aligned}$$

is called a *Euclidean coordinate* or *stereographic projection* of $\mathbb{H}\mathrm{P}^1$. For the inverse of a Euclidean coordinate one gets

$$\begin{aligned} \sigma^{-1}_{\alpha,\beta} \colon \mathbb{H} &\longrightarrow \mathbb{H}\mathrm{P}^1\backslash\{\infty\} \\ \lambda &\longmapsto a\lambda + b, \end{aligned}$$

where $a, b \in \mathbb{H}^2$ denotes the dual basis of α, β.

5.2. Identification of $\mathbb{H}\mathrm{P}^1$ and S^4. If one identifies $\mathbb{H} \cup \{\infty\}$ with $S^4 \subset \mathbb{R}^5$ via stereographic projection, then the composition with a Euclidean coordinate is a bijection between S^4 and $\mathbb{H}\mathrm{P}^1$. To see that this identification of S^4 and $\mathbb{H}\mathrm{P}^1$ induces a well defined *conformal structure* on $\mathbb{H}\mathrm{P}^1$, one needs to show that the transition functions of the Euclidean coordinates are conformal transformations of $\mathbb{H} \cup \{\infty\}$.

Let α, β and $\tilde\alpha, \tilde\beta$ be two bases of $(\mathbb{H}^2)^*$ and $A \in \mathrm{GL}(\mathbb{H}, 2)$ such that $\alpha A = \tilde\alpha$ and $\beta A = \tilde\beta$, then $\sigma_{\alpha,\beta}A = \sigma_{\tilde\alpha,\tilde\beta}$. Thus the transition functions of Euclidean coordinates $\sigma_{\tilde\alpha,\tilde\beta}\sigma^{-1}_{\alpha,\beta} = \sigma_{\alpha,\beta}A\sigma^{-1}_{\alpha,\beta}$ can be identified, via a fixed Euclidean coordinate, with the projective transformations $\mathrm{PGL}(\mathbb{H}, 2) = \mathrm{GL}(\mathbb{H}, 2)/\mathbb{R}^*$ of $\mathbb{H}\mathrm{P}^1$.

The group $\mathrm{GL}(\mathbb{H}, 2)$ is generated by

$$S_{\lambda_1,\lambda_2} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \qquad T_\mu = \begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix}, \qquad R = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

λ_1, λ_2, $\mu \in \mathbb{H} \setminus \{0\}$. These transformations correspond in the Euclidean coordinate $\sigma_{\alpha,\beta}$ to the following transformations of $\mathbb{H} \cup \{\infty\}$:

$$\sigma_{e_1,e_2}S_{\lambda_1,\lambda_2}\sigma^{-1}_{e_1,e_2}(x) = \lambda_1 x \lambda_2^{-1}, \qquad \sigma_{e_1,e_2}T_\mu\sigma^{-1}_{e_1,e_2}(x) = x + \mu,$$

$$\sigma_{e_1,e_2}R\sigma^{-1}_{e_1,e_2}(x) = x^{-1} = \frac{\bar x}{|x|^2},$$

for all $x \in \mathbb{H}$. Hence S_{λ_1,λ_2} induces an orientation preserving similarity of $\mathbb{H}$, T_μ a translation and R the orientation preserving inversion at the unit sphere $S^3 \subset \mathbb{H}$.

These three types of transformations generate the group of orientation preserving *Möbius transformations* of $S^4 = \mathbb{H} \cup \{\infty\}$, see [**KulPi**, p. 13–15]. Consequently, the identification of $\mathbb{HP}^1$ with $S^4 \subset \mathbb{R}^5$ via a Euclidean coordinate with $S^4 = \mathbb{H} \cup \{\infty\}$ is unique up to Möbius transformations. The group of projective transformations of $\mathbb{HP}^1$ is noncanonically identified with the group of Möbius transformations of S^4. The induced conformal structure on $\mathbb{HP}^1$ is independent of the choice of the Euclidean coordinate. The group of projective transformations fixing the pole ∞ of a Euclidean coordinate $\sigma_{\alpha,\beta}$ corresponds to the group of orientation preserving similarities of $\mathbb{HP}^1 \setminus \{\infty\} \cong \mathbb{H}$. Finally, one can define the *standard orientation* of $\mathbb{HP}^1$ to be the pullback of the standard orientation of $\mathbb{H}$, for which $1, \mathbb{i}, \mathbb{j}, \mathbb{k}$ is a positively oriented bases, because the transition functions of the Euclidean coordinates are orientation preserving.

5.3. The Tangent Space of $\mathbb{HP}^1$. One usually identifies the tangent space $T_{[x]}\,\mathbb{HP}^1$ of $\mathbb{HP}^1$ at $[x]$ with $\operatorname{Hom}([x], \mathbb{H}^2/[x])$ as follows: Let $\pi\colon \mathbb{H}^2 \to \mathbb{HP}^1$, $x \mapsto [x]$ be the canonical projection, then $d_x\pi\colon \mathbb{H}^2/[x] \to T_{[x]}\,\mathbb{HP}^1$ is an isomorphism. The map $[x] \to \mathbb{H}^2/[x]$, $y \mapsto d_y\pi^{-1}(v)$ is for all $v \in T_{[x]}\,\mathbb{HP}^1$ quaternionic linear, because $d_y\pi(w) = d_{y\lambda}\pi(w\lambda)$, for all y, $w \in \mathbb{H}^2$ and $\lambda \in \mathbb{H}$. Thus the map

$$\begin{array}{rcl} T_{[x]}\,\mathbb{HP}^1 & \longrightarrow & \operatorname{Hom}([x], \mathbb{H}^2/[x]) \\ v & \longmapsto & (y \mapsto d_x\pi^{-1}(y)) \end{array}$$

is an isomorphism.

Let $f\colon \mathbb{R} \to \mathbb{HP}^1$ and $\tilde{f}\colon \mathbb{R} \to \mathbb{H}^2$ be some lift of f, i.e., $f = \pi\tilde{f}$. The differential of f at $p \in \mathbb{R}$ can, with the above identification, be written as

$$d_pf = (\tilde{f}(p) \mapsto \nabla\tilde{f}_{|p} \mod f(p)),$$

where ∇ is the trivial connection on the trivial bundle $\mathbb{H}^2$ over $\mathbb{R}$.

The differential of a map $f\colon\ \mathbb{HP}^1 \to \mathbb{R}$ at $[x] \in \mathbb{HP}^1$ is in terms of its lift $\tilde{f} = f\pi\colon \mathbb{H}^2 \to \mathbb{R}$ given by

$$d_{[x]}f(v) = d_x\tilde{f}(v(x)),$$

for all $v \in T_{[x]}\,\mathbb{HP}^1 = \operatorname{Hom}([x], \mathbb{H}^2/[x])$. Here the map $d_x\tilde{f}\colon \mathbb{H}^2 \to \mathbb{R}$ and the induced map $d_x\tilde{f}\colon \mathbb{H}^2/[x] \to \mathbb{R}$ are not distinguished notationally. More generally, if $A \in \operatorname{Hom}(V, W)$, then the induced map on $V/\ker A$ is in the following also denoted by A.

For example, the affine coordinate $A_\beta\colon\ \mathbb{HP}^1 \setminus \{\infty\} \to \beta^{-1}(\{1\})$ has at $[x] \in \mathbb{HP}^1$ the differential

$$\begin{array}{rcl} d_{[x]}A_\beta\colon \operatorname{Hom}([x], \mathbb{H}^2/[x]) & \longrightarrow & \infty \subset \mathbb{H}^2 \\ v & \longmapsto & v(x)(\beta(x))^{-1} - x(\beta(x))^{-1}\beta(v(x))(\beta(x))^{-1}. \end{array}$$

Here, $v(x)$ stands, according to the above convention, at both occurrences for the same arbitrary element of the equivalence class $v(x) \subset \mathbb{H}^2$. The formula of the differential of a Euclidean coordinate $\sigma_{\alpha,\beta}$ is then easily derived from $d\sigma_{\alpha,\beta} = \alpha dA_\beta$.

5.4. 3–Spheres in $\mathbb{HP}^1$. A subset of $\mathbb{HP}^1$ is a *3–sphere* if one (and then every) stereographic projection maps it onto a 3–sphere or a real hyperplane plus infinity in $\mathbb{H} \cup \infty$. The 3–spheres in $\mathbb{HP}^1$ can be described with quaternionic Hermitian forms. A *quaternionic Hermitian form* $\langle\,,\rangle$ on $\mathbb{H}^2$ is a real bilinear $\mathbb{H}$–valued map that is quaternionic linear in the right entry and Hermitian, i.e., $\overline{\langle x, y\rangle} = \langle y, x\rangle$. The set of isotropic lines of an indefinite Hermitian form on $\mathbb{H}^2$ is a 3–sphere in $\mathbb{HP}^1$, and every 3–sphere arises this way. The 3–sphere uniquely determines the indefinite Hermitian form, up to a real factor.

This can be seen as follows. If $\langle\,,\rangle$ is an indefinite Hermitian form. Then there exists a basis $a, b \in \mathbb{H}^2$ such that $\langle a, a\rangle = \langle b, b\rangle = 0$ and $\langle a, b\rangle = 1$. If α, β is the dual basis of a, b, then $\langle x, x\rangle = 2\operatorname{Re}(\sigma_{\alpha,\beta}(x))|\beta(x)|^{-2}$. Thus the stereographic projection $\sigma_{\alpha,\beta}$ maps the set of isotropic lines of $\langle\,,\rangle$ onto the real hyperplane $\operatorname{Im}\mathbb{H} \cup \{\infty\}$. Conversely, let a, b be a basis of $\mathbb{H}^2$ such that $\sigma_{\alpha,\beta}$ maps the 3–sphere onto $\operatorname{Im}\mathbb{H} \cup \{\infty\}$, and define the Hermitian form by $\langle a, a\rangle = \langle b, b\rangle = 0$ and $\langle a, b\rangle = 1$. Then $\langle x, x\rangle = 0$ if and only if $\sigma_{\alpha,\beta}(x) \in \operatorname{Im}\mathbb{H}$ as above. Uniqueness follows since $\langle a, a\rangle = \langle b, b\rangle = 0$ and $\langle x, x\rangle = 0 \iff \sigma_{\alpha,\beta}(x) \in \operatorname{Im}\mathbb{H}$ implies $\langle a, b\rangle \in \mathbb{R}$.

5.5. 2–Spheres in $\mathbb{HP}^1$. A subset of $\mathbb{HP}^1$ is a *2–sphere* if one (and then every) stereographic projection maps it onto a 2–sphere or a real plane plus infinity in $\mathbb{H} \cup \infty$. If $S \in \operatorname{End}(\mathbb{H}^2)$ and $S^2 = -1$, then the set of fixed points of the Möbius involution induced by S is a 2–sphere in $\mathbb{HP}^1$, every 2–sphere arises this way, and the 2–sphere determines the endomorphism S up to sign (cf., [**Jeromin**, Lemma 4.8.1]). In what follows the 2–sphere, the endomorphism of $\mathbb{H}^2$ as well as the Möbius involution are denoted by the same letter, usually S.

The endomorphism S can be used to define an orientation of the 2–sphere. If $p \in \mathbb{HP}^1$ is a fixed point of S, i.e., $Sp = p$, then S induces on each of the quaternionic lines p and $\mathbb{H}^2/p$ a complex structure, and $T_pS = \operatorname{Hom}_+(p, \mathbb{H}^2/p)$. Thus S induces, by pre– or post–composition, a complex structure on T_pS, hence an orientation of the 2–sphere. Consequently, oriented 2–spheres in $\mathbb{HP}^1$ are in canonical one–to–one correspondence to the endomorphisms of $\mathbb{H}^2$ that square to minus one.

5.6. Incidence of 2–spheres and 3–spheres in $\mathbb{HP}^1$. A 2–sphere is contained in a 3–sphere if and only if the endomorphism $S \in \operatorname{End}(\mathbb{H}^2)$ that represents the 2–sphere is Hermitian with respect to the Hermitian form representing the 3–sphere, see [**Jeromin**, Lemma 4.8.6].

6. Conformal Immersions into $\mathbb{H}$

A differentiable map $f\colon M \to \mathbb{C}$ of a Riemann surface M to $\mathbb{C}$ is holomorphic if and only if the Cauchy–Riemann equations $*df = \mathbb{i}df$ are satisfied. As a conformal map $f\colon M \to \mathbb{C}$ is either holomorphic or antiholomorphic, this can be reformulated as follows: A smooth map $f\colon M \to \mathbb{C}$ is conformal if and only if there exists a smooth map $N\colon M \to \mathbb{C}$ such that $*df = Ndf$ (clearly N is then $\mathbb{i}$ or $-\mathbb{i}$). This version generalizes verbatim to $\mathbb{H}$–valued immersions of Riemann surfaces.

6.1. Lemma. *Let $f\colon M \to \mathbb{H}$ be an immersion. Then the following three statements are equivalent*

(i) f is conformal.
(ii) There exists $N\colon M \to \mathbb{H}$ such that $*df = Ndf$.
(iii) There exists $R\colon M \to \mathbb{H}$ such that $*df = -dfR$.

In this case N and R are unique, $N^2 = R^2 = -1$, and the tangent and the normal bundle of f can be described by

$$\top f = \{\, x \in \mathbb{H} \mid NxR = x \,\}, \qquad \bot f = \{\, x \in \mathbb{H} \mid NxR = -x \,\}.$$

Proof. The immersion $f\colon M \to \mathbb{H}$ is conformal if and only if $|*df| = |df|$ and $*df \perp df$.

(ii) implies $-df = *^2 df = N^2 df$. Thus $N^2 = -1$, which is equivalent to $|N| = 1$ and $N(p) \in \operatorname{Im}\mathbb{H}$ for all $p \in M$. Hence $|*df| = |Ndf| = |df|$ and $\langle *df, df\rangle = \operatorname{Re}(*df\overline{df}) = \operatorname{Re}(N|df|^2) = 0$. Thus existence of N implies conformality of f.

Suppose now that $|*df| = |df|$ and $*df \perp df$. This implies that $N := *df(X)(df(X))^{-1} : M \to \mathbb{H}$ does not depend on the choice of $X \in TM$. Hence $*df = Ndf$. Thus (i) and (ii) are equivalent. The equivalence of (i) and (iii) can be proven the same way.

At each point $p \in M$ the assignment $x \mapsto NxR$ is a real linear map. It has the two eigenvalues ± 1, and the corresponding eigenspaces are orthogonal, real 2–dimensional subspaces of $\mathbb{H}$. Since $\top f = df(TM) \subset \{\, x \in \mathbb{H} \mid NxR = x \,\}$ is also 2–dimensional, the description of $\top f$ and $\bot f$ follows. $\square$

6.2. Left and Right Normal Vector. The equation $N^2 = R^2 = -1$ means that N and R assume values in $S^2 \subset \operatorname{Im}\mathbb{H} = \mathbb{R}^3$. They are called the *left* respectively *right normal vector* of f. Note that N and R are in general *not* orthogonal to f. In fact N (or R) taking values in the normal bundle of f is equivalent to $N = R$, which is equivalent to f lying in some translation of $\operatorname{Im}\mathbb{H} = \mathbb{R}^3$. In this case N is the Gauss normal of f (see Section 12).

6.3. Degree of the Tangent and Normal Bundle. Left multiplication by N induces complex structures on $\top f$ and $\bot f$, which have the property that the induced orientations on $\top f$ and $\bot f$ together give the standard orientation of $\mathbb{H}$, as defined in 5.2.

Corollary. *On compact M the degree of $\top f$ and $\bot f$ can be calculated from the degrees of N and R:*

$$\deg \top f = \deg N + \deg R \quad \textit{and} \quad \deg \bot f = \deg N - \deg R.$$

Proof. Left multiplication with N makes $\mathbb{H}$ into a complex quaternionic line bundle. As a complex bundle it is the double of the underlying complex line bundle. Thus the degree of $\mathbb{H}$ is $2 \deg N$, by Theorem 3.2. The rank 2 complex vector bundle $\mathbb{H}$ is also the direct sum of $\top f$ and $\bot f$, hence

$$2 \deg N = \deg \top f + \deg \bot f.$$

Left multiplication with R makes $\mathbb{H}$ into a complex quaternionic line bundle. On the conjugate of $\top f$ left multiplication with R induces the complex structure of $\top f$, because $R\bar{x} = -\overline{xR} = \overline{Nx}$ for $x \in \top f$, but on the conjugate of $\perp f$ left multiplication with R induces the negative of its complex structure. Thus as above, one gets

$$2\deg R = \deg \top f - \deg \perp f.$$

$\square$

6.4. Remark. For orientable surfaces it is no restriction to consider only conformal immersions of Riemann surfaces. Because, if M is an oriented 2–dimensional manifold and $f\colon M \to \mathbb{R}^4$ is an immersion, then there is a unique complex holomorphic structure on M, such that f is a conformal immersion. But if one considers two conformal immersions on M, then it is a special property of this pair to introduce the same complex holomorphic structure on M. The Bäcklund and Darboux transformations of chapter IV and V have this as a build in feature.

7. Holomorphic Curves in $\mathbb{HP}^1$

In the previous section the conformality of an immersion into $\mathbb{H}$ was characterized by an equation analogous to the Cauchy–Riemann equations of complex analysis. In the present section this is used to identify conformal immersions into $\mathbb{HP}^1$ with immersed holomorphic curves in $\mathbb{HP}^1$.

7.1. Maps into $\mathbb{HP}^1$. Let $f\colon M \to \mathbb{HP}^1$ be a smooth map, then $f(p)$ is a quaternionic line in $\mathbb{H}^2$. Hence, f can (and will) be identified with the quaternionic line subbundle $L \subset \mathbb{H}^2$ that satisfies $f(p) = L_p$, where $\mathbb{H}^2$ denotes the trivial quaternionic vector bundle of rank 2 over M. Each section $\psi \in \Gamma(L)$ without zeros can be interpreted as a lift $\psi\colon M \to \mathbb{H}^2$ of f. One defines the *derivative* of L to be $\delta := df \in \Omega^1 \operatorname{Hom}(L, \mathbb{H}^2/L)$, thus $\delta_X(\psi) \equiv \nabla_X \psi \mod L$ for all $X \in TM$, see 5.3. Let $\pi\colon \mathbb{H}^2 \to \mathbb{H}^2/L$ be the canonical projection, then

$$\delta := df = \pi \nabla|_L.$$

A quaternionic line subbundle $L \subset \mathbb{H}^2$ over a Riemann surface M is called a *curve in* $\mathbb{HP}^1$.

7.2. Holomorphic Curves in $\mathbb{HP}^1$. A line subbundle $L \subset \mathbb{H}^2$ over a Riemann surface M is called a *holomorphic curve* in $\mathbb{HP}^1$ if there exists a complex structure J on L such that

$$*\delta = \delta J.$$

The zeros of the derivative δ of a holomorphic curve L are isolated branch points of L, by Theorem 10.2 and its corollary. This implies that the complex structure J is uniquely determined by $*\delta = \delta J$. Furthermore, $*\delta = \delta J$ implies that L is immersed away from the zeros of its derivative.

7.3. A form $\beta \in (\mathbb{H}^2)^*$ or basis $\alpha, \beta \in (\mathbb{H}^2)^*$ is called *admissible* for a curve $L \subset \mathbb{H}^2$, if $\infty = \ker \beta$ does not lie on L. That means that the affine lift $\beta^{-1} := A_\beta L \in \Gamma(L)$ and the stereographic projection $\sigma_{\alpha,\beta} L = \alpha(\beta^{-1})$ are well defined. The notation β^{-1} for the affine lift of L with respect to β emphasizes the fact that β^{-1} is the unique section of L satisfying $\beta(\beta^{-1}) = 1$. From Sard's Theorem follows that generic elements of $(\mathbb{H}^2)^*$ are admissible.

7.4. Lemma. *Let $L \subset \mathbb{H}^2$ be a curve in $\mathbb{H}\mathrm{P}^1$ and $\alpha, \beta \in (\mathbb{H}^2)^*$ an admissible basis. Then L is a holomorphic curve if and only if the stereographic projection $\sigma_{\alpha,\beta}L$ of L has a right normal vector. In this situation each of the following three equations determines the right normal vector $R\colon M \to \mathbb{H}$ of $\sigma_{\alpha,\beta}L$.*

(i) $J\beta^{-1} = -\beta^{-1}R$, *i.e.,* $-R$ *is the normal vector of* β^{-1}.

(ii) $*d(\sigma_{\alpha,\beta}L) = -d(\sigma_{\alpha,\beta}L)R$, *i.e.,* R *is the right normal vector of* $\sigma_{\alpha,\beta}L$.

(iii) $d_L\sigma_{\alpha,\beta}(vJ) = -d_L\sigma_{\alpha,\beta}(v)R$ *for all* $v \in \mathrm{Hom}(L, \mathbb{H}^2/L)$, *i.e.,* $d_L\sigma_{\alpha,\beta}$ *is complex linear with respect to* J *and right multiplication with* $-R$.

Note that the curve in the lemma does not need to be immersed.

Proof. The affine lift $\beta^{-1} = A_\beta L$ satisfies $\beta(\beta^{-1}) = 1$. Thus the differential of the stereographic projection $\sigma_{\alpha,\beta}$ along L satisfies

$$d_L\sigma_{\alpha,\beta}(v) = \alpha(v\beta^{-1}) - \alpha(\beta^{-1})\beta(v\beta^{-1})$$

for all $v \in T_L\,\mathbb{H}\mathrm{P}^1 = \mathrm{Hom}(L, \mathbb{H}^2/L)$ (cf., 5.3). Thus

$$d(\sigma_{\alpha,\beta}L) = d_L\sigma_{\alpha,\beta}\delta = \alpha(\delta\beta^{-1}) - \alpha(\beta^{-1})\beta(\delta\beta^{-1}).$$

Hence if L is a holomorphic curve and $-R$ is the normal vector of β^{-1}, then $*d(\sigma_{\alpha,\beta}L) = -d(\sigma_{\alpha,\beta}L)R$. Thus $\sigma_{\alpha,\beta}L$ has a right normal vector. If, conversely, $\sigma_{\alpha,\beta}L$ has a right normal vector R, then the quaternionic linear extension of $J\beta^{-1} = -\beta^{-1}R$ is the complex structure on L that makes L into a holomorphic curve. □

7.5. Dual Curve. With every curve L in $\mathbb{H}\mathrm{P}^1$ comes another curve, namely its *dual curve* $L^\perp \subset (\mathbb{H}^2)^*$, whose fiber at $p \in M$ is the quaternionic line $L^\perp_p = \{\, \beta \in (\mathbb{H}^2)^* \mid \beta_{|L} = 0 \,\}$.

Lemma. *If $L \subset \mathbb{H}^2$ is a holomorphic curve and $\alpha, \beta \in (\mathbb{H}^2)^*$ an admissible basis with dual basis $a, b \in \mathbb{H}$, then b, a is admissible for $L^\perp$ and*

$$\sigma_{b,a}L^\perp = -\overline{\sigma_{\alpha,\beta}L}.$$

This means that the stereographic projections of $L^\perp$ are the Möbius reflections of the stereographic projections of L. Seen as maps into $S^4 \subset \mathbb{R}^5$, the lemma implies that the curve $L^\perp$ is, up to a Möbius transformation, the *antipodal reflection* of L.

Proof. Admissibility of β for L is equivalent to the admissibility of a for $L^\perp$. The affine lift $a^{-1} = A_a L^\perp$ of $L^\perp$ satisfies $a^{-1} = \alpha - \beta\overline{\sigma_{\alpha,\beta}L}$, because $(\alpha - \beta\overline{\sigma_{\alpha,\beta}L})(\beta^{-1}) = \alpha(\beta^{-1}) - \sigma_{\alpha,\beta}L = 0$ and $(\alpha - \beta\overline{\sigma_{\alpha,\beta}L})(a) = 1$. Hence $\sigma_{b,a}L^\perp = a^{-1}(b) = -\overline{\sigma_{\alpha,\beta}L}$. □

7.6. Conformal Immersions into $\mathbb{H}\mathrm{P}^1$. Lemma 6.1 can now be formulated in terms of holomorphic curves instead of normal vectors.

Theorem. *Let $L \subset \mathbb{H}^2$ be an immersed curve. Then the following three statements are equivalent*

(i) L *is conformal.*

(ii) L *is holomorphic.*

(iii) $L^\perp$ *is holomorphic.*

In this case the tangent and the normal bundle of L are given by

$$\top L = \mathrm{Hom}_+(L, \mathbb{H}^2/L) \quad \textit{and} \quad \perp L = \mathrm{Hom}_+(\bar{L}, \mathbb{H}^2/L),$$

as complex line bundles.

Proof. Let $\alpha, \beta \in (\mathbb{H}^2)^*$ be an admissible basis. The definition of the conformal structure of $\mathbb{HP}^1$ in 5.2 then implies that L, seen as a map into $\mathbb{HP}^1$, is conformal if and only if $\sigma_{\alpha,\beta}L$ is conformal. This is by Lemma 6.1 and 7.4 equivalent to L being a holomorphic curve.

The stereographic projection $\sigma_{\alpha,\beta}L$ is conformal if and only if $\sigma_{b,a}L^\perp = -\overline{\sigma_{\alpha,\beta}L}$ is conformal (cf., Lemma 7.5). This is, as before, equivalent to $L^\perp$ being a holomorphic curve. Thus L is a holomorphic curve if and only if $L^\perp$ is a holomorphic curve.

The description of the tangent and normal bundle of L follows from the equalities

$$\top\sigma_{\alpha,\beta}L = \{\, x \in \mathbb{H} \mid NxR = x \,\} \quad \text{and} \quad \perp\sigma_{\alpha,\beta}L = \{\, x \in \mathbb{H} \mid NxR = -x \,\},$$

given in Lemma 6.1, the formulas

$$d_L\sigma_{\alpha,\beta}(vJ) = -d_L\sigma_{\alpha,\beta}(v)R \quad \text{and} \quad d_L\sigma_{\alpha,\beta}(J^\perp v) = N d_L\sigma_{\alpha,\beta}(v),$$

from Lemma 7.4, and the fact that $d_{L_p}\sigma_{\alpha,\beta}$ is by definition a conformal isomorphism between $T_{L_p}\mathbb{HP}^1$ and $\mathbb{H}$, which maps $\top_p L$ onto $\top_p\sigma_{\alpha,\beta}L$. The complex structures on $\top\sigma_{\alpha,\beta}L$ and $\perp\sigma_{\alpha,\beta}L$, as defined in 6.3, are given by right multiplication with $-R$ and R, respectively. This implies that precomposition with J is the complex structure on $\top L = \mathrm{Hom}_+(L, \mathbb{H}^2/L)$ and precomposition with $-J$ is the complex structure on $\perp L = \mathrm{Hom}_+(\bar{L}, \mathbb{H}^2/L)$. □

7.7. Example. If a curve L in $\mathbb{HP}^1$ is not immersed, then it is possible, in contrast to the complex situation, that L is holomorphic, but $L^\perp$ is not. For example, the curve $L = \begin{pmatrix} z^2 \\ \mathbb{j}+z \end{pmatrix}\mathbb{H} \subset \mathbb{H}^2$ over $M = \mathbb{C}$ is an immersion away from $z = 0$. The stereographic projection $\sigma_{e_1^*,e_2^*}L = z^2(\mathbb{j}+z)^{-1}$ has the right normal vector $R = -(\mathbb{j}+z)\mathbb{i}(\mathbb{j}+z)^{-1}$. Hence L is a holomorphic curve, by Lemma 7.4. But the left normal vector of $\sigma_{e_1,e_2}L$ is

$$N = \frac{4 + 3|z|^2 + |z|^4}{4 + 5|z|^2 + |z|^4}\mathbb{i} - \frac{z^3}{|z|^2}\frac{4 + 2|z|^2}{4 + 5|z|^2 + |z|^4}\mathbb{k},$$

which does not extend smoothly into $z = 0$. Hence Lemma 7.4 and 7.5 imply that $L^\perp$ is not a holomorphic curve.

7.8. The Four Complex Quaternionic Line Bundles. There are four complex quaternionic line bundles canonically associated to a conformal immersion $f\colon M \to \mathbb{HP}^1$:

$$L \subset \mathbb{H}^2, \quad L^\perp \subset (H^2)^*, \quad L^{-1} = (\mathbb{H}^2)^*/L^\perp, \quad \mathbb{H}^2/L = (L^\perp)^{-1}.$$

Let δ be the derivative of L, $\delta^\perp$ the derivative of $L^\perp$, and ∇ the trivial connection of $\mathbb{H}^2$ as well as $(\mathbb{H}^2)^*$. Let φ and ψ be sections of $L^\perp$ and L, respectively, then they are also sections of $(\mathbb{H}^2)^*$ and $\mathbb{H}^2$. From $\varphi(\psi) = 0$ follows $\nabla\varphi(\psi) = -\varphi(\nabla\psi)$ and $\delta^\perp\varphi(\psi) = -\varphi(\delta\psi)$. Consequently,

$$\delta^\perp = -\delta^*, \quad *\delta = J^\perp\delta = \delta J, \quad \text{and} \quad *\,\delta^\perp = J\delta^\perp = \delta^\perp J^\perp,$$

where J denotes the complex structures of L as well as L^{-1}, and $J^\perp$ the complex structures of $L^\perp$ as well as $(L^\perp)^{-1}$.

7.9. If M is compact, then there is a corollary to Theorem 7.6, which relates the degrees of the four complex quaternionic line bundles associated to an immersed holomorphic curve L, the degrees of the normal vectors of stereographic projections of L, and the degrees of the tangent and normal bundle of L.

Corollary. *Let L be an immersed holomorphic curve on a compact Riemann surface M of genus g. Let N and R be the left and right normal vectors of some admissible stereographic projection of L, then*

$$\deg R = -\deg L = \deg L^{-1},$$
$$\deg N = -\deg L^\perp = \deg \mathbb{H}^2/L.$$

The degrees of the tangent and normal bundle satisfy

$$\deg \top L = \deg \mathbb{H}^2/L + \deg L^{-1} = 2 - 2g,$$
$$\deg \perp L = \deg \mathbb{H}^2/L - \deg L^{-1} = 2 - 2g - 2\deg L^{-1}.$$

Proof. The first two equations are a consequence of Theorem 3.2 and the fact that $-R$ and $-N$ are the normal vectors of non vanishing sections of L and $L^\perp$, by Lemma 7.4. The formulas for $\deg \top L$ and $\deg \perp L$ follow from Theorem 7.6, because $\deg \mathrm{Hom}_+(L, \mathbb{H}^2/L) = \deg \mathbb{H}^2/L - \deg L$ and $\deg \mathrm{Hom}_+(\bar{L}, \mathbb{H}^2/L) = \deg \mathbb{H}^2/L + \deg L$. The equalities involving the genus follow from the Gauss–Bonnet formula $\deg \top L = 2 - 2g$. □

8. The Möbius Invariant Holomorphic Line Bundle

Let $L \subset \mathbb{H}^2$ be a holomorphic curve in $\mathbb{HP}^1$. The following theorem shows that the complex quaternionic line bundle

$$L^{-1} = (\mathbb{H}^2)^*/L^\perp$$

carries a unique Möbius invariant holomorphic structure such that the coordinate functions of $\mathbb{H}^2$ induce holomorphic sections. If the dual curve $L^\perp \subset (\mathbb{H}^2)^*$ of L is also a holomorphic curve (in particular if L is immersed, by Theorem 7.6), then

$$\mathbb{H}^2/L = (L^\perp)^{-1}$$

also has a canonical Möbius invariant holomorphic structure.

8.1. Theorem. *Let $L \subset \mathbb{H}^2$ be a holomorphic curve. Then there is a unique holomorphic structure on L^{-1} with the following property: The restriction of every $\beta \in (\mathbb{H}^2)^*$ to L is a holomorphic section of L^{-1}.*

This theorem is a special case of a theorem for holomorphic curves in quaternionic projective space of arbitrary dimension (cf., [**FLPP01**, Theorem 2.3] or Section 14).

The holomorphic line bundle $(ML)^{-1}$ of every orientation preserving Möbius transformation ML, $M \in \mathrm{PGL}(\mathbb{H}^2)$, of L is obviously isomorphic to L^{-1}. The holomorphic line bundle L^{-1} with the holomorphic structure of the theorem is called the *Möbius invariant* or *canonical holomorphic line bundle* of the curve L.

PROOF. Let $\beta \in (\mathbb{H}^2)^*$ be admissible for L. Then $\beta_{|_L}$ is a nowhere vanishing section of L^{-1} and $D(\beta_{|_L}) = 0$ uniquely determines a quaternionic holomorphic structure on L^{-1}, by the Leibniz rule. It remains to show that for every $\alpha \in (\mathbb{H}^2)^*$ the restriction $\alpha_{|_L}$ of α to L is holomorphic with respect to D. But $\alpha_{|_L} = \beta_{|_L}\overline{\sigma_{\alpha,\beta}L}$, because $\alpha(\beta^{-1}) = \sigma_{\alpha,\beta}L$. The Leibniz rule then implies that $D\alpha_{|_L} = \beta_{|_L}\overline{d\sigma_{\alpha,\beta}L}'' = 0$, by Lemma 7.4. □

8.2. Corollary. *If R is the right normal vector of an admissible stereographic projection of a holomorphic curve $L \subset \mathbb{H}^2$, then the Möbius invariant holomorphic line bundle L^{-1} of L is isomorphic to $M \times \mathbb{H}$ with complex structure J defined by $J1 = R$ and holomorphic structure D defined by $D1 = 0$.*

If $f\colon M \to \mathbb{H}$ has a right normal vector R, then the holomorphic line bundle defined by R as in the corollary is called the *Möbius invariant holomorphic line bundle* of f. It is isomorphic to the Möbius invariant holomorphic line bundle of any stereographic lift of f to $\mathbb{HP}^1$, and, consequently, invariant under orientation preserving Möbius transformations of $\mathbb{H}$.

PROOF. If β is admissible for L then $\beta_{|_L}$ is a holomorphic section of L without zeros. Lemma 7.4 implies that $J\beta^{-1} = -\beta^{-1}R$. From $J\beta(\beta^{-1}) = \beta(J\beta^{-1}) = \beta(-\beta^{-1}R) = -R = (\beta R)(\beta^{-1})$ follows $J\beta_{|_L} = \beta_{|_L}R$. Eventually, holomorphicity of $\beta_{|_L}$ and Theorem 4.2 yields the isomorphism. □

8.3. Let $L \subset \mathbb{H}^2$ be a holomorphic curve and $\pi\colon (\mathbb{H}^2)^* \to (\mathbb{H}^2)^*/L^\perp$ the canonical projection and ∇ the trivial connection of $(H^2)^*$. The Möbius invariant holomorphic structure D of $L^{-1} = (\mathbb{H}^2)^*/L^\perp$ then satisfies

$$D\pi\psi = \tfrac{1}{2}(\pi\nabla + *J\pi\nabla)\psi,$$

for all $\psi \in \Gamma(\mathbb{H}^2)$. This formula follows since its right hand side, which only depends on $\pi\psi$, defines a holomorphic structure on L^{-1} that trivially satisfies the condition of Theorem 8.1.

8.4. Umbilics. Let $f\colon M \to \mathbb{HP}^1$ be a conformal immersion. The trace free part $\mathring{\mathrm{II}}$ of the second fundamental form of f is independent of the choice of metric in the conformal structure of $\mathbb{HP}^1$ (cf., [**Jeromin**, P.6.4]). This also follows from the Möbius invariance of the holomorphic structure of L^{-1} and $\mathbb{H}^2/L$, since, using the identification $\perp L = \mathrm{Hom}_+(\bar{L}, \mathbb{H}^2/L)$ (cf., 7.6) the trace free part of the second fundamental form satisfies

$$\mathring{\mathrm{II}}(X,Y) = Q_X^{\mathbb{H}^2/L}\delta_Y + \delta_X(Q_Y^{L^{-1}})^*,$$

where δ is the derivative of L, and $Q^{L^{-1}}$ and $Q^{\mathbb{H}^2/L}$ are the Hopf fields of L^{-1} and $\mathbb{H}^2/L$ (cf., [**Boh03**, p. 104]).

This formula also implies that $p \in M$ is an umbilic of f if and only if the Hopf fields of L^{-1} and $\mathbb{H}^2/L$ vanish at p. Consequently, the Hopf fields of the Möbius invariant holomorphic line bundles of a holomorphic curve in $\mathbb{HP}^1$ both vanish identically if and only if L takes values in some 2–sphere.

8.5. Kodaira Correspondence. In addition to the Möbius invariant quaternionic holomorphic structure on L^{-1}, a nonconstant holomorphic curve $L \subset \mathbb{H}^2$ induces a 2–dimensional base point free linear system $\{\, \beta_{|_L} \mid \beta \in (\mathbb{H}^2)^* \,\}$ of holomorphic sections of L^{-1}, which is also Möbius invariant. This system is canonically isomorphic to the vector space $(\mathbb{H}^2)^*$. It is called the *canonical linear system* of L. From Theorem 10.2 bellow follows that the Weierstrass points of the canonical linear system are exactly the zeros of the differential of L.

Each basis $\alpha_{|_L}, \beta_{|_L}$ of H yields a stereographic projection $\sigma_{\alpha,\beta}L \colon M \to \mathbb{H}$ of L, that satisfies, away from the zeros of $\beta_{|_L}$,

$$\alpha_{|_L} = \beta_{|_L}\overline{\sigma_{\alpha,\beta}L},$$

because $\alpha(\beta^{-1}) = \sigma_{\alpha,\beta}L$. A map $f \colon M \to \mathbb{H}$ such that $\varphi = \psi\bar{f}$ holds for two sections of a quaternionic line bundle is called the *quotient of* φ *and* ψ. Thus the stereographic projections of L are the quotients of linearly independent holomorphic sections in $(\mathbb{H}^2)^* \subset H^0(L^{-1})$.

Hence the canonical linear system $(\mathbb{H}^2)^*$ describes L up to projective transformations of $\mathbb{HP}^1$. The curve L can also be directly reobtained from the canonical linear system: If $H \subset H^0(L^{-1})$ is a 2–dimensional base point free linear system, then the *evaluation map* $\mathrm{ev} \colon H \to L^{-1}$, $(p,\beta) \mapsto \beta(p)$ is a smooth bundle homomorphism from the trivial bundle H to L^{-1}. One can check (cf., [**FLPP01**, Section 2.6]) that $\ker(\mathrm{ev})^\perp \subset H^*$ is a holomorphic curve, and if $H = (\mathbb{H}^2)^*$ is the canonical linear system of a holomorphic curve L in $\mathbb{HP}^1$, then $\ker(\mathrm{ev})^\perp = L$.

In summary one has the bijective correspondence:

$$\left\{\begin{array}{c}\text{2–dimensional base point free}\\ \text{linear systems of quaternionic}\\ \text{holomorphic line bundles.}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{Projective equivalence}\\ \text{classes of nonconstant}\\ \text{holomorphic curves in } \mathbb{HP}^1.\end{array}\right\}$$

This is the quaternionic version of the *Kodaira correspondence*[1] for the special case of 2–dimensional linear systems and holomorphic curves in $\mathbb{HP}^1$.

8.6. Willmore Energy. Let $L \subset \mathbb{H}^2$ be a holomorphic curve and $f = \sigma_{\alpha,\beta}L$ an admissible stereographic projection of L. If R is the right normal vector of f, then R is the normal vector of a nowhere vanishing holomorphic section of the Möbius invariant holomorphic line bundle L^{-1} of L, by Corollary 8.2. The Hopf field Q of L^{-1} then satisfies $\langle Q \wedge *Q\rangle = \frac{1}{2}\langle dR''R, dR''\rangle$ (cf., 3.5). Furthermore, $\langle dR''R, dR''\rangle = (|\mathcal{H}|^2 - K + K^\perp)\langle df, dfR\rangle$, where K is the Gauss curvature, $K^\perp$ the curvature of the normal bundle and $\mathcal{H} \colon M \to \mathbb{H}$ the mean curvature vector of f, by [**BFLPP02**, Proposition 11][2]. Hence

$$\langle Q \wedge *Q\rangle = \frac{1}{2}\langle dR''R, dR''\rangle = \tfrac{1}{2}(|\mathcal{H}|^2 - K + K^\perp)dA$$

[1] Cf., [**FLPP01**, Section 2.6] or Section 14 for a more general definition of the quaternionic Kodaira correspondence, and [**GriHa**, Section 1.4] for the complex Kodaira correspondence.

[2] The normal curvature has the opposite sign in [**BFLPP02**]. The sign in the formula above is due to the convention that the orientation of the tangent and normal bundle together should induce the standard orientation of $\mathbb{HP}^1$, see 6.3.

and for the Willmore energy of L^{-1} one obtains

$$W(L^{-1}) = \int_M (|\mathcal{H}|^2 - K + K^\perp) dA,$$

where $dA = \langle df, dfR\rangle$ is the area form induced on M by f.

Suppose that the dual curve $L^\perp$ of L is also a holomorphic curve. The right normal vector of $\sigma_{b,a}L^\perp = -\bar{f}$ (cf., Lemma 7.5) is then the left normal vector N of f. The Hopf field $\tilde{Q}$ of the Möbius invariant holomorphic line bundle $\mathbb{H}^2/L$ of $L^\perp$ then satisfies, as above, $\langle \tilde{Q} \wedge *\tilde{Q}\rangle = \frac{1}{2}\langle dN''N, dN''\rangle$. Since $\langle RdR'', dR''\rangle - 2K^\perp dA = \langle dN''N, dN''\rangle$ (cf., [**BFLPP02**, Proposition 8 & 9]) one gets

$$\langle \tilde{Q} \wedge *\tilde{Q}\rangle = \frac{1}{2}\langle dN''N, dN''\rangle = \tfrac{1}{2}(|\mathcal{H}|^2 - K - K^\perp)dA$$

and for the Willmore energy of $\mathbb{H}^2/L$ one obtains

$$W(\mathbb{H}^2/L) = \int_M (|\mathcal{H}|^2 - K - K^\perp) dA.$$

Suppose now that L is immersed and M is compact. The famous *Willmore functional* $W(f)$ of the stereographic projection f of L is defined by

$$W(f) = \int_M |\mathcal{H}|^2 dA.$$

Using $\deg \top f = \frac{1}{2\pi}\int_M K dA$, $\deg \perp f = \frac{1}{2\pi}\int_M K^\perp dA$ Corollary 7.9 implies the following equations:

$$W(L^{-1}) = W(f) - 4\pi \deg L^{-1}, \qquad W(\mathbb{H}^2/L) = W(f) - 4\pi \deg(\mathbb{H}^2/L),$$
$$W(L^{-1}) - W(\mathbb{H}^2/L) = 4\pi \deg \perp f = 8\pi(1 - g - \deg L^{-1}).$$

for every admissible stereographic projection f of L.

9. The Euclidean Holomorphic Line Bundle

Let $L \subset \mathbb{H}^2$ be a holomorphic curve. If one singles out a point at infinity $\infty \in \mathbb{HP}^1$ that does not lie on L, then one can define a holomorphic structure on L that is invariant under Möbius transformations that fix ∞. This structure is called the Euclidean holomorphic structure of L with respect to ∞. If the dual curve $L^\perp \subset (\mathbb{H}^2)^*$ of L is also a holomorphic curve (in particular if L is immersed, see Theorem 7.6), then also $L^\perp$ has a Euclidean holomorphic structure. If L is an immersed holomorphic curve then the holomorphic line bundles L and $L^\perp$ provide the data for a Weierstrass type representation of its stereographic projections.

9.1. Theorem. *Let $L \subset \mathbb{H}^2$ be a holomorphic curve and $\infty \in \mathbb{HP}^1$ a point that does not lie on L. Then there is a unique holomorphic structure on L, such that the affine lift β^{-1} of L is a holomorphic section of L for all nontrivial $\beta \in \infty^\perp$.*

PROOF. If $\beta \in (\mathbb{H}^2)^* \setminus \{0\}$, then there is a unique holomorphic structure D on L such that $D(\beta^{-1}) = 0$. If $\beta, \tilde{\beta} \in \infty^\perp \setminus \{0\}$, then there exists $\lambda \in \mathbb{H}$ such that $\tilde{\beta} = \beta\lambda$. Hence $\tilde{\beta}^{-1} = (\beta\lambda)^{-1} = \beta^{-1}\bar{\lambda}^{-1}$, which implies that $D(\tilde{\beta}) = 0$. □

The unique holomorphic structure defined by $\infty \in \mathbb{HP}^1$ is called the *Euclidean holomorphic structure* of L with respect to ∞. If $M \in \mathrm{PGL}(\mathbb{H}^2)$ is an orientation preserving similarity of $\mathbb{HP}^1 \setminus \{\infty\} \cong \mathbb{H}$ (cf., 5.2), then M fixes ∞ and the Euclidean holomorphic structures of L and ML with respect to ∞ are obviously isomorphic. If $L^\perp$ is also a holomorphic curve, then the Euclidean holomorphic structure on $L^\perp$ is defined to be the holomorphic structure induced by $\infty^\perp \subset (\mathbb{H}^2)^*$.

9.2. Although the holomorphic structure on L depends on the choice of a point at infinity, its J–commuting part $\bar\partial$, does not depend on that choice: Writing also $\bar\partial$ for the J–commuting part of the Möbius invariant holomorphic structure of L^{-1}, one obtains

$$\bar\partial\alpha(\psi) + \alpha(\bar\partial\psi) = \tfrac{1}{2}(d(\alpha\psi) + *d(\alpha J\psi))$$

for all $\alpha \in \Gamma(L^{-1})$ and $\psi \in \Gamma(L)$. (Because the difference of both sides of this equation is tensorial, it suffices to check it for $\alpha = \beta_{|L}$, $\psi = \beta^{-1}$ and some admissible $\beta \in (\mathbb{H}^2)^*$. But $D\beta_{|L} = 0$ and $D\beta^{-1} = 0$ implies $\bar\partial\beta_{|L} = -Q\beta_{|L} = -\beta_{|L}\frac{1}{2}NdN''$ and $\bar\partial\beta^{-1} = -Q\beta^{-1} = -\beta^{-1}\frac{1}{2}NdN'$ for the normal vector N of $\beta_{|L}$, by 3.3. Thus for the left hand side of the above equation one has $\bar\partial\beta_{|L}(\beta^{-1}) + \beta_{|L}(\bar\partial\beta^{-1}) = -\frac{1}{2} * dN$, which equals the right hand side $\frac{1}{2} * d(\beta_{|L}J\beta^{-1})$.)

Hence the complex holomorphic structure $\bar\partial$ of the Euclidean holomorphic line bundle L is the dual complex holomorphic structure of the J–commuting part of the Möbius invariant holomorphic structure of L^{-1}, which implies that it does not depend on the choice of $\infty \in \mathbb{HP}^1$.

9.3. Corollary. *If R is the right normal vector of an admissible stereographic projection with pole ∞ of a holomorphic curve $L \subset \mathbb{H}^2$, then the Euclidean holomorphic line bundle L with respect to ∞ is isomorphic to $M \times \mathbb{H}$ with complex structure J defined by $J1 = -R$ and holomorphic structure D defined by $D1 = 0$.*

Proof. Let $\sigma_{\alpha,\beta}$ be an admissible stereographic projection. Lemma 7.4 then implies that $J\beta^{-1} = -\beta^{-1}R$. Furthermore, $\beta \in \infty^\perp$, because ∞ is the pole of $\sigma_{\alpha,\beta}$. Thus $\beta^{-1} = A_\beta L$ is a holomorphic section of L. The corollary then follows from Theorem 4.2. □

If $f\colon M \to \mathbb{H}$ has a right normal vector R, then the holomorphic line bundle defined by R as in the corollary is called the *Euclidean holomorphic line bundle* of f. It is isomorphic to the the Euclidean holomorphic line bundle of any stereographic lift of f to $\mathbb{HP}^1$ with holomorphic structure induced by the pole of the stereographic projection. It is consequently invariant under orientation preserving similarities of $\mathbb{H}$.

9.4. Willmore Energy. Let $L \subset \mathbb{H}^2$ be a holomorphic curve and $f = \sigma_{\alpha,\beta}L$ an admissible stereographic projection of L. If R is the right normal vector of f, then $-R$ is the normal vector of a nowhere vanishing holomorphic section in the Euclidean holomorphic line bundle L with pole $\infty = \ker\beta$, by Corollary 9.3. The Hopf field Q of L then satisfies $\langle Q \wedge *Q\rangle = \frac{1}{2}\langle RdR', dR'\rangle$, see 3.5, and because $d(-R)'' = \frac{1}{2}(d(-R) +$

$(-R)d(-R) = -dR'$, where the double prime refers to left multiplication with $-R$ and the single prime to left multiplication with R. The formula $\langle RdR', dR'\rangle = |\mathcal{H}|^2 dA$ from [**BFLPP02**, Proposition 9] implies $\langle Q \wedge *Q\rangle = \frac{1}{2}|\mathcal{H}|^2 dA$ and $W(L) = W(f)$.

If $L^\perp$ is also a holomorphic curve and $a, b \in \mathbb{H}^2$ is the dual basis of α, β, then $f = -\overline{\sigma_{b,a} L^\perp}$ (cf., Lemma 7.5) implies that $\langle \tilde{Q} \wedge *\tilde{Q}\rangle = \frac{1}{2}\langle NdN', dN'\rangle = \frac{1}{2}|\mathcal{H}|^2 dA$ for the Hopf field $\tilde{Q}$ of the Euclidean holomorphic structure of $L^\perp$ with respect to $\infty = [a]$ and the left normal vector N of f.

Hence

$$\langle Q \wedge *Q\rangle = \frac{1}{2}\langle NdN', dN'\rangle = \langle \tilde{Q} \wedge *\tilde{Q}\rangle = \frac{1}{2}\langle RdR', dR'\rangle = \frac{1}{2}|\mathcal{H}|^2 dA$$

and the Willmore energies of the Euclidean holomorphic line bundles satisfy

$$W(L) = W(L^\perp) = W(f) = \int_M |\mathcal{H}|^2 dA.$$

This implies that the Euclidean holomorphic line bundles are complex holomorphic line bundles if and only if the stereographic projection of L with pole ∞ is a minimal surface. The Euclidean holomorphic line bundles are consequently not Möbius invariant.

On compact M the two formulas $W(L^{-1}) = W(f) - 4\pi \deg L^{-1}$ and $W(\mathbb{H}^2/L) = W(f) - 4\pi \deg(\mathbb{H}^2/L)$ from 8.6 yield a relation between the Willmore energies of the Möbius invariant and Euclidean holomorphic line bundles:

$$W(L^{-1}) = W(L) - 4\pi \deg L^{-1}, \quad W(\mathbb{H}^2/L) = W(L^\perp) - \deg \mathbb{H}^2/L.$$

9.5. Paired Complex Quaternionic Line Bundles. Finally, we will see how the Weierstrass representation of a conformal immersion $f\colon M \to \mathbb{H}$ is obtained from its Euclidean holomorphic line bundles. The Weierstrass representation gets a particularly simple form, if it is written with pairings of holomorphic line bundles. A *pairing* of two complex quaternionic line bundles $\tilde{L}$ and L is a pointwise nondegenerate real bilinear form

$$(\,,)\colon \tilde{L} \times L \to T^*M \otimes \mathbb{H}$$

such that for all $p \in M$, $\lambda \in \mathbb{H}$, $\varphi \in \tilde{L}_p$, and $\psi \in L_p$

$$(\varphi, \psi\lambda) = (\varphi, \psi)\lambda, \qquad (\varphi\lambda, \psi) = \bar{\lambda}(\varphi, \psi),$$
$$*(\varphi, \psi) = (J\varphi, \psi) = (\varphi, J\psi).$$

If $\tilde{L}$ and L are paired, then L and $\tilde{L}$ are paired with $(\psi, \varphi) := \overline{(\varphi, \psi)}$. If L is a complex quaternionic line bundle, then KL^{-1} and L are canonically paired via the evaluation pairing $(\varphi, \psi) := \varphi(\psi)$.

Proposition. *If $\tilde{L}$ and L are paired complex quaternionic line bundles, then $\alpha\colon \tilde{L} \to KL^{-1}$, $\varphi \mapsto (\varphi, \cdot)$ is an isomorphism that maps the pairing of $\tilde{L}$ and L onto the evaluation pairing of KL^{-1} and L.*

Proof. If $\varphi \in \tilde{L}_p$, $p \in M$, then $\alpha\varphi$ is quaternionic linear on fibers of L_p, real linear on the fibers of T_pM and $*\alpha\varphi = \alpha\varphi J$, thus $\alpha\varphi \in KL^{-1}$. α is a quaternionic linear bundle homomorphism, because for $\lambda \in \mathbb{H}$: $\alpha(\varphi\lambda)(\psi) =$

$\bar\lambda(\varphi,\psi) = \bar\lambda\alpha\varphi(\psi) = ((\alpha\varphi)\lambda)(\psi)$. Since $(\,,)$ is nondegenerate α is an isomorphism. □

9.6. Paired Holomorphic Line Bundles. Two quaternionic holomorphic line bundles are *paired* if they are paired as complex quaternionic line bundles and if the product of every pair of local holomorphic sections of $\tilde L$ and L is closed.

Theorem. *Let $\tilde L$ and L be paired complex quaternionic line bundles. If there is a holomorphic structure D on L, then there is a unique holomorphic structure $\tilde D$ on $\tilde L$ such that the $\mathbb{H}$–valued 1–form (φ,ψ) is closed for all local holomorphic sections $\varphi \in H^0(\tilde L_{|U})$ and $\psi \in H^0(L_{|U})$. Furthermore, if ψ is a holomorphic section of L and φ is a section of $\tilde L$ such that (φ,ψ) is closed, then φ is a holomorphic section of $\tilde L$.*

Proof. Let $\varphi \in \Gamma(\tilde L)$ and $\psi \in \Gamma(L)$, then there is a unique section $\tilde D\varphi \in \Omega^1(\tilde L)$ such that for all $p \in M$ and $X \in T_pM$

$$(*)\qquad d_p(\varphi,\psi)(X, J_MX) = 2(\tilde D_X\varphi,\psi)(J_MX) + 2(\varphi, D_X\psi)(J_MX).$$

$\tilde D\colon \Gamma(\tilde L) \to \Omega^1(\tilde L)$ is obviously quaternionic linear and $*\tilde D\varphi = -J\tilde D\varphi$. $\tilde D$ satisfies the Leibniz rule, because for $\lambda\colon M \to \mathbb{H}$ one has

$$\begin{aligned}
&d(\varphi\lambda,\psi)(X,J_MX)\\
&\qquad= \bar\lambda d(\varphi,\psi)(X,J_MX) + d\bar\lambda \wedge (\varphi,\psi)(X,J_MX)\\
&\qquad= \bar\lambda d(\varphi,\psi)(X,J_MX) + (\varphi d\bar\lambda(X) + J\varphi{*d\bar\lambda}(X),\psi)(J_MX)\\
&\qquad= \bar\lambda d(\varphi,\psi)(X,J_MX) + 2((\varphi d\bar\lambda)''(X),\psi)(J_MX).
\end{aligned}$$

Hence $\tilde D$ is a holomorphic structure on $\tilde L$.

Let $U \subset M$ be an open set, $\varphi \in H^0(\tilde L_{|U})$ and $\psi \in H^0(L_{|U})$ local holomorphic sections, then $(*)$ implies that (φ,ψ) is closed. Let $\hat D$ be a holomorphic structure on $\tilde L$ with the property that products of local holomorphic sections of $\tilde L$ and L are closed. Then let $\varphi_i \in \Gamma(\tilde L_{|U_i})$ be a covering with holomorphic sections of $\hat D$ and $\psi_i \in H^0(L_{|U_i})$, then (φ_i,ψ_i) is closed and $(*)$ implies that $\tilde D\varphi_i = 0$. Thus the Leibniz rule implies $\tilde D = \hat D$. □

$\tilde D$ is called the *paired holomorphic structure* of D. Two holomorphic line bundles are *paired*, if they have a pairing such that their holomorphic structures are paired. If L is a holomorphic line bundle over a compact Riemann surface M of genus g, and KL^{-1} is equipped with the paired holomorphic structure—which will from now on be assumed without further notice—the Riemann–Roch formula for complex holomorphic line bundles holds verbatim for quaternionic holomorphic line bundles (cf., [**FLPP01**, Theorem 2.2]):

$$\dim H^0(L) - \dim H^0(KL^{-1}) = \deg L - g + 1.$$

9.7. Corollary. *If two holomorphic line bundles are paired as complex quaternionic line bundles, then they are paired as holomorphic line bundles if and only if there exists a covering with local holomorphic sections whose products are closed 1–forms.*

PROOF. This corollary follows from the fact that a holomorphic structure of a line bundle is, by the Leibniz rule, already determined by a covering with local holomorphic sections. $\square$

9.8. Corollary. *All paired holomorphic line bundles of a holomorphic line bundle L are isomorphic to KL^{-1}.*

PROOF. If $\tilde{L}$ and L are paired, then the isomorphism $\alpha\colon \tilde{L} \to KL^{-1}$ of Proposition 9.5 is holomorphic, because of the uniqueness of the paired holomorphic structure on $\tilde{L}$. $\square$

9.9. For $\eta \in \Omega^1(\tilde{L})$, $\psi \in \Gamma(L)$, $\varphi \in \Gamma(\tilde{L})$, $\omega \in \Omega^1(L)$ one defines the following $\mathbb{H}$ valued 2–forms

$$(\eta \wedge \psi)(X,Y) = (\eta(X), \psi)(Y) - (\eta(Y), \psi)(X),$$
$$(\varphi \wedge \omega)(X,Y) = (\varphi, \omega(X))(Y) - (\varphi, \omega(Y))(X).$$

Corollary. *Let $\tilde{L}$ and L be paired complex quaternionic line bundles. If D is a holomorphic structure on L, then the paired holomorphic structure $\tilde{D}$ on $\tilde{L}$ is the unique holomorphic structure on $\tilde{L}$ that satisfies*

$$d(\varphi,\psi) = (\tilde{D}\varphi \wedge \psi) + (\varphi \wedge D\psi),$$

or, equivalently, the complex holomorphic structures and Hopf fields of $\tilde{L}$ and L satisfy

$$d(\varphi,\psi) = (\tilde{\bar{\partial}}\varphi \wedge \psi) + (\varphi \wedge \bar{\partial}\psi) \quad \text{and} \quad 0 = (\tilde{Q}\varphi \wedge \psi) + (\varphi \wedge Q\psi).$$

Furthermore, $W(L) = W(\tilde{L})$.

PROOF. The first equation is equivalent to the equation of 9.6(∗). It is equivalent to the two equations for the complex holomorphic structures and the Hopf fields, because $\bar{\partial} = \frac{1}{2}(D - JDJ)$ and $Q = \frac{1}{2}(D + JDJ)$. Finally, the equation for the Hopf fields implies $\langle \tilde{Q}_X \tilde{Q}_X \rangle = \langle Q_X Q_X \rangle$ for all $X \in TM$, thus $\langle \tilde{Q} \wedge *\tilde{Q} \rangle = \langle Q \wedge *Q \rangle$ and $W(L) = W(\tilde{L})$. $\square$

9.10. Weierstrass Representation. The description of the differential of a conformal immersion $f\colon M \to \mathbb{H}$ as the product of two holomorphic sections of paired quaternionic holomorphic line bundles is called the *Weierstrass representation*[3] of f.

Proposition. *Let $L \subset \mathbb{H}^2$ be an immersed holomorphic curve, $\alpha, \beta \in (\mathbb{H}^2)^*$ an admissible basis, and $a, b \in \mathbb{H}^2$ its dual basis. The Euclidean holomorphic line bundles $L^\perp$ and L with respect to $\infty^\perp := [\beta]$ and $\infty = [a]$ are paired and*

$$d\sigma_{\alpha,\beta} L = (a^{-1}, \beta^{-1})$$

is the Weierstrass representation of $\sigma_{\alpha,\beta} L$.

Corollary 9.8 implies that the holomorphic line bundles $L^\perp$ and KL^{-1} are isomorphic.

[3]This is a generalization of the Weierstrass representation of minimal surfaces. It was introduced by Iskander Taimanov, [**Ta98**], for surface in $\mathbb{R}^3$ and generalized by Franz Pedit and Ulrich Pinkall in [**PP98**] to surfaces in $\mathbb{R}^4$.

PROOF. Let $\delta^\perp$ and δ be the derivatives of $L^\perp$ and L. Then

$$(\varphi, \psi) := \varphi(\delta\psi) = -\delta^\perp\varphi(\psi),$$

(cf., 7.8) defines a pairing between the complex quaternionic line bundles $L^\perp$ and L, because $*(\varphi, \psi) = \varphi(*\delta\psi) = \varphi(\delta J\psi) = (\varphi, J\psi)$ and $*(\varphi, \psi) = - * \delta^\perp\varphi(\psi) = -\delta^\perp(J\varphi)(\psi) = (J\varphi, \psi)$. From $a^{-1} = \alpha - \beta\overline{\sigma_{\alpha,\beta}L}$ and $\beta^{-1} = a\sigma_{\alpha,\beta}L + b$, follows $d\sigma_{\alpha,\beta}L = (a^{-1}, \beta^{-1})$.

As a^{-1} and β^{-1} are by definition (cf., 9.1) nowhere vanishing holomorphic sections of $L^\perp$ and L. Furthermore, the 1–form (a^{-1}, β^{-1}) is closed, hence $L^\perp$ and L are paired, by Corollary 9.7. □

9.11. Theorem. *The Euclidean holomorphic line bundle L of a conformal immersion $f\colon M \to \mathbb{H}$ is, up to isomorphisms, the unique holomorphic line bundle with holomorphic sections $\varphi \in H^0(KL^{-1})$ and $\psi \in H^0(L)$ such that*

$$df = (\varphi, \psi),$$

where KL^{-1} is equipped with the paired holomorphic structure. Furthermore,

$$W(L) = W(KL^{-1}) = W(f).$$

On the other hand, if L is a holomorphic line bundle over simply connected M, $\varphi \in H^0(KL^{-1})$ and $\psi \in H^0(L)$ are nowhere vanishing holomorphic sections, then $\int(\varphi, \psi)$ is a conformal immersion of M into $\mathbb{H}$.

PROOF. The existence of the holomorphic sections $\varphi \in H^0(L)$ and $\psi \in H^0(KL^{-1})$ follows from Proposition 9.10, because KL^{-1} is isomorphic to $L^\perp$, by the same proposition and Corollary 9.8. The uniqueness follows from Theorem 4.2 and Corollary 9.3, because ψ has no zeros, since f is immersed, and the normal vector of ψ is determined by $df = (\varphi, \psi)$.

In 9.4 the equation $W(L) = W(L^\perp) = W(f)$ was shown for the Euclidean line bundles L and $L^\perp$ of a holomorphic curve L and its stereographic projection f. Thus $W(L) = W(KL^{-1}) = W(f)$.

From the definition of a pairing, follows that (φ, ψ) is closed and nowhere vanishing. The normal vectors of φ and ψ provide the normal vectors of $\int(\varphi, \psi)$. Lemma 6.1 then implies that $\int(\varphi, \psi)$ is conformal. □

9.12. If f is not minimal, then KL^{-1} and L have nontrivial Hopf fields, and Theorem 3.4 implies that f also uniquely determines φ and ψ, up to $\varphi \mapsto \varphi c$, $\psi \mapsto \psi c^{-1}$, for $c \in \mathbb{R} \setminus \{0\}$. The triple (L, φ, ψ) is called the *Weierstrass data* of the conformal immersion $f\colon M \to \mathbb{H}$.

The two holomorphic sections φ and ψ have no zeros, thus they determine two 1–dimensional base point free linear systems of KL^{-1} and L, respectively. On the other hand, two 1–dimensional base point free linear systems in KL^{-1} and L, determine, up to similarities of $\mathbb{H}$ (cf., 1.2) a conformal map $f\colon \tilde{M} \to \mathbb{H}$ on the universal covering $\tilde{M}$ of M with translational periods. Hence for simply connected Riemann surfaces M the Weierstrass representation provides the following correspondence:

$$\left\{\begin{array}{c}\text{1–dimensional base point free}\\ \text{linear systems of paired}\\ \text{quaternionic holomorphic line}\\ \text{bundles.}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{Conformal immersions}\\ f\colon M \to \mathbb{H} \text{ modulo}\\ \text{similarities.}\end{array}\right\}$$

9.13. Example: Minimal Surfaces in $\mathbb{R}^4$. Let $f\colon M \to \mathbb{R}^4$ be a conformal immersion. By Theorem 9.11 there exists a holomorphic line bundle L, $\varphi \in H^0(KL^{-1})$ and $\psi \in H^0(L)$ such that $df = (\varphi, \psi)$. From $\langle Q \wedge *Q\rangle = \frac{1}{2}|\mathcal{H}|^2 dA$ (cf., 9.4 and 3.6) follows that f is minimal if and only if L and KL^{-1} have zero Willmore energy. This is a generalization of the well known spinor Weierstrass representation of minimal surfaces in $\mathbb{R}^3$, see 12.6.

10. Branched Conformal Immersions

To define the Möbius invariant and Euclidean holomorphic line bundles of a holomorphic curve in $\mathbb{HP}^1$ the curve does not need to be immersed (cf., Theorems 8.1 and 9.1). Hence the quaternionic holomorphic geometry can also be applied to nonimmersed holomorphic curves. In this section the behavior of a holomorphic curve at the zeros of its differential is described, a Weierstrass representation for branched conformal immersions that admit a smooth left and a smooth right normal vector is derived, and the behavior of the normal vector of a holomorphic section at its zeros is discussed.

10.1. Branch Points of Holomorphic Curves in $\mathbb{HP}^1$. Recall the definition of branch points from [**GOR73**]: Let M be a smooth 2–dimensional manifold, N a smooth manifold of dimension $n \geq 2$ and $f\colon M \to N$ a smooth map. A point $p \in M$ is called a *branch point* of f if and only if there exists an integer $k \geq 1$, and centered coordinates $z\colon M \supset U \to \mathbb{C}$ and $u\colon N \supset V \to \mathbb{R}^n$, such that

$$u_1(f) + \mathbb{i} u_2(f) = z^{k+1} + O(k+2), \qquad u_l(f) = O(k+2),\ l = 3, \dots, n.$$

The integer k is independent of the charts. It is called the *order*, denoted $b_p(f)$, of the branch point p. The map f is called a *branched immersion* if all points at which df fails to be injective are branch points. One immediately deduces from the definition that the differential of f at a branch point equals zero, and that branch points are isolated. On compact M one can define the *total branching* $b(f) := \sum_{p\in M} b_p(f)$ of f.

10.2. Theorem. *A holomorphic curve $L \subset \mathbb{H}^2$ over a Riemann surface M is a branched immersion. It has a branch point $p \in M$ if and only if p is a Weierstrass point of its canonical linear system $H \subset H^0(L^{-1})$. Furthermore, $b_p(L) = \operatorname{ord}_p H$.*

Proof. If $\alpha, \beta \in (\mathbb{H}^2)^*$ is an admissible basis, then the canonical linear system $H \subset H^0(L)$ of L is spanned by $\alpha_{|L}$ and $\beta_{|L}$. Let $f := \overline{\sigma_{\alpha,\beta} L}$, $p \in M$ and $\psi := \beta_{|L}(f - f(p))$. Then $\psi(p) = 0$ and $\psi \in H$, since $\alpha_{|L} = \beta_{|L} f$, see 8.5. Because $\beta_{|L}$ does not vanish, one has

$$\operatorname{ord}_p \psi = \operatorname{ord}_p H + 1.$$

Consequently, there is a centered holomorphic coordinate $z\colon M \supset U \to \mathbb{C}$, $z(p) = 0$ and a nowhere vanishing $\varphi \in \Gamma(L_{|U})$ such that

$$\beta_{|L}(f - f(p)) = \psi = z^{k+1}\varphi + O(k+2),$$

where $k := \operatorname{ord}_p H$. Let N be the normal vector of $\beta_{|L}$, $g\colon M \to \mathbb{H}$ such that $\varphi = \beta_{|L} g$, $z = x + \mathbb{i} y$, and $\tilde{z} = x + Ny$. Then $f - f(p) = \tilde{z}^{k+1} g + O(k+2)$.

Applying a Euclidean motion one can assume $f(p) = 0$, $g(p) = 1$ and $N(p) = \mathbb{i}$. Then $\tilde{z} = z + O(2)$, $g = 1 + O(1)$, and, consequently,

$$f = z^{k+1} + O(k+2).$$

Thus $k = 0$ or f has a branch point of order k at p. □

10.3. Since Weierstrass points are isolated (cf., 3.7) the theorem has the following corollary.

Corollary. *The derivative of a holomorphic curve has isolated zeros.*

10.4. Branched Conformal Immersions with a Normal Vector. A smooth map $f\colon M \to \mathbb{H}$ is called a *branched conformal immersion with a right (left) normal vector*, if there exists a smooth map $R\colon M \to \mathbb{H}$, $R^2 = -1$ ($N\colon M \to \mathbb{H}$, $N^2 = -1$) such that

$$*df = -df R \quad (*df = N df).$$

It follows directly from the definition that the differential of a branched conformal immersion f with a normal vector is either zero or injective. Lemma 6.1 implies that f is conformal away from the zeros of its differential. In 10.14 branched conformal immersions whose normal vector extend continuously but not smoothly into its branch points are constructed.

10.5. Proposition. *Every admissible stereographic projection of a curve L in $\mathbb{HP}^1$ to $\mathbb{H}$ is a branched conformal immersion with a smooth right (left) normal vector if and only if L ($L^\perp$) is holomorphic.*

Proof. This is a direct consequence of the Lemmas 7.4 and 7.5. □

Hence every branched conformal immersion with a right normal vector is the stereographic projection of a holomorphic curve in $\mathbb{HP}^1$. In particular, it possesses a unique Möbius invariant and a unique Euclidean holomorphic line bundle, by Corollary 8.2 and 9.3. The proposition also implies that having a smooth normal vector is a Möbius invariant property for branched conformal immersions $f\colon M \to \mathbb{H}$.

10.6. Holomorphic curves are branched conformal immersions, by Theorem 10.2. Their stereographic projections are also branched conformal immersions, since stereographic projections are conformal and the definition of a branched immersion is invariant under diffeomorphisms. Thus one obtains the following corollary of the preceding proposition.

Corollary. *Branched conformal immersions $f\colon M \to \mathbb{H}$ with a smooth right or left normal vector are branched immersions.*

10.7. Weierstrass Representation: One Normal Vector. Branched conformal immersions that have one normal vector have a Weierstrass representation, in which the section on the side of the missing normal vector has at least one zero. The case of branched conformal immersions with both normal vectors is discussed in 10.10 below.

Proposition. *The Euclidean holomorphic line bundle L of a branched conformal immersion $f\colon M \to \mathbb{H}$ with a right normal vector is the unique* *holomorphic line bundle for which there exists a nowhere vanishing holomorphic section $\psi \in H^0(L)$ and a holomorphic section $\varphi \in H^0(KL^{-1})$ satisfying*

$$df = (\varphi, \psi).$$

Moreover, p is a branch point of f if and only if p is a zero of φ, and the vanishing order of φ is the branching order of f at p.

If f is not minimal, i.e., $W(f) = W(L) = W(KL^{-1}) \neq 0$ (cf., 9.11), then Theorem 3.4 implies that φ and ψ are uniquely determined up to $\varphi \mapsto \varphi c$, $\psi \mapsto \psi \frac{1}{c}$ for some $c \in \mathbb{R} \setminus \{0\}$

Proof. Let $R\colon M \to \mathbb{H}$ be the right normal vector of f. Let $L := M \times \mathbb{H}$ with complex structure J such that $J1 = -R$ and holomorphic structure D such that $D1 = 0$. Then L is the Euclidean holomorphic line bundle of f, by Corollary 9.3. Let $\varphi := (1 \mapsto df) \in \Gamma(KL^{-1})$ and $\psi := 1 \in \Gamma(L)$. Then $df = (\varphi, \psi)$ and $\varphi \in H^0(KL^{-1})$, by Theorem 9.6.

The branch points of f coincide with the zeros of φ, because the pairing is pointwise nondegenerate and ψ has no zeros. If φ has a zero of order n at $p \in M$, then $\varphi = z^n \tilde{\varphi} + O(n+1)$ for some centered coordinate z and some nowhere vanishing section $\tilde{\varphi}$. Thus $df = z^n(\tilde{\varphi}, \psi) + O(n+1)$. This formula and Corollary 10.6 imply that p is a branch point of f of order n. □

10.8. Branched Pairings. Branched conformal immersions with both normal vectors have a Weierstrass representation with two nowhere vanishing holomorphic sections, if one allows the pairing to be branched.

A *branched pairing* of holomorphic line bundles $\tilde{L}$ and L is a smooth map $(\,,)\colon \tilde{L} \times L \to T^*M \otimes \mathbb{H}$ that is a pairing of holomorphic line bundles on some nonempty subset of M and zero on its complement.

Proposition. *A smooth bundle map $(\,,)\colon \tilde{L} \times L \to T^*M \otimes \mathbb{H}$ is a branched pairing of the holomorphic line bundles $\tilde{L}$ and L if and only if $\tilde{B}\colon \tilde{L} \to KL^{-1}$, $\varphi \mapsto (\varphi, \cdot)$, or, equivalently, $B\colon L \to K\tilde{L}^{-1}$, $\psi \mapsto \overline{(\cdot, \psi)}$ is a nontrivial holomorphic bundle homomorphism. Moreover, $\operatorname{ord}_p B = \operatorname{ord}_p \tilde{B}$.*

This proposition implies that the zeros of nontrivial branched pairings are isolated. The vanishing order of the pairing $(\,,)$ can then be defined by $\operatorname{ord}_p(\,,) := \operatorname{ord}_p B = \operatorname{ord}_p \tilde{B}$.

Proof. If $(\,,)$ is a branched pairing, then $*(\tilde{B}\varphi) = \tilde{B}(J\varphi) = (\tilde{B}\varphi)J = J\tilde{B}\varphi$ implies that $\tilde{B}\varphi \in \Gamma(KL^{-1})$ and that $\tilde{B}$ is complex linear. If φ and ψ are local holomorphic sections of $\tilde{L}$ and L, then $\tilde{B}\varphi(\psi) = (\varphi, \psi)$ is closed, because $(\,,)$ is a pairing away from its zeros. Thus $\tilde{B}\varphi$ is a holomorphic section of KL^{-1}, by Theorem 9.6. Lemma 4.1 then implies that $\tilde{B}$ is holomorphic. The proof of the holomorphicity of B is similar. The converse follows because $\tilde{B}$ and B are holomorphic isomorphisms away from their isolated zeros, and KL^{-1} and L as well as $K\tilde{L}^{-1}$ and $\tilde{L}$ are by definition paired with the evaluation pairing. Finally, the equality of the vanishing orders of B and $\tilde{B}$ follows from $\tilde{B}\varphi(\psi) = (\varphi, \psi) = \overline{B\psi(\varphi)}$. □

10.9. Branched Conformal Immersions from Branchedly Paired Holomorphic Line Bundles. Products of holomorphic sections, possibly with zeros, of branchedly paired holomorphic line bundles can be integrated to yield branched conformal immersions.

Theorem. *Let $\tilde{L}$ and L be holomorphic line bundles over a simply connected Riemann surface M with a branched pairing $(\,,\,)$. Let $\varphi \in H^0(\tilde{L})$ and $\psi \in H^0(L)$. Then $f := \int \varphi(\psi)\colon M \to \mathbb{H}$ is a branched conformal immersion, whose branching order satisfies $b_p f = \operatorname{ord}_p(\,,\,) + \operatorname{ord}_p \psi + \operatorname{ord}_p \varphi$.*

Proof. Since φ and ψ may have zeros, one can assume, by Proposition 10.8, that $\tilde{L} = KL^{-1}$ and $\varphi \in H^0(KL^{-1})$. The map $f = \int \varphi(\psi)$ is away from the isolated zeros of ψ and φ a conformal immersion, by Theorem 9.11. Thus it remains to show that f has a branch point of the claimed order at $p \in M$, whenever φ or ψ has a zero at p.

Near $p \in M$ one can write

$$(*) \qquad \varphi = z^m \tilde{\varphi} + O(m+1) \quad \text{and} \quad \psi = z^n \tilde{\psi} + O(n+1),$$

as in 2.4. Multiplying φ by a quaternionic constant one can assume that $J\tilde{\varphi}(p) = \tilde{\varphi}(p)\mathbb{i}$. The section $\hat{\varphi} := \frac{1}{2}(\tilde{\varphi} - J\tilde{\varphi}\mathbb{i})$ then satisfies $J\hat{\varphi} = \hat{\varphi}\mathbb{i}$ and coincides with $\tilde{\varphi}$ at p. Hence it does not vanish near p and $\hat{\varphi} - \tilde{\varphi} = O(1)$. Thus one can replace $\tilde{\varphi}$ by $\hat{\varphi}$ in $(*)$. The same can be done for ψ, and one obtains

$$\begin{aligned} df &= (\varphi, \psi) = (\hat{\psi} z^n + O(n+1), \hat{\varphi} z^m + O(m+1)) \\ &= (\hat{\psi}, \hat{\varphi}) z^{m+n} + O(m+n+1), \end{aligned}$$

because $-\mathbb{i}(\hat{\varphi}, \hat{\psi}) = (J\hat{\varphi}, \hat{\psi}) = *(\hat{\varphi}, \hat{\psi}) = (\hat{\varphi}, J\hat{\psi}) = (\hat{\varphi}, \hat{\psi})\mathbb{i}$. Since $\hat{\varphi} \neq 0$ and $\hat{\psi} \neq 0$, there is near p a smooth $\mathbb{C}$–valued map g such that $(\hat{\varphi}, \hat{\psi}) = \mathbb{j} dz g$ and $g(p) \neq 0$. Thus

$$df (g(p))^{-1} = \mathbb{j} dz z^{m+n} + O(m+n+1),$$

which implies that f has a branch point of order $m+n$ at p. □

10.10. Weierstrass Representation: Two Normal vectors. Replacing the pairing by a branched pairing, Theorem 9.11 holds verbatim for branched conformal immersions that have both normal vectors.

Theorem. *If $f\colon M \to \mathbb{H}$ is a branched conformal immersion with both normal vectors, then the Euclidean holomorphic line bundles $L^\perp$ and L of $\bar{f}$ and f are the unique holomorphic branchedly paired line bundles such that there exist nowhere vanishing holomorphic sections $\varphi \in H^0(L^\perp)$ and $\psi \in H^0(L)$ satisfying*

$$df = (\varphi, \psi).$$

The branching order of f equals the vanishing order of the pairing.

On the other hand, if $\varphi \in H^0(\tilde{L})$ and $\psi \in H^0(L)$ are nowhere vanishing sections of branchedly paired holomorphic line bundles over simply connected M, then $\int(\varphi, \psi)$ is a branched conformal immersion of M into $\mathbb{H}$ with both normal vectors.

As for the Weierstrass representation of conformal immersions, (cf., 9.12) the quadruple $(L^\perp, L, \varphi, \psi)$ is called the *Weierstrass data* of f. If $f = \sigma_{\alpha,\beta} L$ is the stereographic projection of a holomorphic curve in $\mathbb{HP}^1$, then $df = (a^{-1}, \beta^{-1})$ at immersed points of f, by Proposition 9.10, and trivially at the branch points of f.

Proof. Let $N\colon M \to \mathbb{H}$ and $R\colon M \to \mathbb{H}$ be the left and right normal vectors of f. The Euclidean holomorphic line bundles L and $L^\perp$ of f and $\bar{f}$ are then $M \times \mathbb{H}$ with the complex structures $1 \mapsto -R$ and $1 \mapsto -N$ and holomorphic structure such that 1 is holomorphic, by Corollary 9.3 and Lemma 7.5. Then define

$$(\varphi, \psi) := \bar{\varphi} df \psi,$$

for all $p \in M$, $\varphi \in L_p^\perp$ and $\psi \in L_p$, where φ and ψ on the left hand side stand for the corresponding quaternions. One easily checks that $(\,,)$ is a pairing of the complex quaternionic line bundles $L^\perp$ and L away from its zeros, i.e., the branch points of f.

To see that $(\,,)$ is a branched pairing, one needs to check, by Theorem 9.6, that (φ, ψ) is closed for all local holomorphic sections of $\tilde{L}$ and L: If φ is a holomorphic section of $L^\perp$, then the Leibniz rule implies that $*d\varphi = -Nd\varphi$, where φ is interpreted as a map on M with values in $\mathbb{H}$. For the same reason one has $*d\psi = -Rd\psi$ for holomorphic sections ψ of L. Thus $d(\varphi, \psi) = d\bar{\varphi} \wedge df \psi - \varphi df \wedge d\psi = 0$, as both terms vanish by type.

This shows the existence part, since $df = (1, 1)$ for the holomorphic sections $1 \in H^0(L^\perp)$ and $1 \in H^0(L)$. Uniqueness, follows from Theorem 4.2, because if $df = (\varphi, \psi)$, then $J\varphi = -\varphi N$ and $J\psi = -\psi R$.

Corollary 10.6 implies that f is a branched immersion. Hence one only needs to calculate the branching order of f at the branch points of the pairing: If $B\colon L^\perp \to KL^{-1}$ is the holomorphic bundle map $\xi \mapsto (\xi, \cdot)$ of Proposition 10.8, then $k := \operatorname{ord}_p B = \operatorname{ord}_p(B\varphi)$ is the vanishing order of $(\,,)$ at $p \in M$. Writing $B\varphi = z^k \tilde{\varphi} + O(k+1)$, as in 2.4, one gets $df = (\varphi, \psi) = B\varphi(\psi) = z^k \tilde{\varphi}(\psi)$. Hence f has a branch point of order k at p.

The last statement of the theorem follows from the definition of a branched pairing, which assures that (φ, ψ) is closed, and the fact that nowhere vanishing sections of L and $\tilde{L}$ do have normal vectors. $\square$

10.11. Tangent and Normal Bundle. Both the tangent and the normal bundle of a branched conformal immersion that has both normal vectors extend smoothly into its branch points.

Proposition. *If L is a holomorphic curve in $\mathbb{HP}^1$ whose dual curve $L^\perp$ is also a holomorphic curve, then the tangent and the normal bundle of L extend smoothly through the branch points of L. The extensions satisfy*

$$\top L = \operatorname{Hom}_+(L, \mathbb{H}^2/L) \quad \textit{and} \quad \perp L = \operatorname{Hom}_+(\bar{L}, \mathbb{H}^2/L),$$

as complex line bundles.

In terms of normal vectors one obtains: The tangent bundle as well as the normal bundle of a branched conformal immersion $f\colon M \to \mathbb{H}$ with right normal vector N and left normal vector R extends smoothly through *the branch points of f, and*

$$\top f = \{\, x \in \mathbb{H} \mid NxR = x \,\}, \qquad \perp f = \{\, x \in \mathbb{H} \mid NxR = -x \,\}.$$

PROOF. This follows from the description of the tangent and normal bundle away from the branch points of L or f in Theorem 7.6 or Lemma 6.1, respectively. $\square$

10.12. Normal Vectors of Holomorphic Sections with Zeros. In 3.1 normal vectors where only defined for nowhere vanishing sections of complex quaternionic line bundles. The following theorem shows that if the section is holomorphic, then the normal vector extends continuously into the zeros of the section, but the extension of the normal vector is in general not smooth.

Theorem. *Let L be a holomorphic line bundle and $\psi \in H^0(L)$ a holomorphic section of L with a zero at $p \in M$.*

(i) The normal vector of ψ extends continuously into p.
(ii) If the extended normal vector of ψ is continuously differentiable at p, then the Hopf field Q of L vanishes at p.

PROOF. Restricting to some open neighborhood of p, one can assume that p is the only zero of ψ. One can write $\psi = z^n\varphi + O(n+1)$ with $n \in \mathbb{N}$, $n \geq 1$ and a nowhere vanishing section $\varphi \in \Gamma(L)$, as in 2.4. Let N be the normal vector of ψ restricted to $M \setminus \{p\}$.

(i) From $\psi = z^n\varphi + O(n+1)$ follows that there exists a continuous function $g\colon M \to \mathbb{H}$ such that $g(p) = 1$ and $\psi = z^n\varphi g$. Let $\tilde{N}$ be the normal vector of φ. Then

$$J\psi = J(z^n\varphi g) = z^n\varphi\tilde{N}g \quad \text{and} \quad J\psi = \psi N = z^n\varphi g N.$$

on $M \setminus \{p\}$. Hence $N = g^{-1}\tilde{N}g$. Since $g^{-1}\tilde{N}g$ is continuous on all of M, it follows that N extends continuously into p.

(ii) For the Hopf field Q of L one has $Q\psi = \psi\frac{1}{2}NdN''$ (cf., 3.3) on $M \setminus \{p\}$. Thus

$$Q\psi = \psi\tfrac{1}{2}NdN'' = z^n\varphi\tfrac{1}{2}NdN'' + O(n+1) \quad \text{and}$$
$$Q\psi = Q(z^n\varphi + O(n+1)) = \bar{z}^n Q\varphi + O(n+1).$$

Hence $\left(\frac{\bar{z}}{z}\right)^n Q\varphi = \varphi\frac{1}{2}NdN'' + O(1)$ on $M \setminus \{p\}$. This implies that $\left(\frac{\bar{z}}{z}\right)^n Q\varphi$ extends continuously into p. Consequently, $Q\varphi_{|_p} = 0$. But φ does not vanish at p, thus Q is zero at p.

$\square$

10.13. Corollary. *The Hopf fields of branchedly paired holomorphic line bundles vanish at the branch points of the pairing.*

PROOF. Let $\tilde{L}$ and L be branchedly paired holomorphic line bundles, $p \in M$ a zero of the pairing, and $B\colon \tilde{L} \to KL^{-1}$ the holomorphic bundle map $\xi \mapsto (\xi, \cdot)$ of Proposition 10.8. Let φ be a local nowhere vanishing holomorphic section near p and N its normal vector. Then $B\varphi$ is zero at p, but has the smooth normal vector N, thus the Hopf field of KL^{-1} vanishes at p, by Theorem 10.12. Corollary 9.9 implies that the Hopf field of L at p

vanishes. Interchanging $\tilde{L}$ and L, the same argument shows that the Hopf field of $\tilde{L}$ vanishes. $\square$

10.14. Branched Conformal Immersions Without Smooth Normal Vectors. Let L be a quaternionic line bundle whose Hopf field does not vanish at $p \in M$. Then Corollary 9.9 implies that the Hopf field of KL^{-1} does not vanish at p either. Restricting to a neighborhood of p one can assume that there are holomorphic sections φ and ψ of KL^{-1} and L with zeros at p, [**BP**]. Then Theorem 10.9 implies that $f := \int \varphi(\psi)$ is a branched conformal immersion. This branched conformal immersion possesses continuous left and right normal vectors that are not continuously differentiable, by Theorem 10.12.

11. The Ladder of Holomorphic Line Bundles

The relations of the holomorphic structures of the Euclidean and Möbius invariant holomorphic line bundles of an immersed holomorphic curve in $\mathbb{HP}^1$ can be generalized to relations of holomorphic structures on the four complex quaternionic line bundles L, L^{-1}, KL^{-1} and $K^{-1}L$. This situation can be visualized in a diagram, which naturally extends to a ladder whose vertices are the complex quaternionic line bundles K^nL and K^nL^{-1}, $n \in \mathbb{Z}$. The relation of the holomorphic line bundles that occur in this ladder play an important role in the following investigations.

11.1. The lines in the following diagram depict the relations of the holomorphic structures of the Möbius invariant holomorphic line bundles L^{-1} and $\mathbb{H}^2/L$ and the Euclidean holomorphic line bundles L and $L^\perp$ of an immersed holomorphic curve L in $\mathbb{HP}^1$ and its dual curve $L^\perp$:

$$\begin{tikzcd}
L^\perp \arrow[rr, dash, dashed, "9.1"] \arrow[drr, dash, "9.6"] & & \mathbb{H}^2/L \\
L^{-1} \arrow[rr, dash, dashed, "9.1"] & & L
\end{tikzcd}$$

- The *solid* line stands for the paired holomorphic structures (cf., Theorem 9.6). The holomorphic structure at one vertex uniquely determines the holomorphic structure at the other vertex.
- The lower *dashed* line stands for the relation of the Möbius invariant holomorphic structure on L^{-1} and the Euclidean holomorphic structure on L, described in Theorem 9.1. The upper dashed line describes the same relation with L replaced by $L^\perp$ and L^{-1} replaced by $\mathbb{H}^2/L = (L^\perp)^{-1}$. These relations depend on the choice of an admissible point $\infty \in \mathbb{HP}^1$.

11.2. The Horizontal Relation with Connections. The relation depicted by the dashed line can be generalized as follows: Let $L \subset \mathbb{H}^2$ be a holomorphic curve in $\mathbb{HP}^1$ and $\infty \in \mathbb{HP}^1$ a point that does not lie on L. Then there is a unique quaternionic connection ∇ on L^{-1} such that $\nabla\beta_{|L} = 0$ for all $\beta \in (\mathbb{H}^2)^*$ whose kernel is ∞. This connection is trivial and its $(0,1)$ part ∇'' is the holomorphic structure D of L^{-1}, because one has $\nabla''\beta_{|L} = 0 = D\beta_{|L}$ for all $\beta \in (\mathbb{H}^2)^*$ such that $\ker\beta = \infty$. The Euclidean holomorphic structure of L with respect to ∞ is the $(0,1)$ part $(\nabla^*)''$ of the dual connection of ∇ on L, because the affine lift β^{-1} of L is holomorphic, by the definition of the Euclidean holomorphic structure, and $(\nabla^*)''\beta^{-1} = 0$, since $\beta(\beta^{-1}) = 1$.

If ∇ is now an arbitrary flat quaternionic connection on a holomorphic line bundle L^{-1} such that ∇'' is the holomorphic structure of L^{-1}, then $(\nabla^*)''$ is a holomorphic structure on L. There is, at least locally, a curve in $\mathbb{HP}^1$ such that L^{-1} and L are its Möbius invariant and Euclidean holomorphic line bundles: Let β be a local section of L^{-1} and H a 2–dimensional linear system of local holomorphic sections of L^{-1} that contains β. The Kodaira corresponding curve of H then has the Möbius invariant holomorphic line bundle L^{-1} and the Euclidean holomorphic line bundle L.

If one, on the other hand, starts with β^{-1}, instead of β, and a 2–dimensional linear system H of local holomorphic sections of L, then L, instead of L^{-1}, is the Möbius invariant holomorphic line bundle and L^{-1}, instead of L, is the Euclidean holomorphic line bundle of the Kodaira corresponding curve of H.

The relation of the Willmore energies obtained in 9.4 is preserved in the more general situation.

11.3. Proposition. *Let L be a quaternionic holomorphic line bundle over a compact Riemann surface M, ∇ a flat quaternionic connection on L such that ∇'' is the holomorphic structure of L and let $(\nabla^*)''$ be the holomorphic structure of L^{-1}. The Willmore energies of L and L^{-1} then satisfy*

$$W(L) = W(L^{-1}) - 4\pi \deg L$$

Proof. Let $\nabla = \partial + A + \bar\partial + Q$ be the type decomposition of ∇, as in 2.7, then one easily checks that $\nabla^* = \partial^* - Q^* + \bar\partial^* - A^*$, where ∂^* and $\bar\partial^*$ are the dual antiholomorphic and holomorphic structures of ∂ and $\bar\partial$, and A^* and Q^* are the dual endomorphisms of A and Q. Furthermore, $\nabla'' = \bar\partial + Q$ and $(\nabla^*)'' = \bar\partial^* - A^*$. Thus $W(L) = 2\int\langle Q\wedge *Q\rangle$ and $W(L^{-1}) = 2\int\langle A^*\wedge *A^*\rangle = 2\int\langle A\wedge *A\rangle$. The flatness of ∇ and the formula for the complex linear part of the curvature tensor of ∇, given in 2.7, then imply $0 = \mathcal{R}^\nabla_+ = \mathcal{R}^{\hat\nabla} + Q\wedge Q + A\wedge A$, hence $Q\wedge *Q = A\wedge *A - J\mathcal{R}^{\hat\nabla}$. Integration of the real trace of this formula, and the formula $2\pi\deg L = \int\langle J\mathcal{R}^{\hat\nabla}\rangle$, gives the claimed formula. □

11.4. Direct Vertical Relation. The diagonal and the horizontal relation of the holomorphic structures in the diagram (11.1) yield a zigzag relation between the holomorphic structures of L^{-1} and KL^{-1}. This relation can be described by the exterior derivative: Let ∇ be a quaternionic connection on L^{-1}. The exterior derivative d^∇ is then a quaternionic holomorphic structure on KL^{-1} if one identifies the tensor product $\bar KK$ of the canonical and anticanonical bundle and the complex line bundle of $\mathbb{C}$–valued 2–forms $\Lambda^2\otimes\mathbb{C}$ by

$$\bar KK \ni \omega\otimes\eta \leftrightarrow \omega\wedge\eta \in \Lambda^2\otimes\mathbb{C},$$

which induces an identification of $\bar KKL^{-1}$ and $\Lambda^2\otimes L^{-1}$. The exterior derivative $d^\nabla : \Gamma(KL^{-1}) \to \Gamma(\bar KKL^{-1})$ then satisfies the Leibniz rule, because for $\omega \in \Gamma(KL)$ and $\lambda\colon M \to \mathbb{H}$ one has

$$d^\nabla(\omega\lambda) = d^\nabla\omega\lambda - \omega \wedge d\lambda = d^\nabla\omega\lambda + (\omega d\lambda)'',$$

since $w \wedge d\lambda' = 0$ by type and $w \wedge d\lambda''$ is identified with $-(\omega d\lambda)''$.

Theorem. *If L^{-1} is a complex quaternionic line bundle and ∇ a flat quaternionic connection on L^{-1}, then d^∇ and $(\nabla^*)''$ are paired holomorphic structures on KL^{-1} and L. Moreover, ∇ induces a quaternionic linear map from $H^0(L^{-1}, \nabla'')$ to $H^0(KL^{-1}, d^\nabla)$, and on compact M the Willmore energies of KL^{-1}, L and L^{-1} satisfy*

$$W(L^{-1}) = W(L) - 4\pi \deg L^{-1} = W(KL^{-1}) - 4\pi \deg L^{-1}.$$

Proof. Let $U \subset M$ be open, $\omega \in \Gamma(KL^{-1}{}_{|U})$ such that $d^\nabla\omega = 0$, and $\psi \in \Gamma(L_{|U})$ such that $(\nabla^*\psi)'' = 0$. Then

$$d(\omega, \psi) = d(\omega(\psi)) = d^\nabla\omega(\psi) + \omega \wedge \nabla^*\psi = \omega \wedge (\nabla^*\psi)'' = 0,$$

where $\omega \wedge \nabla^*\psi = \omega \wedge (\nabla^*\psi)''$ follows by type (cf., 2.8). Thus d^∇ and $(\nabla^*)''$ are paired, by Theorem 9.6.

If ψ is a holomorphic section in (L^{-1}, ∇''), then $\nabla\psi \in \Gamma(KL^{-1})$, because $(\nabla\psi)'' = 0$, and $\nabla\psi$ is holomorphic, because the flatness of ∇ implies $d^\nabla\nabla\psi = 0$. The equation for the Willmore energies follows from Corollary 9.9 and Proposition 11.3 $\square$

11.5. Remark. The J commuting part of d^∇ is the tensor product of the complex holomorphic structures of K and L^{-1}: The complex holomorphic structure on K is the exterior derivative d, and the complex holomorphic structure of L^{-1} is the J–commuting part of ∇''. For $\omega \in K$ and $\varphi \in \Gamma(L^{-1})$ one gets

$$\begin{aligned} &d^\nabla(\omega\varphi) - Jd^\nabla(\omega J\varphi) \\ &\qquad = d\omega\varphi - \omega \wedge \nabla\varphi - J(d\omega J\varphi) + \omega \wedge J(\nabla J)\varphi + \omega \wedge J^2\nabla\varphi \\ &\qquad = 2d\omega\varphi + 2\omega \wedge (Q + A)\varphi - \underbrace{2\omega \wedge \nabla\varphi}_{=2\omega\wedge(\nabla\varphi)''=2\omega\wedge(\bar\partial+Q)\varphi} \\ &\qquad = 2d\omega\varphi + 2\omega\bar\partial\varphi. \end{aligned}$$

In the second line the formula $\nabla J = 2(*Q - *A) = -2J(Q + A)$ from 2.7 is used, and $\omega \wedge \nabla\varphi = \omega \wedge (\nabla\varphi)''$ as well as $\omega \wedge A\varphi = 0$ by type (cf., 2.8).

11.6. The Quadrilateral of Holomorphic Line Bundles. Theorem 11.4 implies that every flat connection ∇ on L^{-1} induces a commutative triangle of quaternionic holomorphic structures:

$$\begin{tikzcd} (KL^{-1}, d^\nabla) \arrow[drr, dash, "9.6"] & & \\ (L^{-1}, \nabla'') \arrow[u, dashed, "11.4"] \arrow[rr, dashed, dash, "11.3"'] & & (L, (\nabla^*)'') \end{tikzcd}$$

Rotating the triangle of holomorphic line bundles and substituting L by KL^{-1} yields

$$
\begin{tikzcd}
(KL^{-1}, (\tilde{\nabla}^*)'') \arrow[r, dashed, no head, "11.3"] \arrow[dr, no head, "9.6"] & (K^{-1}L, \tilde{\nabla}'') \arrow[d, dashed, "11.4"] \\
& (L, d^{\tilde{\nabla}})
\end{tikzcd}
$$

If one chooses $\tilde{\nabla}$ such that $d^{\nabla} = (\tilde{\nabla}^*)''$ then the triangles fit together, and one obtains the quadrilateral of holomorphic line bundles:

$$
\begin{tikzcd}
(KL^{-1}, d^{\nabla} = (\tilde{\nabla}^*)'') \arrow[r, dashed, no head, "11.3"] \arrow[dr, no head, "9.6"] & (K^{-1}L, \tilde{\nabla}'') \arrow[d, dashed, "11.4"] \\
(L^{-1}, \nabla'') \arrow[u, dashed, "11.4"] \arrow[r, dashed, no head, "11.3"'] & (L, (\nabla^*)'' = d^{\tilde{\nabla}})
\end{tikzcd}
$$

11.7. The Ladder of Holomorphic Line Bundles. KL^{-1} with flat connection $\tilde{\nabla}^*$ is the lower left corner of a new triangle above the quadrilateral, and L with ∇^* is the upper right corner of a rotated triangle below the quadrilateral. Choosing new flat connections whose $(0,1)$ parts equal d^{∇^*} and $d^{\tilde{\nabla}^*}$, respectively, yields two new triangles. This procedure can be continued as long as there are flat connections with appropriate $(0,1)$ parts. Because there are always local nowhere vanishing holomorphic sections, such connections exist at least locally. All in all, one arrives at a *ladder of holomorphic line bundles*:

$$
\begin{tikzcd}
K^nL^{-1} \arrow[d, dotted, no head] \arrow[drr, no head] & & \\
(K^2L^{-1}, d^{\tilde{\nabla}^*}) \arrow[drr, no head] & \vdots & K^{-n+1}L \arrow[d, dotted, no head] \\
(KL^{-1}, d^{\nabla} = (\tilde{\nabla}^*)'') \arrow[u, dashed] \arrow[rr, dashed, no head, "11.3"] \arrow[drr, no head, "9.6"] & & (K^{-1}L, \tilde{\nabla}'') \arrow[d, dashed, "11.4"] \\
(L^{-1}, \nabla'') \arrow[u, dashed, "11.4"] \arrow[rr, dashed, no head, "11.3"'] \arrow[drr, no head] \arrow[d, dotted, no head] & & (L, (\nabla^*)'' = d^{\tilde{\nabla}}) \arrow[d, dashed] \\
K^{-n}L^{-1} \arrow[drr, no head] & \vdots & (KL, d^{\nabla^*}) \arrow[d, dotted, no head] \\
& & K^{n+1}L
\end{tikzcd}
$$

12. Surfaces in $\mathbb{R}^3$

In this section it is shown that the Euclidean holomorphic line bundles of a conformal immersion into $\operatorname{Im}\mathbb{H} = \mathbb{R}^3$ are isomorphic. This means that only one holomorphic line bundle enters into the Weierstrass data of such an immersion. This line bundle is paired to itself, and the conformal immersions into $\operatorname{Im}\mathbb{H}$ are obtained integrating squares of holomorphic sections.

12.1. Surfaces in $\operatorname{Im}\mathbb{H}$. Immersions with values in a parallel hyperplane of $\operatorname{Im}\mathbb{H}$ are characterized by equality of their normal vectors.

Proposition. *If $f\colon M \to \mathbb{H}$ is a branched conformal immersion with a normal vector, then f takes values in a hyperplane parallel to $\operatorname{Im}\mathbb{H}$ if and only if it has both normal vectors and they are equal. In this case, the normal vector is the Gauss map of f.*

Proof. If N is the left normal vector of f, then $*df = -\overline{*df} = -\overline{Ndf} = -dfN$, because the image of df is contained in $\operatorname{Im}\mathbb{H}$. Thus N is also the right normal vector of f. On the other hand, if there exists a map $N\colon M \to \mathbb{H}$ such that $N^2 = -1$ and $*df = Ndf = -dfN$, then $\operatorname{Re}(df) = -\operatorname{Re}(dfN^2) = -\operatorname{Re}(NdfN) = \operatorname{Re}(dfN^2) = -\operatorname{Re}(df)$, because the real part of the quaternionic multiplication is commutative. Thus df takes values in $\operatorname{Im}\mathbb{H}$. Finally, $N(p)$ is for all $p \in M$ orthogonal to the image of d_pf, since $Ndf = -dfN$ implies $\langle N, df\rangle = \operatorname{Re}(Ndf) = 0$. $\square$

12.2. Weierstrass Representation. The preceding proposition and Theorem 10.10 imply that a branched conformal immersion with a normal vector and values in $\operatorname{Im}\mathbb{H}$ has a Weierstrass representation. Its Weierstrass data is symmetric:

Lemma. *If $(L^\perp, L, \varphi, \psi)$ is the Weierstrass data of a branched conformal immersion with a normal vector into $\operatorname{Im}\mathbb{H}$, then $L^\perp$ and L are isomorphic with an isomorphism mapping φ onto ψ.*

Proof. Let $f\colon M \to \operatorname{Im}\mathbb{H}$ be a branched conformal immersion into $\operatorname{Im}\mathbb{H}$ with normal vector $N\colon M \to \mathbb{H}$, then $*df = Ndf = -dfN$, by Proposition 12.1. Thus $df = (\varphi, \psi)$ implies that φ and ψ are nowhere vanishing holomorphic sections of $L^\perp$ and L with the same normal vector $-N$. The bundle map $B\colon L^\perp \to L$ mapping φ onto ψ is a holomorphic isomorphism, by Lemma 4.1. $\square$

12.3. If two paired holomorphic line bundles $\tilde{L}$ and L are isomorphic with an isomorphism $B\colon L \to \tilde{L}$, then $(\cdot,\cdot) := (B\cdot,\cdot)$ is a pairing of L with itself. A holomorphic line bundle L that is paired to itself is called a *spin bundle*, and a *branched spin bundle*, if L is branchedly paired to itself. Proposition 10.8 implies that a holomorphic line bundle L is a branched spin bundle if and only if there is a holomorphic bundle homomorphism from L to KL^{-1}. It is a spin bundle if and only if L and KL^{-1} are isomorphic.

Lemma. *If L is a branched spin bundle with branched pairing $(\cdot,\cdot)$, then $\overline{(\varphi,\psi)} = -(\psi,\varphi)$.*

PROOF. Let $\psi \in L$ not zero and $N \in \mathbb{H}$ such that $J\psi = \psi N$. Then $N^2 = -1$ and $\operatorname{Re}(\psi, \psi) = -\operatorname{Re}(J\psi, J\psi) = \operatorname{Re}(N(\psi,\psi)N) = -\operatorname{Re}(\psi,\psi)$. Thus (ψ, ψ) takes values in $\operatorname{Im}\mathbb{H}$. Let φ be some element in the same fiber as ψ, then there exists $\lambda \in \mathbb{H}$ such that $\varphi = \psi\lambda$. This implies $\overline{(\varphi,\psi)} = \overline{\bar{\lambda}(\psi,\psi)} = -(\psi,\psi)\lambda = -(\psi,\varphi)$. □

12.4. Putting everything together one concludes that the Weierstrass representation of a branched conformal immersion with a normal vector into $\operatorname{Im}\mathbb{H}$ is obtained from one holomorphic section in a branched spin bundle.

Theorem. *If $f\colon M \to \operatorname{Im}\mathbb{H}$ is a branched conformal immersion with a normal vector. Then the Euclidean holomorphic line bundle L is the unique branched spin bundle with a nowhere vanishing holomorphic section $\varphi \in H^0(L)$ such that*

$$df = (\varphi, \varphi).$$

The branching order of f equals the vanishing order of the pairing. If M is compact of genus g, then

$$b(f) = 2g - 2 - 2\deg L.$$

Conversely, if $\varphi \in H^0(L)$ is a nowhere vanishing holomorphic section of a branched spin bundle L over a simply connected Riemann surface, then $\int(\varphi,\varphi)$ is a branched conformal immersion of M into $\mathbb{H}$ with both normal vectors that takes values in some hyperplane parallel to $\operatorname{Im}\mathbb{H} \subset \mathbb{H}$.

The pair (L, φ) is called the *Weierstrass data* of f.

PROOF. The first part follows form Theorem 10.10 and Lemma 12.2. In particular, L is a branched spin bundle, the map $B\colon L \to KL^{-1}$, $\varphi \mapsto (\varphi, \cdot)$ is holomorphic, by Proposition 10.8, and $\operatorname{ord} B$ is the total branching order $b(f)$ of f. Thus $b(f) = \operatorname{ord} B = -2\deg L + \deg K = -2\deg L + 2g - 2$.

The converse is a consequence of Theorem 10.10 and Proposition 12.1, because $df = (\varphi, \varphi)$ implies that the left and the right normal vector of f coincide. □

12.5. Maps into Real Hyperplanes of $\mathbb{H}$. If $V \subset \mathbb{H}$ is a real hyperplane of $\mathbb{H}$ through zero, then $V\bar{\mu} = \operatorname{Im}\mathbb{H}$, for every nonzero $\mu \in \mathbb{H}$ orthogonal to V. Thus the image of a map $f\colon M \to \mathbb{H}$ is contained in a hyperplane parallel to V if and only if the image of $f\bar{\mu}$ is contained in a hyperplane parallel to $\operatorname{Im}\mathbb{H}$. It follows, by Theorem 12.4, that (branched) conformal immersions (with a normal vector) that take values in a real hyperplane of $\mathbb{H}$, have Weierstrass data of the form $(L, L, \varphi, \varphi\bar{\mu})$ with a nowhere vanishing holomorphic section φ in a (branched) spin bundle L, and vice versa. In terms of normal vectors one gets that the equation $N = \mu R \mu^{-1}$, for the normal vectors N and R of f and some constant $\mu \in \mathbb{H}$, is equivalent to f assuming values in a hyperplane parallel to the orthogonal complement of μ. In particular, a smooth map $f\colon M \to \mathbb{H}$ whose image is contained in a real hyperplane has either both or no normal vectors.

12.6. Example: Minimal Surfaces. Let $f\colon M \to \operatorname{Im}\mathbb{H}$ be a branched conformal immersion such that

$$df = (\varphi, \varphi),$$

with a holomorphic section φ of a spin bundle L. The equation $2\langle Q \wedge *Q\rangle = |\mathcal{H}|^2 dA$ (cf., 9.4) and Remark 3.6 imply that f is minimal if and only if the Willmore energy of L vanishes.

This is the quaternionic version of the well known spinor Weierstrass representation of a minimal surface in $\mathbb{R}^3$, see for example [**Bob94**]. To get the standard formula with two complex holomorphic spinors choose a meromorphic section ψ of the underlying complex line bundle $\hat{L}$ of L. Then $*(\psi,\psi) = -\mathbb{i}(\psi,\psi) = (\psi,\psi)\mathbb{i}$. Hence locally, away from the zeros and poles of ψ, there are holomorphic charts $z\colon M \supset U \to \mathbb{C}$ such that $(\psi,\psi) = \mathbb{j}dz$. Writing $\varphi = \psi(\lambda_1\mathbb{k} + \lambda_2)$ with complex holomorphic maps $\lambda_{1,2}\colon U \to \mathbb{C}$ one gets

$$df = \operatorname{Re}(2\lambda_1\lambda_2 dz)\mathbb{i} + \operatorname{Re}((-\lambda_1^2 + \lambda_2^2)dz)\mathbb{j} + \operatorname{Re}(\mathbb{i}(\lambda_1^2 + \lambda_2^2)dz)\mathbb{k}.$$

With the meromorphic map $g = \frac{\lambda_1}{\lambda_2}$ and the holomorphic 1–form $\eta = 2\lambda_2^2 dz$ one obtains the classical Weierstrass representation (cf., [**HK97**, Theorem 2.1]) of minimal surfaces

$$df = \operatorname{Re}(g\eta)\mathbb{i} + \operatorname{Re}(\tfrac{1}{2}(1-g^2)\eta)\mathbb{j} + \operatorname{Re}(\tfrac{1}{2}\mathbb{i}(1+g^2)\eta)\mathbb{k}.$$

The normal vector N of f is minus the normal vector of φ, which implies $N = -(\lambda_1\mathbb{k} + \lambda_2)^{-1}\mathbb{i}(\lambda_1\mathbb{k} + \lambda_2)$. Consequently, $N = \frac{(|g|^2-1)\mathbb{i} + 2g\mathbb{j}}{|g|^2+1}$. Thus g is the stereographic projection of the normal vector of f.

13. Surfaces in S^3

If $L \subset \mathbb{H}^2$ is an immersed holomorphic curve in $\mathbb{HP}^1$ that lies in some 3–sphere of $\mathbb{HP}^1$, then there is a stereographic projection $s_{\alpha,\beta}L$ of L that takes values in the imaginary quaternions $\operatorname{Im}\mathbb{H}$. The Euclidean holomorphic structures of L and $L^\perp$ with respect to $\infty \in S^3$ are then isomorphic, by Lemma 12.2. In this section it is shown that the Möbius invariant holomorphic line bundles are also isomorphic. Because this is a Möbius invariant condition one could ask whether the property that the Euclidean holomorphic line bundles are isomorphic is possibly also invariant under Möbius transformations. The answer is no. In fact, a conformal immersion of a compact Riemann surface into S^3 whose Euclidean holomorphic line bundles are isomorphic is a surface of constant mean curvature in S^3.

13.1. The Möbius Invariant Holomorphic Line Bundles. The Möbius invariant holomorphic line bundles of a holomorphic curve in $\mathbb{HP}^1$ that lies in a 3–sphere are isomorphic. This can be derived from the fact that the Euclidean holomorphic line bundles of such a curve and its dual are isomorphic. Nevertheless an independent proof is given below.

Theorem. *A holomorphic curve L in $\mathbb{HP}^1$ lies in some 3–sphere if and only if $L^\perp$ is a holomorphic curve and there is a holomorphic isomorphism between the Möbius invariant holomorphic line bundles of $L^\perp$ and L that maps the canonical linear systems of $L^\perp$ and L onto each other.*

The author does not know any example of a holomorphic curve in $\mathbb{H}\mathrm{P}^1$ that does not lie in a 3–sphere whose Möbius invariant holomorphic line bundle is isomorphic to the Möbius invariant holomorphic line bundle of its dual curve. Thus the condition on the canonical linear systems in the theorem may be superfluous.

PROOF. Suppose $L \subset \mathbb{H}^2$ lies in some 3–sphere $S^3 \subset \mathbb{H}\mathrm{P}^1$. Then L can be stereographically projected onto a branched conformal immersion in $\operatorname{Im}\mathbb{H}$. This immersion has a smooth right normal vector, by Proposition 10.5. Proposition 12.1 implies that it also has a smooth left normal vector. Thus $L^\perp$ is also a holomorphic curve, by Proposition 10.5 and Lemma 7.5.

Let $\langle\,,\rangle$ be the indefinite Hermitian form on $\mathbb{H}^2$ associated to S^3, as described in 5.4, then $\langle L, L\rangle = 0$. The quaternionic linear homomorphism

$$B\colon \mathbb{H}^2 \to (\mathbb{H}^2)^*, \quad a \mapsto \langle a, \cdot\rangle$$

maps L onto $L^\perp$. Hence it induces a quaternionic linear bundle homomorphism $\tilde{B}\colon (L^\perp)^{-1} = \mathbb{H}^2/L \to L^{-1} = (\mathbb{H}^2)/L^\perp$, which maps the canonical linear system of $L^\perp$ onto the canonical linear system of L. If $a, b \in \mathbb{H}^2$ is an admissible basis, then $a_{|L^\perp}$ and $b_{|L^\perp}$ are holomorphic sections of $(L^\perp)^{-1}$, and there exists $f\colon M \to \mathbb{H}$ such that $a_{|L^\perp} = b_{|L^\perp} f$. From the Leibniz rule follows that the normal vector of $b_{|L^\perp}$ is the left normal vector of f. The left normal vector of f is also the normal vector of $\tilde{B}(b_{|L^\perp})$, because $\tilde{B}(a_{|L^\perp}) = \langle a, \cdot\rangle_{|L}$ and $\tilde{B}(b_{|L^\perp}) = \langle b, \cdot\rangle_{|L}$ are holomorphic and $\tilde{B}(a_{|L^\perp}) = \tilde{B}(b_{|L^\perp})f$. Thus Theorem 4.2 implies that $\tilde{B}$ is holomorphic.

Suppose now that there exists a holomorphic bundle homomorphism $\tilde{B}\colon (L^\perp)^{-1} \to L^{-1}$ that maps the canonical linear system of $L^\perp$ onto the canonical linear system of L. Then $\tilde{B}$ induces a parallel quaternionic linear bundle homomorphism $B\colon \mathbb{H}^2 \to (\mathbb{H}^2)^*$ that satisfies $B(L) = L^\perp$. The latter follows, because $\psi \in L_p$ corresponds to a holomorphic section of $(L^\perp)^{-1}$ with a zero at p whose image under B is also a holomorphic section with a zero at p. If one now proves that $\langle x, y\rangle := B(x)(y)$ is a Hermitian form on $\mathbb{H}^2$, then L lies in the 3–sphere associated to $\langle\,,\rangle$, and the proof is complete.

Let $\psi \in \Gamma(L)$ be a nowhere vanishing section, then $B(\psi)(\psi) = 0$ implies

$$B(\delta\psi)(\psi) = -B(\psi)(\delta\psi).$$

This equation implies $-NB(\delta\psi)(\psi) = B(\delta\psi)(\psi)N$ for the normal vector $N\colon M \to \mathbb{H}$ of ψ. Thus the image of $B(\delta\psi)(\psi)$ is contained in the imaginary quaternions. Let $p, q \in M$ such that $L_p \neq L_q$, $\delta_{|p} \neq 0$, and $X \in T_pM \setminus \{0\}$. Then there exists $b \in L_q \setminus \{0\}$ such that $b \equiv \delta_X\psi_{|p} \mod L_p$. If $a = \psi_{|p}$, then a, b is a basis of $\mathbb{H}^2$ such that $B(a)(a) = B(b)(b) = 0$, because $a \in L_p$ and $b \in L_q$. Furthermore, $B(b)(a) = B(\delta(X)\psi_{|p})(\psi_{|p})$ is imaginary, and

$$B(b)(a) = B(\delta(X)\psi_{|p})(\psi_{|p}) = -B(\psi_{|p})(\delta(X)\psi_{|p}) = -B(a)(b) = \overline{B(a)(b)}.$$

$\square$

13.2. Maps into S^3 and Spin Bundles. The Weierstrass data of a branched conformal immersion into a real hyperplane of $\mathbb{H}$ consists of two linearly dependent sections of a branched spin bundle (cf., 12.5). Linearly independent nowhere vanishing sections induce immersions that do not lie in a real hyperplane of $\mathbb{H}$. The quotient of these sections is a nowhere vanishing holomorphic section of the normal bundle (cf., Lemma 13.4). From this one can derive (cf., Theorem 13.6) that the Weierstrass data of a branched conformal immersion of a compact Riemann surface into S^3 lies in a branched spin bundle if and only if the immersion has constant mean curvature (CMC) in S^3.

13.3. Recall from Proposition 10.11 that the normal bundle $\perp f$ of a branched conformal immersion $f\colon M \to \mathbb{H}$ with right normal vector R and left normal vector N is $\{\, x \in \mathbb{H} \mid NxR = -x \,\}$. It is a complex line bundle, whose complex structure is left multiplication with N, which equals right multiplication with R. The trivial connection d of $\mathbb{H}$ induces a complex connection $\nabla^\perp$ on the normal bundle of f, since for a section $\psi \in \Gamma(\perp f)$ one has $\nabla^\perp(\psi R) = \nabla^\perp(\psi)R$, because ψdR has values in the tangent bundle of f. The $(0,1)$ part of the normal connection $\nabla^\perp$ defines the complex holomorphic structure of the normal bundle of f.

13.4. Lemma. *Let L be a branched spin bundle over simply connected M, $\varphi, \psi \in H^0(L)$ nowhere vanishing holomorphic sections, and $\lambda\colon M \to \mathbb{H}$ the quotient of φ and ψ, i.e., $\varphi = \psi\bar\lambda$. Then λ is a nowhere vanishing holomorphic section of the normal bundle of the branched conformal immersion $\int(\varphi, \psi)$.*

Proof. Let N be the left normal vector and R the right normal vector of the branched conformal immersion $f := \int(\varphi, \psi)$. Then $J\varphi = -\varphi N = -\psi\bar\lambda N$ and $J\varphi = J\psi\bar\lambda = -\psi R\bar\lambda$. Hence $N\lambda R = -\lambda$, which means that λ is a nowhere vanishing holomorphic section of the normal bundle of f. Because φ and ψ are holomorphic, the Leibniz rule implies $*d\bar\lambda = -Rd\bar\lambda$, hence $*d\lambda = d\lambda R$. The $(0,1)$ part of $d\lambda$ with respect to right multiplication by R on $\mathbb{H}$ thus vanishes identically. Its projection onto the normal bundle of f is of course also zero. This means that λ is a holomorphic section of the normal bundle of f. $\square$

13.5. Remark. As $*d\lambda = d\lambda R$ implies $d\lambda \wedge d\bar f = 0$ by type, λ is a 1–step Bäcklund transform of $\bar f$ (cf., 21.5). From $*d(\lambda^{-1}) = d(\lambda^{-1})N$ follows that $d(\lambda^{-1}) \wedge df = 0$, hence λ^{-1} is a 1–step Bäcklund transform of f.

13.6. Theorem. *Let M be a compact Riemann surface, $f\colon M \to S^3$ a branched conformal immersion with both normal vectors, and $(L^\perp, L, \varphi, \psi)$ its Weierstrass data. The holomorphic line bundles $L^\perp$ and L are isomorphic if and only if f is a CMC surface in S^3. The quotient of φ and ψ is in this situation, up to a constant real factor, the parallel CMC surface of f.*

Proof. If L and $L^\perp$ are isomorphic, then let $\lambda\colon M \to \mathbb{H}$ be the quotient of φ and ψ, i.e., $\varphi = \psi\bar\lambda$. Lemma 13.4 then implies that λ is a holomorphic section of the normal bundle of f. Since f takes values in S^3, f is also a section of its normal bundle. By the definition of the normal connection it is parallel, and, consequently, holomorphic. Because M is compact this implies that there are real constants $\lambda_{1,2} \in \mathbb{R}$ such that

$$\lambda = \lambda_1 f + \lambda_2 n,$$

where $n = Nf = fR$ is the normal vector of f in S^3.

Let $\mathrm{I} = \langle df, df\rangle$, $\mathrm{II} = -\langle df, dn\rangle$ and $\mathrm{III} = \langle dn, dn\rangle$ be the first, second, and third fundamental forms of f as a surface in S^3. Let H be the mean curvature of f in S^3 and K the Gauss curvature of f. Then H is half the trace and $(K-1)$ the determinant of II with respect to I. From the Cayley–Hamilton theorem follows

$$0 = (K-1)I - 2H\,\mathrm{II} + \mathrm{III}\,.$$

Furthermore, there exists a smooth map $u\colon M \to \mathbb{R}$ such that $|d\lambda| = u\,\mathrm{I}$, because λ is a conformal immersion, away from the zeros of its differential, by Lemma 6.1 (λ has a right normal vector, see the proof of Lemma 13.4). Consequently,

$$u\,\mathrm{I} = |d\lambda| = \lambda_1^2\,\mathrm{I} - 2\lambda_1\lambda_2\,\mathrm{II} + \lambda_2^2\,\mathrm{III} = (\lambda_1^2 - K + 1)\,\mathrm{I} + 2\lambda_2(-\lambda_1 + H\lambda_2)\,\mathrm{II}\,.$$

Thus one has the following three cases: (1) $\lambda_2 = 0$; (2) $\lambda_2 \neq 0$ and $H = \frac{\lambda_1}{\lambda_2}$; or (3) II is at all $p \in M$ a multiple of I. Case (1) means that λ is a constant real multiple of f, which is a contradiction to $*df = -dfR$ and $*d\lambda = d\lambda R$, see the proof of Lemma 13.4. Case (2) implies that f has constant mean curvature $H = \lambda_1/\lambda_2$ in S^3, and that $\lambda = \lambda_2(Hf + n)$. Hence λ has constant length (because $d\lambda$ is orthogonal to λ) and $\lambda/|\lambda|$ is the parallel CMC surface of f in S^3. Case (3) means that f is totally umbilic. It, consequently, lies in some real hyperplane of $\mathbb{H}$. From 12.5 then follows that ψ and φ are linearly dependent. Thus λ is constant. This implies $dn = -\frac{\lambda_1}{\lambda_2}df$. Hence $H = \frac{\lambda_1}{\lambda_2}$. So, case (3) is contained in case (2).

Suppose now that f has constant mean curvature H in S^3. Then

$$\lambda := Hf + n$$

is, up to a constant real factor, the parallel CMC surface of f in S^3. Then $d\lambda$ is orthogonal to λ, and $d\lambda$ takes values in the tangent bundle of f. Furthermore,

$$|d\lambda| = H^2\,\mathrm{I} - 2H\,\mathrm{II} + \mathrm{III} = (H^2 - K + 1)\,\mathrm{I}\,.$$

Hence λ is conformal. Let N be the left and R the right normal vector of f. Then one arrives at the following three cases: (a) λ is constant; (b) N is its left and R is its right normal vector; or (c) $-N$ is its left and $-R$ is its right normal vector.

Case (a) means that df is orthogonal to the constant λ. Thus f takes values in a real hyperplane of $\mathbb{H}$ parallel to $\lambda^\perp$, and $L^\perp$ and L are holomorphically isomorphic (see 12.5). In case (b) $d\lambda = Hdf + dn$ implies that $*dn = Ndn = -dnR$. From $n = Nf = fR$ one concludes $dN'' = dR'' = 0$. Thus the Willmore energies of the Möbius invariant holomorphic line bundles of f and $\bar{f}$ vanish (cf., 8.6). Consequently f takes values in some 2–sphere (cf., 8.4). Hence as in case (a), f takes values in a real hyperplane of $\mathbb{H}$, and $L^\perp$ and L are holomorphically isomorphic (see 12.5). In case (c) one has $*d\lambda = d\lambda R$. This equation and the Leibniz rule imply that $\psi\bar{\lambda}$ is a holomorphic section of L. From $J\psi = -\psi R$ (since $df = (\varphi, \psi)$)

and $N\lambda = \lambda R$ follows that $J\psi\bar{\lambda} = -\psi R\bar{\lambda} = -\psi\bar{\lambda}N$. So $\psi\lambda$ is a nowhere vanishing holomorphic section of L that has the same normal vector as the nowhere vanishing section φ of $L^\perp$. Thus $L^\perp$ and L are holomorphically isomorphic, by Theorem 4.2. $\square$

CHAPTER III

Equality in the Plücker Estimate

The complex version of the Plücker formula relates basic extrinsic and intrinsic invariants of compact complex holomorphic curves in $\mathbb{C}\mathrm{P}^n$. If b_k is the branching order of the osculating k–plane, g the genus and d the degree of the curve, then

$$0 = (n+1)(n(1-g)-d) + \sum_{k=0}^{n-1}(n-k)b_k,$$

(cf., [**GriHa**, Section 2.4, p. 270f]). In the quaternionic situation the formula holds verbatim if one replaces the zero on the left hand side by the difference of the Willmore energies of the curve and its dual. As a matter of fact, the quaternionic Plücker formula, as formulated in [**FLPP01**] (or 14.7), is about linear systems of holomorphic line bundles and the weighted sum of the branching orders of the osculating k–planes is replaced by the Weierstrass order of the linear system. The connection between these two formulations is the Kodaira correspondence (cf., 14.1) and the formula $b_k = n_{k+1} - n_k - 1$, which implies $\sum_{k=0}^{n-1}(n-k)b_k(p) = \sum_{k=0}^{n}(n_k(p)-k) = \operatorname{ord}_p H$ (cf., 3.7 and Lemma 14.2).

The first section of the present chapter contains important definitions and the formulation of the Plücker formula and the Plücker estimate. The Plücker formula is in the next section applied to 1– and 2–dimensional linear systems. This yields a formula that relates the Willmore energies of different stereographic projections of a holomorphic curve in $\mathbb{H}\mathrm{P}^1$, and a formula for the total curvature of the stereographic projection of a holomorphic curve in $\mathbb{H}\mathrm{P}^1$. Furthermore, it is shown that compact holomorphic curves which project stereographically onto minimal surfaces in $\mathbb{R}^4$ are those curves whose canonical linear system contains a 1–dimensional linear system with equality in the Plücker estimate. Section 16 then contains the most important fact about equality in the Plücker estimate: Linear systems with equality in the Plücker estimate can be described by complex holomorphic data. Then three operations that preserve equality in the Plücker estimate are presented.

In Section 18 soliton spheres are defined as those branched conformal immersion of $\mathbb{C}\mathrm{P}^1$ into $\mathbb{H}$ whose canonical linear system is contained in a linear system with equality. It is then shown that this Möbius invariant definition is equivalent to a definition in Euclidean terms, which was proposed by Iskander Taimanov in [**Ta99**] for immersions of $\mathbb{C}\mathrm{P}^1$ into $\mathbb{R}^3$ with rotationally symmetric potential. This leads to a large class of examples of soliton spheres in $\mathbb{R}^3$, the Taimanov soliton spheres. The last section of the present chapter is concerned with the possible Willmore energies of soliton spheres in $\mathbb{R}^3$.

14. Plücker Formula and Plücker Estimate

In this section holomorphic curves in the Grassmannian manifold of a quaternionic vector space, their osculating k–planes, their dual curve, as well as their canonical complex structure are introduced. It is shown how the osculating k–planes are related to the Weierstrass flag of the canonical linear system of a holomorphic curve in $\mathbb{HP}^n$. Finally, the quaternionic Plücker formula can be formulated. This formula yields an estimate on the Willmore energy, which is called the Plücker estimate.

14.1. Holomorphic Maps into Grassmannians. Let $G_k(H)$ be the *Grassmannian manifold* of k–dimensional subspaces of a quaternionic vector space H. A map $M \to G_k(H)$ is, as for $\mathbb{HP}^1 = G_1(\mathbb{H}^2)$, identified with a rank k subbundle V of the trivial H–bundle over M. In order to keep the notation simple, the trivial H–bundle is again denoted by H. The endomorphism $\delta := \pi\nabla_{|_V} \in \Omega^1 \operatorname{Hom}(V, H/V)$ is then the *derivative* of the map represented by V, analogous to 7.1. Here ∇ denotes the trivial connection of H and $\pi : H \to H/V$ the canonical projection. A subbundle $V \subset H$ is called a *holomorphic curve in* $G_k(H)$, if V posses a complex structure J such that

$$*\delta = \delta J.$$

As for holomorphic curves in $\mathbb{HP}^1$ (cf., Theorem 8.1) there is a unique holomorphic structure on the dual bundle $V^* = H^*/V^\perp$ of V such that the restrictions of the elements of H^* to V are holomorphic sections of V^* (cf., [**FLPP01**, Theorem 2.3]). The holomorphic vector bundle V^* with this holomorphic structure is called the *canonical holomorphic vector bundle* of V. If V is full, i.e., V does not lie in a linear subspace of H, then H^* can be identified with the base point free linear system obtained from H^* restricting its elements to V. $H^* \subset H^0(V^*)$ is called the *canonical linear system* of the curve V. As for curves in $\mathbb{HP}^1$ (cf., Paragraph 8.5) V can be recovered from H^*, because $V = \ker(\mathrm{ev})^\perp \subset H$, where $\mathrm{ev}\colon H^* \to V^*$, $(p, \beta) \mapsto \beta(p)$ is the *evaluation map.* This is the *Kodaira correspondence* between base point free linear systems H^* of holomorphic vector bundles of rank k and projective equivalence classes of full holomorphic curves in $G_k(H)$ (cf., [**FLPP01**, Paragraph 2.6]).

14.2. Osculating k–Planes. Let H be an $(n+1)$–dimensional base point free linear system of a holomorphic line bundle L^{-1}, $\{0\} \subset H_0 \subset \ldots \subset H_n = H$ the Weierstrass flag, $0 = n_0 < n_1 < \ldots < n_n$ the Weierstrass gap sequence of H (cf., 3.7) and $L \subset H^*$ the corresponding holomorphic curve. For $p \in M$ let

$$\nabla^k L_{|_p} := \operatorname{span}\{\, \nabla_{X_1} \ldots \nabla_{X_{\tilde{k}}} \psi_{|_p} \mid \psi \in \Gamma(L),\ X_1, \ldots, X_{\tilde{k}} \in \Gamma(TM),\ \tilde{k} \le k \,\}.$$

If $\dim \nabla^k L_{|_p} = k+1$ for all $p \in M$, then $\nabla^k L$ is the osculating k–plane of L. The $\nabla^k L$ fail to satisfy the condition on the dimension exactly at the Weierstrass points of the linear system H, because of the following relation to the Weierstrass flag of H.

Lemma. *For all $p \in M$ and all $k = 0, \dots, n$ one has*

$$\nabla^{n_k(p)} L_{|_p} = (H_{n-k-1}(p))^\perp \quad \textit{and} \quad \dim\left(\nabla^{n_k(p)} L_{|_p}\right) = k+1.$$

PROOF. Let $\beta \in H$, $\beta = z^l\varphi + O(l+1)$ at p, as in 2.4, $\psi \in \Gamma(L)$, and $X_1, \dots, X_{\tilde{k}} \in \Gamma(TM)$. Then $\beta(\nabla_{X_1} \dots \nabla_{X_{\tilde{k}}}\psi) = X_1 \dots X_{\tilde{k}}(\beta(\psi))$. Thus $\beta(\nabla_{X_1} \dots \nabla_{X_{\tilde{k}}}\psi)_{|_p} = 0$ for all $\psi \in \Gamma(L)$ and $X_1, \dots, X_{\tilde{k}} \in \Gamma(TM)$ if and only if $l > \tilde{k}$. Hence $\beta \in H_{n-k-1}(p) = \{\, \beta \in H \mid \operatorname{ord}_p \beta > n_k \,\}$ if and only if $\beta \in \left(\nabla^{n_k(p)} L_{|_p}\right)^\perp$. The dimension formula follows from $\dim H_k = k+1$ (cf., 3.7). □

The lemma implies that $\dim \nabla^k L_{|_p} \le k+1$, since $\nabla^k L \subset \nabla^{k+1} L$ and $n_k \ge k$. Thus $\dim \nabla^k L_{|_p} = k+1$ holds at p for all k if and only if p is no Weierstrass point. Hence the Weierstrass flag is smooth away from the Weierstrass points of H. It is not smooth at the Weierstrass points. But the Weierstrass flag is continuous on M, by Lemma A.1. This now implies that the subbundle

$$L_k := \nabla^{n_k} L \subset H$$

of rank $k+1$ is continuous on M and smooth away from the Weierstrass points. L_k is the *osculating k–plane* of L.

14.3. Canonical Complex Structure. Let H be an $(n+1)$–dimensional Weierstrass point free linear system of a holomorphic line bundle L^{-1}. Theorems 4.2 and 4.4 of [**FLPP01**] imply that H posses a unique complex structure S such that the following three conditions are satisfied:

(i) S stabilizes the Weierstrass flag of H, i.e., $SH_k = H_k$.
(ii) If $\nabla S = 2(*Q - *A)$ is the type decomposition of ∇S, as in 2.7, then

$$H_{n-1} \subset \ker Q \qquad \text{and} \qquad \operatorname{im} A \subset H_0.$$

(iii) S induces the given complex structure on $H/H_{n-1} = L^{-1}$.

The endomorphism field S is called the *canonical complex structure* of the linear system H. Lemma 14.2 yields $\nabla\Gamma(H_k) \subset \Omega^1(H_{k+1})$, since H is assumed to be base point free. If $\pi_k \colon H \to H/H_{k-1}$ is the canonical projection, then the derivatives

$$\delta_k := \pi_k \nabla \colon H_k/H_{k-1} \to T^*M \otimes H_{k+1}/H_k$$

satisfy

$$*\delta_k = S\delta_k = \delta_k S,$$

because the flatness of ∇ implies $\delta_k \wedge \delta_{k-1} = 0$ and $*\delta_{n-1} = S\delta_{n-1}$ by (iii). The $\delta_k \colon H_k/H_{k-1} \to KH_{k+1}/H_k$ are complex quaternionic bundle isomorphisms. The dual bundle homomorphism $\delta_k^* \colon L_{n-k-1}/L_{n-k-2} \to L_{n-k}/L_{n-k-1}$ of δ_k is the derivative of L_{n-k-1}. The two homomorphism valued 1–forms $A \in \Gamma(\mathrm{KHom}_-(H, L))$ and $Q \in \Gamma(\bar{\mathrm{K}}\mathrm{Hom}_-(H/L, H))$ such that

$$\nabla S = 2(*Q - *A)$$

are called the *Hopf fields* of S.

14.4. If $L = H_{n-1}^{\perp} \subset H^*$ is the Kodaira corresponding holomorphic curve of H in $G_1(H^*)$, then the dual complex structure $S^* \in \Gamma(\mathrm{End}(H^*))$ of S is called the *canonical complex structure* of L. If a holomorphic curve $L \subset \mathbb{H}^{n+1}$ is given then its mean curvature sphere is away from the isolated Weierstrass points of its canonical linear system defined as before. The canonical complex structure in general does not extend continuously into the Weierstrass points (cf., Appendix A).

The differential of S^* satisfies $\nabla S^* = 2(-*A^* + *Q^*)$. One easily checks that $A^* \in \Gamma(\bar{\mathrm{K}}\mathrm{Hom}_-(H/L, H)$ and $Q^* \in \Gamma(\mathrm{KHom}_-(H, L)$. Thus $-A^*$ is the "Q" and $-Q^*$ the "A" of S^*. In particular S^* satisfies and is uniquely determined by analogous conditions to 14.3 (i)–(iii):

(i) $S^* L_k = L_k$, (ii) $L_{n-1} \subset \ker Q^+$, $\mathrm{im}\, A^+ \subset L$, (iii) $S^*{}_{|_L} = J$,

where $Q^+ = -A^*$ and $A^+ = -Q^*$ are the Hopf fields of S^*. The derivative $\delta = \delta_{n-1}^* \in \Gamma(\mathrm{KHom}_+(L, H/L)$ of L satisfies $*\delta = S^*\delta = \delta S^*$. This implies that L is a complex holomorphic subbundle with respect to the complex holomorphic structure $\bar{\partial}$ of the decomposition (2.7) of the trivial connection of H with respect to S^*.

14.5. Dual Curve. Let $L \subset \mathbb{H}^{n+1}$ be a holomorphic curve and $M_0 \subset M$ such that $M \backslash M_0$ are the Weierstrass points of the canonical linear system $H = (\mathbb{H}^{n+1})^*$. Let S be the canonical complex structure of H on M_0. The equation $*\delta_k = \delta_k S$ implies that each member H_k of the Weierstrass flag is on M_0 a holomorphic curve in $G_{k+1}(H)$ with complex structure induced by S, in particular,

$$L^d := H_0{}_{|_{M_0}}$$

is a holomorphic curve in $G_1(H) \cong \mathbb{HP}^n$. This curve is called the *dual curve* of H or L.

On M_0 let $H_k^* \subset H^*$ be the Weierstrass flag of the canonical linear system $H^* = \mathbb{H}^{n+1} \subset H^0((L^d)^{-1})$. If $\varphi \in \Gamma(L^d)$, then $\varphi \in \Gamma(\nabla^{n-1} L_{|_{M_o}})^{\perp}$, by Lemma 14.2. Thus

$$\nabla_{X_1} \dots \nabla_{X_k} \varphi(\nabla_{X_{k+1}} \dots \nabla_{X_{n-1}} \psi) = 0,$$

for all $\psi \in \Gamma(L_{|_{M_0}})$ and $X_1, \dots, X_{n-1} \in \Gamma(TM_{|_{M_0}})$. This equation implies $\nabla^k L^d{}_{|_p} = (\nabla^{n-k-1} L)^{\perp}{}_{|_p}$ for all $p \in M_0$. Hence Lemma 14.2, applied to L and L^d, yields

$$(H_{n-k-1}^*)^{\perp} = \nabla^k L^d = (\nabla^{n-k-1} L_{|_{M_0}})^{\perp} = H_k{}_{|_{M_0}}.$$

This implies that the canonical linear system H^* of L^d is Weierstrass point free, $(L^d)^d = L_{|_{M_0}}$ and S^* is the canonical complex structure of H^*.

14.6. Mean Curvature Sphere of a Holomorphic Curve in $\mathbb{HP}^1$. Let $H \subset H^0(L^{-1})$ be the 2–dimensional Weierstrass point free canonical linear system of an immersed holomorphic curve L in $\mathbb{HP}^1$. Then

$$L^d = L^{\perp},$$

because $L^d = H_0 = H_{n-1} = L^{\perp}$ for $n = 1$.

If $S \in \Gamma(\mathrm{End}(\mathbb{H}^2))$ is the canonical complex structure of L, then the S_p invariant lines in $\mathbb{H}^2$ form a 2–sphere (cf., 5.5). This is the *mean curvature sphere* or *conformal Gauss map* of the immersion $M \to S^4$ represented by

L at p (cf., [**BFLPP02**, Section 5]). The canonical complex structure of an immersed holomorphic curve in $\mathbb{HP}^1$ is, therefore, also called the *mean curvature sphere (congruence)* of the holomorphic curve L. Let $\delta\colon L \to T^*M \otimes \mathbb{H}^2/L$ be the derivative of L, and Q and A the Hopf fields of S. From 14.4 then follows that the mean curvature sphere S of L satisfies (and is uniquely determined by)

$$SL = L, \quad *\delta = S\delta = \delta S, \quad L \subset \ker Q.$$

Note that, given the first equation the second implies $\nabla S(L) \subset \Omega^1(L)$. Given the first two equations, $L \subset \ker Q$ is equivalent to $\operatorname{Im} A \subset L$, because $\nabla S = 2(*Q - *A)$. The condition $SL = L$ means that L_p lies on the 2–sphere S_p. The second condition, $*\delta = S\delta = \delta S$, implies that the tangent space of L at $p \in M$ and S_p at the point L_p coincide, since $T_{L_p}S = \operatorname{Hom}_+(L_p, \mathbb{H}^2/L_p)$ (cf., 5.5). The last condition, $L \subset \ker Q$, ensures that the mean curvature vectors of L at p and S_p at L_p coincide for one and, consequently, every stereographic projection onto $\mathbb{H}$ (cf,. [**BFLPP02**, Remark 9, p. 44]).

For later applications a description of S, Q and A in Euclidean terms will be useful (cf., [**BFLPP02**, Propositions 12 & 15]). Let $\alpha, \beta \in (\mathbb{H}^2)^*$ be an admissible basis and $a, b \in \mathbb{H}^2$ its dual basis. In the frame (a, β^{-1}) one has

$$S = \begin{pmatrix} N & 0 \\ -H & -R \end{pmatrix}, \quad 4{*}Q = \begin{pmatrix} 2dN'' & 0 \\ \omega - 2dH & 0 \end{pmatrix}, \quad 4{*}A = \begin{pmatrix} 0 & 0 \\ \omega & 2dR'' \end{pmatrix},$$

$$4d{*}Q = 4d{*}A = \begin{pmatrix} 0 & 0 \\ \omega & 0 \end{pmatrix}, \qquad \begin{aligned} \omega &= dH + R{*}dH + HNdN'', \\ \bar{H} &= N\mathcal{H} = \mathcal{H}R, \end{aligned}$$

where β^{-1} is the affine lift of L with respect to β, N and R are the left and right normal vectors and $\mathcal{H}$ is the mean curvature vector of $\sigma_{\alpha,\beta}L$. Furthermore, $Hdf = dR'$ and $dfH = dN'$. Geometrically H is (up to quaternionic conjugation) the rotation of $\mathcal{H}$ by $\frac{\pi}{2}$ in the normal bundle of $s_{\alpha,\beta}L$. Note that, if $s_{\alpha,\beta}L$ takes values in $\operatorname{Im}\mathbb{H}$, then H is real valued and $-H$ equals the mean curvature of $s_{\alpha,\beta}L$.

14.7. Plücker Formula. Let $H \subset H^0(L^{-1})$ be an $(n+1)$–dimensional linear system of a holomorphic line bundle L^{-1} over a compact Riemann surface M of genus g. Let $L^d \subset H$ be the dual curve of H on M_0, where $M \setminus M_0$ are the Weierstrass points of H. The Willmore energies of L^{-1} and $(L^d)^{-1}$ are then related by the *Plücker formula* (cf., [**FLPP01**, Theorem 4.7]):

$$\tfrac{1}{4\pi}\Big(W(L^{-1}) - W((L^d)^{-1})\Big) = (n+1)(n(1-g) - \deg L^{-1}) + \operatorname{ord} H.$$

In particular, the Willmore energy $W((L^d)^{-1})$ is finite, although $(L^d)^{-1}$ is only defined on M_0. If S is the canonical complex structure of H, and Q and A are its Hopf fields, then

$$W(L^{-1}) = 2\int_M \langle Q \wedge *Q\rangle \quad \text{and} \quad W((L^d)^{-1}) = 2\int_M \langle A \wedge *A\rangle.$$

14.8. Plücker Estimate. Because $W((L^d)^{-1}) \geq 0$, the Plücker formula yields an estimate for the Willmore energy of L^{-1}, the *Plücker estimate*

$$\tfrac{1}{4\pi}W(L^{-1}) \geq (n+1)(n(1-g) - \deg L^{-1}) + \operatorname{ord} H.$$

14.9. Equality. The main theme of the rest of this thesis is the investigation of linear systems with equality in the Plücker estimate as well as the branched conformal immersions obtained from such systems via Kodaira correspondence and Weierstrass representation. A linear system has equality in the Plücker estimate if and only if the Willmore energy of $(L^d)^{-1}$ vanishes, i.e., the canonical holomorphic line bundle of the dual curve of the linear system is a (doubled) complex holomorphic line bundle.

15. The Plücker Formula for 1– and 2–Dimensional Linear Systems

Before we start investigating linear systems with equality in the Plücker estimate, some immediate consequences of the Plücker formula for curves in $\mathbb{HP}^1$ are derived in this section. First the Plücker formula is applied to the canonical linear system of a holomorphic curve in $\mathbb{HP}^1$. This leads to a generalization of the formula $W(L^{-1}) - W(\mathbb{H}^2/L) = 8\pi(1-g-\deg L^{-1})$ from 8.6, which relates the Willmore energies of the two Möbius invariant holomorphic line bundles of an immersed holomorphic curve in $\mathbb{HP}^1$. Applying the Plücker formula to the 1–dimensional linear subsystem of the canonical linear system associated to a point ∞ in $\mathbb{HP}^1$ yields a generalization of the formula $W(L^{-1}) = W(L) - 4\pi \deg L^{-1}$ from 9.4, which relates the Willmore energies of the Möbius invariant and Euclidean holomorphic line bundle of an immersed holomorphic curve. The generalized formula implies that Euclidean minimal curves in $\mathbb{HP}^1$ are those holomorphic curves whose canonical linear system contains a 1–dimensional linear subsystem with equality in the Plücker estimate. Finally, the formulas are combined and a formula for the total curvature of the (possibly non admissible) stereographic projections of a (possibly branched) holomorphic curve in $\mathbb{HP}^1$ is derived.

15.1. Application of the Plücker Formula to the Canonical Linear System of a Holomorphic Curve in $\mathbb{HP}^1$. If M is compact and H is the 2–dimensional canonical linear system of a holomorphic curve L in $\mathbb{HP}^1$, then the Plücker formula yields

$$W(L^{-1}) - W(\mathbb{H}^2/L) = 4\pi(2 - 2g - 2\deg(L^{-1}) + b(L)),$$

because $\operatorname{ord} H = b(L)$, by Theorem 10.2. For immersed holomorphic curves, i.e., $b(L) = 0$, this formula was already derived in 8.6.

15.2. Application of the Plücker Formula to 1–Dimensional Linear Systems. Let $H = \beta\mathbb{H} \subset H^0(L^{-1})$ be the 1–dimensional linear system of a holomorphic line bundle L^{-1} spanned by some $\beta \in H^0(L^{-1})$. The set M_0 of Weierstrass points of H is then the set of zeros of $\beta\mathbb{H}$. If ∇ is the connection on $L^{-1}{}_{|M_0}$ that makes β parallel, then the canonical holomorphic line bundle $(L^d)^{-1}$ of the dual curve L^d of H is isomorphic to $(L_{|M_0}, (\nabla^*)'')$: $(L^d)^{-1}$ is by definition the trivial bundle $H^{-1} \times M_0$ with the complex structure $J\beta^* = -\beta^* N$, where N is the normal vector of $\beta_{|M_0}$ and $\beta^* \in \Gamma(H^{-1} \times M_0)$ is determined by $\beta^*(\beta) = 1$, and the holomorphic structure that makes β^* holomorphic. The isomorphism between $(L_{|M_0}, (\nabla^*)'')$ and $(L^d)^{-1}$ is then provided by the dual of the evaluation map $\mathrm{ev}_{|M_o} \colon H \times M_0 \to L_{|M_0}$. The Plücker formula applied to $\beta\mathbb{H}$ then reads

$$W(L^{-1}) - W(L_{|M_0}, (\nabla^*)'') = -4\pi \deg(L^{-1}) + 4\pi \sum_{p\in M} \operatorname{ord}_p \beta.$$

15.3. Let H be the canonical linear system of a holomorphic curve $L \subset \mathbb{H}^2$. Let $\infty \in \mathbb{HP}^1$ and

$$H_\infty = \{\, \beta_{|L} \mid \beta \in (\mathbb{H}^2)^*, \ker \beta = \infty \,\} = \infty^\perp$$

be the 1–dimensional linear subsystem of H associated to the point ∞. The Weierstrass points $p \in M$ of H_∞ are the points at which $L_p = \infty$. Hence $M_0 = M \setminus \{\, p \in M \mid L_p = \infty \,\}$.

If $L_{|M_0}$ is equipped with the Euclidean holomorphic structure with respect to ∞, then $W(L_{|M_0}) = W(L_{|M_0}, (\nabla^*)'') = W((L^d)^{-1})$. Because H is base point free, one gets $\operatorname{ord}_p H_\infty = \operatorname{ord}_p H + 1$ if $L_p = \infty$. But $\operatorname{ord}_p H = b_p(L)$, by Theorem 10.2, hence $\operatorname{ord} H_\infty = \sum_{p\in M\setminus M_0}(b_p(L) + 1)$. The Plücker formula then reads

$$W(L^{-1}) - W(L) = -4\pi \deg(L^{-1}) + 4\pi \sum_{p\in M,\, L_p=\infty} (b_p(L) + 1).$$

For immersed L and admissible ∞ this formula was already derived in 9.4.

15.4. Equality and Euclidean Minimal Curves. Let $f \colon M_0 \to \mathbb{H}$ be a stereographic projection of L with respect to ∞, and let $\tilde{f} \colon M \to \mathbb{H}$ be a stereographic projection of L with respect to an admissible point $\tilde{\infty}$. Let D^∞ and $D^{\tilde{\infty}}$ be the Euclidean holomorphic structures of L with respect to ∞ and $\tilde{\infty}$. Then $W(f) = W(L, D^\infty)$, $W(\tilde{f}) = W(L, D^{\tilde{\infty}})$ (cf., 9.4), and $b_p(\tilde{f}) = b_p(L)$ for all $p \in M$. Applying the formula from 15.3 to ∞ and $\tilde{\infty}$, one obtains

$$W(\tilde{f}) - W(f) = 4\pi \sum_{p\in M,\, L_p=\infty} (b_p(\tilde{f}) + 1).$$

As $W(f) \geq 0$, this implies $W(\tilde{f}) \geq 4\pi \sum_{p\in M,\, L_p=\infty}(b_p(\tilde{f}) + 1)$, and equality if and only if f is a minimal surface. For branched immersions into $\mathbb{R}^3$ this estimate can be found in [**Ku89**, Proposition 1.3]. For immersions into $\mathbb{R}^4$ it is contained in [**LiYa82**, Theorem 6].

A holomorphic curve L in $\mathbb{HP}^1$ is called a *Euclidean minimal curve* if L can be stereographically projected onto a branched minimal immersion in $\mathbb{R}^4$. The pole of the stereographic projection that projects a Euclidean minimal curve onto a branched minimal immersion in $\mathbb{R}^4$ is called the *pole* of the Euclidean minimal curve.

Proposition. *A compact holomorphic curve L in $\mathbb{HP}^1$ is a Euclidean minimal curve with pole $\infty \in \mathbb{HP}^1$ if and only if $H_\infty \subset H^0(L^{-1})$ has equality in the Plücker estimate.*

Proof. The Plücker formula for H_∞ reads

$$W(L^{-1}) - W(L) = -4\pi \deg(L^{-1}) + 4\pi \operatorname{ord} H_\infty$$

(cf., 15.3), where $W(L)$ is the Willmore energy of the Euclidean holomorphic structure on L with respect to ∞. Hence equality in the Plücker estimate $W(L^{-1}) \geq -4\pi \deg(L^{-1}) + 4\pi \operatorname{ord} H_\infty$ is equivalent to $W(L) = 0$. But $W(L)$ is the Willmore energy of the stereographic projection of L with pole ∞ (cf., 9.4). □

If S is the mean curvature sphere of L, then the Euclidean formulas of 14.6 for its Hopf fields Q and A imply that $\operatorname{im} Q \subset \infty \subset \ker A$. So that the pole of a Euclidean minimal curve is unique, if S is not constant (or otherwise said, if the corresponding minimal surface in $\mathbb{R}^4$ is not contained in a plane). If, on the other hand, there is a point $\infty \in \mathbb{HP}^1$ such that $\operatorname{im} Q \subset \infty$, $Q \not\equiv 0$ or $\infty \subset \ker A$, $A \not\equiv 0$, then L is a Euclidean holomorphic curve with pole ∞, because all mean curvature spheres of L then pass through ∞.

Note that a compact Euclidean holomorphic curve L obviously projects onto a complete minimal surface in $\mathbb{R}^4$. Furthermore, one can show analogous to [**Br84**, §4], see [**Mu90**, Corollary 6.2 and Proposition 6.5], that all ends are planar, if L is immersed at the end, or of Enneper type, if L is branched at the end. Theorem 15.5 below implies that the total curvature of the minimal surface is finite. So compact Euclidean minimal holomorphic curves correspond to complete minimal surfaces in $\mathbb{R}^4$ with finite total curvature and planar or Enneper type ends.

15.5. Total Curvature. Combining the formulas from 15.1 and 15.3 one can derive a formula for the total curvature of the stereographic projections of a compact holomorphic curve in $\mathbb{HP}^1$.

Theorem. *Let L be a compact holomorphic curve in $\mathbb{HP}^1$, $\infty \in \mathbb{HP}^1$ some point and $\sigma\colon \mathbb{HP}^1 \setminus \{\infty\} \to \mathbb{H}$ a stereographic projection with pole ∞. Then the total curvature of $f := \sigma(L_{|\{p \in M | L_p \neq \infty\}})$, satisfies*

$$\frac{1}{2\pi}\int_M K dA = 2(1-g) + b(f) - \sum_{p\in M,\, L_p = \infty} (b_p(L) + 2).$$

For branched immersions into $\mathbb{R}^3$ that extend smoothly ($C^{1,\alpha}$) to a compact surface in $S^3 = \mathbb{R}^3 \cup \{\infty\}$ this formula can also be found in [**Ku89**, Lemma 1.2].

Proof. Subtracting twice 15.3 from the formula 15.1 yields

$$2W(L) - W(L^{-1}) - W(\mathbb{H}^2/L) = 8\pi(1-g) + 4\pi b(L) - 8\pi \sum_{p\in M,\, L_p=\infty} (b_p(L)+1).$$

The formulas $W(L^{-1}) + W(\mathbb{H}^2/L) = 2\int(|\mathcal{H}|^2 - K) dA$ (8.6) and the formula $W(L) = \int |\mathcal{H}|^2 dA$ (9.4) imply that the left hand side is twice the total curvature of f. Finally, $b(L) = b(f) + \sum_{p\in M,\, L_p=\infty} b_p(L)$. □

16. Twistor Projection and Equality

If for a linear system equality holds in the Plücker estimate, then the Plücker formula implies that the canonical holomorphic line bundle of the dual curve has zero Willmore energy (cf., 14.9). This observation is now used to show that all $(n+1)$–dimensional linear systems with equality are obtained from complex holomorphic curves in $\mathbb{C}\mathrm{P}^{2n+1}$ via twistor projection and dualization.

16.1. Twistor Projection and Twistor Lift. Let H be a quaternionic vector space, and write $(H, \mathbb{i})$ for the complex vector space obtained from H via multiplication by $\mathbb{i}$. Let $G_k^*(H,\mathbb{i}) \subset G_k(H,\mathbb{i})$ be the open set of the Grassmannian manifold of k–dimensional subspaces $W \subset H$ that satisfy $W \cap W\mathbb{j} = \{0\}$. Then

$$\begin{array}{rcl} T\colon G_k^*(H,\mathbb{i}) & \longrightarrow & G_k(H) \\ W & \longmapsto & W \oplus W\mathbb{j} \end{array}$$

is called the *twistor projection.* Note that $G_1^*(H,\mathbb{i}) = G_1(H,\mathbb{i}) = P(H,\mathbb{i}) \cong \mathbb{C}\mathrm{P}^{2\dim H - 1}$ and in the case where $H = \mathbb{H}^2$ one gets $T\colon\ \mathbb{C}\mathrm{P}^3 \to \mathbb{H}\mathrm{P}^1 = S^4$.

If $V \subset H$ is a holomorphic curve in $G_k(H)$, then the underlying complex line bundle $\hat{V} = \{\, \psi \in V \mid J\psi = \psi\mathbb{i}\,\}$ is a lift of V to $G_k^*(H)$, because

$$T(\hat{V}) = \hat{V} \oplus \hat{V}\mathbb{j} = V.$$

The complex curve $\hat{V}$ in $G_k^*(H,\mathbb{i})$ is called the *twistor lift* of the holomorphic curve V in $G_k(H)$. If $H = \mathbb{H}^2$ and $k = 1$, then this is the usual twistor lift (cf., [**Fr84**, p. 259]), because $\hat{V}$ uniquely determines the complex structure J and vice versa, and J is the rotation by $\frac{\pi}{2}$ in the positive (negative) direction in $\top L$ ($\perp L$), by Theorem 7.6.

The twistor projection $T(W)$ of a smooth curve W in $G_k^*(H,\mathbb{i})$ is a holomorphic curve in $G_k(H)$ if and only if W has vertical $\bar{\partial}$ derivative (cf., [**FLPP01**, Lemma 2.7]). The complex structure of $T(W) = W \oplus W\mathbb{j}$ is then the quaternionic linear extension of $x \mapsto x\mathbb{i}$ for $x \in W$, and the twistor lift of $T(W)$ is W again.

16.2. Holomorphic Twistor Lift. Let $E \subset \mathbb{C}^{n+1}$ be a complex holomorphic curve in $\mathbb{C}\mathrm{P}^n$, and let $\nabla^k E$ be defined as $\nabla^k L$ in 14.2. In the complex case $\nabla^k E$ always extends holomorphically into the isolated points at which its rank is not $k+1$ (cf., [**GriHa**, 2.4]). The extended curve E_k is the *osculating k–plane* of E. If the canonical holomorphic line bundle of a holomorphic curve in $\mathbb{H}\mathrm{P}^n$ has zero Willmore energy and the canonical linear system is Weierstrass point free, then the curve and its osculating planes are the twistor projection of a complex holomorphic curve and its osculating planes in $\mathbb{C}\mathrm{P}^{2n+1}$:

Lemma. *Let H be an $(n+1)$–dimensional quaternionic vector space, L a holomorphic curve in $P(H)$ whose canonical linear system is Weierstrass point free. Let S be the canonical complex structure of L with Hopf fields Q and A. Then the following three statements are equivalent:*

(i) The twistor lift $\hat{L}$ of L is holomorphic in $P(H,\mathbb{i})$.
(ii) $A \equiv 0$.

(iii) L^{-1} has zero Willmore energy.
In this situation one has for all $k = 0, \dots, n$:

$$\nabla^k \hat{L} = \widehat{\nabla^k L} := \{\, \psi \in \nabla^k L \mid S\psi = \psi \mathbb{i} \,\}.$$

Contrary to the convention 4.4, holomorphic twistor lift always means *complex* holomorphic twistor lift.

Proof. Using the formula $D\pi = \frac{1}{2}(\pi\nabla + *J\pi\nabla)$ from 8.3 for the holomorphic structure of L^{-1} and the decomposition $\nabla = \partial + A + \bar{\partial} + Q$ of the trivial connection ∇ of H with respect to S, as in 2.7, one sees that $-A^*$ induces the Hopf field of $L^{-1} = H^*/L^{\perp}$. Thus (ii) implies (iii).

If the Hopf field of L^{-1} vanishes, then $A_{|_L} \equiv 0$. Let $\varphi \in \Gamma(\hat{L})$. Then the 1–form $\nabla\varphi + *\nabla\varphi\mathbb{i} = \nabla\varphi + *\nabla(S\varphi) = 2\bar{\partial}\varphi + 2A\varphi = 2\bar{\partial}\varphi$ takes values in $\hat{L}$, because $\bar{\partial}$ commutes with S and $\bar{\partial}L$ takes values in L (cf., 14.4). Hence (iii) implies (i).

Suppose now that $\hat{L}$ is holomorphic and $\varphi \in \Gamma(\hat{L})$, then

$$\Omega^1(\hat{L}) \ni \nabla\varphi + *\nabla\varphi\mathbb{i} = \nabla\varphi + *\nabla(S\varphi) = 2\bar{\partial}\varphi + 2A\varphi.$$

Hence $A\varphi$ is contained in the $\mathbb{i}$–eigenspace of S, because $\hat{L}$ and $\bar{\partial}\varphi$ lie in that space. But, $SA\varphi = -AS\varphi = -A\varphi\mathbb{i}$, hence $A\varphi = 0$. Thus $L = \hat{L} + \hat{L}\mathbb{j} \subset \ker A$. From $L \subset L_{n-1} \subset \ker Q$ (cf., 14.4) now follows $\nabla S_{|_L} = 2(*Q - *A)_{|_L} = 0$, and, consequently, $\nabla^1 \hat{L} = \widehat{\nabla^1 L}$. Because the osculating k–planes of the complex holomorphic curve $\hat{L}$ are again holomorphic, one can proceed inductively to show that $\nabla^k \hat{L} = \widehat{\nabla^k L}$ and $\widehat{\nabla^k L} \subset \ker A$ for all $k = 0, \dots, n$. Since the canonical linear system of L is assumed to be Weierstrass point free, Lemma 14.2 implies $H = \nabla^n L = \widehat{\nabla^n L} \oplus \widehat{\nabla^n L}\mathbb{j}$. Thus $\widehat{\nabla^n L} \subset \ker A$ implies $A \equiv 0$. Therefore (i) implies (ii). □

16.3. Holomorphic Twistor Lift and Equality. The characterization of holomorphic curves with holomorphic twistor lift in Lemma 16.2 yields the following characterization of linear systems with equality in the Plücker estimate.

Theorem. *If M is compact then equality in the Plücker estimate holds for a linear system if and only if the dual curve of the linear system extends to a holomorphic curve on M with holomorphic twistor lift.*

Proof. Let $H \subset H^0(L^{-1})$ be a linear system of holomorphic sections of a holomorphic line bundle L^{-1}. The canonical linear system of the dual curve $L^d = H_{0|_{M_0}} \subset H$ of H is then Weierstrass point free (cf., 14.5). So the Plücker formula and Lemma 16.2 imply that H has equality in the Plücker estimate if and only if L^d has holomorphic twistor lift. The twistor lift $\widehat{L^d}$ of L^d extends continuously to M, by Lemma A.1. Thus $\widehat{L^d}$ extends to a complex holomorphic curve on M. The twistor projection of this curve is, consequently, a holomorphic curve on all of M that coincides with L^d away from the Weierstrass points of H. □

In the situation of the theorem Lemma 14.2 implies that the osculating k–planes E_k of the holomorphic twistor lift E of L^d are, away from the

Weierstrass points, the holomorphic twistor lifts of the members H_k of the Weierstrass flag of the linear system H with equality. If the quaternionic dimension of their twistor projection does not collapse at the Weierstrass points, which is equivalent to $E_n \oplus E_n \mathbb{j} = H$, then the continuity of the Weierstrass flag H_k (cf., Lemma A.1) implies that the H_k are holomorphic curves on all of M with holomorphic twistor lifts E_k. Clearly $E_n \oplus E_n \mathbb{j} = H$ holds if and only if the canonical complex structure S of L^d extends continuously (and then smoothly) into the Weierstrass points.

16.4. Let H be an $(n+1)$–dimensional quaternionic vector space and E a compact complex holomorphic curve in $P(H, \mathbb{i})$. If the dual curve $L := T(E)^d$ of E's twistor projection extends to a full holomorphic curve on all of M, then the canonical linear system $H \subset H^0(L^{-1})$ of L is a base point free linear system with equality in the Plücker estimate, by Theorem 16.3.

$T(E)^d$ extends smoothly to a full holomorphic curve if the osculating n–plane E_n of E satisfies $E_n \oplus E_n \mathbb{j} = H$: Let $L^d := T(E) = E \oplus E\mathbb{j}$. Then $L^d \subset H$ is a holomorphic curve with holomorphic twistor lift E. Its canonical complex structure S satisfies $Sx = x\mathbb{i}$ for all $x \in E_n$, by Lemma 16.2. Thus, S extends smoothly onto all of M, since E_n is smooth on M and $E_n \oplus E_n \mathbb{j} = H$, by assumption. From $E_n \oplus E_n \mathbb{j} = H$ also follows that $H_{n-1} = T(E_{n-1}) = E_{n-1} \oplus E_{n-1}\mathbb{j} \subset H$ is a smooth quaternionic vector subbundle of rank n. Hence $H_{n-1}^{\perp} \subset H^*$ defines a holomorphic curve in $P(H^*)$ that extends $T(E)^d$ onto all of M.

Moreover, if the canonical complex structure S of $L^d = T(E)$ extends smoothly into the Weierstrass points of L^d, then $E_n \oplus E_n \mathbb{j} = H$ follows, because E_n is the $\mathbb{i}$–eigenspace of S. Thus S extends smoothly into the Weierstrass points of L^d if and only if $E_n \oplus E_n \mathbb{j} = H$.

Remark. It is not clear to the author whether there exists a linear system with equality in the Plücker estimate whose canonical complex structure does not extend continuously into the Weierstrass points. See Lemma 20.2 for a first result in this direction.

16.5. More Details. A thorough investigation of the Weierstrass gap sequence yields a more detailed correspondence (cf., [**FLPP01**, Paragraph 4.4]) between

(i) $(n+1)$–dimensional base point free linear systems of holomorphic line bundles of degree d over compact M of genus g with equality in the Plücker estimate and Weierstrass gap sequence $(n_k)_{0 \le k \le n}$ whose canonical complex structure extends smoothly into the Weierstrass points, and

(ii) compact complex holomorphic curves of genus g in $\mathbb{C}\mathrm{P}^{2n+1}$ of degree $\sum_{p \in M}(n_n(p) - n) - d - 2n(g-1)$ whose Weierstrass gap sequence starts with $(n_n - n_{n-k})_{0 \le k \le n}$ together with a quaternionic structure on $\mathbb{C}^{2n+2}$ such that the quaternionic span of the osculating n–plane of the curve is $\mathbb{C}^{2n+2}$.

17. Three Equality Preserving Operations

In this section three operations that preserve equality in the Plücker estimate are discussed. The first is about holomorphic bundle homomorphisms, the second is along vertical arrows in the ladder of holomorphic line bundles (11.7), and the third is concerned with paired holomorphic line bundles.

17.1. Proposition. *If $B\colon \tilde{L} \to L$ is a nontrivial holomorphic bundle homomorphism, $\tilde{H} \subset H^0(\tilde{L})$ a linear system and $H := B\tilde{H} \subset H^0(L)$ its image under B, then $W(\tilde{L}) = W(L)$, $\dim \tilde{H} = \dim H$, and for the Weierstrass gap sequences of $\tilde{H}$ and H at a point $p \in M$ one obtains $n_k(p) = \tilde{n}_k(p) + \operatorname{ord}_p B$. If M is compact, then $\deg L = \deg \tilde{L} + \operatorname{ord} B$ and $\tilde{H}$ has equality in the Plücker estimate if and only if the linear system H has equality.*

Proof. As B is not identically zero, its zeros are isolated and $\dim \tilde{H} = \dim H$. Because $B\tilde{Q} = QB$, one has $W(\tilde{L}) = W(L)$. Let $\hat{B}\colon \hat{\tilde{L}} \to \hat{L}$ be the induced complex holomorphic bundle homomorphism. On compact M one then gets $\deg L = \deg \hat{L} = \deg \hat{\tilde{L}} + \operatorname{ord} \hat{B} = \deg \tilde{L} + \operatorname{ord} B$.

Let $\tilde{n}_k(p)$ be the Weierstrass gap sequence of $\tilde{H}$ at $p \in M$ and $\tilde{\psi}_k$ a basis of $\tilde{H}$ such that $\operatorname{ord}_p \tilde{\psi}_k = \tilde{n}_k(p)$. Then $\psi_k := B\tilde{\psi}_k$ is a basis of H and $\operatorname{ord}_p \psi_k = \operatorname{ord}_p \tilde{\psi}_k + \operatorname{ord}_p B$. Thus $n_k(p) = \tilde{n}_k(p) + \operatorname{ord}_p B$ is the Weierstrass gap sequence of H. All together one gets for the right hand side of the Plücker estimate:

$$\begin{aligned}
&(\tilde{n}+1)(\tilde{n}(1-g) - \deg \tilde{L}) + \operatorname{ord} \tilde{H} \\
&\qquad = (n+1)(n(1-g) - \deg L + \operatorname{ord} B) + \operatorname{ord} H - (n+1) \operatorname{ord} B \\
&\qquad = (n+1)(n(1-g) - \deg L) + \operatorname{ord} H.
\end{aligned}$$

□

17.2. Equality along Vertical Arrows. The operation, described in 11.4, to transport holomorphic structures and linear system along vertical arrows in the ladder of holomorphic line bundles preserves equality in the Plücker estimate. Recall that this relation was obtained from a flat connection ∇ on a holomorphic line bundle L whose $(0,1)$ part ∇'' is the holomorphic structure of L. The connection ∇ then induces a quaternionic linear map from $H^0(L)$ to $H^0(KL, d^\nabla)$. If ψ is a nowhere vanishing holomorphic section of L, then the $(0,1)$ part ∇'' of the unique connection ∇ on L that makes ψ parallel is the holomorphic structure of L.

Theorem. *Let L be a holomorphic line bundle, $H \subset H^0(L)$ a linear system with a nowhere vanishing holomorphic section $\psi_0 \in H$, ∇ the trivial connection for which ψ_0 is parallel and suppose that KL is equipped with the holomorphic structure d^∇.*

The linear system $\nabla H \subset H^0(KL)$ then satisfies $\dim \nabla H = \dim H - 1$ and the Weierstrass gap sequences satisfy $n_k^{\nabla H}(p) = n_{k+1}^H(p) - 1$ for all $p \in M$ and $k = 0, \dots \dim H - 2$. If M is compact, then $W(KL) = W(L) + 4\pi \deg L$ and the linear system $H \subset H^0(L)$ has equality in the Plücker estimate if and only if $\nabla H = \{\, \nabla\psi \mid \psi \in H \,\} \subset H^0(L)$ has equality.

Recall that a linear system has a nowhere vanishing holomorphic section if and only if it is base points free, by Theorem 3.8.

Proof. The kernel of the linear map $H \to \nabla H$, $\varphi \mapsto \nabla\varphi$ is $\psi_0\mathbb{H}$, thus $\dim \nabla H = \dim H - 1$. On compact M one has $W(KL) = W(L) + 4\pi \deg L$, by Theorem 11.4. To prove the equality of the Weierstrass orders let $\psi \in H$ with $k := \operatorname{ord}_p \psi > 0$ for $p \in M$, then $\psi = z^k\varphi + O(k+1)$, as in 2.4. Let $\nabla = \hat{\nabla} + \omega$ be the decomposition of ∇ into its J–commuting and J–anticommuting part, then $\omega \in \Omega^1(\operatorname{End}_- L)$. Hence

$$\nabla\psi = kz^{k-1}dz\varphi + z^k\hat{\nabla}\varphi + \bar{z}^k\omega\varphi + O(k) = kz^{k-1}dz\varphi + O(k).$$

This means $\operatorname{ord}_p(\nabla\psi) = \operatorname{ord}_p(\psi) - 1$. If $\psi_0, \dots, \psi_n \in H$ is a basis of H such that $(\operatorname{ord}_p \psi_j)_{0\le j\le n}$ is the Weierstrass gap sequence of H at p, then $(\operatorname{ord}_p \nabla\psi_j)_{1\le j\le n}$ is the gap sequence of ∇H at p. Consequently, $\operatorname{ord} \nabla H = \operatorname{ord} H$. Finally, if $n+1 = \dim H$, then

$$\begin{aligned}
&\frac{1}{4\pi}W(L) = (n+1)(n(1-g) - \deg L) + \operatorname{ord} H \\
&\iff \frac{1}{4\pi}W(KL) = (n+1)n(1-g) - n\deg L + \operatorname{ord}\nabla H \\
&\iff \frac{1}{4\pi}W(KL) = (n+1)n(1-g) - n\deg KL - 2n(1-g) + \operatorname{ord}\nabla H \\
&\iff \frac{1}{4\pi}W(KL) = n((n-1)(1-g) - \deg KL) + \operatorname{ord}\nabla H.
\end{aligned}$$

Thus equality for H is equivalent to equality for ∇H. $\square$

17.3. Equality and Paired Holomorphic Line Bundles. Let H be a base point free linear system of a holomorphic line bundle L^{-1}, and suppose that the canonical complex structure S of H extends smoothly into the Weierstrass points of H. If Q is the Hopf field of S, then $Q^*b \in \Gamma(KL)$ for all $b \in H^*$. If H has equality in the Plücker estimate, then $*Q^*b$ is in fact holomorphic, see the proof of the following theorem. Moreover, the holomorphic sections of the form $*Q^*b$ constitute a linear system with equality in $H^0(KL)$, and the branched conformal immersions that have Weierstrass data in H and $*Q^*(H^*)$ are obtained from S without integration.

Theorem. *Let L^{-1} be a holomorphic line bundle over compact M with nonzero Willmore energy. If $H \subset H^0(L^{-1})$ is a base point free linear system with equality in the Plücker estimate whose canonical complex structure S extends smoothly into the Weierstrass points of H and $Q \in \Gamma(\bar{K}\operatorname{End}_-(H))$ is its Hopf field, then $*Q^*(H^*) \subset H^0(KL)$ is a linear system with equality in the Plücker estimate. Moreover, if $b \in H^*$ and $\beta \in H$, then the $\mathbb{H}$–valued 1–form $*Q^*b(\beta)$ is exact, indeed*

$$S^*b(\beta)\colon M \to \mathbb{H}$$

is a branched conformal immersion with Weierstrass data $(KL, L^{-1}, *Q^*b, \beta)$.

Proof. Equality for H and smoothness of S implies that the Weierstrass flag $(H_k)_{0\le k\le n}$ of H is smooth on M (cf., 16.3). Let $L \subset H^*$ be the Kodaira corresponding curve of H. Then S^* induces the complex structure of L, $\operatorname{im} Q^* \subset L$, and $*Q^* = S^*Q^*$ (cf., 14.4). Thus $Q^*b \in \Gamma(KL)$ for all $b \in H^*$.

Equality for H is equivalent to $A \equiv 0$, by Lemma 16.2 and Theorem 16.3. Hence $\nabla S = 2{*}Q$, which yields $d^\nabla {*}Q = 0$. Thus for every $b \in H^*$ and $\beta \in H$ one has $d({*}Q^* b(\beta)) = 0$, which implies

$$*Q^* b \in H^0(KL),$$

by Theorem 9.6.

Lemma 16.2 implies that Q is not identically zero, since $-Q^*$ is the "A" of S^* and the Willmore energy of L^{-1} is not zero, by assumption. Lemma 23.2 below then implies that there exists a holomorphic curve $\tilde{L} \subset H$ on M with complex structure $-S_{|\operatorname{im} Q}$ such that $\operatorname{im} Q = \tilde{L}$ away from the isolated zeros of Q. Since $\tilde{L}^{-1} = H/\tilde{L}^\perp$ and $\tilde{L}^\perp \subset \ker Q^*$, Q^* induces a quaternionic bundle homomorphism $Q^*\colon \tilde{L}^{-1} \to KL$. It is complex linear, because $S^*Q^* = -Q^*S^*$, and the complex quaternionic bundle homomorphism

$$*Q^*\colon \tilde{L}^{-1} \to KL$$

is holomorphic, by Lemma 4.1, since ${*}Q^* b \in H^0(KL)$ for all $b \in H^* \subset H^0(\tilde{L}^{-1})$. The smooth map $S^* b(\beta)\colon M \to \mathbb{H}$ satisfies

$$d(S^* b(\beta)) = 2{*}Q^* b(\beta).$$

Thus $S^* b(\beta)\colon M \to \mathbb{H}$ is a branched conformal immersion with Weierstrass data $(KL, L^{-1}, {*}Q^* b, \beta)$, by Theorem 10.9.

It remains to show that the linear system ${*}Q^*(H^*) \subset H^0(KL)$ has equality in the Plücker estimate. To this end we show that $-S$ is the canonical complex structure of $\tilde{L}$. S clearly stabilizes $\tilde{L}$, because $\tilde{L} = \operatorname{im} Q$ away from the isolated zeros of Q and S anticommutes with Q. Let $\psi \in \Gamma(H)$ and $X, Y \in \Gamma(TM)$. Then

$$S\nabla_X(Q_Y\psi) = \nabla_X S(Q_Y\psi) - \nabla_X(SQ_Y\psi) = 2{*}Q_X Q_Y \psi - \nabla_X(SQ_Y\psi).$$

Hence $\operatorname{im}(S\nabla_X(Q_Y\psi)) \subset \Gamma(\nabla^1(\operatorname{im} Q))$. This implies $S\nabla^1\tilde{L} = \nabla^1\tilde{L}$. Proceeding inductively one concludes that S stabilizes $\nabla^k\tilde{L}$ for all $k = 0, \dots, n$. Hence $-S$ satisfies condition 14.4 (i). Let δ be the derivative of $\tilde{L}$, then $d^\nabla {*}Q = 0$ implies $\delta \wedge {*}Q = 0$, thus ${*}\delta = -\delta S$, which is condition 14.4 (iii). Finally $-A$ is the "Q" and $-Q$ is the "A" of $-S$, thus $A \equiv 0$ and $\operatorname{im} Q \subset \tilde{L}$ implies that $-S$ satisfies condition 14.4 (ii).

Consider now the linear system $\tilde{H}^* := \{\, b_{|\tilde{L}} \mid b \in H^* \,\} \subset H^0(\tilde{L}^{-1})$. If $\tilde{L}$ is full, then $H^* = \tilde{H}^*$ and $-S^*$ is the canonical complex structure of $\tilde{H}^*$, because $-S$ is the canonical complex structure of $\tilde{L}$. If $\tilde{L}$ is not full, then there is a subspace $\tilde{H}$ of H such that $\tilde{L} \subset \tilde{H}$ is full, and the linear system $\tilde{H}^*$ is the canonical linear system of $\tilde{L} \subset \tilde{H}$ and the canonical complex structure of $\tilde{H}^*$ is again induced by $-S^*$. As $A^* \equiv 0$ and $-A^*$ is the "A" of $-S^*$, $\tilde{H}^*$ has equality in the Plücker estimate, by Lemma 16.2 and Theorem 16.3. The holomorphicity of ${*}Q^*$ and Proposition 17.1 implies that the linear system ${*}Q^*(\tilde{H}^*) \subset H^0(KL)$ has equality. This completes the proof, because ${*}Q^*(\tilde{H}^*) = {*}Q^*(H^*)$. □

Note that $\dim {*}Q^*(H^*) \leq \dim H$ and equality holds if and only if $\operatorname{im} Q$ does not lie in some linear subspace of H.

18. Soliton Spheres

A holomorphic curve in $\mathbb{HP}^1$ over $M = \mathbb{CP}^1$ is called a *soliton sphere* if its canonical linear system is contained in a linear system with equality. A branched conformal immersion of $\mathbb{CP}^1$ with values in $\mathbb{H}$ is called a *soliton sphere* if it is the stereographic projection of a soliton sphere in $\mathbb{HP}^1$.

18.1. Weierstrass Representation of Soliton Spheres. Iskander Taimanov calls in [**Ta99**] a conformal immersion of $\mathbb{CP}^1$ into $\mathbb{R}^3$ with rotationally symmetric Hopf field a soliton sphere if its Weierstrass data lies in a linear system with equality (cf., Section 19). The following theorem shows that this is equivalent to the definition in Möbius invariant terms given above.

Theorem. *If $f\colon \mathbb{CP}^1 \to \mathbb{H}$ is a branched conformal immersion with a smooth right normal vector and Weierstrass data (L, φ, ψ), as in 10.7, then φ is contained in a linear system with equality if and only if f is a soliton sphere.*

Proof. Proposition 10.5 implies that there is a holomorphic curve $L \subset \mathbb{H}^2$ and a stereographic projection $s_{\alpha,\beta}\colon \mathbb{HP}^1 \backslash \{\infty\} \to \mathbb{H}$ such that

$$f = s_{\alpha,\beta} L.$$

The canonical linear system $H \subset H^0(L^{-1})$ of L is spanned by $\alpha_{|_L}$ and $\beta_{|_L}$ (cf., 8.5) and $\alpha_{|_L} = \beta_{|_L}\bar{f}$. Let ∇ be the trivial quaternionic connection on L^{-1} such that $\nabla\beta_{|_L} = 0$. Then $(L, (\nabla^*)'')$ is the Euclidean holomorphic line bundle of L with respect to $\infty = \ker\beta$. The uniqueness of the Weierstrass representation (cf., 10.7) and $\nabla\alpha(\beta^{-1}) = df = (\varphi, \psi)$ imply that $\varphi, \nabla\alpha \in H^0(KL^{-1}, d^\nabla)$ are linearly dependent over the reals. For holomorphic line bundles over $\mathbb{CP}^1$ the quaternionic linear map $\nabla\colon H^0(L^{-1}) \to H^0(KL^{-1})$ is surjective, because the sphere is simply connected. Theorem 17.2 then implies that the canonical linear system, which is spanned by α and β, is contained in a linear system with equality if and only if $\nabla\alpha$ lies in a linear system with equality. $\square$

18.2. Remark. Let $f = \sigma_{\alpha,\beta} L\colon \mathbb{CP}^1 \to \mathbb{H}$ be a soliton sphere, H the linear system with equality that contains the canonical linear system of L, and ∇H the corresponding linear system in KL^{-1} as in the proof above. In addition to ∇H having equality in the Plücker estimate, Theorem 17.2 implies that $\dim H = \dim \nabla H + 1$, and if $0 = n_0 \leq n_1 \leq \ldots \leq n_n$ is the Weierstrass gap sequence of H then $n_1 - 1 \leq \ldots \leq n_n - 1$ is the one of ∇H.

18.3. Rational Parametrizations. Let $z\colon \mathbb{CP}^1 \backslash \{\infty\} \to \mathbb{C}$ be a rational coordinate of $\mathbb{CP}^1$. Then every holomorphic line subbundle $E \subset \mathbb{C}^n$ over $\mathbb{CP}^1$ has a meromorphic section that has a pole at ∞ and is holomorphic elsewhere. This section provides homogeneous coordinates of E consisting of n polynomials in z. If $f : \mathbb{CP}^1 \to \mathbb{H}$ is a soliton sphere, then f is by definition the quotient of two sections in a linear system with equality. Theorem 16.3 then implies that f is rational in z, because twistor projection and dualization only involves differentiation and algebraic manipulations of the polynomials that describe the dual curve. So soliton spheres are given by conformal and rational parametrizations.

19. Taimanov Soliton Spheres

Iskander Taimanov provides in [**Ta99**] explicit formulas for all soliton spheres in $\mathbb{R}^3$ with rotationally symmetric potentials. These formulas were used by the author to provide pictures (see 19.6) of the simplest nontrivial examples. The similarity of these pictures with pictures of the catenoid cousins led to the investigations on the relation between soliton spheres and catenoid cousins. The result of these investigations is Theorem 30.3. The present section is a collection of results from [**Ta99**] in terms of quaternionic holomorphic spin bundles and equality in the Plücker estimate.

19.1. Spin Bundles over $\mathbb{C}\mathrm{P}^1$ in Rational Coordinates. Let L be a quaternionic holomorphic spin bundle over $M = \mathbb{C}\mathrm{P}^1$ (cf., 12.3). Then $\deg L = \frac{1}{2}\deg K = -1$. This implies that the underlying complex holomorphic line bundle $\hat{L}$ of L is isomorphic to the tautological bundle $\Sigma_{\mathbb{C}\mathrm{P}^1}$ of $\mathbb{C}\mathrm{P}^1$, because there is exactly one complex holomorphic line bundle over $\mathbb{C}\mathrm{P}^1$ for every degree. Consequently, L is the doubled complex holomorphic line bundle $\Sigma_{\mathbb{C}\mathrm{P}^1} \oplus \Sigma_{\mathbb{C}\mathrm{P}^1}\mathbb{j}$ plus a Hopf field. Fix some point $\infty \in \mathbb{C}\mathrm{P}^1$. Then there exists a meromorphic section ψ_∞ in $\hat{L}$ with divisor $-\infty$ and

$$d(\psi_\infty,\psi_\infty) = (\bar\partial\psi_\infty \wedge \psi_\infty) + (\psi_\infty \wedge \bar\partial\psi_\infty) = 0$$

on $\mathbb{C}\mathrm{P}^1 \setminus \{\infty\}$, by Corollary 9.9. Thus $(\psi_\infty,\psi_\infty)$ is closed, and $*(\psi_\infty,\psi_\infty) = -\mathbb{i}(\psi_\infty,\psi_\infty) = (\psi_\infty,\psi_\infty)\mathbb{i}$ implies that there exists a holomorphic map

$$z\colon\ \mathbb{C}\mathrm{P}^1 \setminus \{\infty\} \to \mathbb{C}, \quad dz = \mathbb{j}(\psi_\infty,\psi_\infty).$$

This is, up to $z \mapsto az + b$, the *stereographic projection* or *rational coordinate* of $S^2 \cong \mathbb{C}\mathrm{P}^1$ with pole ∞, since z is holomorphic with a simple pole at ∞.

19.2. Lemma. *Let L be a spin bundle over $\mathbb{C}\mathrm{P}^1$, $z\colon \mathbb{C}\mathrm{P}^1 \setminus \{\infty\} \to \mathbb{C}$ a rational coordinate and ψ_∞ a meromorphic section of $\hat{L}$ such that $dz = \mathbb{j}(\psi_\infty,\psi_\infty)$.*

(i) If Q is the Hopf field of L, then there is a smooth map $q\colon \mathbb{C} \to \mathbb{R}$ such that

$$Q\psi_\infty = \psi_\infty \mathbb{k} dz\, q(z)$$

and $\frac{1}{|w|^2} q\left(\frac{1}{w}\right)$ extends smoothly into zero. On the other hand, every such q determines a Hopf field Q on $L = \Sigma_{\mathbb{C}\mathrm{P}^1} \oplus \Sigma_{\mathbb{C}\mathrm{P}^1}\mathbb{j}$ such that L with $D = \bar\partial + Q$ is a quaternionic holomorphic spin bundle.

(ii) If $\psi \in H^0(L)$ and $\mu_{1,2}\colon \mathbb{C} \to \mathbb{C}$ such that $\psi = \psi_\infty(\mu_1(z) + \mathbb{k}\mu_2(z))$ on $\mathbb{C}\mathrm{P}^1 \setminus \{\infty\}$, then the maps $\mu_{1,2}$ satisfy

$$\begin{pmatrix} q & \partial \\ -\bar\partial & q \end{pmatrix}\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} = 0, \quad |\mu_1|^2 + |\mu_2|^2 = O(|z|^{-2}) \text{ as } |z| \to \infty.$$

Every pair $\mu_{1,2}\colon \mathbb{C} \to \mathbb{C}$ that satisfies this equation gives rise to a holomorphic section of L.

(iii) The Willmore energy of L satisfies

$$W(L) = 4\int_{\mathbb{C}} q^2(x + \mathbb{i}y)dx \wedge dy.$$

PROOF. (i) Identify $\mathbb{C}\mathrm{P}^1 \setminus \{\infty\}$ and $\mathbb{C}$ via z. As $Q \in \Gamma(\bar{K}\,\mathrm{End}_-\, L)$ one knows $Q\psi_\infty \in \Gamma(\bar{K}\hat{L}\mathbb{j}_{|\mathbb{C}})$. Hence there exists a smooth map $q\colon \mathbb{C} \to \mathbb{C}$ such that $Q\psi_\infty = \psi_\infty \mathbb{k} dz q$, because $\psi_\infty \mathbb{k} dz$ is a nowhere vanishing section of $\bar{K}\hat{L}\mathbb{j}_{|\mathbb{C}}$. Then

$$(*) \qquad 0 = (Q\psi_\infty \wedge \psi_\infty) + (\psi_\infty \wedge Q\psi_\infty) = (q - \bar{q})\mathbb{i} dz \wedge d\bar{z},$$

by Corollary 9.9. This implies that q is real valued. Let $p_0 \in \mathbb{C}\mathrm{P}^1$ such that $z(p_0) = 0$ and $w\colon \mathbb{C}\mathrm{P}^1 \setminus \{p_0\} \to \mathbb{C}$, $w = \frac{1}{z}$. Then $\tilde{\psi}_\infty := \psi_\infty w \mathbb{i}$ is a meromorphic section of $\hat{L}$ such that $dw = \mathbb{j}(\tilde{\psi}_\infty, \tilde{\psi}_\infty)$ and

$$(**) \qquad Q\tilde{\psi}_\infty = \psi_\infty \mathbb{k} dz q(z) w \mathbb{i} = \tilde{\psi}_\infty \mathbb{k} dw \tfrac{1}{|w|^2} q\left(\tfrac{1}{w}\right).$$

Because $w(\infty) = 0$ and $\tilde{\psi}_\infty \mathbb{k} dw$ is not zero at ∞ it follows that $\frac{1}{|w|^2} q\left(\frac{1}{w}\right)$ is smooth at $w = 0$.

If, on the other hand, $q\colon \mathbb{C} \to \mathbb{R}$ is a smooth map such that $\frac{1}{|w|^2} q\left(\frac{1}{w}\right)$ is smooth at $w = 0$, then the equations $Q\psi_\infty = \psi_\infty \mathbb{k} dz q$ and $(**)$ define a smooth Q on the complex quaternionic line bundle $L = \Sigma_{\mathbb{C}\mathrm{P}^1} \oplus \Sigma_{\mathbb{C}\mathrm{P}^1}\mathbb{j}$. The complex holomorphic line bundle $\Sigma_{\mathbb{C}\mathrm{P}^1} \otimes \Sigma_{\mathbb{C}\mathrm{P}^1}$ is isomorphic to the canonical bundle $K = \{\, \omega \in T^*\,\mathbb{C}\mathrm{P}^1 \otimes \mathbb{C} \mid *\omega = \mathbb{i}\omega \,\}$ of $\mathbb{C}\mathrm{P}^1$. The bundle homomorphism $\Sigma_{\mathbb{C}\mathrm{P}^1} \times \Sigma_{\mathbb{C}\mathrm{P}^1} \to \mathbb{H} \otimes_{\mathbb{C}} K \subset T^*M \otimes \mathbb{H}$, $(\varphi, \psi) \mapsto \mathbb{j}\varphi \otimes \psi$ can be extended to define a complex quaternionic pairing of L with itself. This pairing satisfies $d(\varphi, \psi) = (\bar{\partial}\varphi, \psi) + (\varphi, \bar{\partial}\psi)$ for all $\varphi, \psi \in \Gamma(L)$, because the holomorphic structure of $K \subset T^*\,\mathbb{C}\mathrm{P}^1 \otimes \mathbb{C}$ is d. The complex quaternionic line bundle L with the holomorphic structure $D = \bar{\partial} + Q$ is a quaternionic spin bundle, because the pairing defines a holomorphic pairing of (L, D) with itself, by Corollary 9.9 and equation $(*)$.

(ii) $D\psi_\infty = Q\psi_\infty$, as ψ_∞ is a complex holomorphic section of $\hat{L}_{|\mathbb{C}}$. Then

$$D(\psi_\infty(\mu_1 + \mathbb{k}\mu_2)) = \psi_\infty d\bar{z}(-q\mu_2 + \bar{\partial}\mu_1) + \psi_\infty \mathbb{k} dz(q\mu_1 + \partial\mu_2).$$

Thus it remains to show that $|\mu_1|^2 + |\mu_2|^2 = O\left(\frac{1}{|z|^2}\right)$ as $|z| \to \infty$ is equivalent to the smoothness of ψ at ∞. In the coordinate w one obtains $\psi = \psi_\infty\left(\mu_1\left(\frac{1}{w}\right) + \mathbb{k}\mu_2\left(\frac{1}{w}\right)\right)$. Because ψ_∞ has a pole at $w = 0$, smoothness of ψ at $w = 0$ implies $\mu_{1,2}\left(\frac{1}{w}\right) = O(|w|)$ at $w = 0$, which implies $|\mu_1|^2 + |\mu_2|^2 = O\left(\frac{1}{|z|^2}\right)$ as $z \to \infty$. If now μ_1 and μ_2 satisfy this condition, then $\psi = \psi_\infty(\mu_1 + \mathbb{k}\mu_2)$ is bounded at p_0 and holomorphic on $\mathbb{C}\mathrm{P}^1 \setminus \{p_0\}$. Hence ψ extends smoothly into p_0.

(iii) The equation for the Willmore energy follows since

$$Q \wedge *Q\psi_\infty = \psi_\infty \mathbb{k} dz q \wedge \mathbb{k}\mathbb{i} dz q = -\psi_\infty |q|^2 \mathbb{i} d\bar{z} \wedge dz.$$

□

19.3. Explicit Formulas for Taimanov Soliton Spheres. Iskander Taimanov investigates in [**Ta99**] spin bundles of $\mathbb{C}\mathrm{P}^1$ with *rotationally symmetric Hopf fields*, i.e., Hopf fields that are represented in some rational coordinate $z\colon \mathbb{C}\mathrm{P}^1 \setminus \{\infty\} \to \mathbb{C}$ by a q (cf., Lemma 19.2) that satisfies $q(z) = q(|z|)$. Taimanov translates the problem of finding the holomorphic sections of such a spin bundle to the "simplest reduction of the Zakharov–Shabat linear problem". In the case of reflectionless potentials he describes the space of all holomorphic sections explicitly. In the language of the present text this means:

Theorem. *Let L be a spin bundle over $\mathbb{CP}^1$ with rotationally symmetric Hopf field. Then there is a linear system $H \subset H^0(L)$ with equality in the Plücker estimate if and only if there are integers $0 \leq n_0 < n_1 \ldots < n_n$ and real numbers $\lambda_0, \ldots, \lambda_n \in \mathbb{R}$ such that*

$$q(z) = -\langle \Phi(|z|), T(|z|)\rangle |z|^{-1},$$

where $\langle\, , \rangle$ is the standard scalar product of $\mathbb{R}^{n+1}$,

$$T(x) = \left(x^{-\frac{1}{2}(2n_0+1)}, \ldots, x^{-\frac{1}{2}(2n_n+1)}\right),$$
$$\Psi(x) = -\left(\lambda_0 x^{-\frac{1}{2}(2n_0+1)}, \ldots, \lambda_n x^{-\frac{1}{2}(2n_n+1)}\right),$$
$$\Phi(x) = \Psi(x)(1 + M^2(x))^{-1},$$

and $M(x)$ is the $(n+1) \times (n+1)$ matrix valued function with coefficients

$$M_{jk}(x) = \frac{-\lambda_k x^{-(n_j+n_k+1)}}{n_j + n_k + 1}$$

for $0 \leq j, k \leq n$.

Furthermore, $H = H^0(L)$, $\dim H^0(L) = n+1$, and $H^0(L)$ is spanned by the holomorphic sections $\psi_j = \psi_\infty(\mu_{1j}(z) + \Bbbk\mu_{2j}(z))$, where

$$\mu_{1j}(z) = \langle \Phi(|z|), W_j(|z|)\rangle \left(\frac{z}{|z|}\right)^{n_j} |z|^{-\frac{1}{2}},$$
$$\mu_{2j}(z) = (|z|^{-\frac{1}{2}(2n_j+1)} - \langle \Phi(|z|)M(|z|), W_j(|z|)\rangle) \left(\frac{z}{|z|}\right)^{n_j+1} |z|^{-\frac{1}{2}},$$
$$W_j(x) = \left(\frac{x^{-(n_0+n_j+1)}}{n_0 + n_j + 1}, \ldots, \frac{x^{-(n_n+n_j+1)}}{n_n + n_j + 1}\right).$$

The linear system $H^0(L)$ has Weierstrass points at $z = 0$ and $z = \infty$, if any, with Weierstrass gap sequence $(n_j)_{0 \leq j \leq n}$ at both points. The Willmore energy of L is

$$W(L) = 4\pi \sum_{j=0}^{n} (2n_j + 1).$$

Integrating the closed one forms $\left(\sum_{j=0}^n \psi_j c_j, \sum_{j=0}^n \psi_j c_j\right)$ yields for all $a_j \in \mathbb{H}$ soliton spheres, by Theorem 18.1, because $H^0(L) = \operatorname{span}\{\psi_j\}_{0 \leq j \leq n}$ has equality in the Plücker estimate. The soliton spheres $f_j \colon \mathbb{CP}^1 \to \mathbb{R}^3$ that satisfy

$$\begin{aligned} df_j &= (\psi_j, \psi_j) \\ &= -\operatorname{Re}(2\mu_1\bar{\mu}_2 dz)\mathbb{i} + \operatorname{Re}((\mu_1^2 - \bar{\mu}_2^2)dz))\mathbb{j} + \operatorname{Re}(\mathbb{i}(\mu_1^2 + \bar{\mu}_2^2)dz))\Bbbk \end{aligned}$$

are surfaces of revolution on the cylinder $\mathbb{C}/(2\pi(2n_j+1)\mathbb{i}\mathbb{Z})$, see 19.6.

Collection of Results from [**Ta99**]. If L has a rotationally symmetric Hopf field, then Lemma 19.4 implies that $H = H^0(L)$. [**Ta99**, Lemma 4] together with Lemma 19.2(ii) implies that there are integers

$0 \leq n_0 < n_1 < \ldots < n_n$ and some smooth functions[1] $\varphi_j \colon \mathbb{R}_{>0} \to \mathbb{C}^2 \backslash \{0\}$ for $0 \leq j \leq n$ such that the $(n+1)$–dimensional space of holomorphic sections of L is spanned by the holomorphic sections $\psi_j = \psi_\infty(\mu_{1j}(z) + \Bbbk\mu_{2j}(z))$, where

$$\begin{pmatrix}\mu_{1j}(z)\\ \mu_{2j}(z)\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & \frac{z}{|z|}\end{pmatrix}\varphi_j(|z|)\left(\frac{z}{|z|}\right)^{n_j}|z|^{-\frac{1}{2}}.$$

From [**Ta99**, equations 45, 47 and 53] follows that the φ_j have the asymptotics

$$\varphi_j(x) = O(x^{-\frac{1}{2}(2n_j+1)}), \quad x \to \infty,$$
$$\varphi_j(x) = O(x^{\frac{1}{2}(2n_j+1)}), \quad x \to 0,$$

which implies

$$|\mu_{1j}(x)|^2 + |\mu_{2j}(x)|^2 = O(x^{-2n_j-2}), \quad x \to \infty,$$
$$|\mu_{1j}(x)|^2 + |\mu_{2j}(x)|^2 = O(x^{2n_j}), \quad x \to 0.$$

Because ψ_∞ has divisor $-\infty$, one concludes from the asymptotic behavior of the μ's that

$$\mathrm{ord}_0\,\psi_j = \mathrm{ord}_\infty\,\psi_j = n_j.$$

Thus n_j is the Weierstrass gap sequence of $H^0(L)$ at 0 and ∞. The Plücker estimate for $M = \mathbb{C}\mathrm{P}^1$ then implies

$$\frac{1}{4\pi}W(L) \geq (n+1)^2 + 2\sum_{j=0}^{n}(n_j - j) = \sum_{j=0}^{n}(2n_j+1),$$

as $g = 0$ and $\deg L = -1$. [**Ta99**, equation (25)] implies for $\tilde{q}(t) = q(e^t)e^t$:

$$\int_{-\infty}^{\infty}\tilde{q}^2(t)dt \geq \frac{1}{2}\sum_{j=0}^{n}(2n_j+1),$$

and equality in this equation is equivalent to equality in the Plücker estimate for the linear system $H^0(L)$, because Lemma 19.2(iii) implies

$$\frac{1}{4\pi}W(L) = \frac{1}{\pi}\int_{\mathbb{C}} q^2(x+\mathbb{i}y)dx\wedge dy = 2\int_{-\infty}^{\infty}\tilde{q}^2(t)dt.$$

In this case Iskander Taimanov [**Ta99**, §4] derives the explicit formulas for q and φ_j given in the theorem. □

19.4. Lemma. *Let L be a spin bundle over $\mathbb{C}\mathrm{P}^1$ with rotationally symmetric Hopf field. If $H \subset H^0(L)$ is a linear system with equality in the Plücker estimate, then $H = H^0(L)$.*

PROOF. If $z\colon \mathbb{C}\mathrm{P}^1 \to \mathbb{C}\cup\{\infty\}$ is a rational coordinate in which the Hopf field of L is rotationally symmetric, then linear systems of L only have Weierstrass points at $z = 0$ and $z = \infty$, because Weierstrass points are isolated (cf., 3.7). The Plücker estimate then implies that H already contains all holomorphic sections of L: Otherwise, if $n+1 = \dim H$, then H would be contained in an $(n+2)$–dimensional linear system $\tilde{H}$ of holomorphic

[1] We write $\varphi_j(x)$ for the function $\varphi_1^+(\ln x, \frac{1}{2}(2n_j+1)\mathbb{i})$ from [**Ta99**].

sections. $\tilde{H}$ would have Weierstrass order $\operatorname{ord}\tilde{H} \geq \operatorname{ord} H - 2(n+1)$. The Plücker estimate then yields

$$\begin{aligned}\frac{1}{4\pi}W(L) &\geq (n+2)((n+1)(1-g) - \deg(L) + \operatorname{ord}\tilde{H}\\ &\geq (n+2)^2 + \operatorname{ord} H - 2(n+1) = (n+1)^2 + \operatorname{ord} H + 1 \geq \frac{1}{4\pi}W(L) + 1.\end{aligned}$$

This is a contradiction. □

19.5. Note that the theorem does not answer the question which integers $0 \leq n_0 < n_1 \ldots < n_n$ and real numbers $\lambda_0, \ldots, \lambda_n \in \mathbb{R}$ correspond to quaternionic holomorphic spin bundles over $\mathbb{CP}^1$. This is, by Lemma 19.2, equivalent to the question for which parameters the functions $q(z)$ and $\frac{1}{|w|^2} q\left(\frac{1}{w}\right)$ extend smoothly into zero. The formula for $q(z)$ can be written as follows

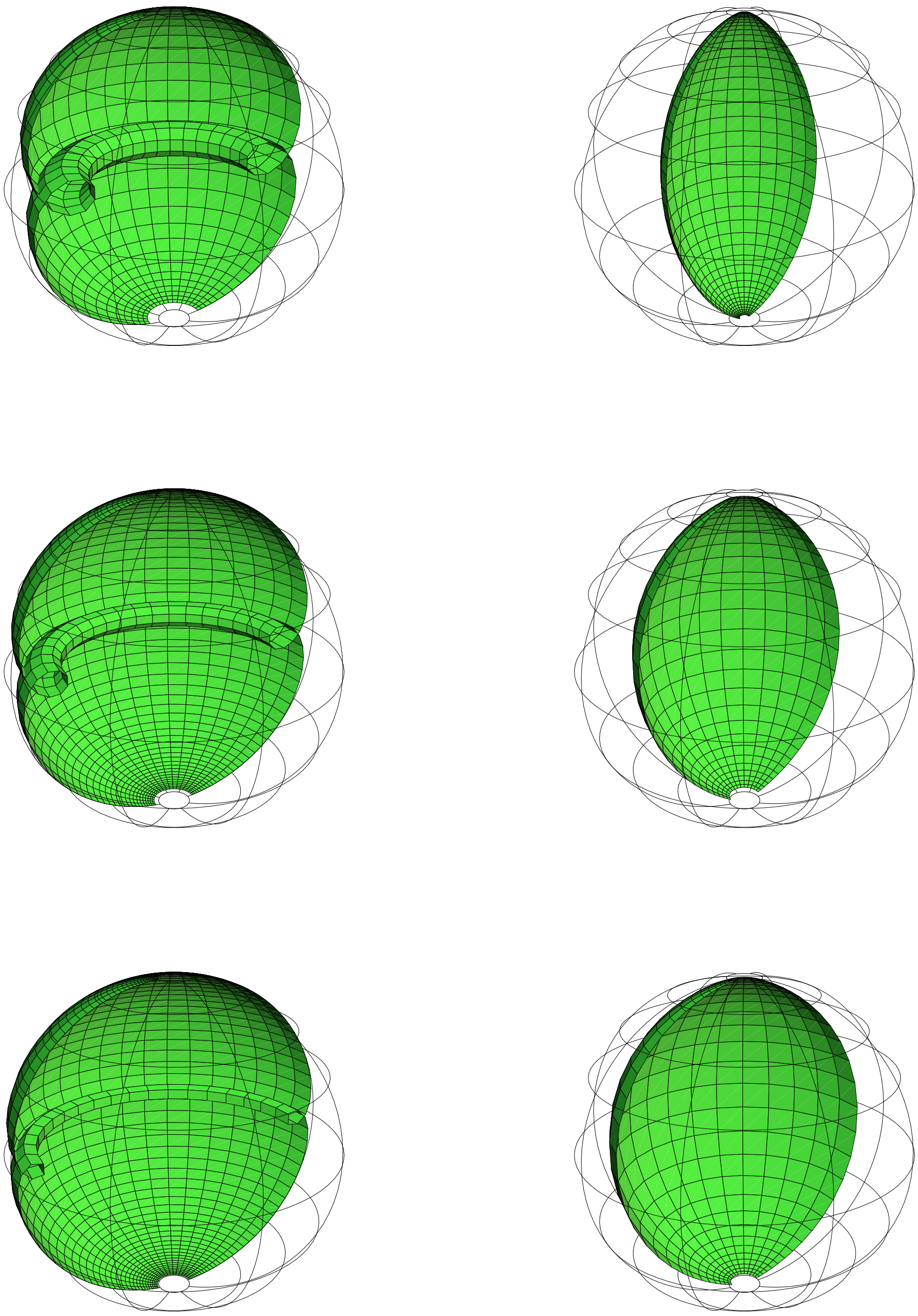
$$q(z) = \frac{1}{\det(A_{jk})} \sum_{0\leq j,k\leq n} A^*_{jk}\lambda_j |z|^{-n_j-n_k-2},$$

where

$$A_{jk} = (1+M^2)_{jk} = \delta_{jk} + |z|^{-n_j-n_k} \sum_{l=0}^{n} \frac{\lambda_l \lambda_k |z|^{-2n_l-2}}{(n_j+n_l+1)(n_l+n_k+1)}$$

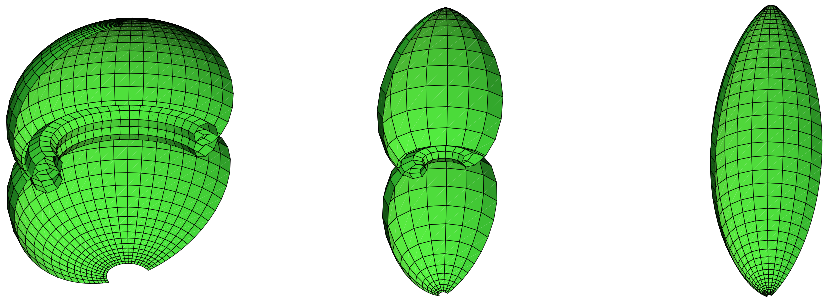
and A^*_{jk} is $(-1)^{j+k}$ times the determinant of the matrix obtained from A_{jk} after canceling the k–th row and the j–th column. One sees that $\det A_{jk}$ as well as $|z|^{-n_j-n_k}A^*_{jk}$ are polynomials in $|z|^{-2}$. Hence q is the quotient of polynomials in $|z|^2$. It, consequently, suffices to distinguish the parameters for which the functions $q(z)$ and $\frac{1}{|w|^2} q\left(\frac{1}{w}\right)$ are both bounded at zero. Using Mathematica the author checked that this is true for small n and n_j $(0 \leq n, n_j \leq 6)$ and arbitrary λ_j. The problem to derive a general answer from the formulas is the fact that the coefficients of the highest and lowest powers coming into the formula for the numerator and denominator of q may cancel. See also the remark in the last paragraph of [**Ta99**] about the general problem of determining the decay of the potential $U(x) = q(e^x)e^x$ for $x \to \pm\infty$ from the spectral data.

19.6. Pictures. The Taimanov soliton spheres for $n = 1$, $n_0 = 0$, $n_1 = \mu$, $\lambda_0 = \frac{\mu+1}{\mu}$, and $\lambda_1 = \frac{(\mu+1)(2\mu+1)}{\mu}$ are catenoid cousins, see 30.4. The first picture in each row is $\int(\psi_0, \psi_0)$ and the second is $\int(\psi_1, \psi_1)$. The rows show the surfaces corresponding to $\mu = 1, 2, 4$.

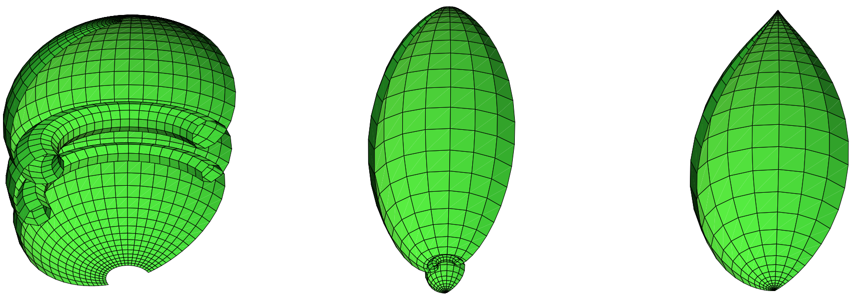


Pictures of the rotationally symmetric Taimanov soliton spheres for $n = 2, 3$:

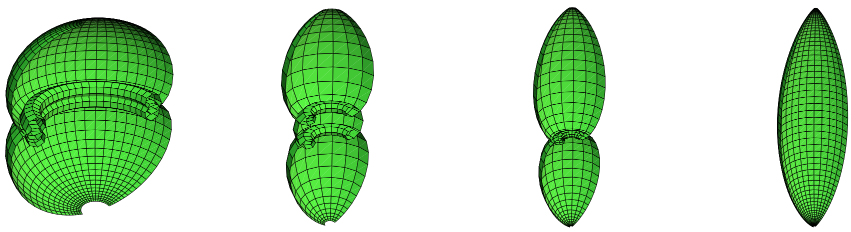
$\int(\psi_0, \psi_0)$, $\int(\psi_1, \psi_1)$, and $\int(\psi_2, \psi_2)$ for $n = 2$, $n_0 = 0$, $n_1 = 1$, $n_2 = 2$, $\lambda_0 = 2$, $\lambda_1 = 6$, and $\lambda_2 = 3$:



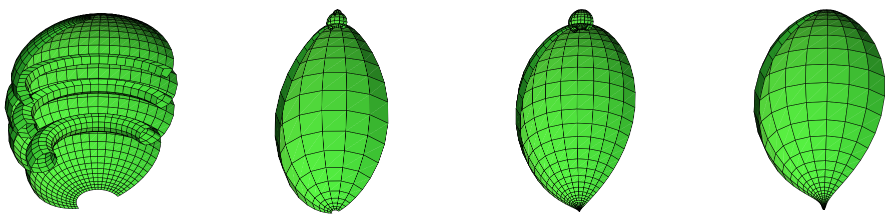
$\int(\psi_0, \psi_0)$, $\int(\psi_1, \psi_1)$, and $\int(\psi_2, \psi_2)$ for $n = 2$, $n_0 = 0$, $n_1 = 1$, $n_2 = 2$, $\lambda_0 = 2$, $\lambda_1 = 6$, and $\lambda_2 = 120$:



$\int(\psi_0, \psi_0)$, $\int(\psi_1, \psi_1)$, $\int(\psi_2, \psi_2)$, and $\int(\psi_3, \psi_3)$ for $n = 3$, $n_0 = 0$, $n_1 = 1$, $n_2 = 2$, $n_3 = 3$, $\lambda_0 = 4$, $\lambda_1 = 48$, $\lambda_2 = 120$, and $\lambda_3 = 120$:

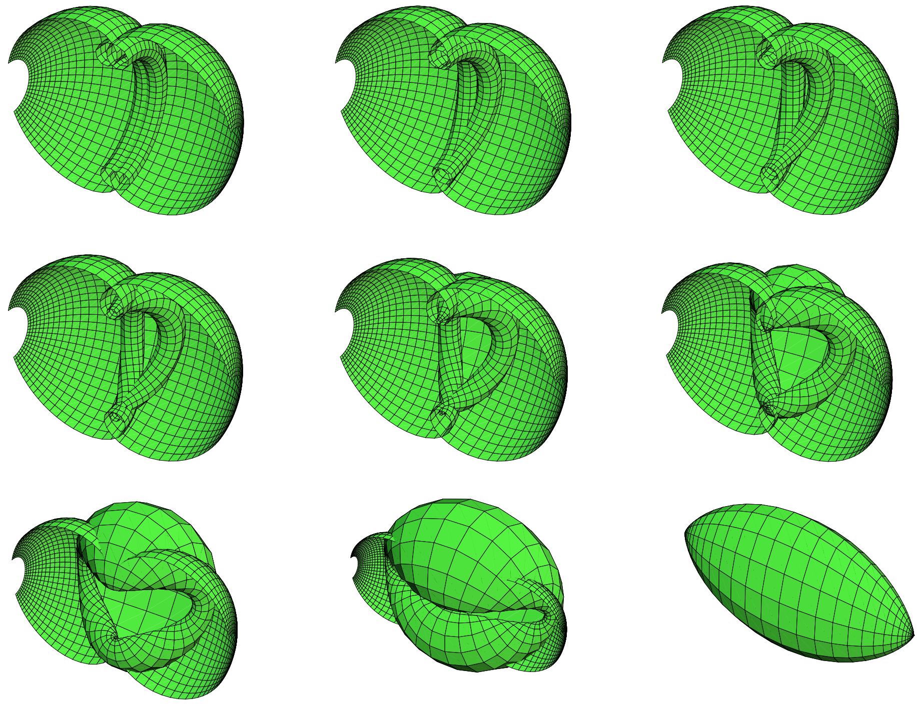


$\int(\psi_0, \psi_0)$, $\int(\psi_1, \psi_1)$, $\int(\psi_2, \psi_2)$, and $\int(\psi_3, \psi_3)$ for $n = 3$, $n_0 = 0$, $n_1 = 1$, $n_2 = 2$, $n_3 = 3$, $\lambda_0 = 6$, $\lambda_1 = 720$, $\lambda_2 = 120$, and $\lambda_3 = 1$:

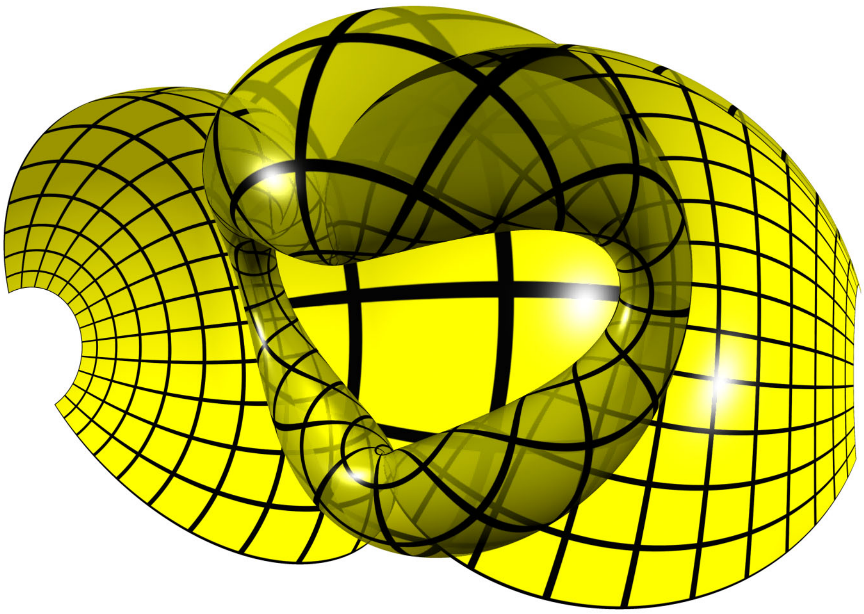


Finally, some pictures of non rotationally symmetric Taimanov soliton spheres. To get such pictures one needs to integrate the square of linear combinations of the ψ_i. This has been done by Jörg Richter [**Ri97**] for the Dirac spheres, which are Taimanov soliton sphere for $n_k = k$, $k = 0, \ldots, n$, and some λ's that only depend on n (cf., [**Ta99**, 4.2.3]).

The following are pictures of Dirac Spheres for $n = 2$:



The surface in the lower left corner:



20. Willmore Numbers of Soliton Spheres in $\mathbb{R}^3$

Equality in the Plücker estimate implies that the Willmore energy is an integer multiple of 4π. The Willmore energy of the immersed Taimanov soliton spheres is of the form $4\pi\sum_{j=0}^{n}(2n_j+1)$ with integers $0 = n_0 < \ldots < n_n$ (cf., Theorem 19.3). Thus, Taimanov soliton spheres have Willmore energy $4\pi n$ for some $n \in \mathbb{N} \setminus \{2,3,5,7\}$. Interestingly, this set coincides with the possible Willmore energies of immersed Willmore spheres in $\mathbb{R}^3$ (cf., [**Br88**]), which are soliton spheres (cf., 25.3). In the present section it is shown that immersed soliton spheres in $\mathbb{R}^3$ with Willmore energy 8π, 12π, and 20π do not exist. The case 28π is left open.

20.1. Let $f\colon \mathbb{CP}^1 \to \operatorname{Im}\mathbb{H}$ be an immersed soliton sphere, and L its Euclidean holomorphic line bundle. Then L is a spin bundle and $\deg L = -1$, by Theorem 12.4. The spinor $\varphi \in H^0(L)$ such that $df = (\varphi, \varphi)$ is then, by Theorem 18.1, contained in a linear system $H \subset H^0(L)$ with equality in the Plücker estimate, i.e.,

$$\frac{1}{4\pi}W(L) = (n+1)^2 + \operatorname{ord} H,$$

where $\dim H = n+1$. The spinor φ has no zeros, because f is immersed, thus H is base point free. If $\frac{1}{4\pi}W(L)$ is 2 or 3, then $n = 1$ and $\operatorname{ord} H > 0$. This is a contradiction to H being base point free.

Theorem. *The Willmore energy of an immersed soliton sphere in $\mathbb{R}^3$ is $4\pi n$ for some $n \in \mathbb{N} \setminus \{2,3,5\}$.*

Proof. It remains to show that the Willmore energy of an immersed soliton sphere in $\mathbb{R}^3$ is not 20π. Let L be the Euclidean holomorphic line bundle of an immersed soliton sphere with Willmore energy 20π. Then $\deg L = -1$, the linear system $H = H^0(L)$ is 2–dimensional, base point free, has equality in the Plücker estimate, and $\operatorname{ord} H = 1$. Let $E \subset (H, \mathbb{i}) \cong \mathbb{C}^4$ be the twistor lift of the dual curve $L^d \subset H$ of the linear system H extended to a compact complex holomorphic curve on $\mathbb{CP}^1$ (cf., Theorem 16.3).

The proof now goes as follows: a) The first step is to show that E has degree 4 and exactly one branch point of order 1, i.e., a cusp. b) The isomorphism of L and KL^{-1} is then used to deduce a symmetry of the branching of E and its dual curve E^d. This symmetry and the complex version of the Plücker formula imply that E can not be full in $\mathbb{CP}^3$. c) This is used to show that L^d is a Euclidean minimal surface with four planar ends. These ends are inflection points of E, and the curve of tangent lines E_1 of E does not have other branch points. Hence E is a rational curve in $\mathbb{CP}^2$ of degree 4 with exactly one cusp and exactly four inflections points, which lie on one projective line. d) In the last step it is shown that such a curve does not exist.

a) The derivative $\delta \in \Gamma(\mathrm{KHom}_+(L^d, H/L^d))$ of L^d is complex holomorphic, by Lemma 20.3, and has exactly one zero of order 1 at the Weierstrass point of H, by Theorem 10.2. Thus

$$\deg E = \deg(L^d) = \deg(H/L^d) + \deg K - \operatorname{ord}\delta = \deg(L) - 3 = -4.$$

(Note that $\deg E$ is the degree of the complex holomorphic line bundle E, which is minus the degree of the holomorphic curve E.)

Because the twistor projection L^d of E has only one branch point of order 1, E has at most one branch point of order 1. If E had no branch point, then the branch point $p_0 \in \mathbb{C}\mathrm{P}^1$ of L^d would come from the twistor projection. That means that the osculating 1–plane $E_1 \subset (H, \mathbb{i})$ of E is a quaternionic subspace of H at p_0. But S is smooth and E_1 is its $\mathbb{i}$–eigenspace near p_0, by Lemma 16.2. Hence E_1 is the $\mathbb{i}$–eigenspace of S at p_0, which can not be a quaternionic subspace of H. Contradiction. The branch point of L^d is, consequently, the only branch point of E.

b) The canonical complex structure S of H extends smoothly into the Weierstrass point of H, by Lemma 20.2. Its Hopf field A vanishes, by Lemma 16.2. Hence $d^\nabla * Q = \frac{1}{2} d^\nabla \nabla S = 0$. Let E^d be the dual curve of E in $P(H, \mathbb{i})$. Then the twistor projection of E^d satisfies $T(E^d) \subset \ker Q^* = (\operatorname{im} Q)^\perp \subset H^*$ and $-S^*$ is the canonical complex structure of $T(E^d)$, by Lemma 20.4. Hence

$$*Q^* \colon H^*/T(E^d) \to KL^{-1}$$

is a complex quaternionic bundle homomorphism. It is holomorphic, by Corollary 9.7, for if $\beta \in H^*$ and $b \in H$ one gets $d(*Q^*\beta(b)) = d^\nabla * Q^* \beta(b) = 0$. Since L is the Euclidean holomorphic line bundle of a conformal immersion into $\mathbb{R}^3$, KL^{-1} is isomorphic to L, by Theorem 12.4. Hence, Q induces a holomorphic bundle homomorphism

$$*Q^* \colon H^*/T(E^d) \to L.$$

The holomorphicity of $*Q^*$ induces a symmetry of the branching of E and E^d, which implies that E lies in a projective plane: The complex version of the Plücker formula reads

$$0 = (n+1)(n(1-g) - d) + \sum_{k=0}^{n-1} (n-k) b_k$$

(cf., [**GriHa**, Section 2.4, p. 271f]) for a compact holomorphic curve of genus g in $\mathbb{C}\mathrm{P}^n$, where b_k is the total branching of the osculating k–plane. If E is full, then this formula yields $0 = 4(3-4) + 3 + 2b_1 + b_2$, since $g = 0$, $n = 3$, $d = 4$ and $b_0 = 1$. Hence $b_1 = 0$ and $b_2 = 1$. As E is a holomorphic curve of degree 4 in $\mathbb{C}\mathrm{P}^3$, it follows that the branch point $p_0 \in \mathbb{C}\mathrm{P}^1$ of E, which is the Weierstrass point of H, and the branch point $p_\infty \in \mathbb{C}\mathrm{P}^1$ of its osculating 2–plane are different points. The point p_∞ is a branch point of E^d. Hence p_∞ is a Weierstrass point of the canonical linear system $H^* \subset H^0(H^*/T(E^d))$ of $T(E^d)$, by Theorem 10.2. But this implies that p_∞ is the Weierstrass point p_0 of H, by Proposition 17.1, since $*Q^* \colon H^*/T(E^d) \to L$ is holomorphic and $H = H^0(L)$ is 2–dimensional. This is a contradiction to $p_0 \neq p_\infty$. Hence E is not full.

c) If E is contained in a projective line, then S is constant and Q vanishes identically, which implies $W(L) = 0 \neq 20\pi$. Thus E lies in a projective plane and Q does not vanish identically. The complex Plücker formula then implies $0 = (n+1)(n-d) + 2b_0 + b_1 = -4 + b_1$. The tangent line congruence E_1 of E thus has branching order $b_1 = 4$.

Lemma 20.5 implies that $T(E) = L^d$ is Euclidean minimal. In particular $\operatorname{im} Q$ is constant. This and Theorem 17.3 implies that $*Q^*(H^*) \subset H = H^0(L)$ is a 1–dimensional linear system with equality in the Plücker estimate. Hence L^d is Euclidean minimal with pole $*Q^*(H^*)$, by Proposition 15.4. Hence $\operatorname{im} Q = *Q^*(H^*)$, by the uniqueness of the pole of a Euclidean minimal curve (cf., 15.4). Hence the zeros of Q are the points at which L^d equals $\operatorname{im} Q$. But the zeros of Q are branch points of E_1, by Lemma 20.4. As $b_1 = 4$, E_1 has no other branch points and its 4 branch points are of order 1, so they are inflection points of E, and they all lie on the complex projective line $\operatorname{im} Q$.

d) Suppose that

$$E = [f]_{\mathbb{C}} \subset \mathbb{C}^3, \quad f = (f_1, f_2, f_3)\colon \mathbb{C} \to \mathbb{C}^3,$$

where the $f_i\colon \mathbb{C} \to \mathbb{C}$ are polynomials in z of degree at most 4. Assume that the cusp of E lies at $z = 0$ and $z = \infty$ is one of the 4 inflection points. Changing the coordinates of $\mathbb{C}^3$ one can assume that the line $g = \operatorname{im} Q$ is given by the vanishing of the third coordinate. Since E has degree 4 the intersection order of E and g is 4, hence E intersects g only at the 4 inflection points and the intersection is transversal. Hence f_3 is a cubic polynomial with nonvanishing constant term. Scaling the coordinate z and the third coordinate one can assume that

$$f_3(z) = z^3 + c_2 z^2 + c_1 z + 1.$$

Changing again the coordinates of $\mathbb{C}^3$ one can assume that the cusp is given by the vanishing of the first two coordinates and the tangent at the cusp by the vanishing of the first coordinate. Scaling the first two coordinates and adding some multiple of the first coordinate to the second, one can assume that

$$\begin{aligned} f_1(z) &= a_4 z^4 - \frac{1}{6} z^3, \\ f_2(z) &= b_4 z^4 + z^2. \end{aligned}$$

The zeros of

$$\begin{aligned} &\det(f, f', f'') z^{-2} \\ &\qquad = (2a_4 - \frac{b_4 c_2}{3}) z^4 - c_1 b_4 z^3 - 2(3c_1 a_4 + b_4) z^2 + (\frac{c_1}{3} - 16 a_4) z + 1 \end{aligned}$$

are the inflection points of E. These zeros coincide with the zeros of f_3, since the inflection points of E lie on g. Hence $f_3 = \det(f, f', f'') z^{-2}$, which is a contradiction.

□

20.2. Lemma. *The canonical complex structure of a 2–dimensional base point free linear system with equality in the Plücker estimate extends smoothly into its Weierstrass points.*

Proof. Let $H \subset H^0(L)$ be a 2–dimensional base point free linear system with equality in the Plücker estimate. Let $L^d \subset H$ be the dual curve extended into the Weierstrass points of H (cf., Theorem 16.3). Let S be the canonical complex structure of H on M_0, where $M \setminus M_0$ are the Weierstrass points of H. Let $p \in M \setminus M_0$, z a centered coordinate at p, and $\varphi \in \Gamma(\widehat{L^d})$ a holomorphic section such that $\varphi(p)$ and $\tilde\varphi(p)$ are linearly independent over $\mathbb{C}$, where $\tilde\varphi$ is the holomorphic extension of $\varphi' z^{-k}$, $k = \operatorname{ord}_p \varphi'$. If $\varphi(p)$ and $\tilde\varphi(p)$ are linearly independent over $\mathbb{H}$, then the quaternionic linear extension of $\varphi(p) \mapsto \varphi(p)\mathbb{i}$ and $\tilde\varphi(p) \mapsto \tilde\varphi(p)\mathbb{i}$ is a quaternionic linear endomorphism that extends S smoothly into p, because $\operatorname{Eig}(S, \mathbb{i}) = \operatorname{span}_{\mathbb{C}}\{\varphi, \varphi'\}$ near p, by Lemma 16.2.

Suppose now that there exists $\lambda \in \mathbb{H} \setminus \mathbb{C}$ such that $\tilde\varphi(p) = \varphi(p)\lambda$. Let $e_1, e_2 \in H$ be a basis of H such that $e_1 = \varphi(p)$. Let φ_1 and φ_2 be $\mathbb{H}$ valued functions defined near p such that $\varphi = e_1\varphi_1 + e_2\varphi_2$. Then φ_1 and φ_2 are complex holomorphic functions into $(\mathbb{H}, \mathbb{i})$. Furthermore,

$$\varphi_1(p) = 1, \quad \varphi_2(p) = 0, \quad \varphi_1' z^{-k} =_{|_p} \lambda \in \mathbb{H} \setminus \mathbb{C}, \quad \text{and} \quad \varphi_2' z^{-k} =_{|_p} 0.$$

The second and third equation follows from $e_1\lambda = \tilde\varphi(p) = (e_1\varphi_1' + e_2\varphi_2')z^{-k}$. Since H is base point free, $(L^d)^\perp \subset H^*$ is the curve that corresponds to H via Kodaira correspondence. In particular, it is a holomorphic curve on all of M and $L = H/L^d$ is its canonical holomorphic line bundle. Thus, the derivative $\delta\colon L^d \to KH/L^d$ is complex holomorphic, by Lemma 20.3 below. This and

$$\nabla_{\frac{\partial}{\partial z}} \varphi = \varphi' = e_1\varphi_1' + e_2\varphi_2' = e_2(\varphi_2' - \varphi_2\varphi_1^{-1}\varphi_1') + \varphi\varphi_1^{-1}\varphi_1'$$

implies that $\psi = e_2(\varphi_2' - \varphi_2\varphi_1^{-1}\varphi_1') \mod L^d \in \Gamma(\widehat{H/L^d})$ is a complex holomorphic section of $\widehat{H/L^d}$. ψ has a zero of order strictly greater than k at p, because $\varphi_2(p) = 0$ and $\varphi_2' z^{-k} =_{|_p} 0$. Hence ψz^{-k-1} extends smoothly into p. But ψz^{-k-1} does not extend smoothly into p, since $e_2 \notin L^d{}_{|_p}$, $\varphi_1^{-1}\varphi_1' z^{-k} =_{|_p} \lambda \in \mathbb{H} \setminus \mathbb{C}$ and $\varphi_2 \bar z^{-1}$ is not smooth at p, because φ_2 is a complex holomorphic map into $(H, \mathbb{i})$. $\square$

20.3. Lemma. *If L is a holomorphic curve in $\mathbb{HP}^1$ whose dual curve is also holomorphic, then the derivative δ of L is a complex holomorphic section of $\operatorname{KHom}_+(L, \mathbb{H}^2/L)$.*

Proof. There is a sphere congruence, i.e., $S \in \Gamma(\operatorname{End}\mathbb{H}^2)$, $S^2 = -1$, such that S preserves L and induces the complex structures of L and $\mathbb{H}^2/L$. Let $\nabla = \partial + A + \bar\partial + Q$ be the decomposition (2.7) of the trivial connection ∇ on $\mathbb{H}^2$ with respect to S. Let $\pi\colon \mathbb{H}^2 \to \mathbb{H}^2/L$ be the canonical projection. Then $*\delta = J\delta = \delta J$ implies that $\delta = \pi\partial_{|_L}$ and $\bar\partial\Gamma(L) \subset \Omega^1(L)$. In particular, $L \subset (\mathbb{H}^2, \bar\partial, S)$ is a complex holomorphic subbundle, and $\bar\partial$ induces a complex holomorphic structure on L. It is straight forward to check that this complex holomorphic structure is the complex holomorphic structure of L (as defined in 9.2). Furthermore, $\bar\partial$ induces the complex holomorphic structure of the Möbius invariant holomorphic line bundle $\mathbb{H}^2/L$, because $D\pi\psi = \frac{1}{2}(\pi\nabla + *J\pi\nabla)\psi$, for all $\psi \in \Gamma(\mathbb{H}^2)$ (cf., 8.3).

The formula for $\mathcal{R}^\nabla_+$ in 2.7 implies $\pi\,\mathcal{R}^{\partial+\bar\partial}{}_{|_L} = -\pi(A\wedge A + Q\wedge Q)_{|_L} = 0$. Hence for $\psi \in \Gamma(L)$ and a (local) holomorphic vector field $X \in \Gamma(TM)$, i.e., $[X, JX] = 0$, one gets $0 = \pi\,\mathcal{R}^{\partial+\bar\partial}_{(X,JX)}\,\psi = \pi(-2J\partial_X\bar\partial_X + 2J\bar\partial_X\partial_X)\psi$. Hence

$\pi\partial_X\bar\partial_X\psi = \pi\bar\partial_X\partial_X\psi$, which implies

$$\delta_X\bar\partial_X\psi = \pi\bar\partial_X\delta_X\psi.$$

Thus δ is complex holomorphic. □

20.4. Lemma. *Let H be a 2–dimensional base point free linear system with equality in the Plücker estimate, S its canonical complex structure and Q its Hopf field. Let E be the holomorphic twistor lift of the dual curve of H and suppose Q is not identically zero.*

Then $\operatorname{im} Q$ extends to a holomorphic curve on M, the dual curve E^d of E is the holomorphic twistor lift of $\tilde L = (\operatorname{im} Q)^\perp \subset H^*$, $-S^*$ is the canonical complex structure of $\tilde L$, and $Q_p = 0$ if and only if $p \in M$ is a branch point of the tangent line congruence E_1 of E.

Note that the image of Q may be constant, then E lies in some projective plane $\mathbb{C}\mathrm{P}^2 \subset \mathbb{C}\mathrm{P}^3$, since $\tilde L$ as well as E^d are in this case constant. The condition that Q is not identically zero assures that E is not contained in a projective line $\mathbb{C}\mathrm{P}^1 \subset \mathbb{C}\mathrm{P}^3$.

Proof. The canonical complex structure is smoothly defined on all of M, by Lemma 20.2. The dual curve L^d of H has holomorphic twistor lift E, by Theorem 16.3, and $\nabla S = 2 * Q$, since $A \equiv 0$, by Lemma 16.2. The same lemma says that E_1, the tangent line congruence of E, is the $\mathbb{i}$–eigenspace of S. In particular, $E_1 \oplus E_1\mathbb{j} = H$. Let $\varphi \in \Gamma(E_1)$ and $\omega_1, \omega_2 \in \Omega^1(E_1)$ such that $\nabla\varphi = \omega_1 + \omega_2\mathbb{j}$. Then $\omega_2\mathbb{j} = \nabla\varphi - \omega_1 \in \Omega^1(E_2)$ and

$$\begin{aligned} 2{*}Q\varphi &= \nabla S\varphi = \nabla(S\varphi) - S\nabla\varphi \\ &= \omega_1\mathbb{i} + \omega_2\mathbb{j}\mathbb{i} - \omega_1\mathbb{i} - \omega_2\mathbb{i}\mathbb{j} = -2\omega_2\mathbb{k} \in \Omega^1(E_2). \end{aligned}$$

Hence $\operatorname{im} Q \subset E_2$. The dual curve E^d of E is by definition $E_2^\perp \subset (H, \mathbb{i})^*$. With the complex linear isomorphism

$$(H, \mathbb{i})^* \to (H^*, \mathbb{i}), \quad \alpha \mapsto (x \mapsto \overline{\alpha(x\mathbb{j})} + \mathbb{j}\alpha(x)),$$

one sees $T(E^d) \subset (\operatorname{im} Q)^\perp$ and $\tilde L = T(E^d)$ smoothly extends the line subbundle $\operatorname{im} Q \subset H$ into the zeros of Q. The $-\mathbb{i}$ eigenspace of S^* is the osculating 1–plane E_1^d of E^d, because $E_1^d = E_1^\perp$ and E_1 is the $\mathbb{i}$–eigenspace of S. Thus, as $-S^*$ is the canonical complex structure of $T(E^d)$, by Lemma 16.2, $-S$ is the canonical complex structure of $\tilde L$ and $\tilde L$ is a holomorphic curve on all of M whose dual curve is $T(E^d)$.

Q is zero at a point $p \in M$ if and only if $Q\varphi = 0$ for all $\varphi \in E_{1|_p}$, since $E_1 \oplus E_1\mathbb{j} = H$. Let $\varphi \in \Gamma(E_1)$. Then $2{*}Q\varphi = \nabla S\varphi = \nabla\varphi\mathbb{i} - S\nabla\varphi$. This is zero at p if and only if $\operatorname{im} \nabla\varphi_{|_p} \subset E_{1|_p}$, because E_1 is the $\mathbb{i}$–eigenspace of S. But $\operatorname{im} \nabla\varphi_{|_p} \subset E_{1|_p}$ means that p is a branch point of E_1. □

20.5. Lemma. *A holomorphic curve in $\mathbb{H}\mathrm{P}^1$ with holomorphic twistor lift is a Euclidean minimal curve if and only if its twistor lift is planar.*

Proof. Let $L \subset \mathbb{H}^2$ be a holomorphic curve with holomorphic twistor lift $E \subset (\mathbb{H}^2, \mathbb{i})$. L is Euclidean minimal if and only if $\operatorname{im} Q$ is constant (cf.,15.4). But $\operatorname{im} Q$ is constant if and only if the dual curve E^d of E is constant, by Lemma 20.4, which is equivalent to E being planar. □

20.6. Willmore Energy 28π. The author does not know any example of an immersed soliton sphere with Willmore energy 28π. Applying the arguments of the proof of the preceding theorem one arrives at the following situation:

Let L be the Euclidean holomorphic line bundle of an immersed soliton sphere in $\mathbb{R}^3$ with Willmore energy 28π. Then $H^0(L)$ is a 2–dimensional base point free linear system with equality in the Plücker estimate and $\operatorname{ord} H^0(L) = 3$. In the notation of the proof of Theorem 20.1 one gets $\deg E = -6$ and $b_0 = 3$. If E is full then the complex Plücker formula implies $b_1 = 0$ and $b_2 = 3$, or $b_1 = b_2 = 1$. Again, using the fact that $*Q^*$ provides a holomorphic bundle homomorphism from $H^*/T(E^d)$ to L, one concludes that $b_0(p) = b_2(p) + \operatorname{ord}_p Q$ for all $p \in \mathbb{C}\mathrm{P}^1$. In the case $b_1 = b_2 = 1$ this formula implies that Q has 2 zeros. But Lemma 20.4 implies that Q has exactly 1 zero in this case. Contradiction. Hence $b_1 = 0$, $b_2 = 3$, and $b_0(p) = b_2(p)$. Since E is a curve of degree 6 in $\mathbb{C}\mathrm{P}^3$, one knows $b_0(p) + b_2(p) \leq 3$ for all $p \in \mathbb{C}\mathrm{P}^1$. Hence there are three distinct points $p_0, p_1, p_\infty \in \mathbb{C}\mathrm{P}^1$ such that $b_0(p) = b_2(p) = 1$, and $b_0 = b_1 = b_2 = 0$ at all other points.

CHAPTER IV

Bäcklund Transformation and Equality

The main purpose of this chapter is to show that Willmore spheres in $\mathbb{HP}^1$ are soliton spheres and to derive a procedure to construct all Willmore spheres in $\mathbb{HP}^1$ from holomorphic curves in $\mathbb{CP}^3$ using only algebraic operations. This is done in the Sections 23–25. The first step is a theorem which states that a holomorphic Willmore sphere in $\mathbb{HP}^1$ is a Euclidean minimal curve, or the curve or its dual has holomorphic twistor lift [**Ej88, Mu90, Mo00**]. The second step is to show that the dual curve of the 1–step Willmore–Bäcklund transform of a Euclidean minimal sphere has holomorphic twistor lift. Because the composition of the 1–step Willmore–Bäcklund transformation with dualization is an involutive transformation, all Willmore spheres in $\mathbb{HP}^1$ are the twistor projection of a holomorphic curve in $\mathbb{CP}^3$ or its dual, or the 1–step Willmore–Bäcklund transform of the twistor projection of a holomorphic curve in $\mathbb{CP}^3$. The observation that the 1–step Willmore–Bäcklund transform of the twistor projection of a holomorphic curve in $\mathbb{CP}^3$ can, in contrast to the general situation, be obtained without integration completes the algebraic construction.

The Sections 26 and 27 deal with the question how this procedure can be used to construct the Willmore spheres in $\mathbb{R}^3$. This is done by applying a theorem by Jörg Richter [**Ri97**] which says that the 1–step Willmore–Bäcklund transform of a Willmore holomorphic curve in S^3 is minimal in hyperbolic 4–space. To apply this theorem to our construction, the condition of hyperbolic minimality needs to be slightly relaxed to allow compact hyperbolic minimal surfaces. This is natural for curves in $\mathbb{HP}^1$, as $\mathbb{HP}^1$ can be understood as two hyperbolic spaces glued together at their ideal boundary 3–sphere. With the new definition one can show that hyperbolic minimal spheres are superminimal, which is analogous to a result for minimal spheres in S^4 obtained by Robert Bryant in [**Br82**]. It is shown, that the 1–step Willmore–Bäcklund transforms of hyperbolic superminimal curves are branched conformal Willmore immersions into $\mathbb{R}^3$ (in fact they are Euclidean minimal curves), and, conversely, every Willmore sphere in $\mathbb{R}^3$ is the 1–step Willmore–Bäcklund transform of a hyperbolic superminimal sphere.

An important tool in the present chapter is the Willmore–Bäcklund transformation, which was introduced in [**BFLPP02**]. In Section 21 a generalization of this transformation is proposed. As it concentrates on the involved holomorphic line bundles, it is assumed to be helpful to keep track of the different holomorphic line bundles involved in the proofs of this and the next chapter. It shows how the relations of the Willmore energies and the preservation of equality in the Plücker estimate on the ladder of holomorphic line bundles can be applied to the Willmore–Bäcklund transformation. Moreover, in Section 22 it is shown that the Christoffel transformation of isothermic surfaces is also an example of the generalized Bäcklund transformation. This transformation is applied in the last chapter of this thesis.

21. Bäcklund Transformation

In this section a definition of a Bäcklund transformation for holomorphic curves in $\mathbb{HP}^n$ is proposed. It is shown that the $(n+1)$–step Bäcklund transforms are projective invariants of the transformed curve. Before we give the definition of the Bäcklund transformation in 21.3 some explanation seems useful. A first approximation of the definition of the $(n+1)$–step Bäcklund transformation is the following. If L is a holomorphic curve in $\mathbb{HP}^n$ then the canonical holomorphic line bundle $\tilde{L}^{-1}$ of the Bäcklund transform $\tilde{L}$ is related to the canonical holomorphic line bundle L^{-1} of L along the ladder of holomorphic line bundles (11.7). This means that there exist flat connections $(\nabla_i)_{0\leq i\leq n}$ on K^iL^{-1} such that ∇_0'' is the holomorphic structure of L^{-1} and $\nabla_i'' = d^{\nabla_{i-1}}$ for all $i = 1, \dots, n$, and a quaternionic holomorphic bundle homomorphism $\tilde{L}^{-1} \to (K^{n+1}L^{-1}, d^{\nabla_n})$.

21.1. $(n+1)$**–Step Bäcklund Transformation I.** The next step is to require that the holomorphic structure d^{∇_n} is related to the canonical linear system of L. This goes as follows: Let $H \subset H^0(L^{-1})$ be an $(n+1)$–dimensional linear system without Weierstrass points. Then there exists a nowhere vanishing holomorphic section $\beta_0 \in H$, by Theorem 3.8. The equation $\nabla_0\beta_0 = 0$ then defines a trivial connection on L^{-1} such that ∇_0'' is the holomorphic structure of L^{-1}. $\nabla_0 H \subset H^0(KL^{-1}, d^{\nabla_0})$ is an n–dimensional linear system without Weierstrass points, by Theorem 17.2. Consequently, one can proceed successively choosing $\beta_i \in H$ such that $\nabla_{i-1}\dots\nabla_0\beta_i \in \nabla_{i-1}\dots\nabla_0 H \subset H^0(K^iL^{-1}, d^{\nabla_{i-1}})$ has no zeros and trivial connections ∇_i of K^iL^{-1} such that $\nabla_i(\nabla_{i-1}\dots\nabla_0\beta_i) = 0$ until $i = n$, because the linear systems $\nabla_{i-1}\dots\nabla_0 H$ are $(n+1-i)$–dimensional and Weierstrass point free.

This procedure provides holomorphic structures on K^iL^{-1} for all $i = 0, \dots, n+1$. The following theorem shows that the induced holomorphic structure d^{∇_n} on $K^{n+1}L^{-1}$ only depends on the linear system H and not on the choice of the basis $(\beta_i)_{0\leq i\leq n}$.

Theorem. *If $H \subset H^0(L^{-1})$ is an $(n+1)$–dimensional linear system without Weierstrass points and the connection ∇_n on K^nL^{-1} is defined as described above, then $(K^{n+1}L^{-1}, d^{\nabla_n})$ is paired with $(L^d)^{-1}$.*

In the case of the canonical 2–dimensional linear system of an immersed holomorphic curve L in $\mathbb{HP}^1$ this theorem reduces to the fact that the upper right vertex $(K^{-1}L, (\nabla_1^*)'')$ of the quadrilateral of holomorphic line bundles is isomorphic to the canonical holomorphic line bundle of the dual curve L^d (cf., 11.6). The lemma below shows that the isomorphism is given by the dual of the derivative $\delta\colon L^d \to KL^{-1}$ of L^d.

PROOF. It suffices to show that $(K^{-n}L, (\nabla_n^*)'')$ and $(L^d)^{-1}$ are isomorphic, because $(K^{-n}L, (\nabla_n^*)'')$ and $(K^{n+1}L^{-1}, d^{\nabla_n})$ are paired by Theorem 11.4. Let $L^d = H_0 \subset \dots \subset H_n = H$ be the Weierstrass flag of H. The composition of all its derivatives $\delta := \delta_n \circ \dots \circ \delta_0\colon L^d \to K^nH/H_{n-1} =$

K^nL^{-1} (cf., 14.3) is a complex quaternionic isomorphism. Thus it suffices to show that $\delta^*\colon K^{-n}L \to (L^d)^{-1}$ maps a nontrivial holomorphic section onto a holomorphic section, by Lemma 4.1. If $(\beta_i)_{0\le i\le n}$ is the basis of H that defines the ∇_n's and $(\beta_i^*)_{0\le i\le n}$ is its dual basis, then $(\nabla_{n-1}\dots\nabla_0\beta_n)^{-1}$ and $\beta^*_{n|L^d}$ are holomorphic sections of $(K^{-n}L,(\nabla_n^*)'')$ and $(L^d)^{-1}$. The following lemma thus finishes the proof. $\square$

21.2. Lemma. $\delta((\beta_n^*)^{-1}) = (-1)^n\nabla_{n-1}\dots\nabla_0\beta_n$.

Proof. Let β_i^0 be the constant section of the trivial bundle $H = H_n$ that corresponds to β_i. Construct successively for $k = 0,\dots,n$ sections $(\beta_i^k)_{k\le i\le n}$ of $H_{n-k}\subset H$ such that

$$(*)\qquad \delta_{n-1}\dots\delta_{n-k}\beta_i^k = (-1)^k\nabla_{k-1}\dots\nabla_0\beta_i \quad\text{and}\quad \beta_l^*(\beta_i^k) = \delta_{li}$$

for all $i,l = k,\dots,n$ as follows:

Suppose that the all $(\beta_i^k)_{k\le i\le n}$ are constructed up to some k, then $H_{n-k} = H_{n-k-1}\oplus\beta_k^k\mathbb{H}$, because $\nabla_{k-1}\dots\nabla_0\beta_k\in\Gamma(K^kL^{-1})$ has no zeros, by the definition of β_k. Consequently, there are sections $\beta_i^{k+1}\in\Gamma(H_{n-k-1})$ and smooth maps $f_i\colon M\to\mathbb{H}$ such that $\beta_i^{k+1} = \beta_i^k+\beta_k^kf_i$ for all $i = k+1,\dots,n$. These sections clearly satisfy for all $l,i = k+1,\dots,n$ the second condition $\beta_l^*(\beta_i^{k+1}) = \delta_{li}$ in $(*)$.

Because $\nabla H_{n-k-1}\subset\Omega^1(H_{n-k})$, and since $\beta_k^*(\beta_i^k) = \delta_{ki}$ for $i = k,\dots,n$ implies $\beta_k^*(\nabla\beta_i^k) = 0$, one gets $\nabla\beta_i^{k+1}-\beta_k^kdf_i = \nabla\beta_i^k-\nabla\beta_k^kf_i\in\Omega^1(H_{n-k-1})$. Hence

$$\begin{aligned}\delta_{n-1}\dots\delta_{n-k-1}\beta_i^{k+1} &= \delta_{n-1}\dots\delta_{n-k}(\pi_{n-k-1}\nabla\beta_i^{k+1})\\ (**)\qquad &= \delta_{n-1}\dots\delta_{n-k}(\beta_k^kdf_i) = (-1)^k\nabla_{k-1}\dots\nabla_0\beta_idf_i.\end{aligned}$$

But

$$\begin{aligned}0 = \delta_{n-1}\dots\delta_{n-k}\beta_i^{k+1} &= \delta_{n-1}\dots\delta_{n-k}(\beta_i^k+\beta_k^kf_i)\\ &= (-1)^k\nabla_{k-1}\dots\nabla_0\beta_i+(-1)^k\nabla_{k-1}\dots\nabla_0\beta_kf_i.\end{aligned}$$

This equation and the definition of ∇_i imply

$$\nabla_k\nabla_{k-1}\dots\nabla_0\beta_i = -\nabla_{k-1}\dots\nabla_0\beta_kdf_i,$$

which together with $(**)$ yields the first condition in $(*)$.

For $k = n$ one gets $\delta(\beta_n^n) = (-1)^n\nabla_{n-1}\dots\nabla_0\beta_n$ and $\beta_n^*(\beta_n^n) = 1$, thus $\beta_n^n = (\beta_n^*)^{-1}$ and $\delta((\beta_n^*)^{-1}) = (-1)^n\nabla_{n-1}\dots\nabla_0\beta_n$. $\square$

21.3. Bäcklund Transformation. Let L be a holomorphic curve in $\mathbb{HP}^n$ and $H\subset H^0(L^{-1})$ its canonical linear system. Let $\tilde H\subset H^0(L^{-1})$ be a $(k+1)$–dimensional linear system, such that $\tilde H\subset H$ or $H\subset\tilde H$. Let $M_0 = M\setminus\{\text{Weierstrass points of }\tilde H\}$ and ∇_k be a flat connection of $K^kL_{|M_0}$ as in 21.1 for some basis of $\tilde H$. A holomorphic curve $\tilde L$ is called a $(k+1)$*–step Bäcklund transform* of L with respect to $\tilde H$, if

(i) its canonical holomorphic line bundle restricted to M_0 admits a nontrivial holomorphic map to $(K^{k+1}L^{-1}{}_{|M_0}, d^{\nabla_k})$, and
(ii) the image of the canonical linear system of $\tilde L$ is included in or includes the linear system $\nabla_k\dots\nabla_0H$.

Theorem 21.1 and Proposition 10.8 imply that the canonical holomorphic line bundle of a $(k+1)$–step Bäcklund transform is on M_0 branchedly paired to the canonical holomorphic line bundle $(L^d)^{-1}$ of the dual curve L^d of $\tilde{H}$. The Willmore energies of L and of its $(k+1)$–step Bäcklund transforms $\tilde{L}$ are in the case that M is compact of genus g related by the Plücker formula:

$$\frac{1}{4\pi}(W(L^{-1}) - W(\tilde{L}^{-1})) = (k+1)(k(1-g) - \deg L^{-1}) + \operatorname{ord}\tilde{H},$$

because branchedly paired holomorphic line bundles have the same Willmore energies.

21.4. $(n+1)$**–Step Bäcklund Transformation II.** For the $(n+1)$–step Bäcklund transformation of a holomorphic curve in $\mathbb{HP}^n$ one has no choice for $\tilde{H}$ but $\tilde{H} = H = (\mathbb{H}^{n+1})^*$. Then (ii) is trivially satisfied. Theorem 21.1 thus implies that the set of $(n+1)$–step Bäcklund transforms of L only depends on the projective equivalence class of L. More precisely, one gets the following corollary.

Corollary (of Theorem 21.1)**.** *Let L be a holomorphic curve in $\mathbb{HP}^n$ and $M_0 = M \setminus \{$Weierstrass points of $(\mathbb{H}^{n+1})^*\}$. A holomorphic curve $\tilde{L}$ in $\mathbb{HP}^{\tilde{n}}$ is then an $(n+1)$–step Bäcklund transform of L if and only if the canonical holomorphic line bundle of $\tilde{L}$ is on M_0 branchedly paired with $(L^d)^{-1}$. If M is compact of genus g, then the Willmore energies of L and $\tilde{L}$ are related by*

$$\frac{1}{4\pi}(W(L^{-1}) - W(\tilde{L}^{-1})) = (n+1)(n(1-g) - \deg L^{-1}) + \operatorname{ord}(\mathbb{H}^{n+1})^*.$$

In particular, the 2–step Bäcklund transformation of holomorphic curves in $\mathbb{HP}^1$ is Möbius invariant.

21.5. 1–Step Bäcklund Transformation of Holomorphic Curves in $\mathbb{HP}^1$**.** The 1–step Bäcklund transformation involves the choice of a 1–dimensional linear subsystem of the canonical linear system. For holomorphic curves in $\mathbb{HP}^1$ this can be interpreted as the choice of a point $\infty \in \mathbb{HP}^1$. The 1–step Bäcklund transformation of a holomorphic curve in $\mathbb{HP}^1$ with respect to a fixed point at infinity then yields a transformation of branched conformal immersions $f\colon M \to \mathbb{H}$ that respects the geometry of similarities of $\mathbb{HP}^1 \setminus \{\infty\} = \mathbb{H}$.

Proposition. *Let $L \subset \mathbb{H}^2$ be a holomorphic curve in $\mathbb{HP}^1$, whose dual curve is also holomorphic. If $f := \sigma_{\alpha,\beta}L\colon M \to \mathbb{H}$ is an admissible stereographic projection and $g\colon M \to \mathbb{H}$ is a nonconstant smooth map satisfying*

$$(*) \qquad dg \wedge df = 0,$$

then g is a 1–step Bäcklund transform of L with respect to $\beta_{|L}\mathbb{H}$.

If, conversely, $\tilde{g}\colon M \to \mathbb{H}$ is a 1–step Bäcklund transform of L with respect to $\beta_{|L}\mathbb{H}$, then there is a Möbius transform g of $\tilde{g}$ in $\mathbb{H} \cup \{\infty\}$ that satisfies $(*)$, away from the points that are mapped to ∞.

If one transforms f by a similarity $x \mapsto \lambda x \mu + c$ of $\mathbb{H}$ and g by $x \mapsto \overline{\lambda \bar{x} \mu + c}$, then $(*)$ is preserved. Thus the transformation $f \mapsto \bar{g}$ commutes with similarities. Furthermore, this modified transformation is involutive.

A smooth map $g\colon M \to \mathbb{H}$ is called a *1–step Bäcklund transform* of a branched conformal immersion $f\colon M \to \mathbb{H}$ with a left normal vector, if $(*)$ is satisfied.

Proof. Since the dual curve of L is by assumption a holomorphic curve, it follows that f has a left normal vector N, by Lemma 7.4 and 7.5. Let ∇_0 be the trivial connection on L^{-1} such that $\nabla_0 \beta_{|L} = 0$. Then $\alpha_{|L} = \beta_{|L}\bar{f}$ (cf., 8.5) implies $\nabla_0 \alpha_{|L} = \beta_{|L} d\bar{f}$. Hence $-N$ is the normal vector of $\nabla_0\alpha \in H^0(KL^{-1}, d^{\nabla_0})$.

If $(*)$ holds, then g has right normal vector $-N$. Hence the canonical linear system of g contains a nowhere vanishing holomorphic section φ with normal vector $-N$, because g is a quotient of elements of the canonical linear system (cf., 8.5). The quaternionic bundle homomorphism from the Möbius invariant holomorphic line bundle of g to KL^{-1} that maps φ to $\nabla_0\alpha$ is complex linear, because φ and $\nabla_0\alpha$ have the same normal vector, and holomorphic, by Lemma 4.1, because φ and $\nabla_0\alpha$ are both holomorphic. Thus g is a 1–step Bäcklund transform of L.

If, on the other hand, $\tilde{g}$ is a 1–step Bäcklund transform of L with respect to $\beta_{|L}\mathbb{H}$, then its canonical linear system must contain a section with normal vector $-N$, by condition (ii) of Definition 21.3. Hence there is a Möbius transform g of $\tilde{g}$ that has right normal vector $-N$, by 8.5. Then, away from the points that are mapped to ∞, g satisfies $(*)$ by type. $\square$

21.6. Successive Bäcklund Transformations. One easily sees from the definition of the Bäcklund transformation: If $k_1, k_2 \in \mathbb{N} \setminus \{0\}$, then every $(k_1 + k_2)$–step Bäcklund transform is, besides isolated points, a k_2–step Bäcklund transform of some k_1–step Bäcklund transform.

22. Christoffel Transformation

The aim of this section is to discuss the Christoffel transformation as an example for the Bäcklund transformation defined in the previous section as well as to establish some facts that are needed in Chapter V. For more on the theory of isothermic surfaces see [**Jeromin**] or the survey [**Bu00**].

22.1. Isothermic Holomorphic Curves. A holomorphic curve L in $\mathbb{HP}^1$ is called *isothermic* if and only if its dual curve $L^d = L^\perp$ is a holomorphic curve and the Möbius invariant holomorphic line bundles $L^{-1} = (\mathbb{H}^2)^*/L^d$ and $(L^d)^{-1} = \mathbb{H}^2/L$ of L and L^d are branchedly paired. A holomorphic curve in $\mathbb{HP}^1$ whose dual curve is also holomorphic is, by Corollary 21.4, isothermic if and only if it is a 2–step Bäcklund transform of itself.

If L is a holomorphic curve that is not contained in a 2–sphere, then at least one of the Hopf fields of L^{-1} and $(L^d)^{-1}$ does not vanish identically, by 8.4. If L is isothermic the both Hopf fields do not vanish, by Corollary 9.9. Then Proposition 10.8 and Theorem 4.3 imply that the pairing of L^{-1} and $(L^d)^{-1}$ is unique up to multiplication by a real constant.

22.2. Christoffel Transformation. If L is a holomorphic curve in $\mathbb{HP}^1$ whose dual curve is also holomorphic, then every nonconstant smooth map $g\colon M \to \mathbb{H}$ such that

$$(**) \qquad dg \wedge df = df \wedge dg = 0$$

for some admissible stereographic projection $f = \sigma_{\alpha,\beta} L$ of L is called a *Christoffel transform* of L. The equation $(**)$ is equivalent to saying that the left / right normal vector of f equals minus the right / left normal vector of g. Proposition 21.5 implies that the Christoffel transforms of a holomorphic curve L are 1–step Bäcklund transforms of the curve. Furthermore, L is a 1–step Bäcklund transform of every one of its Christoffel transforms g.

22.3. Isothermic holomorphic curves L have Christoffel transforms. Their Weierstrass data is obtained from the Möbius invariant holomorphic line bundles of L.

Lemma. *Let L be a holomorphic curve in $\mathbb{HP}^1$ whose dual curve is also holomorphic. A conformal immersion $g\colon M \to \mathbb{H}$ is a Christoffel transform of L if and only if L^{-1} and $(L^d)^{-1}$ are the paired Euclidean holomorphic line bundles of g, and there exists an admissible $\beta \in (\mathbb{H}^2)^*$ and $a \in \mathbb{H}^2$ with $\beta(a) = 0$ such that $(L^{-1}, (L^d)^{-1}, \beta_{|L}, a_{|L^d})$ is the Weierstrass data of g.*

Proof. Let $f = \sigma_{\alpha,\beta} L$ be some admissible stereographic projection of L. From Theorem 10.10 and Proposition 9.10 then follows that $df = (a^{-1}, \beta^{-1})$. If N and R are the left and right normal vectors of f, then N and R are the normal vectors of $\alpha_{|L^d}$ and $\beta_{|L}$, respectively.

If $g\colon M \to \mathbb{H}$ is a Christoffel transform of L such that $(**)$ holds for f, then $*dg = -Rdg = dgN$, and $(\beta_{|L}, a_{|L^d}) := dg$ defines a pairing of L^{-1} and $(L^d)^{-1}$, by Corollary 9.7, because dg is closed and

$$(\beta_{|L}, Ja_{|L^d}) = (\beta_{|L}, a_{|L^d})N = *(\beta_{|L}, a_{|L^d}) = -R(\beta_{|L}, a_{|L^d}) = (J\beta_{|L}, a_{|L^d}).$$

Suppose now that $g\colon M \to \mathbb{H}$ has Weierstrass data $(L^{-1}, (L^d)^{-1}, \beta_{|L}, a_{|L^d})$ for some admissible $\beta \in (\mathbb{H}^2)^*$ and $a \in \mathbb{H}^2$ with $\beta(a) = 0$. Choose $\alpha \in (\mathbb{H}^2)^*$ such that $\alpha(a) = 1$. Then and $df = d\sigma_{\alpha,\beta} L = (a^{-1}, \beta^{-1})$ and $dg = (\beta_{|L}, a_{|L^d})$ imply $(**)$ by type. $\square$

22.4. Proposition. *A holomorphic curve L in $\mathbb{HP}^1$ whose dual curve is also a holomorphic curve is isothermic if and only if L has a Christoffel transform on the universal covering of M with translational periods.*

Proof. If L is isothermic and $f = \sigma_{\alpha,\beta} L$ is an admissible stereographic projection, then L^{-1} and $(L^d)^{-1}$ are paired and Lemma 22.3 implies that $\int(\beta_{|L}, a_{|L^\perp})\colon \tilde{M} \to \mathbb{H}$ is a smooth Christoffel transform on the universal covering $\tilde{M}$ of M with translational periods only. If, on the other hand, L has a Christoffel transform $g\colon \tilde{M} \to \mathbb{H}$ with only translational periods, then dg is well defined on M and $(\beta_{|L}, a_{|L^\perp}) := dg$ defines a pairing of L^{-1} and $\mathbb{H}^2/L$, as in the proof of Lemma 22.3. $\square$

22.5. The Retraction Form. If L is an isothermic holomorphic curve in $\mathbb{HP}^1$, then there is, by Proposition 10.8, a holomorphic bundle homomorphism $\tau\colon (L^d)^{-1} \to KL$. As $(L^d)^{-1} = \mathbb{H}^2/L$ and $L \subset \mathbb{H}^2$ one can interpret τ as a 1–form with values in $\operatorname{End}(\mathbb{H}^2)$ satisfying $\operatorname{im}\tau \subset L \subset \ker\tau$ and $d\tau = 0$, because $\beta(\tau a)$ is closed for all $\beta \in (\mathbb{H}^2)^*$ and $a \in \mathbb{H}^2$, by the definition of the paired holomorphic structure of KL (cf., 9.6). A nontrivial closed 1–form τ with values in $\operatorname{End}(\mathbb{H}^2)$ such that $\operatorname{im}\tau \subset L \subset \ker\tau$ is called a *retraction form* of the holomorphic curve L.

Proposition. *A holomorphic curve L in $\mathbb{HP}^1$, whose dual curve is also a holomorphic curve, is isothermic if and only if L has a retraction form. A retraction form τ defines and is uniquely determined by a branched pairing of L^{-1} and $(L^d)^{-1}$ via $(\varphi, \psi) = \varphi\tau\psi$. Furthermore, τ induces a holomorphic bundle homomorphism $(L^d)^{-1} \to KL$.*

If L is isothermic but not contained in a 2–sphere, then we have already seen in 22.1 that the pairing of L^{-1} and $(L^d)^{-1}$ is unique up to multiplication by a real constant. Hence the retraction form of an isothermic holomorphic curve that is not contained in a 2–sphere is unique up to multiplication by a real constant.

Proof. The fact that isothermic holomorphic curves have retraction forms is already shown in the preliminary remark. If now τ is a retraction form of L, then $0 = \pi d\tau = \delta \wedge \tau$, for the derivative δ of L and the canonical projection $\pi\colon \mathbb{H}^2 \to \mathbb{H}^2/L$. Moreover, for $\psi \in \Gamma(L)$ one gets $0 = d(\tau\psi) = \tau \wedge \nabla\psi = \tau \wedge \delta\psi$. Hence τ induces a complex quaternionic bundle homomorphism $(L^d)^{-1} \to KL$. Furthermore, every holomorphic section $a_{|L^\perp}$ for $a \in \mathbb{H}^2$ is mapped by τ onto a holomorphic section of KL, by Theorem 9.6, since $\beta(\tau a)$ is closed for all $\beta \in (\mathbb{H}^2)^*$. Thus τ is holomorphic, by Lemma 4.1. Hence L is isothermic, because τ defines via $(\varphi, \psi) := \varphi\tau\psi$ a branched pairing of L^{-1} and $(L^d)^{-1}$, by Proposition 10.8. $\square$

22.6. The Christoffel transforms $g\colon M \to \mathbb{H}$ of L with Weierstrass data $(L^{-1}, (L^d)^{-1}, \beta_{|L}, a_{|L^d})$, as in Lemma 22.3, satisfies

$$dg = \beta\tau a,$$

if the pairing of L^{-1} and $(L^d)^{-1}$ is induced by τ. If L is isothermic that is not contained in a round 2–sphere and τ is a retraction form of L, then the uniqueness of τ, up to a real factor, and Lemma 22.3 imply that $g\colon M \to \mathbb{H}$ is a Christoffel transform of L if and only if there exists an admissible $\beta \in (\mathbb{H}^2)^*$ and $a \in \mathbb{H}^2$ with $\beta(a) = 0$ such that $dg = \lambda\beta\tau a$ for some $\lambda \in \mathbb{R}$.

22.7. Remark. Usually an immersion $f\colon M \to \mathbb{H}$ is called isothermic, if f admits, away from the umbilics of f, local conformal curvature line parameters, see [**Jeromin**, Definition 5.1.1]. In [**Jeromin**, Lemma 5.2.6] it is shown that an immersed Christoffel transform assures the existence of local conformal curvature line parameters. Thus Proposition 22.4 implies that, away from the branch points of L and the branch points of the pairing of L^{-1} and $(L^d)^{-1}$, the branched conformal immersion $f = \sigma_{\alpha,\beta}L$ admits conformal curvature line parameters. The branch points of the pairing are the umbilics of f: Corollary 10.13 implies that $p \in M$ is a branch point of the pairing if and only if the Hopf fields $Q^{L^{-1}}$ and $Q^{\mathbb{H}^2/L}$ of L^{-1} and $\mathbb{H}^2/L$ vanish at p, i.e., p is an umbilic of f (cf., 8.4).

Consequently, if L is an isothermic holomorphic curve, then every stereographic projection $f = s_{\alpha,\beta}L$ restricted to $M \setminus \{\text{branch points of } f\}$ is an isothermic immersion in the usual sense. The definition in the present text is stronger than the usual definition, because it is required that the pairing of $(L^d)^{-1}$ and L^{-1} is globally defined. This difference is discussed in more detail in [**Boh03**, 9.3].

In particular, our definition of isothermic holomorphic curves is to restrictive in case $M = \mathbb{C}\mathrm{P}^1$, because the composition of a holomorphic homomorphism from $(L^d)^{-1}$ to KL with the derivative δ of L, yields a complex holomorphic homomorphism from L to K^2L, i.e., a holomorphic section of K^2. But K^2 has degree -4 and, consequently, no holomorphic sections. A solution to this problem is to allow the retraction form to be meromorphic. The smooth catenoid cousins in Section 30 are an example of Darboux transforms of the round sphere with respect to a meromorphic retraction form.

23. Willmore–Bäcklund Transformation

A Bäcklund transformation for Willmore surfaces in S^4 was introduced in [**BFLPP02**]. In the present section it is shown that this Bäcklund transformation is another example of the Bäcklund transformation of Section 21. Important for the following investigations is the fact that the 1–step Willmore–Bäcklund transforms of a holomorphic curve with holomorphic twistor lift can be obtained without integration.

23.1. Willmore Holomorphic Curves in $\mathbb{H}\mathrm{P}^1$. A holomorphic curve in $\mathbb{H}\mathrm{P}^1$ is called *Willmore* if it is a critical point of the Willmore energy for all variations with compact support in $M_0 = M \setminus \{\text{Weierstrass points of } L\}$. Katrin Leschke and Franz Pedit show in [**LP03**, Theorem 3.4] that a holomorphic curve is Willmore if and only if its mean curvature sphere S is harmonic. If A and Q are the Hopf fields of S, then S is harmonic if and only if $d^\nabla{*}Q = 0$, or, equivalently, $d^\nabla{*}A = 0$. In particular, all holomorphic curves with holomorphic twistor lift and their duals are Willmore holomorphic curves, since A or Q vanishes identically in this case (cf., Lemma 16.2).

The Willmore–Bäcklund transform of a Willmore holomorphic curve can only be defined for Willmore holomorphic curves whose mean curvature sphere extends smoothly into the branch points of L. Such a Willmore holomorphic curve is called a *regular Willmore holomorphic curve*.

23.2. 2–Step Willmore–Bäcklund Transformation. Let $L \subset \mathbb{H}^2$ be a holomorphic curve in $\mathbb{H}\mathrm{P}^1$ and A and Q the Hopf fields of its mean curvature sphere S. Then $\operatorname{im} A \subset L$ and $L \subset \ker Q$ implies that $\ker A$ and $\operatorname{im} Q$ define, besides the branch points of L and the zeros of A and Q, new S invariant line subbundles of $\mathbb{H}^2$. If ${*}Q$ is closed, i.e., L is Willmore, then $0 = \pi d^\nabla{*}Q = \delta^{\operatorname{im} Q} \wedge {*}Q$ implies that $\operatorname{im} Q$ is a holomorphic curve whose complex structure is induced by $-S$, again only away from the branch points of L and the zeros of Q. Dualizing, the same holds for A, as $-A^*$ is the "Q" of S^*. The following lemma shows that $\operatorname{im} Q$ extends into the zeros of Q as a holomorphic curve.

Lemma. *If S is a complex structure of a trivial quaternionic vector space H such that the $(0,1)$ part $2{*}Q$ of its derivative is closed and not identically zero, then $Q\colon H \to K\bar{H}$ is complex holomorphic and there is a unique holomorphic curve $V \subset H$ of rank $\max_{p\in M}(\dim \operatorname{im} Q_p)$ with complex structure $-S_{|V}$ such that $V = \operatorname{im} Q$ besides isolated points of M.*

If L is a regular Willmore holomorphic curve, then the holomorphic curve defined by $\operatorname{im} Q$ as in the lemma is called the 2*–step Willmore–Bäcklund transform*[1] of L, denoted $\mathrm{WBT}(L) = \operatorname{im} Q$. Dualization yields $\mathrm{WBT}(L^d) = (\ker A)^{\perp}$. Moreover, if $\mathrm{WBT}(L)$ is not constant, then $L = \mathrm{WBT}(\mathrm{WBT}(L)^d)^d$, by [**BFLPP02**, Theorem 8, p. 59].

Proof. The bundle map $Q\colon H \to K\bar{H}$ is complex quaternionic, because ${*}Q = -SQ = QS$. Let ∇ be the trivial connection of H, then d^∇ is a holomorphic structure on $K\bar{H}$, analogous to 11.4. Then $d^\nabla{*}Q = 0$ implies that ${*}Q$ maps constant sections of H, which are holomorphic with respect to ∇'', to holomorphic sections of $K\bar{H}$. Thus, ${*}Q$ is holomorphic, by Lemma 4.1. In particular, Q is a complex holomorphic bundle homomorphism. Thus, $\dim \operatorname{im} Q_p = \max_{p\in M}(\dim \operatorname{im} Q_p)$ besides isolated points and there is a subbundle $V \subset H$ (by a standard argument for the image of complex holomorphic bundle homomorphisms, see for example [**BFLPP02**, Lemma 23] or [**GriHa**, Section 2.4]) such that V coincides with $\operatorname{im} Q$ besides the same isolated points. S clearly stabilizes V and V is a holomorphic curve whose complex structure is induced by $-S$, because $d^\nabla{*}Q = 0$ implies

$$0 = \pi d^\nabla{*}Q = \delta \wedge {*}Q,$$

where $\pi\colon H \to H/V$ is the the canonical projection and δ is the derivative of V. Thus ${*}\delta = -\delta S$. $\square$

Remark. Suppose L is a regular Willmore holomorphic curve that lies in some 3–sphere. Write † for the dual with respect to the Hermitian form that describes this 3–sphere. Then $S^\dagger = S$ and $Q^\dagger = -A$ (cf., 5.6). Hence $\operatorname{im} Q = \ker A$ and $\mathrm{WBT}(L) = \mathrm{WBT}(L^d)^d$ lies in the same 3–sphere. The Euclidean formulas (14.6) for Q and A then imply that $\mathrm{WBT}(L)_{|p}$ is the unique point in that 3–sphere, such that the mean curvature of the stereographic projection of L with pole $\infty = \mathrm{WBT}(L)_{|p}$ vanishes to second order. Thus $\mathrm{WBT}(L)$ is the dual Willmore surface as defined by Robert Bryant in [**Br84**].

23.3. Proposition. *If $L \subset \mathbb{H}^2$ is a regular Willmore holomorphic curve in $\mathbb{HP}^1$ and Q its Hopf field, then the bundle map*

$$(\,,)\colon\ \mathrm{WBT}(L)^{-1} \times (L^d)^{-1} \to T^*M \otimes \mathbb{H}, \quad (\varphi, \psi) = {*}Q^*\beta(b) = {*}\beta(Qb)$$

is a branched pairing of holomorphic line bundles.

[1] In [**BFLPP02**, Section 9.2] this is called the backward Bäcklund transform of L. The forward Bäcklund transform of [**BFLPP02**] is the holomorphic curve obtained extending $\ker A$ into the zeros of A. With the notation of the present text this would be $\mathrm{WBT}(L^d)^d$.

Corollary 21.3 thus implies that the holomorphic curve $\mathrm{WBT}(L)$ is a 2–step Bäcklund transform of L and

$$\frac{1}{4\pi}(W(L^{-1}) - W(\mathrm{WBT}(L)^{-1})) \\ = (n+1)(n(1-g) - \deg L^{-1}) + \mathrm{ord}(\mathbb{H}^{n+1})^*.$$

Proof. Let $S \in \Gamma(\mathrm{End}\,\mathbb{H}^2)$ be the canonical complex structure of L and Q and A its Hopf differentials. Then $L \subset \ker Q$ implies $\mathrm{im}\,Q^* \subset L^d$, and $\mathrm{im}\,Q \subset \mathrm{WBT}(L)$ implies $\mathrm{WBT}(L)^\perp \subset \ker Q^*$. Since $-S^*$ induces the complex structure of $\mathrm{WBT}(L)^{-1}$, by Lemma 23.2, the quaternionic bundle homomorphism $Q^*\colon\ \mathrm{WBT}(L)^{-1} \to KL^d$ is a complex linear. If $\beta \in (\mathbb{H}^2)^* \subset H^0(\mathrm{WBT}(L)^{-1})$ and $b \in \mathbb{H}^2 \subset H^0((L^d)^{-1})$, then $d^\nabla{*Q} = 0$ implies $d({*Q^*}\beta(b)) = d{*\beta}(Qb) = 0$. Thus ${*Q^*}\beta$ is a holomorphic section of KL^d, by Theorem 9.6. Hence

$$*Q^*\colon\ \mathrm{WBT}(L)^{-1} \to KL^d$$

is a holomorphic bundle homomorphism, by Lemma 4.1, and Proposition 10.8 implies that the pairing of the proposition is indeed a pairing of holomorphic line bundles. □

23.4. 1–Step Willmore–Bäcklund Transformation. Let $L \subset \mathbb{H}^2$ be a regular Willmore holomorphic curve, $\beta \in (\mathbb{H}^2)^* \subset H^0(\mathrm{WBT}(L)^{-1})$ and $a \in \mathbb{H}^2 \subset H^0((L^d)^{-1})$ such that $\beta(a) = 0$ and β is admissible for L and $\mathrm{WBT}(L)$. A branched conformal immersion $g\colon \tilde{M} \to \mathbb{H}$ on the universal covering $\tilde{M}$ of M with Weierstrass representation $(\mathrm{WBT}(L)^{-1}, (L^d)^{-1}, \beta, a)$, which means that

$$dg = 2{*\beta}(Qa),$$

is called a *1–step Willmore–Bäcklund transform of L*. β is admissible for L and $\mathrm{WBT}(L)$ if and only if the holomorphic sections induced by β and a have no zeros. Hence g is a branched conformal immersion with both normal vectors and g's branching order equals the vanishing order of Q, by Theorem 10.10. Furthermore, g is Willmore (cf., [**BFLPP02**, Theorem 6, p. 55]).

23.5. Choose some $\alpha \in (\mathbb{H}^2)^*$ such that $\alpha(a) = 1$ and let $f := \sigma_{\alpha,\beta}L$. If N is the left normal vector of f, then $Ja = aN$, because $df = (a^{-1}, \beta^{-1})$ by Proposition 9.10, and ${*dg} = 2{*\beta}(Qa) = dgN$. Hence $dg \wedge df = 0$ and g is a 1–step Bäcklund transform of f, by Proposition 21.5.

23.6. In [**BFLPP02**, Section 9.1][2] the following is shown: Suppose that g is immersed, and let $L_g \subset \mathbb{H}^2$ be the immersed Willmore holomorphic curve such that $\sigma_{\alpha,\beta}L_g = g$. Let A_g be the Hopf field of the mean curvature sphere of L_g, then

$$d\sigma_{\alpha,\beta}L = 2{*\beta}(A_g a).$$

Hence $\overline{\sigma_{\alpha,\beta}L}$ is a 1–step Willmore–Bäcklund transform of L_g^d.

[2] In [**BFLPP02**] the 1–step Willmore–Bäcklund transform goes by the name backward Bäcklund transform, the 1–step Willmore–Bäcklund transform of the dual curve is called a forward Bäcklund transform, and the formula $d\sigma_{\alpha,\beta}L = 2{*\beta}(A_g a)$ reads $df = \frac{1}{2}\omega_h$ and h is our g.

23.7. 1–Step WBT of Curves with Holomorphic Twistor Lift. Let L be a holomorphic curve in $\mathbb{HP}^1$ with holomorphic twistor lift. Then L is Willmore (cf., 23.1). L is regular if and only if the curve of tangent lines $\hat{L}_1$ of the twistor lift $\hat{L}$ of L satisfies $\hat{L}_1 \oplus \hat{L}_1 \mathbb{j} = \mathbb{H}^2$, see 16.4. The 1–step Willmore–Bäcklund transforms of L can then be obtained from the mean curvature sphere of L without integration.

Proposition. *Let L be a holomorphic curve in $\mathbb{HP}^1$ with holomorphic twistor lift $\hat{L}$, suppose that $\hat{L}_1 \oplus \hat{L}_1 \mathbb{j} = \mathbb{H}^2$, and let S be the mean curvature sphere of L extended to M. If $\beta \in (\mathbb{H}^2)^*$ and $a \in \mathbb{H}^2$ such that $\beta(a) = 0$ and β admissible for L and* $\mathrm{WBT}(L)$*, then*

$$g = \beta(Sa)\colon M \to \mathbb{H}$$

is a 1–step Willmore–Bäcklund transform of L. The branch points of g are the branch points of $\hat{L}_1$. Furthermore, g is the Möbius transform of a minimal surface in $\mathbb{H}$ or the stereographic projection of a holomorphic curve in $\mathbb{HP}^1$ with holomorphic twistor lift. If M is compact of genus g, then the Willmore energy of g satisfies

$$W(g) = 4\pi(2g - 2 + 2\deg(L^{-1}) - b(L)).$$

The map $H = -\beta(Sa)$ has a geometric meaning. Let $f = \sigma_{\alpha,\beta} L$, for $\alpha \in \mathbb{H}^2$ such that $\alpha(a) = 1$. H is then the quaternionic conjugate of the rotation by $\frac{\pi}{2}$ of the mean curvature vector of f in the normal bundle of f (cf., 14.6). This implies that if $f\colon M \to \mathbb{H}$ is the stereographic projection of a curve with holomorphic twistor lift, then the mean curvature vector of f rotated by $\frac{\pi}{2}$ in the normal bundle of f is again a Willmore surface.

Proof. Because L has holomorphic twistor lift, one gets $\nabla S = 2{*}Q$, by Lemma 16.2. Hence $d(\beta(Sa)) = 2\beta({*}Qa)$ and $g = \beta(Sa)$ is a 1–step Willmore–Bäcklund transform of L.

If β is admissible for L and $\mathrm{WBT}(L)$, then $dg = 2\beta({*}Qa)$ is zero at $p \in M$ if and only if Q is zero at $p \in M$. But the zeros of Q are the branch points of $\hat{L}_1$: Let $\varphi \in \Gamma(\hat{L}_1)$, then $2{*}Q\varphi = \nabla S\varphi = \nabla\varphi\mathbb{i} - S\nabla\varphi$. Thus Q is zero at $p \in M$ if and only if the image of $\nabla\varphi_{|p}$ is for all $\varphi \in \Gamma(\hat{L}_1)$ contained in the $\mathbb{i}$–eigenspace of S. For $\hat{L}_1$ is the $\mathbb{i}$–eigenspace of S, by Lemma 16.2, $\hat{L}_1$ is branched at p if and only if Q_p is zero.

Let $\tilde{L} \subset \mathbb{H}^2$ be a holomorphic curve that stereographically projects onto g. If $\tilde{L}$ does not have holomorphic twistor lift, then the Hopf field $\tilde{A}$ of $\tilde{L}$ does not vanish, by Lemma 16.2. From [**BFLPP02**, Lemma 10] then follows that $\ker \tilde{A}$ stereographically projects onto a 1–step Willmore–Bäcklund transform of L^d. Hence it is constant, because the Hopf field A of L vanishes. Hence $\ker A$ is constant and $\tilde{L}$ is a Euclidean minimal curve with pole $\ker A$, see 24.1.

The Euclidean holomorphic line bundle of g is the Möbius invariant holomorphic line bundle $(L^d)^{-1}$ of L^d, by Theorem 10.10, because the Weierstrass data of g is $(\mathrm{WBT}(L)^{-1}, (L^d)^{-1}, \beta, a)$. The Willmore energy of the Möbius invariant holomorphic line bundle L^{-1} of L vanishes, by Lemma 16.2, because L has holomorphic twistor lift. If M is compact of genus g, then one derives from the Plücker formula, as in 15.1, that $W(g) = W((L^d)^{-1}) = -4\pi(2-2g-2\deg(L^{-1})+b(L))$. $\square$

24. Euclidean Minimal Curves and Equality

The 2–step Willmore–Bäcklund transform of a holomorphic curve L in $\mathbb{HP}^1$ is a constant point if and only if the curve is Euclidean minimal (cf., the next paragraph). This is in the present section used to show that the canonical linear system of the 1–step Willmore–Bäcklund transform of such a curve has equality in the Plücker estimate. If $M = \mathbb{CP}^1$ one can go downwards on the ladder of holomorphic line bundles (applying Theorem 17.2) to get a 3–dimensional linear system with equality in the Plücker estimate that contains the canonical linear system of the Euclidean minimal curve.

24.1. Let L be a Euclidean minimal curve in $\mathbb{HP}^1$ (cf., 15.4) and S its mean curvature sphere on $M_0 = M \setminus \{\text{Branch points of } L\}$. The Euclidean (14.6) formula for $d^\nabla{*}A = d^\nabla{*}Q$ and the fact that the mean curvature vector of some stereographic projection of L vanishes identically imply $d^\nabla{*}A = d^\nabla{*}Q = 0$. Hence every Euclidean minimal curve in $\mathbb{HP}^1$ is Willmore (cf., 23.1). If ∞ is the pole of L then $\operatorname{im} Q \subset \infty \subset \ker A$. This follows again from the Euclidean formulas for A and Q of 14.6. Hence the 2–step Willmore–Bäcklund transform of a Euclidean minimal curve is its pole. If, on the other hand, Q (or A) does not vanish identically and there exists a point $\infty \in \mathbb{HP}^1$ such that $\operatorname{im} Q \subset \infty$ (or $\infty \subset \ker A$), then $S\infty = S$ and L is, consequently, a Euclidean minimal curve with pole ∞.

As for Willmore holomorphic curves, a Euclidean minimal curve is called *regular* if its mean curvature sphere extends smoothly into its branch points.

24.2. Theorem. *If a compact regular Euclidean minimal curve L in $\mathbb{HP}^1$ has a nonconstant closed 1–step Willmore–Bäcklund transform g, then the canonical linear system of g has equality in the Plücker estimate.*

Proof. The proof can be divided into two steps. The first is to show that ${*}Q^*((\mathbb{H}^2)^*) \subset H^0(KL^d)$ is a linear system with equality in the Plücker estimate. This in fact follows from Theorem 17.3. But we prefer to give the proof for the special case that is needed here. The second step is to go one step down on the ladder of holomorphic line bundles (applying Theorem 17.2) to show equality for the canonical linear system of g.

The 2–step Willmore–Bäcklund transform of the Euclidean minimal curve L is the pole ∞ of L. Hence $\infty^\perp \subset \ker Q^*$. Thus $Q^*\colon (\mathbb{H}^2)^*/\infty^\perp \to KL^d$ is a quaternionic bundle homomorphism. It is nonzero because g is nonconstant. If one equips $(\mathbb{H}^2)^*/\infty^\perp$ with the complex structure J that satisfies $-\pi S^* = J\pi$, where $\pi\colon (\mathbb{H}^2)^* \to (\mathbb{H}^2)^*/\infty^\perp$ is the canonical projection, then Q^* is complex linear.

Contemplate the 1–dimensional space $H = \{\, \alpha_{|\infty} \mid \alpha \in (\mathbb{H}^2)^* \,\}$ of sections of $(\mathbb{H}^2)^*/\infty^\perp$ and let $\tilde\nabla$ be the connection on $(\mathbb{H}^2)^*/\infty^\perp$ that makes H parallel. Then H is a linear system of holomorphic sections of $((\mathbb{H}^2)^*/\infty^\perp, \tilde\nabla'')$ with equality in the Plücker estimate: J, understood as an endomorphism of the trivial H–bundle, is the canonical complex structure of H. Since π is ∇^*–$\tilde\nabla$–parallel and $-\pi S^* = J\pi$ it follows that

$\nabla J\pi = -\pi(*Q^* - *A^*) = -\pi(*Q^*)$, since $\infty \subset \ker A$. Hence the "A" of J vanishes and the Plücker formula (14.7) implies that H has equality in the Plücker estimate.

Since $d^\nabla *Q = 0$ the map

$$*Q^*\colon ((\mathbb{H}^2)^*/\infty^\perp, \tilde{\nabla}'') \to KL^d$$

is holomorphic, as is shown in the proof of Proposition 23.3. The linear system $*Q^*(H) \subset H^0(KL^d)$, consequently, has equality in the Plücker estimate, by Proposition 17.1.

Let now $\beta \in (\mathbb{H}^2)^*$ and $a \in \mathbb{H}^2$ such that $\beta(a) = 0$ and $dg = 2*Q^*\beta(a)$. Let ∇_1 be the trivial connection of L^d such that $\nabla_1 a^{-1} = 0$. Then (L^d, ∇_1'') is Möbius invariant holomorphic line bundle of g, by Corollary 8.2, since $dg = 2*Q^*\beta(a)$ implies that the right normal vector of g is the normal vector of a^{-1}. Furthermore, $\tilde{H} := \operatorname{span}\{a^{-1}, a^{-1}\bar{g}\} \subset H^0(L^d)$ is the canonical linear system of the stereographic lift of g (cf., 8.5). Then

$$\nabla_1\tilde{H} = a^{-1}d\bar{g}\mathbb{H} = *Q^*\beta\mathbb{H} = *Q^*(H),$$

because $a^{-1}d\bar{g}(a) = dg$. d^{∇_1} is the holomorphic structure of KL^d, by Theorem 11.4, since $(\nabla_1^*)''$ by definition of ∇_1 the holomorphic structure of $(L^d)^{-1}$. Now Theorem 17.2 implies that the canonical linear system $\tilde{H}$ of g has equality in the Plücker estimate. □

24.3. Euclidean Minimal Spheres and Equality. If $M = \mathbb{C}\mathrm{P}^1$ the assumption in the theorem above that the 1–step Willmore–Bäcklund transform be closed is trivially satisfied. Moreover, it is possible to apply Theorem 17.2 once again and descend one step further on the ladder of holomorphic line bundles, i.e., from L^d to L^{-1}, which yields equality for a 3–dimensional linear system in the Möbius invariant holomorphic line bundle of L.

Theorem. *The regular Euclidean minimal spheres in $\mathbb{H}\mathrm{P}^1$ are soliton spheres. More precisely, the canonical linear system of a regular Euclidean minimal sphere has equality in the Plücker estimate or it is contained in a 3–dimensional linear system with equality.*

Proof. If the Hopf field Q of the mean curvature sphere of $L \subset \mathbb{H}^2$ is identically zero, then the canonical linear system of L has equality in the Plücker estimate. This follows from the Plücker formula (14.7) since $-Q^*$ is the "A" of the complex structure S^* of the canonical linear system of L (cf., 14.4). Assume that Q is not identically zero. Then L has a nontrivial and closed, as $\mathbb{C}\mathrm{P}^1$ is simply connected, Willmore–Bäcklund transform g.

Use the notation of the proof of Theorem 24.2 and choose α and b such that $\alpha, \beta \in (\mathbb{H}^2)^*$ is a basis and $a, b \in \mathbb{H}^2$ its dual. Let ∇_0 be the flat connection on L^{-1} such that $\nabla_0\beta_{|L} = 0$, and $\delta\colon L^d \to KL^{-1}$ be the derivative of L^d. Then $\delta(a^{-1}) = -\nabla_0\alpha_{|L}$, by Lemma 21.2, and δ is holomorphic, by Lemma 4.1. The connection ∇_0 induces a surjective quaternionic linear map $\nabla_0\colon H^0(L^{-1}) \to H^0(KL^{-1})$, because $\mathbb{C}\mathrm{P}^1$ is simply connected. It maps the canonical linear system of L onto $\delta(a^{-1})\mathbb{H} = \nabla_0\alpha_{|L}\mathbb{H} \subset \delta(\tilde{H})$, where $\tilde{H}$ is the canonical linear system of g as in the proof of Theorem 24.2. Consequently, there is a linear system $\tilde{\tilde{H}} \subset H^0(L^{-1})$ that contains the canonical linear system of L such that $\nabla_0\tilde{\tilde{H}} = \delta(\tilde{H})$. Because δ is holomorphic and $\tilde{H}$ has equality, Proposition 17.1 and Theorem 17.2 imply equality for $\tilde{\tilde{H}}$. $\square$

With the notation of the preceding two proofs, the 3–dimensional linear system with equality is spanned by $(\beta, \alpha = \beta\bar{f}, \beta\int(d\bar{f})\bar{g})$, where $f := \sigma_{\alpha,\beta}L$, because $\delta(a^{-1}) = -\nabla_0\alpha = -\beta d\bar{f}$ and $\delta(a^{-1}\bar{g}) = -\beta(d\bar{f})\bar{g}$.

24.4. The author found the following diagram helpful to keep track of the ingredients of the preceding two proofs:

$$
\begin{tikzcd}
((\mathbb{H}^2)^*/\infty^\perp, \tilde{\nabla}'') \arrow[r, "*Q^*", "\mathbf{17.1}"'] & (KL^d, d^{\nabla_1}) \arrow[dr, dash, "9.6"] & & \\
& (L^d, \nabla_1'') \arrow[u, dashed, "11.4"', "\mathbf{17.2}"] \arrow[r, dashed, dash, "11.3"] \arrow[dr, "\delta", "\mathbf{17.1}"'] & ((L^d)^{-1}, (\nabla_1^*)'') & \\
& & (KL^{-1}, d^{\nabla_0}) \arrow[dr, dash, "9.6"] & \\
& & (L^{-1}, \nabla_0'') \arrow[u, dashed, "11.4"', "\mathbf{17.2}"] \arrow[r, dashed, dash, "11.3"'] & (L, (\nabla_0^*)'')
\end{tikzcd}
$$

25. Willmore Spheres in S^4 and Equality

25.1. Robert Bryant in [**Br84**] showed that every immersed Willmore sphere in $\mathbb{R}^3$ has Willmore energy $4\pi n$ for some integer n. Furthermore, it has an n–fold point (which is unique if $n > 1$), and if one applies a Möbius transformation to the Willmore sphere that sends the n–fold point to infinity, then one gets a complete minimal surface with finite total curvature and n embedded planar ends. Conversely, every bounded Möbius transform of a complete immersed minimal sphere with finite total curvature and n embedded planar ends in $\mathbb{R}^3$ extends to a compact immersed Willmore sphere in $\mathbb{R}^3$ with Willmore energy $4\pi n$. In the language of the present text this can be rephrased as follows: If $M = \mathbb{C}\mathrm{P}^1$ and L is an immersed holomorphic sphere in $\mathbb{H}\mathrm{P}^1$ that lies in some 3–sphere, then L is Willmore if and only if L is Euclidean minimal. This result was generalized to S^4 by Nori Ejiri, [**Ej88**], Emilio Musso [**Mu90**], and Sebastián Montiel, [**Mo00**]: If $M = \mathbb{C}\mathrm{P}^1$ and L is an immersed Willmore holomorphic sphere in $\mathbb{H}\mathrm{P}^1$, then L or L^d have holomorphic twistor lift in $\mathbb{C}\mathrm{P}^3$, or L is Euclidean minimal. The proof of this fact in [**BFLPP02**, Section 11] applies verbatim to regular Willmore holomorphic curves and arbitrary compact Riemann surfaces M, and one gets the following theorem.

Theorem. *If M is compact of genus g and L is a regular Willmore holomorphic curve in $\mathbb{H}\mathrm{P}^1$ with total branching order $\operatorname{ord}\delta_L > 8(g-1)$, then L or L^d have holomorphic twistor lift or L is a Euclidean minimal curve with pole* $\mathrm{WBT}(L)$.

25.2. Willmore Spheres in $\mathbb{H}\mathrm{P}^1$. Because the condition on the total branching of L in Theorem 25.1 is vacuous for $g = 0$, one obtains the following corollary.

Corollary. *If $M = \mathbb{C}\mathrm{P}^1$ and L is a regular Willmore holomorphic sphere in $\mathbb{H}\mathrm{P}^1$, then L or L^d have holomorphic twistor lift or L is a regular Euclidean minimal sphere with pole* $\mathrm{WBT}(L)$*.*

Note that the three cases in the corollary are not disjoint. If L and L^d have holomorphic twistor lift, then both Hopf fields of the mean curvature sphere vanish identically, by Lemma 16.2. Thus $\nabla S = 2(*Q - *A) \equiv 0$ and L is a branched covering of a totally geodesic 2–sphere in $\mathbb{H}\mathrm{P}^1$. Lemma 20.5 implies that a Euclidean holomorphic sphere L with holomorphic twistor lift is the twistor projection of a planar holomorphic curve.

25.3. Theorem. *If $M = \mathbb{C}\mathrm{P}^1$ and L is a regular Willmore holomorphic sphere in $\mathbb{H}\mathrm{P}^1$, then L is a soliton sphere. More precisely, a regular Willmore holomorphic sphere in $\mathbb{H}\mathrm{P}^1$ has holomorphic twistor lift, its canonical linear system has equality in the Plücker estimate, or it is contained in a 3–dimensional linear system with equality.*

Proof. The theorem follows from Corollary 25.2 and Theorem 24.3. □

25.4. Construction of Willmore Spheres in $\mathbb{H}\mathrm{P}^1$ from Holomorphic Data. Theorem 25.3 implies that every regular Willmore holomorphic sphere in $\mathbb{H}\mathrm{P}^1$ can be obtained from a rational curve in $\mathbb{C}\mathrm{P}^5$ via twistor projection, dualization, and projection from $\mathbb{H}\mathrm{P}^2$ to $\mathbb{H}\mathrm{P}^1$. But it is hard to say which curves in $\mathbb{C}\mathrm{P}^5$ yield Willmore holomorphic curves in $\mathbb{H}\mathrm{P}^1$.

On the other hand, Theorem 24.2 together with Corollary 25.2 implies that the canonical linear system of every 1–step Willmore–Bäcklund transform g of a regular Willmore holomorphic sphere in $\mathbb{H}\mathrm{P}^1$ whose Hopf fields are nontrivial is a linear system with equality in the Plücker estimate. This and the fact that the 1–step Willmore–Bäcklund transformation composed with dualization is involutive (cf., 23.6) means that every regular Willmore holomorphic sphere in $\mathbb{H}\mathrm{P}^1$ can be obtained from a rational curve in $\mathbb{C}\mathrm{P}^3$. The corresponding construction yields for all rational complex holomorphic curves in $\mathbb{C}\mathrm{P}^3$ Willmore holomorphic spheres in $\mathbb{H}\mathrm{P}^1$. Moreover, the construction is, by Proposition 23.7, algebraic, i.e., only differentiation of polynomials and algebraic operations are involved:

Theorem. *Let $E \subset (\mathbb{H}^2, \mathbb{i})$ be a complex holomorphic curve in $P(\mathbb{H}^2, \mathbb{i})$ such that $E_1 \oplus E_1\mathbb{j} = \mathbb{H}^2$. Then L and L^d are regular Willmore holomorphic curves in $\mathbb{H}\mathrm{P}^1$. Let $L = T(E)$ be its twistor projection and S the mean curvature sphere of L extended into the branch points of L. If $\beta \in (\mathbb{H}^2)^*$, $a \in (\mathbb{H}^2)$ such that $\beta(a) = 0$, and β is admissible for L and* $\mathrm{WBT}(L)$*, then*

$$\beta(Sa)\colon\ \mathbb{C}\mathrm{P}^1 \to \mathbb{H}$$

is a branched conformal Willmore immersion with both normal vectors. Its branch points are the branch points of E_1.

If $M = \mathbb{C}\mathrm{P}^1$ and L is a regular Willmore holomorphic sphere in $\mathbb{H}\mathrm{P}^1$, then L or L^d has holomorphic twistor lift, or L can be obtained as described above.

See 23.7 for a geometric interpretation of the branched conformal immersion $\beta(Sa)$. See 27.3 for an explicit example of the construction of this theorem. If M is compact of genus g, then the Willmore energy of $\beta(Sa)$ is

$$W(\beta(Sa)) = 8\pi(g - 1 + \deg(L^{-1})) - 4\pi b(L),$$

by Proposition 23.7

Proof. In 16.4 it is shown that the mean curvature sphere of the twistor projection $L = T(E)$ of E extends smoothly into the branch points of L if and only if $E_1 \oplus E_1\mathbb{j} = \mathbb{H}^2$. Proposition 23.7 and the fact that the 1–step Willmore–Bäcklund transforms are branched conformal Willmore immersions (cf., 23.4) then imply the statement about $\beta(Sa)$. So it remains to show that if L is a regular Willmore holomorphic sphere such that neither L nor L^d has holomorphic twistor lift, then some stereographic projection of L^d is of the form $\beta(Sa)$, as described in the theorem.

Corollary 25.2 implies that L is a regular Euclidean minimal curve. The Hopf field Q of L does not vanish identically, since the twistor lift of L^d is by assumption not holomorphic. Let $g\colon \mathbb{CP}^1 \to \mathbb{H}$ be defined by $dg = 2{*}\beta(Qa)$. Then g is a nontrivial closed 1–step Willmore–Bäcklund transform of L and the canonical linear system of g has equality in the Plücker estimate, by Theorem 24.2.

Choose $\alpha \in (\mathbb{H}^2)^*$ and $b \in \mathbb{H}^2$ such that α, β is the dual basis of a, b. Let $L_g \subset \mathbb{H}^2$ be the Willmore holomorphic curve that stereographically projects onto g, i.e., $\sigma_{\alpha,\beta} L_g = g$. Then L_g is regular, by Lemma 20.2, and

$$d\sigma_{\alpha,\beta} L = 2{*}\beta(A_g \beta) = 2{*}d\beta(Sa)$$

for the mean curvature sphere S of L. The first equality follows from 23.6 and the second because $Q \equiv 0$, since L_g^d has holomorphic twistor lift, by Theorem 16.3. Thus up to a constant of integration $\sigma_{\alpha,\beta} L = 2{*}\beta(Sa)$. This implies

$$\sigma_{b,a} L^d = -\overline{\sigma_{\alpha,\beta} L} = -2{*}\overline{(S^*\beta)(a)} = -2{*}a(S^*\beta),$$

by Lemma 7.5. As S^* is the mean curvature sphere of L_g^d. S^* extends smoothly into the branch points of L_g^d, because L^d is regular. The holomorphic twistor lift $E := \widehat{L_g^d}$ of L_g^d thus satisfies $E_1 \oplus E_1\mathbb{j} = (\mathbb{H}^2)^*$. $\square$

26. $\mathbb{HP}^1$–Models of the 4–Dimensional Space Forms

In order to construct Willmore spheres in $\mathbb{R}^3$ with the technique of Theorem 25.4, one needs to understand the condition on the holomorphic curve E in $\mathbb{CP}^3$ which ensures that the 1–step Willmore–Bäcklund transform of its twistor projection takes values in $\mathbb{R}^3$. This is done in two steps. The first step is Jörg Richter's theorem (cf., 27.1) which says that the twistor projection of E has to be hyperbolic minimal. The second step is the description of the twistor lift of hyperbolic minimal spheres in $\mathbb{HP}^1$.

Although the Euclidean and spherical case is not needed for the present purpose, all three 4–dimensional space forms are treated in this section. This is done to show the remarkable similarity of the three cases in the quaternionic description.

26.1. The Metric 4–Spaces: $\mathbb{R}^4$, S^4, and H^4 Modeled on $\mathbb{HP}^1$. Let $\langle\,,\rangle$ be a nondegenerate Hermitian form on $\mathbb{H}^2$. The real part of $\langle\,,\rangle$ then induces a (pseudo–)Riemannian metric on $N = \{\, x \in \mathbb{H}^2 \mid |\langle x, x\rangle| = 1 \,\}$. This can be used, as in the definition of the Fubini–Study metric of $\mathbb{CP}^1$, to define a metric on $\mathbb{HP}^1$. In fact, there is a unique metric on $\mathbb{HP}^1 \setminus I$, where $I := \{\, [x] \in \mathbb{HP}^1 \mid \langle x, x\rangle = 0 \,\}$, such that $\pi\colon N \to \mathbb{HP}^1 \setminus I$, $x \mapsto [x]$ is a Riemannian submersion, i.e., $d_x\pi\colon (\ker d_x\pi)^\perp \to T_{\pi(x)}\,\mathbb{HP}^1$ is an isometry for all $x \in N$, where the orthogonal complement $(\ker d_x\pi)^\perp$ is meant in T_xN with respect to the real part of $\langle\,,\rangle$.

Lemma. *If $x \in \mathbb{HP}^1 \setminus I$ and $v, w \in T_{[x]}\,\mathbb{HP}^1 = \operatorname{Hom}([x], \mathbb{H}^2/[x])$, then*

$$(*)\qquad g_{[x]}(v,w) := \frac{1}{\langle x,x\rangle^2}\operatorname{Re}\big(\langle v(x), w(x)\rangle\langle x,x\rangle - \langle v(x), x\rangle\langle x, w(x)\rangle\big).$$

is the metric defined above, up to a sign which is constant on the components of $\mathbb{HP}^1 \setminus I$.

Proof. It is straight forward to check that the right hand side of $(*)$ does not depend on the choice of the vector in $\mathbb{H}^2$ that represents $[x]$, $v(x)$ and $w(x)$. It is, consequently, a well defined real bilinear form on $T_{[x]}\,\mathbb{HP}^1$. If $x \in N$ and $y \in (\ker d_x\pi)^\perp$, then $\langle x, y\rangle = 0$, because $\ker d_x\pi = [x] \cap T_xN$. Since $d_x\pi(y)(x) \equiv y \mod [x]$, one gets

$$g_{[x]}(d_x\pi(y), d_x\pi(y)) = \frac{1}{\langle x,x\rangle^2}\operatorname{Re}\big(\langle y,y\rangle\langle x,x\rangle - \langle y,x\rangle\langle x,y\rangle\big) = \pm\langle y,y\rangle.$$

$\square$

26.2. A Hermitian form is called *degenerate*, *definite*, or *indefinite* if its real part is a degenerate, definite, or indefinite symmetric form on $\mathbb{R}^8 = \mathbb{H}^2$. Multiplying a Hermitian form by a real nonzero constant does neither change its type nor the induced metric g on $\mathbb{HP}^1$. In what follows Hermitian forms are therefore considered equal, if they only differ by a real nonzero multiplicative constant.

Proposition. *Let $\langle\,,\rangle$ be a nontrivial Hermitian form and define $I := \{\, [x] \in \mathbb{HP}^1 \mid \langle x, x\rangle = 0 \,\}$ to be the set of isotropic points in $\mathbb{HP}^1$. If $\langle\,,\rangle$ is*

(i) degenerate, then $I = \{\infty\}$ for some point $\infty \in \mathbb{HP}^1$, and the stereographic projections with pole ∞ provide an identification $\mathbb{HP}^1 \setminus I \cong \mathbb{H}$ up to orientation preserving similarities of $\mathbb{H}$.

(ii) definite, then $I = \emptyset$ and there is a stereographic projection that maps $(\mathbb{HP}^1, 4g)$ isometrically onto $\mathbb{H}$ equipped with the metric induced by the stereographic projection of $S^4 \subset \mathbb{R}^5 = \mathbb{H} \times \mathbb{R}$ onto $\mathbb{H}$.

(iii) indefinite, then I is a 3–sphere in $\mathbb{HP}^1$ and there is a stereographic projection that maps the two components of $(\mathbb{HP}^1 \setminus I, -4g)$ isometrically onto $\mathbb{H} \setminus \operatorname{Im}\mathbb{H}$ if both components are equipped with the metric of the Poincaré half space model of hyperbolic 4–space.

The metrics in the cases (ii) and (iii) induce the standard conformal structure of $\mathbb{HP}^1$.

The last statement implies that the mean curvature sphere of a surface has the same mean curvature vector as the surface at the point of tangency in all three cases. Hence the mean curvature sphere of the curve in $\mathbb{HP}^1$ is the same as the mean curvature sphere in the metric (sub)space defined by a Hermitian form. Note that in case (iii) the Riemannian manifold $(\mathbb{HP}^1 \setminus I, -4g)$ consists of two hyperbolic 4–spaces which are glued together at their *ideal boundary*, i.e., the 3–sphere $S^3_\infty := I$.

PROOF. (i) As $\langle\,,\rangle$ is nontrivial and degenerate, there is exactly one point $\infty \in \mathbb{HP}^1$ such that $x \in \infty$ is equivalent to $\langle x, x\rangle = 0$. The rest follows from 5.2.

(ii) Assuming without loss of generality that $\langle\,,\rangle$ is positive definite, there exists a basis $a, b \in \mathbb{H}^2$ such that $\langle a, a\rangle = \langle b, b\rangle = 1$ and $\langle a, b\rangle = 0$. Let $\alpha, \beta \in (\mathbb{H}^2)^*$ be its dual basis. Then $\sigma^{-1}_{\alpha,\beta}\colon \mathbb{H} \to \mathbb{HP}^1 \setminus \{[a]\}$ satisfies $\sigma^{-1}_{\alpha,\beta}(\lambda) = [a\lambda + b]$, and for $\lambda, \mu \in \mathbb{H}$ one gets

$$d_\lambda \sigma^{-1}_{\alpha,\beta}(\mu)(a\lambda + b) \equiv a\mu \mod [a\lambda + b],$$

by 5.3, and

$$(\sigma^{-1}_{\alpha,\beta})^* g_\lambda(\mu, \mu) = \tfrac{\langle a\mu, a\mu\rangle\langle a\lambda+b, a\lambda+b\rangle - \langle a\mu, a\lambda+b\rangle\langle a\lambda+b, a\mu\rangle}{\langle a\lambda+b, a\lambda+b\rangle^2} = \frac{|\mu|^2}{(1 + |\lambda|^2)^2}.$$

The assertion follows, since $\frac{4|\mu|^2}{(1+|\lambda|^2)^2}$ is the induced metric of the stereographic projection of S^4 to $\mathbb{R}^4 = \mathbb{H}$.

(iii) In this case I is a 3–sphere in $\mathbb{HP}^1$ (cf., 5.4) and there exists a basis $a, b \in \mathbb{H}^2$ such that $\langle a, a\rangle = \langle b, b\rangle = 0$ and $\langle a, b\rangle = 1$. For $\lambda \in \mathbb{H} \setminus \operatorname{Im}\mathbb{H}$, $\mu \in \mathbb{H}$ one then obtains as before

$$(\sigma^{-1}_{\alpha,\beta})^* g_\lambda(\mu, \mu) = -\frac{|\mu|^2}{4\operatorname{Re}^2(\lambda)}.$$

The assertion follows, since $\frac{|\mu|^2}{\operatorname{Re}^2(\lambda)}$ is the metric of the Poincaré half space model of H^4.

□

26.3. Isometries of $\mathbb{R}^4$, S^4, and H^4. If $\langle\,,\rangle$ is a nondegenerate Hermitian form and $M \in \mathrm{GL}(\mathbb{H}, 2)$, then the Möbius transformation of $\mathbb{HP}^1$ represented by M is an isometry of $(\mathbb{HP}^1 \setminus I, g)$ if and only if M preserves the Hermitian form, i.e., there exists a constant $c \in \mathbb{R}$ such that $\langle Mx, Mx\rangle = c\langle x, x\rangle$ for all $x \in \mathbb{H}^2$, see Lemma 26.1. All orientation preserving isometries of $(\mathbb{HP}^1 \setminus I, \pm g)$ are obtained this way (cf., [**Jeromin**, Theorem 1.3.14 & Lemma 1.4.13]). If the Hermitian form is degenerate, then M induces a similarity of $\mathbb{HP}^1 \setminus \{\infty\} = \mathbb{H}$ if and only if M fixes ∞ (cf., 5.2), which is equivalent to M preserving the Hermitian form. All similarities of $\mathbb{HP}^1 \setminus \{\infty\} = \mathbb{H}$ are of this form (cf., 5.2).

26.4. Euclidean, Spherical and Hyperbolic Minimal Curves. Proposition 26.2 justifies the following definitions. A curve L in $\mathbb{HP}^1$ is called a *spherical minimal curve*, if there is a definite Hermitian form such that L is minimal in $(\mathbb{HP}^1, g)$. A curve L in $\mathbb{HP}^1$ is called a *hyperbolic minimal curve*, if there is an indefinite Hermitian form such that L is not completely contained in its 3–sphere S^3_∞ of isotropic lines and L is minimal in $(\mathbb{H}\mathrm{P}^1 \setminus S^3_\infty, -g)$. Euclidean minimal curves are defined in 15.4. In all three cases minimality of the curve is equivalent to its mean curvature sphere being skew Hermitian.

Theorem. *Let $L \subset \mathbb{H}^2$ be a holomorphic curve in $\mathbb{H}\mathrm{P}^1$ and S its mean curvature sphere. L is Euclidean, spherical or hyperbolic minimal, if and only if there exists a degenerate, definite, or indefinite Hermitian form, respectively, for which S is skew Hermitian. In all three cases L is a Willmore holomorphic curve. In the hyperbolic case L intersects the ideal boundary orthogonally.*

PROOF. Assume, without loss of generality, that L has no branch points.

If S is skew for a degenerate Hermitian form $\langle\, , \rangle$ with isotropic line $\infty = [a] \in \mathbb{H}\mathrm{P}^1$, then $\langle Sa, x\rangle = -\langle a, Sx\rangle = 0$ for all $x \in \mathbb{H}^2$. Thus ∞ is a fixed point of S and, consequently, lies for all $p \in M$ on the mean curvature sphere S_p. This implies that the stereographic projection of L with pole ∞ is a minimal conformal immersion. If L is a Euclidean minimal curve with pole ∞, then there is a degenerate Hermitian form with degenerate direction ∞. The mean curvature sphere fixes $\infty = [a]$, see 14.6. For every $x \in \mathbb{H}^2 \setminus \{\infty\}$ there exists $\lambda \in \mathbb{H}$ such that $Sx \equiv x\lambda \mod [a]$. Then $\lambda^2 = -1$ and $\langle Sx, x\rangle = \langle x\lambda, x\rangle = -\langle x, x\lambda\rangle = -\langle x, Sx\rangle$, thus S is skew. The fact that Euclidean minimal curves are Willmore is proven in 24.1. This finishes the proof of the Euclidean case.

Suppose now that S is skew for a nondegenerate form. The skewness of S implies $\langle Sx, Sx\rangle = \langle x, x\rangle$ for all $x \in \mathbb{H}^2$. Fix a point $p \in M$. The Möbius involution induced by S_p is then an isometry for the metric g of 26.1. The sphere defined by S_p is the fixed point set of this isometry. It is not the identity, squares to the identity and preserves orientation. Thus, S_p is minus the identity on the normal space of S_p. Consequently, the sphere defined by S_p is totally geodesic in $(\mathbb{H}\mathrm{P}^1, g)$ or $(\mathbb{H}\mathrm{P}^1 \setminus S^3_\infty, g)$. This implies that if the Hermitian form is definite, then L is minimal in $(\mathbb{H}\mathrm{P}^1, g)$, and if the form is indefinite, then L is minimal on the preimage of $\mathbb{H}\mathrm{P}^1 \setminus S^3_\infty$. In the indefinite case let now $p \in M$ such that $L_p \in S^3_\infty$. Because S_p is skew Hermitian for $\langle\, , \rangle$, it does not lie in S^3_∞ (cf., 5.6). Hence it intersects S^3_∞ orthogonally. Since, S_p is tangent to L at L_p, L also intersects S^3_∞ orthogonally at p .

If L is now a spherical minimal curve with respect to the Hermitian form $\langle\, , \rangle$, then its mean curvature spheres S_p are totally geodesic for all $p \in M$. Let $[a] \in S_p$ and $[b] \in \mathbb{H}\mathrm{P}^1$ such that $\langle a, a\rangle = \langle b, b\rangle = 1$ and $\langle a, b\rangle = 0$. Let $\alpha, \beta \in (\mathbb{H}^2)^*$ be its dual basis. The stereographic projection $\sigma_{\alpha,\beta}$ maps S_p onto a two plane in $\mathbb{H}$. Since S_p is totally geodesic its projection is the stereographic projection of a great sphere. Hence the two plane contains the origin. Thus $[b] \in S_p$ and there are $N, R \in \mathbb{H}$ such that $Sa = aN$, $Sb = bR$ and $N^2 = R^2 = -1$. Hence S_p is skew Hermitian with respect to the Hermitian form $\langle\, , \rangle$.

If L is hyperbolic minimal with respect to the indefinite Hermitian form $\langle\, , \rangle$, then its mean curvature spheres S_p are totally geodesic for all $p \in M$ such that $L_p \notin S^3_\infty$. Hence S_p intersects S^3_∞ in a circle. This implies that there are $[a], [b] \in S_p$ such that $\langle a, a\rangle = \langle b, b\rangle = 0$ and $\langle a, b\rangle = 1$. The stereographic projection $\sigma_{\alpha,\beta}$ maps S_p onto a 2–plane orthogonal to $\operatorname{Im}\mathbb{H}$. In particular, $[a+b] \in S_p$. Let $N, R \in \mathbb{H}$ such that $S_p a = aN$, $S_p b = bR$, and $N^2 = R^2 = -1$. Then $[a+b] \in S_p$ implies $N = R$. Hence S_p is skew for all $p \in M$ such that $L_p \notin S^3_\infty$. Suppose now that there is $p \in M$ such that S_p is not skew, then there exists a whole open neighborhood $U \subset M$ of p such that S_q is not skew for all $q \in U$. Hence $L_q \in S^3_\infty$ for all $q \in U$. But the mean curvature sphere S_p of L at p is then contained in S^3_∞. So S_p is Hermitian (cf., 5.6). Thus for all $p \in M$ either S_p is Hermitian or skew Hermitian. Since M is connected and there is at least one point $p \in M$ for which $L_p \notin S^3_\infty$, by the definition of a hyperbolic minimal curve, S_p has to be skew for all $p \in M$.

It is now easy to see that the spherical and hyperbolic minimal curves are Willmore: Write $()^*$ for the dual of an endomorphism or orthogonal complement of a subspace with respect to the Hermitian form. Then $S^* = -S$ implies $Q^* = -Q$ and $A^* = -A$ for the Hopf fields of S. Hence $d^\nabla {*A^*} = -d^\nabla {*A}$. Consequently, $(\ker d^\nabla {*A})^* = \operatorname{im} d^\nabla {*A}$. This and $\operatorname{im} d^\nabla {*A} \subset L \subset \ker d^\nabla {*A}$, see the Euclidean formula for $d^\nabla {*A}$ in 14.6, implies $L^* + L \subset \ker d^\nabla {*A}$. In the spherical case $L^* + L = \mathbb{H}^2$, and in the hyperbolic case $L_p^* + L_p = \mathbb{H}^2$ for all $p \in M$ such that $L_p \notin S^3_\infty$. But the points p with $L_p \in S^3_\infty$ form a submanifold of dimension one, because L and S^3_∞ intersect transversally. Thus in both cases $d^\nabla {*A}$ vanishes on all of M. This implies that L is a Willmore holomorphic curve (cf., 23.1). □

26.5. As in the Euclidean case, although there are no compact hyperbolic minimal surfaces, there are compact hyperbolic minimal curves in $\mathbb{HP}^1$. These curves have to pass through the ideal boundary of the two hyperbolic spaces defined by the indefinite Hermitian form. An example of such a surface is given in 27.3.

26.6. Superminimal Surfaces in $\mathbb{R}^4$, S^4, and H^4. A surface in S^4 is called superconformal (holomorphic or t–holomorphic) if its twistor lift is holomorphic, see [**Fr84**] and [**Fr88**]. A surfaces in S^4 is superconformal if and only if its mean curvature ellipse is a circle (cf., [**BFLPP02**, section 8.2]). A surface is called superminimal if it is superconformal and minimal. There is a nice overview article on superminimal surface by Thomas Friedrich [**Fr97**]. In this article the hyperbolic superminimal surfaces are discussed in detail as an example of the general construction.

The following precise definition is adopted in the present text: A holomorphic curve L in $\mathbb{HP}^1$ is called Euclidean, spherical, or hyperbolic *superminimal* with positive (or negative) spin[3], if L is Euclidean, spherical, or hyperbolic minimal, and L (or L^d) has holomorphic twistor lift.

If $M = \mathbb{CP}^1$ one gets the following result.

26.7. Theorem. *If $M = \mathbb{CP}^1$, then every regular spherical or hyperbolic minimal sphere in $\mathbb{HP}^1$ is superminimal.*

[3]The notion of spin of a superminimal surface was introduced by Robert Bryant, [**Br82**]. In [**Fr84**] positive spin is build into the definition of superminimal surfaces.

For immersed minimal spheres in S^4 this result was obtained by Robert Bryant (cf., [**Br82**, Theorem C]).

PROOF. If L is a regular spherical or hyperbolic minimal sphere in $\mathbb{HP}^1$, then it is a regular Willmore holomorphic sphere whose mean curvature sphere is skew with respect to some nondegenerate Hermitian form $\langle\,,\rangle$, by Theorem 26.4. Suppose that neither L nor $L^\perp$ has holomorphic twistor lift. Then the Hopf fields Q and A of S do not vanish identically, and L is a Euclidean minimal curve with pole $\operatorname{im} Q$, by Corollary 25.2. The skewness of S implies $Q^* = -Q$, where * denotes the dual endomorphism with respect to $\langle\,,\rangle$. But then $L \subset \ker Q = (\operatorname{im} Q)^*$. This implies that L is constant, contradiction. $\square$

Using Theorem 25.1 instead of Corollary 25.2 one gets that every regular spherical or hyperbolic minimal sphere in $\mathbb{HP}^1$ whose total branching order satisfies $\operatorname{ord}\delta_L > 8(g-1)$ is superminimal.

26.8. The Complexified Light Cone Model of S^4. To describe superminimality in terms of the twistor lift, the following link between the quaternionic model and the light cone model of the conformal 4–spheres is used in the next theorem. Every element of the Grassmannean $G_2(\mathbb{H}^2, \mathbb{i})$ is either a quaternionic 1–dimensional subspaces, i.e., a point in $\mathbb{HP}^1$, or the $\mathbb{i}$–eigenspace of a quaternionic endomorphism S that squares to -1, i.e., a 2–sphere in $\mathbb{HP}^1$. The *Plücker embedding* (cf., [**Harris**, Lecture 6])

$$\begin{array}{rccc} Pl\colon & G_2(\mathbb{H}^2,\mathbb{i}) & \longrightarrow & P\Lambda^2(\mathbb{H}^2,\mathbb{i}) \cong \mathbb{CP}^5 \\ & \operatorname{span}\{x,y\} & \longmapsto & [x\wedge y]_{\mathbb{C}} \end{array}$$

embeds $G_2(\mathbb{H}^2, \mathbb{i})$ as the *Plücker quadric*

$$Q^4 = P\{\, v \in \Lambda^2(\mathbb{H}^2,\mathbb{i}) \mid v\wedge v = 0\,\},$$

into the 5–dimensional complex projective space $P\Lambda^2(\mathbb{H}^2, \mathbb{i})$.

The quaternionic structure of $(\mathbb{H}^2, \mathbb{i})$ induces a real structure on $\Lambda^2(\mathbb{H}^2, \mathbb{i})$

$$\overline{x\wedge y} := x\mathbb{j}\wedge y\mathbb{j}.$$

The real part of Q^4 then corresponds to $\mathbb{HP}^1$, $Q^4\setminus \operatorname{Re} Q^4$ corresponds to the set of oriented 2–spheres in $\mathbb{HP}^1$, and the $\wedge$–product defines a symmetric bilinear form on the real 6–dimensional vector space $\operatorname{Re}\Lambda^2(\mathbb{H}^2, \mathbb{i})$. This product is a Minkowski product: If $a, b \in \mathbb{H}^2$ is a basis, then the $\wedge$–product is represented by the diagonal matrix $(-1, 1, 1, 1, 1, 1)$ in the basis

$$\begin{aligned} &a\wedge a\mathbb{j} + b\wedge b\mathbb{j}, \quad a\wedge a\mathbb{j} - b\wedge b\mathbb{j}, \quad a\wedge b + a\mathbb{j}\wedge b\mathbb{j}, \\ &\qquad -(a\wedge b - a\mathbb{j}\wedge b\mathbb{j})\mathbb{i}, \quad a\wedge b\mathbb{j} - a\mathbb{j}\wedge b, \quad -(a\wedge b\mathbb{j} + a\mathbb{j}\wedge b)\mathbb{i} \end{aligned}$$

of $\operatorname{Re}\Lambda^2(\mathbb{H}^2, \mathbb{i})$ and $2a\wedge b\wedge a\mathbb{j}\wedge b\mathbb{j}$ of $\operatorname{Re}\Lambda^4(\mathbb{H}^2, \mathbb{i})$. Thus $\operatorname{Re} Q^4$ is the projectivised light cone of the 6–dimensional Minkowski space $(\operatorname{Re}\Lambda^2(\mathbb{H}^2, \mathbb{i}), \wedge)$.

26.9. The Twistor Lift of a Superminimal Curve in $\mathbb{HP}^1$. The $\mathbb{i}$–eigenspaces of the mean curvature sphere of a holomorphic curve L in $\mathbb{HP}^1$ with holomorphic twistor $\hat{L}$ are, by Lemma 16.2, the osculating lines (or the tangent line congruence) $\hat{L}_1$ of the twistor lift $\hat{L}$. Minimality of L can thus be described in terms of the tangent line congruence $\hat{L}_1$ of $\hat{L}$:

Theorem. *A holomorphic curve L is superminimal with positive spin if and only if its twistor lift $\hat{L}$ is holomorphic and there exists $[h]$, $h \in \operatorname{Re}\Lambda^2(\mathbb{H}^2, \mathbb{i})$ polar to the Plücker embedding $Pl(\hat{L}_1)$ of $\hat{L}_1$, i.e., $h \wedge v = 0$ holds for all $p \in M$ and $v \in Pl(\hat{L}_1)_{|p}$. If h is light (time, space) like then L is Euclidean (spherical, hyperbolic) minimal with respect to the Hermitian form*

$$(*) \qquad \langle x, y\rangle = -h \wedge x\mathbb{j} \wedge y + \mathbb{j}(h \wedge x \wedge y).$$

Proof. If one identifies $\Lambda^4(\mathbb{H}^2, \mathbb{i}) \cong \mathbb{C} \cong \mathbb{R} \oplus \mathbb{R}\mathbb{i} \subset \mathbb{H}$ such that $\operatorname{Re}\Lambda^4(\mathbb{H}^2, \mathbb{i}) \cong \mathbb{R}$, then $(*)$ defines a Hermitian form on $\mathbb{H}^2$ (up to a real factor), because h is real. The light like vectors $v \in \operatorname{Re}\Lambda^2(\mathbb{H}^2, \mathbb{i})$ can be written $v = x \wedge x\mathbb{j}$ for some $x \in \mathbb{H}^2$, because $\operatorname{Re} Q^4$ corresponds to $\mathbb{HP}^1$. Then $h \in \operatorname{Re}\Lambda^2(\mathbb{H}^2, \mathbb{i})$ is orthogonal to $v = x \wedge x\mathbb{j}$ if and only if $\langle x, x\rangle = 0$. Thus h is light, time, or space like if and only if the set of isotropic lines of the Hermitian form $(*)$ contains one, zero, or infinitely many points, respectively. Proposition 26.2 then implies that the form $(*)$ is degenerate, definite, or indefinite.

Suppose that the twistor lift $\hat{L}$ of L is holomorphic and that $Pl(\hat{L}_1)$ is polar to $[h]$, $h \in \operatorname{Re}\Lambda^2(\mathbb{H}^2, \mathbb{i})$. The quaternionic extension of $x \mapsto x\mathbb{i}$ on $\hat{L}_1$ is then the mean curvature sphere S of L, by Lemma 16.2. To see that S is skew with respect to the Hermitian form $(*)$ fix a point $p \in M$ and choose $x \in \mathbb{H}^2$. Then one can write $x = x_1 + x_2\mathbb{j}$ with $x_i \in \hat{L}_{1|p}$, and $h \wedge x_1 \wedge x_2 = 0$, since $[h]$ is polar to $Pl(\hat{L}_1)_{|p}$. Hence $\langle x_i, x_j\rangle \in \mathbb{C}$, for $i, j = 1, 2$. This implies $\langle S_p x_i, x_j\rangle = -\mathbb{i}\langle x_i, x_j\rangle = -\langle x_i, S_p x_j\rangle$. Hence S is skew Hermitian.

To see the converse, suppose that L is superminimal with respect to the Hermitian form $\langle\,,\rangle$. Then there exists $h \in \operatorname{Re}\Lambda^2(\mathbb{H}^2, \mathbb{i})$ such that $(*)$ holds, because $\operatorname{Re}\Lambda^2(\mathbb{H}^2, \mathbb{i})$ and the space of Hermitian forms on $\mathbb{H}^2$ are real 6–dimensional vector spaces and the map described by $(*)$ is an injective linear map between these spaces. Let now $x_{1,2} \in \hat{L}_1 atp \subset (\mathbb{H}^2, \mathbb{i})$ for some $p \in M$, then $S_p x_{1,2} = x_{1,2}\mathbb{i}$ and $\mathbb{i}\langle x_1, x_2\rangle = -\langle S_p x_1, x_2\rangle = \langle x_1, x_2\rangle\mathbb{i}$, since S_p is skew Hermitian. Hence $\langle x_1, x_2\rangle \in \mathbb{C}$, which implies $h \wedge x_1 \wedge x_2 = 0$. Consequently, $[h]$ is polar to $Pl(\hat{L}_1)$. □

26.10. Remark. Holomorphic curves in $\mathbb{CP}^3$ whose tangent lines all lie in the intersection of the Plücker quadric $Q^4 \subset \mathbb{CP}^5$ with some projective hyperplane $\mathbb{CP}^4$, as in the theorem, are well studied, see for example [**Bol**, §54]. Choosing a suitable local coordinate z and suitable homogeneous coordinates of $\mathbb{CP}^3$, all such curves are locally of the form

$$(1, z, F'(z), zF'(z) - 2F(z))$$

for some holomorphic function F.

A curve in Q^4 is locally the curve of osculating lines of a curve in $\mathbb{CP}^3$ if and only if it is null, i.e., its tangent lines are contained in Q^4. So hyperbolic and spherical superminimal curves correspond to holomorphic null curves in the nondegenerate 3–dimensional quadric $Q^3 = Q^4 \cap \mathbb{CP}^4 \subset \mathbb{CP}^5 \cong P\Lambda^2(\mathbb{H}^2, \mathbb{i})$. In the Euclidean case $Q^4 \cap \mathbb{CP}^4$ is a degenerate quadric.

The condition that the tangent lines of a holomorphic curve $E \subset \mathbb{C}^4$ are as points in the Plücker quadric $Q^4 \subset \mathbb{CP}^5$ polar to a fixed point in

$\mathbb{C}\mathrm{P}^5 \setminus Q^4$ implies that the dual curve $E^d \subset (\mathbb{C}^4)^*$ is the same curve as E: Let φ be a local holomorphic section of E and $h \in \Lambda^2(\mathbb{C}^4) \setminus Q^4$ such that $h \wedge \varphi \wedge \varphi' = 0$. Then $h\wedge\colon \mathbb{C}^4 \to \Lambda^3(\mathbb{C}^4)$ is an isomorphism. If one identifies $\Lambda^4(\mathbb{C}^4)$ with $\mathbb{C}$, then $\Lambda^3(\mathbb{C}^4) \cong (\mathbb{C}^4)^*$ and $h\wedge$ maps φ onto a section of the dual curve, because $h \wedge \varphi \wedge \varphi = h \wedge \varphi \wedge \varphi' = h \wedge \varphi \wedge \varphi'' = 0$. In particular, E is either full or takes values in some projective line.

27. Willmore Spheres in $\mathbb{R}^3$

The results of the previous section and a theorem by Jörg Richter [**Ri97**] are now applied to specialize the construction of Willmore spheres in $\mathbb{H}\mathrm{P}^1$ from 25.4 to Willmore spheres in $\mathbb{R}^3$.

27.1. Hyperbolic Minimal Curves in $\mathbb{H}\mathrm{P}^1$ and Branched Conformal Willmore Immersions into $\mathbb{R}^3$. Let L be a regular hyperbolic minimal curve in $\mathbb{H}\mathrm{P}^1$, $\langle\, ,\rangle$ the corresponding indefinite Hermitian form and $S^3_\infty \subset \mathbb{H}\mathrm{P}^1$ the 3–sphere of isotropic lines of $\langle\, ,\rangle$. Let $[a] \in S^3_\infty$ be some point in this 3–sphere and $g\colon M \to \mathbb{H}$ such that

$$dg = 2{*}\langle a, Qa\rangle,$$

where Q is the Hopf field of L. Then g is a 1–step Willmore–Bäcklund transform of L, since $\beta := \langle a, \cdot\rangle \in (\mathbb{H}^2)^*$ satisfies $\beta(a) = 0$. Because the mean curvature sphere of L is skew, by Theorem 26.4, one gets

$$\overline{dg} = 2{*}\overline{\langle a, Qa\rangle} = 2{*}\langle Qa, a\rangle = -2{*}\langle a, Qa\rangle = -dg.$$

Hence, g takes values in $\operatorname{Im}\mathbb{H}$, up to some translation. Conversely, if L is a regular Willmore holomorphic curve in S^3_∞, then its 1–step Willmore–Bäcklund transforms with $[a] \in S^3_\infty$ are up to a translation hyperbolic minimal in the two Poincaré half spaces $\mathbb{H} \setminus \operatorname{Im}\mathbb{H}$., by Jörg Richter's theorem (cf., [**Ri97**] or [**BFLPP02**, Theorem 9]).

27.2. 1–Step Willmore–Bäcklund Transformation of Hyperbolic Superminimal Curves in $\mathbb{H}\mathrm{P}^1$. The observation of the previous paragraph implies that the 1–step Willmore–Bäcklund transforms of hyperbolic superminimal curves, which can be obtained without integration, by Proposition 23.7, take values in $\mathbb{R}^3$, and if $M = \mathbb{C}\mathrm{P}^1$ then all regular Willmore spheres in S^3 are obtained this way.

Theorem. *Let L be a regular hyperbolic superminimal curve in $\mathbb{H}\mathrm{P}^1$ with positive spin, $\langle\, ,\rangle$ the corresponding indefinite Hermitian form, S the mean curvature sphere of L, and S^3_∞ the ideal boundary of the hyperbolic spaces. Let $[a] \in S^3_\infty \setminus (L \cup L^\perp)$. Then*

$$\langle a, Sa\rangle\colon M \to \operatorname{Im}\mathbb{H}$$

is a branched conformal Willmore immersion into $\operatorname{Im}\mathbb{H}$ *with both normal vectors. Its branch points are the branch points of the tangent line congruence of the holomorphic twistor lift of L. If M is compact of genus g, then its Willmore energy is*

$$W(\langle a, Sa\rangle) = 4\pi(2g - 2 + 2\deg(L^{-1}) - b(L)),$$

and its total branching

$$b(\langle a, Sa\rangle) = 6g - 6 + 2\deg(L^{-1}) - 2b(L).$$

Furthermore, $\langle a, Sa\rangle$ is the Möbius inversion of a complete minimal surface of finite total curvature with $n = 2g-2+2\deg(L^{-1})-b(L)$ planar or Enneper type ends (counted with multiplicity). Its n–fold point is the origin.

Every regular Willmore holomorphic sphere in S^3, besides coverings of the round sphere, arises this way.

The map $\langle a, Sa\rangle$ allows the following geometric interpretation: If $\beta = \langle a, \cdot\rangle$ and $\alpha \in (\mathbb{H}^2)^*$ such that $\alpha(a) = 1$, then $\langle a, Sa\rangle$ is the mean curvature vector of $f = \sigma_{\alpha,\beta}L$ rotated by $\frac{\pi}{2}$ in the normal bundle of f (cf., 14.6 and use $\bar{H} = -H$).

This theorem in connection with Theorem 26.9 describes a construction for all regular branched conformal Willmore immersions of $\mathbb{C}\mathrm{P}^1$ into $\mathbb{R}^3$ from rational curves in $\mathbb{C}\mathrm{P}^3$ whose tangent lines satisfy a linear equation, in other words, from 4 polynomials whose derivatives satisfy a linear equation. The construction is algebraic, since the $\mathbb{i}$–eigenspaces of the mean curvature spheres S_p, $p \in M$, are the tangent lines of the holomorphic twistor lift L, and differentiation of polynomials is an algebraic operation. Moreover, the Willmore spheres in $\mathbb{R}^3$ can be constructed directly from the $\mathbb{i}$–eigenspaces of the mean curvature sphere congruence (cf., 27.3), whose Plücker embedding is a holomorphic null curve in $Q^3 = Q^4 \cap \mathbb{C}\mathrm{P}^4 \subset P(\Lambda^2(\mathbb{H}^2, \mathbb{i}))$, see 26.10. A similar construction was used by Robert Bryant (cf., [**Br84**] and [**Br88**]) to investigate the moduli space of immersed Willmore spheres of low Willmore energy.

Proof. Since S is skew, by Theorem 26.4, one gets $\overline{\langle a, Sa\rangle} = -\langle a, Sa\rangle$. Thus $\langle a, Sa\rangle$ takes values in $\operatorname{Im}\mathbb{H}$. Furthermore, $\operatorname{im} Q = (\ker Q) \perp$, where $\perp$ denotes the orthogonal complement with respect to $\langle\, ,\rangle$. Hence $\mathrm{WBT}(L) = L^{\perp}$. Thus $\langle a, Sa\rangle$ is a branched conformal Willmore immersion into $\operatorname{Im}\mathbb{H}$ with both normal vectors whose branch points are the branch points of the tangent line congruence of the holomorphic twistor lift of L, by Proposition 23.7. Furthermore, if M is compact of genus g, then the Willmore energy of $\langle a, Sa\rangle$ satisfies

$$W(\langle a, Sa\rangle) = 4\pi(2g-2+2\deg(L^{-1})-b(L)).$$

The derivative δ of L is a holomorphic section of $K\mathrm{Hom}_+(L, (L^d)^{-1})$, by Lemma 20.3. Thus

$$b(L) = 2g-2+\deg(L^d)^{-1}+\deg(L^{-1}).$$

$(L^d)^{-1}$ is the Euclidean holomorphic line bundle of $\langle a, Sa\rangle$, by Proposition 23.7 and the definition of the 1–step Willmore–Bäcklund transformation (cf., 23.4). Thus

$$\deg(L^d)^{-1} = g-1-\frac{1}{2}b(\langle a, Sa\rangle),$$

by Theorem 12.4. Hence $b(L) = 3g-3+\deg(L^{-1})-\frac{1}{2}b(\langle a, Sa\rangle)$.

If one shows that the origin is an n–fold point,

$$n = 2g-2+2\deg(L^{-1})-b(L),$$

of $\langle a, Sa\rangle$, then $\langle a, Sa\rangle$ is the Möbius inversion of a complete minimal surface of finite total curvature with n planar or Enneper type ends (counted with multiplicity), by 2.4, because $W(\langle a, Sa\rangle) = 4\pi(2g-2+2\deg(L^{-1})-b(L))$.

Let $\beta := \langle a, \cdot\rangle$ and $\alpha \in (\mathbb{H}^2)^*$ such that $\alpha(a) = 1$. Let R be the right normal vector of $\sigma_{\alpha,\beta}L$, then $H = -\langle Sa, a\rangle$ and $dR' = Hdf$ (cf., 14.6). But since L has holomorphic twistor lift, the Hopf field A of L vanishes and $dR'' = 0$. Thus $dR = Hdf$ and R is a holomorphic map from M to $\mathbb{CP}^1$. Furthermore, $\deg R = \deg(L^{-1})$, by 7.4 and 3.2. Hence the Riemann–Hurwitz formula [**GriHa**] implies that the branching order of R satisfies

$$b(R) = 2g - 2 + 2\deg(L^{-1}),$$

and $dR = Hdf$ implies for the total vanishing order of H satisfies $\operatorname{ord} H = 2g - 2 + 2\deg(L^{-1}) - b(L) = n$.

A holomorphic curve in S^3 with holomorphic twistor lift takes values in some round sphere, because $S = S^*$ (with respect to the Hermitian form that corresponds to S^3) implies that if one Hopf field of S vanishes then the other one vanishes too. Thus, Theorem 25.4 and Jörg Richter's Theorem (cf., 27.1) imply that all regular Willmore holomorphic spheres in S^3, besides coverings of the round sphere, can be obtained by the construction of the theorem. $\square$

27.3. Example: Immersed Willmore Spheres in $\mathbb{R}^3$ with Willmore Energy 16π. According to a result of Robert Bryant (cf., [**Br88**]) the immersed Willmore spheres in $\mathbb{R}^3$ have Willmore energy $4\pi n$ for $n \in \mathbb{N}\backslash\{0, 2, 3, 5, 7\}$. The value 4π corresponds to the round sphere. So 16π is the lowest possible Willmore energy for nontrivial immersed Willmore spheres in $\mathbb{R}^3$. If one wants to construct such an immersion from a regular hyperbolic minimal curve L in $\mathbb{HP}^1$, as in Theorem 27.2, then $W(\langle a, Sa\rangle) = 16\pi$, $g = 0$, and $b(\langle a, Sa\rangle) = 0$ imply $\deg(L^{-1}) = 3$ and $b(L) = 0$. Its twistor lift $\hat{L}$ is thus a rational curve in $\mathbb{CP}^3$ of degree 3. It is full by Theorem 26.9 and Remark 26.10. Hence it is the rational normal curve. Let $e_1, e_2 \in \mathbb{H}^2$ be some basis and $z\colon \mathbb{CP}^1 \backslash\{\infty\} \to \mathbb{C}$ a rational coordinate of $\mathbb{CP}^1$. The complex holomorphic curve

$$\hat{L} := [\varphi], \quad \varphi := e_1 + e_1\mathbb{j}\tfrac{1}{6}z^3 + e_2 z + e_2\mathbb{j}\tfrac{1}{2}z^2$$

is a parametrization of the rational normal curve. It satisfies the assumption of Theorem 26.9 for

$$h = e_1 \wedge e_1\mathbb{j} - e_2 \wedge e_2\mathbb{j},$$

because

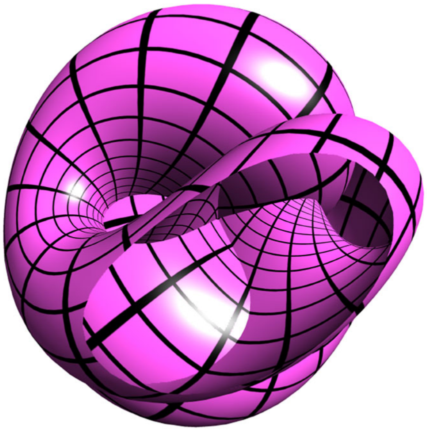
$$\varphi \wedge \varphi' = \tfrac{1}{24}(12z^2, 0, 12 - z^4, 12\mathbb{i} + z^4\mathbb{i}, 12z + 4z^3, 12z\mathbb{i} - 4z^3\mathbb{i})$$

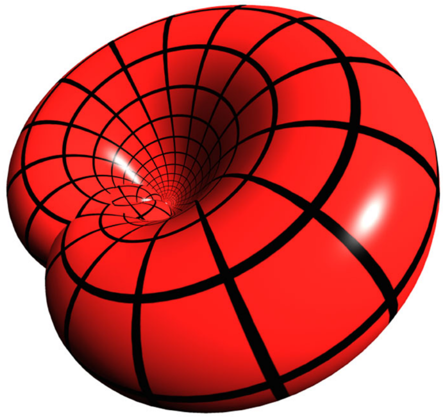
in the basis of $\operatorname{Re}\Lambda^2(\mathbb{H}^2, \mathbb{i})$ given in 26.8. This formula also implies that the Plücker embedding $Pl(\hat{L}_1)$ of $\hat{L}_1$ not pass through $\operatorname{Re} Q^4 \cong \mathbb{HP}^1$, which means that $\hat{L}_1 \oplus \hat{L}_1\mathbb{j} = \mathbb{H}^2$. The twistor projection $L = \hat{L} \oplus \hat{L}\mathbb{j}$ of L is thus a regular hyperbolic superminimal curve in $\mathbb{HP}^1$ with respect to the Hermitian form $\left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right)$ (in the basis e_1, e_2). Write now $a = a_1 + a_2$ with two sections $a_1 \in \Gamma(\hat{L}_1)$ and $a_2 \in \Gamma(\hat{L}_1)\mathbb{j}$. Then $\langle a, Sa\rangle = \langle a, a_1\mathbb{i} - a_2\mathbb{i}\rangle$.

For $a = (1, \mathbb{j})$ one gets the following formula

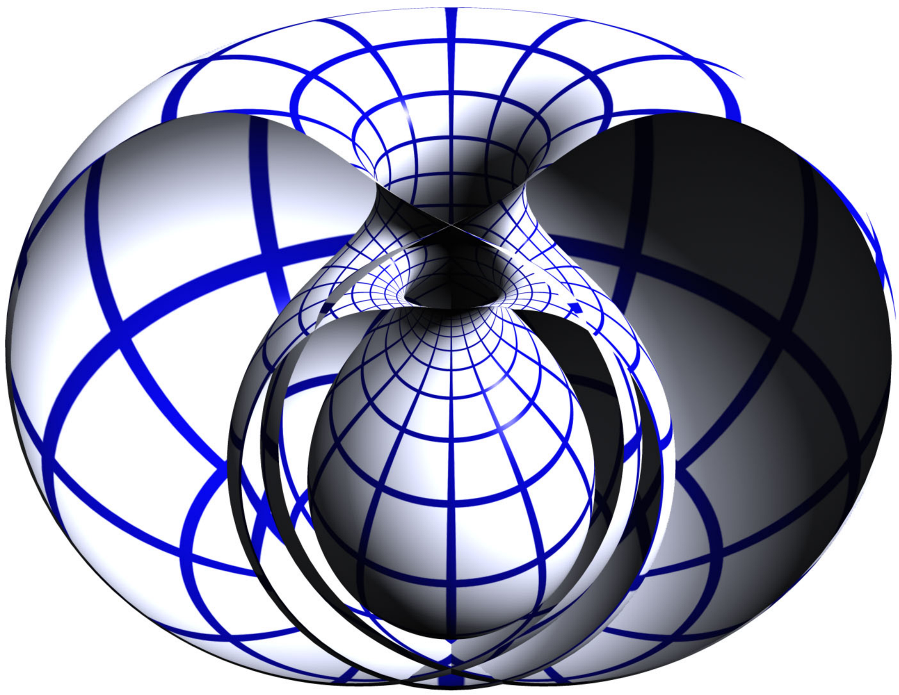
$$\langle a, Sa\rangle = \frac{\mathbb{i}\operatorname{Re}(144+144z^2+12|z|^4z^2-|z|^8)+4\mathbb{k}(-36\bar{z}-36|z|^2z+12z^3+3|z|^2z^3+12|z|^4z-|z|^6\bar{z})}{144+144|z|^2-72|z|^4+16|z|^6+|z|^8}$$

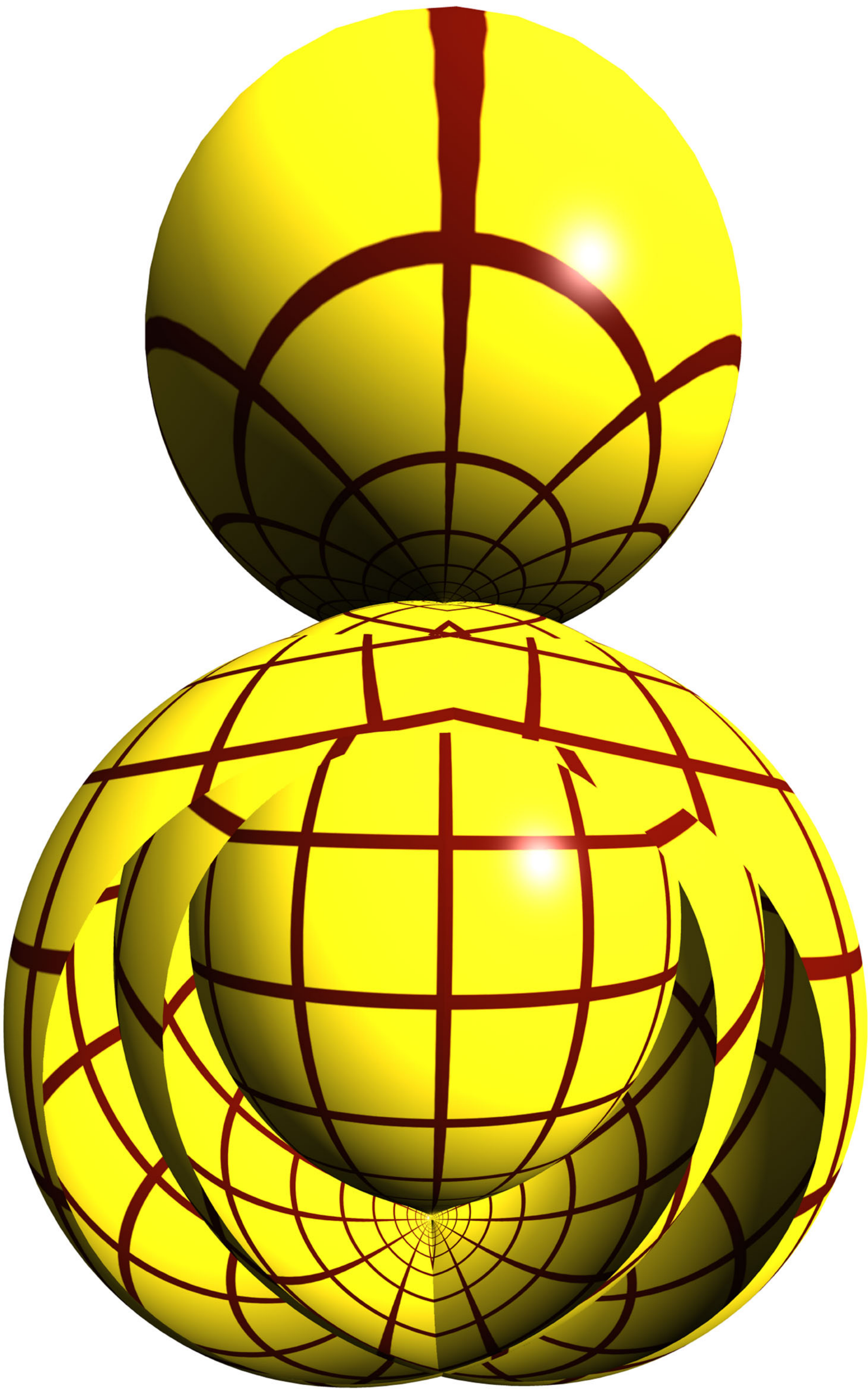
The surfaces for $a = (1, \mathbb{j})$ (left) and $a = (1, 1)$ (right):





For $\varphi \mapsto \varphi e_1 + e_2\mathbb{j}3z$ and $a = (1, \mathbb{j})$ (below), and $\varphi = e_1 + e_2\mathbb{j}7z$ and $a = (1, 1)$ (next page) one gets:





CHAPTER V

Darboux Transformation and Equality

Surfaces of constant mean curvature one in hyperbolic 3–space of curvature minus one where intensively studied in the last two decades. The starting point was Robert Bryant's representation (cf., [**Br87**]) of these surfaces in terms of holomorphic data. In this article Robert Bryant studies an important example: the catenoid cousins. These surfaces are spheres punctured at two points, at which the surface approaches the ideal boundary of the hyperbolic 3–space. The two punctures are called the ends of the surface. The ends of the catenoid cousins behave nicely in the sense that the surface continuously extends to the ideal boundary. In this chapter it is shown that the members of a countable subset of the family of catenoid cousins extend to smooth conformal immersions of $\mathbb{C}\mathrm{P}^1$, and that these immersions are soliton spheres. For the proof it is useful to describe Robert Bryant's representation of surfaces of constant mean curvature one in hyperbolic 3–space as Darboux transformation of the round 2–sphere.

28. Darboux Transformation

In this section the Darboux transformation for isothermic holomorphic curves in $\mathbb{H}\mathrm{P}^1$ is described in the language of a generalized Darboux transformation for arbitrary holomorphic curves in $\mathbb{H}\mathrm{P}^1$ (see [**Boh03**][1]). As this is in terms of quaternionic holomorphic geometry, relations between the involved quaternionic holomorphic line bundles are easily deduced, see for example Lemma 28.4.

28.1. Splitting. A *splitting* of $\mathbb{H}^2 = L \oplus L^\sharp$ by two holomorphic curves L and $L^\sharp$ in $\mathbb{H}\mathrm{P}^1$ induces a decomposition of the trivial connection ∇ of $\mathbb{H}^2$:

$$\nabla = \begin{pmatrix} \nabla^L & \delta^\sharp \\ \delta^L & \nabla^\sharp \end{pmatrix}.$$

One easily checks that ∇^L and $\nabla^\sharp$ are connections on L and $L^\sharp$, and that $\delta^L \in \Omega^1 \operatorname{Hom}(L, L^\sharp)$ and $\delta^\sharp \in \Omega^1 \operatorname{Hom}(L^\sharp, L)$. The splitting induces quaternionic bundle isomorphisms $L^\sharp \cong \mathbb{H}^2/L$ and $L \cong \mathbb{H}^2/L^\sharp$. With these isomorphisms δ^L and $\delta^\sharp$ can be interpreted as the derivatives of the holomorphic curves L and $L^\sharp$. The flatness of ∇ implies

$$(*) \qquad 0 = \mathcal{R}^\nabla = \begin{pmatrix} \mathcal{R}^{\nabla^L} + \delta^\sharp \wedge \delta^L & d^{\nabla^L, \nabla^\sharp} \delta^\sharp \\ d^{\nabla^\sharp, \nabla^L} \delta^L & \mathcal{R}^{\nabla^\sharp} + \delta^L \wedge \delta^\sharp \end{pmatrix}.$$

[1] In devising the proofs of the present chapter Christoph Bohle's formulas, especially the equations (40) and (41) in [**Boh03**], proved to be very useful.

28.2. Darboux Transformation. Let L be a holomorphic curve in $\mathbb{HP}^1$. A holomorphic curve $L^\sharp$ in $\mathbb{HP}^1$ is called a *Darboux transform* of L if $L \oplus L^\sharp = \mathbb{H}^2$, $\delta^\sharp \wedge \delta^L = 0$, and $\delta^L \wedge \delta^\sharp = 0$. The pair $(L, L^\sharp)$ is called a *Darboux pair.* Equation 28.1($*$) implies that $\delta^\sharp \wedge \delta^L = 0$ if and only if ∇^L is flat, and $\delta^L \wedge \delta^\sharp = 0$ if and only if $\nabla^\sharp$ is flat. Thus, a splitting $\mathbb{H}^2 = L \oplus L^\sharp$ with holomorphic curves is a Darboux pair if and only if the trivial connection of $\mathbb{H}^2$ induces flat connections on L and $L^\sharp$.

If L and $L^\sharp$ are nonconstant holomorphic curves and $\mathbb{H}^2 = L \oplus L^\sharp$, then $\delta^\sharp \wedge \delta^L = 0$ is equivalent to $*\delta^L = J^\sharp \delta^L$, and $\delta^L \wedge \delta^\sharp = 0$ is equivalent to $*\delta^\sharp = J\delta^\sharp$. This implies that two nonconstant holomorphic curves L and $L^\sharp$ such that $\mathbb{H}^2 = L \oplus L^\sharp$ form a Darboux pair if and only if $\delta^L \in \Gamma(\mathrm{KHom}_+(L, L^\sharp))$ and $\delta^\sharp \in \Gamma(\mathrm{KHom}_+(L^\sharp, L))$. The 2–sphere congruence

$$S = \begin{pmatrix} J^L & 0 \\ 0 & J^\sharp \end{pmatrix}$$

then satisfies $SL = L$, $SL^\sharp = L^\sharp$, $*\delta = S\delta^L = \delta^L S$ and $*\delta^\sharp = S\delta^\sharp = \delta^\sharp S$. Hence it touches L and $L^\sharp$, and the orientation of S coincides with the one of L and $L^\sharp$ at the points of tangency. If, on the other hand, L and $L^\sharp$ are holomorphic curves such that there exists a 2-sphere congruence S which touches L and $L^\sharp$ with the right orientation, then $*\delta = S\delta^L = \delta^L S$ and $*\delta^\sharp = S\delta^\sharp = \delta^\sharp S$. Hence L and $L^\sharp$ form a Darboux pair of isothermic holomorphic curves.

28.3. The following proposition connects the definition of the Darboux transformation given above with the definition in [**Jeromin**, 5.4.8].

Proposition. *If a holomorphic curve L in $\mathbb{HP}^1$ has a nonconstant Darboux transform $L^\sharp$, then L is isothermic and the derivative of $L^\sharp$ interpreted as a section τ of $T^*M \otimes \mathrm{End}(\mathbb{H}^2)$ is a retraction from of L. Furthermore, $L^\sharp$ is $(\nabla - \tau)$–parallel. Conversely, if L is a nonconstant isothermic holomorphic curve with retraction form τ, then every $(\nabla - \tau)$–parallel line subbundle of $\mathbb{H}^2$ is, away from the isolated points $p \in M$ at which $L_p = L^\sharp_p$, a Darboux transform of L.*

Proof. Suppose that L has a Darboux transform $L^\sharp$. In the splitting $\mathbb{H}^2 = L \oplus L^\sharp$ the derivative $\delta^\sharp$ is then a section of $K\,\mathrm{Hom}_+(L^\sharp, L)$ and

$$\tau := \begin{pmatrix} 0 & \delta^\sharp \\ 0 & 0 \end{pmatrix}$$

obviously satisfies $\operatorname{im}\tau \subset L \subset \ker\tau$. From $\delta^\sharp \wedge \delta^L = 0$, $\delta^L \wedge \delta^\sharp = 0$, and $d^{\nabla^L,\nabla^\sharp}\delta^\sharp$ (cf., 28.1) follows $d\tau = \begin{pmatrix} \delta^\sharp\wedge\delta^L & d^{\nabla^L,\nabla^\sharp}\delta^\sharp \\ 0 & \delta^L\wedge\delta^\sharp \end{pmatrix} = 0$. Hence τ is a retraction form of L. L is then isothermic, by Proposition 22.5, because the complex structure of $L^\sharp$ defines a complex structure on $(L^d)^{-1} = \mathbb{H}^2/L \cong L^\sharp$ such that L^d is a holomorphic curve.

Suppose now that τ is a retraction form of L. Then $d\tau + \tau \wedge \tau = 0$, hence the connection $\nabla - \tau$ is flat. Let ψ be a local $(\nabla - \tau)$–parallel section of a $(\nabla - \tau)$–parallel subbundle $L^\sharp \subset \mathbb{H}^2$ and $\pi\colon \mathbb{H}^2 \to \mathbb{H}^2/L$ the canonical projection. The section $\pi\psi \in \Gamma(\mathbb{H}^2/L)$ is then holomorphic, because if D is the Möbius invariant holomorphic structure of $\mathbb{H}^2/L$ then $D\pi\psi = \frac{1}{2}(\pi\nabla + *J\pi\nabla)\psi = \frac{1}{2}(\pi\tau + *J\pi\tau)\psi = 0$ (cf., 8.3). In particular, $\pi\psi$ has isolated zeros. Hence the points at which $L^\sharp$ coincides with L are isolated.

Away from these points L and $L^\sharp$ are a splitting of $\mathbb{H}^2$, and $\nabla^\sharp$ is flat, because $\nabla\psi = \tau\psi \in \Gamma(KL)$ implies that ψ is $\nabla^\sharp$–parallel. Thus $L^\sharp$ is a Darboux transform of L. $\Box$

28.4. The observation that the projection of a $\nabla^\sharp$ parallel section of $L^\sharp$ is a holomorphic section of $\mathbb{H}^2/L$ has the following consequence.

Lemma. *If $L^\sharp$ is a Darboux transform of an isothermic holomorphic curve L in $\mathbb{HP}^1$ and $L^\sharp$ is endowed with the holomorphic structure $(\nabla^\sharp)''$, then the bundle isomorphism $L^\sharp \cong \mathbb{H}^2/L$ is holomorphic. Furthermore, $(\nabla^\sharp)^{*''}$ is the holomorphic structure of the Möbius invariant holomorphic line bundle $(L^\sharp)^{-1}$ of $L^\sharp$.*

Proof. Let $\pi\colon \mathbb{H}^2 \to \mathbb{H}^2/L$ be the canonical projection and D the Möbius invariant holomorphic structure of $\mathbb{H}^2/L$, then $D\pi\psi = \frac{1}{2}(\pi\nabla + *J\pi\nabla)\psi$ holds for every section $\psi \in \Gamma(\mathbb{H}^2)$ (cf., 8.3). Let ψ be a local $\nabla^\sharp$–parallel section of $L^\sharp$, then $\nabla^\sharp\psi = 0$ and $D\pi\psi = \frac{1}{2}(\pi\nabla + *J\pi\nabla)\psi = \frac{1}{2}(\delta^\sharp + *J\delta^\sharp)\psi = 0$. Lemma 4.1 then implies that π induces a holomorphic bundle isomorphism between $L^\sharp$ and $\mathbb{H}^2/L$.

The second statement of the lemma follows from the first, because $L^\perp$ and $(L^\sharp)^\perp$ also form a Darboux pair and the canonical projection induces a $(\nabla^*)^{L^\perp}$–$(\nabla^\sharp)^*$–parallel isomorphism $L^\perp \cong (\mathbb{H}^2)^*/(L^\sharp)^\perp$, where $(\nabla^*)^{L^\perp}$ is the connection induced on $L^\perp$ by the dual connection ∇^* of ∇ and the splitting $(\mathbb{H}^2)^* = L^\perp \oplus (L^\sharp)^\perp$, and $(\nabla^\sharp)^*$ is the dual connection of $\nabla^\sharp$ on $(\mathbb{H}^2)^*/(L^\sharp)^\perp$. $\Box$

28.5. For a retraction form τ of an isothermic holomorphic curve L the differential equation

$$\nabla F = \tau F, \quad F\colon M \to \mathrm{SL}(2,\mathbb{H}),$$

has for every initial condition a unique local solution, because $\nabla - \tau$ is flat and the trace of τ is zero on all of M. Here $\mathrm{SL}(2,\mathbb{H})$ is the group of quaternionic 2 by 2 matrices whose Study determinant equals 1. The Study determinant of a quaternionic 2 by 2 matrix is the determinant of the corresponding complex 4 by 4 matrix (cf., [**Jeromin**, 4.2]).

The Darboux transforms of L that correspond to the retraction form τ, i.e., the $(\nabla - \tau)$–parallel ones, are locally all of the form

$$L^\sharp = [Fa], \quad a \in \mathbb{H}^2,$$

because F is a parallel Frame for $\nabla - \tau$.

28.6. Calapso Transformation, Associated Family. Lemma 28.4 implies that the $(\nabla - \tau)$–parallel sections $Fb \in \Gamma(\mathbb{H}^2)$, $b \in \mathbb{H}^2$, project onto a 2–dimensional base point free linear system of holomorphic sections of $\mathbb{H}^2/L$. The dual curve of the Kodaira embedding of this linear system, is called a *Calapso transform* of L. In other words, L viewed as a holomorphic curve in $\mathbb{HP}^1$ with respect to a $(\nabla - \tau)$–parallel trivialization of $\mathbb{H}^2$ is a Calapso transform of L. In the standard ∇–parallel trivialization of $\mathbb{H}^2$ the Calapso transform of L becomes

$$F^{-1}L.$$

The Calapso transforms of an isothermic surface L are isothermic, because if τ is a retraction from of L, then $F^{-1}\tau F$ is a retraction from of $F^{-1}L$. If L is not totally umbilic, then all retraction forms of L are a real multiples of τ (see 22.5). Hence locally there is a real 1–parameter family of Calapso transforms, which is also called the *associated family* of L.

29. CMC–1 Surfaces in Hyperbolic Space

In this section Robert Bryant's representation of surfaces of constant mean curvature ± 1 in hyperbolic 3–space of curvature -1, CMC–1 surfaces for short, is interpreted as Darboux transformation of holomorphic curves that take values in the ideal boundary of the hyperbolic 3–space. This exposes the similarity of Bryant's representation of CMC–1 surfaces to the Weierstrass representation of minimal surfaces in $\mathbb{R}^3$, which can be interpreted as Christoffel transformation of holomorphic curves that take values in a 2–sphere. Furthermore, there naturally arises for every minimal surface a family of CMC–1 surfaces that has the minimal surface as a limit. The members of this family are called the CMC–1 cousins of the minimal surface. They are related by Lawson's correspondence (cf., [**La70**, § 12]). The description of minimal surfaces and its hyperbolic CMC–1 cousins as Christoffel and Darboux transforms of conformal parametrizations of the 2–sphere was pointed out to the author by Udo Hertrich–Jeromin (see also [**JMN01**] and [**Jeromin**, Chapter 5, in particular the Examples 5.3.21 and 5.5.29]).

29.1. Minimal Surfaces in $\mathbb{R}^3$ via Christoffel Transformation. The Weierstrass representation of minimal surfaces in $\mathbb{R}^3$ can be interpreted as Christoffel transform of the round sphere: If L is a holomorphic curve in $\mathbb{HP}^1$ that takes values in a 2–sphere, then L^d is also a holomorphic curve, and the Hopf fields of the Möbius invariant holomorphic line bundles L^{-1} and $\mathbb{H}^2/L = (L^d)^{-1}$ vanish (cf., 8.4). Then every choice of meromorphic sections φ_1 of $\hat{L}$ and φ_2 of $\widehat{(L^d)^{-1}}$ of the underlying complex line bundles of L and $(L^d)^{-1}$, and meromorphic 1–form η on M defines on M_0 a holomorphic bundle homomorphism from $(L^d)^{-1}$ to KL via $\varphi_2 \mapsto \varphi_1\eta$, where $M \setminus M_0$ is the set of poles of φ_1,φ_2, and η. If $M \setminus M_0$ is the set of poles of φ_1, φ_2, and η, then $(L^d)^{-1}$ and L^{-1} are paired on M_0, by Proposition 10.8, and L is isothermic on M_0. Moreover, every Christoffel transform of L is a minimal surface, by 9.13 and Lemma 22.3. This is the Weierstrass representation of minimal surfaces in $\mathbb{R}^3$ (12.6):

Fix the 2–sphere in $\mathbb{HP}^1$ to be

$$\mathbb{CP}^1 = \{ \left(\begin{smallmatrix} z \\ w \end{smallmatrix}\right)\mathbb{H} \subset \mathbb{H}^2 \mid z, w \in \mathbb{C} \}.$$

If $g\colon M \to \mathbb{C} \cup \{\infty\}$ is a meromorphic function, then $L = \left(\begin{smallmatrix} g \\ 1 \end{smallmatrix}\right)\mathbb{H}$ is a holomorphic curve that takes values in $\mathbb{CP}^1$. Let η be a meromorphic 1–form on M. Then

$$\tau := \operatorname{Ad}\left(\begin{smallmatrix} 1 & g \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & 0 \\ \eta & 0 \end{smallmatrix}\right) = \begin{pmatrix} g & -g^2 \\ 1 & -g \end{pmatrix}\eta$$

is a retraction form of $L_{|M_0}$ (closedness follows from the fact that there are no nontrivial $(2,0)$ forms on a Riemann surface). Hence if $\beta \in (\mathbb{H}^2)^*$ and

$a \in \mathbb{H}^2$ such that $\beta(a) = 0$, then every local potential f of the closed 1–form $\beta\tau a$ is a Christoffel transform of L (cf., 22.6). If one chooses the basis $a = \begin{pmatrix} \Bbbk \\ 1 \end{pmatrix}$, $b = \begin{pmatrix} 1 \\ \Bbbk \end{pmatrix}$ of $\mathbb{H}^2$, and its dual basis $\alpha, \beta \in (\mathbb{H}^2)^*$, then

$$df = \beta\tau a = \mathrm{Re}(g\eta)\Bbbk + \mathrm{Re}(\tfrac{1}{2}(1-g^2)\eta) + \mathrm{Re}(\tfrac{1}{2}\mathbb{i}(1+g^2)\eta)\mathbb{i}.$$

This is, up to the rotation $x \mapsto -\mathbb{i}x\Bbbk$, the formula of the usual Weierstrass representation of minimal surfaces (cf., 12.6).

29.2. CMC–1 Surfaces via Darboux Transformation. Bryants representation of CMC-1 surfaces can be interpreted as Darboux transform of the round sphere: To see this one describes hyperbolic three space as follows. Let $\langle\,,\rangle$ be a Hermitian form on $\mathbb{H}^2$ and think of it to be a 3–sphere in $\mathbb{HP}^1$. Then choose some fixed 2–sphere in this 3–sphere and call the corresponding endomorphism S. Then S is Hermitian with respect to $\langle\,,\rangle$ (cf., 5.4–5.6). Hence $\langle S\cdot,\cdot\rangle$ is also a Hermitian form. Its corresponding 3–sphere intersects the 3–sphere $\langle\,,\rangle$ orthogonally in the 2–sphere S. (This can be seen from 5.4 choosing $[a], [b] \in S$.) In particular with the hyperbolic metric induced by $\langle S\cdot,\cdot\rangle$ via Proposition 26.2(iii) the two components of the 3–sphere $\langle\,,\rangle$ minus the 2–sphere S both are isomorphic to the hyperbolic 3–space of curvature -1.

Let $L^\sharp$ be a holomorphic curve that takes values in the 3–sphere $\langle\,,\rangle$, i.e., $L^\sharp$ is isotropic for $\langle\,,\rangle$. $L^\sharp$ is a CMC–1 surface in the hyperbolic 3–space determined by $\langle\,,\rangle$ and S, if and only if its mean curvature sphere is a horosphere, i.e., a sphere that touches the ideal boundary S of the hyperbolic space, because the horospheres are the spheres of mean curvature ± 1 in hyperbolic space of curvature -1. The intersection point of the mean curvature sphere of $L^\sharp$ with S is called the *hyperbolic Gauss map*[2] of $L^\sharp$. Robert Bryant's representation of CMC–1 surfaces in [**Br87**] can now be described as follows (see also [**JMN01**]).

29.3. Theorem. *Let $L^\sharp$ be a holomorphic curve in $\mathbb{HP}^1$ that is not contained in a 2–sphere. Then $L^\sharp$ is a CMC–1 surface if and only if it has a Darboux transform L that takes values in a fixed 2–sphere. L is then the hyperbolic Gauss map of $L^\sharp$.*

Proof. Let $L^\sharp$ be a Darboux transform of a holomorphic curve L that takes values in the 2–sphere S. Let τ be the corresponding retraction form. Then locally $L^\sharp = [Fa]$ for some $a \in \mathbb{H}^2$ and some $F\colon M \to \mathrm{SL}(2,\mathbb{H})$ that solves $\nabla F = \tau F$ (cf., 28.5). Lemma 29.4 then implies that if $\langle\,,\rangle$ is a 3–sphere that contains S, then $d\langle Fa, Fa\rangle = 0$. Hence if $\langle\,,\rangle$ is a 3–sphere that contains S and one point of $L^\sharp$, then $L^\sharp$ is contained in this 3–sphere $\langle\,,\rangle$. Thus $L^\sharp$ and S do not intersect, because otherwise $L^\sharp$ would be contained in S.

[2] The geodesic that passes through a point p of the surface and the image of the Gauss map at p intersects the surface orthogonally at p, since this geodesic is orthogonal to the mean curvature sphere at the point of tangency with the ideal boundary S of the hyperbolic space. From this one sees that if the mean curvature of the surface is $+1$, then the Gauss map is the same as the one of Robert Bryant [**Br87**, p. 326]. In the case of mean curvature -1 Robert Bryant chooses the opposite intersection point of the orthogonal geodesic with the ideal boundary.

So $\mathbb{H}^2 = L \oplus L^\sharp$ and $L^\sharp$ is contained in one of the two hyperbolic 3–spaces obtained intersecting the hyperbolic 4–space $\langle S\cdot,\cdot\rangle$ with the 3–sphere $\langle\,,\rangle$. Let $S^\sharp = \begin{pmatrix} J^L & 0 \\ 0 & J^\sharp \end{pmatrix}$ be the 2–sphere congruence that touches L and $L^\sharp$ (cf., 28.2). The Hopf field $Q^\sharp$ of $S^\sharp$ then satisfies

$$(*) \qquad Q^\sharp = \begin{pmatrix} Q^L & 0 \\ 0 & Q^{L^\sharp} \end{pmatrix},$$

where Q^L and $Q^{L^\sharp}$ are the Hopf fields of $(\nabla^L)''$ and $(\nabla^\sharp)''$. Since the holomorphic line bundles $(L^\sharp, (\nabla^\sharp)'')$ and $\mathbb{H}^2/L$ are isomorphic, by Lemma 28.4, and because the Hopf field of $\mathbb{H}^2/L$ vanishes identically (cf., 8.4), one concludes that $Q^{L^\sharp}$ vanishes identically. Hence $L^\sharp \subset \ker Q^\sharp$ and $S^\sharp$ is the mean curvature sphere of $L^\sharp$ (cf., 14.6). Since $S^\sharp$ is tangent to S it is a horosphere, and, consequently, $S^\sharp$ as well as $L^\sharp$ have mean curvature ± 1.

If, on the other hand, $L^\sharp$ is a CMC–1 surface in the hyperbolic 3–space determined by the 3–sphere $\langle\,,\rangle$ and the 2–sphere S, then its mean curvature spheres $S^\sharp$ are horospheres, i.e., for all $p \in M$ there is a point $L_p \in \mathbb{HP}^1$ at which $S^\sharp$ and S intersect and have the same tangent space. Without loss of generality one can assume that $S^\sharp_p$ and S_p induce the same orientation on their common tangent space. The description of the tangent space in 5.5 then implies that $R = S^\sharp - S \in \Gamma(\operatorname{End}\mathbb{H}^2)$ satisfies $\operatorname{im} R \subset L \subset \ker R$. Furthermore R anticommutes with S and $S^\sharp$. Let $\psi \in \Gamma(L)$ then

$$S\nabla\psi = \nabla(S\psi) = \nabla(S^\sharp\psi) = (\nabla S^\sharp)\psi + S^\sharp\nabla\psi = (\nabla S^\sharp)\psi + S\nabla\psi + R\nabla\psi$$

implies that

$$R\delta\psi = (\nabla S^\sharp)\psi,$$

where $\delta \in \Omega^1(\operatorname{End}(L, \mathbb{H}^2/L))$ is the derivative of L. Let $Q^\sharp$ and $A^\sharp$ be the Hopf fields of $S^\sharp$. Then $(\nabla S^\sharp)\psi \in \Omega^1(L)$ implies $A^\sharp\psi = \frac{1}{4}(S^\sharp\nabla S^\sharp + *\nabla S^\sharp)\psi \in \Omega^1(L)$. Hence $A^\sharp\psi = 0$, because $\operatorname{im} A^\sharp \subset L^\sharp$ (cf., 14.6). Thus

$$R\delta\psi = 2{*}Q^\sharp\psi.$$

Hence $*\delta = S^\sharp\delta = \delta S^\sharp$, and L is a Darboux transform of $L^\sharp$. □

29.4. Lemma. *If τ is a retraction form of an isothermic holomorphic curve $L \subset \mathbb{H}^2$ that takes values in a 3–sphere $\langle\,,\rangle$, and $\nabla F = \tau F$, then*

$$d\langle F\cdot, F\cdot\rangle = 0.$$

Proof. Since $d\langle F\cdot, F\cdot\rangle = \langle \nabla F\cdot, F\cdot\rangle + \langle F\cdot, \nabla F\cdot\rangle = \langle \tau F\cdot, F\cdot\rangle + \langle F\cdot, \tau F\cdot\rangle$, it suffices to show that the Hermitian form $\widetilde{\langle\cdot,\cdot\rangle} = \langle \tau_X\cdot,\cdot\rangle + \langle\cdot,\tau_X\cdot\rangle$ vanishes for all $p \in M$ and $X \in T_pM$. If $[a] = L_p$ then $\widetilde{\langle a, b\rangle} = 0$ for all $b \in \mathbb{H}^2$, since $\langle a, a\rangle = 0$, $\tau_X a = 0$, and $[\tau_X b] = L_p = [a]$. Hence it suffices to show that there exists $b \in \mathbb{H}^2$ linearly independent of a such that $\widetilde{\langle b, b\rangle} = 0$. Since the branch points of L are isolated one can assume that L is immersed near p. Let S_p be the mean curvature sphere of L at p. Then S_p is Hermitian with respect to $\langle\cdot,\cdot\rangle$ (cf., 5.6) and commutes with τ (cf., 22.5). Hence S_p is Hermitian with respect to $\widetilde{\langle\cdot,\cdot\rangle}$ and all points of the sphere S_p are isotropic lines of $\widetilde{\langle\cdot,\cdot\rangle}$. □

29.5. The Standard Formulas. Let us describe the construction of CMC–1 surfaces in the setup of 29.1, i.e.,

$$S = \left(\begin{smallmatrix} \mathbb{i} & 0 \\ 0 & \mathbb{i} \end{smallmatrix}\right), \quad L = [\left(\begin{smallmatrix} g \\ 1 \end{smallmatrix}\right)], \quad \text{and} \quad \tau = \begin{pmatrix} g & -g^2 \\ 1 & -g \end{pmatrix}\eta.$$

Furthermore, fix the 3–sphere S^3 to be

$$\langle \left(\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right), \left(\begin{smallmatrix} y_1 \\ y_2 \end{smallmatrix}\right) \rangle = \bar{x}_1 \mathbb{j} y_2 - \bar{x}_2 \mathbb{j} y_1.$$

Then S is Hermitian with respect to $\langle\cdot,\cdot\rangle$, and the hyperbolic metric on $S^3 \setminus S$ is induced by $\langle S\cdot,\cdot\rangle$. Let now F be a solution of

$$\text{(A)} \qquad \nabla F = \tau F = \begin{pmatrix} g & -g^2 \\ 1 & -g \end{pmatrix}\eta F, \quad F\colon M \to \mathrm{SL}(2,\mathbb{C}),$$

(cf., 28.5). Note that in contrast to F taking values in $\mathrm{SL}(2,\mathbb{H})$ as in 28.5 it is now assumed that F takes values in $\mathrm{SL}(2,\mathbb{C})$. This can be done, because $\nabla[F,S] = \tau[F,S]$ implies that if F is in $\mathrm{SL}(2,\mathbb{C})$ at some point then it is in $\mathrm{SL}(2,\mathbb{C})$ at all points. So it is a matter of choosing the right initial condition. This implies that the Darboux transform

$$L^\sharp = Fa,$$

for $a \in \mathbb{H}^2$ such that $\langle a,a\rangle = 0$ takes values in the 3–sphere $\langle\,,\rangle$, because $F \in \Gamma(\mathrm{SL}(2,\mathbb{C}))$ implies $\langle F\cdot, F\cdot\rangle = \langle\cdot,\cdot\rangle$.

Let $a = \left(\begin{smallmatrix} \mathbb{k} \\ 1 \end{smallmatrix}\right)$, $b = \left(\begin{smallmatrix} 1 \\ \mathbb{k} \end{smallmatrix}\right) \in \mathbb{H}^2$, and $\alpha, \beta \in (\mathbb{H}^2)^*$ its dual basis as in 29.1. The stereographic projection $\sigma_{\beta,\alpha}$ of S is then the unit sphere of $\mathbb{R} \oplus \mathbb{R}\mathbb{i} \oplus \mathbb{R}\mathbb{k}$, since $\sigma_{\beta,\alpha}\left(\begin{smallmatrix} z \\ 1 \end{smallmatrix}\right) = \frac{2z + (|z|^2-1)\mathbb{k}}{|z|^2+1}$. The stereographic projection $\sigma_{\beta,\alpha}$ of the Darboux transform $L^\sharp = [Fa]$ of L is then a CMC–1 surface in the corresponding Poincaré ball model of hyperbolic 3–space, by Theorem 29.3. More explicitly one gets

$$\sigma_{\beta,\alpha} L^\sharp = \frac{1}{x_0+1}(x_1 + x_2\mathbb{i} + x_3\mathbb{k}), \text{ where } \begin{pmatrix} x_0 + x_3 & x_1 + x_2\mathbb{i} \\ x_1 - x_2\mathbb{i} & x_0 - x_3 \end{pmatrix} := F\bar{F}^t.$$

This is the version of Robert Bryant's representation of CMC–1 surfaces used in [**BPS02**] (see also [**RUY97**]).

Instead of solutions of (A) Robert Bryant considers solutions of

$$\text{(B)} \qquad \nabla\tilde{F} = -\tilde{F}\tau, \quad \tilde{F}\colon M \to \mathrm{SL}(2,\mathbb{C}).$$

If $\tilde{F} = F^{-1}$ then $\tilde{F}$ solves (B) if and only if F solves (A). The relation of $\tilde{L}^\sharp = [\tilde{F}a]$ and $L^\sharp$ is the following. The form $\tilde{\tau} = F^{-1}\tau F$ is a retraction form of the Calapso transform $\tilde{L} := \tilde{F}L = F^{-1}L$ of L (cf., 28.6). The holomorphic curve $\tilde{L}$ takes values in S, since F and S commute. Because $\tilde{F}$ solves (A) for $\tilde{\tau}$, it follows that $\tilde{L}^\sharp$ is a Darboux transform of $\tilde{L}$. Hence in Robert Bryant's representation of CMC–1 surfaces, the surface is obtained as a Darboux transform of a Calapso transform of L.

Robert Bryant's representation has the disadvantage that the CMC–1 surface $\tilde{L}^\sharp = [\tilde{F}a]$ may be well defined on the Riemann surface M, although τ is only defined on some covering of M (for example for the catenoid cousins in [**Br87**, p. 341]). In the representation with solutions of (A) the form τ is well defined on the domain of $L^\sharp$, since its hyperbolic Gauss map L is well defined on that domain and τ is the derivative of $L^\sharp$ in the splitting $\mathbb{H}^2 = L \oplus L^\sharp$ (cf., 28.3).

29.6. Cousin relation. In 22.5 it was observed that in general the retraction form of an isothermic holomorphic curve is unique up to multiplication by a real constant. Contemplate the corresponding 1–parameter family of CMC–1 surfaces

$$f_\lambda = \sigma_{\beta,\alpha}[F_\lambda a], \quad \text{where} \quad \nabla F_\lambda = \lambda \tau F_\lambda, \quad a = \begin{pmatrix} \mathbb{k} \\ 1 \end{pmatrix}.$$

Let $p_0 \in M$ be some point and choose the initial condition at p_0 such that $F_\lambda(p_0)$ is differentiable with respect to λ and $F_0(p_0) = \mathrm{Id}$ (The reason not to choose $F_\lambda(p_0) = \mathrm{Id}$ for all $\lambda \in \mathbb{R}$ is the fact that this is not done for the catenoid cousins in [**Br87**], and the formula for F of [**Br87**] is used in Section 30.). Then f_λ converges to the constant point $f_0 = \sigma_{\beta,\alpha} a = 0$, as $F_0 \equiv \mathrm{Id}$. If one takes a magnifying glass that multiplies f_λ by $\frac{1}{\lambda}$, then the family of scaled surfaces converges to $\hat{f}_0 = \frac{\partial f_\lambda}{\partial \lambda}|_0$. The differential of $\hat{f}_0$ satisfies

$$d\hat{f}_0 = \frac{\partial}{\partial \lambda}\Big|_0 df_\lambda = \frac{\partial}{\partial \lambda}\Big|_0 d\left(\beta(F_\lambda a)(\alpha(F_\lambda a))^{-1}\right) = \beta\left(\frac{\partial \nabla F_\lambda a}{\partial \lambda}\Big|_0\right) = \beta \tau a.$$

Hence $\hat{f}_0$ is the minimal surface of 29.1. The CMC–1 surfaces f_λ are called the *CMC–1 cousins* of the minimal surface $\hat{f}_0$.

30. Catenoid Cousins

The best known example of the cousin relation are the catenoid cousins (cf., [**Br87**, p. 341]). In this section it is shown that some of these catenoid cousins extend smoothly into both ends. It turns out that the smoothly extendable catenoid cousins are immersed Taimanov 1–soliton spheres. Robert Bryant notes that one can see from the family of catenoid cousins that there is no “quantization” of the total curvature of CMC–1 surfaces. But Theorem 30.3 implies that if one adds the reasonable boundary condition that the CMC–1 surface extends through its ends to an immersion of a compact surface, then the total curvature or Willmore energy of the catenoid cousins is quantized.

30.1. In [**Br87**, p. 341] one finds the following multi–valued functions $F_\mu \colon \mathbb{C} \setminus \{0\} \to \mathrm{SL}(2, \mathbb{C})$,

$$F_\mu(z) = \frac{1}{\sqrt{2\mu+1}} \begin{pmatrix} (\mu+1)z^\mu & \mu z^{-(\mu+1)} \\ \mu z^{\mu+1} & (\mu+1)z^{-\mu} \end{pmatrix},$$

for $\mu > -\frac{1}{2}$. They satisfy the equation (A) of 29.5 for

$$\tau = \mu(\mu+1)\begin{pmatrix} z^{-1} & -z^{-2} \\ 1 & -z^{-1} \end{pmatrix} dz.$$

So with the notation of 29.1, 29.5, and 29.6 one gets $\lambda = \mu(\mu+1)$, $g = \frac{1}{z}$, $\eta = dz$, and the minimal surface

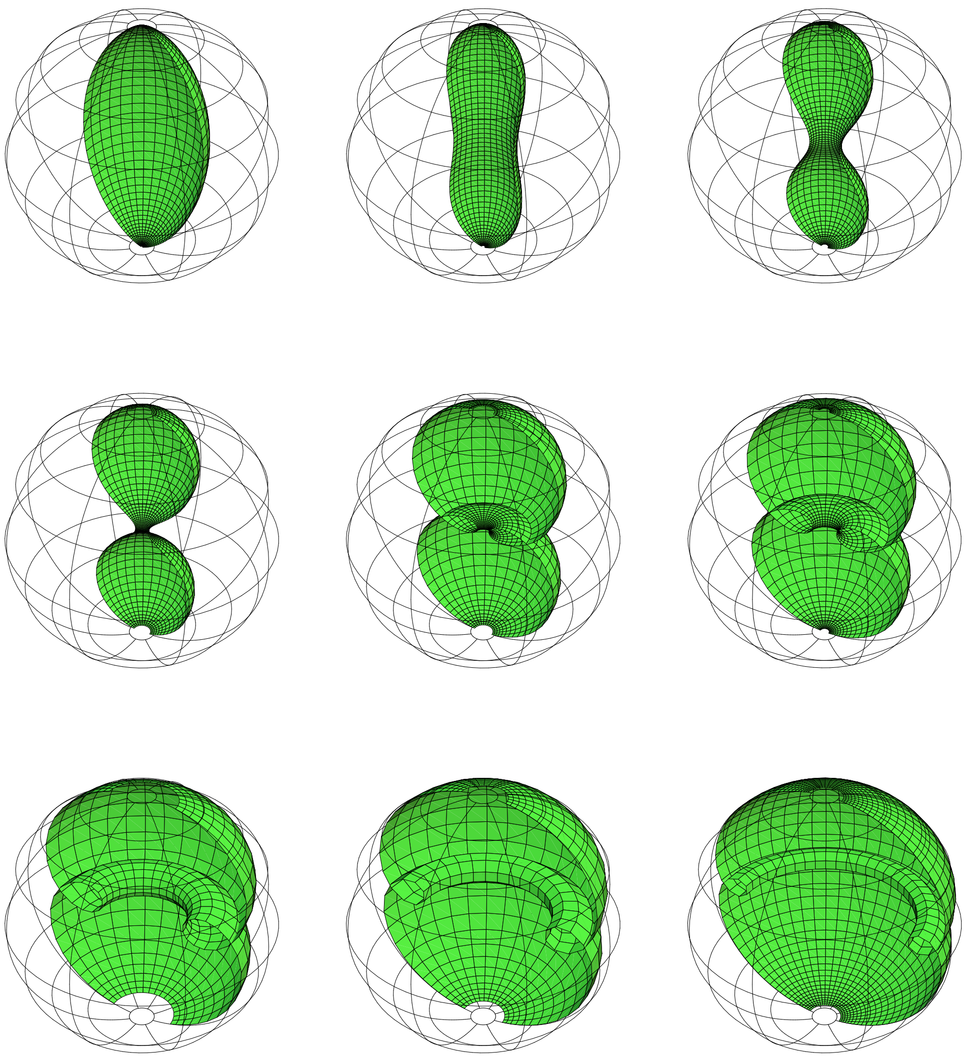
$$\hat{f}_0 = x\mathbb{k} + \cosh(x)(\cos(y) - \mathbb{i}\sin(y)),$$

where $z = e^{x+\mathbb{i}y}$. Hence $\hat{f}_0$ is the catenoid. Let $\alpha, \beta \in (\mathbb{H}^2)^*$ be the dual basis of $a = \binom{\mathbb{k}}{1}$, $b = \binom{1}{\mathbb{k}} \in \mathbb{H}^2$ as in 29.5. The immersions

$$\begin{aligned} f_\mu &= \sigma_{\beta,\alpha}[F_\mu a] \\ &= \frac{2\mu(\mu+1)(|z|^{2\mu}+|z|^{-2(\mu+1)})\bar{z}+\left[(\mu+1)^2(|z|^{2\mu}-|z|^{-2\mu})+\mu^2(-|z|^{2(\mu+1)}+|z|^{-2(\mu+1)})\right]\mathbb{k}}{(\mu+1)^2(|z|^{2\mu}+|z|^{-2\mu})+\mu^2(|z|^{2(\mu+1)}+|z|^{-2(\mu+1)})+4\mu+2} \end{aligned}$$

are called the *catenoid cousins*. They are surfaces of revolution in $\mathbb{R}\oplus\mathbb{R}\mathbb{i}\oplus\mathbb{R}\mathbb{k}$ with axis $\mathbb{R}\mathbb{k}$, since $f_\mu(e^{2\varphi\mathbb{i}}z) = e^{-\varphi\mathbb{i}}f_\mu(z)e^{-\varphi\mathbb{i}}$.

Here are pictures of the f_μ for $\mu = -\frac{1}{3}, -\frac{1}{5}, -\frac{1}{9}, -\frac{1}{20}, \frac{1}{20}, \frac{1}{9}, \frac{1}{2}, 1, 2$:



30.2. Smoothness at the Ends for $\mu \in \mathbb{N}$. The catenoid cousins are stereographic projections of the holomorphic curves

$$L^\sharp_\mu = [\psi], \quad \psi := \sqrt{2\mu+1}F_\mu a = \begin{pmatrix} (\mu+1)z^\mu \mathbb{k} + \mu z^{-(\mu+1)} \\ \mu z^{\mu+1}\mathbb{k} + (\mu+1)z^{-\mu} \end{pmatrix}.$$

Let

$$\psi^0 := \psi z^{\mu+1} = \begin{pmatrix} (\mu+1)|z|^{2\mu}\bar{z}\mathbb{k} + \mu \\ \mu|z|^{2\mu+2}\mathbb{k} + (\mu+1)z \end{pmatrix},$$

$$\psi^\infty := \psi \bar{z}^{-(\mu+1)} = \begin{pmatrix} (\mu+1)z^{-1}\mathbb{k} + \mu|z|^{-2\mu-2} \\ \mu\mathbb{k} + (\mu+1)|z|^{-2\mu}\bar{z}^{-1} \end{pmatrix}.$$

The sections ψ^0, ψ^∞, and, consequently, $L^\sharp_\mu$ are in contrast to F_μ well defined on $\mathbb{C}\setminus\{0\}$. Furthermore, $L^\sharp_\mu$ extends smoothly into 0 and ∞ if and only if $\mu \in \mathbb{N}$. The extended holomorphic curve on $\mathbb{CP}^1 = \mathbb{C}\cup\{\infty\}$ for $\mu \in \mathbb{N}\setminus\{0\}$ is immersed. In the following theorem $L^\sharp_\mu$ stands for the extended curve.

30.3. Theorem. *The catenoid cousins $L^\sharp_\mu$ for $\mu \in \mathbb{N}\setminus\{0\}$ are immersed soliton spheres. More precisely, the Willmore energy of the Möbius invariant holomorphic line bundle $(L^\sharp_\mu)^{-1}$ of the extension of $L^\sharp_\mu$ to $\mathbb{CP}^1$ equals $4\pi(2\mu+1)$, $\dim H^0\big((L^\sharp_\mu)^{-1}\big) = 3$, $H^0\big((L^\sharp_\mu)^{-1}\big)$ has equality in the Plücker estimate, the Weierstrass points of $H^0\big((L^\sharp_\mu)^{-1}\big)$ are the ends of $L^\sharp_\mu$, and the Weierstrass gap sequence at both ends is $0, 1, \mu+1$.*

Proof. Let $a = \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$, $b = \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right) \in \mathbb{H}^2$ and $\alpha, \beta \in H^0\big((L^\sharp_\mu)^{-1}\big)$ the projection of its dual basis to $(L^\sharp_\mu)^{-1} = (\mathbb{H}^2)^*/(L^\sharp_\mu)^\perp$. The section ψ is well defined on $M_0 := \mathbb{C}\setminus\{0\}$, since $\mu \in \mathbb{N}$, and has no zeros. Let $\gamma := \psi^{-1} \in \Gamma\big((L^\sharp_\mu)^{-1}{}_{|M_0}\big)$. From $\nabla^\sharp\psi = 0$ and $\gamma(\psi) = 1$ follows $(\nabla^\sharp)^*\gamma = 0$. Thus γ is a holomorphic section of $(L^\sharp_\mu)^{-1}{}_{|M_0}$, by Lemma 28.4. Since

$$\gamma(\psi^0) = z^{\mu+1} \quad \text{and} \quad \gamma(\psi^\infty) = \bar{z}^{-(\mu+1)}$$

γ extends smoothly into $z = 0, \infty$ with the vanishing orders

$$\operatorname{ord}_0 \gamma = \mu+1 \quad \text{and} \quad \operatorname{ord}_\infty \gamma = \mu+1,$$

and has no other zeros. Applying α and β to ψ^0 and ψ^∞ yields the vanishing orders

$$\operatorname{ord}_0 \alpha = 0,\ \operatorname{ord}_\infty \alpha = 1, \quad \text{and} \quad \operatorname{ord}_0 \beta = 1,\ \operatorname{ord}_\infty \beta = 0.$$

Contemplate the 1–dimensional linear system $H = \gamma\mathbb{H}$ of holomorphic sections of $(L^\sharp_\mu)^{-1}$. The formula of 15.2 then reads

$$\begin{aligned} W((L^\sharp_\mu)^{-1}) - W(L^\sharp_{\mu|M_0}, (\nabla^\sharp)'') &= -4\pi\deg((L^\sharp_\mu)^{-1}) + 4\pi\sum_{p\in M}\operatorname{ord}_p\gamma \\ &= 4\pi(2\mu+1), \end{aligned}$$

since $\deg((L^\sharp_\mu)^{-1}) = -\deg(L^\sharp_\mu) = 1$, by Theorem 12.4. Furthermore, since $L^\sharp_\mu$ is the Darboux transform of a holomorphic curve L that takes values in some 2–sphere, the Willmore energy of $\mathbb{H}^2/L$ vanishes, by 8.4, and Lemma 28.4 then implies $W((L^\sharp_\mu)_{|M_0}, (\nabla^\sharp)'') = W(\mathbb{H}^2/L) = 0$. Hence

$$W((L^\sharp_\mu)^{-1}) = 4\pi(2\mu+1).$$

Let now H be the linear system spanned by α, β and γ. It has the Weierstrass points 0 and ∞ with gap sequence $0, 1, \mu+1$. Hence

$$\operatorname{ord}_0 H = \operatorname{ord}_\infty H = \mu - 1.$$

Thus $\operatorname{ord} H \geq 2(\mu - 1)$. The Plücker estimate (14.8) applied to H then yields:

$$\begin{aligned} 2\mu + 1 = \tfrac{1}{4\pi} W((L^\sharp_\mu)^{-1}) &\geq (n+1)(n(1-g) - \deg((L^\sharp_\mu)^{-1})) + \operatorname{ord} H \\ &\geq 3(2-1) + 2(\mu - 1) = 2\mu + 1. \end{aligned}$$

Both inequalities are, consequently, equalities. The points $z = 0$ and $z = \infty$ are thus the only Weierstrass points of H and H has equality in the Plücker estimate. Since $L^\sharp_\mu$ is rotationally symmetric, Lemma 19.4 implies that $H = H^0(L)$. □

30.4. The Catenoid Cousins and Taimanov 1–Soliton Spheres. The catenoid cousins for $\mu \in \mathbb{N} \setminus \{0\}$ are immersed rotationally symmetric soliton spheres in $\mathbb{R}^3$. Their Euclidean holomorphic line bundle is thus a spin bundle with rotationally symmetric Hopf field. The full linear system of their Möbius invariant holomorphic line bundle is a 3–dimensional linear system with equality in the Plücker estimate. The full linear system of the Euclidean holomorphic line bundle is, consequently, a 2–dimensional linear system with equality in the Plücker estimate (see 18.2). Hence the catenoid cousins for $\mu \in \mathbb{N} \setminus \{0\}$ are Taimanov soliton spheres, by Theorem 19.3. If one chooses

$$n = 1, \quad n_0 = 0, \quad n_1 = \mu, \quad \lambda_0 = \frac{\mu + 1}{\mu}, \quad \lambda_1 = \frac{(\mu+1)(2\mu+1)}{\mu}$$

in Theorem 19.3, then f_μ and $\frac{2(\mu+1)}{\mu} \int (\psi_1, \psi_1)$ coincide up to some Euclidean motion (see 19.6 for pictures of these surfaces).

30.5. Conjecture. The author conjectures that other surfaces of constant mean curvature 1 in hyperbolic 3–space with catenoidal ends besides the catenoid cousins, in particular a countable subset of the trinoids studied in [**BPS02**], extend to immersed spheres. Successive Darboux transformations of these spheres provide via permutability (cf., [**Boh03**, Theorem 3, p. 57]) a procedure to construct a large class of soliton spheres (all?) with algebraic operations from branched holomorphic coverings of the round 2–sphere.

APPENDIX A

Continuity of the Dual Curve

In Section 16, the fact that the twistor lift of the dual curve extends continuously into the Weierstrass points, is used to show that linear systems with equality are described by complex holomorphic data. If the Weierstrass flag is continuous and the canonical complex structure extends continuously into the Weierstrass points, then the twistor lift of the members of the Weierstrass flag also extends continuously, by Lemma 16.2. But in general the canonical complex structure does not, in contrast to the false statement of [**FLPP01**, Lemma 4.10], extend continuously into the Weierstrass points. To see this consider the following example.

The line bundle $L = \begin{pmatrix} z^2 \\ \mathbb{j}+z \end{pmatrix}\mathbb{H} \subset \mathbb{H}^2$ is a holomorphic curve on $M = \mathbb{C}$ with holomorphic twistor lift $\hat{L} = \begin{pmatrix} z^2 \\ \mathbb{j}+z \end{pmatrix}\mathbb{C}$. It has exactly one branch point at $z = 0$. Away from $z = 0$, the mean curvature sphere $S \in \Gamma(\operatorname{End} \mathbb{H}^2|_{\mathbb{C}\setminus\{0\}})$ of L satisfies

$$S\begin{pmatrix} z^2 \\ \mathbb{j}+z \end{pmatrix} = \begin{pmatrix} z^2 \\ \mathbb{j}+z \end{pmatrix}\mathbb{i}, \qquad S\begin{pmatrix} 2z \\ 1 \end{pmatrix} = \begin{pmatrix} 2z \\ 1 \end{pmatrix}\mathbb{i},$$

by Lemma 16.2. S does not extend continuously into $z = 0$, because the sections $\begin{pmatrix} z^2 \\ \mathbb{j}+z \end{pmatrix}$ and $\begin{pmatrix} 2z \\ 1 \end{pmatrix}$ are at $z = 0$ linearly independent over $\mathbb{C}$ and linearly dependent over $\mathbb{H}$. The problem in the proof of [**FLPP01**, Lemma 4.10] is in the last formula, where $I_{n+1}\mathbb{i} + O(1)$ should read $I_{n+1}\mathbb{i} + W^{-1}O(1)W$, which in general is not continuous at p_α.

Nevertheless, the arguments in the proof of [**FLPP01**, Lemma 4.10] can still be used to show the continuity of the Weierstrass flag as well as the continuity of the twistor lift of the dual curve.

A.1. Lemma. *Let L be a holomorphic line bundle and $H \subset H^0(L)$ a linear system. The Weierstrass flag of H is then continuous at the Weierstrass points, and the twistor lift of the dual curve extends continuously into the Weierstrass points.*

PROOF. First we need to recall some facts from [**FLPP01**, Lemma 4.9 and the discussion preceding this lemma]. Let p be a Weierstrass point of H. Then there exists a basis ψ_k, $k = 0, \ldots, n := \dim H - 1$ that realizes the Weierstrass gap sequence $n_k(p)$ of H at p such that:

(i) There exists an open neighborhood V of p that does not contain another Weierstrass point and a matrix valued smooth function $B\colon V \to M(n+1, \mathbb{H})$ that is invertible on the punctured neighborhood $V_0 := V \setminus \{p\}$ of p and $\underline{\psi}B^{-1}$ is an adapted frame of the Weierstrass flag $H_k|_{V_0} \subset H$.

(ii) The matrix valued function B has the form
$$B = Z(B_0 + O(1))W,$$
where $Z = \operatorname{diag}(1, z^{-1}, \dots, z^{-n})$, $W = \operatorname{diag}(z^{n_0(p)}, \dots, z^{n_n(p)})$, in a centered holomorphic coordinate $z\colon V \to \mathbb{C}$, $z(p) = 0$, and B_0 is a constant invertible matrix with integer coefficients, in fact B_0 is the Wronskian matrix of the functions $x \mapsto x^{n_k(p)}$ evaluated at $x = 1$.

(iii) The canonical complex structure S satisfies $S\underline{\psi}B^{-1} = \underline{\psi}B^{-1}\mathbb{i}$ on U_0.

The principle minors of B_0 do not vanish. Hence one can assume that the principle minors of $B_0+O(1)$ do not vanish on V, and one can decompose
$$B_0 + O(1) = \tilde{L}\tilde{U}$$
into a lower and an upper triangular matrix such that the upper triangular matrix has ones on its diagonal. The upper triangular matrix
$$U := W^{-1}\tilde{U}W$$
converges to the identity matrix as z goes to 0, because $n_k(p)$ is strictly increasing. With the lower triangular matrix
$$L := Z\tilde{L}W$$
one obtains the LU decomposition
$$B = LU$$
of B. Since L is lower diagonal, the frame
$$\underline{\psi}B^{-1}L = \underline{\psi}U^{-1}$$
is adapted to the Weierstrass flag on V_0, and, because ψU^{-1} converges to $\underline{\psi}$ as z goes to 0, it is continuous and adapted at p. Thus the Weierstrass flag is continuous at p.

Hence the last section
$$\varphi := \underline{\psi}B^{-1}Le_{n+1} = \underline{\psi}U^{-1}e_{n+1}$$
of the adapted frame $\underline{\psi}U^{-1}$ is a continuous section of $V \times H$. It spans the dual curve L^d on V_0 and $\varphi\mathbb{H}$ extends L^d continuously into p. The complex structure of L^d on V_0 is given by S. From fact (iii) follows
$$S\varphi = S\underline{\psi}B^{-1}Le_{n+1} = \underline{\psi}B^{-1}\mathbb{i}Le_{n+1} = \varphi(L^{-1}\mathbb{i}L)_{(n+1,n+1)}$$
on V_0, where $(L^{-1}\mathbb{i}L)_{(n+1,n+1)}\colon V_0 \to \mathbb{H}$ denotes the lower right entry of the matrix $L^{-1}\mathbb{i}L$. Let $\lambda := (\tilde{L})_{(n+1,n+1)}$, then
$$(L^{-1}\mathbb{i}L)_{(n+1,n+1)} = (W^{-1}\tilde{L}^{-1}\mathbb{i}\tilde{L}W)_{(n+1,n+1)} = z^{-n_n(p)}\lambda^{-1}\mathbb{i}\lambda z^{n_n(p)}.$$
Because B_0 is an invertible real matrix, λ is not zero, real and continuous at $z = 0$, hence
$$S\varphi = \varphi(\mathbb{i} + O(1)),$$
which shows that the complex structure of the dual curve is continuous on V and $\varphi(p)\mathbb{C} \subset (\mathbb{H}, \mathbb{i})$ continuously extends the twistor lift $\widehat{L^d} = \{\, \psi \in L^d \mid S\psi = \psi\mathbb{i} \,\}$ of the dual curve. $\square$

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