Convergence of Binomial Large Investor Models and General Correlated Random Walks

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Abstract

This thesis studies the problem of option pricing via replication by a large investor whose trading affects the stock price. We formulate and solve this question first in a binomial setting. Then we consider a suitably scaled sequence of such binomial large investor models and prove their convergence towards a continuous-time diffusion. This requires that we analyze carefully both the convergence of the large investor's strategy functions and the stochastic process of the underlying fundamentals. The convergence of the latter is derived from a new convergence result for general correlated random walks.

In each single time step, we model the stock price as a function of time, some fundamentals and the large investor's stock holdings, and we assume that the fundamentals describe a random walk. We analyze in detail the price mechanism which models how the large investor's trades affect prices and elaborate on the importance of a "fair" price system as a theoretical benchmark. This can be used to define implicit transaction losses and the real value of a large investor's portfolio. We derive conditions which prevent paper-value and real-value arbitrage opportunities for the large investor and show the existence and uniqueness of a replication strategy for a given contingent claim. As a consequence of its feedback on the stock price, this strategy is in general only given implicitly by a fixed point theorem.

To study the convergence of a sequence of binomial large investor models, we rescale the fundamentals as in Donsker's theorem. In a first step, we then show that the convergence of the large investor's strategy functions is implied by their convergence at maturity. The limit function is identified as the solution of a non-linear final value problem. By a suitable strategy transform, this can be simplified to a perturbation of a linear problem in a "fair" market. We then prove the convergence in distribution of the binomial large investor models under two different regimes of martingale measures. Because the transition probabilities for the fundamentals under these measures typically depend on the large investor's stock holdings before and after his trade, we have to extend classical convergence results to a setting with general correlated random walks.

For general correlated random walks, the direction of the next move depends on time, the current position and the direction of the previous move. Using Donsker's scaling, we prove the convergence in distribution of a sequence of such walks towards a diffusion limit, and we explicitly identify the diffusion coefficients. It turns out that in comparison to the classical case, both volatility and drift are reinforced due to the correlation between the increments of the discrete walks. In particular, we obtain a convergence result for existing large investor models from the literature. Moreover, our study highlights the importance and influence of the choice of price mechanism.

Zusammenfassung

Die vorliegende Arbeit betrachtet das Problem der Optionsbewertung durch Replikation für einen Großinvestor, der den Aktienpreis durch sein Handeln beeinflußt. Diese Frage wird zuerst in einem binomialen Rahmen formuliert und gelöst. Im Anschluß untersuchen wir eine geeignet skalierte Folge von solchen binomialen Großinvestormodellen und beweisen ihre Konvergenz gegen ein zeitstetiges Diffusionsmodell. Dazu müssen wir sowohl die Konvergenz der Strategiefunktionen des Großinvestors als auch den stochastischen Prozeß, der die zugrundeliegenden Fundamentaldaten modelliert, sorgfältig beschreiben. Die Konvergenz der Modelle erhalten wir aus einem neuen Konvergenzresultat für allgemeine korrelierte Irrfahrten.

Für jeden einzelnen Zeitpunkt modellieren wir den Aktienpreis als eine Funktion von Zeit, gewissen Fundamentaldaten und dem Aktienbesitz des Großinvestors, und wir beschreiben die Fundamentaldaten durch eine binäre Irrfahrt. Wir untersuchen detailliert den Preismechanismus, der den Einfluß des Großinvestors auf den Aktienkurs modelliert, und arbeiten die Bedeutung eines "fairen" Preissystems als theoretischer Benchmark heraus. Dieser kann dann benutzt werden, um implizite Transaktionsverluste und den Realwert eines Großinvestorportefeuilles zu definieren. Wir entwickeln Bedingungen, die Papierwert- und Realwert-Arbitrage ausschließen, und beweisen die Existenz und Eindeutigkeit von Replikationsstrategien für ein gegebenes Endportefeuille. Wegen ihrer rückkoppelnden Wirkung auf den Aktienpreis ist diese Strategie im allgemeinen nur implizit durch einen Fixpunktsatz gegeben.

Um die Konvergenz einer Folge von binomialen Großinvestormodellen zu betrachten, reskalieren wir den Prozeß der Fundamentaldaten wie im Satz von Donsker. Zunächst zeigen wir dann, daß die Konvergenz der Strategiefunktionen des Großinvestors aus ihrer Konvergenz am Fälligkeitstermin folgt. Die Grenzfunktion ergibt sich als Lösung eines nicht-linearen Endwertproblems, welches durch eine geeignete Strategietransformation auf eine Störung eines linearen Problems in einem "fairen" Markt reduziert werden kann. Im Anschluß beweisen wir die Verteilungskonvergenz der binomialen Großinvestorenmodelle unter zwei verschiedenen Regimen von Martingalmaßen. Weil die Übergangswahrscheinlichkeiten für den Fundamentaldatenprozeß unter diesen Maßen in der Regel vom Aktienbestand des Großinvestors vor und nach seiner Transaktion abhängen, müssen wir dazu klassische Konvergenzresultate auf allgemeine korrelierte Irrfahrten erweitern.

Für allgemeine korrelierte Irrfahrten hängt die Richtung des nächsten Schrittes von Zeit, momentaner Position und der Richtung des letzten Schrittes ab. Wenn eine Folge solcher Irrfahrten wie bei Donsker skaliert wird, zeigen wir, daß sie in Verteilung gegen einen Diffusionsprozeß konvergiert, dessen Diffusionskoeffizienten wir explizit beschreiben. Dabei stellt sich heraus, daß im Vergleich zum klassischen Fall sowohl Volatilität als auch Drift durch die Korrelation zwischen den Zuwächsen der Irrfahrten verstärkt werden. Insbesondere erhalten wir ein Konvergenzresultat für bestehende Großinvestormodelle aus der Literatur. Darüber hinaus unterstreicht unsere Arbeit die Bedeutung und den Einfluß, den die Wahl des Preismechanismus hat.

Contents

Introduction

Stochastic models for option pricing can be traced back to the thesis of Bachelier (1900) from the beginning of the last century. Bachelier modelled the stock price by a Brownian motion with drift and then calculated option prices as expected values under the real-world probability measure. To prevent negative stock prices, Samuelson (1965) proposed to model returns by a Brownian motion so that the stock price itself becomes a geometric Brownian motion. In that setup, Black and Scholes (1973) argued that the price for an option on the stock must coincide with the price of a replicating portfolio in stock and bond. By showing that replicating strategies for European calls and puts exist, they derived their celebrated pricing formula. Merton (1973, 1977) extended the results of Black and Scholes in many directions. Cox and Ross (1976) and more generally Harrison and Kreps (1979) showed that in contrast to Bachelier's approach, the Black-Scholes option price can be computed as the expectation of the final payoff under the risk-neutral measure under which the stock earns the riskless rate of return.

Around the same time Cox, Ross and Rubinstein (1979) developed a discrete approximation of the Black-Scholes model. They showed that the Black-Scholes price for a European call can be obtained as the limit of the unique arbitrage-free call option prices in a sequence of suitably scaled binomial models if the time step goes to zero. In each binomial model, the option price can be found by elementary mathematics because at any node of the binomial tree, the required stock and bond holdings are determined by a self-financing condition from the two possible option values at the next nodes. This gives a recursion to calculate simultaneously the option values and the hedging strategy from the final payoff values at maturity.

The classical literature assumes that all investors have the same information, that the market is complete in the sense that every contingent claim is attainable by some replicating trading strategy, that the market is frictionless, and that *all investors act as price takers*. Of course, these assumptions only give a very idealized picture of reality. To amend this, the Black-Scholes analysis has been extended in numerous ways to accomodate for example incompleteness, transaction costs, short-sale constraints, or asymmetric information. Also the price-taking assumption has been relaxed, first in a discrete binomial model by Jarrow (1994), and then in a continuous-time model by Frey (1998) and others. In this thesis, we first extend the class of price mechanisms considered by these authors and then show in this more general setting that a sequence of discrete binomial models similar to Jarrow (1994) converges to a continuous model which generalizes Frey (1998). If the large investor acts as a price taker, this reduces to the Cox-Ross-Rubinstein (1979) result.

Jarrow (1992, 1994) starts with a general model for the stock price process in a discrete binomial large investor model. To exclude market manipulation generated by trading in stock and bond, he assumes that the stock price depends on time, some fundamentals, and on the current but not on the previous stock holdings of the large investor. If the large investor is allowed to trade in a derivative of the stock, the markets for the stock and its derivative must be in synchrony to prevent market manipulation strategies for the large investor. Jarrow (1994) then shows by example that in such a synchronous market, the price

and the large investor's hedging strategy for a European call can still be derived as in the Cox-Ross-Rubinstein model by a backward recursion. In contrast to the standard binomial model, however, the stock prices in the recursive formulæ depend on the large investor's hedging strategy, which induces additional volatility.
Frey (1998) starts with a general reaction function which describes in a temporary equilibrium the stock price as a function of time, some fundamental value modelled as a geometric Brownian motion, and the large investor's stock holdings. He concentrates on the replication price at which a large investor can perfectly replicate an option with a sufficiently smooth payoff. The replicating strategy is given via a martingale representation for the final option value, where the trading strategy appears not only as integrand, but also in the integrator of the stochastic integral which describes the large investor's cumulative gains from trade. Frey (1998) transforms this stochastic representation into a quasi-linear final value problem for the large investor's strategy function, parametrized by time and fundamentals. He proves existence and uniqueness of solutions to that final value problem and discusses the qualitative difference of the large investor's replicating strategy compared to the corresponding hedging strategy in the classical Black-Scholes model.
While Jarrow and Frey work with an external fundamental state variable, Schönbucher and Wilmott (2000) and Sircar and Papanicolaou (1998) use the feedback perturbed price process as the observable process. Schönbucher and Wilmott (2000) study the price dynamics in illiquid markets and its reaction to the trading strategy of a large investor. Their analysis leads to a partial differential equation for the replication paper value of an option, which is equivalent to Frey's (1998) description via the associated strategy. Sircar and Papanicolaou (1998) derive the same partial differential equation and perform an extensive asymptotic and numerical study by considering the nonlinearity as a perturbation to the classical Black-Scholes partial differential equation.
Weak convergence questions for discrete large investor models have already been examined by Frey and Stremme (1997) and Bierbaum (1997). These authors assume in their results the convergence both of the strategy functions used by the large investor and of the discrete fundamental price processes. In contrast, we derive here the convergence of these two sequences directly from the option replication result in the discrete binomial models.

In relation to the existing literature on *large investor models*, this thesis makes two main contributions. We first introduce and analyze in detail an extensive class of discrete binomial models for option replication and option valuation with a large investor. Then we show that these discrete models converge in distribution to certain diffusion models.
Similarly as in Jarrow (1994) or Frey (1998), the equilibrium price on intervals where the large investor does not trade is modelled as a function of time, some fundamental value and the large investor's stock holdings. In discrete time, we carefully explore the price mechanism which determines the stock price at which the large investor actually trades. This includes an investigation of trades at the initial and final trading dates, and covers both permanent and temporary price impacts which may result from the large investor's activity. The corresponding continuous-time limits provide new insights into the assumptions about the price mechanism in existing large investor models.
In the discrete model, the binomial tree for the relevant stock prices is still recombining if the large investor uses path-independent trading strategies. We derive existence and uniqueness of such a strategy which replicates a given contingent claim. If the stock price does not completely adjusts to an order of the large investor before it is executed, the large investor's replicating strategy is not given explicitly as in Cox, Ross and Rubinstein (1979), but only as a solution to a fixed point problem. Its non-linearity makes the subsequent convergence analysis more difficult to handle and forces us to derive quite precise asymptotic error estimates.

One important insight that emerges from our analysis is that in large investor models, the main focus should be on the trading strategy and not on the value process. In particular, the self-financing condition is a condition on the strategy, and it uniquely determines the latter from a given final position. But as already observed by Jarrow (1992) and Schönbucher and Wilmott (2000), there are at least two different methods to assess a large investor's strategy: its paper value and its real value. We give a new interpretation of the real value as the portfolio liquidation value under a "fair" price system in the large investor market. We also derive conditions on the trading strategy which exclude paper-value arbitrage, and we show that a large class of price systems does not allow real-value arbitrage opportunities.
The *convergence* in distribution of our binomial large investor models to a continuous-time diffusion is proved in two steps. We first show that the large investor's strategy functions converge towards some limit function and then use this to show that the sequence of fundamental processes converges in distribution. The above limit function is the solution of a generally highly non-linear final value problem; this can be substantially simplified by a suitable integral transform. The transformed problem can then be viewed as a perturbation of a linear problem in a large investor market with a "fair" pricing system.
To obtain the convergence of the fundamental processes in a sequence of large investor models, we have to study *general correlated random walks*, for which the direction of the next move depends on time, the current position and the direction of the previous move. We prove that a sequence of such correlated random walks which are scaled as in Donsker's theorem converges to a diffusion limit, and we identify the diffusion coefficients. The volatility and drift of the limit process are reinforced due to the correlation between the increments of the discrete random walks. These results are of independent mathematical interest and constitute another main contribution of the thesis.
The basic convergence theorem for general correlated random walks is first applied to prove the convergence in distribution of a sequence of binomial models under the associated p-martingale measures under which both the large investor stock price and the paper value of the large investor's portfolio are martingales. The transition probabilities for the fundamentals can here depend on the two previous values, since the large investor stock price is a function of the large investor's stock holdings both before and after his trade. Hence we need the full strength of the theorem. For the particular class of models where the stock price completely adjusts to an order of the large investor before that is executed, the transition probabilities only depend on the last value of the fundamentals. Here we establish that a suitably scaled sequence of Jarrow's (1994) binomial models converges to Frey's (1998) model under the p-martingale measures.

We now give a more detailed overview of the various chapters in this thesis.
In **Chapter 1**, we present the discrete-time binomial model of a large investor market. Like Frey (1998), we model the equilibrium stock price in the market as a function of time, of some fundamental value, and of the current stock holdings of the large investor. However, this price is only valid if the large investor is inactive at this point in time, and one must model very precisely what happens when the large investor trades a non-infinitesimal number of shares. The *large investor price* at which the large investor can actually settle his trades is defined as a weighted average of equilibrium prices so that the price system in the large investor market is described by a pair (ψ, μ) of an equilibrium price function ψ and a price determining measure μ. The choice of μ represents the way that the market reacts to the large investor's order. Our setting covers the mechanisms used in the papers of Frey and Stremme (1997), Bierbaum (1997), Frey (1998), Sircar and Papanicolaou (1998), Schönbucher and Wilmott (2000), Jonsson, Keppo and Meng (2004) and Baum (2001), who all assume that the market immediately and fully adjusts to an order of the large investor. It also includes the

other extreme where the large investor can trade at the old equilibrium price with the price only adjusting directly after the large investor's trade; this mechanism is implied in Platen and Schweizer (1998) when switching from the ith to the $(i+1)$th model. In addition to the above permanent price impacts we can also model temporary price impacts in the spirit of Bertsimas and Lo (1998), Bertsimas, Hummel and Lo (2000), Almgren and Chriss (1999, 2000), Huberman and Stanzl (2004, 2003), Bakstein (2001), or Çetin, Jarrow and Protter (2004). For a practitioner's view on the market mechanism in the presence of a large investor see also Taleb (1996).

A key role is played by the (theoretical) *benchmark price* defined as the arithmetic average of all equilibrium stock prices which correspond to a fixed stock position of the large investor lying between the stock holdings before and after his trade. It satisfies the properties of a fair price and can be used to specify implied *transaction losses* relative to the benchmark price. This provides a link to small investor models with transaction costs as studied in discrete time by Boyle and Vorst (1992) and Opitz (1999).

Since the stock price is affected by the large investor's stock holdings, it is not a priori clear how to value a portfolio of the large investor. Chapter 1 introduces two different concepts for this. One is a mark-to-market approach which simply uses the last price seen on the market *by the large investor* to assess his complete stock holdings. The portfolio value obtained from this valuation is called *paper value* as in Jarrow (1992) and Schönbucher and Wilmott (2000). Frey (1998) also implicitly values the large investor's portfolio by means of the paper value. The second approach considers the *real value* of the large investor's portfolio, defined as the theoretical liquidation price the large investor could achieve if he sold his portfolio without any transaction losses. This coincides with the "real value" of Schönbucher and Wilmott (2000) who define this as the limit of successive small block trades. Hence we obtain a new interpretation of why the real value is a good proxy for the actual value of the large investor's portfolio.

After fixing the one-period price mechanism, we extend the large investor model to a dynamic multi-period setting where the fundamentals are given by a binomial random walk. This extends the binomial models of Jarrow (1994) and Bakstein (2001). For path-independent trading strategies, the binomial tree described by the vector of possible large investor and equilibrium stock prices is still recombining. For a suitable class of contingent claims, this allows one to determine a self-financing option replication strategy along the lines of Cox, Ross and Rubinstein (1979). But in contrast to the explicit Cox-Ross-Rubinstein case, if the stock price does not completely adjust to an order of the large investor before its execution, the large investor's strategy is only given as the solution to a fixed point problem for which we show existence and uniqueness. Moreover, one must carefully investigate the behavior of the large investor stock price at the initial and final trading dates.

In the Cox-Ross-Rubinstein model, the (discounted) value process of any self-financing trading strategy is a martingale under the unique measure which makes the (discounted) price process a martingale. This gives a recursive formula for the value function and also shows that this small investor market is free of arbitrage. **Chapter 2** contains similar results for our large investor market, but we now need to distinguish between paper value and real value arbitrage. For a natural class of self-financing trading strategies, the paper value process is a martingale under the *p-martingale measure*, the unique measure which turns the large investor price process into a martingale, and so this class is free of paper-value arbitrage. However, the p-martingale measure is highly dependent on the large investor's trading strategy. If the equilibrium price function has a multiplicative structure, the real value process is always a supermartingale under the *s-martingale measure*, which is the martingale measure for the associated small investor price process. Hence such a market structure permits no

real-value arbitrage opportunities.
Sufficient conditions to exclude arbitrage for the large investor have been given in different models by Jarrow (1992, 1994), Bank (1999), Baum (2001), Bank and Baum (2004), Huberman and Stanzl (2004), Çetin, Jarrow and Protter (2004). The martingale property of the paper value has already been used by Frey (1998) to determine the replicating strategy for a contingent claim in his continuous-time model.
In Chapter 2, we also derive an implicit difference equation of second order for the large investor's strategy function ξ^n in the nth binomial model. This forms the basis of the subsequent convergence analysis in Chapter 3. We then explain the similarities with the proportional transaction cost models of Boyle and Vorst (1992) and Musiela and Rutkowski (1998). These suggest to focus on multiplicative price systems, where the impact from the large investor's stock holdings enters the equilibrium price in a multiplicative way. For such price systems the recursions for the strategy function and the real value process simplify considerably. If the large investor does not face any transaction losses, the real value process even becomes a martingale under the s-martingale measure, every contingent claim is attainable, and we can explicitly calculate its replicating strategy.

Chapter 3 is devoted to the convergence of a sequence $\{\xi^n\}_{n\in\mathbb{N}}$ of strategy functions from binomial large investor models as the time step goes to zero. Under suitable assumptions, the limit function φ must satisfy a partial differential equation which is the continuous analogue of the difference equation for ξ^n. Together with the convergence at the final date, this gives a final value problem for φ, and we prove existence and uniqueness of a solution. Then we show that the convergence of the discrete strategy functions $\{\xi^n\}_{n\in\mathbb{N}}$ to φ follows from the convergence of their values immediately before and at maturity to the corresponding values of φ.
In general, the final value problem for φ is highly non-linear, but it can be simplified to a quasi-linear problem by working with transformed functions $g^n = g \circ \xi^n$ and $\gamma = g \circ \varphi$. If the price system excludes any instantaneous transaction gains or losses, the transformed problem is even linear, each g^n can be calculated by an explicit recursive scheme from its values at and immediately before maturity, and the limit γ satisfies a *linear* final value problem. Thus existence and uniqueness of solutions to the final value problem as well as the convergence of the transformed strategy functions follow from classical results. If the price system does not prevent transaction losses, however, the recursive scheme for g^n remains implicit and the final value problem for γ is only quasi-linear. We adapt a proof by Frey (1998) to show that even in this setting the final value problem for γ still has a solution if the boundary values at maturity do not become too large. We then generalize the convergence result for $\{g^n\}_{n\in\mathbb{N}}$ to this general case and transform the results back into corresponding results for the strategy functions $\{\xi^n\}_{n\in\mathbb{N}}$ and their limit φ.
The detailed investigation of the strategy as a *function* of time and fundamentals is not needed in the standard Cox-Ross-Rubinstein model; since the stock price is not affected by the strategy, the convergence of the value process can be shown without using the convergence of the strategy. But in the large investor model, convergence of the strategy functions is essential to deduce the convergence in distribution of the binomial large investor models. Convergence of the strategy functions is also a key assumption for the convergence results of Frey and Stremme (1997) and Bierbaum (1997). Once the convergence of the strategy functions is shown, we use this to derive a similar convergence result for the real value functions. The partial differential equation satisfied by their limit $\bar{v}$ resembles the structure known from continuous-time models with proportional transaction costs as the continuous-time limits of the models of Leland (1985) and Boyle and Vorst (1992) or the continuous-time model of Barles and Soner (1998). This structural resemblance has also been observed by Frey (2000).

And like in the Black-Scholes model, the limiting strategy function φ is (a transform of) the spatial derivative of the real value function $\bar{v}$. If the price system does not induce transaction losses, the partial differential equation for $\bar{v}$ again becomes linear and basically reduces to the Black-Scholes equation. Such a behavior was also discovered by Jonsson, Keppo and Meng (2004).

In **Chapter 4**, we investigate the convergence in distribution of a sequence of binomial large investor models. Apart from the convergence of the strategy functions, the other key assumption for the convergence result in Frey and Stremme (1997) is the convergence of the discrete-time fundamental processes to a continuous-time diffusion. We give conditions on the parameters of the binomial models which actually imply the convergence in distribution of the fundamental processes, and we explicitly determine the coefficients of the limiting diffusion in terms of the price system (ψ, μ) and the limiting strategy function φ. It is then straightforward to prove by the continuous mapping theorem the convergence of all other model-relevant processes like price, strategy and value.
Of course, convergence in distribution always depends on the underlying probability measures, and this becomes an issue for the large investor model. We show convergence under two different regimes: the p-martingale measures and the s-martingale measures. Under essentially the same assumptions which guarantee the convergence of the strategy functions in Chapter 3, we can apply a convergence theorem for *general correlated random walks* from Chapter 5 to deduce the desired convergence. In the particular case where the equilibrium price completely adjusts to an order of the large investor *before* the order is actually executed, the convergence under the p-martingale measures implies the convergence of a suitably scaled version of Jarrow's (1994) model to the model of Frey (1998). By writing the paper value in the limit model as a function of time and stock price, we also obtain a non-linear partial differential equation for the continuous paper value function which generalizes the corresponding partial differential equations of Schönbucher and Wilmott (1996, 2000), Sircar and Papanicolaou (1998), Frey (2000), and Frey and Patie (2002). The situation under the s-martingale measures is considerably simpler, and the limit of the fundamentals is just a Brownian motion with drift. In the absence of a large investor, the results under the p- and s-martingale measures coincide and we recover the convergence of the Cox-Ross-Rubinstein models to the Black-Scholes model as a special case.

The key ingredient for the results in Chapter 4 is a convergence theorem for general correlated random walks. This is a mathematical contribution of independent interest which is presented in **Chapter 5**. For a correlated random walk, the direction of the next move depends on its tilt, i.e., on the direction of the move in the previous step. But for our application in Chapter 4, we need *general correlated random walks* where the direction of the next move can also depend on time and the current position in space. If a sequence of such walks is scaled as in Donsker's theorem and if for each possible direction of the random walk's previous move, the transition probabilities converge to a (possibly different) limit function, our main convergence theorem for general correlated random walks states that the rescaled sequence converges in distribution to the solution of a stochastic differential equation. The generator of the latter is explicitly given, and a positive correlation between the direction of two successive moves increases the volatility of the limit process. As a corollary, we show that general correlated random walks can be used to approximate general diffusion processes via a recombining binomial tree.
While the proof of the main convergence theorem is based on standard ideas, the details become rather tricky and involved because the correlation between two successive increments of the random walk need not vanish asymptotically. It is essential to carefully select proper lenses to look at our random walk, since its behavior "at a microscopic level" on time intervals

of order $O(\delta^2)$ is very different from the "large picture" on time intervals of length $O(\delta)$ which prevails in the limit.
Correlated random walks, which were introduced by Gillis (1955) and Mohan (1955), are important objects on their own and have a variety of applications outside mathematical finance. An overview of some of the literature is given in Section 5.1. But up to now, research has almost exclusively focused on time- and space-homogeneous correlated random walks which are much easier to handle than the general inhomogeneous case. Thus our convergence results noticeably extend results of Renshaw and Henderson (1981), Szász and Tóth (1984), Tóth (1986), Opitz (1999) and Mauldin, Monticino and Weizsäcker (1996) on the convergence of homogeneous correlated random walks.

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Chapter 1

The Large Investor in Discrete Time

In this chapter we present the discrete, binomial model of a large investor market, which forms the basis of our convergence analysis in Chapter 3 and 4. At the beginning, we have to describe the market mechanism in some more detail. The market is supposed to be in a Walrasian equilibrium as long as the large investor does not trade. It is then essential to model very precisely the stock price movements when the large investor trades a non-infinitesimal number of shares, and because of its importance, we start with such a model in a static world. Our discussion will reveal the significance of a certain *benchmark price* which can then be used in order to specify implied *transaction losses* in the large investor market model. After having described the price mechanism in a single time point, we turn to a general dynamic multi-period large investor market model, which also depends on time and on the evolution of some external fundamentals given by a binomial random walk, and define self-financing trading strategies and portfolio values in a way similar to small investor market models. However, we have to differentiate between the *paper value* and the *real value*. Under certain assumptions on the price system of our large investor model – stated in terms of the associated transaction losses – we then present the class of *star-convex contingent claims*, which are defined in terms of the large investor's final stock holdings at maturity, and we show that all those contingent claims are *attainable* by a replicating trading strategy. A similar result holds for the replication of a certain paper value. In contrast to the standard Cox-Ross-Rubinstein model, the corresponding replicating strategies will in general only be given as solutions to a fixed point equation, and the derivation of an existence and uniqueness result for this fixed point equation is a central result of this chapter. Last but not least we give examples of large investor price systems which satisfy the assumptions needed for these results, and we show that our large investor model contains the Cox-Ross-Rubinstein model as a special case.

1.1 The Market Mechanism in a Single Time Point

In a large investor financial market there exists one (large) investor who can affect the stock prices by his trading. Because of the large investor's influence on the stock prices, the stock price will vary depending on the trades of the large investor, even if the time and the fundamental information at this time is kept constant. Especially, allowing the large investor to perform several subsequent trades at one point in time, the large investor might even realize immediate arbitrage opportunities due to price manipulation techniques, which a small investor cannot apply.

Before we develop a fully dynamic large investor model where the large investor's stock price also depends on time and some fundamental information at this time, we will first focus on the price mechanism in a single time point, i.e. before some new fundamental information arrives. In such a situation we carve out conditions on general large investor's price functions which ensure that any round-trip of the large investor excludes any transaction gains or losses. These conditions are satisfied by the *benchmark price*, which is constructed as the mean of equilibrium prices. For the important class of exponential equilibrium price functions, the benchmark price even is the unique price function which excludes both immediate transaction gains and immediate transaction losses.

Thus, let us assume that at some fixed time (or in some time interval) in which no new information arrives the market there exist some function $S : I\!R^2 \to I\!R$ such that for each $\xi_1, \xi_2 \in I\!R$ the large investor is faced with a per-share price of $S(\xi_1, \xi_2)$ when shifting his stock position from ξ_1 to ξ_2. Supposing that the trades of the large investor are wound off much faster than new information appears in the market, we can take for granted that the large investor can conduct several transactions according to this price building mechanism. In an idealized world, all transactions do not take time at all, such that the large investor can perform infinitely many transactions.

1.1.1 Round-Trips and a Fair Price

Depending on the particular form of the price mechanism described by $S : I\!R^2 \to I\!R$ the large investor might profit or suffer from buying and selling stocks. In this section we look for conditions which a price function $S^* : I\!R^2 \to I\!R$ has to satisfy to be a "fair price" in that the large investor does not gain or lose any money from round-trips.

In order to start, suppose that the large investor initially holds ξ shares of stock. Then he could buy $\alpha \geq 0$ shares at a total price of $\alpha S(\xi, \xi+\alpha)$ and then sell these α shares immediately at a total price of $\alpha S(\xi+\alpha, \xi)$. Overall, this investment costs him

$$C^+(\xi, \alpha) := \alpha\big(S(\xi, \xi+\alpha) - S(\xi+\alpha, \xi)\big).$$

If the costs $C^+(\xi, \alpha)$ are negative, this means that the large investor receives the amount $\left|C^+(\xi, \alpha)\right|$ as a result of his two transactions. In this case the large investor could repeat the game over and over and basically earn any positive amount one can imagine; this would be an immediate arbitrage opportunity for the large investor.

Similarly, the large investor could also sell α stocks and then re-buy them leading to total costs of

$$C^-(\xi, \alpha) := \alpha\big(S(\xi-\alpha, \xi) - S(\xi, \xi-\alpha)\big).$$

The two strategies described above are simple forms of round-trips. A round-trip is a strategy to buy and sell stocks in such a way that the initial stock position is re-attained at the end. In mathematical terms we use the following definition:

Definition 1.1. A **(deterministic)** k**-step round-trip** is a vector $\alpha \in I\!R^k$ which satisfies $\sum_{i=1}^k \alpha_i = 0$. For all $k \in I\!N$ we denote the space of all k-step round-trips by $\mathfrak{R}^k$. The **costs** associated with a round-trip $\alpha \in \mathfrak{R}^k$ starting at level $\xi \in I\!R$ are given by

$$C_k(\xi, \alpha) := \sum_{i=1}^k \alpha_i S\bigg(\xi + \sum_{j=1}^{i-1} \alpha_j, \xi + \sum_{j=1}^{i} \alpha_j\bigg).$$

Remark. By the definition of $\mathfrak{R}^k$ it is obvious that $\mathfrak{R}^k$ is the $(k-1)$-dimensional space orthogonal to the vector $(1, 1, \ldots, 1)^{tr} \in I\!R^k$. □

A large investor with an initial stock holding ξ who applies the round-trip $\alpha \in \mathfrak{R}^k$ changes his stock holdings according to the scheme

$$\xi \to \xi + \alpha_1 \to \xi + \alpha_1 + \alpha_2 \to \cdots \to \xi + \sum_{j=1}^{k-1} \alpha_j \to \xi.$$

Like in the exemplary buy-and-sell case, it is clear that if the large investor starts with a stock position $\xi \in I\!R$ and there exist some $k \in I\!N$ and some round-trip $\alpha \in \mathfrak{R}^k$ such that the associated costs $C_k(\xi, \alpha)$ are strictly negative, then the large investor has an arbitrage opportunity.
A "fair price" mechanism in a large investor market would be a price mechanism which excludes any instantaneous transaction gains and transaction losses from round-trips. Thus, a fair price function $S^* : I\!R^2 \to I\!R$ should satisfy

$$C_k^*(\xi, \alpha) := \sum_{i=1}^{k} \alpha_i S^* \left(\xi + \sum_{j=1}^{i-1} \alpha_j, \xi + \sum_{j=1}^{i} \alpha_j \right) = 0 \quad \text{for all } \xi \in I\!R,\ \alpha \in \mathfrak{R}^k, \text{ and } k \in I\!N \quad (1.1)$$

Instead of verifying (1.1) for all $k \in I\!N$ it suffices to check it for $k = 3$, as the following proposition shows:

Proposition 1.2. *Condition* (1.1) *holds for all $k \in I\!N$ if and only if*

$$\rho S^*\big(\xi, \xi + \rho d\big) + (1 - \rho) S^*\big(\xi + \rho d, \xi + d\big) = S^*\big(\xi + d, \xi\big) \quad (1.2)$$

for all $\rho \in [0, 1]$ and $\xi, d \in I\!R$.

Proof. It is clear that (1.2) is necessary for (1.1), since the former follows from the latter by taking $\alpha = \big(\rho d, (1 - \rho)d, -d\big)^{tr} \in \mathfrak{R}^3$. It remains to show that (1.2) for all $\rho \in [0, 1]$ and $\xi, d \in I\!R$ is also sufficient for (1.1). For this reason let us fix $k \in I\!N$ and suppose that (1.2) holds for all $\rho \in [0, 1]$ and $\xi, d \in I\!R$.
Since the only round-trip $\alpha \in \mathfrak{R}^1$ is $\alpha = 0$, it is clear that (1.1) holds for $k = 1$. For $k = 2$ a round-trip $\alpha \in \mathfrak{R}^2$ must have the form $\alpha = (d, -d)$ for some $d \in I\!R$. Then (1.1) is implied by (1.2) with $\rho = 1$ which gives the symmetry

$$S^*\big(\xi, \xi + d\big) = S^*\big(\xi + d, \xi\big) \quad \text{for all } \xi, d \in I\!R. \quad (1.3)$$

Let us now come to the case $k = 3$ and take some $\alpha \in \mathfrak{R}^3$. Without loss of generality we can assume $\alpha_i \neq 0$ for all $i \in \{1, 2, 3\}$, otherwise we are back in the case $k = 2$. By the definition of $\mathfrak{R}^3$ we have $\alpha_1 + \alpha_2 + \alpha_3 = 0$. Then one of the α_i's has the opposite sign of the two others, i.e. there exist an $i^* \in \{1, 2, 3\}$ with $\operatorname{sgn}\big(\alpha_{i^*}\big) = -\operatorname{sgn}\big(\alpha_j\big)$ for $j \neq i^*$. We will first assume $i^* = 3$ and set $d = -\alpha_3$. Since $\sum_{i=1}^{3} \alpha_i = 0$, $\alpha_i \neq 0$ for $i \in \{1, 2, 3\}$, and $i^* = 3$ we have

$$\alpha_1 = \rho d \quad \text{and} \quad \alpha_2 = (1 - \rho)d \quad \text{for some } \rho \in (0, 1),$$

hence we see that

$$C_3^*(\xi, \alpha) = \rho d S^*(\xi, \xi + \rho d) + (1 - \rho) d S^*(\xi + \rho d, \xi + d) - d S^*(\xi + d, \xi).$$

Due to (1.2) this term vanishes, thus we have proved (1.1) for $k = 3$ if $i^* = 3$. The cases $i^* = 1$ and $i^* = 2$ follow similarly. For example if $i^* = 1$ we can set $\hat{\alpha}_1 = \alpha_2$, $\hat{\alpha}_2 = \alpha_3$, and $\hat{\alpha}_3 = \alpha_1$ as well as $\hat{\xi} = \xi + \alpha_1$ to conclude from $\sum_{i=1}^{3} \hat{\alpha}_i = 0$ that

$$C_3^*(\xi, \alpha) = \hat{\alpha}_3 S^*\big(\hat{\xi} + \hat{\alpha}_1 + \hat{\alpha}_2, \hat{\xi}\big) + \hat{\alpha}_1 S^*\big(\hat{\xi}, \hat{\xi} + \hat{\alpha}_1\big) + \hat{\alpha}_2 S^*\big(\hat{\xi} + \hat{\alpha}_1, \hat{\xi} + \hat{\alpha}_1 + \hat{\alpha}_2\big) = 0$$

which of course simplifies to

$$C_3^*(\xi,\alpha) = \sum_{i=1}^{3} \hat{\alpha}_i S^* \left(\hat{\xi} + \sum_{j=1}^{i-1} \hat{\alpha}_j, \hat{\xi} + \sum_{j=1}^{i} \hat{\alpha}_j \right) = C_3^*(\hat{\xi},\hat{\alpha}). \tag{1.4}$$

It still remains to prove (1.1) for $k > 3$. Here we are going to use an inductive argument. Let us assume that for some $k > 3$ we have shown

$$\sum_{i=1}^{k-1} \alpha_i S^* \left(\xi + \sum_{j=1}^{i-1} \alpha_j, \xi + \sum_{j=1}^{i} \alpha_j \right) = 0 \quad \text{for all } \xi \in I\!R \text{ and } \alpha \in \mathfrak{R}^{k-1}. \tag{1.5}$$

Then we have to show that (1.1) even holds for all $\xi \in I\!R$ and $\alpha \in \mathfrak{R}^k$. Thus, let us fix $\xi \in I\!R$ and some round-trip $\alpha^k \in \mathfrak{R}^k$. We then fragment this k-step round-trip into one $(k-1)$-step round-trip α^{k-1} and a 3-step round-trip β^3 by defining $\alpha^{k-1} \in \mathfrak{R}^{k-1}$ by

$$\alpha_i^{k-1} = \alpha_i^k \quad \text{for } 1 \le i \le k-2 \quad \text{and} \quad \alpha_{k-1}^{k-1} = -\sum_{j=1}^{k-2} \alpha_j^k,$$

and the vector $\beta^3 \in \mathfrak{R}^3$ by

$$\beta_1^3 := \sum_{j=1}^{k-2} \alpha_j^k, \quad \beta_2^3 = \alpha_{k-1}^k, \quad \text{and } \beta_3^3 = \alpha_k^k.$$

Then we have for $1 \le i \le k-2$:

$$\alpha_i^k S^* \left(\xi + \sum_{j=1}^{i-1} \alpha_j^k, \xi + \sum_{j=1}^{i} \alpha_j^k \right) = \alpha_i^{k-1} S^* \left(\xi + \sum_{j=1}^{i-1} \alpha_j^{k-1}, \xi + \sum_{j=1}^{i} \alpha_j^{k-1} \right),$$

for $i = k-1$:

$$\alpha_{k-1}^k S^* \left(\xi + \sum_{j=1}^{k-2} \alpha_j^k, \xi + \sum_{j=1}^{k-1} \alpha_j^k \right) = \beta_2^3 S^* \left(\xi + \sum_{j=1}^{1} \beta_j^3, \xi + \sum_{j=1}^{2} \beta_j^3 \right),$$

and for $i = k$:

$$\alpha_k^k S^* \left(\xi + \sum_{j=1}^{k-1} \alpha_j^k, \xi + \sum_{j=1}^{k} \alpha_j^k \right) = \beta_3^3 S^* \left(\xi + \sum_{j=1}^{2} \beta_j^3, \xi + \sum_{j=1}^{3} \beta_j^3 \right).$$

Finally, by the definitions of α_{k-1}^{k-1} and β_1^3 and the two-dimensional case (1.3), we get

$$\alpha_{k-1}^{k-1} S^* \left(\xi + \sum_{j=1}^{k-2} \alpha_j^{k-1}, \xi + \sum_{j=1}^{k-1} \alpha_j^{k-1} \right) = -\beta_1^3 S^* \left(\xi + \beta_1^3, \xi \right) = -\beta_1^3 S^* \left(\xi, \xi + \beta_1^3 \right).$$

Thus (1.1) with $\alpha = \alpha^k$ is equivalent to

$$\sum_{i=1}^{k-1} \alpha_i^{k-1} S^* \left(\xi + \sum_{j=1}^{i-1} \alpha_j^{k-1}, \xi + \sum_{j=1}^{i} \alpha_j^{k-1} \right) + \sum_{i=1}^{3} \beta_i^3 S^* \left(\xi + \sum_{j=1}^{i-1} \beta_j^3, \xi + \sum_{j=1}^{i} \beta_j^3 \right) = 0$$

and this holds true because of the induction hypothesis (1.5) and the case for $k = 3$. **q.e.d.**

Remark. Jarrow (1992) derives sufficient conditions for the non-existence of market manipulation strategies in discrete multi-period large investor markets. One of these conditions prevents any trading strategies where the large investor establishes a trend and then trades against it before the market collapses. In our model which was restricted to a single time point, or at least a time interval in which no new information occurs and solely the large investor has a significant impact on the stock price, the condition (1.1) excludes such market manipulating trading strategies. □

Of course, the constant price function $S^* : \mathbb{R}^2 \to \mathbb{R}$ with $S^* \equiv c$ for some $c \in \mathbb{R}$ satisfies the conditions (1.1) and (1.2). Before we present non-trivial price functions which satisfy these conditions as well, we should take a closer look at how we want to model the price mechanism in the large investor market so as to search for functions within the proper class.

1.1.2 The Class of General Price Functions

In this section we will present a class of price functions $S : \mathbb{R}^2 \to \mathbb{R}$ which is used to describe the stock price mechanism in a single time point. The class presented is motivated by an analysis of the market reaction to trades of the large investor, and allows for various information structures between the small and the large investor.

Thus, let us assume that $S : \mathbb{R}^2 \to \mathbb{R}$ is *some* price function, such that the large investor is faced with a price of $S(\xi_1, \xi_2)$ when he shifts his stock holdings from ξ_1 to ξ_2. If the large investor holds $\xi \in \mathbb{R}$ shares and does not trade, the stock price in the market will be $f(\xi) := S(\xi, \xi)$. This price $f(\xi)$ can be viewed as the *Walrasian equilibrium price* in a market where ξ shares are held by the large investor and the large investor has no additional demand or supply. The rest of the shares is assumed to be held by the small investors, and thus the price $f(\xi)$ could have been derived from the cumulative excess demand function of the small investors for any given constant stock position $\xi \in \mathbb{R}$ of the large investor.

Now suppose that for some $\xi_1, \xi_2 \in \mathbb{R}$ the large investor changes his stock position from ξ_1 to ξ_2 shares. In this case the old Walrasian equilibrium at the price $f(\xi_1)$ is disturbed and the market will move towards the new equilibrium at the price $f(\xi_2)$. It remains to model in more detail how the transition takes place from the old to the new equilibrium, and in particular from the old equilibrium price $f(\xi_1)$ to the new equilibrium price $f(\xi_2)$. Especially, we are interested at which per-share price the large investor can trade the $|\xi_2 - \xi_1|$ stocks needed to shift his stock holdings from ξ_1 to ξ_2 shares. This question basically leads back to the question how the information about the large investor's trade is noticed by the market participants. Depending on the information structure the small investors (can) adapt their stock holdings more or less quickly to the stock holdings which they prefer in the new equilibrium. We will illustrate this with two simple examples.

Example 1.1. Suppose that our market consists of the large investor and "infinitely many" "infinitesimally small" investors. As long as the large investor holds $\xi_1 \in \mathbb{R}$ stocks, we are in the old equilibrium at the stock price $f(\xi_1)$; thus each small investor is willing to buy and sell an infinitesimal share of the stock at a price $f(\xi_1)$ per stock. Without loss of generality let us assume that $\xi_1 < \xi_2$, i.e. the large investor wants to buy $\xi_2 - \xi_1$ stocks. He could achieve this goal by entering separate contracts with all the small investors to buy an "infinitesimal small" amount of stocks from each of the "infinitely many" small investors, such that overall he has bought $\xi_2 - \xi_1$ stocks.

In this case the small investors notice the disturbance of the old equilibrium with a certain delay, and the large investor can realize a per-share price of $f(\xi_1)$. After the large investor's trade the small investors have to adjust their individual stock holdings according to their individual excess demand functions, such that the new equilibrium price $f(\xi_2)$ will quickly emerge. The large investor can realize the price $f(\xi_1)$ since the stocks are exchanged before the small investors are aggregated. □

In the next example the information structure is reversed and the demand of the small investors is aggregated before the large investor's trade is executed.

Example 1.2. We once again consider a market with one large investor and "infinitely many" "infinitesimally small" investors, but now suppose that the large investor does not or cannot enter into contracts with each small investor separately, but buys the $\xi_2 - \xi_1 > 0$ stocks needed to shift his stock holdings from ξ_1 to ξ_2 at the stock exchange. Since the large investor wants to shift his portfolio regardless of the stock price he can obtain, he has to place an unlimited order. Due to the additional demand for stocks at the stock exchange there will be much more small investors whose sell orders can be accepted to maximize the volume of sales, and the price fixed by the broker will be $f(\xi_2)$.
Since a similar reasoning works if the large investor sells stocks, we can conclude that if the small investors are aggregated before the stocks are exchanged the large investor can only realize a price $f(\xi_2)$. □

Depending on the realistic problem, it is easy to think of cases where we have a situation in between the two extremes of the preceding examples: Perhaps not all of the small investors are of the same size, there might be a few larger ones with whom the large investor is willing to enter into separate contracts on a part of the stocks he is going to buy or sell, and he might buy/sell the rest at possibly different stock exchanges.
If we know for each $\xi \in I\!R$ the equilibrium stock price $f(\xi)$ which would appear in the market whenever the large investor held the fixed amount of ξ shares, then we can model the price mechanism in the large investor market by introducing a **price-determining (probability) measure** μ on $I\!R$, which reflects the information structure between the small investors and the large investor. Namely, we then model the price function $S : I\!R^2 \to I\!R$ which describes the per-share price of the large investor's transaction of $\xi_2 - \xi_1$ shares to change his stock holdings from ξ_1 to ξ_2 by

$$S(\xi_1, \xi_2) = \int f\big((1-\theta)\xi_1 + \theta\xi_2\big)\mu(d\theta) \quad \text{for all } \xi_1, \xi_2 \in I\!R. \tag{1.6}$$

Within this setting we recover the price-building mechanism of Example 1.1, where the price-determining measure is the Dirac measure δ_0 concentrated in 0, and the mechanism of Example 1.2 where $\mu = \delta_1$, i.e. the Dirac measure concentrated in 1. Most natural are price-determining measures which lie "between" these two Dirac measures, i.e. which are probability measures on $[0, 1]$. However in taking for example $\mu = \delta_x$ for some $x > 1$, we can also model price dynamics, where the market at first overreacts because of the sudden additional large investor's supply or demand, since there is not enough instantaneous liquidity in the market.

Remark. The representation (1.6) can be rewritten as

$$S(\xi, \xi + \alpha) = \int f(\xi + \theta\alpha)\, \mu(d\theta) \quad \text{for all } \xi, \alpha \in I\!R. \tag{1.7}$$

We will use both representations in the sequel. □

In order to guarantee the existence of the price functions of the form (1.6) we suppose that the function $f : I\!R \to I\!R$ describing the equilibrium stock prices is Lebesgue-measurable and locally bounded, and that the price-determining measure μ is such that the integral in (1.6) exists for all $\xi_1, \xi_2 \in I\!R$. This leads to the following definitions:

Definition 1.3. Let $\mathcal{M}(\mathbb{R})$ denote the set which contains all probability measures on $\mathbb{R}$. For every Lebesgue-measurable and locally bounded function $f : \mathbb{R} \to \mathbb{R}$ the **family $\mathfrak{S}(f)$ of general price functions based on** f is given by the set of all functions $S : \mathbb{R}^2 \to \mathbb{R}$ satisfying (1.6) for some price-determining measure

$$\mu \in \mathfrak{M}(f) := \left\{ \mu \in \mathcal{M}(\mathbb{R}) \ : \ \int \big| f\big((1-\theta)\xi_1 + \theta\xi_2\big) \big| \mu(d\theta) < \infty \quad \text{for all } \xi_1, \xi_2 \in \mathbb{R} \right\}. \quad (1.8)$$

The **class of general price functions** is then given by

$$\mathfrak{S} := \big\{ S \,\big|\, S \in \mathfrak{S}(f) \text{ for some Lebesgue-measurable and locally bounded } f : \mathbb{R} \to \mathbb{R} \big\}.$$

Moreover, we define the **class $\mathfrak{S}_e$ of exponential price functions** as the set

$$\mathfrak{S}_e := \big\{ S \,\big|\, S \in \mathfrak{S}(f) \text{ for some } f : \mathbb{R} \to \mathbb{R} \text{ with } f(\xi) = a + be^{c\xi} \text{ for all } \xi \in \mathbb{R} \big\}.$$

The class of exponential price function turns out to be a well-suited subclass of general price functions. Especially, all exponential price functions are either bounded from above (if $b \leq 0$) or from below (if $b \geq 0$), and in particular if both $b \geq 0$ and $a \geq 0$, then each large investor price $S \in \mathfrak{S}(f)$ which is generated from the equilibrium price function $f(\xi) = a + be^{c\xi}$ for all $\xi \in \mathbb{R}$ is nonnegative. Provided that even $a = 0$ and $b > 0$, the stock price is either constant (if $c = 0$) or for every $x > 0$ there exists some position ξ of shares held by the large investor, such that the equilibrium price in the market becomes $f(\xi) = x$. Of course, the equilibrium function f associated to an exponential price function $S \in \mathfrak{S}_e$ is always monotone, and it is strictly monotone if $b, c \in \mathbb{R}\backslash\{0\}$.

Remark. In Section 1.3.1 we will generalize the equilibrium stock price f so that it also depends on time and some stochastic process which describes market fundamentals. The defining equation (1.6) for the large investor's stock price is generalized accordingly. This will give us a stochastic and dynamic model for the stock price which is comparable to the usual models of the stock price in large investor models as for example in Jarrow (1992, 1994) or Frey (1998). Compared to these models we have modelled more precisely the price mechanism at a trading date for the large investor, and we have substantially extended the class of price mechanism considered. For this reason, we will already pause here to discuss how the stock price in a large investor market is modelled in the literature.

Jarrow (1992, 1994) models the stock price as a reaction function to the large investor's trades and thus assumes that the market completely adjust to an order of the large investor before this order is executed. While Jarrow (1992, 1994) starts his discussion with a very general discrete stock price process which can depend on the entire history of the large investor's strategy, he can only prove absence of arbitrage for the large investor if the stock price process is independent of the large investor's past holdings. In a market in continuous time, Frey and Stremme (1997) use a market clearing condition of zero total excess demand as introduced by Föllmer and Schweizer (1993) to obtain a Walrasian equilibrium stock price. By this means, they implictly suppose that the market adjusts as well to the large investor's order before it is actually executed. The model of Frey and Stremme (1997) has been applied and extended by Frey (1998), Sircar and Papanicolaou (1998), Platen and Schweizer (1998), Bierbaum (1997), Baum (2001) and Bank and Baum (2004). Schönbucher and Wilmott (1996, 2000) describe in detail the price mechanism; they assume that first the small investors and the large investor simultaneously submit an order and that then the equilibrium price is determined. Closely related are the models of Kyle (1985) and Back (1992) for a financial market with an insider. In all these models the implied price-determining measure is given by $\mu = \delta_1$.

In Cvitanić and Ma (1996) and the successional papers of Buckdahn and Hu (1998) and Cuoco and Cvitanić (1998) the large investor's stock holdings do not affect the stock price

immediately, but only in the long run via the drift and volatility coefficients of a stochastic differential equation which describes the stock price. In order to be able to apply the theory of forward-backward stochastic differential equations these authors suppose that the stochastic differential equation for the stock price depends only on the number of stocks held by the large investor, but not on the instantaneous change of the number. This limits the feedback of the large investor's trading strategy on the stock price. Bank (1999) combines elements of the diffusion approach of Cuoco and Cvitanić (1998) with the reaction function framework of Jarrow (1992) and Frey and Stremme (1997).
The above models only consider the permanent price impact of the large investor's trade. Temporary price impacts where the market first overreacts and then recovers are also a common feature in real-world large investor markets, and sometimes these price impacts are much larger than the permanent impacts. For example, a sizable order of the large trader fills up the best quotes on the market, and it will take some time until new orders from the small investors arrive. Recent studies have shown that institutional investors often break their larger trades into several smaller packages that they execute successively. Bertsimas and Lo (1998) have set up an additive model which incorporates both permanent and temporary price impact in order to study optimal execution strategies for a large investor who has to build up or liquidate a certain portfolio. Their model has been subsequently extended by Almgren and Chriss (1999, 2000), Almgren (2003) and Huberman and Stanzl (2004, 2003). A multiplicative version of Bertsimas and Lo (1998) has been studied in Bertsimas et al. (2000), Bakstein (2001), and Bakstein and Howison (2002). Çetin et al. (2004) and Çetin et al. (2002) model temporary price impacts by hypothesizing a stochastic supply curve for the stock price as a function of trade size. A different form of price impact is modelled by Subramanian and Jarrow (2001), who derive optimal liquidation strategies for a large investor who can only execute his orders with a certain time delay. Taleb (1996) describes the market behavior in an illiquid financial market from a practitioner's point of view. □

1.1.3 Existence and Uniqueness of a Fair Price

We will now show that any general price function $S \in \mathfrak{S}$ for which the price-determining measure μ is the Lebesgue measure λ on $[0,1]$ satisfies the "fair price" condition (1.1), and thus give a sufficient condition for (1.1) to hold. For all non-degenerate exponential price functions $S \in \mathfrak{S}_e$ we even prove that $\mu = \lambda$ is necessary for (1.1). We will start with the first statement:

Proposition 1.4. *Let $f : I\!R \to I\!R$ be Lebesgue-measurable and locally bounded, and define $S^* \in \mathfrak{S}$, $S^* : I\!R^2 \to I\!R$ by*

$$S^*\big(\xi_1, \xi_2\big) = \int_0^1 f\big((1-\theta)\xi_1 + \theta\xi_2\big)\lambda(d\theta) \quad \text{for all } \xi_1, \xi_2 \in I\!R. \tag{1.9}$$

Then the "fair price" condition (1.1) *holds.*

Proof. By Proposition 1.2 it suffices to prove that (1.2) holds, i.e. we have to show for all $\xi, \alpha \in I\!R$ and $\rho \in [0,1]$

$$\rho \int_0^1 f(\xi+\rho\alpha\theta)\lambda(d\theta) + (1-\rho) \int_0^1 f\big(\xi+\rho\alpha+(1-\rho)\alpha\theta\big)\lambda(d\theta) = \int_0^1 f(\xi+\alpha-\alpha\theta)\lambda(d\theta). \tag{1.10}$$

In order to do so, we fix $\xi, \alpha \in I\!R$ and then use a probabilistic argument which will be recycled in Proposition 1.5. To this end let $\big(\Omega, \mathcal{F}, \mathbf{P}\big)$ be some probability space on which we can define for each $\rho \in [0,1]$ a random variable U^ρ by $\mathbf{P}\big(U^\rho = 1\big) = \rho = 1 - \mathbf{P}\big(U^\rho = 0\big)$ and some random variable Z which is uniformly distributed on $[0,1]$ and independent of $\{U^\rho\}_{\rho\in[0,1]}$. If we now set $Y := 1 - Z$ and define the family of random variables $\{X^\rho\}_{\rho\in[0,1]}$ in terms of U^ρ and Z as

$$X^\rho := U^\rho \rho Z + \big(1 - U^\rho\big)\big(\rho + (1-\rho)Z\big) \quad \text{for } \rho \in [0,1],$$

then (1.10) can be rewritten as

$$\mathbf{E}\Big[f\big(\xi + \alpha X^\rho\big)\Big] = \mathbf{E}\Big[f\big(\xi + \alpha Y\big)\Big] \quad \text{for all } \rho \in [0,1]. \tag{1.11}$$

A sufficient condition for this to hold is that X^ρ and Y are identically distributed. Since Z is uniformly distributed on $[0,1]$, this is also the case for $Y = 1 - Z$. Moreover, by the law of total probability we have for all $\rho \in (0,1)$ and $x \in (0,1)$:

$$\begin{aligned}
\mathbf{P}\big(X^\rho \le x\big) &= \mathbf{P}\Big(U^\rho \rho Z + \big(1-U^\rho\big)\big(\rho + (1-\rho)Z\big) \le x\Big)\\
&= \rho\mathbf{P}\big(\rho Z \le x\big) + (1-\rho)\mathbf{P}\big(\rho + (1-\rho)Z \le x\big)\\
&= \rho\mathbf{P}\left(Z \le \frac{x}{\rho}\right) + (1-\rho)\mathbf{P}\left(Z \le \frac{x-\rho}{1-\rho}\right)\\
&= \begin{cases} \rho\frac{x}{\rho} & 0 < x \le \rho \\ \rho + (1-\rho)\frac{x-\rho}{1-\rho} & \rho \le x < 1 \end{cases}\\
&= x,
\end{aligned}$$

and due to $X^\rho = Z$ for all $\rho \in \{0,1\}$ the equality $\mathbf{P}\big(X^\rho \le x\big) = x$ even holds for all $x \in (0,1)$ and $\rho \in [0,1]$. Thus, for all $\rho \in [0,1]$ the random variable X^ρ is uniformly distributed on $[0,1]$ as is Y. This proves (1.11), and since ξ and α have been fixed arbitrarily the proposition follows. **q.e.d.**

The "fair" price functions given in Proposition 1.4 all employ the rather special price-determining measure λ. Of course, if the equilibrium price function $f : I\!R \to I\!R$ does not depend on the large investor's stock holdings but is constant, i.e. if there exists some $c \in I\!R$ such that $f(\xi) = c$ for all $\xi \in I\!R$, *every* price-determining measure leads to the same "fair" price function $S^* : I\!R^2 \to I\!R$ given by $S(\xi_1, \xi_2) = c$ for all $\xi_1, \xi_2 \in I\!R$. Thus, the price-determining measure associated to a "fair" price function need not be λ. Of course, in the given example of a constant equilibrium price function, the fair price function S^* can be represented as in (1.9) as well, since S^* does not depend on the price-determining measure at all.
We will now show that every "fair" *exponential* price function can be represented like (1.9). For non-constant price functions this implies that the price-determining measure μ *has to be* the Lebesgue measure on $[0,1]$, while for constant price functions it only implies that μ *can be chosen* as the Lebesgue measure on $[0,1]$.

Proposition 1.5. *For any $S^* \in \mathfrak{S}_e$ the "fair price" condition* (1.1) *holds if and only if S^* has a representation of the form* (1.9).

Proof. Let $S^* \in \mathfrak{S}_e$. We have already seen in Proposition 1.4 that (1.1) holds if S^* has a representation of the form (1.9). It remains to show that the existence of such a representation is also necessary for "fair price" functions. Due to Proposition 1.2 it suffices to show that the simplified condition (1.2) implies that S^* can be represented as in (1.9).
By the definition of $\mathfrak{S}_e$ there exist some $a, b, c \in I\!R$ and some probability measure μ such that

$$S^*\big(\xi_1, \xi_2\big) = a + be^{c\xi_1} \int e^{c\theta(\xi_2 - \xi_1)} \mu(d\theta) \quad \text{for all } \xi_1, \xi_2 \in I\!R. \tag{1.12}$$

If $b = 0$ or $c = 0$, then the price $S^*(\xi_1, \xi_2)$ is constant and depends neither on ξ_1 nor on ξ_2, and also not on the price-determining measure μ. In these cases (1.2) trivially holds and we can replace the original price-determining measure μ in (1.12) by the Lebesgue measure λ on $[0,1]$ to obtain a representation of the form (1.9). Thus, we can assume without loss of generality that $b, c \in I\!R \backslash \{0\}$.
Next we recall from the definition of $\mathfrak{S}_e \subset \mathfrak{S}$ that the price-determining measure μ in (1.12) satisfies

$$\int \left| a + b e^{c((1-\theta)\xi_1 + c\theta\xi_2)} \right| \mu(d\theta) < \infty \quad \text{for all } \xi_1, \xi_2 \in I\!R.$$

By the triangular inequality this is equivalent to imposing $\int |b| e^{c((1-\theta)\xi_1 + c\theta\xi_2)} \mu(d\theta) < \infty$ for all $\xi_1, \xi_2 \in I\!R$. Dividing the latter bound by $|b| e^{c\xi_1} > 0$ and then substituting $s = c(\xi_2 - \xi_1)$, shows that the price-determining measure satisfies

$$\int e^{s\theta} \mu(d\theta) < \infty \quad \text{for all } s \in I\!R. \tag{1.13}$$

As in the proof of Proposition 1.4 we now rewrite our problem in stochastic terms. Therefore, let us take a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ on which we can define some random variable Z with distribution μ. Because of (1.13) we have $E\left[e^{sZ}\right] = \int e^{s\theta} \mu(d\theta) < \infty$ for all $s \in I\!R$. In terms of Z the change of variable formula restates (1.12) as

$$S^*(\xi_1, \xi_2) = a + b e^{c\xi_1} \mathbf{E}\left[e^{c(\xi_2 - \xi_1)Z} \right] \quad \text{for all } \xi_1, \xi_2 \in I\!R,$$

and hence, due to $b \neq 0$, the simplified "fair price" condition (1.2) is equivalent to

$$\rho \mathbf{E}\left[e^{cd\rho Z} \right] + (1-\rho) \mathbf{E}\left[e^{cd(\rho + (1-\rho)Z)} \right] = \mathbf{E}\left[e^{cd(1-Z)} \right] \quad \text{for all } \rho \in [0,1] \text{ and } d \in I\!R. \tag{1.14}$$

We continue to follow the stochastic description of the problem in the proof of Proposition 1.4 and take a family $\{U^\rho\}_{\rho \in [0,1]}$ of random variables which are independent of Z and satisfy $\mathbf{P}(U^\rho = 1) = \rho = 1 - \mathbf{P}(U^\rho = 0)$ for all $\rho \in [0,1]$. Then we can again define the random variable $Y = 1 - Z$ and the family $\{X^\rho\}_{\rho \in [0,1]}$ of random variables given by

$$X^\rho := U^\rho \rho Z + (1 - U^\rho)(\rho + (1-\rho)Z) \quad \text{for } \rho \in [0,1].$$

Substituting $d = \frac{s}{c}$ we now see that equation (1.14) is equivalent to

$$\mathbf{E}\left[e^{sX^\rho} \right] = \mathbf{E}\left[e^{sY} \right] \quad \text{for all } s \in I\!R. \tag{1.15}$$

Since $\mathbf{E}\left[e^{sZ}\right] < \infty$ for all $s \in I\!R$, we also have $\mathbf{E}\left[e^{sY}\right] = e^s \mathbf{E}\left[e^{-sZ}\right] < \infty$ for all $s \in I\!R$. Hence the moment generating function $s \to \mathbf{E}\left[e^{sY}\right]$ determines the distribution of Y, and it follows from (1.15) that (1.2) holds if and only if each X^ρ for $\rho \in [0,1]$ and Y are identically distributed.
It remains to show that X^ρ and Y can only be identically distributed for all $\rho \in [0,1]$ if $\mu = \lambda$. Thus, let us assume that for each $\rho \in [0,1]$ the random variable X^ρ has the same distribution as Y, i.e. we assume that

$$\mathbf{P}(X^\rho \leq x) = \mathbf{P}(Y \leq x) \tag{1.16}$$

for all $x \in I\!R$ and $\rho \in [0,1]$. Since the two random variables U^ρ and Z which generate X^ρ are independent, we can apply for each $\rho \in (0,1)$ the law of total probability to express the distribution of X^ρ as in Proposition 1.4 by

$$\mathbf{P}(X^\rho \leq x) = \rho \mathbf{P}\left(Z \leq \frac{x}{\rho} \right) + (1-\rho) \mathbf{P}\left(Z \leq \frac{x - \rho}{1 - \rho} \right) \quad \text{for all } x \in I\!R.$$

If we now recall the definition of Z we can rewrite the distribution of X^ρ in terms of the price-determining measure μ and the parameter $\rho \in (0,1)$ as

$$\mathbf{P}\big(X^\rho \le x\big) = \rho\mu\Big(\Big(-\infty, \frac{x}{\rho}\Big]\Big) + (1-\rho)\mu\Big(\Big(-\infty, \frac{x-\rho}{1-\rho}\Big]\Big) \quad \text{for all } x \in \mathbb{R}. \tag{1.17}$$

Moreover, since $U^0 = 0$ and $U^1 = 1$ a.s., it follows directly that

$$\mathbf{P}\big(X^\rho \le x\big) = \mathbf{P}\big(Z \le x\big) = \mu\big((-\infty, x]\big) \quad \text{for all } x \in \mathbb{R} \text{ and } \rho \in \{0,1\}. \tag{1.18}$$

On the other hand, by the definition of Y we can also express the distribution of Y in terms of μ, namely we get

$$\mathbf{P}\big(Y \le x\big) = \mathbf{P}\big(1 - Z \le x\big) = \mathbf{P}\big(Z \ge 1 - x\big) = \mu\big([1-x, \infty)\big). \tag{1.19}$$

We will now employ (1.16) for different x and ρ to show that we indeed have $\mu = \lambda$. At first we fix $\rho = 1$ to show that μ is symmetric around the point $\frac{1}{2}$. Namely, we get from (1.16), (1.18), and (1.19) with $x = \frac{1}{2} - z$:

$$\mu\Big(\Big(-\infty, \frac{1}{2} - z\Big]\Big) = \mu\Big(\Big[\frac{1}{2} + z, \infty\Big)\Big) \qquad \text{for all } z \in \mathbb{R}. \tag{1.20}$$

Now we fix $x = 1$, to conclude from (1.16), (1.17), and (1.19):

$$\rho\mu\Big(\big(-\infty, \rho^{-1}\big]\Big) + (1-\rho)\mu\big((-\infty, 1]\big) = \mu\big([0,\infty)\big) \qquad \text{for all } \rho \in (0,1). \tag{1.21}$$

Since the symmetry (1.20) in particular implies that $\mu\big((-\infty,1]\big) = \mu\big([0,\infty)\big)$, we can simplify (1.21) to

$$\rho\mu\Big(\big(-\infty, \rho^{-1}\big]\Big) = \rho\mu\big([0,\infty)\big) \qquad \text{for all } \rho \in (0,1),$$

hence after a division by ρ and a subtraction of $\mu\big((-\infty,1]\big) = \mu\big([0,\infty)\big)$ we arrive at

$$\mu\big((1, \rho^{-1}]\big) = 0 \qquad \text{for all } \rho \in (0,1).$$

If we take the limit as $\rho \searrow 0$, we conclude $\mu\big((1,\infty)\big) = 0$, and by the symmetry (1.20) we even get

$$\mu\big((-\infty, 0) \cap (1, \infty)\big) = 0. \tag{1.22}$$

Now we employ (1.16) for $x = 0$, and obtain by (1.17), (1.19) and (1.22) the equality

$$\rho\mu\big(\{0\}\big) = \mu\big(\{1\}\big) \quad \text{for all } \rho \in (0,1),$$

since $\frac{-\rho}{1-\rho} < 0$ for these ρ. Taking again $\rho \searrow 0$, we conclude $\mu(\{1\}) = 0$ and therefore also $\mu(\{0\}) = 0$. Finally, we can consider (1.16) for $x \in (0,1)$ and $\rho = x$. In this case (1.17) and (1.19) imply

$$x\mu\big((-\infty,1]\big) + (1-x)\mu\big((-\infty,0]\big) = \mu\big([1-x,\infty)\big) \quad \text{for all } x \in (0,1). \tag{1.23}$$

Due to $\mu\big((-\infty,0] \cap [1,\infty)\big) = 0$, equation (1.23) can now be rewritten in terms of $z = 1 - x$ as $(1-z)\mu\big((0,1)\big) = \mu\big([z,1)\big)$ for all $z \in (0,1)$. Taking into account that μ is a probability measure, this implies

$$\mu\big((0,z)\big) = z \quad \text{for all } z \in (0,1),$$

hence, indeed, the price-determining measure μ has to be the Lebesgue measure λ on $[0,1]$, which was left to show. **q.e.d.**

1.1.4 The Benchmark Price

Let now $S \in \mathfrak{S}$ be some general price function for the large investor in the single-period market. If we want to valuate the large investor's transactions – or also his total portfolio – in an unbiased fashion, the price function S is not of great use. For example it does not take into account that the price paid by the large investor to reach a certain number of shares might have been excessively high, since the large investor's demand had led to a market squeeze. In such a case the price given by S would be higher than the "fair" value for the stocks bought.
In view of Proposition 1.4 and Proposition 1.5 we will use a "fair" valuation principle, which is based on the benchmark price function associated to a general price function $S \in \mathfrak{S}$.

Definition 1.6. Let $f : \mathbb{R} \to \mathbb{R}$ be some Lebesgue-measurable and locally bounded function, and $S \in \mathfrak{S}(f)$. Then the associated **benchmark price function** $S^* : \mathbb{R}^2 \to \mathbb{R}$ is given by

$$S^*(\xi_1, \xi_2) := \int f\big((1-\theta)\xi_1 + \theta\xi_2\big)\lambda(d\theta) \quad \text{for all } \xi_1, \xi_2 \in \mathbb{R}, \tag{1.24}$$

where of course λ is again the Lebesgue measure on $[0, 1]$.

Because of Proposition 1.4 the benchmark price function satisfies the "fair price" condition (1.1), and moreover, by Proposition 1.5 it is the unique price function $S^* \in \mathfrak{S}(f)$ which satisfies this condition if f is of exponential form.

Remark. Note that for each $S \in \mathfrak{S}$ the function $f : \mathbb{R} \to \mathbb{R}$ of the representation (1.6) can be recovered as $f(\xi) = S(\xi, \xi)$ for all $\xi \in \mathbb{R}$. Hence we can calculate the benchmark price function $S^* : \mathbb{R}^2 \to \mathbb{R}$ from the sole knowledge of the (real-world-)price function $S : \mathbb{R}^2 \to \mathbb{R}$. Especially, the price function $S \in \mathfrak{S}$ and the associated benchmark price function S^* coincide if the large investor does not trade, i.e. we have $S(\xi, \xi) = S^*(\xi, \xi)$ for all $\xi \in \mathbb{R}$. □

For a large investor who has initially held ξ_1 shares and now changes his stock holdings to ξ_2 shares, the benchmark price $S^*(\xi_1, \xi_2)$ represents a fair per-share price for this transaction, in particular the price is symmetric in that $S^*(\xi_1, \xi_2) = S^*(\xi_2, \xi_1)$ for all $\xi_1, \xi_2 \in \mathbb{R}$. In most cases however, the benchmark price $S^*(\xi_1, \xi_2)$ will not coincide with the actual price $S(\xi_1, \xi_2)$ at which the large investor shifts his portfolio from ξ_1 to ξ_2.
If for example the large investor has more information than the average small investor, then the large investor might buy stocks more cheaply than for the benchmark price and sell stocks at a higher price. Such a situation has been depicted in Example 1.1 if the equilibrium price function $f : \mathbb{R} \to \mathbb{R}$ is chosen to be strictly increasing. In such a market environment it is easy to see that the large investor has arbitrage opportunities. However, the arbitrage opportunity does not necessarily mean that the large investor can make any profit he desires: If the large investor can only buy a limited number of shares for a price below the benchmark price, the arbitrage opportunities might be limited.
On the other hand, if the large investor faces prices higher than the benchmark price whenever he buys shares, and if at the same time for all his sales he can only attain prices lower than the benchmark price, then the large investor does not have any arbitrage opportunities at all. This happens to be the case in the market of Example 1.2 if the equilibrium price function function $f : \mathbb{R} \to \mathbb{R}$ is increasing. If the actual prices are strictly worse than the corresponding benchmark prices, then the large investor is even exposed to some transaction losses with regard to the benchmark price, as we will describe in detail in Section 1.2.

Under the regime of the benchmark price system, all round-trips lead neither to costs nor to profits, thus especially we conclude from (1.1) for $k = 2$ that the price to buy an additional amount of $\xi_2 - \xi_1$ shares to ξ_1 shares originally held by the large investor equals the price the large investor would get for selling $\xi_2 - \xi_1$ out of a total number of ξ_2 shares. Similar considerations for $k = 3$ show that under the benchmark price regime there is no difference between buying an extra amount of $\xi_2 - \xi_1$ shares in addition to the ξ_1 shares originally held by the large investor, or selling all ξ_1 shares and then buying the total amount of ξ_2 immediately thereafter. For later reference, we will put this observation into a small lemma:

Lemma 1.7. *Let $f : I\!R \to I\!R$ be a Lebesgue-measurable and locally bounded function and $\mu \in \mathcal{M}(f)$ some associated price determining measure. Then the benchmark price function $S^* : I\!R^2 \to I\!R$ in the market described by f and μ satisfies*

$$S^*\big(\xi_1, \xi_2\big) = S^*\big(\xi_2, \xi_1\big) \tag{1.25}$$

and

$$\big(\xi_2 - \xi_1\big) S^*\big(\xi_1, \xi_2\big) = \xi_2 S^*\big(\xi_2, 0\big) - \xi_1 S^*\big(\xi_1, 0\big) \quad \text{for all } \xi_1, \xi_2 \in I\!R. \tag{1.26}$$

Proof. Since the benchmark price function satisfies the "fair price" condition (1.1), the statements (1.25) and (1.26) follow directly from (1.1) for $k = 2$ and $k = 3$. **q.e.d.**

Often we will find it useful to fix some Lebesgue-measurable and locally bounded equilibrium price function $f : I\!R \to I\!R$ and then consider the various large investor prices which result for different price-determining measures. In this case we parametrize all $S \in \mathfrak{S}(f)$ in terms of the associated price-determining measures $\mu \in \mathfrak{M}(f)$ by setting

$$S_\mu\big(\xi_1, \xi_2\big) = \int f\big((1-\theta)\xi_1 + \theta\xi_2\big)\mu(d\theta) \quad \text{for all } \xi_1, \xi_2 \in I\!R \text{ and } \mu \in \mathfrak{M}(f). \tag{1.27}$$

Thus, we have $\mathfrak{S}(f) = \big\{S_\mu : \mu \in \mathfrak{M}(f)\big\}$. Of course the benchmark price function associated to the whole class $\mathfrak{S}(f)$ is identified as $S^* = S_\lambda$, and since this particular price function satisfies the "fair price" condition, it will often turn out to be the price function which is most easy to deal with, and which gives the basic intuition to deal with the more general price functions from the class $\mathfrak{S}(f)$.

1.1.5 A Translation Invariance for Exponential Price Functions

We have already mentioned some advantages of exponential price functions in Section 1.1.2. In this section we will present another specific feature of exponential price functions: They are basically the only interesting functions for which a certain translation invariance holds. The results of this section will not be used in the rest of this thesis, thus the section may be omitted at first reading.

Suppose that we are given an equilibrium price function $f : I\!R \to I\!R$ of true exponential form, i.e. there exist some $a \in I\!R$ and $b, c \in I\!R\backslash\{0\}$ such that

$$f(\xi) = a + be^{c\xi} \quad \text{for all } \xi \in I\!R. \tag{1.28}$$

Parametrize now all associated large investor price functions $S \in \mathfrak{S}(f)$ by $\mu \in \mathfrak{M}(f)$ as in (1.27), fix for a moment the amount $\alpha \in I\!R$ of shares by which the large investor shifts his portfolio, and set

$$\Delta(\mu) := \Delta^\alpha(\mu) := \frac{1}{c}\log\int e^{c\alpha\theta}\mu(d\theta) \quad \text{for all } \mu \in \mathfrak{M}(f).$$

Then for each price-determining measure $\mu \in \mathfrak{M}(f)$ and an arbitrary initial stock holding of ξ shares we can calculate the large investor price $S_\mu(\xi, \xi + \alpha)$ which the large investor has to pay per share of the α shares he wants to buy as

$$S_\mu(\xi, \xi + \alpha) = a + b \int e^{c(\xi + \alpha\theta)} \mu(d\theta) = a + b e^{c(\xi + \Delta(\mu))},$$

and hence we have

$$S_\mu(\xi, \xi + \alpha) = f\big(\xi + \Delta(\mu)\big) \quad \text{for all } \xi \in I\!R \text{ and } \mu \in \mathfrak{M}(f), \tag{1.29}$$

where $\Delta(\mu)$ depends on α, but not on ξ. This means that for each fixed transaction size $\alpha \in I\!R$ the (real-world-)price $S_\mu(\xi, \xi + \alpha)$, given that the large investor had initially held ξ shares, can be obtained by evaluating the equilibrium price function f at the initial stock position shifted by an amount $\Delta(\mu)$, which does not depend on the initial stock position ξ. The next proposition shows that a condition of the form (1.29) basically limits f to be of exponential form.

Lemma 1.8. *Assume that the equilibrium price function $f \in C^2(I\!R)$ is strictly monotone and fix $\alpha \in I\!R \backslash \{0\}$. Then there exists some function $\Delta = \Delta^\alpha : \mathfrak{M}(f) \to I\!R$ with*

$$S_\mu(\xi, \xi + \alpha) = f\big(\xi + \Delta(\mu)\big) \quad \textit{for all } \xi \in I\!R \textit{ and } \mu \in \mathfrak{M}(f) \tag{1.30}$$

if and only if

$$f(\xi) = a + b e^{c\xi} \quad \textit{or} \quad f(\xi) = a + b\xi \quad \textit{for all } \xi \in I\!R \tag{1.31}$$

for some $a, b, c \in I\!R$.

Proof. We have already shown above that (1.30) holds if f is of exponential form. To show that it also holds for all affine functions $f : I\!R \to I\!R$ of the form $f(\xi) = a + b\xi$, let us fix such a function $f : I\!R \to I\!R$ and define $\Delta(\mu) = \alpha \int \theta \mu(d\theta)$ for all $\mu \in \mathfrak{M}(f)$. Then (1.30) holds since we have for all $\xi \in I\!R$ and $\mu \in \mathfrak{M}(f)$

$$S_\mu(\xi, \xi + \alpha) = a + b\xi + b\alpha \int \theta \mu(d\theta) = a + b\big(\xi + \Delta(\mu)\big).$$

It remains to show that a representation of the form (1.30) implies (1.31). For a proof let us note that $f \in C^2(I\!R)$ implies that both the Lebesgue measure λ and the Dirac measure δ_1 concentrated in 1 belong to the class $\mathfrak{M}(f)$. Moreover, $\mathfrak{M}(f)$ is convex, thus we conclude

$$\mu(\rho) := (1 - \rho)\lambda + \rho\delta_1 \in \mathfrak{M}(f) \quad \text{for all } \rho \in [0, 1],$$

and therefore (1.30) provides the existence of some function $\hat{\Delta} : [0, 1] \to I\!R$ which satisfies

$$S_{\mu(\rho)}(\xi, \xi + \alpha) = f\big(\xi + \hat{\Delta}(\rho)\big) \quad \text{for all } \xi \in I\!R \text{ and } \rho \in [0, 1]. \tag{1.32}$$

By the definition of S_μ and $\mu(\rho)$ we can rewrite this expression for all $\xi \in I\!R$ and $\rho \in [0, 1]$ as

$$f\big(\xi + \hat{\Delta}(\rho)\big) = (1 - \rho) \int_0^1 f\big(\xi + \theta\alpha\big)\lambda(d\theta) + \rho \int f\big(\xi + \theta\alpha\big)\delta_1(d\theta). \tag{1.33}$$

Since $f \in C^2(I\!R)$ is strictly monotone, the last equation can be solved for $\hat{\Delta}(\rho)$, and so we see that the function $\hat{\Delta} : [0, 1] \to I\!R$ is twice differentiable. Hence (1.33) can be differentiated with respect to ρ and we obtain for all $\xi \in I\!R$ and $\rho \in (0, 1)$:

$$\hat{\Delta}'(\rho) f'\big(\xi + \hat{\Delta}(\rho)\big) = \int f\big(\xi + \theta\alpha\big)\big(\delta_1 - \lambda\big)(d\theta). \tag{1.34}$$

A second differentiation of this equation gives

$$\hat{\Delta}''(\rho) f'\big(\xi + \hat{\Delta}(\rho)\big) + \big(\hat{\Delta}'(\rho)\big)^2 f''\big(\xi + \hat{\Delta}(\rho)\big) = 0. \tag{1.35}$$

Now fix an arbitrary $\rho \in (0,1)$. Since f is strictly monotone and $\alpha \neq 0$, we conclude that the right hand side of (1.34) does not vanish. Then (1.34) implies $\hat{\Delta}'(\rho) \neq 0$, and thus (1.35) becomes

$$\frac{f''\big(\xi + \hat{\Delta}(\rho)\big)}{f'\big(\xi + \hat{\Delta}(\rho)\big)} = -\frac{\hat{\Delta}''(\rho)}{\alpha\big(\hat{\Delta}'(\rho)\big)^2}.$$

Since this holds for all $\xi \in I\!R$ we conclude

$$\frac{f''(\xi)}{f'(\xi)} = \frac{f''(0)}{f'(0)} \quad \text{for all } \xi \in I\!R,$$

and this is only fulfilled if either $f'' \equiv 0$, i.e. $f(\xi) = a + b\xi$, or if $f(\xi) = a + be^{c\xi}$ for some $a \in I\!R$ and $b, c \in I\!R \backslash \{0\}$ and all $\xi \in I\!R$. **q.e.d.**

Remark. Gerber (1979) uses a similar proof to show that in actuarial mathematics a mean value principle is consistent if and only if it is an exponential principle or the net premium principle. Gerber points out that the net premium principle is of limited usefulness since it does not produce any security loading. Similarly, we do not work with a linear equilibrium price function $f : I\!R \to I\!R$, $\xi \mapsto a + b\xi$, since for $b \neq 0$ such an equilibrium price function leads to possible unbounded negative prices. The constant case $f(\xi) = a$ can also be covered as a special case of an exponential equilibrium price functions, once we allow that b and c in the representation (1.28) may also become zero. □

The translation invariance (1.30) which we have expressed in terms of the underlying equilibrium price function $f : I\!R \to I\!R$ can also be expressed in terms of any large investor price function $S \in \mathfrak{S}(f)$. For example, for any Lebesgue-measurable and locally bounded equilibrium price function $f : I\!R \to I\!R$ for which the translation invariance (1.30) holds we can define

$$D(\mu) := D^\alpha(\mu) := \Delta^\alpha(\mu) - \Delta^\alpha(\lambda) \quad \text{for all } \mu \in \mathfrak{M}(f),$$

since for all those functions we have $\lambda \in \mathfrak{M}(f)$. Then (1.30) and $S_\lambda = S^*$ imply

$$S_\mu(\xi, \xi + \alpha) = f\big(\xi + \Delta(\mu) - \Delta(\lambda) + \Delta(\lambda)\big) = S^*\big(\xi + D(\mu), \xi + D(\mu) + \alpha\big) \tag{1.36}$$

for all $\xi \in I\!R$ and $\mu \in \mathfrak{M}(f)$. Hence, for a fixed transaction size of α shares the actual price $S_\mu(\xi, \xi + \alpha)$ the large investor would be faced with if he had initially held ξ shares equals the benchmark price $S^*\big(\xi + D(\mu), \xi + D(\mu) + \alpha\big)$ for the same transaction volume, but now starting from an initial endowment of $\xi + D(\mu)$. Here the shift $D(\mu)$ does not depend on the initial position ξ.

Even if the equilibrium price function $f : I\!R \to I\!R$ does not have a structure of the form (1.31), there exists of course *sub*classes of $\mathfrak{M}(f)$ such that a representation like the one in (1.36) still holds. The following example will present such a class which works for almost every given equilibrium price functions. The particular feature of *exponential* and *linear* price functions is that *all* the associated price-determining measures μ satisfy (1.30) and (1.36).

Example 1.3. Let $f : I\!R \to I\!R$ be Lebesgue-measurable and locally bounded and $\mu \in \mathfrak{M}(f)$ some price-determining measure associated to f. Then for each $s \in I\!R$ for which the shifted measure μ_s defined by $\mu_s\big((-\infty, x]\big) = \mu\big((-\infty, x - s]\big)$ for all $x \in I\!R$ is also a price-determining measure associated to μ (i.e. $\mu_s \in \mathfrak{M}(f)$), it is easy to see from (1.7) that

$$S_{\mu_s}\big(\xi, \xi + \alpha\big) = S_\mu\big(\xi + \alpha s, \xi + \alpha s + \alpha\big) \quad \text{for all } \xi, \alpha \in I\!R, \tag{1.37}$$

i.e. a shift of the measure μ by s is equivalent to a shift in the large investor's initial stock holdings by αs. If the equilibrium price function $f : \mathbb{R} \to \mathbb{R}$ is strictly increasing, then a positive shift of the measure implies that the large investor can only obtain a lower price than under the original price-determining measure μ if he sells some or all of his shares, and at the same time he has to pay more for any share he buys.
If the original price-determining measure μ is the Lebesgue measure λ on $[0, 1]$, the representation (1.37) becomes

$$S_{\mu_s}\big(\xi, \xi + \alpha\big) = S^*\big(\xi + \alpha s, \xi + \alpha s + \alpha\big) \quad \text{for all } \xi, \alpha \in \mathbb{R},$$

and thus it resembles the representation (1.36). □

1.2 Transaction Losses

Having defined a benchmark price, we can now relate the actual price paid by the large investor for a certain transaction to the corresponding benchmark price. In doing so we will introduce the notation of *transaction losses*. These transaction losses provide a link between our large investor model and small investor models with transaction costs. We then outline some basic properties which a true transaction loss function should satisfy and give conditions such that these hold.

1.2.1 The Transaction Loss Function

For each large investor price function $S : \mathbb{R}^2 \to \mathbb{R}$ based on a weighted mean of equilibrium prices as in (1.6) we can quantify the transaction losses which the large investor has to bear because he cannot buy shares for the “fair” benchmark price. We do this by taking the difference between the total actual costs and the theoretical costs which the large investor would pay if he could use the benchmark price for his transaction. Similarly, we can calculate the transaction loss which the large investor suffers when he sells shares. Formally, we introduce the transaction loss in terms of the *transaction loss function*:

Definition 1.9. For any $S \in \mathfrak{S}$ the implied **transaction loss function** $c : \mathbb{R}^2 \to \mathbb{R}$ is defined by

$$c\big(\xi_1, \xi_2\big) := \big(\xi_2 - \xi_1\big)\Big(S\big(\xi_1, \xi_2\big) - S^*\big(\xi_1, \xi_2\big)\Big) \quad \text{for all } \xi_1, \xi_2 \in \mathbb{R}.$$

If we want to parametrize the whole class $\mathfrak{S}(f) = \big\{S_\mu \,:\, \mu \in \mathfrak{M}(f)\big\}$ of large investor price functions associated to some given Lebesgue-measurable and locally bounded equilibrium price function $f : \mathbb{R} \to \mathbb{R}$, we will parametrize the associated transaction loss functions accordingly, i.e. we then define the family $\big\{c_\mu \,:\, \mu \in \mathfrak{M}(f)\big\}$ of transaction loss functions $c_\mu : \mathbb{R}^2 \to \mathbb{R}$ by

$$c_\mu\big(\xi_1, \xi_2\big) := \big(\xi_2 - \xi_1\big)\Big(S_\mu\big(\xi_1, \xi_2\big) - S^*\big(\xi_1, \xi_2\big)\Big) \quad \text{for all } \xi_1, \xi_2 \in \mathbb{R} \text{ and } \mu \in \mathfrak{M}(f). \tag{2.1}$$

Remark. By the definitions of the set $\mathfrak{S}(f)$ and the benchmark price function $S^* = S_\lambda$ we immediately see that (2.1) reads as

$$c_\mu\big(\xi, \xi + \alpha\big) = \alpha \int f(\xi + \theta\alpha)\big(\mu - \lambda\big)(d\theta) \quad \text{for all } \xi, \alpha \in \mathbb{R} \text{ and } \mu \in \mathfrak{M}(f). \tag{2.2}$$

Thus, the transaction loss $c_\mu(\xi, \xi+\alpha)$ made by a large investor who initially held ξ shares and then buys another α shares is given in terms of an integral of the equilibrium price function with respect to the (signed) measure $\mu - \lambda$. □

Let us now fix some $S_\mu \in \mathfrak{S}$. If we write the price paid for $\xi_2 - \xi_1$ shares to shift the large investor's portfolio from ξ_1 to ξ_2 as

$$(\xi_2 - \xi_1) S_\mu(\xi_1, \xi_2) = (\xi_2 - \xi_1) S^*(\xi_1, \xi_2) + c_\mu(\xi_1, \xi_2) \quad \text{for all } \xi_1, \xi_2 \in I\!R, \tag{2.3}$$

the actual price $(\xi_2 - \xi_1)S_\mu(\xi_1, \xi_2)$ paid by the large investor can be recovered as the sum of the "fair" price $(\xi_2 - \xi_1)S^*(\xi_1, \xi_2)$ for that transaction plus the transaction loss loading $c_\mu(\xi_1, \xi_2)$. This resembles the structure of a market with transaction costs, and these similarities will be often explored in this thesis. However, we want to point out that there is neither a bid-ask spread, nor is there any transaction fee in our large investor market. As long as the large investor behaves like a small investor and trades only infinitesimally many shares, he can buy and sell them for the same (equilibrium) price. The large investor is exposed to transaction losses only because of the lack of liquidity in the market due to the disproportionately large size of his trades. Moreover, since the market power of the large investor changes the equilibrium price in the market, the transaction losses of the large investor are not immediately realized. Only if the large investor performs a round-trip he has to admit that the transaction losses occurred are really transaction *costs*, otherwise the transaction losses are hidden behind the change of the equilibrium price. That is why the large investor's transaction losses are really only *losses* and not *costs*.
In order to exclude immediate arbitrage opportunities, we need to require that for all round-trips of the large investor, starting from an arbitrary initial stock holding, the sum of transaction losses incurred is always nonnegative, i.e. we have to suppose that

$$\sum_{i=1}^{k} c_\mu\left(\xi + \sum_{j=1}^{i-1} \alpha_j, \xi + \sum_{j=1}^{i} \alpha_j\right) \geq 0 \quad \text{for all } \xi \in I\!R,\ \alpha \in \mathfrak{R}^k \text{ and } k \in I\!N.$$

Though this slightly complicated condition already rules out any possibility that the large investor becomes rich at a single point in time by performing a sequence of successive trades at this single time point, we will usually impose a slightly stronger condition on the transaction loss function, which is much easier to check. Namely, we will require that the whole transaction loss function $c_\mu : I\!R^2 \to I\!R$ is nonnegative. This condition also simplifies the exclusion of arbitrage opportunities in a dynamic framework which we will set up in Section 1.3. Moreover, nonnegative transaction losses make the connection to small investor models with transaction costs much more transparent.
In order to draw a parallel to transaction costs models we will also state conditions on the large investor market in terms of the transaction loss function $c_\mu : R^2 \to I\!R$ and not in terms of the equilibrium price function $f : I\!R \to I\!R$ and the associated price-determining measure $\mu \in \mathfrak{M}(f)$ which lie behind the large investor market.

Remark. For any fixed locally bounded and Lebesgue-measurable equilibrium price function $f : I\!R \to I\!R$ we can define the relation $\preccurlyeq_f$ on the set $\mathfrak{M}(f)$ of associated price-determining measures by setting

$$\mu_1 \preccurlyeq_f \mu_2 \quad \Longleftrightarrow \quad \alpha \int f(\xi + \alpha\theta)\mu_1(d\theta) \leq \alpha \int f(\xi + \alpha\theta)\mu_2(d\theta) \quad \text{for all } \xi, \alpha \in I\!R. \tag{2.4}$$

Because of (2.2) it follows that for any $\mu \in \mathfrak{M}(f)$ the transaction loss function $c_\mu : I\!R^2 \to I\!R$ is nonnegative if and only if $\mu \preccurlyeq_f \lambda$. The latter condition can sometimes be easier to check. If for example the equilibrium price function $f : I\!R \to I\!R$ is nondecreasing, then $\mu_1 \preccurlyeq_f \mu_2$ holds for all measures $\mu_1 \in \mathcal{M}(f)$ which are stochastically smaller than μ_2.
The relation $\preccurlyeq_f$ is indeed a pre-order, since it is reflexive (i.e. $\mu \preccurlyeq_f \mu$) and transitive (i.e. $\mu_1 \preccurlyeq_f \mu_2 \preccurlyeq_f \mu_3$ implies $\mu_1 \preccurlyeq_f \mu_3$). In general the pre-order $\preccurlyeq_f$ does not define a

partial order on $\mathfrak{M}(f)$, since it need not be antisymmetric (i.e. $\mu_1 \preccurlyeq_f \mu_2 \preccurlyeq_f \mu_1$ need not imply $\mu_1 = \mu_2$). However, if the equilibrium price function $f : I\!\!R \to I\!\!R$ is of strict exponential form, i.e. if there exist some $a \in I\!\!R$ and $b, c \in I\!\!R \setminus \{0\}$, such that $f(\xi) = a + be^{c\xi}$ for all $\xi \in I\!\!R$, then the pre-order $\preccurlyeq_f$ is even antisymmetric and hence a partial order on $\mathfrak{M}(f)$, since (2.4) and $\mu_1 \preccurlyeq_f \mu_2 \preccurlyeq_f \mu_1$ imply

$$\alpha b \int e^{\alpha c\theta} \mu_1(d\theta) = \alpha b \int e^{\alpha c\theta} \mu_2(d\theta) \quad \text{for all } \alpha \in I\!\!R,$$

from which we can conclude that the moment generating functions of μ_1 and μ_2, and therefore also the measures themselves coincide. □

1.2.2 Two Desirable Properties for Transaction Loss Functions

Common sense would easily state a whole bunch of properties which a transaction loss function should satisfy. We have already argued in the previous section why we prefer to work with nonnegative transaction loss functions. Moreover, in a perfect market it should not be beneficial to buy more shares than necessary and then immediately sell back the excess part of these shares. Additionally, the total transaction losses should not decrease with the size of the transaction. In this section we will give a precise definition of those two properties and then find conditions on the equilibrium price function and the price-determining measure in the large investor market under which these properties hold.

Definition 1.10. Let a large investor market be described by an equilibrium price function $f : I\!\!R \to I\!\!R$ which is Lebesgue-measurable and locally bounded and by an associated price-determining measure $\mu \in \mathfrak{M}(f)$.

(*i*) The market (or also the stock price S_μ) implies a **natural loss structure** if the associated transaction loss function $c_\mu : I\!\!R^2 \to I\!\!R$ satisfies

$$c_\mu\big(\xi, \xi + \alpha + \beta\big) + c_\mu\big(\xi + \alpha + \beta, \xi + \alpha\big) \geq c_\mu\big(\xi, \xi + \alpha\big) \tag{2.5}$$

for all $\xi, \alpha, \beta \in I\!\!R$ with $\alpha\beta \geq 0$.

(*ii*) The market (or also the stock price S_μ) implies **nondecreasing total transaction losses** if the associated transaction loss function $c_\mu : I\!\!R^2 \to I\!\!R$ satisfies

$$c_\mu\big(\xi \pm \alpha, \xi\big) \leq c_\mu\big(\xi \pm \beta, \xi\big) \tag{2.6}$$

for all $\xi, \alpha, \beta \in I\!\!R$ with $0 \leq \alpha \leq \beta$.

The condition (2.5) for the natural loss structure can be viewed as a weak no-arbitrage statement. If this condition even held for all $\alpha, \beta \in I\!\!R$, then the large investor would never benefit from breaking one transaction into two or more successive transactions. For a market with a natural loss structure we only require this condition for $\alpha\beta \geq 0$. In this case, it might be beneficial for the large investor to split one large transaction into several smaller ones, but the large investor cannot prevent or reduce losses by buying more stocks than he ultimately wants to buy and then selling the excess of his demand immediately. Similarly, it is not advantageous for him to sell more stocks than he really wants to sell and then to buy back those which he needs to reach his target stock holdings.

The second condition on the large investor market, namely the condition of nondecreasing *total* transaction losses, states that the total transaction losses can only increase, but never decrease with increasing transaction volume. Of course, this does not mean that the *relative* transaction losses per share cannot decrease. Since the definition of the transaction loss

function implies that there are no transaction losses if the large investor does not trade, i.e. $c_\mu(\xi,\xi) = 0$ for all $\xi \in I\!R$, a market with nondecreasing total transaction losses always implies that the transaction loss function $c_\mu : I\!R^2 \to I\!R$ is nonnegative.
In Section 1.4 we will employ the condition of a natural loss structure in order to prove existence, and the condition of nondecreasing total transaction losses in order to prove uniqueness, of discrete trading strategies which replicate a certain class of contingent claims in multi-period markets. For this reason, we will provide now some lemmata, which give sufficient conditions on both the equilibrium price function and the associated price-determining measure, such that the resulting market implies a natural loss structure and nondecreasing total transaction losses. These conditions are far from being necessary and should only be viewed as exemplary.

At first we focus on the natural loss structure. For practical purposes it helps to rewrite the loss-oriented Definition 1.10(i) in terms of a condition on the stock price:

Lemma 1.11. *Let $S_\mu \in \mathfrak{S}$ be a large investor price function. Then for any $\xi, \alpha, \beta \in I\!R$ the condition* (2.5) *holds if and only if*

$$(\alpha+\beta)S_\mu\big(\xi,\xi+\alpha+\beta\big) \geq \beta S_\mu\big(\xi+\alpha+\beta,\xi+\alpha\big) + \alpha S_\mu\big(\xi,\xi+\alpha\big). \qquad (2.7)$$

In particular, S_μ implies a natural loss structure if and only if (2.7) *holds for all $\xi, \alpha, \beta \in I\!R$ with $\alpha\beta \geq 0$.*

Proof. Let us recall that the benchmark price function $S^* : I\!R^2 \to I\!R$ was defined so that it satisfies the "fair price" condition (1.1). Applying this equation to the 2-step round-trip $(\alpha, -\alpha) \in \mathfrak{R}^2$, starting from any initial stock holding $\xi \in I\!R$, we obtain the equality $S^*(\xi, \xi+\alpha) = S^*(\xi+\alpha, \xi)$, and just another application of the same equation now to the 3-step round-trip $\big(\alpha+\beta, -\beta, -\alpha\big) \in \mathfrak{R}^3$ yields

$$(\alpha+\beta)S^*\big(\xi,\xi+\alpha+\beta\big) - \beta S^*\big(\xi+\alpha+\beta,\xi+\alpha\big) - \alpha S^*\big(\xi,\xi+\alpha\big) = 0.$$

If the definition (2.1) of the transaction loss function $c_\mu : I\!R^2 \to I\!R$ is plugged into expression (2.5) it immediately follows from the equation above that (2.5) and (2.7) are equivalent for all $\xi, \alpha, \beta \in I\!R$. **q.e.d.**

Example 1.4. Let us assume that the large investor market is specified by some strictly increasing equilibrium function $f : I\!R \to I\!R$ and the price-determining measure δ_1, i.e. the Dirac measure concentrated at 1. Then the market switches to the new equilibrium before the large investor has traded any stocks, so that for all $\xi_1, \xi_2 \in I\!R$ the large investor price is given by $S_\mu(\xi_1,\xi_2) = f(\xi_2)$. Since $f : I\!R \to I\!R$ is strictly increasing, (2.7) holds for all $\xi, \alpha, \beta \in I\!R$ with $\alpha\beta \geq 0$. However, for $-\alpha < \beta < 0$ it is easy to see that (2.7) fails to hold. Hence we conclude from Lemma 1.11 that the market implies a natural loss structure, but (2.5) does not hold for all $\xi, \alpha, \beta \in I\!R$, i.e. it can be advantageous for the large investor to break up one transaction into smaller subtransactions. □

The next lemma gives sufficient conditions on the equilibrium price function $f : I\!R \to I\!R$ and the associated price-determining measure μ in order to ensure that the resulting market represented by the large investor price function $S_\mu : I\!R^2 \to I\!R$ implies a natural loss structure.

Lemma 1.12. *Let us suppose that the price function $S_\mu : I\!R^2 \to I\!R$ of the large investor has the form*

$$S_\mu\big(\xi,\xi+\alpha\big) = \int f\left(\xi+\theta\alpha\right)\mu(d\theta) \quad \textit{for all } \xi, \alpha \in I\!R, \qquad (2.8)$$

where the equilibrium price function $f : \mathbb{R} \to \mathbb{R}$ *is nondecreasing and the associated price-determining measure* $\mu \in \mathfrak{M}(f)$ *is a probability measure* μ *on* $[0, 1]$ *which satisfies*

$$\mu\big([0, px]\big) \leq p\mu\big([0, x]\big) \qquad \text{for all } p, x \in [0, 1]. \tag{2.9}$$

Then S_μ *implies a natural loss structure.*
Similarly, a price function $S_\mu : \mathbb{R}^2 \to \mathbb{R}$ *of the form* (2.8) *implies a natural loss structure if the equilibrium price function* $f : \mathbb{R} \to \mathbb{R}$ *is nonincreasing and the price-determining measure* $\mu \in \mathfrak{M}(f)$ *is a probability measure on* $[0, 1]$ *which has no point mass in* $(0, 1]$ *and satisfies*

$$\mu\big([0, px]\big) \geq p\mu\big([0, x]\big) \quad \text{for all } p, x \in [0, 1].$$

Proof. Let us start with the proof of the first statement. Due to Lemma 1.11 it suffices to show (2.7) for all $\xi, \alpha, \beta \in \mathbb{R}$ with $\alpha\beta \geq 0$. Since (2.7) trivially holds if $\alpha = \beta = 0$, we can exclude this case. In all other cases we have to consider, the sum $c := \alpha + \beta$ will not vanish so that $\rho = \frac{\alpha}{\alpha+\beta}$ is well defined and $\rho \in [0, 1]$. Thus, we only have to show

$$cS_\mu(\xi, \xi + c) \geq c\rho S_\mu(\xi, \xi + c\rho) + c(1 - \rho)S_\mu(\xi + c, \xi + c\rho) \tag{2.10}$$

for all $\rho \in [0, 1]$ and $c \in \mathbb{R}$. We will recover (2.10) as an inequality between expectations of f with respect to certain random variables based on μ as the one used in Proposition 1.5. In order to prove the inequality between these expectations we utilize the properties imposed on μ. Namely, we will prove that (2.9) and the concentration of the probability measure μ on $[0, 1]$ imply

$$\mu\big((a, 1]\big) \geq \rho\mu\bigg(\Big(\frac{a}{\rho}, 1\Big]\bigg) + (1 - \rho)\mu\bigg(\Big[0, \frac{1 - a}{1 - \rho}\Big)\bigg) \tag{2.11}$$

for all $a \in \mathbb{R}$ and $\rho \in [0, 1]$, where we understand the two cases $\rho \in \{0, 1\}$ as limiting cases of $\rho \searrow 0$ and $\rho \nearrow 1$, respectively. For this purpose, we distinguish four cases: $a < 0$, $0 \leq a < \rho \leq 1$, then $0 \leq \rho \leq a \leq 1$, and as the forth case $a \geq 1$. In the first and forth case, i.e. if $a < 0$ or $a \geq 1$, we see that (2.11) holds with equality due to $\mu\big([0, 1]\big) = 1$. If $0 \leq a < \rho \leq 1$ we can apply (2.9) with $p = \rho$ and $x = \frac{a}{\rho}$ to obtain

$$\mu\big([0, a]\big) \leq \rho\mu\big([0, \rho^{-1}a]\big).$$

Using once again the concentration of μ on $[0, 1]$ we get from subtracting $\mu\big([0, 1]\big) = 1$ on both sides and rearranging terms

$$1 - \rho + \rho\mu\big((\rho^{-1}a, 1]\big) \leq \mu\big((a, 1]\big).$$

If we finally note that due to $\frac{1-a}{1-\rho} > 1$ we have $\mu\big(\big[0, \frac{1-a}{1-\rho}\big)\big) = 1$, we see that (2.11) holds for $0 \leq a < \rho \leq 1$ as well. It just remains to prove (2.11) for the case $0 \leq \rho \leq a \leq 1$. In this case we notice that (2.9) with $x = 1$ implies

$$\mu([0, p]) \leq p \quad \text{for all } p \in [0, 1], \tag{2.12}$$

hence, using (2.12) at first with $p = a$ and then with $p = \frac{1-a}{1-\rho}$, we conclude

$$\mu\big((a, 1]\big) = 1 - \mu([0, a]) \geq 1 - a = (1 - \rho)\frac{1 - a}{1 - \rho} \geq (1 - \rho)\mu\bigg(\Big[0, \frac{1 - a}{1 - \rho}\Big]\bigg).$$

Due to $\frac{a}{\rho} \geq 1$ we now have $\mu\left(\left(\frac{a}{\rho}, 1\right]\right) = 0$, hence the inequality (2.11) follows also for all $0 \leq \rho \leq a \leq 1$, and thus it indeed holds for all $\rho \in [0, 1]$ and $a \in \mathbb{R}$.

Let us now take a random variable Z with distribution μ. Then (2.11) is equivalent to

$$\mathbf{P}\big(Z > a\big) \geq \rho\mathbf{P}\big(\rho Z > a\big) + (1-\rho)\mathbf{P}\big(1-(1-\rho)Z > a\big) \quad \text{for all } a \in I\!R \text{ and } \rho \in [0,1]. \tag{2.13}$$

If we also introduce the family of random variables $\{U^\rho\}_{\rho\in[0,1]}$ in such a way that each U^ρ is independent of Z and satisfies

$$\mathbf{P}\big(U^\rho = 1\big) = \rho = 1 - \mathbf{P}\big(U^\rho = 0\big) \quad \text{for } \rho \in [0,1],$$

and then define the family $\{X^\rho\}_{\rho\in[0,1]}$ of random variables as in Proposition 1.4 and 1.5 by

$$X^\rho := U^\rho \rho Z + \big(1-U^\rho\big)\big(1-(1-\rho)Z\big) \quad \text{for all } \rho \in [0,1],$$

we can rewrite (2.13) in terms of X^ρ as

$$\mathbf{P}\big(Z > a\big) \geq \mathbf{P}\big(X^\rho > a\big) \quad \text{for all } a \in I\!R \text{ and } \rho \in [0,1].$$

But this is equivalent to saying that for all $\rho \in [0,1]$ the random variable Z is stochastically larger than X^ρ (i.e. $Z \geq_{st} X^\rho$). Therefore we have

$$\mathbf{E}\big[g(Z)\big] \geq \mathbf{E}\big[g\big(X^\rho\big)\big] \quad \text{for all } \rho \in [0,1]$$

and for all nondecreasing functions $g : I\!R \to I\!R$. Especially, the function $g^c : I\!R \to I\!R$, $z \mapsto cf(\xi + cz)$ inherits its monotonicity from $f : I\!R \to I\!R$, and thus we get

$$c\mathbf{E}\big[f(\xi + cZ)\big] \geq c\mathbf{E}\big[f\big(\xi + cX^\rho\big)\big] \quad \text{for all } \rho \in [0,1] \text{ and } c \in I\!R.$$

By the definitions of X^ρ and U^ρ, we can now use the law of total probability to rewrite the last inequality as

$$c\mathbf{E}\big[f(\xi + cZ)\big] \geq c\rho\mathbf{E}\big[f\big(\xi + c\rho Z\big)\big] + c(1-\rho)\mathbf{E}f\big[\big(\xi + c(1-(1-\rho)Z)\big)\big]$$

and recalling the definitions of the random variable Z and the large investor price function $S_\mu : I\!R^2 \to I\!R$ this is indeed for all $\rho \in [0,1]$ and $c \in I\!R$ equivalent to (2.10). Hence it follows that S_μ implies a natural loss structure.
For the second statement one can use similar arguments to conclude that $X^\rho \geq_{st} Z$ for all $\rho \in [0,1]$, from which we draw the same conclusion as in the first case. **q.e.d.**

Remark. Two remarks of (im)possible extensions of Lemma 1.12 should be made:

(i) The inequality (2.12) shows that for any given nondecreasing equilibrium price functions $f : I\!R \to I\!R$ all price-determining measures $\mu \in \mathfrak{M}(f)$ considered in Lemma 1.12 are stochastically larger than the Lebesgue measure λ on $[0,1]$. However, the condition $\mu \geq_{st} \lambda$ alone is too weak to imply (2.11) for all $a \in I\!R$ and all $\rho \in [0,1]$, and it was this condition which was essential to prove the natural loss structure without any additional restrictions on f. In order to see that (2.11) need not hold if $\mu \geq_{st} \lambda$, let us define the probability measure μ by $\mu\big(\{1/2\}\big) = \mu\big(\{1\}\big) = 1/2$. Then we have $\mu \geq_{st} \lambda$, but for $\frac{1}{2} \leq a < \rho \leq 1$ we have $\mu\big(\big(\frac{a}{\rho},1\big]\big) = \mu\big(\{1\}\big) = \frac{1}{2}$ and $\mu\big(\big[0,\frac{1-a}{1-\rho}\big)\big) = \mu\big(\{1/2,1\}\big) = 1$, hence we get

$$\rho\mu\bigg(\bigg(\frac{a}{\rho},1\bigg]\bigg) + (1-\rho)\mu\bigg(\bigg[0,\frac{1-a}{1-\rho}\bigg)\bigg) = \frac{1}{2}\rho + (1-\rho) = \frac{1}{2} + \frac{1}{2}(1-\rho) > \frac{1}{2} = \mu\big((a,1]\big).$$

To this price-determining measure μ one can now easily find examples of nondecreasing equilibrium price functions $f : I\!R \to I\!R$ (e.g. the identity) for which (2.7) does not hold for all $\xi, \alpha, \beta \in I\!R$ with $\alpha\beta \geq 0$, i.e. where the market described by f and μ does not imply a natural loss structure.

(ii) If Z is a random variable with values in $(1,\infty)$, we have of course $\mathbf{P}(Z > a) = 1$ for all $a \le 1$. On the other hand, for all $a > 1$ and $\rho \in [0,1]$ such a random variable satisfies $\mathbf{P}\big(1-(1-\rho)Z > a\big) = \mathbf{P}\big(-(1-\rho)Z > a-1\big) = 0$ and $\mathbf{P}(\rho Z > a) \le \mathbf{P}(Z > a)$. This shows that (2.13) also holds for all random variables with values in $(1,\infty)$, and we can use the same arguments as in the proof of Lemma 1.12 to conclude that the market described by f and μ still implies a natural loss structure if the equilibrium price function $f : I\!R \to I\!R$ is nondecreasing and if the price-determining measure $\mu \in \mathfrak{M}(f)$ is a mixture of a probability measure on $[0,1]$ satisfying (2.9) and a probability measure on $(1,\infty)$. $\square$

We now give an example of a probability measure μ with values in $[0,1]$ which satisfies (2.9).

Example 1.5. Let λ denote as always in this thesis the Lebesgue measure on $[0,1]$, and δ_1 the Dirac measure concentrated in 1. Then for any fixed $\rho \in [0,1]$ the measure $\mu = \rho\lambda + (1-\rho)\delta_1$ satisfies (2.9), since

$$\mu\big([0,px]\big) = \rho px + (1-\rho)\delta_1\big(\{px\}\big) \le p\big(\rho x + (1-\rho)\delta_1(\{x\})\big) = p\big([0,x]\big)$$

for all $p, x \in [0,1]$. Especially, a large investor market described by any nondecreasing equilibrium price function $f : I\!R \to I\!R$ and by this price-determining measure μ implies a natural loss structure. $\square$

We may consider the Dirac measure δ_1 as the limit of uniform distributions on $[1-h,1]$ as $h \to 0$. In the next lemma we will give conditions on the equilibrium price function f and the associated price-determining measure μ which guarantee that the market does not only imply a natural loss structure, but also nondecreasing total transaction losses. For the price-determining measure μ we allow for a generalized version of the measures considered in Example 1.5, for the equilibrium price function $f : I\!R \to I\!R$ we require some smoothness in addition to the monotonicity of Lemma 1.12.

Lemma 1.13. *Let $f : I\!R \to I\!R$ be some nondecreasing and continuously differentiable equilibrium price function, and consider for $\rho, h \in [0,1]$ the associated price-determining measure $\mu = \mu_{\rho,h} \in \mathfrak{M}(f)$ on $\big([0,1], \mathcal{B}([0,1])\big)$ defined by*

$$\mu_{\rho,h}(A) = \rho\lambda(A) + (1-\rho)\frac{1}{h}\lambda\big(A \cap [1-h,1]\big) \quad \textit{for all } A \in \mathcal{B}([0,1]), \tag{2.14}$$

where we shall interpret the last term on the right-hand side as the Dirac measure in 1 if $h = 0$. Then the price function $S_\mu : I\!R^2 \to I\!R$ given by (2.8) implies both a natural loss structure and nondecreasing total transaction losses.

Proof. Let us first show that S_μ implies a natural loss structure. Due to Lemma 1.12 it suffices for this to show (2.9). In the case $h = 0$ this was already shown in Example 1.5, so we can assume without loss of generality that $0 < h \le 1$. But also in these cases (2.9) holds, since the inequality $(px-c)^+ \le p(x-c)^+$ for all $c, p, x \in [0,1]$ implies

$$\begin{aligned}\mu_{\rho,h}\big([0,px]\big) &= \rho px + (1-\rho)\frac{1}{h}\lambda\big([1-h,px]\big) = \rho px + (1-\rho)\frac{1}{h}\big(px-1+h\big)^+ \\ &\le \rho px + (1-\rho)p\frac{1}{h}\big(x-1+h\big)^+ = p\mu_{\rho,h}\big([0,x]\big)\end{aligned}$$

for all $\rho, p, x \in [0,1]$. Thus, for all $\rho, h \in [0,1]$ the market described by f and $\mu = \mu_{\rho,h}$, or also the associate price function S_μ, indeed implies a natural loss structure.

Now let us proceed to show that it also implies nondecreasing transaction losses, i.e. we have to show that (2.6) holds for all $\xi, \alpha, \beta \in I\!R$ with $\alpha \le \beta$. We first employ representation (2.2) of c_μ and the definition of $\mu = \mu_{\rho,h}$ in (2.14) to write $c_\mu(\xi + \alpha, \xi)$ for all $\alpha, \xi \in I\!R$ as

$$\begin{aligned} c_\mu\big(\xi+\alpha,\xi\big) &= -\alpha \int f\big(\xi + (1-\theta)\alpha\big)\big(\mu - \lambda\big)(d\theta) \\ &= \alpha(1-\rho)\Bigg(\int_0^1 f\big(\xi+(1-\theta)\alpha\big)d\theta - \frac{1}{h}\int_{1-h}^1 f\big(\xi+(1-\theta)\alpha\big)d\theta\Bigg). \end{aligned}$$

Without loss of generality let us assume $\rho = 0$. Then $\alpha \mapsto c_\mu(\xi+\alpha,\xi)$ is differentiable, and its derivative is for all $\xi, \alpha \in I\!R$ given by

$$\begin{aligned} \frac{d}{d\alpha}c_\mu\big(\xi+\alpha,\xi\big) = &\int_0^1 f\big(\xi+(1-\theta)\alpha\big)d\theta + \alpha\int_0^1 (1-\theta)f'\big(\xi+(1-\theta)\alpha\big)d\theta \\ &- \frac{1}{h}\int_{1-h}^1 f\big(\xi+(1-\theta)\alpha\big)d\theta - \alpha\frac{1}{h}\int_{1-h}^1 (1-\theta)f'\big(\xi+(1-\theta)\alpha\big)d\theta. \end{aligned}$$

Using partial integration, this derivative simplifies for all $\xi, \alpha \in I\!R$ to

$$\begin{aligned} \frac{d}{d\alpha}c_\mu\big(\xi+\alpha,\xi\big) &= \Big[-(1-\theta)f\big(\xi+(1-\theta)\alpha\big)\Big]_0^1 - \frac{1}{h}\Big[-(1-\theta)f\big(\xi+(1-\theta)\alpha\big)\Big]_{1-h}^1 \\ &= f(\xi+\alpha) - f(\xi+h\alpha) \end{aligned}$$

Therefore, it follows from $f : I\!R \to I\!R$ being nondecreasing and $h \in [0,1]$ that

$$\frac{d}{d\alpha}c_\mu\big(\xi+\alpha,\xi\big)\begin{cases} \le 0 & \text{for } \alpha \le 0 \\ \ge 0 & \text{for } \alpha \ge 0, \end{cases} \tag{2.15}$$

and consequently S_μ indeed implies nondecreasing total transaction losses. **q.e.d.**

Remark. As in Lemma 1.12 the conclusion of Lemma 1.13 holds as well if the equilibrium price function $f : I\!R \to I\!R$ in the market is nonincreasing instead of nondecreasing and if on the other hand the price-determining measure $\mu = \mu_{\rho,h}$ on $\big([0,1], \mathcal{B}([0,1])\big)$ is for some $\rho, h \in [0,1]$ given by

$$\mu_{\rho,h} = \rho\lambda(A) + (1-\rho)\frac{1}{h}\lambda\big(A \cap [0,h]\big) \quad \text{for all } A \in \mathcal{B}([0,1])$$

instead of (2.14). Moreover, rewriting (2.2) as

$$c_\mu(\xi+\alpha,\xi) = \alpha\left(\int_0^1 f\big(\xi+(1-\theta)\alpha\big)d\theta - \int f(\xi+(1-\theta)\alpha\big)\mu(d\theta)\right) \quad \text{for all } \alpha, \xi \in I\!R.$$

it can easily be seen that for all nondecreasing equilibrium price functions $f : I\!R \to I\!R$ and all price-determining measures μ which are concentrated on $(1, \infty)$ the function $\alpha \mapsto c_\mu(\xi+\alpha,\xi)$ is nondecreasing as long as $\alpha > 0$ and it is nonincreasing for $\alpha < 0$. Together with the remark after Lemma 1.12 this shows that we could draw the same conclusions as in Lemma 1.13 if we relaxed the form of price-determining measures considered to mixtures of measures of the form (2.14) and probability measures on $\big((1,\infty), \mathcal{B}((1,\infty))\big)$. $\square$

1.2.3 The Local Transaction Loss Rate

The transaction loss function $c_\mu : I\!R^2 \to I\!R$ of (2.1) models the transaction losses in an additive way, and therefore can also be used if the equilibrium stock price becomes 0 or even negative. However, in ordinary stock markets – even in the presence of a large investor – the stock price will always stay positive, since the liability of each shareholder is limited to the amount which he had invested in the stock. In such markets we can introduce a (local) transaction loss *rate* function which describes the transaction losses per traded share as a fraction of the benchmark price. A Taylor expansion of the transaction loss rate function will then give necessary conditions on the market to guarantee nonnegative transaction losses. The multiplicative representation of the transaction losses by means of the transaction loss rate will be used in the subsequent chapters to exploit similarities with small investor markets models with proportional transaction costs as described for example by Boyle and Vorst (1992) or Musiela and Rutkowski (1998).

Definition 1.14. Let the large investor market be described by some positive, locally bounded, and Lebesgue-measurable function $f : I\!R \to (0, \infty)$ and some associated price determining measure $\mu \in \mathfrak{M}(f)$. Then the **local (implied) transaction loss rate function** $k_\mu : I\!R^2 \to I\!R$ is given by

$$k_\mu\big(\xi_1, \xi_2\big) = \operatorname{sgn}\big(\xi_2 - \xi_1\big) \left(\frac{\int f\big((1-\theta)\xi_1 + \theta\xi_2\big)\, \mu(d\theta)}{\int f\big((1-\theta)\xi_1 + \theta\xi_2\big)\, \lambda(d\theta)} - 1 \right) \quad \text{for all } \xi_1, \xi_2 \in I\!R, \tag{2.16}$$

where $\operatorname{sgn}\big(\xi_1, \xi_2\big) = 0$ if $\xi_1 = \xi_2$.

Remark. For each fixed $\xi_1, \xi_2 \in I\!R$ we can use (2.16) and (2.2) to write the transaction loss incurred by the large investor when shifting his stock holdings from ξ_1 shares to ξ_2 shares as

$$c_\mu\big(\xi_1, \xi_2\big) = \big|\xi_2 - \xi_1\big| S^*\big(\xi_1, \xi_2\big) k_\mu\big(\xi_1, \xi_2\big). \tag{2.17}$$

This shows that $k_\mu(\xi_1, \xi_2)$ indeed describes the transaction loss rate as a fraction of the total transaction price $|\xi_2 - \xi_1| S^*(\xi_1, \xi_2)$ which would be necessary if the transaction were made at the benchmark price of $S^*(\xi_1, \xi_2)$ per share.
Whenever k_μ is well-defined, the previous equality also shows that the transaction loss rate $k_\mu(\xi_1, \xi_2)$ is nonnegative if and only if the transaction loss $c_\mu(\xi_1, \xi_2)$ is nonnegative, since k_μ is only well-defined if the equilibrium price function f and hence also the benchmark price function S^* is positive. □

Example 1.6. Let us fix some $b > 0$ and $c \in I\!R$ and consider the equilibrium price function $f : I\!R \to (0, \infty)$ given by $f(\xi) = be^{c\xi}$ for all $\xi \in I\!R$. Then for any price-determining measure $\mu \in \mathfrak{M}(f)$ we have

$$k_\mu\big(\xi, \xi + \alpha\big) = \operatorname{sgn}(\alpha) \left(\frac{\int be^{c(\xi+\alpha\theta)} \mu(d\theta)}{\int be^{c(\xi+\alpha\theta)} \lambda(d\theta)} - 1 \right) = \operatorname{sgn}(\alpha) \left(\frac{\int e^{c\alpha\theta} \mu(d\theta)}{\int e^{c\alpha\theta} \lambda(d\theta)} - 1 \right) \text{ for all } \alpha, \xi \in I\!R.$$

Hence for this exponential equilibrium price function the transaction loss rate does not depend on the initial stock holding ξ of the large investor. But in contrast to small investor models with proportional transaction costs the transaction loss rate $k_\mu\big(\xi, \xi + \alpha\big)$ still depends on the size α of the transaction, so it is really only a *local* rate. □

When we consider in Chapters 3 and 4 continuous-time limits of discrete multi-period large investor markets it will become necessary to study the behavior of $k_\mu(\xi + \alpha, \xi)$ for small values of α. The following lemma gives the key insights by providing a Taylor expansion of $k_\mu(\xi+\alpha,\xi)$. This expansion can also be utilized to state necessary conditions on the equilibrium price function f and the associated price-determining measure μ such that the transaction loss rate function $k_\mu : \mathbb{R}^2 \to \mathbb{R}$, or equivalently the transaction loss function $c_\mu : \mathbb{R}^2 \to \mathbb{R}$, in the large investor market described by f and μ does not become negative.

Proposition 1.15. *Let a large investor market be described by some equilibrium price function f and an associated price-determining measure μ. Suppose that the equilibrium price function $f : \mathbb{R} \to (0,\infty)$ is positive and twice continuously differentiable, and that there exist some constants $L_1, L_2 \in \mathbb{R}$ such that*

$$\left|\frac{f'(\xi)}{f(\xi)}\right| \le L_1 \quad \textit{and} \quad \left|\frac{f''(\xi)}{f(\xi)}\right| \le L_2 \quad \textit{for all } \xi \in \mathbb{R}.$$

Moreover, assume that the associated price-determining measure μ satisfies $\int e^{\eta|\theta|}\mu(d\theta) < \infty$ for some $\eta > 0$. Then for all $\xi \in \mathbb{R}$ the transaction loss rate function satisfies

$$k_\mu(\xi+\alpha,\xi) = |\alpha|\frac{f'(\xi)}{f(\xi)}\int \theta\big(\mu-\lambda\big)(d\theta) + O\big(\alpha^2\big) \quad \textit{as } \alpha \to 0. \tag{2.18}$$

Especially, $k_\mu : \mathbb{R}^2 \to \mathbb{R}$ can only be nonnegative if either $\int \theta\mu(d\theta) = \frac{1}{2}$

or $f : \mathbb{R} \to \mathbb{R}$ is nondecreasing and $\int \theta\mu(d\theta) > \frac{1}{2}$

or $f : \mathbb{R} \to \mathbb{R}$ is nonincreasing and $\int \theta\mu(d\theta) < \frac{1}{2}$.

Proof. By the definition of $k_\mu : \mathbb{R}^2 \to \mathbb{R}$ in (2.16) we have

$$k_\mu(\xi+\alpha,\xi) = -\mathrm{sgn}(\alpha)\left(\frac{\frac{1}{f(\xi)}\int f\big(\xi+(1-\theta)\alpha\big)\mu(d\theta)}{\frac{1}{f(\xi)}\int f\big(\xi+(1-\theta)\alpha\big)\lambda(d\theta)} - 1\right) \quad \text{for all } \xi,\alpha \in \mathbb{R}. \tag{2.19}$$

In order to reach the expansion (2.18), we are going to utilize several Taylor expansions. At first, note that for all $\xi,\alpha \in \mathbb{R}$ there exist functions $\gamma = \gamma_{\xi,\alpha} : \mathbb{R} \to [0,1]$ such that

$$f\big(\xi+(1-\theta)\alpha\big) = f(\xi) + (1-\theta)\alpha f'(\xi) + \frac{1}{2}(1-\theta)^2\alpha^2 f''\big(\xi+(1-\theta)\gamma(\theta)\alpha\big). \tag{2.20}$$

Let ν be an arbitrary probability measure on $\big(\mathbb{R},\mathcal{B}(\mathbb{R})\big)$ which satisfies $\int e^{|\theta|\eta}\nu(d\theta) < \infty$ for some $\eta > 0$. Integrating (2.20), dividing the result by $f(\xi) > 0$, and utilizing $\int \nu(d\theta) = 1$ leads to

$$\frac{1}{f(\xi)}\int f\big(\xi+(1-\theta)\alpha\big)\nu(d\theta) = 1 - \alpha h_\nu(\xi,\alpha) \quad \text{for all } \xi,\alpha \in \mathbb{R}, \tag{2.21}$$

where the function $h_\nu : \mathbb{R}^2 \to \mathbb{R}$ is for all $\xi,\alpha \in \mathbb{R}$ defined by

$$h_\nu(\xi,\alpha) = \frac{f'(\xi)}{f(\xi)}\left(\int \theta\nu(d\theta) - 1\right) - \frac{1}{2}\alpha\int (1-\theta)^2\frac{f''\big(\xi+(1-\theta)\gamma(\theta)\alpha\big)}{f(\xi)}\nu(d\theta).$$

We now want to show that, uniformly for all $\xi \in \mathbb{R}$,

$$h_\nu(\xi,\alpha) = \frac{f'(\xi)}{f(\xi)}\left(\int \theta\nu(d\theta) - 1\right) + O(\alpha) \quad \text{as } \alpha \to 0. \tag{2.22}$$

In order to do so we notice that for all $\xi,\beta \in \mathbb{R}$ there exists some $\gamma_1 \in [0,1]$ such that

$$\frac{f''(\xi+\beta)}{f(\xi)} = \frac{f''(\xi+\beta)}{f(\xi+\beta)}\exp\big(\log f(\xi+\beta) - \log f(\xi)\big)$$

$$= \frac{f''(\xi+\beta)}{f(\xi+\beta)} \exp\left(\beta \frac{d}{d\xi} \log f(\xi+\gamma_1\beta)\right) = \frac{f''(\xi+\beta)}{f(\xi+\beta)} \exp\left(\beta \frac{f'(\xi+\gamma_1\beta)}{f(\xi+\gamma_1\beta)}\right).$$

Thus, we can use the bounds on the ratios of derivatives of $f : \mathbb{R} \to (0,\infty)$ to deduce

$$\left|\frac{f''(\xi+\beta)}{f(\xi)}\right| \leq \left|\frac{f''(\xi+\beta)}{f(\xi+\beta)}\right| \exp\left(|\beta| \left|\frac{f'(\xi+\gamma_1\beta)}{f(\xi+\gamma_1\beta)}\right|\right) \leq L_2 e^{|\beta| L_1} \quad \text{for all } \xi, \beta \in \mathbb{R},$$

and since the latter bound implies for all $\xi, \alpha \in \mathbb{R}$ with $|\alpha| L_1 < \eta$ that

$$\left|\int (1-\theta)^2 \frac{f''\big(\xi + (1-\theta)\gamma_{\xi,\alpha}(\theta)\alpha\big)}{f(\xi)}\right| \leq L_2 \int (1-\theta)^2 e^{|(1-\theta)\alpha| L_1} \nu(d\theta) < \infty, \tag{2.23}$$

the desired expansion (2.22) follows uniformly for all $\xi \in \mathbb{R}$ from the definition of h_ν. Moreover, we can also write $h_\nu(\xi,\alpha) = O(1)$ as $\alpha \to 0$, uniformly for all $\xi \in \mathbb{R}$, since

$$\left|\frac{f'(\xi)}{f(\xi)}\left(\int \theta\nu(d\theta) - 1\right)\right| \leq L_1 \left(\left|\int \theta\nu(d\theta)\right| + 1\right) < \infty. \tag{2.24}$$

Especially, (2.23) and (2.24) hold uniformly for all $\xi \in \mathbb{R}$ and both measures $\nu = \lambda$ and $\nu = \mu$. As the last step preparatory to prove (2.18), we apply another Taylor expansion to note that for all $c_\mu, c_\lambda \in \mathbb{R}$ and $\alpha < \frac{1}{|c_\lambda|}$, there exists some $\gamma_2 \in [0,1]$ so that

$$\frac{1-\alpha c_\mu}{1-\alpha c_\lambda} = 1 + \big(c_\lambda - c_\mu\big)\alpha + c_\lambda \frac{c_\lambda - c_\mu}{(1-\gamma_2\alpha c_\lambda)^3}\alpha^2 = 1 + \big(c_\lambda - c_\mu\big)\alpha + O\big(\alpha^2\big) \quad \text{as } \alpha \to 0. \tag{2.25}$$

If we now apply (2.25) to $c_\nu = h_\nu(\xi,\alpha)$ for $\nu \in \{\lambda,\mu\}$ then (2.21) and the bounds (2.23) and (2.24) imply

$$\frac{\frac{1}{f(\xi)} \int f\big((1-\theta)(\xi+\alpha)+\theta\xi\big)\mu(d\theta)}{\frac{1}{f(\xi)} \int f\big((1-\theta)(\xi+\alpha)+\theta\xi\big)\lambda(d\theta)} = 1 - \alpha\frac{f'(\xi)}{f(\xi)} \int \theta\big(\mu-\lambda\big)(d\theta) + O\big(\alpha^2\big) \quad \text{as } \alpha \to 0,$$

uniformly for all $\xi \in \mathbb{R}$. Hence (2.18) follows from (2.19).
The second statement of the proposition is a direct consequence of (2.18): Let us assume that the transaction loss rate function $k_\mu : \mathbb{R}^2 \to \mathbb{R}$ is nonnegative, and suppose that for example $\int \theta\mu(d\theta) > \frac{1}{2}$, but that there exists some $\xi \in \mathbb{R}$ for which $f'(\xi) < 0$. Then (2.18) and $\int \theta\lambda(d\theta) = \frac{1}{2}$ imply

$$0 \leq \lim_{\alpha \downarrow 0} \frac{k_\mu(\xi+\alpha,\xi)}{\alpha} = \frac{f'(\xi)}{f(\xi)} \int \theta\big(\mu-\lambda\big)(d\theta) < 0,$$

which gives a contradiction. **q.e.d.**

Proposition 1.15 gives rise to the following definition:

Definition 1.16. For any probability measure μ on $\big(\mathbb{R}, \mathcal{B}(\mathbb{R})\big)$ for which the first moment is well-defined we introduce the **non-linearity parameter**

$$d(\mu) := \int \theta\big(\mu-\lambda\big)(d\theta) = \int \theta\mu(d\theta) - \frac{1}{2}. \tag{2.26}$$

Remark. Let us assume that there do no exist any negative transaction losses. Because of Proposition 1.15 we can then restrict our attention without losing much generality to a large investor market with nondecreasing equilibrium price function $f : \mathbb{R} \to \mathbb{R}$ and a price-determining measure μ satisfying $d(\mu) \geq 0$. In such a situation the non-linearity parameter $d(\mu)$ describes the distance between two pricing mechanisms, namely between the actually experienced one and the benchmark price. The larger $d(\mu)$, the larger the local transaction loss rate $k_\mu : \mathbb{R}^2 \to \mathbb{R}$, and also the larger the transaction loss $c_\mu : \mathbb{R}^2 \to \mathbb{R}$. If $d(\mu) = 0$ the local transaction loss rate $k_\mu(\xi+\alpha,\xi)$ is only a term of order $O\big(\alpha^2\big)$ as $\alpha \to 0$. □

In a continuous-time limit of multi-period discrete large investor market models as we will describe in Chapters 3 and 4, the parameter $d(\mu)$ gives the key information about the market's price building mechanism. Especially, $d(\mu)$ describes the degree of non-linearity of the partial differential equation satisfied by (a transform of) all self-financing strategy functions in those markets. But before we come to this in any more detail, we first have to build up a discrete multi-period large investor model.

1.3 The Binomial Multi-Period Large Investor Market Model

In this section we will lift the large investor market model from a model at one single point in time to a stochastic multi-period model. The market is once again described by an equilibrium price function and an associated price-determining measure, but now the equilibrium price function depends not only on the large investor's stock holdings, but also on time and on some stochastic *fundamentals*. In order to capture these additional dependences we have to extend the various price functions and the transaction loss function of Sections 1.1 and 1.2, respectively. We then restrict our attention to discrete markets where the large investor can only trade at a finite number of equidistant points in time. The process of the fundamentals at the trading dates is modelled by a binomial random walk. Since the large investor affects the stock price by his trading, the stock price in the market is also influenced by the particular *portfolio strategy* of the large investor. As in small investor markets we stipulate that the large investor uses *self-financing* trading strategies, so that no funds are given to or taken away from the market between the first and last trade. An example will show in which sense the stock prices appearing in such a market are still recombining. Having set up the large investor market model, we come to the natural question what the value of a portfolio held by the large investor is. Depending on the objective, there are different valuation principles for such a portfolio. Section 1.3.5 will introduce the two valuation concepts used in the sequel, namely the concept of the *paper value* and the concept of the *real value* of a portfolio.

1.3.1 The General Dynamic Large Investor Price System

In order to start with the description of the multi-period model, we fix some time point $T > 0$. At first we will then introduce the large investor price system on the whole time interval $[0, T]$. The financial market described by this price system contains many small investors and one large investor, and we assume that there are two primary traded securities on this market: a risky asset, referred to as a stock, and a risk-free asset, referred to as a bank account. While the market power of each single small investor (given by the size of his transactions) is so small that his transactions do not significantly influence the market prices, the *large* investor's trades can move the stock prices. The bond market is supposed to be much more liquid than the stock market such that at any time $t \in [0, T]$ even the large investor can borrow or lend any cash amount for the same interest rate. Without loss of generality we then may even suppose that the risk-free interest rate is 0, i.e. the value of a unit of cash is constantly 1 on the whole time interval $[0, T]$; otherwise we could take the bank account as a numeraire and consider discounted stock prices.

The basic parameter driving the stock price is given by some fundamental value process. It accounts for all relevant (or "fundamental") stochastic influences on the stock price which are not caused by the large investor's trades, and it may represent the aggregated income of the small investors (as in Frey and Stremme (1997)), or stock-relevant news, or any other nondeterministic influence on the stock price. In addition, we assume that at any fixed time point and for any fixed value of the fundamentals at this time, the stock price is influenced by the large investor's trades as in Section 1.1. Last but not least, we also allow for other

factors, which have a deterministic impact on the stock prices as time proceeds.
Thus, instead of the Walrasian equilibrium price function $f : \mathbb{R} \to \mathbb{R}$ which was the starting point of our simplified model in Section 1.1 and depended only on the large investor's stock holdings, we now start with an equilibrium price function $\psi : [0,T] \times \mathbb{R}^2 \to \mathbb{R}$ which depends on time, the fundamentals, and the stock holdings of the large investor, i.e. we assume that for any time $t \in [0,T]$, any fixed value $u \in \mathbb{R}$ for the fundamentals at time t, and any number ξ of stocks held by the large investor at time t there exists a Walrasian equilibrium price for the stock *if the large investor does not trade* at that time, and this equilibrium price is given by $\psi(t,u,\xi)$. We will always assume that the fundamentals are modelled in such a way that – ceteris paribus – an increase in the fundamentals leads to a higher stock price.

Definition 1.17. A Lebesgue-measurable function $\psi : [0,T] \times \mathbb{R}^2 \to \mathbb{R}$, $(t,u,\xi) \mapsto \psi(t,u,\xi)$ is called an **equilibrium price function** if it is locally bounded and strictly increasing in u. In this case we define the **associated small investor price function** $\overline{\psi} : [0,T] \times \mathbb{R} \to \mathbb{R}$ by

$$\overline{\psi}(t,u) = \psi(t,u,0) \quad \text{for all } (t,u) \in [0,T] \times \mathbb{R}.$$

The equilibrium price function $\psi : [0,T] \times \mathbb{R}^2 \to \mathbb{R}$ is called **multiplicative** if there exists some function $f : \mathbb{R} \to \mathbb{R}$ such that ψ can be represented as

$$\overline{\psi}(t,u,\xi) = \overline{\psi}(t,u) f(\xi) \quad \text{for all } (t,u,\xi) \in [0,T] \times \mathbb{R}^2. \tag{3.1}$$

Remark. The associated small investor price function relates the large investor market to an associated small investor market. The name goes back to Baum (2001), who was the first who has seen the importance of the associated small investor market ("assozierter Finanzmarkt") for the investigation of the large investor market. We shall also exploit the relationship between both markets. □

As we proceed we will require different degrees of smoothness for the function ψ, and for the largest part of this thesis we will also suppose that ψ is strictly positive. For the beginning, however, we will stay with the very general equilibrium price function of Definition 1.17 and transfer the price mechanism introduced in the single-period model of Section 1.1.2 in a straightforward way to the dynamic model which comes with such a general equilibrium price function, by condensing the analogues of Definitions 1.3 and 1.6 to the dynamic model in the following definition:

Definition 1.18. Let $\psi : [0,T] \times \mathbb{R}^2 \to \mathbb{R}$ be some equilibrium price function. A probability measure $\mu \in \bigcap_{(t,u) \in [0,T] \times \mathbb{R}} \mathfrak{M}\big(\psi(t,u,\,\cdot\,)\big)$ is called **price-determining measure** for ψ. The **large investor price function** $S_\mu : [0,T] \times \mathbb{R}^3 \to \mathbb{R}$ associated to such a pair of equilibrium price function ψ and corresponding price-determining measure μ is given by

$$S_\mu\big(t,u,\xi_1,\xi_2\big) = \int \psi\big(t,u,(1-\theta)\xi_1 + \theta\xi_2\big)\mu(d\theta) \quad \text{for all } (t,u,\xi_1,\xi_2) \in [0,T] \times \mathbb{R}^3. \tag{3.2}$$

In such a case, for all $(t,u,\xi_1,\xi_2) \in [0,T] \times \mathbb{R}^3$ the associated **benchmark price function** $S^* : [0,T] \times \mathbb{R}^3 \to \mathbb{R}$ is defined by

$$S^*(t,u,\xi_1,\xi_2) := S_\lambda(t,u,\xi_1,\xi_2) = \int_0^1 \psi\big(t,u,(1-\theta)\xi_1 + \theta\xi_2\big)\lambda(d\theta), \tag{3.3}$$

where λ denotes – as always in the sequel – the Lebesgue measure on $[0,1]$.

For each $(t,u,\xi_1,\xi_2) \in [0,T] \times \mathbb{R}^3$ the large investor price $S_\mu(t,u,\xi_1,\xi_2)$ gives the actual price per share which the large investor is faced with when shifting his stock holdings at time t from ξ_1 to ξ_2 shares and if at that time the fundamentals are u.

Remark. If the equilibrium price function ψ is multiplicative as in (3.1) the model becomes particularly nice, since then the term $\overline{\psi}(t,u)$, which depends solely on time and fundamentals, can be pulled out of the integrals in (3.2) and (3.3), respectively. Moreover, in this case the set $\bigcap_{(t,u)\in[0,T]\times\mathbb{R}} \mathfrak{M}\big(\psi(t,u,\,\cdot\,)\big)$ of admissible price-determining measures associated to ψ simplifies to $\mathfrak{M}(f)$. $\square$

As in Section 1.1 the large investor price is completely determined by the equilibrium price function and the associated price-determining measure; thus we may state:

Definition 1.19. The tuple (ψ,μ) is called a **(large investor) price system** or even a **large investor market** if $\psi : [0,T]\times\mathbb{R}^2 \to \mathbb{R}$ is an equilibrium price function and μ is a price-determining measure for ψ.

Remark. Of course the tuple (ψ,μ) only provides the framework of the large investor market, namely the price system, but it does not specify the evolution of the fundamentals nor the trades of the large investor, though they are essential for the actual appearance of the large investor market. $\square$

Now we can also transfer Definition 1.9 of a transaction loss function and the related properties of Section 1.2.2 to the general large investor market described by (ψ,μ):

Definition 1.20. Let (ψ,μ) be a large investor price system. Then the (implied) **transaction loss function** $c_\mu : [0,T]\times\mathbb{R}^3 \to \mathbb{R}$ is given by

$$c_\mu\big(t,u,\xi_1,\xi_2\big) := \big(\xi_2-\xi_1\big)\int \psi\big(t,u,(1-\theta)\xi_1+\theta\xi_2\big)\big(\mu-\lambda\big)(d\theta) \tag{3.4}$$

for all $t\in[0,T]$ and $u,\xi_1,\xi_2\in\mathbb{R}$. We say that the price system (ψ,μ) (or the large investor price function $S_\mu : [0,T]\times\mathbb{R}^3\to\mathbb{R}$) **implies a natural loss structure** or **nondecreasing total transaction losses**, if $S_\mu(t,u,\,\cdot\,,\,\cdot\,) : \mathbb{R}^2\to\mathbb{R}$ implies a natural loss structure or nondecreasing total transaction loss, respectively, for all $t\in[0,T]$ and $u\in\mathbb{R}$.

While the general dynamic price system introduced in this section is suitable for continuous trading in time, we will assume for the rest of this chapter that the large investor trades only at a finite number of trading times. The general formulation of the price system as we have set it up in this section will become important in Chapter 3 and 4 when we look at the convergence of a sequence of discrete models which are all based on the same underlying price system.

1.3.2 A Binomial Model for the Fundamentals

The price system (ψ,μ) determines only the basic pricing mechanism in the large investor market. If we now want to construct a stochastic model, we have to specify when and how the large investor trades in this market and how the fundamentals evolve between the trading times. In this section we specify the time points at which the large investor can trade and the evolution of the fundamentals between these time points. The large investor's *portfolio strategy*, which describes the large investor's trades in stocks and cash, will then be defined in Section 1.3.3.

In order to start with a discrete model, we fix an arbitrary $n\in\mathbb{N}$, divide the interval $[0,T]$ into $\lceil nT\rceil$ subintervals by picking out the $\lceil nT\rceil+1$ equidistant time points $\big\{t_k^n\big\}_{0\le k\le\lceil nT\rceil}$ given by

$$t_k^n = \frac{kT}{\lceil nT\rceil} \quad \text{for all } k\in\mathbb{N}_0 \text{ with } 0\le k\le\lceil nT\rceil, \tag{3.5}$$

and assume that the large investor can only trade at these dates. On the set of time points given by $\big\{t_k^n\big\}_{0\le k\le\lceil nT\rceil}$ we model the evolution of the fundamentals as a random walk, in the sense that between two successive time points the fundamentals can either increase or decrease by an amount δ, where δ is used here and for the whole remainder of the thesis as the reciprocal square root of n, i.e. we will always (and for all $n \in I\!N$) use the shorthand

$$\delta := \delta_n := \frac{1}{\sqrt{n}}.$$

Remark. For the most part of this thesis we assume for simplicity that $T = 1$ such that the kth time point simplifies to $t_k^n = \frac{k}{n}$ for all $0 \le k \le n$. □

In order to formally set up the random walk which describes the values of the fundamentals at each of the time points $\{t_k^n\}_{0 \le k \le \lceil nT \rceil}$ we define for each $n \in I\!N$ the set $\Omega^n := \{-1, 1\}^{\lceil nT \rceil + 1}$ of outcomes and associate to it the corresponding power set $\mathcal{F}^n_{\lceil nT \rceil} := \mathcal{P}(\Omega^n)$ as σ-field, such that $(\Omega^n, \mathcal{F}^n_{\lceil nT \rceil})$ is a measurable space, on which we can introduce the probability measure $\mathbf{P}^n$ by

$$\mathbf{P}^n(\omega) := \frac{1}{2^{\lceil nT \rceil + 1}} \quad \text{for all } \omega \in \Omega^n. \tag{3.6}$$

For each $n \in I\!N$ and any fixed initial value $u_0 \in I\!R$ of the fundamentals at time $t_0^n = 0$ we can then define the **fundamental process** $U^n = \{U_k^n\}_{0 \le k \le \lceil nT \rceil}$ and the associated **tilt process** $Z^n = \{Z_k^n\}_{0 \le k \le \lceil nT \rceil}$ on the probability space $(\Omega^n, \mathcal{F}^n_{\lceil nT \rceil}, \mathbf{P}^n)$ by setting

$$Z_k^n := \omega_{k+1}^n \quad \text{and} \quad U_k^n := u_0 + \delta_n \sum_{j=1}^{k} Z_j^n \quad \text{for all } 0 \le k \le \lceil nT \rceil, \tag{3.7}$$

where ω_{k+1}^n denotes the $(k+1)$st component of $\omega^n \in \Omega^n$. On $(\Omega^n, \mathcal{F}^n_{\lceil nT \rceil}, \mathbf{P}^n)$ we can then introduce the filtration $\mathcal{F}^n = \{\mathcal{F}_k^n\}_{0 \le k \le \lceil nT \rceil}$ generated by the tilt process, i.e. we define

$$\mathcal{F}_k^n := \sigma(Z_0^n, Z_1^n, \ldots, Z_k^n) \quad \text{for all } 0 \le k \le \lceil nT \rceil.$$

Remark. Actually, (after a certain change of measure) the process $\{U_k^n\}_{0 \le k \le \lceil nT \rceil}$ is a general correlated random walk in the sense of Definition 5.2 in Chapter 5. For each $0 \le k \le \lceil nT \rceil$ the tilt Z_k^n depicts the direction of the last move of the fundamentals leading to the fundamental value of U_k^n at time t_k^n. Especially, if the stock price adjusts with a certain delay to the large investor's trades such that the price-determining measure μ is not the Dirac measure δ_1 concentrated in 1, the tilt Z_0^n at time $t_0^n = 0$ will become important for our general convergence results in Chapter 4, once we suppose a certain relationship between the tilt Z_0^n and the stock holdings ξ_{-1}^n immediately before time 0.

On the other hand, the choice of the probability measure $\mathbf{P}^n$ in (3.6) is not essential for our proceedings. We could have taken any probability measure $\widetilde{\mathbf{P}}^n$ on $(\Omega^n, \mathcal{F}^n_{\lceil nT \rceil})$ under which all states $\omega^n \in \Omega^n$ have a strictly positive probability to occur, since we will see that like in the Cox-Ross-Rubinstein model the original probability measure has no effect on the large investor's replication price of an option. □

We will now introduce some more notation to denote the possible realizations of the fundamental process over time. Since the random walk U^n lives on a triangular grid, we define at first a compact notation to address each node of the grid:

Definition 1.21. For each $m \in I\!N_0$ the **triangular grid of indices** $I(m)$ is given by

$$I(m) := \{(k, i) \,|\, k \in \{0, 1, \ldots, m\} \text{ and } i \in \mathcal{I}_k\},$$

where the set $\mathcal{I}_k$ of possible indices at step k is defined by

$$\mathcal{I}_k := \{-k, 2-k, \ldots, k\} \quad \text{for all } k \in I\!N_0.$$

In order to avoid some case distinctions in Section 1.4 we also set $\mathcal{I}_{-1} := \{\}$.

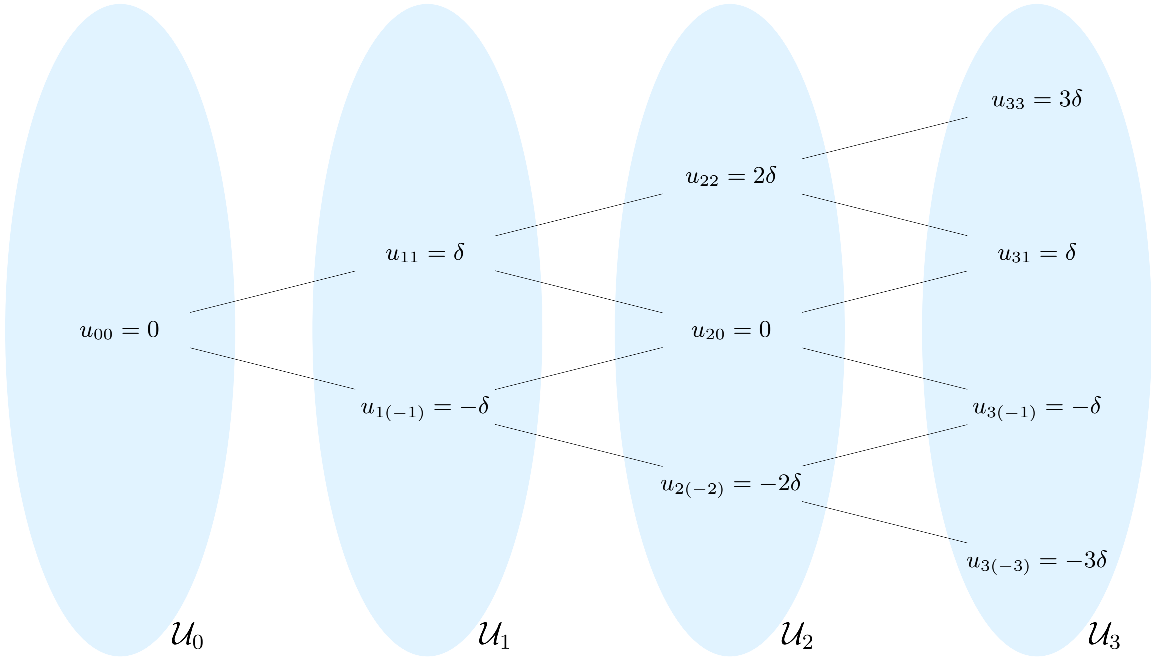

Figure 1.1: Possible realizations up to time t_3^n with $u_0 = 0$

If we now take $m = \lceil nT \rceil$ for some fixed $n \in I\!N$, then $I\big(\lceil nT \rceil\big)$ denotes the complete triangular grid of indices needed to describe all possible realizations of U^n, and we can adopt for each $n \in I\!N$ the notation

$$u_{ki}^n := u_0 + i\delta_n \quad \text{for all } (k,i) \in I\big(\lceil nT \rceil\big) \tag{3.8}$$

to denote all possible realizations of U^n between $t_0^n = 0$ and $t_{\lceil nT \rceil}^n = T$.

Definition 1.22. The set $\mathcal{U}_k^n$ of **possible realizations of** U^n **at time** t_k^n (or: at the kth step) is given by

$$\mathcal{U}_k^n := \big\{u_{ki}^n \;:\; i \in \mathcal{I}_k\big\} \quad \text{for all } 0 \le k \le \lceil nT \rceil \text{ and } n \in I\!N. \tag{3.9}$$

Moreover, the set of all **possible time-space realizations** $\mathcal{A}^n(m)$ up to time t_m^n is defined by

$$\mathcal{A}^n(m) := \big\{\big(t_k^n, u_{ki}^n\big) \,\big|\, (k,i) \in I(m)\big\} \quad \text{for all } m, n \in I\!N_0 \text{ with } m \le \lceil nT \rceil. \tag{3.10}$$

For the set of all possible time-space realizations up to time $t_{\lceil nT \rceil}^n = T$ we write $\mathcal{A}^n$ instead of $\mathcal{A}^n\big(\lceil nT \rceil\big)$.

Figure 1.1 depicts the possible realizations of the fundamentals up to time t_3^n. In this figure we have set $u_0 = 0$ and suppressed the index n.

1.3.3 The Large Investor's Portfolio Strategy

Having specified the discretization of the time axis and having modelled the behavior of the fundamentals along the so-defined time grid, we still need to exactly specify the model for the large investor's stock and cash holdings in order to have a full description of the large investor market. In Section 1.3.2 we have determined for each fixed $n \in I\!N$ the time points $\big\{t_k^n\big\}_{0 \le k \le \lceil nT \rceil}$ at which the large investor can trade, now we have to specify the *portfolio strategy* used by the large investor, i.e. we have to describe how the large investor shifts his portfolio from stocks into the bank account and vice versa as time proceeds. Since the large investor does not trade between any two successive time points t_k^n and t_{k+1}^n, his portfolio in stocks and cash will remain constant in between. At each time point t_k^n with $0 \leq k \leq \lceil nT \rceil$ the large investor will take into account all the fundamentals or news which are known by time t_k^n for the set-up of his revised portfolio structure. Formally, we define the large investor's portfolio strategy as a sequence of instantaneous portfolios:

Definition 1.23. Let $n \in I\!N$ and the fundamental process $U^n = \{U_k^n\}_{0 \leq k \leq \lceil nT \rceil}$ be defined as in (3.7).

(*i*) For each $0 \leq k \leq \lceil nT \rceil$ a **portfolio at time** t_k^n is a two-dimensional $\mathcal{F}_k^n$-measurable random variable (ξ_k^n, b_k^n). The portfolio (ξ_k^n, b_k^n) is called **path-independent** if there exist two functions $\xi^n(t_k^n, \cdot\,)$, $b^n(t_k^n, \cdot\,) : \mathcal{U}_k^n \to I\!R$ such that

$$\xi_k^n = \xi^n(t_k^n, U_k^n) \quad \text{and} \quad b_k^n = b^n(t_k^n, U_k^n), \tag{3.11}$$

and in this case we adopt the shorthands

$$\xi_{ki}^n = \xi^n(t_k^n, u_{ki}^n) \quad \text{and} \quad b_{ki}^n = b^n(t_k^n, u_{ki}^n) \quad \text{for all } i \in \mathcal{I}_k \tag{3.12}$$

to denote all possible realizations of ξ_k^n and b_k^n, respectively.

(*ii*) A **portfolio strategy** or **trading strategy** (ξ^n, b^n) is an $\mathcal{F}^n$-adapted two-dimensional stochastic process. A portfolio strategy is **path-independent** if for each $0 \leq k \leq \lceil nT \rceil$ the portfolio (ξ_k^n, b_k^n) is path-independent. In this case we introduce the **strategy function** $\xi^n : \mathcal{A}^n \to I\!R$ and the **cash holdings function** $b^n : \mathcal{A}^n \to I\!R$ as the two functions which satisfy (3.11) for all $0 \leq k \leq \lceil nT \rceil$, and once again adopt for all $0 \leq k \leq \lceil nT \rceil$ the shorthands (3.12).

(*iii*) The portfolio held by the large investor immediately before time $t_0^n = 0$ is denoted by (ξ_{-1}^n, b_{-1}^n) and supposed to be deterministic.

If the large investor uses the portfolio strategy (ξ^n, b^n) he holds for each $0 \leq k \leq \lceil nT \rceil - 1$ a total number of ξ_k^n stocks and the cash amount b_k^n between time t_k^n and time t_{k+1}^n. At time $T = t_{\lceil nT \rceil}^n$ he holds a portfolio of $\xi_{\lceil nT \rceil}^n$ stocks and $b_{\lceil nT \rceil}^n$ in cash.

Remark. Note that our definition of a portfolio strategy differs from the standard definition used in the Cox-Ross-Rubinstein model, since we only require that the strategy (ξ^n, b^n) is adapted, but not predictable. In the Cox-Ross-Rubinstein model the portfolio strategy $(\phi^n, \beta^n) = \{\phi_k^n, \beta_k^n\}_{1 \leq k \leq \lceil nT \rceil}$ is normally introduced in such a way that the portfolio (ϕ_k^n, β_k^n) describes the number of stocks held between t_{k-1}^n and t_k^n for each $1 \leq k \leq \lceil nT \rceil$. If we take our definition of a portfolio strategy (ξ^n, b^n) and set $(\phi_k^n, \beta_k^n) = (\xi_{k-1}^n, b_{k-1}^n)$ for all $1 \leq k \leq \lceil nT \rceil$, then (ϕ^n, β^n) is of course predictable and fulfills the ordinary definition of the portfolio strategy in the Cox-Ross-Rubinstein model. However, since the large investor price for the stock depends on the large investor's stock holdings, and in general even both before and after his trade, it is much easier for us to think of ξ_k^n as the number of stocks and b_k^n as the cash amount held between time t_k^n and time t_{k+1}^n, for all $0 \leq k \leq \lceil nT \rceil - 1$. □

As in the standard small investor models like the Cox-Ross-Rubinstein model we concentrate our analysis on those trading strategies of the large investor for which the large investor does not input or withdraw any funds from the market between the time points 0 and T. This leads to the definition of a self-financing strategy:

Definition 1.24. For any $n \in I\!N$ a portfolio strategy (ξ^n, b^n) is called **self-financing** if

$$b_{k-1}^n = b_k^n + (\xi_k^n - \xi_{k-1}^n) S_\mu(t_k^n, U_k^n, \xi_{k-1}^n, \xi_k^n) \quad \text{for all } 1 \leq k \leq \lceil nT \rceil. \tag{3.13}$$

Under the self-financing condition any increase in the number of shares of stock held by large investor at any of the trading dates $\{t_k^n\}_{1\le k\le \lceil nT\rceil}$ is completely financed by a reduction of the cash amount held in his portfolio and vice versa.

1.3.4 The Evolution of the Stock Price

For every $n \in I\!N$ the evolution of the stock price in the discrete large investor market with the trading dates $\{t_k^n\}_{0\le k\le \lceil nT\rceil}$ is now completely determined by the price system (ψ, μ), the fundamental process U^n, the stock-related part ξ^n of the portfolio strategy (ξ^n, b^n), and – depending on the price system – also on the large investor's stock holdings ξ_{-1}^n immediately before time $t_0^n = 0$. Of course, the critical moments in the evolution of the stock price are the trading times $\{t_k^n\}_{0\le k\le \lceil nT\rceil}$ where the large investor adjusts his portfolio, since here the stock price obtained in the market switches from the old Walrasian equilibrium, which holds as long as the large investor keeps his old stock position, to a new equilibrium which takes into account the large investor's new stock holdings. Such a critical moment has been described in detail for a fixed time point t_k^n and fixed fundamentals U_k^n in Section 1.1.2.

Figure 1.2 depicts the possible evolution of the stock price up to the trading time t_3^n if the large investor uses a path-independent portfolio strategy (ξ^n, b^n). For the sake of clarity the dependence of ψ on t and any superscript n is suppressed in this figure. In order to explain the figure, let us start at the first point in time which is used for trading, namely at $t_0^n = 0$. At this time the fundamentals are given by $U_0^n = u_{00}^n = u_0$, and according to his portfolio strategy (ξ^n, b^n) the large investor has to shift his stock holdings from the ξ_{-1}^n shares of stock which he had initially held to $\xi_{00}^n = \xi^n\big(t_0^n, u_{00}^n\big)$ shares. Before the large investor places any order, the equilibrium price for the stock at time $t_0^n = 0$ is based on the large investor's initial ξ_{-1}^n shares of stock, so it is given by $\psi\big(t_0^n, u_{00}^n, \xi_{-1}^n\big)$. In the figure we now suppose that $\xi_{-1}^n < \xi_{00}^n$, such that the large investor has to buy $\xi_{00}^n - \xi_{-1}^n$ additional shares of stock. Because of the price system (ψ, μ) the average price per share for this transaction is given by $S_{00}^+ := S_\mu\big(t_0^n, u_{00}^n, \xi_{-1}^n, \xi_{00}^n\big)$. (If $\xi_{-1}^n \ge \xi_{00}^n$, the large investor would have to sell $\xi_{-1}^n - \xi_{00}^n$ shares for the price S_{00}^+ per share.) With the large investor having shifted his stock holdings to ξ_{00}^n stocks, the new equilibrium stock price on the market at time t_0^n becomes $\psi\big(t_0^n, u_{00}^n, \xi_{00}^n\big)$.

As time proceeds, the fundamentals change, for example because of some news arriving. At time t_1^n the fundamentals have changed either to u_{11}^n or to $u_{1(-1)}^n$, and the corresponding equilibrium prices before the large investor acts on the market are $\psi\big(t_1^n, u_{1(\pm 1)}^n, \xi_{00}^n\big)$. But at time t_1^n the large investor has to shift his stock position again, namely to ξ_{11}^n or $\xi_{1(-1)}^n$, respectively. In our figure we assume $\xi_{11}^n > \xi_{00}^n > \xi_{1(-1)}^n$, which implies that the large investor has to buy $\xi_{11}^n - \xi_{00}^n$ shares at an average stock price of $S_{11}^+ := S_\mu\big(t_1^n, u_{11}^n, \xi_{00}^n, \xi_{11}^n\big)$ if the fundamentals have increased, and he has to sell $\xi_{00}^n - \xi_{1(-1)}^n$ shares at an average price of $S_{1(-1)}^- := S_\mu\big(t_1^n, u_{1(-1)}^n, \xi_{00}^n, \xi_{1(-1)}^n\big)$ if the fundamentals have decreased between time t_0^n and time t_1^n.

Figure 1.2 illustrates that the binomial tree for the stock prices is recombining as long as the large investor's portfolio strategy is path-independent. However, compared to a standard Cox-Ross-Rubinstein model, the influence of the large investor's stock holdings on the stock price itself complicates the method how the stock prices recombine. In the standard Cox-Ross-Rubinstein model the stock price at each trading time is exogenously given as a function $\overline{\psi} : [0,T] \times I\!R \to I\!R$ of time and fundamentals and does not depend on the actual stock holdings of one particular investor. Therefore, the stock price which appears in the market if the fundamentals go up first and then down is the same as if the fundamentals go down first and then up again. In both cases the price would be given by $\overline{\psi}\big(t_2^n, u_{20}^n\big)$.

In Figure 1.2 we can see the additional complexity of the recombining property in our large

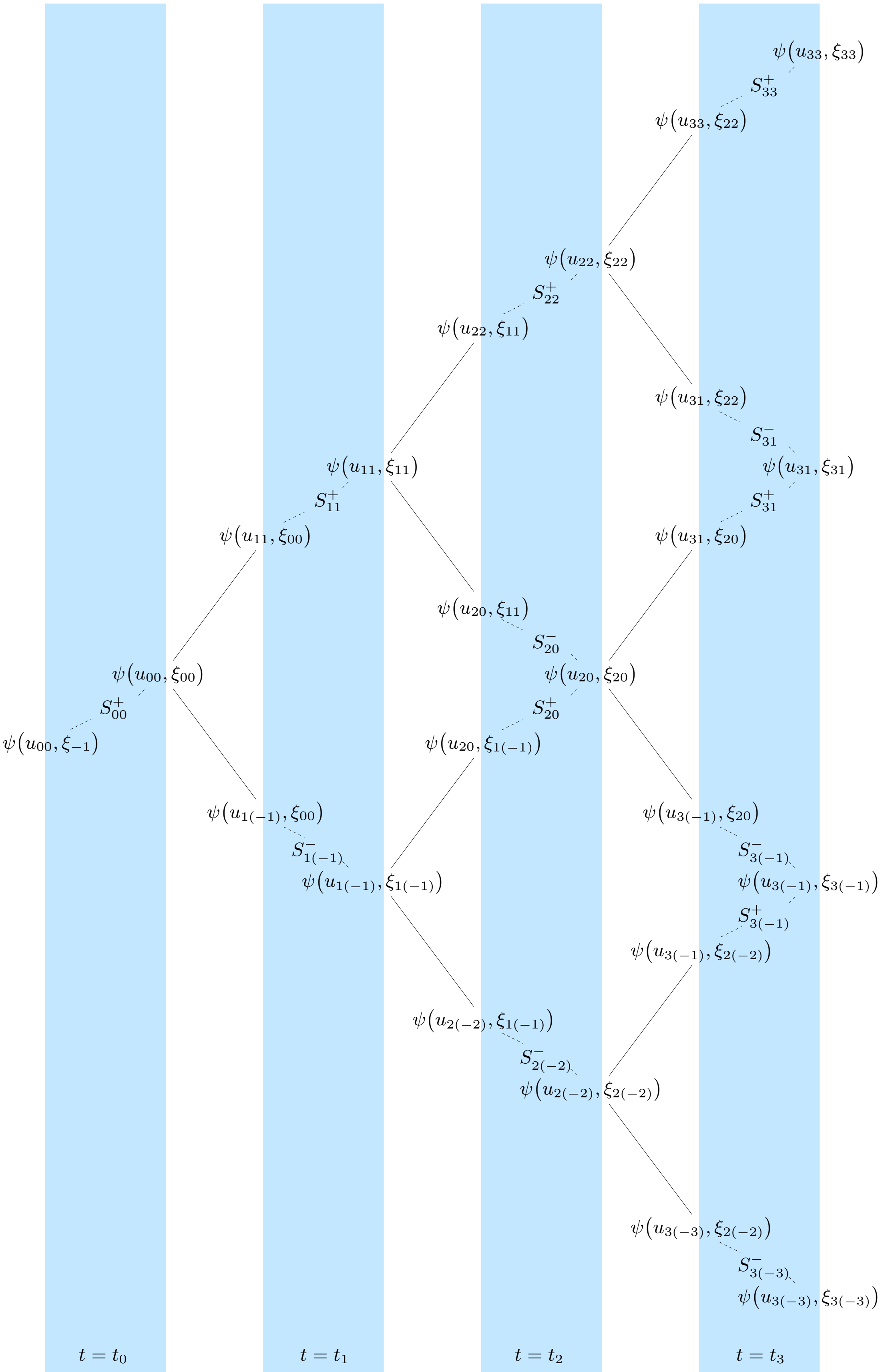

Figure 1.2: Possible stock prices up to time t_3^n

investor model: The equilibrium price at time t_2^n before the large investor has traded is given by $\psi\big(t_2^n, u_{20}^n, \xi_{11}^n\big)$ if the fundamentals went up and then down, and by $\psi\big(t_2^n, u_{20}^n, \xi_{1(-1)}^n\big)$ if the fundamentals went down first and then up. If the prices really depend on the holdings of the large investor, these two equilibrium prices do not coincide, and as long as the price-determining measure μ of the price system (ψ, μ) is not the Dirac measure δ_1, the two large investor prices $S_{20}^{\mp} := S_\mu\big(t_2^n, u_{20}^n, \xi_{1(\pm1)}^n, \xi_{20}^n\big)$, i.e. the average stock prices the large investor might be faced with when performing the necessary transaction at time t_2^n, do not coincide either. However, the new equilibrium price which appears in the market after the large investor's transaction will be $\psi\big(t_2^n, u_{20}^n, \xi_{20}^n\big)$ in both cases, and thus, in the sense described above, the binomial tree for the stock prices in the large investor model is indeed recombining as well.

Remark. Bakstein (2001) has developed a binomial large investor model which incorporates in addition to the random price impact known from the Cox-Ross-Rubinstein model both a short-term price impact due to the lack of liquidity at the large investor's *trades* and a permanent slippage due to the large investor's stock *holdings*. If the large investor uses a path-independent trading strategy, the stock price process lives on a recombining tree which is similar to the one of Figure 1.2. □

1.3.5 The Value of a Portfolio Strategy

If an investor holds a certain portfolio, he is inclined to assess his portfolio by some valuation rule. For a large investor who affects the stock price by his own trading it is not a priori clear how to value his stock holdings. In this section we develop two different valuation concepts for a large investor's portfolio or his portfolio strategy. Therefore, let us once again fix some price system (ψ, μ), some discretization parameter $n \in \mathbb{N}$ and the large investor's stock holdings ξ_{-1}^n immediately before time $t_0^n = 0$, and model the evolution of the fundamentals by a random walk U^n as in (3.7).

There is no doubt how the cash in the large investor's portfolio should be valued. Since the investor could borrow or lend an arbitrary amount of cash for the same interest rate, which we suppose to be zero, the only reasonable price per unit of cash in the large investor's portfolio is 1. In the standard Cox-Ross-Rubinstein model the shares of stock in the investor's portfolio are priced along the same reasoning by the actual stock price in the market. In this case the stock price is also exogenously given and does not depend on the investor's particular stock holdings. However, the discussion at the end of Section 1.3.4 has once again shown that in a true large investor market there is no longer one unique stock price for a given combination of time and fundamentals like in the Cox-Ross-Rubinstein model. Hence every valuation of the large investor's stock holdings will depend on the particular stock price selected to value his share of stock, and like in small investor models with transaction costs the values of the large investor's portfolio before and after his transaction need not coincide. Our two valuation concepts for the value of the large investor's portfolio strategy at a certain point t_k^n in time evaluate the portfolio after the large investor's trade at time t_k^n.

The first concept uses the most recent price experienced by the large investor, i.e. the average per-share price for his transaction at this point in time. If for example the large investor has held ξ_{k-1}^n shares of stock immediately before time t_k^n and shifts his stock holdings at time t_k^n to ξ_k^n shares, then the average stock price achieved by that transaction is given by $S_\mu\big(t_k^n, U_k^n, \xi_{k-1}^n, \xi_k^n\big)$. This approach to valuate the large investor's stock holdings leads to the concept of the *paper value* of a portfolio strategy.

Definition 1.25. The **paper value** V_k^n of a portfolio strategy $\big(\xi^n, b^n\big)$ at time t_k^n is given by

$$V_k^n := \xi_k^n S_\mu\big(t_k^n, U_k^n, \xi_{k-1}^n, \xi_k^n\big) + b_k^n \quad \text{for all } 0 \le k \le \lceil nT \rceil.$$

We will write $V^n = \{V_k^n\}_{0 \le k \le \lceil nT \rceil}$ to denote the paper value process between time 0 and T.

Remark. The paper value is a mark-to-market approach in the sense that it values the whole stock holdings of the large investor by the last price seen on the market by the large investor. A similar valuation concept of paper value was first introduced in discrete time by Jarrow (1992) and in continuous time by Schönbucher and Wilmott (2000). Implicitly, the paper valuation concept is also assumed in the continuous-time models of Frey (1998, 2000) and Sircar and Papanicolaou (1998), as we shall see when we investigate the limits of our discrete models in Chapter 4. A related mark-to-market concept in discrete time has been developed by Bakstein (2001) and Bakstein and Howison (2002). These two authors use the mid-market price, which prevails in the market after the transaction of the large investor has been executed and the stock price has found its new equilibrium, as their mark-to-market value of the large investor's stock position. If we suppose a negligible bid-ask spread the mid-market price is the mark-to-market price for a *small* investor. Our paper value is also similar to the "marked-to-market value" of Çetin et al. (2004). In a continuous-time economy where the large investor affects the stock price only temporarily through the number of shares traded in a particular point in time, this mark-to-market value uses the marginal stock price to evaluate the large investor's portfolio. □

The stock price used for determining the paper value is an observed market price, so that the paper value can be calculated by the large investor without knowing the detailed structure of the price system (ψ, μ). This simplicity and transparency makes the paper value an important valuation rule, especially in a real-life market, where the actual price system (ψ, μ) is not known at all, and where in general two agents will have two different beliefs on the "true" price system; if these agents had to find a common valuation principle for the large investor's portfolio, they could still agree on the paper value.
However, the paper value has some serious drawbacks: The stock price of $S_\mu(t_k^n, U_k^n, \xi_{k-1}^n, \xi_k^n)$ per share is only the per-share price for the transaction which changed the large investor's stock holdings from ξ_{k-1}^n to ξ_k^n shares. It is not guaranteed that the large investor can achieve the same price when immediately switching his stock holdings back from ξ_k^n to ξ_{k-1}^n shares, nor that he could sell all his ξ_k^n shares for that price. Moreover, even if the large investor could sell all his ξ_k^n shares for that price, this would not imply that at time t_k^n the large investor could also build up his stock holdings from zero for the same price, so the paper value need not reflect the strategic value of the large investor's stock holdings at all. Thirdly, as a function of the large investor's stock holdings ξ_{k-1}^n *before* time t_k^n, the stock price $S_\mu(t_k^n, U_k^n, \xi_{k-1}^n, \xi_k^n)$ used for the calculation of the paper value is a retrospective stock price, though a decision-oriented valuation rule should rather use a topical or even a prospective price.

Remark. The third drawback could be circumvented by valuing the large investor's stock holdings with the new Walrasian equilibrium price $\psi(t_k^n, U_k^n, \xi_k^n)$ which appears immediately after the large investor's transaction. However, except for an infinitesimal amount of shares the large investor can not trade shares for this price. In reality the large investor might even be unaware of this equilibrium price, while he will always know the price $S_\mu(t_k^n, U_k^n, \xi_{k-1}^n, \xi_k^n)$ at which he has traded. □

In Section 1.1 we have explained in great detail the usefulness of the benchmark price function as a large investor price function which satisfies the "fair" price condition (1.1). The benchmark price gives the "fair" price per share for a transaction of the large investor in the sense that any (instantaneous) round-trip of the large investor does not lead to any transaction loss or profit. For example, the "fair" price for the transaction necessary to switch the large investor's stock holdings at time t_k^n from ξ_{k-1}^n to ξ_k^n shares is given by $S^*(t_k^n, U_k^n, \xi_{k-1}^n, \xi_k^n)$,

and this price coincides with the benchmark price $S^*\big(t_k^n, U_k^n, \xi_k^n, \xi_{k-1}^n\big)$ for the immediate cancellation of that transaction. In order to assess the whole position of ξ_k^n shares held by the large investor at time t_k^n, the benchmark price $S^*\big(t_k^n, U_k^n, \xi_{k-1}^n, \xi_k^n\big)$ is still unsuitable, since it is a retrospective price as well, as it depends on the large investor's stock holdings ξ_{k-1}^n before time t_k^n. However, we can easily use the benchmark price function $S^*\big(t_k^n, U_k^n, \cdot\,, \cdot\,\big) : I\!R^2 \to I\!R$ to construct an objective and topical valuation for the ξ_k^n shares held by the large investor at time t_k^n, namely just by taking the benchmark price $S^*\big(t_k^n, U_k^n, \xi_k^n, 0\big)$ which corresponds to the liquidation of all the ξ_k^n shares held by the large investor at time t_k^n. Of course, this price only depends on the ξ_k^n shares held by the large investor at time t_k^n , and since the benchmark price is a "fair" price, it even coincides with the benchmark price $S^*\big(t_k^n, U_k^n, 0, \xi_k^n\big)$ which corresponds to the transaction needed by the large investor if he wants to change his stock holdings at time t_k^n from 0 to ξ_k^n shares.
Thus, it is sensible to add another price function to the price functions of Section 1.3.1:

Definition 1.26. Let (ψ, μ) be a large investor price system, and $S^* : I\!R^3 \to I\!R$ the benchmark price function in the market described by (ψ, μ). Then the **(loss-free) liquidation price function** $\bar{S} : [0, T] \times I\!R^2 \to I\!R$ is for all $(t, u, \xi) \in [0, T] \times I\!R \times I\!R$ given by

$$\bar{S}(t, u, \xi) := S^*(t, u, \xi, 0) = S^*(t, u, 0, \xi) = \int_0^1 \psi(t, u, \theta\xi)\, \lambda(d\theta). \tag{3.14}$$

Remark. Note that the liquidation price is only a theoretical liquidation price, which cannot be observed in the market. In a "normal" large investor market, the large investor price and the benchmark price will not coincide, but the large investor is exposed to transaction losses. In this case the large investor would only achieve a worse price per share than the liquidation price if he would immediately liquidate his stock holdings.
Though the liquidation price $\bar{S}\big(t_k^n, U_k^n, \xi_k^n\big)$ itself cannot be observed in the market, it can be calculated like in (3.3) from the new Walrasian equilibrium price $\psi\big(t_k^n, U_k^n, \xi_k^n\big)$ at time t_k^n, if both the price system (ψ, μ) and the large investor's stock holdings ξ_k^n are known, since the only missing variable, the value U_k^n of the fundamentals at time t_k^n, can be calculated from $\psi\big(t_k^n, U_k^n, \xi_k^n\big)$ by inverting the strictly increasing function $\psi\big(t_k^n, \cdot, \xi_k^n\big) : I\!R \to I\!R$. $\square$

If we now use the liquidation price to value the stock holdings of the large investor, we have found a second valuation principle for the portfolio strategy of a large investor, and since the liquidation price depends only on the large investor's stock holdings at a single point in time, this second valuation concept even works for a portfolio of the large investor at any one point in time:

Definition 1.27. The **real value** $\bar{V}_k^n$ of a portfolio $\big(\xi_k^n, b_k^n\big)$ at time t_k^n is given by

$$\bar{V}_k^n := \xi_k^n \bar{S}\big(t_k^n, U_k^n, \xi_k^n\big) + b_k^n \quad \text{for all } 0 \le k \le \lceil nT \rceil. \tag{3.15}$$

If $\big(\xi^n, b^n\big)$ is a portfolio strategy, then for any $0 \le k \le \lceil nT \rceil$ the **real value** of $\big(\xi^n, b^n\big)$ at time t_k^n is the real value of the corresponding portfolio $\big(\xi_k^n, b_k^n\big)$ at time t_k^n. In this case we will write $\bar{V}^n = \big\{\bar{V}_k^n\big\}_{0 \le k \le \lceil nT \rceil}$ to denote the real value process between time 0 and T.

Remark. The introduction of the real value shows the self-financing property of (3.13) in a new light. Recalling the definition of the transaction loss function $c_\mu : [0, T] \times I\!R^3 \to I\!R$ of (3.4), we see that the self-financing condition (3.13) is equivalent to

$$b_{k-1}^n = b_k^n + \big(\xi_k^n - \xi_{k-1}^n\big) S^*\big(t_k^n, U_k^n, \xi_{k-1}^n, \xi_k^n\big) + c_\mu\big(t_k^n, U_k^n, \xi_{k-1}^n, \xi_k^n\big) \quad \text{for all } 1 \le k \le \lceil nT \rceil.$$

Moreover, the time- and fundamental-dependent analogue of (1.26) gives us the formula $(\xi_k^n - \xi_{k-1}^n)S^*(t_k^n, U_k^n, \xi_{k-1}^n, \xi_k^n) = \xi_k^n \bar{S}(t_k^n, U_k^n, \xi_k^n) - \xi_{k-1}^n \bar{S}(t_k^n, U_k^n, \xi_{k-1}^n)$, and thus the self-financing condition is also equivalent to

$$\xi_{k-1}^n \bar{S}(t_k^n, U_k^n, \xi_{k-1}^n) + b_{k-1}^n = \xi_k^n \bar{S}(t_k^n, U_k^n, \xi_k^n) + b_k^n + c_\mu(t_k^n, U_k^n, \xi_{k-1}^n, \xi_k^n) \tag{3.16}$$

for all $1 \leq k \leq \lceil nT \rceil$. Hence the self-financing condition can be reinterpreted as the rule that for any trading time t_k^n from $\{t_k^n\}_{1 \leq k \leq \lceil nT \rceil}$ the "real value" of the "old" portfolio (ξ_{k-1}^n, b_{k-1}^n) has to compensate both the real value of the "new" portfolio (ξ_k^n, b_k^n) and the loss induced by the portfolio shift from ξ_{k-1}^n to ξ_k^n shares of stock.
The equation (3.16) also shows that even for a self-financing strategy the "real value" before and after the large investor's trade need not coincide. If the implied transaction losses $c_\mu(t_k^n, U_k^n, \xi_{k-1}^n, \xi_k^n)$ are positive, then the "real value" of the portfolio strategy at time t_k^n evaluated before the large investor shifts his portfolio from ξ_{k-1}^n to ξ_k^n shares of stock would be larger then the real value after the transaction. We have introduced the real value as the value *after* the large investor's trade since this definition ensures that the real value of a path-independent self-financing portfolio strategy is recombining. □

As we proceed we will often have to distinguish between the different possible future values of the large investor's portfolio given the information which is available at a certain point in time. In order to describe these possible outcomes, we will now introduce functional analogues of the two value processes V^n and $\bar{V}^n$, i.e. we want to express the real and the paper value of a given trading strategy (ξ^n, b^n) as functions of time and the possible outcomes of the fundamental process U^n.
If we restrict our attention to path-independent portfolio strategies, we can introduce such a function for the real value by means of the strategy and cash holdings function of Definition 1.23:

Definition 1.28. Let us suppose that (ξ^n, b^n) is a path-independent portfolio with corresponding strategy and cash holdings functions $\xi^n : \mathcal{A}^n \to I\!R$ and $b^n : \mathcal{A}^n \to I\!R$, respectively. Then the **real value function** $\bar{v}^n = \bar{v}^{n,(\xi^n, b^n)} : \mathcal{A}^n \to I\!R$ is given by

$$\bar{v}^n(t,u) := \xi^n(t,u)\bar{S}(t,u,\xi^n(t,u)) + b^n(t,u) \quad \text{for all } (t,u) \in \mathcal{A}^n. \tag{3.17}$$

With this definition we can recover the real value at time t_k^n of a path-independent portfolio strategy (ξ^n, b^n) as

$$\bar{V}_k^n = \bar{v}^n(t_k^n, U_k^n) \quad \text{for all } 0 \leq k \leq \lceil nT \rceil. \tag{3.18}$$

Since the paper value does not only depend on the stock holdings ξ_k^n after the trade at time t_k^n, but also on the stock holdings ξ_{k-1}^n at time t_{k-1}^n, an analogous functional representation of the paper value function needs an additional dimension.

Definition 1.29. Under the assumptions of Definition 1.28 the **paper value function** $v^n = v^{n,(\xi^n, b^n)} : \mathcal{A}^n \times I\!R \to I\!R$ is given by

$$v^n(t,u,\xi) := \xi^n(t,u)S_\mu(t,u,\xi,\xi^n(t,u)) + b^n(t,u) \quad \text{for all } (t,u,\xi) \in \mathcal{A}^n \times I\!R. \tag{3.19}$$

With this definition we can recover the paper value V_k^n of a path-independent portfolio strategy (ξ^n, b^n) at time t_k^n as

$$V_k^n = v^n(t_k^n, U_k^n, \xi_{k-1}^n) \quad \text{for all } 0 \leq k \leq \lceil nT \rceil. \tag{3.20}$$

Moreover, for all those path-independent trading strategies and all $1 \leq k \leq \lceil nT \rceil$ the large investor's stock holdings ξ_{k-1}^n between time t_{k-1}^n and time t_k^n can be written in the functional form $\xi^n_{k-1} = \xi^n\big(t^n_{k-1}, U^n_{k-1}\big)$, hence we can rewrite (3.20) as $V^n_k = v^n\big(t^n_k, U^n_k, \xi^n\big(t^n_{k-1}, U^n_{k-1}\big)\big)$. This shows that as opposed to the real value the paper value at time t^n_k will in general depend not only on the fundamental value U^n_k at time t^n_k, but also on the fundamentals U^n_{k-1} just before time t^n_k.

Remark. The real value of a portfolio was first proposed by Schönbucher and Wilmott (2000) in the context of a continuous-time market where the large investor can trade at any time during some time interval $[0, T]$. Their market mechanism corresponds to a price system (ψ, μ) where the price-determining measure μ either is or can at least be chosen to be the Dirac measure δ_1 concentrated in 1. Like in our derivation Schönbucher and Wilmott (2000) obtain the real value of a portfolio $\big(\xi^n_k, b^n_k\big)$ as the sum of the price of the cash amount b^n_k and a *theoretical* liquidation value for the ξ^n_k shares of stocks, but they motivate the liquidation value for the stock position differently. Namely, they define the liquidation value as the price which could be obtained by an infinitely rapid, yet not instantaneous, liquidation of the stock position via infinitely many infinitely small transaction blocks, such that – given that the price function $\psi : [0, T] \times I\!R^2 \to I\!R$, $(t, u, \xi) \mapsto \psi(t, u, \xi)$ is increasing in ξ – the real value describes the maximal cash amount the large investor could receive for his portfolio. When using the limit of their liquidation strategies, Schönbucher and Wilmott (2000) also notice that a round-trip does not induce any transaction losses, and conclude that it cannot induce any transaction gains either for no-arbitrage reasons. Such a statement was made precise by means of the benchmark price in Section 1.1. However, despite their findings with regards to the real value, Schönbucher and Wilmott (2000) only consider the replication of the paper value of an option and look at the paper value of the replicating strategy when it comes to the replication of options. Thus, they do not fully employ the real value concept for replication purposes as we will do in Chapter 2. In a general semimartingale setting, Schönbucher and Wilmott's real value concept has been analyzed and exploited for superreplication by Baum (2001) and Bank and Baum (2004).
The real value is also implicitly used by Jonsson and Keppo (2001), who assume in their pricing model for a European call option written by a large investor that at maturity the holder of the option has the right to immediately sell all shares of stock back to the large investor and receive instead an amount which equals the real value of the large investor's stock position. In the most recent version of this paper, Jonsson et al. (2004) argue that the delivery is taking place in infinitesimal packages, and hence they really employ the real value.
A slightly different version of a liquidation price in discrete time was already introduced by Jarrow (1992). In opposition to our valuation concept Jarrow's "real wealth" considers the *real* liquidation price and implicitly accounts for transaction losses, ignoring the strategic advantages of holding the stock position for hedging purposes. □

1.4 Replication

In this section we consider replication problems in our binomial large investor market. The natural replication problem in large investor market concerns the replication of certain portfolios, especially of *contingent claims*. Since the possible stock and cash positions of a replicating trading strategy have to satisfy a sequence of (in general) non-trivial fixed point equations it is not a priori clear that replicating strategies for the large investor exist at all. However, we will introduce the notation of *star-convex* portfolios and show that under some regularity assumptions on the price system (ψ, μ) and the structure of the transaction losses in this market there exists for each star-convex contingent claim a self-financing replicating strategy, which is described by the sequence of fixed point problems, and if the price system implies nondecreasing transaction losses, this replicating strategy even is unique. While the more

natural replication problem in the large investor market is to replicate a given number of shares of stock and a given cash amount, the large investor might also want to replicate a certain paper value. In Section 1.4.3 we will give conditions under which the large investor can solve such a problem as well.
Thus, let us fix some price system (ψ, μ) and some $n \in I\!N$ and define the trading dates $\{t_k^n\}_{0 \leq k \leq \lceil nT \rceil}$ on $[0, T]$ and the binomial random walk $U^n = \{U_k^n\}_{0 \leq k \leq \lceil nT \rceil}$ describing the fundamentals at these time points as in Section 1.3.2.

1.4.1 Definitions

At first we have to specify the definition of a contingent claim and its attainability in a large investor environment. Then we will introduce the concept of star-convex portfolios in our market and show that the long European call is an example of a star-convex contingent claim. In the standard Cox-Ross-Rubinstein model the investor can shift his portfolio without any transaction losses from a pure stock portfolio to one which consists of cash only, and vice versa. Thus in that model one does not have to distinguish between physical delivery and cash settlement, and a contingent claim is defined as a portfolio value at maturity T, or – put in mathematical terms – as an $\mathcal{F}^n_{\lceil nT \rceil}$-measurable random variable.
However, in an environment with a large investor a round trip will in general induce some transaction losses, and thus the large investor cannot transfer cash positions into stock positions with the same real value (and vice versa) at no costs. Therefore, we have to distinguish in large investor markets between physical delivery and cash settlement of an option – or more generally: between the different combinations in stocks and cash at maturity, even though they might lead to the same real value.
This situation differs from the standard Cox-Ross-Rubinstein or Black-Scholes setting, but it is similar to the replication of contingent claims in a market with transaction costs. Hence our definition of a contingent claim can parallel the definition used in binomial transaction costs models as considered by Boyle and Vorst (1992) and others.

Definition 1.30. A **contingent claim** (ξ_n, b_n) at maturity T is a portfolio at time $t^n_{\lceil nT \rceil} = T$.

Definition 1.31. A contingent claim (ξ_n, b_n) is called **attainable** if there exists a self-financing portfolio strategy (ξ^n, b^n) such that $\xi^n_{\lceil nT \rceil} = \xi_n$ and $b^n_{\lceil nT \rceil} = b_n$. In this case we say that (ξ^n, b^n) **replicates** the contingent claim (ξ_n, b_n).

Remark. If (ξ^n, b^n) is the replicating strategy of some attainable contingent claim (ξ_n, b_n), then the large investor needs ξ_0^n shares of stock and the cash amount b_0^n at time $t_0^n = 0$ in order to be able to replicate the option. Under the assumption that the large investor has held ξ_{-1}^n shares of stock immediately before time 0, the large investor would need a total cash amount of

$$v(0, u_{00}^n, \xi_{-1}^n) = \xi_0^n S_\mu(0, u_{00}^n, \xi_{-1}^n, \xi_0^n) + b_0^n, \tag{4.1}$$

at time 0 in order to build up his portfolio at this point in time. This amount is just the paper value of the portfolio (ξ_0^n, b_0^n) at time 0. However, as in small investor markets with transaction costs we have to be careful with the interpretation of $v(0, u_{00}^n, \xi_{-1}^n)$ as the "fair" price of the contingent claim, since there might exist other self-financing trading strategies which lead to at least ξ_n shares of stock and a cash amount of at least b_n at time $t^n_{\lceil nT \rceil}$, but which are cheaper to set up than the perfect replicating strategy (ξ^n, b^n).

Not every contingent claim need be attainable. Therefore, when we want to find replicating strategies, we have to limit the class of contingent claims considered. It turns out that the class of star-convex contingent claims is a good choice, since every star-convex contingent claim can be replicated by the large investor.

Definition 1.32. For each $0 \le k \le \lceil nT \rceil$ a path-independent portfolio $\big(\xi^n_k, b^n_k\big)$ at time t^n_k is called **convex** if the function $\xi^n\big(t^n_k, \cdot\big) : \mathcal{U}^n_k \to I\!R$ of representation (3.11) is nondecreasing; it is called **concave** if $\xi^n\big(t^n_k, \cdot\big) : \mathcal{U}^n_k \to I\!R$ is nonincreasing.
Recalling the shorthand (3.12) we say that a convex portfolio $\big(\xi^n_k, b^n_k\big)$ at time t^n_k is **star-convex** if for all $i \in \mathcal{I}_{k-1} = \big\{1-k, 3-k, \dots, k-1\big\}$ we have

$$\begin{aligned} &\xi^n_{k(i\pm1)} S_\mu\big(t^n_k, u^n_{k(i\pm1)}, \xi^n_{k(i\mp1)}, \xi^n_{k(i\pm1)}\big) + b^n_{k(i\pm1)} \\ &\qquad \ge \xi^n_{k(i\mp1)} S_\mu\big(t^n_k, u^n_{k(i\pm1)}, \xi^n_{k(i\mp1)}, \xi^n_{k(i\pm1)}\big) + b^n_{k(i\mp1)}. \end{aligned} \tag{4.2}$$

A concave portfolio $\big(\xi^n_k, b^n_k\big)$ at time t^n_k is called **star-concave** if for all $i \in \mathcal{I}_{k-1}$

$$\begin{aligned} &\xi^n_{k(i\pm1)} S_\mu\big(t^n_k, u^n_{k(i\pm1)}, \xi^n_{k(i\mp1)}, \xi^n_{k(i\pm1)}\big) + b^n_{k(i\pm1)} \\ &\qquad \le \xi^n_{k(i\mp1)} S_\mu\big(t^n_k, u^n_{k(i\pm1)}, \xi^n_{k(i\mp1)}, \xi^n_{k(i\pm1)}\big) + b^n_{k(i\mp1)}. \end{aligned}$$

A portfolio strategy $\big(\xi^n, b^n\big)$ is **(star-)convex** if the portfolios $\big(\xi^n_k, b^n_k\big)$ are (star-)convex for all $0 \le k \le \lceil nT \rceil$, and similarly the portfolio strategy is called **(star-)concave** if all the portfolios $\big\{\big(\xi^n_k, b^n_k\big)\big\}_{0 \le k \le \lceil nT \rceil}$ are (star-)concave.

Remark. Since $\mathcal{U}^n_0 = \{u_0\}$ consists of a single element and since $\mathcal{I}_{-1} := \{\}$, every portfolio at time 0 is star-convex. Moreover, by the same arguments as in the remark after Definition 1.27 we see that (4.2) can for all $0 \le k \le \lceil nT \rceil$ and all $i \in \mathcal{I}_{k-1}$ be rewritten as

$$\begin{aligned} &\xi^n_{k(i\pm1)} \bar S\big(t^n_k, u^n_{k(i\pm1)}, \xi^n_{k(i\pm1)}\big) + b^n_{k(i\pm1)} \\ &\qquad \ge \xi^n_{k(i\mp1)} \bar S\big(t^n_k, u^n_{k(i\pm1)}, \xi^n_{k(i\mp1)}\big) + b^n_{k(i\mp1)} - c_\mu\big(t^n_k, u^n_{k(i\pm1)}, \xi^n_{k(i\mp1)}, \xi^n_{k(i\pm1)}\big). \end{aligned} \tag{4.3}$$

Hence, in the presence of an ordinary price system (ψ, μ) with a nonnegative transaction loss function $c_\mu : [0,T] \times I\!R^3 \to [0,\infty)$, the condition of star-convexity means that at any trading time t^n_k and for any particular outcome $u^n_{k(i\pm1)}$ of the fundamentals at this time, the large investor will always turn down an offer to exchange his portfolio for the deterministic portfolio $\big(\xi^n_{k(i\mp1)}, b^n_{k(i\mp1)}\big)$, which his trading strategy would require if at one time point between $t^n_0 = 0$ and t^n_k the fundamentals had gone in the opposite direction . Namely, (4.3) says that the real value of the target portfolio $\big(\xi^n_{k(i\pm1)}, b^n_{k(i\pm1)}\big)$ required by the large investor's trading strategy under the time-space realization $\big(t^n_k, u^n_{k(i\pm1)}\big)$ is never lower than the real value of the "neighboring" portfolio $\big(\xi^n_{k(i\mp1)}, b^n_{k(i\mp1)}\big)$ reduced by the losses necessary to shift this portfolio to the target portfolio. □

In the next example we show that a long European call with physical settlement is star-convex.

Example 1.7 *(European Call).* Let us assume for simplicity $T = 1$. The *(long) European call* $C = C^\alpha$ of $\alpha \ge 0$ shares of stock with strike $K \in I\!R$ is defined as the portfolio $C = \big(\xi_n, b_n\big)$ at time $t^n_n = T$ which consists of $\xi_n = \alpha \mathbf{1}_{\{\bar S(T, U^n_n, \alpha) > K\}}$ shares of stock and the cash amount $b_n = -\alpha K \mathbf{1}_{\{\bar S(T, U^n_n, \alpha) > K\}}$. If we look at the real value of this contingent claim as we have defined it in Definition 1.28 we see that it is given by

$$\bar V^n_n = \alpha\big(\bar S(T, U^n_n, \alpha) - K\big)^+,$$

and hence the real value of the European call has (almost) the form of an excess claim if the (in this case: liquidation) stock price exceeds the strike K which we are accustomed to from small investor market models.
We will now show that nonnegative transaction losses imply that the call C is star-convex. In order to do so, we will identify the contingent claim $\big(\xi_n, b_n\big)$ as in Definition 1.31 with the portfolio $\big(\xi^n_n, b^n_n\big)$ at time $t^n_n = T$ in order to access its possible outcomes $\big(\xi^n_i, b^n_i\big)$ for all

$i \in \mathcal{I}_n$. Because of the previous remark, we only have to show that (4.3) holds with $k = n$ and all $i \in \mathcal{I}_{n-1}$, and since the transaction loss function $c_\mu : [0,T] \times I\!R^3 \to I\!R$ is supposed to be nonnegative, it suffices to show

$$\xi^n_{n(i\pm1)} \bar{S}\big(T, u^n_{n(i\pm1)}, \xi^n_{n(i\pm1)}\big) + b^n_{n(i\pm1)} \geq \xi^n_{n(i\mp1)} \bar{S}\big(T, u^n_{n(i\pm1)}, \xi^n_{n(i\mp1)}\big) + b^n_{n(i\mp1)} \tag{4.4}$$

for all $i \in \mathcal{I}_{n-1}$. Let us now fix such an $i \in \mathcal{I}_{n-1}$ and note that by the definition of the loss-free liquidation price function $\bar{S} : [0,T] \times I\!R^2 \to I\!R$ in (3.14) we have

$$\begin{aligned} \bar{S}\big(T, u^n_{n(i-1)}, \alpha\big) &= \int_0^1 \psi\big(T, u^n_{n(i-1)}, (1-\theta)\alpha\big)\, \lambda(d\theta) \\ &< \int_0^1 \psi\big(T, u^n_{n(i+1)}, (1-\theta)\alpha\big)\, \lambda(d\theta) = \bar{S}\big(T, u^n_{n(i+1)}, \alpha\big), \end{aligned}$$

since the equilibrium price function $\psi : [0,T] \times I\!R^2 \to I\!R$, $(t,u,\xi) \mapsto \psi(t,u,\xi)$, is strictly increasing in u. Three possible cases can occur. Either we have $\bar{S}\big(T, u^n_{n(i+1)}, \alpha\big) \leq K$, or $\bar{S}\big(T, u^n_{n(i-1)}, \alpha\big) > K$, or we have

$$\bar{S}\big(T, u^n_{n(i-1)}, \alpha\big) \leq K < \bar{S}\big(T, u^n_{n(i+1)}, \alpha\big). \tag{4.5}$$

In the first two cases it follows that $\xi^n_{n(i-1)} = \xi^n_{n(i+1)}$ and $b^n_{n(i-1)} = b^n_{n(i+1)}$, and hence (4.4) trivially holds. In the third case the definition of the call implies that $\big(\xi^n_{n(i-1)}, b^n_{n(i-1)}\big) = (0,0)$ and $\big(\xi^n_{n(i+1)}, b^n_{n(i+1)}\big) = \big(\alpha, -\alpha K\big)$. Thus, the two inequalities in (4.4) follow from (4.5) as well, and the long European call is indeed a star-convex contingent claim. □

Remark. Example 1.7 reveals the main reason why we call the class of portfolios introduced in Definition 1.32 *convex*: In the example of the long European call its real value is a convex function of the loss-free liquidation price $\bar{S}(T, U^n_n, \alpha)$. However, as we have indicated in the beginning of this section the value of a contingent claim is not the right tool to describe contingent claims in a market with a large investor, and so we have to transfer the convexity from functions to portfolios. This brings some difficulties, but still relates back to the terms used in standard small investor market models. □

1.4.2 Replication of Star-Convex Contingent Claims

Under some regularity assumptions on the price system and the implied loss structure we will show in this section that in our binomial model every star-convex contingent claim is attainable, and we will also give conditions such that the replicating strategy is unique.

In order to find the replicating strategy we use the same approach as in the Cox-Ross-Rubinstein model and construct it step by step by calculating the portfolios necessary to replicate the contingent claim backwards in time. In the Cox-Ross-Rubinstein model this recursively leads to explicit equations for the values of the strategy and cash holdings at the different points in time. Similarly we can find formulæ for these values in the large investor case. However, the values of the strategy function are only given as solutions of a non-trivial fixed point equation. Thus, the existence and uniqueness results for the replicating strategy become much more involved than in a small investor model.

Let us now assume we are given some star-convex contingent claim (ξ_n, b_n). In order to replicate this claim by a portfolio strategy (ξ^n, b^n) we need $\xi^n_{\lceil nT\rceil} = \xi_n$ shares of stock and the cash amount $b^n_{\lceil nT\rceil} = b_n$ at maturity, so let us define the portfolio of the replicating strategy (ξ^n, b^n) at time $t^n_{\lceil nT\rceil} = T$ like this. Since (ξ_n, b_n) is star-convex, the portfolio $\big(\xi^n_{\lceil nT\rceil}, b^n_{\lceil nT\rceil}\big)$ is star-convex as well.

Let us now assume that for some $1 \leq k \leq \lceil nT \rceil$ and all $k \leq j \leq \lceil nT \rceil$ we have already constructed star-convex portfolios $\left(\xi_j^n, b_j^n\right)$ at time t_j^n as part of a potential replicating strategy (ξ^n, b^n). Then we have to find out how the large investor's portfolio between the trading times t_{k-1}^n and t_k^n has to look like in order to allow the large investor to (exactly) finance the portfolio $\left(\xi_k^n, b_k^n\right)$ at time t_k^n. Hence we are looking for a portfolio $\left(\xi_{k-1}^n, b_{k-1}^n\right)$ at time t_{k-1}^n which satisfies the self-financing condition (3.13). If the fundamentals at time t_{k-1}^n are U_{k-1}^n, then the fundamentals U_k^n at time t_k^n will either be $U_{k-1}^n + \delta_n$ or $U_{k-1}^n - \delta_n$, and the self-financing condition reads

$$\begin{aligned} b_{k-1}^n &= b^n\left(t_k^n, U_{k-1}^n \pm \delta_n\right) \\ &\quad + \Big(\xi^n\left(t_k^n, U_{k-1}^n \pm \delta_n\right) - \xi_{k-1}^n\Big) S_\mu\Big(t_k^n, U_{k-1}^n \pm \delta_n, \xi_{k-1}^n, \xi^n\left(t_k^n, U_{k-1}^n \pm \delta_n\right)\Big). \end{aligned} \tag{4.6}$$

From these two equations it follows that if some portfolio $\left(\xi_{k-1}^n, b_{k-1}^n\right)$ satisfies (4.6), then there exists a path-independent one. Thus, we can use the shorthands of Definition 1.23, namely

$$\xi_{(k-1)i}^n = \xi^n\left(t_{k-1}^n, u_{(k-1)i}^n\right) \quad \text{and} \quad b_{(k-1)i}^n = b^n\left(t_{k-1}^n, u_{(k-1)i}^n\right) \quad \text{for all } i \in \mathcal{I}_{k-1}, \tag{4.7}$$

where $\xi^n\left(t_{k-1}^n, \cdot\,\right) : \mathcal{U}_{k-1}^n \to I\!R$ and $b^n\left(t_{k-1}^n, \cdot\,\right) : \mathcal{U}_{k-1}^n \to I\!R$ are the two functions that allow us to represent the stock and bank account holdings at time t_{k-1}^n in terms of the fundamentals U_{k-1}^n as

$$\xi_{k-1}^n = \xi^n\left(t_{k-1}^n, U_{k-1}^n\right) \quad \text{and} \quad b_{k-1}^n = b^n\left(t_{k-1}^n, U_{k-1}^n\right), \tag{4.8}$$

respectively, and rewrite (4.6) in terms of these shorthands. Hence the self-financing condition implies that the possible stock and cash holdings (4.7), which the potential replicating strategy prescribes at time t_{k-1}^n depending on the particular realization $u_{(k-1)i}^n$ of the fundamentals U_{k-1}^n, have to satisfy

$$b_{(k-1)i}^n = b_{k(i\pm1)}^n + \left(\xi_{k(i\pm1)}^n - \xi_{(k-1)i}^n\right) S_\mu\left(t_k^n, u_{k(i\pm1)}^n, \xi_{(k-1)i}^n, \xi_{k(i\pm1)}^n\right) \quad \text{for all } i \in \mathcal{I}_{k-1}. \tag{4.9}$$

If we subtract the two equations in (4.9) from each other we get for all $i \in \mathcal{I}_{k-1}$:

$$\begin{aligned} 0 = b_{k(i+1)}^n - b_{k(i-1)}^n &+ \left(\xi_{k(i+1)}^n - \xi_{(k-1)i}^n\right) S_\mu\left(t_k^n, u_{k(i+1)}^n, \xi_{(k-1)i}^n, \xi_{k(i+1)}^n\right) \\ &+ \left(\xi_{(k-1)i}^n - \xi_{k(i-1)}^n\right) S_\mu\left(t_k^n, u_{k(i-1)}^n, \xi_{(k-1)i}^n, \xi_{k(i-1)}^n\right). \end{aligned} \tag{4.10}$$

This is a fixed point equation for $\xi_{(k-1)i}^n$, and the derivation of this fixed point equation did not employ the star-convexity at all.

Remark. In the particular case where the price-determining measure μ of the price system (ψ, μ) is the Dirac measure δ_1 concentrated in 1, such that the equilibrium price always immediately adjusts to an order of the large investor before it is executed, the large investor price function $S_\mu : [0,T] \times I\!R^3 \to I\!R$ of (3.2) simplifies to $S_\mu\big(t, u, \xi_1, \xi_2\big) = \psi(t, u, \xi_2)$ for all $(t, u, \xi_1, \xi_2) \in [0,T] \times I\!R^3$. If $\psi(t_k^n, u_{k(i+1)}^n, \xi_{k(i+1)}^n) - \psi(t_k^n, u_{k(i-1)}^n, \xi_{k(i-1)}^n)$ is positive, the equation (4.10) can be transformed into an explicit equation for $\xi_{(k-1)i}^n$, and it becomes

$$\xi_{(k-1)i}^n = \frac{b_{k(i+1)}^n + \xi_{k(i+1)}^n \psi(t_k^n, u_{k(i+1)}^n, \xi_{k(i+1)}^n) - \big(b_{k(i-1)}^n + \xi_{k(i-1)}^n \psi(t_k^n, u_{k(i-1)}^n, \xi_{k(i-1)}^n)\big)}{\psi(t_k^n, u_{k(i+1)}^n, \xi_{k(i+1)}^n) - \psi(t_k^n, u_{k(i-1)}^n, \xi_{k(i-1)}^n)}$$

for all $i \in \mathcal{I}_{k-1}$. The synchronous market condition of Jarrow (1994) implicitly imposes a price mechanism which corresponds to a price-determining measure $\mu = \delta_1$. Whenever the market does only perceive the large investor's stock position, but not his position in the option contract, Jarrow derives the replication strategy of a contingent claim in his two-step binomial model by equivalent formulæ. Also Frey's (1998) model in a continuous-time framework implicitly uses a price system (ψ,μ) with a price-determining measure $\mu=\delta_1$, since it assumes that an order of the large investor is executed at the equilibrium price which already reflects the new stock position of the large investor. Though the fixed point problem becomes explicit for the particular choice of the price-determining measure which has been used in the literature, for our general analysis we need to consider more general price-determining measures, and thus we proceed in finding solutions for the general fixed point problem (4.10). $\square$

Our next important step is to find conditions under which (4.10) has a solution for all $i\in\mathcal{I}_{k-1}$. If such solutions exist we can use (4.9) as a definition for the possible cash holdings $\{b^n_{(k-1)i}\}_{i\in\mathcal{I}_{k-1}}$, and the portfolio $\left(\xi^n_{k-1},b^n_{k-1}\right)$ given by (4.7) and (4.8) satisfies the self-financing condition (4.6). Hence we can use this portfolio at time t^n_{k-1} as part of the replicating strategy (ξ^n,b^n) we want to construct.

The next lemma states conditions under which the fixed point equation (4.10) has a solution. Under some reasonable assumptions on the loss structure implied by the price system (ψ,μ) we can even show that the portfolio $\left(\xi^n_{k-1},b^n_{k-1}\right)$ inherits the star-convexity of the portfolio $\left(\xi^n_k,b^n_k\right)$, which becomes important if we want to repeat the preceding argument to construct the portfolio $\left(\xi^n_{k-2},b^n_{k-2}\right)$ of the replicating strategy (ξ^n,b^n) as well.

Proposition 1.33. *Let us assume that the equilibrium price function $\psi:[0,T]\times I\!R^2\to I\!R$, $(t,u,\xi)\mapsto\psi(t,u,\xi)$, is continuous in ξ, and suppose that for some $1\le k\le\lceil nT\rceil$ the portfolio (ξ^n_k,b^n_k) is star-convex. Then there exists some function $\xi^n\left(t^n_{k-1},\,\cdot\,\right):\mathcal{U}^n_{k-1}\to I\!R$ with*

$$\xi^n_{(k-1)i}:=\xi^n\left(t^n_{k-1},u^n_{(k-1)i}\right)\in\left[\xi^n_{k(i-1)},\xi^n_{k(i+1)}\right]\quad\text{for all } i\in\mathcal{I}_{k-1}\tag{4.11}$$

which solves (4.10).

Moreover, if the price system (ψ,μ) implies a natural loss structure and if for all $\xi_1,\xi_2\in I\!R$ and $i\in\mathcal{I}_{k-2}$ the large investor price function $S_\mu:[0,T]\times I\!R^3\to I\!R$ satisfies

$$S_\mu\left(t^n_{k-1},u^n_{(k-1)(i-1)},\xi_1,\xi_2\right)\le S_\mu\left(t^n_k,u^n_{ki},\xi_1,\xi_2\right)\le S_\mu\left(t^n_{k-1},u^n_{(k-1)(i+1)},\xi_1,\xi_2\right),\tag{4.12}$$

then the portfolio $\left(\xi^n_{k-1},b^n_{k-1}\right)$ at time t^n_{k-1} implied by (4.10) *and* (4.9) *is again star-convex.*

Proof. For the proof of the first statement, let us fix $1\le k\le n$ and $i\in\mathcal{I}_{k-1}$, and define the function $g^n_{ki}:\left[\xi^n_{k(i-1)},\xi^n_{k(i+1)}\right]\to I\!R$ by

$$\begin{aligned}g^n_{ki}(\xi):=b^n_{k(i+1)}-b^n_{k(i-1)}&+\left(\xi^n_{k(i+1)}-\xi\right)S_\mu\left(t^n_k,u^n_{k(i+1)},\xi,\xi^n_{k(i+1)}\right)\\&+\left(\xi-\xi^n_{k(i-1)}\right)S_\mu\left(t^n_k,u^n_{k(i-1)},\xi,\xi^n_{k(i-1)}\right)\end{aligned}\tag{4.13}$$

for all $\xi\in\left[\xi^n_{k(i-1)},\xi^n_{k(i+1)}\right]$. Since $\left(\xi^n_k,b^n_k\right)$ is star-convex, we get by (4.2)

$$g^n_{ki}\left(\xi^n_{k(i-1)}\right)=b^n_{k(i+1)}-b^n_{k(i-1)}+\left(\xi^n_{k(i+1)}-\xi^n_{k(i-1)}\right)S_\mu\left(t^n_k,u^n_{k(i+1)},\xi^n_{k(i-1)},\xi^n_{k(i+1)}\right)\ge 0$$

and

$$g^n_{ki}\left(\xi^n_{k(i+1)}\right)=b^n_{k(i+1)}-b^n_{k(i-1)}+\left(\xi^n_{k(i+1)}-\xi^n_{k(i-1)}\right)S_\mu\left(t^n_k,u^n_{k(i-1)},\xi^n_{k(i+1)},\xi^n_{k(i-1)}\right)\le 0.$$

Now ψ is continuous in its third component, hence g^n_{ki} is continuous as well, and there has to be some $\xi\in\left[\xi^n_{k(i-1)},\xi^n_{k(i+1)}\right]$ with $g^n_{ki}(\xi)=0$, i.e. the fixed point problem (4.10) has at least one solution $\xi^n_{(k-1)i}$. Since such a solution exist for every $i\in I_{k-1}$ the first statement of Proposition 1.33 is shown.

The implied portfolio $\left(\xi^n_{k-1}, b^n_{k-1}\right)$ at time t^n_{k-1} is convex, since for any $j \in \mathcal{I}_{k-2}$ an application of (4.11) to $i = j \pm 1$ yields

$$\xi^n_{(k-1)(j+1)} \geq \xi^n_{kj} \geq \xi^n_{(k-1)(j-1)}.$$

Now let us suppose that the assumptions of the second statement in Proposition 1.33 hold. We then have to show that the portfolio $\left(\xi^n_{k-1}, b^n_{k-1}\right)$ at time t^n_{k-1} inherits its star-convexity from $\left(\xi^n_k, b^n_k\right)$. By the two inequalities (4.2) in Definition 1.32 this requires to show that for all $i \in \mathcal{I}_{l-1}$ the two inequalities

$$\left(\xi^n_{l(i+1)} - \xi^n_{l(i-1)}\right) S_\mu\left(t^n_l, u^n_{l(i+1)}, \xi^n_{l(i-1)}, \xi^n_{l(i+1)}\right) + b^n_{l(i+1)} - b^n_{l(i-1)} \geq 0 \tag{4.14}$$

and

$$\left(\xi^n_{l(i+1)} - \xi^n_{l(i-1)}\right) S_\mu\left(t^n_l, u^n_{l(i-1)}, \xi^n_{l(i+1)}, \xi^n_{l(i-1)}\right) + b^n_{l(i+1)} - b^n_{l(i-1)} \leq 0 \tag{4.15}$$

hold, where we have set for expository reasons $l = k-1$. If $l = 0$ there is nothing to show, since every portfolio at time $t^n_0 = 0$ is star-convex. Thus we may assume without loss of generality $l \geq 1$, and fix some $i \in \mathcal{I}_{l-1}$. In this case we obtain from an application of both inequalities in (4.9):

$$\begin{aligned} b^n_{l(i-1)} - b^n_{l(i+1)} = b^n_{(l+1)i} &+ \left(\xi^n_{(l+1)i} - \xi^n_{l(i-1)}\right) S_\mu\left(t^n_{l+1}, u^n_{(l+1)i}, \xi^n_{l(i-1)}, \xi^n_{(l+1)i}\right) \\ &- b^n_{(l+1)i} - \left(\xi^n_{(l+1)i} - \xi^n_{l(i+1)}\right) S_\mu\left(t^n_{l+1}, u^n_{(l+1)i}, \xi^n_{l(i+1)}, \xi^n_{(l+1)i}\right). \end{aligned} \tag{4.16}$$

Hence the left-hand side of inequality (4.14) can be rewritten as

$$\left(\xi^n_{l(i+1)} - \xi^n_{l(i-1)}\right) S_\mu\left(t^n_l, u^n_{l(i+1)}, \xi^n_{l(i-1)}, \xi^n_{l(i+1)}\right) + b^n_{l(i+1)} - b^n_{l(i-1)} = I + II,$$

where the terms I and II are given by

$$I := \left(\xi^n_{l(i+1)} - \xi^n_{l(i-1)}\right)\Big(S_\mu\left(t^n_l, u^n_{l(i+1)}, \xi^n_{l(i-1)}, \xi^n_{l(i+1)}\right) - S_\mu\left(t^n_{l+1}, u^n_{(l+1)i}, \xi^n_{l(i-1)}, \xi^n_{l(i+1)}\right)\Big)$$

and

$$\begin{aligned} II := &\left(\xi^n_{l(i+1)} - \xi^n_{l(i-1)}\right) S_\mu\left(t^n_{l+1}, u^n_{(l+1)i}, \xi^n_{l(i-1)}, \xi^n_{l(i+1)}\right) \\ &- \left(\xi^n_{l(i+1)} - \xi^n_{(l+1)i}\right) S_\mu\left(t^n_{l+1}, u^n_{(l+1)i}, \xi^n_{l(i+1)}, \xi^n_{(l+1)i}\right) \\ &- \left(\xi^n_{(l+1)i} - \xi^n_{l(i-1)}\right) S_\mu\left(t^n_{l+1}, u^n_{(l+1)i}, \xi^n_{l(i-1)}, \xi^n_{(l+1)i}\right) \end{aligned}$$

Because of the natural loss structure we can apply Lemma 1.11 with

$$\xi = \xi^n_{l(i-1)}, \quad \alpha = \xi^n_{(l+1)i} - \xi^n_{l(i-1)} \geq 0, \quad \text{and} \quad \beta = \xi^n_{l(i+1)} - \xi^n_{(l+1)i} \geq 0$$

in order to conclude that $II \geq 0$, and due to $\alpha + \beta \geq 0$ the second inequality in (4.12) implies $I \geq 0$ as well, hence we have shown (4.14).
Similarly, we can write the left-hand side of (4.15) as

$$\left(\xi^n_{l(i+1)} - \xi^n_{l(i-1)}\right) S_\mu\left(t^n_l, u^n_{l(i-1)}, \xi^n_{l(i+1)}, \xi^n_{l(i-1)}\right) + b^n_{l(i+1)} - b^n_{l(i-1)} = III + IV,$$

where the terms III and IV are given by

$$III := \left(\xi^n_{l(i+1)} - \xi^n_{l(i-1)}\right)\Big(S_\mu\left(t^n_l, u^n_{l(i-1)}, \xi^n_{l(i+1)}, \xi^n_{l(i-1)}\right) - S_\mu\left(t^n_{l+1}, u^n_{(l+1)i}, \xi^n_{l(i+1)}, \xi^n_{l(i-1)}\right)\Big)$$

and

$$\begin{aligned} IV := {} & \big(\xi^n_{l(i+1)} - \xi^n_{l(i-1)}\big) S_\mu\big(t^n_{l+1}, u^n_{(l+1)i}, \xi^n_{l(i+1)}, \xi^n_{l(i-1)}\big) \\ & - \big(\xi^n_{l(i+1)} - \xi^n_{(l+1)i}\big) S_\mu\big(t^n_{l+1}, u^n_{(l+1)i}, \xi^n_{l(i+1)}, \xi^n_{(l+1)i}\big) \\ & - \big(\xi^n_{(l+1)i} - \xi^n_{l(i-1)}\big) S_\mu\big(t^n_{l+1}, u^n_{(l+1)i}, \xi^n_{l(i-1)}, \xi^n_{(l+1)i}\big), \end{aligned}$$

and we can conclude from the first inequality in (4.12) and Lemma 1.11 applied to

$$\xi = \xi^n_{l(i+1)}, \quad \alpha = -\big(\xi^n_{l(i+1)} - \xi^n_{(l+1)i}\big) \le 0 \quad \text{and} \quad \beta = -\big(\xi^n_{(l+1)i} - \xi^n_{l(i-1)}\big) \le 0$$

that $III \le 0$ and $IV \le 0$, respectively, hence (4.15) holds as well. Since $i \in \mathcal{I}_{l-1}$ can be chosen arbitrarily, we thus conclude that the portfolio $\big(\xi^n_l, b^n_l\big)$ is indeed star-convex. **q.e.d.**

Remark. In Section 1.5 we will give non-trivial examples for price systems (ψ, μ) which satisfy the assumptions of Proposition 1.33, at least for all sufficiently large $n \in \mathbb{N}$.

In general there might exist more than one solution $\xi^n_{(k-1)i} \in \big[\xi^n_{k(i-1)}, \xi^n_{k(i+1)}\big]$ to the fixed point problem (4.10). The next proposition states that under some intuitively convincing assumptions on the loss structure there is at most one solution to the fixed point problem.

Proposition 1.34. *Assume again that the equilibrium price function $\psi : [0,T] \times \mathbb{R}^2 \to \mathbb{R}$, $(t,u,\xi) \mapsto \psi(t,u,\xi)$, is continuous in ξ, but now suppose also that the price system (ψ, μ) implies nondecreasing total transaction losses. Then for any convex portfolio $\big(\xi^n_k, b^n_k\big)$ at some time t^n_k with $1 \le k \le \lceil nT \rceil$ and for all $i \in \mathcal{I}_{k-1}$ a solution $\xi^n_{(k-1)i} \in \big[\xi^n_{k(i-1)}, \xi^n_{k(i+1)}\big]$ to the fixed point equation* (4.10) *is always unique.*

Proof. Let us fix $1 \le k \le n$ and $i \in \mathcal{I}_{k-1}$ and recall from the proof of Proposition 1.33 the function $g^n_{ki} : \big[\xi^n_{k(i-1)}, \xi^n_{k(i+1)}\big] \to \mathbb{R}$ given by (4.13). Now it is easy to see that the representation (2.1) of the static transaction loss function can be transferred to the dynamic transaction loss function of (3.4), hence (4.13) can be rewritten in terms of the transaction loss function $c_\mu : [0,T] \times \mathbb{R}^3 \to \mathbb{R}$ and the benchmark value function $S^* : [0,T] \times \mathbb{R}^3 \to \mathbb{R}$ as

$$\begin{aligned} g^n_{ki}(\xi) = {} & b^n_{k(i+1)} - b^n_{k(i-1)} + c_\mu\big(t^n_k, u^n_{k(i+1)}, \xi, \xi^n_{k(i+1)}\big) - c_\mu\big(t^n_k, u^n_{k(i-1)}, \xi, \xi^n_{k(i-1)}\big) \\ & + \big(\xi^n_{k(i+1)} - \xi\big) S^*\big(t^n_k, u^n_{k(i+1)}, \xi, \xi^n_{k(i+1)}\big) + \big(\xi - \xi^n_{k(i-1)}\big) S^*\big(t^n_k, u^n_{k(i-1)}, \xi, \xi^n_{k(i-1)}\big). \end{aligned}$$

Since S_μ implies nondecreasing total transaction losses, $\xi \mapsto c_\mu\big(t^n_k, u^n_{k(i+1)}, \xi, \xi^n_{k(i+1)}\big)$ decreases and $\xi \mapsto c_\mu\big(t^n_k, u^n_{k(i-1)}, \xi, \xi^n_{k(i-1)}\big)$ increases on $\big[\xi^n_{k(i-1)}, \xi^n_{k(i+1)}\big]$. Thus, if we can show that the function $h^n_{ki} : \big[\xi^n_{k(i-1)}, \xi^n_{k(i+1)}\big] \to \mathbb{R}$ given by

$$h^n_{ki}(\xi) = \big(\xi^n_{k(i+1)} - \xi\big) S^*\big(t^n_k, u^n_{k(i+1)}, \xi, \xi^n_{k(i+1)}\big) + \big(\xi - \xi^n_{k(i-1)}\big) S^*\big(t^n_k, u^n_{k(i-1)}, \xi, \xi^n_{k(i-1)}\big)$$

for all $\xi \in \big[\xi^n_{k(i-1)}, \xi^n_{k(i+1)}\big]$ is strictly decreasing, then g^n_{ki} strictly decreases on $\big[\xi^n_{k(i-1)}, \xi^n_{k(i+1)}\big]$ as well, and thus there can exist at most one $\xi \in \big[\xi^n_{k(i-1)}, \xi^n_{k(i+1)}\big]$ with $g^n_{ki}(\xi) = 0$.

In order to show that h^n_{ki} is indeed strictly decreasing, let us first note that by the time-space dependent analogue of (1.26) and the definition of the loss-free liquidation price function $\bar{S} : [0,T] \times \mathbb{R}^2 \to \mathbb{R}$ in (3.14) we can rewrite $h^n_{ki}(\xi)$ for all $\xi \in \big[\xi^n_{k(i-1)}, \xi^n_{k(i+1)}\big]$ as

$$\begin{aligned} h^n_{ki}(\xi) = {} & \xi^n_{k(i+1)} \bar{S}\big(t^n_k, u^n_{k(i+1)}, \xi^n_{k(i+1)}\big) - \xi^n_{k(i-1)} \bar{S}\big(t^n_k, u^n_{k(i-1)}, \xi^n_{k(i-1)}\big) \\ & - \xi\Big(\bar{S}\big(t^n_k, u^n_{k(i+1)}, \xi\big) - \bar{S}\big(t^n_k, u^n_{k(i-1)}, \xi\big)\Big). \end{aligned}$$

The first two terms of this expression are constant in ξ, and since Lebesgue and Riemann integrals over continuous functions on a finite interval coincide, we can rewrite the third term of this expressions as

$$\begin{aligned}\xi\Big(\bar S\big(t_k^n,u_{k(i+1)}^n,\xi\big)-\bar S\big(t_k^n,u_{k(i-1)}^n,\xi\big)\Big)&=\xi\int_0^1\psi\big(t_k^n,u_{k(i+1)}^n,\theta\xi\big)-\psi\big(t_k^n,u_{k(i-1)}^n,\theta\xi\big)\lambda(d\theta)\\&=\xi\int_0^1\psi\big(t_k^n,u_{k(i+1)}^n,\theta\xi\big)-\psi\big(t_k^n,u_{k(i-1)}^n,\theta\xi\big)d\theta\\&=\int_0^\xi\psi\big(t_k^n,u_{k(i+1)}^n,x\big)-\psi\big(t_k^n,u_{k(i-1)}^n,x\big)dx.\end{aligned}$$

But due to our general assumption that the equilibrium price function $\psi:[0,T]\times I\!R^2\to I\!R$, $(t,u,\xi)\mapsto\psi(t,u,\xi)$ is strictly increasing in u, it follows that the integrand of the last integral is strictly positive for all possible values $x\in I\!R$, thus the integral itself is strictly increasing, and hence the function $h_{ki}^n:\big[\xi_{k(i-1)}^n,\xi_{k(i+1)}^n\big]\to I\!R$, $\xi\mapsto h_{ki}^n(\xi)$ strictly decreases, which was left to prove. **q.e.d.**

In our attempt to find a replicating strategy for a star-convex contingent claim Proposition 1.33 has shown that if for some $1\le k\le\lceil nT\rceil$ we have already constructed star-convex portfolios $\big\{\big(\xi_j^n,b_j^n\big)\big\}_{k\le j\le\lceil nT\rceil}$ as part of a potential replicating strategy $\big(\xi^n,b^n\big)$, then we can also construct a star-convex portfolio $\big(\xi_{k-1}^n,b_{k-1}^n\big)$ at time t_{k-1}^n of such a (potential) replicating strategy – at least as long as the conditions given in Proposition 1.33 are satisfied. Having found this portfolio we can repeat our arguments until we finally have constructed a full sequence of star-convex portfolios at all trading times, from $t_{\lceil nT\rceil}^n=T$ down to $t_0^n=0$. But in that case we have constructed a full self-financing trading strategy (ξ^n,b^n) which replicates (ξ_n,b_n). Since the condition (4.12), which assures that $\big(\xi_{k-1}^n,b_{k-1}^n\big)$ inherits the star-convexity from $\big(\xi_k^n,b_k^n\big)$, does not depend on the portfolio $\big(\xi_k^n,b_k^n\big)$ itself, the following corollary is an immediate consequence of Propositions 1.33 and 1.34:

Corollary 1.35. *Let us assume that the large investor price system (ψ,μ) satisfies the following properties:*

(i) *The equilibrium price function $\psi:[0,T]\times I\!R^2\to I\!R$, $(t,u,\xi)\mapsto\psi(t,u,\xi)$ is continuous in ξ.*

(ii) *The price system (ψ,μ) implies a natural cost structure.*

(iii) *For all $1\le k\le\lceil nT\rceil$, all $i\in\mathcal{I}_{k-1}$ and all $\xi_1,\xi_2\in I\!R$ the large investor price function $S_\mu:[0,T]\times I\!R^3\to I\!R$ generated by (ψ,μ) satisfies (4.12).*

Then for every star-convex contingent claim $\big(\xi_n,b_n\big)$ there exists a self-financing trading strategy $\big(\xi^n,b^n\big)=\big\{\big(\xi_k^n,b_k^n\big)\big\}_{0\le k\le\lceil nT\rceil}$ which replicates $\big(\xi_n,b_n\big)$. Moreover, if (ψ,μ) also implies nondecreasing total transaction losses, then the replicating strategy is unique among all strategies which satisfy

$$\xi_{(k+1)(i-1)}^n\le\xi_{ki}^n\le\xi_{(k+1)(i+1)}^n\quad\text{for all }(k,i)\in I(\lceil nT\rceil-1).$$

Remark. As in the first part of the proof of Proposition 1.33 it is easy to see that if the portfolio $\big(\xi_k^n,b_k^n\big)$ is star-concave, then there exists a portfolio $\big(\xi_{k-1}^n,b_{k-1}^n\big)$ which satisfies the self-financing condition (4.9). Unfortunately, if S_μ implies a natural loss structure then the star-concavity of $\big(\xi_k^n,b_k^n\big)$ is not necessarily inherited by $\big(\xi_{k-1}^n,b_{k-1}^n\big)$. We will discuss star-concave portfolios later in Section 1.4.4. □

Let us first show that star-convex contingent claims also appear in certain problems where the large investor wants to replicate his paper wealth.

1.4.3 Paper Value Replication

Most of the replication problems in small investor markets are written in terms of value replication, and even Frey (1998) states his replication problem in a continuous large investor market as a problem to achieve a certain paper value at maturity. In order to clarify the relation of these replication problems, and especially of Frey's problem, to our replication problem of Section 1.4.2, we will now present a link between the replication of a certain option value by the paper value at maturity of a self-financing portfolio strategy on the one hand and the replication of star-convex contingent claims as considered in Section 1.4.2 on the other hand. This section might be skipped on first reading.

Since the large investor price depends on the large investor's stock holdings, the stock holdings at and immediately before maturity needed to replicate the option value can only be derived as a solution of a multidimensional fixed point equation. If the option value is given as a convex function of the large investor price at maturity, we will prove the existence of a solution to this fixed point problem under some regularity conditions on the price system (ψ, μ) and the option's payoff function. Once we have also shown that the implied portfolio immediately before maturity is star-convex, we can proceed as in Section 1.4.2 to construct a trading strategy which replicates the option value.

For ease of presentation we will assume in this section that $T = 1$ so that the index $\lceil nT \rceil$ of the different processes and shorthands simplifies to $\lceil nT \rceil = n$.

We now suppose that the large investor wants to find a self-financing strategy (ξ^n, b^n) such that in any state of the world the induced paper value at maturity coincides with the value of a certain option which is determined as some function $h : \mathbb{R} \to \mathbb{R}$ of the large investor price at time T. For example the large investor might be an investment fund that has to achieve a certain target at time T, which is measured by the stock price at this particular date. The large investor price $S_\mu\big(T, U_n^n, \xi_{n-1}^n, \xi_n^n\big)$ at time T depends not only on the fundamentals U_n^n at this time, but also on the ξ_{n-1}^n and ξ_n^n shares of stock held by the large investor at the time points t_{n-1}^n and $t_n^n = T$, respectively, and so we have to look for a self-financing trading strategy (ξ^n, b^n) such that the corresponding paper value V_n^n at time T satisfies $V_n^n = h\big(S_\mu\big(T, U_n^n, \xi_{n-1}^n, \xi_n^n\big)\big)$ in order to replicate the option value by the paper value of the large investor's portfolio strategy. Due to the definition of the paper value V_n^n this means that we have to find a self-financing trading strategy (ξ^n, b^n) with

$$\xi_n^n S_\mu\big(T, U_n^n, \xi_{n-1}^n, \xi_n^n\big) + b_n^n = h\Big(S_\mu\big(T, U_n^n, \xi_{n-1}^n, \xi_n^n\big)\Big). \tag{4.17}$$

Under suitable conditions on the price system (ψ, μ) and the convex option payoff $h : \mathbb{R} \to \mathbb{R}$ we will derive in three steps a self-financing strategy (ξ^n, b^n) which satisfies the final condition (4.17). At first, we will simultaneously show the existence of two portfolios $\big(\xi_n^n, b_n^n\big)$ and $\big(\xi_{n-1}^n, b_{n-1}^n\big)$ at and immediately before time $t_n^n = T$, such that (4.17) holds and such that at time T the portfolio $\big(\xi_n^n, b_n^n\big)$ can be generated from the portfolio $\big(\xi_{n-1}^n, b_{n-1}^n\big)$ in a self-financing fashion, meaning that

$$b_{n-1}^n = b_n^n + \big(\xi_n^n - \xi_{n-1}^n\big) S_\mu\big(T, U_n^n, \xi_{n-1}^n, \xi_n^n\big). \tag{4.18}$$

Then we will show that the portfolio $\big(\xi_{n-1}^n, b_{n-1}^n\big)$ at time t_{n-1}^n is star-convex, and finally use our results of Section 1.4.2 to construct a full self-financing trading strategy with the final portfolios $\big(\xi_{n-1}^n, b_{n-1}^n\big)$ and $\big(\xi_n^n, b_n^n\big)$ at time t_{n-1}^n and t_n^n, respectively.

The first step is the most demanding step, since it involves an existence result on a high-dimensional fixed point problem. Instead of the original fixed point problem (4.17) we will solve in the following lemma a related fixed point problem, which does not involve the cash position b_n^n. We write it in terms of the possible realizations as introduced by the shorthands (3.12).

Proposition 1.36. *Let us assume that the equilibrium function* $\psi : [0,T] \times I\!R^2 \to I\!R$, $(t,u,\xi) \mapsto \psi(t,u,\xi)$ *of the price system* (ψ,μ) *is nondecreasing in* ξ *and that the price-determining measure* μ *is a probability measure on* $[0,1]$*. Moreover, suppose that the payoff function* $h : I\!R \to I\!R$ *in* (4.17) *is convex and that its left- and right-hand first derivatives are bounded. Then there exist some nondecreasing functions* $\xi^n\big(t^n_{n-1},\cdot\big) : \mathcal{U}^n_{n-1} \to I\!R$ *and* $\xi^n\big(t^n_n,\cdot\big) : \mathcal{U}^n_{n-1} \to I\!R$*, such that*

$$\xi^n_{(n-1)i} = \frac{h\big(S_\mu\big(T,u^n_{n(i+1)},\xi^n_{(n-1)i},\xi^n_{n(i+1)}\big)\big) - h\big(S_\mu\big(T,u^n_{n(i-1)},\xi^n_{(n-1)i},\xi^n_{n(i-1)}\big)\big)}{S_\mu\big(T,u^n_{n(i+1)},\xi^n_{(n-1)i},\xi^n_{n(i+1)}\big) - S_\mu\big(T,u^n_{n(i-1)},\xi^n_{(n-1)i},\xi^n_{n(i-1)}\big)} \tag{4.19}$$

for all $i \in \mathcal{I}_{n-1}$,

$$\xi^n_{n(\pm n)} = h'\Big(\psi\big(T,u^n_{n(\pm n)},\xi^n_{n(\pm n)}\big)+\Big), \tag{4.20}$$

and

$$\xi^n_{ni} = \begin{cases} \frac{h(S_\mu(T,u^n_{ni},\xi^n_{(n-1)(i+1)},\xi^n_{ni})) - h(S_\mu(T,u^n_{ni},\xi^n_{(n-1)(i-1)},\xi^n_{ni}))}{S_\mu(T,u^n_{ni},\xi^n_{(n-1)(i+1)},\xi^n_{ni}) - S_\mu(T,u^n_{ni},\xi^n_{(n-1)(i-1)},\xi^n_{ni})} & \text{if denominator} \neq 0 \\ h'\Big(\psi\big(T,u^n_{ni},\xi^n_{ni}\big)+\Big) & \text{otherwise.} \end{cases} \tag{4.21}$$

for all $i \in \mathcal{I}_{n-2}$.

The proof of the proposition will be based on Brouwer's fixed point theorem, thus let us briefly restate it:

Brouwer's Fixed Point Theorem . *Let* D *be a non-empty compact convex subset of a finite-dimensional normed space. If* L *is a continuous operator which maps* D *into itself, then* L *has at least one fixed point.*

For a proof of Brouwer's fixed point theorem, see for example (Aliprantis and Border, 1999, Chapter 16.9).

Proof of Proposition 1.36. Let us choose some constant $M > 0$ which bounds the left- and right-hand derivatives of the option payoff function $h : I\!R \to I\!R$, i.e. M satisfies $\big|h'(x-)\big| \le M$ and $\big|h'(x+)\big| \le M$ for all $x \in I\!R$. The left- and right-hand derivatives of h exist on the whole real line, since for any convex function there is at most a countable number of points where the function is not differentiable. Then define the set D of ranked vectors (α_{n-1},α_n) by

$$D := \Big\{\big(\alpha_{n-1},\alpha_n\big) \in \mathcal{D} \,\Big|\, -M \le \alpha_{n(-n)} \le \alpha_{(n-1)(1-n)} \le \alpha_{n(2-n)} \le \cdots \le \alpha_{(n-1)(n-1)} \le \alpha_{nn} \le M\Big\}$$

where the set $\mathcal{D}$ of $(2n+1)$-dimensional vectors is given by

$$\mathcal{D} = \Big\{\big(\alpha_{n-1},\alpha_n\big) \,\Big|\, \alpha_k = \big(\alpha_{k(-k)},\alpha_{k(2-k)},\ldots,\alpha_{kk}\big) \in I\!R^{k+1} \text{ for } k \in \{n-1,n\}\Big\}.$$

The set D is not empty, and it is obviously closed, bounded, and convex. Now define the mapping $f : D \to \mathcal{D}$ by setting

$$f_{(n-1)i}\big(\alpha_{n-1},\alpha_n\big) := \frac{h\big(S_\mu\big(T,u^n_{n(i+1)},\alpha_{(n-1)i},\alpha_{n(i+1)}\big)\big) - h\big(S_\mu\big(T,u^n_{n(i-1)},\alpha_{(n-1)i},\alpha_{n(i-1)}\big)\big)}{S_\mu\big(T,u^n_{n(i+1)},\alpha_{(n-1)i},\alpha_{n(i+1)}\big) - S_\mu\big(T,u^n_{n(i-1)},\alpha_{(n-1)i},\alpha_{n(i-1)}\big)}$$

for all $i \in I_{n-1}$,

$$f_{ni}\big(\alpha_{n-1},\alpha_n\big) := \begin{cases} \frac{h(S_\mu(T,u^n_{ni},\alpha_{(n-1)(i+1)},\alpha_{ni})) - h(S_\mu(T,u^n_{ni},\alpha_{(n-1)(i-1)},\alpha_{ni}))}{S_\mu(T,u^n_{ni},\alpha_{(n-1)(i+1)},\alpha_{ni}) - S_\mu(T,u^n_{ni},\alpha_{(n-1)(i-1)},\alpha_{ni})} & \text{if denominator} \neq 0 \\ h'\Big(\psi\big(T,u^n_{ni},\alpha_{ni}\big)+\Big) & \text{otherwise} \end{cases}$$

for all $i \in I_{n-2}$, and $f_{n(\pm n)}\big(\alpha_{n-1},\alpha_n\big) := h'\big(\psi(T, u^n_{n(\pm n)}, \alpha_{n(\pm n)})+\big)$. If we can show that $f\big(\alpha_{n-1},\alpha_n\big) \in D$ for all $\big(\alpha_{n-1},\alpha_n\big) \in D$, Brouwer's fixed point theorem will yield the desired assertion.

As a first step to show that f maps D into itself let us note that the definition of the large investor price function $S_\mu : [0,T] \times I\!R^3 \to I\!R$ in (3.2), the assumed monotonicity of the function $\xi \mapsto \psi(t,u,\xi)$, and the concentration of the probability measure μ on the interval $[0,1]$ imply for all $t \in [0,T]$ and $u, \xi_1, \xi_2 \in I\!R$ the implication

$$\xi_1 \le \xi_2 \le \xi_3 \quad \Rightarrow \quad \begin{cases} S_\mu\big(t,u,\xi_2,\xi_1\big) \le \psi\big(t,u,\xi_2\big) \le S_\mu\big(t,u,\xi_2,\xi_3\big) \\ \text{and} \\ S_\mu\big(t,u,\xi_1,\xi_2\big) \le \psi\big(t,u,\xi_2\big) \le S_\mu\big(t,u,\xi_3,\xi_2\big). \end{cases} \tag{4.22}$$

Let us now fix some arbitrary $\big(\alpha_{n-1},\alpha_n\big) \in D$. Then $f\big(\alpha_{n-1},\alpha_n\big) \in D$ if and only if

$$\begin{array}{rcccl} -M & \le & f_{n(-n)}\big(\alpha_{n-1},\alpha_n\big) & \le & f_{(n-1)(1-n)}\big(\alpha_{n-1},\alpha_n\big), \\ f_{(n-1)(n-1)}\big(\alpha_{n-1},\alpha_n\big) & \le & f_{nn}\big(\alpha_{n-1},\alpha_n\big) & \le & M, \end{array}$$

and if for all $i \in \mathcal{I}_{n-2}$ we have

$$f_{(n-1)(i-1)}\big(\alpha_{n-1},\alpha_n\big) \;\le\; f_{ni}\big(\alpha_{n-1},\alpha_n\big) \;\le\; f_{(n-1)(i+1)}\big(\alpha_{n-1},\alpha_n\big). \tag{4.23}$$

We will start with the proof of (4.23) for all $i \in \mathcal{I}_{n-2}$. For this purpose we fix an arbitrary $i \in \mathcal{I}_{n-2}$ and introduce the shorthands

$$\begin{aligned} \psi_1 &= S_\mu\big(T, u_{n(i-2)}, \alpha_{(n-1)(i-1)}, \alpha_{n(i-2)}\big) \\ \psi_2 &= S_\mu\big(T, u_{ni}, \alpha_{(n-1)(i-1)}, \alpha_{ni}\big) \\ \psi_3 &= S_\mu\big(T, u_{ni}, \alpha_{(n-1)(i+1)}, \alpha_{ni}\big) \\ \psi_4 &= S_\mu\big(T, u_{n(i+2)}, \alpha_{(n-1)(i+1)}, \alpha_{n(i+2)}\big). \end{aligned}$$

Since $\big(\alpha_{n-1},\alpha_n\big) \in D$ implies the ordering

$$\alpha_{n(i-2)} \le \alpha_{(n-1)(i-1)} \le \alpha_{ni} \le \alpha_{(n-1)(i+1)} \le \alpha_{n(i+2)}$$

and since the equilibrium price function $\psi : [0,T] \times I\!R^2 \to I\!R$, $(t,u,\xi) \mapsto \psi(t,u,\xi)$, is strictly increasing in the fundamental value u, we then get from (4.22) that $\psi_1 < \psi_2 \le \psi_3 < \psi_4$. If we even have $\psi_2 < \psi_3$, the convexity of h implies

$$\frac{h(\psi_1) - h(\psi_2)}{\psi_1 - \psi_2} \le \frac{h(\psi_2) - h(\psi_3)}{\psi_2 - \psi_3} \le \frac{h(\psi_3) - h(\psi_4)}{\psi_3 - \psi_4},$$

which is by the definition of $f : D \to \mathcal{D}$ seen to be equivalent to (4.23). In the other case, i.e. if $\psi_1 < \psi_2 = \psi_3 < \psi_4$ we can still deduce from the convexity of h that

$$\begin{aligned} \frac{h(\psi_1) - h(\psi_2)}{\psi_1 - \psi_2} &\le \lim_{\varepsilon \searrow 0} \frac{h(\psi_2) - h(\psi_2 - \varepsilon)}{\psi_2 - (\psi_2 - \varepsilon)} = h'(\psi_2-) = h'(\psi_3-) \\ &\le h'(\psi_3+) = \lim_{\varepsilon \searrow 0} \frac{h(\psi_3 + \varepsilon) - h(\psi_3)}{\psi_3 + \varepsilon - \psi_3} \le \frac{h(\psi_4) - h(\psi_3)}{\psi_4 - \psi_3}, \end{aligned}$$

and hence $\alpha_{(n-1)(i-1)} \le \alpha_{ni} \le \alpha_{(n-1)(i+1)}$ and (4.22) even imply

$$S_\mu\big(T, u_{ni}, \alpha_{(n-1)(i\pm1)}, \alpha_{ni}\big) = \psi\big(T, u_{ni}, \alpha_{ni}\big).$$

Looking once again at the definition of the multi-dimensional function $f : D \to \mathcal{D}$ this shows that (4.23) holds as well if $\psi_2 = \psi_3$, and since $i \in \mathcal{I}_{n-2}$ could be chosen arbitrarily, (4.23) holds for all $i \in \mathcal{I}_{n-2}$.
By similar considerations we can show that $f_{n(-n)}\big(\alpha_{n-1}, \alpha_n\big) \le f_{(n-1)(1-n)}\big(\alpha_{n-1}, \alpha_n\big)$ and $f_{(n-1)(n-1)}\big(\alpha_{n-1}, \alpha_n\big) \le f_{nn}\big(\alpha_{n-1}, \alpha_n\big)$. Finally, the two bounds $-M \le f_{n(-n)}\big(\alpha_{n-1}, \alpha_n\big)$ and $f_{nn}\big(\alpha_{n-1}, \alpha_n\big) \le M$ follow directly from the definition of M as a bound on the derivative of $h : I\!R \to I\!R$.
Thus, we have proved that $f : D \to \mathcal{D}$ maps D into itself, and hence we can indeed apply Brouwer's fixed point theorem to conclude that there exists at least one fixed point $\big(\alpha^*_{n-1}, \alpha^*_n\big) \in D$ of f. This enables us to define the functions $\xi^n\big(t^n_{n-1}, \cdot\big) : \mathcal{U}^n_{n-1} \to I\!R$ and $\xi^n\big(t^n_n, \cdot\big) : \mathcal{U}^n_n \to I\!R$ by

$$\xi^n_k\big(t^n_k, u^n_{ki}\big) = \alpha^*_{ki} \quad \text{for } k \in \{n-1, n\} \text{ and all } i \in \mathcal{I}_k,$$

and these functions obviously satisfy the equations (4.19) to (4.21) because of the definition of $f : D \to \mathcal{D}$. Moreover, the two functions $\xi^n\big(t^n_{n-1}, \cdot\big)$ and $\xi^n\big(t^n_n, \cdot\big)$ are nondecreasing because of the structure of the set D. This completes our proof of Proposition 1.36. **q.e.d.**

Once we have shown the existence of the two functions $\xi^n\big(t^n_{n-1}, \cdot\big)$ and $\xi^n\big(t^n_n, \cdot\big)$, we can use these functions to define the amount of shares in the two portfolios held by the large investor immediately before and at time $t^n_n = T$ by evaluating the functions at the fundamentals U^n_{n-1} and U^n_n, respectively, and setting $\xi^n_{n-1} = \xi^n\big(t^n_{n-1}, U^n_{n-1}\big)$ and $\xi^n_n = \xi^n\big(t^n_n, U^n_n\big)$.
To complete our definitions at times t^n_{n-1} and t^n_n of the large investor's portfolios $\big(\xi^n_{n-1}, b^n_{n-1}\big)$ and $\big(\xi^n_n, b^n_n\big)$ for which the replication condition (4.17) and the self-financing condition (4.18) hold, we still have to determine the cash holdings b^n_{n-1} and b^n_n at these two points in time. In order to make sure that (4.17) and (4.18) hold in any state of the world, we will define these cash holdings as well as functions of the possible fundamental values at these time points, and introduce at first the function $b^n\big(t^n_n, \cdot\big) : \mathcal{U}^n_n \to I\!R$ by

$$b^n\big(t^n_n, u^n_{ni}\big) = h\Big(S_\mu\big(T, u^n_{ni}, \xi^n_{(n-1)(i+1)}, \xi^n_{ni}\big)\Big) - \xi^n_{ni} S_\mu\big(T, u^n_{ni}, \xi^n_{(n-1)(i+1)}, \xi^n_{ni}\big) \;\text{ for } i \in \mathcal{I}_n \backslash \{n\}$$

and

$$b^n\big(t^n_n, u^n_{ni}\big) = h\Big(S_\mu\big(T, u^n_{ni}, \xi^n_{(n-1)(i-1)}, \xi^n_{ni}\big)\Big) - \xi^n_{ni} S_\mu\big(T, u^n_{ni}, \xi^n_{(n-1)(i-1)}, \xi^n_{ni}\big) \;\text{ for } i \in \mathcal{I}_n \backslash \{-n\}.$$

It can be easily seen that this definition of the function $b^n\big(t^n_n, \cdot\big)$ is consistent, though the definitions of the values $b^n\big(t^n_n, u^n_{ni}\big)$ overlap for all $i \in \mathcal{I}_{n-2} = \mathcal{I}_n \backslash \{-n, n\}$: Because of the particular form of ξ^n_{ni} as given in (4.21), for all $i \in \mathcal{I}_{n-2}$ both definitions of $b^n_n\big(t^n_n, u^n_{ni}\big)$ are equivalent to

$$b^n\big(t^n_n, u^n_{ni}\big) = \begin{cases} \frac{h(S^-_{ni})S^+_{ni} - h(S^+_{ni})S^-_{ni}}{S^+_{ni} - S^-_{ni}} & \text{if } S^+_{ni} \ne S^-_{ni}, \\ h(S^+_{ni}) - h'(S^+_{ni}+)S^+_{ni} & \text{if } S^+_{ni} = S^-_{ni}, \end{cases}$$

where $S^\pm_{ni}$ is used as a shorthand for $S_\mu\big(T, u^n_{ni}, \xi^n_{(n-1)(i\pm1)}, \xi^n_{ni}\big)$.
Similarly, it follows from the definition of the function $\xi^n\big(t^n_{n-1}, \cdot\big)$ by (4.19) that the function $b^n\big(t^n_{n-1}, \cdot\big) : \mathcal{U}^n_{n-1} \to I\!R$ which is for all $i \in \mathcal{I}_{n-1}$ given by

$$b^n\big(t^n_{n-1}, u^n_{(n-1)i}\big) = b^n\big(t^n_n, u^n_{n(i\pm1)}\big) + \big(\xi^n_{n(i\pm1)} - \xi^n_{(n-1)i}\big) S_\mu\big(t^n_n, u^n_{n(i\pm1)}, \xi^n_{(n-1)i}, \xi^n_{n(i\pm1)}\big)$$

is consistently defined despite the double definition for each possible value $b^n\big(t^n_{n-1}, u^n_{(n-1)i}\big)$.

If we now determine the cash holdings of the two portfolios $\left(\xi_n^n, b_n^n\right)$ and $\left(\xi_{n-1}^n, b_{n-1}^n\right)$ at and immediately before time $t_n^n = T$ by $b_n^n = b^n\left(t_n^n, U_n^n\right)$ and $b_{n-1}^n = b^n\left(t_{n-1}^n, U_{n-1}^n\right)$, it follows from Proposition 1.36 that on the one hand the condition (4.17) for the paper value replication holds in any state of the world, and that on the other hand the self-financing condition (4.18) holds as well, i.e. the portfolio $\left(\xi_n^n, b_n^n\right)$ can be financed at time T by the portfolio $\left(\xi_{n-1}^n, b_{n-1}^n\right)$ held by the large investor between time t_{n-1}^n and t_n^n.

Having found portfolios at time t_{n-1}^n and at time $t_n^n = T$, our replication problem (4.17) now basically becomes a replication problem for the portfolio $\left(\xi_{n-1}^n, b_{n-1}^n\right)$ at time t_{n-1}^n. In order to apply our results of Section 1.4.2 to this problem, we have to show that the portfolio $\left(\xi_{n-1}^n, b_{n-1}^n\right)$ is star-convex. The next lemma shows that this is indeed the case under the same additional conditions as in Lemma 1.33.

Lemma 1.37. *In addition to the assumptions of Proposition 1.36 let us suppose that the price system (ψ, μ) implies a natural loss structure and that the large investor price function $S_\mu : [0,T] \times I\!R^3 \to I\!R$ generated by (ψ, μ) satisfies*

$$S_\mu\left(t_{n-1}^n, u_{(n-1)(i-1)}^n, \xi_1, \xi_2\right) \le S_\mu\left(t_n^n, u_{ni}^n, \xi_1, \xi_2\right) \le S_\mu\left(t_{n-1}^n, u_{(n-1)(i+1)}^n, \xi_1, \xi_2\right)$$

for all $i \in \mathcal{I}_{n-2}$ and all $\xi_1, \xi_2 \in I\!R$. Then the portfolio $\left(\xi_{n-1}^n, b_{n-1}^n\right)$ is star-convex.

Proof. The portfolio $\left(\xi_{n-1}^n, b_{n-1}^n\right)$ is convex, since the function $\xi^n\left(t_{n-1}^n, \cdot\right) : \mathcal{U}_{n-1}^n$ is nondecreasing by Proposition 1.36. In order to show that it is even star-convex, we will once again use our shorthands (3.12) and note that the self-financing condition (4.18) implies

$$b_{(n-1)j}^n = b_{n(j\pm1)}^n + \left(\xi_{n(j\pm1)}^n - \xi_{(n-1)j}^n\right) S_\mu\left(t_n^n, u_{n(j\pm1)}^n, \xi_{(n-1)j}^n, \xi_{n(j\pm1)}^n\right) \quad \text{for all } j \in \mathcal{I}_{n-1}.$$

Let us then set $l = n-1$ and use the upper (+) equation for $j = i-1$ and the lower (-) one for $j = i+1$. Then we obtain:

$$\begin{aligned} b_{l(i-1)}^n - b_{l(i+1)}^n = &\left(\xi_{l(i+1)}^n - \xi_{(l+1)i}^n\right) S_\mu\left(t_n^n, u_{(l+1)i}^n, \xi_{l(i+1)}^n, \xi_{(l+1)i}^n\right) \\ &+ \left(\xi_{(l+1)i}^n - \xi_{l(i-1)}^n\right) S_\mu\left(t_n^n, u_{(l+1)i}^n, \xi_{l(i-1)}^n, \xi_{(l+1)i}^n\right) \quad \text{for all } i \in \mathcal{I}_{l-1}. \end{aligned}$$

From here we can proceed exactly as in the proof of Proposition 1.33. Thus, the portfolio $\left(\xi_{n-1}^n, b_{n-1}^n\right)$ at time t_{n-1}^n is indeed star-convex. **q.e.d.**

As the third step of constructing the replicating trading strategy to (4.17), it just remains to find a trading strategy up to the time point t_{n-1}^n which replicates the star-convex portfolio $\left(\xi_{n-1}^n, b_{n-1}^n\right)$ at time t_{n-1}^n. Such conditions were derived in Proposition 1.33, and so we can conclude by an analogue to Corollary 1.35:

Corollary 1.38. *Let us assume that the large investor price system (ψ, μ) satisfies the following properties:*

(i) *The equilibrium price function $\psi : [0,T] \times I\!R^2 \to I\!R$, $(t, u, \xi) \mapsto \psi(t, u, \xi)$ is continuous in ξ and nondecreasing in ξ.*

(ii) *The price-determining measure μ is a probability measure concentrated on $[0,1]$.*

(iii) *The price system (ψ, μ) implies a natural cost structure.*

(iv) *For all $1 \le k \le n$, all $i \in \mathcal{I}_{k-1}$ and all $\xi_1, \xi_2 \in I\!R$ the large investor price function $S_\mu : [0,T] \times I\!R^3 \to I\!R$ generated by (ψ, μ) satisfies (4.12).*

Then for each convex payoff function $h : I\!R \to I\!R$ for which the left- and right-hand derivatives are bounded there exists some self-financing trading strategy $\big(\xi^n, b^n\big)$ such that the paper value replication (4.17) *holds.*

In Section 1.4.2 we have also derived a uniqueness result for the replicating strategy of a star-convex contingent claim $\big(\xi_n, b_n\big)$. However, since Brouwer's fixed point theorem only gives the existence and not the uniqueness of a fixed point, we cannot draw the same conclusions for the problem of paper value replication as we did in Section 1.4.2. Actually, in the particular case where the large investor price does not depend on the large investor's strategy it is easy to see that there exist infinitely many self-financing trading strategies (ξ^n, b^n) which satisfy (4.17), but which only differ in the (not necessarily star-convex) portfolio $\big(\xi^n_n, b^n_n\big)$ at time $t^n_n = T$.

Remark. Because of the particular features of the large investor model and the various possible stock prices which might be used for the valuation of the large investor's stock holdings and the value of the option at maturity, there are various other possible models for value replication: For example, instead of using the average price of $S_\mu\big(T, U^n_n, \xi^n_{n-1}, \xi^n_n\big)$ paid for the $\xi^n_n - \xi^n_{n-1}$ shares of stock bought by the large investor at time $t^n_n = T$, we could use the equilibrium price $\psi\big(T, U^n_n, \xi^n_n\big)$, which *any* investor in the market would pay for an infinitesimal amount of stock immediately after the transaction of these $\xi^n_n - \xi^n_{n-1}$ shares, in order to calculate the option value. Of course, in this case one should use the same stock price to calculate the "value" of the large investor's portfolio at time T, so that our condition for the replicating self-financing strategy (ξ^n, b^n) would become

$$\xi^n_n \psi\big(T, U^n_n, \xi^n_n\big) + b^n_n = h\Big(\psi\big(T, U^n_n, \xi^n_n\big)\Big). \tag{4.24}$$

In the special case where the price-determining measure μ of the price system (ψ, μ) is the Dirac measure δ_1 concentrated in 1, the large investor price function $S_\mu : [0, T] \times I\!R^3 \to I\!R$ of (3.2) simplifies to $S_\mu\big(t, u, \xi_1, \xi_2\big) = \psi(T, u, \xi_2)$ for all $(t, u, \xi_1, \xi_2) \in [0, T] \times I\!R^3$, so that (4.17) and (4.24) coincide. As pointed out in the remark following equation (4.10), this special price-determining measure is used in the discrete model of Jarrow (1994) and in the continuous model of Frey (1998). In particular, Frey's approach of paper value replication, which he writes in terms of the equilibrium price function like (4.24), is at the same time the (continuous-time analogue of the) paper value replication (4.17) which we have considered in this section.
However, in general the two conditions (4.17) and (4.24) differ. In such a situation there are two reasons why we do not work with condition (4.24): First of all, in practice it is unlikely that the value of some option is determined by one single spot price in the market at which only an *infinitesimal* small amount of shares is traded. But even if it is we might still argue that the large investor will buy immediately before appraisal such that his price $S_\mu\big(T, u^n_{n(i\pm1)}, \xi^n_{(n-1)i}, \xi^n_{n(i\pm1)}\big)$ will be the assessed price.
Secondly, (4.24) is mathematically undesirable, since the price $\psi\big(T, U^n_n, \xi^n_n\big)$ would become an additional price which we have to consider in order to find a replicating strategy, in addition to the large investor price $S_\mu\big(T, U^n_n, \xi^n_{n-1}, \xi^n_n\big)$ at which the large investor *really* trades.

Of course, instead of replicating a certain option by some trading strategy's paper value at maturity, we could also consider the problem where the large investor wants to find a trading strategy (ξ^n, b^n) such that the real value of this strategy at time T matches with some prescribed option value. In order to determine the real value as introduced in Definition 1.28, the large investor's stock holdings of ξ^n_n shares at time T have to be valued by using the loss-free liquidation price $\bar{S}\big(T, U^n_n, \xi^n_n\big)$, and so it makes sense that the same price is used for determining the option value at maturity. Thus, we could also look for a self-financing trading strategy (ξ^n, b^n) which satisfies

$$\xi_n^n \bar{S}\big(T, U_n^n, \xi_n^n\big) + b_n^n = h\Big(\bar{S}\big(T, U_n^n, \xi_n^n\big)\Big). \tag{4.25}$$

We have already touched on this sort of replication problem in Example 1.7 where we have seen that the real value of the long European call of α shares at time T has such a representation if the payoff function $h : I\!R \to I\!R$ is given by $h(x) = \alpha(x-K)^+$ for all $x \in I\!R$.
Basically this real-value approach splits the trades at the time point T in two parts: At first the large investor buys the $\xi_n^n - \xi_{n-1}^n$ shares of stock which he needs to buy at time $t_n^n = T$ according to his trading strategy (ξ^n, b^n), and the average price at which these stocks are traded is the large investor price $S_\mu\big(T, U_n^n, \xi_{n-1}^n, \xi_n^n\big)$. Thereafter, the large investor sells all the ξ_n^n shares of his portfolio for the loss-free liquidation price $\bar{S}\big(T, U_n^n, \xi_n^n\big)$. This liquidation price only is a *realizable* liquidation price if the transaction does not cause any transaction losses. In a similar fashion Boyle and Vorst (1992) model option replication in a small investor market with proportional transaction costs, since they implicitly value the stock and cash holdings at maturity without any transaction cost charge. (Musiela and Rutkowski, 1998, Section 2.5) explicitly mention the absence of transaction costs at maturity as one of the their key assumptions for a small investor model with transaction costs, which slightly extends Boyle's and Vorst's work.
However we do not want to exclude implied transaction losses at maturity, since we do not see why the price mechanism at time T should conceptually differ from the price mechanism at all other points in time. Moreover, the loss-free liquidation price cannot be observed in the market, so that the statement of the replication problem in terms of the loss-free liquidation price will be of limited use for real-world markets. Especially, a market participant who does not know the fundamental value and the large investor's stock holdings is in general unable to find out the loss-free liquidation price, even if he knows the price system (ψ, μ). □

1.4.4 Star-Concave Portfolios

In this section we shortly describe the problems which occur if we want to transfer our derivation of replicating strategies for star-convex contingent claims as in Section 1.4.2 to the derivation of similar replicating strategies for *star-concave* contingent claims. Basically, in a non-degenerate large investor market the recursive derivation of a self-financing trading strategy (ξ^n, b^n) which replicates a certain star-concave contingent claim (ξ_n, b_n) along the lines of Section 1.4.2 will only be successful if the discrete spatial derivative of the associated strategy function $\xi^n : \mathcal{A}^n \to I\!R$ does not become too large, and an a-priori bound on this derivative is needed in order to prove the attainability of (ξ_n, b_n). Section 1.4.4 may be omitted at the first reading.
Let us assume that we are given some star-concave contingent claim (ξ_n, b_n), and that in our attempt to construct a self-financing trading strategy (ξ^n, b^n) which replicates (ξ_n, b_n) we have constructed star-concave portfolios $\big(\xi_j^n, b_j^n\big)$ for some $1 \le k \le \lceil nT \rceil$ and all $k \le j \le \lceil nT \rceil$. If there exists a portfolio $\big(\xi_{k-1}^n, b_{k-1}^n\big)$ at time t_{k-1}^n such that the portfolio $\big(\xi_k^n, b_k^n\big)$ at time t_k^n can be (perfectly) financed by the portfolio $\big(\xi_{k-1}^n, b_{k-1}^n\big)$ in that the self-financing condition (3.13) holds, our discussion in Section 1.4.2 has already shown that the possible realizations $\xi_{(k-1)i}^n = \xi^n\big(t_{k-1}^n, u_{(k-1)i}^n\big)$ and $b_{(k-1)i}^n = \xi^n\big(t_{k-1}^n, u_{(k-1)i}^n\big)$ of the large investor's shares in stocks and cash amounts in the portfolio $\big(\xi_{k-1}^n, b_{k-1}^n\big)$ satisfy the fixed point equation (4.10) for the stock holdings and the explicit equation (4.9) for the cash amounts.
In order to prove the existence of a solution to the fixed point problem (4.10) *for this time step*, we can proceed along the same lines as in the proof of Proposition 1.33. For the replication of star-convex contingent claims in Section 1.4.2 we then have shown that the

star-convexity of the portfolio at time t_k^n will be passed on to the portfolio at time t_{k-1}^n if the price system implies a natural loss structure and if the large investor price function $S_\mu : [0,T] \times I\!R^3 \to I\!R$ satisfies the ordering (4.12) for all $i \in \mathcal{I}_{k-2}$ and all $\xi_1, \xi_2 \in I\!R$. As we will show in Section 1.5 under very limited restrictions on the price system (ψ, μ) this condition (4.12) can be guaranteed for all $1 \le k \le \lceil nT \rceil$, all $i \in \mathcal{I}_{k-2}$ and *all* $\xi_1, \xi_2 \in I\!R$. However, if we want to prove that the star-*concavity* of (ξ_k^n, b_k^n) is inherited by the portfolio (ξ_{k-1}^n, b_{k-1}^n) at time t_{k-1}^n, we have to impose a different condition on the large investor price function. Namely, we then have to guarantee that the large investor price function is of such a form that for all possible outcomes of the fundamental value at time t_{k-1}^n the (ask-)price at which the large investor sells shares of stock at time t_k^n can be higher than the (bid-)price at which the large investor has bought the same amount of stocks at the time t_{k-1}^n, and vice versa, that the (ask-)price at which the large investor has sold shares of stock at time t_{k-1}^n can be lower than the (bid-)price at which the large investor could re-buy the same amount of stocks at time t_k^n. In general, such a condition on the price system (ψ, μ) is much more difficult to satisfy than the condition (4.12), which only compares bid-prices with bid-prices and ask-prices with ask-prices, and if we do not want to stick with degenerate price systems which exclude transaction losses, we cannot require such a condition to hold for all possible transactions, but we have to limit the size of the transactions. Assuming that for all $i \in \mathcal{I}_{k-2}$ the difference between the two possible stock holdings $\xi_{(k-1)(i+1)}^n$ and $\xi_{(k-1)(i-1)}^n$ given the fundamental value $U_{k-2}^n = u_{(k-2)i}^n$ at time t_{k-2}^n is lower than the maximal transaction size, we can then proceed as in Proposition 1.33 to show that the portfolio (ξ_{k-1}^n, b_{k-1}^n) is star-concave as well.

Proposition 1.39. *Let us assume that the equilibrium price function $\psi : [0,T] \times I\!R^2 \to I\!R$, $(t,u,\xi) \mapsto \psi(t,u,\xi)$ is continuous in ξ, and suppose that for some $1 \le k \le \lceil nT \rceil$ the portfolio (ξ_k^n, b_k^n) is star-concave. Then there exists some function $\xi^n(t_{k-1}^n, \cdot) : \mathcal{U}_{k-1}^n \to I\!R$ with*

$$\xi_{(k-1)i}^n := \xi^n\big(t_{k-1}^n, u_{(k-1)i}^n\big) \in \big[\xi_{k(i+1)}^n, \xi_{k(i-1)}^n\big] \quad \text{for all } i \in \mathcal{I}_{k-1} \tag{4.26}$$

which solves the fixed point problem (4.10). *Assume now that the price system (ψ, μ) also implies a natural loss structure and that there exists some $B^n \in [0, \infty]$ such that for all $i \in \mathcal{I}_{k-2}$ and all $\xi_1, \xi_2 \in I\!R$ with $\xi_1 \le \xi_2 \le \xi_1 + B^n$ the large investor price function satisfies*

$$S_\mu\big(t_{k-1}^n, u_{(k-1)(i-1)}^n, \xi_1, \xi_2\big) \le S_\mu\big(t_k^n, u_{ki}^n, \xi_2, \xi_1\big) \tag{4.27}$$

and

$$S_\mu\big(t_k^n, u_{ki}^n, \xi_1, \xi_2\big) \le S_\mu\big(t_{k-1}^n, u_{(k-1)(i+1)}^n, \xi_2, \xi_1\big). \tag{4.28}$$

Then the portfolio (ξ_{k-1}^n, b_{k-1}^n) at time t_{k-1}^n which is implied by (4.10) *and* (4.9) *is again star-concave if for all $i \in \mathcal{I}_{k-2}$ we have $\big|\xi_{(k-1)(i+1)}^n - \xi_{(k-1)(i-1)}^n\big| \le B^n$.*

Proof. The existence of the function $\xi^n(t_{k-1}^n, \cdot)$ can be shown as in Proposition 1.33, and because of the interlocked structure $\xi_{k(i+1)}^n \le \xi_{(k-1)i}^n \le \xi_{k(i-1)}^n$ for all $i \in \mathcal{I}_{k-1}$ the function $\xi^n(t_{k-1}^n, \cdot) : \mathcal{U}_{k-1}^n \to I\!R$ has to be nonincreasing because the function $\xi^n(t_k^n, \cdot) : \mathcal{U}_k^n \to I\!R$ is nonincreasing. Let us now set $l = k-1$ and consider the implied portfolio (ξ_l^n, b_l^n) given by $\xi_l^n = \xi^n(t_l^n, U_l^n)$ and $b_l^n = b^n(t_l^n, U_l^n)$, where the function $b^n(t_l^n, \cdot)$ is defined via (4.9). Under our conditions we have to show that (ξ_l^n, b_l^n) inherits the *star*-concavity from (ξ_{l+1}^n, b_{l+1}^n) as well. Also this proof parallels the proof of Proposition 1.33. In order to show that

$$\big(\xi_{l(i+1)}^n - \xi_{l(i-1)}^n\big) S_\mu\big(t_l^n, u_{l(i+1)}^n, \xi_{l(i-1)}^n, \xi_{l(i+1)}^n\big) + b_{l(i+1)}^n - b_{l(i-1)}^n \le 0 \quad \text{for all } i \in \mathcal{I}_{l-1} \tag{4.29}$$

we notice that by assumption we have $\left|\xi^n_{l(i+1)} - \xi^n_{l(i-1)}\right| \leq B^n$, hence it follows from the inequality (4.28) that

$$S_\mu\big(t^n_{l+1}, u^n_{(l+1)i}, \xi^n_{l(i+1)}, \xi^n_{l(i-1)}\big) \leq S_\mu\big(t^n_l, u^n_{l(i+1)}, \xi^n_{l(i-1)}, \xi^n_{l(i+1)}\big) \quad \text{for all } i \in \mathcal{I}_{l-1}. \tag{4.30}$$

Together with the monotonicity of $\xi^n\big(t^n_l, \cdot\big)$ and (4.16) this implies that

$$\begin{aligned}
\big(\xi^n_{l(i+1)} - \xi^n_{l(i-1)}\big) S_\mu\big(t^n_l, u^n_{l(i+1)}, \xi^n_{l(i-1)}, \xi^n_{l(i+1)}\big) + b^n_{l(i+1)} - b^n_{l(i-1)} \\
\leq \big(\xi^n_{l(i+1)} - \xi^n_{l(i-1)}\big) S_\mu\big(t^n_{l+1}, u^n_{(l+1)i}, \xi^n_{l(i+1)}, \xi^n_{l(i-1)}\big) \\
- \big(\xi^n_{l(i+1)} - \xi^n_{(l+1)i}\big) S_\mu\big(t^n_{l+1}, u^n_{(l+1)i}, \xi^n_{l(i+1)}, \xi^n_{(l+1)i}\big) \\
- \big(\xi^n_{(l+1)i} - \xi^n_{l(i-1)}\big) S_\mu\big(t^n_{l+1}, u^n_{(l+1)i}, \xi^n_{l(i-1)}, \xi^n_{(l+1)i}\big).
\end{aligned}$$

The right-hand side of this inequality can now be bounded from above by 0 because of Lemma 1.11 applied with $\xi = \xi^n_{l(i+1)}$, $\alpha = \xi^n_{(l+1)i} - \xi^n_{l(i+1)} \geq 0$, and $\beta = \xi^n_{l(i-1)} - \xi^n_{(l+1)i} \geq 0$. Thus, the bound (4.29) holds, and by analogous arguments we can also show

$$\big(\xi^n_{l(i+1)} - \xi^n_{l(i-1)}\big) S_\mu\big(t^n_l, u^n_{l(i-1)}, \xi^n_{l(i+1)}, \xi^n_{l(i-1)}\big) + b^n_{l(i+1)} - b^n_{l(i-1)} \geq 0 \quad \text{for all } i \in \mathcal{I}_{l-1}.$$

Hence $\big(\xi^n_l, b^n_l\big)$ is indeed star-concave. **q.e.d.**

Now we can use the same sort of recursive arguments as in Section 1.4.2 to obtain an attainability result for star-concave contingent claims:

Corollary 1.40. *Let us assume that the large investor price system (ψ, μ) satisfies the following properties:*

(i) *The equilibrium price function $\psi : [0,T] \times I\!R^2 \to I\!R$, $(t,u,\xi) \mapsto \psi(t,u,\xi)$ is continuous in ξ.*

(ii) *The price system (ψ, μ) implies a natural cost structure.*

(iii) *There exists some $B^n \in [0,\infty]$ such that for all $1 \leq k \leq \lceil nT \rceil$, all $i \in \mathcal{I}_{k-1}$, and all $\xi_1, \xi_2 \in I\!R$ with $\xi_1 \leq \xi_2 \leq \xi_1 + B^n$ the large investor price function $S_\mu : [0,T] \times I\!R^3 \to I\!R$ generated by (ψ, μ) satisfies (4.27) and (4.28).*

Then for every star-concave contingent claim $\big(\xi_n, b_n\big)$ there exists some trading strategy $\big(\xi^n, b^n\big) = \big\{\big(\xi^n_k, b^n_k\big)\big\}_{0 \leq k \leq \lceil nT \rceil}$ which replicates $\big(\xi_n, b_n\big)$ and for which the corresponding strategy function $\xi^n : \mathcal{A}^n \to I\!R$ satisfies

$$\xi^n_{(k+1)(i-1)} \leq \xi^n_{ki} \leq \xi^n_{(k+1)(i+1)} \quad \textit{for all } (k,i) \in I(\lceil nT \rceil - 1), \tag{4.31}$$

if we can guarantee by some other means that in the case of existence every such replicating strategy satisfies $\left|\xi^n_{(k+1)(i+1)} - \xi^n_{(k+1)(i-1)}\right| \leq B^n$ for all $(k,i) \in I(\lceil nT \rceil - 1)$.

In order to apply Corollary 1.40 to show the attainability of a sequence $\big\{(\xi_n, b_n)\big\}_{n \in I\!N}$ of star-concave contingent claims, we shall find some global bound $B \in [0,\infty]$ such that on the one hand for all $n \in I\!N$, all $1 \leq k \leq \lceil nT \rceil$, all $i \in \mathcal{I}_{k-1}$, and all $\xi_1 \leq \xi_2 \leq \xi_1 + 2\delta_n B$ the inequalities (4.27) and (4.28) for the large investor price function hold, and such that on the other hand for all $n \in I\!N$ the (possible) existence of a replicating strategy (ξ^n, b^n) for the contingent claim (ξ_n, b_n) implies that the discrete derivative

$$\Delta^n_u\, \xi^n\big(t^n_{k+1}, u^n_{ki}\big) := \frac{1}{2\delta_n}\Big(\xi^n\big(t^n_{k+1}, u^n_{ki} + \delta_n\big) - \xi^n\big(t^n_{k+1}, u^n_{ki} - \delta_n\big)\Big) = \frac{1}{2\delta_n}\big(\xi^n_{(k+1)(i+1)} - \xi^n_{(k+1)(i-1)}\big)$$

of the associated strategy function $\xi^n : \mathcal{A}^n \to I\!R$ is bounded by B for all $(k,i) \in I(\lceil nT \rceil - 1)$. Those a-priori bounds on $\Delta^n_u\, \xi^n$ will only be derived at the end of Section 3.3.3, where we will use bounds on the derivative of the (candidate) continuous limiting function of the discrete strategy functions $\big\{\xi^n\big\}_{n \in I\!N}$ to obtain a bound on the discrete derivatives as well.

Remark. Because of the interlocked structure there is always a natural bound on all possible star-concave trading strategies (ξ^n, b^n) which satisfy (4.31), since this condition implies $\xi^n_{(k+1)(i-1)} - \xi^n_{(k+1)(i+1)} \leq \xi^n_{k(i-2)} - \xi^n_{k(i+2)}$ for all $0 \leq k \leq \lceil nT \rceil$ and all $i \in \mathcal{I}_{k-2}$, and hence they especially meet

$$\left|\xi^n_{(k+1)(i+1)} - \xi^n_{(k+1)(i-1)}\right| \leq \xi^n_{\lceil nT \rceil(-\lceil nT \rceil)} - \xi^n_{\lceil nT \rceil \lceil nT \rceil} \quad \text{for all } (k,i) \in I(\lceil nT \rceil - 1).$$

Since the stock holdings $\xi^n_{\lceil nT \rceil}$ at maturity $t^n_{\lceil nT \rceil} = T$ of any trading strategy (ξ^n, b^n) which replicates (ξ_n, b_n) are determined by $\xi^n_{\lceil nT \rceil} = \xi_n$, this observation gives us also a directly available a-priori bound B_n on the trading strategies in Corollary 1.40.
However, if the large investor price function $S_\mu : [0,T] \times I\!R^3 \to I\!R$ is not degenerate, for employing this bound the difference $\xi^n_{[nT](-[nT])} - \xi^n_{[nT][nT]}$ between the largest and the smallest possible number of shares held by the large investor at time $t^n_{[nT]} = T$ normally needs to be too small to be of any practical use. For example, if we fix such a non-degenerate large investor price function and consider a sequence of replication problems by successively increasing the number of points on the binomial grid through the parameter $n \in I\!N$ and looking at a corresponding family $\{(\xi_n, b_n)\}_{n \in I\!N}$ of star-concave contingent claims, the inequalities (4.27) and (4.28) will only hold for all (sufficiently large) $n \in I\!N$, all $1 \leq k \leq \lceil nT \rceil$, all $i \in \mathcal{I}_{k-1}$, and all $\xi_1 \leq \xi_2 \leq \xi_1 + B^n$ if $B^n = O(\delta_n)$ as $n \to \infty$. Hence, the choice of $B^n = \xi^n_{[nT](-[nT])} - \xi^n_{[nT][nT]}$ would restrict us to star-concave contingent claims for which the difference between the largest and smallest possible number of shares held by the large investor at time T is of order $O(\delta_n)$ and thus diminishes as $n \to \infty$. At least in the limit, this approach would exclude all interesting cases of star-concave contingent claims.
Another, totally different approach to circumvent the problems which occur when replicating star-concave contingent claims as the number of time points on the grid tends to infinity makes the price system (ψ, μ) depend on $n \in I\!N$. If the corresponding sequence $\{(\psi^n, \mu^n)\}_{n \in I\!N}$ of price systems converges to a degenerate price system (ψ, μ) as $n \to \infty$ in that the limiting price system (ψ, μ) excludes any transaction losses, one can guarantee that in the limit the ordering conditions (4.27) and (4.28) hold for all $\xi_1, \xi_2 \in I\!R$. In this case, the sequence of bounds $\{B^n\}_{n \in I\!N}$ need not be chosen to be of order $O(\delta_n)$ as $n \to \infty$ any more, and one can once again consider more general star-concave contingent claims. However, since this model would limit the candidate limiting price system, we will not further pursue this idea. $\square$

For the largest part of our thesis we will from now on stick to the replication of star-convex contingent claims (ξ_n, b_n) and their corresponding replication strategies (ξ^n, b^n). Even in this case we will need to find bounds on the discrete derivatives $\Delta^n_u \xi^n$ when it comes to the convergence of a sequence of strategy functions $\xi^n : \mathcal{A}^n \to I\!R$ towards a continuous-time limit. But the problem is much easier to handle since the existence result for the replicating strategies (ξ^n, b^n) as given in Corollary 1.35 does not depend on these bounds.

1.5 Examples of Large Investor Price Functions

To finish this chapter we now give examples of large investor price systems (ψ, μ) for which the various conditions in Section 1.4 hold. We start with a parametrized family of large investor price systems (ψ, μ) where the equilibrium price function ψ is non-negative and multiplicative and where the price-determining measure μ is concentrated on the unit interval, and then give conditions on the parameters which guarantee that the associated price system satisfies the conditions in Section 1.4. In particular, we will show that our large-investor model contains the standard Cox-Ross-Rubinstein model as a special case.

Example 1.8. Let $\overline{\psi} : [0,T] \times I\!R \to (0, \infty)$ be a positive and differentiable function which is strictly increasing in u and satisfies the bounds $L := \left\| \frac{\overline{\psi}}{\overline{\psi}_u} \right\| := \sup_{(t,u) \in [0,T] \times I\!R} \left| \frac{\overline{\psi}(t,u)}{\overline{\psi}_u(t,u)} \right| < \infty$

and $K := \left\|\frac{\overline{\varphi}_t}{\overline{\varphi}_u}\right\| < \infty$. Then for some fixed $c \geq 0$ a feasible equilibrium price function $\psi : [0,T] \times \mathbb{R}^2 \to (0,\infty)$ can be defined by

$$\psi(t,u,\xi) = \overline{\varphi}(t,u)e^{c\xi} \quad \text{for all } (t,u,\xi) \in [0,T] \times \mathbb{R} \times \mathbb{R}.$$

By the multiplicative structure of ψ we have $\left\|\frac{\psi}{\psi_u}\right\| = L < \infty$ and $\left\|\frac{\psi_t}{\psi_u}\right\| = K < \infty$ as well. As a price-determining measure associated to ψ, we take again some probability measure $\mu = \mu_{\rho,h}$ from the family $\left(\mu_{\rho,h}\right)_{\rho,h\in[0,1]}$ of probability measures on $\big([0,1], \mathcal{B}([0,1])\big)$ which were for each $h, \rho \in [0,1]$ introduced in (2.14) as

$$\mu_{\rho,h}(A) = \rho\lambda(A) + (1-\rho)\frac{1}{h}\lambda\big(A \cap [1-h,1]\big) \quad \text{for all } A \in \mathcal{B}([0,1]). \tag{5.1}$$

Recall that the last term on the right-hand side is interpreted as the Dirac measure in 1 if $h = 0$. The large investor price function $S_\mu : [0,T] \times \mathbb{R}^3 \to (0,\infty)$ associated to ψ and μ is then for all $(t,u,\xi_1,\xi_2) \in [0,T] \times \mathbb{R}^3$ given by

$$S_\mu(t,u,\xi_1,\xi_2) = \int \psi\big(t,u,(1-\theta)\xi_1 + \theta\xi_2\big)\mu(d\theta) = \overline{\varphi}(t,u)\int_0^1 e^{c(1-\theta)\xi_1 + c\theta\xi_2}\mu(d\theta). \tag{5.2}$$

Because of the special form of the price-determining measure $\mu = \mu_{\rho,h}$ of (5.1) we can compute the integral $\int_0^1 e^{c(1-\theta)\xi_1+c\theta\xi_2}\mu(d\theta) = e^{c\xi_1}\big(\rho\int_0^1 e^{c\theta(\xi_2-\xi_1)}d\theta + (1-\rho)\frac{1}{h}\int_{1-h}^1 e^{c\theta(\xi_2-\xi_1)}d\theta\big)$ to conclude

$$S_\mu\big(t,u,\xi_1,\xi_2\big) = \overline{\varphi}(t,u)e^{c\xi_1}\left(\rho\frac{e^{c(\xi_2-\xi_1)} - 1}{c(\xi_2-\xi_1)} + (1-\rho)\frac{e^{c(\xi_2-\xi_1)} - e^{c(1-h)(\xi_2-\xi_1)}}{c(\xi_2-\xi_1)h}\right) \tag{5.3}$$

for all $(t,u,\xi_1,\xi_2) \in [0,T] \times \mathbb{R}^3$, where the cases $\xi_1 = \xi_2$, $h = 0$, and $c = 0$ have to be understood as the corresponding limits. $\square$

The family of large investor price systems (ψ,μ) described by Example 1.8 is a very rich family. By choosing the parameters ρ and h in an appropriate way, we can construct from this family of price systems certain sub-families which satisfy the various conditions imposed on price systems in Section 1.4, especially the conditions of Corollary 1.35, Corollary 1.38, and Corollary 1.40.

Lemma 1.41. *Fix a large investor price system (ψ,μ) with some equilibrium price function $\psi : [0,T] \times \mathbb{R}^2 \to (0,\infty)$ and some price-determining measure $\mu = \mu_{\rho,h}$ as specified in Example 1.8. Then:*

(i) (ψ,μ) implies a natural loss structure and nondecreasing total transaction losses.

Now let us suppose that $n > K^2$ and $t_1, t_2 \in [0,T]$ satisfy $\big|t_1 - t_2\big| \leq \delta_n^2$. Then we also have:

(ii) For all $u, \xi_1, \xi_2 \in \mathbb{R}$ the large investor price function $S_\mu : [0,T] \times \mathbb{R}^3 \to (0,\infty)$ satisfies

$$S_\mu\big(t_1, u-\delta_n, \xi_1, \xi_2\big) < S_\mu\big(t_2, u, \xi_1, \xi_2\big) < S_\mu\big(t_1, u+\delta_n, \xi_1, \xi_2\big). \tag{5.4}$$

(iii) If for some $R \in [0,\infty]$ and $\rho, h \in [0,1]$ we have

$$chR + \log\Big((1-\rho) + \rho e^{c(1-h)R}\Big) < \delta_n\frac{1}{L}\big(1 - \delta_n K\big), \tag{5.5}$$

then inequalities like the one in (5.4) hold even if the large investor's stock holdings are changed a little bit between t_1 and t_2, namely then S_μ satisfies

$$S_\mu\big(t_1, u-\delta_n, \xi_1, \xi_2\big) < S_\mu\big(t_2, u, \xi_3, \xi_1\big) \qquad \text{for all } \xi_2 \leq \xi_1 \leq \xi_3 + R \tag{5.6}$$

and

$$S_\mu\big(t_1, u+\delta_n, \xi_1, \xi_2\big) > S_\mu\big(t_2, u, \xi_3, \xi_1\big) \qquad \text{for all } \xi_2 \geq \xi_1 \geq \xi_3 - R. \tag{5.7}$$

(*iv*) *If* $\xi_2 = \xi_3$ *then* (5.6) *holds for all* $\xi_2 \le \xi_1$ *and* (5.7) *hold for all* $\xi_2 \ge \xi_1$, *regardless of whether* (5.5) *is fulfilled.*

(*v*) *On the other hand, if*

$$\min\left\{\log\frac{\rho+(1-\rho)e^{\frac{1}{2}c(1-h)R}}{\rho+(1-\rho)e^{-\frac{1}{2}c(1-h)R}}, \log\left(1+\frac{1-\rho}{\rho h}\right)\right\} < \delta_n\frac{1}{L}\big(1-\delta_n K\big), \tag{5.8}$$

then for all $u \in I\!R$ *and all* $\xi_1, \xi_2 \in I\!R$ *with* $\big|\xi_2 - \xi_1\big| \le R$ *we have*

$$S_\mu\big(t_1, u-\delta_n, \xi_1, \xi_2\big) < S_\mu\big(t_2, u, \xi_2, \xi_1\big) < S_\mu\big(t_1, u+\delta_n, \xi_1, \xi_2\big). \tag{5.9}$$

If all strict inequalities in (*ii*) *to* (*v*) *are replaced by weak ones, the statements hold for all* $n \ge K^2$ *and* $t_1, t_2 \in [0,T]$ *satisfying* $|t_1 - t_2| \le \delta_n^2$.

Remark. In (5.5) and (5.8) we use the convention $0 \cdot x = 0$ and $\frac{x}{0} = \infty$ for all $x \in [0,\infty]$.

Proof. Statement (*i*) follows directly from Lemma 1.13, thus let us move on to (*ii*). Owing to the multiplicative structure (5.2) of $S_\mu : [0,T] \times I\!R^3 \to (0,\infty)$, for all $t_1, t_2 \in [0,T]$ and all $u, \xi_1, \xi_2 \in I\!R$ the quotient

$$\frac{S_\mu\big(t_1, u-\delta_n, \xi_1, \xi_2\big)}{S_\mu\big(t_2, u, \xi_1, \xi_2\big)} = \frac{\overline{\psi}(t_1, u-\delta_n)}{\overline{\psi}(t_2, u)} \tag{5.10}$$

does not depend on the large investor's initial and final stock holdings of ξ_1 and ξ_2 shares, respectively. An application of Taylor's rule then shows that there exist some t^*, u^* with t^* being located between t_1 and t_2 and $u - \delta_n \le u^* \le u$ such that

$$\log\frac{\overline{\psi}(t_1, u-\delta_n)}{\overline{\psi}(t_2, u)} = \log\overline{\psi}\big(t_1, u-\delta_n\big) - \log\overline{\psi}\big(t_2, u\big) = -\delta_n\frac{\overline{\psi}_u\big(t^*, u^*\big)}{\overline{\psi}\big(t^*, u^*\big)} + (t_1 - t_2)\frac{\overline{\psi}_t\big(t^*, u^*\big)}{\overline{\psi}\big(t^*, u^*\big)}.$$

The last term can now be rewritten as $-\delta_n\frac{\overline{\psi}_u(t^*,u^*)}{\overline{\psi}(t^*,u^*)}\Big(1 - \frac{t_1-t_2}{\delta_n}\frac{\overline{\psi}_t(t^*,u^*)}{\overline{\psi}_u(t^*,u^*)}\Big)$. Since $\overline{\psi}$ and $\overline{\psi}_u$ are positive, we conclude from the definition of L that $\frac{\overline{\psi}_u(t^*,u^*)}{\overline{\psi}(t^*,u^*)} \ge \frac{1}{L}$. On the other hand, the condition $|t_1 - t_2| \le \delta_n^2$ and the definition of K imply $\frac{t_1-t_2}{\delta_n}\frac{\overline{\psi}_t(t^*,u^*)}{\overline{\psi}_u(t^*,u^*)} \le \delta_n K$, which is strictly less than 1 because $n > K^2$. If we finally take the exponential, we can bound (5.10) from above by

$$\frac{S_\mu\big(t_1, u-\delta_n, \xi_1, \xi_2\big)}{S_\mu\big(t_2, u, \xi_1, \xi_2\big)} \le \exp\Big(-\delta_n\frac{1}{L}\big(1-\delta_n K\big)\Big) \quad \text{for all } u, \xi_1, \xi_2 \in I\!R. \tag{5.11}$$

Since the expression on the right-hand side is strictly less than 1, this proves the lower inequality of (5.4). It is clear that for $n = K^2$ we would still get the weak inequality $S_\mu\big(t_1, u-\delta_n, \xi_1, \xi_2\big) \le S_\mu\big(t_2, u, \xi_1, \xi_2\big)$, since in this case the right-hand side of (5.11) is still not larger than 1. The upper inequality of (5.4) follows similarly.
In order to prove (*iii*) assume that $R \in [0,\infty]$ and $\rho, h \in [0,1]$ satisfy (5.5). We will only prove (5.6) for the case where $h > 0$ and $c > 0$, but it is easy to see that our argument works for $h = 0$ as well, and the assertion for $c = 0$ is almost trivial. Since we have already shown (5.11) it suffices to show

$$\frac{S_\mu\big(t, u, \xi_1, \xi_2\big)}{S_\mu\big(t, u, \xi_3, \xi_1\big)} < \exp\Big(\delta_n\frac{1}{L}(1-\delta_n K)\Big) \tag{5.12}$$

for all $t \in [0,T]$, $u \in I\!R$ and $\xi_2 \leq \xi_1 \leq \xi_3 + R$; then (5.6) follows from

$$\frac{S_\mu\big(t_1, u-\delta_n, \xi_1, \xi_2\big)}{S_\mu\big(t_2, u, \xi_3, \xi_1\big)} = \frac{S_\mu\big(t_1, u-\delta_n, \xi_1, \xi_2\big)}{S_\mu\big(t_2, u, \xi_1, \xi_2\big)} \frac{S_\mu\big(t_2, u, \xi_1, \xi_2\big)}{S_\mu\big(t_2, u, \xi_3, \xi_1\big)} < 1. \tag{5.13}$$

Let us fix $t \in [0,T]$, $u \in I\!R$, and $\xi_j \in I\!R$ for $j \in \{1,2,3\}$. As a shorthand let us set $\alpha_j = \xi_j - \xi_1$ for $j \in \{2,3\}$. Without loss of generality we assume that $\alpha_j \neq 0$. By (5.3) we have:

$$S_\mu\big(t, u, \xi_1, \xi_2\big) = \overline{\psi}(t,u) e^{c\xi_1} \left(\rho \frac{1}{c\alpha_2} \big(e^{c\alpha_2} - 1\big) + (1-\rho) \frac{1}{c\alpha_2 h} \big(e^{c\alpha_2} - e^{c(1-h)\alpha_2}\big) \right).$$

Using now this equation and factoring out $e^{c(\xi_1 - \xi_3)}$ we also get

$$S_\mu\big(t, u, \xi_3, \xi_1\big) = \overline{\psi}(t,u) e^{c\xi_1} \left(\rho \frac{1}{c\alpha_3} \big(e^{c\alpha_3} - 1\big) + (1-\rho) \frac{1}{c\alpha_3 h} \big(e^{ch\alpha_3} - 1\big) \right).$$

Thus, the quotient of $S_\mu\big(t, u, \xi_1, \xi_2\big)$ and $S_\mu\big(t, u, \xi_3, \xi_1\big)$ is given by

$$\frac{S_\mu\big(t, u, \xi_1, \xi_2\big)}{S_\mu\big(t, u, \xi_3, \xi_1\big)} = \frac{\rho \frac{1}{c\alpha_2} \big(e^{c\alpha_2} - 1\big) + (1-\rho) \frac{1}{c\alpha_2 h} \big(e^{c\alpha_2} - e^{c(1-h)\alpha_2}\big)}{\rho \frac{1}{c\alpha_3} \big(e^{c\alpha_3} - 1\big) + (1-\rho) \frac{1}{c\alpha_3 h} \big(e^{ch\alpha_3} - 1\big)} \tag{5.14}$$

and only depends on the differences $\alpha_2 = \xi_2 - \xi_1$ and $\alpha_3 = \xi_3 - \xi_1$. Since the condition $\xi_2 \leq \xi_1 \leq \xi_3 + R$ is equivalent to $\alpha_2 \leq 0$ and $-R \leq \alpha_3$, for $\{\xi_i\}_{1 \leq i \leq 3}$ satisfying this condition the quotient (5.14) can be bounded by

$$\frac{S_\mu\big(t, u, \xi_1, \xi_2\big)}{S_\mu\big(t, u, \xi_3, \xi_1\big)} \leq \frac{\rho e^0 + (1-\rho) e^0}{\rho e^{-cR} + (1-\rho) e^{-chR}} = e^{chR} \frac{1}{(1-\rho) + \rho e^{-c(1-h)R}} \tag{5.15}$$

due to the mean value theorem and the monotonicity of $x \mapsto e^x$. Now $x \mapsto e^x$ is also convex, and hence $\frac{1}{2}\big(e^{c(1-h)R} + e^{-c(1-h)R}\big) \geq 1$, which leads to the bound

$$\begin{aligned} &\big((1-\rho) + \rho e^{c(1-h)R}\big)\big((1-\rho) + \rho e^{-c(1-h)R}\big) \\ &\qquad = 1 + 2\rho(1-\rho)\left(\frac{e^{c(1-h)R} + e^{-c(1-h)R}}{2} - 1 \right) \geq 1. \end{aligned}$$

If we then replace the fraction in the upper bound (5.15) of $\frac{S_\mu(t,u,\xi_1,\xi_2)}{S_\mu(t,u,\xi_3,\xi_1)}$ by $(1-\rho) + \rho e^{c(1-h)R}$ and apply (5.5) we conclude

$$\frac{S_\mu\big(t, u, \xi_1, \xi_2\big)}{S_\mu\big(t, u, \xi_3, \xi_1\big)} \leq e^{chR}\big((1-\rho) + \rho e^{c(1-h)R}\big) < \exp\left(\delta_n \frac{1}{L} \big(1 - \delta_n K\big) \right). \tag{5.16}$$

Thus (5.12) holds for all $t \in [0,T]$, $u \in I\!R$, and $\xi_2 \leq \xi_1 \leq \xi_3 + R$, and hence (5.6) holds as well. It is clear from (5.16) that we still have a weak inequality in (5.6) if we only have a weak inequality in (5.5). The proof of (5.7) goes along the same lines.
If $\xi_2 = \xi_3$ we can derive the inequalities in (5.6) and (5.7) under less stringent conditions and thus show (iv). Let us again concentrate on the lower inequality (5.6), and consider without loss of generality only the case $h, c > 0$ and $\xi_1 \neq \xi_2$. With $\alpha := \alpha_2 = \alpha_3$ the quotient (5.14) simplifies for any fixed $t \in [0,T]$ and $u \in I\!R$ to

$$\frac{S_\mu\big(t, u, \xi_1, \xi_2\big)}{S_\mu\big(t, u, \xi_2, \xi_1\big)} = \frac{\rho + (1-\rho) x_h(c\alpha) e^{\frac{1}{2}c(1-h)\alpha}}{\rho + (1-\rho) x_h(c\alpha) e^{-\frac{1}{2}c(1-h)\alpha}}, \tag{5.17}$$

where the function $x_h : I\!R \to I\!R$ is given by

$$x_h(z) = \frac{1}{h} e^{\frac{1}{2}(1-h)z} \frac{e^{hz}-1}{e^z-1} \quad \text{for all } z \in I\!R.$$

Since $h \in (0,1]$, it is clear that $x_h(z) \geq 0$ for all $z \in I\!R$, and since $c \in (0,1]$ as well, we get

$$\frac{S_\mu\big(t,u,\xi_1,\xi_2\big)}{S_\mu\big(t,u,\xi_2,\xi_1\big)} \leq \frac{\rho + (1-\rho)x_h(c\alpha)}{\rho + (1-\rho)x_h(c\alpha)} = 1 \quad \text{as long as } \alpha \leq 0. \tag{5.18}$$

If we combine this inequality for $t = t_2$ with the lower inequality of (5.4), this gives (5.6). Combining that equation with (5.11) we derive (5.6) for all $\xi_2 = \xi_3 \leq \xi_1$. The second part of (iv) can be shown similarly.
Finally, we come to (v). Again, we will only show the lower inequality in (5.9) for the special case $h, c > 0$. Because of (iv) it just remains to show that this inequality holds for $\xi_2 - R \leq \xi_1 < \xi_2$, i.e. if $0 < \alpha := \xi_2 - \xi_1 \leq R$. Looking once again at the function $x_h : I\!R \to [0,\infty)$ we notice that for all $z \in I\!R$

$$x_h^2(z) = \frac{\big(e^z - e^{(1-h)z}\big)\big(e^{hz}-1\big)}{h^2\left(e^z-1\right)^2} = \frac{e^{(1+h)z} - 2e^z + e^{(1-h)z}}{h^2\left(e^{2z}-2e^z+1\right)} = \frac{\sum_{k=1}^\infty \frac{1}{(2k)!}(hz)^{2k}}{\sum_{k=1}^\infty \frac{1}{(2k)!}z^{2k}} \leq 1.$$

Thus, we can use the monotonicity of $x \mapsto \frac{d+bx}{d+ax}$ for $a, b, d \geq 0$ and $b \geq a$ to bound the fraction (5.17) as long as $0 \leq \alpha \leq R$ by

$$\frac{S_\mu\big(t,u,\xi_1,\xi_2\big)}{S_\mu\big(t,u,\xi_2,\xi_1\big)} \leq \frac{\rho + (1-\rho)e^{\frac{1}{2}c(1-h)\alpha}}{\rho + (1-\rho)e^{-\frac{1}{2}c(1-h)\alpha}} \leq \frac{\rho + (1-\rho)e^{\frac{1}{2}c(1-h)R}}{\rho + (1-\rho)e^{-\frac{1}{2}c(1-h)R}}. \tag{5.19}$$

If ρ is close to 1, the bound (5.19) is not very tight. In this case a better bound for the fraction (5.17) can be obtained by noting that

$$he^{\frac{1}{2}(1-h)z}x_h(z) = \frac{e^z - e^{(1-h)z}}{e^z-1} \leq 1 \quad \text{for all } z \geq 0,$$

since this inequality and $x_h(z) \geq 0$ allow us to bound (5.17) by

$$\frac{S_\mu\big(t,u,\xi_1,\xi_2\big)}{S_\mu\big(t,u,\xi_2,\xi_1\big)} \leq \frac{\rho + (1-\rho)\frac{1}{h}}{\rho} = 1 + \frac{1-\rho}{\rho h}. \tag{5.20}$$

Combining (5.19) and (5.20) and then applying condition (5.8) leads to the inequality

$$\frac{S_\mu\big(t,u,\xi_1,\xi_2\big)}{S_\mu\big(t,u,\xi_2,\xi_1\big)} \leq \min\left\{\frac{\rho + (1-\rho)e^{\frac{1}{2}c(1-h)R}}{\rho + (1-\rho)e^{-\frac{1}{2}c(1-h)R}}, 1 + \frac{1-\rho}{\rho h}\right\} < \exp\Big(\delta_n \frac{1}{L}(1-\delta_n K)\Big).$$

Therefore, we can conclude as in (5.13) that under (5.8) we have

$$\frac{S_\mu\big(t_1,u-\delta_n,\xi_1,\xi_2\big)}{S_\mu\big(t_2,u,\xi_2,\xi_1\big)} < 1. \tag{5.21}$$

This shows that the lower inequality in (5.9) holds not only for all $\xi_1 \leq \xi_2$ as in (iv), but also for all $\xi_2 - R \leq \xi_1 < \xi_2$. Again it is easy to see that we still have the weak inequality $S_\mu\big(t_2,u,\xi_2,\xi_1\big) \leq S_\mu\big(t_1,u-\delta_n,\xi_1,\xi_2\big)$ if (5.8) holds with weak inequality only or if $n = K^2$. Similarly to (5.21) one can show that the second inequality in (5.9) holds as long as $\xi_2 \leq \xi_1 \leq \xi_2 + R$. **q.e.d.**

Lemma 1.41(i) and (ii) give conditions such that the assumptions on the large investor price system (ψ, μ) in Corollary 1.35, which shows the attainability of star-convex contingent claims, and in Corollary 1.38, which shows the existence of a paper value replicating strategy, are fulfilled. Moreover, under the conditions imposed by Lemma 1.41(i) and (v) the assumptions on the large investor price system in the attainability result for star-concave contingent claims, namely in Corollary 1.40, are satisfied as well. In Chapter 2 we will also need price systems (ψ, μ) for which the associated large investor price function S_μ satisfies inequalities of the form (5.6) and (5.7).

Remark. Note that Lemma 1.41(ii) does not hinge on the multiplicative structure of the equilibrium price function $\psi : [0,T] \times I\!R^2 \to I\!R$ nor on the particular form (5.1) of the price-determining measure μ. It is straightforward to prove (ii) for any (not necessarily positive) equilibrium price function $\psi : [0,T] \times I\!R^2 \to I\!R$ which is differentiable in t and u and which satisfies $K := \left\| \frac{\psi_t}{\psi_u} \right\| < \infty$, and for *any* associated price-determining measure μ. However, the multiplicative structure of the ψ and the specific form of μ were essential in order to bound the ratios $\frac{S_\mu(t,u,\xi_1,\xi_2)}{S_\mu(t,u,\xi_3,\xi_1)}$ and $\frac{S_\mu(t,u,\xi_1,\xi_2)}{S_\mu(t,u,\xi_2,\xi_1)}$, respectively, which led to the statements (iii) to (v).□

The next example shows that the Cox-Ross-Rubinstein model of a small investor market is indeed a special case of our large investor market model.

Example 1.9. If in Example 1.8 we choose $c = 0$, such that for all $(t, u, \xi) \in [0,T] \times I\!R^2$ the equilibrium price function $\psi : [0,T] \times I\!R^2 \to I\!R$ is given by $\psi(t,u,\xi) = \overline{\psi}(t,u)$, not only this function, but also the large investor price function $S_\mu : [0,T] \times I\!R^3 \to (0,\infty)$ and the benchmark price function $S^* : [0,T] \times I\!R^3 \to (0,\infty)$ do not depend any more on the large investor's stock holdings, and we have

$$\psi(t,u,\xi_1) = \psi(t,u,\xi_2) = S_\mu(t,u,\xi_1,\xi_2) = S^*(t,u,\xi_1,\xi_2) = \overline{\psi}(t,u)$$

for all $(t,u,\xi_1,\xi_2) \in [0,T] \times I\!R^3$. In particular, the large investor price function S_μ is also independent of the price-determining measure μ.
If we then fix some $S_0, \sigma > 0$ and $\mu, r \in I\!R$ and further specify the (small investor price) function $\overline{\psi} : [0,T] \times I\!R \to (0,\infty)$ by

$$\overline{\psi}(t,u) = S_0 e^{\sigma u + (\mu - r)t} \quad \text{for all } (t,u) \in [0,T] \times I\!R,$$

all the three price functions ψ, S_μ, and S^* used in our large investor model coincide with the discounted price function in the Cox-Ross-Rubinstein model where the initial price at time 0 is given by S_0, the risk free interest rate by r, and the volatility and drift parameters of the stock price process by σ and μ, respectively. The bounds L and K then simplify to $L = \left\| \frac{\overline{\psi}}{\overline{\psi}_u} \right\| = \frac{1}{\sigma}$ and $K = \left\| \frac{\overline{\psi}_t}{\overline{\psi}_u} \right\| = \frac{|\mu - r|}{\sigma}$.
Because of Lemma 1.41(i) and (ii) the large investor price system (ψ, μ) satisfies the assumptions imposed on the price system in Corollary 1.35 and Corollary 1.38 once the discretization parameter $n \in I\!N$ is chosen to be larger than K^2. Moreover, because of the special form of the equilibrium price function $\psi : [0,T] \times I\!R^2 \to (0,\infty)$, the condition (4.17) for the paper value replication reduces to

$$\xi_n^n \overline{\psi}(T, U_n^n) + b_n^n = h\big(\overline{\psi}(T, U_n^n)\big).$$

Because of the self-financing condition (3.13) the paper value replication condition then implies

$$\xi_{n-1}^n \overline{\psi}(T, U_n^n) + b_{n-1}^n = h\big(\overline{\psi}(T, U_n^n)\big)$$

as well, and recalling from the remark following Definition 1.23 that we use (ξ_{n-1}^n, b_{n-1}^n) to denote the (large) investor's portfolio between t_{n-1}^n and t_n^n, the latter replication condition is seen to be the usual replication condition in the Cox-Ross-Rubinstein model. In the *standard* Cox-Ross-Rubinstein model one considers the replication of a unit European call with a strike price $X \in I\!R$, i.e. the payoff function $h : I\!R \to I\!R$ is given by $h(x) = (x - X)^+$ for all $x \in I\!R$. This payoff function satisfies the conditions imposed in Corollary 1.38, since $h : I\!R \to I\!R$ is obviously convex, and since its left- and right-hand derivatives are bounded by 1. Thus, our model indeed contains the Cox-Ross-Rubinstein model as a (very) special case. □

Remark. Since the large investor price function $S_\mu : [0, T] \times I\!R^3 \to (0, \infty)$ of the price system described in Example 1.9 does not depend on the large investor's stock holdings, for all $u, \xi_1, \xi_2 \in I\!R$ and all $t_1, t_2 \in [0, T]$ with $\left|t_1 - t_2\right| \leq \delta_n^2$ the condition (5.4) immediately implies (5.9). This shows that not only the assumptions of Corollary 1.35 and Corollary 1.38, but also the assumptions of Corollary 1.40 are satisfied by the price system (ψ, μ) described in Example 1.9, even if the bound B^n used for that assumptions is set to be $B_n = \infty$. Hence, under the price system of Example 1.9 which corresponds to the Cox-Ross-Rubinstein model we can also replicate all star-concave contingent claims. □

Chapter 2

Recursive Equations for Value and Strategy

In the previous chapter, we have shown the existence and uniqueness of replicating strategies in a binomial large investor market model. This chapter takes a closer look at the large investor's trading strategies and the corresponding paper and real values.
We first focus on extending classical results from the theory of small investor binomial market models to the large investor model. These include the determination of a martingale measure, a recursive formula for the value function, and last but not least a no-arbitrage result. The analogues of these results in a large investor model depend on the particular valuation concept used, and so we have to distinguish between the real and the paper value concept here as well. Above all, we have to differentiate between the corresponding two notions of arbitrage. In order to draw the same conclusion as in the Cox-Ross-Rubinstein model that the market is free of arbitrage, we have to restrict the class of *admissible* trading strategies. We also find backward recursions for the two value functions, but in contrast to the Cox-Ross-Rubinstein model these recursive formulæ now depend on the large investor's particular trading strategy. Likewise, there still exists a martingale measure which turns both the large investor price process and the paper value process into martingales, but this measure depends on the large investor's strategy as well. For the concept of the real value, we present a measure under which the real value process is a supermartingale and the loss-free liquidation price process at least almost a martingale.
Since the recursion formulæ for the value functions and the martingale measures strongly depend on the large investor's strategy, it is essential to consider in detail the strategy, and especially the strategy function, of the large investor. This is done in Section 2.2 where we derive an implicit difference equation of second order for the large investor's strategy function. This will later be a starting point for a convergence analysis as $n \to \infty$. Before we come to any convergence results in Chapter 3, however, we shortly turn our attention from large investor models to small investor market models with proportional transaction costs. After translating into our notation we shall see that large investor models and small investor models with transaction costs have many similarities, and some of these similarities will be exploited in Chapter 3 and 4.
In these two later chapters we shall only work with large investor markets where the equilibrium price function has a multiplicative structure as in Definition 1.17. For those large investor markets, the similarities with small investor markets with transaction costs turn out to be even more pronounced. Moreover, the recursive formula for the real value function then simplifies considerably, and in particular, the real value process now becomes a supermartingale under the martingale measure in the associated small investor market. In the specific case where the large investor trades at the benchmark price the real value is even a

martingale under this measure, every contingent claim is attainable, and we can explicitly calculate the replicating strategy.

For simplicity, we will assume in the whole chapter that $T = 1$ so that we have $t_n^n = T$ in the nth binomial model. The generalization to an arbitrary $T > 0$ is straightforward and just requires frequent replacements of n by $\lceil nT \rceil$.

2.1 No Arbitrage and Martingale Measures

In the usual Cox-Ross-Rubinstein setting of a *small investor* financial market there exists only one price at any trading time $\left\{t_k^n\right\}_{0 \le k \le n}$, and only one reasonable valuation principle for pricing a portfolio $\left(\xi_k^n, b_k^n\right)$ at time t_k^n. Under the *risk-neutral* measure in such a model not only the stock price process but also the value process of every self-financing trading strategy is a martingale. The martingale representation for the value *process* can then be used to recursively calculate all possible values of the discrete value *function* on the binomial grid, starting with the possible values at maturity T. Moreover, since the value process of every self-financing portfolio strategy is a martingale under the risk neutral measure, it is easily seen that there are no arbitrage opportunities in the Cox-Ross-Rubinstein model. In this section we want to explore how these properties of small investor binomial models generalize in the presence of a large investor.

Now we have introduced several stock prices and two different valuation principles for a portfolio strategy of the large investor in such a market, and each of these has its own relevance, depending on the particular intention of the large investor. A large investor who wants to replicate the *paper value* of a given option would be more interested in the paper value process; a large investor who valuates his portfolio by the *real value*, which he could achieve by selling his whole portfolio without any transaction losses, would be inclined towards using the real value process. For this reason we will deal both with the concept of the large investor stock price and the paper value of a portfolio, and with the concept of the loss-free liquidation price and the associated valuation concept of the real value.

Depending on the price and valuation concept chosen, several properties of small investor markets will only hold in a relaxed form as soon as a large investor is introduced. For example, it will turn out that the paper value process remains a martingale under the martingale measure for the large investor price process, but this measure now becomes highly dependent on the large investor's strategy, and in general it will even depend on his pre-trading endowment.

However, the martingale representation of the paper value does not take into account the implied transaction losses caused by the large investor's transactions. The generalization of the Cox-Ross-Rubinstein model to transaction costs shows that in such a *small investor* market the value process is only a supermartingale under the martingale measure for the stock price. We will also find a measure under which the *real value* process in a large investor market becomes a supermartingale, but this measure is an ordinary martingale measure for the loss-free liquidation price process only between the transactions of the large investor. Our findings will then be used to prove that there are no paper and no real value arbitrage opportunities within certain classes of admissible trading strategies.

In order to find the martingale representations for the value processes, and as a preparation of the continuous-time limit, we also give recursive representations for both value functions if the large investor's trading strategy is known.

Let us consider a large investor market which is specified in terms of some price system (ψ, μ) consisting as usual of an equilibrium price function $\psi : [0, T] \times \mathbb{R}^2 \to \mathbb{R}$ and a price determining measure μ associated to ψ. Immediately before time 0 the large investor is supposed to hold ξ^n_{-1} shares of stock. The large investor trades on this market according to some self-financing trading strategy (ξ^n, b^n).

2.1.1 No Arbitrage for the Large Investor

In the first subsection we introduce two different arbitrage concepts which are based on the two different value concepts, and define admissible trading strategies. Moreover, as the main result of this section we state a no-arbitrage result for the large investor model.

Definition 2.1. A self-financing trading strategy (ξ^n, b^n) is a **paper-value arbitrage opportunity** if

$$\mathbf{P}\Big(v^n\big(T, U^n_n, \xi^n_{n-1}\big) \geq v^n\big(0, U^n_0, \xi^n_{-1}\big)\Big) = 1 \quad \text{and} \quad \mathbf{P}\Big(v^n\big(T, U^n_n, \xi^n_{n-1}\big) > v^n\big(0, U^n_0, \xi^n_{-1}\big)\Big) > 0.$$

A self-financing trading strategy (ξ^n, b^n) is a **real-value arbitrage opportunity** if

$$\mathbf{P}\Big(\bar{v}^n\big(T, U^n_n\big) \geq \bar{v}^n\big(0, U^n_0\big)\Big) = 1 \quad \text{and} \quad \mathbf{P}\Big(\bar{v}^n\big(T, U^n_n\big) > \bar{v}^n\big(0, U^n_0\big)\Big) > 0.$$

Since the large investor's trades affect the stock prices in the market, we cannot expect that the whole class of self-financing portfolio strategies is free of arbitrage opportunities, even if we only consider discrete binomial models. However, for each of the two arbitrage concepts introduced in Definition 2.1 we can single out a large subclass of self-financing *admissible* strategies which do not lead to arbitrage opportunities.
In order to minimize the notational ballast we will only look for arbitrage opportunities within the class of path-independent portfolio strategies $(\xi^n, b^n) = \big\{\big(\xi^n_k, b^n_k\big)\big\}_{0\leq k\leq n}$. In this case we can introduce the strategy function $\xi^n : \mathcal{A}^n \to I\!R$ which corresponds to the trading strategy (ξ^n, b^n) as in Definition 1.23, and then define some condensed notation to denote the large investor price and the transaction loss at time t, respectively, where the large investor switches his stock holdings ξ in the presence of the fundamentals u to the amount required by the strategy function ξ^n, i.e. we define the functions $S^{\xi^n}_\mu : \mathcal{A}^n \times I\!R \to I\!R$ and $c^{\xi^n}_\mu : \mathcal{A}^n \times I\!R \to I\!R$ for all $(t, u, \xi) \in \mathcal{A}^n \times I\!R$ by

$$S^{\xi^n}_\mu(t, u, \xi) := S_\mu\big(t, u, \xi, \xi^n(t, u)\big) = \int \psi\big(t, u, (1-\theta)\xi + \theta\xi^n(t, u)\big)\mu(d\theta) \tag{1.1}$$

and

$$c^{\xi^n}_\mu(t, u, \xi) := c_\mu\big(t, u, \xi, \xi^n(t, u)\big) = \big(\xi^n(t, u) - \xi\big)\int \psi\big(t, u, (1-\theta)\xi + \theta\xi^n(t, u)\big)\big(\mu - \lambda\big)(d\theta). \tag{1.2}$$

Within the class of path-independent self-financing strategies we can spot two different classes of admissible strategies.

Definition 2.2. The set $\mathcal{Z}^n_P = \mathcal{Z}^n_P\big(\psi, \mu, \xi^n_{-1}\big)$ consists of all path-independent and self-financing strategies (ξ^n, b^n) which satisfy

$$S^{\xi^n}_\mu\big(t^n_k, U^n_{k-1} - \delta, \xi^n_{k-1}\big) < S^{\xi^n}_\mu\big(t^n_{k-1}, U^n_{k-1}, \xi^n_{k-2}\big) < S^{\xi^n}_\mu\big(t^n_k, U^n_{k-1} + \delta, \xi^n_{k-1}\big) \tag{1.3}$$

for all $1 \leq k \leq n$, where we use as usual the convention $\delta = \delta_n = \frac{1}{\sqrt{n}}$. The elements $(\xi^n, b^n) \in \mathcal{Z}^n_P$ are called ***p*-admissible**.
Moreover, we introduce the set $\mathcal{Z}^n_R$ of all path-independent and self-financing trading strategies (ξ^n, b^n) which satisfy

$$\bar{S}\big(t^n_k, U^n_{k-1} - \delta, \xi^n_{k-1}\big) < \bar{S}\big(t^n_{k-1}, U^n_{k-1}, \xi^n_{k-1}\big) < \bar{S}\big(t^n_k, U^n_{k-1} + \delta, \xi^n_{k-1}\big) \tag{1.4}$$

for all $1 \leq k \leq n$. A strategy $(\xi^n, b^n) \in \mathcal{Z}^n_R$ is called ***r*-admissible**.

If the large investor does not affect the prices and thus acts as a small investor, the equilibrium price function $\psi : [0,T] \times \mathbb{R}^2 \to \mathbb{R}$ satisfies $\psi(t,u,\xi) = \overline{\psi}(t,u)$ for all $(t,u,\xi) \in [0,T] \times \mathbb{R}^2$. Then each of the sequences of inequalities (1.3) and (1.4) holds if and only if we have

Assumption A. *The associated small investor price function* $\overline{\psi} : [0,T] \times \mathbb{R} \to \mathbb{R}$ *satisfies*

$$\overline{\psi}\big(t_k^n, U_{k-1}^n - \delta\big) < \overline{\psi}\big(t_{k-1}^n, U_{k-1}^n\big) < \overline{\psi}\big(t_k^n, U_{k-1}^n + \delta\big) \quad \textit{for all } 1 \le k \le n. \tag{1.5}$$

Thus under Assumption A, the sets of p- and r-admissible strategies coincide, and all path-independent self-financing strategies are admissible.

Remark. Note that Assumption A is the usual no-arbitrage condition in general binomial small investor markets. In order to replicate a contingent claim in a discrete large investor market, Jarrow (1994) assumes that the speculator has only "local" price adjustment power. This corresponds to the condition on p-admissibility in (1.3). □

We can easily state conditions on the small investor price function $\overline{\psi} : [0,T] \times \mathbb{R} \to \mathbb{R}$ such that Assumption A holds:

Lemma 2.3. *Suppose that the small investor price function* $\overline{\psi} : [0,T] \times \mathbb{R} \to \mathbb{R}$ *is continuously differentiable and satisfies* $K := \big\|\frac{\psi_t}{\psi_u}\big\| < \infty$. *Then Assumption A holds for all* $n > K^2$.

Proof. Let $n > K^2$ and fix some $1 \le k \le n$ and $u \in \mathbb{R}$. Since $t_k^n - t_{k-1}^n = \delta^2$, it follows from the mean value theorem that there exist some $t_{k-1}^n \le t^* \le t_k^n$ and some $u \le u^* \le u + \delta$ such that

$$\overline{\psi}\big(t_k^n, u+\delta\big) - \overline{\psi}\big(t_{k-1}^n, u\big) = \delta^2 \overline{\psi}_t\big(t^*, u^*\big) + \delta \overline{\psi}_u\big(t^*, u^*\big) = \delta^2 \overline{\psi}_u\big(t^*, u^*\big) \left(1 + \delta^{-1} \frac{\overline{\psi}_t\big(t^*, u^*\big)}{\overline{\psi}_u\big(t^*, u^*\big)}\right).$$

Since $\overline{\psi}$ is a small investor price function, it is strictly increasing in u like the underlying large investor price function, and hence we can employ the definition $\delta = \frac{1}{\sqrt{n}}$ and $n > K^2$ to bound

$$\overline{\psi}\big(t_k^n, u+\delta\big) - \overline{\psi}\big(t_{k-1}^n, u\big) \ge \delta^2 \overline{\psi}_u\big(t^*, u^*\big) \left(1 - \sqrt{n}\left\|\frac{\overline{\psi}_t}{\overline{\psi}_u}\right\|\right) > 0.$$

This proves the upper inequality in (1.5). The lower inequality follows analogously. **q.e.d.**

In Section 2.1.5 we will prove:

Proposition 2.4. *For every large investor market* (ψ, μ) *we have the following no-arbitrage statements:*

(i) *There is no paper-value arbitrage opportunity for the large investor within the class of p-admissible trading strategies.*

(ii) *If (ψ, μ) excludes instantaneous transaction gains, i.e. if the transaction loss function $c_\mu : [0,T] \times \mathbb{R}^3 \to \mathbb{R}$ is nonnegative, then there is no real-value arbitrage opportunity within the class of r-admissible trading strategies.*

If the large investor is forced to use only admissible trading strategies, we therefore can transfer the principles of arbitrage-free pricing from small investor models to the large investor market.

2.1.2 Examples of Admissible Trading Strategies

Before we move on to prove that there are indeed no arbitrage opportunities among the class of admissible trading strategies, we should rather convince ourselves that the degenerate large investor market (ψ, μ) where $\psi(t, u, \xi) = \overline{\psi}(t, u)$ for all $(t, u) \in [0, T] \times I\!R$ is not the only large investor market, for which p- and r-admissible trading strategies exist. For that reason, we will use this section to give examples of p- and r-admissible trading strategies.
It will turn out that the condition (1.3) for p-admissibility in general limits the distance of the large investor's stock holdings at two subsequent time points t_k^n and t_{k+1}^n. However, whenever the large investor price system (ψ, μ) is multiplicative, the condition (1.4) for r-admissibility does not restrict the choice of strategies at all.

For ease of presentation we will restrict our search of p-admissible trading strategies to those large investor price systems (ψ, μ) which were introduced in Example 1.8. For these large investor price systems we have shown in Lemma 1.41(i) and (ii) that they satisfy both the assumptions on the price system needed in Section 1.4.2 to show the existence and uniqueness of replication strategies for star-convex contingent claims, and the assumptions needed for the existence result for paper value replication in Section 1.4.3.
In both sections we have constructed replicating strategies $\left(\xi^n, b^n\right)$ which are interlocked in the sense that the associated strategy *functions* $\xi^n : \mathcal{A}^n \to I\!R$ satisfy

$$\xi^n_{k(i-1)} \le \xi^n_{(k-1)i} \le \xi^n_{k(i+1)} \quad \text{for all } 1 \le k \le n \text{ and } i \in \mathcal{I}_{k-1}. \tag{1.6}$$

Thus we may be satisfied with considering only strategies which have such an interlocked structure.

Proposition 2.5. *Let (ψ, μ) be a price system as specified in Example 1.8, and let ξ^n_{-1} be the large trader's endowment in stocks immediately before time $t_0^n = 0$. Using the convention $0 \cdot \infty = 0$, take now some $B \in (0, \infty]$ such that*

$$chB + \frac{1}{\delta} \log\Big((1-\rho) + \rho e^{c(1-h)B\delta}\Big) < \frac{1}{L}\big(1 - \delta K\big). \tag{1.7}$$

Then each self-financing and path-independent portfolio strategy (ξ^n, b^n) which is interlocked as in (1.6) *and which satisfies*

$$\left|\xi^n_k - \xi^n_{k-1}\right| \le \delta B \quad \textit{for all } 0 \le k \le n-1 \tag{1.8}$$

is p-admissible.

Proof. Let us take some price system (ψ, μ) and some self-financing trading strategy (ξ^n, b^n) as described in the proposition. We then have to show that the two inequalities in (1.3) hold for all $1 \le k \le n$.
Fix $1 \le k \le n$. Due to the interlocked structure we have $\xi^n\big(t_k^n, U_{k-1}^n - \delta\big) \le \xi^n_{k-1}$, and due to (1.8) we also have $\xi^n_{k-1} \le \xi^n_{k-2} + \left|\xi^n_{k-1} - \xi^n_{k-2}\right| \le \xi^n_{k-2} + \delta B$. Since $t_k^n - t_{k-1}^n = \delta^2$, an application of Lemma 1.41(iii) and the definition of our shorthand $S_\mu^{\xi^n} : \mathcal{A}^n \times I\!R \to I\!R$ in (1.1) yields the lower inequality in (1.3). The upper inequality follows by symmetric arguments. **q.e.d.**

Remark. In the specific case where either $c = 0$ or $h = \rho = 0$ the condition (1.7) holds for each $B \in (0, \infty]$. In the first case, the large investor acts as a small investor and does not influence the market price at all. In the second case the price-determining measure $\mu = \mu_{\rho,h}$ is the Dirac measure δ_1 concentrated in 1 such that the stock price moves to the new equilibrium at the moment where the large investor announces his trade, but before he can execute a transaction. In both cases all trading strategies with the interlocked structure (1.6) are p-admissible if $\delta K < 1$. In general, however, only moderately fluctuating portfolio strategies

will satisfy (1.8), since B has to be chosen sufficiently small. Apart from the two cases mentioned above, (1.8) and thus (1.3) for $k = 1$ depend on the large investor's endowment ξ^n_{-1} in stocks before time 0. This is not a parameter which can be manipulated arbitrarily to match the bound in (1.8). If we required (1.3) only for all $2 \le k \le n$ the dependence on ξ^n_{-1} would vanish. However, in order that (ξ^n, b^n) is p-admissible, (1.3) has to hold for all $1 \le k \le n$, and hence the dependence on ξ^n_{-1} reduces the possible trading strategies to those strategies where the large investor's portfolio does not drift too far away from the original stock holdings ξ^n_{-1}.
We may allow a manipulation of ξ^n_{-1} if the buyer of the contingent claims pays the large investor at time $t^n_0 = 0$ with a portfolio $\left(\xi^n_*, b^n_*\right)$ of stocks and cash. In such a situation we may set $\xi^n_{-1} = \xi^n_*$. Especially, if the large investor receives the portfolio $\left(\xi^n_0, b^n_0\right)$ at time $t^n_0 = 0$, then we would set $\xi^n_{-1} = \xi^n_0$, and (1.8) for $k = 0$ is trivially satisfied.
Note also that the left-hand side in (1.7) is of order $c\big(\rho + (1-\rho)h\big)B$ as $n \to \infty$, which shows that the condition (1.8) is really only an $O(\delta)$-condition as $n \to \infty$. □

For $1 \le k \le n-1$ the condition (1.8) can be rewritten in terms of the "discrete derivatives"

$$\frac{\xi^n(t+\delta^2, u \pm \delta) - \xi^n(t,u)}{\delta} \quad \text{for all } (t,u) \in \mathcal{A}^n(n-2) \tag{1.9}$$

of the strategy *function* $\xi^n : \mathcal{A}^n \to I\!R$. Except in the cases $c = 0$ and $h = \rho = 0$ condition (1.8) requires global bounds on these discrete derivatives. Global bounds on the derivatives of the strategy function play a major rule in dealing with existence and uniqueness of the replicating strategies in continuous times, as we shall see in Section 3.3.3.

Let us now turn to r-admissible trading strategies: For a large class of large investor price systems (ψ, μ), which includes all the price systems covered in Example 1.8, all self-financing trading strategies are r-admissible.

Proposition 2.6. *Let $\psi : [0,T] \times I\!R^2 \to I\!R$, $(t,u,\xi) \mapsto \psi(t,u,\xi)$, be some equilibrium price function which is differentiable with respect to t and u and for which $K := \left\| \frac{\psi_t}{\psi_u} \right\| < \infty$. If μ is any price-determining measure for ψ and if $n > K^2$, then all self-financing trading strategies in the large investor market described by (ψ, μ) are r-admissible.*

Proof. Since $\bar{S}(t,u,\xi) = S^*(t,u,\xi,0) = S_\lambda(t,u,\xi,0)$ with $\lambda = \mu_{1,1}$, the statement for those price systems (ψ, μ) which were introduced in Example 1.8 follows from Lemma 1.41(ii). The general case follows from the remark following that lemma. **q.e.d.**

Remark. For $n \le K^2$ it may be that there are some self-financing trading strategies which are r-admissible and others which are not. However, this cannot happen if ψ is multiplicative, since then the condition of r-admissibility simplifies to (1.5), which does not depend on the strategy at all. □

2.1.3 Three Kinds of Martingale Measures

In a small investor market it is known that the absence of arbitrage is basically equivalent to the existence of an equivalent martingale measure, i.e. a measure $\mathbf{P}^*_n \approx \mathbf{P}^n$ under which the price process is a martingale. Since there are different possible price processes which one may consider in a large investor market, and since most of these price processes depend on the large investor's actual trading strategy, the situation in a large investor market becomes slightly more complicated. However, if we single out certain price processes for admissible trading strategies, we can identify several meaningful martingale measures: We derive one martingale measure for the associated small investor price process, another one for the large investor price process, and finally find a measure under which the loss-free liquidation price behaves at least almost like a martingale.

Definition 2.7. Suppose a large investor market is described by the price system (ψ, μ) and let the fundamental process $U^n = \{U_k^n\}_{0 \le k \le n}$ and the associated tilt process $Z^n = \{Z_k^n\}_{0 \le k \le n}$ on $(\Omega^n, \mathcal{F}_n^n)$ be defined as in (1.3.7).

(*i*) Under Assumption A the *s*-**martingale weight function** $\bar{p}^n : \mathcal{A}^n(n-1) \to (0,1)$ is defined by

$$\bar{p}^n(t,u) = \frac{\overline{\psi}(t,u) - \overline{\psi}(t+\delta^2, u-\delta)}{\overline{\psi}(t+\delta^2, u+\delta) - \overline{\psi}(t+\delta^2, u-\delta)} \quad \text{for all } (t,u) \in \mathcal{A}^n(n-1). \tag{1.10}$$

In terms of this weight function we introduce on the measurable space $(\Omega^n, \mathcal{F}_n^n)$ the **martingale measure $\bar{\mathbf{P}}^n$ in the associated small investor market** by

$$\bar{\mathbf{P}}^n\Big(U_k^n = U_{k-1}^n + \delta \,\Big|\, U_{k-1}^n\Big) = \bar{p}^n\big(t_{k-1}^n, U_{k-1}^n\big) \quad \text{for all } 1 \le k \le n, \tag{1.11}$$

and by $\bar{\mathbf{P}}^n\big(Z_0^n = 1\big) = \mathbf{P}^n\big(Z_0^n = 1\big)$.

(*ii*) Let (ξ^n, b^n) be a *p*-admissible trading strategy and the *p*-**martingale weight function** $p_n^{\xi^n} : \mathcal{A}^n(n-1) \times I\!R \to I\!R$ be given by

$$p_n^{\xi^n}(t,u,\xi) := \frac{S_\mu^{\xi^n}\big(t,u,\xi\big) - S_\mu^{\xi^n}\big(t+\delta^2, u-\delta, \xi^n(t,u)\big)}{S_\mu^{\xi^n}\big(t+\delta^2, u+\delta, \xi^n(t,u)\big) - S_\mu^{\xi^n}\big(t+\delta^2, u-\delta, \xi^n(t,u)\big)} \tag{1.12}$$

for all $(t,u,\xi) \in \mathcal{A}^n(n-1) \times I\!R$. Then the *p*-**martingale measure** $\mathbf{P}_n^{\xi^n}$ on $(\Omega^n, \mathcal{F}_n^n)$ is the probability measure which is defined in terms of the initial distribution given by $\mathbf{P}_n^{\xi^n}\big(Z_0^n = 1\big) = \mathbf{P}^n\big(Z_0^n = 1\big)$ and $\mathbf{P}_n^{\xi^n}\big(U_1^n = U_0^n + \delta \,\big|\, Z_0^n\big) = p_n^{\xi^n}\big(t_0^n, U_0^n, \xi_{-1}^n\big)$, and by the transition probabilities

$$\mathbf{P}_n^{\xi^n}\Big(U_k^n = U_{k-1}^n + \delta \,\Big|\, U_{k-1}^n,\, U_{k-2}^n\Big) = p_n^{\xi^n}\Big(t_{k-1}^n, U_{k-1}^n, \xi^n\big(t_{k-2}^n, U_{k-2}^n\big)\Big) \tag{1.13}$$

for all $2 \le k \le n$.

(*iii*) Let (ξ^n, b^n) be some *r*-admissible trading strategy and the *r*-**martingale weight function** $\bar{p}_n^{\xi^n} : \mathcal{A}^n(n-1) \to (0,1)$ for all $(t,u) \in \mathcal{A}^n(n-1)$ be defined by

$$\bar{p}_n^{\xi^n}(t,u) := \frac{\bar{S}\big(t,u,\xi^n(t,u)\big) - \bar{S}\big(t+\delta^2, u-\delta, \xi^n(t,u)\big)}{\bar{S}\big(t+\delta^2, u+\delta, \xi^n(t,u)\big) - \bar{S}\big(t+\delta^2, u-\delta, \xi^n(t,u)\big)}. \tag{1.14}$$

Then the *r*-**martingale measure** $\bar{\mathbf{P}}_n^{\xi^n}$ is the unique measure on $(\Omega^n, \mathcal{F}_n^n)$ which satisfies $\bar{\mathbf{P}}_n^{\xi^n}\big(Z_0^n = 1\big) = \mathbf{P}^n\big(Z_0^n = 1\big)$ and

$$\bar{\mathbf{P}}_n^{\xi^n}\Big(U_k^n = U_{k-1}^n + \delta \,\Big|\, U_{k-1}^n\Big) = \bar{p}_n^{\xi^n}\big(t_{k-1}^n, U_{k-1}^n\big) \quad \text{for all } 1 \le k \le n. \tag{1.15}$$

Note that for any *p*-admissible trading strategy (ξ^n, b^n) the *p*-martingale measure $\mathbf{P}_n^{\xi^n}$ is indeed a probability measure equivalent to the original measure $\mathbf{P}^n$, despite the fact that the *p*-martingale weight function was introduced as a function which maps into the whole domain of real numbers. Namely, by the definition of *p*-admissibility, it is guaranteed that the weights $p_n^{\xi^n}\big(t_k^n, U_k^n, \xi_{k-1}^n\big)$ take on only values in $(0,1)$ for each $0 \le k \le n-1$.

Remark. If the large investor does not affect the prices and thus acts as a small investor, so that the equilibrium price function $\psi : [0,T] \times I\!R^2 \to I\!R$ satisfies $\psi(t,u,\xi) = \overline{\psi}(t,u)$ for all $(t,u,\xi) \in [0,T] \times I\!R$, all three martingale measures $\bar{\mathbf{P}}^n$, $\bar{\mathbf{P}}_n^{\xi^n}$, and $\bar{\mathbf{P}}_n^{\xi^n}$ are well defined if and only if Assumption A holds, and then these measures coincide. In general, however, the p- and r-martingale measures will depend on the particular trading strategy (ξ^n, b^n) used by the large investor (respectively on his strategy function $\xi^n : \mathcal{A}^n \to I\!R$), while the martingale measure in the associated small investor market never depends on the large investor's trades. Note that the strategy influences the p- and the r-martingale measure in a rather different way: Like the martingale measure in the associated small investor market, the r-martingale measure keeps the fundamental process $\{U_k^n\}_{0\le k\le n}$ a one-step Markov process, but for all $1 \le k \le n$ the transition probability $\bar{p}_n^{\xi^n}(t_{k-1}^n, U_{k-1}^n)$ from U_{k-1}^n to U_k^n now depends on the large investor's *static* endowment $\xi_{k-1}^n = \xi^n(t_{k-1}^n, U_{k-1}^n)$ in stocks between time t_{k-1}^n and t_k^n. On the p-martingale measure, however, the influence of the large investor's strategy is so strong that it will in general destroy the Markov property of the fundamental process $\{U_k^n\}_{0\le k\le n}$. Only the two-dimensional process $\{(U_k^n, U_{k-1}^n)\}_{1\le k\le n}$ remains Markov, but its transition probabilities will now strongly depend on the evolution of the actual strategy; for all $2 \le k \le n$ its transition probability $p_n^{\xi^n}(t_{k-1}^n, U_{k-1}^n, \xi^n(t_{k-2}^n, U_{k-2}^n)) = p_n^{\xi^n}(t_{k-1}^n, U_{k-1}^n, \xi_{k-2}^n)$ to move from (U_{k-1}^n, U_{k-2}^n) to (U_k^n, U_{k-1}^n) depends not only on the large investor's stock holdings ξ_{k-1}^n between time t_{k-1}^n and t_k^n, but also on the previous endowment ξ_{k-2}^n and the two possible values $\xi^n(t_k^n, U_{k-1}^n \pm \delta)$ of the new endowment $\xi_k^n = \xi^n(t_k^n, U_k^n)$ at time t_k^n, given the information $\mathcal{F}_{k-1}^n$ up to time t_{k-1}^n. This can be seen from (1.12) as soon as the definition of $\tilde{S}_\mu$ in (1.1) is employed. Moreover, $\mathbf{P}_n^{\xi^n}$ even depends on the large investor's pre-trading endowment ξ_{-1}^n in stocks via the initial distribution of U_1^n. □

If the equilibrium price function $\psi : [0,T] \times I\!R^2 \to I\!R$ is multiplicative so that there is some function $f : I\!R \to I\!R$ which can be used to factorize ψ as $\psi(t,u,\xi) = \overline{\psi}(t,u)f(\xi)$ for all $(t,u,\xi) \in [0,T] \times I\!R^2$, the definitions of the loss-free liquidation function $\bar{S}$ and the benchmark price function S^* imply that the loss-free liquidation price function $\bar{S} : [0,T] \times I\!R \times I\!R \to I\!R$ is multiplicative as well, namely

$$\bar{S}(t,u,\xi) = \overline{\psi}(t,u) \int_0^1 f(\theta\xi)\, d\lambda(\theta) \quad \text{for all } (t,u,\xi) \in [0,T] \times I\!R \times I\!R. \tag{1.16}$$

Then all path-independent self-financing strategies are r-admissible if and only if Assumption A holds, and in this case the r-martingale measures $\bar{\mathbf{P}}_n^{\xi^n}$ for all self-financing trading strategies (ξ^n, b^n) coincide with the martingale measure $\bar{\mathbf{P}}^n$ in the associated small investor market.

While the p-martingale measure is a martingale measure in the strict sense in that the large investor price process is a martingale, the r-martingale measure does not make the loss-free liquidation price process into a martingale, because it does not take into account the jumps invoked by the large investor's trades. However, it turns out that even in the non-multiplicative case a *similar* property remains valid.

Proposition 2.8. *Fix a large investor price system (ψ, μ).*

(i) Under Assumption A the associated small investor martingale measure $\bar{\mathbf{P}}^n$ is the unique probability measure on $(\Omega^n, \mathcal{F}_n^n)$ under which the associated small investor price process $\{\overline{\psi}(t_k^n, U_k^n)\}_{0\le k\le n}$ is a martingale.

(ii) Let (ξ^n, b^n) be a p-admissible trading strategy. Then the p-martingale measure $\mathbf{P}_n^{\xi^n}$ is the unique probability measure on $(\Omega^n, \mathcal{F}_n^n)$ under which the large investor price process $\{S_\mu(t_k^n, U_k^n, \xi_{k-1}^n, \xi_k^n)\}_{0\le k\le n}$ is a martingale.

(*iii*) *For every r-admissible trading strategy* (ξ^n, b^n) *the r-martingale measure* $\bar{\mathbf{P}}_n^{\xi^n}$ *is the unique probability measure on* $(\Omega^n, \mathcal{F}_n^n)$ *under which the loss-free liquidation price after the transaction at time* t_{k-1}^n *can be calculated from the loss-free liquidation price* before *the transaction at time* t_k^n *as*

$$\bar{S}\big(t_{k-1}^n, U_{k-1}^n, \xi_{k-1}^n\big) = \bar{\mathbf{E}}_n^{\xi^n}\Big[\bar{S}\big(t_k^n, U_k^n, \xi_{k-1}^n\big)\,\Big|\,\mathcal{F}_{k-1}^n\Big] \quad \text{for all } 1 \le k \le n. \tag{1.17}$$

Moreover the martingale measures $\bar{\mathbf{P}}^n$, $\mathbf{P}_n^{\xi^n}$, *and* $\bar{\mathbf{P}}_n^{\xi^n}$ *are equivalent to the original measure* $\mathbf{P}^n$ *on* $(\Omega^n, \mathcal{F}_n^n)$.

Proof. The proof of the statements (*i*) to (*iii*) follows directly from the definition of the measures $\bar{\mathbf{P}}^n$, $\mathbf{P}_n^{\xi^n}$, and $\bar{\mathbf{P}}_n^{\xi^n}$ in Definition 2.7. For example the definition of the p-martingale weight function $p_n^{\xi^n} : \mathcal{A}^n(n-1) \times I\!R \to I\!R$ in (1.12) implies for all $1 \le k \le n$ that

$$\begin{aligned} S_\mu^{\xi^n}\big(t_{k-1}^n, U_{k-1}^n, \xi_{k-2}^n\big) &= p_n^{\xi^n}\big(t_{k-1}^n, U_{k-1}^n, \xi_{k-2}^n\big) S_\mu^{\xi^n}\big(t_k^n, U_{k-1}^n + \delta, \xi_{k-1}^n\big) \\ &\quad + \Big(1 - p_n^{\xi^n}\big(t_{k-1}^n, U_{k-1}^n, \xi_{k-2}^n\big)\Big) S_\mu^{\xi^n}\big(t_k^n, U_{k-1}^n - \delta, \xi_{k-1}^n\big), \end{aligned} \tag{1.18}$$

and by the definition of $\mathbf{P}_n^{\xi^n}$ this is equivalent to

$$S_\mu\big(t_{k-1}^n, U_{k-1}^n, \xi_{k-2}^n, \xi_{k-1}^n\big) = \mathbf{E}_n^{\xi^n}\Big[S_\mu\big(t_k^n, U_k^n, \xi_{k-1}^n, \xi_k^n\big)\,\Big|\,\mathcal{F}_{k-1}^n\Big] \quad \text{for all } 1 \le k \le n, \tag{1.19}$$

which indeed means that $\{S_\mu(t_k^n, U_k^n, \xi_{k-1}^n, \xi_k^n)\}_{0 \le k \le n}$ is a martingale under $\mathbf{P}_n^{\xi^n}$. On the other hand (1.19) implies (1.18) and this equation determines for all p-admissible functions the probabilities $\{p_n^{\xi^n}(t_{k-1}^n, U_{k-1}^n, \xi_{k-2}^n)\}_{1 \le k \le n}$ which define $\mathbf{P}_n^{\xi^n}$. The statements (*i*) and (*iii*) can be proved similarly.
Each of the three measures $\bar{\mathbf{P}}^n$, $\mathbf{P}_n^{\xi^n}$, and $\bar{\mathbf{P}}_n^{\xi^n}$ is equivalent to $\mathbf{P}^n$ on $(\Omega^n, \mathcal{F}_n^n)$, since all three weight functions $\bar{p}^n$, $p_n^{\xi^n}$ and $\bar{p}_n^{\xi^n}$ take on values in the open interval $(0,1)$ only. **q.e.d.**

Remark. The martingale-like condition (1.17) states that the loss-free liquidation price process is a martingale between the large investor's transactions. Note that both the loss-free liquidation price $\bar{S}(t_k^n, U_k^n, \xi_{k-1}^n)$ *before* and the loss-free liquidation price $\bar{S}(t_k^n, U_k^n, \xi_k^n)$ *after* the large investor has executed his transaction of $\xi_k^n - \xi_{k-1}^n$ stocks at time t_k^n are only theoretical prices, which are not apparent on the market. As opposed to the small investor prices, the loss-free liquidation prices depend on the large investor's stock holding.
Of course, we could easily construct modified stock prices which preserve the martingale property under $\mathbf{P}_n^{\xi^n}$: We have to offset the jumps in the loss-free liquidation price process which occur whenever the large investor's portfolio has to be adjusted. This could either be achieved additively by defining the price process $\check{S}^n = \{\check{S}_k^n\}_{0 \le k \le n}$ as

$$\check{S}_k^n := \bar{S}\big(t_k^n, U_k^n, \xi_k^n\big) - \sum_{j=0}^{k}\Big(\bar{S}\big(t_j^n, U_j^n, \xi_j^n\big) - \bar{S}\big(t_j^n, U_j^n, \xi_{j-1}^n\big)\Big) \quad \text{for all } 0 \le k \le n, \tag{1.20}$$

or in a multiplicative way – at least as long as the equilibrium price function ψ and hence $\bar{S}$ is strictly positive. In this case one would define the price process $\hat{S}^n = \{\hat{S}_k^n\}_{0 \le k \le n}$ by

$$\hat{S}_k^n := \bar{S}\big(t_k^n, U_k^n, \xi_k^n\big) \prod_{j=0}^{k} \frac{\bar{S}\big(t_j^n, U_j^n, \xi_{j-1}^n\big)}{\bar{S}\big(t_j^n, U_j^n, \xi_j^n\big)} \quad \text{for all } 0 \le k \le n. \tag{1.21}$$

Though both price processes $\check{S}^n$ and $\hat{S}^n$ adjust the loss-free liquidation price process by the immediate leverage of the large investor's transaction, they will in general not remove all influence of the strategy (ξ^n, b^n) on the price process. However, if ψ has a multiplicative structure then by (1.16) the product on the right hand side of (1.21) becomes a telescoping product, and thus the defining equation (1.21) for $\hat{S}^n_k$ simplifies to

$$\hat{S}^n_k = \overline{\psi}\big(t^n_k, U^n_k\big) \int_0^1 f\big(\theta \xi^n_{-1}\big)\, d\theta = \bar{S}\big(t^n_k, U^n_k, \xi^n_{-1}\big) \quad \text{for all } 0 \le k \le n, \tag{1.22}$$

i.e. the adjusted stock price $\hat{S}^n$ is the liquidation price in the same market, which would appear if the large investor did not trade during the whole time interval $[0, T]$, but kept his stock holdings at the position ξ^n_{-1} he had immediately before time $t^n_0 = 0$. □

2.1.4 Recursive Schemes for the Value Functions

We now give schemes to recursively calculate the paper and the real value function for a replicating strategy at all nodes of the binomial tree from the set of possible final values. On the one hand, such representations will imply that the value processes are (super-)martingales under the p- and r-martingale measure, respectively, and on the other hand, such a representation will be used in Section 3.4 to derive a PDE for the continuous-time limit of the value functions. As opposed to the corresponding recursive scheme in the Cox-Ross-Rubinstein model, the recursions for the paper and real value function in the large investor model will normally depend on the large investor's strategy, so, in general, the recursions for the value functions will not be of any use to find a replicating trading strategy for a given contingent claim.

Proposition 2.9. *Consider a large investor market described by the price system (ψ, μ).*

(i) *If (ξ^n, b^n) is a p-admissible trading strategy, then the associated paper value function $v^n : \mathcal{A}^n \times I\!R \to I\!R$ along the process $\big\{\big(t^n_k, U^n_k, \xi^n_{k-1}\big)\big\}_{0 \le k \le n}$ can be calculated from the possible realizations of the final paper values $V^n_n = v^n\big(T, U^n_n, \xi^n_{n-1}\big)$ by the recursive scheme*

$$\begin{aligned} v^n\big(t^n_{k-1}, U^n_{k-1}, \xi^n_{k-2}\big) = p^{\xi^n}_n\big(t^n_{k-1}, U^n_{k-1}, \xi^n_{k-2}\big) v^n\big(t^n_k, U^n_{k-1} + \delta, \xi^n_{k-1}\big) \\ + \Big(1 - p^{\xi^n}_n\big(t^n_{k-1}, U^n_{k-1}, \xi^n_{k-2}\big)\Big) v^n\big(t^n_k, U^n_{k-1} - \delta, \xi^n_{k-1}\big) \end{aligned} \tag{1.23}$$

for all $1 \le k \le n$. Moreover, the large investor's stock holdings ξ^n_{k-1} between time t^n_{k-1} and time t^n_k satisfy the fixed point equation

$$\xi^n_{k-1} = \frac{v^n\big(t^n_k, U^n_{k-1} + \delta, \xi^n_{k-1}\big) - v^n\big(t^n_k, U^n_{k-1} - \delta, \xi^n_{k-1}\big)}{S^{\xi^n}_\mu\big(t^n_k, U^n_{k-1} + \delta, \xi^n_{k-1}\big) - S^{\xi^n}_\mu\big(t^n_k, U^n_{k-1} - \delta, \xi^n_{k-1}\big)} \quad \textit{for all } 1 \le k \le n. \tag{1.24}$$

(ii) *If (ξ^n, b^n) is an r-admissible trading strategy, then the associated real value function $\bar{v}^n : \mathcal{A}^n \to I\!R$ can be calculated from the possible realizations of the final real values $\bar{V}^n_n = \bar{v}^n\big(T, U^n_n\big)$ by the recursive scheme*

$$\begin{aligned} \bar{v}^n\big(t^n_{k-1}, U^n_{k-1}\big) = \bar{p}^{\xi^n}_n\big(t^n_{k-1}, U^n_{k-1}\big) \Big(\bar{v}^n\big(t^n_k, U^n_{k-1} + \delta\big) + c^{\xi^n}_\mu\big(t^n_k, U^n_{k-1} + \delta, \xi^n_{k-1}\big)\Big) \\ + \Big(1 - \bar{p}^{\xi^n}_n\big(t^n_{k-1}, U^n_{k-1}\big)\Big) \Big(\bar{v}^n\big(t^n_k, U^n_{k-1} - \delta\big) + c^{\xi^n}_\mu\big(t^n_k, U^n_{k-1} - \delta, \xi^n_{k-1}\big)\Big) \end{aligned} \tag{1.25}$$

for all $1 \le k \le n$. *Moreover, in this case the number* ξ_{k-1}^n *of shares of stock held by the large investor between time* t_{k-1}^n *and* t_k^n *satisfies the fixed point equation*

$$\begin{aligned}\xi_{k-1}^n = &\frac{\bar v\big(t_k^n, U_{k-1}^n + \delta\big) - \bar v\big(t_k^n, U_{k-1}^n - \delta\big)}{\bar S\big(t_k^n, U_{k-1}^n + \delta, \xi_{k-1}^n\big) - \bar S\big(t_k^n, U_{k-1}^n - \delta, \xi_{k-1}^n\big)} \\ &+ \frac{c_\mu^{\xi^n}\big(t_k^n, U_{k-1}^n + \delta, \xi_{k-1}^n\big) - c_\mu^{\xi^n}\big(t_k^n, U_{k-1}^n - \delta, \xi_{k-1}^n\big)}{\bar S\big(t_k^n, U_{k-1}^n + \delta, \xi_{k-1}^n\big) - \bar S\big(t_k^n, U_{k-1}^n - \delta, \xi_{k-1}^n\big)}\end{aligned} \tag{1.26}$$

for all $1 \le k \le n$.

Proof. Basically, the proof of both parts follows the usual reasoning in standard Cox-Russ-Rubinstein models.

(*i*) Since (ξ^n, b^n) is p-admissible, it is in particular self-financing, i.e. (1.3.13) holds. Plugging Definition 1.25 into this equation we can rewrite (1.3.13) in terms of the paper value $V^n = \big\{V_k^n\big\}_{0\le k\le n}$ of (ξ^n, b^n), which leads to the system of equations

$$V_{k-1}^n = V_k^n - \xi_{k-1}^n\Big(S_\mu\big(t_k^n, U_k^n, \xi_{k-1}^n, \xi_k^n\big) - S_\mu\big(t_{k-1}^n, U_{k-1}^n, \xi_{k-2}^n, \xi_{k-1}^n\big)\Big) \tag{1.27}$$

for $1 \le k \le n$. Now we employ (1.3.20) to express the paper value by means of the paper value function, replace U_k^n by its two possible realizations $U_{k-1}^n \pm \delta$, and make use of (1.1), to rewrite (1.27) for all $1 \le k \le n$ as

$$\begin{aligned}v^n\big(t_{k-1}^n, U_{k-1}^n, \xi_{k-2}^n\big) &= v^n\big(t_k^n, U_{k-1}^n \pm \delta, \xi_{k-1}^n\big) \\ &\quad - \xi_{k-1}^n\Big(S_\mu^{\xi^n}\big(t_k^n, U_{k-1}^n \pm \delta, \xi_{k-1}^n\big) - S_\mu^{\xi^n}\big(t_{k-1}^n, U_{k-1}^n, \xi_{k-2}^n\big)\Big).\end{aligned} \tag{1.28}$$

Subtracting one equation in (1.28) from the other and then dividing the result by the term $S_\mu^{\xi^n}\big(t_k^n, U_{k-1}^n + \delta, \xi_{k-1}^n\big) - S_\mu^{\xi^n}\big(t_k^n, U_{k-1}^n - \delta, \xi_{k-1}^n\big) > 0$ yields (1.24). The denominator of (1.24) is strictly larger than zero, since (ξ^n, b^n) is p-admissible.

In order to derive the recursive scheme (1.23) we now just have to plug (1.24) into (1.28), recall $\xi_{k-1}^n = \xi^n\big(t_{k-1}^n, U_{k-1}^n\big)$, and apply the definition of the p-martingale weight function $p_n^{\xi^n}$.

(*ii*) In order to prove the second part, we will once again rewrite the self-financing condition, now in terms of the real value process $\bar V^n$. By Definition 1.27 the real value $\bar V^n$ satisfies

$$b_k^n = \bar V_k^n - \xi_k^n \bar S\big(t_k^n, U_k^n, \xi_k^n\big) \quad \text{for all } 0 \le k \le n. \tag{1.29}$$

Now Propositions 1.4 and 1.2 assure that the fair price condition (1.1.1) holds, hence an application of that equation with $\alpha_1 = -\xi_k^n$, $\alpha_2 = \xi_{k-1}^n$, and $\alpha_3 = \xi_k^n - \xi_{k-1}^n$ shows that the revenue of selling ξ_k^n shares meets the price paid for buying at first ξ_{k-1}^n shares and then $\xi_k^n - \xi_{k-1}^n$ shares, if all transactions are based on the corresponding benchmark prices. Therefore, (1.29) can be rewritten as

$$b_k^n = \bar V_k^n - \xi_{k-1}^n \bar S\big(t_k^n, U_k^n, \xi_{k-1}^n\big) - \big(\xi_k^n - \xi_{k-1}^n\big) S^*\big(t_k^n, U_k^n, \xi_{k-1}^n, \xi_k^n\big) \quad \text{for all } 0 \le k \le n,$$

and by the remark following Definition 1.27 a self-financing portfolio strategy fulfills

$$b_{k-1}^n = \bar V_k^n - \xi_{k-1}^n \bar S\big(t_k^n, U_k^n, \xi_{k-1}^n\big) + c_\mu^{\xi^n}\big(t_k^n, U_k^n, \xi_{k-1}^n\big) \quad \text{for all } 1 \le k \le n.$$

Applying (1.29) we can get rid of the cash holdings b_{k-1}^n and obtain for all $1 \le k \le n$:

$$\bar V_{k-1}^n = \bar V_k^n + c_\mu^{\xi^n}\big(t_k^n, U_k^n, \xi_{k-1}^n\big) - \xi_{k-1}^n\Big(\bar S\big(t_k^n, U_k^n, \xi_{k-1}^n\big) - \bar S\big(t_{k-1}^n, U_{k-1}^n, \xi_{k-1}^n\big)\Big). \tag{1.30}$$

Now we can represent the real value process by means of the real value functions as in (1.3.18) and replace U_k^n by its two possible outcomes $U_{k-1}^n \pm \delta$ based on the information at time t_{k-1}^n. Then (1.30) implies

$$\begin{aligned}\bar{v}^n\big(t_{k-1}^n, U_{k-1}^n\big) = \bar{v}^n\big(t_k^n, U_{k-1}^n \pm \delta\big) + c_\mu^{\xi^n}\big(t_k^n, U_{k-1}^n \pm \delta, \xi_{k-1}^n\big) \\ - \xi_{k-1}^n\Big(\bar{S}\big(t_k^n, U_{k-1}^n \pm \delta, \xi_{k-1}^n\big) - \bar{S}\big(t_{k-1}^n, U_{k-1}^n, \xi_{k-1}^n\big)\Big)\end{aligned} \tag{1.31}$$

for all $1 \le k \le n$. As in the first part we now subtract one equation of (1.31) from the other, which gives us for all $1 \le k \le n$ the representation (1.26).

Last but not least, we insert (1.26) into (1.31) and apply the definition of the r-martingale weight function $\bar{p}_n^{\xi^n}$ to arrive at (1.25).

Thus, both statements of the proposition have been proved. **q.e.d.**

The right-hand sides of the recursive equations (1.23) and (1.25) still depend on the strategy function $\xi^n : \mathcal{A}^n \to I\!R$ via the probability $p_n^{\xi^n}$ and $\bar{p}_n^{\xi^n}$ and the function $c_\mu^{\xi^n}$. For this reason, we have to determine the strategy function *before* we can analyze the corresponding value process if we want to replicate a certain contingent claim or if we want to show the convergence of our discrete-time model towards a continuous-time limit. This order seems to be in opposition to the order in deriving a replication strategy in the Cox-Ross-Rubinstein model. But it is the more natural approach, since the *strategy* is the fundamental quantity for the replication of a contingent claim, in particular if the large investor is faced with transaction losses or gains when shifting some of his stock holdings into cash or vice versa.

Remark. If the equilibrium price function $\psi : [0,T] \times I\!R^2 \to I\!R$ does not depend on the large investor's stock holdings, such that the large investor acts like a small investor, the (paper) value function simplifies to $v^n(t,u,\xi) = v^n(t,u,0)$ for all $(t,u,\xi) \in \mathcal{A}^n \times I\!R$, according to its definition in (1.3.19). In this case the p-martingale weights $\{p_n^{\xi^n}\big(t_{k-1}^n, U_{k-1}^n, \xi_{k-2}^n\big)\}_{1 \le k \le n}$ do not depend on the investor's strategy either, and one can recursively calculate the value function of a self-financing strategy (ξ^n, b^n) along the process $\{\big(t_k^n, U_k^n\big)\}$ solely from knowing its possible final values, since the recursive scheme (1.23) *separates* the calculation of the value function from the corresponding strategy function $\xi^n : \mathcal{A}^n \to I\!R$.
In general, however, the recursive scheme given by (1.23) strongly depends on the particular trading strategy of the large investor: For each of the weights $\{p_n^{\xi^n}\big(t_{k-1}^n, U_{k-1}^n, \xi_{k-2}^n\big)\}_{1 \le k \le n}$ the strategy function $\xi^n : \mathcal{A}^n \to I\!R$ has to be evaluated at four different nodes. In particular, it is unpleasant that the formula (1.23) to calculate the paper value *at* time t_{k-1}^n depends on the large investor's stock holdings ξ_{k-2}^n immediately *before* t_{k-1}^n. The formula in (1.24) does not help to calculate these stock holdings, since we would need to know the paper value function $v^n\big(t_{k-1}^n, \cdot, \cdot\big) : \mathcal{U}_{k-1}^n \times I\!R \to I\!R$ at time t_{k-1}^n. Hence we will end up with a vicious cycle if $\xi^n : \mathcal{A}^n \to I\!R$ is not known in advance.
In the special case where the price-determining measure μ is the Dirac measure δ_1 concentrated in 1, such that at any trading time the stock price in the market jumps from the old to the new equilibrium before the large investor can start his transactions and the large investor price function simplifies to $S_\mu(t,u,\xi_1,\xi_2) = \psi(t,u,\xi_2)$ for all $(t,u,\xi_1,\xi_2) \in [0,T] \times I\!R^3$, both the value function and the p-martingale weight in (1.23) do not depend on ξ_{k-2}^n, but the p-martingale weight $p_n^{\xi^n}\big(t_{k-1}^n, U_{k-1}^n, \xi_{k-2}^n\big)$ still depends on the values of the strategy function at time t_k^n and t_{k+1}^n. Thus, (1.23) still does not separate the recursive calculation of the paper value function from the associated strategy function.
Compared to the recursive scheme for the paper value, the recursion (1.25) for the real value is less intertwined with the strategy function $\xi^n : \mathcal{A}^n \to I\!R$. By definition, the real value depends

only on the current stock holdings of the large investor, and the same holds true for the r-martingale weights $\{\bar{p}_n^{\xi^n}(t_{k-1}^n, U_{k-1}^n)\}_{1\le k\le n}$. If the equilibrium price function is multiplicative it follows from the representation (1.16) that the r-martingale weights (1.14) do not depend on the strategy at all. However, in general the implied transaction losses $c_\mu^{\xi^n}(t_k^n, U_{k-1}^n \pm \delta, \xi_{k-1}^n)$, which depend on both the large investors stock holdings ξ_{k-1}^n immediately before and the holdings $\xi^n(t_k^n, U_{k-1}^n \pm \delta)$ at (the end of) time t_k^n, will still prevent us to separate the recursive calculation of the real value from the associated strategy.
The recursive scheme (1.25) does not depend on the strategy if the equilibrium price function ψ is multiplicative *and* if the large investor always trades at the benchmark price, such that no implied transaction (gains or) losses can occur. In that case, the recursive scheme to calculate the real value is separated from the strategy function in exactly the same way as the calculation of the value function in a small investor market. We will discuss this important special case in some more detail in Section 2.4.3. □

The missing separability of the recursive value calculation in (1.23) and (1.25) from the corresponding strategy function $\xi^n : \mathcal{A}^n \to I\!R$ for most large investor markets (ψ, μ) will complicate our approach to approximate the value functions v^n and $\bar{v}^n$ for large $n \in I\!N$ by some continuous functions; it forces us to find some approximations for the strategy function first. For that reason, we will derive in Section 2.2 recursive schemes for the strategy function $\xi^n : \mathcal{A}^n \to I\!R$, which basically resemble the form of (1.23) and (1.25), but which now depend only on the strategy function itself.

2.1.5 The Value Processes as (Super-)Martingales

As a consequence of the recursive equations for the value *functions* in Proposition 2.9 we can easily characterize the value *processes*, which are associated to a certain self-financing trading strategy of the large investor, in terms of the p- and r-martingale measures, respectively: Under the right measure the value processes become (super-)martingales. These characterizations can then be used to prove the no-arbitrage statement of Proposition 2.4.

Corollary 2.10. *Consider a large investor market described by the price system (ψ, μ).*

(i) For every p-admissible trading strategy (ξ^n, b^n) the paper value process $V^n = \{V_k^n\}_{0\le k\le n}$ is a martingale under the p-martingale measure.

(ii) If the price system (ψ, μ) excludes instantaneous transaction gains, then for every r-admissible trading strategy (ξ^n, b^n) the real value process $\bar{V}^n = \{\bar{V}_k^n\}_{0\le k\le n}$ is a super-martingale under the r-martingale measure.

(iii) If the price system (ψ, μ) excludes instantaneous transaction gains and *transaction losses (so that the large investor always trades at the benchmark price), then for every r-admissible trading strategy (ξ^n, b^n) the real value process $\bar{V}^n$ is even a martingale under the r-martingale measure.*

Proof. The proofs of the three statements are straightforward consequences of Proposition 2.9 and Definition 2.7.

(i) By the definition of the p-martingale measure $\mathbf{P}_n^{\xi^n}$ we can rewrite (1.23) in terms of $\mathbf{P}_n^{\xi^n}$ as

$$V_{k-1}^n = v(t_{k-1}^n, U_{k-1}^n, \xi_{k-2}^n) = \mathbf{E}_n^{\xi^n}\left[v(t_k^n, U_k^n, \xi_{k-1}^n)\,\middle|\mathcal{F}_{k-1}^n\right] = \mathbf{E}_n^{\xi^n}\left[V_k^n\,\middle|\mathcal{F}_{k-1}^n\right], \quad (1.32)$$

for all $1 \le k \le n$, which shows the martingale property of V^n.

(ii) Since (ψ, μ) is assumed to exclude instantaneous transaction gains, the transaction loss function $c^n_\mu : [0,T] \times I\!R^3 \to I\!R$ is nonnegative, and so is the reduced function $\tilde{c}^n_\mu : \mathcal{A}^n \times I\!R \to I\!R$ of (1.2). Moreover the r-martingale weights $\{\bar{p}^n(t^n_{k-1}, U^n_{k-1})\}_{1\le k\le n}$ are probability weights, hence (1.25) implies for all $1 \le k \le n$:

$$\begin{aligned} \bar{v}^n\big(t^n_{k-1}, U^n_{k-1}\big) \ge \bar{p}^{\xi^n}_n\big(t^n_{k-1}, U^n_{k-1}\big)\bar{v}^n\big(t^n_k, U^n_{k-1} + \delta\big) \\ + \Big(1 - \bar{p}^{\xi^n}_n\big(t^n_{k-1}, U^n_{k-1}\big)\Big)\bar{v}^n\big(t^n_k, U^n_{k-1} - \delta\big). \end{aligned} \tag{1.33}$$

Of course, this inequality can be rewritten in terms of the r-martingale measure $\bar{\mathbf{P}}^{\xi^n}_n$, and we get the supermartingale property

$$\bar{V}^n_{k-1} = \bar{v}^n\big(t^n_{k-1}, U^n_{k-1}\big) \ge \bar{\mathbf{E}}^{\xi^n}_n\Big[\bar{v}^n\big(t^n_k, U^n_k\big)\,\Big|\,\mathcal{F}^n_{k-1}\Big] = \bar{\mathbf{E}}^{\xi^n}_n\big[\bar{V}^n_k \,\big|\, \mathcal{F}^n_{k-1}\big] \tag{1.34}$$

for all $1 \le k \le n$. In general, $\bar{V}^n$ will not be a martingale under $\bar{\mathbf{P}}^{\xi^n}_n$ due to the transaction loss terms in (1.25).

(iii) However, if the market price mechanism excludes both instantaneous transaction gains and instantaneous transaction losses, so that the large investor can always trade at the benchmark price, the implied transaction losses $c^{\xi^n}_\mu\big(t^n_k, U^n_{k-1} \pm \delta, \xi^n_{k-1}\big)$ in (1.25) vanish, and equality holds in (1.33) and (1.34) for all $1 \le k \le n$, so that $\bar{V}^n$ is even a $\bar{\mathbf{P}}^{\xi^n}_n$-martingale.

Thus, all three statements of the corollary have been proved. **q.e.d.**

Remark. Since both the large investor price process $\big\{S_\mu\big(t^n_k, U^n_k, \xi^n_{k-1}, \xi^n_k\big)\big\}_{0\le k\le n}$ *and* the paper value process $V^n = \big\{V^n_k\big\}_{0\le k\le n}$ are martingales under the p-martingale measure, the fundamental property of small investor markets *without* transaction costs that both the stock price and the value process of any self-financing trading strategy are martingales under the same measure, is preserved if the large investor stock price is taken as the relevant price for valuation. However, the p-martingale measure $\mathbf{P}^{\xi^n}_n$ is by far more complex than in the small investor case, since it strongly depends on the particular trading strategy of the large investor. In continuous-time, Frey (1998) uses the fact that both the price process and the paper value process are martingales under the p-martingale measure in order to determine the replicating strategy of a contingent claim.
If the large investor's portfolio is evaluated by means of the loss-free liquidation price, our large investor model is more reminiscent of a small investor model *with* transaction costs: Like the value process in such a model, the real value is only a supermartingale under the r-martingale measure $\bar{\mathbf{P}}^{\xi^n}_n$, and a $\mathbf{P}^{\xi^n}_n$-martingale only if there are no transaction losses. Also Baum (2001) noted that in his general semimartingale model the real value process is a supermartingale, and then employed the supermartingale property for his no-arbitrage result. In the subsequent paper of Bank and Baum (2004), the finite variation part of the real value is interpreted as the induced transaction costs due to limited liquidity. However, since Baum (2001) and Bank and Baum (2004) only considered price mechanisms which correspond to a price determining measure $\mu = \delta_1$, these authors did not obtain a martingale property analogously to Corollary 2.10*(iii)* as a special case. Instead, they focus on superreplication with respect to the real value to obtain reasonable prices for a contingent claim. Baum (2001) then transforms theses results into corresponding results for superreplication with respect to the paper value.
We will come back to similarities with transaction costs models in Section 2.3 and we continue the discussion of similarities to the model of Baum (2001) and Bank and Baum (2004) at the end of Section 2.4. □

Now we can finally devote our attention to the proof of the no-arbitrage statement of Proposition 2.4.

Proof of Proposition 2.4. Focusing only on admissible trading strategies as defined in Definition 2.2, the proof of both parts follows from Corollary 2.10 and the usual arguments of no-arbitrage theory in small investor markets.

(i) Assume $(\xi^n, b^n) \in \mathcal{Z}_P^n$ is an arbitrage opportunity. Then we will have under the original measure $\mathbf{P}^n$ the strict inequality $\mathbf{E}^n\big[v^n\big(T, U_n^n, \xi_{n-1}^n\big) - v^n\big(0, U_0^n, \xi_{-1}^n\big)\big] > 0$. Since $\mathbf{P}_n^{\xi^n}$ is equivalent to $\mathbf{P}^n$, the strict inequality is preserved by the change from the original measure $\mathbf{P}^n$ to the paper value martingale measure $\mathbf{P}_n^{\xi^n}$. On the other hand, Corollary 2.10 implies that $\mathbf{E}_n^{\xi^n}\big[v^n\big(T, U_n^n, \xi_{n-1}^n\big)\big] = v^n\big(0, U_0^n, \xi_{-1}^n\big)$, which leads to a contradiction.

(ii) The proof of (ii) follows along the same lines.

Thus, within the classes of p- and r-admissible strategies there are no paper-value and real-value arbitrage opportunities, respectively. **q.e.d.**

Remark. The no-arbitrage statement of Proposition 2.4 has forced us to restrict the class of strategies used by the large investor to p- and r-admissible trading strategies, respectively. However, one might argue that a realistic model for a large investor model has to allow (limited) arbitrage opportunities by the large investor.
Then it may make sense to relax the class of p- or r-admissible trading strategies which the large investor can pick. For example Definition 2.2 could be relaxed such that all path-independent and self-financing strategies (ξ^n, b^n) which satisfy for all $1 \le k \le n$ the inequality $S_\mu^{\xi^n}\big(t_k^n, U_{k-1}^n - \delta, \xi_{k-1}^n\big) < S_\mu^{\xi^n}\big(t_k^n, U_{k-1}^n + \delta, \xi_{k-1}^n\big)$ are p-admissible. Besides the no-arbitrage statement of Proposition 2.4 all statements of the two previous section, especially the recursive representation of Proposition 2.9(i), would transfer to the more general situation, but the p-martingale weight function $p_n^{\xi^n}$ of Definition 2.7 may then take on arbitrary real values so that the p-martingale measure $\mathbf{P}_n^{\xi^n}$ might be only a signed measure, and not necessarily a probability measure equivalent to $\mathbf{P}^n$. □

2.2 Recursive Schemes for the Strategy Function

As we have seen in the discussions in Section 2.1.4, the strategy function $\xi^n : \mathcal{A}^n \to I\!R$ is the most important object in large investor market models, and in general, we will only be able to derive some results for the value functions if we have first derived similar results for the strategy function. This especially holds true if we look for some convergence results for our large investor models as $n \to \infty$. Therefore, we will start the convergence results in Chapter 3 with convergence results for the sequence $\big\{\xi^n\big\}_{n \in I\!N}$ of strategy functions $\xi^n : \mathcal{A}^n \to I\!R$, and thus we need for each (sufficiently large) $n \in I\!N$ a suitable representation for the strategy function $\xi^n : \mathcal{A}^n \to I\!R$ which can support us for the proof of convergence. The representations given in this section will be difference equations of second order.
In view of finding representations for $\xi^n : \mathcal{A}^n \to I\!R$ and some fixed $n \in I\!N$, we start with the fixed point equation (1.4.10). Next to the original fixed point problem (1.4.10) we have already derived two alternate representations of this fixed point problem in Section 2.1.4, namely the representations (1.24) and (1.26). Like the original fixed point problems, both representations have the same drawback that they involve not only the strategy function $\xi^n : \mathcal{A}^n \to I\!R$, but also some additional function: either the function $b^n : \mathcal{A}^n \to I\!R$ describing the cash amount held by the large investor, or one of the two sorts of value functions for the large investor's portfolio.

But in order to take limits it is necessary to find a representation of $\xi^n : \mathcal{A}^n \to I\!R$ which does not involve any other function for which the limit as $n \to \infty$ is not known yet. Fortunately, we can easily get rid of the large investor's cash holdings in the representation (1.4.10), and thus arrive at the following two representations, which for the sake of clarity are written in terms of our shorthands (1.3.12).

Proposition 2.11. *Consider a large investor market described by the price system (ψ, μ).*

(i) If the trading strategy (ξ^n, b^n) is p-admissible, then the associated strategy function $\xi^n : \mathcal{A}^n \to I\!R$ satisfies the recursive relation

$$\begin{aligned}\xi^n_{(k-1)i} = \xi^n_{k(i+1)} & \frac{S^{\xi^n}_\mu\big(t^n_k, u^n_{k(i+1)}, \xi^n_{(k-1)i}\big) - S^{\xi^n}_\mu\big(t^n_{k+1}, u^n_{(k+1)i}, \xi^n_{k(i+1)}\big)}{S^{\xi^n}_\mu\big(t^n_k, u^n_{k(i+1)}, \xi^n_{(k-1)i}\big) - S^{\xi^n}_\mu\big(t^n_k, u^n_{k(i-1)}, \xi^n_{(k-1)i}\big)} \\ + \xi^n_{(k+1)i} & \frac{S^{\xi^n}_\mu\big(t^n_{k+1}, u^n_{(k+1)i}, \xi^n_{k(i+1)}\big) - S^{\xi^n}_\mu\big(t^n_{k+1}, u^n_{(k+1)i}, \xi^n_{k(i-1)}\big)}{S^{\xi^n}_\mu\big(t^n_k, u^n_{k(i+1)}, \xi^n_{(k-1)i}\big) - S^{\xi^n}_\mu\big(t^n_k, u^n_{k(i-1)}, \xi^n_{(k-1)i}\big)} \\ + \xi^n_{k(i-1)} & \frac{S^{\xi^n}_\mu\big(t^n_{k+1}, u^n_{(k+1)i}, \xi^n_{k(i-1)}\big) - S^{\xi^n}_\mu\big(t^n_k, u^n_{k(i-1)}, \xi^n_{(k-1)i}\big)}{S^{\xi^n}_\mu\big(t^n_k, u^n_{k(i+1)}, \xi^n_{(k-1)i}\big) - S^{\xi^n}_\mu\big(t^n_k, u^n_{k(i-1)}, \xi^n_{(k-1)i}\big)}\end{aligned} \tag{2.1}$$

for all $1 \le k \le n-1$ and all $i \in \mathcal{I}_{k-1}$.

(ii) If the trading strategy (ξ^n, b^n) is r-admissible, then the associated strategy function $\xi^n : \mathcal{A}^n \to I\!R$ solves the system of equations given by

$$\begin{aligned}\xi^n_{(k-1)i} = \xi^n_{k(i+1)} & \frac{\bar S\big(t^n_k, u^n_{k(i+1)}, \xi^n_{k(i+1)}\big) - \bar S\big(t^n_{k+1}, u^n_{(k+1)i}, \xi^n_{k(i+1)}\big)}{\bar S\big(t^n_k, u^n_{k(i+1)}, \xi^n_{(k-1)i}\big) - \bar S\big(t^n_k, u^n_{k(i-1)}, \xi^n_{(k-1)i}\big)} \\ + \xi^n_{k(i-1)} & \frac{\bar S\big(t^n_{k+1}, u^n_{(k+1)i}, \xi^n_{k(i-1)}\big) - \bar S\big(t^n_k, u^n_{k(i-1)}, \xi^n_{k(i-1)}\big)}{\bar S\big(t^n_k, u^n_{k(i+1)}, \xi^n_{(k-1)i}\big) - \bar S\big(t^n_k, u^n_{k(i-1)}, \xi^n_{(k-1)i}\big)} \\ + & \frac{D^{\xi^n}_\mu\big(t^n_k, u^n_{(k-1)i}\big)}{\bar S\big(t^n_k, u^n_{k(i+1)}, \xi^n_{(k-1)i}\big) - \bar S\big(t^n_k, u^n_{k(i-1)}, \xi^n_{(k-1)i}\big)}\end{aligned} \tag{2.2}$$

for all $1 \le k \le n-1$ and all $i \in \mathcal{I}_{k-1}$, where the nominator $D^{\xi^n}_\mu\big(t^n_k, u^n_{(k-1)i}\big)$ of the last term denotes the spread

$$\begin{aligned}D^{\xi^n}_\mu\big(t^n_k, u^n_{(k-1)i}\big) &= c_\mu\big(t^n_k, u^n_{k(i+1)}, \xi^n_{(k-1)i}, \xi^n_{k(i+1)}\big) + c_\mu\big(t^n_{k+1}, u^n_{(k+1)i}, \xi^n_{k(i+1)}, \xi^n_{(k+1)i}\big) \\ &\quad - c_\mu\big(t^n_k, u^n_{k(i-1)}, \xi^n_{(k-1)i}, \xi^n_{k(i-1)}\big) - c_\mu\big(t^n_{k+1}, u^n_{(k+1)i}, \xi^n_{k(i+1)}, \xi^n_{(k+1)i}\big)\end{aligned}$$

between the transaction losses along the two possible paths which lead from fundamentals of $U^n_{k-1} = u^n_{(k-1)i}$ at time t^n_{k-1} to fundamentals of $U^n_{k+1} = u^n_{(k+1)i}$ at time t^n_{k+1}.

Proof. Let us fix $1 \le k \le n-1$ and $i \in \mathcal{I}_{k-1}$. By (1.4.9) applied for $(k+1, i \pm 1)$ instead of (k, i), we know that if the fundamentals have moved from $u^n_{k(i\pm1)}$ to $u^n_{(k+1)i}$, the large investor's portfolios $\big(\xi^n_{k(i\pm1)}, b^n_{k(i\pm1)}\big)$ and $\big(\xi^n_{(k+1)i}, b^n_{(k+1)i}\big)$ before and after his self-financing transaction at time t^n_{k+1} satisfy

$$b^n_{k(i\pm1)} = b^n_{(k+1)i} + \big(\xi^n_{(k+1)i} - \xi^n_{k(i\pm1)}\big) S_\mu\big(t^n_{k+1}, u^n_{(k+1)i}, \xi^n_{k(i\pm1)}, \xi^n_{(k+1)i}\big). \tag{2.3}$$

If we plug these two equations into (1.4.10), we get

$$\begin{aligned}0 &= \big(\xi^n_{k(i+1)} - \xi^n_{(k-1)i}\big) S_\mu\big(t^n_k, u^n_{k(i+1)}, \xi^n_{(k-1)i}, \xi^n_{k(i+1)}\big) \\ &\quad + \big(\xi^n_{(k-1)i} - \xi^n_{k(i-1)}\big) S_\mu\big(t^n_k, u^n_{k(i-1)}, \xi^n_{(k-1)i}, \xi^n_{k(i-1)}\big) \\ &\quad - \big(\xi^n_{k(i+1)} - \xi^n_{(k+1)i}\big) S_\mu\big(t^n_{k+1}, u^n_{(k+1)i}, \xi^n_{k(i+1)}, \xi^n_{(k+1)i}\big) \\ &\quad - \big(\xi^n_{(k+1)i} - \xi^n_{k(i-1)}\big) S_\mu\big(t^n_{k+1}, u^n_{(k+1)i}, \xi^n_{k(i-1)}, \xi^n_{(k+1)i}\big).\end{aligned} \tag{2.4}$$

This equation can now easily be rewritten in representations similar to (1.23) and (1.25):

(i) Collecting similar terms in (2.4) and then dividing the result by the stock price difference $S_\mu\big(t_k^n, u_{k(i+1)}^n, \xi_{(k-1)i}^n, \xi_{k(i+1)}^n\big) - S_\mu\big(t_k^n, u_{k(i-1)}^n, \xi_{(k-1)i}^n, \xi_{k(i-1)}^n\big)$ leads to (2.1). For all p-admissible trading strategies the denominator in (2.1) is strictly positive.

(ii) Recalling the definition of the implied transaction loss function, we have

$$(\xi_2 - \xi_1)S_\mu(t, u, \xi_1, \xi_2) = (\xi_2 - \xi_1)S^*(t, u, \xi_1, \xi_2) + c_\mu(t, u, \xi_1, \xi_2) \tag{2.5}$$

for all $(t, u, \xi_1, \xi_2) \in [0, T] \times I\!R^3$. Moreover, due to the fair price condition (1.1.1) and the definition of the loss-free liquidation price $\bar{S}(t, u, \xi) = S^*(t, u, \xi, 0) = S^*(t, u, 0, \xi)$ for all $(t, u, \xi) \in [0, T] \times I\!R^2$ it follows that

$$(\xi_2 - \xi_1)S^*(t, u, \xi_1, \xi_2) = \xi_2\bar{S}(t, u, \xi_2) - \xi_1\bar{S}(t, u, \xi_1). \tag{2.6}$$

If (2.6) is inserted in (2.5) and then the latter equation utilized to replace the S_μ-terms in (2.4), we obtain

$$\begin{aligned}
0 = {} & \xi_{k(i+1)}^n\bar{S}\big(t_k^n, u_{k(i+1)}^n, \xi_{k(i+1)}^n\big) - \xi_{(k-1)i}^n\bar{S}\big(t_k^n, u_{k(i+1)}^n, \xi_{(k-1)i}^n\big) \\
& + \xi_{(k-1)i}^n\bar{S}\big(t_k^n, u_{k(i-1)}^n, \xi_{(k-1)i}^n\big) - \xi_{k(i-1)}^n\bar{S}\big(t_k^n, u_{k(i-1)}^n\xi_{k(i-1)}^n\big) \\
& - \xi_{k(i+1)}^n\bar{S}\big(t_{k+1}^n, u_{(k+1)i}^n, \xi_{k(i+1)}^n\big) + \xi_{(k+1)i}^n\bar{S}\big(t_{k+1}^n, u_{(k+1)i}^n, \xi_{(k+1)i}^n\big) \\
& - \xi_{(k+1)i}^n\bar{S}\big(t_{k+1}^n, u_{(k+1)i}^n, \xi_{(k+1)i}^n\big) + \xi_{k(i-1)}^n\bar{S}\big(t_{k+1}^n, u_{(k+1)i}^n, \xi_{k(i-1)}^n\big) \\
& + D_\mu^{\xi^n}\big(t_k^n, u_{(k-1)i}^n\big).
\end{aligned}$$

If we now rearrange terms and then divide the resulting equation by the stock price difference $\bar{S}\big(t_k^n, u_{k(i-1)}^n, \xi_{(k-1)i}^n\big) - \bar{S}\big(t_k^n, u_{k(i-1)}^n, \xi_{(k-1)i}^n\big)$, which is strictly positive for all r-admissible trading strategies, we indeed obtain (2.2).

This conclude our proof. **q.e.d.**

Both recursive schemes (2.1) and (2.2) are implicit difference equations of second order in space and time for the strategy function $\xi^n : \mathcal{A}^n \to I\!R$, since they both depend on the values of ξ^n at the four points $\big(t_{k\pm1}^n, u_{(k\pm1)i}^n\big)$ and $\big(t_k^n, u_{k(i\pm1)}^n\big)$. Despite the involvement of three points in time, we will see in Chapter 3 that the difference equation converges towards a differential equation of second order in space and *first* order in time.

Remark. To some extend, the recursive schemes (2.1) and (2.2) for calculating the strategy function $\xi^n : \mathcal{A}^n \to I\!R$ resemble the schemes to calculate the value functions in Proposition 2.9. However, there are several differences. If the pre-trading stock endowment ξ_{-1}^n and the strategy function $\xi^n : \mathcal{A}^n \to I\!R$ are known, the recursions of Proposition 2.9 can be used to calculate the value functions as soon as the corresponding final values, $V_n^n = v_n^n\big(T, U_n^n, \xi_{n-1}^n\big)$ and $\bar{V}_n^n = \bar{v}^n\big(T, U_n^n\big)$, respectively, are known in any state of the world. In contrast, the recursions of Proposition 2.11 require the knowledge of the large investors stock holdings $\xi_{n-1}^n = \xi^n\big(t_{n-1}^n, U_{n-1}^n\big)$ and $\xi_n^n = \xi^n\big(T, U_n^n\big)$ *immediately before and* at maturity $t_n^n = T$, since for each $1 \le k \le n-1$ and each particular realization $u_{(k-1)i}^n$ of U_{k-1}^n both equations for $\xi^n\big(t_{k-1}^n, U_{k-1}^n\big) = \xi_{(k-1)i}^n$ depend not only on the two possible stock holdings $\xi^n\big(t_k^n, U_{k-1}^n \pm \delta\big) = \xi_{k(i\pm1)}^n$ which the strategy function ξ^n will prescribe at time t_k^n, depending on the outcome of $U_k^n = U_{k-1}^n \pm \delta$, but also on the stock holdings $\xi^n\big(t_{k+1}^n, U_{k-1}^n\big) = \xi_{(k+1)i}^n$ which will be prescribed at time t_{k+1}^n if the fundamental process will return to the value U_{k-1}^n at this time.

In Section 1.4.2, the stock holdings ξ^n_{n-1} immediately before maturity have been determined from the final stock and cash holdings $\xi^n_n = \xi_n$ and $b^n_n = b_n$ via the fixed point equation (1.4.10); in Section 1.4.3, the stock holdings ξ^n_{n-1} and ξ^n_n immediately before and at maturity have been determined simultaneously in Proposition 1.36 such that they satisfy the condition (1.4.17) of paper value replication.
A second difference relates to the implicit form of the determining equations for $\xi^n_{(k-1)i}$ in Proposition 2.11. Given that the strategy function $\xi^n : \mathcal{A}^n \to I\!R$ is known, the recursive schemes for the value functions in Proposition 2.9 provide explicit equations for the possible values $v^n\big(t^n_{k-1}, U^n_{k-1}, \xi^n_{k-2}\big)$ and $\bar{v}^n\big(t^n_{k-1}, U^n_{k-1}\big)$ of the value functions at time t^n_{k-1} from the possible values of these functions at time t^n_k. The recursion schemes for the strategy function $\xi^n : \mathcal{A}^n \to I\!R$ of Proposition 2.11, however, remain implicit schemes, since the weights and the spread $D^{\xi^n}_\mu\big(t^n_k, u^n_{(k-1)i}\big)$ between the transaction losses on the right-hand side of (2.1) and (2.2) depend on $\xi^n_{(k-1)i}$ as well.
In the special situation where the large investor price function has the particular form $S_\mu(t, u, \xi_1, \xi_2) = \psi(t, u, \xi_2)$ for all $(t, u, \xi_1, \xi_2) \in [0, T] \times I\!R^3$ (as it is for example the case if the price-determining measure μ is the Dirac measure δ_1 concentrated in 1) the first representation (2.1) of Proposition 2.11 becomes an explicit equation for the value $\xi^n_{(k-1)i} = \xi\big(t^n_{k-1}, u^n_{(k-1)i}\big)$ for all $1 \le k \le n-1$ and $i \in \mathcal{I}_{k-1}$, while the second representation may still remain an implicit equation. □

Example 2.1. Let us assume that the price system (ψ, μ) excludes any instantaneous transaction gains or losses, since either the price-determining measure μ is the Lebesgue measure λ on $[0, 1]$, or since the equilibrium price function ψ does not depend on the large investor's stock holdings. In such a market, the spread $D^{\xi^n}_\mu\big(t^n_k, u^n_{(k-1)i}\big)$ between the possible transaction losses vanishes, and (2.2) can be used to calculate the restriction $\xi^n\big|_{\mathcal{A}^n(n-1)}$ of the strategy function $\xi^n : \mathcal{A}^n \to I\!R$ to the possible time-space combinations before time $t^n_n = T$, solely from the possible stock holdings $\xi^n_{n-1} = \xi^n\big(t^n_{n-1}, U^n_{n-1}\big)$ immediately before time T, without knowing the possible values $\xi^n_n = \xi^n\big(T, U^n_n\big)$ of ξ^n at maturity. Since for all $(t, u, \xi_1, \xi_2) \in [0, T] \times I\!R^3$ Propositions 1.4 and 1.2 imply that

$$(\xi - \xi_1) S^*(t, u, \xi_1, \xi) + (\xi_2 - \xi) S^*(t, u, \xi_2, \xi)$$

does not depend on $\xi \in I\!R$, it follows that the knowledge of ξ^n_n is also not necessary if the restriction $\xi^n\big|_{\mathcal{A}^n(n-1)}$ is calculated from (2.1). □

Example 2.2. Let the price system (ψ, μ) be determined by some equilibrium price function $\psi : [0, T] \times I\!R^2 \to I\!R$, $(t, u, \xi) \mapsto \psi(t, u, \xi)$ which is nondecreasing in ξ, and an associated price-determining measure μ which is concentrated on $[0, 1]$. Then for every p-admissible *convex* trading strategy we have

$$S^{\xi^n}_\mu\big(t^n_{k+1}, u^n_{(k+1)i}, \xi^n_{k(i-1)}\big) \le S^{\xi^n}_\mu\big(t^n_{k+1}, u^n_{(k+1)i}, \xi^n_{k(i+1)}\big) \text{ for all } 1 \le k \le n-1 \text{ and all } i \in \mathcal{I}_{k-1}.$$

Let us now fix some $1 \le k \le n-1$ and $i \in \mathcal{I}^n_{k-1}$, and assume that the fundamentals at time t^n_{k-1} are given by $U^n_{k-1} = u^n_{(k-1)i}$. Since the definition of p-admissibility implies that the weights of $\xi^n_{k(i+1)}$ and $\xi^n_{k(i-1)}$ in the representation (2.1) are probability weights, the weight of $\xi^n_{(k+1)i}$ is also a probability weight, and (2.1) represents the stock holdings $\xi^n\big(t^n_{k-1}, U^n_{k-1}\big) = \xi^n_{(k-1)i}$ between time t^n_{k-1} and time t^n_k as a convex combination of the values of the strategy function $\xi^n : \mathcal{A}^n \to I\!R$ at the three points $\big(t^n_k, U^n_{k-1} + \delta\big) = \big(t^n_k, u^n_{k(i+1)}\big)$, $\big(t^n_{k+1}, U^n_{k-1}\big) = \big(t^n_{k+1}, u^n_{(k+1)i}\big)$, and $\big(t^n_k, U^n_{k-1} - \delta\big) = \big(t^n_k, u^n_{k(i-1)}\big)$.

Remark. Note that the weights of $\xi^n_{k(i+1)}$ and $\xi^n_{k(i-1)}$ in (2.1) are probability weights for all p-admissible trading strategies (ξ^n, b^n), even if the price system (ψ, μ) does not satisfy the conditions of Example 2.2. □

For practical purposes and for the determination of a continuous approximating function of a discrete strategy function, the representation (2.2) proves to be more useful than the representation (2.1), especially if the equilibrium price function is multiplicative.

2.3 Connections to Models with Transaction Costs

In this section we show that our large investor market model has many similarities with certain binomial models for *small* investor markets *with transaction costs*. Especially we will see that the p-martingale measure $\mathbf{P}_n^{\xi^n}$, which turns the paper value into a martingale, has a counterpart in a small investor model with transaction costs. Since transaction costs models are widely known and since they have been analyzed in great detail, the similarity to transaction costs models gives an indication how the non-linearity in large investor markets affects martingale measures and replicating strategies. In particular, this has proved useful to find the limit of the strategy functions $\xi^n : \mathcal{A}^n \to I\!R$ as $n \to \infty$ as it will be derived in Chapter 3.

One of the best-known binomial models with transaction costs is the model with fixed proportional transaction costs as presented e.g. in Boyle and Vorst (1992), the textbook of Musiela and Rutkowski (1998), or – in a slightly generalized version allowing for different transaction costs rates for buying and selling – in the diploma thesis of Opitz (1999). If we look at Boyle and Vorst's binomial model with n steps, at each trading time $t \in (0, T]$ and each possible state $u \in I\!R$ of the fundamentals, the small investor has to pay for each share of stock that he buys not only the exogenously given stock price $\overline{\psi}(t, u)$, but the amount $\big(1+\kappa_n\big)\overline{\psi}(t, u)$, where $\kappa_n \geq 0$ is some fixed transaction cost rate. Similarly the investor receives only $\big(1-\kappa_n\big)\overline{\psi}(t, u)$ for each share of stock he sells. Boyle and Vorst (1992) assume that there are no transaction costs at time 0, so that the large investor's pre-trading endowment does not affect the stock price.

In order to compare this model with our large investor model, we can now write the average price per share which the small investor actually has to pay when shifting his portfolio at time t from ξ_1 to ξ_2 shares of stock, given that the fundamentals are u, in terms of the price function $S^n : [0, T] \times I\!R^3 \to I\!R$ which for all $t \in [0, T]$ and $u, \xi_1, \xi_2 \in I\!R$ is defined by

$$S^n(t, u, \xi_1, \xi_2) := \begin{cases} \overline{\psi}(0, u) & \text{if } t = 0 \\ \overline{\psi}(t, u)\big(1 + \operatorname{sgn}(\xi_2 - \xi_1)\kappa_n\big) & \text{if } t \in (0, T]. \end{cases} \tag{3.1}$$

The price function S^n in the small investor market with transaction costs corresponds to the large investor price function S_μ, but the influence of the two stock positions ξ_1 and ξ_2 before and after the trade at time t is noticeably simpler than their influence on the function S_μ, since ξ_1 and ξ_2 affect the price function S^n only through $\operatorname{sgn}\big(\xi_2 - \xi_1\big)$. However, in contrast to the large investor price function S_μ, the price function S^n is discontinuous on the hyperplane $\xi_1 = \xi_2$. This discontinuity will restrict some of the analogies with large investor models to a formal level.

Since for each fixed $(t, u) \in [0, T] \times I\!R$ and non-vanishing stock prices $\overline{\psi}(t, u)$ the price function $S^*(t, u, \cdot, \cdot) : I\!R^2 \to I\!R$ determined by $S^*(t, u, \xi_1, \xi_2) = \overline{\psi}(t, u)$ for all $\xi_1, \xi_2 \in I\!R$ is the only function of the form (3.1) which satisfies the fair-price condition (1.1.1), it is quite clear that in the small investor model with proportional transaction costs, the price $\overline{\psi}(t, u)$ plays the role of the benchmark price. Then the transaction costs function $c^n : [0, T] \times I\!R^3 \to I\!R$ in the model with transaction costs can be introduced like the transaction losses in the large investor model, namely by $c^n(t, u, \xi_1, \xi_2) = \big(\xi_2 - \xi_1\big)\big(S^n(t, u, \xi_1, \xi_2) - \overline{\psi}(t, u)\big)$ for all

$(t,u,\xi_1,\xi_2) \in [0,T] \times I\!R^3$, which leads to the transaction costs of

$$c^n\big(t,u,\xi_1,\xi_2\big) = \begin{cases} 0 & \text{for } (t,u,\xi_1,\xi_2) \in \{0\} \times I\!R^3 \\ \big|\xi_2 - \xi_1\big|\kappa_n \overline{\psi}(t,u) & \text{for } (t,u,\xi_1,\xi_2) \in (0,T] \times I\!R^3, \end{cases} \tag{3.2}$$

as we would expect.
Like in our large investor market, we can then define trading strategies (ξ^n, b^n) in the market with transaction costs. If such a trading strategy is path-independent, we can also introduce the strategy function $\xi^n : \mathcal{A}^n \to I\!R$ as in the large investor case. In analogy to Definition 1.24 a trading strategy $(\xi^n, b^n) = \big\{\big(\xi^n_k, b^n_k\big)\big\}_{0 \le k \le n}$ in the market with transaction costs is self-financing if

$$b^n_{k-1} = b^n_k + \big(\xi^n_k - \xi^n_{k-1}\big) S^n\big(t^n_k, U^n_k, \xi^n_{k-1}, \xi^n_k\big) \quad \text{for all } 1 \le k \le n.$$

Moreover, we can introduce the paper value process $\tilde{V}^n = \big\{\tilde{V}^n_k\big\}$ of a trading strategy (ξ^n, b^n) in the market with transaction costs by

$$\tilde{V}^n_k = b^n_k + \xi^n_k S^n\big(t^n_k, U^n_k, \xi^n_{k-1}, \xi^n_k\big) \quad \text{for all } 0 \le k \le n,$$

where again ξ^n_{-1} denotes the initial endowment of the (small) investor immediately before time $t^n_0 = 0$. Note, however, that the paper value process does not depend on the particular value of ξ^n_{-1}, since $S^n\big(0, U^n_0, \xi^n_{-1}, \xi^n_0\big) = \overline{\psi}\big(0, U^n_0\big)$.
Now suppose that for some *fixed* $n \in I\!N$ the small investor price function $\overline{\psi} : [0,T] \times I\!R \to I\!R$ satisfies Assumption A. Then for all sufficiently small $\kappa_n > 0$ we even have

$$\frac{1+\kappa_n}{1-\kappa_n}\overline{\psi}\big(t^n_k, U^n_{k-1} - \delta\big) < \overline{\psi}\big(t^n_k, U^n_{k-1}\big) < \frac{1-\kappa_n}{1+\kappa_n}\overline{\psi}\big(t^n_k, U^n_{k-1} + \delta\big) \quad \text{for all } 1 \le k \le n.$$

Under this condition we can, for *each* path-independent and self-financing trading strategy (ξ^n, b^n), copy the Definition 2.7(*ii*) of the p-martingale measure $\mathbf{P}^{\xi^n}_n$ in the large investor market, and analogously define a $\mathbf{P}^n$-equivalent probability measure $\tilde{\mathbf{P}}^{\xi^n}_n$ in the market with transaction costs. Namely, we can define some weight function $\tilde{p}^{\xi^n}_n : \mathcal{A}^n(n-1) \times I\!R \to (0,1)$ for all $(t,u,\xi) \in \mathcal{A}^n(n-1) \times I\!R$ by

$$\tilde{p}^{\xi^n}_n\big(t,u,\xi\big) = \frac{S^n\big(t,u,\xi,\xi^n(t,u)\big) - S^n\big(t+\delta^2, u-\delta, \xi^n(t,u), \xi^n(t+\delta^2, u-\delta)\big)}{S^n\big(t+\delta^2, u+\delta, \xi^n(t,u), \xi^n(t+\delta^2, u+\delta)\big) - S^n\big(t+\delta^2, u-\delta, \xi^n(t,u), \xi^n(t+\delta^2, u-\delta)\big)},$$

and then define the new probability measure $\tilde{\mathbf{P}}^{\xi^n}_n$ on $\big(\Omega^n, \mathcal{F}^n_n\big)$ via the initial distributions $\tilde{\mathbf{P}}^{\xi^n}_n\big(Z^n_0 = 1\big) = \mathbf{P}^n\big(Z^n_0 = 1\big)$ and $\tilde{\mathbf{P}}^{\xi^n}_n\big(U^n_1 = U^n_0 + \delta \,\big|\, Z^n_0\big) = \tilde{p}^{\xi^n}_n\big(0, U^n_0, \xi^n_{-1}\big) = \tilde{p}^{\xi^n}_n\big(0, U^n_0, 0\big)$ and the transition probabilities

$$\tilde{\mathbf{P}}^{\xi^n}_n\big(U^n_k = U^n_{k-1} + \delta \,\big|\, U^n_{k-1}, U^n_{k-2}\big) = \tilde{p}^{\xi^n}_n\Big(t^n_{k-1}, U^n_{k-1}, \xi^n\big(t^n_{k-2}, U^n_{k-2}\big)\Big) \quad \text{for all } 2 \le k \le n$$

for the fundamental process U^n under $\tilde{\mathbf{P}}^{\xi^n}_n$.
By the same arguments as in the large investor model it follows that the paper value process $\tilde{V}^n$ associated to the trading strategy (ξ^n, b^n) becomes a martingale under the measure $\tilde{\mathbf{P}}^{\xi^n}_n$. Hence the initial wealth $\tilde{V}^n_0 = b^n_0 + \xi^n_0 \overline{\psi}\big(0, U^n_0\big)$ needed at time 0 to replicate the final contingent claim $\big(\xi^n_n, b^n_n\big)$ with paper value $\tilde{V}^n_n = b^n_n + \xi^n_n \overline{\psi}\big(T, U^n_n\big)\big(1 + \operatorname{sgn}(\xi^n_n - \xi^n_{n-1})\kappa_n\big)$ can be calculated as the expectation $\tilde{V}^n_0 = \tilde{\mathbf{E}}^{\xi^n}_n\big[\tilde{V}^n_n\big]$. For the special case where $\big(\xi^n_n, b^n_n\big)$ is a long European call which is settled by delivery, such a representation can already be

found in Boyle and Vorst (1992) or Musiela and Rutkowski (1998), and they show that the corresponding strategy function $\xi^n : \mathcal{A}^n \to I\!R$ satisfies

$$\xi^n\big(t+\delta^2, u-\delta\big) \le \xi^n(t,u) \le \xi^n\big(t+\delta^2, u+\delta\big) \quad \text{for all } (t,u) \in \mathcal{A}^n(n-1). \tag{3.3}$$

If for some $1 \le k^* \le n$ and all $0 \le t < t^n_{k^*}$ the inequalities in (3.3) are strict, then the distribution of U^n_1 under $\mathbf{P}^{\xi^n}_n$ simplifies to

$$\tilde{\mathbf{P}}^{\xi^n}_n\big(U^n_1 = U^n_0 + \delta\big) = \frac{\overline{\psi}(0,U^n_0) - \overline{\psi}\big(t^n_1, U^n_0+\delta\big)(1-\kappa_n)}{\overline{\psi}\big(t^n_1, U^n_0+\delta\big)(1+\kappa_n) - \overline{\psi}\big(t^n_1, U^n_0-\delta\big)(1-\kappa_n)}$$

and for all $2 \le k \le k^*$ the conditional distribution of U^n_k given its two predecessors becomes

$$\tilde{\mathbf{P}}^{\xi^n}_n\big(U^n_k = U^n_{k-1} + \delta \,\big|\, U^n_{k-1},\, U^n_{k-2}\big) = \frac{\overline{\psi}(t^n_{k-1}, U^n_{k-1})(1+\eta^n_{k-1}\kappa_n) - \overline{\psi}\big(t^n_k, U^n_{k-1}+\delta\big)(1-\kappa_n)}{\overline{\psi}\big(t^n_k, U^n_{k-1}+\delta\big)(1+\kappa_n) - \overline{\psi}\big(t^n_k, U^n_{k-1}-\delta\big)(1-\kappa_n)},$$

where $\eta^n_{k-1} = \operatorname{sgn}\big(\xi^n\big(t^n_{k-1}, U^n_{k-1}\big) - \xi^n\big(t^n_{k-2}, U^n_{k-2}\big)\big)$.
On the set $\mathcal{F}^n_{k^*}$ our measure $\tilde{\mathbf{P}}^{\xi^n}_n$ coincides with the martingale measure used by Boyle and Vorst (1992) and Musiela and Rutkowski (1998). Note, however, that for the European call the two measures do not coincide on all of Ω^n, since there are normally some states $(t,u) \in \mathcal{A}^n(n-1)$ on the binomial tree, with t being sufficiently close to maturity, where it is clear that the call is going to end in the money, say, regardless how the fundamentals will evolve between time t and T, just because of the binomial restrictions on the price process. In such states the associated replicating strategy function of the large investor stays constant, and the inequalities in (3.3) become equalities. For these states our transition probabilities for the fundamentals will not match with those given in Musiela and Rutkowski (1998). However, since the choice of these probability weights only reallocates the probability weights among paths which all lead to the same final paper value of $\tilde{V}^n_n$, the (paper) value at time 0 calculated by $\tilde{V}^n_0 = \tilde{\mathbf{E}}^{\xi^n}_n\big[\tilde{V}^n_n\big]$ does indeed coincide with the value calculated by Boyle and Vorst (1992) or Musiela and Rutkowski (1998).

Remark. We should conclude this section with a discussion on the absence of transaction costs at time $t^n_0 = 0$. Musiela and Rutkowski (1998) motivate this assumption by the aversion to dealing with pre-trading endowment in stocks. This is basically the only reason to exclude transaction costs at time 0, since there is no other convincing reason why the market structure at time 0 should be conceptually different from the one at all other time points $t \in (0,T]$. Of course, it is straightforward to adjust the transaction cost model to include transaction costs at time 0 as well.
The dependence of the pre-trading endowment is a bother in our large investor model as well, even more than in the small investor model with transaction costs: In the large investor model it strongly restricts the class of p-admissible trading strategies, and hence the class of contingent claims which are attainable by such p-admissible trading strategies. In contrast to the small investor model with transaction costs, however, it does not suffice to exclude any transaction gains or losses at time $t^n_0 = 0$ in order to remove the influence of the large investor's stock holdings ξ^n_{-1} immediately before t^n_0 on the model, since for any equilibrium price function $\psi : [0,T] \times I\!R^2 \to I\!R$, for which $\xi \mapsto \psi(0, U^n_0, \xi)$ is not constant, the benchmark price $S^*\big(0, U^n_0, \xi^n_{-1}, \xi^n_0\big)$ depends on both the large investor's stock holdings ξ^n_{-1} and ξ^n_0 immediately before and after t^n_0.
Of course, by the definition of the large investor price function in (1.3.2) the large investor price $S_\mu\big(0, U^n_0, \xi^n_{-1}, \xi^n_0\big)$ does not depend on ξ^n_{-1} if the price-determining measure μ is the Dirac measure δ_1 concentrated in 1. If $\mu = \delta_1$ the market price jumps to its new equilibrium

as soon as the large investor announces, but before he can execute, his trade, and we get $S_\mu\big(t,u,\xi_1,\xi_2\big) = \psi\big(t,u,\xi_2\big)$ for all $(t,u,\xi_1,\xi_2) \in [0,T] \times I\!R^3$.
In order to avoid the dependence on the large investor's pre-trading endowment we could also employ the price-determining measure $\mu = \delta_1$ only to describe the price mechanism at time 0, and some more suitable measures for all other trading times in $(0,T]$. However, since there is no obvious reason why the price mechanism at time 0 should conceptually differ from the price mechanisms at the other time points, we have decided not to pursue such a model. Thus, in our large investor model we have to remain careful about the large investor's initial endowment in the stock. □

2.4 Markets with a Multiplicative Equilibrium Price Function

The connections between small investor models with transaction costs and large investor models become especially noticeable if the equilibrium price function in the large investor market has a multiplicative structure. In this case the large investor market model can basically be written in terms of a small investor market model with transaction costs, where the small investor uses a transformed trading strategy. However, the large investor model remains more complex than the ordinary small investor models with transaction costs, since the transaction loss rate depends on the absolute size of the large investor's stock holdings. In addition to the similarities with transaction cost models, the multiplicative structure considerably simplifies the shape of the r-martingale measure and the recursion for the real value function, since the r-martingale weight function now coincides with the s-martingale weight function and hence does not depend on the large investor's strategy any more. The simplification is maximal for multiplicative large investor models if the price system prevents any instantaneous transaction gains and losses. In this case, the large investor's trading strategy is just a transform of the small investor's trading strategy in the associated small investor market and all path-independent contingent claims are attainable.

For the remainder of the thesis, we will only consider large investor price systems (ψ,μ) where the equilibrium price function $\psi : [0,T] \times I\!R^2 \to I\!R$ has a multiplicative structure, and we will always assume

Assumption B (Multiplicative structure of ψ). *There exists a locally bounded function $f : I\!R \to (0,\infty)$ which is continuous a.e. (with respect to the Lebesgue measure on $I\!R$) such that the equilibrium price function $\psi : [0,T] \times I\!R^2 \to I\!R$ can be written as*

$$\psi(t,u,\xi) = \overline{\psi}(t,u)f(\xi) \quad \textit{for all } (t,u,\xi) \in [0,T] \times I\!R \times I\!R. \tag{4.1}$$

Remark. Since every equilibrium price function $\psi : [0,T] \times I\!R^2 \to I\!R$, $(t,u,\xi) \mapsto \psi(t,u,\xi)$ is increasing in u, any function $f : I\!R \to I\!R$ that satisfies (4.1) cannot become zero on $I\!R$. By the definition of $\overline{\psi} : [0,T] \times I\!R \to I\!R$ in Definition 1.17 we also have $f(0) = 1$. In Proposition 1.33, where we have proved the existence of a replicating strategy, we employed the condition that ψ is continuous in ξ. In this case $f : I\!R \to I\!R$ is of course continuous as well, and the positivity of f follows already from the representation (4.1). □

2.4.1 The Strategy Transform

We will see that under Assumption B a large investor market can be basically viewed as a small investor market with transaction costs, where the small investor uses a transformed strategy. This point of view will simplify the analysis of large investor markets considerably.

In Section 2.1.3 the multiplicative form (4.1) of ψ was seen to carry over to the loss-free liquidation price function $\bar{S} : [0,T] \times I\!R^2 \to I\!R$, which then can be written as in (1.16). Since for all a.e. continuous functions the Lebesgue integral over $[0,1]$ coincides with the Riemann integral, the real value of ξ shares of stock at time t given fundamentals of u now becomes

$$\xi\bar{S}\big(t,u,\xi\big) = \xi\overline{\psi}(t,u)\int_0^1 f(\theta\xi)\,\lambda(d\theta) = \overline{\psi}(t,u)\int_0^\xi f(x)\,dx \text{ for all } (t,u,\xi)\in[0,T]\times I\!R^2, \quad (4.2)$$

which is the same as the price of $\int_0^\xi f(x)\,dx$ shares in the associated small investor market described by the small investor price function $\overline{\psi} : [0,T]\times I\!R \to I\!R$. More generally, the benchmark price for buying $\xi_2-\xi_1$ shares of stock turns out to be

$$\big(\xi_2-\xi_1\big)S^*\big(t,u,\xi_1,\xi_2\big) = \overline{\psi}(t,u)\big(\xi_2-\xi_1\big)\int f\big((1-\theta)\xi_1+\theta\xi_2\big)\lambda(d\theta) = \overline{\psi}(t,u)\int_{\xi_1}^{\xi_2} f(x)dx, \quad (4.3)$$

i.e. it corresponds to the price of $\int_{\xi_1}^{\xi_2} f(x)dx = \int_0^{\xi_2} f(x)dx - \int_0^{\xi_1} f(x)dx$ shares in the associated small investor market.
This connection between the benchmark prices and the prices in the associated small investor market gives us a first hint why the following transform $g : I\!R \to I\!R$ becomes extremely useful in multiplicative large investor markets.

Definition 2.12. If the large investor market (ψ,μ) satisfies Assumption B, then the **strategy transform** $g : I\!R \to I\!R$ is defined by

$$g(\xi) = \int_0^\xi f(x)\,dx. \quad (4.4)$$

If $\xi^n : \mathcal{A}^n \to I\!R$ denotes some strategy function in such a market, the **transformed strategy function** $g^n : \mathcal{A}^n \to I\!R$ is given by

$$g^n(t,u) = g\big(\xi^n(t,u)\big) = \int_0^{\xi^n(t,u)} f(x)\,dx \quad \text{for all } (t,u)\in\mathcal{A}^n. \quad (4.5)$$

Since $f : I\!R \to (0,\infty)$ is positive, $g : I\!R \to I\!R$ is strictly increasing and hence invertible.

Besides the loss-free liquidation and the benchmark price, we would also like to write the transaction loss function $c_\mu : [0,T]\times I\!R^3 \to I\!R$ in a way which is familiar from small investor models. Because of the nature of c_μ, we cannot expect to find an analogue in the standard Cox-Ross-Rubinstein model without transaction costs, but we have to allow for models with transaction costs, and since we have presented the small investor market model with proportional transaction costs in detail in Section 2.3, we want to mirror the multiplicative structure (3.2) of the transaction cost function in such a model. For this purpose let us recall the local transaction loss rate function $k_\mu : I\!R^2 \to I\!R$, which we have introduced so far only for static large investor markets. Under Assumption B the definition of Section 1.2.3 can be transferred one-to-one to a multiplicative dynamic large investor market:

Definition 2.13. For any large investor market (ψ,μ) for which Assumption B holds the **local (implied) transaction loss rate function** $k_\mu : I\!R^2 \to I\!R$ is given by

$$k_\mu\big(\xi_1,\xi_2\big) = \operatorname{sgn}\big(\xi_2-\xi_1\big)\left(\frac{\int f\big((1-\theta)\xi_1+\theta\xi_2\big)\,\mu(d\theta)}{\int f\big((1-\theta)\xi_1+\theta\xi_2\big)\,\lambda(d\theta)} - 1\right) \quad \text{for all } \xi_1,\xi_2\in I\!R. \quad (4.6)$$

Remark. As usual, we use $\operatorname{sgn}(0)=0$. □

Like in the static case, the local transaction loss rate indicates the average transaction loss per share as a percentage of the benchmark price. More precisely, we have for all tuples $(t, u, \xi_1, \xi_2) \in [0, T] \times I\!R^3$ that

$$c_\mu(t, u, \xi_1, \xi_2) = \left|\xi_2 - \xi_1\right| S^*(t, u, \xi_1, \xi_2) k_\mu(\xi_1, \xi_2), \tag{4.7}$$

since $c_\mu : [0, T] \times I\!R^3$ and $S^* : [0, T] \times I\!R^3$ inherit the multiplicative structure of ψ so that the proof of (1.2.17) can be copied. In this representation, we can recognize the cost structure (3.2) of the small investor model with proportional transaction costs, where now the rate $k_\mu(\xi_1, \xi_2)$ may depend on the large investor's stock holdings, and where the "fair", exogenously given price $\overline{\psi}(t, u)$ in the small investor market is replaced by the benchmark price $S^*(t, u, \xi_1, \xi_2)$ in the large investor market. Due to (4.3) we can also express the transaction losses in terms of the *small* investor price $\overline{\psi}(t, u)$ by writing

$$c_\mu(t, u, \xi_1, \xi_2) = \left|\int_{\xi_1}^{\xi_2} f(x)\, dx\right| \overline{\psi}(t, u) k_\mu(\xi_1, \xi_2) \quad \text{for all } (t, u, \xi_1, \xi_2) \in [0, T] \times I\!R^3. \tag{4.8}$$

This shows that even the transaction losses in our multiplicative large investor model can (locally) be represented as the costs which a small investor in a market with proportional transaction costs of $k_\mu(\xi_1, \xi_2)$ is faced when buying $g(\xi_2) - g(\xi_1) = \int_{\xi_1}^{\xi_2} f(x)\, dx$ shares of stock in order to adjust his share in stocks from $g(\xi_1)$ to $g(\xi_2)$.

Remark. The transformation of the original strategy (function) $\xi^n : \mathcal{A}^n \to I\!R$ by the strategy transform $g : I\!R \to I\!R$ will help us to compare our large investor model with small investor models with transaction costs. However, it should be noted that there remains an essential difference to those transaction cost models, especially when it comes to convergence: In the standard n-step small investor models with transaction costs as presented by Boyle and Vorst (1992) or Musiela and Rutkowski (1998), the transaction cost rate κ_n is a constant which only depends on n. Opitz (1999) slightly extends this model by allowing two different rates for buying and selling, but all these models do not allow for a dependence of the transaction cost rate on the absolute size of the large investor's stock holdings. Thus, the dependence of the transaction loss rate $k_\mu : I\!R^2 \to I\!R$ on the large investor's stock holdings as described by (4.6) substantially differs from the known transaction cost models. Moreover, the convergence results for small investor models with transaction costs rely on the assumption that the transaction cost rate in the n-step model is scaled by $\kappa_n = \kappa\delta_n$ for some fixed $\kappa > 0$. In our large investor model the transaction loss rate $k_\mu(\xi_1^n, \xi_2^n)$ tends to 0 as $n \to \infty$ only if the size $\left|\xi_1^n - \xi_2^n\right|$ of the large investor's trade tends to 0. For this reason, the derivation of limits for the large investor model will remain much more complicated than the derivation of limits in the small investor model with transaction costs. □

2.4.2 The Recursive Schemes Revisited

As we have already stated in Section 2.1.3, in a multiplicative market the r-martingale measure $\bar{\mathbf{P}}_n^{\xi^n}$ and the associated small-investor martingale measure $\bar{\mathbf{P}}^n$ coincide for any self-financing and path-independent trading strategy (ξ^n, b^n), and the results of Proposition 2.9 and Corollary 2.10 on the real value simplify as well:

Corollary 2.14. *Consider a large investor market described by (ψ, μ), and suppose that both Assumptions A and B hold. Then for all path-independent and self-financing trading strategies (ξ^n, b^n) we have:*

(i) The trading strategy (ξ^n, b^n) is r-admissible.

(*ii*) *The associated real value function* $\bar{v}^n : \mathcal{A}^n \to I\!R$ *can be calculated from the possible realizations of the final real value* $\bar{V}_n^n = \bar{v}^n(T, U_n^n)$ *by the recursive scheme*

$$\begin{aligned}\bar{v}^n(t_{k-1}^n, U_{k-1}^n) = \bar{p}^n(t_{k-1}^n, U_{k-1}^n)\Big(\bar{v}^n(t_k^n, U_{k-1}^n+\delta) + c_\mu^{\xi^n}(t_k^n, U_{k-1}^n+\delta, \xi_{k-1}^n)\Big) \\ +\Big(1-\bar{p}^n(t_{k-1}^n, U_{k-1}^n)\Big)\Big(\bar{v}^n(t_k^n, U_{k-1}^n-\delta) + c_\mu^{\xi^n}(t_k^n, U_{k-1}^n-\delta, \xi_{k-1}^n)\Big)\end{aligned} \tag{4.9}$$

for all $1 \le k \le n$. *Moreover, in this case the number of stocks* ξ_{k-1}^n *held by the large investor between time* t_{k-1}^n *and* t_k^n *satisfies the fixed point equation*

$$\begin{aligned}\int_0^{\xi_{k-1}^n} f(x)\,dx = \frac{\bar{v}(t_k^n, U_{k-1}^n+\delta) - \bar{v}(t_k^n, U_{k-1}^n-\delta)}{\overline{\psi}(t_k^n, U_{k-1}^n+\delta) - \overline{\psi}(t_k^n, U_{k-1}^n-\delta)} \\ + \frac{c_\mu^{\xi^n}(t_k^n, U_{k-1}^n+\delta, \xi_{k-1}^n) - c_\mu^{\xi^n}(t_k^n, U_{k-1}^n-\delta, \xi_{k-1}^n)}{\overline{\psi}(t_k^n, U_{k-1}^n+\delta) - \overline{\psi}(t_k^n, U_{k-1}^n-\delta)}\end{aligned} \tag{4.10}$$

for all $1 \le k \le n$.

(*iii*) *If the market* (ψ, μ) *excludes instantaneous transaction gains, then the real value process* $\bar{V}^n$ *is a supermartingale under the martingale measure* $\bar{\mathbf{P}}^n$ *in the associated small investor market.*

(*iv*) *If* (ψ, μ) *excludes instantaneous transaction gains* and *losses, then* $\bar{V}^n$ *even is a martingale under* $\bar{\mathbf{P}}^n$.

Proof. As we have already noticed in Section 2.1.3, the conditions (1.4) for all $1 \le k \le n$ on r-admissibility are equivalent to the conditions enforced by Assumption A, because of the multiplicative structure (1.16) and the positivity of $f : I\!R \to (0, \infty)$.
In Section 2.1.3 we have also seen that for all self-financing trading strategies (ξ^n, b^n) the r-martingale weight functions $\bar{p}_n^{\xi^n} : \mathcal{A}^n(n-1) \to I\!R$ do not depend on the strategy and coincide with the s-martingale weight function $\bar{p}^n : \mathcal{A}^n(n-1) \to I\!R$. Hence (*ii*) follows directly from Proposition 2.9(*ii*) and (4.2).
With the r- and s-martingale weight functions coinciding, of course all r-martingale measures $\bar{\mathbf{P}}^{\xi^n}$ coincide with the martingale measure $\bar{\mathbf{P}}^n$ in the associated small investor market, so that (*iii*) and (*iv*) are immediate consequences of Corollary 2.10(*ii*) and (*iii*). **q.e.d.**

Remark. Note that by (4.8) the transaction losses in (4.9) and (4.10) could also be represented as

$$c_\mu^{\xi^n}(t_k^n, U_{k-1}^n \pm \delta, \xi_{k-1}^n) = \left|\int_{\xi_{k-1}^n}^{\xi^n(t_k^n, U_{k-1}^n \pm \delta)} f(x)\,dx\right| \overline{\psi}(t_k^n, U_{k-1}^n \pm \delta) k_\mu\big(\xi_{k-1}^n, \xi^n(t_k^n, U_{k-1}^n \pm \delta)\big)$$

for all $1 \le k \le n$. We have avoided this representation to keep the equations (4.9) and (4.10) more clearly arranged. □

As we have seen in Section 2.2, for all $1 \le k \le n-1$ the stock holdings at time t_{k-1}^n can also be represented in terms of a fixed point equation which solely depends on the possible stock holdings at time t_k^n and t_{k+1}^n. In a multiplicative setting the statement of Proposition 2.11(*ii*) simplifies as well and now becomes:

Corollary 2.15. *Let* $\psi : [0,T] \times I\!R^2 \to I\!R$ *be some equilibrium price function which is multiplicative and which satisfies Assumptions A and B. Then for any path-independent* *and self-financing trading strategy the associated strategy function $\xi^n : \mathcal{A}^n \to I\!R$ solves the recursive scheme*

$$
\begin{aligned}
\int_0^{\xi^n_{(k-1)i}} f(x)\,dx = {} & \frac{\overline{\psi}\big(t^n_k, u^n_{k(i+1)}\big) - \overline{\psi}\big(t^n_{k+1}, u^n_{(k+1)i}\big)}{\overline{\psi}\big(t^n_k, u^n_{k(i+1)}\big) - \overline{\psi}\big(t^n_k, u^n_{k(i-1)}\big)} \int_0^{\xi^n_{k(i+1)}} f(x)\,dx \\
& + \frac{\overline{\psi}\big(t^n_{k+1}, u^n_{(k+1)i}\big) - \overline{\psi}\big(t^n_k, u^n_{k(i-1)}\big)}{\overline{\psi}\big(t^n_k, u^n_{k(i+1)}\big) - \overline{\psi}\big(t^n_k, u^n_{k(i-1)}\big)} \int_0^{\xi^n_{k(i-1)}} f(x)\,dx \\
& + \frac{D^{\xi^n}_\mu\big(t^n_k, u^n_{(k-1)i}\big)}{\overline{\psi}\big(t^n_k, u^n_{k(i+1)}\big) - \overline{\psi}\big(t^n_k, u^n_{k(i-1)}\big)}
\end{aligned}
\tag{4.11}
$$

for all $1 \le k \le n-1$ and all $i \in \mathcal{I}_{k-1}$.

Proof. This is an immediate consequence of Proposition 2.11(*ii*) and Assumption B. **q.e.d.**

If the large investor can always trade at the benchmark price S^*, such that the transaction loss function $c_\mu : [0,T] \times I\!R^3 \to I\!R$ and hence its spread $D^{\xi^n}_\mu$ vanishes, the transformed strategy $g\big(\xi^n_{(k-1)i}\big) = \int_0^{\xi^n_{(k-1)i}} f(x)\,dx$ at the grid point $(k-1,i)$ is an explicit linear combination of the two possible successors $g\big(\xi^n_{k(i\pm1)}\big)$. Especially, in terms of the transform, (4.11) resembles the recursive formula for the strategy function in a small investor market, where the transform $g : I\!R \to I\!R$ is just the identity. If we recall the introduction to Section 2.4.1, it is not surprising that the strategy transform in a large investor market without transaction losses and transaction gains plays exactly the same role as the original strategy in a small investor market.

For the general case with non-vanishing transaction losses (or gains), we combine (4.8) with the definition of the spread $D^{\xi^n}_\mu\big(t^n_k, u^n_{(k-1)i}\big)$ in Proposition 2.11(*ii*), and thus obtain for later reference the equality

$$
\begin{aligned}
D^{\xi^n}_\mu\big(t^n_k, u^n_{(k-1)i}\big) = {} & \overline{\psi}\big(t^n_{k+1}, u^n_{k+1}\big)\Bigg(\left|\int_{\xi^n_{(k+1)i}}^{\xi^n_{k(i+1)}} f(x)\,dx\right| k_\mu\big(\xi^n_{k(i+1)}, \xi^n_{(k+1)i}\big) \\
& \qquad - \left|\int_{\xi^n_{(k+1)i}}^{\xi^n_{k(i-1)}} f(x)\,dx\right| k_\mu\big(\xi^n_{k(i-1)}, \xi^n_{(k+1)i}\big)\Bigg) \\
& + \overline{\psi}\big(t^n_k, u^n_{k(i+1)}\big)\left|\int_{\xi^n_{(k-1)i}}^{\xi^n_{k(i+1)}} f(x)\,dx\right| k_\mu\big(\xi^n_{(k-1)i}, \xi^n_{k(i+1)}\big) \\
& - \overline{\psi}\big(t^n_k, u^n_{k(i-1)}\big)\left|\int_{\xi^n_{(k-1)i}}^{\xi^n_{k(i-1)}} f(x)\,dx\right| k_\mu\big(\xi^n_{(k-1)i}, \xi^n_{k(i-1)}\big)
\end{aligned}
\tag{4.12}
$$

for all $1 \le k \le n-1$ and $i \in \mathcal{I}_{k-1}$.

2.4.3 Trading at the Benchmark Price

The results of Corollaries 2.14 and 2.15 become particularly handy if, in addition to the Assumptions A and B on the equilibrium price function, the price system (ψ, μ) of the large investor market excludes any instantaneous transaction gains or losses. In this case all path-independent contingent claims are attainable and the replicating strategy of such a claim is a simple transform of a related replicating strategy in the associated small-investor market model.

Before we state this result formally, let us shortly discuss how a price system (ψ, μ) which excludes any instantaneous transaction gains and losses will look like. Since the transaction loss function $c_\mu : [0,T] \times I\!R^3 \to I\!R$ can for all $(t, u, \xi_1, \xi_2) \in [0,T] \times I\!R^3$ be written as

$$c_\mu\big(t, u, \xi_1, \xi_2\big) = \big(\xi_2 - \xi_1\big)\Big(S_\mu\big(t, u, \xi_1, \xi_2\big) - S^*\big(t, u, \xi_1, \xi_2\big)\Big), \tag{4.13}$$

the absence of any instantaneous transaction gains or losses implies that the large investor executes all his trades (where $\xi_1 \neq \xi_2$) at the benchmark price, and due to the definitions of the large investor and the benchmark price function S_μ and S^* in Definition 1.18 the values of these two price functions coincide if $\xi_1 = \xi_2$, so $c_\mu \equiv 0$ even implies

$$S_\mu\big(t, u, \xi_1, \xi_2\big) = S^*\big(t, u, \xi_1, \xi_2\big) \quad \text{for all } (t, u, \xi_1, \xi_2) \in [0,T] \times I\!R^3.$$

Thus, if the large investor market (ψ, μ) excludes instantaneous transaction gains and losses, we may always assume without loss of generality that the price-determining measure μ is the Lebesgue measure λ on $[0,1]$.

Proposition 2.16. *Let us assume that the large investor market (ψ, μ) excludes any instantaneous transaction gains or losses and satisfies Assumptions A and B. Then every path-independent contingent claim (ξ_n, b_n) is attainable by a unique replicating strategy (ξ^n, b^n). If $(\bar\gamma^n, \bar b^n)$ is the replicating strategy for the contingent claim $\big(g(\xi_n), b_n\big)$ in the associated* small *investor market, then the replicating strategy (ξ^n, b^n) for the claim (ξ_n, b_n) in the* large *investor market is given by $\xi^n_k = g^{-1}\big(\bar\gamma^n_k\big)$ and $\bar b^n_k = b^n_k$ for all $0 \le k \le n$.*

Proof. Let (ξ_n, b_n) be an arbitrary path-independent contingent claim. We will construct a replicating trading strategy $(\xi^n, b^n) = \big\{\big(\xi^n_k, b^n_k\big)\big\}_{0 \le k \le n}$ similar to the replicating trading strategy of a star-convex contingent claim in Section 1.4.2.
Of course, the replicating portfolio at time $t^n_n = T$ has to satisfy $\xi^n_n = \xi_n$ and $b^n_n = b_n$, so we will define ξ^n_n and b^n_n that way. This definition allows us to determine the real value $\bar V^n_n = \bar v^n\big(T, U^n_n\big) = b^n_n + \xi^n_n \bar S\big(T, U^n_n, \xi^n_n\big)$ at time T in each state of the world. Since the transaction loss terms in (4.9) vanish, we can then recursively calculate the values of the real value function $\bar v^n : \mathcal{A}^n \to I\!R$ solely from its final values $v^n(T, \cdot\,) : \mathcal{U}^n_n \to I\!R$ via

$$\bar v^n\big(t^n_k, U^n_k\big) = \bar p^n\big(t^n_k, U^n_k\big)\bar v^n\big(t^n_{k+1}, U^n_k + \delta\big) + \big(1 - \bar p^n\big(t^n_k, U^n_k\big)\big)\bar v^n\big(t^n_{k+1}, U^n_k - \delta\big) \tag{4.14}$$

for all $0 \le k \le n-1$.
Now we have seen in Section 2.4.1 that the strategy transform $g : I\!R \to I\!R$, $\xi \mapsto \int_0^\xi f(x)\,dx$ is invertible, hence the stock positions $\big\{\xi^n_k\big\}_{0 \le k \le n-1}$ of the large investor's trading strategy can be *explicitly* calculated from (4.10) as

$$\xi^n_k = g^{-1}\left(\frac{\bar v\big(t^n_{k+1}, U^n_k + \delta\big) - \bar v\big(t^n_{k+1}, U^n_k - \delta\big)}{\overline{\psi}\big(t^n_{k+1}, U^n_k + \delta\big) - \overline{\psi}\big(t^n_{k+1}, U^n_k - \delta\big)}\right) \quad \text{for all } 0 \le k \le n-1. \tag{4.15}$$

Last but not least, because of Definition 1.27 the large investor's cash position b^n_k between time t^n_k and t^n_{k+1} can be calculated from the real value $\bar V^n_k = \bar v^n\big(t^n_k, U^n_k\big)$ and the stock holdings ξ^n_k by

$$b^n_k = \bar v^n\big(t^n_k, U^n_k\big) - \xi^n_k \bar S\big(t^n_k, U^n_k, \xi^n_k\big) \quad \text{for all } 0 \le k \le n-1. \tag{4.16}$$

It is easy to see that the so-defined trading strategy (ξ^n, b^n) is path-independent. In order to check the self-financing condition, note that the definitions of b^n_k and of the strategy transform $g : I\!R \to I\!R$ imply for all $0 \le k \le n-1$ that

$$\xi^n_k \bar S\big(t^n_{k+1}, U^n_{k+1}, \xi^n_k\big) + b^n_k = g\big(\xi^n_k\big)\overline{\psi}\big(t^n_{k+1}, U^n_{k+1}\big) + \bar v^n\big(t^n_k, U^n_k\big) - g\big(\xi^n_k\big)\overline{\psi}\big(t^n_k, U^n_k\big). \tag{4.17}$$

Due to (4.15), (4.14), and the definition of the s-martingale weights in (1.10), the right-hand side of (4.17) equals just the portfolio value $\bar{v}^n\big(t_{k+1}^n, U_{k+1}^n\big) = \xi_{k+1}^n \bar{S}\big(t_{k+1}^n, U_{k+1}^n, \xi_{k+1}^n\big) + b_{k+1}^n$ at time t_{k+1}^n, regardless of the outcome of the fundamental value $U_{k+1}^n = U_k^n \pm \delta$ at time t_{k+1}^n. Hence we have

$$\xi_k^n \bar{S}\big(t_{k+1}^n, U_{k+1}^n, \xi_k^n\big) + b_k^n = \xi_{k+1}^n \bar{S}\big(t_{k+1}^n, U_{k+1}^n, \xi_{k+1}^n\big) + b_{k+1}^n \quad \text{for all } 0 \le k \le n-1,$$

and by the remark following Definition 1.27, this and $c_\mu \equiv 0$ indeed implies that (ξ^n, b^n) is self-financing.

The trading strategy (ξ^n, b^n) is the unique self-financing trading strategy which replicates (ξ_n, b_n), since every replicating strategy has to fulfill (4.15) and (4.16).

In order to prove the second statement, let us first note that the modified equilibrium function $\overline{\psi}^* : [0,T] \times I\!R^2 \to I\!R$ defined by

$$\overline{\psi}^*(t,u,\xi) = \overline{\psi}(t,u) \quad \text{for all } (t,u,\xi) \in [0,T] \times I\!R^2 \tag{4.18}$$

satisfies the Assumptions A and B as well. Of course $(\overline{\psi}^*, \mu)$ just describes the associated small investor market, since $\overline{\psi}^*$ does not depend on ξ, and the loss-free liquidation price function in the market $(\overline{\psi}^*, \mu)$ coincides with the equilibrium price function $\overline{\psi}^*$ in this market.

An application of the proposition's first part to the market $(\overline{\psi}^*, \mu)$ yields that the contingent claim $\big(g(\xi_n), b_n\big)$ is replicable by some replicating strategy $(\bar{\gamma}^n, \bar{b}^n)$. Since $\overline{\psi}^*$ is the loss-free liquidation price function in $(\overline{\psi}^*, \mu)$, the real value function $\bar{v}^* = \bar{v}^{n,*} : \mathcal{A}^n \to I\!R$ of the replicating strategy $(\bar{\gamma}^n, \bar{b}^n)$ has the final value

$$\bar{v}^*\big(T, U_n^n\big) = \bar{b}_n^n + \bar{\gamma}_n^n \overline{\psi}^*\big(T, U_n^n, g(\xi_n)\big) = b_n + g(\xi_n)\overline{\psi}\big(T, U_n^n\big) = b_n + \xi_n \bar{S}\big(T, U_n^n, \xi_n\big), \tag{4.19}$$

where the second equality stems from the replication condition $\big(\bar{\gamma}_n^n, \bar{b}_n^n\big) = \big(g(\xi_n), b_n\big)$ and the definition of $\overline{\psi}^*$, and the third equality from the specific form (4.2) of the loss-free liquidation price function $\bar{S} : [0,T] \times I\!R^2 \to I\!R$ under Assumption B. But the expression on the right-hand side of (4.19) is just the final value $\bar{v}^n\big(T, U_n^n\big)$ of the real value function of the strategy (ξ^n, b^n) in the large investor market, and since both $\bar{v}^n$ and $\bar{v}^*$ can be completely recovered from their final values by the recursion (4.14) for $0 \le k \le n-1$, it follows that both value functions coincide, i.e. $\bar{v}^* \equiv \bar{v}^n$.

Due to (4.18) and Definition 2.12, the strategy transform in the small investor market $(\overline{\psi}^*, \mu)$ is the identity, hence (4.15) applied to the replicating strategy $(\bar{\gamma}^n, \bar{b}^n)$ in $(\overline{\psi}^*, \mu)$ implies that the (small) investors stock position $\bar{\gamma}_k^n$ between time t_k^n and t_{k+1}^n is given by

$$\bar{\gamma}_k^n = \frac{\bar{v}\big(t_{k+1}^n, U_k^n + \delta\big) - \bar{v}\big(t_{k+1}^n, U_k^n - \delta\big)}{\overline{\psi}\big(t_{k+1}^n, U_k^n + \delta\big) - \overline{\psi}\big(t_{k+1}^n, U_k^n - \delta\big)} \quad \text{for all } 0 \le k \le n-1. \tag{4.20}$$

Comparing this definition with the definition of the stock holdings ξ_k^n in the large investor market (ψ, μ), shows indeed $\xi_k^n = g^{-1}\big(\bar{\gamma}_k^n\big)$ for all $0 \le k \le n-1$. Of course, this equality holds for $k = n$ as well, since the replicating conditions of both strategies imply the equalities $\xi_n^n = \xi_n = g^{-1}\big(g(\xi_n)\big) = g^{-1}\big(\bar{\gamma}_n^n\big)$.

In order to show that the cash holdings of both trading strategies coincide, we once again recall the definition of the real value and note that the loss-free liquidation price function in the small investor market $(\overline{\psi}^*, \mu)$ is given by $\overline{\psi}^*$, such that the equality of the two real value functions and $\xi_k^n = g^{-1}\big(\bar{\gamma}_k^n\big)$ imply

$$\bar{b}_k^n = \bar{v}^*\big(t_k^n, U_k^n\big) - \bar{\gamma}_k^n \overline{\psi}^*\big(t_k^n, U_k^n, \bar{\gamma}_k^n\big) = v^n\big(t_k^n, U_k^n\big) - \xi_k^n \bar{S}\big(t_k^n, U_k^n, \xi_k^n\big) = b_k^n$$

for all $0 \le k \le n$. This concludes our proof. **q.e.d.**

In small investor models, it is common to parametrize the replicating strategy and the corresponding value function in terms of time t and the stock price $\overline{\psi}(t,u)$, and not in terms of time and the fundamentals u at this time. In order to make the results of Proposition 2.16 more comparable with the usual parametrization of small investor models, we would like to parametrize the real value function in terms of time and stock price as well.

We will see that for the small investor market such a parametrization is no problem. If (ψ,μ) is a true large investor market such that the equilibrium price function depends on the large investor's stock holdings, the key question is once again *which* stock price we should choose for the parametrization. Since we have restricted us in this section to large investor models where the large investor trades at the benchmark price, we have the advantage that large investor price and benchmark price coincide, but there are still several other candidates of possible stock prices. For example, let us assume that at time t the fundamental value is u, and that in this situation the large investor's strategy tells him to switch his stock holdings from ξ_1 to ξ_2 shares. Then we might use as the parameter for the reparametrized real value function either the small investor stock price $\overline{\psi}(t,u)$, the benchmark price $S^*(t,u,\xi_1,\xi_2)$ effectively paid by the large investor for the transaction of the missing $\xi_2-\xi_1$ stocks, one of the two equilibrium prices $\psi(t,u,\xi_1)$ and $\psi(t,u,\xi_2)$ immediately before and after the large investor's transaction has been executed, or also one of the corresponding loss-free liquidation prices $\bar{S}(t,u,\xi_2)$ and $\bar{S}(t,u,\xi_2)$.

Though the small investor stock price is in general not visible in the market, we will take this price to parametrize the value function, since all other candidates depend on the number of shares held by the large investor, and can therefore be manipulated. Moreover, since no instantaneous transaction losses and gains occur, we might argue that at the trading time t the large investor first liquidates his whole stock holdings ξ_1 at the per-share liquidation price $\bar{S}(t,u,\xi_1)$, such that at least for an infinitesimally short time the equilibrium price on the market is actually given by $\overline{\psi}(t,u)$, and that the large investor then buys ξ_2 shares of stocks at the per-share liquidation price $\bar{S}(t,u,\xi_2)$, even if this leads to unnecessarily large transactions.

Thus, in order to clarify the connections with the usual small investor binomial model, we will now shortly explain how the value function can be parametrized in terms of time and *small investor* stock price as well. Let us again take some large investor price system (ψ,μ) as in Proposition 2.16, where the large investor always trades at the benchmark price, and assume that the contingent claim (ξ_n,b_n) is replicated by the trading strategy (ξ^n,b^n) with associated real value function $\bar{v}^n:\mathcal{A}^n\to I\!R$. In order to capture the possible combinations of time t and small investor stock price $x=\overline{\psi}(t,u)$ in our discrete model, we introduce the set $\bar{\mathcal{D}}^n$ as the image of the trace function $r:\mathcal{A}^n\to I\!R$, $(t,u)\mapsto\big(t,\overline{\psi}(t,u)\big)$, i.e.

$$\bar{\mathcal{D}}^n := \big\{(t,x)\in[0,T]\times I\!R \,\big|\, (t,x)=\big(t,\overline{\psi}(t,u)\big) \text{ for some } (t,u)\in\mathcal{A}^n\big\}. \tag{4.21}$$

By Definition 1.17 the equilibrium price function $\psi:[0,T]\times I\!R^2\to I\!R$, and therefore also the associated small investor price function $\overline{\psi}:[0,T]\times I\!R\to I\!R$, is strictly increasing in the fundamentals. Hence for each fixed $t\in[0,T]$ the function $u\mapsto\overline{\psi}(t,u)$ is invertible and there exists a uniquely defined function $u^n:\bar{\mathcal{D}}^n\to I\!R$ such that $\overline{\psi}\big(t,u^n(t,x)\big)=x$ for all $(t,x)\in\bar{\mathcal{D}}^n$. Then the real value of the portfolio strategy (ξ^n,b^n) can be parametrized in terms of time t and the corresponding small investor stock price x by the function $\bar{w}^n:\bar{\mathcal{D}}^n\to I\!R$ which we define as

$$\bar{w}^n(t,x)=\bar{v}^n\big(t,u^n(t,x)\big) \quad \text{for all } (t,x)\in\bar{\mathcal{D}}^n. \tag{4.22}$$

In particular, the real value of the contingent claim at time T can be written as a function $h(x)$ of the small investor stock price $x=\overline{\psi}(T,u)$ at this time by setting $h(x)=\bar{w}^n(T,x)$ for all $x\in I\!R$ which satisfy $x=\overline{\psi}(T,u)$ for some $u\in\mathcal{U}_n^n$.

Because of (4.14) the value at time t_k^n of the replicating strategy of *any* contingent claim with a real value of $h\big(\overline{\psi}(T, U_n^n)\big)$ can then be calculated by

$$\bar{w}^n\Big(t_k^n, \overline{\psi}\big(t_k^n, U_k^n\big)\Big) = \bar{\mathbf{E}}_n\Big[h\Big(\overline{\psi}\big(T, U_n^n\big)\Big) \,\Big|\, \mathcal{F}_k^n\Big] \quad \text{for } 0 \le k \le n, \tag{4.23}$$

and rewriting (4.15) in terms of $\bar{w}^n$ we see that for any $0 \le k \le n-1$ the large investor's stock holdings between time t_k^n and t_{k+1}^n of such a replicating strategy are given by

$$\xi_k^n = g^{-1}\left(\frac{\bar{w}\big(t_{k+1}^n, \overline{\psi}\big(t_{k+1}^n, U_k^n + \delta\big)\big) - \bar{w}\big(t_{k+1}^n, \overline{\psi}\big(t_{k+1}^n, U_k^n - \delta\big)\big)}{\overline{\psi}\big(t_{k+1}^n, U_k^n + \delta\big) - \overline{\psi}\big(t_{k+1}^n, U_k^n - \delta\big)}\right).$$

If (ψ, μ) is a small investor market, the strategy transform $g : \mathbb{R} \to \mathbb{R}$ will be the identity, and it is easily seen that our large investor strategies really generalize the hedging strategies known from small investor markets.

Remark. For caution, we recall that in small investor markets, the notation ξ_k^n is often used to denote the small investor's strategy between time t_{k-1}^n and t_k^n, while we use it to denote the large investor's strategy between time t_k^n and t_{k+1}^n. □

The importance of the associated small investor market for the analysis of the large investor market was first noted by Baum (2001). In a general semimartingale model he supposes that the equilibrium price process $\big\{\psi\big(t_k^n, U_k^n, \xi\big)\big\}_{0 \le k \le n}$ is a (local) martingale for all constant stock holdings $\xi \in \mathbb{R}$ of the large investor. This corresponds to the martingale-like property (1.17), which shows that the loss free-liquidation price would become a martingale under the r-martingale measure if the large investor's trading itself did not affect the stock prices. Under the multiplicative structure of Assumption B, the r-martingale measure coincides with the s-martingale measure of the associated small investor market, and the representation (1.17) is equivalent to saying that the associated small investor prices process $\big\{\overline{\psi}\big(t_k^n, U_k^n\big)\big\}_{0 \le k \le n}$ is a martingale.

Baum (2001) and Bank and Baum (2004) also noted that the real value process is a supermartingale under the s-martingale measure, and they employed this supermartingale property to derive a no-arbitrage result and superreplication strategies. However, because they only focused on price mechanisms which correspond to a price determining measure $\mu = \delta_1$, they did not notice that the real value becomes a martingale if the large investor trades at the benchmark price.

The simplified behavior of the large investor model, and especially of a multiplicative model, where the large investor trades at the benchmark price can serve as the starting point of an analysis of more complex large investor models where another price mechanism applies. The special case where the large investor trades at the benchmark price still contains much more information on a large investor model than the very special case where the large investor trades like an investor in the associated small investor market.

Chapter 3

Convergence of the Strategy Functions

In the standard Cox-Ross-Rubinstein model, the stock price does not depend on the investor's trading strategy, and hence it is possible to show the convergence in distribution of the discrete binomial models without a detailed investigation of the strategy. However, if the large investor becomes so large that his trades actually affect the stock price, we first need to show that his discrete strategy functions converge before we can derive in Chapter 4 results on the convergence in distribution of our large investor models. Thus, the following chapter is devoted to the convergence of a sequence $\{\xi^n\}_{n\in\mathbb{N}}$ of strategy functions in the discrete binomial large investor models. The limit of the strategy functions has to satisfy a certain final value problem. If a solution φ to the candidate final value problem exists, the convergence of the discrete strategy functions follows from their convergence immediately before and at maturity to the corresponding values of φ. The final value problem for φ is highly non-linear, but it can be transformed into a simpler *quasi-linear* problem by means of the strategy transform $g : \mathbb{R} \to \mathbb{R}$. Therefore, we shall rather study the convergence of the transformed strategy functions $g^n = g \circ \xi^n$ towards some continuous-time limit γ, and then transform our results back into corresponding results for ξ^n and the limit $\varphi = g^{-1} \circ \gamma$. Once the convergence of the strategy functions is shown, we can employ this convergence in order to derive a similar statement for the real value functions.

The existence and uniqueness results for solutions to the final value problems which have to be satisfied by the limit functions γ and φ are stated in terms of certain Hölder spaces. Those function spaces are introduced in Section 3.1. In Section 3.2 we focus on large investor models where the price system (ψ, μ) excludes any instantaneous transaction gains or losses. In this particular case, each of the transformed strategy functions $\{g^n\}_{n\in\mathbb{N}}$ can be calculated from its values at and immediately before maturity by means of an explicit recursive scheme, and the limit γ satisfies a *linear* final value problem. Thus, existence and uniqueness of solutions to the final value problem as well as the convergence of the transformed strategy functions follow from classical results.

If the price system does not prevent transaction losses, however, the recursive schemes for $\{g^n\}_{n\in\mathbb{N}}$ remain implicit schemes, and the final value problem for the limit γ is only quasi-linear. In Section 3.3 we adapt a related proof by Frey (1998) to show that even in this more general setting the final value problem for γ still has a solution if the boundary values at maturity do not become too large. Under this condition we then sketch how the methods used to prove the convergence statement of Section 3.2 can be generalized in order to prove the convergence of $\{g^n\}_{n\in\mathbb{N}}$ even in the presence of transaction losses. Our convergence results are then transformed back into the corresponding results for the strategy functions $\{\xi^n\}_{n\in\mathbb{N}}$ and their limit φ.

Finally, we come to the convergence of the discrete real value functions $\{\bar{v}^n\}_{n\in I\!N}$ towards a continuous-time limit in Section 3.4. If the large investor does not always trade at the benchmark price, our convergence result for the real value functions relies on the convergence of the strategy functions, and the final value problem which describes the limit function $\bar{v}$ also depends on the limit φ of the strategy functions. It turns out that φ is a transform of $\bar{v}$'s spatial derivative. A first discussion of the final value problem for $\bar{v}$ and a comparison with corresponding problems for value functions in standard small investor models concludes the chapter.

Throughout this chapter, we work with a large investor price system (ψ, μ) which satisfies the multiplicative structure of Assumption B. Moreover, as in Chapter 2 we only consider the case $T = 1$ in order to avoid lengthy indices for the discrete time points $\{t_k^n\}_{0\le k\le \lceil nT\rceil}$.

3.1 Hölder Spaces and Discrete Derivatives

Of course, we will only be able to prove the convergence of a family $\{\xi^n\}_{n\in I\!N}$ of discrete strategy functions $\xi^n : \mathcal{A}^n \to I\!R$ to a continuous function $\varphi : [0,T] \times I\!R \to I\!R$ as $n \to \infty$ if we assume certain regularity conditions, and since the candidate limiting function φ itself is given as a solution to a (non-linear) partial differential equation (PDE), we even need some regularity conditions just to show the existence of such a solution. It turns out that certain *Hölder spaces* are appropriate function spaces both for the limiting function φ and for the two components of the multiplicative equilibrium price function in order to derive existence and convergence results. In this first section we introduce the various Hölder spaces which we employ in Chapters 3 and 4, and we also define some abridged notation for discrete derivatives which help us to keep the complexity of our formulæ at a moderate level.

Let us first recall the definition of Hölder continuity:

Definition 3.1. Let $D \subset I\!R$. Then a function $h : D \to I\!R$ is **Hölder continuous** with exponent $\beta \in (0,1)$ if there exists a nonnegative constant K such that

$$|h(x) - h(y)| \le K|x-y|^\beta \quad \text{for all } x, y \in D.$$

Then we can define Hölder spaces as in Ladyženskaja, Solonnikov and Ural′ceva (1968):

Definition 3.2. For $k, l \in I\!N_0$ and $\beta \in (0,1)$, the **Hölder space** $H^{\frac{1}{2}(k+\beta),k+l+\beta}\big([0,T]\times I\!R\big)$ consists of all continuous and bounded functions $h : [0,T]\times I\!R \to I\!R$, $(t,x) \mapsto h(t,x)$, such that

(*i*) for all $\eta_1, \eta_2 \in I\!N_0$ with $2\eta_1 \le k$ and $2\eta_1 + \eta_2 \le k + l$ the derivatives

$$\left(\frac{\partial}{\partial t}\right)^{\eta_1}\left(\frac{\partial}{\partial x_j}\right)^{\eta_2} h : (0,T)\times I\!R \to I\!R \tag{1.1}$$

exist and are both continuous and bounded,

(*ii*) all of the derivatives in (*i*) with $2\eta_1 = k-1$ or $2\eta_1+\eta_2 = k+l-1$ are Hölder continuous in t with exponent $\frac{1}{2}(1+\beta)$, and

(*iii*) all of the derivatives in (*i*) with $2\eta_1 = k$ or $2\eta_1 + \eta_2 = k + l$ are Hölder continuous in t with exponent $\frac{1}{2}\beta$ and Hölder continuous in x with exponent β.

We then write $h \in H^{\frac{1}{2}(k+\beta),k+l+\beta}\big([0,T]\times I\!R\big)$, and since the derivatives in (1.1) are continuous and bounded, we can extend them in a continuous fashion to the whole domain $[0,T]\times I\!R$.

Therefore, for $t=0$ and $t=T$ we will understand any derivative $\left(\frac{\partial}{\partial t}\right)^{\eta_1}\left(\frac{\partial}{\partial x_j}\right)^{\eta_2} h(t,u)$ as the right- or left-hand limit as $t \searrow 0$ or $t \nearrow T$, respectively.
A function $h : [0,T] \times I\!R \to I\!R$ belongs to the **Hölder space** $H_{loc}^{\frac{1}{2}(k+\beta),k+l+\beta}\big([0,T) \times I\!R\big)$ if for all $\eta_1, \eta_2 \in I\!N_0$ with $2\eta_1 \le k$ and $2\eta_1 + \eta_2 \le k+l$ the derivatives (1.1) exist and satisfy the conditions (i) to (iii) for each compact subinterval of $[0,T) \times I\!R$. In this case we can extend the derivatives of (1.1) in a continuous fashion to the domain $[0,T) \times I\!R$.

Remark. Normally we consider "symmetric" Hölder spaces $H^{\frac{1}{2}(k+\beta),k+\beta}\big([0,T] \times I\!R\big)$. In this case we can simplify each pair of conditions on η_1 and η_2 in (i) to (iii) to a single condition, since then the first condition of each pair is seen to be redundant. □

Similarly to the Hölder spaces for time-dependent functions $h : [0,T] \times I\!R \to I\!R$ we can also define the corresponding Hölder spaces for functions $f : I\!R \to I\!R$:

Definition 3.3. For any $k \in I\!N_0$ and $\beta \in (0,1)$ the **Hölder space** $H^{k+\beta}(I\!R)$ consists of all bounded functions $f : I\!R \to I\!R$ which are k times continuously differentiable, with bounded derivatives up to order k, and for which the kth derivative $f^{(k)} : I\!R \to I\!R$ is Hölder continuous with exponent β.
Finally, we introduce the Hölder space $H_{loc}^{k+\beta}\big(I\!R\big)$ of all functions $f : I\!R \to I\!R$ which are k times differentiable, and for which the kth derivative $f^{(k)} : I\!R \to I\!R$ is Hölder continuous with exponent β on any compact subset of $I\!R$.

Using a norm which basically sums up the (minimal) bounds on the derivatives (1.1) and the minimal Hölder constants for the derivatives considered in (ii) and (iii), it can be shown that the space $H^{\frac{1}{2}(k+\beta),k+l+\beta}\big([0,T] \times I\!R\big)$ is complete, and likewise it follows that $H^{k+\beta}(I\!R)$ is complete as well.
It is clear that for any $k, l \in I\!N_0$ and $\beta \in (0,1)$ the Hölder space $H^{\frac{1}{2}(2k+\beta),2k+l+\beta}\big([0,T] \times I\!R\big)$ is a subspace of the space $C^{k,2k+l}\big([0,T] \times I\!R\big)$ of continuously differentiable functions of order k and $2k+l$, respectively:

Definition 3.4. Let $k, l \in I\!N_0$. Then the space $C^{k,l}\big([0,T] \times I\!R\big)$ consists of all continuous functions $h : [0,T] \times I\!R \to I\!R$, $(t,x) \mapsto h(t,x)$, which are k times continuously differentiable with respect to t and l times continuously differentiable with respect to x. The subspace $C_b^{k,l}\big([0,T] \times I\!R\big)$ consists of all $h \in C^{k,l}\big([0,T] \times I\!R\big)$ which are bounded together with their partial derivatives $\left(\frac{\partial}{\partial t}\right)^{\eta_1}\left(\frac{\partial}{\partial x}\right)^{\eta_2} h : (0,T) \times I\!R \to I\!R$ for all $\eta_1, \eta_2 \in I\!N_0$ with $\eta_1 \le k$, $\eta_2 \le l$ and either $\eta_1\eta_2 = 0$ or $\eta_1 + \eta_2 \le k \wedge l$.
We write $h \in C^{k,l}\big([0,T) \times I\!R\big)$ if $h : [0,T] \times I\!R \to I\!R$, $(t,x) \mapsto h(t,x)$, is continuous on $[0,T) \times I\!R$ and if it is k times continuously differentiable with respect to t and l times continuously differentiable with respect to x.
Last but not least, for each $k \in I\!N_0$ we denote as usual the space of k times continuously differentiable functions by $C^k(I\!R)$, and the subspace of all functions in $C^k(I\!R)$ which are bounded together with all their derivatives up to order k by $C_b^k(I\!R)$.

Actually, we will need that ratios of the small investor price function $\overline{\psi} : [0,T] \times I\!R \to I\!R$ like $\frac{\frac{1}{2}\overline{\psi}_{uu} - \overline{\psi}_t}{\overline{\psi}_u}$ lie in certain Hölder spaces. In order to restate such conditions in terms of the underlying function $\overline{\psi}$ itself, it is useful to define another class of functions which need not be bounded, but for which some ratios of derivatives belong to certain Hölder spaces.

Definition 3.5. Let $k \in I\!N$, $l \in I\!N_0$, and $\beta \in (0,1)$. A function $h : [0,T] \times I\!R \to I\!R$, $(t,x) \mapsto h(t,x)$, belongs to the **class** $\widehat{H}^{\frac{1}{2}(k+\beta),k+l+\beta}\big([0,T] \times I\!R\big)$ if and only if

(i) for all $\eta_1, \eta_2 \in I\!N_0$ with $2\eta_1 \le k$ and $2\eta_1 + \eta_2 \le k+l$ the derivatives (1.1) exist,

(ii) the first spatial derivative $\frac{\partial}{\partial x}h : (0,T)\times I\!R \to I\!R$ is strictly positive,

(iii) for all $\eta_1, \eta_2 \in I\!N_0$ with $2\eta_1 \le k$ and $1 \le 2\eta_1 + \eta_2 \le k+l$ the ratio functions

$$\left(\frac{\partial}{\partial x}h(t,x)\right)^{-1}\left(\frac{\partial}{\partial t}\right)^{\eta_1}\left(\frac{\partial}{\partial x_j}\right)^{\eta_2} h : (0,T)\times I\!R \to I\!R$$

are bounded, Hölder continuous in x with exponent β, and Hölder continuous in t with exponent $\frac{1}{2}(1+\beta)$ if $2\eta_1+\eta_2 = k+l-1$ or $2\eta_1 = k-1$, and with exponent $\frac{1}{2}\beta$ otherwise.

In this case we write again $h \in \widehat{H}^{\frac{1}{2}(k+\beta),k+l+\beta}\big([0,T]\times I\!R\big)$.

For our convergence results, we will frequently deal with maximum norms over different domains. We find it convenient to denote all these norms by $\|\cdot\|$, and distinguish between the norms by specifying the set over which the maximum is taken as a subscript:

Definition 3.6. Let $d \in I\!N$, and suppose we are given some functions $h_1 : D_1 \to I\!R$ and $h_2 : D_2 \to I\!R$ mapping from some subsets $D_1 \subset I\!R^d$ and $D_2 \subset [0,T]\times I\!R^d$ to $I\!R$. Then we write $\|h_1\|_{D_1} := \sup_{x\in D_1} |h_1(x)|$ and $\|h_2\|_{D_2} := \sup_{(t,x)\in D_2} |h_2(t,x)|$. In the special case where $D_1 = I\!R^d$ or $D_2 = [0,T]\times I\!R^d$, we often suppress the subscript D_1 or D_2, respectively.

We have already mentioned discrete derivatives of the discrete strategy function $\xi^n : \mathcal{A}^n \to I\!R$ when discussing Corollary 1.40 in Section 1.4.4. When proving the convergence of a sequence $\{\xi^n\}_{n\in I\!N}$ of strategy functions towards a continuous-time limit $\varphi : [0,T]\times I\!R \to I\!R$ which is given as a solution to a certain PDE, we will definitely need to work with certain discrete derivatives of ξ^n which converge towards the corresponding partial derivatives of φ. Now we have to take into account that for each $n \in I\!N$ the strategy function $\xi^n : \mathcal{A}^n \to I\!R$ is only defined a discrete binomial grid, so we need to be careful with the arguments of the discrete derivatives. For example, it is useful to define the discrete first spatial derivative of a function $h : \mathcal{A}^n \to I\!R$ for all those points $(t,u) \in [0,T]\times I\!R$ which lie exactly between two points $(t, u\pm\delta) \in \mathcal{A}^n$. Since these points do not lie on the binomial grid $\mathcal{A}^n$, we first need to define a reasonable domain for the discrete derivatives.

Definition 3.7. As a generalization of the set of possible time-space realizations $\mathcal{A}^n(m)$ in (1.3.10), we define the **set of possible arguments of the lth discrete derivative** as

$$\mathcal{A}^n_l(m) = \Big\{ \big(t^n_k, u\big) \;\Big|\; \text{for } k \in \{l, l+1, \dots, m\} \text{ and } u \in \mathcal{U}^n_{k-l} \Big\} \quad \text{for all } 0 \le l \le m \le n,$$

and we again write $\mathcal{A}^n_l$ instead of $\mathcal{A}^n_l(n)$. Then for any $n \in I\!N$ and any function $h : \mathcal{A}^n \to I\!R$ on the grid $\mathcal{A}^n$ the three **discrete derivatives** $\Delta^n_u h : \mathcal{A}^n_1 \to I\!R$, $\Delta^n_{uu} h : \mathcal{A}^n_2 \to I\!R$, and $\Delta^n_t h : \mathcal{A}^n_1(n-1) \to I\!R$ are defined by

$$\Delta^n_u h(t,u) = \frac{h(t,u+\delta) - h(t,u-\delta)}{2\delta} \qquad \text{for } (t,u) \in \mathcal{A}^n_1\ ,$$

$$\Delta^n_{uu} h\big(t,u\big) = \frac{\Delta^n_u h(t,u+\delta) - \Delta^n_u h(t,u-\delta)}{2\delta} \qquad \text{for } (t,u) \in \mathcal{A}^n_2,$$

and

$$\Delta^n_t h\big(t,u\big) = \frac{h(t+\delta^2,u) - h(t-\delta^2,u)}{2\delta^2} \qquad \text{for } (t,u) \in \mathcal{A}^n_1(n-1).$$

Remark. If the function h even is defined on the whole slab $[0,T]\times I\!R$ and belongs to the space $C^{1,2}_b\big([0,T]\times I\!R\big)$, it is clear that $\big\|\Delta^n_u h - h_u\big\|_{\mathcal{A}^n_1} \to 0$, $\big\|\Delta^n_{uu} h - h_{uu}\big\|_{\mathcal{A}^n_2} \to 0$, and $\big\|\Delta^n_t h - h_t\big\|_{\mathcal{A}^n_1(n-1)} \to 0$ as $n\to\infty$, so in this sense the discrete derivatives of Definition 3.7 are indeed approximations of the partial derivatives of $h : [0,T]\times I\!R \to I\!R$. Note also that for all $0 \le m \le n$ the set $\mathcal{A}^n_0(m)$ of possible arguments for the 0th discrete derivative is equal to the set $\mathcal{A}^n(m)$ of possible time-space realizations up to time t^n_m as introduced in (1.3.10). □

3.2 The Case without Transaction Losses

Before we shall treat in Section 3.3 the convergence of a sequence of strategy functions in binomial markets which are based on a general multiplicative price system, we confine ourselves in this section to multiplicative price systems which exclude any instantaneous transaction gains or losses. For such a price system (ψ, μ), we build for each $n \in \mathbb{N}$ the nth binomial market model as described in Section 1.3, and choose a self-financing strategy (ξ^n, b^n) with corresponding strategy function $\xi^n : \mathcal{A}^n \to \mathbb{R}$. We then give conditions on the convergence of the values of the large investor's strategy functions immediately before time T which imply that the sequence $\{\xi^n\}_{n\in\mathbb{N}}$ of discrete strategy functions converges uniformly for all trading times before maturity to the solution $\varphi : [0, T] \times \mathbb{R} \to \mathbb{R}$ of a certain semi-linear parabolic final value problem. Such a convergence result for the strategy functions can be easily shown once we have proved an analogous statement for the convergence of the corresponding sequence $\{g^n\}_{n\in\mathbb{N}}$ of transformed trading strategy functions $g^n : \mathcal{A}^n \to \mathbb{R}$, given by $g^n = g \circ \xi^n$, towards the transform $\gamma = g \circ \varphi : [0, T] \times \mathbb{R} \to \mathbb{R}$.
The section is divided into four parts. In the first part we present the final value problem for γ and then show its equivalence with the final value problem for the corresponding original strategy function φ. While the PDE for φ is only semi-linear, the PDE for the transform γ is linear, hence existence and uniqueness follow from classical results. In the subsequent part, i.e. in Section 3.2.2, we use standard techniques for the approximation of *linear* PDEs by difference schemes like certain maximum principles in order to prove that – roughly speaking – the transformed strategy functions $\{g^n\}_{n\in\mathbb{N}}$ converge uniformly on $[0, T) \times \mathbb{R}$ to γ with order $O\big(\delta^{2+\beta}\big)$ if the values of the transformed strategy function $\gamma^n\big(t^n_{n-1}, \cdot\big)$ immediately before time T converge uniformly to the corresponding values of γ with order $O\big(\delta^{4+\beta}\big)$. In Section 3.2.3 the results of Section 3.2.2 are then translated into corresponding results for the original strategy functions $\{\xi^n\}_{n\in\mathbb{N}}$. While we shall take for granted in Section 3.2.2 the existence of a sufficiently smooth solution γ to the final value problem, we will sketch another proof for the convergence of *sub*sequences of $\{g^n\}_{n\in\mathbb{N}}$ in Section 3.2.4, which does not presuppose the existence of γ. Section 3.2 gives us much insight into the tools used to handle the general case, where transaction losses are not prevented by the price system (ψ, μ), but it does not overwhelm us with bounds on the non-linearities caused by these losses.

We have already seen in Section 2.4.3 that for any multiplicative price system (ψ, μ) which excludes any instantaneous transaction gains *and* losses we may assume without loss of generality that the price-determining measure μ is the Lebesgue measure λ concentrated on the unit interval. Thus, in this section we will fix an equilibrium price function $\psi : [0, T] \times \mathbb{R}^2 \to \mathbb{R}$ which satisfies Assumption B and consider for each $n \in \mathbb{N}$ the binomial market of Section 1.3 which is based on the price system $(\psi, \mu) = (\psi, \lambda)$. As we proceed, we will require different degrees of smoothness for the two components $\overline{\psi}$ and f of the function ψ. However, we will always suppose that the small investor price function $\overline{\psi} : [0, T] \times \mathbb{R} \to \mathbb{R}$, $(t, u) \mapsto \overline{\psi}(t, u)$, is continuously differentiable with respect to t, two times continuously differentiable with respect to u, and that it satisfies $\big\|\frac{\overline{\psi}_t}{\overline{\psi}_u}\big\| < \infty$. The function $f : \mathbb{R} \to (0, \infty)$ needs to be at least continuously differentiable.
Since we know from Lemma 2.3 that under these conditions Assumption A holds for all $n > \big\|\frac{\overline{\psi}_t}{\overline{\psi}_u}\big\|^2$, we can conclude from Proposition 2.16 that for all those $n \in \mathbb{N}$ every path-independent contingent claim $\big(\xi_n, b_n\big)$ is attainable by a unique path-independent and self-financing trading strategy $\big(\xi^n, b^n\big)$. Moreover, we recall from Corollary 2.15 and the subsequent discussion that in this case the transformed strategy function $g^n : \mathcal{A}^n \to \mathbb{R}$, which was introduced in Definition 2.12 as

$$g^n(t, u) := g\big(\xi^n(t, u)\big) = \int_0^{\xi^n(t,u)} f(x)\, dx \quad \text{for all } (t, u) \in \mathcal{A}^n,$$

can be recursively calculated as

$$g^n\big(t_{k-1}^n, u\big) = \hat{p}^n\big(t_{k-1}^n, u\big)g^n\big(t_k^n, u+\delta\big) + \Big(1 - \hat{p}^n\big(t_{k-1}^n, u\big)\Big)g^n\big(t_k^n, u-\delta\big), \tag{2.1}$$

where the weight function $\hat{p}^n : \mathcal{A}^n(n-2) \to (0,1)$ is given by

$$\hat{p}^n\big(t_{k-1}^n, u\big) = \frac{\overline{\psi}\big(t_k^n, u+\delta\big) - \overline{\psi}\big(t_{k+1}^n, u\big)}{\overline{\psi}\big(t_k^n, u+\delta\big) - \overline{\psi}\big(t_k^n, u-\delta\big)} \quad \text{for } 1 \le k \le n-1 \text{ and } u \in \mathcal{U}_{k-1}^n. \tag{2.2}$$

The recursive scheme (2.1) will be the starting point for our convergence analysis in Section 3.2.2.

3.2.1 The Limiting PDEs for the Strategy Functions and their Transforms

In this section we first introduce the final value problem for the potential limiting function $\gamma : [0,T] \times I\!R \to I\!R$ of a sequence $\{g^n\}_{n \in I\!N}$ of transformed strategy functions $g^n : \mathcal{A}^n \to I\!R$. Existence and uniqueness of solutions to this final value problems follow from standard results. We then proceed and define the corresponding final value problem for the limit φ of the original strategy functions and show the equivalence of both final value problems.

In case of convergence, the limit $\gamma : [0,T] \times I\!R \to I\!R$ of a sequence of transformed strategy functions will solve a final value problem of the form

$$\gamma_t(t,u) + \frac{1}{2}\gamma_{uu}(t,u) = \gamma_u(t,u)\frac{\overline{\psi}_t(t,u) - \frac{1}{2}\overline{\psi}_{uu}(t,u)}{\overline{\psi}_u(t,u)} \qquad \text{for all } (t,u) \in [0,T) \times I\!R, \tag{2.3}$$

with final condition

$$\gamma(T,u) = \int_0^{\zeta(u)} f(x)\,dx \qquad \text{for all } u \in I\!R, \tag{2.4}$$

where the function $\zeta : I\!R \to I\!R$ describes the required stock holdings of the large investor immediately before maturity. Since this problem is a *linear* final value problem, we can use standard results from PDE theory to prove existence and uniqueness of (classical) solutions to this final value problem. For example, we have

Lemma 3.8. *Suppose that for some $k \ge 2$ we have $\overline{\psi} \in \widehat{H}^{\frac{1}{2}(k+\beta),k+\beta}\big([0,T] \times I\!R\big)$. If the boundary function $\zeta : I\!R \to I\!R$ is continuous and chosen such that $\big\|\int_0^{\zeta(\cdot)} f(x)\,dx\big\|$ is finite, then the final value problem (2.3), (2.4) has a unique solution $\gamma : [0,T] \times I\!R \to I\!R$ in the space $C_b^{0,0}\big([0,T] \times I\!R\big) \cap C^{1,2}\big([0,T) \times I\!R\big)$. If we even have $f \in H_{loc}^{k-1+\beta}(I\!R)$ and $\zeta \in H^{k+\beta}\big(I\!R\big)$, this solution even belongs to the Hölder space $H^{\frac{1}{2}(k+\beta),k+\beta}\big([0,T] \times I\!R\big)$.*

Proof. Let us define the drift coefficient $b : [0,T] \times I\!R \to I\!R$ by

$$b(t,u) = \frac{\frac{1}{2}\overline{\psi}_{uu}(t,u) - \overline{\psi}_t(t,u)}{\overline{\psi}_u(t,u)} \qquad \text{for all } (t,u) \in [0,T] \times I\!R,$$

where we identify the derivatives in $t = 0$ and $t = T$ as the right- and left-hand limits, respectively. By the definition of the Hölder class $\widehat{H}^{\frac{1}{2}(k+\beta),k+\beta}\big([0,T] \times I\!R\big)$, it follows that b belongs to the Hölder space $H^{\frac{1}{2}(k-2+\beta),k-2+\beta}\big([0,T] \times I\!R\big)$. Moreover, the boundary function $\gamma(T,\cdot\,) : I\!R \to I\!R,\ u \mapsto \int_0^{\zeta(u)} f(x)\,dx$ is continuous and bounded. So, after a time inversion $\tilde{t} = T - t$, the existence of a solution $\gamma \in C_b^{0,0}\big([0,T] \times I\!R\big) \cap C^{1,2}\big([0,T) \times I\!R\big)$ to the final value problem (2.3), (2.4) follows either from Theorem 12, Sec. 7, Chap. 1 in Friedman

(1964), or from Theorem IV.16.2 in Ladyženskaja et al. (1968), and the uniqueness follows by Theorem 16, Sec. 9, Chap. 1 in Friedman (1964) or Theorem I.2.6 in Ladyženskaja et al. (1968).
If we suppose $f \in H_{loc}^{k-1+\beta}(I\!R)$ and $\zeta \in H^{k+\beta}(I\!R)$, the boundary function $\gamma(T,\,\cdot\,)$ even belongs to the Hölder space $H^{k+\beta}(I\!R)$, and $\gamma \in H^{\frac{1}{2}(k+\beta),k+\beta}\big([0,T]\times I\!R\big)$ follows from Theorem IV.5.1 in Ladyženskaja et al. (1968). **q.e.d.**

Remark. The proof shows that the statement of Lemma 3.8 still holds if we only require that the function $\int_0^{\zeta(\,\cdot\,)} f(x)\,dx$ is continuous instead of requiring that ζ itself is continuous. □

The next proposition yields that the linear parabolic final value problem (2.3), (2.4) for $\gamma : [0,T]\times I\!R \to I\!R$ is equivalent to the semi-linear parabolic final value problem

$$\varphi_t(t,u) + \frac{1}{2}\varphi_{uu}(t,u) = \varphi_u(t,u)\frac{\psi_t\big(t,u,\varphi(t,u)\big) - \frac{1}{2}\frac{d}{du}\psi_u\big(t,u,\varphi(t,u)\big)}{\psi_u\big(t,u,\varphi(t,u)\big)} \quad \text{on } [0,T)\times I\!R, \tag{2.5}$$

where the final condition is now given by

$$\varphi(T,u) = \zeta(u) \quad \text{for all } u \in I\!R. \tag{2.6}$$

It will turn out later in Corollary 3.14 that such a final value problem is satisfied by the limit of a converging sequence $\{\xi^n\}_{n\in I\!N}$ of strategy functions $\xi^n : \mathcal{A}^n \to I\!R$.

Proposition 3.9. *Let $k \geq 2$ and $\beta \in (0,1)$, and suppose that there exist some functions $\overline{\psi} \in C^{1,2}\big([0,T]\times I\!R\big)$ and $f \in H_{loc}^{k-1+\beta}(I\!R)$ such that*

$$\psi(t,u,\xi) = \overline{\psi}(t,u)f(\xi) \quad \textit{for all } (t,u,\xi) \in [0,T]\times I\!R^2.$$

Then there exists a solution $\varphi \in H^{\frac{1}{2}(k+\beta),k+\beta}\big([0,T]\times I\!R\big)$ of the final value problem (2.5), (2.6) if and only if there exists a solution $\gamma \in H^{\frac{1}{2}(k+\beta),k+\beta}\big([0,T]\times I\!R\big)$ of the final value problem (2.3), (2.4), and two such solutions are connected via

$$\gamma(t,u) = g\big(\varphi(t,u)\big) = \int_0^{\varphi(t,u)} f(x)\,dx \quad \textit{for all } (t,u) \in [0,T]\times I\!R. \tag{2.7}$$

Proof. Let us assume that $\varphi \in H^{\frac{1}{2}(k+\beta),k+\beta}\big([0,T]\times I\!R\big)$ is a solution of the final value problem (2.5), (2.6), and define $\gamma : [0,T]\times I\!R \to I\!R$ via (2.7). Differentiating this equation we get for all $(t,u) \in [0,T]\times I\!R$:

$$\gamma_t(t,u) = f\big(\varphi(t,u)\big)\varphi_t(t,u) \tag{2.8}$$
$$\gamma_u(t,u) = f\big(\varphi(t,u)\big)\varphi_u(t,u), \tag{2.9}$$

and

$$\gamma_{uu}(t,u) = f\big(\varphi(t,u)\big)\varphi_{uu}(t,u) + f'\big(\varphi(t,u)\big)\varphi_u^2(t,u). \tag{2.10}$$

As a first step we prove that a solution of the final value problem (2.5), (2.6) is also a solution of (2.3), (2.4). The boundary condition (2.4) is obviously implied by the boundary condition (2.6). For the purpose of obtaining the PDE (2.3) as well, we fix $(t,u) \in [0,T)\times I\!R$ and notice that (2.8) and (2.10) induce

$$\gamma_t(t,u) + \frac{1}{2}\gamma_u(t,u) = f\big(\varphi(t,u)\big)\left(\varphi_t(t,u) + \frac{1}{2}\varphi_{uu}(t,u) + \frac{1}{2}\frac{f'(\varphi(t,u))}{f(\varphi(t,u))}\varphi_u^2(t,u)\right). \tag{2.11}$$

Since the equilibrium price function $\psi : [0,T] \times \mathbb{R}^2 \to \mathbb{R}$ has a multiplicative structure, its first order derivatives are given by $\psi_t(t,u,\xi) = \overline{\psi}_t(t,u) f(\xi)$ and $\psi_u(t,u,\xi) = \overline{\psi}_u(t,u) f(\xi)$ for all $\xi \in \mathbb{R}$, and, moreover, we obtain

$$\frac{d}{du}\psi_u\big(t,u,\varphi(t,u)\big) = \overline{\psi}_{uu}(t,u) f\big(\varphi(t,u)\big) + \overline{\psi}_u(t,u) f'\big(\varphi(t,u)\big)\varphi_u(t,u).$$

Hence the PDE (2.5) can be rewritten as

$$\varphi_t(t,u) + \frac{1}{2}\varphi_{uu}(t,u) + \frac{1}{2}\frac{f'(\varphi(t,u))}{f(\varphi(t,u))}\varphi_u^2(t,u) = \varphi_u(t,u)\frac{\overline{\psi}_t(t,u) - \frac{1}{2}\overline{\psi}_{uu}(t,u)}{\overline{\psi}_u(t,u)}.$$

If we plug this equation into our formula (2.11), we get

$$\gamma_t(t,u) + \frac{1}{2}\gamma_u(t,u) = f\big(\varphi(t,u)\big)\varphi_u(t,u)\frac{\overline{\psi}_t(t,u) - \frac{1}{2}\overline{\psi}_{uu}(t,u)}{\overline{\psi}_u(t,u)},$$

and upon identifying the factor in front of the fraction as $\gamma_u(t,u)$ due to (2.9), it follows that $\gamma : [0,T] \times \mathbb{R} \to \mathbb{R}$ indeed solves the PDE (2.3), and hence the final value problem (2.3), (2.4).

In order to show that $\gamma \in H^{\frac{1}{2}(k+\beta),k+\beta}\big([0,T]\times\mathbb{R}\big)$ as well, we remark that by the boundedness of φ there exists a compact interval $I \subset \mathbb{R}$ such that $\varphi(t,u) \in I$ for all $(t,u) \in [0,T] \times \mathbb{R}$. Since $f \in H^{k-1+\beta}_{loc}(\mathbb{R})$, its derivatives up to order $k-1$ are continuous, and hence there exists some $K_1 \in \mathbb{R}$ such that

$$\big|f^{(i)}(x)\big| \leq K_1 \quad \text{for all } x \in I \text{ and all } 0 \leq i \leq k-1. \tag{2.12}$$

Additionally $f^{(k-1)} : \mathbb{R} \to \mathbb{R}$ is Hölder continuous, so there exists some K_2 in $\mathbb{R}$ such that $\big|f^{(k-1)}(x) - f^{(k-1)}(y)\big| \leq K_2|x-y|^\beta$ for all $x,y \in I$. Then it is easily seen from (2.7) – (2.10) and similar formulæ for the higher derivatives of γ (if $k > 2$) that the boundedness of γ and its derivatives is implied by the boundedness of the derivatives of φ up to the same order, and a similar statement holds for the Hölder continuity of derivatives of γ. Hence $\gamma \in H^{\frac{1}{2}(k+\beta),k+\beta}\big([0,T] \times \mathbb{R}\big)$ is implied by $\varphi \in H^{\frac{1}{2}(k+\beta),k+\beta}\big([0,T] \times \mathbb{R}\big)$.

Now assume that $\gamma \in H^{\frac{1}{2}(k+\beta),k+\beta}\big([0,T] \times \mathbb{R}\big)$ satisfies the final value problem (2.3), (2.4), and define $\varphi(t,u) := h\big(\gamma(t,u)\big)$, where $h : \mathbb{R} \to \mathbb{R}$ is the inverse function to $g : \mathbb{R} \to \mathbb{R}$, $\xi \mapsto \int_0^\xi f(x)\,dx$. Using the derivatives $h'(x) = \frac{1}{f(h(x))}$ and $h''(x) = -\frac{f'(h(x))}{f^3(h(x))}$ for all $x \in \mathbb{R}$, we can calculate for all $(t,u) \in [0,T] \times \mathbb{R}$ the derivatives

$$\begin{aligned}
\varphi_t(t,u) &= \frac{1}{f\big(h(\gamma(t,u))\big)}\gamma_t(t,u),\\
\varphi_u(t,u) &= \frac{1}{f\big(h(\gamma(t,u))\big)}\gamma_u(t,u),
\end{aligned}$$

and

$$\varphi_{uu}(t,u) = \frac{1}{f\big(h(\gamma(t,u))\big)}\gamma_{uu}(t,u) - \frac{f'\big(h(\gamma(t,u))\big)}{f^3\big(h(\gamma(t,u))\big)}\gamma_u^2(t,u),$$

and it follows that $\varphi : [0,T] \times \mathbb{R}$ satisfies (2.5), (2.6). Again the boundedness of γ yields some compact interval $I \subset \mathbb{R}$ such that $\gamma(t,u) \in I$ for all $(t,u) \in \mathbb{R}$. The function $f : \mathbb{R} \to \mathbb{R}$ is strictly positive, and continuous since $f \in H^{k-1+\beta}_{loc}\big(\mathbb{R}\big)$. Thus there exists some $c > 0$ such that $f(h(x)) \geq c$ for all $x \in I$. Now the proof proceeds as in the first part and shows that indeed $\varphi \in H^{\frac{1}{2}(k+\beta),k+\beta}\big([0,T] \times \mathbb{R}\big)$. **q.e.d.**

3.2.2 Convergence of the Transformed Strategy Functions

In this section we show the convergence result for the transformed strategy functions $\{g^n\}_{n\in I\!N}$. We assume that a solution $\gamma \in H^{2+\frac{1}{2}\beta,4+\beta}\big([0,T]\times I\!R\big)$ of the final value problem (2.3), (2.4) exists, as it will be the case if for example $\overline{\psi} \in \widehat{H}^{2+\frac{1}{2}\beta,4+\beta}\big([0,T]\times I\!R\big)$, $f \in H^{3+\beta}_{loc}(I\!R)$, and $\zeta \in H^{4+\beta}\big(I\!R\big)$ (see Lemma 3.8).
Basically, we will show that the transformed strategies $\{g^n\}$ converge uniformly on $[0,T)\times I\!R$ with order $O\big(\delta^{2+\beta}\big)$ to γ as $n\to\infty$ if the values of the transformed strategy functions immediately before T converge to the corresponding values of T uniformly with order $O\big(\delta^{4+\beta}\big)$. We leave aside the convergence of the values of the transformed strategies $\{g^n\}_{n\in I\!N}$ at time T for the moment, but we shall discuss this shortfall in Section 3.2.3, where we deal with the convergence of the original strategy functions.
Let us now start with the precise statement of the result:

Theorem 3.10. *Let (ψ,μ) be a large investor price system which satisfies $\mu=\lambda$ and Assumption B, and suppose that $\overline{\psi} \in \widehat{H}^{2+\frac{1}{2}\beta,4+\beta}\big([0,T]\times I\!R\big)$. If $\gamma \in H^{2+\frac{1}{2}\beta,4+\beta}\big([0,T]\times I\!R\big)$ solves the final value problem (2.3), (2.4) and if there exists some $K\in I\!R$ such that*

$$\big\|g^n\big(t^n_{n-1},\,\cdot\,\big)-\gamma\big(t^n_{n-1},\,\cdot\,\big)\big\|_{\mathcal{U}^n_{n-1}} \le K\delta^{4+\beta} \quad \text{for all } n\in I\!N, \tag{2.13}$$

then the sequence $\{g^n\}_{n\in I\!N}$ of discrete functions $g^n:\mathcal{A}^n\to I\!R$ converges to the function $\gamma:[0,T]\times I\!R\to I\!R$ in the sense that

$$\|g^n-\gamma\|_{\mathcal{A}^n(n-1)} = O\big(\delta^2\big) \tag{2.14}$$

and

$$\left\|g^n(\cdot+\delta^2,\cdot\pm\delta)-g^n\mp\delta\gamma_u-\delta^2\left(\gamma_t+\frac{1}{2}\gamma_{uu}\right)\right\|_{\mathcal{A}^n(n-2)} = O\big(\delta^{2+\beta}\big) \quad \text{as } n\to\infty. \tag{2.15}$$

Remark. The theorem's proof will indicate that we could replace condition (2.13) by the slightly weaker condition that both the inequality $\big\|g^n\big(t^n_{n-1},\,\cdot\,\big)-\gamma\big(t^n_{n-1},\,\cdot\,\big)\big\|_{\mathcal{U}^n_{n-1}}\le K\delta^4$ and the inequality $\big\|\Delta^n_u\, g^n\big(t^n_{n-1},\,\cdot\,\big)-\Delta^n_u\,\gamma\big(t^n_{n-1},\,\cdot\,\big)\big\|_{\mathcal{U}^n_{n-2}}\le K\delta^{3+\beta}$ hold for all $n\in I\!N$. □

The theorem's proof follows from certain maximum principles for $g^n-\gamma$ and $\Delta^n_u\, g^n-\Delta^n_u\,\gamma$. Before we come to the proof itself, we will sketch the idea.
In order to show that $\|g^n-\gamma\|_{\mathcal{A}^n(n-1)}=O\big(\delta^2\big)$ as $n\to\infty$, we will show that the function $\gamma:[0,T]\times I\!R\to I\!R$ approximately solves the recursive equation (2.1) with the function $g^n:\mathcal{A}^n\to I\!R$ replaced by the function $\gamma:[0,T]\times I\!R\to I\!R$, up to an error term of order $O\big(\delta^4\big)$. Thus we get for all sufficiently large $n\in I\!N$, all $1\le k\le n-1$ and $u\in\mathcal{U}^n_{k-1}$:

$$\begin{aligned} g^n\big(t^n_{k-1},u\big)-\gamma\big(t^n_{k-1},u\big) &= \hat{p}^n\big(t^n_{k-1},u\big)\Big(g^n\big(t^n_k,u+\delta\big)-\gamma\big(t^n_k,u+\delta\big)\Big)\\ &\quad+\Big(1-\hat{p}^n\big(t^n_{k-1},u\big)\Big)\Big(g^n\big(t^n_k,u-\delta\big)-\gamma\big(t^n_k,u-\delta\big)\Big)+O\big(\delta^4\big). \end{aligned}$$

For all $n>\big\|\frac{\overline{\psi}_t}{\overline{\psi}_u}\big\|^2$ the function $\hat{p}^n:\mathcal{A}^n(n-2)\to(0,1)$ of (2.2) is well defined, and since it takes values in $(0,1)$, it follows that for all sufficiently large $n\in I\!N$ and each $1\le k\le n-1$ the difference $\big\|g^n\big(t^n_{k-1},\,\cdot\,\big)-\gamma\big(t^n_{k-1},\,\cdot\,\big)\big\|_{\mathcal{U}^n_{k-1}}$ can be bounded in terms of $\big\|g^n\big(t^n_k,\,\cdot\,\big)-\gamma\big(t^n_k,\,\cdot\,\big)\big\|_{\mathcal{U}^n_k}$ plus an $O\big(\delta^4\big)$-term. Starting at $k=n-1$ and working backward until $k=1$ gives a bound in terms of $\big\|g^n\big(t^n_{n-1},\,\cdot\,\big)-\gamma\big(t^n_{n-1},\,\cdot\,\big)\big\|_{\mathcal{U}^n_{n-1}}$ plus $n-1$ error terms of order $O\big(\delta^4\big)$, which sum up to a term of order $O\big(\delta^2\big)$. This is a standard argument for approximating a linear PDE by a difference scheme.

In order to prove (2.15), we show that a similar maximum principle holds for the maximum norm $\left\|\Delta_u^n g^n\left(t_{k-1}^n, \cdot\right) - \Delta_u^n \gamma\left(t_{k-1}^n, \cdot\right)\right\|_{\mathcal{U}_{k-2}^n}$ by using the difference of (2.1) evaluated for $u' := u+\delta$ and $u' := u-\delta$ and some $2 \leq k \leq n-1$, $u \in \mathcal{U}_{k-2}^n$. This allows us to conclude that for $1 \leq k \leq n-1$ the maximum norm $\left\|\Delta_u^n g^n\left(t_k^n, \cdot\right) - \Delta_u^n \gamma\left(t_k^n, \cdot\right)\right\|_{\mathcal{U}_{k-1}^n}$ can be bounded by $\left\|\Delta_u^n g^n\left(t_{n-1}^n, \cdot\right) - \Delta_u^n \gamma\left(t_{n-1}^n, \cdot\right)\right\|_{\mathcal{U}_{n-2}^n}$ plus $n-k-1$ terms of order $O\left(\delta^{3+\beta}\right)$, and therefore by a term of order $O\left(\delta^{1+\beta}\right)$. Then a suitable application of (2.1) and (2.3) will yield (2.15). Thus, we first have to make sure that γ really solves (2.1) up to a term of order $O\left(\delta^4\right)$. This requires an order approximation of $\hat{p}^n : \mathcal{A}^n(n-2) \to I\!R$. Next to the order approximations for $\hat{p}^n$ which is needed now, we also state two similar order approximations in the following lemma, which we need later when we prove the convergence of $\Delta_u^n g^n$ as well.

Lemma 3.11. *Suppose that the assumptions of Theorem 3.10 hold. Then*

$$2\hat{p}^n\left(t_{k-1}^n, u\right) - 1 = \delta \frac{\frac{1}{2}\overline{\psi}_{uu}\left(t_k^n, u\right) - \overline{\psi}_t\left(t_k^n, u\right)}{\overline{\psi}_u\left(t_k^n, u\right)} + O\left(\delta^3\right) \quad \textit{as } n \to \infty, \tag{2.16}$$

uniformly for all $1 \leq k \leq n-1$ *and* $u \in \mathcal{U}_{k-1}^n$. *Moreover, uniformly for all* $2 \leq k \leq n-1$ *and* $u \in \mathcal{U}_{k-2}^n$ *we have*

$$\hat{p}^n\left(t_{k-1}^n, u+\delta\right) + \hat{p}^n\left(t_{k-1}^n, u-\delta\right) - 1 = \delta \frac{\frac{1}{2}\overline{\psi}_{uu}\left(t_k^n, u\right) - \overline{\psi}_t\left(t_k^n, u\right)}{\overline{\psi}_u\left(t_k^n, u\right)} + O\left(\delta^3\right) \tag{2.17}$$

and

$$\hat{p}^n\left(t_{k-1}^n, u+\delta\right) - \hat{p}^n\left(t_{k-1}^n, u-\delta\right) = \delta^2 \frac{d}{du} \frac{\frac{1}{2}\overline{\psi}_{uu}\left(t_k^n, u\right) - \overline{\psi}_t\left(t_k^n, u\right)}{\overline{\psi}_u\left(t_k^n, u\right)} + O\left(\delta^{3+\beta}\right) \tag{2.18}$$

as $n \to \infty$.

Proof. We will prove (2.16) in detail, since it exemplarily reveals the techniques used to get order approximations for functionals of $\overline{\psi} : [0,T] \times I\!R \to I\!R$. It draws on several Taylor expansions.

At first, let us fix $\delta > 0$ and suppose that $0 \leq t \leq T$ and $u \in I\!R$. By Taylor's rule there exist some $u^-, u^+ \in I\!R$ with $u - \delta \leq u^- \leq u \leq u^+ \leq u + \delta$ such that

$$\overline{\psi}(t, u \pm \delta) - \overline{\psi}(t,u) = \pm\delta\overline{\psi}_u(t,u) + \frac{1}{2}\delta^2\overline{\psi}_{uu}(t,u) \pm \frac{1}{6}\delta^3\overline{\psi}_{uuu}(t,u) + \frac{1}{24}\delta^4\overline{\psi}_{uuuu}\left(t,u^{\pm}\right).$$

If we divide the sum of these two equations by $2\delta\overline{\psi}_u(t,u)$ and apply the intermediate value theorem to the mean of $\overline{\psi}_{uuuu}\left(t,u^{\pm}\right)$, we get for some $u - \delta \leq u_1 \leq u + \delta$:

$$\frac{\overline{\psi}(t,u+\delta) - 2\overline{\psi}(t,u) + \overline{\psi}(t,u-\delta)}{2\delta\overline{\psi}_u(t,u)} = \frac{1}{2}\delta\frac{\overline{\psi}_{uu}(t,u)}{\overline{\psi}_u(t,u)} + \frac{1}{24}\delta^3\frac{\overline{\psi}_{uuuu}\left(t,u_1\right)}{\overline{\psi}_u(t,u)}. \tag{2.19}$$

Now notice that by the mean value theorem there exists some u_2 between u and u_1 such that $\log\overline{\psi}_u(t,u_1) - \log\overline{\psi}_u(t,u) = (u_1 - u)\frac{d}{du}\log\overline{\psi}_u(t,u_2) = (u_1 - u)\frac{\overline{\psi}_{uu}(t,u_2)}{\overline{\psi}_u(t,u_2)}$, and therefore

$$\left|\frac{\overline{\psi}_{uuuu}(t,u_1)}{\overline{\psi}_u(t,u)}\right| = \left|\frac{\overline{\psi}_{uuuu}(t,u_1)}{\overline{\psi}_u(t,u_1)}\right| e^{\log\overline{\psi}_u(t,u_1) - \log\overline{\psi}_u(t,u)} \leq \left\|\frac{\overline{\psi}_{uuuu}}{\overline{\psi}_u}\right\| \exp\left(\delta\left\|\frac{\overline{\psi}_{uu}}{\overline{\psi}_u}\right\|\right), \tag{2.20}$$

where we have suppressed the subscript $[0,T] \times I\!R$ of the norm $\|\cdot\|$. Since $\overline{\psi} : [0,T] \times I\!R \to I\!R$ belongs to the Hölder class $\widehat{H}^{2+\frac{1}{2}\beta, 4+\beta}\left([0,T] \times I\!R\right)$, we have $\left\|\frac{\overline{\psi}_z}{\overline{\psi}_u}\right\| < \infty$ for $z \in \{uu, uuuu\}$, and thus (2.19) and (2.20) imply uniformly for all $0 \leq t \leq T$ and $u \in I\!R$:

$$\frac{\overline{\psi}(t,u+\delta) - 2\overline{\psi}(t,u) + \overline{\psi}(t,u-\delta)}{2\delta\overline{\psi}_u(t,u)} = \frac{1}{2}\delta\frac{\overline{\psi}_{uu}(t,u)}{\overline{\psi}_u(t,u)} + O\left(\delta^3\right) \quad \text{as } \delta \to 0.$$

Similarly we can show uniformly for all $(t,u)\in[0,T]\times I\!R$ that

$$\frac{\overline{\psi}(t,u+\delta)-\overline{\psi}(t,u-\delta)}{2\delta\overline{\psi}_u(t,u)}=1+O\big(\delta^2\big)\quad\text{as }\delta\to 0,$$

and uniformly for all $0\le t\le T-\delta^2$ and $u\in I\!R$ that

$$\frac{\overline{\psi}(t+\delta^2,u)-\overline{\psi}(t,u)}{2\delta\overline{\psi}_u(t,u)}=\delta\frac{\overline{\psi}_t(t,u)}{\overline{\psi}_u(t,u)}+O\big(\delta^3\big)\quad\text{as }\delta\to 0.$$

Now recall the definition of $\hat{p}^n:\mathcal{A}^n(n-2)\to I\!R$ in (2.2) for all $n\in I\!N$ with $n>\big\|\frac{\overline{\psi}_t}{\overline{\psi}_u}\big\|^2$. We get for all those $n\in I\!N$, all $1\le k\le n-1$ and all $u\in\mathcal{U}^n_{k-1}$:

$$2\hat{p}\big(t^n_{k-1},u\big)-1=\frac{\overline{\psi}\big(t^n_k,u+\delta\big)-2\overline{\psi}\big(t^n_k,u\big)+\overline{\psi}\big(t^n_k,u-\delta\big)-2\big(\overline{\psi}\big(t^n_{k+1},u\big)-\overline{\psi}\big(t^n_k,u\big)\big)}{\overline{\psi}\big(t^n_k,u+\delta\big)-\overline{\psi}\big(t^n_k,u-\delta\big)}.$$

Inserting the three previous expansions and using once again Taylor's rule, now applied to the function $x\mapsto\frac{1}{1+x}$, we conclude from the boundedness of the ratios $\frac{\overline{\psi}_t}{\overline{\psi}_u}$ and $\frac{\overline{\psi}_{uu}}{\overline{\psi}_u}$ that

$$2\hat{p}\big(t^n_{k-1},u\big)-1=\frac{\frac{1}{2}\delta\frac{\overline{\psi}_{uu}(t^n_k,u)}{\overline{\psi}_u(t^n_k,u)}-\delta\frac{\overline{\psi}_t(t^n_k,u)}{\overline{\psi}_u(t^n_k,u)}+O\big(\delta^3\big)}{1+O\big(\delta^2\big)}=\delta\frac{\frac{1}{2}\overline{\psi}_{uu}\big(t^n_k,u\big)-\overline{\psi}_t\big(t^n_k,u\big)}{\overline{\psi}_u\big(t^n_k,u\big)}+O\big(\delta^3\big),$$

and because of $\delta=\delta_n=n^{-\frac{1}{2}}$, we arrive at (2.16).
The proofs of (2.17) and (2.18) follow similarly to the proof of (2.16). For (2.18) we have to be a little bit careful, since we have only assumed $\overline{\psi}\in\widehat{H}^{2+\frac{1}{2}\beta,4+\beta}$ and therefore cannot use Taylor approximations for the function $\overline{\psi}$ of any order with more than two derivatives in time or more than four space derivatives, nor derivatives like $\overline{\psi}_{tuuu}$. However, if we first develop a Taylor series for $\overline{\psi}\big(t+\delta^2,u\pm\delta\big)$ around $(t,u\pm\delta)$ and then expand $\overline{\psi}(t,u\pm\delta)$ and $\overline{\psi}_t(t,u\pm\delta)$ around (t,u), we can show that a suitable expansion for the left-hand side of (2.18) exists. Moreover, the highest derivatives which appear, namely $\overline{\psi}_{uuuu}$, $\overline{\psi}_{tuu}$, and $\overline{\psi}_{tt}$, appear pairwise and offset each other due to the Hölder continuity of the derivative ratios.
To make this clearer, let us first convince ourselves that due to the bounds $e^x-1\le xe^x$ and $1-e^{-x}\le x$ for $x\ge 0$, and due to the considerations leading to (2.20), we have

$$\left|\frac{\overline{\psi}_u\big(t,u+\eta\big)}{\overline{\psi}_u(t,u)}-1\right|\le|\eta|\exp\Big(|\eta|\Big\|\frac{\overline{\psi}_{uu}}{\overline{\psi}_u}\Big\|\Big)\quad\text{for all }t\in[0,T]\text{ and }u,\eta\in I\!R.\tag{2.21}$$

Our Taylor series for the left-hand side of (2.18) will involve for example a term of the form $\frac{\overline{\psi}_z(t,u^+)-\overline{\psi}_z(t,u^-)}{\overline{\psi}_u(t,u)}$ with $z=uuuu$ and $u-2\delta\le u^-\le u\le u^+\le u+2\delta$. This term can be bounded by

$$\left|\frac{\overline{\psi}_z(t,u^+)}{\overline{\psi}_u(t,u^+)}\right|\left|\frac{\overline{\psi}_u(t,u^+)}{\overline{\psi}_u(t,u)}-1\right|+\left|\frac{\overline{\psi}_z(t,u^+)}{\overline{\psi}_u(t,u^+)}-\frac{\overline{\psi}_z(t,u^-)}{\overline{\psi}_u(t,u^-)}\right|+\left|\frac{\overline{\psi}_z(t,u^-)}{\overline{\psi}_u(t,u^-)}\right|\left|1-\frac{\overline{\psi}_u(t,u^-)}{\overline{\psi}(t,u)}\right|,$$

and since $\big\|\frac{\overline{\psi}_z}{\overline{\psi}_u}\big\|<\infty$, our bound in (2.21) shows that the first and the last terms are of order $O\big(\delta\big)$, while the term in the middle can be seen to be of order $O\big(\delta^\beta\big)$ as $\delta\to 0$ by the Hölder continuity of the derivative $\frac{\overline{\psi}_z}{\overline{\psi}_u}$.
Full details of the proof of (2.17) and (2.18), however, are omitted to save the reader from another bunch of Taylor approximations. **q.e.d.**

Remark. The above notes to the proof of (2.18) could also be used to show that for any $k \in \mathbb{N}$ the condition $\log \overline{\psi} \in H^{\frac{1}{2}(k+\beta),k+\beta}\big([0,T] \times \mathbb{R}\big)$ together with the additional condition $\frac{d}{du} \log \overline{\psi}(t,u) \geq c > 0$ for all $(t,u) \in [0,T] \times \mathbb{R}$ implies that $\overline{\psi} \in \widehat{H}^{\frac{1}{2}(k+\beta),k+\beta}\big([0,T] \times \mathbb{R}\big)$. However, even for the Cox-Ross-Rubinstein model of Example 1.9, where the small investor price function $\overline{\psi} : [0,T] \times \mathbb{R} \to \mathbb{R}$ is given by $\overline{\psi}(t,u) = S_0 e^{\sigma u + (\mu - r)t}$ for all $(t,u) \in [0,T] \times \mathbb{R}$ and some fixed $S_0, \sigma > 0$ and $\mu, r \in \mathbb{R}$, the condition $\log \overline{\psi} \in H^{\frac{1}{2}(k+\beta),k+\beta}\big([0,T] \times \mathbb{R}\big)$ is not satisfied, since the function $\overline{\psi}$ itself is not bounded. This is the prime reason why we consider equilibrium price functions $\overline{\psi}$ from the (larger) class $\widehat{H}^{\frac{1}{2}(k+\beta),k+\beta}\big([0,T] \times \mathbb{R}\big)$. □

As promised, we can now show that $\big\|g^n - \gamma\big\|_{\mathcal{A}^n(n-1)} = O\big(\delta^2\big)$ as $n \to \infty$.

Lemma 3.12. *Let us suppose that the price system (ψ, μ) satisfies the assumptions of Theorem 3.10, and suppose that the solution $\gamma : [0,T] \times \mathbb{R} \to \mathbb{R}$ of the final value problem (2.3), (2.4) lies in the space $C_b^{2,4}\big([0,T] \times \mathbb{R}\big)$. Then $\big\|g^n - \gamma\big\|_{\mathcal{A}^n(n-1)} = O\big(\delta^2\big)$ as $n \to \infty$ is implied by the condition $\big\|g^n\big(t_{n-1}^n, \cdot\big) - \gamma\big(t_{n-1}^n, \cdot\big)\big\|_{\mathcal{U}_{n-1}^n} \leq K\delta^4$ for all $n \in \mathbb{N}$ and some $K \geq 0$.*

Proof. Similarly to the proof of Lemma 3.11, Taylor's rule shows for all $t \in \big[\delta^2, T\big]$ and $u \in \mathbb{R}$ that there exist some $u^-, u^+ \in \mathbb{R}$ with $u - \delta \leq u^- \leq u \leq u^+ \leq u + \delta$ and some $t^* \in [0,T]$ with $t - \delta^2 \leq t^* \leq t$ such that

$$\gamma\big(t, u \pm \delta\big) = \gamma(t,u) \pm \delta\gamma_u(t,u) + \frac{1}{2}\delta^2\gamma_{uu}(t,u) \pm \frac{1}{6}\delta^3\gamma_{uuu}(t,u) + \frac{1}{24}\delta^4\gamma_{uuuu}\big(t,u^{\pm}\big)$$

and

$$\gamma\big(t - \delta^2, u\big) = \gamma(t,u) - \delta^2\gamma_t(t,u) + \frac{1}{2}\delta^4\gamma_{tt}\big(t^*, u\big).$$

Since all appearing derivatives of γ are globally bounded on $[0,T] \times \mathbb{R}$, and due to formula (2.16) of Lemma 3.11, we can now write uniformly for all $n > \big\|\frac{\overline{\psi}_t}{\overline{\psi}_u}\big\|^2$, all $1 \leq k \leq n-1$, and all $u \in \mathcal{U}_{k-1}^n$:

$$\begin{aligned}
&\hat{p}^n\big(t_{k-1}^n, u\big)\gamma\big(t_k^n, u+\delta\big) + \Big(1 - \hat{p}^n\big(t_{k-1}^n, u\big)\Big)\gamma\big(t_k^n, u-\delta\big) - \gamma\big(t_{k-1}^n, u\big) \\
&\quad = \frac{1}{2}\Big(\gamma\big(t_k^n, u+\delta\big) + \gamma\big(t_k^n, u-\delta\big)\Big) - \gamma\big(t_{k-1}^n, u\big) \\
&\qquad + \Big(2\hat{p}^n\big(t_{k-1}^n, u\big) - 1\Big)\frac{1}{2}\Big(\gamma\big(t_k^n, u+\delta\big) - \gamma\big(t_k^n, u-\delta\big)\Big) \\
&\quad = \frac{1}{2}\delta^2\gamma_{uu}\big(t_k^n, u\big) + \delta^2\gamma_t\big(t_k^n, u\big) + O\big(\delta^4\big) \\
&\qquad + \left(\delta\frac{\frac{1}{2}\overline{\psi}_{uu}\big(t_k^n, u\big) - \overline{\psi}_t\big(t_k^n, u\big)}{\overline{\psi}_u\big(t_k^n, u\big)} + O\big(\delta^3\big)\right)\Big(\delta\gamma_u\big(t_k^n, u\big) + O\big(\delta^3\big)\Big) \\
&\quad = \delta^2\left(\gamma_t\big(t_k^n, u\big) + \frac{1}{2}\gamma_{uu}\big(t_k^n, u\big) + \gamma_u\big(t_k^n, u\big)\frac{\frac{1}{2}\overline{\psi}_{uu}\big(t_k^n, u\big) - \overline{\psi}_t\big(t_k^n, u\big)}{\overline{\psi}_u\big(t_k^n, u\big)}\right) + O\big(\delta^4\big),
\end{aligned}$$

and since $\gamma : [0,T] \times \mathbb{R}$ solves the parabolic partial differential equation (2.3), the $O\big(\delta^2\big)$-term vanishes. Hence for each fixed $N > \big\|\frac{\overline{\psi}_t}{\overline{\psi}_u}\big\|^2$ there exists an $L \in \mathbb{R}$ such that for all $n \geq N$, $1 \leq k \leq n-1$ and $u \in \mathcal{U}_{k-1}^n$ we have

$$\Big|\hat{p}^n\big(t_{k-1}^n, u\big)\gamma\big(t_k^n, u+\delta\big) + \Big(1 - \hat{p}^n\big(t_{k-1}^n, u\big)\Big)\gamma\big(t_k^n, u-\delta\big) - \gamma\big(t_{k-1}^n, u\big)\Big| \leq \delta^4 L. \qquad (2.22)$$

Without loss of generality we may assume $L \geq K$. The statement of the lemma will follow once we have shown that

$$\left\| g^n\big(t_k^n, \cdot\big) - \gamma^n\big(t_k^n, \cdot\big)\right\|_{\mathcal{U}_k^n} \leq (n-k)\delta^4 L, \tag{2.23}$$

for all $n \geq N$ and all $0 \leq k \leq n-1$, since then $\|g^n - \gamma^n\|_{\mathcal{A}^n(n-1)} \leq n\delta^4 L$, and the definition of $\delta = \delta_n = n^{-\frac{1}{2}}$ leads to $\|g^n - \gamma^n\|_{\mathcal{A}^n(n-1)} \leq \delta^2 L$.
The proof of (2.23) follows from a backward induction over k. For $k = n-1$ the bound (2.23) holds because of our assumption and $L \geq K$. Let us now assume that $n \geq N$ and that (2.23) holds for some $1 \leq k \leq n-1$. Then from (2.1) and $0 < \hat{p}^n(t,u) < 1$ for all $(t,u) \in \mathcal{A}^n(n-2)$ it follows that for all $u \in \mathcal{U}_{k-1}^n$:

$$\begin{aligned}\left|g^n\big(t_{k-1}^n, u\big) - \gamma^n\big(t_{k-1}^n, u\big)\right| &\leq \hat{p}^n\big(t_{k-1}^n, u\big)\left|g^n\big(t_k^n, u+\delta\big) - \gamma\big(t_k^n, u+\delta\big)\right| \\ &\quad + \Big(1 - \hat{p}^n\big(t_{k-1}^n, u\big)\Big)\left|g^n\big(t_k^n, u-\delta\big) - \gamma\big(t_k^n, u-\delta\big)\right| + \left|R_{k-1}^n(u)\right|,\end{aligned}$$

where $\left|R_{k-1}^n(u)\right|$ is given by the left hand side of (2.22). Hence we can conclude from (2.22) and our induction hypothesis that

$$\left\| g^n\big(t_{k-1}^n, \cdot\big) - \gamma^n\big(t_{k-1}^n, \cdot\big)\right\|_{\mathcal{U}_{k-1}^n} \leq \left\| g^n\big(t_k^n, \cdot\big)\right\|_{\mathcal{U}_k^n} + \delta^4 L \leq \delta^4(n-k+1)L,$$

which proves the induction step. **q.e.d.**

The next lemma uses similar methods to bound the error incurred by approximating the discrete derivative $\Delta_u^n g^n : \mathcal{A}_1^n \to I\!R$ with the discrete derivative $\Delta_u^n \gamma : \mathcal{A}_1^n \to I\!R$, which itself is an $O\big(\delta^2\big)$-approximation on the grid $\mathcal{A}_1^n$ of the continuous derivative $\gamma_u : [0,T] \times I\!R$, as $n \to \infty$.

Lemma 3.13. *Suppose that the large investor price system (ψ, μ) and the solution γ of the final value problem* (2.3), (2.4) *satisfy the assumptions of Theorem 3.10. Then*

$$\left\| \Delta_u^n g^n\big(t_{n-1}^n, \cdot\big) - \Delta_u^n \gamma\big(t_{n-1}^n, \cdot\big)\right\|_{\mathcal{U}_{n-2}^n} \leq K\delta^{3+\beta} \quad \textit{for all } n \in I\!N \tag{2.24}$$

implies $\left\|\Delta_u^n g^n - \Delta_u^n \gamma\right\|_{\mathcal{A}_1^n(n-1)} = O\big(\delta^{1+\beta}\big)$ *as* $n \to \infty$.

Proof. Let us first generate a recursive formula for the restriction of $\Delta_u^n g^n : \mathcal{A}_1^n \to I\!R$ to $\mathcal{A}_1^n(n-1)$, analogous to the recursive equation (2.1). By the definition of $\Delta_u^n g^n$ in Definition 3.7 and by a twofold application of (2.1) we obtain for all $n > \left\|\frac{\overline{\psi}_t}{\overline{\psi}_u}\right\|^2$, all $2 \leq k \leq n-1$ and $u \in \mathcal{U}_{k-2}^n$:

$$\begin{aligned}2\delta\Delta_u^n g^n\big(t_{k-1}^n, u\big) &= g^n\big(t_{k-1}^n, u+\delta\big) - g^n\big(t_{k-1}^n, u-\delta\big) \\ &= \hat{p}^n\big(t_{k-1}^n, u+\delta\big) g^n\big(t_k^n, u+2\delta\big) - \Big(1 - \hat{p}^n\big(t_{k-1}^n, u-\delta\big)\Big) g^n\big(t_k^n, u-2\delta\big) \\ &\quad + \Big(1 - \hat{p}^n\big(t_{k-1}^n, u+\delta\big) - \hat{p}^n\big(t_{k-1}^n, u-\delta\big)\Big) g^n\big(t_k^n, u\big).\end{aligned}$$

After a rearrangement of terms we obtain the following recursive formula for the restriction of $\Delta_u^n g^n$ to $\mathcal{A}_1^n(n-1) \to I\!R$:

$$\begin{aligned}\Delta_u^n g^n\big(t_{k-1}^n, u\big) &= \hat{p}^n\big(t_{k-1}^n, u+\delta\big)\Delta_u^n g^n\big(t_k^n, u+\delta\big) \\ &\quad + \Big(1 - \hat{p}^n\big(t_{k-1}^n, u-\delta\big)\Big)\Delta_u^n g^n\big(t_k^n, u-\delta\big).\end{aligned} \tag{2.25}$$

Next we investigate the error we make if we replace in the previous equation g^n by γ, i.e. we consider the term

$$\begin{aligned} R^n\big(t^n_{k-1},u\big) &:= \hat{p}^n\big(t^n_{k-1},u+\delta\big)\Delta^n_u\gamma_u(t^n_k,u+\delta) \\ &\quad + \Big(1-\hat{p}^n\big(t^n_{k-1},u-\delta\big)\Big)\Delta^n_u\gamma\big(t^n_k,u-\delta\big) - \Delta^n_u\gamma\big(t^n_{k-1},u\big) \end{aligned}$$

for $n > \big\|\frac{\bar{\psi}_t}{\bar{\psi}_u}\big\|^2$, all $2 \le k \le n-1$ and $u \in \mathcal{U}^n_{k-2}$. To find order approximations for this, let us fix $\delta^2 \le t \le T$ and $u \in I\!R$. Similar to the Taylor expansions of the previous lemma, but now employing the Hölder continuity of γ_{uuuu} as well, we get for $s = t-\delta^2$ and some $u-\delta \le u^- \le u \le u^+ \le u+\delta$:

$$\begin{aligned} \Delta^n_u\gamma\big(s,u\big) &= \gamma_u\big(s,u\big) + \frac{1}{6}\delta^2\gamma_{uuu}(s,u) + \frac{1}{48}\delta^3\Big(\gamma_{uuuu}\big(s,u^+\big) - \gamma_{uuuu}\big(s,u^-\big)\Big) \\ &= \gamma_u\big(s,u\big) + \frac{1}{6}\delta^2\gamma_{uuu}(s,u) + O\big(\delta^{3+\beta}\big) \quad \text{as } \delta \to 0. \end{aligned}$$

If we use this equation, expand $\gamma_u(s,u)$ around (t,u), and also recall that γ_{tu} and γ_{uuu} are Hölder continuous in t with exponent $\frac{1}{2}\big(1+\beta\big)$ we see that

$$\Delta^n_u\gamma\big(t-\delta^2,u\big) = \gamma_u(t,u) - \delta^2\left(\gamma_{tu}(t,u) - \frac{1}{6}\gamma_{uuu}(t,u)\right) + O\big(\delta^{3+\beta}\big) \quad \text{as } \delta \to 0.$$

Moreover, by the Hölder continuity of γ_{uuuu} we get for all $(t,u) \in [0,T] \times I\!R$ that

$$\frac{1}{2}\big(\Delta^n_u\gamma(t,u+\delta) + \Delta^n_u\gamma(t,u-\delta)\big) = \gamma_u(t,u) + \frac{2}{3}\delta^2\gamma_{uuu}(t,u) + O\big(\delta^{3+\beta}\big) \quad \text{as } \delta \to 0$$

and

$$\frac{1}{2}\big(\Delta^n_u\gamma(t,u+\delta) - \Delta^n_u\gamma(t,u-\delta)\big) = \delta\gamma_{uu}(t,u) + O\big(\delta^3\big) \quad \text{as } \delta \to 0,$$

and all the preceding convergence statements are seen to hold uniformly for all $\delta^2 \le t \le T$ and $u \in I\!R$. Then we rewrite $R^n\big(t^n_{k-1},u\big)$ as

$$\begin{aligned} R^n\big(t^n_{k-1},u\big) = &\Big(1+\hat{p}^n\big(t^n_{k-1},u+\delta\big) - \hat{p}^n\big(t^n_{k-1},u-\delta\big)\Big)\frac{\Delta^n_u\gamma(t^n_k,u+\delta) + \Delta^n_u\gamma(t^n_k,u-\delta)}{2} \\ &+ \Big(\hat{p}^n\big(t^n_{k-1},u+\delta\big) + \hat{p}^n\big(t^n_{k-1},u-\delta\big) - 1\Big)\frac{\Delta^n_u\gamma(t^n_k,u+\delta) - \Delta^n_u\gamma(t^n_k,u-\delta)}{2} \\ &- \Delta^n_u\gamma\big(t^n_{k-1},u\big) \end{aligned}$$

and apply the expansion for $\Delta^n_u\gamma$ and the expansions (2.17) and (2.18) of Lemma 3.11 to that equation; this shows that uniformly for all $2 \le k \le n-1$ and $u \in \mathcal{U}^n_{k-2}$ we can approximate $R^n : \mathcal{A}^n_1(n-1) \to I\!R$ by

$$R^n\big(t^n_{k-1},u\big) = \delta^2\frac{d}{du}\left(\gamma_t\big(t^n_k,u\big) + \frac{1}{2}\gamma_{uu}\big(t^n_k,u\big) + \frac{\frac{1}{2}\bar{\psi}_{uu}(t^n_k,u) - \bar{\psi}_t(t^n_k,u)}{\bar{\psi}_u(t^n_k,u)}\gamma_u\big(t^n_k,u\big)\right) + O\big(\delta^{3+\beta}\big),$$

as $n \to \infty$, and as in the previous lemma the $O\big(\delta^2\big)$-term vanishes, since $\gamma : [0,T] \times I\!R \to I\!R$ solves the partial differential equation (2.3). Thus, for each $N > \big\|\frac{\bar{\psi}_t}{\bar{\psi}_u}\big\|^2$ there exists some L_1 such that for all $n \ge N$ we have $\|R^n\|_{\mathcal{A}^n_1(n-1)} \le L_1\delta^{3+\beta}$. We may again assume without loss of generality that $L_1 \ge K$.

In contrast to (2.1), the recursive equation (2.25) does not present $\Delta_u^n g^n\big(t_{k-1}^n, u\big)$ as a linear combination of the two potential successors $\Delta_u^n g^n\big(t_k^n, u \pm \delta\big)$, since the two weights for $\Delta_u^n g^n\big(t_k^n, u \pm \delta\big)$ do not add up to 1. However, the amount $\hat{p}^n\big(t_{k-1}^n, u+\delta\big) - \hat{p}^n\big(t_{k-1}^n, u-\delta\big)$, by which the sum of the weights misses the value 1, is seen to be only of order $O\big(\delta^2\big)$, due to (2.18) in Lemma 3.11. Thus, for each $N > \big\|\frac{\overline{\varphi}_t}{\overline{\psi}_u}\big\|^2$ there exist some L_2 such that for all $n \geq N$, all $2 \leq k \leq n-1$ and $u \in \mathcal{U}_{k-2}^n$ we have

$$\Big|\hat{p}^n\big(t_{k-1}^n, u+\delta\big) - \hat{p}^n\big(t_{k-1}^n, u-\delta\big)\Big| \leq \delta^2 L_2. \tag{2.26}$$

Let us now fix some $N > \|\frac{\overline{\varphi}_t}{\overline{\psi}_u}\|^2$ and the corresponding constants L_1 and L_2. We will show that for all $n \geq N$ and $1 \leq k \leq n-1$:

$$\Big\|\Delta_u^n g\big(t_k^n, \cdot\big) - \Delta_u^n \gamma\big(t_k^n, \cdot\big)\Big\|_{\mathcal{U}_{k-1}^n} \leq \delta^{3+\beta} L_1 \sum_{j=0}^{n-k-1} (1+\delta^2 L_2)^j. \tag{2.27}$$

Again, this will be done via an inductive argument over k. It is clear by assumption (2.24) that (2.27) holds for $k = n-1$. For the induction step let us suppose that (2.27) holds for some $2 \leq k \leq n-1$. Then from the recursive formula (2.25) and from the definition of $R^n\big(t_{k-1}^n, u\big)$ we get for all $u \in U_{k-2}^n$ that

$$\begin{aligned}
&\Big|\Delta_u^n g^n\big(t_{k-1}^n, u\big) - \Delta_u^n \gamma\big(t_{k-1}^n, u\big)\Big| \\
&\quad \leq \hat{p}^n\big(t_{k-1}^n, u+\delta\big)\Big|\Delta_u^n g^n\big(t_k^n, u+\delta\big) - \Delta_u^n \gamma\big(t_k^n, u+\delta\big)\Big| \\
&\qquad + \Big(1 - \hat{p}^n\big(t_{k-1}^n, u-\delta\big)\Big)\Big|\Delta_u^n g^n\big(t_k^n, u-\delta\big) - \Delta_u^n \gamma\big(t_k^n, u-\delta\big)\Big| + \big|R^n\big(t_{k-1}^n, u\big)\big|,
\end{aligned}$$

since the two appearing weights are nonnegative. Due to $\big\|R^n\big\|_{\mathcal{A}_1^n(n-1)} \leq L_1 \delta^{3+\beta}$ we thus recursively obtain a bound on the maximum norms of the differences $\Delta_u^n g^n - \Delta_u^n \gamma$:

$$\begin{aligned}
\Big\|\Delta_u^n g^n\big(t_{k-1}^n, \cdot\big) - \Delta_u^n \gamma\big(t_{k-1}^n, \cdot\big)\Big\|_{\mathcal{U}_{k-2}^n} &\leq \Big(1 + \hat{p}^n\big(t_{k-1}^n, u+\delta\big) - \hat{p}^n\big(t_{k-1}^n, u-\delta\big)\Big) \\
&\qquad \cdot \Big\|\Delta_u^n g^n\big(t_k^n, \cdot\big) - \Delta_u^n \gamma\big(t_k^n, \cdot\big)\Big\|_{\mathcal{U}_k^n} + L_1 \delta^{3+\beta}.
\end{aligned}$$

Hence (2.26) and our induction assumption (2.27) imply

$$\begin{aligned}
\Big\|\Delta_u^n g^n\big(t_{k-1}^n, \cdot\big) - \Delta_u^n \gamma\big(t_{k-1}^n, \cdot\big)\Big\|_{\mathcal{U}_{k-2}^n} &\leq \big(1+\delta^2 L_2\big) L_1 \delta^{3+\beta} \sum_{j=0}^{n-k-1} \big(1+\delta^2 L_2\big)^j + L_1 \delta^{3+\beta} \\
&= \delta^{3+\beta} L_1 \sum_{j=0}^{n-k} \big(1+\delta^2 L_2\big)^j,
\end{aligned}$$

and so indeed (2.27) holds for all $1 \leq k \leq n-1$ and all $n \geq N$. To complete the proof, notice that by the definition of $\delta = \delta_n = n^{-\frac{1}{2}}$ and the monotonicity of $n \mapsto \big(1 + \frac{1}{n} L_2\big)^n$ we have for all $1 \leq k \leq n-1$:

$$\delta^{3+\beta} \sum_{j=0}^{n-k-1} \big(1+\delta^2 L_2\big)^j \leq \delta^{3+\beta} n \big(1+\delta^2 L_2\big)^n \leq \delta^{1+\beta} e^{L_2},$$

and therefore the lemma is implied by the availability of (2.27) for all $1 \leq k \leq n-1$ and $n \geq N$. **q.e.d.**

Remark. We could also replace the discrete derivative $\Delta_u^n \gamma$ by the continuous derivative γ_u and show that $\big\|\Delta_u^n g^n - \gamma_u\big\|_{\mathcal{A}_1^n(n-1)} = O\big(\delta^{1+\beta}\big)$ as $n \to \infty$ follows from the condition that $\big\|\Delta_u^n g^n\big(t_{n-1}^n, \cdot\big) - \gamma_u\big(t_{n-1}^n, \cdot\big)\big\|_{\mathcal{U}_{n-2}^n} \le K\delta^{3+\beta}$ for all $n \in I\!N$. However, if the third spatial derivative γ_{uuu} does not vanish at $t = T$, it is easy to see that for any $\alpha > 3$ the latter condition and $\big\|g^n\big(t_{n-1}^n, \cdot\big) - \gamma\big(t_{n-1}^n, \cdot\big)\big\|_{\mathcal{U}_{n-1}^n} \le K\delta^\alpha$ cannot hold simultaneously, and hence we could not derive the statement of Lemma 3.12 at the same time. □

There are only minor steps left to prove Theorem 3.10:

Proof of Theorem 3.10. Since (2.14) is already implied by Lemma 3.12, we only have to validate that (2.15) holds as $n \to \infty$, and we are content to prove the lower (minus) case, since the upper (plus) case follows by similar arguments. For the proof notice that due to condition (2.13) and the definition of the discrete derivatives $\Delta_u^n g^n : \mathcal{A}_1^n \to I\!R$ and $\Delta_u^n \gamma : \mathcal{A}_1^n \to I\!R$, we have

$$\Big|\Delta_u^n g^n\big(t_{n-1}^n, u\big) - \Delta_u^n \gamma\big(t_{n-1}^n, u\big)\Big| \le K\delta^{3+\beta} \quad \text{for all } u \in \mathcal{U}_{n-2}^n.$$

Hence the assumptions of Lemma 3.13 hold, and thus $\big\|\Delta_u^n g^n - \Delta_u^n \gamma\big\|_{\mathcal{A}_1^n(n-1)} = O\big(\delta^{1+\beta}\big)$ as $n \to \infty$.

By calculations analogous to the one in Lemma 3.11 it can be shown that uniformly for all $0 \le t \le T - \delta^2$ and $u \in I\!R$:

$$\frac{\frac{1}{2}\overline{\psi}_{uu}(t+\delta^2, u) - \overline{\psi}_t(t+\delta^2, u)}{\overline{\psi}_u(t+\delta^2, u)} = \frac{\frac{1}{2}\overline{\psi}_{uu}(t, u) - \overline{\psi}_t(t, u)}{\overline{\psi}_u(t, u)} + O\big(\delta^2\big) \quad \text{as } \delta \to 0.$$

Together with the approximation (2.16) of Lemma 3.11 and $\gamma \in H^{2+\frac{1}{2}\beta, 4+\beta}\big([0,T] \times I\!R\big)$, this implies uniformly for all $0 \le k \le n-2$ and $u \in \mathcal{U}_k^n$ that

$$\begin{aligned} R^n\big(t_k^n, u\big) &:= 2\hat{p}^n\big(t_k^n, u\big)\Delta_u^n \gamma\big(t_{k+1}^n, u\big) - \gamma_u\big(t_k^n, u\big) + \delta\left(\gamma_t\big(t_k^n, u\big) + \frac{1}{2}\gamma_{uu}\big(t_k^n, u\big)\right) \\ &= \delta\left(\gamma_t\big(t_k^n, u\big) + \frac{1}{2}\gamma_{uu}\big(t_k^n, u\big) + \frac{\frac{1}{2}\overline{\psi}_{uu}(t_k^n, u) - \overline{\psi}_t(t_k^n, u)}{\overline{\psi}_u(t_k^n, u)}\gamma_u\big(t_k^n, u\big)\right) + O\big(\delta^2\big) \end{aligned}$$

as $n \to \infty$, and since $\gamma : [0,T] \times I\!R \to I\!R$ satisfies (2.3), the first term on the right-hand side drops out and $\big\|R^n\big\|_{\mathcal{A}^n(n-2)} = O\big(\delta^2\big)$ as $n \to \infty$.

Now fix some $n > \big\|\frac{\overline{\psi}_t}{\overline{\psi}_u}\big\|^2$, some $0 \le k \le n-2$, and some $u \in \mathcal{U}_k^n$, and recall the recursive equation (2.1) with $k-1$ replaced by k. Subtracting $g^n\big(t_{k+1}^n, u-\delta\big)$ on both sides, we get

$$g^n\big(t_k^n, u\big) - g^n\big(t_{k+1}^n, u-\delta\big) = \hat{p}^n\big(t_k^n, u\big)2\delta\Delta_u^n g^n\big(t_{k+1}^n, u\big).$$

If we now subtract $\delta\gamma_u(t_k^n, u) - \delta^2\big(\gamma_t(t_k^n, u) + \frac{1}{2}\gamma_{uu}(t_k^n, u)\big)$ on both sides and use the triangle inequality and the definition of $R^n\big(t_k^n, u\big)$, we obtain:

$$\begin{aligned} &\Big|g^n\big(t_k^n, u\big) - g^n\big(t_{k+1}^n, u-\delta\big) - \delta\gamma_u\big(t_k^n, u\big) + \delta^2\Big(\gamma_t\big(t_k^n, u\big) + \frac{1}{2}\gamma_{uu}\big(t_k^n, u\big)\Big)\Big| \\ &\quad \le 2\delta\hat{p}^n\big(t_k^n, u\big)\big|\Delta_u^n g^n\big(t_{k+1}^n, u\big) - \Delta_u^n \gamma\big(t_{k+1}^n, u\big)\big| + \delta\big|R^n\big(t_k^n, u\big)\big| \\ &\quad \le 2\delta\big\|\Delta_u^n g^n - \Delta_u^n \gamma\big\|_{\mathcal{A}_1^n(n-1)} + \delta\big\|R^n\big\|_{\mathcal{A}^n(n-2)} = O\big(\delta^{2+\beta}\big) \qquad \text{as } n \to \infty, \end{aligned}$$

where the last line follows from $\hat{p}^n : \mathcal{A}^n(n-2) \to (0,1)$, $\|R^n\|_{\mathcal{A}^n(n-2)} = O\big(\delta^2\big)$ as $n \to \infty$, and Lemma 3.13. Changing the sign within the absolute value and taking the norm $\|\cdot\|_{\mathcal{A}^n(n-2)}$ we see that the lower (minus) case approximation of (2.15) holds. This concludes the proof of the theorem. **q.e.d.**

Remark. The proof suggests that it would perhaps be more natural to use a Taylor expansion around $\big(t_k^n, u\big)$ for the second-order approximation of $g^n\big(t_k^n, u \pm \delta\big) - g^n\big(t_{k-1}^n, u\big)$, as opposed to the Taylor expansion around $\big(t_{k-1}^n, u\big)$. We prefer the expansion around $\big(t_{k-1}^n, u\big)$ because it simplifies the representation (2.15) in terms of the sup-norm on the space $\mathcal{A}^n(n-2)$. $\square$

3.2.3 Convergence of the Strategy Functions

As a corollary of Theorem 3.10, we can now use the convergence of the transformed strategy functions to derive an analogous result for the convergence of the sequence $\{\xi^n\}_{n\in\mathbb{N}}$ of strategy functions $\xi^n : \mathcal{A}^n \to \mathbb{R}$ as $n \to \infty$ to the solution $\varphi : [0,T] \times \mathbb{R} \to \mathbb{R}$ of the final value problem (2.5), (2.6).

Corollary 3.14. *Let (ψ, μ) be a large investor price system which satisfies $\mu = \lambda$ and Assumption B, and suppose that the two components $\overline{\psi}$ and f of ψ in the representation $\psi(t,u,\xi) = \overline{\psi}(t,u)f(\xi)$ belong to the Hölder spaces $\widehat{H}^{2+\frac{1}{2}\beta,4+\beta}\big([0,T]\times\mathbb{R}\big)$ and $H^{3+\beta}_{loc}(\mathbb{R})$, respectively. If $\varphi \in H^{2+\frac{1}{2}\beta,4+\beta}\big([0,T]\times\mathbb{R}\big)$ solves the final value problem (2.5), (2.6) and there exists some $K \in \mathbb{R}$ such that*

$$\big\|\xi^n\big(t_{n-1}^n, \cdot\big) - \varphi\big(t_{n-1}^n, \cdot\big)\big\|_{\mathcal{U}_{n-1}^n} \le K\delta^{4+\beta} \quad \text{for all } n \in \mathbb{N}, \tag{2.28}$$

then the discrete strategy functions $\xi^n : \mathcal{A}^n \to \mathbb{R}$ converge to $\varphi : [0,T]\times\mathbb{R} \to \mathbb{R}$ in the sense that

$$\|\xi^n - \varphi\|_{\mathcal{A}^n(n-1)} = O\big(\delta^2\big) \tag{2.29}$$

and

$$\left\|\xi^n(\cdot + \delta^2, \cdot \pm \delta) - \xi^n \mp \delta\varphi_u - \delta^2\left(\varphi_t + \frac{1}{2}\varphi_{uu}\right)\right\|_{\mathcal{A}^n(n-2)} = O\big(\delta^{2+\beta}\big) \quad \text{as } n \to \infty. \tag{2.30}$$

Proof. Since $\varphi \in H^{2+\frac{1}{2}\beta,4+\beta}\big([0,T]\times\mathbb{R}\big)$, it is in particular bounded. As a first step towards the proof of the corollary, we will show that condition (2.28) guarantees the existence of a uniform bound on $\big\|\xi^n\big\|_{\mathcal{A}^n(n-1)}$ for all $n \in \mathbb{N}$ as well.

Let us fix some $n > \big\|\frac{\overline{\psi}_t}{\overline{\psi}_u}\big\|^2$. Then the weight function $\hat{p}^n : \mathcal{A}^n(n-2) \to (0,1)$ of (2.2) is well defined, and since the weights take only values in $(0,1)$, the recursive equation (2.1) implies:

$$\min_{v\in\mathcal{U}_k^n}\big\{g^n\big(t_k^n, v\big)\big\} \le g^n\big(t_{k-1}^n, u\big) \le \max_{v\in\mathcal{U}_k^n}\big\{g^n\big(t_k^n, v\big)\big\} \quad \text{for all } 1 \le k \le n-1 \text{ and } u \in \mathcal{U}_{k-1}^n.$$

If we nest these bounds for all $1 \le k \le n-1$, we finally realize that the minimum and the maximum of $g^n\big(\mathcal{A}^n(n-1)\big)$ are already determined by the minimum and maximum of $g^n\big(t_{n-1}^n, \mathcal{U}_{n-1}^n\big)$, the possible values of the transformed strategy function g^n at time t_{n-1}^n, i.e.

$$\min_{v\in\mathcal{U}_{n-1}^n}\big\{g^n\big(t_{n-1}^n, v\big)\big\} \le g^n(t,u) \le \max_{v\in\mathcal{U}_{n-1}^n}\big\{g^n\big(t_{n-1}^n, v\big)\big\} \quad \text{for all } (t,u) \in \mathcal{A}^n(n-1). \tag{2.31}$$

Now recall the definition of $g^n : \mathcal{A}^n \to \mathbb{R}$ as $g^n(t,u) = g^n\big(\xi^n(t,u)\big)$ for all $(t,u) \in \mathcal{A}^n$. Since the strategy transform $g : \mathbb{R} \to \mathbb{R}$ is strictly increasing and invertible, it follows that $\xi^n(t,u) = g^{-1}\big(g^n(t,u)\big)$ for all $(t,u) \in \mathcal{A}^n$, and we derive from (2.31) similar bounds for the range of $\xi^n\big(\mathcal{A}^n(n-1)\big)$, namely

$$\min_{v\in\mathcal{U}_{n-1}^n}\big\{\xi^n\big(t_{n-1}^n, v\big)\big\} \le \xi^n(t,u) \le \max_{v\in\mathcal{U}_{n-1}^n}\big\{\xi^n\big(t_{n-1}^n, v\big)\big\} \quad \text{for all } (t,u) \in \mathcal{A}^n(n-1). \tag{2.32}$$

Combining (2.32) with (2.28) and using the boundedness of φ, we can find some compact interval $I \subset I\!R$ such that not only $\varphi(t,u) \in I$ for all $(t,u) \in [0,T] \times I\!R$, but even $\xi^n(t,u) \in I$ for all $(t,u) \in \mathcal{A}^n(n-1)$ and all $n \in I\!N$.
Next, we set $\gamma(t,u) := g\big(\varphi(t,u)\big)$ for all $(t,u) \in [0,T] \times I\!R$. By Proposition 3.9 the transformed function $\gamma : [0,T] \times I\!R \to I\!R$ solves the final value problem (2.3), (2.4) and belongs to the Hölder space $H^{2+\frac{1}{2}\beta, 4+\beta}\big([0,T] \times I\!R\big)$. To apply Theorem 3.10 it remains to show (2.13) for some $\widetilde{K} \in I\!R$. Since $f : I\!R \to I\!R$ is continuous and strictly positive and since I is compact, we conclude that there exist some $0 < c \le C$ such that $c \le f(\xi) \le C$ for all $\xi \in I$. But then it follows from the definition of g^n and γ for all $u \in \mathcal{U}^n_{n-1}$ and all $n \in I\!N$ that

$$\Big|g^n\big(t^n_{n-1},u\big) - \gamma\big(t^n_{n-1},u\big)\Big| = \left|\int_{\varphi(t^n_{n-1},u)}^{\xi^n(t^n_{n-1},u)} f(x)\,dx\right| \le C\Big|\xi^n\big(t^n_{n-1},u\big) - \varphi^n\big(t^n_{n-1},u\big)\Big|,$$

and after taking the norm $\|\cdot\|_{\mathcal{U}^n_{n-1}}$ and using the assumption (2.28), we can infer that (2.13) holds with the constant K replaced by $\widetilde{K} = KC$. Hence we deduce from Theorem 3.10 the convergence results (2.14) and (2.15).
Now we change the direction of our arguments and show that these two statements in terms of g^n and γ imply the statements (2.29) and (2.30) given in terms of ξ^n and φ. For this purpose, let us consider the inverse $g^{-1} : g(I\!R) \to I\!R$ of the strategy transform $g : I\!R \to I\!R$, $\xi \mapsto \int_0^\xi f(z)\,dz$. Since $f \in H^{3+\beta}_{loc}\big(I\!R\big)$, the first three derivatives of $h := g^{-1}$ exist, are continuous and for all $x \in g(I\!R)$ given by

$$h'(x) = \frac{1}{f(h(x))}, \quad h''(x) = -\frac{f'(h(x))}{f^3(h(x))} \quad \text{and} \quad h^{(3)}(x) = 3\frac{(f')^2(h(x))}{f^5(h(x))} - \frac{f''(h(x))}{f^4(h(x))}.$$

In particular, all three derivatives are bounded on the compact interval $g(I)$. Let us now take $(t,u) \in \mathcal{A}^n(n-1)$ for some $n \in I\!N$. On account of the mean value theorem there exists some $x^* \in g(I)$, lying between $g^n(t,u) = g\big(\xi^n(t,u)\big)$ and $\gamma(t,u) = g\big(\varphi(t,u)\big)$, such that

$$\xi^n(t,u) - \varphi(t,u) = h\big(g^n(t,u)\big) - h\big(\gamma(t,u)\big) = \frac{1}{f(h(x^*))}\big(g^n(t,u) - \gamma(t,u)\big)$$

and after taking the norms $\|\cdot\|_{\mathcal{A}^n(n-1)}$ and using the definition of $c > 0$ we conclude from (2.14):

$$\big\|\xi^n - \varphi\big\|_{\mathcal{A}^n(n-1)} \le \frac{1}{c}\big\|g^n - \gamma\big\|_{\mathcal{A}^n(n-1)} = O\big(\delta^2\big) \quad \text{as } n \to \infty.$$

This proves (2.29). In order to prove (2.30) let us note that a Taylor expansion of h around $\gamma(t,u)$, the boundedness of $h'' : I\!R \to I\!R$ on $g(I)$, and the convergence in (2.14) imply uniformly for all $(t,u) \in \mathcal{A}^n(n-1)$ that

$$h\big(g^n(t,u)\big) - h\big(\gamma(t,u)\big) = \frac{1}{f\big(\varphi(t,u)\big)}\big(g^n(t,u) - \gamma(t,u)\big) + O\big(\delta^4\big) \quad \text{as } n \to \infty, \tag{2.33}$$

where we again took advantage of $h\big(\gamma(t,u)\big) = \varphi(t,u)$ for all $(t,u) \in [0,T] \times I\!R$. Another application of (2.14) together with the boundedness of γ_t and γ_{uu} enables us to write uniformly for all $(t,u) \in \mathcal{A}^n(n-2)$ that

$$\begin{aligned} g^n\big(t+\delta^2, u \pm \delta\big) - \gamma(t,u) &= g^n\big(t+\delta^2, u \pm \delta\big) - \gamma\big(t+\delta^2, u \pm \delta\big) + \gamma\big(t+\delta^2, u \pm \delta\big) - \gamma(t,u) \\ &= \delta\gamma_u(t,u) + O\big(\delta^2\big) \end{aligned}$$

for $n \to \infty$, and therefore we can use a second Taylor expansion of h around $\gamma(t,u)$, this time up to the third derivative, to conclude from the boundedness of the second and third derivative of $h : I\!R \to I\!R$ on $g(I)$ that uniformly for all $(t,u) \in \mathcal{A}^n(n-2)$

$$\begin{aligned} h\big(g^n(t+\delta^2, u \pm \delta)\big) - h\big(\gamma(t,u)\big) &= \frac{1}{f(\varphi(t,u))}\big(g^n(t+\delta^2, u\pm\delta) - \gamma(t,u)\big) \\ &\quad - \frac{1}{2}\delta^2 \frac{f'(\varphi(t,u))}{f^3(\varphi(t,u))}\gamma_u^2(t,u) + O\big(\delta^3\big) \quad \text{as } n \to \infty. \end{aligned} \tag{2.34}$$

If we now subtract (2.33) from (2.34) and notice that $h\big(g^n(t,u)\big) = \xi^n(t,u)$ for all $(t,u) \in \mathcal{A}^n$ we get uniformly for all $(t,u) \in \mathcal{A}^n(n-2)$

$$\begin{aligned} &\xi^n\big(t+\delta^2, u\pm\delta\big) - \xi^n(t,u) \\ &= \frac{1}{f\big(\varphi(t,u)\big)}\left(g^n\big(t+\delta^2, u\pm\delta\big) - g^n(t,u) - \frac{1}{2}\delta^2\frac{f'\big(\varphi(t,u)\big)}{f^2\big(\varphi(t,u)\big)}\gamma_u^2(t,u)\right) + O\big(\delta^3\big) \text{ as } n \to \infty, \end{aligned}$$

and by the approximation (2.15) and the lower bound $f\big(\varphi(t,u)\big) \geq c$ for all $(t,u) \in [0,T] \times I\!R$, we can proceed and rewrite the last term as

$$= \frac{1}{f\big(\varphi(t,u)\big)}\left(\pm\delta\gamma_u(t,u) + \delta^2\left(\gamma_t(t,u) + \frac{1}{2}\gamma_{uu}(t,u)\right) - \frac{1}{2}\delta^2\frac{f'\big(\varphi(t,u)\big)}{f^2\big(\varphi(t,u)\big)}\gamma_u^2(t,u)\right) + O\big(\delta^{2+\beta}\big).$$

Last but not least let us recall that the derivatives of $\gamma : [0,T] \times I\!R \to I\!R$ are given by (2.8) – (2.10). If we plug these formulæ into the previous line, we get uniformly for all $(t,u) \in \mathcal{A}^n(n-2)$ that

$$\xi^n\big(t+\delta^2, u\pm\delta\big) - \xi^n(t,u) = \pm\delta\varphi_u(t,u) + \delta^2\left(\varphi_t + \frac{1}{2}\varphi_{uu}(t,u)\right) + O\big(\delta^{2+\beta}\big) \quad \text{as } n \to \infty,$$

and (2.30) follows immediately. **q.e.d.**

Remark. As Theorem 3.10 does not contain a convergence statement for the boundary function $g^n(T,\,\cdot\,) : \mathcal{U}_n^n \to I\!R$, our convergence statement of Corollary 3.14 does not say anything either about the convergence of the values at maturity of the strategy function $\xi^n : \mathcal{A}^n \to I\!R$; we have formulated our convergence results in Section 3.2.2 and 3.2.3 only for the maximum norms over the truncated grids $\mathcal{A}^n(n-1) = \mathcal{A}^n \cap [0,T) \times I\!R$. With this restriction, our convergence statements are much more powerful than they would be if we were to include the convergence of the large investor's stock holdings (or its transforms) *at* time T in our statements.

In order to see why this is so, let us fix some $n > \left\| \frac{\overline{\varphi}_t}{\overline{\varphi}_u} \right\|^2$ and consider all path-independent contingent claims which lead to the same real value, i.e. we take some $\bar{V}_n^n \in \mathcal{F}_n^n$ and look at the set $\big\{\big(\xi_{n,\alpha}, b_{n,\alpha}\big)\big\}_{\alpha \in I^n}$ of all path-independent contingent claims $\big(\xi_{n,\alpha}, b_{n,\alpha}\big)$ with real value $\bar{V}_n^n = \xi_{n,\alpha}\bar{S}\big(T, U_n^n, \xi_{n,\alpha}\big) + b_{n,\alpha}$, where I^n is a suitable index set. By Lemma 2.3 and Proposition 2.16 any of these contingent claims $\big(\xi_{n,\alpha}, b_{n,\alpha}\big)$ is attainable by a unique portfolio strategy $\big(\xi^{n,\alpha}, b^{n,\alpha}\big)$. As we have seen in (2.4.15) for each discrete time point $t_k^n < T$, the values of the adjoining strategy functions $\xi^{n,\alpha} : \mathcal{A}^n \to I\!R$ are uniquely determined by the possible values of the real value function $\bar{v}^n : \mathcal{A}^n \to I\!R$ at time point t_{k+1}^n. Thus, for all $\alpha \in I^n$ the strategy functions $\xi^{n,\alpha} : \mathcal{A}^n \to I\!R$ coincide on all time points before time T, i.e. on $\mathcal{A}^n(n-1)$.

However, there are infinitely many contingent claims $\big\{\big(\xi_{n,\alpha}, b_{n,\alpha}\big)\big\}_{\alpha \in I^n}$ and hence infinitely many portfolio strategies $\big\{\big(\xi^{n,\alpha}, b^{n,\alpha}\big)\big\}_{\alpha \in I^n}$ which lead to the same real value $\bar{V}_n^n = \bar{v}^n\big(T, U_n^n\big)$

at maturity; the corresponding strategy functions $\xi^{n,\alpha} : \mathcal{A}^n \to I\!R$ differ only on the set $\{T\} \times \mathcal{U}_n^n$, which determines the final stock holdings of the large investor *at* maturity, as required by the associated contingent claim $\big(\xi_{n,\alpha}, b_{n,\alpha}\big)$. Since the large investor trades at the benchmark price, he can arbitrarily re-shuffle his portfolio between his bank account holdings and his stock holdings without any transaction losses, and hence, from a replication-oriented point of view, all the contingent claims $\big(\xi_{n,\alpha}, b_{n,\alpha}\big)$ are equivalent.

In order to derive a convergence statement for the strategy functions $\{\xi^n\}_{n\in I\!N}$ which includes the convergence at time T, we would need to specify the particular contingent claim $\big(\xi_{n,\alpha}, b_{n,\alpha}\big)$ which we replicate, and not only its real value. For example, we cannot expect the uniform convergence as $n \to \infty$ of $\xi^n : \mathcal{A}^n \to I\!R$, including the values at maturity, towards the solution $\varphi : [0,T] \times I\!R \to I\!R$ of a final value problem of the form (2.5), (2.6), if the sequence $\{\xi^n\}_{n\in I\!N}$ of strategy functions prescribes cash settlement at time T. In fact, we should then have $\xi^n(T,u) = 0$ for all $u \in \mathcal{U}_n^n$ and $n \in I\!N$, hence the limit φ would also satisfy $\varphi(T,u) = 0$ for all $u \in I\!R$, and a maximum principle would imply that φ vanishes on the whole domain $[0,T] \times I\!R$.

This indicates why we really ought to limit our investigation to the convergence results as stated in Theorem 3.10 and in Corollary 3.14. Within this formulation, we have some degree of freedom to choose the large investor's stock holdings at time T. The particular stock holdings at time T will only become important for the convergence results of Section 3.3, where we consider price systems which allow for implied transaction losses. We shall see that in such a general setting we need both the convergence of the strategy functions at time T and immediately before time T to obtain uniform convergence results for the strategy functions $\{\xi^n\}_{n\in I\!N}$ similar to the ones of (2.29) and (2.30). $\square$

3.2.4 Convergence of a Subsequence of Strategy Functions

In the last subsection of Section 3.2, we will sketch the proof of a second convergence statement for a sequence $\{g^n\}_{n\in I\!N}$ of transformed strategy functions. Like Theorem 3.10, this convergence statement will again only show convergence on the domain $[0,T) \times I\!R$, but the result is weaker than Theorem 3.10 in that it only proves convergence of a subsequence of $\{g^n\}_{n\in I\!N}$. On the other hand, the proof does not rely on the existence of a solution to the final value problem (2.3), (2.4), but proves its existence as a by-product of the convergence of subsequences of the transformed strategy functions $g^n : \mathcal{A}^n \to I\!R$ and their discrete derivatives. Compared to the situation where we first have to guarantee the existence of a solution $\gamma \in H^{2+\frac{1}{2}\beta,4+\beta}\big([0,T] \times I\!R\big)$ via Lemma 3.8, the convergence statement which we present here requires less restrictive assumptions on the functions $f : I\!R \to I\!R$ and $\zeta : I\!R \to I\!R$.

We will limit our attention to the convergence statement for the sequence $\{g^n\}_{n\in I\!N}$ of transformed strategies, but we could of course formulate an equivalent result for the sequence $\{\xi^n\}_{n\in I\!N}$ of associated original strategy functions $\xi^n : \mathcal{A}^n \to I\!R$. Our proof adapts an existence proof of Section 7.2 in John (1978) for solutions to an initial value problem of the form $u_t(t,x) - a(t,x)u_{xx}(t,x) - b(t,x)u_x(t,x) = 0$ for all $(t,x) \in [0,T] \times I\!R$ with initial condition given by $u(0,x) = h(x)$ for all $x \in I\!R$. John proves the existence only under the assumption that $h \in C_b^4(I\!R)$, but his argument can be generalized to situations where this condition is relaxed to $h \in H^{2+\beta}(I\!R)$ for some $\beta \in (0,1)$.

We have the following convergence statement for $\{g^n\}_{n\in I\!N}$:

Proposition 3.15. *Let the price system $(\psi,\mu) = (\psi,\lambda)$ satisfy the multiplicative structure of Assumption B, and suppose that the small investor function $\overline{\psi} : [0,T] \times I\!R \to I\!R$ belongs to the class $\widehat{H}^{2+\frac{1}{2}\beta,4+\beta}\big([0,T] \times I\!R\big)$. Moreover, suppose that there exists some function $\zeta : I\!R \to I\!R$*

such that the function $\int_0^{\zeta(\,\cdot\,)} f(x)\,dx$ *belongs to* $H^{2+\beta}(I\!R)$ *and*

$$\left\| g^n\big(t^n_{n-1},\,\cdot\,\big) - \int_0^{\zeta(\,\cdot\,)} f(x)dx \right\|_{\mathcal{U}^n_{n-1}} = O\big(\delta^{2+\beta}\big) \quad \text{as } n \to \infty. \tag{2.35}$$

Then there exist a solution $\gamma : [0,T]\times I\!R \to I\!R$ *of* (2.3), (2.4) *and a subsequence* $\{n_j\}_{j\in I\!N} \subset I\!N$ *such that*

$$\begin{aligned}
\lim_{j\to\infty} \big\| g^{n_j} - \gamma \big\|_{\mathcal{A}^{n_j}(n_j-1)} &= 0,\\
\lim_{j\to\infty} \big\| \Delta^{n_j}_t\, g^{n_j} - \gamma_t \big\|_{\mathcal{A}^{n_j}_1(n_j-2)} &= 0,\\
\lim_{j\to\infty} \big\| \Delta^{n_j}_u\, g^{n_j} - \gamma_u \big\|_{\mathcal{A}^{n_j}_1(n_j-1)} &= 0,
\end{aligned}$$

and

$$\lim_{j\to\infty} \big\| \Delta^{n_j}_{uu}\, g^{n_j} - \gamma_{uu} \big\|_{\mathcal{A}^{n_j}_2(n_j-1)} = 0.$$

Remark. Of course, $\int_0^{\zeta(\,\cdot\,)} f(x)\,dx \in H^{2+\beta}(I\!R)$ is implied by $f \in H^{1+\beta}_{loc}(I\!R)$ and $\zeta \in H^{2+\beta}(I\!R)$. This shows indeed that Proposition 3.15 requires less stringent assumptions on the regularity of f and ζ than the existence result for $\gamma \in H^{2+\frac{1}{2}\beta,4+\beta}\big([0,T]\times I\!R\big)$ by means of Lemma 3.8.

Proof (sketched). We will only sketch the proof of Proposition 3.15. Let us note first that (2.35) together with $\int_0^{\zeta(\,\cdot\,)} f(x)\,dx \in H^{2+\beta}(I\!R)$ implies the existence of some $K_i \in I\!R$ for $i \in \{0,1,2,3\}$ such that

$$\big\|g^n\big(t^n_{n-1},\,\cdot\,\big)\big\|_{\mathcal{U}^n_{n-1}} \le K_0, \quad \big\|\Delta_u\, g^n\big(t^n_{n-1},\,\cdot\,\big)\big\|_{\mathcal{U}^n_{n-2}} \le K_1, \quad \big\|\Delta_{uu}\, g^n\big(t^n_{n-1},\,\cdot\,\big)\big\|_{\mathcal{U}^n_{n-3}} \le K_2,$$

$$\text{and} \quad \frac{1}{\delta^\beta}\big\|\Delta_{uu}\, g^n\big(t^n_{n-1},\,\cdot+\delta\big) - \Delta_{uu}\, g^n\big(t^n_{n-1},\,\cdot-\delta\big)\big\|_{\mathcal{U}^n_{n-4}} \le K_3 \quad \text{for all } n \ge 4.$$

Using these bounds we can conclude step by step that the restriction of $g^n : \mathcal{A}^n \to I\!R$ to the set $\mathcal{A}^n(n-1) = \mathcal{A}^n \cap [0,T) \times I\!R$, and similar restrictions of the discrete derivatives $\Delta_u\, g^n : \mathcal{A}^n_1 \to I\!R$, $\Delta_{uu}\, g^n : \mathcal{A}^n_2 \to I\!R$ and $\Delta_t\, g^n : \mathcal{A}^n_1(n-1) \to I\!R$, can be bounded for all $n \in I\!N$. Since for each $n \in I\!N$ the domains of these functions contain only a finite number of elements, it suffices to show that the restrictions are bounded for all sufficiently large $n \in I\!N$. In order to do so we recollect the recursive inequalities (2.1) for g^n and (2.25) for $\Delta^n_u\, g^n$ for all $n > \big\|\frac{\overline{\psi}_t}{\overline{\psi}_u}\big\|^2$. Since the weight function $\hat{p}^n : \mathcal{A}^n(n-2) \to (0,1)$ of (2.2) takes only values in $(0,1)$, an iterated application of (2.1) and the definition of K_0 imply $\big\|g^n\big\|_{\mathcal{A}^n(n-1)} \le K_0$ for all $n > \big\|\frac{\overline{\psi}_t}{\overline{\psi}_u}\big\|^2$. Similarly, we use the recursive equation (2.25) for $\Delta^n_u\, g^n$ and the bound (2.26) to obtain for all $n > \big\|\frac{\overline{\psi}_t}{\overline{\psi}_u}\big\|^2$ and $1 \le k \le n-1$ the inequality

$$\big\|\Delta^n_u\, g^n\big(t^n_k,\,\cdot\,\big)\big\|_{\mathcal{U}^n_{k-1}} \le \sum_{j=0}^{n-k-1} \big(1+\delta^2 R\big)^j \big\|\Delta^n_u\, g^n\big(t^n_{n-1},\,\cdot\,\big)\big\|_{\mathcal{U}^n_{n-2}}. \tag{2.36}$$

As a consequence, we get from the definition of K_1 the upper bound $\big\|\Delta^n_u\, g^n\big\|_{\mathcal{A}^n_1(n-1)} \le e^R K_1$. From the recursive equation (2.25) we can also construct some recursion formula for $\Delta^n_{uu}\, g^n$, namely we get for all $n > \big\|\frac{\overline{\psi}_t}{\overline{\psi}_u}\big\|^2$ and $3 \le k \le n-1$:

$$\Delta^n_{uu}\, g^n\big(t^n_{k-1}, u\big) = \hat{p}^n\big(t^n_{k-1}, u+2\delta\big)\Delta^n_{uu}\, g^n\big(t^n_k, u+\delta\big) + 2\delta\Delta_{uu}\,\hat{p}^n\big(t^n_{k-1}, u\big)\Delta^n_u\, g^n\big(t^n_k, u\big)$$

$$+ \big(1 - \hat{p}^n\big(t^n_{k-1}, u - 2\delta\big)\big)\Delta^n_{uu}\, g^n\big(t^n_k, u - \delta\big).$$

Arguments similar to the one leading to (2.26) show that for any $R' > \big\|\frac{d^2}{du^2}\frac{\frac{1}{2}\overline{\psi}_{uu}-\overline{\psi}_t}{\overline{\psi}_u}\big\|$ we obtain the bound $\frac{1}{\delta}\big\|\Delta^n_{uu}\,\hat{p}^n\big\|_{\mathcal{A}^n_2(n-2)} \leq R'$ for all sufficiently large $n \in I\!N$. If, in addition, we use $\big\|\Delta^n_u\, g^n\big\|_{\mathcal{A}^n_1(n-1)} \leq e^R K_1$, we can adjust and slightly generalize the proof of (2.36) to show that the sup-norm $\big\|\Delta^n_{uu}\, g^n\big\|_{\mathcal{A}^n_2(n-1)}$ of the second discrete space derivatives can be bounded by some constant C_1 depending only on K_2, K_1, and global bounds on functionals of $\overline{\psi} : [0,T] \times I\!R \to I\!R$, which do not depend on n either. A last iteration of that procedure shows that

$$\frac{1}{2\delta^\beta}\Big\|\Delta^n_{uu}\, g^n\big(\cdot, \cdot + \delta\big) - \Delta^n_{uu}\, g^n\big(\cdot, \cdot - \delta\big)\Big\|_{\mathcal{A}^n_3(n-1)} \leq C_2$$

for some constant C_2 which only depends on K_3, K_2, K_1 and on global bounds of functionals of $\overline{\psi} : [0,T] \times I\!R \to I\!R$.

If we now consider the discrete time derivative $\Delta^n_t\, g^n$, a twofold application of the recursive equation (2.1) and rearranging terms shows that for all $n > \big\|\frac{\overline{\psi}_t}{\overline{\psi}_u}\big\|^2$ and $1 \leq k \leq n-2$ we have

$$\begin{aligned} 0 &= \Delta^n_t\, g^n\big(t^n_k, u\big) + a^n\big(t^n_k, u\big)\Delta^n_{uu}\, g^n\big(t^n_{k+1}, u\big) \\ &\quad + b^n\big(t^n_k, u\big)\frac{1}{2}\Big(\Delta^n_u\, g^n\big(t^n_{k+1}, u + \delta\big) + \Delta^n_u\, g^n\big(t^n_{k+1}, u - \delta\big)\Big), \end{aligned} \tag{2.37}$$

where the function $a^n : \mathcal{A}^n_1(n-2) \to I\!R$ is for all $(t,u) \in \mathcal{A}^n_1(n-2)$ given by

$$\begin{aligned} a^n(t,u) &:= \hat{p}^n(t - \delta^2, u)\hat{p}^n(t, u + \delta) + \big(1 - \hat{p}^n(t - \delta^2, u)\big)\big(1 - \hat{p}^n(t, u - \delta)\big) \\ &= \frac{1}{2} + \delta\Delta^n_u\, \hat{p}^n(t,u) + \frac{1}{2}\big(2\hat{p}^n(t - \delta^2, u) - 1\big)\Big(\hat{p}^n(t, u + \delta) + \hat{p}^n(t, u - \delta) - 1\Big), \end{aligned}$$

and where similarly the function $b^n : \mathcal{A}^n_1(n-2) \to I\!R$ satisfies for all $(t,u) \in \mathcal{A}^n_1(n-2)$

$$\begin{aligned} b^n(t,u) &:= \frac{1}{\delta}\Big(\hat{p}^n(t - \delta^2, u)\hat{p}^n(t, u + \delta) - \big(1 - \hat{p}^n(t - \delta^2, u)\big)\big(1 - \hat{p}^n(t, u - \delta)\big)\Big) \\ &= \frac{1}{2\delta}\Big(\big(2\hat{p}^n(t - \delta^2, u) - 1\big)\big(1 + 2\delta\Delta^n_u\, \hat{p}^n(t,u)\big) + \hat{p}^n(t, u + \delta) + \hat{p}^n(t, u - \delta) - 1\Big). \end{aligned}$$

By approximations like in Lemma 3.11 we get uniformly for all $(t,u) \in \mathcal{A}^n_1(n-2)$

$$a^n(t,u) = \frac{1}{2} + O(\delta) \quad \text{and} \quad b^n(t,u) = \frac{\frac{1}{2}\overline{\psi}_{uu}(t,u) - \overline{\psi}_t(t,u)}{\overline{\psi}_u(t,u)} + O\big(\delta^\beta\big) \quad \text{as } n \to \infty. \tag{2.38}$$

Thus, for all $n > \big\|\frac{\overline{\psi}_t}{\overline{\psi}_u}\big\|^2$ and $1 \leq k \leq n-2$, the norm $\big\|\Delta^n_t\, g^n\big(t^n_k, \cdot\big)\big\|_{\mathcal{U}^n_{k-1}}$ can be bounded in terms of the norms $\big\|\Delta^n_{uu}\, g^n\big(t^n_{k+1}, \cdot\big)\big\|_{\mathcal{U}^n_{k-1}}$, $\big\|\Delta^n_u\, g^n\big(t^n_{k+1}, \cdot\big)\big\|_{\mathcal{U}^n_k}$ and of global bounds on functionals of $\overline{\psi} : [0,T] \times I\!R \to I\!R$; hence $\big\|\Delta^n_t\, g^n\big\|_{\mathcal{A}^n_1(n-2)}$ can be bounded uniformly for all $n \in I\!N$ as well. Similarly it follows that

$$\begin{aligned} &\frac{1}{2\delta^\beta}\Big\|\Delta^n_t\, g^n(\cdot, \cdot + \delta) - \Delta^n_t\, g^n(\cdot, \cdot - \delta)\Big\|_{\mathcal{A}^n_2(n-2)} \\ &\qquad = \frac{1}{2\delta^{1+\beta}}\Big\|\Delta^n_u\, g^n\big(\cdot + \delta^2, \cdot\big) - \Delta^n_u\, g^n\big(\cdot - \delta^2, \cdot\big)\Big\|_{\mathcal{A}^n_2(n-2)} \leq C_3 \end{aligned}$$

for some C_3, and last but not least we also conclude that there is some $C_4 \in I\!R$ such that $\frac{1}{2\delta^\beta}\big\|\Delta^n_t\, g^n(\cdot - \delta^2, \cdot) - \Delta^n_t\, g^n(\cdot + \delta^2, \cdot)\big\|_{\mathcal{A}^n_2(n-3)} \leq C_4$, where both C_3 and C_4 only depend on K_1, K_2, K_3 and on global bounds on functionals of $\overline{\psi} : [0,T] \times I\!R \to I\!R$.

We then restrict our attention to the subsequence $\{4^i\}_{i\geq i_0} \subset \mathbb{N}$, starting at some sufficiently large $i_0 \in \mathbb{N}$. Without loss of generality we may assume $i_0 = 1$. The sequence of grids $\{\mathcal{A}^{4^i}\}_{i\in\mathbb{N}}$ is an increasing sequence of lattices, and so are the sequences $\{\mathcal{A}^{4^i}(4^i-1)\}_{i\in\mathbb{N}}$, $\{\mathcal{A}_1^{4^i}(4^i-1)\}_{i\in\mathbb{N}}$, and all the other similar sequences of sets of possible arguments for the discrete derivatives. Now the convergence of a subsequence of the $\{g^n\}$ and of the corresponding discrete derivatives of Definition 3.7 follows from a straightforward application of the Bolzano-Weierstraß theorem: Let us start with g^{4^i} and define $\mathcal{A} := \bigcup_{k=1}^{\infty} \mathcal{A}^{4^k}(4^k-1)$. Since $\|g^n\|_{\mathcal{A}^n(n-1)} \leq K_0$ for all sufficiently large $n \in \mathbb{N}$, there exists a subsequence $\{n_j\}_{j\in\mathbb{N}}$ of $\{4^i\}_{i\in\mathbb{N}}$ such that $\gamma(t,u) := \lim_{j\to\infty} g^{n_j}(t,u)$ exists for all $(t,u) \in \mathcal{A}$. Now the discrete derivatives $\Delta_u^n\, g^n$, $\Delta_{uu}^n\, g^n$, and $\Delta_t^n\, g^n$ are bounded as well, uniformly in $n \in \mathbb{N}$. Hence we find, possibly after extracting a further subsequence of $\{n_j\}_{j\in\mathbb{N}}$, which we will call again $\{n_j\}_{j\in\mathbb{N}}$ for simplicity, that

$$\begin{aligned}
\gamma'(t,u) &:= \lim_{j\to\infty} \Delta_u^{n_j}\, g^{n_j}(t,u) \quad \text{for all } (t,u) \in \mathcal{A}' := \bigcup_{i=1}^{\infty} \mathcal{A}_1^{4^i}(4^i-1), \\
\gamma''(t,u) &:= \lim_{j\to\infty} \Delta_{uu}^{n_j}\, g^{n_j}(t,u) \quad \text{for all } (t,u) \in \mathcal{A}'' := \bigcup_{i=1}^{\infty} \mathcal{A}_2^{4^i}(4^i-1), \\
\text{and} \quad \dot{\gamma}(t,u) &:= \lim_{j\to\infty} \Delta_t^{n_j}\, g^{n_j}(t,u) \quad \text{for all } (t,u) \in \dot{\mathcal{A}} := \bigcup_{i=1}^{\infty} \mathcal{A}_1^{4^i}(4^i-2).
\end{aligned}$$

Since g^n, $\Delta_u^n\, g^n$, $\Delta_{uu}^n\, g^n$, and $\Delta_t^n\, g^n$ can be bounded on the intersection of $[0,T) \times \mathbb{R}$ with their respective domains uniformly for all $n \in \mathbb{N}$, the functions γ, γ', γ'', and $\dot{\gamma}$ are bounded on their respective domains as well.

Let (t,u) and (t,v) be two elements of $\mathcal{A}$ with $v > u$. Then for all sufficiently large $j \in \mathbb{N}$ we have $(t,u), (t,v) \in \mathcal{A}^{n_j}(n_j-1)$, and $v - u = 2k_j\delta_{n_j}$ for some $k_j \in \mathbb{N}$, since, by its definition, $\delta = \delta_n = n^{-\frac{1}{2}}$. Thus we can conclude from $\|\Delta_u^n\, g^n\|_{\mathcal{A}_1^n(n-1)} \leq e^R K_1$ for all sufficiently large n that

$$\begin{aligned}
\left|g^{n_j}(t,v) - g^{n_j}(t,u)\right| &= \sum_{l=1}^{k_j} \left| g^{n_j}\big(t, u + 2l\delta_{n_j}\big) - g^{n_j}\big(t, u + 2(l-1)\delta_{n_j}\big)\right| \\
&\leq 2k_j\delta_{n_j} e^R K_1 = e^R K_1 |u - v|.
\end{aligned}$$

Taking the limit $j \to \infty$ implies that $\gamma : \mathcal{A} \to \mathbb{R}$ is Lipschitz in u. Similarly, we can show that $\gamma : \mathcal{A} \to \mathbb{R}$ satisfies a Lipschitz condition in t as well, that $\gamma' : \mathcal{A}' \to \mathbb{R}$ is Lipschitz in u and satisfies a Hölder condition with exponent $\frac{1}{2}(1+\beta)$ in t, and that $\gamma'' : \mathcal{A}'' \to \mathbb{R}$ and $\dot{\gamma} : \dot{\mathcal{A}} \to \mathbb{R}$ satisfy Hölder conditions with exponents β in u and exponent $\frac{1}{2}\beta$ in t. Then due to the Arzelà-Ascoli theorem the functions γ, γ', γ'', and $\dot{\gamma}$ can be extended to continuous bounded functions on the whole closed slab $[0,T] \times \mathbb{R}$.

Actually, we can show a little bit more than just Lipschitz and Hölder continuity. For example, let us suppose that (t,u), $(t,v) \in \mathcal{A}$. Then for sufficiently large $j \in \mathbb{N}$ we have (t,u), $(t,v) \in \mathcal{A}^{n_j}\big(n_j - 1\big)$, and for $v > u$ it follows that $\big(t, u + \delta_{n_j}\big) \in \mathcal{A}_1^{n_j}\big(n_j - 1\big)$. It can now be shown that

$$\left| \frac{g^{n_j}(t,u) - g^{n_j}(t,v)}{u - v} - \Delta_u^{n_j}\, g^{n_j}\big(t, u + \delta_{n_j}\big)\right| \leq C_1 |u - v|,$$

where C_1 is the uniform bound on $\|\Delta_{uu}^{n_j}\, g^n\|_{\mathcal{A}_2^n(n-1)}$, and thus we get in the limit

$$\left| \frac{\gamma(t,u) - \gamma(t,v)}{u - v} - \gamma'(t,u)\right| \leq C_5 |u - v| \quad \text{for all } t \in [0,T] \text{ and } u, v \in \mathbb{R}$$

for some constant $C_5 \in I\!R$. If we let $v \to u$, the derivative $\gamma_u(t,u)$ is seen to exist and $\gamma_u(t,u) = \gamma'(t,u)$ for all $(t,u) \in [0,T] \times I\!R$. In the same manner one can show for some $C_6 \in I\!R$ that

$$\left| \frac{\gamma'(t,u) - \gamma'(t,v)}{u-v} - \gamma''(t,u) \right| \le C_6 |u-v|^\beta \quad \text{for all } t \in [0,T] \text{ and } u,v \in I\!R$$

to conclude that $\gamma_{uu}(t,u) = \gamma''(t,u)$ for all $(t,u) \in [0,T] \times I\!R$, and analogously it follows that $\gamma_t(t,u) = \dot{\gamma}(t,u)$. Last but not least, due to (2.37) and (2.38) it can be seen that $\gamma : [0,T] \times I\!R \to I\!R$ satisfies the partial differential equation (2.3). The final condition (2.4) is implied by (2.35), $\gamma(t,u) = \lim_{j\to\infty} g^{n_j}(t,u)$ for all $(t,u) \in \mathcal{A}$, and the boundedness of γ_t. **q.e.d.**

In the formulation of Proposition 3.15 the subsequence $\{n_j\}_{j\in I\!N}$ for which the associated subsequence $\left\{g^{n_j}\right\}_{j\in I\!N}$ of transformed strategy functions converges is not explicitly given. If we want to approximate a solution of the final value problem (2.3), (2.4) for instance on a computer, we have to know an explicit subsequence for which the discrete transforms converge.

Note however that we have actually provided the essentials of a slightly stronger statement than the one of Proposition 3.15. Namely, our proof indicates that the assumptions of the proposition imply that for *any* subsequence of $\{4^i\}_{i\in I\!N}$ there exists some subsequence $\{n_j\}_{j\in I\!N}$ such that the sequence $\left\{g^{n_j}\right\}_{j\in I\!N}$ converges as $j \to \infty$ to a solution γ of the linear final value problem (2.3), (2.4) in the sense of the statement in the proposition, and by the remark to Lemma 3.8 the solution to that partial differential equation is unique. Hence the convergence of the subsequences implies that the whole sequence $\left\{g^{4^i}\right\}_{i\in I\!N}$ of transformed strategy functions $g^{4^i} : \mathcal{A}^{4^i} \to I\!R$ converges on $[0,T) \times I\!R$ in the same sense to this solution as $i \to \infty$, i.e. we have

Corollary 3.16. *Let $n_j = 4^j$ for all $j \in I\!N$. Under the assumptions of Proposition 3.15 the subsequence $\left\{g^{n_j}\right\}_{j\in I\!N}$ of the discrete transformed strategies $\{g^n\}_{n\in I\!N}$ converges on $[0,T) \times I\!R$ to the solution $\gamma : [0,T] \times I\!R \to I\!R$ of* (2.3), (2.4) *in the form stated in the proposition.*

Remark. This subsection has indicated how we can weaken the conditions of Theorem 3.10 such the (transformed) strategy functions still converge in a sufficiently accurate way, at least when restricting the attention to n-step binomial models where n is a power of 4. Nevertheless, in the remainder of this thesis we will focus on situations where the whole sequence $\{g^n\}_{n\in I\!N}$ of transformed strategy functions $g^n : \mathcal{A}^n(n-1) \to I\!R$ converges as $n \to \infty$, and accept the stronger differentiability conditions in order to assure such a convergence. The notation becomes simpler under these stronger conditions. In particular, our restriction simplifies the treatment of the non-linear case, which will be considered in the next section. □

3.3 The General Case

Having shown the uniform convergence of the strategy functions $\{\xi^n\}_{n\in I\!N}$ in the special case of a large investor market model which excludes any immediate transaction gains and losses, we now shift our attention to a general large investor market (ψ, μ), which still satisfies a multiplicative structure as in Assumption B, but which might induce transaction *losses*.

In the loss-free case of Section 3.2 we were able to turn the implicit recursive scheme for the strategy function ξ^n into an explicit scheme for the transformed strategy function g^n. Then we first proved the convergence of the sequence $\{g^n\}_{n\in I\!N}$ to a solution of a *linear* partial differential equation. In the general case, however, the recursive schemes for the transformed

strategy functions remain implicit, and the candidate limit function γ is only given as the solution of a *quasi-linear* partial differential equation.
Like in Section 3.2 we will first look at the existence and uniqueness of a solution to the final value problem for the potential limit of the transformed strategy function. Because of the non-linearity of the partial differential equation, the existence and uniqueness is not as straightforward as in the special case of Section 3.2, but we can adapt a proof of Frey (1998) and show that a unique solution to the final value problem still exists if the values in the final condition stay sufficiently small. Having obtained this first important result in Section 3.3.1, we explain in the following section how the methods used in Section 3.2.2 can be generalized in order to prove that the sequence $\{g^n\}_{n\in\mathbb{N}}$ of transformed strategy functions $g^n : \mathcal{A}^n \to \mathbb{R}$ converges uniformly to the solution $\gamma : [0,T]\times\mathbb{R} \to \mathbb{R}$ of the quasi-linear final value problem if only the values of the transformed strategies immediately before and at maturity converge towards the corresponding values of γ.
In Section 3.3.3 we finally use the convergence result for the transformed strategy functions to derive an analogous convergence result for the sequence $\{\xi^n\}_{n\in\mathbb{N}}$ of original strategy functions. This convergence result for the strategy functions provides exactly the kind of convergence which we need in Chapter 4 in order to guarantee the weak convergence of the value and price processes, but we have to make the detour via the transformed strategy function, since the final value problem which is solved by the limit of the strategy functions fails to be quasi-linear.

In order to get started, let us consider a multiplicative price system (ψ,μ) for which the equilibrium price function $\psi : [0,T]\times\mathbb{R}^2 \to \mathbb{R}$ satisfies Assumption B. For the general case we now also require that the equilibrium price function ψ (or equivalently $\overline{\psi}$, because of Assumption B) stays strictly positive and that the price determining measure μ is sufficiently good-natured. Moreover, as for the special case dealt with in Section 3.2 we need some smoothness and boundedness for the two components $\overline{\psi}$ and f of ψ. We will impose

Assumption C. *The small investor price function $\overline{\psi} : [0,T]\times\mathbb{R} \to \mathbb{R}$, $(t,u)\mapsto\overline{\psi}(t,u)$ is continuously differentiable with respect to t and twice continuously differentiable with respect to u. The function $\overline{\psi}$ itself and its spatial derivative $\overline{\psi}_u$ are strictly positive, and the function satisfies $\left\|\frac{\overline{\psi}_t}{\overline{\psi}_u}\right\| < \infty$ and $L_0 := \left\|\frac{\overline{\psi}}{\overline{\psi}_u}\right\| < \infty$. For the function $f : \mathbb{R} \to (0,\infty)$ we assume that it is at least twice continuously differentiable. Finally we assume that the price determining measure $\mu \in \mathcal{M}(f)$ has a finite first moment.*

As we proceed, we will further strengthen the regularity assumptions on ψ and μ. Especially, we will once again assume that both components of ψ belong to certain Hölder spaces. In addition to that, we will prevent any immediate arbitrage opportunity as described in Section 1.2.1 by

Assumption D. *The price system (ψ,μ) excludes any immediate transaction gains, i.e. the local transaction loss rate function $k_\mu : \mathbb{R}^2 \to \mathbb{R}$ of Definition 2.13 is nonnegative.*

If we now recall Definitions 1.16 and 2.12 of the non-linearity parameter $d(\mu)$ and the strategy transform g, respectively, we can introduce a function which plays the role of the transaction loss rate function in the corresponding continuous-time model, and which for simplicity is defined as a function of the large investor's transformed stock holdings $x = g(\xi)$:

Definition 3.17. The **transformed loss function** $\kappa : g(\mathbb{R}) \to \mathbb{R}$ is given by

$$\kappa(x) = 2d(\mu)\frac{f'\big(g^{-1}(x)\big)}{f^2\big(g^{-1}(x)\big)} \quad \text{for all } x \in g(\mathbb{R}). \tag{3.1}$$

Remark. Proposition 1.15 and the Definition 1.16 of $d(\mu)$ imply that Assumption D implies $\kappa(x) \geq 0$ for all $x \in \mathbb{R}$.

3.3.1 Existence of a Solution to the Limiting PDE

As in Section 3.2 we start with the final value problem which is solved by the limit of the transformed strategy functions $g^n : \mathcal{A}^n \to I\!R$. In contrast to the case without transaction losses, this final value problem becomes non-linear; more specifically, it is quasi-linear. First we introduce the final value problem with a scaling factor for the final condition. Then we present in Proposition 3.18 this section's main result, saying that a unique solution to the non-linear final value problem still exists if the scaling factor is chosen sufficiently small.
The proof of Proposition 3.18 fills out most of this section and it is, at least to a large extent, very technical. The most demanding part is to prove the existence of a solution. We intend to follow a proof of Frey (1998) who shows the existence of continuous hedging strategies for convex options in a continuous-time large-investor model where the price building mechanism is determined by the Dirac measure δ_1 concentrated in 1 as in Example 1.2 of Chapter 1. For that reason, we first show that Proposition 3.18 follows from a second proposition, i.e. from Proposition 3.19, which states a similar existence and uniqueness result for a more suitable initial value problem. In the initial value problem of Proposition 3.19 the scaling parameter controls the non-linearity of the partial differential equation instead of the boundary condition. The proof of Proposition 3.19 can then parallel the proof of Frey's Theorem 4.2 and involves two more lemmata to transfer existence and uniqueness results for parabolic quasi-linear Cauchy problems as stated in Ladyženskaja et al. (1968) via some modified initial value problem to our somewhat more general case.
At the very end of this section, we slightly extend the results of Proposition 3.18 in two corollaries by reconsidering the regularity assumptions used.

We shall see in Section 3.3.2 that in case of convergence the limit $\gamma : [0,T] \times I\!R \to I\!R$ of the transformed strategy functions will solve a final value problem which is given by the quasi-linear partial differential equation

$$\begin{aligned} \gamma_t(t,u) + \frac{1}{2}\frac{d}{du}\left(\left(1 + \frac{\overline{\psi}(t,u)}{\overline{\psi}_u(t,u)}\kappa\big(\gamma(t,u)\big)\gamma_u(t,u)\right)\gamma_u(t,u)\right) \\ = \gamma_u(t,u)\left(\frac{\overline{\psi}_t(t,u)}{\overline{\psi}_u(t,u)} - \frac{1}{2}\frac{\overline{\psi}_{uu}(t,u)}{\overline{\psi}_u(t,u)}\left(1 + \frac{\overline{\psi}(t,u)}{\overline{\psi}_u(t,u)}\kappa\big(\gamma(t,u)\big)\gamma_u(t,u)\right)\right) \end{aligned} \tag{3.2}$$

for all $(t,u) \in (0,T) \times I\!R$, and a final condition of the form

$$\gamma(T,u) = \alpha \int_0^{\zeta(u)} f(x)\,dx \quad \text{for all } u \in I\!R, \tag{3.3}$$

where $\alpha \in I\!R$ is some scaling parameter. Apart from this scaling parameter, the function $\zeta : I\!R \to I\!R$, $u \mapsto \zeta(u)$ describes for every fundamental value $u \in I\!R$ immediately before maturity the corresponding stock holdings of the large investor at this time.

Remark. If the transformed loss function $\kappa : I\!R \to I\!R$ vanishes (either via $d(\mu) = 0$ or via $f' \equiv 0$), then the non-linear PDE (3.2) reduces to the linear PDE (2.3). In this case Lemma 3.8 has shown that for *any* $\alpha \in I\!R$ there exists a solution $\gamma \in H^{1+\frac{1}{2}\beta,2+\beta}\big([0,T] \times I\!R\big)$ to the final value problem (3.2), (3.3) as long as the two components $\overline{\psi}$ and f of ψ belong to the Hölder spaces $\hat{H}^{1+\frac{1}{2}\beta,2+\beta}\big([0,T] \times I\!R\big)$ and $H^{1+\beta}_{loc}(I\!R)$, respectively, and if also $\zeta \in H^{2+\beta}(I\!R)$. In general, however, the terms which involve the transformed loss function $\kappa : I\!R \to I\!R$ lead to non-linear effects, and we can prove existence of solutions to (3.2), (3.3) only for $|\alpha| > 0$ sufficiently small and, of course, for $\alpha = 0$. □

Let us now fix some bounded function $\zeta : I\!R \to I\!R$ and define the two bounds

$$b := \min\left\{0, \inf_{u\in I\!R}\int_0^{\zeta(u)} f(x)\,dx\right\} \quad \text{and} \quad B := \max\left\{0, \sup_{u\in I\!R}\int_0^{\zeta(u)} f(x)\,dx\right\},$$

which are basically bounds on the initial condition in (3.3). Due to Assumption C, the function f is in particular continuous and it follows that the two bounds b and B are finite since ζ is bounded.
We also define the constants ρ_1 and ρ_2 by

$$\rho_1 := \limsup_{\xi\to\infty} \max\left\{\frac{g(-\xi)}{g(B)}, -\left|\frac{g(\xi)}{g(b)}\right|\right\} \quad \text{and} \quad \rho_2 := \liminf_{\xi\to\infty} \min\left\{\frac{g(\xi)}{g(B)}, \left|\frac{g(-\xi)}{g(b)}\right|\right\},$$

where we use the convention $\frac{|x|}{0} = \infty$ for all $x \neq 0$. Since Assumption B implies that $f : I\!R \to I\!R$ is strictly positive, the strategy transform $g : I\!R \to I\!R$, $\xi \mapsto \int_0^\xi f(x)\,dx$ is strictly increasing and satisfies $g(0) = 0$. Thus we conclude from $-\infty < b \le B < \infty$ that $\rho_1 \in [-\infty, 0)$ and $\rho_2 \in (1, \infty]$.
From the definition of ρ_1 and ρ_2 it follows that for all $\rho_1 < \alpha < \rho_2$ and all $b \le x \le B$ we have $\alpha x \in g(I\!R)$, and especially the inverse $g^{-1} : g(I\!R) \to I\!R$, and therefore also $\kappa : g(I\!R) \to I\!R$, is well defined on the set $\big(\rho_2 g(b), \rho_2 g(B)\big)$. Hence if $f : I\!R \to I\!R$ is twice continuously differentiable, we can define the functions $L_\kappa : I\!R^+ \to I\!R^+$ and $L'_\kappa : I\!R^+ \to I\!R^+$, which bound the influence of implied transaction losses, by

$$L_\kappa(\alpha) := \|\kappa\|_{[\alpha b, \alpha B]} \text{ and } L'_\kappa(\alpha) := \max\left\{L_\kappa(\alpha), |\alpha|\|\kappa'\|_{[\alpha b,\alpha B]}\right\} \text{ for all } 0 \le \alpha < \rho_2. \tag{3.4}$$

Note that L'_κ is a bound on the derivative of κ, but is not the derivative of L_κ. We may extend the definitions of (3.4) to $\rho_1 < \alpha \le 0$ if we define the interval $[y, x]$ for $x < y$ as the interval $[x, y]$, as we will do for the rest of this chapter. Since f is strictly positive by Assumption B and continuously differentiable by Assumption C it is also bounded away from 0 on each compact interval $[\alpha b, \alpha B]$ with $\rho_1 < \alpha < \rho_2$. Hence we conclude $L_\kappa(\alpha) < \infty$ and $L'_\kappa(\alpha) < \infty$ for all $\rho_1 < \alpha < \rho_2$.
In the present section, we will show the following result:

Proposition 3.18. *In addition to the Assumptions B, C, and D suppose that the two components $\overline{\psi}$ and f of ψ belong to the Hölder spaces $\widehat{H}^{1+\frac{1}{2}\beta,3+\beta}\big([0,T]\times I\!R\big)$ and $H^{3+\beta}_{loc}(I\!R)$, respectively, and suppose that $\zeta \in H^{2+\beta}(I\!R)$. Then there exist some constants $\alpha_1, \alpha_2 \in I\!R$ with $\rho_1 < \alpha_1 < 0 < \alpha_2 < \rho_2$ such that for all $\alpha \in (\alpha_1, \alpha_2)$, the final value problem* (3.2), (3.3) *has a solution $\gamma \in H^{1+\frac{1}{2}\beta,2+\beta}\big([0,T]\times I\!R\big)$ with*

$$\inf_{v\in I\!R} \alpha \int_0^{\zeta(v)} f(x)\,dx \le \gamma(t,u) \le \sup_{v\in I\!R} \alpha \int_0^{\zeta(v)} f(x)\,dx \quad \textit{for all } (t,u) \in [0,T]\times I\!R \tag{3.5}$$

and

$$2L_0 L'_\kappa(\alpha) \inf_{(t,u)\in[0,T]\times I\!R} \gamma_u(t,u) > -1. \tag{3.6}$$

Moreover, for all $\alpha \in (\rho_1, \rho_2)$ there exists at most one solution $\gamma \in C_b^{1,2}\big([0,T]\times I\!R\big)$ of (3.2), (3.3) *which satisfies $\gamma(t,u) \in \big[\alpha b, \alpha B\big]$ for all $(t,u) \in [0,T]\times I\!R$, and* (3.6).

Remark. Since the space $H^{1+\frac{1}{2}\beta,2+\beta}\big([0,T]\times I\!R\big)$ is a subspace of $C_b^{1,2}\big([0,T]\times I\!R\big)$, the stated conditions imply that there exists for all $\alpha \in (\alpha_1, \alpha_2)$ a unique solution within the class of functions $\gamma \in C_b^{1,2}\big([0,T]\times I\!R\big)$ which satisfy (3.6). □

Proof. As a first step, we will explain why it suffices to show that for all $\zeta \in H^{2+\beta}(I\!R)$ there exists some $\bar{\alpha} > 0$ such that for all $0 \leq \alpha < \bar{\alpha}$ the final value problem (3.2), (3.3) has a solution as stated in the proposition, and why we only need to show uniqueness for the special case $\alpha = 1$.

Note that $\zeta \in H^{2+\beta}(I\!R)$ is bounded, so that the discussion after the definition of ρ_1 and ρ_2 implies $\rho_1 < 0$ and $\rho_2 > 1$. Since $f : I\!R \to I\!R$ is strictly positive, the strategy transform $g : I\!R \to I\!R$, $\xi \mapsto \int_0^\xi f(x)\,dx$ is strictly increasing and thus it follows that for all $\rho_1 < \alpha < \rho_2$ we have $\alpha \int_0^{\zeta(u)} f(x)\,dx \in g(I\!R)$, and we can define for all those α the function $\zeta^\alpha : I\!R \to I\!R$ by $\zeta^\alpha(u) = g^{-1}\big(\alpha \int_0^{\zeta(u)} f(x)\,dx\big)$ for all $u \in I\!R$, where again $g^{-1} : g(I\!R) \to I\!R$ is the inverse function of $g : I\!R \to I\!R$. By the monotonicity of $g : I\!R \to I\!R$ the final condition (3.3) is for all $\rho_1 < \alpha < \rho_2$ equivalent to

$$\gamma(T,u) = 2\frac{\alpha}{\rho_1} \int_0^{\zeta^{\rho_1/2}(u)} f(x)\,dx \quad \text{for all } u \in I\!R, \tag{3.7}$$

or also to

$$\gamma(T,u) = \int_0^{\zeta^{\alpha}(u)} f(x)\,dx \quad \text{for all } u \in I\!R. \tag{3.8}$$

Once we have proved that $\zeta^{\rho_1/2} \in H^{2+\beta}(I\!R)$ and that for all $\zeta \in H^{2+\beta}(I\!R)$ there exists some $\bar{\alpha} = \bar{\alpha}(\zeta) > 0$ such that for all $0 \leq \alpha < \bar{\alpha}$ a solution of the final value problem (3.2), (3.3) exists which satisfies the conditions (3.5) and (3.6), we can apply this result to the final value problem (3.2), (3.7) to conclude that a solution to (3.2), (3.3) with (3.5) and (3.6) also exists for all $\alpha \in I\!R$ with $\max\big\{\frac{1}{2}\rho_1\bar{\alpha}\big(\zeta^{\rho_1/2}\big), \rho_1\big\} < \alpha \leq 0$.

Moreover, noting that for any $\rho_1 < \alpha < \rho_2$ the uniqueness of a solution to (3.2), (3.3) is equivalent to the uniqueness of the problem (3.2), (3.8), we see that once we have shown $\zeta^\alpha \in H^{2+\beta}(I\!R)$ for all $\rho_1 < \alpha < \rho_2$ the uniqueness statement of Proposition 3.18 already follows from the statement for $\alpha = 1$ and all $\zeta \in H^{2+\beta}(I\!R)$.

Thus, let us fix $\rho_1 < \alpha < \rho_2$ and prove that $f \in H^{1+\beta}_{loc}(I\!R)$ and $\zeta \in H^{2+\beta}(I\!R)$ imply $\zeta^\alpha \in H^{2+\beta}(I\!R)$: We have to show the boundedness of ζ^α, the existence and boundedness of the derivatives ζ^α_u and ζ^α_{uu}, and the Hölder continuity of $\zeta^\alpha_{uu} : I\!R \to I\!R$.

In order to show that $\zeta^\alpha : I\!R \to I\!R$ stays bounded, we note that the boundedness of $\zeta : I\!R \to I\!R$ and the continuity of $f : I\!R \to I\!R$ imply $\big\|\int_0^\zeta f(x)\,dx\big\| < \infty$. If we now recall the definitions of the bounds b and B and ρ_1 and ρ_2 we see that on the compact set $\big[\alpha b, \alpha B\big]$ the function $g^{-1} : g(I\!R) \to I\!R$ is well defined and bounded, since it is continuous. Hence, by definition, $\zeta^\alpha : I\!R \to I\!R$ is bounded as well.

Secondly, the differentiability properties of $f : I\!R \to I\!R$ and $\zeta : I\!R \to I\!R$ imply that the function $\zeta^\alpha : I\!R \to I\!R$ is twice differentiable and its derivatives are given by

$$\zeta^\alpha_u(u) = \alpha \frac{f(\zeta(u))}{f(\zeta^\alpha(u))} \zeta_u(u)$$

and

$$\zeta^\alpha_{uu}(u) = \alpha \frac{f'(\zeta(u))\zeta_u^2(u) + f(\zeta(u))\zeta_{uu}(u)}{f(\zeta^\alpha(u))} - \alpha^2 \frac{f^2(\zeta(u))}{f^3(\zeta^\alpha(u))} \zeta_u^2(u) \quad \text{for all } u \in I\!R.$$

These derivatives are bounded, since the appearing derivatives of f and ζ are bounded, and since the continuity and strict positivity of $f : I\!R \to I\!R$ imply that f is bounded away from 0 on the bounded range of $\zeta^\alpha : I\!R \to I\!R$. The Hölder continuity of $\zeta^\alpha_{uu} : I\!R \to I\!R$ can be shown similarly.

It is obvious that $\gamma \equiv 0$ is a solution of (3.2), (3.3) for $\alpha = 0$. Hence, for the proof of Proposition 3.18 it is sufficient to consider only solutions of (3.2), (3.3) with $0 < \alpha < \rho_2$. Let us normalize this family of final value problems by replacing every appearing γ in these two equations by $\alpha\gamma^\alpha$. Note that here and in the sequel, the superscript α stands for an index, not for an exponent. After dividing both equations by α, the family of final value problems (3.2), (3.3) can be transformed to the family given by the partial differential equation

$$\gamma_t^\alpha(t,u) + \frac{d}{du} a_1^\alpha\big(t,u,\gamma^\alpha(t,u),\gamma_u^\alpha(t,u)\big) - a^\alpha\big(t,u,\gamma^\alpha(t,u),\gamma_u^\alpha(t,u)\big) = 0 \tag{3.9}$$

on $[0,T) \times I\!R$, with the final condition

$$\gamma^\alpha(T,u) = \int_0^{\zeta(u)} f(x)\,dx \quad \text{for all } u \in I\!R \tag{3.10}$$

being the same for all $0 < \alpha < \rho_2$. Here the two coefficients $a_1^\alpha : [0,T] \times I\!R^3 \to I\!R$ and $a^\alpha : [0,T] \times I\!R^3 \to I\!R$ in (3.9) are for all $0 < \alpha < \rho_2$ and all $(t,u,\gamma,p) \in [0,T] \times I\!R^3$ given by

$$a_1^\alpha(t,u,\gamma,p) := \frac{1}{2}\left(1 + \frac{\overline{\psi}(t,u)}{\overline{\psi}_u(t,u)}\kappa(\alpha\gamma)\alpha p\right)p = \frac{1}{2}p + \frac{\overline{\psi}(t,u)}{\overline{\psi}_u(t,u)}\kappa(\alpha\gamma)\frac{\alpha}{2}p^2 \tag{3.11}$$

and

$$a^\alpha(t,u,\gamma,p) := \left(\frac{\overline{\psi}_t(t,u)}{\overline{\psi}_u(t,u)} - \frac{1}{2}\frac{\overline{\psi}_{uu}(t,u)}{\overline{\psi}_u(t,u)}\left(1 + \frac{\overline{\psi}(t,u)}{\overline{\psi}_u(t,u)}\kappa(\alpha\gamma)\alpha p\right)\right)p. \tag{3.12}$$

In order to apply standard results from the theory of quasi-linear Cauchy problems, we perform a time inversion by considering the PDE for $\bar{\gamma}^\alpha : [0,T] \times I\!R \to I\!R$ defined by $\bar{\gamma}^\alpha(t,u) = \gamma^\alpha(T-t,u)$ for all $(t,u) \in [0,T] \times I\!R$ and all $0 < \alpha < \rho_2$. We get for all $0 < \alpha < \rho_2$ and $(t,u) \in (0,T] \times I\!R$:

$$\bar{\gamma}_t^\alpha(t,u) - \frac{d}{du} a_1^\alpha\big(T-t,u,\bar{\gamma}^\alpha(t,u),\bar{\gamma}_u^\alpha(t,u)\big) + a^\alpha\big(T-t,u,\bar{\gamma}^\alpha(t,u),\bar{\gamma}_u^\alpha(t,u)\big) = 0 \tag{3.13}$$

and

$$\bar{\gamma}^\alpha(0,u) = \int_0^{\zeta(u)} f(x)\,dx \quad \text{for all } u \in I\!R. \tag{3.14}$$

Hence Proposition 3.18 immediately follows from the following proposition, stated in terms of the corresponding initial value problem (3.13), (3.14). **q.e.d.**

Proposition 3.19. *In addition to the Assumptions B, C, and D suppose that the two components $\overline{\psi}$ and f of ψ belong to the Hölder spaces $\widehat{H}^{1+\frac{1}{2}\beta,3+\beta}\big([0,T] \times I\!R\big)$ and $H_{loc}^{3+\beta}(I\!R)$, respectively, and suppose that $\zeta \in H^{2+\beta}(I\!R)$. Then there exists some $0 < \bar{\alpha} \le \rho_2$ such that for all $0 \le \alpha < \bar{\alpha}$ the initial value problem (3.13), (3.14) has a solution $\bar{\gamma}^\alpha \in H^{1+\frac{1}{2}\beta,2+\beta}\big([0,T] \times I\!R\big)$ with*

$$\inf_{v \in I\!R} \int_0^{\zeta(v)} f(x)\,dx \le \bar{\gamma}^\alpha(t,u) \le \sup_{v \in I\!R} \int_0^{\zeta(v)} f(x)\,dx \quad \text{for all } (t,u) \in [0,T] \times I\!R \tag{3.15}$$

and

$$2\alpha L_0 L_\kappa'(\alpha) \inf_{(t,u) \in [0,T] \times I\!R} \bar{\gamma}_u^\alpha(t,u) > -1. \tag{3.16}$$

Moreover, for all $0 \le \alpha < \rho_2$ there exists at most one solution $\bar{\gamma}^\alpha \in C_b^{1,2}\big([0,T] \times I\!R\big)$ of (3.13), (3.14) which satisfies (3.16) and

$$b \le \bar{\gamma}^\alpha(t,u) \le B \quad \text{for all } (t,u) \in [0,T] \times I\!R. \tag{3.17}$$

We will prove this second proposition with the help of two lemmas. As in Frey (1998) we will introduce a modified initial value problem, where the diffusion coefficients are truncated so that the non-linearity stays bounded. For a good choice of the truncation, the modified partial differential equation becomes uniformly parabolic, and an existence and uniqueness result for quasi-linear partial differential equations can be applied. For sufficiently small $\alpha \geq 0$ it then can be shown that the solutions $\bar{\gamma}^\alpha$ to the modified initial value problem do not reach regions where the diffusion coefficients have been genuinely truncated, and thus, they are also solutions to the unrestricted initial value problem (3.13), (3.14).

At first, we have to truncate the unbounded diffusion coefficients $a_1 : [0,T] \times \mathbb{R}^3 \to \mathbb{R}$ and $a : [0,T] \times \mathbb{R}^3 \to \mathbb{R}$. In order to define truncated versions of these coefficients, let us recall the definition of $L_\kappa : \mathbb{R}^+ \to \mathbb{R}^+$ and $L'_\kappa : \mathbb{R}^+ \to \mathbb{R}^+$ in (3.4). Then for all $0 < \varepsilon < 1$, $M > 0$, and $0 \leq \alpha < \rho_2$ we can define a smooth cutoff function

$$c^\alpha = c^\alpha_{\varepsilon,M} : \mathbb{R} \to \left[\frac{1}{2}\frac{\varepsilon - 1}{L_0 L'_\kappa(\alpha)}, \frac{M}{L_0 L'_\kappa(\alpha)}\right] \tag{3.18}$$

which satisfies the two conditions

$$c^\alpha(x) = x \quad \text{for} \quad \frac{1}{2}\frac{2\varepsilon - 1}{L_0 L'_\kappa(\alpha)} \leq x \leq \frac{1}{2}\frac{M}{L_0 L'_\kappa(\alpha)} \qquad \text{and} \qquad 0 \leq \frac{d}{dx}c^\alpha(x) \leq 1 \quad \text{for all } x \in \mathbb{R}.$$

Here and for the rest of this section we set $\frac{x}{0} = \operatorname{sgn}(x)\infty$ for $x \neq 0$. The cutoff function will be used to cut off those terms in the diffusion coefficients a^α_1 and a^α which lead to a non-linear appearance of the first derivative $\bar{\gamma}^\alpha_u$ in the PDE (3.13).

In order to show uniqueness of the modified initial value problem, we also introduce for all $0 < \varepsilon < 1$ and all $0 \leq \alpha < \rho_2$ truncated versions $\bar{\kappa}^\alpha = \bar{\kappa}^\alpha_\varepsilon$ of the transformed loss function $\kappa : g(\mathbb{R}) \to \mathbb{R}$, by taking some function $\bar{\kappa}^\alpha : \mathbb{R} \to \big[0, (1+\varepsilon)L_\kappa(\alpha)\big]$ which is smooth outside of $[\alpha b, \alpha B]$ and satisfies both

$$\bar{\kappa}^\alpha(x) = \kappa(x) \quad \text{for all } x \in [\alpha b, \alpha B] \quad \text{and} \quad 0 \leq \bar{\kappa}^\alpha(x) \leq (1+\varepsilon)L_\kappa(\alpha) \quad \text{for all } x \in \mathbb{R}. \tag{3.19}$$

By the definition of ρ_2 the transformed loss function is well defined for all $x \in [\alpha b, \alpha B]$ and any $0 \leq \alpha < \rho_2$. Finally, we can define the truncated diffusion coefficients $\bar{a}^\alpha_1 : [0,T] \times \mathbb{R}^3 \to \mathbb{R}$ and $\bar{a}^\alpha : [0,T] \times \mathbb{R}^3 \to \mathbb{R}$ by

$$\bar{a}^\alpha_1(t,u,\gamma,p) := \frac{1}{2}p + \frac{\overline{\psi}(t,u)}{\overline{\psi}_u(t,u)}\bar{\kappa}^\alpha(\alpha\gamma)\int_0^p c^\alpha(\alpha q)\,dq \tag{3.20}$$

and

$$\bar{a}^\alpha(t,u,\gamma,p) := \frac{\overline{\psi}_t(t,u) - \frac{1}{2}\overline{\psi}_{uu}(t,u)}{\overline{\psi}_u(t,u)}p - \frac{\overline{\psi}_{uu}(t,u)}{\overline{\psi}_u(t,u)}\frac{\overline{\psi}(t,u)}{\overline{\psi}_u(t,u)}\bar{\kappa}^\alpha(\alpha\gamma)\int_0^p c^\alpha(\alpha q)\,dq \tag{3.21}$$

for all $(t,u,\gamma,p) \in [0,T] \times \mathbb{R}^3$ and all $0 \leq \alpha < \rho_2$.

We will then consider the initial value problem which for all $(t,u) \in (0,T] \times \mathbb{R}$ is given by

$$\bar{\gamma}^\alpha_t(t,u) - \frac{d}{du}\bar{a}^\alpha_1\big(T-t,u,\bar{\gamma}^\alpha(t,u),\bar{\gamma}^\alpha_u(t,u)\big) + \bar{a}^\alpha\big(T-t,u,\bar{\gamma}^\alpha(t,u),\bar{\gamma}^\alpha_u(t,u)\big) = 0 \tag{3.22}$$

and

$$\bar{\gamma}^\alpha(0,u) = \int_0^{\zeta(u)} f(x)\,dx \quad \text{for all } u \in \mathbb{R}. \tag{3.23}$$

The first lemma proves existence and uniqueness of solutions to the modified problem (3.22), (3.23).

Lemma 3.20. *In addition to the Assumptions B, C, and D suppose that the two components $\overline{\psi}$ and f of ψ belong to the Hölder spaces $\widehat{H}^{1+\frac{1}{2}\beta,2+\beta}\big([0,T]\times I\!R\big)$ and $H^{2+\beta}_{loc}(I\!R)$, respectively, and suppose that $\zeta \in H^{2+\beta}(I\!R)$. Then for all $0 \le \alpha < \rho_2$ the modified initial value problem (3.22), (3.23) has a solution $\bar{\gamma}^\alpha \in H^{1+\frac{1}{2}\beta,2+\beta}\big([0,T]\times I\!R\big)$, and for any specific choice of the cutoff function (3.18), there exist some constants $K(\alpha,\varepsilon,M)$ depending on $0<\varepsilon<1$, and $M>0$, such that $\|\bar{\gamma}^\alpha_u\| \le K(\alpha,\varepsilon,M) \le K(\alpha_0,\varepsilon,M)$ for all $0\le\alpha\le\alpha_0<\rho_2$.*
If $f \in C^3(I\!R)$ then for all $0\le\alpha<\rho_2$, $0<\varepsilon<1$ and $M>0$, the solution $\bar{\gamma}^\alpha$ of (3.22), (3.23) is unique within the space $C^{1,2}_b\big([0,T]\times I\!R\big)$.

Proof. The proof will follow from Theorem V.8.1 in Ladyženskaja et al. (1968) (shortly denoted by Theorem V.8.1), and parallels the proof of Proposition 4.3 in Frey (1998). Let us fix $0\le\alpha<\rho_2$. For the existence part we have to show that the assumptions a) – c) of Theorem V.8.1 are fulfilled. Condition a) holds, since $\zeta\in H^{2+\beta}(I\!R)$ and $f\in H^{2+\beta}_{loc}(I\!R)$ implies that $\int_0^\zeta f(x)\,dx \in H^{2+\beta}(I\!R)$ (actually, at this point it suffices that f belongs only to the space $H^{1+\beta}_{loc}(I\!R)$), and hence especially $\big\|\int_0^\zeta f(x)\,dx\,\big\|<\infty$. For condition b), let us likewise define the function $A^\alpha:[0,T]\times I\!R^3\to I\!R$ by setting for all $(t,u,\gamma,p)\in[0,T]\times I\!R^3$

$$A^\alpha(t,u,\gamma,p) := \bar{a}^\alpha(t,u,\gamma,p) - \frac{\partial}{\partial\gamma}\bar{a}^\alpha_1(t,u,\gamma,p)p - \frac{\partial}{\partial u}\bar{a}^\alpha_1(t,u,\gamma,p).$$

Since our assumptions and the definition of $\bar{\kappa}^\alpha$ imply that the truncated transformed loss function $\bar{\kappa}^\alpha: I\!R\to I\!R$ is differentiable, taking the derivatives in (3.20) yields

$$\frac{\partial}{\partial\gamma}\bar{a}^\alpha_1(t,u,\gamma,p) = \frac{\overline{\psi}(t,u)}{\overline{\psi}_u(t,u)}\alpha\big(\bar{\kappa}^\alpha\big)'(\alpha\gamma)\int_0^p c^\alpha(\alpha q)\,dq, \tag{3.24}$$

$$\frac{\partial}{\partial u}\bar{a}^\alpha_1(t,u,\gamma,p) = \left(1-\frac{\overline{\psi}(t,u)}{\overline{\psi}_u(t,u)}\frac{\overline{\psi}_{uu}(t,u)}{\overline{\psi}_u(t,u)}\right)\bar{\kappa}^\alpha(\alpha\gamma)\int_0^p c^\alpha(\alpha q)\,dq, \tag{3.25}$$

and plugging the three equations (3.21), (3.24), and (3.25) into the definition of the function $A^\alpha:[0,T]\times I\!R^3\to I\!R$ leads to

$$A^\alpha(t,u,\gamma,p) = -b^\alpha(t,u,\gamma,p)p, \tag{3.26}$$

where

$$\begin{aligned} b^\alpha(t,u,\gamma,p) = {} & \frac{\frac{1}{2}\overline{\psi}_{uu}(t,u)-\overline{\psi}_t(t,u)}{\overline{\psi}_u(t,u)} \\ & + \left(\bar{\kappa}^\alpha(\alpha\gamma)+\frac{\overline{\psi}(t,u)}{\overline{\psi}_u(t,u)}\alpha\big(\bar{\kappa}^\alpha\big)'(\alpha\gamma)\right)\int_0^p c^\alpha(\alpha q)\,dq \end{aligned} \tag{3.27}$$

for all $(t,u,\gamma,p)\in[0,T]\times I\!R^3$. Especially, we get $A^\alpha(t,u,\gamma,0)=0$ for all $(t,u,\gamma)\in[0,T]\times I\!R^2$. To verify c) in Theorem V.8.1 we have to check the conditions b) and c) of Theorem V.6.1 in Ladyženskaja et al. (1968), which from now on will just be called Theorem V.6.1. As in Frey (1998), we will prove slightly weaker conditions, since we do not show the conditions b) and c) for $|\gamma|\le M$, but for $\gamma\in[b,B]$, since it easily follows from the discussion leading to (V.6.8) and its transference to the Cauchy problem in (V.8.2), (V.8.3) that by the remarks leading to Theorem I.2.9 of Ladyženskaja et al. (1968) we might apply a generalized version of their Theorem I.2.1 (or their Corollary I.2.1) to replace in our case the range $[-M,M]$ of possible solutions $\big\{\bar{\gamma}^{\alpha,N}\big\}_{N\in I\!N}$ of the first boundary problems in the expanding cylinders $\big\{\Omega^N\times(0,T)\big\}_{N\in I\!N}$, and thus also of the initial value problem (3.22), (3.23), by $[b,B]$ or even its subset $\big[\inf_{v\in I\!R}\int_0^{\zeta(v)} f(x)\,dx, \sup_{v\in I\!R}\int_0^{\zeta(v)} f(x)\,dx\,\big]$ (see Ladyženskaja et al. (1968), p. 493).

At first, we are going to check condition b) of Theorem V.6.1. By our assumptions on the functions $\overline{\psi} : [0,T] \times I\!R \to I\!R$ and $f : I\!R \to I\!R$, the functions $\bar{a}_1^\alpha : [0,T] \times I\!R^3 \to I\!R$ and $\bar{a}^\alpha : [0,T] \times I\!R^3 \to I\!R$ are continuous, the derivatives $\frac{\partial}{\partial z}\bar{a}_1^\alpha$ for $z \in \{p, \gamma, u\}$ exist and are given by (3.24), (3.25), and

$$\frac{\partial}{\partial p}\bar{a}_1^\alpha(t,u,\gamma,p) = \frac{1}{2} + \frac{\overline{\psi}(t,u)}{\overline{\psi}_u(t,u)}\bar{\kappa}^\alpha(\alpha\gamma)c^\alpha(\alpha p) \quad \text{for all } (t,u,\gamma,p) \in [0,T] \times I\!R^3. \tag{3.28}$$

Now we have to check that the two statements of (V.6.9) in Ladyženskaja et al. (1968) hold, i.e. in our one-dimensional setting we have to show that there exist some strictly positive constants $l, L \in I\!R$ which may depend on the choice of α, ε, and M, such that

$$l \le \frac{\partial}{\partial p}\bar{a}_1^\alpha(t,u,\gamma,p) \le L \tag{3.29}$$

and

$$\begin{aligned}\Big(\big|\bar{a}_1^\alpha(t,u,\gamma,p)\big| + \Big|\frac{\partial}{\partial\gamma}\bar{a}_1^\alpha(t,u,\gamma,p)\Big|\Big)\,(1+|p|)& \\ + \Big|\frac{\partial}{\partial u}\bar{a}_1^\alpha(t,u,\gamma,p)\Big| + \big|a^\alpha(t,u,\gamma,p)\big| &\le L(1+|p|)^2\end{aligned} \tag{3.30}$$

for all $(t,u,\gamma,p) \in [0,T] \times I\!R \times [b,B] \times I\!R$. Let us fix $(t,u,\gamma,p) \in [0,T] \times I\!R \times [b,B] \times I\!R$. Note that such a choice of γ implies by the definition of $\bar{\kappa}^\alpha : I\!R \to \big[0,(1+\varepsilon)L_\kappa(\alpha)\big]$ in (3.19) that $\bar{\kappa}^\alpha(\alpha\gamma) = \kappa(\alpha\gamma)$ and $\big(\bar{\kappa}^\alpha\big)'(\alpha\gamma) = \kappa'(\alpha\gamma)$. Let us also notice that the strict positivity of $\overline{\psi} : [0,T] \times I\!R \to I\!R$ implies $L_0 > 0$.
Let us start with (3.29). Since $\frac{\overline{\psi}}{\overline{\psi}_u} : [0,T] \times I\!R \to I\!R$ and $\bar{\kappa}^\alpha : I\!R \to \big[0,(1+\varepsilon)L_\kappa(\alpha)\big]$ are nonnegative, since for $\gamma \in [b,B]$ the truncated $\bar{\kappa}^\alpha$ satisfies $\bar{\kappa}^\alpha(\alpha\gamma) = \kappa(\alpha\gamma) \le L_\kappa(\alpha)$, and since $c^\alpha : I\!R \to \Big[\frac{1}{2}\frac{\varepsilon-1}{L_0L'_\kappa(\alpha)}, \frac{M}{L_0L'_\kappa(\alpha)}\Big]$ is nondecreasing and satisfies $c^\alpha(0) = 0$, we can bound (3.28) on $[0,T] \times I\!R \times [b,B] \times I\!R$ by

$$\frac{1}{2} + L_0L_\kappa(\alpha)c^\alpha\big(-\alpha|p|\big) \le \frac{\partial}{\partial p}\bar{a}_1^\alpha(t,u,\gamma,p) \le \frac{1}{2} + L_0L_\kappa(\alpha)c^\alpha\big(\alpha|p|\big). \tag{3.31}$$

Hence we get $\frac{\partial}{\partial p}\bar{a}_1^\alpha(t,u,\gamma,p) = \frac{1}{2}$ if $L_\kappa(\alpha) = 0$. If $L_\kappa(\alpha) > 0$ the definition of $L'_\kappa(\alpha)$ implies that $L'_\kappa(\alpha) \ge L_\kappa(\alpha) > 0$. Hence the range of c^α is bounded, and since $c^\alpha(0) = 0$ and $\frac{d}{dx}c^\alpha \in [0,1]$ we now get from the mean value theorem and the definition of the range of c^α that

$$\max\Big\{-\alpha|p|, \frac{1}{2}\frac{\varepsilon-1}{L_0L'_\kappa(\alpha)}\Big\} \le c^\alpha\big(-\alpha|p|\big) \le 0 \quad \text{and} \quad 0 \le c^\alpha\big(\alpha|p|\big) \le \min\Big\{\alpha|p|, \frac{M}{L_0L'_\kappa(\alpha)}\Big\}.$$

Thus we get, again using $L'_\kappa(\alpha) \ge L_\kappa(\alpha)$, for all $(t,u,\gamma,p) \in [0,T] \times I\!R \times [b,B] \times I\!R$ that

$$0 < \max\Big\{\frac{1}{2} - \alpha L_0L_\kappa(\alpha)|p|, \frac{\varepsilon}{2}\Big\} \le \frac{\partial}{\partial p}\bar{a}_1^\alpha(t,u,\gamma,p) \le \frac{1}{2} + \min\big\{\alpha L_0L_\kappa(\alpha)|p|, M\big\}. \tag{3.32}$$

This proves (3.29), since we may define l and L by $l = l(\alpha,\varepsilon,M) = \max\big\{\frac{1}{2} - \alpha L_0L_\kappa(\alpha)|p|, \frac{\varepsilon}{2}\big\}$ and $L = L(\alpha,\varepsilon,M) = \frac{1}{2} + \min\big\{\alpha L_0L_\kappa(\alpha)|p|, M\big\}$. For (3.30) note that by the monotonicity of c^α and due to $c^\alpha(0) = 0$ we have $\big|\int_0^p c^\alpha(\alpha q)dq\big| \le |p|\big|c^\alpha(\alpha p)\big|$ for all $p \in I\!R$. Without loss of generality we assume that $M \ge \frac{1}{2}$. Then we have $|\varepsilon - 1| \le 1 \le 2M$, and using the bounds on $c^\alpha\big(\pm\alpha|p|\big)$ we get

$$\Big|\int_0^p c^\alpha(\alpha q)\,dq\Big| \le |p|\min\Big\{\alpha|p|, \frac{M}{L_0L'_\kappa(\alpha)}\Big\} \quad \text{for all } p \in I\!R.$$

Hence, starting from (3.20), the same bounds that led to (3.32) now imply

$$\left|\bar{a}_1^\alpha(t,u,\gamma,p)\right| \le \frac{1}{2}|p| + L_0 L_\kappa(\alpha)\left|\int_0^p c^\alpha(\alpha q)\,dq\right| \le |p|\left(\frac{1}{2} + \min\bigl\{\alpha L_0 L_\kappa(\alpha)|p|, M\bigr\}\right), \quad (3.33)$$

for any $(t,u,\gamma,p) \in [0,T] \times I\!R \times [b,B] \times I\!R$. Similarly we get for those (t,u,γ,p) from (3.24), (3.25), and (3.21), respectively,

$$\left|\frac{\partial}{\partial\gamma}\bar{a}_1^\alpha(t,u,\gamma,p)\right| \le |p| \min\bigl\{\alpha L_0 L_\kappa'(\alpha)|p|, M\bigr\}, \quad (3.34)$$

$$\left|\frac{\partial}{\partial u}\bar{a}_1^\alpha(t,u,\gamma,p)\right| \le |p|\left(\frac{1}{L_0} + \left\|\frac{\overline{\psi}_{uu}}{\overline{\psi}_u}\right\|\right)\min\bigl\{\alpha L_0 L_\kappa(\alpha)|p|, M\bigr\} \quad (3.35)$$

and

$$\left|a^\alpha(t,u,\gamma,p)\right| \le |p|\left(\left\|\frac{\overline{\psi}_t}{\overline{\psi}_u}\right\| + \frac{1}{2}\left\|\frac{\overline{\psi}_{uu}}{\overline{\psi}_u}\right\|\Bigl(1 + 2\min\bigl\{\alpha L_0 L_\kappa(\alpha)|p|, M\bigr\}\Bigr)\right). \quad (3.36)$$

The previous four bounds validate the second condition of (V.6.9), i.e. there exists some $L = L(\alpha,\varepsilon,M) \in I\!R$ such that not only (3.29) but also (3.30) holds. This proves that Assumption b) of Theorem V.6.1 is satisfied as well. As explained on p. 451 and p. 493 in Ladyženskaja et al. (1968), the norm $\|\bar{\gamma}_u^\alpha\|$ can now be bounded by some constant $K := K(\alpha,\varepsilon,M)$, which only depends on the lower and upper bounds $l = l(\alpha,\varepsilon,M)$ and $L = L(\alpha,\varepsilon,M)$, and on $\bigl\|\frac{d}{du}\int_0^{\zeta(\cdot)} f(x)\,dx\bigr\| = \bigl\|f(\zeta)\zeta_u\bigr\|$. Since the bounds in (3.32) to (3.36) are monotone in α, the bound $K := K(\alpha,\varepsilon,M)$ can be chosen to be nondecreasing in α, i.e. for all $0 \le \alpha \le \alpha_0 < \rho_2$ we have $\bigl\|\bar{\gamma}_u^\alpha\bigr\| \le K(\alpha,\varepsilon,M) \le K(\alpha_0,\varepsilon,M)$.

To validate condition c) in Theorem V.6.1, we have to show that the restrictions to the set $[0,T] \times I\!R \times [b,B] \times [-K,K]$ of the functions $\bar{a}_1^\alpha : [0,T] \times I\!R^3 \to I\!R$ and $\bar{a}^\alpha : [0,T] \times I\!R^3 \to I\!R$ and of the derivatives $\frac{\partial}{\partial z}\bar{a}_1^\alpha : [0,T] \times I\!R^3 \to I\!R$, with $z \in \{p,\gamma,u\}$, satisfy Hölder conditions with exponent $\frac{1}{2}\beta$ in t, and with exponent β in u, γ, and p. This follows easily from our assumptions, and since our choice of the cutoff function c^α of (3.18) implies that these functions are even Lipschitz in p on sets of the form $[0,T] \times I\!R \times [b,B] \times [-\tilde{K},\tilde{K}]$ for all $\tilde{K} \ge 0$.

Thus, by Theorem V.8.1 for each $0 \le \alpha < \rho_2$ a solution $\bar{\gamma}^\alpha \in H^{1+\frac{1}{2}\beta,2+\beta}\bigl([0,T] \times I\!R\bigr)$ of the truncated initial value problem (3.22), (3.23) exists.

It remains to prove the uniqueness under the additional assumption that $f \in C^3(I\!R)$. This requires to check the three conditions given at the end of Theorem V.8.1. For this purpose, let us fix $0 \le \alpha < \rho_2$. Similarly to the lower bound in (3.32), which is valid only as long as $\gamma \in [b,B]$, we can show from (3.28) and the definitions of L_0, the cutoff function c^α given in (3.18), and the truncated transformed loss function $\bar{\kappa}^\alpha : I\!R \to \bigl[0,(1+\varepsilon)L_\kappa(\alpha)\bigr]$, that for all $(t,u,\gamma,p) \in [0,T] \times I\!R^3$ we have

$$\frac{\partial}{\partial p}\bar{a}_1^\alpha(t,u,\gamma,p) \ge \frac{1}{2} - \frac{1}{2}(1+\varepsilon)(1-\varepsilon) \ge \frac{1}{2}\varepsilon^2 \ge 0. \quad (3.37)$$

Hence condition (V.8.6) in Ladyženskaja et al. (1968) holds. Moreover, it is easily seen from (3.28) and $0 \le \frac{d}{dx}c^\alpha(x) \le 1$ for all $x \in I\!R$ that both derivatives $\frac{\partial^2}{\partial p^2}\bar{a}_1^\alpha(t,u,\gamma,p)$ and $\frac{\partial^2}{\partial\gamma\partial p}\bar{a}_1^\alpha(t,u,\gamma,p)$ exist for all $(t,u,\gamma,p) \in [0,T] \times I\!R^3$ and that for all $N > 0$ they can be bounded on sets of the form $[0,T] \times I\!R \times [-N,N]^2$ by arguments similar to the one used to derive (3.32) to (3.36). From (3.26) and (3.27) we obtain the same statement for the derivative $\frac{\partial}{\partial p}A^\alpha(t,u,\gamma,p)$. This gives the second condition at the end of Theorem V.8.1.

Finally, our additional condition $f \in C^3(I\!R)$ assures that $\kappa : g(I\!R) \to I\!R$ belongs to $C^2(I\!R)$, and especially that $\sup_{x\in[\alpha b,\alpha B]}\left|\kappa''(x)\right|$ is bounded. Since $\bar{\kappa}^\alpha$ is smooth outside $[\alpha b, \alpha B]$, we can also conclude that $\sup_{\gamma\in[-N,N]}\left|\left(\bar{\kappa}^\alpha\right)''(\alpha\gamma)\right|$ can be bounded for each $N > 0$, and hence it follows from (3.26) and (3.27) that $\frac{\partial}{\partial\gamma}A^\alpha(t,u,\gamma,p)$ exists for all $(t,u,\gamma,p) \in [0,T] \times I\!R^3$ and that it is bounded on sets of the form $[0,T] \times I\!R \times [-N,N]^2$ for any $N > 0$ as well. Thus, the third condition at the end of Theorem V.8.1 is also satisfied and Theorem V.8.1 in Ladyženskaja et al. (1968) gives us for all $0 \le \alpha < \rho_2$ the uniqueness of the solution $\bar{\gamma}^\alpha$ to (3.22), (3.23) within the class $C_b^{1,2}\big([0,T] \times I\!R\big)$. **q.e.d.**

Remark. Instead of using the second half of Theorem V.8.1 in Ladyženskaja et al. (1968), Frey (1998) uses their Theorem V.6.1 to prove the uniqueness of his truncated non-linear parabolic initial value problem (33), (30). However, it is not clear to us how he transfers the uniqueness result of the first boundary problem in a bounded domain, which is treated in Theorem V.6.1, to the uniqueness of the Cauchy problem (33), (30). For that reason, we rather use the uniqueness results for Cauchy problems given at the end of Theorem V.8.1, which require the additional postulate that $f : I\!R \to I\!R$ is three times differentiable with bounded derivatives on compact sets. □

For the proof of Proposition 3.19 we have to find explicit bounds for $\bar{\gamma}^\alpha : [0,T] \times I\!R \to I\!R$, and we need to strengthen the bounds on the derivatives $\bar{\gamma}^\alpha_u : [0,T] \times I\!R \to I\!R$ of Lemma 3.20 a little bit. In analogy to the definition of b and B let us define the bounds b' and B' on the derivatives of the initial condition (3.23) by

$$b' := \min\left\{0, \inf_{u\in I\!R}\left\{f\big(\zeta(u)\big)\zeta_u(u)\right\}\right\} \quad \text{and} \quad B' := \max\left\{0, \sup_{u\in I\!R}\left\{f\big(\zeta(u)\big)\zeta_u(u)\right\}\right\}.$$

Moreover, we define the bounds $\underline{K}$ and $\overline{K}$, depending on α, ε, and M, by

$$\underline{K}(\alpha,\varepsilon,M) := \begin{cases} -K(\alpha,\varepsilon,M) & \text{if } b' < 0 \\ 0 & \text{if } b' = 0 \end{cases} \quad \text{and} \quad \overline{K}(\alpha,\varepsilon,M) := \begin{cases} K(\alpha,\varepsilon,M) & \text{if } B' > 0 \\ 0 & \text{if } B' = 0 \end{cases}$$

Since $K(\cdot,\varepsilon,M) : [0,\rho_2) \to [0,\infty)$ is nondecreasing, $\underline{K}(\cdot,\varepsilon,M) : [0,\rho_2) \to (-\infty,0]$ is nonincreasing and $\overline{K}(\cdot,\varepsilon,M) : [0,\rho_2) \to [0,\infty)$ is nondecreasing. Under slightly stronger conditions than the ones of Lemma 3.20, we arrive at the following result:

Lemma 3.21. *Let us consider the family of modified Cauchy problems* (3.22), (3.23) *for* $0 \le \alpha < \rho_2$ *and some fixed cutoff levels* $0 < \varepsilon < 1$ *and* $M > 0$. *Under the assumptions of the existence part of Lemma 3.20 we have for all* $0 \le \alpha < \rho_2$

$$\inf_{v\in I\!R}\int_0^{\zeta(v)} f(x)\,dx \le \bar{\gamma}^\alpha(t,u) \le \sup_{v\in I\!R}\int_0^{\zeta(v)} f(x)\,dx \quad \textit{for all } (t,u) \in [0,T] \times I\!R. \tag{3.38}$$

Moreover, if $\overline{\psi} \in \widehat{H}^{1+\frac{1}{2}\beta,3+\beta}\big([0,T] \times I\!R\big)$ *and* $f \in H^{3+\beta}_{loc}(I\!R)$, *then for all* $0 \le \alpha < \rho_2$ *the bounds on the derivative* $\bar{\gamma}^\alpha_u : [0,T] \times I\!R \to I\!R$ *of Lemma 3.20 can be sharpened to*

$$\underline{K}(\alpha,\varepsilon,M) \le \bar{\gamma}^\alpha_u(t,u) \le \overline{K}(\alpha,\varepsilon,M) \quad \textit{for all } (t,u) \in [0,T] \times I\!R. \tag{3.39}$$

Proof. Let us fix $0 < \varepsilon < 1$, $M > 0$, and $0 \le \alpha \le \alpha_0 < \rho_2$. As pointed out in Frey (1998), the existence of a solution $\gamma_\alpha \in H^{1+\frac{1}{2}\beta,2+\beta}\big([0,T] \times I\!R\big)$ to (3.22), (3.23) implies that this solution also solves the linear parabolic equation

$$\bar{\gamma}^\alpha_t(t,u) = \hat{a}^\alpha(t,u)\bar{\gamma}^\alpha_{uu}(t,u) + \hat{b}^\alpha(t,u)\bar{\gamma}^\alpha_u(t,u) \quad \text{for all } (t,u) \in [0,T] \times I\!R, \tag{3.40}$$

where the diffusion coefficients $\hat{a}^\alpha : [0,T] \times I\!R \to I\!R$ and $\hat{b}^\alpha : [0,T] \times I\!R \to I\!R$ are given by $\hat{a}^\alpha(t,u) = \frac{\partial}{\partial p}\bar{a}_1^\alpha\big(T-t, u, \bar{\gamma}^\alpha(t,u), \bar{\gamma}_u^\alpha(t,u)\big)$ and $\hat{b}^\alpha(t,u) = b^\alpha\big(T-t, u, \bar{\gamma}^\alpha(t,u), \bar{\gamma}_u^\alpha(t,u)\big)$ for all $(t,u) \in [0,T] \times I\!R$, and $\frac{\partial}{\partial p}\bar{a}_1^\alpha : [0,T] \times I\!R^3$ and $b^\alpha : [0,T] \times I\!R^3$ are given by (3.28) and (3.27), respectively.
The validity of the estimate (3.38) was already derived in the proof of Lemma 3.20, in the comments before the demonstration of condition c) in Theorem V.8.1. Moreover, due to (3.37) and the boundedness of $\hat{a}^\alpha : [0,T] \times I\!R \to I\!R$ and $\hat{b}^\alpha : [0,T] \times I\!R \to I\!R$, the bounds on $\bar{\gamma}^\alpha : [0,T] \times I\!R \to I\!R$ in (3.38) now also follow directly either from the maximum principle for linear parabolic partial differential equations (see for example Corollary I.2.1 in Ladyženskaja et al. (1968)), or the Feynman-Kac formula, as Frey (1998) proposes.
Under the additional differentiability assumptions on the two functions $\overline{\psi} : [0,T] \times I\!R \to I\!R$ and $f : I\!R \to I\!R$, the derivatives $\hat{a}_u^\alpha : [0,T] \times I\!R \to I\!R$ and $\hat{b}_u^\alpha : [0,T] \times I\!R \to I\!R$ exist and belong to the class $H^{\frac{1}{2}\beta,\beta}\big([0,T] \times I\!R\big)$. Thus we can conclude from Theorem 10, Sec. 5, Chap. 3 in Friedman (1964), in combination with the remark after Theorem 11 of the same section, that the derivatives $\bar{\gamma}_{ut}^\alpha$ and $\bar{\gamma}_{uuu}^\alpha$ exist and are continuous on $(0,T] \times I\!R$, and that they are bounded on each slab of the form $(t_0, T] \times I\!R$ with $0 < t_0 < T$. Especially, we can differentiate the linear PDE (3.40), and obtain as in the proof of Proposition 4.4 in Frey (1998) for all $(t,u) \in (0,T] \times I\!R$ that

$$\bar{\gamma}_{ut}^\alpha(t,u) = \hat{a}^\alpha(t,u)\bar{\gamma}_{uuu}^\alpha(t,u) + \big(\hat{a}_u^\alpha(t,u) + \hat{b}^\alpha(t,u)\big)\bar{\gamma}_{uu}^\alpha(t,u) + \hat{b}_u^\alpha(t,u)\bar{\gamma}_u^\alpha(t,u). \tag{3.41}$$

Since $\zeta : I\!R \to I\!R$ is differentiable, upon differentiating the initial condition, we see that $\bar{\gamma}_u^\alpha(0,u) = \frac{d}{du}\int_0^{\zeta(u)} f(x)\,dx = f\big(\zeta(u)\big)\zeta_u(u)$. Hence, after taking the limit of the solutions $\big\{\bar{\gamma}^{\alpha,N}\big\}$ of the first boundary problem of the form (V.0.1), (V.8.2) in Ladyženskaja et al. (1968), we get from their uniform bound (V.8.4) and the maximum principle Theorem I.2.1 in Ladyženskaja et al. (1968) (or from the Feynman-Kac formula) that

$$b' e^{t\|\hat{b}_u^\alpha\|} \le \bar{\gamma}_u^\alpha(t,u) \le B' e^{t\|\hat{b}_u^\alpha\|} \quad \text{for all } (t,u) \in [0,T] \times I\!R. \tag{3.42}$$

If we now use (3.42) to determine whether $\bar{\gamma}_u^\alpha : [0,T] \times I\!R \to I\!R$ either stays nonnegative or nonpositive on $[0,T] \times I\!R$, and the bound $\big\|\bar{\gamma}_u^\alpha\big\| \le K(\alpha, \varepsilon, M)$ of Lemma 3.20 to bound the modulus, we obtain (3.39). **q.e.d.**

Remark. We do not use (3.42) directly since this would require us to bound $\|\hat{b}_u^\alpha\|$, and that would involve bounds for the second derivative $\bar{\gamma}_{uu}^\alpha : [0,T] \times I\!R \to I\!R$, which we know is bounded, but for which we did not derive a uniform bound for all $0 \le \alpha \le \alpha_0 < \rho_2$.
Also notice that our proof of (3.39) implies that for all $0 < t_0 \le T$ the solution $\bar{\gamma}^\alpha$ to the Cauchy problem (3.22), (3.23) belongs to $H^{1+\frac{1}{2}\beta,3+\beta}\big([t_0,T] \times I\!R\big)$, and if $\zeta \in H^{3+\beta}(I\!R)$ we even get $\bar{\gamma}^\alpha \in H^{1+\frac{1}{2}\beta,3+\beta}\big([0,T] \times I\!R\big)$. □

Finally, we can accomplish the proof of Proposition 3.19, which mimics Frey's (1998) proof of his Theorem 4.2.

Proof of Proposition 3.19. For the existence part, we will consider the modified Cauchy problem (3.22), (3.23) instead of (3.13), (3.14) and show that for sufficiently small $\alpha > 0$, the cutoff which was introduced by our truncated diffusion coefficients $\bar{a}_1 : [0,T] \times I\!R^3 \to I\!R$ and $\bar{a} : [0,T] \times I\!R^3 \to I\!R$ does not take place.
Let us set

$$\bar{\alpha} := \sup_{\substack{0<\varepsilon<0.5, M>0 \\ 0<\alpha<\rho_2}} \left\{ \min\left\{ \alpha, \frac{2\varepsilon - 1}{2L_0 L'_\kappa(\alpha)} \frac{1}{\underline{K}(\alpha,\varepsilon,M)}, \frac{1}{2} \frac{M}{L_0 L'_\kappa(\alpha)} \frac{1}{\overline{\overline{K}}(\alpha,\varepsilon,M)} \right\} \right\},$$

and fix $\alpha < \bar{\alpha}$. Then there exist some $0 < \alpha_0 < \rho_2$, some $M > 0$ sufficiently large, and some $0 < \varepsilon < \frac{1}{2}$ sufficiently small, such that

$$\alpha \leq \min\left\{\alpha_0, \frac{2\varepsilon - 1}{2L_0 L'_\kappa(\alpha_0)} \frac{1}{\underline{K}(\alpha_0, \varepsilon, M)}, \frac{1}{2} \frac{M}{L_0 L'_\kappa(\alpha_0)} \frac{1}{\overline{K}(\alpha_0, \varepsilon, M)}\right\}.$$

Especially, we get $\alpha \leq \alpha_0$. Since $L'_\kappa : [0, \infty) \to [0, \infty)$ and $\overline{K}(\cdot, \varepsilon, M) : [0, \rho_2) \to [0, \infty)$ are nondecreasing, and $\underline{K}(\cdot, \varepsilon, M) : [0, \rho_2) \to (-\infty, 0]$ is nonincreasing, it follows from $2\varepsilon - 1 < 0$ and $M > 0$ that

$$\alpha \leq \min\left\{\frac{2\varepsilon - 1}{2L_0 L'_\kappa(\alpha)} \frac{1}{\underline{K}(\alpha, \varepsilon, M)}, \frac{1}{2} \frac{M}{L_0 L'_\kappa(\alpha)} \frac{1}{\overline{K}(\alpha, \varepsilon, M)}\right\}.$$

Lemma 3.20 guarantees the existence of a solution $\bar{\gamma}^\alpha \in H^{1+\frac{1}{2}\beta, 2+\beta}\big([0,T] \times I\!R\big)$ to the modified initial value problem (3.22), (3.23). By Lemma 3.21 the solution satisfies (3.38) and (3.39). Now we first note that by the bounds (3.38) and the definition of $\bar{\kappa}^\alpha : I\!R \to \big[0, (1+\varepsilon)L_\kappa(\alpha)\big]$ in (3.19) we have $\bar{\kappa}^\alpha\big(\alpha\gamma(t,u)\big) = \kappa\big(\alpha\gamma(t,u)\big)$ for all $(t,u) \in [0,T] \times I\!R$. Secondly, the left-hand side of (3.39), $2\varepsilon - 1 < 0$ and $\underline{K}(\alpha, \varepsilon, M) \leq 0$ imply $\alpha\bar{\gamma}^\alpha_u(t,u) \geq \frac{1}{2}\frac{2\varepsilon-1}{L_0 L'_\kappa(\alpha)}$, and similarly it follows that $\alpha\bar{\gamma}^\alpha_u(t,u) \leq \frac{1}{2}\frac{M}{L_0 L'_\kappa(\alpha)}$. Hence the definition of $c^\alpha : I\!R \to \big[\frac{1}{2}\frac{\varepsilon-1}{L_0 L'_\kappa(\alpha)}, \frac{M}{L_0 L'_\kappa(\alpha)}\big]$ in (3.18) yields $\int_0^{\bar{\gamma}^\alpha_u(t,u)} c^\alpha(\alpha q)\, dq = \int_0^{\bar{\gamma}^\alpha_u(t,u)} \alpha q\, dq = \frac{1}{2}\alpha\big(\bar{\gamma}^\alpha_u(t,u)\big)^2$ for all $(t,u) \in [0,T] \times I\!R$. Comparing now the definitions (3.20) and (3.21) with (3.11) and (3.12) respectively, we find that for all $(t,u) \in [0,T] \times I\!R$ the diffusion coefficients

$$a^\alpha_1\big(T-t, u, \bar{\gamma}^\alpha(t,u), \bar{\gamma}^\alpha_u(t,u)\big) = \bar{a}^\alpha_1\big(T-t, u, \bar{\gamma}^\alpha(t,u), \bar{\gamma}^\alpha_u(t,u)\big)$$

and

$$a^\alpha\big(T-t, u, \bar{\gamma}^\alpha(t,u), \bar{\gamma}^\alpha_u(t,u)\big) = \bar{a}^\alpha\big(T-t, u, \bar{\gamma}^\alpha(t,u), \bar{\gamma}^\alpha_u(t,u)\big)$$

coincide if $\bar{\gamma}^\alpha$ is plugged in. Hence $\bar{\gamma}^\alpha : [0,T] \times I\!R \to I\!R$ solves the unrestricted Cauchy problem (3.13), (3.14) as well. The property (3.16) now follows from (3.37) for the tuple $\big(T-t, u, \bar{\gamma}^\alpha(t,u), \bar{\gamma}^\alpha_u(t,u)\big)$, and all $(t,u) \in [0,T] \times I\!R$.
Now assume that $\bar{\gamma}^{\alpha,1} : [0,T] \times I\!R \to I\!R$ and $\bar{\gamma}^{\alpha,2} : [0,T] \times I\!R \to I\!R$ are two solutions of the unrestricted problem (3.13), (3.14), both satisfying (3.17) and (3.16). By (3.17) and the definition of $\bar{\kappa}^\alpha = \bar{\kappa}^\alpha_\varepsilon : I\!R \to \big[0, (1+\varepsilon)L_\kappa(\alpha)\big]$ in (3.19) we obtain for each $\varepsilon > 0$ the equality $\bar{\kappa}^\alpha\big(\alpha\bar{\gamma}^{\alpha,i}(t,u)\big) = \kappa\big(\alpha\bar{\gamma}^{\alpha,i}(t,u)\big)$ for all $(t,u) \in [0,T] \times I\!R$ and $i \in \{1,2\}$. Secondly, due to (3.16) there exists some $\varepsilon_0 > 0$ such that

$$2\alpha L_0 L'_\kappa(\alpha)\bar{\gamma}^{\alpha,i}_u(t,u) \geq \varepsilon_0 - 1 \quad \text{for all } (t,u) \in [0,T] \times I\!R \text{ and } i \in \{1,2\}.$$

If we now set $\varepsilon = \frac{1}{2}\varepsilon_0$ and $M = 2\alpha L_0 L'_\kappa(\alpha) \max_{i \in \{1,2\}} \big\|\bar{\gamma}^{\alpha,i}_u\big\|$, the definition of c^α given by (3.18) implies again that $\int_0^{\alpha\bar{\gamma}^{\alpha,i}_u(t,u)} c^\alpha\big(\alpha q\big)\, dq = \frac{1}{2}\alpha\big(\bar{\gamma}^\alpha_u(t,u)\big)^2$ for all $(t,u) \in [0,T] \times I\!R$ and $i \in \{1,2\}$, and hence $\bar{\gamma}^{\alpha,1}$ and $\bar{\gamma}^{\alpha,2}$ are also solutions of the truncated problem (3.22), (3.23), and therefore they must coincide by the uniqueness statement of Lemma 3.20. **q.e.d.**

Using Remark V.8.1 in Ladyženskaja et al. (1968) and revising our proofs of Proposition 3.18 and Proposition 3.19 carefully, we can state

Corollary 3.22. *If the condition on $\zeta : I\!R \to I\!R$ in Proposition 3.18 is relaxed to $\zeta \in C^1_b(I\!R)$, there still exist some $\alpha_1 < 0 < \alpha_2$ such that for all $\alpha \in (\alpha_1, \alpha_2)$, the initial value problem* (3.2), (3.3) *has a solution $\gamma \in C^{1,2}\big([0,T) \times I\!R\big) \cap C^{0,1}_b\big([0,T] \times I\!R\big)$ which satisfies* (3.5) *and* (3.6).

Moreover, another application of Theorem 10, Sec. 5, Chap. 3 in Friedman (1964) as in the proof of Lemma 3.21, or of Theorem IV.5.1 in Ladyženskaja et al. (1968), gives us as a second corollary:

Corollary 3.23. *In addition to the Assumptions B, C, and D suppose now that for some* $k \geq 2$ *the two components* $\overline{\psi}$ *and* f *of* ψ *belong to the Hölder spaces* $\widehat{H}^{k+\frac{1}{2}\beta,2k+\beta}\big([0,T] \times I\!R\big)$ *and* $H^{2k+\beta}_{loc}(I\!R)$*, respectively, and that* $\zeta \in H^{2k+\beta}(I\!R)$*. Then for all* $\alpha \in (\alpha_1, \alpha_2)$*, the solution* $\gamma : [0,T] \times I\!R \to I\!R$ *of* (3.2), (3.3) *satisfies* $\gamma \in H^{k+\frac{1}{2}\beta,2k+\beta}\big([0,T] \times I\!R\big)$*.*

3.3.2 Convergence of the Transformed Strategy Functions

In this subsection, we consider the convergence of the sequence $\{g^n\}_{n\in I\!N}$ of discrete transformed strategy functions $g^n : \mathcal{A}^n \to I\!R$ to the continuous solution of the final value problem (3.2), (3.3). The central result is Theorem 3.24. Under the assumption that a solution γ to the final value problem exists and that the price system (ψ, μ) is sufficiently regular, the theorem states that the functions $\{g^n\}_{n\in I\!N}$ converge with a certain order towards this solution if the function values of g^n in the last two time steps $t = t^n_{n-1}$ and $t = t^n_n$ converge to the corresponding values of γ.
Because of the non-linearities the proof of Theorem 3.24 is technically involved and will only be sketched by indicating how the methods used for Theorem 3.10 in the setting *without* transaction losses can be carried over to the case *with* transaction losses.

Under the assumptions of Proposition 3.18 we have shown in the previous section that the initial value problem (3.2), (3.3) has a solution $\gamma : [0,T] \times I\!R \to I\!R$, and this solution satisfies (3.5) and (3.6). Especially, by the definitions of $L_0 := \big\|\frac{\overline{\psi}}{\overline{\psi}_u}\big\|$ and $L'_\kappa(\alpha)$ in (3.4) it is easily seen that (3.6) implies

$$1 + \frac{\overline{\psi}(t,u)}{\overline{\psi}_u(t,u)}\kappa\big(\gamma(t,u)\big)\gamma_u(t,u) \geq \varepsilon \quad \text{for all } (t,u) \in [0,T] \times I\!R \text{ and some } \varepsilon > 0, \tag{3.43}$$

since the infimum over $[0,T] \times I\!R$ of the left-hand side must be strictly larger than $\frac{1}{2}$.
We will outline a proof of the following analogue of Theorem 3.10:

Theorem 3.24. *Let* (ψ, μ) *be a large investor price system which satisfies Assumptions B, C, and D. Moreover, suppose that the two components* $\overline{\psi}$ *and* f *of the equilibrium price function* ψ *belong to the Hölder spaces* $\widehat{H}^{2+\frac{1}{2}\beta,4+\beta}\big([0,T] \times I\!R\big)$ *and* $H^{4+\beta}_{loc}(I\!R)$*, respectively, and that there exists some* $\eta > 0$ *such that* $\int e^{\eta|\theta|}\mu(d\theta) < \infty$*.*
If the final value problem (3.2), (3.3) *has a solution* $\gamma : [0,T] \times I\!R \to I\!R$ *which belongs to the Hölder space* $H^{2+\frac{1}{2}\beta,4+\beta}\big([0,T] \times I\!R\big)$ *and satisfies* (3.43)*, and if there exists some* $L \in I\!R$ *such that for all sufficiently large* $n \in I\!N$ *we have*

$$\big\|g^n\big(t^n_k, \cdot\big) - \gamma\big(t^n_k, \cdot\big)\big\|_{\mathcal{U}^n_k} \leq L\delta^{4+\beta} \quad \textit{for } k \in \{n-1, n\}, \tag{3.44}$$

then the sequence $\{g^n\}_{n\in I\!N}$ *of discrete transformed strategy functions* $g^n : \mathcal{A}^n \to I\!R$ *converges to the function* $\gamma : [0,T] \times I\!R \to I\!R$ *in the sense that*

$$\|g^n - \gamma\|_{\mathcal{A}^n} = O\big(\delta^2\big) \tag{3.45}$$

and

$$\left\|g^n(\cdot + \delta^2, \cdot \pm \delta) - g^n \mp \delta\gamma_u - \delta^2\left(\gamma_t + \frac{1}{2}\gamma_{uu}\right)\right\|_{\mathcal{A}^n(n-1)} = O\big(\delta^{2+\beta}\big) \quad \textit{as } n \to \infty. \tag{3.46}$$

Remark 1. As in Theorem 3.10 we might relax the condition (3.44) slightly, and replace it by the conditions $\|g^n(t_k^n, \cdot) - \gamma(t_k^n, \cdot)\| \leq L\delta^{4+\beta}$ and $\|\Delta_u^n g^n(t_k^n, \cdot) - \Delta_u^n \gamma(t_k^n, \cdot)\|_{\mathcal{U}_k^n} \leq L\delta^{3+\beta}$ for $k \in \{n-1, n\}$. □

Remark 2. Besides the additional condition (3.43), which trivially holds in the case without transaction losses (where $\kappa \equiv 0$), there are three differences to the statement of Theorem 3.10. First of all, we need some more assumptions on the price system (ψ, μ). This is caused by the non-linear loss terms. Note that apart from $\int e^{\eta|\theta|}\mu(d\theta) < \infty$ all these additional conditions have also been used in Corollary 3.23 in order to show the existence of a solution $\gamma : [0,T] \times I\!R \to I\!R$ to the non-linear final value problem.
Secondly, we now have to require the convergence of the (transformed) strategy functions not only at the time points $\{t_{n-1}^n\}_{n\in I\!N}$, which are all strictly smaller than T, increase monotonously, and converge to T as $n \to \infty$, but also at the time point $t_n^n = T$ for all $n \in I\!N$. This is due to the implied transaction losses, which we assume to occur even at time $t = T$.
Since we have to require the convergence at the time point t_n^n, we can use this additional assumption to include the possible time-space realizations $\{t_n^n\} \times \mathcal{U}_n^n$ in the approximations for g^n and the time-space realizations $\{t_{n-1}^n\} \times \mathcal{U}_{n-1}^n$ in the approximations for $g^n(\cdot+\delta^2, \cdot+\delta) - g^n$. This constitutes the third (but minor) difference to the statement of Theorem 3.10. □

In the remainder of this section we will sketch a proof of Theorem 3.24 which follows to the largest possible extent the proof of Theorem 3.10 in Section 3.2.2. For this reason let us suppose that the theorem's assumptions hold. We should like to write for each $1 \leq k \leq n-1$ the transformed strategy $g^n(t_{k-1}^n, u)$ at time t_{k-1}^n as a convex combination of its two possible successors $g^n(t_k^n, u \pm \delta)$ at time t_k^n as we did in (2.1), i.e. we should like to find for all sufficiently large $n \in I\!N$ some probability weight function $\check{p}^n : \mathcal{A}^n(n-2) \to [0,1]$ such that for all $1 \leq k \leq n-1$ and all $u \in \mathcal{U}_{k-1}^n$ we have

$$g^n(t_{k-1}^n, u) = \check{p}^n(t_{k-1}^n, u) g^n(t_k^n, u+\delta) + \big(1 - \check{p}^n(t_{k-1}^n, u)\big) g^n(t_k^n, u-\delta). \tag{3.47}$$

If the weights $\check{p}^n(t_{k-1}^n, u)$ are sufficiently well-behaved, we can use at least basically the same arguments as in the proof of Theorem 3.10 to derive the convergence statements of Theorem 3.24 from (3.47).
However, the crucial issue in the general case is the derivation of a sufficiently well-behaved weight function $\check{p}^n$. The transaction losses induced by the large investor's trading strategy have already prevented us in Section 2.4 from obtaining an explicit recursive scheme for the transformed strategy function, and so we cannot find a representation (3.47) where the weight $\check{p}^n(t_{k-1}^n, u)$ does not depend on the transformed strategy $g^n(t_{k-1}^n, u)$. Thus, our goal can only be to find weights $\check{p}^n(t_{k-1}^n, u)$ for all $1 \leq k \leq n-1$ and $u \in \mathcal{U}_{k-1}^n$ which satisfy (3.47) and which do not depend *too* strongly on the transformed strategy so that we can still control the weights sufficiently well. Actually, it will turn out that such a weight $\check{p}^n(t_{k-1}^n, u)$ depends on the values $g^n(t_{k-1}^n, u)$, $g^n(t_k^n, u \pm \delta)$, and $g^n(t_{k+1}^n, u)$ of the transformed strategy function. With suitable controls on the influence of the transformed strategy at hand, the arguments of the proof of Theorem 3.10 can then be transferred to this section's general setting.

As a first step to the proof of Theorem 3.24 we write the fixed point equation (2.4.11) as a difference equation which is seen to be the discrete analogue of the partial differential equation (3.2). However, keeping our goal to obtain a suitable representation (3.47), we should like to rewrite (3.2) in another form. Therefore let us suppose that $\gamma : [0,T] \times I\!R \to I\!R$ solves the PDE (3.2) and satisfies (3.43). If we subtract

$$\gamma_u(t,u) \frac{1}{2} \frac{d}{du} \left(\frac{\overline{\psi}(t,u)}{\overline{\psi}_u(t,u)} \kappa\big(\gamma(t,u)\big) \gamma_u(t,u) \right) - \gamma_t(t,u) \frac{\overline{\psi}(t,u)}{\overline{\psi}_u(t,u)} \kappa\big(\gamma(t,u)\big) \gamma_u(t,u)$$

from both sides of (3.2), and divide the resulting equation by $1+\frac{\overline{\psi}(t,u)}{\overline{\psi}_u(t,u)}\kappa\big(\gamma(t,u)\big)\gamma_u(t,u)$, we can transform (3.2) into

$$\gamma_t(t,u)+\frac{1}{2}\gamma_{uu}(t,u)+b^\gamma(t,u)\gamma_u(t,u)=0 \quad \text{for all } (t,u)\in[0,T]\times I\!R, \tag{3.48}$$

where the function $b^\gamma:[0,T]\times I\!R\to I\!R$ is defined by

$$b^\gamma(t,u)=\frac{\frac{1}{2}\frac{d}{du}\big(\overline{\psi}_u(t,u)+\overline{\psi}(t,u)\kappa(\gamma(t,u))\gamma_u(t,u)\big)-\big(\overline{\psi}_t(t,u)+\overline{\psi}(t,u)\kappa(\gamma(t,u))\gamma_t(t,u)\big)}{\overline{\psi}_u(t,u)+\overline{\psi}(t,u)\kappa(\gamma(t,u))\gamma_u(t,u)}.$$

Then we shall transform the fixed point equation (2.4.11) into a difference equation for the discrete transformed strategy $g^n:\mathcal{A}^n\to I\!R$, which is a discrete analogue of the partial differential equation (3.48).

If we look at the candidate limiting equation (3.48), it is clear that our difference equation for $g^n:\mathcal{A}^n\to I\!R$ should contain terms which approximate the transformed loss function $\kappa:I\!R\to I\!R$ of (3.1), evaluated at g^n, and also terms which approximate its first derivative. In order to find those approximations, we fix the measure μ and introduce some more notation for all $n\in I\!N$ and any function $\rho:\mathcal{A}^n\to G:=g(I\!R)$ (and hence in particular for $\rho=g^n=g\circ\xi^n$). We introduce the function $J:G^2\to I\!R$ given by

$$J(x_1,x_2)=\operatorname{sgn}(x_2-x_1)k_\mu\big(g^{-1}(x_1),g^{-1}(x_2)\big) \quad \text{for all } x_1,\xi_2\in I\!R, \tag{3.49}$$

and define for all $(t,u)\in\mathcal{A}_1^n(n-1)$ the shorthands

$$J_-^\pm\big(\rho,t,u\big)=J\big(\rho(t-\delta^2,u),\rho(t,u\pm\delta)\big)$$

and

$$J_+^\pm\big(\rho,t,u\big)=J\big(\rho(t,u\pm\delta),\rho(t+\delta^2,u)\big).$$

For all $(t,u)\in\mathcal{A}_1^n(n-1)$ we shall use three different approximations of $\kappa\big(\Sigma_u^n\,\rho(t,u)\big)$, where

$$\Sigma_u^n\,\rho(t,u)=\frac{1}{2}\rho(t,u+\delta)+\frac{1}{2}\rho(t,u-\delta) \quad \text{for all } (t,u)\in\mathcal{A}_1^n.$$

For $\Delta_u^n\,\rho(t,u)\neq 0$, our approximations of $\kappa\big(\Sigma_u^n\,\rho(t,u)\big)$ are given by

$$K_\pm^n\big(\rho,t,u\big)=\mp\frac{J_\pm^+(\rho,t,u)-J_\pm^-(\rho,t,u)}{\delta\Delta_u^n\,\rho(t,u)},$$

and

$$\bar{K}^n\big(\rho,t,u\big)=\frac{\big(\rho(t,u+\delta)-\rho(t-\delta^2,u)\big)J_-^+(\rho,t,u)-\big(\rho(t-\delta^2,u)-\rho(t,u-\delta)\big)J_-^-(\rho,t,u)}{\delta^2\big(\Delta_u^n\,\rho(t,u)\big)^2}.$$

If $\Delta_u^n\,\rho(t,u)=0$, we set $K_\pm^n(\rho,t,u)=\bar{K}^n(\rho,t,u)=\kappa\big(\Sigma_u^n\,\rho(t,u)\big)$.

In order to approximate $\kappa'\big(\Sigma_u^n\,\rho(t,u)\big)$, we define for all $(t,u)\in\mathcal{A}_1^n(n-1)$ the two expressions

$$\dot{K}_\pm^n\big(\rho,t,u\big)=\mp\frac{\big(\rho(t\pm\delta^2,u)-\rho(t,u-\delta)\big)J_\pm^+(\rho,t,u)+\big(\rho(t,u+\delta)-\rho(t\pm\delta^2,u)\big)J_\pm^-(\rho,t,u)}{\delta^3\big(\Delta_u^n\,\rho(t,u)\big)^3},$$

if $\Delta_u^n \rho(t,u) \neq 0$ and $\dot{K}_\pm^n(\rho,t,u) = \frac{1}{2}\kappa'\big(\Sigma_u^n \rho(t,u)\big) \pm \tau\big(\Sigma_u^n \rho(t,u)\big)$ otherwise. We shall see in Lemma 3.25 that for all $(t,u) \in \mathcal{A}_1^n(n-1)$ the term $\dot{K}_\pm^n(\rho,t,u)$ approximates the expression $\frac{1}{2}\kappa'\big(\Sigma_u^n \rho(t,u)\big) \pm \tau\big(\Sigma_u^n \rho(t,u)\big)$, respectively, where the function $\tau : G \to I\!R$ is given by

$$\tau(x) = \frac{f''\big(g^{-1}(x)\big)}{f^3\big(g^{-1}(x)\big)} \int \theta(1-\theta)\big(\mu - \lambda\big)(d\theta) \quad \text{for all } x \in G.$$

Thus, the sum of the terms $\dot{K}_\pm^n\big(\rho,t,u\big)$ approximates $\kappa'\big(\Sigma_u^n \rho(t,u)\big)$.

Remark. The reader might be puzzled why we talk about approximations of e.g. $\kappa\big(\Sigma_u^n \rho(t,u)\big)$ and not of approximations for $\kappa\big(\rho(t,u)\big)$. The reason is simple: It follows from the definition of $\mathcal{A}^n$ and $\mathcal{A}_1^n(n-1)$ in Definition 3.7 that $(t,u) \in \mathcal{A}_1^n(n-1)$ implies $(t,u) \notin \mathcal{A}^n$, hence $\rho(t,u)$ is not defined if ρ is only defined on the grid $\mathcal{A}^n$. On the other hand $\Sigma_u^n \rho(t,u)$ is defined for all functions $\rho : \mathcal{A}^n \to I\!R$, and if for all $n \in I\!N$ the functions $\rho = \rho^n : \mathcal{A}^n \to I\!R$ can be extended to the same smooth function on $[0,T] \times I\!R$, then the expression $\Sigma_u^n \rho(t,u)$ will give an order $O\big(\delta^2\big)$-approximation for $\rho(t,u)$ as $n \to \infty$. □

With the definitions of $K_\pm^n\big(\rho,t,u\big)$ and so forth in mind, we should assure ourselves that our approximations really approximate those values which we claim that they approximate. This will be carried out in the next lemma.

Lemma 3.25. *Assume $f \in H_{loc}^{4+\beta}(I\!R)$ and that the measure μ fulfills $\int e^{\eta|\theta|}\mu(d\theta) < \infty$ for some $\eta > 0$. Moreover, suppose that $\rho \in C_b^{2,4}\big([0,T] \times I\!R\big)$ is some function with a range $\rho\big([0,T] \times I\!R\big)$ which is contained in $G := g(I\!R)$. Then we have*

$$K_\pm^n\big(\rho,t^n,u^n\big) = \kappa\big(\rho(t^n,u^n)\big) + O\big(\delta^2\big), \tag{3.50}$$

$$\bar{K}^n\big(\rho,t^n,u^n\big) = \kappa\big(\rho(t^n,u^n)\big) + O\big(\delta^2\big), \tag{3.51}$$

and

$$\dot{K}_\pm^n(\rho,t^n,u^n) = \frac{1}{2}\kappa'\big(\rho(t^n,u^n)\big) \pm \tau\big(\rho(t^n,u^n)\big) + O\big(\delta^2\big) \tag{3.52}$$

as $n \to \infty$ uniformly for all sequences $\big\{(t^n,u^n)\big\}_{n\in I\!N}$ for which $(t^n,u^n) \in \mathcal{A}_1^n(n-1)$ for all $n \in I\!N$.

Proof. Let us set $h = g^{-1}$ to denote the inverse $h : G \to I\!R$ of the strategy transform $g : I\!R \to I\!R$, and introduce the integral function $I : I\!R^2 \to I\!R$ by

$$I(\xi_1,\xi_2) = \int f\big((1-\theta)\xi_1 + \theta\xi_2\big)\big(\mu - \lambda\big)(d\theta) \quad \text{for all } \xi_1,\xi_2 \in I\!R.$$

We will first show that the function $J : G^2 \to I\!R$ of (3.49) satisfies

$$J(x_1,x_2) = \frac{h(x_2)-h(x_1)}{x_2-x_1} I\big(h(x_1),h(x_2)\big) \quad \text{for all } x_1,x_2 \in I\!R \text{ with } x_1 \neq x_2. \tag{3.53}$$

For this purpose note that by the definition of $g : I\!R \to I\!R$, $\xi \mapsto \int_0^\xi f(x)\,dx$ we have

$$\frac{g(\xi_2)-g(\xi_1)}{\xi_2-\xi_1} = \frac{1}{\xi_2-\xi_1}\int_{\xi_1}^{\xi_2} f(q)\,dq = \int_0^1 f\big((1-\theta)\xi_1 + \theta\xi_2\big)\,d\theta = \int f\big((1-\theta)\xi_1 + \theta\xi_2\big)\lambda(d\theta)$$

for all $\xi_1,\xi_2 \in I\!R$ with $\xi_1 \neq \xi_2$, since λ is the Lebesgue measure concentrated on $[0,1]$. Now $g : I\!R \to I\!R$ is strictly increasing, hence the representation (3.53) follows from the definition of $h = g^{-1}$ and the local transaction loss rate function $k_\mu : I\!R^2 \to I\!R$ in (2.4.6). Moreover, we have $I(\xi,\xi) = f(\xi)\int(\mu-\lambda)(d\theta) = 0$ for all $\xi \in I\!R$ since μ and λ are probability measures, and thus we may also write $J(x,x) = 0 = h'(x)I\big(h(x),h(x)\big)$ for all $x \in G$.

Next we calculate the derivatives of the function $I : I\!R^2 \to I\!R$. Since $f \in H^{4+\beta}(I\!R)$, since there exists exists some $\eta > 0$ such that $\int e^{\eta|\theta|}\mu(d\theta) < \infty$, and since λ is concentrated on the compact set $[0,1]$, we conclude as in Theorem A.9.1 of Durrett (1996) that the function I is four times continuously differentiable, and for $0 \le k \le 4$ the partial derivatives with respect to one variable only can be calculated as

$$\frac{\partial^k}{\partial \xi_2^k} I(\xi_1,\xi_2) = \int \theta^k f^{(k)}\big((1-\theta)\xi_1 + \theta\xi_2\big)\big(\mu-\lambda\big)(d\theta)$$

and

$$\frac{\partial^k}{\partial \xi_1^k} I(\xi_1,\xi_2) = \int (1-\theta)^k f^{(k)}\big((1-\theta)\xi_1 + \theta\xi_2\big)\big(\mu-\lambda\big)(d\theta) \quad \text{for all } \xi_1,\xi_2 \in I\!R.$$

Since $\int 1\big(\mu-\lambda\big)(d\theta) = 0$, the first derivatives at the point (ξ,ξ) become:

$$\frac{\partial}{\partial \xi_2} I(\xi_1,\xi_2)\bigg|_{\substack{\xi_1=\xi\\ \xi_2=\xi}} = -\frac{\partial}{\partial \xi_1} I(\xi_1,\xi_2)\bigg|_{\substack{\xi_1=\xi\\ \xi_2=\xi}} = f'(\xi)\int \theta\big(\mu-\lambda\big)(d\theta) = f'(\xi)d(\mu) \quad \text{for all } \xi \in I\!R.$$

Similarly we see from $\int \theta^2\big(\mu-\lambda\big)(d\theta) = \int \theta - \theta(1-\theta)\big(\mu-\lambda\big)(d\theta)$ that for all $\xi \in I\!R$

$$\frac{\partial^2}{\partial \xi_2^2} I(\xi_1,\xi_2)\bigg|_{\substack{\xi_1=\xi\\ \xi_2=\xi}} = f''(\xi)\int \theta^2\big(\mu-\lambda\big)(d\theta) = f''(\xi)d(\mu) - f''(\xi)\int \theta(1-\theta)\big(\mu-\lambda\big)(d\theta),$$

and by analogy the second derivative with respect to the first variable becomes

$$-\frac{\partial^2}{\partial \xi_1^2} I(\xi_1,\xi_2)\bigg|_{\substack{\xi_1=\xi\\ \xi_2=\xi}} = f''(\xi)d(\mu) + f''(\xi)\int \theta(1-\theta)\big(\mu-\lambda\big)(d\theta) \quad \text{for all } \xi \in I\!R.$$

After this preparatory work we can show the convergence results (3.50)-(3.52). Without loss of generality we assume that $\Delta_u^n\,\rho(t^n,u^n) \neq 0$ for all sufficiently large $n \in I\!N$. Let us start with the expression $K_-^n(\rho,t,u)$, and take any sequence $\big\{(t^n,u^n)\big\}_{n\in I\!N}$ with $(t^n,u^n) \in \mathcal{A}_1^n(n-1)$ for all $n \in I\!N$. By the definitions of $K_-^n(\rho,t,u)$ and $\Delta_u^n\,\rho(t^n,u^n)$ we see that for all sufficiently large $n \in I\!N$

$$\begin{aligned} K_-^n(\rho,t^n,u^n) &= \frac{J_-^+(\rho,t^n,u^n) - J_-^-(\rho,t^n,u^n)}{\delta\Delta_u^n\,\rho(t^n,u^n)} \\ &= 2\frac{J\big(\rho(t^n-\delta^2,u^n),\rho(t^n,u^n+\delta)\big) - J\big(\rho(t^n-\delta^2,u^n),\rho(t^n,u^n-\delta)\big)}{\rho(t^n,u^n+\delta)-\rho(t^n,u^n-\delta)}. \end{aligned}$$

Since $I : I\!R^2 \to I\!R$ is four times continuously differentiable, it is easily seen from the representation (3.53) that $J : G^2 \to I\!R$ is four times continuously differentiable as well. Since $\rho : [0,T]\times I\!R \to I\!R$ is bounded, there exists some compact interval $D \subset I\!R$ such that $\rho(t,u) \in D$ for all $(t,u) \in [0,T]\times I\!R$. Hence we especially get that the maximum of the third derivative of $J : G^2 \to I\!R$ in the compact set D^2 is attained, and by making a Taylor expansion around the point $\rho(t^n,u^n)$ we can conclude

$$K_-^n(\rho,t^n,u^n) = 2\frac{\partial}{\partial x_2}J(x_1,x_2)\bigg|_{\substack{x_1=\rho(t^n,u^n)\\ x_2=\rho(t^n,u^n)}} + O\big(\delta^2\big) = 2\frac{f'\big(g^{-1}(\rho(t^n,u^n))\big)}{f^2\big(g^{-1}(\rho(t^n,u^n))\big)}d(\mu) + O\big(\delta^2\big),$$

as $n \to \infty$, uniformly for all sequences $\{(t^n, u^n)\}_{n \in I\!N}$ with $(t^n, u^n) \in \mathcal{A}_1^n(n-1)$ for all $n \in I\!N$. By the definition of $\kappa : I\!R \to I\!R$ in (3.1) the leading term on the right-hand side can be identified as $\kappa\big(\rho(t^n, u^n)\big)$, hence the lower (minus) case of (3.50) is shown.
The statements for $K_+^n(\rho, t^n, u^n)$ and $\bar{K}^n(\rho, t^n, u^n)$ follow by similar means. Now let us consider (3.52) and exemplarily show the convergence of $\dot{K}_-^n(\rho, t^n, u^n)$. By definition, we obtain for all $(t, u) \in \mathcal{A}_1^n(n-1)$:

$$\dot{K}_-^n(\rho, t, u) = 8 \frac{\rho(t-\delta^2, u) - \rho(t, u-\delta)}{\big(\rho(t, u+\delta) - \rho(t, u-\delta)\big)^3} J\big(\rho(t-\delta^2, u), \rho(t, u+\delta)\big) + 8 \frac{\rho(t, u+\delta) - \rho(t-\delta^2, u)}{\big(\rho(t, u+\delta) - \rho(t, u-\delta)\big)^3} J\big(\rho(t-\delta^2, u), \rho(t, u-\delta)\big).$$

If we expand the right-hand side of this equation around $\rho(t, u)$, and use $J(x_1, x_2) = 0$ and $\frac{\partial}{\partial x_2} J(x_1, x_2) = -\frac{\partial}{\partial x_1} J(x_1, x_2)$ for all $x_1, x_2 \in G$ with $x_1 = x_2$, we obtain

$$\dot{K}_-^n(\rho, t, u) = \frac{\partial^2}{\partial x_2^2} J(x_1, x_2)\bigg|_{\substack{x_1 = \rho(t^n, u^n) \\ x_2 = \rho(t^n, u^n)}} + O\big(\delta^2\big) \quad \text{as } n \to \infty.$$

Now it can be shown that for all $x \in G$ we have

$$\frac{\partial^2}{\partial x_2^2} J(x_1, x_2)\bigg|_{\substack{x_1 = x \\ x_2 = x}} = 2h'(x)h''(x) \frac{\partial}{\partial \xi_2} I\big(\xi_1, \xi_2\big)\bigg|_{\substack{\xi_1 = h(x) \\ \xi_2 = h(x)}} + \big(h'(x)\big)^3 \frac{\partial^2}{\partial \xi_2^2} I\big(\xi_1, \xi_2\big)\bigg|_{\substack{\xi_1 = h(x) \\ \xi_2 = h(x)}}.$$

Since $h'(x) = \frac{1}{f(h(x))}$ and $h''(x) = -\frac{f'(h(x))}{f^3(h(x))}$ for all $x \in G$, the formulas for $\frac{\partial}{\partial \xi_2} I(\xi_1, \xi_2)$ and $\frac{\partial^2}{\partial \xi_2^2} I(\xi_1, \xi_2)$ on the diagonal $\xi_1 = \xi_2$ lead to

$$\frac{\partial^2}{\partial x_2^2} J(x_1, x_2)\bigg|_{\substack{x_1 = x \\ x_2 = x}} = \left(\frac{f''(h(x))}{f^3(h(x))} - 2 \frac{\big(f'(h(x))\big)^2}{f^4(h(x))} \right) d(\mu) - \frac{f''(h(x))}{f^3(h(x))} \int \theta(1-\theta)\big(\mu - \lambda\big)(d\theta),$$

and if we differentiate the function $\kappa : I\!R \to I\!R$ in (3.1) and use the definitions of $h : G \to I\!R$ and $\tau : G \to I\!R$, the last line becomes $\frac{1}{2}\kappa'(x) - \tau(x)$. Thus we have derived the convergence statement for $\dot{K}_-^n(\rho, t^n, u^n)$ as stated in (3.52). **q.e.d.**

Before we can rewrite the fixed point equation (2.4.11) into a discrete analogue of the partial differential equation (3.48) we still need to introduce some more definitions to bring the analogy out most clearly. For this reason, let us recall Definition 3.7, and by slightly extending this definition to functions which are defined on the whole time slab $[0, T] \times I\!R$, let us define the discrete derivative $\Delta_u^n \overline{\psi} : \mathcal{A}_1^n \to I\!R$ as the discrete derivative of the restriction of $\overline{\psi} : [0, T] \times I\!R \to I\!R$ on $\mathcal{A}^n$, i.e.

$$\Delta_u^n \overline{\psi}(t, u) = \frac{\overline{\psi}\big(t, u+\delta\big) - \overline{\psi}(t, u-\delta)}{2\delta} \quad \text{for all } (t, u) \in \mathcal{A}_1^n.$$

In a similar manner we also define

$$\Sigma_u^n \overline{\psi}(t, u) = \frac{\overline{\psi}\big(t, u+\delta\big) + \overline{\psi}(t, u-\delta)}{2} \quad \text{for all } (t, u) \in \mathcal{A}_1^n.$$

Additionally, we will now define the operator $\Delta_{uu,t}^n$ on the space of functions $h : \mathcal{A}^n \to I\!R$ by setting for all $(t, u) \in \mathcal{A}_1^n(n-1)$

$$\Delta_{uu,t}^n h(t, u) = \frac{1}{\delta^2}\Big(h(t, u+\delta) - 2h(t+\delta^2, u) + h(t, u-\delta)\Big), \tag{3.54}$$

and then $\Delta^n_{uu,t}\,\overline{\psi}(t,u)$ analogously. It is clear that $\Delta^n_{uu,t}$ is a discrete version of the differential operator $\frac{\partial^2}{\partial u^2} - 2\frac{\partial}{\partial t}$.

Now let us choose for each $n \in I\!N$ some $k = k(n)$ and $i = i(n)$ with $1 \le k \le n-1$ and $i \in \mathcal{I}_{k-1}$, and set $(t^n, u^n) := \big(t^n_k, u^n_{(k-1)i}\big)$. By the definition of $\mathcal{A}^n_1(n-1)$ in Definition 3.7 it follows that $(t^n, u^n) \in \mathcal{A}^n_1(n-1)$. Observe that by the definition of $u^n_{(k\pm1)i}$ in (1.3.8) we also have $\big(t^n_k, u^n_{(k+1)i}\big) = (t^n, u^n)$. We will omit the subscript n in t^n and u^n from now on, but notice that the point $(t,u) \in \mathcal{A}^n_1(n-1)$ changes for varying $n \in I\!N$.

If we look back to Corollary 2.15 and recall our shorthands (1.3.12) and the definition of the strategy transform $g^n : \mathcal{A}^n \to I\!R$ in (2.4.5), we just bring $\int_0^{\xi^n_{(k-1)i}} f(x)\,dx = g^n(t-\delta^2, u)$ to the other side of the equality sign to rewrite the fixed point equation (2.4.11) as

$$0 = \frac{\overline{\psi}(t,u+\delta) - \overline{\psi}(t+\delta^2,u)}{\overline{\psi}(t,u+\delta) - \overline{\psi}(t,u-\delta)}\Big(g^n(t,u+\delta) - g^n(t-\delta^2,u)\Big)$$
$$+ \frac{\overline{\psi}(t+\delta^2,u) - \overline{\psi}(t,u-\delta)}{\overline{\psi}(t,u+\delta) - \overline{\psi}(t,u-\delta)}\Big(g^n(t,u-\delta) - g^n(t-\delta^2,u)\Big) + \frac{D^{\xi^n}_\mu(t,u)}{\overline{\psi}(t,u+\delta) - \overline{\psi}(t,u-\delta)}.$$

Here the spread $D^{\xi^n}_\mu(t,u)$ of the transaction losses involved is given by (2.4.12).

Let us first consider the first two summands which do not involve the term $D^{\xi^n}_\mu(t,u)$. Upon using the equality $ac - bd = \frac{1}{2}(a+b)(c-d) + \frac{1}{2}(a-b)(c+d)$ and our definitions of the discrete derivatives, we can rewrite this part as

$$L^n(t,u) := \Big(g^n(t,u+\delta) - 2g^n\big(t-\delta^2,u\big) + g^n(t,u-\delta)\Big) + \delta^2\frac{\Delta^n_{uu,t}\,\overline{\psi}(t,u)}{2\Delta^n_u\overline{\psi}(t,u)}\Delta^n_u\,g^n(t,u). \tag{3.55}$$

Notice that the fraction $\frac{\Delta^n_{uu,t}\,\overline{\psi}(t,u)}{\Delta^n_u\,\overline{\psi}(t,u)}$ is an $O\big(\delta^2\big)$-approximation of $\frac{\overline{\psi}_{uu}(t,u) - 2\overline{\psi}_t(t,u)}{\overline{\psi}_u(t,u)}$ as $n \to \infty$. If $\|g^n - \gamma\|_{\mathcal{A}^n} = O\big(\delta^{2+\beta}\big)$ as $n \to \infty$ for some function $\gamma \in C^{1,2}_b\big([0,T] \times I\!R\big)$, we also have $\big\|\Delta^n_u\,g^n(t,u) - \gamma_u(t,u)\big\|_{\mathcal{A}^n_1} = O\big(\delta^{1+\beta}\big)$ and

$$\left\|\frac{1}{2\delta^2}\Big(g^n(\cdot,\cdot+\delta) - 2g^n\big(\cdot-\delta^2,\cdot\big) + g^n(\cdot,\cdot-\delta)\Big) - \Big(\gamma_t + \frac{1}{2}\gamma_{uu}\Big)\right\|_{\mathcal{A}^n_1} = O\big(\delta^\beta\big) \quad \text{as } n \to \infty.$$

Let us now suppose for a moment that we are in the linear setting of Section 3.2 such that there are no implied transaction losses, i.e. that the term $D^{\xi^n}_\mu(t,u)$ vanishes. Then we have $L^n(t,u) = 0$, and $\|g^n - \gamma\|_{\mathcal{A}^n} = O\big(\delta^{2+\beta}\big)$ as $n \to \infty$ and (3.55) imply that the function $\gamma : [0,T] \times I\!R \to I\!R$ has to satisfy the partial differential equation (2.3).

For the general non-linear case, we now show that the term $D^{\xi^n}_\mu(t,u)$ can equally be expressed in terms similar to the ones in (3.55), so that we can indeed rewrite (2.4.11) into a discrete version of (3.48). To this end, we have to separate another term from $D^{\xi^n}_\mu(t,u)$ which also involves the expression $g^n(t,u+\delta) - 2g^n\big(t-\delta^2,u\big) + g^n(t,u-\delta)$. For this purpose, recall the definition of $D^{\xi^n}_\mu(t,u)$ in (2.4.12) and the function $J : G^2 \to I\!R$ of (3.49). Then we can rewrite

$$\frac{D^{\xi^n}_\mu(t,u)}{\overline{\psi}(t,u+\delta) - \overline{\psi}(t,u-\delta)} = T^n_1(t,u) + T^n_2(t,u), \tag{3.56}$$

where the terms $T^n_1(t,u)$ and $T^n_2(t,u)$ are given by

$$T^n_1(t,u) = \frac{\overline{\psi}(t+\delta^2,u)}{2\delta\Delta^n_u\,\overline{\psi}(t,u)}\Big(J\big(g^n(t,u+\delta), g^n(t+\delta^2,u)\big)\big(g^n(t+\delta^2,u) - g^n(t,u+\delta)\big)$$
$$- J\big(g^n(t,u-\delta), g^n(t+\delta^2,u)\big)\big(g^n(t+\delta^2,u) - g^n(t,u-\delta)\big)\Big)$$

and

$$T_2^n(t,u) = \frac{1}{2\delta\Delta_u^n\,\overline{\psi}(t,u)}\Big(\overline{\psi}\big(t,u{+}\delta\big)J\big(g^n(t{-}\delta^2,u),g^n(t,u{+}\delta)\big)\big(g^n(t,u{+}\delta)-g^n(t{-}\delta^2,u)\big)$$
$$-\,\overline{\psi}\big(t,u{-}\delta\big)J\big(g^n(t{-}\delta^2,u),g^n(t,u{+}\delta)\big)\big(g^n(t,u{-}\delta)-g^n(t{-}\delta^2,u)\big)\Big).$$

With our approximations for κ and κ' on hand, we can find difference representations for $T_1^n(t,u)$ and $T_2^n(t,u)$ which resemble (3.55). Let us first consider the term $T_1^n(t,u)$. By an application of the equality $ac-bd=(a-b)(c+d)-ad+bc$, we can express it in terms of $K_+^n(g^n,t,u)$, $\dot K_+^n(g^n,t,u)$ and $\Delta_{uu,t}^n\,g^n(t,u)$ from (3.54) by

$$T_1^n(t,u) = \frac{1}{2}\delta^2\frac{\overline{\psi}(t{+}\delta^2,u)}{\Delta_u^n\,\overline{\psi}(t,u)}\Delta_u^n\,g^n(t,u)\Big(K_+^n(g^n,t,u)\Delta_{uu,t}^n\,g^n(t,u)+\dot K_+^n(g^n,t,u)\big(\Delta_u^n\,g^n(t,u)\big)^2\Big).$$

The term $T_2^n(t,u)$ is slightly more complicated since it involve the two prices $\overline{\psi}(t,u\pm\delta)$. However, if we use the equality $ac-bd=\frac{1}{2}(a+b)(c-d)+\frac{1}{2}(a-b)(c+d)$ and the definition of $\bar K^n(\rho,t,u)$, we get

$$\begin{aligned}T_2^n(t,u) := {}&\frac{\Sigma_u^n\,\overline{\psi}(t,u)}{2\delta\Delta_u^n\,\overline{\psi}(t,u)}\Big(J\big(g^n(t{-}\delta^2,u),g^n(t,u{+}\delta)\big)\big(g^n(t,u{+}\delta)-g^n(t{-}\delta^2,u)\big)\\ &\qquad-J\big(g^n(t{-}\delta^2,u),g^n(t,u{+}\delta)\big)\big(g^n(t,u{-}\delta)-g^n(t{-}\delta^2,u)\big)\Big)\\ &+\frac{1}{2}\delta^2\bar K^n(g^n,t,u)\big(\Delta_u\,g^n(t,u)\big)^2\end{aligned}$$

The first line of the last equation has now the same structure as the term D_1, and we can again use the equality $ac-bd=(c+d)(a-b)+bc-ad$ to achieve:

$$\begin{aligned}T_2^n(t,u) = {}&\frac{1}{2}\frac{\Sigma_u^n\,\overline{\psi}(t,u)}{\Delta_u^n\,\overline{\psi}(t,u)}K_-^n(g^n,t,u)\Delta_u^n\,g^n(t,u)\big(g^n(t,u+\delta)-2g^n(t-\delta^2,u)+g^n(t,u-\delta)\big)\\ &+\frac{1}{2}\delta^2\big(\Delta_u^n\,g^n(t,u)\big)^2\left(\bar K^n(g^n,t,u)+\frac{\Sigma_u^n\,\overline{\psi}(t,u)}{\Delta_u^n\,\overline{\psi}(t,u)}\dot K_-^n(g^n,t,u)\Delta_u^n\,g^n(t,u)\right).\end{aligned}$$

Since we have $L^n(t,u)+T_1^n(t,u)+T_2^n(t,u)=0$, we can divide this equation by δ^2 to see that it is equivalent to the difference equation

$$0 = D^n(t,u)\frac{1}{2\delta^2}\big(g^n(t,u+\delta)-2g^n(t-\delta^2,u)+g^n(t,u-\delta)\big)+\frac{1}{2}\Delta_u^n\,g^n(t,u)N^n(t,u), \quad (3.57)$$

where the two functions $D^n:\mathcal{A}_1^n(n-1)\to I\!R$ and $N^n:\mathcal{A}_1^n(n-1)\to I\!R$ are for all $n\in I\!N$ and all $(t,u)\in\mathcal{A}_1^n(n-1)$ defined by setting

$$D^n(t,u) = 1+\frac{\Sigma_u^n\,\overline{\psi}(t,u)}{\Delta_u^n\,\overline{\psi}(t,u)}K_-^n(g^n,t,u)\Delta_u^n\,g^n(t,u)$$

and

$$\begin{aligned}N^n(t,u) = {}&\frac{\Delta_{uu,t}^n\,\overline{\psi}(t,u)}{\Delta_u^n\,\overline{\psi}(t,u)}+\bar K^n(g^n,t,u)\Delta_u^n\,g^n(t,u)+\frac{\overline{\psi}(t+\delta^2,u)}{\Delta_u^n\,\overline{\psi}(t,u)}K_+^n(g^n,t,u)\Delta_{uu,t}^n\,g^n(t,u)\\ &+\left(\frac{\Sigma_u^n\,\overline{\psi}(t,u)}{\Delta_u^n\,\overline{\psi}(t,u)}\dot K_-^n(g^n,t,u)+\frac{\overline{\psi}(t+\delta^2,u)}{\Delta_u^n\,\overline{\psi}(t,u)}\dot K_+^n(g^n,t,u)\right)\big(\Delta_u^n\,g^n(t,u)\big)^2.\end{aligned}$$

The expression $D^n(t,u)$ approximates the denominator of $b^\gamma(t,u)$ in (3.48). Once we have shown that $D^n(t,u) > 0$ for all sufficiently large n, we can divide the equation (3.57) by $D^n(t,u)$ and obtain

$$\frac{1}{2\delta^2}\big(g^n(t,u+\delta) - 2g^n(t-\delta^2,u) + g^n(t,u-\delta)\big) + \Delta_u^n g^n(t,u)B^n(t,u) = 0, \tag{3.58}$$

where $B^n : \mathcal{A}_1^n(n-1) \to I\!R$ is for all $(t,u) \in \mathcal{A}_1^n(n-1)$ defined as the ratio $B^n(t,u) = \frac{1}{2}\frac{N^n(t,u)}{D^n(t,u)}$. The difference equation (3.58) is now easily seen to be a discrete version of the PDE (3.48): If $\|g^n - \gamma\|_{\mathcal{A}^n} = O\big(\delta^{2+\beta}\big)$ as $n \to \infty$, then we could conclude from Taylor's rule and Lemma 3.25 that uniformly for all $(t,u) \in [0,T] \times I\!R$, we have

$$\begin{aligned}
\frac{1}{\delta^2}\big(g^n(t,u+\delta) - 2g^n(t-\delta^2,u) + g^n(t,u-\delta)\big) &= \gamma_t(t,u) + \frac{1}{2}\gamma_{uu}(t,u) + O\big(\delta^\beta\big),\\
\Delta_u^n g^n(t,u) &= \gamma_u(t,u) + O\big(\delta^{1+\beta}\big),\\
B^n(t,u) &= b^\gamma(t,u) + O\big(\delta^\beta\big),
\end{aligned}$$

and it would also suffice to have (3.46) and $\|g^n - \gamma\|_{\mathcal{A}^n} = O\big(\delta^2\big)$ as $n \to \infty$.

In order to show that $D^n(t,u) > 0$ for all sufficiently large $n \in I\!N$, note that by the definitions of $K_-^n(\rho,t,u)$ we have $K_-^n\big(g^n,t,u\big) = \kappa\big(\Sigma_u^n g^n(t,u)\big)$ if $\Delta_u^n g^n(t,u) = 0$, and otherwise the definitions of $J : G^2 \to I\!R$ in (3.49), and of $g^n = g \circ \xi^n : \mathcal{A}^n \to I\!R$ in (2.4.5) lead to

$$\begin{aligned}
K_-^n\big(g^n,t,u\big) = \operatorname{sgn}&\left(\frac{g^n(t,u+\delta) - g^n(t-\delta^2,u)}{g^n(t,u+\delta) - g^n(t,u-\delta)}\right)\frac{k_\mu\big(\xi^n(t-\delta^2,u),\xi^n(t,u+\delta)\big)}{\delta\big|\Delta_u^n g^n(t,u)\big|}\\
+\operatorname{sgn}&\left(\frac{g^n(t-\delta^2,u) - g^n(t,u-\delta)}{g^n(t,u+\delta) - g^n(t,u-\delta)}\right)\frac{k_\mu\big(\xi^n(t-\delta^2,u),\xi^n(t,u+\delta)\big)}{\delta\big|\Delta_u^n g^n(t,u)\big|}.
\end{aligned}$$

Hence Assumption D implies that K_-^n is nonnegative as long as

$$g^n(t,u-\delta) \le g^n(t-\delta^2,u) \le g^n(t,u+\delta)$$

or equivalently

$$\xi^n(t,u-\delta) \le \xi^n(t-\delta^2,u) \le \xi^n(t,u+\delta). \tag{3.59}$$

Since $\overline{\psi} : [0,T] \times I\!R \to I\!R$ is nonnegative as well, we can conclude that $D^n(t,u) \ge 1$ for all $(t,u) \in \mathcal{A}_1^n(n-1)$ if the strategy function $\xi^n : \mathcal{A}^n \to I\!R$ satisfies the interlocking property (2.1.6), which in particular holds for the replication strategy of all star-convex contingent claims.

The replication strategy of a star-concave contingent claim, however, satisfies

$$\xi^n(t,u-\delta) \ge \xi^n(t-\delta^2,u) \ge \xi^n(t,u+\delta) \tag{3.60}$$

for all $(t,u) \in \mathcal{A}_1^n(n-1)$ and we get $D^n(t,u) \le 1$ for all $(t,u) \in \mathcal{A}_1^n(n-1)$. If the contingent claim is neither star-convex nor star-concave we might have (3.59) for some $(t,u) \in \mathcal{A}_1^n(n-1)$ and (3.60) for others. In all these cases we know that for all sufficiently small α, a continuous solution to (3.2), (3.3) exists. Starting with this solution it can then be shown that for some sufficiently large $M \in I\!N$ we can bound $D^n(t,u) \ge \frac{1}{2}\varepsilon$, uniformly for all $n \ge M$ and all $(t,u) \in \mathcal{A}_1^n(n-1)$. We will come back to star-concave trading strategies at the end of Section 3.3.3.

The difference equation (3.58) can now also serve as the starting point to derive for all sufficiently large $n \in I\!N$ suitable probability weight functions $\check{p}^n : \mathcal{A}^n(n-2) \to [0,1]$

which satisfy the representation (3.47): We can define such weight functions $\check{p}^n$ by setting $\check{p}^n(t,u) = \frac{1}{2}\big(1 + \delta B^n(t+\delta^2,u)\big)$ for all $(t,u) \in \mathcal{A}^n(n-2)$. If we multiply (3.58) again by δ^2, bring $g^n(t-\delta^2,u)$ to the other side of the equality sign, and use the definition of $\Delta_u^n\, g(t,u)$ we see that (3.58) is indeed equivalent to

$$g^n(t-\delta^2,u) = \check{p}^n\big(t-\delta^2,u\big)g^n(t,u+\delta) + \big(1-\check{p}^n(t-\delta^2,u)\big)g^n(t,u-\delta),$$

for all $(t,u) \in \mathcal{A}_1^n(n-1)$. Moreover, for all sufficiently large $n \in I\!N$ the range of $\check{p}^n$ is indeed contained in $[0,1]$, since it can be inductively shown that $\big\|B^n\big\|_{\mathcal{A}_1^n(n-1)}$ is uniformly bounded for all $n \in I\!N$ for which it is defined because of the boundedness of the solution $\gamma : [0,T] \times I\!R \to I\!R$ to (3.48) and the boundedness of its derivatives γ_u, γ_{uu}, and γ_t.
Using now basically the same arguments as in Section 3.2.2 we arrive at the representation

$$\begin{aligned} g^n(t_{k-1}^n,u) - \gamma(t_{k-1}^n,u) &= \check{p}^n(t_{k-1}^n,u)\big(g^n(t_k^n,u+\delta) - \gamma(t_k^n,u+\delta)\big) \\ &\quad + \big(1-\check{p}^n(t_{k-1}^n,u)\big)\big(g^n(t_k^n,u-\delta) - \gamma(t_k^n,u-\delta)\big) + O\big(\delta^4\big), \end{aligned}$$

as $n \to \infty$, uniformly for all $1 \le k \le n-1$ and $u \in \mathcal{U}_{k-1}^n$, and this allows us to argue again that the convergence of g^n immediately before maturity implies the convergence of g^n to γ in all prior time points. Thus we get (3.45). Similarly, we can transfer the arguments used to derive (2.15) to the general case with transaction losses in order to prove (3.46), but an exact proof becomes technically cumbersome.
This ends our outline of the proof of Theorem 3.24.

3.3.3 Convergence of the Strategy Functions

Having obtained a convergence result for a sequence of transformed strategy functions, we can now use the strategy transform to transfer that convergence result back into a convergence statement for the associated sequence of original strategy functions. We first state the non-linear final value problem for the limiting strategy function. Existence and uniqueness results for this problem are immediately derived from the corresponding results for the limiting transformed strategy. Then the convergence result for the strategy functions follows as a corollary to Theorem 3.24.

In order to rewrite the final value problem (3.2), (3.3) for the limit $\gamma : [0,T] \times I\!R \to I\!R$ of the transformed strategies $g^n : \mathcal{A}^n \to I\!R$ into a final value problem for the limit $\varphi : [0,T] \times I\!R \to I\!R$ of the strategy functions $\xi^n : \mathcal{A}^n \to I\!R$ we proceed as in Section 3.2.1, and therefore recall the strategy transform $g : I\!R \to I\!R$ of Definition 2.12. Since we have $g^n = g \circ \xi^n$ for all $n \in I\!N$ we obtain for the limits

$$\gamma(t,u) = g\big(\varphi(t,u)\big) = \int_0^{\varphi(t,u)} f(x)\,dx \quad \text{for all } (t,u) \in [0,T] \times I\!R. \tag{3.61}$$

If we now apply Assumption B, the quasi-linear partial differential equation (3.2) can be rewritten in terms of the limiting strategy function φ as

$$\begin{aligned} &\varphi_t(t,u) + \frac{1}{2}\frac{d}{du}\left(\left(1+2d(\mu)\frac{\psi_\xi\big(t,u,\varphi(t,u)\big)}{\psi_u\big(t,u,\varphi(t,u)\big)}\varphi_u(t,u)\right)\varphi_u(t,u)\right) \\ &= \varphi_u(t,u)\left(\frac{\psi_t\big(t,u,\varphi(t,u)\big)}{\psi_u\big(t,u,\varphi(t,u)\big)} - \frac{1}{2}\frac{\frac{d}{du}\psi_u\big(t,u,\varphi(t,u)\big)}{\psi_u\big(t,u,\varphi(t,u)\big)}\left(1+2d(\mu)\frac{\psi_\xi\big(t,u,\varphi(t,u)\big)}{\psi_u\big(t,u,\varphi(t,u)\big)}\varphi_u(t,u)\right)\right). \end{aligned} \tag{3.62}$$

It is obvious that (3.62) generalizes the linear PDE (2.5), and we can directly state the non-linear analogue of Proposition 3.9:

Proposition 3.26. *In addition to the Assumptions B, C, and D suppose that* $f \in H^{k+\beta}_{loc}(\mathbb{R})$ *for some* $k \geq 2$ *and* $\beta \in (0,1)$. *Then there exists a solution* $\gamma \in H^{\frac{1}{2}(k+\beta),k+\beta}\big([0,T] \times \mathbb{R}\big)$ *of the final value problem* (3.2), (3.3) *with scaling parameter* $\alpha = 1$ *if and only if there exists a solution* $\varphi \in H^{\frac{1}{2}(k+\beta),k+\beta}\big([0,T] \times \mathbb{R}\big)$ *of* (3.62) *with final condition given by* $\varphi(T,u) = \zeta(u)$ *for all* $u \in \mathbb{R}$, *and two such solutions are connected via* (3.61).

Remark 1. Note that we need one more derivative of $f : \mathbb{R} \to \mathbb{R}$ than in Proposition 3.9, since the derivative of the transformed loss function $\kappa : \mathbb{R} \to \mathbb{R}$ contains the second derivative of $f : \mathbb{R} \to \mathbb{R}$. □

Remark 2. Of course, we can also rewrite the PDE (3.62), which is given in divergence form, into a PDE of a form similar to (3.48). For this purpose, let us define the drift coefficient $b^\varphi : [0,T] \times \mathbb{R} \to \mathbb{R}$ for all $(t,u) \in [0,T] \times \mathbb{R}$ by

$$\begin{aligned} b^\varphi(t,u) = \; & \frac{\frac{1}{2}\frac{d}{du}\big(\psi_u(t,u,\varphi(t,u)) + 2d(\mu)\psi_\xi(t,u,\varphi(t,u))\varphi_u(t,u)\big)}{\psi_u(t,u,\varphi(t,u)) + 2d(\mu)\psi_\xi(t,u,\varphi(t,u))\varphi_u(t,u)} \\ & - \frac{\psi_t(t,u,\varphi(t,u)) + 2d(\mu)\psi_\xi(t,u,\varphi(t,u))\varphi_t(t,u)}{\psi_u(t,u,\varphi(t,u)) + 2d(\mu)\psi_\xi(t,u,\varphi(t,u))\varphi_u(t,u)}. \end{aligned}$$

Then it can be shown by performing the same operations which led to (3.48) that the non-linear PDE (3.62) can be rewritten as

$$\varphi_t(t,u) + \frac{1}{2}\varphi_{uu}(t,u) + b^\varphi(t,u)\varphi_u(t,u) = 0 \quad \text{for all } (t,u) \in [0,T] \times \mathbb{R},$$

and this representation indeed resembles the form of (3.48). □

We consider once again the final value problem for (3.62) with the final condition chosen sufficiently small, and hence for fixed $\zeta : \mathbb{R} \to \mathbb{R}$ we look at solutions $\varphi : [0,T] \times \mathbb{R} \to \mathbb{R}$ to the PDE (3.62) which satisfy a boundary condition of the form

$$\varphi(T,u) = \tilde{\alpha}\zeta(u) \quad \text{for all } u \in \mathbb{R} \tag{3.63}$$

for some $\tilde{\alpha} \in \mathbb{R}$ which is sufficiently small. The parameter $\tilde{\alpha}$ serves the same purpose as the scaling parameter α of (3.3): If $|\tilde{\alpha}|$ is chosen sufficiently small, the norms $\big\| \int_0^{\tilde{\alpha}\zeta(u)} f(x) \big\|$ and $\big\| \frac{d}{du} \int_0^{\tilde{\alpha}\zeta(u)} f(x)\,dx \big\|$ become small enough so that the final value problem (3.2) with final condition $\gamma(T,u) = \int_0^{\tilde{\alpha}\zeta(u)} f(x)\,dx$ has a solution.

In order to state this in exact terms, let us define the bounds $\tilde{b}$ and $\widetilde{B}$ on the range of $\frac{1}{\tilde{\alpha}}\varphi : [0,T] \times \mathbb{R} \to \mathbb{R}$ by analogy with the bounds b and B by

$$\tilde{b} = \min\Big\{0, \inf_{u \in \mathbb{R}} \zeta(u)\Big\} \quad \text{and} \quad \widetilde{B} = \max\Big\{0, \sup_{u \in \mathbb{R}} \zeta(u)\Big\}.$$

Last but not least, let us also introduce the function $\tilde{L}'_\kappa : \mathbb{R} \to \mathbb{R}$ as an analogue of $L'_\kappa(\alpha)$ by setting

$$\tilde{L}'_\kappa(\tilde{\alpha}) := \big|2d(\mu)\big| \max\left\{ \left\| \frac{f'(\xi)}{f(\xi)} \right\|_{[\tilde{\alpha}\tilde{b},\tilde{\alpha}\widetilde{B}]}, |\tilde{\alpha}| \left\| \frac{d}{d\xi}\frac{f'(\xi)}{f(\xi)} \right\|_{[\tilde{\alpha}\tilde{b},\tilde{\alpha}\widetilde{B}]} \right\} \quad \text{for all } \tilde{\alpha} \in \mathbb{R}. \tag{3.64}$$

Then we can use Proposition 3.18 and its proof to derive a similar statement for the final value problem (3.62), (3.63), which again says that for sufficiently small values on the boundary $\{T\} \times \mathbb{R}$, the final value problem has a solution:

Proposition 3.27. *In addition to the Assumptions B, C, and D suppose that the two components* $\overline{\psi}$ *and* f *of* ψ *belong to the Hölder spaces* $\widehat{H}^{1+\frac{1}{2}\beta,3+\beta}\big([0,T]\times I\!R\big)$ *and* $H^{3+\beta}_{loc}(I\!R)$, *respectively, and suppose that* $\zeta \in H^{2+\beta}(I\!R)$. *Then there exist some* $\tilde{\alpha}_1, \tilde{\alpha}_2 \in I\!R$ *with* $\tilde{\alpha}_1 < 0 < \tilde{\alpha}_2$ *such that for all* $\tilde{\alpha} \in (\tilde{\alpha}_1, \tilde{\alpha}_2)$, *the final value problem* (3.62), (3.63) *has a solution* $\varphi \in H^{1+\frac{1}{2}\beta,2+\beta}\big([0,T]\times I\!R\big)$ *which satisfies*

$$\inf_{v\in I\!R} \tilde{\alpha}\zeta(v) \le \varphi(t,u) \le \sup_{v\in I\!R} \tilde{\alpha}\zeta(v) \quad \textit{for all } (t,u)\in[0,T]\times I\!R \tag{3.65}$$

and

$$2L_0\tilde{L}'_\kappa(\tilde{\alpha}) \inf_{(t,u)\in[0,T]\times I\!R} \varphi_u(t,u) > -1. \tag{3.66}$$

Moreover, for all $\tilde{\alpha} \in I\!R$ *there exists at most one solution* $\varphi \in C_b^{1,2}\big([0,T]\times I\!R\big)$ *of* (3.62), (3.63), *which satisfies* (3.66) *and* $\varphi(t,u) \in \big[\tilde{\alpha}\tilde{b}, \tilde{\alpha}\widetilde{B}\big]$ *for all* $(t,u)\in[0,T]\times I\!R$.

Remark. In Proposition 3.18 we had to limit the range of α to (ρ_1,ρ_2), since the transformed loss function $\kappa : g(I\!R) \to I\!R$ is only defined on $g(I\!R)$, which might be a true subset of $I\!R$. In (3.62) such a problem cannot occur, since $f : I\!R \to I\!R$ is defined for all $\xi \in I\!R$, hence we can state for example the uniqueness result for all $\tilde{\alpha}\in I\!R$. The definition of $\tilde{L}'_\kappa(\tilde{\alpha})$ can be justified if we once again run through the proof of Proposition 3.18 and adjust it in order to prove the existence of a solution of (3.62), (3.63) directly. As an indication why we have to replace $L'_\kappa(\alpha)$ by $\tilde{L}'_\kappa(\tilde{\alpha})$, let us note that for sufficiently small $|\alpha| > 0$ the expression $L'_\kappa(\alpha)$ of (3.4) will be dominated by $L_\kappa(\alpha) = \|\kappa\|_{[\alpha b,\alpha B]} = \big\|\frac{f(g^{-1}(\,\cdot\,))}{f^2(g^{-1}(\,\cdot\,))}\big\|_{[\alpha b,\alpha B]}$. Likewise, for small $\big|\tilde{\alpha}\big| > 0$ the maximum in the definition of $L_\kappa(\tilde{\alpha})$ will be attained by the term $\big\|\frac{f'}{f}\big\|$. Since $\gamma_u(t,u) = f\big(\varphi(t,u)\big)\varphi_u(t,u)$ for all $(t,u)\in[0,T]\times I\!R$, a comparison of (3.66) with (3.6) makes the different denominator at least plausible. □

As in Section 3.2.3 we can now rephrase the convergence statement for the transformed strategy functions $g^n : \mathcal{A}^n \to I\!R$ of Theorem 3.24 into a statement in terms of the strategy functions $\xi^n : \mathcal{A}^n \to I\!R$ and their limit $\varphi : [0,T]\times I\!R \to I\!R$:

Corollary 3.28. *Let* (ψ,μ) *be a large investor price system which satisfies Assumptions B, C, and D. Moreover, suppose that the two components* $\overline{\psi}$ *and* f *of the equilibrium price function* ψ *belong to the Hölder spaces* $\widehat{H}^{2+\frac{1}{2}\beta,4+\beta}\big([0,T]\times I\!R\big)$ *and* $H^{4+\beta}_{loc}(I\!R)$, *respectively, and that there exists some* $\eta > 0$ *such that* $\int e^{\eta|\theta|}\mu(d\theta) < \infty$. *If the function* $\varphi \in H^{2+\frac{1}{2}\beta,4+\beta}\big([0,T]\times I\!R\big)$ *solves the final value problem* (3.62), (3.63) *and satisfies* $1+2d(\mu)\frac{\psi_\xi(t,u,\varphi(t,u))}{\psi_u(t,u,\varphi(t,u))}\varphi_u(t,u) \ge \varepsilon > 0$ *for all* $(t,u)\in[0,T]\times I\!R$, *and if there exists some* $L\in I\!R$ *such that for all sufficiently large* $n \in I\!N$ *and for* $k\in\{n-1,n\}$

$$\big\|\xi^n\big(t^n_k,\,\cdot\,\big) - \varphi\big(t^n_k,\,\cdot\,\big)\big\|_{\mathcal{U}^n_k} \le L\delta^{4+\beta}, \tag{3.67}$$

then the discrete strategy functions $\xi^n : \mathcal{A}^n \to I\!R$ *converge to the function* $\varphi : [0,T]\times I\!R \to I\!R$ *in the sense that*

$$\|\xi^n - \varphi\|_{\mathcal{A}^n} = O\big(\delta^2\big) \tag{3.68}$$

and

$$\left\|\xi^n(\cdot+\delta^2,\cdot\pm\delta) - \xi^n \mp \delta\varphi_u - \delta^2\left(\varphi_t + \frac{1}{2}\varphi_{uu}\right)\right\|_{\mathcal{A}^n(n-1)} = O\big(\delta^{2+\beta}\big) \quad \textit{as } n\to\infty. \tag{3.69}$$

Remark. In view of the first remark following Theorem 3.24 it is possible to replace the final condition (3.67) by the combination of the two conditions $\left\|\xi^n\left(t_k^n,\,\cdot\,\right)-\varphi\left(t_k^n,\,\cdot\,\right)\right\|_{\mathcal{U}_k^n} \leq L\delta^4$ and $\left\|\Delta_u^n\,\xi^n\left(t_k^n,\,\cdot\,\right)-\Delta_u^n\,\varphi^n\left(t_k^n,\,\cdot\,\right)\right\|_{\mathcal{U}_k^n} \leq L\delta^{3+\beta}$ for all $k \in \{n-1, n\}$. □

The convergence result of Corollary 3.28 not only proves the convergence of a sequence of given strategy functions $\xi^n : \mathcal{A}^n \to I\!R$. In combination with the existence result of Proposition 3.27, it is also a helpful tool if we want to construct discrete replicating strategies for contingent claims $\left(\xi_n, b_n\right)$ as we did in Section 1.4, at least if n is sufficiently large.
To illustrate this, let us come back to the replication of star-concave contingent claims of Section 1.4.4, and take some nonincreasing function $\zeta : I\!R \to I\!R$. Supposing the assumptions of Proposition 3.27 to hold, we choose some $\tilde{\alpha}_1 < \tilde{\alpha} < \tilde{\alpha}_2$, and denote by $\varphi : [0,T] \times I\!R \to I\!R$ the solution of (3.62), (3.63). If we want to replicate a discrete star-concave contingent claim $\left(\xi_n, b_n\right)$ with $\xi_n = \tilde{\alpha}\zeta\left(U_n^n\right)$, we start by defining $\xi^n\left(t_n^n,\,\cdot\,\right) : \mathcal{U}_n^n \to I\!R$ by $\xi^n\left(t_n^n, u\right) = \tilde{\alpha}\zeta(u)$ for all $u \in \mathcal{U}_n^n$, and use Proposition 1.33 to calculate the function $\xi^n\left(t_{n-1}^n,\,\cdot\,\right) : \mathcal{U}_{n-1}^n \to I\!R$, which gives us the necessary stock holdings for a self-financing replicating strategy at time t_{n-1}^n. Then it can be checked whether for all sufficiently large $n \in I\!N$ this function satisfies $\left\|\xi^n\left(t_{n-1}^n,\,\cdot\,\right)-\varphi\left(t_{n-1}^n,\,\cdot\,\right)\right\|_{\mathcal{U}_{n-1}^n} \leq L\delta^{4+\beta}$ for some $L > 0$. For example, this condition will hold if $b_n = b^n\left(t_n^n, U_n^n\right)$ is chosen as

$$b^n\left(t_n^n, u_{ni}^n\right) = b_0^\alpha - \sum_{\substack{j\in\mathcal{I}_{n-1}\\ j\leq i}} \Big(\left(\varphi_{n(j+1)}^n - \varphi_{(n-1)j}^n\right) S_\mu\left(t_k^n, u_{n(j+1)}^n, \varphi_{(n-1)i}^n, \varphi_{n(i+1)}^n\right) + \left(\varphi_{(n-1)j}^n - \varphi_{n(j-1)}^n\right) S_\mu\left(t_k^n, u_{n(j-1)}^n, \varphi_{(n-1)i}^n, \varphi_{n(i-1)}^n\right)\Big) \tag{3.70}$$

for some $b_0^\alpha \in I\!R$, where in analogy to the shorthand $\xi_{ki}^n = \xi^n\left(t_k^n, u_{ki}^n\right)$ of (1.3.12) the expression φ_{ki}^n is a shorthand for $\varphi\left(t_k^n, u_{ki}^n\right)$ for all $0 \leq k \leq n$ and $i \in \mathcal{I}_k$. Namely, if $\xi_n = \xi^n\left(t_n^n, U_n^n\right)$ and $b_n = b^n\left(t_n^n, U_n^n\right)$ is chosen as in (3.70), then the definition of $\left(\xi_n, b_n\right)$ implies that $\xi_{(n-1)i}^n = \varphi_{(n-1)i}^n$ solves the fixed point equation (1.4.10) for all $k = n$ and all $i \in \mathcal{I}_{n-1}$.
If the trading strategies $\xi^n : \mathcal{A}^n \to I\!R$ from time $t_0^n = 0$ up to time $t_n^n = T$ exist for all sufficiently large $n \in I\!N$, we can apply Corollary 3.28 to prove the convergence of those replicating strategies towards the continuous-time solution $\varphi : [0,T] \times I\!R \to I\!R$. Moreover, since in particular the derivative $\varphi_u : [0,T] \times T \to I\!R$ is bounded, we can conclude that for any $\varepsilon > 0$ and all sufficiently large $n \in I\!N$ we have $\left\|\Delta_u^n\,\xi^n\right\|_{\mathcal{A}_1^n} \leq \left\|\varphi_u\right\| + \varepsilon$. This bound on the derivative can now serve as an a priori estimate on $\left\|\Delta_u^n\,\xi^n\right\|_{\mathcal{A}_1^n}$ when we construct a star-concave replicating trading strategy as in Section 1.4.4.

3.4 The Limit of the Real Value Functions

For the convergence in distribution of our discrete binomial large investor models we shall require in Chapter 4 not only the convergence of the discrete *strategy functions*, but also the convergence of the associated *real value functions* towards a continuous-time limit. Such a convergence statement for the real value functions is derived in Section 3.4.1. For a general price system (ψ, μ) with transaction losses this convergence result draws on the convergence of the discrete strategy functions as shown in Section 3.3, and like the limit functions of the convergence statements in Sections 3.2 and 3.3, the limit function $\bar{v}$ is given as the solution to a final value problem. In the presence of transaction losses the final value problem for $\bar{v}$ also depends on the limit φ of the strategy functions. In Section 3.4.2 we derive minimal regularity assumptions which simultaneously guarantee the existence of solutions to the final value problems for φ and $\bar{v}$. The limit strategy φ is seen to be a transform of the spatial derivative of $\bar{v}$. Especially, in the absence of transaction losses we use this relationship to weaken the regularity assumptions on the boundary condition for φ which we had to impose in Sections 3.2 and 3.3, so that our limit model can cope for example with a European call. In Section 3.4.3 we introduce a new parametrization for $\bar{v}$ in order to compare the final value problem for the limiting real value function with the corresponding final value problems in standard small investor models like the Black-Scholes model and small investor models with proportional transaction costs.

Some results of this section could be presented in more detail, and in particular the convergence of the value functions could be proved rigorously by using tools of Section 3.2 and 3.3. However, we do not always go into every detail and rather give an overview of possible extensions and immediate consequences of our model, in order to quickly come to the convergence results in Chapter 4, where we show convergence in distribution of our discrete binomial models.

3.4.1 Convergence of the Real Value Functions

In this section we show the convergence of the discrete real value functions along the lines of the convergence statements for the (transformed) strategy functions in Section 3.2 and 3.3. It is seen that the limit of the real value functions satisfies a *linear* final value problem. If the price system does not exclude transaction losses, this final value problem involves the limit φ of the strategy functions, and in noticeable contrast to small investor models, we can only show the convergence of the discrete value functions if we likewise suppose the convergence of the discrete strategy functions.

Again we suppose that the price system (ψ, μ) satisfies Assumptions B and D. In order to get started and quickly come to the convergence result for the real value functions, we will also suppose Assumption C, though we can slightly relax that assumption if (ψ, μ) does not induce any transaction losses. In particular, we recall from Lemma 2.3 that Assumption C implies Assumption A for all sufficiently large $n \in I\!N$.

We now assume that for each $n \in I\!N$ the large investor replicates some contingent claim (ξ_n, b_n) by using some path-independent self-financing portfolio strategy (ξ^n, b^n) with associated strategy function $\xi^n : \mathcal{A}^n \to I\!R$ and cash holdings function $b^n : \mathcal{A}^n \to I\!R$ as given by Definition 1.23. We have already shown in great detail in Sections 3.2 and 3.3 under which conditions the strategy functions $\{\xi^n\}_{n \in I\!N}$ converge to a limit function $\varphi : [0,T] \times I\!R \to I\!R$. In this section we give additional conditions on the convergence of the cash holdings functions at maturity, such that the sequence $\{\bar{v}^n\}_{n \in I\!N}$ of real value functions converges towards a limit function $\bar{v} : [0,T] \times I\!R \to I\!R$ as well, where for each $n \in I\!N$ the real value function $\bar{v}^n : \mathcal{A}^n \to I\!R$ was introduced in Definition 1.28 by

$$\bar{v}^n(t,u) := \xi^n(t,u)\bar{S}\big(t,u,\xi^n(t,u)\big) + b^n(t,u) \quad \text{for all } (t,u) \in \mathcal{A}^n.$$

For all sufficiently large $n \in I\!N$ (for which Assumption A holds) we have shown in Corollary 2.14(ii) that the real value function can be recursively calculated from its range of final values by

$$\begin{aligned} \bar{v}^n(t,u) = \bar{p}^n(t,u)\Big(&\bar{v}^n\big(t+\delta^2, u+\delta\big) + c_\mu^{\xi^n}\big(t+\delta^2, u+\delta, \xi^n(t,u)\big)\Big) \\ &+ \big(1-\bar{p}^n(t,u)\big)\Big(\bar{v}^n\big(t+\delta^2, u-\delta\big) + c_\mu^{\xi^n}\big(t+\delta^2, u+\delta, \xi^n(t,u)\big)\Big) \end{aligned} \tag{4.1}$$

for all $(t,u) \in \mathcal{A}^n(n-1)$, and for all those (t,u) we have derived the representation

$$\begin{aligned} g^n(t,u) = g\big(\xi^n(t,u)\big) &= \frac{\bar v(t+\delta^2,u+\delta) - \bar v(t+\delta^2,u-\delta)}{\overline{\psi}(t+\delta^2,u+\delta) - \overline{\psi}(t+\delta^2,u-\delta)} \\ &\quad + \frac{c_\mu^{\xi^n}\big(t+\delta^2,u+\delta,\xi^n(t,u)\big) - c_\mu^{\xi^n}\big(t+\delta^2,u-\delta,\xi^n(t,u)\big)}{\overline{\psi}(t+\delta^2,u+\delta) - \overline{\psi}(t+\delta^2,u-\delta)}. \end{aligned} \tag{4.2}$$

for the transformed strategy function $g^n = g \circ \xi^n : \mathcal{A}^n \to I\!R$. Note that the s-martingale weight function $\bar p^n : \mathcal{A}^n(n-1) \to I\!R$ and the shorthand $c_\mu^{\xi^n}$ used in the two representations were defined in (2.1.10) and (2.1.2), respectively.

Let us now take some function $\zeta : I\!R \to I\!R$ and some constant $\alpha \in I\!R$ so that the limiting strategy function $\varphi : [0,T] \times I\!R \to I\!R$ solves the final value problem

$$\begin{aligned} &\varphi_t(t,u) + \frac{1}{2}\frac{d}{du}\left(\left(1+2d(\mu)\frac{\psi_\xi\big(t,u,\varphi(t,u)\big)}{\psi_u\big(t,u,\varphi(t,u)\big)}\varphi_u(t,u)\right)\varphi_u(t,u)\right) \\ &= \varphi_u(t,u)\left(\frac{\psi_t\big(t,u,\varphi(t,u)\big)}{\psi_u\big(t,u,\varphi(t,u)\big)} - \frac{1}{2}\frac{\frac{d}{du}\psi_u\big(t,u,\varphi(t,u)\big)}{\psi_u\big(t,u,\varphi(t,u)\big)}\left(1+2d(\mu)\frac{\psi_\xi\big(t,u,\varphi(t,u)\big)}{\psi_u\big(t,u,\varphi(t,u)\big)}\varphi_u(t,u)\right)\right) \end{aligned} \tag{4.3}$$

for all $(t,u) \in [0,T) \times I\!R$ with final condition

$$\varphi(T,u) = \alpha\zeta(u) \quad \text{for all } u \in I\!R, \tag{4.4}$$

and let us set again $\gamma = g \circ \varphi$. From the discussion in the previous section we know that under these conditions γ solves the final value problem (3.2) with $\gamma(T,u) = \int_0^{\alpha\zeta(u)} f(x)\,dx$ for all $u \in I\!R$. Now fix some $b_0^\alpha \in I\!R$ and define the function $b^\alpha : I\!R \to I\!R$ by

$$b^\alpha(u) = b_0^\alpha - \int_0^u \overline{\psi}(T,\bar u)\, d\left(\int_0^{\alpha\zeta(\bar u)} f(z)\,dz\right) \quad \text{for all } u \in I\!R. \tag{4.5}$$

It turns out that under suitable conditions the discrete real value functions $\{\bar v^n\}_{n\in I\!N}$ will converge towards the solution $\bar v : [0,T] \times I\!R \to I\!R$ of the *linear* final value problem

$$\begin{aligned} &\bar v_t(t,u) + \frac{1}{2}\left(1 + \frac{\overline{\psi}(t,u)}{\overline{\psi}_u(t,u)}\kappa\big(\gamma(t,u)\big)\gamma_u(t,u)\right)\bar v_{uu}(t,u) \\ &= \bar v_u(t,u)\left(\frac{\overline{\psi}_t(t,u)}{\overline{\psi}_u(t,u)} + \frac{1}{2}\frac{\overline{\psi}_{uu}(t,u)}{\overline{\psi}_u(t,u)}\left(1 + \frac{\overline{\psi}(t,u)}{\overline{\psi}_u(t,u)}\kappa\big(\gamma(t,u)\big)\gamma_u(t,u)\right)\right) \end{aligned} \tag{4.6}$$

for all $(t,u) \in [0,T) \times I\!R$, where the boundary condition at time T is given by

$$\bar v(T,u) = \alpha\zeta(u)\bar S\big(T,u,\alpha\zeta(u)\big) + b^\alpha(u) \quad \text{for all } u \in I\!R. \tag{4.7}$$

If the transaction loss function κ does not vanish, (4.6) depends on the limiting strategy function φ via $\gamma = g \circ \varphi$. Moreover, we shall see that $\gamma(t,u) = \frac{\bar v_u(t,u)}{\overline{\psi}_u(t,u)}$ for all $(t,u) \in [0,T] \times I\!R$. Hence γ, which we already know, is also completely determined by the solution $\bar v$ of (4.6), (4.7). If we plugged in this relation in (4.6), we would obtain a non-linear final value problem for $\bar v$. Due to our knowledge of γ, however, we only need to solve the linear problem (4.6), (4.7).

Remark. The final condition on the right-hand side of (4.7) can be interpreted as the sum of the loss-free liquidation value $\alpha\zeta(u)\bar S\big(T,u,\alpha\zeta(u)\big)$ of $\varphi(T,u) = \alpha\zeta(u)$ shares of stock and an amount of $b^\alpha(u)$ in cash. If $\zeta \in C_b^1(I\!R)$, we can rewrite (4.5) as

$$b^\alpha(u) = b_0^\alpha - \alpha\int_0^u \psi\big(T,\tilde u,\alpha\zeta(\tilde u)\big)\zeta_u(\tilde u)\,d\tilde u \quad \text{for all } u \in I\!R,$$

and (3.70) becomes the limit of (4.5) as $n \to \infty$. □

The next lemma gives fairly general conditions under which the function $b^\alpha : I\!R \to I\!R$ is well defined and under which the final value condition (4.7) is continuous in u. It will soon become clear in Section 3.4.2 why we allow ζ to be non-continuous, despite the fact that we have required much more smoothness for it in Sections 3.2 and 3.3.

Lemma 3.29. *Assume that $\overline{\psi} \in C^{0,1}\big([0,T]\times I\!R\big)$ and $f \in C(I\!R)$, and suppose that $\zeta : I\!R \to I\!R$ is bounded and can be written as $\zeta = \zeta^{ac} + \zeta^d$, where ζ^{ac} is absolutely continuous with respect to the Lebesgue measure on $I\!R$ and where ζ^d consists only of (left and right) jumps. Then the function $b^\alpha : I\!R \to I\!R$ of (4.5) is well-defined, and the final condition for $\bar{v}$ in (4.7) is continuous in u and differentiable in all continuity points of ζ.*

Proof. We first have to show that the Riemann-Stieltjes integral in (4.5) is well-defined for all $u \in I\!R$. Therefore, we have to show that, like the function $\zeta : I\!R \to I\!R$, the function $\bar{\gamma} : I\!R \to I\!R$, given by $\bar{\gamma}(u) := \int_0^{\alpha\zeta(u)} f(z)dz$ for all $u \in I\!R$, can be written as the sum of an absolutely continuous and a pure jump function, i.e. that it does not contain a singularly continuous part. To limit the notational burden, let us assume without loss of generality that ζ^d is right continuous. The pure jump part $\bar{\gamma}^d : I\!R \to I\!R$ of $\bar{\gamma}$ is given as

$$\bar{\gamma}^d(u) := \int_0^{\alpha\zeta(0)} f(z)dz + \sum_{0<\bar{u}\le u} \int_{\alpha\zeta(\bar{u}-)}^{\alpha\zeta(\bar{u})} f(z)\,dz - \sum_{u<\bar{u}\le 0} \int_{\alpha\zeta(\bar{u}-)}^{\alpha\zeta(\bar{u})} f(z)\,dz \quad \text{for all } u \in I\!R,$$

where of course (at least) one of the two sums is always empty. For the remainder term $\bar{\gamma}^{ac} : I\!R \to I\!R$, given by $\bar{\gamma}^{ac}(u) := \bar{\gamma}(u) - \bar{\gamma}^d(u)$ for all $u \in I\!R$, we obtain from considering the cases $0 \le v \le u$, $v \le 0 \le u$ and $v \le u \le 0$ separately

$$\begin{aligned}\bar{\gamma}^{ac}(u) - \bar{\gamma}^{ac}(v) &= \int_{\alpha\zeta(v)}^{\alpha\zeta(u)} f(z)\,dz - \sum_{v<\bar{u}\le u} \int_{\alpha\zeta(\bar{u}-)}^{\alpha\zeta(\bar{u})} f(z)\,dz \\ &= \int_{\alpha\zeta(v)}^{\alpha\zeta(u)} f(z)\mathbf{1}_{\left\{I\!R\setminus\bigcup_{\bar{u}\in I\!R}[\alpha\zeta(\bar{u}-),\alpha\zeta(\bar{u})]\right\}}(z)\,dz \quad \text{for all } 0 \le v \le u,\end{aligned}$$

where we again set $\big[\alpha\zeta(\bar{u}-), \alpha\zeta(\bar{u})\big] := \big[\alpha\zeta(\bar{u}), \alpha\zeta(\bar{u}-)\big]$ if $\alpha\zeta(\bar{u}-) > \alpha\zeta(\bar{u})$. The function $\zeta : I\!R \to I\!R$ is bounded, thus there exists some $R > 0$ such that $\|\zeta\| \le R$, and since $f : I\!R \to (0,\infty)$ is continuous, $\overline{f} := \max_{[-R,R]} f(x)$ exists as well. Hence we get

$$\big|\bar{\gamma}^{ac}(u) - \bar{\gamma}^{ac}(v)\big| \le \overline{f}\left|\int_{\alpha\zeta(v)}^{\alpha\zeta(u)} \mathbf{1}_{\left\{I\!R\setminus\bigcup_{\bar{u}\in I\!R}[\alpha\zeta(\bar{u}-),\alpha\zeta(\bar{u})]\right\}}(z)\,dz\right| = \overline{f}|\alpha|\big|\zeta^{ac}(u) - \zeta^{ac}(v)\big|.$$

By definition, a function z is absolutely continuous, if for any $\varepsilon > 0$ there exists some $\delta > 0$ such that $\sum_{i=1}^n \big|z(x_i) - z(y_i)\big| < \varepsilon$ for every finite collection $\big\{(x_i, y_i)\big\}_{1\le i\le n}$ of non-overlapping intervals with $\sum_{i=1}^n |x_i - y_i| < \delta$. Thus it is clear that the absolute continuity of ζ^{ac} implies the absolute continuity of $\bar{\gamma}^{ac}$, and hence $\bar{\gamma} : I\!R \to I\!R$ can indeed by written as the sum $\bar{\gamma} = \bar{\gamma}^{ac} + \bar{\gamma}^d$ of an absolutely continuous function $\bar{\gamma}^{ac} : I\!R \to I\!R$ and some pure jump part $\bar{\gamma}^d : I\!R \to I\!R$. Since the function $\overline{\psi} : [0,T] \times I\!R \to I\!R$, $(t,u) \mapsto \overline{\psi}(t,u)$ is continuous in u, the Riemann-Stieltjes integral in (4.5) is well defined.

By partial integration we then obtain

$$b^\alpha(u) = b_0^\alpha - \overline{\psi}(T,u)\int_0^{\alpha\zeta(u)} f(x)\,dx + \overline{\psi}(T,0)\int_0^{\alpha\zeta(0)} f(z)\,dz + \int_0^u \int_0^{\alpha\zeta(\bar{u})} \overline{\psi}_u(T,\bar{u})f(z)\,dz\,d\bar{u},$$

and by (2.4.2) the final condition (4.7) becomes

$$\bar{v}(T,u) = \alpha\zeta(0)\bar{S}\big(T,0,\alpha\zeta(0)\big) + b_0^\alpha + \int_0^u \int_0^{\alpha\zeta(\bar{u})} \overline{\psi}_u(T,\bar{u})f(z)\,dz\,d\bar{u} \quad \text{for all } u \in I\!R, \qquad (4.8)$$

which is of course continuous in u and differentiable in all continuity points of ζ. **q.e.d.**

If the strategy function $\varphi : [0,T] \times I\!R \to I\!R$ and hence also the transform $\gamma = g \circ \varphi$ is *known*, the partial differential equation (4.6) is linear, hence sufficient conditions for the existence of uniqueness of solutions can be derived from classical arguments. For example, we might basically use the same arguments as for the final value problem (2.3), (2.4) to conclude from Theorem IV.5.1 in connection with the discussion on p. 389 in Ladyženskaja et al. (1968) that for any $k \geq 2$ and $\beta \in (0,1)$ the final value problem (4.6), (4.7) has a unique solution $\bar{v} : [0,T] \times I\!R \to I\!R$ from the class $H_{loc}^{\frac{1}{2}(k+\beta),k+\beta}\big([0,T] \times I\!R\big)$ for which $\big\|\frac{\bar{v}}{\overline{\psi}_u}\big\| < \infty$ and $\big\|\frac{\bar{v}_z}{\overline{\psi}_u}\big\| < \infty$ for $z \in \{t, u, uu\}$ if the functions $\overline{\psi}$, f, and φ belong to the Hölder spaces $\widehat{H}^{\frac{1}{2}(k+\beta),k+\beta}\big([0,T] \times I\!R\big)$, $H^{k-1+\beta}(I\!R)$, and $H^{\frac{1}{2}(k-1+\beta),k-1+\beta}\big([0,T] \times I\!R\big)$, respectively. Note that we need not state an extra condition for the boundary, since $\alpha\zeta = \varphi(T,\,\cdot\,)$ and hence $\alpha\zeta \in H^{k-1+\beta}(I\!R)$ is already implied by $\varphi \in H^{\frac{1}{2}(k-1+\beta),k-1+\beta}\big([0,T] \times I\!R\big)$.

Especially, a unique solution $\bar{v} \in H_{loc}^{\frac{1}{2}(4+\beta),4+\beta}\big([0,T] \times I\!R\big)$ exists under the assumptions of Corollary 3.28, where we have stated conditions which guarantee that the convergence of the strategy functions $\xi^n : \mathcal{A}^n \to I\!R$ at the times $t_{n-1}^n = T - \delta^2$ and $t_n^n = T$ implies their convergence towards φ on their whole domain.

If we now additionally suppose that the cash holdings functions $b^n : \mathcal{A}^n \to I\!R$ of Definition 1.23 converge to the function b^α at time T, at which for each $n \in I\!N$ the cash holdings function just describes the cash position b_n prescribed by the replicated contingent claim (ξ_n, b_n), then by the definition of $\bar{v}^n$ this is equivalent to additionally supposing that the final values $\bar{v}^n(T,\,\cdot\,) : \mathcal{U}_n^n \to I\!R$ of the real value functions converge to $\bar{v}(T,\,\cdot\,) : I\!R \to I\!R$, and we can show the following result:

Proposition 3.30. *In addition to the assumptions of Corollary 3.28 suppose that the final values $b^n(T,\,\cdot\,) : \mathcal{U}_n^n \to I\!R$ of the sequence $\{b^n\}_{n\in I\!N}$ of cash holdings functions $b^n : \mathcal{A}^n \to I\!R$ converge to the function $b^\alpha : I\!R \to I\!R$ of* (4.5) *in the sense that*

$$\big\|b^n(T,\,\cdot\,) - b^\alpha\big\|_{\mathcal{U}_n^n} \leq K\delta^{2+\beta} \quad \textit{for all sufficiently large } n \in I\!N. \tag{4.9}$$

Then the real value functions $\bar{v}^n : \mathcal{A}^n \to I\!R$ converge to the solution $\bar{v} : [0,T] \times I\!R \to I\!R$ of (4.6)*,* (4.7) *in the sense that $\|\bar{v}^n - \bar{v}\|_{\mathcal{A}^n} = O\big(\delta^\beta\big)$ as $n \to \infty$.*

Remark. A proof of Proposition 3.30 is based on the recursive formula (4.1) in combination with the representation (4.2), which both hold for all sufficiently large $n \in I\!N$, namely for all those $n \in I\!N$ for which Assumption C implies Assumption A, and the proof follows the ideas of Theorem 3.10 and Theorem 3.24. Especially note that by the remark following Corollary 2.14 and the definition of the function function $J : G^2 \to I\!R$ in (3.49) we have

$$c_\mu^{\xi^n}\big(t+\delta^2, u\pm\delta, \xi^n(t,u)\big) = \big(g^n(t+\delta^2, u\pm\delta) - g^n(t,u)\big) J\big(g^n(t,u), g^n\big(t+\delta^2, u\pm\delta\big)\big)\overline{\psi}(t+\delta^2, u\pm\delta)$$

for all $(t,u) \in \mathcal{A}^n(n-1)$ and hence the expansions of J in Lemma 3.25 yield that the transaction loss term in (4.2) vanishes if $n \to \infty$, and therefore

$$\gamma(t,u) = \frac{\bar{v}_u(t,u)}{\overline{\psi}_u(t,u)} \quad \text{for all } (t,u) \in [0,T] \times I\!R. \tag{4.10}$$

And indeed, if we differentiate the PDE (4.6) with respect to u, divide it by $\overline{\psi}_u(t,u)$, and use (4.10) to replace the derivatives of $\bar{v}_u : [0,T] \times I\!R \to I\!R$ by derivatives of $\gamma : [0,T] \times I\!R \to I\!R$, we arrive at the PDE (3.2). $\square$

3.4.2 The Final Value Problems for Strategy and Real Value Revisited

In this section we derive minimal regularity assumptions which simultaneously guarantee the existence of a solution φ to the final value problem (4.3), (4.4) and of a solution $\bar{v}$ to the final value problem (4.6), (4.7). Our findings will indicate advances which can be obtained in the continuous model at almost no additional efforts.

We are guided by the results of the Black-Scholes model, where $f \equiv 1$. In such a model the value function $\bar{v} : [0,T] \times I\!R \to I\!R$ of a unit European call is continuous, but the final strategy function $\zeta : I\!R \to I\!R$, $u \mapsto \zeta(u)$, which describes the required stock holdings at maturity T, jumps from 0 to 1, i.e. from not holding the stock at time T to holding it, depending on the fundamental value u. However, all the existence results for solutions $\varphi : [0,T] \times I\!R \to I\!R$ to the final value problem (4.3), (4.4) which we have presented in Sections 3.2 and 3.3, require at least continuity of the function $\zeta : I\!R \to I\!R$. Hence we are especially interested in weakening our conditions on the continuity of ζ in order to include the European call in our analysis.

We will now introduce a tuple $\big(\alpha\zeta, b^\alpha\big)$ of functions which describes a European call of α shares of stock in the continuous model:

Example 3.1 *(European Call).* Let the price system (ψ, μ) satisfy Assumption B and suppose that the two components $\overline{\psi}$ and f of ψ belong to the spaces $C^{0,1}\big([0,T] \times I\!R\big)$ and $C(I\!R)$, respectively. Let $K \in I\!R$ be some strike price so that

$$\lim_{u \to -\infty} \overline{\psi}(t,u) \leq \left(\int_0^1 f(\theta\alpha)\, d\theta\right)^{-1} K \leq \lim_{u \to \infty} \overline{\psi}(t,u).$$

Then there exists some $u^* \in I\!R$ such that $\overline{\psi}(T,u^*) = \big(\int_0^1 f(\theta\alpha)\, d\theta\big)^{-1} K$, and by (2.4.2) the real value $\alpha\bar{S}(T,u^*,\alpha)$ of a position of α shares of stock satisfies

$$\alpha\bar{S}(T,u^*,\alpha) = \overline{\psi}(T,u^*) \int_0^\alpha f(z)\, dz = \alpha \int_0^1 \overline{\psi}(T,u^*) f(\theta\alpha)\, d\theta = \alpha K.$$

Let us now define the functions $\zeta : I\!R \to I\!R$ and $b^\alpha : I\!R \to I\!R$ by $\zeta(u) = \mathbf{1}_{\{u \geq u^*\}}$ and

$$b^\alpha(u) = -\alpha K - \int_{u^*}^u \overline{\psi}(T,\bar{u})\, d\left(\int_0^{\alpha\zeta(\bar{u})} f(z)\, dz\right) \quad \text{for all } u \in I\!R,$$

respectively. Since the function $u \mapsto \int_0^{\alpha\zeta(u)} f(z)\, dz$ is a pure jump function with only one jump from the left at u^*, we have

$$\int_{u^*}^u \overline{\psi}(T,\bar{u})\, d\left(\int_0^{\alpha\zeta(\bar{u})} f(z)\, dz\right) = \begin{cases} -\overline{\psi}(T,u^*) \int_0^\alpha f(z) dz = -\alpha K & \text{if } u < u^* \\ 0 & \text{if } u \geq u^*, \end{cases} \tag{4.11}$$

and hence b^α simplifies to $b^\alpha(u) = -\alpha K \mathbf{1}_{\{u \geq u^*\}}$. The tuple $(\alpha\zeta, b^\alpha)$ is a (functional) description of a European call of (long) α shares of stock in the continuous limit model. In order to see its connection to the European call which we have introduced in the discrete setting of Example 1.7, note that the definition of ζ implies

$$\alpha\zeta(u)\bar{S}\big(T,u,\alpha\zeta(u)\big) = \begin{cases} 0 & \text{if } u < u^* \\ \alpha\bar{S}\big(T,u,\alpha\big) & \text{if } u \geq u^*. \end{cases}$$

The real value of the European call at time T can therefore be written as

$$\bar{v}(T,u) = \alpha\zeta(u)\bar{S}\big(T,u,\alpha\zeta(u)\big) + b^\alpha(u) = \alpha\big(\bar{S}(T,u,\alpha) - K\big)^+ \quad \text{for all } u \in I\!R, \tag{4.12}$$

where the last equality employs the fact that $\overline{\psi} : [0,T] \times I\!R$, $(t,u) \mapsto \overline{\psi}(t,u)$ and hence also $\bar{S} : [0,T] \times I\!R \times I\!R$, $(t,u,\xi) \mapsto \bar{S}(t,u,\xi)$ is strictly increasing in u. Now (4.12) reflects the real value $\bar{v}(T,U_n^n) = \bar{V}_n^n = \alpha\big(\bar{S}(T,U_n^n,\alpha) - K\big)^+$ of the European Call in the discrete Example 1.7.
Note that we would obtain the same real value $\bar{v}(T,u)$ if we set $\zeta(u) = 1_{\{u>u^*\}}$ and

$$b^\alpha(u) = -\int_{u^*}^{u} \overline{\psi}(T,\bar{u})\, d\bigg(\int_0^{\alpha\zeta(\bar{u})} f(z)\,dz\bigg) = -\alpha K \mathbf{1}_{\{u>u^*\}} \quad \text{for all } u \in I\!R.$$

However, the first choice of ζ corresponds to the large investor's final stock holdings ξ_n for the discrete European call of Example 1.7. □

For our goal to state minimal conditions which simultaneously guarantee the existence of solutions to the final value problems for φ and $\bar{v}$, we first consider price systems (ψ,μ) without transaction losses. In this case the final value problem (4.6), (4.7) for $\bar{v}$ does not depend on the solution φ (or on its transform $\gamma = g \circ \varphi$) to the final value problem (4.3), (4.4). Thus, we can first obtain an existence result for the final value problem for $\bar{v}$ and then look for a solution φ to (4.3), (4.4). We find that the function $\zeta : I\!R \to I\!R$ in the boundary condition (4.7) need not be continuous in order to guarantee the existence of a solution $\bar{v}$ to the final value problem (4.6), (4.7). Once a solution to that final value problem has been found, this solution can be used to construct a solution φ to (4.3), (4.4). In particular, our results will allow us to treat European calls.
We suppose again that Assumptions B and C hold. In addition to that, we suppose

Assumption E. *The transformed loss function $\kappa : g(I\!R) \to I\!R$ of (3.1) vanishes, i.e. we have $d(\mu)f'(\xi) = 0$ for all $\xi \in I\!R$. Moreover, suppose that the two components $\overline{\psi}$ and f of ψ belong to the Hölder spaces $\overline{\psi} \in \widehat{H}^{1+\frac{1}{2}\beta,3+\beta}\big([0,T] \times I\!R\big)$ and $H^{1+\beta}_{loc}(I\!R)$, respectively. The function $\zeta : I\!R \to I\!R$ for the final condition is bounded and can be written as $\zeta = \zeta^{ac} + \zeta^d$, where ζ^{ac} is absolutely continuous with respect to the Lebesgue measure on $I\!R$ and ζ^d consists only of (left and right) jumps. The parameter $\alpha \in I\!R$ is some arbitrary real number.*

If Assumptions B, C, and E hold, we can conclude by the same sort of arguments as in Lemma 3.8 that the final problem (4.6), (4.7) for the continuous benchmark price function $\bar{v} : [0,T] \times I\!R \to I\!R$ has a unique solution $\bar{v} : [0,T] \times I\!R \to I\!R$ which belongs to the class $H^{1+\frac{1}{2}\beta,3+\beta}_{loc}\big([0,T) \times I\!R\big) \cap C^{0,0}\big([0,T] \times I\!R\big)$. For this solution $\bar{v}$ we can define the function $\gamma : [0,T] \times I\!R \to I\!R$ by

$$\gamma(t,u) = \frac{\bar{v}_u(t,u)}{\overline{\psi}_u(t,u)} \quad \text{for all } (t,u) \in [0,T) \times I\!R \tag{4.13}$$

and

$$\gamma(T,u) = \int_0^{\alpha\zeta(u)} f(z)\,dz \quad \text{for all } u \in I\!R. \tag{4.14}$$

Note that the representation (4.8) implies that $\bar{v}(T,\cdot)$ is almost everywhere differentiable and

$$\gamma(T,u) = \frac{\bar{v}_u(T,u)}{\overline{\psi}_u(T,u)} \quad \text{for all continuity points of } \zeta : I\!R \to I\!R. \tag{4.15}$$

Then $\gamma \in H^{1+\frac{1}{2}\beta,2+\beta}_{loc}\big([0,T) \times I\!R\big)$ solves the linear final value problem (2.3), (4.14). If we now set $\varphi := g^{-1} \circ \gamma$ we can conclude as in Proposition 3.9 that φ solves the final value problem

(2.5) with the (not necessarily continuous) final condition $\varphi(T,u) = \alpha\zeta(u)$ for all $u \in I\!R$, and this solution is unique in the class $C^{1,2}\big([0,T) \times I\!R\big)$. Thus we have found weaker existence conditions than the ones stated in terms of the transform γ in Lemma 3.8. Moreover, it still follows from a maximum principle (Corollary I.2.1 in Ladyženskaja et al. (1968)) for the linear parabolic final value problem (3.2.3), (3.2.4) that the range of γ is determined by the final condition, i.e. we still have

$$\inf_{\bar u \in I\!R} \left\{ \int_0^{\alpha\zeta(\bar u)} f(x)\,dx \right\} \le \gamma(t,u) \le \sup_{\bar u \in I\!R} \left\{ \int_0^{\alpha\zeta(\bar u)} f(x)\,dx \right\},$$

and by the monotonicity of the strategy transform $g : I\!R \to I\!R$, $x \mapsto \int_0^x f(z)\,dz$ it follows that

$$\inf_{\bar u \in I\!R} \alpha\zeta(\bar u) \le \varphi(t,u) \le \sup_{\bar u \in I\!R} \alpha\zeta(\bar u) \quad \text{for all } (t,u) \in [0,T] \times I\!R.$$

Assumption E has the great advantage that it allows us to extend the continuous-time model, which we derived as a limit of our discrete models, so that we can include a European call or other "options" for which the prescribed stock holdings at maturity can still be represented as a function of the fundamentals at that time, but for which this function is either not differentiable or even not continuous. The drawback, however, is that Assumption E restricts the class of price systems (ψ, μ) drastically, since it only allows for price systems without transaction losses. If we want to include transaction losses in our analysis, we cannot expect the same freedom in choosing ζ any more: In this case the final value problem (4.6), (4.7) depends via $\gamma = g \circ \varphi$ on the solution φ of the final value problem (4.3), (4.4), and therefore we first have to find a solution φ to the later problem. We note that minimal conditions for the existence of a solution to the non-linear final value problem (4.3), (4.4) were given in Corollary 3.22, though they were only stated in terms of the transformed strategy function $\gamma = g \circ \varphi$. It turns out that these conditions are also sufficient to guarantee the existence of a solution to the final value problem (4.6), (4.7). Hence, in the general case with transaction losses, we suppose in addition to Assumptions B and C:

Assumption F. *The transaction loss function $\kappa : g(I\!R) \to I\!R$ of (3.1) is nonnegative, i.e. $d(\mu)f'(\xi) \ge 0$ for all $\xi \in I\!R$. Moreover, suppose that the two components $\overline{\psi}$ and f of ψ belong to the Hölder spaces $\overline{\psi} \in \widehat{H}^{1+\frac{1}{2}\beta,3+\beta}\big([0,T] \times I\!R\big)$ and $H^{3+\beta}_{loc}(I\!R)$, respectively. The function $\zeta : I\!R \to I\!R$ for the final condition belongs to the class $C^1_b\big(I\!R\big)$. The parameter $\alpha \in I\!R$ is sufficiently close to 0 so that a solution $\varphi : [0,T] \times I\!R \to I\!R$ to the final problem (4.3), (4.4) exists and so that this solution satisfies the constraint*

$$1 + 2d(\mu)\frac{\psi_\xi\big(t,u,\varphi(t,u)\big)}{\psi_u\big(t,u,\varphi(t,u)\big)}\varphi_u(t,u) \ge \varepsilon \quad \textit{for all } (t,u) \in [0,T] \times I\!R \textit{ and some } \varepsilon > 0.$$

Under Assumptions B, C, and F we get $\varphi \in H^{1+\frac{1}{2}\beta,2+\beta}_{loc}\big([0,T) \times I\!R\big) \cap C^{0,1}_b\big([0,T] \times I\!R\big)$, and thus it follows once again from the theory of linear partial differential equations that the final value problem (4.6), (4.7) has a solution $\bar v \in H^{1+\frac{1}{2}\beta,3+\beta}_{loc}\big([0,T) \times I\!R\big) \cap C^{0,2}\big([0,T] \times I\!R\big)$.

3.4.3 Comparison with Standard Models

We now compare the final value problem (4.6), (4.7) for the limiting real value function in our large investor model with the corresponding final value problems for the value function in the Black-Scholes model and in some more general *small* investor models with transaction costs. Since the value function in the standard small investor models is written in terms of time and (small investor) stock price, and not in terms of time and fundamentals, a comparison with

these models becomes much more transparent if we reparametrize the real value function $\bar{v} : [0,T] \times I\!R \to I\!R$, $(t,u) \mapsto \bar{v}(t,u)$ as a function $\bar{w}$ of time t and small investor stock price $\overline{\psi}(t,u)$.
If the large investor price system (ψ, μ) excludes any transaction losses, we shall see that the transform $\bar{w}$ satisfies the Black-Scholes equation and the real value (function) of a European call in the continuous large investor market will be seen to be just the Black-Scholes price of the same European call with a modified strike price which reflects the market power of the large investor. In the general case, where (ψ, μ) does not necessarily prevent transaction losses, we do not have a closed-form solution for $\bar{w}$ any more, but we can still make qualitative statements. Besides comparisons with the associated model without transaction losses, we show structural analogies to small investor models with transaction costs. However, in this section we only consider the final value problem for the limiting real value function. The distributional limit model and the limit for the paper value function are discussed in Chapter 4.

We now require the two Assumptions B and C and one of the Assumptions E and F. At first we want to transform the real value function $\bar{v} : [0,T] \times I\!R \to I\!R$, $(t,u) \mapsto \bar{v}(t,u)$, of time t and fundamental value u into some function $\bar{w} : \bar{\mathcal{D}} \to I\!R$, $(t,x) \mapsto \bar{w}(t,x)$ of time t and small investor stock price $x = \overline{\psi}(t,u)$. We have already encountered such a transformation in the particular discrete setting of Section 2.4.3. Note that the small investor stock price $\overline{\psi}(t,u)$, which would occur if the large investor did not trade in stocks at all, might not be observable by the small investors, who do not know the actual stock holdings of the large investor. However, the large investor can derive and employ this price. Since he knows his own stock holdings $\varphi = \varphi(t,u)$ at time t, he can – and should – deduct his own leverage on the stock price by dividing the observed market price $\psi(t,u,\varphi)$ by $f(\varphi)$ in order to obtain the small investor price $\overline{\psi}(t,u)$.
Before we can formally define the function $\bar{w}$, we have to define its domain $\bar{\mathcal{D}}$; hence let us introduce the set of possible time-space combinations of small investor prices in continuous time as

$$\bar{\mathcal{D}} := \big\{(t,x) \in [0,T] \times I\!R \,\big|\, x = \overline{\psi}(t,u) \text{ for some } u \in I\!R\big\},$$

which is the continuous analogue of (2.4.21). Under our standing assumptions, the next lemma shows that $\bar{\mathcal{D}} = [0,T] \times (0,\infty)$.

Lemma 3.31. *Under Assumptions B and C we have for all $t \in [0,T]$:*

$$\lim_{u\to-\infty} \overline{\psi}(t,u) = 0 \quad \textit{and} \quad \lim_{u\to\infty} \overline{\psi}(t,u) = \infty.$$

Especially, for any bounded strategy function $\varphi : [0,T] \times I\!R \to I\!R$ we have

$$\lim_{u\to-\infty} \psi\big(t,u,\varphi(t,u)\big) = 0 \quad \textit{and} \quad \lim_{u\to\infty} \psi\big(t,u,\varphi(t,u)\big) = \infty.$$

Proof. Since $\frac{d}{du}\log\overline{\psi}(t,u) = \frac{\overline{\psi}_u(t,u)}{\overline{\psi}(t,u)} = \frac{\psi_u(t,u,\xi)}{\psi(t,u,\xi)} \geq \frac{1}{L_0} =: \sigma_0 > 0$ for all $(t,u,\xi) \in [0,T] \times I\!R^2$ we have

$$\log\overline{\psi}(t,u) = \log\overline{\psi}(t,0) + \int_0^u \frac{d}{du}\log\overline{\psi}(t,\bar{u})d\bar{u} \begin{cases} \geq \log\overline{\psi}(t,0) + \sigma_0 u & \text{if } u \geq 0 \\ \leq \log\overline{\psi}(t,0) + \sigma_0 u & \text{if } u \leq 0. \end{cases} \tag{4.16}$$

Taking the exponential on both sides and noting that $\overline{\psi}(t,u) > 0$ for all $(t,u) \in [0,T] \times I\!R$, we get for any fixed $t \in [0,T]$:

$$\overline{\psi}(t,u) \geq \overline{\psi}(t,0)e^{\sigma_0 u} \to \infty \quad \text{as } u \to \infty$$

$$\text{and} \quad 0 < \overline{\psi}(t,u) \leq \overline{\psi}(t,0)e^{\sigma_0 u} \to 0 \qquad \text{as } u \to -\infty.$$

This shows the first statement. The second part of the lemma follows immediately from the multiplicative structure of $\psi : [0,T] \times I\!R^2 \to I\!R$, and the boundedness of $\varphi : [0,T] \times I\!R \to I\!R$, since it implies that there exist some $0 < c \leq C$ such that $c \leq f\big(\varphi(t,u)\big) \leq C$ for all $(t,u) \in [0,T] \times I\!R$. **q.e.d.**

By Definition 1.17 the equilibrium price function $\psi : [0,T] \times I\!R^2 \to I\!R$ is strictly increasing in u, and hence this monotonicity also holds for the associated small investor price function $\overline{\psi} : [0,T] \times I\!R \to I\!R$. In particular, the function $u : \bar{\mathcal{D}} \to I\!R$ is well defined by $\overline{\psi}\big(t, u(t,x)\big) = x$ for all $(t,x) \in \bar{\mathcal{D}}$, and we can then define the (transformed) real value function $\bar{w} : \bar{\mathcal{D}} \to I\!R$ in analogy to (2.4.22) by

$$\bar{w}(t,x) = \bar{v}\big(t, u(t,x)\big) \quad \text{for all } (t,x) \in \bar{\mathcal{D}}. \tag{4.17}$$

Noting that $\bar{v}_t(t,u) = \bar{w}_t\big(t, \overline{\psi}(t,u)\big) + \bar{w}_x\big(t, \overline{\psi}(t,u)\big)\overline{\psi}_u(t,u)$ for all $(t,u) \in [0,T] \times I\!R$ and using similar equations for $\bar{v}_u(t,u)$ and $\bar{v}_{uu}(t,u)$, we can rewrite the partial differential equation (4.6) for all $(t,u) \in [0,T) \times I\!R$ as:

$$\bar{w}_t\big(t, \overline{\psi}(t,u)\big) + \frac{1}{2}\left(1 + \frac{\overline{\psi}(t,u)}{\overline{\psi}_u(t,u)}\kappa\big(\gamma(t,u)\big)\gamma_u(t,u)\right)\overline{\psi}_u^2(t,u)\bar{w}_{xx}\big(t, \overline{\psi}(t,u)\big) = 0. \tag{4.18}$$

In order to express (4.18) in terms of t and $x = \overline{\psi}(t,u)$ only, we next define the volatility function $\bar{\sigma} : \bar{\mathcal{D}} \to (0,\infty)$ by

$$\bar{\sigma}(t,x) = \frac{\overline{\psi}_u(t, u(t,x))}{\overline{\psi}(t, u(t,x))} \quad \text{for all } (t,x) \in \bar{\mathcal{D}}. \tag{4.19}$$

If we also recall (4.10), we see that

$$\gamma(t,u) = \bar{w}_x\big(t, \overline{\psi}(t,u)\big) \quad \text{for all } (t,u) \in [0,T) \times I\!R, \tag{4.20}$$

and hence we get $\gamma_u(t,u) = \bar{w}_{xx}\big(t, \overline{\psi}(t,u)\big)\overline{\psi}_u(t,u)$. Plugging these expressions into (4.18) we obtain the generalized Black-Scholes equation

$$\bar{w}_t(t,x) + \frac{1}{2}\Big(1 + x\kappa\big(\bar{w}_x(t,x)\big)\bar{w}_{xx}(t,x)\Big)\bar{\sigma}^2(t,x)x^2\bar{w}_{xx}(t,x) = 0 \tag{4.21}$$

for all $(t,x) \in [0,T) \times (0,\infty) = \mathcal{D}\backslash\big(\{T\} \times (0,\infty)\big)$. For the boundary condition (4.7) we can once again apply (2.4.2) and substitute $x = \overline{\psi}(t,u)$ to rewrite it as

$$\bar{w}(T,x) = h^\alpha(x) := x\int_0^{\alpha\eta(x)} f(z)\,dz + b^\alpha\big(u(T,x)\big) \quad \text{for all } x \in \psi(T, I\!R) = (0,\infty), \tag{4.22}$$

where the function $\eta : \overline{\psi}(T, I\!R) \to I\!R$ is given by $\eta(x) = \zeta\big(u(T,x)\big)$ for all $x \in \overline{\psi}(T, I\!R)$. By the final condition (4.22) the large investor has to hold $\alpha\eta(x)$ shares of stock and a cash amount of $b^\alpha\big(u(T,x)\big)$ at time T if the small investor stock price at that time is given by x. The non-linear partial differential equation (4.21) generalizes the standard Black-Scholes equation by the additional transaction loss term involving κ. Under Assumption E the transformed loss function $\kappa : g(I\!R) \to I\!R$ vanishes on $I\!R$, and (4.21) basically reduces to the Black-Scholes equation. In this case we can easily transfer results from the standard Black-Scholes analysis and the theory of stochastic volatility models to the final value problem (4.21), (4.22), e.g. in order to derive existence conditions or to find optimal super-replication strategies for an associated continuous-time large investor model.

In this thesis we will concentrate our investigation of the case where the large investor's final stock holdings are not described by a smooth function of the fundamentals, but where the price system (ψ, μ) excludes any transaction losses, on our standard example, the European call. If the volatility function $\bar{\sigma}$ is constant, we are able to explicitly calculate the corresponding large investor's replication price for that call:

Example 3.2 *(European Call).* We continue with the European call $\big(\alpha\zeta, b^\alpha\big)$ of Example 3.1 in the special case where the loss function $\kappa : g(I\!R) \to I\!R$ vanishes, and where the small investor price function $\overline{\psi} : [0,T] \times I\!R \to I\!R$ can be written as

$$\overline{\psi}(t,u) = e^{a(t)+\sigma u} \quad \text{for all } (t,u) \in [0,T] \times I\!R$$

and some constant volatility $\sigma > 0$ and some drift function $a : [0,T] \to I\!R$. In this case the volatility function $\bar{\sigma} : \bar{\mathcal{D}} \to I\!R$ of (4.19) is constant and equals σ, so that (4.21) becomes the classical Black-Scholes equation.

Because of (4.12) in Example 3.1 we can rewrite the final condition (4.22) as

$$\bar{w}(T,x) = \alpha \int_0^1 f(\alpha\theta)\, d\theta \big(x - K^*\big)^+ \quad \text{for all } x \in \overline{\psi}(T, I\!R),$$

where $K^* = K\big(\int_0^1 f(\alpha\theta)\, d\theta\big)^{-1}$. This shows that in terms of $\bar{w}$ the final value problem (4.21), (4.22) for the replication of a European call with final real value $\alpha\big(\bar{S}(T,u,\alpha) - K\big)^+$ in the continuous limit model becomes a standard Black-Scholes problem in the associated small investor market, where a *small* investor has to replicate $g(\alpha) = \int_0^\alpha f(z)\, dz = \alpha \int_0^1 f(\alpha\theta) d\theta$ unit European calls $(x - K^*)^+$ with the modified strike price K^*. Now it follows from the standard Black-Scholes formula and the definition of K^* that

$$\bar{w}(t,x) = \alpha C\left(t, x\int_0^1 f(\alpha\theta) d\theta\right). \tag{4.23}$$

In this formula $C : [0,T] \times (0,\infty) \to I\!R$ is the Black-Scholes price for a European call of one share of stock with strike K, given by

$$C(t,x) = x\Phi\left(\frac{\log\frac{x}{K} + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}\right) - K\Phi\left(\frac{\log\frac{x}{K} - \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}\right),$$

and the function $\Phi : I\!R \to [0,1]$ is the standard normal cumulative distribution function $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-z^2/2}\, dz$ for all $x \in I\!R$.

Recall from (2.4.2) that $\overline{\psi}(t,u) \int_0^1 f(\alpha\theta) d\theta = \bar{S}(t,u,\alpha)$ for all $(t,u) \in [0,T] \times I\!R$. Hence we conclude from (4.23) that at each date $t \in [0,T] \times I\!R$, the replication price of the European call of α shares of stock with total final payoff of $\alpha\big(\bar{S}(T,u,\alpha) - K\big)^+$ for all $u \in I\!R$ can be calculated by the Black-Scholes formula if we plug in the loss-free liquidation price $\bar{S}(t,u,\alpha)$ at time t given fundamentals of u. Note that this liquidation price is only a theoretical liquidation price, since the large investor does not hold α stocks at time t. More precisely, upon differentiating (4.23) and substituting $z = \alpha\theta$ we see that

$$\bar{w}_x(t,x) = \int_0^\alpha f(z) dz\, \Phi\left(\frac{\log\big(\frac{x}{K}\int_0^1 f(\alpha\theta) d\theta\big) + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}\right) \quad \text{for all } (t,x) \in [0,T) \times (0,\infty).$$

Because of $g(x) = \int_0^x f(z) dz$ and $\Phi(x) < 1$ for all $x \in I\!R$ it then follows for all $\alpha \neq 0$ that

$$0 < \varphi(t,u) = g^{-1}\big(\bar{w}_x\big(t, \overline{\psi}(t,u)\big)\big) < \alpha \quad \text{or} \quad \alpha < \varphi(t,u) < 0 \quad \text{for all } (t,u) \in [0,T) \times I\!R.$$

This means that as in the small investor case, the large investor holds at any time before maturity a stock position which lies strictly between the two extreme payment obligations of 0 and α stocks, which the large investor might face at maturity. □

Remark. In the particular case where $\psi(t,u,\xi) = S_0 e^{\sigma u + (\mu - r)t + g\xi}$ for all $(t,u,\xi) \in [0,T] \times I\!R^2$, Jonsson et al. (2004) have independently derived the same formula (4.23) for the value of α European call options if the large investor always trades at the benchmark price. As discussed in the remark concluding Section 1.3.5, in order to justify the validity of the benchmark price at maturity, these authors assume that the holder of the option has the right to immediately sell the α shares of stock back to the large investor and receive in exchange the corresponding real value in cash. □

The previous example works especially well because it only considers the special setting of Assumption E where the large investor does not have to bear any transaction losses. Under Assumption F, however, the large investor is exposed to some nonnegative implied transaction losses, and the term $x\kappa\big(\bar{w}_x(t,x)\big)\bar{w}_{xx}(t,x)$ in (4.21) does not vanish. In that case we can still prove the existence of solutions to (4.21), (4.22) via the final value problem (4.6), (4.7) whenever the second derivative $\frac{d^2}{dx^2}h^\alpha$ of the boundary function $h^\alpha : \psi(T, I\!R) \to I\!R$, $x \mapsto h^\alpha(x)$, in (4.22) stays sufficiently small. Of course, this restriction excludes all standard calls and puts, but as noted by Frey (1998), this is more a technical issue than a strong limitation on the applicability of our model, since we might smooth out the kink in the payoff of a call or put by replacing for example the payoff condition $h^\alpha(x) = g(\alpha)(x-K)^+$ by $h^\alpha(x) = \frac{1}{2}g(\alpha)\big(\sqrt{\varepsilon + (x-K)^2} + x - K\big)$ for some small $\varepsilon > 0$. This smoothing would also accommodate the fact that traders will stop their delta hedging of a call close to maturity if its gamma becomes too large, and in particular if the stock price is close to the strike price. The term for the implied transaction losses in the generalized Black-Scholes equation (4.21) will change the volatility from $\bar{\sigma} : \bar{\mathcal{D}} \to (0,\infty)$ in the setting without transaction losses to $\sigma : \bar{\mathcal{D}} \to (0,\infty)$ in the setting of Assumption F, where

$$\sigma^2(t,x) = \Big(1 + x\kappa\big(\bar{w}_x(t,x)\big)\bar{w}_{xx}(t,x)\Big)\bar{\sigma}^2(t,x) \quad \text{for all } (t,x) \in \bar{\mathcal{D}}. \tag{4.24}$$

Under our standing assumptions we have $L_0 := \big\|\frac{\overline{\psi}}{\overline{\psi}_u}\big\| < \infty$, hence we get $\bar{\sigma}(t,x) \geq \frac{1}{L_0}$ for all $(t,x) \in \bar{\mathcal{D}}$. On the other hand, under Assumption F we conclude from $\zeta \in C_b^1(I\!R)$ that (4.20) even holds for all $(t,u) \in [0,T] \times I\!R$, and together with Definition 3.17, the definition $\gamma = g \circ \varphi$ and the multiplicative structure of ψ we obtain

$$1 + \overline{\psi}(t,u)\kappa\big(\bar{w}_x(t,\overline{\psi}(t,u))\big)\bar{w}_{xx}\big(t,\overline{\psi}(t,u)\big) = 1 + 2d(\mu)\frac{\psi_\xi\big(t,u,\varphi(t,u)\big)}{\psi_u\big(t,u,\varphi(t,u)\big)}\varphi_u(t,u) > \varepsilon.$$

Under Assumption E the left-hand side is 1, because of $\kappa \equiv 0$, thus we have proved that under our standing assumptions the function $\sigma^2 : \bar{\mathcal{D}} \to I\!R$ can be bounded away from 0.
Since the loss function $\kappa : g(I\!R) \to I\!R$ is nonnegative and since $\bar{\mathcal{D}} = [0,T] \times (0,\infty)$, it follows that $\sigma^2(t,x) \geq \bar{\sigma}^2(t,x)$ if $\bar{w}_{xx}(t,x) \geq 0$, or equivalently if $\gamma_u\big(t,u(t,x)\big) \geq 0$. Now Lemma 3.21 yields that $\gamma_u(t,u) \geq 0$ for all $(t,u) \in [0,T] \times I\!R$ is already implied by $\alpha\zeta_u(t,u) \geq 0$ for all $(t,u) \in I\!R$, hence it follows from the definition of $\eta : \overline{\psi}(T,I\!R) \to I\!R$ and the boundary condition (4.22) that $\bar{w}_{xx}(t,x) \geq 0$ for all $(t,x) \in \bar{\mathcal{D}}$ if only $\alpha\eta : \overline{\psi}(T,I\!R) \to I\!R$ is nondecreasing, or equivalently if $h^\alpha : \overline{\psi}(T,I\!R) \to I\!R$ is convex. Similarly, it follows that $\bar{w}_{xx}(t,x) \leq 0$ for all $(t,x) \in \bar{\mathcal{D}}$ if $\alpha\eta : \overline{\psi}(T,I\!R) \to I\!R$ is nonincreasing, which is equivalent to requiring that $h^\alpha : \overline{\psi}(T,I\!R) \to I\!R$ is concave. These observations can be used to show that the implied transaction losses raise the real value of the replicating portfolio of a convex contingent claim in the continuous limit model, and likewise they reduce the limiting real value of portfolios which replicate concave contingent claims. Especially, in the limit model, the real value for replicating a smoothed long call is higher and the one for a smoothed short call is lower than the corresponding replication value in the associated small investor model of Black-Scholes type. This coincides with our intuition, since in replicating a long call the large investor

has to buy stocks when the stock price rises, which again leads to a further rise of the stock prices, while in replicating a short call, the large investor's strategy is anticyclic and reduces the volatility in the market.

The additional implied term induced by the transaction losses might also be a good explanation for the smile effect, which says that if the Black-Scholes formula is used to derive market volatilities from European call prices in real markets, the implied volatility is normally a U-shaped function of the option's strike.

Remark 1. We have already encountered the connection between our discrete large investor model and standard small investor models with transaction losses in Section 2.3. So it is no surprise that the final value problem for the limit of the real value functions of the discrete models, if the discrete grid becomes finer and finer, resembles the corresponding final value problems for the replication value of options in continuous-time small investor models with transaction costs. In particular, our results on the function $\bar{w}$ and the volatility in the limit model match up with the results on proportional transaction costs model such as the ones of Leland (1985), Boyle and Vorst (1992) and Opitz (1999), and our limiting non-linear PDE for $\bar{w}$ is similar to the non-linear Black-Scholes equation derived by Barles and Soner (1998) as a limit of utility-maximization-based option prices in certain proportional transaction cost models.

Upon transforming Boyle and Vorst's scaling $h = \frac{T}{n}$ into our scaling $\delta^2 = \frac{1}{n}$, we recall from Boyle and Vorst (1992) that in a Cox-Ross-Rubinstein model with n time steps, constant volatility $\bar{\sigma}_0$ and some constant transaction cost rate k, the replication prices of a long and a short European call are for large values of n and small values of k approximately given by the Black-Scholes prices with increased or decreased volatility σ given by

$$\sigma^2 = \bar{\sigma}_0^2\Big(1 + \frac{1}{\bar{\sigma}_0}2k\sqrt{n}\Big) \quad \text{and} \quad \sigma^2 = \bar{\sigma}_0^2\Big(1 - \frac{1}{\bar{\sigma}_0}2k\sqrt{n}\Big), \tag{4.25}$$

respectively. Opitz (1999) extends this model and shows that if the transaction cost rates for purchase and sale differ, the factor $2k$ has to be replaced by the sum of the rates for purchase and sale.

Comparing the equations for σ^2 in (4.25) with those in (4.24) we observe that the transaction cost rate k plays the same role as the expression $\frac{1}{2}\delta\kappa\big(w_x(t,x)\big)\big|xw_{xx}(t,x)\big|$ in (4.24). By (4.20), by the definition of the loss function $\kappa : I\!R \to I\!R$ in (3.1) and by $\gamma = g \circ \varphi$ the last expression becomes $\frac{1}{2}\delta\kappa\big(w_x(t,x)\big)\big|xw_{xx}(t,x)\big| = \delta d(\mu)\frac{f'(\varphi(t,u))}{f(\varphi(t,u))}|\varphi_u(t,u)|$, which again is for large n approximately given by $k_\mu\big(\varphi(t-\delta^2,u),\varphi(t,u\pm\delta)\big)$ due to the expansion (1.2.18) in Proposition 1.15. Hence the transaction cost rate k in Boyle and Vorst (1992) corresponds to the local implied transaction losses $k_\mu\big(\varphi(t-\delta^2,u),\varphi(t,u\pm\delta)\big)$. This actually was the main motivation to introduce the local implied transaction loss rate $k_\mu : [0,T] \times I\!R^2 \to I\!R$ in (2.4.6).

Barles and Soner (1998) investigate the utility maximization approach of Hodges and Neuberger (1989) to show that if the product γN of risk aversion and the number of shares of stocks sold tends to 0 and if the proportional transaction costs are given by $\mu = a\sqrt{\gamma N}$ for some $a \in I\!R$, then in the limit the value function for a short European call (either with physical delivery or with cash settlement) will satisfy a non-linear PDE similar to (4.21): only the expression $x\kappa\big(\bar{w}_x(t,x)\big)\bar{w}_{xx}(t,x)$ is replaced by some more complicated function of $a^2x^2\bar{w}_{xx}(t,x)$.

The final value problem (4.21), (4.22) also resembles the final value problem for a European call in the large investor model of Jonsson and Keppo (2002). These authors suppose that the large investor's trades change the relative excess return of the stock by some exponential factor, and for the value function associated to a European call they then obtain a non-linear

partial differential equation of the form (4.21), where the term $1 + x\kappa\big(\bar{w}_x(t,x)\big)\bar{w}_{xx}(t,x)$ is replaced by some exponential of the derivative $\bar{w}_x(t,x)$.
Platen and Schweizer (1998), Frey and Patie (2002) and Liu and Yong (2004) have explained the smile pattern in various large investor models by the feedback of the large investor's trading strategy on the stock price. □

Remark 2. As a last remark in this section, we should spend some more time on our choice of parametrizing the real value function $\bar{v} : [0,T] \times I\!R \to I\!R$, $(t,u) \mapsto \overline{\psi}(t,u)$, in terms of the small investor stock prices $\overline{\psi}(t,u)$ by introducing the transform $\bar{w} : \bar{\mathcal{D}} \to I\!R$ which satisfies $\bar{w}\big(t, \overline{\psi}(t,u)\big) = \bar{v}(t,u)$ for all $(t,u) \in [0,T] \times I\!R$. The small investor prices would only appear in the market if the large investor did not trade in the stock at all. However, since the large investor actively trades in the market according to the limit strategy function $\varphi : [0,T] \times I\!R \to I\!R$, it would also make sense to parametrize the value function in terms of time t and the equilibrium stock price $\psi\big(t,u,\varphi(t,u)\big)$ at that time. This would give a parametrization in terms of a price which is not only observable by the large investor, but also by the small investors.
There are several reasons which have prevented us from using such a parametrization for the real value function. On the one hand, we would need to further restrict the possible strategy functions $\varphi : [0,T] \times I\!R \to I\!R$ for the large investor in order to guarantee that the price function $u \mapsto \psi\big(T,u,\phi(t,u)\big)$ is invertible. In particular, we would have to put more restrictions on α and ζ of the boundary condition (4.4); for example we would need to exclude any short European call, where $\zeta : I\!R \to I\!R$ is given as in Example 3.1 and $\alpha < 0$. But even if the derivative $\frac{d}{du}\psi\big(t,u,\varphi(t,u)\big)$ is strictly positive for all $(t,u) \in [0,T] \times I\!R$ so that we can indeed invert $u \mapsto \psi\big(T,u,\phi(t,u)\big)$, the analysis of the real value and strategy functions in terms of some functions $\tilde{w}$ and $\tilde{\varphi}$ which satisfy $\tilde{w}\big(t,\psi(t,u,\varphi(t,u))\big) = \bar{v}(t,u)$ and $\tilde{\varphi}\big(t,\psi(t,u,\varphi(t,u))\big) = \varphi(t,u)$ for all $(t,u) \in [0,T] \times I\!R$ becomes very unpleasant, since the parameter $x = \psi\big(t,u,\varphi(t,u)\big)$ of $\tilde{w}$ and $\tilde{\varphi}$ itself depends on $\varphi(t,u) = \tilde{\varphi}(t,x)$. In particular, this complicates the relationship between the parametrization $\tilde{\varphi}$ of the large investor's strategy and the parametrization $\tilde{w}$ of the associated real value.
If the large investor always trades at the benchmark price so that the loss function κ vanishes, the transfer from the partial differential equation (4.6) or (4.18) to a partial differential equation for $\tilde{w}$ remains comparatively simple. In such a situation we get for all $(t,u) \in [0,T) \times I\!R$ that

$$
\begin{aligned}
0 = \tilde{w}_t(t,x) &+ \frac{1}{2}\left(\frac{d}{du}\psi\big(t,u,\varphi(t,u)\big)\right)^2 \tilde{w}_{xx}(t,x) \\
&+ \tilde{w}_x(t,x)\left(\frac{d}{dt}\psi\big(t,u,\varphi(t,u)\big) + \frac{1}{2}\frac{d^2}{du^2}\psi\big(t,u,\varphi(t,u)\big) - \frac{\psi_t + \frac{1}{2}\psi_{uu}}{\psi_u}\frac{d}{du}\psi\big(t,u,\varphi(t,u)\big)\right),
\end{aligned}
$$

where $x = x(t,u) = \psi\big(t,u,\varphi(t,u)\big)$, and where the missing arguments for the partial derivatives of $\psi : [0,T] \times I\!R^2 \to I\!R$ are also given by $\big(t,u,\varphi(t,u)\big)$. Note that the drift term $\tilde{w}_x(t,x)$ disappears if $\frac{\psi_\xi}{\psi_u} \equiv const$, since then $\psi_{u\xi} = \frac{\psi_{uu}}{\psi_u}\psi_\xi$ and $\psi_{\xi\xi} = \frac{\psi_{u\xi}}{\psi_u}\psi_\xi$, and the term in brackets is seen to become 0 due to the partial differential equation (4.3) for $\varphi : [0,T] \times I\!R \to I\!R$. In this particular case $\tilde{w}$ satisfies an equation of Black-Scholes type with a time-space-dependent volatility which depends on the strategy $\varphi : [0,T] \times I\!R \to I\!R$ and its derivative φ_u, or equivalently on the transform $\tilde{\varphi}$ of the strategy function φ and its derivative $\tilde{\varphi}_x$. Though this case is the easiest, without reparametrization the standard Black-Scholes analysis fails to find solutions of the corresponding final value problem for $\tilde{w}$ or to derive the limiting replicating strategy function $\tilde{\varphi}$. This does not give much hope for more general situations, where the drift term in the partial differential equation for $\tilde{w}$ need not vanish at all.

Although we reject the parametrization in terms of $\overline{\psi}\big(t, u, \varphi(t, u)\big)$ for the limiting *real* value function, we shall see in Section 4.2.5 that exactly this parametrization is beneficial for analyzing the limit of the *paper* value functions $\{v^n\}_{n\in\mathbb{N}}$ of (1.3.19). $\square$

Chapter 4

Convergence of the Binomial Model

In this chapter we investigate the convergence in distribution of the sequence of binomial large investor models under two different regimes of martingale measures.
Our convergence results rely on a convergence theorem for general correlated random walks. These random walks, for which the direction of the next move depends on time, space, and the direction of the previous move, are introduced in Section 4.1. The convergence theorem for this class of random walks, stated in Section 4.1 and proved in Chapter 5, shows that a sequence of such (re-scaled) general correlated random walks converges to a diffusion process with explicitly given coefficients if the one-step transition probabilities can be approximated for large $n \in \mathbb{N}$ by a suitable function of time, space and the direction of the previous move.
Section 4.2 deals with the convergence in distribution under the p-martingale measure which is implied by the large investor's strategy. Under each p-martingale measure the fundamental process is a general correlated random walk. If we impose conditions as in Chapter 3 which imply the convergence of the large investor's discrete strategy and real value functions to solutions of certain final value problems, then the transition probabilities for the fundamental process have the asymptotic behavior required for an application of the convergence theorem for correlated random walks. From this theorem, it follows that the sequence of (rescaled) fundamental processes converges in distribution to a diffusion limit. The coefficients of the limit process are explicitly given and depend on the limit of the large investor's strategy functions. From the convergence of the fundamental process we can then deduce the convergence in distribution of all other model-relevant processes like price, strategy and value. We find that our limit model does not only extend the Black-Scholes model, but also extends many of the standard continuous-time large investor models found in the literature.
In Section 4.3 we consider the converge in distribution of the sequence of binomial large investor models under the s-martingale measures. Under each of these measures, the fundamental processes is again a general correlated random walks, but of a very simple structure, since its increments are not correlated at all. We again apply the convergence theorem for general correlated random walks and show that the fundamentals converge to a Brownian motion with drift. From this, we derive the convergence of strategy, price, and real value. Like in the discrete case, the real value is in general a supermartingale under the s-martingale measure , and it is a martingale if the price system excludes transaction losses.
As in Chapters 2 and 3 we again take $T = 1$ to limit the notational burden.

4.1 Convergence for General Correlated Random Walks

In order to prove the convergence of our discrete binomial large investor models, we appeal to a powerful convergence theorem for a certain class of random walks, which we shall refer to as *general correlated random walks.* These random walks are called correlated, since their transition probabilities depend on the direction of the random walk's previous move, and they are called general, since their one-step transition probabilities can also depend on time and the random walk's previous position in space. The convergence result is remarkable, since it does not assume that the random walk is asymptotically uncorrelated in the sense that the dependence of the transition probabilities on the direction of the random walk's previous move vanishes asymptotically.

Because the theory of general correlated random walks is a topic on its own, we defer a precise definition and a discussion of those random walks to Chapter 5. In the current section we only define a sequence of correlated random walks which have the same structure as the fundamental processes in our discrete binomial models, and then state a convergence theorem for this class of random walks which directly follows from the main convergence theorem for general correlated random walks in Chapter 5. This will be sufficient to show the convergence in distribution of our discrete large investor models in Sections 4.2 and 4.3.

Let us fix some $u_0, \mu_0 \in I\!R$ and some $\sigma \geq 0$, and as in the previous chapters let us denote $\delta = \delta_n = n^{-\frac{1}{2}}$ and $t_k^n = \frac{k}{n}$ for all $0 \leq k \leq n$ and $n \in I\!N$. Suppose now that for each $n \in I\!N$ we are given some function $p^n : [0,T) \times I\!R \times \{\pm 1\} \to [0,1]$. Then for each $n \in I\!N$ we can recursively define a *general correlated random walk* $X^n = \{X_k^n\}_{0 \leq k \leq n}$ with values in $I\!R$ and its associated *tilt process* $Z^n = \{Z_k^n\}_{0 \leq k \leq n}$ on some probability space $(\Omega^n, \mathcal{F}^n, \mathbf{P}^n)$ by taking some $\{\pm 1\}$-valued random variable Z_0^n, setting $X_0^n = u_0$, and then defining step by step the random variables $Z_k^n \in \{\pm 1\}$ denoting the tilt at step k (or at time t_k^n) by

$$\mathbf{P}^n\big(Z_k^n = 1 \,\big|\, \{Z_i^n\}_{0 \leq i \leq k-1}\big) = \mathbf{P}^n\big(Z_k^n = 1 \,\big|\, X_{k-1}^n,\, Z_{k-1}^n\big) = p^n\big(t_{k-1}^n, X_{k-1}^n, Z_{k-1}^n\big) \tag{1.1}$$

and the random variable X_k^n, which denotes the position of the correlated random walk X^n at time t_k^n, by

$$X_k^n = X_0^n + \mu_0 t_k^n + \sigma\delta \sum_{i=1}^{k} Z_i^n, \quad \text{for all } 1 \leq k \leq n.$$

Thus the direction of the random walk's move at time t_k^n, which is indicated by the tilt Z_k^n, depends on the direction Z_{k-1}^n of the previous move. Moreover, in contrast to a homogeneous random walk, the tilt Z_k^n may also depend on the time t_{k-1}^n and the position X_{k-1}^n of the random walk at t_{k-1}^n. This explains why we call the random walk X^n a *general correlated random walk*. An extensive discussion of correlated random walks and similar concepts in the literature is given in the introductory Sections 5.1 and 5.2 of Chapter 5.

Our convergence theorem for general correlated random walks is formulated in terms of continuous-time stochastic processes with paths in the space $D[0,T]$ of càdlàg functions $f : [0,T] \to I\!R$, i.e. of functions that are right-continuous and have left limits. Here the space $D[0,T]$ is endowed with the Skorohod topology. In order to transform our sequence $\{X^n\}_{n \in I\!N}$ of discrete general correlated random walks into a sequence of processes in $D[0,T]$ we define for each $n \in I\!N$ the continuous-time stochastic processes $U^n = \{U_t^n\}_{t \in [0,T]}$ in terms of the correlated random walk $X^n = \{X_k^n\}_{0 \leq k \leq n}$ by setting

$$U_t^n = X_{\lceil nt \rceil}^n \quad \text{for all } 0 \leq t \leq T. \tag{1.2}$$

Remark. Recall our standing assumption $T = 1$. For a general $T > 0$, we would set $U_t^n = X_{\lceil \tilde{n}t \rceil}^n$, where $\tilde{n}$ is for all $n \in I\!N$ given by $\tilde{n} = T^{-1}\lceil nT \rceil$. Then the following convergence theorem also covers the general case. □

Now we can state the convergence theorem for general correlated random walks which we shall need in Sections 4.2 and 4.3:

Theorem 4.1. *Suppose the functions $a : [0,T] \times I\!R \to I\!R$ and $b : [0,T] \times I\!R \to I\!R$ belong to the Hölder spaces $H^{\frac{1}{2}(1+\beta),1+\beta}\big([0,T] \times I\!R\big)$ and $H^{\frac{1}{2}\beta,\beta}\big([0,T] \times I\!R\big)$, respectively, and assume*

$\|a\| < 1$. *If the probability functions* $p^n : [0,T) \times I\!R \times \{\pm 1\} \to [0,1]$ *of* (1.1) *converge uniformly for all* $(t,u) \in [0,T) \times I\!R$ *in such a way that*

$$p^n(t,u,\pm 1) = \frac{1}{2}\Big(1 \pm a(t,u) + \delta b(t,u)\Big) + O\big(\delta^{1+\beta}\big) \quad \textit{as } n \to \infty, \tag{1.3}$$

then the sequence of processes $\{U^n\}_{n\in I\!N}$ *of* (1.2)*, starting in* u_0 *at time* 0*, converges in distribution to the process* $U = \{U_t\}_{t\in[0,T]}$ *given by* $U_0 = u_0$ *and*

$$dU_t = \left(\mu_0 + \frac{\sigma b(t,U_t)}{1-a(t,U_t)} + \frac{\sigma^2 a_u(t,U_t)}{\big(1-a(t,U_t)\big)^2}\right) dt + \sigma\sqrt{\frac{1+a(t,U_t)}{1-a(t,U_t)}}\, dW_t, \tag{1.4}$$

where $W = \{W_t\}_{t\in[0,T]}$ *is a standard Brownian motion on* $[0,T]$*. In particular, there exists a weak solution of* (1.4)*.*

As usual, we write $U^n \Rightarrow U$ in order to denote that a sequence $\{U^n\}_{n\in I\!N}$ of stochastic processes in $D[0,T]$ (or more generally in $D^d[0,T]$) converges in distribution to U. Of course, convergence in distribution depends on the underlying probability measures. If we want to emphasize the corresponding sequence $\{\mathbf{P}^n\}_{n\in I\!N}$ of probability measures under which $\{U^n\}_{n\in I\!N}$ converges in distribution, we shall adopt the notation

$$\big(U^n \,\big|\, \mathbf{P}^n\big) \Rightarrow \big(U \,\big|\, \mathbf{P}\big) \quad \text{as } n \to \infty,$$

where $\mathbf{P}$ is the distribution of the limit process U.
We shall come back to correlated random walks in Chapter 5, where we also provide the reader with a proof of Theorem 4.1. For now, we immediately proceed to an application of this theorem in order to show the convergence of our binomial large investor models.

4.2 Convergence under the p-Martingale Measures

The p-martingale measure is the unique measure under which both the large investor price process and the paper value process are martingales. In this section we show that our discrete binomial models converge in distribution to a stochastic diffusion model if the large investor asymptotically replicates the same contingent claim. Here the convergence means that the sequence of distributions of the fundamental processes, the stock price processes and the paper value processes under the associated p-martingale measures converge weakly.
The assumptions necessary to prove such a convergence statement are summarized in Section 4.2.1. Especially, we suppose that the large investor's discrete strategy functions converge at *and* immediately before maturity to the corresponding values of the continuous strategy function φ of Chapter 3, so that the strategy functions converge as in Chapter 3 on the whole interval $[0,T]$. Whenever the stock price does not immediately adjust to an order of the large investor, we also require the convergence of the large investor's stock holdings just before time 0. Section 4.2.2 yields that for all sufficiently large $n \in I\!N$ the large investor's discrete trading strategy $\big(\xi^n, b^n\big)$ is p-admissible so that the associated p-martingale measure is well-defined. In Section 4.2.3 we find that under the p-martingale measure the fundamental process describes a general correlated random walk. Then an application of Theorem 4.1 shows that the sequence of fundamental processes converges in distribution to a diffusion process. The diffusion coefficients are given explicitly and depend on φ. In Section 4.2.4 we deduce the convergence of the sequences of tuples of fundamental, price, value and strategy process from the convergence of the fundamentals, since these processes are all more or less complicated functions of the fundamentals. Section 4.2.5 considers the limiting paper value

function as a function of time and large investor stock price. We shall see that this function solves a generalized Black-Scholes equation. Other continuous-time large investor models of the literature turn out to be special cases of our limit model. Finally, in Section 4.2.6 we explore the resulting limit model; we investigate existence and uniqueness of the stochastic differential equation for the fundamentals under weaker regularity conditions, give martingale representations for the stock price and paper value process, and discuss the problems of our model with regard to the large investor's trades at time 0 and at maturity.

4.2.1 General Assumptions and Definitions

In this section, we state the general setting used for our convergence results under the p-martingale measures. First we describe the evolution of the fundamentals over time in the discrete binomial models. Then we present the Assumptions G to L under which we derive our convergence result. Some technical definitions conclude the section.

We start with expanding our definition of the fundamental process in the discrete models to corresponding processes in the space $D[0,T]$ of càdlàg functions on $[0,T]$. Therefore, let us recall from Section 1.3.2 for each $n \in I\!N$ the filtered probability space $\left(\Omega^n, \mathcal{F}_n^n, \mathcal{F}^n, \mathbf{P}^n\right)$, the tilt process $Z^n = \{Z_k^n\}_{0\leq k\leq n}$ on $\left(\Omega^n, \mathcal{F}_n^n\right)$, consisting of $\{\pm 1\}$-valued random variables, and the fundamental process $U^n = \{U_k^n\}_{0\leq k\leq n}$, which is given by

$$U_k^n = u_0 + \delta \sum_{j=1}^{k} Z_j^n \quad \text{for all } 0 \leq k \leq n. \tag{2.1}$$

Here $u_0 \in I\!R$ is some arbitrary fixed real number, and δ is as usual given by $\delta = \delta_n = n^{-\frac{1}{2}}$ for each $n \in I\!N$. As in (1.2) we now also define the continuous-time process $U^n = \left\{U_t^n\right\}_{t\in[0,T]}$ with paths in the space $D[0,T]$ by setting

$$U_t^n := U_{\lceil nt\rceil}^n \quad \text{for all } t \in [0,T] \tag{2.2}$$

and all $n \in I\!N$. Note that the definition of U^n leads to some ambiguities, but it will be clear from the context whether we consider the continuous-time jump process $U^n = \left\{U_t^n\right\}_{t\in[0,T]}$ or the discrete random walk $U^n = \left\{U_k^n\right\}_{0\leq k\leq n}$.

For the entire sequence of discrete large investor models, we have one underlying price system (ψ, μ). In order to show the convergence in distribution of our discrete models we need to impose:

Assumption G (On the price system (ψ, μ)). *There exist some strictly positive functions $\overline{\psi} \in \widehat{H}^{2+\frac{1}{2}\beta,4+\beta}\big([0,T]\times I\!R\big)$ and $f \in H_{loc}^{4+\beta}(I\!R)$ such that the equilibrium price function $\psi : [0,T]\times I\!R^2 \to I\!R$ satisfies the multiplicative structure*

$$\psi(t,u,\xi) = \overline{\psi}(t,u)f(\xi) \quad \textit{for all } (t,u,\xi) \in [0,T]\times I\!R^2.$$

We also have $L_0 := \left\|\frac{\overline{\psi}}{\overline{\psi}_u}\right\| < \infty$. For the measure μ there exists some $\eta > 0$ such that $\int e^{\eta|\theta|}\mu(d\theta) < \infty$. The price system (ψ, μ) excludes any immediate transaction gains, i.e. by the remark following Definition 3.17 we have in particular $d(\mu)f'(\xi) \geq 0$ for all $\xi \in I\!R$.

Having agreed on the price system, we now consider the large investor's strategy. We start with fixing the shape $\zeta : I\!R \to I\!R$ of the large investor's stock holdings at maturity in the continuous limit model as a function of the fundamental value. For a sufficiently small scaling parameter $|\alpha|$, Proposition 3.27 guarantees the existence of a sufficiently smooth function $\varphi : [0,T]\times I\!R \to I\!R$ which coincides with $\alpha\zeta$ at maturity and which satisfies the partial differential equation (3.4.3). This function φ will be our candidate for the large investor's limiting strategy function in continuous time:

Assumption H (Solvability of the non-linear PDE for φ). *For $\zeta \in H^{4+\beta}(I\!R)$ the parameter $\alpha \in I\!R$ is chosen so close to 0 that there exists some $\varphi \in H^{2+\frac{1}{2}\beta,4+\beta}\big([0,T] \times I\!R\big)$ which solves the final value problem (3.4.3), (3.4.4), which satisfies*

$$\alpha \inf_{z\in I\!R} \zeta(z) \le \varphi(t,u) \le \alpha \sup_{z\in I\!R} \zeta(z) \quad \textit{for all } (t,u) \in [0,T] \times I\!R, \tag{2.3}$$

and for which there exists some $\varepsilon > 0$ such that

$$2d(\mu) \frac{\psi_\xi\big(t,u,\varphi(t,u)\big)}{\psi_u\big(t,u,\varphi(t,u)\big)} \varphi_u(t,u) \ge -1+\varepsilon \quad \textit{for all } (t,u) \in [0,T] \times I\!R. \tag{2.4}$$

Assumption I (The p-martingale measures are well-defined). *The scaling parameter $\alpha \in I\!R$ from Assumption H is also chosen so close to 0 that*

$$\frac{\psi_\xi\big(t,u,\varphi(t,u)\big)}{\psi_u\big(t,u,\varphi(t,u)\big)} \varphi_u(t,u) \ge -1+\varepsilon \quad \textit{for all } (t,u) \in [0,T] \times I\!R. \tag{2.5}$$

Remark. Note that we have shown in Proposition 3.27 that for all sufficiently small $|\alpha| > 0$ the final value problem for φ given by (3.4.3), (3.4.4) has a solution which satisfies (2.3) and $2L_0 \tilde{L}'_\kappa(\alpha) \inf_{(t,u)\in[0,T)\times I\!R} \varphi_u(t,u) > -1$. By the definition of L_0 and of $\tilde{L}'_\kappa$ in (3.3.64), the latter inequality implies that there exists some $\varepsilon > 0$ such that

$$1 + 4d(\mu) \frac{\psi_\xi\big(t,u,\varphi(t,u)\big)}{\psi_u\big(t,u,\varphi(t,u)\big)} \varphi_u(t,u) \ge \varepsilon. \tag{2.6}$$

This obviously implies (2.4). If $4d(\mu) \ge 1$, we can conclude from (2.6) that (2.5) holds as well. On the other hand, if we have the opposite inequality $4d(\mu) < 1$, the interval $(\tilde{\alpha}_1, \tilde{\alpha}_2)$ from which we can choose the scaling parameter α to guarantee the existence of a solution φ to (3.4.3), (3.4.4) might be too big to guarantee that φ satisfies (2.5) as well. However, (2.5) will hold if the scaling parameter is taken small enough: It can be checked from the proof of Proposition 3.27 and Proposition 3.19 that we can still find an open subinterval $(\bar{\alpha}_1, \bar{\alpha}_2)$ with $0 \in (\bar{\alpha}_1, \bar{\alpha}_2) \subset (\tilde{\alpha}_1, \tilde{\alpha}_2)$ such that all solutions to the final value problem (3.4.3), (3.4.4) with $\alpha \in (\bar{\alpha}_1, \bar{\alpha}_2)$ satisfy condition (2.5).
Of course, condition (2.5) ensures that the derivative $\frac{d}{du}\psi\big(t,u,\varphi(t,u)\big)$ remains bounded away from 0. In particular, it guarantees that we can invert the function $u \mapsto \psi\big(t,u,\varphi(t,u)\big)$. □

For each of our discrete large investor models, we can now take some path-independent portfolio strategy $\big(\xi^n, b^n\big)$ as introduced in Definition 1.23, and we define the associated strategy function $\xi^n : \mathcal{A}^n \to I\!R$ and the cash holdings function $b^n : \mathcal{A}^n \to I\!R$ accordingly. We only need to ensure the convergence of the strategy functions $\{\xi^n\}_{n\in I\!N}$ to the continuous limit function φ immediately before and at maturity in order to conclude from Corollary 3.28 that the sequence $\{\xi^n\}_{n\in I\!N}$ converges to φ everywhere. In order to apply this corollary, we therefore require:

Assumption J (Stock holdings converge close to maturity). *Immediately before and at maturity, the large investor's stock holdings converge in the sense that*

$$\max_{k\in\{n-1,n\}} \big\|\xi^n\big(t_k^n, \cdot\big) - \varphi\big(t_k^n, \cdot\big)\big\|_{\mathcal{U}_k^n} = O\big(\delta^{4+\beta}\big) \quad \textit{as } n \to \infty. \tag{2.7}$$

If we now suppose in addition to Assumptions G to J that the final values of the sequence $\{b^n\}_{n\in I\!N}$ of the discrete cash holdings functions $b^n : \mathcal{A}^n \to I\!R$ converge to a certain function $b^\alpha : I\!R \to I\!R$, we can likewise conclude from Proposition 3.30 that the cash holdings functions converge on their full domain to a continuous limit, or equivalently that the sequence $\{\bar{v}^n\}_{n\in I\!R}$ of real value functions $\bar{v}^n : \mathcal{A}^n \to I\!R$, $(t,u) \mapsto \bar{v}^n(t,u) = \xi^n(t,u)\bar{S}\big(t,u,\xi^n(t,u)\big) + b^n(t,u)$, converges to a continuous limit $\bar{v}$. Thus, we also impose

Assumption K (Cash holdings converge at maturity). *For some $b_0^\alpha \in \mathbb{R}$ the function $b^\alpha : \mathbb{R} \to \mathbb{R}$ is given by*

$$b^\alpha(u) = b_0^\alpha - \alpha \int_{u_0}^{u} \psi\big(T, \tilde{u}, \alpha\zeta(\tilde{u})\big)\zeta_u(\tilde{u})\, d\tilde{u},$$

and $\bar{v} : [0,T] \times \mathbb{R} \to \mathbb{R}$ is the solution to the linear final value problem (3.4.6), (3.4.7) which corresponds to the continuous strategy function $\varphi : [0,T] \times \mathbb{R} \to \mathbb{R}$ via $\gamma(t,u) = \int_0^{\varphi(t,u)} f(z)\, dz$ for all $(t,u) \in [0,T] \times \mathbb{R}$. The cash holdings at maturity satisfy

$$\big\| b^n(t_n^n, \cdot\,) - b^\alpha(\cdot\,) \big\|_{\mathcal{U}_n^n} = O\big(\delta^{2+\beta}\big) \quad \text{as } n \to \infty. \tag{2.8}$$

In general, for each $n \in \mathbb{N}$, the large investor stock price $S_\mu\big(t_0^n, U_0^n, \xi_{-1}^n, \xi_0^n\big)$ at time $t_0^n = 0$ depends not only on the large investor's stock holdings ξ_0^n at time $t_0^n = 0$, but also on his stock holdings ξ_{-1}^n immediately before time 0. Only in the special situation where either the price determining measure is the Dirac measure δ_1 concentrated in 1 or the function $f : \mathbb{R} \to \mathbb{R}$ is constant, we can be sure that the large investor price at time 0 does not depend on ξ_{-1}^n. In the second case, we could also assume without loss of generality that $\mu = \delta_1$, since in this case the price determining measure has no influence on the large investor stock price at all. In order to obtain convergence in distribution of our discrete large investor models under the p-martingale measures, we therefore impose

Assumption L (Pre-trading behavior of the stock holdings). *One of the two following conditions holds:*

(*i*) *The price determining measure μ is (or can be chosen as) the Dirac measure δ_1 concentrated in 1, so that for all $(t, u, \xi_1, \xi_2) \in [0,T] \times \mathbb{R}^3$ the large investor stock price equals the equilibrium stock price directly after his trades, i.e. the large investor price function $S_\mu : [0,T] \times \mathbb{R}^3 \to \mathbb{R}$ is for all $(t, u, \xi_1, \xi_2) \in [0,T] \times \mathbb{R}^3$ given by the equation $S_\mu(t, u, \xi_1, \xi_2) = \psi(t, u, \xi_2)$.*

(*ii*) *For each $n \in \mathbb{N}$ there is some $Z_0^n \in \{\pm 1\}$ such that the large investor's stock holdings ξ_{-1}^n immediately before time $t_0^n = 0$ satisfy*

$$\xi_{-1}^n = \xi^n(0,u_0) - \delta Z_0^n \varphi_u(0,u_0) + \delta^2 \left(\frac{1}{2}\varphi_{uu}(0,u_0) - \varphi_t(0,u_0)\right) + O\big(\delta^{2+\beta}\big) \;\; \text{as } n \to \infty.$$

Remark. Assumption L(ii) is a delicate issue for several reasons. First of all, it is somewhat unsatisfactory that we have to worry about the large investor's stock holdings before time 0. However, this is clearly forced by our price building mechanism and the definition of the large investor stock price.

Secondly, if we do reluctantly have to take into account the large investor's stock holdings immediately before time 0, the natural approach would be to assume that the large investor did not trade at all in stocks before time 0, and therefore to assume $\xi_{-1}^n = 0$. Unfortunately, this approach does not yield meaningful results. Namely, by taking $\xi_{-1}^n = 0$ for all $n \in \mathbb{N}$, the asymptotic evolution in Assumption L(ii) implies that $\xi^n(0,u_0) \to 0$ as $n \to \infty$, and the convergence of $\{\xi^n\}_{n \in \mathbb{N}}$ in the sense that $\|\xi^n - \varphi\|_{\mathcal{A}^n} \to 0$ as $n \to \infty$ leads to $\varphi(0,u_0) = 0$ as well. If the final condition $\phi(T, \cdot\,) = \alpha\zeta$ is nonnegative, and if there exists some compact interval on which $\varphi(T, \cdot\,)$ is strictly positive, it can be easily seen from the Feynman-Kac formula that $\varphi(0,u_0) > 0$. Thus, if $\xi^n(T,\cdot) : \mathcal{U}_n^n \to \mathbb{R}$ were nonnegative and $\xi_{-1}^n = 0$ for all $n \in \mathbb{N}$, then Assumption L(ii) and the convergence of ξ^n to φ would imply that the large investor would asymptotically hold no shares at all between time 0 to time T, i.e. $\varphi \equiv 0$. Due

to the convergence of ξ^n to φ this would mean that also the strategy function $\xi^n : \mathcal{A}^n \to I\!R$ would asymptotically vanish as $n \to \infty$.
The same conclusion would follow if the final values $\xi^n(T, \cdot\,)$ of the discrete strategy function $\xi^n : \mathcal{A}^n \to I\!R$ were nonpositive for all $n \in I\!N$. This shows that under Assumption L(ii), at least for a large class of strategy functions $\xi^n : \mathcal{A}^n \to I\!R$, we can only have $\xi^n_{-1} = 0$ for all $n \in I\!N$ if the stock holdings $\xi^n : \mathcal{A}^n \to I\!R$ vanish as $n \to \infty$. But eliminating any stock trades of the large investor on the time interval $[0, T]$ in the limiting model is a constraint on the strategy functions which is far more restrictive than desirable.
However, if we want to show convergence on the open time interval $(0, T]$ only, we could drop Assumption L. In doing so we fade out how the hedging portfolio of the large investor is built up at time 0. Basically, Assumption L does nothing else: Under Assumption L(i) the large investor stock price at which the large investor can buy shares at time $t = 0$ does not depend on his portfolio before time 0. Under Assumption L(ii) the large investor needs only asymptotically small adjustments to his portfolio at time 0, and for this reason the issue how the large investor arrived at the stock position ξ^n_{-1} is put back to the distant past. In the situation we are most interested in, the strategy function $\xi^n : \mathcal{A}^n \to I\!R$ is used to replicate a certain contingent claim (ξ^n, b^n) at time T over the time interval $[0, T]$. In such a model it is important what happens at the time point 0, where the replicating portfolio is built up. Thus, we are really interested in the convergence on the closed time interval $[0, T]$ and hence, for $\mu \neq \delta_1$, we need to employ Assumption L(ii) on the stock holdings immediately before time 0. This reveals the difficulties which occur at time 0 even in the limit. We will come back to this point at the end of Section 4.2.6. □

Under Assumption L(ii) it is reasonable to extend for each $n \in I\!N$ the definition of the large investor's strategy function $\xi^n : \mathcal{A}^n \to I\!R$ to the time point $-\delta^2$ by defining the pre-trading fundamentals' value as $u_0 - \delta Z^n_0$ and setting $\xi^n\big(-\delta^2, u_0 - \delta Z^n_0\big) := \xi^n_{-1}$. If Assumption L$(ii)$ does not hold, but Assumption L(i) does, we also define $\xi^n\big(-\delta^2, u_0 - \delta Z^n_0\big)$ by setting

$$\xi^n\big(-\delta^2, u_0 - \delta Z^n_0\big) := \xi^n(0, u_0) - \delta Z^n_0 \varphi_u(0, u_0) + \delta^2 \left(\frac{1}{2}\varphi_{uu}(0, u_0) - \varphi_t(0, u_0)\right)$$

in order to avoid some distinction of cases as we proceed. By this definition we have guaranteed that both cases of Assumption L imply

$$\xi^n\big(-\delta^2, u_0 - \delta Z^n_0\big) - \xi^n(0, u_0) = -\delta Z^n_0 \varphi_u(0, u_0) + \delta^2 \left(\frac{1}{2}\varphi_{uu}(0, u_0) - \varphi_t(0, u_0)\right) + O\big(\delta^{2+\beta}\big) \quad (2.9)$$

as $n \to \infty$. We then define for all $n \in I\!N$ and all $0 \le m \le n$ the set $\widehat{\mathcal{A}}^n(m)$ by

$$\widehat{\mathcal{A}}^n(m) = \big\{(t, u, z) \in \mathcal{A}^n(m) \times \{\pm 1\} \,:\, \xi^n(t - \delta^2, u - z\delta) \text{ is defined}\big\}, \quad (2.10)$$

and in analogy to $\mathcal{A}^n = \mathcal{A}^n(n)$ of Definition 1.22 we may also write $\widehat{\mathcal{A}}^n$ instead of $\widehat{\mathcal{A}}^n(n)$.

4.2.2 Existence of the p-Martingale Measures

After the statement of all our assumptions, we first have to make sure that under these assumptions the p-martingale measures $\mathbf{P}^{\xi^n}_n$ of Definition 2.7 are well-defined, at least for all sufficiently large $n \in I\!N$, so that we can indeed consider the convergence in distribution under the p-martingale measures. Therefore, we show in this section by some asymptotic analysis that for all sufficiently large $n \in I\!N$ the large investor's trading strategy (ξ^n, b^n) is p-admissible. Our asymptotic analysis can then also be used to obtain the same asymptotic properties for the one-step transition probabilities of the tilt and fundamental value processes under $\mathbf{P}_n^{\xi^n}$ as they are required by the conditions of the convergence theorem for general correlated random walks in Section 4.1.

The following lemma yields that the large investor's portfolio strategy (ξ^n, b^n) is indeed p-admissible for all sufficiently large $n \in I\!N$.

Lemma 4.2. *Under Assumptions G to J and L there exists some $n_0 \in I\!N$ such that for each $n \geq n_0$ the self-financing trading strategy (ξ^n, b^n) is p-admissible.*

Proof. In order to show that a self-financing strategy (ξ^n, b^n) is p-admissible, it suffices to show that

$$S_\mu^{\xi^n}\big(t+\delta^2, u-\delta, \xi^n(t,u)\big) < S_\mu^{\xi^n}\big(t, u, \xi^n(t-\delta^2, u-z\delta)\big) < S_\mu^{\xi^n}\big(t+\delta^2, u+\delta, \xi^n(t,u)\big) \quad (2.11)$$

for all $(t,u,z) \in \widehat{\mathcal{A}}^n(n-1)$. We will show that

$$\begin{aligned} &\pm \frac{S_\mu^{\xi^n}\big(t, u, \xi^n(t-\delta^2, u-z\delta)\big) - S_\mu^{\xi^n}\big(t+\delta^2, u\mp\delta, \xi^n(t,u)\big)}{\delta\psi_u\big(t,u,\xi^n(t,u)\big)} \\ &\qquad = 1 + \left((1\pm z)d(\mu) + (1\mp z)\frac{1}{2}\right)\frac{\psi_\xi\big(t,u,\varphi(t,u)\big)}{\psi_u\big(t,u,\varphi(t,u)\big)}\varphi_u(t,u) + O(\delta) \quad \text{as } n\to\infty, \end{aligned} \quad (2.12)$$

uniformly for all $(t,u,z) \in \widehat{\mathcal{A}}^n(n-1)$. Because of $z \in \{\pm 1\}$ and the two bounds (2.4) and (2.5), we can then find some $n_0 \in I\!N$ such that for all $(t,u,z) \in \widehat{\mathcal{A}}^n(n-1)$ and $n \geq n_0$ we have

$$\pm \frac{S_\mu^{\xi^n}\big(t, u, \xi^n(t-\delta^2, u-z\delta)\big) - S_\mu^{\xi^n}\big(t+\delta^2, u\mp\delta, \xi^n(t,u)\big)}{\delta\psi_u\big(t,u,\xi^n(t,u)\big)} \geq \frac{1}{2}\varepsilon.$$

Since ψ_u is strictly positive, this leads to (2.11), and thus, for all $n \geq n_0$ all the trading strategies (ξ^n, b^n) are p-admissible if (2.12) holds uniformly for all $(t,u,z) \in \widehat{\mathcal{A}}^n(n-1)$.

In order to show that (2.12) actually holds, we have to employ the convergence of the strategy functions $\{\xi^n\}_{n\in I\!N}$ towards φ as guaranteed by Corollary 3.28, and then basically apply the techniques used in the proof of Lemma 3.11 to approximate the terms on the left-hand side of (2.12) by $\psi\big(t,u,\xi^n(t,u)\big)$. We develop these approximations up to such accuracy that they can also be used in Lemma 4.3, which is a little bit more than needed for the proof of (2.12) alone.

Let us recall from Corollary 3.28 that (3.3.69) holds, i.e. we have

$$\xi^n\big(t+\delta^2, u\pm\delta\big) - \xi^n(t,u) = \pm\delta\varphi_u(t,u) + \delta^2\left(\varphi_t(t,u) + \frac{1}{2}\varphi_{uu}(t,u)\right) + O\big(\delta^{2+\beta}\big) \quad (2.13)$$

as $n \to \infty$, uniformly for all $(t,u) \in \mathcal{A}^n(n-1)$. Applying Taylor's rule to expand the derivatives of $\varphi : [0,T] \times I\!R \to I\!R$ around $(t+\delta^2, u\pm\delta)$, we thus obtain uniformly for all $(t,u) \in \mathcal{A}^n(n-1)$:

$$\begin{aligned} \xi^n\big(t+\delta^2, u\pm\delta\big) - \xi^n(t,u) &= \pm\delta\varphi_u\big(t+\delta^2, u\pm\delta\big) \\ &\quad + \delta^2\left(\varphi_t(t+\delta^2, u\pm\delta) - \frac{1}{2}\varphi_{uu}(t+\delta^2, u\pm\delta)\right) + O\big(\delta^{2+\beta}\big) \end{aligned}$$

as $n \to \infty$. If we now substitute in the previous equation $t+\delta^2$ and $u\pm\delta$ by t and u, respectively, and if we also apply (2.9), then we get, in addition to (2.13), for all $(t,u,z) \in \widehat{\mathcal{A}}^n$, the expansion

$$\begin{aligned} &\xi^n(t-\delta^2, u-z\delta) - \xi^n(t,u) \\ &\qquad = -z\delta\varphi_u(t,u) - \delta^2\left(\varphi_t(t,u) - \frac{1}{2}\varphi_{uu}(t,u)\right) + O\big(\delta^{2+\beta}\big) \quad \text{as } n\to\infty. \end{aligned} \quad (2.14)$$

Due to the definition of the shorthand $S_\mu^{\xi^n}(t,u,\xi) := S_\mu\big(t,u,\xi,\xi^n(t,u)\big)$ in (2.1.1) we have

$$S_\mu^{\xi^n}(t,u,\xi) = \int \psi\big(t,u,(1-\theta)\xi + \theta\xi^n(t,u)\big)\mu(d\theta)$$

for all $(t,u,\xi) \in [0,T] \times I\!R^2$. By Taylor's rule we then obtain that for all $(t,u,\xi) \in \mathcal{A}^n \times I\!R$ there exists some function $\gamma : I\!R \to I\!R$, with $\gamma(\theta)$ lying between θ and 1 for all $\theta \in I\!R$, such that

$$\begin{aligned} S_\mu^{\xi^n}(t,u,\xi) &= \psi\big(t,u,\xi^n(t,u)\big) + \psi_\xi\big(t,u,\xi^n(t,u)\big)\big(\xi - \xi^n(t,u)\big)\int (1-\theta)\,\mu(d\theta) \\ &\quad + \frac{1}{2}\big(\xi-\xi^n(t,u)\big)^2 \int (1-\theta)^2 \psi_{\xi\xi}\big(t,u,(1-\gamma(\theta))\xi + \gamma(\theta)\xi^n(t,u)\big)\,\mu(d\theta). \end{aligned}$$

If we now set $\xi = \xi^n(t-\delta^2, u-z\delta)$ and divide the previous equation by $\delta\psi_u\big(t,u,\xi^n(t,u)\big)$, we can apply the techniques of the proof of Lemma 3.11 combined with (2.14) to conclude from $\psi \in \widehat{H}^{1+\frac{1}{2}\beta,2+\beta}\big([0,T]\times I\!R^2\big)$, $\varphi \in H^{1+\frac{1}{2}\beta,2+\beta}\big([0,T]\times I\!R\big)$, $\int e^{\eta|\theta|}\mu(d\theta) < \infty$ for some $\eta > 0$, and the boundedness of $\xi^n : \mathcal{A}^n \to I\!R$, which is induced by the convergence of $\|\xi^n - \varphi\|_{\mathcal{A}^n}$, that uniformly for all $(t,u,z) \in \widehat{\mathcal{A}}^n(n-1)$, we have

$$\begin{aligned} &\frac{S_\mu^{\xi^n}\big(t,u,\xi^n(t-\delta^2,u-z\delta)\big) - \psi\big(t,u,\xi^n(t,u)\big)}{\delta\psi_u\big(t,u,\xi^n(t,u)\big)} \\ &\quad = \int (\theta-1)\,\mu(d\theta)\frac{\psi_\xi\big(t,u,\xi^n(t,u)\big)}{\psi_u\big(t,u,\xi^n(t,u)\big)}\left(z\varphi_u(t,u) + \delta\left(\varphi_t(t,u) - \frac{1}{2}\varphi_{uu}(t,u)\right)\right) \\ &\qquad + \frac{1}{2}\delta\int (1-\theta)^2\,\mu(d\theta)\frac{\psi_{\xi\xi}\big(t,u,\xi^n(t,u)\big)}{\psi_u\big(t,u,\xi^n(t,u)\big)}\varphi_u^2(t,u) + O\big(\delta^{1+\beta}\big) \end{aligned}$$

as $n \to \infty$. Since $\|\xi^n - \varphi\|_{\mathcal{A}_n^n} = O\big(\delta^2\big)$ as $n \to \infty$, we can replace the $\xi^n(t,u)$'s on the right-hand side of the equation by $\varphi(t,u)$, and so we finally obtain

$$\begin{aligned} &\frac{S_\mu^{\xi^n}\big(t,u,\xi^n(t-\delta^2,u-z\delta)\big) - \psi\big(t,u,\xi^n(t,u)\big)}{\delta\psi_u\big(t,u,\xi^n(t,u)\big)} \\ &\quad = \int (\theta-1)\,\mu(d\theta)\frac{\psi_\xi\big(t,u,\varphi(t,u)\big)}{\psi_u\big(t,u,\varphi(t,u)\big)}\left(z\varphi_u(t,u) + \delta\left(\varphi_t(t,u) - \frac{1}{2}\varphi_{uu}(t,u)\right)\right) \\ &\qquad + \frac{1}{2}\delta\int (1-\theta)^2\,\mu(d\theta)\frac{\psi_{\xi\xi}\big(t,u,\varphi(t,u)\big)}{\psi_u\big(t,u,\varphi(t,u)\big)}\varphi_u^2(t,u) + O\big(\delta^{1+\beta}\big) \end{aligned} \tag{2.15}$$

as $n \to \infty$. Analogously, using (2.13) instead of (2.14) we can expand uniformly for all $(t,u,z) \in \widehat{\mathcal{A}}^n(n-1)$

$$\begin{aligned} &\frac{S_\mu^{\xi^n}\big(t+\delta^2,u\pm\delta,\xi^n(t,u)\big) - \psi\big(t,u,\xi^n(t,u)\big)}{\delta\psi_u\big(t,u,\xi^n(t,u)\big)} \\ &\quad = \pm 1 + \int \theta\,\mu(d\theta)\frac{\psi_\xi\big(t,u,\varphi(t,u)\big)}{\psi_u\big(t,u,\varphi(t,u)\big)}\left(\pm\varphi_u(t,u) + \delta\left(\varphi_t(t,u) + \frac{1}{2}\varphi_{uu}(t,u)\right)\right) \\ &\qquad + \delta\left(\frac{\psi_t\big(t,u,\varphi(t,u)\big) + \frac{1}{2}\psi_{uu}\big(t,u,\varphi(t,u)\big)}{\psi_u\big(t,u,\varphi(t,u)\big)} + \int \theta\,\mu(d\theta)\frac{\psi_{u\xi}\big(t,u,\varphi(t,u)\big)}{\psi_u\big(t,u,\varphi(t,u)\big)}\varphi_u(t,u)\right. \\ &\qquad\quad \left. + \frac{1}{2}\int \theta^2\mu(d\theta)\frac{\psi_{\xi\xi}\big(t,u,\varphi(t,u)\big)}{\psi_u\big(t,u,\varphi(t,u)\big)}\varphi_u^2(t,u)\right) + O\big(\delta^{1+\beta}\big) \end{aligned} \tag{2.16}$$

as $n \to \infty$, since by the definition of $\widehat{\mathcal{A}}^n(n-1)$ in (2.10) we have $(t,u) \in \mathcal{A}^n(n-1)$ for all $(t,u,z) \in \widehat{\mathcal{A}}^n(n-1)$. Subtracting (2.16) from (2.15) and noting that by the definition of the non-linearity parameter $d(\mu)$ in (1.2.26) we have $\int \theta\, \mu(d\theta) = d(\mu) + \frac{1}{2}$, we obtain (2.12), uniformly for all $(t,u,z) \in \widehat{\mathcal{A}}^n(n-1)$. This concludes our proof. **q.e.d.**

For all $n \geq n_0$ the p-martingale weight function $p_n^{\xi^n} : \mathcal{A}^n(n-1) \to I\!R$ of Definition 2.7(ii) is well-defined, and we can introduce the probability weight function $p^n : \widehat{\mathcal{A}}^n(n-1) \to (0,1)$ in terms of $p_n^{\xi^n}$ by setting

$$\begin{aligned} p^n(t,u,z) &= p_n^{\xi^n}\big(t,u,\xi^n(t-\delta^2, u-z\delta)\big) \\ &= \frac{S_\mu^{\xi^n}\big(t,u,\xi^n(t-\delta^2,u-z\delta)\big) - S_\mu^{\xi^n}\big(t+\delta^2,u-\delta,\xi^n(t,u)\big)}{S_\mu^{\xi^n}\big(t+\delta^2,u+\delta,\xi^n(t,u)\big) - S_\mu^{\xi^n}\big(t+\delta^2,u-\delta,\xi^n(t,u)\big)} \end{aligned} \tag{2.17}$$

for all $(t,u,z) \in \widehat{\mathcal{A}}^n(n-1)$. In the second line we have again employed the shorthand $S_\mu^{\xi^n}(t,u,\xi) := S_\mu\big(t,u,\xi,\xi^n(t,u)\big)$ of (2.1.1). In terms of the weight function p^n, the p-martingale measure $\mathbf{P}_n^{\xi^n}$ of Definition 2.7(ii) is for all $n \geq n_0$ given by the initial distribution $\mathbf{P}_n^{\xi^n}\big(Z_0^n = 1\big) = \mathbf{P}^n\big(Z_0^n = 1\big)$ and by

$$\mathbf{P}_n^{\xi^n}\big(Z_{k+1}^n = 1 \,\big|\, U_k^n, Z_k^n\big) = p^n\big(t_k^n, U_k^n, Z_k^n\big) \quad \text{for all } 0 \leq k \leq n-1, \tag{2.18}$$

and it is clear that for all those n the measure $\mathbf{P}_n^{\xi^n}$ is equivalent to the original measure $\mathbf{P}^n$, since the range of p^n is contained in the interval $(0,1)$. Recalling Proposition 2.8 and Corollary 2.10, we note that for each $n \geq n_0$ both the large investor price process $\big\{S_\mu\big(t_k^n, U_k^n, \xi_{k-1}^n, \xi_k^n\big)\big\}_{0\leq k\leq n}$ and the paper value process $V^n = \big\{V_k^n\big\}_{0\leq k\leq n}$ are martingales under the p-martingale measure $\mathbf{P}_n^{\xi^n}$.

Remark. If the price determining measure μ is the Dirac measure δ_1 concentrated in 1, the stock price immediately adjusts to an order of the large investor before it is executed and we have $S_\mu^{\xi^n}(t,u,\xi) = \psi\big(t,u,\xi^n(t,u)\big)$ for all $(t,u,\xi) \in [0,T] \times I\!R^2$. Hence the probability function $p^n : \hat{\mathcal{A}}^n(n-1) \to (0,1)$ does not depend on the tilt z any more, and the random walk $U^n = \big\{U_k^n\big\}_{0\leq k\leq n}$ describing the fundamentals becomes a Markov process under $\mathbf{P}_n^{\xi^n}$. If the equilibrium price function $\psi : [0,T] \times I\!R^2 \to I\!R$ does not depend on the (large) investor's stock holdings, such that $\psi(t,u,\xi) = \overline{\psi}(t,u)$ for all $(t,u) \in [0,T] \times I\!R$, then the function $p^n : \hat{\mathcal{A}}^n(n-1) \to (0,1)$ becomes $p^n(t,u,z) = \frac{\overline{\psi}(t,u)-\overline{\psi}(t+\delta^2,u-\delta)}{\overline{\psi}(t+\delta^2,u+\delta)-\overline{\psi}(t+\delta^2,u-\delta)}$ for all $(t,u,z) \in \hat{\mathcal{A}}^n(n-1)$, and thus, p^n and the p-martingale measure $\mathbf{P}_n^{\xi^n}$ do not depend on the particular trading strategy of the large investor. Especially, recall from Example 1.9 that the (discounted) equilibrium price function in the Cox-Ross-Rubinstein model is given by

$$\psi\big(t,u,\xi\big) = S_0 e^{\sigma u + (\mu - r)t} \quad \text{for all } (t,u,\xi) \in [0,T] \times I\!R^2$$

with some fixed constants $S_0, \sigma > 0$ and $\mu, r \in I\!R$. In this case the probability function $p^n : \hat{\mathcal{A}}^n(n-1) \to (0,1)$ further simplifies to

$$p^n(t,u,z) = \frac{e^{(\mu-r)\delta^2} - e^{-\sigma\delta}}{e^{\sigma\delta} - e^{-\sigma\delta}} \quad \text{for all } (t,u,z) \in \widehat{\mathcal{A}}^n(n-1),$$

i.e. it is even constant, so that for each $n > n_0$ the fundamental process U^n is a (space- and time-) homogeneous Markov process. This special structure of U^n under the (p-)martingale measure makes the proof of convergence for a sequence of Cox-Ross-Rubinstein models like in Duffie (1988) so easy. However, with the convergence theorem for general correlated random walks of Section 4.1 at hand, we shall see that we can extend this convergence statement to general large investor models. □

Let us now define the two functions $a : [0,T] \times I\!R \to I\!R$ and $b : [0,T] \times I\!R \to I\!R$ by

$$a(t,u) = \frac{\big(d(\mu) - \frac{1}{2}\big)\psi_\xi\big(t,u,\varphi(t,u)\big)\varphi_u(t,u)}{\psi_u\big(t,u,\varphi(t,u)\big) + \big(d(\mu)+\frac{1}{2}\big)\psi_\xi\big(t,u,\varphi(t,u)\big)\varphi_u(t,u)}$$

and

$$b(t,u) = -\frac{\frac{1}{2}\frac{d}{du}\big(\psi_u(t,u,\varphi(t,u)) + 2d(\mu)\psi_\xi(t,u,\varphi(t,u))\varphi_u(t,u)\big) + \frac{d}{dt}\psi\big(t,u,\varphi(t,u)\big)}{\psi_u\big(t,u,\varphi(t,u)\big) + \big(d(\mu)+\frac{1}{2}\big)\psi_\xi\big(t,u,\varphi(t,u)\big)\varphi_u(t,u)}$$

for all $(t,u) \in [0,T] \times I\!R$. These two functions are well-defined, since their common denominator stays strictly positive because of the conditions (2.4) and (2.5) of Assumption H and I, respectively.
As $n \to \infty$, we can express the weight function $p^n : \widehat{\mathcal{A}}^n(n-1) \to (0,1)$ in terms of the functions a and b like in (1.3). Moreover, it can be seen that the functions a and b satisfy the Assumptions of Theorem 4.1:

Lemma 4.3. *Under Assumptions G to J and L the weight function $p^n : \widehat{\mathcal{A}}^n(n-1) \to I\!R$ can be asymptotically expanded as*

$$p^n(t,u,z) = \frac{1}{2}\big(1 + za(t,u) + \delta b(t,u)\big) + O\big(\delta^{1+\beta}\big) \quad \text{as } n \to \infty, \tag{2.19}$$

uniformly for all $(t,u,z) \in \widehat{\mathcal{A}}^n(n-1)$.
Moreover the functions $a : [0,T] \times I\!R \to I\!R$ and $b : [0,T] \times I\!R \to I\!R$ belong to the Hölder spaces $H^{\frac{1}{2}(1+\beta),1+\beta}\big([0,T]\times I\!R\big)$ and $H^{\frac{1}{2}\beta,\beta}\big([0,T]\times I\!R\big)$, respectively, and $\|a\| < 1$.

Proof. In order to show (2.19) we proceed where we stopped with the proof of Lemma 4.2. Dividing the difference of the upper (+) and the lower (−) case of (2.16) by 2, and using again $\int \theta\,\mu(d\theta) = d(\mu) + \frac{1}{2}$ from (1.2.26), we get for the divided denominator of the function $p^n : \widehat{\mathcal{A}}^n(n-1) \to I\!R$ given by (2.17)

$$\begin{aligned}&\frac{S_\mu^{\xi^n}\big(t+\delta^2, u+\delta, \xi^n(t,u)\big) - S_\mu^{\xi^n}\big(t+\delta^2, u-\delta, \xi^n(t,u)\big)}{2\delta\psi_u\big(t,u,\varphi(t,u)\big)}\\ &\qquad\qquad = 1 + \left(d(\mu)+\frac{1}{2}\right)\frac{\psi_\xi\big(t,u,\varphi(t,u)\big)}{\psi_u\big(t,u,\varphi(t,u)\big)}\varphi_u(t,u) + O\big(\delta^{1+\beta}\big) \quad \text{as } n\to\infty,\end{aligned}$$

uniformly for all $(t,u,z) \in \widehat{\mathcal{A}}^n(n-1)$. For $n \geq n_0$ this term is uniformly bounded away from 0, and if we now also use (2.15) and again the lower (−) case of (2.16) to expand the divided numerator $\frac{S_\mu^{\xi^n}(t,u,\xi^n(t-\delta^2,u-z\delta)) - S_\mu^{\xi^n}(t+\delta^2,u-\delta,\xi^n(t,u))}{2\delta\psi_u(t,u,\varphi(t,u))}$ of (2.17), it easily follows from the definitions of $p^n : \widehat{\mathcal{A}}^n(n-1) \to I\!R$ and of the two continuous functions $a : [0,T] \times I\!R \to I\!R$ and $b : [0,T] \times I\!R \to I\!R$ that indeed (2.19) holds.
Next we investigate the functions $a : [0,T] \times I\!R \to I\!R$ and $b : [0,T] \times I\!R \to I\!R$. Dividing numerator and denominator by $\psi_u\big(t,u,\varphi(t,u)\big)$, it is easily seen that the functions a and b belong to the Hölder spaces $H^{\frac{1}{2}(1+\beta),1+\beta}\big([0,T]\times I\!R\big)$ and $H^{\frac{1}{2}\beta,\beta}\big([0,T]\times I\!R\big)$, respectively, since $\psi \in \widehat{H}^{1+\frac{1}{2}\beta,2+\beta}\big([0,T]\times I\!R^2\big)$ and $\varphi \in H^{1+\frac{1}{2}\beta,2+\beta}\big([0,T]\times I\!R\big)$, and since the divided denominator is bounded away from 0 by the two bounds (2.4) and (2.5). In particular, there exists some $M \in (0,\infty)$ such that the divided denominator satisfies

$$\varepsilon \leq 1 + \left(d(\mu)+\frac{1}{2}\right)\frac{\psi_\xi\big(t,u,\varphi(t,u)\big)}{\psi_u\big(t,u,\varphi(t,u)\big)}\varphi_u(t,u) \leq M \quad \text{for all } (t,u) \in [0,T]\times I\!R. \tag{2.20}$$

In order to show that $\|a\| < 1$, let us now note that for the ε in Assumptions H and I we have

$$1 + \left(d(\mu) + \frac{1}{2} \right) \frac{\psi_\xi\big(t, u, \varphi(t,u)\big)}{\psi_u\big(t, u, \varphi(t,u)\big)} \varphi_u(t,u) - \left| \left(d(\mu) - \frac{1}{2} \right) \frac{\psi_\xi\big(t, u, \varphi(t,u)\big)}{\psi_u\big(t, u, \varphi(t,u)\big)} \varphi_u(t,u) \right| \geq \varepsilon,$$

as can be easily seen by considering separately the cases where the term within the absolute value is nonnegative and nonpositive. If we now rearrange the terms such that the absolute value stands alone on the right hand side of the inequality sign, divide the inequality by $1 + \big(d(\mu) + \frac{1}{2}\big) \frac{\psi_\xi(t,u,\varphi(t,u))}{\psi_u(t,u,\varphi(t,u))} \varphi_u(t,u)$, and then use the upper bound in (2.20), we obtain for all $(t,u) \in [0,T] \times I\!R$

$$\big|a(t,u)\big| = \frac{\big|\big(d(\mu) - \frac{1}{2}\big)\psi_\xi\big(t,u,\varphi(t,u)\big)\varphi_u(t,u)\big|}{\psi_u\big(t,u,\varphi(t,u)\big) + \big(d(\mu) + \frac{1}{2}\big)\psi_\xi\big(t,u,\varphi(t,u)\big)\varphi_u(t,u)} \leq 1 - \frac{\varepsilon}{M} < 1. \qquad (2.21)$$

This concludes our proof. **q.e.d.**

4.2.3 Convergence of the Fundamentals

In this section, we state our first distributional convergence result for our discrete large investor models under the paper value martingale measures by showing that the sequence of fundamental processes converges in distribution towards some continuous diffusion process. The limit process depends on the large investor's limiting strategy function φ.
In order to become more precise, let us fix again $n_0 \in I\!N$ as specified by Lemma 4.2 and consider for each $n \geq n_0$ the distribution of the fundamental value process U^n on the probability space $\big(\Omega^n, \mathcal{F}^n_n, \mathbf{P}^{\xi^n}_n\big)$. We will show that the sequence $\{U^n\}_{n \geq n_0}$ of (2.2) converges in distribution to some diffusion process $U = \{U_t\}_{t \in [0,T]}$, i.e. we show that there exists a probability measure $\mathbf{P}^\varphi$ on $\big(D[0,T], \mathcal{B}(D[0,T])\big)$ such that

$$\big(U^n \,\big|\, \mathbf{P}^{\xi^n}_n\big) \Rightarrow \big(U \,\big|\, \mathbf{P}^\varphi\big) \quad \text{as } n \to \infty.$$

The limit process U will be characterized by its initial value $U(0) = u_0$ at time 0, which is the same as the initial value U^n_0 of each of the discrete random walks $\{U^n\}_{n \geq n_0}$, and by the volatility and drift parameters $\sigma_\varphi : [0,T] \times I\!R \to [0,\infty)$ and $\mu_\varphi : [0,T] \times I\!R \to I\!R$, which for all $(t,u) \in [0,T] \times I\!R$ are given by

$$\sigma^2_\varphi(t,u) := \frac{\psi_u\big(t,u,\varphi(t,u)\big) + 2d(\mu)\psi_\xi\big(t,u,\varphi(t,u)\big)\varphi_u(t,u)}{\psi_u\big(t,u,\varphi(t,u)\big) + \psi_\xi\big(t,u,\varphi(t,u)\big)\varphi_u(t,u)} \qquad (2.22)$$

and

$$\mu_\varphi(t,u) := -\frac{\frac{d}{dt}\psi\big(t,u,\varphi(t,u)\big) + \frac{1}{2}\sigma^2_\varphi(t,u)\frac{d^2}{du^2}\psi\big(t,u,\varphi(t,u)\big)}{\frac{d}{du}\psi\big(t,u,\varphi(t,u)\big)}. \qquad (2.23)$$

Remark. Note that the denominators of σ^2_φ and μ_φ coincide. We have applied the chain rule in (2.22) in order to emphasize that $\sigma^2_\varphi \equiv 1$ if $d(\mu) = \frac{1}{2}$. By the definition of $d(\mu)$ in Definition 1.16 this especially holds true if the price-determining measure μ is the Dirac measure δ_1 concentrated in 1 as under Assumption L(i), i.e. if the market totally adjusts the stock price to the changed demand of the large investor *before* he can enter into any transaction. In this case the large investor price function $S_\mu : [0,T] \times I\!R^3$ of (1.3.2) is given by $S_\mu(t,u,\xi_1,\xi_2) = \psi(t,u,\xi_2)$ for all $(t,u,\xi_1,\xi_2) \in I\!R$. □

Now we can state the convergence theorem for the paper-value models:

Theorem 4.4. *Suppose that the Assumptions G- J, and L hold. Then under the respective p-martingale measures* $\mathbf{P}_n^{\xi^n}$*, the sequence of processes* $\{U^n\}_{n\geq n_0}$ *starting in* u_0 *at time* 0 *converges weakly to the process* $U = \{U_t\}_{t\in[0,T]}$ *given by*

$$dU_t = \sigma_\varphi(t, U_t)dW_t + \mu_\varphi(t, U_t)dt \quad \text{for all } t \in [0, T], \quad U_0 = u_0, \tag{2.24}$$

i.e. there exists some measure $\mathbf{P}^\varphi$ *on* $\big(D[0, T], \mathcal{B}(D[0, T])\big)$ *such that* W *is a* $\mathbf{P}^\varphi$*-Brownian motion and*

$$\big(U^n \,\big|\, \mathbf{P}_n^{\xi^n}\big) \Rightarrow \big(U \,\big|\, \mathbf{P}^\varphi\big) \quad \text{as } n \to \infty.$$

Proof. We check that the sequence $\{U^n\}_{n\geq n_0}$ satisfies the assumptions of Theorem 4.1. For each $n \geq n_0$ the stochastic process U^n lives on the probability space $\big(\Omega^n, \mathcal{F}_n^n, \mathbf{P}_n^{\xi^n}\big)$. By the definition of $U^n = \{U_t^n\}_{t\in[0,T]}$ we have (2.2), where the discrete random walk $\{U_k^n\}_{0\leq k\leq n}$ is for all $n \in I\!N$ given by (2.1) in terms of the $\{\pm 1\}$-valued tilt process $\{Z_k^n\}_{0\leq k\leq n}$. In particular, the process U^n maps into the space $D[0, T]$. For each $n \geq n_0$ the processes $\{U_k^n\}_{0\leq k\leq n}$ and $\{Z_k^n\}_{0\leq k\leq n}$ satisfy (2.18). Hence the sequence $\{U^n\}_{n\geq n_0}$ has the form of (1.2) and fits into the framework of Theorem 4.1.
In order to apply the theorem, we also need to check (1.3). By Lemma 4.3 the weight function $p^n : \widehat{\mathcal{A}}^n(n-1) \to I\!R$ satisfies (2.19) uniformly for all $(t, u, z) \in \widehat{\mathcal{A}}^n(n-1)$. In order to apply Theorem 4.1 we just extend the weight function $p^n : \widehat{\mathcal{A}}^n(n-1) \to I\!R$ to $[0, T] \times I\!R \times \{\pm 1\}$, such that (1.3) holds uniformly for all $(t, u) \in [0, T] \times I\!R$. This can be easily arranged by interpolation. Actually, the proof of Theorem 4.1 will show that in our case such an extension is unnecessary, since $\big(t_k^n, U_k^n, Z_k^n\big) \in \widehat{\mathcal{A}}^n(n-1)$ for all $0 \leq k \leq n-1$.
Moreover, the functions $a, b : [0, T] \times I\!R \to I\!R$ of Lemma 4.3 satisfy the requirements of Theorem 4.1. Hence we can apply the theorem to obtain the weak convergence of the sequence $\{U^n\}_{n\in I\!N}$ to the diffusion process $U = \{U_t\}_{t\in[0,T]}$ given by (1.4) (with $\mu_0 = 0$ and $\sigma = 1$). Straightforward calculation in terms of the expressions $f_1(u) := \psi_u\big(t, u, \varphi(t, u)\big)$ and $f_2(u) := \psi_\xi\big(t, u, \varphi(t, u)\big)\varphi_u(t, u)$ yields for all $(t, u) \in [0, T] \times I\!R$

$$\frac{1 + a(t, u)}{1 - a(t, u)} = \sigma_\varphi^2(t, u) \quad \text{and} \quad \frac{b(t, u)}{1 - a(t, u)} + \frac{a_u(t, u)}{\big(1 - a(t, u)\big)^2} = \mu_\varphi(t, u).$$

This conclude the proof of the theorem. **q.e.d.**

Remark 1. For each $n \geq n_0$ the measure $\mathbf{P}_n^{\xi^n}$ on the measurable space $\big(\Omega^n, \mathcal{F}_n^n\big)$ induces a probability measure on the measurable space $\big(D[0, T], \mathcal{B}(D[0, T])\big)$ via the distribution $\mathbf{P}_n^{\xi^n} \circ (U^n)^{-1}$ of U^n, which is the "canonical counterpart" of the measure $\mathbf{P}_n^{\xi^n}$ on the space $\big(D[0, T], \mathcal{B}(D[0, T])\big)$. Thus, the statement of Theorem 4.4 is equivalent to saying that the measures $\big\{\mathbf{P}_n^{\xi^n} \circ (U^n)^{-1}\big\}_{n\geq n_0}$ converge weakly to some measure $\mathbf{P}^\varphi$ as $n \to \infty$. We will see from the proof of Theorem 4.1 that the measure $\mathbf{P}^\varphi$ is the unique solution to the martingale problem starting in $(0, u_0)$ for the generator $L^\varphi : C^2(I\!R) \to C(I\!R)$, which is given by

$$\big(L^\varphi f\big)(t, u) = \frac{1}{2}\sigma_\varphi^2(t, u)f''(u) + \mu_\varphi(t, u)f'(u) \quad \text{for all } (t, u) \in [0, T] \times I\!R \tag{2.25}$$

and all $f \in C^2(I\!R)$. □

Remark 2. In Section 3.2.4 we have discussed how we can derive the convergence of a subsequence of the strategy functions $\{\xi^n\}_{n\in I\!N}$ under less restrictive regularity assumptions on the price system (ψ, μ) in the special case where (ψ, μ) excludes any transaction losses. We are confident that in principle we could generalize such convergence results to the general setting *with* transaction losses.

If we have already found a subsequence $\{n_m\}_{m\in I\!N} \subset I\!N$ such that for $\delta = \delta_{n_m} = n_m^{-1/2}$ we have $\|\xi^{n_m} - \varphi\|_{\mathcal{A}^{n_m}} = O(\delta^2)$ and

$$\left\|\xi^{n_m}(\cdot + \delta^2, \cdot \pm \delta) - \xi^{n_m} \mp \delta\varphi_u - \delta^2\left(\varphi_t + \frac{1}{2}\varphi_{uu}\right)\right\|_{\mathcal{A}^{n_m}(n_m-1)} = O(\delta^{2+\beta}) \quad \text{as } m \to \infty,$$

then the proofs of Lemma 4.3 and Theorem 4.4 indicate that we could still show the convergence

$$\big(U^{n_m} \,\big|\, \mathbf{P}_{n_m}^{\xi^{n_m}}\big) \Rightarrow \big(U \,\big|\, \mathbf{P}^{\varphi}\big) \quad \text{as } m \to \infty,$$

even if we relax the Hölder continuity of the functions $\overline{\psi}$, f, and φ of Assumptions G and H to $\overline{\psi} \in \widehat{H}^{1+\frac{1}{2}\beta,2+\beta}\big([0,T] \times I\!R\big)$, $f \in H_{loc}^{2+\beta}(I\!R)$ and $\varphi \in H^{1+\frac{1}{2}\beta,2+\beta}\big([0,T] \times I\!R\big)$, and if we drop the strong Assumption J on the convergence of the large investor's stock holdings around maturity, which is needed to derive the convergence of the strategy functions $\{\xi^n\}_{n\in I\!N}$ to φ from their convergence at the very end of the time interval $[0,T]$. □

In the large investor models of Frey and Stremme (1997) and Bierbaum (1997), like the convergence of the strategy functions, the convergence of the fundamental value process towards a diffusion limit is just assumed, but not proved. Because of the convergence of the fundamentals, these authors then conclude that the uniform convergence of the equilibrium price *functions* also implies the convergence in distribution of the associated sequence of equilibrium price *processes* by using the Continuous Mapping Theorem. The coefficients of the limit distribution are explicitly given as functions of time and equilibrium price. In the next section we show a slightly stronger result in our setting.

4.2.4 Convergence of the Large Investor Price and the Paper Value Processes

The convergence result of Theorem 4.4 for the fundamentals is the main step in proving the convergence of *all* relevant stochastic processes in the discrete large investor models to continuous-time limit processes under the paper value martingale measure.

In this section we construct stochastic processes in $D[0,T]$ which describe the strategy, stock price, and value processes in the discrete binomial models. Then we show that not only the fundamental processes of Theorem 4.4, but also the strategy, stock price, and value processes converge in distribution to some continuous limit processes, even jointly, when viewed as one process with paths in the space $D^d[0,T]$. In order to show this general convergence result, we need to extend a proposition of Duffie and Protter (1992) which yields the simultaneous convergence in distribution of a process X and other discrete stochastic processes for which the stochastic influence is – at least basically – described by X as well.

Results on the simultaneous convergence of the different stochastic processes in the discrete analogues of the Black-Scholes models are common in the financial literature. Simple variants go back to He (1990). More advanced results can be found in Duffie and Protter (1992), who apply results of Kurtz and Protter (1991, 1995) to mathematical finance. For an overview of different convergence concepts see also Willinger and Taqqu (1991). Duffie and Protter (1992) show the convergence of the financial gains process by checking that the sequence of discrete price processes satisfies the technical condition of being good. This convergence then implies that the discrete value processes converge as well. In contrast, we show directly that the sequence of discrete paper value processes converges, and in this respect are closer to the approach of He (1990).

For the convergence results in this section, we work on the spaces $D^d[0,T]$ of càdlàg functions $f : [0,T] \to I\!R^d$, with $d \in I\!N$, and we endow those spaces with the corresponding Skorohod topology. For a detailed discussion of these spaces, we refer the reader to Chapter 3 in Billingsley (1968) or Section 6 in Kurtz and Protter (1995). However, it is worth mentioning that $D^d[0,T] \neq \big(D[0,T]\big)^d$ as Example 6.4 in Kurtz and Protter (1995) reveals. For a sequence $\{X^n\}_{n\in I\!N}$ of random elements in $D^d[0,T]$ we write $X^n \xrightarrow{P} 0$ if $\{X^n\}_{n\in I\!N}$ converges to 0 in probability as $n \to \infty$. Moreover, we denote the Euclidean norm in $I\!R^d$ by $|\cdot|$, so that we can distinguish it from sup-norms which are denoted by $\|\cdot\|$.

In (2.2) we have already introduced an extension $U^n = \big\{U_t^n\big\}_{t\in[0,T]}$ of the discrete fundamental process $\{U_k^n\}_{0\le k\le n}$ to the space $D[0,T]$. We now introduce similar $D[0,T]$-versions of the discrete (non-stochastic) sequence $\{t_k^n\}_{0\le k\le n}$ of time points and of the (stochastic) strategy, price, and value processes in the discrete large investor markets. At the same time, we define the associated notations for the limit processes as well. In order to keep our notation compact, we drop the time interval $[0,T]$ in our notation of a stochastic process and write for example $U^n = \{U_t^n\}$ instead of $U^n = \big\{U_t^n\big\}_{t\in[0,T]}$.

Definition 4.5. For each $n \in I\!N$, we define the following processes in $D[0,T]$:

(i) The deterministic process $\tau^n = \{\tau_t^n\}$, describing the latest trading time point in the nth binomial model, is given by

$$\tau_t^n := t_{[nt]}^n \quad \text{for all } t \in [0,T].$$

(ii) The **strategy process** $\phi^n = \{\phi_t^n\}$ and the auxiliary strategy process $\tilde{\phi}^n = \big\{\tilde{\phi}_t^n\big\}$, describing in the nth binomial model the large investor's current stock holdings and his stock holdings before his last trade, are defined by

$$\phi_t^n := \xi^n\big(\tau_t^n, U_t^n\big) \quad \text{and} \quad \tilde{\phi}_t^n := \xi^n\big(\tau_t^n - \delta^2, U_t^n - Z_{[nt]}^n\delta\big) \quad \text{for all } t \in [0,T],$$

where the large investor's strategy function $\xi^n : \mathcal{A}^n \to I\!R$ is extended to the point $(-\delta^2, u_0 - Z_0^n\delta)$ by setting $\xi^n(-\delta^2, u_0 - Z_0^n\delta) := \xi_{-1}^n$ in order to incorporate his stock holdings immediately before time $t_0^n = 0$.

(iii) The **large investor price process** $S^n = \big\{S_t^n\big\}$, which describes the (average) stock price paid at the large investor's latest trade in the nth binomial model, is defined in terms of the large investor's price function $S_\mu : [0,T] \times I\!R^3 \to I\!R$ of (1.3.2) by

$$S_t^n := S_\mu\big(\tau_t^n, U_t^n, \tilde{\phi}_t^n, \phi_t^n\big) \quad \text{for all } t \in [0,T].$$

It depends on the strategy process ϕ^n and the auxiliary strategy process $\tilde{\phi}^n$.

(iv) The **loss-free liquidation price process** $\bar{S}^n = \big\{\bar{S}_t^n\big\}$, which describes in the nth binomial model the latest average liquidation price per share of the large investor's stock holdings if he could execute his trades at the benchmark price, is defined in terms of the loss-free liquidation price function $\bar{S} : [0,T] \times I\!R^2 \to I\!R$ given by (1.3.14) as

$$\bar{S}_t^n = \bar{S}\big(\tau_t^n, U_t^n, \phi_t^n\big) \quad \text{for all } t \in [0,T].$$

It depends only on the large investor's strategy process ϕ^n.

(v) The **real value processes** $\bar{V}^n = \big\{\bar{V}_t^n\big\}$, quoting the value of the large investor's portfolio in the nth binomial model in terms of the actual liquidation price, is defined as

$$\bar{V}_t^n = \bar{v}^n\big(\tau_t^n, U_t^n\big) = \phi_t^n \bar{S}_t^n + b^n\big(\tau_t^n, U_t^n\big) \quad \text{for all } t \in [0,T],$$

where $\bar{v}^n : \mathcal{A}^n \to I\!R$ is the real value function of (1.3.17).

(*vi*) Last but not least, we define the **paper value process** $V^n = \{V_t^n\}$, which values the large investor's portfolio in the nth binomial model at the last price experienced by the large investor, by

$$V_t^n = v^n(\tau_t^n, U_t^n, \tilde{\phi}_t^n) = \phi_t^n S_t^n + b^n(\tau_t^n, U_t^n) \quad \text{for all } t \in [0,T].$$

where $v^n : \mathcal{A}^n \times I\!R \to I\!R$ is the paper value function of (1.3.19).

Of course, the process τ^n converges in $D[0,T]$ to the identity on $[0,T]$. We now define the (candidate) limit processes for the stochastic processes of Definition 4.5. In Proposition 4.8 below we then show that under the paper martingale measures, the processes of Definition 4.5 indeed converge in distribution towards their candidate limits, even simultaneously when viewed as one process with path in $D^d[0,T]$.

Definition 4.6. We define the following processes in $D[0,T]$:

(*i*) The **strategy process** $\phi = \{\phi_t\}$, describing the large investor's stock holdings in the limit model, is given by $\phi_t := \varphi(t, U_t)$ for all $t \in [0,T]$, where the limiting **strategy function** $\varphi : [0,T] \times I\!R \to I\!R$ is the solution of the final value problem (3.4.3), (3.4.4).

(*ii*) The (large investor) **price process** $S = \{S_t\}$ in the limit model is given by

$$S_t = \psi(t, U_t, \phi_t) = S_\mu(T, U_t, \phi_t, \phi_t) \quad \text{for all } t \in [0,T].$$

(*iii*) The **loss-free liquidation price process** $\bar{S} = \{\bar{S}_t\}$ in the limit model which is implied by the strategy process ϕ is given by

$$\bar{S}_t := \bar{S}(t, U_t, \phi_t) \quad \text{for all } t \in [0,T].$$

(*iv*) The **real value process** $\bar{V} = \{\bar{V}_t\}$ in the limit model is given as $\bar{V}_t = \bar{v}(t, U_t)$ for all $t \in [0,T]$, where $\bar{v} : [0,T] \times I\!R \to I\!R$ is the **(continuous-time) real value function**, i.e. the solution of the final value problem (3.4.6), (3.4.7).

(*v*) Last but not least, the **paper value process** $V = \{V_t\}$ in the limit model is given as $V_t = v(t, U_t)$ for all $t \in [0,T]$, where $v : [0,T] \times I\!R \to I\!R$ is the **(continuous-time) paper value function** which is for all $(t,u) \in [0,T] \times I\!R$ given by

$$v(t,u) := \bar{v}(t,u) + \varphi(t,u)\Big(\psi\big(t,u,\varphi(t,u)\big) - \bar{S}\big(t,u,\varphi(t,u)\big)\Big).$$

Remark. The limit S of the large investor price processes $\{S^n\}$ is just called price process, since this price is not an exclusive price for the large investor any more. In the limit, the large investor price $S_\mu(t, U_t, \phi_t, \phi_t)$ coincides with the equilibrium price $\psi(t, U_t, \phi_t)$ at which the small investors in the market trade, given that the large investor trades according to his strategy ϕ. □

However, before we can prove the joint convergence of the processes in Definition 4.5 to the corresponding limit processes of Definition 4.6, we first need to extend Corollary 5.2 of Duffie and Protter (1992), which yields the joint convergence of the tuples $\{(X_t^n, h^n(\tau_t^n, X_t^n))\}_{n \in I\!N}$ if the $D^d[0,T]$-valued processes $\{X^n\}_{n \in I\!N}$ converge in distribution. For example, their convergence result does not cover the convergence of the auxiliary strategy processes $\{\tilde{\phi}^n\}_{n \in I\!N}$, since the auxiliary strategy process ϕ^n at time t depends not only on time τ_t^n and fundamentals U_t^n at this time, but also on the current tilt $Z_{[nt]}^n$ of the fundamental process. This problem is solved by the following lemma:

Lemma 4.7. *For each $n \in I\!N$ let X^n and Z^n be random elements in $D^d[0,T]$, defined on the same probability space $(\Omega^n, \mathcal{F}^n, \mathbf{P}^n)$, and suppose $X^n \Rightarrow X$ for some continuous X, and $\delta_n Z^n \xrightarrow{P} 0$ as $n \to \infty$. We also assume that for each $n \in I\!N$ and all $t \in [0,T]$ the realizations of (τ^n_t, X^n_t, Z^n_t) are contained in some set $\mathcal{D}^n \subset [0,T] \times I\!R^{2d}$. If the sequence $\{h^n\}_{n\in I\!N}$ of functions $h^n : \mathcal{D}^n \to I\!R^k$ converges uniformly to a continuous function $h : [0,T] \times I\!R^d \to I\!R^k$ in the sense that $\sup_{(\tau,x,z)\in\mathcal{D}^n} |h^n(\tau,x,z) - h(\tau,x)| \to 0$ as $n \to \infty$, then we have*

$$\Big\{\Big(X^n_t, h^n\big(\tau^n_t, X^n_t, Z^n_t\big)\Big)\Big\} \Rightarrow \Big\{\Big(X_t, h\big(t, X_t\big)\Big)\Big\} \quad \text{as } n \to \infty. \tag{2.26}$$

Especially, we have for all continuous functions $h : [0,T] \times I\!R^d \to I\!R^k$ that

$$\Big\{\Big(X^n_t, h\big(\tau^n_t, X^n_t\big)\Big)\Big\} \Rightarrow \Big\{\Big(X_t, h\big(t, X_t\big)\Big)\Big\} \quad \text{as } n \to \infty. \tag{2.27}$$

Proof. Let us set $Y^n := \delta_n Z^n$ for all $n \in I\!N$. By Theorem 4.4 in Billingsley (1968) the conditions on X^n and Z^n imply $(X^n, Y^n) \Rightarrow (X, 0)$ as $n \to \infty$. The remainder of the proof follows by arguments analogous to the ones proving Lemma 5.2 and Proposition 5.1 in Kurtz and Protter (1991): For any continuous $x^0 \in D^d[0,T]$ and any convergent sequence $(x^n, y^n) \to (x^0, 0)$ in $D^{2d}[0,T]$, there exist some continuous, strictly increasing functions $\lambda_n : [0,T] \to [0,T]$ such that $\lambda_n(t) \to t$ and $\big(x^n_{\lambda_n(t)}, y^n_{\lambda_n(t)}\big) \to \big(x^0_t, 0\big)$ uniformly on $[0,T]$ as $n \to \infty$. Moreover, the bound

$$\big|x^n_t\big| \le \Big|x^n_t - x^0_{\lambda_n^{-1}(t)}\Big| + \Big|x^0_{\lambda_n^{-1}(t)}\Big| \quad \text{for all } t \in [0,T]$$

and the continuity of $x^0 : [0,T] \to I\!R$ imply that $R := \sup_{n\in I\!N} \sup_{t\in[0,T]} \big|x^n_t\big| < \infty$.
Without loss of generality we take $\mathcal{D}^n = [0,T] \times I\!R^{2d}$. Since $h : [0,T] \times I\!R^d \to I\!R^k$ is continuous, it is uniformly continuous on the compact intervals $[0,T] \times [-R,R]^d$, and due to $\sup_{(\tau,x,z)\in\mathcal{D}^n} |h^n(\tau,x,z) - h(\tau,x)| \to 0$, we get

$$\begin{aligned}
&\big|h^n\big(\tau^n_{\lambda_n(t)}, x^n_{\lambda_n(t)}, \delta_n^{-1} y^n_{\lambda_n(t)}\big) - h\big(t, x^0_t\big)\big| \\
&\quad \le \big|h^n\big(\tau^n_{\lambda_n(t)}, x^n_{\lambda_n(t)}, \delta_n^{-1} y^n_{\lambda_n(t)}\big) - h\big(\tau^n_{\lambda_n(t)}, x^n_{\lambda_n(t)}\big)\big| + \big|h\big(\tau^n_{\lambda_n(t)}, x^n_{\lambda_n(t)}\big) - h\big(t, x^0_t\big)\big| \to 0
\end{aligned}$$

as $n \to \infty$, uniformly on $[0,T]$. By the extension of the Continuous Mapping Theorem as stated in Theorem 5.5 of Billingsley (1968), we can now conclude from $(X^n, Y^n) \Rightarrow (X, 0)$ and $Z^n = \delta_n^{-1} Y^n$ that (2.26) holds. The choice $Z^n \equiv 0$ gives (2.27). **q.e.d.**

Now we are well-equipped to show that under the paper value martingale measures not only the fundamental processes, but also all other relevant processes of our discrete large investor models converge in distribution towards their continuous counterparts as stated in Definitions 4.5 and 4.6, and these processes even converge in distribution if they are viewed as tuples in the space $D^7[0,T]$:

Proposition 4.8. *Suppose that the Assumptions G-L hold. Then we have*

$$\Big(\big(U^n, \tilde{\phi}^n, \phi^n, \bar{S}^n, S^n, \bar{V}^n, V^n\big) \Big| \mathbf{P}^{\xi^n}_n\Big) \Rightarrow \Big(\big(U, \phi, \phi, \bar{S}, S, \bar{V}, V\big) \Big| \mathbf{P}^{\varphi}\Big) \quad \text{as } n \to \infty. \tag{2.28}$$

Proof. We prove the assertion by two applications of Lemma 4.7; the first application will show that

$$\Big(\big(U^n, \tilde{\phi}^n, \phi^n, \bar{V}^n\big) \Big| \mathbf{P}^{\xi^n}_n\Big) \Rightarrow \Big(\big(U, \phi, \phi, \bar{V}\big) \Big| \mathbf{P}^{\varphi}\Big) \quad \text{as } n \to \infty, \tag{2.29}$$

while the second application, using (2.29), yields the proposition. By Assumption H the limit strategy function $\varphi \in [0,T] \times I\!R \to I\!R$ belongs to the Hölder space $H^{1+\frac{1}{2}\beta, 2+\beta}\big([0,T] \times I\!R\big)$, hence it is in particular continuous on $[0,T]\times I\!R$. Moreover, due to Corollary 3.28 the discrete strategy functions $\xi^n : \mathcal{A}^n \to I\!R$ satisfy $\|\xi^n - \varphi\|_{\mathcal{A}^n} = O(\delta^2)$ as $n \to \infty$, and thus, using our definition of $\widehat{\mathcal{A}}^n$ in (2.10), we get $\sup_{(t,u,z)\in\widehat{\mathcal{A}}^n}\big|\xi^n(t,u) - \varphi(t,u)\big| \to 0$, since nothing depends on z. We now want to show that similarly

$$\sup_{(t,u,z)\in\widehat{\mathcal{A}}^n}\big|\xi^n\big(t-\delta^2, u - z\delta\big) - \varphi(t,u)\big| \to 0 \quad \text{as } n \to \infty,$$

and for this purpose we recall the expansion (2.14). It allows us to conclude that

$$\big|\xi^n\big(t-\delta^2,u-z\delta\big) - \varphi(t,u)\big| \le \big|\xi^n\big(t-\delta^2,u-z\delta\big) - \xi^n(t,u)\big| + \big|\xi^n(t,u)-\varphi(t,u)\big| = O(\delta)$$

as $n \to \infty$, uniformly for all $(t,u,z) \in \widehat{\mathcal{A}}^n$, since φ belongs in particular to the Hölder space $H^{1+\frac{1}{2}\beta,2+\beta}\big([0,T]\times I\!R\big)$ and $\big\|\xi^n - \varphi\big\|_{\mathcal{A}^n} = O\big(\delta^2\big)$.
Moreover, we recall from Proposition 3.30 that $\big\|\bar v^n - \bar v\big\|_{\mathcal{A}^n} = O\big(\delta^\beta\big)$ as $n \to \infty$. Hence the three-dimensional functions $h^n : \widehat{\mathcal{A}}^n \to I\!R^3$ given by

$$h^n(t,u,z) = \Big(\xi^n\big(t-\delta^2,u-z\delta\big), \xi^n(t,u), \bar v^n(t,u)\Big) \quad \text{for all } (t,u,z) \in \widehat{\mathcal{A}}^n \text{ and } n \in I\!N$$

converge in the sense of Lemma 4.7 to the continuous function $h : [0,T]\times I\!R \to I\!R^3$ given by

$$h(t,u) = \big(\varphi(t,u), \varphi(t,u), \bar v(t,u)\big) \quad \text{for all } (t,u) \in [0,T]\times I\!R,$$

i.e. $\sup_{(t,u,z)\in\widehat{\mathcal{A}}^n}\big|h^n(t,u,z) - h(t,u)\big| \to 0$ as $n \to \infty$. The process $Z^n = \{Z^n_{[nt]}\}$ satisfies $\delta Z^n = \delta_n Z^n \xrightarrow{P} 0$ as $n \to \infty$, since all elements of $\big\{Z^n_k\big\}_{0\le k\le n}$ are 1 in modulus, and hence Z^n is uniformly bounded. Finally, Theorem 4.4 has shown that $\big(U^n \,\big|\, \mathbf{P}^{\xi^n}_n\big) \Rightarrow \big(U \,\big|\, \mathbf{P}^\varphi\big)$, hence Lemma 4.7 yields

$$\Big(\big\{\big(U^n_t, h^n(\tau^n_t, U^n_t, Z^n_{[nt]})\big)\big\} \,\Big|\, \mathbf{P}^{\xi^n}_n\Big) \Rightarrow \Big(\big\{\big(U_t, h(t,U_t)\big)\big\} \,\Big|\, \mathbf{P}^\varphi\Big) \quad \text{as } n \to \infty.$$

Thus, an application of our definition of the functions h^n and h as well as Definitions 4.5(ii) and (v) and Definitions 4.6(i) and (iv) prove (2.29).
We now use the joint convergence (2.29) of the fundamental price processes $\{U^n\}_{n\in I\!N}$, the strategy processes $\{\tilde\phi^n\}_{n\in I\!N}$ and $\{\phi^n\}_{n\in I\!N}$, and of the real value processes $\{\bar V^n\}_{n\in I\!N}$ to obtain the convergence of the two price processes and the paper value processes, which all use the strategy processes as their arguments.
Since the large investor price function $S_\mu : [0,T]\times I\!R^3 \to I\!R$ of (1.3.2) and the loss-free liquidation price function $\bar S : [0,T]\times I\!R^2 \to I\!R$ of (1.3.14) are continuous, it is clear that the function $h : [0,T]\times I\!R^4 \to I\!R^3$ which we define for all $(t,u,\xi_1,\xi_2,v) \in [0,T]\times I\!R^4$ by

$$h(t,u,\xi_1,\xi_2,v) := \Big(S_\mu(t,u,\xi_1,\xi_2), \bar S(t,u,\xi_2), v + \xi_2\big(S_\mu(t,u,\xi_1,\xi_2) - \bar S(t,u,\xi_2)\big)\Big)$$

is continuous as well. Hence it follows from the second part of Lemma 4.7 that

$$\Big(\Big\{\Big(U^n_t, \tilde\phi^n_t, \phi^n_t, \bar V^n_t, h\big(\tau^n_t, U^n_t, \tilde\phi^n_t, \phi^n_t, \bar V^n_t\big)\Big)\Big\} \,\Big|\, \mathbf{P}^{\xi^n}_n\Big)$$
$$\Rightarrow \Big(\Big\{\Big(U_t, \phi_t, \phi_t, \bar V_t, h\big(t, U_t, \phi_t, \phi_t, \bar V_t\big)\Big)\Big\} \,\Big|\, \mathbf{P}^\varphi\Big)$$

as $n \to \infty$. Finally note that we can get rid of the cash holdings function $b^n : \mathcal{A}^n \to I\!R$ in the defining equation (1.3.19) for the paper value function $v^n : \mathcal{A}^n \times I\!R \to I\!R$ by substituting it by the remaining terms of (1.3.17), the definition of the real value function $\bar v^n : \mathcal{A}^n \to I\!R$.

Using also the shorthand $\bar{S}$ for the loss-free liquidation price introduced in (1.3.14), we get for all $(t,u,\xi) \in [0,T] \times I\!R^2$

$$\begin{aligned} v^n(t,u,\xi) &= \xi^n(t,u) S_\mu\big(t,u,\xi,\xi^n(t,u)\big) + b^n(t,u) \\ &= \bar{v}^n(t,u) + \xi^n(t,u)\Big(S_\mu\big(t,u,\xi,\xi^n(t,u)\big) - \bar{S}\big(t,u,\xi^n(t,u)\big)\Big), \end{aligned}$$

and especially $V_t^n = v^n\big(t, U_t^n, \tilde{\phi}_t^n\big) = \bar{V}_t^n + \phi_t^n\big(S_t^n - \bar{S}_t^n\big)$. Thus we get from Definition 4.5 and Definition 4.6 that $h\big(\tau_t^n, U_t^n, \tilde{\phi}_t^n, \phi_t^n, \bar{V}_t^n\big) = \big(S_t^n, \bar{S}_t^n, V_t^n\big)$ and $h\big(t, U_t, \phi_t, \phi_t, \bar{V}_t\big) = \big(S_t, \bar{S}_t, V_t\big)$. This concludes the proof. **q.e.d.**

Remark. The last remark of Section 4.2.3 transfers to Proposition 4.8: We can discard the two conditions (2.7) and (2.8) and weaken the differentiability assumptions on $\varphi : [0,T] \times I\!R \to I\!R$ to $\varphi \in H^{1+\frac{1}{2}\beta, 2+\beta}\big([0,T] \times I\!R\big)$ if we know from some other arguments that the strategy functions $\{\xi^n\}_{n \in I\!N}$ converge to φ in the sense that (3.3.68) and (3.3.69) hold, and that the real value functions $\bar{v}^n$ converge to $\bar{v}$ such that $\big\|\bar{v}^n - \bar{v}\big\|_{\mathcal{A}^n} = O\big(\delta^\beta\big)$ as $n \to \infty$. Similar statements for a subsequence are straightforward. □

4.2.5 The Continuous-Time Paper Value Function

In this section we investigate the limiting paper value function in some more detail. For this reason, we reparametrize the paper value function in terms of time and stock price. We then derive a final value problem for the transformed paper value function. This final value problem shows that our continuous-time limit model covers the models of Schönbucher and Wilmott (2000), Frey (1998, 2000) and Sircar and Papanicolaou (1998) as special cases.

We would like to state the final problem for the transformed paper value function in such a generality that we can also deal with trading strategies which replicate non-smooth contingent claims like European calls, even though the convergence result of Proposition 4.8 does not cover those cases. In Section 3.4.2 we have seen that it suffices to require Assumptions B and C and one of the Assumptions E or F in order to guarantee the existence of solutions φ and v to the final value problems (3.4.3), (3.4.4) and (3.4.6), (3.4.7), respectively. Under both sets of assumptions the (continuous-time) paper value function $v : [0,T] \times I\!R \to I\!R$, which is for all $(t,u) \in [0,T] \times I\!R$ given by

$$v(t,u) = \bar{v}(t,u) + \varphi(t,u)\Big(\psi\big(t,u,\varphi(t,u)\big) - \bar{S}\big(t,u,\varphi(t,u)\big)\Big), \tag{2.30}$$

is still well-defined. For that reason we suppose in this section that either Assumptions B, C and E or Assumptions B, C and F hold. In order to explore the paper value function in some more detail, and in order to compare it with the value function in the ordinary Black-Scholes model, we reparametrize the paper value function in terms of the stock price in the limiting market model. In Section 3.4 we have transformed the real value function into a function of time and the small investor stock price. However, for the paper value function such a transformation does not give much new insight, and it is more natural to reparametrize the paper value function into a function of the equilibrium price $\psi\big(t,u,\varphi(t,u)\big)$ in the limit model, at which the small investors and the large investor buy or sell the next infinitesimal amount of shares. For this reparametrization we have to make sure that the function $u \mapsto \psi\big(t,u,\varphi(t,u)\big)$ is invertible for every fixed time point $t \in [0,T]$; hence we impose:

Assumption M. *Let φ be the solution of the final value problem (3.4.3), (3.4.4). For each fixed $t \in [0,T]$ the function $X_\varphi : I\!R \to (0,\infty)$, $u \mapsto \psi\big(t,u,\varphi(t,u)\big)$, is strictly increasing.*

Assumption M is a weaker formulation of Assumption I. Under Assumption M we can reparametrize the paper value function $v : [0,T] \times I\!R \to I\!R$ from a function of time and *fundamentals* into a function $w : \mathcal{D} \to I\!R$ of time and *stock price*. Here the set $\mathcal{D}$ denotes the set of all possible time-price combinations if the large investor's stock holdings are determined by the strategy function $\varphi : [0,T] \times I\!R \to I\!R$, i.e. $\mathcal{D}$ is given by

$$\mathcal{D} := \big\{(t,x) \in [0,T] \times I\!R \,\big|\, x = \psi\big(t,u,\varphi(t,u)\big) \text{ for some } u \in I\!R\big\}. \tag{2.31}$$

On the set $\mathcal{D}$ we then introduce the reparametrized paper value function $w : \mathcal{D} \to I\!R$ by $w(t,x) = v\big(t, u_\varphi(t,x)\big)$ for all $(t,x) \in \mathcal{D}$, where the reparametrization function $u_\varphi : \mathcal{D} \to I\!R$ is uniquely determined by

$$x = \psi\big(t,u,\varphi(t,u)\big)\big|_{u=u_\varphi(t,x)} \quad \text{for all } (t,x) \in \mathcal{D} \tag{2.32}$$

because of Assumption M. By Definition 4.6(ii) and (v) the paper value process V in the limit model can now be expressed in terms of time t and stock price $S_t = \psi\big(t, U_t, \varphi(t,U_t)\big)$ as $V_t = w\big(t, S_t\big)$ for all $t \in [0,T]$.

We pause here shortly to describe the structure of the set $\mathcal{D}$ of (2.31) under our standing assumptions. If Assumption F holds, then $\varphi : [0,T] \times I\!R \to I\!R$ is continuous and bounded on the whole domain $[0,T] \times I\!R$, and Lemma 3.31 implies $\mathcal{D} = [0,T] \times (0,\infty)$. The situation is more complicated if instead of Assumption F we only have Assumption E. In this case φ is still bounded on $[0,T] \times I\!R$ and continuous on $[0,T) \times I\!R$, but it need not be continuous on the boundary $\{T\} \times I\!R$, since the boundary function $\zeta : I\!R \to I\!R$ might have jumps. If $\zeta : I\!R \to I\!R$ jumps at some point $u^* \in I\!R$ and we do not have $f\big(\alpha\zeta(u^*-)\big) = f\big(\alpha\zeta(u^*)\big) = f\big(\alpha\zeta(u^*+)\big)$, then the function $X : I\!R \to I\!R$ given by

$$X(u) = \psi\big(T,u,\varphi(T,u)\big) = \psi\big(T,u,\alpha\zeta(u)\big) \quad \text{for all } u \in I\!R \tag{2.33}$$

jumps as well. Since $X : I\!R \to I\!R$ is strictly increasing by Assumption M, every jump of X leads to a hole in the domain $X(I\!R)$, i.e. the set $X(I\!R)$ is given by $X(I\!R) = (0,\infty)\backslash\mathcal{E}$, where the exception set $\mathcal{E}$ is given as the union

$$\mathcal{E} = \bigcup_{u \in I\!R} \big[X(u-), X(u)\big) \cup \bigcup_{u \in I\!R} \big(X(u), X(u+)\big] \tag{2.34}$$

of half-open intervals. In particular, $\mathcal{D} = \big([0,T] \times (0,\infty)\big)\backslash\big(\{T\} \times \mathcal{E}\big)$, and the set $\mathcal{D}$ is fuzzy at the boundary if $\mathcal{E}$ is nonempty.

Remark. Recall from Section 3.4.2 that we need an assumption like Assumption E in order to be able to incorporate European calls and other contingent claims in our analysis for which the final stock holdings are not smooth functions of the fundamentals. For that reason it makes sense to deal with the additional complications incurred by switching from Assumption F to Assumption E. □

In analogy to our transformation of the final value problem (3.4.6), (3,4.7) for the real value function $\bar{v} : [0,T] \times I\!R \to I\!R$ into the corresponding final value problem (3.4.21), (3.4.22) for its transform $\bar{w} : \bar{\mathcal{D}} \to I\!R$, we now derive a final value problem for the transformed paper value function $w : \mathcal{D} \to I\!R$ from (2.30) by using our results of Section 3.4.3. We start with rewriting the partial differential equation for $\bar{w}$ into a PDE for w. The diffusion coefficient of the latter PDE can be better expressed in terms of two new functions:

Definition 4.9. Let $u_\varphi : \mathcal{D} \to I\!R$ denote again the reparametrization function which is uniquely determined by (2.32). Then the **Black-Scholes volatility** $\hat{\sigma}_\varphi : \mathcal{D} \to (0,\infty)$ corresponding to our model is given by

$$\hat{\sigma}_\varphi(t,x) := \frac{\psi_u\big(t,u,\varphi(t,u)\big)}{\psi\big(t,u,\varphi(t,u)\big)}\bigg|_{u=u_\varphi(t,x)} \qquad \text{for all } (t,x) \in \mathcal{D},$$

while the **liquidity effect** $c_\varphi : \mathcal{D} \to I\!R$ on the stock prices is defined by

$$c_\varphi(t,x) := \frac{\psi_\xi\big(t,u,\varphi(t,u)\big)}{\psi\big(t,u,\varphi(t,u)\big)}\bigg|_{u=u_\varphi(t,x)} \qquad \text{for all } (t,x) \in \mathcal{D}.$$

Remark. In the specific situation where the equilibrium price function $\psi : [0,T] \times I\!R^2 \to I\!R$ is given by $\psi(t,u,\xi) = S_0 e^{a(t)+\sigma u + c\xi}$ for all $(t,u,\xi) \in [0,T] \times I\!R^2$, we have $\hat{\sigma}_\varphi(t,x) = \sigma$ and $c_\varphi(t,x) = c$ for all $(t,x) \in \mathcal{D}$. In particular, in the Cox-Ross-Rubinstein model of Example 1.9, where $c = 0$ and $a(t) = (\mu - r)t$ for all $t \in [0,T]$, we have $\hat{\sigma}_\varphi \equiv \sigma$ and $c_\varphi \equiv 0$. In general, however, $\hat{\sigma}_\varphi$: $\mathcal{D} \to I\!R$ and $c_\varphi : \mathcal{D} \to I\!R$ still depend on $\varphi : [0,T] \times I\!R \to I\!R$ via the reparametrization function u_φ, even if $\psi : [0,T] \times I\!R^2 \to I\!R$ has a multiplicative structure. □

As a first step, we can now derive the partial differential equation for $w : \mathcal{D} \to I\!R$.

Lemma 4.10. *Suppose that Assumptions B, C and M and either Assumption E or F hold. Then the transformed paper value function $w : \mathcal{D} \to I\!R$ solves the partial differential equation*

$$w_t(t,x) + \frac{1}{2}\hat{\sigma}_\varphi^2(t,x)\frac{1 + \big(2d(\mu)-1\big)c_\varphi(t,x)xw_{xx}(t,x)}{\big(1 - c_\varphi(t,x)xw_{xx}(t,x)\big)^2}x^2 w_{xx}(t,x) = 0 \tag{2.35}$$

for all $(t,x) \in [0,T) \times (0,\infty)$.

Proof. We basically want to use the defining equation (2.30) of the paper value function and (3.4.20) in order to rewrite the PDE (3.4.21) for the transformed real value function $\bar{w}$ into a PDE which only depends on the transformed paper value function w, the price function ψ, the strategy function φ and their derivatives. Here the transformed real value function $\bar{w} : \bar{\mathcal{D}} \to I\!R$ of (3.4.17) is uniquely determined by $\bar{w}\big(t,\overline{\psi}(t,u)\big) = \bar{v}(t,u)$ for all $(t,u) \in [0,T] \times I\!R$. The proof of the lemma itself requires quite tedious calculations. By a clever reshuffling of terms, these calculations can be simplified at least to some extent.
As a first step of the proof, we rewrite the defining equation (2.30) of the paper value function $v : [0,T] \times I\!R \to I\!R$ in terms of the price function ψ, the strategy function φ and the transformed real value function $\bar{w} : D \to I\!R$, and then calculate its first derivatives. For this purpose let us fix $(t,u) \in [0,T) \times I\!R$ and recall from Section 3.4 that

$$\int_0^{\varphi(t,u)} f(z)\,dz = \gamma(t,u) = \frac{\bar{v}_u(t,u)}{\overline{\psi}_u(t,u)} = \bar{w}_x\big(t,\overline{\psi}(t,u)\big). \tag{2.36}$$

Differentiating this equation with respect to t and u, respectively, we see that

$$\gamma_t(t,u) = f\big(\varphi(t,u)\big)\varphi_t(t,u) = \bar{w}_{xt}\big(t,\overline{\psi}(t,u)\big) + \bar{w}_{xx}\big(t,\overline{\psi}(t,u)\big)\overline{\psi}_t(t,u) \tag{2.37}$$

and

$$\gamma_u(t,u) = f\big(\varphi(t,u)\big)\varphi_u(t,u) = \bar{w}_{xx}\big(t,\overline{\psi}(t,u)\big)\overline{\psi}_u(t,u). \tag{2.38}$$

Due to Assumption B and the definition of the liquidation price function $\bar{S} : [0,T] \times \mathbb{R}^2 \to \mathbb{R}$ in (1.3.14), it follows from (2.4.2) and (2.36) that $\varphi(t,u)\bar{S}\big(t,u,\varphi(t,u)\big) = \overline{\varphi}(t,u)\bar{w}_x\big(t,\overline{\varphi}(t,u)\big)$. If we now use $\bar{v}(t,u) = \bar{w}\big(t,\overline{\varphi}(t,u)\big)$, we can rewrite the defining equation (2.30) of the paper value function $v : [0,T] \times \mathbb{R} \to \mathbb{R}$ for our fixed $(t,u) \in [0,T) \times \mathbb{R}$ by

$$v(t,u) = \bar{w}\big(t,\overline{\varphi}(t,u)\big) + \varphi(t,u)\psi\big(t,u,\varphi(t,u)\big) - \overline{\varphi}(t,u)\bar{w}_x\big(t,\overline{\varphi}(t,u)\big).$$

Upon differentiation with respect to u the two terms $\bar{w}_x\big(t,\overline{\varphi}(t,u)\big)\overline{\varphi}_u(t,u)$ offset each other, and we get

$$v_u(t,u) = \varphi(t,u)\frac{d}{du}\psi\big(t,u,\varphi(t,u)\big) + \psi\big(t,u,\varphi(t,u)\big)\varphi_u(t,u) - \overline{\varphi}(t,u)\bar{w}_{xx}\big(t,\overline{\varphi}(t,u)\big)\overline{\varphi}_u(t,u).$$

If we now use the multiplicative structure of ψ for the middle term and apply (2.38), we arrive at

$$v_u(t,u) = \varphi(t,u)\frac{d}{du}\psi\big(t,u,\varphi(t,u)\big), \tag{2.39}$$

and similarly we get from (2.37)

$$v_t(t,u) = \bar{w}_t\big(t,\overline{\varphi}(t,u)\big) + \varphi(t,u)\frac{d}{dt}\psi\big(t,u,\varphi(t,u)\big). \tag{2.40}$$

By definition of $w : \mathcal{D} \to \mathbb{R}$ we have $v(t,u) = w\big(t,\psi(t,u,\varphi(t,u))\big)$. If we differentiate this expression with respect to u and divide the result by $\frac{d}{du}\psi\big(t,u,\varphi(t,u)\big) > 0$, a comparison with (2.39) shows that as in the standard Black-Scholes model we have

$$\varphi(t,u) = w_x\big(t,\psi(t,u,\varphi(t,u))\big) \quad \text{for all } (t,u) \in [0,T) \times \mathbb{R}. \tag{2.41}$$

Now we can also differentiate $v(t,u) = w\big(t,\psi(t,u,\varphi(t,u))\big)$ with respect to t, replace the spatial derivative of w by (2.41) and then compare with (2.40). This yields

$$w_t\big(t,\psi(t,u,\varphi(t,u))\big) = \bar{w}_t\big(t,\overline{\varphi}(t,u)\big) \quad \text{for all } (t,u) \in [0,T) \times \mathbb{R}.$$

Last but not least another differentiation of (2.41) and an application of (2.38) gives for all $(t,u) \in [0,T) \times \mathbb{R}$:

$$w_{xx}\big(t,\psi(t,u,\varphi(t,u))\big)\frac{d}{du}\psi\big(t,u,\varphi(t,u)\big) = \varphi_u(t,u) = \frac{\overline{\varphi}_u(t,u)}{f(\varphi(t,u))}\bar{w}_{xx}\big(t,\overline{\varphi}(t,u)\big). \tag{2.42}$$

Let us now recall the function $u_\varphi : \mathcal{D} \to \mathbb{R}$ of (2.32) and implicitly define the function $\rho_\varphi : [0,T) \times (0,\infty) \to (0,\infty)$ for all $(t,x) \in [0,T) \times (0,\infty)$ by

$$\rho_\varphi^2(t,x) = \frac{\psi_u + 2d(\mu)\psi_\xi\varphi_u(t,u)}{\psi(t,u,\varphi(t,u))}\frac{\psi_u + \psi_\xi\varphi_u(t,u)}{\psi(t,u,\varphi(t,u))}\bigg|_{u=u_\varphi(t,x)}, \tag{2.43}$$

where the arguments $\big(t,u,\varphi(t,u)\big)$ of the derivatives of ψ have been skipped. With this notation, (2.42), another application of (2.38) and $\psi_\xi = \overline{\varphi}f'$ imply for all $(t,u) \in [0,T) \times \mathbb{R}$

$$\begin{aligned}
&w_t\Big(t,\psi\big(t,u,\varphi(t,u)\big)\Big) + \frac{1}{2}\rho_\varphi^2\Big(t,\psi\big(t,u,\varphi(t,u)\big)\Big)\psi^2\big(t,u,\varphi(t,u)\big)w_{xx}\Big(t,\psi\big(t,u,\varphi(t,u)\big)\Big) \\
&\quad = \bar{w}_t\big(t,\overline{\varphi}(t,u)\big) + \frac{1}{2}\Big(\psi_u + 2d(\mu)\psi_\xi\varphi_u(t,u)\Big)\frac{\overline{\varphi}_u(t,u)}{f(\varphi(t,u))}\bar{w}_{xx}\big(t,\overline{\varphi}(t,u)\big) \\
&\quad = \bar{w}_t\big(t,\overline{\varphi}(t,u)\big) + \frac{1}{2}\left(1 + 2d(\mu)\overline{\varphi}(t,u)\frac{f'(\varphi(t,u))}{f^2(\varphi(t,u))}\bar{w}_{xx}\big(t,\overline{\varphi}(t,u)\big)\right)\overline{\varphi}_u^2(t,u)\bar{w}_{xx}\big(t,\overline{\varphi}(t,u)\big).
\end{aligned}$$

After plugging in the definitions of the transformed loss function $\kappa : g(I\!R) \to I\!R$ in (3.3.1) and of the volatility function $\bar{\sigma} : \mathcal{D} \to (0,\infty)$ in (3.4.19), the last line equals the left-hand side of the partial differential equation (3.4.21) for $x = \overline{\psi}(t,u)$. Thus, it follows that

$$w_t(t,x) + \frac{1}{2}\rho_\varphi^2(t,x)x^2 w_{xx}(t,x) = 0 \quad \text{for all } (t,x) \in [0,T) \times (0,\infty). \tag{2.44}$$

In order to prove (2.35) it only remains to show that $\rho_\varphi : [0,T) \times (0,\infty) \to (0,\infty)$ can be rewritten as

$$\rho_\varphi^2(t,x) = \hat{\sigma}_\varphi^2(t,x)\frac{1 + \big(2d(\mu) - 1\big)c_\varphi(t,x)xw_{xx}(t,x)}{\big(1 - c_\varphi(t,x)xw_{xx}(t,x)\big)^2} \quad \text{for all } (t,x) \in [0,T) \times (0,\infty). \tag{2.45}$$

For that purpose we first rewrite (2.43) for all $(t,x) \in [0,T) \times (0,\infty)$ in terms of the Black-Scholes volatility $\hat{\sigma}_\varphi : \mathcal{D} \to (0,\infty)$ and the liquidity effect $c_\varphi : \mathcal{D} \to I\!R$ of Definition 4.9. This gives us

$$\rho_\varphi^2(t,x) = \hat{\sigma}_\varphi^2(t,x)\left(1 + 2d(\mu)\frac{c_\varphi(t,x)}{\hat{\sigma}_\varphi(t,x)}\varphi_u(t,u)\right)\left(1 + \frac{c_\varphi(t,x)}{\hat{\sigma}_\varphi(t,x)}\varphi_u(t,u)\right)\Bigg|_{u=u_\varphi(t,x).} \tag{2.46}$$

Calculating the derivative $\frac{d}{du}\psi\big(t,u,\varphi(t,u)\big)$ and using once again the definitions of $\hat{\sigma}_\varphi$ and c_φ we get from the left equality in (2.42)

$$\varphi_u(t,u) = xw_{xx}(t,x)\big(\hat{\sigma}_\varphi(t,x) + c_\varphi(t,x)\varphi_u(t,u)\big)\Big|_{x=\psi(t,u,\varphi(t,u))} \quad \text{for all } (t,u) \in [0,T) \times I\!R.$$

If we now collect the terms with $\varphi_u(t,u)$ we obtain

$$\Big(1 - c_\varphi(t,x)xw_{xx}(t,x)\Big)\varphi_u(t,u) = \hat{\sigma}_\varphi(t,x)xw_{xx}(t,x). \tag{2.47}$$

Since $\hat{\sigma}_\varphi$ is strictly positive, the assumption $c_\varphi(t,x)xw_{xx}(t,x) = 1$ leads to a contradiction, hence we can divide (2.47) by the factor in front of $\varphi_u(t,u)$ and finally arrive at the relation

$$\varphi_u(t,u) = \hat{\sigma}_\varphi(t,x)\frac{xw_{xx}(t,x)}{1 - c_\varphi(t,x)xw_{xx}(t,x)}\Bigg|_{x=\psi(t,u,\varphi(t,u))} \quad \text{for all } (t,u) \in [0,T) \times I\!R. \tag{2.48}$$

Plugging this representation of φ_u into (2.46), we obtain the representation (2.45) which was left to show. **q.e.d.**

For later reference, we formally define the function $\rho_\varphi : [0,T) \times (0,\infty) \to (0,\infty)$ which represents the diffusion coefficient of the PDE (2.35).

Definition 4.11. The function $\rho_\varphi : [0,T) \times (0,\infty) \to (0,\infty)$ which satisfies (2.43) and (2.45) is called **large investor volatility** for paper replication induced by the final strategy function $\varphi(T,\cdot\,) = \alpha\zeta$.

Remark. It is clear from our proof of Lemma 4.10 that under the stated assumptions the function ρ_φ is well-defined. We have already seen from (2.47) that $c_\varphi(t,x)xw_{xx}(t,x) \neq 1$. Using (2.48) and Definition 4.9, this statement can now be strengthened to $c_\varphi(t,x)xw_{xx}(t,x) < 1$ for all $(t,x) \in [0,T) \times (0,\infty)$, since we have

$$c_\varphi(t,x)xw_{xx}(t,x) = \frac{c_\varphi(t,x)\varphi_u(t,u)}{\hat{\sigma}_\varphi(t,x) + c_\varphi(t,x)\varphi_u(t,u)}\Bigg|_{u=u_\varphi(t,x)} = \frac{\frac{\psi_\xi(t,u,\varphi(t,u))}{\psi_u(t,u,\varphi(t,u))}\varphi_u(t,u)}{1 + \frac{\psi_\xi(t,u,\varphi(t,u))}{\psi_u(t,u,\varphi(t,u))}\varphi_u(t,u)}\Bigg|_{u=u_\varphi(t,x)},$$

where the denominator of the last fraction is strictly positive due to Assumption M. In the "linear" case $d(\mu) = 0$, where the partial differential equation for the strategy function is a linear equation, the pole of the fraction in (2.45) reduces by one order, and (2.45) becomes

$$\rho_\varphi^2(t,x) = \hat{\sigma}_\varphi^2(t,x)\frac{1}{1 - c_\varphi(t,x)xw_{xx}(t,x)} \quad \text{for all } (t,x) \in [0,T) \times (0,\infty).$$

This gives us yet another indication why we need less stringent boundary conditions if the price determining measure μ satisfies $d(\mu) = 0$.
In the special case where μ is the Dirac measure in δ_1, we have $2d(\mu) = 1$, and hence

$$\rho_\varphi^2(t,x) = \hat{\sigma}_\varphi^2(t,x)\left(\frac{1}{1 - c_\varphi(t,x)xw_{xx}(t,x)}\right)^2 \quad \text{for all } (t,x) \in [0,T) \times (0,\infty).$$

Under Assumptions B, C and F we have seen in Section 3.4.2 that the solution φ to the final value problem (3.4.3), (3.4.4) belongs to the space $C_b^1(I\!R)$, and by Lemma 3.31 we have $\mathcal{D} = [0,T] \times (0,\infty)$. If in addition to these conditions the ratio $\frac{\psi}{\psi_u}$ is bounded away from 0, it is easily seen from (2.43) that the continuous large investor volatility function ρ_φ remains bounded on its domain $[0,T) \times (0,\infty)$. Therefore, it can be extended to a bounded continuous function on $\mathcal{D} = [0,T] \times (0,\infty)$.
If Assumption F is replaced by Assumption E, we cannot guarantee $\varphi(T,\,\cdot\,) = \alpha\zeta \in C_b^1(I\!R)$, and the function $\rho_\varphi : [0,T) \times (0,\infty) \to (0,\infty)$ need not be smoothly continuable on $\mathcal{D}$; the large investor volatility ρ_φ might explode at all the points (T,x) where x is a boundary point of the set $\mathcal{E}$ of (2.34). $\square$

In order to complete the final value problem for the transformed paper value function we still need to state the final condition for w. Due to possible jumps of the function $X : I\!R \to I\!R$, $u \mapsto \psi(T,u,\alpha\zeta(u))$ of (2.33), this boundary condition becomes more challenging than in small investor models. Before we come to the general final condition, we consider the final condition for the transformed paper value of the European call which was introduced in Example 3.1.

Example 4.1 *(European Call).* Let us suppose the price system (ψ,μ) satisfies Assumptions B, C and the corresponding conditions of Assumption E. Let us also suppose that $f : I\!R \to I\!R$ is nondecreasing. Fix some $K \in I\!R$ and $\alpha \geq 0$, and as in Example 3.1 take $u^* \in I\!R$ such that $\psi(T,u^*) = \big(\int_0^1 f(\theta\alpha)\,d\theta\big)^{-1}K$. Let us now recall the European call $\big(\alpha\zeta, b^\alpha\big)$ of Example 3.1, which is given by $\zeta(u) = \mathbf{1}_{\{u \geq u^*\}}$ and $b^\alpha(u) = -\alpha K\mathbf{1}_{\{u \geq u^*\}}$ for all $u \in I\!R$. If $\alpha \geq 0$ is chosen small enough, all conditions of Assumption E and Assumption M hold.
The paper value function $v : [0,T] \times I\!R \to I\!R$ at maturity T of the European call is given by

$$v(T,u) = \alpha\zeta(u)\psi(T,u,\alpha\zeta(u)) + b^\alpha(u) = \alpha\big(\psi(T,u,\alpha\zeta(u)) - K\big)^+ \quad \text{for all } u \in I\!R. \qquad (2.49)$$

In order to restate this final condition in terms of the transformed paper value function $w : \mathcal{D} \to I\!R$, we first need to determine the set $X(I\!R)$ of possible final stock prices. Since $X : I\!R \to I\!R$ has only one jump from the left at u^*, the exception set $\mathcal{E}$ of (2.34) is given by $\mathcal{E} = \big[X(u^*-), X(u^*)\big) = \big[\psi(T,u^*,0), \psi(T,u^*,\alpha)\big)$, and hence

$$X(I\!R) = (0,\infty)\backslash\big[\psi(T,u^*,0), \psi(T,u^*,\alpha)\big).$$

On the set $\{T\} \times X(I\!R)$ the transformed paper value function $w : \mathcal{D} \to I\!R$ is now given by

$$w(T,x) = \alpha(x-K)^+ \quad \text{for all } x \in X(I\!R).$$

If the contingent claim $\big(\alpha\zeta, b^\alpha\big)$ were given by $\zeta(u) = \mathbf{1}_{\{u > u^*\}}$ and $b^\alpha(u) = -\alpha K\mathbf{1}_{\{u > u^*\}}$ for all $u \in I\!R$, the paper value function $v : [0,T] \times I\!R \to I\!R$ would also be given by (2.49), but the set $X(I\!R)$ of possible stock price at maturity would differ in that

$$X(I\!R) = (0,\infty)\backslash\big(\psi(T,u^*,0), \psi(T,u^*,\alpha)\big].$$

Hence the domain $\mathcal{D}$ of the transformed paper value function w for the second version of the European call differs on the boundary from the domain of the transformed paper value function for the European call in the initially specified version. $\square$

Now let us derive the final value condition for the transformed strategy function w for a more general contingent claim $(\alpha\zeta, b^\alpha)$. For this purpose, we first investigate the possible stock prices at maturity T. By (3.4.4), the large investor's strategy at time T is prescribed by $\varphi(T,u) = \alpha\zeta(u)$ for each possible value $u \in I\!R$ of the fundamentals at that time. Thus, the function $X : I\!R \to (0,\infty)$ given by

$$X(u) = \psi\big(T, u, \alpha\zeta(u)\big) \quad \text{for all } u \in I\!R$$

associates to each fundamental value the corresponding stock price at time T. We have already seen in the discussion following the definition of X in (2.33) that $X(I\!R) = (0,\infty)\backslash\mathcal{E}$, where the exception set $\mathcal{E}$ is defined in (2.34). According to Assumption M the function $X : I\!R \to X(I\!R)$ is strictly increasing and hence invertible. In addition to the inverse X^{-1}, which is only defined on the domain $X(I\!R)$, we also introduce the generalized inverse $X^- : (0,\infty) \to I\!R$, which is defined for all positive real numbers by

$$X^-(x) = \sup\big\{u \in I\!R \,\big|\, X(u) < x\big\} \quad \text{for all } x \in (0,\infty).$$

If $x \in X(I\!R)$ we get $X^-(x) = X^{-1}(x)$, so X^- indeed generalizes the classical inverse X^{-1}. In order to derive the final condition for the transformed paper value function, we start again with (2.30). Setting $t = T$ we obtain

$$v(T,u) = \bar{v}(T,u) + \varphi(T,u)\Big(\psi\big(T,u,\alpha\zeta(u)\big) - \bar{S}\big(T,u,\alpha\zeta(u)\big)\Big) \quad \text{for all } u \in I\!R.$$

If we now apply the final conditions (3.4.7) and (3.4.4) of the final value problems for $\bar{v}$ and φ, respectively, we get

$$v(T,u) = \alpha\zeta(u)\psi\big(T,u,\varphi(T,u)\big) + b^\alpha(u) \quad \text{for all } u \in I\!R, \tag{2.50}$$

where $b^\alpha : I\!R \to I\!R$ was defined in (3.4.5) by $b^\alpha(u) = b_0^\alpha - \int_0^u \overline{\varphi}(T,\bar{u})\, d\big(\int_0^{\alpha\zeta(\bar{u})} f(z)\,dz\big)$ for all $u \in I\!R$. Now $w\big(T,X(u)\big) = v(T,u)$ implies

$$w(T,x) = \alpha\zeta\big(X^{-1}(x)\big)x + b^\alpha\big(X^{-1}(x)\big) \quad \text{for all } x \in X(I\!R). \tag{2.51}$$

This is a valid boundary condition for the partial differential equation (2.35). However, we would like to state the final condition in some more intuitive form, which resembles the structure of a European call or put as it is given in the Black-Scholes model. The following lemma gives us such a representation.

Lemma 4.12. *Under the Assumptions of Lemma 4.10, we can rewrite the final condition* (2.51) *for the transformed paper value function into the condition*

$$w(T,x) = w_0^\alpha + \alpha h(x) + \alpha H^-(x) + \alpha H^+(x) \quad \text{for all } x \in X(I\!R), \tag{2.52}$$

where the constant $w_0^\alpha = b_0^\alpha + \alpha\zeta(0)\psi\big(T,u,\alpha\zeta(0)\big)$ *denotes the large investor's final paper value* $w\big(T,X(0)\big) = v(T,0)$ *when the fundamentals at time T vanish, the differentiable function* $h : (0,\infty) \to I\!R$ *is defined by*

$$h(x) = \int_{X(0)}^x \zeta^d(0) + \zeta^{ac}\big(X^-(z)\big)\,dz \quad \text{for all } x \in I\!R,$$

and the càdlàg function $H^- : (0,\infty) \to \mathbb{R}$ *defined by*

$$H^-(x) = \begin{cases} \sum_{X^-(x)<u\le 0}\big(\zeta(u)-\zeta(u-)\big)\Big(S^*\big(T,u,\alpha\zeta(u),\alpha\zeta(u-)\big)-x\Big) & \text{if } x < X(0), \\ \sum_{0<u\le X^-(x)}\big(\zeta(u)-\zeta(u-)\big)\Big(x-S^*\big(T,u,\alpha\zeta(u-),\alpha\zeta(u)\big)\Big) & \text{if } x \ge X(0) \end{cases}$$

and the càglàd function $H^+ : (0,\infty) \to \mathbb{R}$ *given by*

$$H^+(x) = \begin{cases} \sum_{X^-(x)\le u<0}\big(\zeta(u+)-\zeta(u)\big)\Big(S^*\big(T,u,\alpha\zeta(u+),\alpha\zeta(u)\big)-x\Big) & \text{if } x < X(0) \\ \sum_{0\le u<X^-(x)}\big(\zeta(u+)-\zeta(u)\big)\Big(x-S^*\big(T,u,\alpha\zeta(u),\alpha\zeta(u+)\big)\Big) & \text{if } x \ge X(0), \end{cases}$$

take account of the influence of ζ*'s jumps from the left and right, respectively, on the final paper value.*

Proof. Without loss of generality, suppose that $\zeta : \mathbb{R} \to \mathbb{R}$ is right continuous, so that H^+ vanishes. In order to show that the two final conditions (2.51) and (2.52) are equivalent we have to consider the function $\bar\gamma : \mathbb{R} \to \mathbb{R}$ given by $\bar\gamma(u) = \gamma(T,u) = \int_0^{\alpha\zeta(u)} f(z)\,dz$ for all $u \in \mathbb{R}$ in some more detail; this function is used as the integrator in the Riemann-Stieltjes integral which appears in the definition of $b^\alpha : \mathbb{R} \to \mathbb{R}$.
In the proof of Lemma 3.29 we have already seen that the decomposition of ζ into an absolutely continuous part and a pure jump part implies the same property for $\bar\gamma$, i.e. we can write $\bar\gamma = \bar\gamma^{ac} + \bar\gamma^d$, where the pure jump part $\bar\gamma^d$ is given by

$$\bar\gamma^d(u) = \int_0^{\alpha\zeta(0)} f(z)dz + \sum_{0<\bar u\le u}\int_{\alpha\zeta(\bar u-)}^{\alpha\zeta(\bar u)} f(z)\,dz - \sum_{u<\bar u\le 0}\int_{\alpha\zeta(\bar u-)}^{\alpha\zeta(\bar u)} f(z)\,dz \quad \text{for all } u \in \mathbb{R},$$

and the absolutely continuous part $\bar\gamma^{ac} : \mathbb{R} \to \mathbb{R}$ by

$$\bar\gamma^{ac}(u) = \int_0^u \frac{d}{d\bar u}\int_0^{\alpha\zeta(\bar u)} f(z)\,dz\,d\bar u = \int_0^u f\big(\alpha\zeta(\bar u)\big)\alpha\zeta_u^{ac}(\bar u)d\bar u = \alpha\int_0^u f\big(\alpha\zeta(\bar u)\big)\,d\zeta^{ac}(\bar u).$$

If we first apply the definitions of $X : \mathbb{R} \to (0,\infty)$ and $b^\alpha : \mathbb{R} \to \mathbb{R}$ to (2.50), use the definition of w_0^α and then split each of the functions ζ and $\bar\gamma$ into an absolutely continuous and a pure jump component, we get

$$v(T,u) = b_0^\alpha + \alpha\zeta(u)X(u) - \int_0^u \overline{\psi}(T,\bar u)\,d\bar\gamma(\bar u) = w_0^\alpha + D^{ac}(u) + D^d(u) \tag{2.53}$$

for all $u \in \mathbb{R}$, where the functions $D^{ac} : \mathbb{R} \to \mathbb{R}$ and $D^d : \mathbb{R} \to \mathbb{R}$ are given by

$$D^{ac}(u) = \alpha\zeta^{ac}(u)X(u) - \alpha\zeta^{ac}(0)X(0) - \int_0^u \overline{\psi}(T,\bar u)\,d\bar\gamma^{ac}(\bar u) + \alpha\zeta^d(0)\big(X(u)-X(0)\big)$$

and

$$D^d(u) = \alpha\zeta^d(u)X(u) - \alpha\zeta^d(0)X(u) - \int_0^u \overline{\psi}(T,\bar u)\,d\bar\gamma^d(\bar u) \quad \text{for all } u \in \mathbb{R},$$

respectively. Now standard calculation yields

$$\int_0^u \overline{\psi}(T,\bar u)\,d\bar\gamma^{ac}(\bar u) = \alpha\int_0^u \overline{\psi}(T,\bar u)f\big(\alpha\zeta(\bar u)\big)\,d\zeta^{ac}(\bar u) = \alpha\int_0^u X(\bar u)\,d\zeta^{ac}(\bar u) \quad \text{for all } u \in \mathbb{R}.$$

Hence we can conclude from an integration by parts, the substitution $dz = dX(\bar{u})$, and the definition of $h : (0,\infty) \to I\!R$ that

$$D^{ac}(u) = \alpha \int_0^u \zeta^{ac}(\bar{u})\, dX(\bar{u}) = \alpha \int_{X(0)}^{X(u)} \zeta^{ac}\big(X^-(z)\big)\, dz = \alpha h\big(X(u)\big) \quad \text{for all } u \in I\!R.$$

In order to calculate $D^d(u)$ we employ a telescoping sum for $\zeta^d(u) - \zeta^d(0)$ to obtain

$$\begin{aligned} D^d(u) = \alpha &\sum_{u<\bar{u}\le 0} \big(\zeta(\bar{u}) - \zeta(\bar{u}-)\big)\Big(S^*\big(T,\bar{u},\alpha\zeta(\bar{u}-),\alpha\zeta(\bar{u})\big) - X(u)\Big) \\ &+ \alpha \sum_{0<\bar{u}\le u} \big(\zeta(\bar{u}) - \zeta(\bar{u}-)\big)\Big(X(u) - S^*\big(T,\bar{u},\alpha\zeta(\bar{u}-),\alpha\zeta(\bar{u})\big)\Big) \quad \text{for all } u \in I\!R, \end{aligned}$$

and by comparing this expression with the definition of $H^- : (0,\infty) \to I\!R$ we see that $D^d(u) = \alpha H^-\big(X(u)\big)$. Hence (2.53) implies

$$v(T,u) = w_0^\alpha + \alpha h\big(X(u)\big) + \alpha H^-\big(X(u)\big) \quad \text{for all } u \in I\!R,$$

and the assertion follows immediately. **q.e.d.**

Although the functions h, H^- and H^+ of (2.52) are defined as functions of the stock price $X(u) = \psi\big(T,u,\alpha\zeta(u)\big)$ at time T, they are still determined in terms of the boundary function $\alpha\zeta : [0,T] \times I\!R \to I\!R$ of the strategy function φ. In the standard Black-Scholes model, where the shares of stock held by any particular investor do not have an effect on the stock price, an "option" is defined by its payoff function $C : (0,\infty) \to I\!R$ in terms of the stock price at maturity T, and this payoff function is used as the boundary condition for the value function of the replicating strategy.

In the discrete setting of Section 1.4.3 we have used the large investor stock price as the input of an "option" which describes the large investor's paper value at maturity. In order to find a trading strategy which replicates the paper value of such an "option", we have solved a fixed point problem for the large investor's stock holdings at *and* immediately before maturity, and then applied our usual recursive scheme to calculate the large investor's replicating strategy for all time points between 0 and T.

In a similar fashion, we can ask in the continuous time model for a trading strategy of the large investor for which the paper value at maturity can be written as a given function $C : (0,\infty) \to I\!R$ of the (large investor) stock price at that time. For this purpose, let us assume that the large investor replicates $\alpha \neq 0$ "options" with the same payoff $C(x)$, depending on the stock price x at time T, such that the transformed paper value function $w : \mathcal{D} \to I\!R$ of his replicating portfolio at maturity satisfies

$$w(T,x) = \alpha C(x) \quad \text{for all } x \in X(I\!R). \tag{2.54}$$

Suppose for a moment that we already know his final stock holdings $\varphi(T,u)$ at time T for all possible fundamental values $u \in I\!R$. Then at maturity the large investor holds on average $\zeta(u) = \frac{1}{\alpha}\varphi(T,u)$ shares per option. In addition, let us suppose that the price system (ψ,μ), the function ζ and the constant α satisfy the Assumptions B, C, F and M.

Especially, Assumption F implies that ζ is differentiable and $\zeta = \zeta^{ac}$ since ζ has no jumps. Therefore the function $X : I\!R \to (0,\infty)$, $u \mapsto \psi\big(T,u,\alpha\varphi(T,u)\big)$ maps $I\!R$ onto $(0,\infty)$, and its inverse $X^{-1} : (0,\infty) \to I\!R$ is differentiable. Moreover, the functions H^- and H^+ of Lemma 4.12 vanish on $I\!R$, and $h : (0,\infty) \to I\!R$ simplifies to $h(x) = \int_{X(0)}^x \zeta\big(X^{-1}(z)\big)\, dz$ for all $x > 0$. By Lemma 4.12 the large investor's (transformed) paper value at time T can now be written as $w(T,x) = w_0^\alpha + \alpha \int_{X(0)}^x \zeta\big(X^{-1}(z)\big)\,dz$ for all $x > 0$ for some constant $w_0^\alpha \in I\!R$. Because of (2.54), this also gives the same representation for $\alpha C(x)$ and differentiation yields

$$w_x(T,x) = \alpha C'(x) = \alpha\zeta\big(X^{-1}(x)\big) \quad \text{for all } x \in (0,\infty). \tag{2.55}$$

Since we assume $\alpha \neq 0$, we can divide the right hand side of (2.55) by α, and plugging in $x = X(u) = \psi\big(T, u, \alpha\zeta(u)\big)$ we obtain the fixed point equation

$$C'\big(\psi(T,u,\alpha\zeta(u))\big) = \zeta(u) \quad \text{for all } u \in I\!R \tag{2.56}$$

for the shape $\zeta : I\!R \to I\!R$ of the large investor's strategy function at time T.
On the other hand, if we are given an arbitrary function $C : (0,\infty) \to I\!R$ and look for a trading strategy such that the associated transformed paper value function satisfies $w(T,x) = \alpha C(x)$, then the normalized stock holdings $\zeta = \frac{1}{\alpha}\varphi(T,\,\cdot\,)$ of the large investor at time T have to satisfy (2.56). Thus, if we do not know ζ, we can specify it as the solution to the fixed point problem (2.56). The next proposition states conditions on the function $C : (0,\infty) \to I\!R$ which ensure that for all sufficiently small $|\alpha|$ the fixed point problem (2.56) has a unique and sufficiently smooth solution $\zeta : I\!R \to I\!R$, so that α and ζ satisfy for example the conditions of Assumption F. Once ζ is specified, the strategy function $\varphi : [0,T] \times I\!R \to I\!R$ which is necessary to achieve final stock holdings of $\varphi(T,u) = \alpha\zeta(u)$ for all $u \in I\!R$ is determined as the solution to the final value problem (3.4.3), (3.4.4). The constant b_0^α, which describes the large investor's bond holdings at time T when the fundamentals are 0, is also completely specified by C and α and the solution $\zeta : I\!R \to I\!R$ of (2.56), since (2.54) implies $b_0^\alpha = \alpha C\big(\psi(T,0,\alpha\zeta(0))\big) - \zeta(0)\psi\big(T,0,\alpha\zeta(0)\big)$. Due to the fixed point equation (2.56) this expression can be written just in terms of C and the stock price $x_0 = X(0) = \psi\big(T,0,\alpha\zeta(0)\big)$ as

$$b_0^\alpha = \alpha\big(C(x_0) - C'(x_0)x_0\big).$$

Thus, the next proposition will show that under suitable regularity conditions we can replicate the paper value of given "options" $C : (0,\infty) \to I\!R$ in the continuous model.

Proposition 4.13. *In addition to Assumptions B and C suppose that for some $k \in I\!N$ the functions $\overline{\psi}$ and f belong to the Hölder spaces $\widehat{H}^{\frac{1}{2}(k+\beta),k+\beta}\big([0,T] \times I\!R\big)$ and $H_{loc}^{k+\beta}(I\!R)$, respectively. If the function $C : (0,\infty) \to I\!R$ is $(k+1)$ times differentiable and if there exists some $B \geq 0$ such that*

$$\left| x^j \left(\frac{d}{dx}\right)^{j+1} C(x) \right| \leq B \quad \text{for all } x \in (0,\infty) \text{ and all } 0 \leq j \leq k, \tag{2.57}$$

then there exists some open interval $(\alpha_1,\alpha_2) \supset \{0\}$ such that for each $\alpha \in (\alpha_1,\alpha_2)$ the fixed point equation (2.56) *has a unique solution $\zeta : I\!R \to I\!R$, and the solution ζ belongs to the space $C_b^k(I\!R)$.*

Proof. Our argument follows the proof of an analogous statement in Lemma 3.6 of Frey (1998). For each $\alpha \in I\!R$ and $u \in I\!R$ the function $a_{\alpha,u} : I\!R \to I\!R$, $\xi \mapsto C'\big(\psi(T,u,\alpha\xi)\big) - \xi$, is continuous and surjective, since (2.57) implies $\|C'\| \leq B$. Hence for all $\alpha \in I\!R$ there exists some function $\zeta = \zeta^\alpha : I\!R \to I\!R$ for which (2.56) holds. Of course, ζ is bounded by B as well. If we now write $\psi_\xi(T,u,\alpha\xi) = \psi(T,u,\alpha\xi)\frac{f'(\alpha\xi)}{f(\alpha\xi)}$ and apply (2.57) with $j=1$ we obtain

$$\Big|C''\big(\psi(T,u,\alpha\xi)\big)\psi_\xi(T,u,\alpha\xi)\Big| \leq B \sup_{-|\alpha|B \leq z \leq |\alpha|B} \left|\frac{f'(z)}{f(z)}\right| \quad \text{for all } \alpha, u \in I\!R \text{ and } \xi \in [-B,B].$$

Since f is continuously differentiable and strictly positive, the fraction $\frac{f'}{f}$ is bounded on each compact interval I. Hence there exists some open interval $(\alpha_1, \alpha_2) \supset \{0\}$ such that for all fixed $\alpha \in (\alpha_1, \alpha_2)$ we obtain the strict inequality

$$\sup\Big\{\alpha C''\big(\psi(T,u,\alpha\xi)\big)\psi_\xi(T,u,\alpha\xi)\,\Big|\, u \in I\!R,\ \xi \in [-B,B]\Big\} < 1.$$

Hence for all $\alpha \in (\alpha_1, \alpha_2)$ and all $u \in I\!R$ the function $a_{\alpha,u} : I\!R \to I\!R$, $\xi \mapsto C'\big(\psi(T,u,\alpha\xi)\big) - \xi$ is strictly increasing. This implies the uniqueness of $\zeta : I\!R \to I\!R$. The implicit function theorem gives the differentiability of ζ, and (2.56) yields

$$\zeta_u(u) = \frac{C''\big(\psi(T,u,\alpha\zeta(u))\big)\psi_u(T,u,\alpha\zeta(u))}{1 - \alpha C''\big(\psi(T,u,\alpha\zeta(u))\big)\psi_\xi(T,u,\alpha\zeta(u))} \quad \text{for all } u \in I\!R.$$

It now follows easily that ζ belongs to the space $C_b^k(I\!R)$. **q.e.d.**

Remark. Recall the discussion of the replication of the paper wealth in the discrete setting of Section 1.4.3. In that section we have shown how one can construct a discrete trading strategy in the n-step binomial model, which replicates the value of a given (convex) "option" $C : (0,\infty) \to I\!R$.
It is natural to start with some sufficiently smooth "option" $C : (0,\infty) \to I\!R$, and some sufficiently small $\alpha > 0$ such that the continuous-time strategy $\varphi : [0,T] \times I\!R \to I\!R$ implied by C and α via (2.56) and $\varphi(T,u) = \alpha\zeta(u)$ exists, and ask whether the sequence of the n-step discrete models of Section 1.4.3 will converge to the continuous model for the paper replication of the same option $C : (0,\infty) \to I\!R$ as $n \to \infty$, in the sense of the present section. Unfortunately, in the large investor setting the dependence of the stock prices on the strategy destroys the most straightforward approach. The values of the discrete strategy functions $\xi^n : \mathcal{A}^n \to I\!R$ at the time points t_{n-1}^n and t_n^n, which are calculated in Proposition 1.36, do not satisfy the condition (3.3.67) of Corollary 3.28, and they also do not satisfy the weaker condition stated in the remark to that corollary. Thus, we cannot show that the sequence of discrete strategy functions $\xi^n : \mathcal{A}^n \to I\!R$ converges in the sense of Corollary 3.28 to the continuous-time strategy function $\varphi : [0,T] \times I\!R \to I\!R$, because Assumption K is not satisfied and for example Proposition 4.8 cannot be applied.
However, the situation is not totally hopeless. In view of our ansatz in Section 3.2.4 we believe that it is still possible to find a subsequence $\{n_j\}_{j\in I\!N} \subset I\!N$ such that along this subsequence we have $\big\|\xi^{n_j} - \varphi\big\|_{\mathcal{A}^{n_j}} = O\big(\delta^2\big)$ and

$$\left\|\xi^{n_j}(\cdot + \delta^2, \cdot \pm \delta) - \xi^{n_j} \mp \delta\varphi_u - \delta^2\left(\varphi_t + \frac{1}{2}\varphi_{uu}\right)\right\|_{\mathcal{A}^{n_j}(n_j-1)} = O\big(\delta^{2+\beta}\big) \quad \text{as } j \to \infty,$$

where $\delta = \delta_{n_j} = n_j^{-\frac{1}{2}}$ for all $j \in I\!N$. Using Remark 2 at the end of Section 4.2.3, we find that at least for this subsequence the discrete models converge to the continuous model in the sense of Proposition 4.8. □

Let us now recall the large investor volatility $\rho_\varphi : [0,T) \times (0,\infty) \to (0,\infty)$ of Definition 4.11. If we write it in terms of the derivative w_{xx} as in (2.45), it is given by

$$\rho_\varphi^2(t,x) = \hat\sigma_\varphi^2(t,x)\frac{1 + \big(2d(\mu) - 1\big)c_\varphi(t,x)xw_{xx}(t,x)}{\big(1 - c_\varphi(t,x)xw_{xx}(t,x)\big)^2} \quad \text{for all } (t,x) \in [0,T) \times (0,\infty).$$

The canonical generalization of the final value problem for the value function in the Black-Scholes model to the paper value function of the large investor is then given by the partial differential equation

$$w_t(t,x) + \frac{1}{2}\rho_\varphi(t,x)x^2 w_{xx}(t,x) = 0 \quad \text{for all } (t,x) \in [0,T) \times (0,\infty) \tag{2.58}$$

together with the final condition

$$w(T,x) = \alpha C(x) \quad \text{for all } x \in X(\mathbb{R}). \tag{2.59}$$

for some payoff function $C : (0,\infty) \to \mathbb{R}$. Whenever C satisfies the assumptions of Proposition 4.13 and α is chosen small enough, or – more generally – whenever we are given some $\alpha \in \mathbb{R}$ and $\zeta : \mathbb{R} \to \mathbb{R}$ such that Assumption M and either Assumption E or F hold so that the functions h, H^- and H^+ of Lemma 4.12 and hence $C : (0,\infty) \to \mathbb{R}$ given by

$$C(x) = w_0^\alpha + \alpha h(x) + \alpha H^-(x) + \alpha H^+(x) \quad \text{for all } x \in (0,\infty), \tag{2.60}$$

are well-defined, a solution to (2.58), (2.59) exists.

Remark. If $\psi_\xi \equiv 0$ so that the large investor does not effect the stock price at all, then the liquidity effect c_φ of the large investor vanishes, the Black-Scholes volatility $\hat{\sigma}_\varphi$ of Definition 4.9 does not depend on the large investor's strategy either, and we obtain $\rho_\varphi \equiv \hat{\sigma}_\varphi \equiv \bar{\sigma}$. Here the function $\bar{\sigma} : [0,T) \times (0,\infty) \to \mathbb{R}$ is given by $\bar{\sigma}(t,x) = \frac{\overline{\psi}_u(t,u)}{\overline{\psi}(t,u)}\big|_{u=u(t,x)}$ for all $(t,x) \in [0,T] \times (0,\infty)$, where $u(t,x)$ is the unique solution of $\overline{\psi}\big(t,u(t,x)\big) = x$. In addition to that simplification, the possible jumps of $\zeta : \mathbb{R} \to \mathbb{R}$ do not lead to any jumps of the function $X : \mathbb{R} \to (0,\infty)$, $u \mapsto \psi\big(T,u,\alpha\zeta(u)\big)$, and hence $X(\mathbb{R}) = (0,\infty)$. It follows that the final value problem for the transformed paper value coincides with the final value problem (3.4.21), (3.4.22) for the corresponding transformed real value function. This is what we expect, since it can be seen directly from the definition of the two functions that the real value function coincides with the paper value function if $\psi_\xi \equiv 0$.

In general, however, the volatility ρ_φ in the partial differential equation (2.58) fundamentally differs from the corresponding volatility term in the standard Black-Scholes model or in the partial differential equation for the real value function. In the final value problem of the value function in the standard Black-Scholes model, the volatility is constant or a function of time and stock price. In Section 3.4.3 we have seen that the actual volatility in the partial differential equation (3.4.21) for the real value function in a large investor model with transaction losses also depends on the Gamma $\bar{w}_{xx}$ and – if κ is not constant – on the Delta $\bar{w}_x$ of the transformed real value function $\bar{w} : [0,T] \times (0,\infty) \to \mathbb{R}$, $(t,u) \mapsto \bar{w}(t,x)$ which describes the value of the large investor's portfolio as a function of time t and associated small investor stock price $x = \overline{\psi}(t,u)$. However, $\bar{w}_{xx}$ has only a linear influence on the volatility. If we now look at the large investor volatility ρ_φ in (2.58), we see that the dependence on the Gamma w_{xx} is much more pronounced than in the partial differential equation for the real value function, since w_{xx} appears both in the numerator and in the denominator of ρ_φ. In contrast to the volatility in (3.4.21) the large investor volatility ρ_φ still depends on w_{xx} if we are in the "linear" case $d(\mu) = 0$, where no implied transaction losses occur and where the two partial differential equations for the real value function and the strategy function become linear. We should also note that the large investor volatility in general also depends on the Delta w_x of the paper value, since the functions $\hat{\sigma}_\varphi$ and c_φ depend on w_x via the equality (2.41).

We also emphasize once again that the domain $X(\mathbb{R}) = (0,\infty)\backslash\mathcal{E}$ of the final condition might consist of several disconnected parts due to the jumps of $X : \mathbb{R} \to (0,\infty)$, $u \mapsto \psi\big(t,u,\alpha\zeta(u)\big)$. According to our model the large investor's stock price at maturity can never take on any of the values in $\mathcal{E}$. However, the functions $h : (0,\infty) \to \mathbb{R}$ and $H^\pm : (0,\infty) \to \mathbb{R}$ in (2.60) are defined on the whole ray, so we can extend the final condition (2.60) to the whole interval $(0,\infty)$. By this extension w is defined on the whole domain $[0,T] \times (0,\infty)$ and not only on the fuzzy domain $\mathcal{D} = \big([0,T] \times (0,\infty)\big)\backslash\big(\{T\} \times \mathcal{E}\big)$. □

Let us now consider the special situation of a large investor price system (ψ,μ) where the price determining measure μ is the Dirac measure δ_1 concentrated in 1, such that the stock price immediately reacts to an announced change of the large investor's portfolio before the large investor can execute any trades. This price mechanism corresponds to the price mechanism in the papers of Schönbucher and Wilmott (1996, 2000), Frey (1998, 2000), Frey and Patie (2002) and Sircar and Papanicolaou (1998), who all consider the replication of the large investor's paper value in a continuous model. Except for Frey (1998), all these papers derive a final value problem for the large investor's paper value function which is equivalent to (2.58), (2.59). In order to see the connection let us first note that $\mu = \delta_1$ implies that $2d(\mu) = 1$, hence the large investor volatility function $\rho_\varphi : [0, T] \times (0, \infty) \to (0, \infty)$ of (2.43) and (2.45) simplifies to

$$\rho_\varphi(t, x) = \left. \frac{\psi_u + \psi_\xi \varphi_u(t, u)}{\psi(t, u, \varphi(t, u))} \right|_{u=u_\varphi(t,x)} = \hat{\sigma}_\varphi(t, x) \frac{1}{1 - c_\varphi(t, x) x w_{xx}(t, x)} \tag{2.61}$$

for all $(t, x) \in [0, T) \times (0, \infty)$. Schönbucher and Wilmott (2000) derive the partial differential equation for the large investor's paper value function from an economic partial equilibrium argument by supposing that the (large investor) stock price can be written as some unspecified continuous diffusion process with time and space dependent diffusion parameters. Sircar and Papanicolaou (1998) base their work on Frey and Stremme (1997), who study the influence of an a priori determined trading strategy on the dynamics of the Walrasian equilibrium price, and derive the same partial differential equation. In Schönbucher and Wilmott (2000) the partial differential equation for the paper value is formulated in terms of the excess demand function and in Sircar and Papanicolaou (1998) in terms of the relative demand function for the small investors; both of these functions are assumed to be sufficiently smooth. For example, we can transfer Sircar's and Papanicolaou's notation in terms of the relative demand function $D : [0, T] \times (0, \infty)^2 \to I\!R$ and the constant supply $S_0 > 0$ of stock into our notation by defining the equilibrium price function $\psi : [0, T] \times I\!R^2 \to I\!R$ as the unique solution to

$$S_0 D\big(t, \psi(t, u, \xi), e^u\big) + \xi = S_0 \quad \text{for all } (t, u, \xi) \in [0, T] \times I\!R^2. \tag{2.62}$$

For the bulk of their paper, Sircar and Papanicolaou (1998) consider a certain class of demand functions, where the relative demand is reciprocal to the stock price and a power function of the aggregated income of the small traders. Basically, this class of demand functions was already proposed by Frey and Stremme (1997) because functions from that class have simple homogeneity properties. The corresponding family of equilibrium price functions $\psi : [0, T] \times I\!R^2 \to I\!R$ is given by

$$\psi(t, u, \xi) = x^* e^{\sigma u} \frac{1}{\max\{\eta, 1 - \frac{1}{S_0}\xi\}} \quad \text{for all } (t, u, \xi) \in [0, T] \times I\!R^2, \tag{2.63}$$

for some scaling variable $x^* > 0$, some volatility parameter $\sigma > 0$ and some sufficiently small $\eta > 0$. Here the constant η should be so small that all "realistic" values ξ for the large investor's stock position satisfy $\frac{1}{S_0}\xi \geq 1 - \eta$. For the class of models described by (2.63), the corresponding Black-Scholes volatility $\hat{\sigma}_\varphi$ is constant and equals σ. The liquidity effect $c_\varphi(t, x)$ is given by $c_\varphi(t, x) = \frac{1}{S_0}\big(1 - \frac{1}{S_0} w_x(t, x)\big)^{-1}$ for all $(t, x) \in [0, T) \times (0, \infty)$ for which the solution of the final value problem (2.58), (2.59) corresponding to the price function (2.63) satisfies $\frac{1}{S_0} w_x(t, x) \leq 1 - \eta$. Sircar and Papanicolaou (1998) then provide an asymptotic analysis of the per-share price $\frac{1}{\alpha} w(t, x)$ of the replicating portfolio of $\alpha > 0$ (smoothed) European calls or more general convex options if the ratio $\rho = \frac{\alpha}{S_0}$ of replicated options to the total supply of options is small.

Frey (1998) also investigates the paper value replication of a contingent claim in a continuous-time market model where the price determining measure of the underlying price system is

given by $\mu = \delta_1$. However, in this paper Frey does not explicitly state the non-linear Black-Scholes equation for the paper value function, but only derives the final value problem for the corresponding strategy function $\varphi : [0,T] \times I\!R \to I\!R$ and discusses the existence and uniqueness of solutions to it. Especially, the partial differential equation for φ is equivalent to (3.4.3) with $2d(\mu) = 1$. While the stock price in Frey (1998) is determined in terms of a reaction function which solves a market clearing condition analogous to (2.62), the model in Frey (2000) directly starts with a diffusion model for the stock price in terms of the stock price process $\{S_t\}$, the large investor's strategy process $\{\phi_t\}$, some volatility parameter $\bar{\sigma} > 0$ which reflects the volatility in the market without the large investor, and some market liquidity parameter $\rho \geq 0$, which – in contrast the corresponding parameters in Sircar and Papanicolaou (1998) and Frey (1998) – is exogenously determined and does not depend on the number α of replicated options. Under these assumptions, Frey (2000) derives a final value problem for the paper value function which corresponds to (2.58), (2.59), where the large investor volatility ρ_φ in (2.61) only depends on the stock price x and the Gamma $w_{xx}(t,x)$ since $\hat{\sigma}_\varphi \equiv \sigma$ and $c_\varphi \equiv \rho$. This model is further extended in Frey and Patie (2002) by introducing some liquidity profile $\lambda : I\!R^+ \to I\!R^+$, which results in terms of our notation in a liquidation effect $c_\varphi : [0,T) \times (0,\infty) \to I\!R$ given by $c_\varphi(t,x) = \rho\lambda(x)$ for all $(t,x) \in [0,T) \times (0,\infty)$. Frey and Patie (2002) argue that a non-constant liquidity profile λ translates the common knowledge that the market liquidity in falling markets is lower than in rising markets and that large moves in either direction limit the market liquidity. Then they use the liquidity profile λ to model and explain a volatility smile observed in real data. Last but not least, Frey and Patie (2002) provide an extensive numerical simulation study for the paper value, its Delta and its Gamma, for a (smoothed) European call and a call spread. Also note that due to $\mu = \delta_1$ the models of these authors can be obtained as limits of discrete binomial models as in Section 4.2.4 without assuming the convergence of the large investor's stock holdings before time 0, since these models satisfy Assumption L(i).
Since the discrete binomial model with $\mu = \delta_1$ is a generalization of Jarrow's (1994) model, we may summarize our discussion of the situation $\mu = \delta_1$ with the observation that we have shown in Proposition 4.8 that properly scaled versions of Jarrow's model converge in distribution towards Frey's (1998) model.

4.2.6 The Continuous-Time Stochastic Model

In order to conclude the discussion of the limit model under the p-martingale measure, we treat in this section the existence and uniqueness of weak and strong solutions to the stochastic differential equations for the fundamentals in a more general setting than in Section 4.2.3. Then we apply Itô's formula to the price and paper value processes to obtain a martingale representation for these two processes. Finally, we discuss the problems which appear if our continuous model is used to describe a large investor investor market where the large investor follows our trading strategy between time 0 and time T, but where he trades a non-infinitesimal amount of shares at the beginning and at maturity.

Recall from Theorem 4.4 the stochastic differential equation

$$dU_t = \sigma_\varphi(t,U_t)dW_t + \mu_\varphi(t,U_t)dt \quad \text{for all } t \in [0,T], \quad U_0 = u_0, \tag{2.64}$$

which is solved by the distributional limit $\left(U \,\middle|\, \mathbf{P}^\varphi\right)$ of the discrete fundamental processes under the p-martingale measures. Here W is a $\mathbf{P}^\varphi$-Brownian motion and the functions $\sigma_\varphi : [0,T] \times I\!R \to I\!R$ and $\mu_\varphi : [0,T] \times I\!R \to I\!R$ have been defined in (2.22) and (2.23), respectively.
For any strategy function $\varphi : [0,T] \times I\!R \to I\!R$ for which the volatility and drift parameters $\sigma_\varphi : [0,T] \times I\!R \to I\!R$ and $\mu_\varphi : [0,T] \times I\!R \to I\!R$ are well-defined, we can introduce the

infinitesimal generator $\boldsymbol{L}^\varphi$ by setting

$$\big(\boldsymbol{L}^\varphi h\big)(t,u) = \frac{1}{2}\sigma_\varphi^2(t,u)h''(u) + \mu_\varphi(t,u)h'(u) \quad \text{for all } (t,u) \in [0,T] \times I\!R \text{ and } h \in C_b^2(I\!R).$$

Let $U = \{U_t\}_{t\in[0,T]}$ be the coordinate process given by $U_t(\omega) = \omega(t)$ for all $t \in [0,T]$ and $\omega \in C[0,T]$ and let $\{\bar{\mathcal{F}}_t\}$ denote the natural filtration of U. Recalling the martingale problem of Chapter 6 in Stroock and Varadhan (1979), we can now give existence and uniqueness conditions for solutions to the SDE (2.64). Since there exists two different existence and uniqueness concepts of solutions to a stochastic differential equation, we state conditions for both concepts.

Lemma 4.14. *Let us assume that the function $\psi : [0,T] \times I\!R^2 \to I\!R$ belongs to the space $C^{1,2}\big([0,T]\times I\!R^2\big)$ with $\psi_u > 0$, $\big\|\frac{\psi}{\psi_u}\big\|$, $\big\|\frac{\psi_t}{\psi_u}\big\|$ and $\big\|\frac{\psi_{uu}}{\psi_u}\big\|$ being finite, and $|d(\mu)| < \infty$. Moreover, suppose that $\varphi : [0,T] \times I\!R \to I\!R$ belongs to $C_b^{1,2}([0,T] \times I\!R)$ and that there exists some $\varepsilon > 0$ such that the inequalities (2.4) and (2.5) hold. Finally, let $(t_0,u_0) \in [0,T] \times I\!R$.*

(i) On the filtered measurable space $\big\{C[0,T], \mathcal{B}(C[0,T]), \{\bar{\mathcal{F}}_t\}_{t\in[0,T]}\big\}$ there exists a solution $\mathbf{P}^\varphi_{t_0,u_0}$ to the martingale problem for $\boldsymbol{L}^\varphi$ starting from (t_0,u_0).

(ii) Let $\{\mathcal{F}_t\}_{t\in[0,T]}$ be the $\mathbf{P}^\varphi_{t_0,u_0}$-completion of $\{\bar{\mathcal{F}}_t\}_{t\in[0,T]}$. On the filtered probability space $\big(C[0,T], \mathcal{B}(C[0,T]), \mathbf{P}^\varphi_{t_0,u_0}, \{\mathcal{F}_t\}_{t\in[0,T]}\big\}$ there exists some Brownian motion W such that the coordinate process $U_t(\omega) = \omega(t)$ solves the SDE

$$dU_t = \sigma_\varphi(t,U_t)dW_t + \mu_\varphi(t,U_t)dt \quad \text{for all } t \in [t_0,T], \quad U_{t_0} = u_0. \tag{2.65}$$

(iii) If additionally ψ has a multiplicative structure

$$\psi(t,u,\xi) = \overline{\psi}(t,u)f(\xi) \quad \text{for all } (t,u,\xi) \in [0,T] \times I\!R^2$$

with both functions $\overline{\psi} \in \widehat{H}^{1+\frac{1}{2}\beta,2+\beta}\big([0,T] \times I\!R\big)$ and $f \in H^{2+\beta}_{loc}(I\!R)$ being strictly positive and $\varphi \in H^{1+\frac{1}{2}\beta,2+\beta}\big([0,T] \times I\!R\big)$, the solution to (2.65) is unique in the sense of probability law, and the martingale problem (and hence the SDE) is well-posed.

(iv) If furthermore the derivative ratios $\frac{\overline{\psi}_t}{\overline{\psi}_u}$ and $\frac{\overline{\psi}_{uu}}{\overline{\psi}_u}$ and the derivatives φ_t and φ_{uu} satisfy not only a global Hölder condition, but even a global Lipschitz condition in u, and if the second derivative f'' is locally Lipschitz on $I\!R$ as well, then there exists a unique strong solution to the SDE (2.65) on $\big(C[0,T], \mathcal{B}(C[0,T]), \mathbf{P}^\varphi_{t_0,u_0}, \{\mathcal{F}_t\}_{t\in[0,T]}\big)$ with the given Brownian motion W of (ii).

Proof. For the first statement, it suffices to recognize that the stated conditions imply that $\sigma_\varphi^2 : [0,T] \times I\!R \to I\!R$ and $\mu_\varphi : [0,T] \times I\!R \to I\!R$ of (2.22) and (2.23) are continuous and bounded. Hence we can conclude with (Stroock and Varadhan, 1979, Theorem 6.1.7) that for any $(t,u) \in [0,T] \times I\!R$ the martingale problem for $\boldsymbol{L}^\varphi$ starting at any $(t,u) \in [0,T] \times I\!R$ has at least one solution $\mathbf{P}^\varphi_{t,u}$.
In order to prove (ii) we notice that due to (2.4) and the boundedness assumptions on φ and ψ, the generator $\boldsymbol{L}^\varphi$ is uniformly elliptic, i.e. there exists some $c > 0$ such that $\sigma_\varphi^2(t,u) \geq c$ for all $(t,u) \in [0,T]$. Now (ii) follows from (i) by Proposition 5.3.1 in Ethier and Kurtz (1986). For (iii) we note that both functions $\sigma_\varphi^2 : [0,T] \times I\!R \to I\!R$ and $\mu_\varphi : [0,T] \times I\!R \to I\!R$ also belong to the Hölder class $H^{\frac{1}{2}\beta,\beta}\big([0,T] \times I\!R\big)$; as in Chapter 3 we only need to assume that $f \in H^{2+\beta}_{loc}(I\!R)$, since f is only evaluated at values in the bounded set $\varphi\big([0,T] \times I\!R\big)$. Due to the uniform ellipticity of $\boldsymbol{L}^\varphi$ we can apply Theorem 6.3.2(i) of Stroock and Varadhan (1979)

in connection with their Theorem 3.2.1 to conclude that the martingale problem for $\boldsymbol{L}^\varphi$ is well-posed, i.e. for any $(t,u) \in [0,T] \times I\!R$ there exists a unique solution $\mathbf{P}^\varphi = \mathbf{P}^\varphi_{t,u}$ for the martingale problem for $\boldsymbol{L}^\varphi$ starting at time t in u. Then, by Corollary 5.3.4 in Ethier and Kurtz (1986), the solution of the SDE (2.65) is unique in the sense of probability law.
Now let us come to (iv). It follows that $\sigma^2_\varphi : [0,T] \times I\!R \to I\!R$ and $\mu_\varphi : [0,T] \times I\!R \to I\!R$ satisfy the Lipschitz condition that there exists some $K \in I\!R$ such that

$$\big|\sigma^2_\varphi(t,u_1) - \sigma^2_\varphi(t,u_2)\big| + \big|\mu_\varphi(t,u_1) - \mu_\varphi(t,u_2)\big| \le K|u_1 - u_2| \text{ for all } t \in [0,T] \text{ and } u_1,u_2 \in I\!R.$$

Then Theorem 5.3.7 and Remark 5.3.9 in Ethier and Kurtz (1986) guarantee the pathwise uniqueness of solutions to the SDE (2.65). **q.e.d.**

Remark. Note that the assumptions of Lemma 4.14 (i) to (iv) hold in particular under Assumptions G and H.
By the same means as in the proof of Lemma 4.14 we can show that (2.65) is well-posed under slightly weaker assumptions on the function $\varphi : [0,T] \times I\!R \to I\!R$ than in (iii). Namely, we may replace condition (2.4) by

$$1 + 2d(\mu)\frac{\psi_\xi\big(t,u,\varphi(t,u)\big)}{\psi_u\big(t,u,\varphi(t,u)\big)}\varphi_u(t,u) \ge 0 \qquad \text{for all } (t,u) \in [0,T] \times I\!R$$

to allow for degenerate volatilities σ_φ. However, in order to obtain uniqueness in the sense of probability law in these degenerate cases we then have to impose stronger regularity conditons on ψ and φ, for example $\psi \in C^{2,4}\big([0,T] \times I\!R^2\big)$ and $\varphi \in C^{2,4}_b\big([0,T] \times I\!R\big)$, and the additional bounds $\big\|\frac{\psi_z}{\psi_u}\big\| < \infty$ for $z \in \{tuu, uuu, uuuu\}$. In fact, under these conditions the functions $\sigma^2_\varphi : [0,T] \times I\!R \to I\!R$ and $\mu_\varphi : [0,T] \times I\!R \to I\!R$ have two bounded continuous spatial derivatives, and it follows from Theorems 6.3.2(i) and 3.2.6 in Stroock and Varadhan (1979) that the martingale problem associated with the SDE (2.65) has a unique solution for each $(t_0,u_0) \in [0,T] \times I\!R$. The rest follows as in the proof of (iii). □

We have seen in Proposition 2.8(ii) and Corollary 2.10(i) that the discrete large investor price process $S^n = \{S^n_t\}$ and the discrete paper value process $V^n = \{V^n_t\}$ are martingales under the p-martingale measure $\mathbf{P}^{\xi^n}_n$, whenever the discrete p-martingale measure is well defined. Due to Lemma 4.3 our assumptions of Proposition 4.8 imply that for all sufficiently large $n \in I\!N$ the trading strategies (ξ^n, b^n) are p-admissible and hence the associated p-martingales are indeed well-defined. Therefore, we can expect that the martingale property of both processes is preserved by the limit processes $S = \{S_t\}$ and $V = \{V_t\}$. The next proposition shows that our intuition is correct, and S and V are indeed $\mathbf{P}^\varphi$-martingales if for each $t \in [0,T]$ the mapping $u \mapsto \overline{\psi}(t,u)$ grows at most exponentially. A local martingale representation for S and V can even be derived under weaker regularity conditions than assumed in Proposition 4.8. For that purpose, let us recall that the continuous-time (large investor) price process $S = \{S_t\}$ has been defined in Definition 4.6(ii) as $S_t = \psi\big(t,U_t,\varphi(t,U_t)\big)$ for all $t \in [0,T]$, and that the large investor volatility function $\rho_\varphi : [0,T) \times (0,\infty) \to (0,\infty)$ of Definition 4.11 satisfies

$$\rho^2_\varphi\big(t,\psi(t,u,\varphi(t,u))\big) = \frac{\psi_u + 2d(\mu)\psi_\xi\varphi_u(t,u)}{\psi(t,u,\varphi(t,u))}\frac{\psi_u + \psi_\xi\varphi_u(t,u)}{\psi(t,u,\varphi(t,u))} \tag{2.66}$$

for all $(t,u) \in [0,T) \times I\!R$, where we have skipped again the arguments $\big(t,u,\varphi(t,u)\big)$ of the derivatives of ψ.

Proposition 4.15. *In addition to Assumptions B, C, F and M suppose that $\zeta \in H^{2+\beta}(I\!R)$. Then the large investor price process $S = \{S_t\}$ and the paper value process $V = \{V_t\}$ associated to the solution $\big(U \,\big|\, \mathbf{P}^\varphi\big)$ to the SDE* (2.64) *are local $\mathbf{P}^\varphi$-martingales and satisfy*

$$S_t = S_0 + \int_0^t \rho_\varphi(\tau,S_\tau)S_\tau\,dW_\tau \qquad \textit{for all } t \in [0,T] \tag{2.67}$$

and

$$V_t = V_0 + \int_0^t \phi_\tau \rho_\varphi(\tau, S_\tau) S_\tau \, dW_\tau \qquad \text{for all } t \in [0,T]. \tag{2.68}$$

If $\left\| \frac{\overline{\psi}_u}{\overline{\psi}} \right\| < \infty$*, then* S *and* V *are true* $\mathbf{P}^\varphi$*-martingales.*

Proof. By Proposition 3.27, our assumptions imply that the solution φ to the final value problem (3.4.3), (3.4.4) belongs to the space $H^{1+\frac{1}{2}\beta, 2+\beta}\big([0,T] \times I\!R\big)$. Then according to Lemma 4.14 the SDE (2.64) is well-posed and there exists a unique weak solution $\big(U \,\big|\, \mathbf{P}^\varphi\big)$. Recall that the continuous-time price process $S = \{S_t\}$ has been introduced in Definition 4.6(ii) as $S_t = \psi\big(t, U_t, \varphi(t, U_t)\big)$; hence an application of Itô's rule to the mapping $(t,u) \mapsto \psi\big(t, u, \varphi(t,u)\big)$ yields

$$S_t = S_0 + \int_0^t \sigma_\varphi(\tau, u) \frac{d}{du} \psi\big(\tau, u, \varphi(\tau, u)\big) \bigg|_{u=U_\tau} dW_\tau \quad \text{for all } t \in [0,T], \tag{2.69}$$

since the definition of $\mu_\varphi : [0,T] \times I\!R \to I\!R$ in (2.23) implies

$$\mu_\varphi(t,u) \frac{d}{du} \psi\big(t, u, \varphi(t,u)\big) + \frac{d}{dt} \psi\big(t, u, \varphi(t,u)\big) + \frac{1}{2} \sigma_\varphi^2(t,u) \frac{d^2}{du^2} \psi\big(t, u, \varphi(t,u)\big) = 0$$

for all $(t,u) \in [0,T] \times I\!R$. Due to the definition of $\sigma_\varphi : [0,T] \times I\!R \to I\!R$ in (2.22) and the representation (2.43) for the large investor volatility $\rho_\varphi : [0,T) \times (0,\infty) \to (0,\infty)$ of Definition 4.11 we can rewrite

$$\sigma_\phi(t,u) \frac{d}{du} \psi\big(t, u, \phi(t,u)\big) = \rho_\varphi\big(t, \psi(t, u, \varphi(t,u))\big) \psi\big(t, u, \varphi(t,u)\big) \tag{2.70}$$

for all $(t,u) \in [0,T) \times I\!R$. If we now apply the definition of S_t, we see that (2.69) is equivalent to (2.67).

Moreover, we have seen in Section 3.4 that our assumptions imply that the paper value function $v : [0,T] \times I\!R \to I\!R$ of (2.30) belongs to the space $C^{1,2}\big([0,T] \times I\!R\big)$, and we can invoke the implicit function theorem to conclude that the transform $w : [0,T] \times (0,\infty) \to I\!R$, which satisfies $w\big(t, \psi(t, u, \varphi(t,u))\big) = v(t,u)$ for all $(t,u) \in [0,T] \times I\!R$, belongs to $C^{1,2}\big([0,T] \times (0,\infty)\big)$ as well. Hence we can apply Itô's formula a second time and obtain

$$dV_t = dw(t, S_t) = w_x(t, S_t) \, dS_t + w_t(t, S_t) \, dt + \frac{1}{2} w_{xx}(t, S_t) \, d\langle S \rangle_t = w_x(t, S_t) \rho_\varphi(t, S_t) S_t \, dW_t$$

for all $t \in [0,T]$, where the second equality follows from the partial differential equation (2.58). If we now substitute (2.41) into this equation and use the Definition 4.6(i) of the strategy process $\phi = \{\phi_t\}$ as $\phi_t = \varphi(t, U_t)$ for all $t \in [0,T]$, we arrive at (2.68).

By (2.67) and (2.68), the processes S and V are local martingales. In order to show that they are actually true martingales if $\left\| \frac{\overline{\psi}_u}{\overline{\psi}} \right\| < \infty$, let us define the continuous local martingale $X = \{X_t\}$ by $X_t = \int_0^t \rho_\varphi(\tau, S_\tau) \, dW_\tau$ for all $t \in [0,T]$, and consider the Doléans-Dade exponential $\mathcal{E}(X)$ given by $\mathcal{E}(X)_t = \exp\big(X_t - \frac{1}{2}\langle X \rangle_t\big)$ for all $t \in [0,T]$. Since $Z = \mathcal{E}(X)$ is the unique solution of $Z_t = 1 + \int_0^t Z_\tau dX_\tau$, the unique solution of

$$S'_t = S_0 + \int_0^t \rho_\varphi(\tau, S_\tau) S'_\tau \, dW_\tau \quad \text{for all } t \in [0,T] \tag{2.71}$$

is given by $S' := S_0 \mathcal{E}(X)$. But $S' = S$ also solves (2.71), hence we have $S = S_0 \mathcal{E}(X)$.

Now the representation (2.70) implies that $\rho_\varphi : [0,T) \times (0,\infty) \to (0,\infty)$ is continuous, and due to the boundedness of φ_u and of the fractions $\frac{\overline{\psi}_u}{\overline{\psi}}$ and $\frac{\overline{\psi}_\xi}{\overline{\psi}_u}$, the function ρ_φ is also bounded. Thus, ρ_φ can be extended to a bounded and continuous function on the space $\mathcal{D} = [0,T] \times (0,\infty)$. In particular, Novikov's condition implies that $\mathcal{E}(X)$ is a martingale, hence S is a martingale as well. Since $\phi = \varphi(t,U_t)$ is also bounded on $[0,T]$, we can use the same type of argument to show that V is a martingale. **q.e.d.**

Remark. Under Assumptions B, C and F, we have seen in Section 3.4.2 that φ still belongs to the space $H_{loc}^{1+\frac{1}{2}\beta,2+\beta}\big([0,T)\times I\!R\big)$ even if we cannot guarantee $\zeta \in H^{2+\beta}(I\!R)$. In particular, this implies $\varphi \in H^{1+\frac{1}{2}\beta,2+\beta}\big([0,T^*] \times I\!R\big)$ for all $T^* < T$. Hence the function $\sigma_\varphi : [0,T] \times I\!R \to I\!R$ of (2.22) is still well-defined, and μ_φ of (2.23) is at least well-defined on the semi-open slab $[0,T) \times I\!R$, but it might explode if $t \to T$. Similarly, the function $v : [0,T] \times I\!R \to I\!R$ is still well-defined on $[0,T] \times I\!R$, and due to (3.4.13) and (3.4.15) v_{uu}, and hence also v_t, still exists not only on $[0,T) \times I\!R$, but on the closed slab $[0,T] \times I\!R$. If we now apply Lemma 4.14 with T replaced by $T^* < T$ we find that there still exists a unique weak solution $\big(U \,\big|\, \mathbf{P}^\varphi\big)$ to the SDE

$$dU_t = \sigma_\varphi(T,U_t)\,dW_t + \mu_\varphi(T,U_t)\,dt \qquad \text{for all } t \in [0,T^*], \quad U_0 = 0,$$

Thus, we can construct step-by-step a unique weak solution on the whole semi-open time interval $[0,T)$. If we also suppose Assumption M, then for all $t \in [0,T)$ the representations of (2.67) and (2.68) still hold, and the processes $\{S_t\}_{t\in[0,T)}$ and $\{V_t\}_{t\in[0,T)}$ of stock price and paper value *before* maturity are local martingales. The same conclusions would hold if Assumption F were replaced by Assumption E. Now suppose that in addition to Assumption F we have $\big\|\frac{\overline{\psi}_u}{\overline{\psi}}\big\| < \infty$. Then the large investor volatility function $\rho_\varphi : [0,T) \times (0,\infty) \to (0,\infty)$ of (2.66) is still bounded, and the Novikov condition

$$\mathbf{E}^\varphi\left[\exp\left(\frac{1}{2}\int_0^T \rho_\varphi^2(t,S_t)dt\right)\right] < \infty \tag{2.72}$$

implies that the process $\{S_t\}_{t\in[0,T)}$ is a true martingale. Thus, $S_T = \lim_{t\nearrow T} S_t$ exists and $S = \{S_t\}_{t\in[0,T]}$, is a continuous martingale as well. Due to $\|\varphi\| = \|\zeta\| < \infty$ and $\phi_t = \varphi(t,U_t)$ for all $t \in [0,T]$, the Novikov condition (2.72) for S also implies that $V = \lim_{t\nearrow T} V_t$ exists and that $V = \{V_t\}_{t\in[0,T]}$ is a martingale.
However, under Assumption E the function $\varphi : I\!R \to I\!R$, $(t,u) \mapsto \varphi(t,u)$, might have jumps at the boundary $t = T$. If $U_{T-} \in I\!R$ is a jump point of $\varphi(T,\,\cdot\,) = \alpha\zeta$, then it is not clear whether $S_T = S_{T-} = \psi\big(T,U_{T-},\varphi(T-,U_{T-})\big)$. □

Initial and Final Stock Holdings

At this point, we also come back to the question of the initial and the final stock holdings. Recall from Proposition 4.8 that our continuous-time model arose as a limit of discrete models for which the stock holdings immediately before time 0 are almost the same as at time 0 (see Assumption L), and similarly we had to require that the stock holdings at time T are almost the same as immediately before time T (see Assumption J). From a practical point of view, we might like to model the trading of the large investor over the time interval $[0,T]$ in such a way that at time 0 he has to build up his portfolio from $\phi_{0-} = 0$ initial stock holdings, and at time T he has to liquidate it again, i.e. $\phi_T = 0$. Only for the time in between, we would like to assume that $\phi_t = \varphi(t,U_t)$ for all $0 \le t < T$. If we recall our general price mechanism given by the large investor stock price $S_\mu : [0,T] \times I\!R^3 \to I\!R$ of (1.3.2), these assumptions would mean that the large investor buys at time 0 an amount of $\varphi(0, U_0)$ shares at an average stock price of

$$\tilde{S}_0 = S_\mu\big(0, U_0, 0, \varphi(0, U_0)\big) = \int \psi\big(0, U_0, \theta\varphi(0, U_0)\big)\mu(d\theta), \tag{2.73}$$

and at time T he sells his $\varphi(T, U_T)$ shares which he holds immediately before time T, having used the continuous trading strategy $\phi = \{\varphi(t, U_t)\}_{t\in[0,T)}$. For this sale he would get an average price of

$$\tilde{S}_T = S_\mu\big(T, U_T, \varphi(T, U_T), 0\big) = \int \psi\big(T, U_T, (1-\theta)\varphi(T, U_T)\big)\mu(d\theta). \tag{2.74}$$

In each intermediate time point t in the open interval $(0, T)$, the large investor would trade only an infinitesimal amount of stocks, and the large investor price would still be given by $\tilde{S}_t = S_t = \psi\big(t, U_t, \varphi(t, U_t)\big)$ for all $t \in (0, T)$.
Hence, we might find it more realistic to work with the tilded price process $\tilde{S} = \{\tilde{S}_t\}$. The drawbacks which we would have to accept under these circumstances are the jumps in the stock price at the time boundaries 0 and T. In the existing literature, the jump at time 0 does not occur, since Frey (1998, 2000), Frey and Patie (2002), Schönbucher and Wilmott (2000), and Sircar and Papanicolaou (1998) all use the Dirac measure δ_1 concentrated in 1 as their price-determining measure μ. In that situation, (2.73) simplifies to $\tilde{S}_0 = S_0$. But another choice of the price-determining measure will in general lead to a jump at time 0 from $\tilde{S}_0$ to $\tilde{S}_{0+} = S_0$. Especially, at time 0 the price process $\{\tilde{S}_t\}_{t\in[0,T]}$ is not right-continuous any more, and $\tilde{S}$ cannot be a martingale in the usual sense. In a way we have already encountered such a problem in Section 2.1 and especially in Proposition 2.5, where we could only show that a discrete trading strategy $\big(\xi^n, b^n\big)$ is p-admissible if the stock holdings ξ^n_{-1} immediately before time 0 almost coincide with the stock holdings ξ^n_0 at time 0.
The other difficulty occurs at time T. At this point the process $\tilde{S}$ is at least right-continuous and has a left-hand limit $\tilde{S}_{T-} = S_T$ as long as the process $S = \{S_t\}$ is continuous, i.e. for example under the conditions of Proposition 4.15. However, as noticed by Frey (1998), the price process $\bar{S} = \{\bar{S}_t\}$ does not admit an equivalent local martingale measure whenever $\mathbf{P}^\varphi\big(\tilde{S}_T = S_T\big) < 1$, since then there exist free lunches with vanishing risk for the small investors.
In view of the above problems, we might be tempted to choose the price-determining measure μ non-constant over time in such a way that $\mu = \delta_1$ at time 0 and $\mu = \delta_0$ at time 1, in order to exclude the difficulties mentioned. Unfortunately, Proposition 1.15 and the multiplicative structure of $\psi : [0, T] \times I\!R^2 \to I\!R$ imply that this does not solve our problems: For non-constant $f : I\!R \to I\!R$, one of these two cases will lead to negative implied transaction losses, and hence to a potential arbitrage opportunity for the large investor.
It is also not totally satisfying to restrict the time of observation to the smaller time window $(0, T)$ where the large investor's stock position changes continuously, which is basically what we did in this section. For realistic models, it is important for the large investor to know what happens at the boundary, how to build up the portfolio and how to liquidate it at the end, since otherwise the applicability of the model will suffer.
A reasonable continuous-time large investor theory would rather give the large investor some (though little) time to build up his stock holdings continuously and also some time around T to liquidate them again. This fits with the intuition of the liquidation value concept of Schönbucher and Wilmott (2000) and that of our benchmark price of Section 1.1. However, in contrast to Schönbucher and Wilmott's (2000) approach it is crucial *not* to take the limit in time in Schönbucher and Wilmott's derivation of the liquidation price, since otherwise we are exactly in the unfavorable situation that the limiting price process jumps at the time points 0 and/or T.

Remark. Bakstein and Howison (2002) have also observed that Bakstein's (2001) binomial model crucially depends on the initial stock holdings of the large investor; they notice that the replication price for a contingent claim becomes a function of the large investor's initial endowment. However, they argue that no-arbitrage will prevent the large investor from manipulating the replication price. Without a rigorous proof, they then employ similarities with transaction cost models to derive a continuous limit for the (super-)replication price in terms of a partial differential equation. However, here they ignore again the dependence on the large investor's initial stock holdings. □

4.3 Convergence under the s-Martingale Measures

In this section we show that our binomial large investor models also converge to a continuous diffusion model if we consider convergence under the s-martingale measures. If the equilibrium price function is multiplicative, this is equivalent to looking at the convergence under the r-martingale measures. Under the s-martingale measures the fundamentals describe a much simpler general correlated random walk as under the p-martingale measure, since they are not correlated at all. Therefore, the convergence of the sequence of fundamental processes is deduced much more easily than in Section 4.2. The convergence of price, strategy, and real value processes follows. Last but not least, we consider the stochastic limit model in some more detail and see that as in the discrete case the real value process is a supermartingale, and a martingale only if the large investor always trades at the benchmark price.

Due to Lemma 2.3, the discrete s-martingale measure $\bar{\mathbf{P}}^n$ of Definition 2.7(i) is well-defined whenever $n > n_0 := \left\| \frac{\overline{\psi}_t}{\overline{\psi}_u} \right\|^2$, and in this case it is equivalent to the original measure $\mathbf{P}^n$ on $(\Omega^n, \mathcal{F}^n)$. We have already seen in Section 2.1.3 that the s-martingale measure coincides with the r-martingale measure if the equilibrium price function is multiplicative as under Assumption B.

For $n > n_0$ let us recall the s-martingale weight function $\bar{p}^n : \mathcal{A}^n(n-1) \to I\!R$ which has been defined in (2.1.10) as

$$\bar{p}^n(t,u) = \frac{\overline{\psi}(t,u) - \overline{\psi}(t+\delta^2, u-\delta)}{\overline{\psi}(t+\delta^2, u+\delta) - \overline{\psi}(t+\delta^2, u-\delta)} \quad \text{for all } (t,u) \in \mathcal{A}^n(n-1).$$

Under the s-martingale measure $\bar{\mathbf{P}}^n$ the distribution of the tilt Z_k^n at time t_k^n only depends on time and the fundamental value U_{k-1}^n immediately before time t_k^n, but not on the tilt Z_{k-1}^n, since the definition of $\bar{\mathbf{P}}^n$ yields

$$\bar{\mathbf{P}}^n\big(Z_k^n = 1 \big| U_{k-1}^n,\, Z_{k-1}^n\big) = \bar{p}^n\big(t_{k-1}^n, U_{k-1}^n\big) \quad \text{for all } 1 \le k \le n. \tag{3.1}$$

Thus, $U^n = \big\{U_k^n\big\}_{0 \le k \le n}$ is a special case of a general correlated random walk as presented in Section 4.1; it is *general*, but *not* correlated.

For our convergence results we rather work with processes in the space $D[0,T]$, so we recall the definition of the càdlàg version $U^n = \{U_t^n\}$ from (2.2). We shall show that the sequence $\{U^n\}_{n>n_0}$ of fundamental processes $U^n = \{U_t^n\}$ in $D[0,T]$ satisfies the assumptions of Theorem 4.1. It turns out that proving the convergence of $U^n = \big\{U_t^n\big\}$ under the s-martingale measure is much easier than proving the analogous statement under the p-martingale measure in Section 4.2.3, since for each $n > n_0$ the measure $\bar{\mathbf{P}}^n$ does not depend on the strategy function $\xi^n : \mathcal{A}^n \to I\!R$ and since the random walk $\big\{U_k^n\big\}_{0 \le k \le n}$ is Markov under $\bar{\mathbf{P}}^n$ because of (3.1).

Similarly to Lemma 4.3 we can now expand the s-martingale weight function as $n \to \infty$ into terms up to order $O\big(\delta^{1+\beta}\big)$.

Lemma 4.16. *If* $\overline{\psi} \in \widehat{H}^{1+\frac{1}{2}\beta,2+\beta}\big([0,T] \times I\!R\big)$ *then*

$$\bar{p}^n(t,u) = \frac{1}{2}\left(1 - \delta\frac{\frac{1}{2}\overline{\psi}_{uu}(t,u) + \overline{\psi}_t(t,u)}{\overline{\psi}_u(t,u)}\right) + O\big(\delta^{1+\beta}\big) \quad \text{as } n \to \infty, \tag{3.2}$$

uniformly for all $(t,u) \in \mathcal{A}^n(n-1)$.

Proof. This follows like Lemma 3.11 (or Lemma 4.3). **q.e.d.**

It follows that the assumptions of Lemma 4.16 imply the conditions of the convergence theorem for general correlated random walks as stated in Theorem 4.1, and we obtain:

Theorem 4.17. *Suppose* $\overline{\psi} \in \widehat{H}^{1+\frac{1}{2}\beta,2+\beta}\big([0,T] \times I\!R\big)$. *Under the equivalent s-martingale measures* $\bar{\mathbf{P}}^n$, *the sequence of fundamental processes* $\{U^n\}_{n\in I\!N}$ *starting in* $u_0 \in I\!R$ *at time* 0 *converges weakly to the process* $U = \{U_t\}$ *given by*

$$dU_t = -\frac{\overline{\psi}_t(t,U_t) + \frac{1}{2}\overline{\psi}_{uu}(t,U_t)}{\overline{\psi}_u(t,U_t)}dt + dW_t \quad \text{for all } t \in [0,T], \text{ and } U_0 = u_0, \tag{3.3}$$

i.e. there exists some measure $\bar{\mathbf{P}}$ *on* $\big(D[0,T], \mathcal{B}(D[0,T])\big)$ *such that* W *is a* $\bar{\mathbf{P}}$*-Brownian motion and*

$$\big(U^n \,\big|\, \bar{\mathbf{P}}^n\big) \Rightarrow \big(U \,\big|\, \bar{\mathbf{P}}\big) \quad \text{as } n \to \infty.$$

Proof. The statement follows directly from Lemma 4.16 and Theorem 4.1 with $a \equiv 0$ and $b \equiv -\frac{\frac{1}{2}\overline{\psi}_{uu}+\overline{\psi}_t}{\overline{\psi}_u} \in H^{\frac{1}{2}\beta,\beta}\big([0,T] \times I\!R\big)$ once we note that for each $n > n_0$ the definition of the functions $\bar{p}^n$ can be extended from the space $\mathcal{A}^n(n-1)$ to the space $[0,T] \times I\!R$ such that (3.2) holds uniformly for all $(t,u) \in [0,T] \times I\!R$. **q.e.d.**

Remark. Under the assumptions of Theorem 4.17, we do not only get the existence of some solution of (3.3) for all $u_0 \in I\!R$. Due to $\big\|\frac{\overline{\psi}_t}{\overline{\psi}_u}\big\| < \infty$ and $\big\|\frac{\overline{\psi}_{uu}}{\overline{\psi}_u}\big\| < \infty$, it immediately follows from Proposition 5.3.10 in Karatzas and Shreve (1996) that a solution to (3.3) is unique in the sense of probability law, hence the SDE (3.3) is well-posed. □

Like in Section 4.2.4 we now employ the convergence of the fundamentals to show that also the strategy, price, and real value processes converge under the s-martingale measure, even when considered as a tuple in $D^d[0,T]$. In Definition 4.5 we have introduced $D[0,T]$-versions of the large investor's discrete strategy process $\phi^n = \{\phi^n_t\}$, the associated loss-free liquidation price process $\bar{S}^n = \{\bar{S}^n_t\}$ and the real value process $\bar{V}^n = \{\bar{V}_t\}$ for each $n \in I\!N$. The corresponding limit processes $\phi = \{\phi_t\}$, $\bar{S} = \{\bar{S}_t\}$ and $V = \{V_t\}$ of Definition 4.6 are well-defined under Assumptions B, C, and either Assumption E or F.
By Proposition 2.8(i) the s-martingale measure is the unique probability measure which turns the associated small investor price process $\big\{\overline{\psi}\big(t^n_k, U^n_k\big)\big\}_{0\le k\le n}$ into a martingale, so it makes sense to consider the convergence of the sequence of small investor price processes as well. In order to introduce $D[0,T]$-versions of these processes, let us recall the deterministic process $\tau^n = \{\tau^n_t\}$ of Definition 4.5(i). For any time $t \in [0,T]$ the time point $\tau^n_t = t^n_{[nt]}$ is the latest trading point for the large investor in the nth binomial model. We also introduce the limiting small investor price process.

Definition 4.18. For all $n \in I\!N$ the discrete **small-investor price process** $X^n = \{X^n_t\}$ in $D[0,T]$ associated to our large investor market is defined as $X^n_t = \overline{\psi}(\tau^n_t, U^n_t)$. Analogously, the associated continuous small-investor price process $X = \{X_t\}$ is given by $X_t = \overline{\psi}(t, U_t)$ for all $t \in [0,T]$.

Remark. Since $\{\overline{\psi}(t_k^n, U_k^n)\}_{0\le k\le n}$ is a martingale under the s-martingale measure, the process X^n is a $\bar{\mathbf{P}}^n$-martingale as well. $\square$

Now it is straightforward to prove:

Proposition 4.19. *Suppose Assumptions G, H, J and K. Under the s-martingale measures $\bar{\mathbf{P}}^n$ for the associated small-investor market we have*

$$\Big((U^n, \phi^n, X^n, \bar{S}^n, \bar{V}^n)\,\Big|\,\bar{\mathbf{P}}^n\Big) \Rightarrow \Big((U, \phi, X, \bar{S}, \bar{V})\,\Big|\,\bar{\mathbf{P}}\Big) \quad \text{as } n \to \infty. \tag{3.4}$$

Proof. The proof follows from Theorem 4.17 and Lemma 4.7 in analogy to the proof of Proposition 4.8, thus it is omitted. **q.e.d.**

Remark. As opposed to the p-martingale measures, the s-martingale measures do not depend on the large investor's stock holdings. For this reason Proposition 4.19 does not require Assumptions I and L. Especially, we do not need to assume any convergence of the large investor's stock holdings before time 0. $\square$

In Section 3.4.2 we have seen that solutions $\varphi : [0,T]\times I\!R \to I\!R$ and $\bar{v} : [0,T]\times I\!R \to I\!R$ to the final value problems (3.4.3), (3.4.4) and (3.4.6), (3.4.7), respectively, exist even if we only require Assumptions B, C and either Assumption E or F. Of course, these conditions are implied by the prerequisites of Proposition 4.19. Under this weaker set of assumptions we now take another look at the limiting stochastic model. In particular, we show that as in the discrete case of Section 2.1.5 the real value process is a supermartingale, and a martingale if the large investor does not face any transaction gains or losses at all.
We again transform the continuous real value function $\bar{v} : [0,T]\times I\!R \to I\!R$ into a function which depends on time and small investor stock price. For this purpose, let us recall from Lemma 3.31 that under Assumptions B and C we have

$$\bar{\mathcal{D}} = \big\{(t,x) \in [0,T]\times I\!R \,\big|\, x = \overline{\psi}(t,u) \text{ for some } u \in I\!R\big\} = [0,T]\times(0,\infty).$$

Then we can define the transformed real value function $\bar{w} : [0,T]\times I\!R \to I\!R$ as in Section 3.4.3 by setting $\bar{w}(t,x) = \bar{v}\big(t, u(t,x)\big)$ for all $(t,x)\in[0,T]\times(0,\infty)$, where $u = u(t,x)$ is the unique solution of $x = \overline{\psi}\big(t, u(t,x)\big)$. Note that $u = u(t,x)$ is well defined since Assumption C implies $\overline{\psi}_u > 0$. Let us also recall the volatility function $\bar{\sigma} : [0,T]\times(0,\infty) \to (0,\infty)$ of (3.4.19), which is given by

$$\bar{\sigma}(t,x) = \frac{\overline{\psi}_u(t,u(t,x))}{\overline{\psi}(t,u(t,x))} \quad \text{for all } (t,x)\in[0,T]\times(0,\infty), \tag{3.5}$$

and the transformed loss function $\kappa : g(I\!R) \to [0,\infty)$ of (3.3.1), which has been defined in terms of $g : I\!R \to I\!R$, $\xi \mapsto \int_0^\xi f(z)\,dz$ as

$$\kappa(x) = 2d(\mu)\frac{f'\big(g^{-1}(x)\big)}{f^2\big(g^{-1}(x)\big)} \quad \text{for all } x \in g(I\!R). \tag{3.6}$$

Now we can express the dynamics of the small investor price and the real value process under $\bar{\mathbf{P}}$ in terms of σ, κ, and the spatial derivatives of $\bar{w}$. As the direct counterpart of Proposition 4.15, this gives us a martingale representation for the price process, and a supermartingale representation for the real value process.

Proposition 4.20. *Let us suppose Assumptions B, C and either Assumption E or Assumption F, and let $\big(U\,\big|\,\bar{\mathbf{P}}\big)$ be the unique weak solution to* (3.3)*. Then the small investor price process $X = \{X_t\}$ associated to the large investor market and the real value process $\bar{V} = \{\bar{V}_t\}$*

of the large investor's trading strategy solve for all $t \in [0,T]$ the stochastic differential equations

$$dX_t = \bar{\sigma}(t, X_t) X_t \, dW_t \tag{3.7}$$

and

$$d\bar{V}_t = \bar{w}_x(t, X_t)\bar{\sigma}(t, X_t) X_t \, dW_t - \frac{1}{2}\kappa\big(\bar{w}_x(t, X_t)\big)\Big(\bar{\sigma}(t, X_t) X_t \bar{w}_{xx}(t, X_t)\Big)^2 X_t \, dt. \tag{3.8}$$

In particular, if $l_0 := \big\|\frac{\overline{\psi}_u}{\overline{\psi}}\big\| < \infty$, then the associated small-investor price process X is a $\bar{\mathbf{P}}$-martingale. The real value process $\bar{V}$ is a $\bar{\mathbf{P}}$-supermartingale, and a $\bar{\mathbf{P}}$-martingale if in addition to $l_0 < \infty$ the price system (ψ, μ) excludes transaction losses, i.e. if $d(\mu) f' \equiv 0$.

Proof. Let us first consider the stochastic differential equation for the continuous small investor price $X_t = \overline{\psi}(t, U_t)$. By Itô's formula and the stochastic differential equation (3.3) we obtain

$$dX_t = d\overline{\psi}(t, U_t) = \overline{\psi}_u(t, U_t)\, dU_t + \left(\overline{\psi}_t(t, U_t) + \frac{1}{2}\overline{\psi}_{uu}(t, U_t)\right) dt = \overline{\psi}_u(t, U_t)\, dW_t \tag{3.9}$$

for all $t \in [0,T]$. Now (3.7) follows directly from the definition of $\bar{\sigma} : [0,T] \times (0, \infty) \to (0, \infty)$ in (3.5). If $l_0 < \infty$, the volatility function $\bar{\sigma} : \bar{\mathcal{D}} \to (0, \infty)$ is bounded. Hence we can apply Novikov's condition as in the proof of Proposition 4.15 to conclude from (3.7) that X is a martingale under $\bar{\mathbf{P}}$.
In order to prove (3.8) we again apply Itô's formula; we write the real value process of Definition 4.6(iv) by means of the transformed real value function $\bar{w} : [0,T] \times (0, \infty) \to I\!R$ in terms of the associated small investor price and obtain $\bar{V}_t = \bar{v}(t, U_t) = \bar{w}(t, X_t)$ for all $t \in [0,T]$. As opposed to the transformed *paper* value function $w : \mathcal{D} \to I\!R$ of Section 4.2.5, the transformed *real* value function $\bar{w} : [0,T] \times (0, \infty) \to I\!R$ is continuous on its whole domain, even if the final condition $\alpha\zeta : I\!R \to I\!R$ for φ has jumps. Thus, Itô's rule and (3.7) imply for all $t \in [0,T]$:

$$\begin{aligned} d\bar{V}_t &= \bar{w}_x(t, X_t)\, dX_t + \bar{w}_t(t, X_t)\, dt + \frac{1}{2}\bar{w}_{xx}(t, X_t)\, d\langle X\rangle_t \\ &= \bar{w}_x(t, X_t)\bar{\sigma}(t, X_t) X_t\, dW_t + \left(\bar{w}_t(t, X_t) + \frac{1}{2}\bar{\sigma}^2(t, X_t) X_t^2 \bar{w}_{xx}(t, X_t)\right) dt. \end{aligned}$$

Now the SDE (3.8) follows from the generalized Black-Scholes equation (3.4.21) for the transformed value function $\bar{w} : [0,T] \times (0, \infty) \to I\!R$.
Next we want to show that the local martingale $\bar{V}^* = \{\bar{V}_t^*\}$ which is for all $t \in [0,T]$ given by $\bar{V}_t^* = \int_0^t \bar{w}_x(\tau, X_\tau)\bar{\sigma}(\tau, X_\tau) X_\tau \, dW_\tau$ is a true martingale if $l_0 < \infty$. Let us recall from our discussion in Section 3.4.2 that the function $\gamma : [0,T] \times I\!R \to I\!R$, $(t,u) \mapsto \int_0^{\varphi(t,u)} f(z)\, dz$ is bounded if either Assumption E or Assumption F holds. We have also seen in Section 3.4, in particular in (3.4.15) and (3.4.13), that under our assumptions

$$\bar{w}_x\big(t, \overline{\psi}(t,u)\big) = \gamma(t,u) \quad \text{for all } (t,u) \in [0,T) \times I\!R \tag{3.10}$$

and

$$\bar{w}_x\big(T, \overline{\psi}(T,u)\big) = \gamma(T,u) \quad \text{for all continuity points of } \zeta : I\!R \to I\!R. \tag{3.11}$$

Since the boundary values of the integrand do not affect the $d\tau$-integral, the local martingale $\bar{V}^*$ has the quadratic variation

$$\langle \bar{V}^* \rangle_t = \int_0^t \Big(\bar{w}_x(\tau, X_\tau) \bar{\sigma}(\tau, X_\tau) X_\tau \Big)^2 \, d\tau = \int_0^t \Big(\gamma(\tau, U_\tau) \bar{\sigma}(\tau, X_\tau) X_\tau \Big)^2 \, d\tau$$

not only for all $t \in [0,T)$, but even for all $t \in [0,T]$. If $l_0 < \infty$, the volatility function $\bar{\sigma}$ is bounded by l_0, and together with the boundedness of γ, we conclude that there exists some constant $c \geq 0$ such that

$$\bar{\mathbf{E}}\big[\langle \bar{V}^* \rangle_T\big] \leq c \bar{\mathbf{E}}\bigg[\int_0^T X_\tau^2 \, d\tau\bigg]. \tag{3.12}$$

Because X is given by (3.7) and $\bar{\sigma}$ is bounded by l_0, for each stopping time with $T^* \leq T$ it follows that $\bar{\mathbf{E}}\big[X_{T^*}^2\big] \leq \exp\big(T l_0^2\big)$, and (3.12) gives $\bar{\mathbf{E}}\big[\langle \bar{V}^* \rangle_T\big] < \infty$. Thus, the Burkholder-Davis-Gundy inequality yields $\bar{\mathbf{E}}\big[\big(\sup_{0 \leq t \leq T} |\bar{V}_t^*|\big)^2\big] < \infty$, and hence $\bar{V}^*$ is indeed a true martingale.
If $d(\mu) f' \equiv 0$, than the transformed loss function $\kappa : g(\mathbb{R}) \to [0,\infty)$ of (3.6) vanishes, hence we have $\bar{V} = \bar{V}^*$, and if $l_0 < \infty$, the real value process $\bar{V}$ is indeed a martingale. Especially, $d(\mu) f' \equiv 0$ is implied by Assumption E.
Under Assumption F the function κ need not vanish, but then we have $\zeta \in C_b^1(\mathbb{R})$ and $\varphi, \gamma \in C^{1,2}\big([0,T) \times \mathbb{R}\big) \cap C_b^{0,1}\big([0,T] \times \mathbb{R}\big)$, and it follows from (3.11) that (3.10) holds for all $(t,u) \in [0,T] \times \mathbb{R}$ and $\bar{w}_x : [0,T] \times (0,\infty) \to \mathbb{R}$ is bounded. Differentiating this expression and employing the definition of $\bar{\sigma} : [0,T] \times (0,\infty) \to \mathbb{R}$ in (3.5) we see that

$$\bar{\sigma}(t,x) x \bar{w}_{xx}(t,x)\Big|_{x = \overline{\psi}(t,u)} = \bar{w}_{xx}\big(t, \overline{\psi}(t,u)\big) \overline{\psi}_u(t,u) = \gamma_u(t,u) \quad \text{for all } (t,u) \in [0,T] \times \mathbb{R} \tag{3.13}$$

is also bounded. Finally, the function $\kappa \circ \bar{w}_x : [0,T] \times (0,\infty) \to [0,\infty)$ is bounded, since $\bar{w}_x : [0,T] \times \mathbb{R} \to g(\mathbb{R})$ is bounded and $\kappa : g(\mathbb{R}) \to \big[0,\infty)$ is continuous. Thus, the process $A = \{A_t\}$ given by

$$A_t := \frac{1}{2} \int_0^t \kappa(\bar{w}_x(\tau, X_\tau)) \big(\bar{\sigma}(\tau, X_\tau) X_\tau \bar{w}_{xx}(\tau, X_\tau)\big)^2 X_\tau \, d\tau \quad \text{for all } t \in [0,T] \tag{3.14}$$

satisfies $\int_0^T d|A|_t = A_T < \infty$, and the mapping $t \mapsto A_t$ is pathwise nondecreasing on $[0,T]$. Thus we conclude that $\bar{V} = \bar{V}^* - A$ is a $\bar{\mathbf{P}}$-supermartingale. **q.e.d.**

Remark. Under Assumption F we have $\gamma \in C^{1,2}\big([0,T) \times \mathbb{R}\big) \cap C_b^{0,1}\big([0,T] \times \mathbb{R}\big)$, and we get from (3.10) and Itô's rule

$$d\bar{w}_x(t, X_t) = d\gamma(t, U_t) = \gamma_u(t, U_t) dU_t + \left(\gamma_t(t, U_t) + \frac{1}{2} \gamma_{uu}(t, U_t) \right) dt \quad \text{for all } t \in [0,T] \times \mathbb{R}.$$

Thus, from the stochastic differential equation (3.3) we obtain $d\big\langle \bar{w}_x(\cdot, X_\cdot) \big\rangle_t = \gamma_u^2(t, U_t) dt$, and using the representation (3.13) we can express the dynamics of the nondecreasing process A of (3.14) as $dA_t = \frac{1}{2} \kappa\big(\bar{w}_x(t, X_t)\big) X_t \, d\big\langle \bar{w}_x(\cdot, X_\cdot) \big\rangle_t$. This demonstrates that the process A accounts for proportional (implied) transaction losses in terms of the transformed strategy $\big\{\gamma(t, U_t)\big\} = \big\{\bar{w}_x(t, X_t)\big\}$. If we now use once again the definition $\gamma : [0,T] \times \mathbb{R} \to \mathbb{R}$, $(t,u) \mapsto g\big(\varphi(t,u)\big) = \int_0^{\varphi(t,u)} f(z) \, dz$, we can express the quadratic variation of the transformed trading strategy in terms of the original strategy ϕ as introduced in Definition 4.6(i), by writing

$$d\big\langle \bar{w}_x(\cdot, X_\cdot) \big\rangle_t = f^2\big(\varphi(t, U_t)\big) \, d\big\langle \varphi(\cdot, U_\cdot) \big\rangle_t = f^2(\phi_t) \, d\langle \phi \rangle_t \quad \text{for all } t \in [0,T].$$

Due to the definition of the transformed loss function $\kappa : g(\mathbb{R}) \to [0, \infty)$ in (3.6) the SDE (3.8) becomes $d\bar{V}_t = \gamma(t, U_t)\, dX_t - d(\mu) X_t f'(\phi_t)\, d\langle\phi\rangle_t$, and recalling the loss-free liquidation price function $\bar{S}$ of Definition 1.26 and its representation in (2.4.2) the real value process satisfies

$$d\bar{V}_t = \bar{S}_u(t, U_t, \phi_t)\, dW_t - d(\mu)\bar{S}_{\xi\xi}(t, U_t, \phi_t)\, d\langle\phi\rangle_t \quad \text{for all } t \in [0, T]. \tag{3.15}$$

This generalizes Satz 2.5 in Baum (2001) and Lemma 2.2 in Bank and Baum (2004). These authors consider only price systems which correspond to a price-determining measure $\mu = \delta_1$, where the large investor's orders always affect the stock price before being exercised, so that $d(\mu) = \frac{1}{2}$. Baum (2001) and Bank and Baum (2004), and in a less constructive way also Çetin et al. (2004), conclude from their analogue of (3.15) that the large investor should rather use trading strategies of bounded variation, since then the transaction loss term in (3.15) vanishes. Therefore, instead of looking for perfect replication of contingent claims, they discuss how to approximate the real value of a contingent claim by self-financing trading strategies with continuous paths of bounded variation.

If the market price smoothly adjusts to orders of the large investor in the sense that the price-determining measure is given by $\mu = \lambda$, then $d(\mu) = 0$, and the quadratic variation of ϕ does not affect the dynamics of the real value process, which then becomes a (local) martingale. Together with the representations (3.7) and (3.8) this allows to work basically as in the Black-Scholes model, as we have already seen in Section 3.4 when we derived the partial differential equation for the continuous-time real value function. $\square$

Chapter 5

Diffusion Limits for General Correlated Random Walks

This chapter deals with the convergence of general correlated random walks. A class of those random walks, for which the direction of the next step depends on time, space, and the direction of the previous move, has already been introduced in Section 4.1, and an associated convergence theorem has been stated without proof. In this Chapter, we introduce correlated random walks systematically in a slightly more general setting, and then prove a convergence theorem for those general correlated random walks, which covers the theorem of Section 4.1. In Section 5.1 we give an overview of the literature on *homogeneous* correlated random walks, for which the direction of the next step only depends on the direction of the previous move, but not on time or space. In Section 5.2 we extend the definition of a homogeneous correlated random walk to the definition of a *general* correlated random walk, where the direction of the next step might also depend on time and the random walk's position in space. If time and space variables of a sequence of general correlated random walks are scaled like in Donsker's theorem and if for each possible direction of the random walk's previous move the transition probabilities converge to a (different) limit function, then our main convergence theorem for general correlated random walks states that the sequence of random walks converges in distribution to the solution of a stochastic differential equation. The generator of this stochastic differential equation is explicitly given. As a corollary to this main convergence theorem, we prove Theorem 4.1 of Section 4.1. Moreover, we show that general correlated random walks can be used to approximate general diffusion processes by a recombining binomial tree. Section 5.3 is devoted to the proof of the main convergence theorem. While the proof is based on standard ideas, the details of the proof become rather tricky and involved since the correlation between two successive increments of the random walk may not vanish asymptotically.

5.1 Results on Homogeneous Correlated Random Walks

We start this chapter with a short discussion on homogeneous correlated random walks as they have been considered in the literature during the last 90 years. In order to introduce them let us denote the jth unit vector on $\mathbb{Z}^d$ by e_j, and set $E = \{\pm e_j \,|\, 1 \leq j \leq d\}$ and $I = \{j \in \mathbb{Z}\backslash\{0\} \,|\, -d \leq j \leq d\}$. Then we can define correlated random walks as in Chen and Renshaw (1994):

Definition 5.1. Suppose $\{Z_k\}_{k\in I\!N}$ is an E-valued homogeneous Markov chain with one-step transition matrix $P = (p_{ij})_{i,j\in I}$ and initial distribution $p^{(1)} = (p_j)_{j\in I}$. Then for any fixed $x_0 \in \mathbb{Z}^d$ the discrete random process $\{X_k\}_{k\in I\!N_0}$ given by $X_k = x_0 + \sum_{j=1}^{k} Z_j$ for all $k \in I\!N_0$ is called a d-dimensional **homogeneous correlated random walk**.

Since $X_k - X_{k-1} = Z_k$ for all $k \in I\!N$ it follows that the probability that the random walk moves in the $(k+1)$-th step in a certain direction, given that a move in the direction e_i brought the random walk to the point x in the kth step, is

$$\mathbf{P}\big(X_{k+1} = x + e_j \,\big|\, X_k = x,\, X_{k-1} = x - e_i\big) = \mathbf{P}\big(Z_{k+1} = e_j \,\big|\, X_k = x,\, Z_k = e_i\big) = p_{ij} \quad (1.1)$$

for all $i, j \in I$ and all $k \in I\!N$. If the $\{Z_k\}_{k \in I\!N}$ are independent, then all the rows of the transition matrix P are identical, and $\{X_k\}_{k \in I\!N}$ reduces to a common random walk on the d-dimensional lattice.

Remark. In Section 5.2 we will introduce more general correlated random walks, which are inhomogeneous in time and space. For this reason we cannot adapt Chen and Renshaw's (1994) notation of "general correlated random walks", but use the term "homogeneous correlated random walks" for the walks of Definition 5.1. Chen and Renshaw (1994) denote their correlated random walks as "general" mainly to distinguish their model from the subclass of *symmetric* homogeneous correlated random walks, for which the one-step transition matrix P is symmetric, and because they allow for general initial distributions $p^{(1)}$. In applied papers, symmetric correlated random walks are often called *persistent* random walks as they were considered in the book of Weiss (1994). However, the notion of persistence is not used consistently. Szász and Tóth (1984) and Tóth (1986) introduce persistent random walks as space-inhomogeneous correlated random walks, though they immediately thereafter restrict themselves to homogeneous ones. □

The notion of correlated random walks was introduced at the same time by Gillis (1955) and Mohan (1955). Weiss (1994) traces such models back to articles by Fürth (1917) and Taylor (1921), who proposed a one-dimensional symmetric correlated random walk to model turbulent diffusions. The work of Taylor was resumed and extended in Goldstein (1951).

Correlated random walks have been used to model a large variety of phenomena in natural science. Next to the modeling of the diffusion of particles as in the early papers of Fürth (1917), Taylor (1921), and Goldstein (1951), Renshaw and Henderson (1981) avail themselves of correlated random walks to model a pin-ball machine, where marbles pass through a triangular system of nails. This machine itself was constructed in order to imitate behavior patterns in genetics. Correlated random walks have also been used to model the growth of the Sitka spruce's roots, animal diffusion and patterns in polymer chemistry (see the references in Chen and Renshaw (1994)). Last but not least, Baloga and Glaze (2003) apply correlated random walks to volcanology to obtain a model for the emplacement of pahoehoe (i.e. basaltic lava) toes.

Let us now give a short overview of the mathematical results on correlated random walks as they have occurred so far in the literature. For further references consider the surveys in Lal and Bhat (1989) and Böhm (2000). In the existing literature the term "correlated random walk" is only used to designate a homogeneous correlated random walk. Thus, unless otherwise stated, all the models described in this section consider homogeneous correlated random walks only.

A large part of the existing work is devoted to transience and recurrence of correlated random walks. Gillis (1955) investigates a symmetric correlated random walk on a d-dimensional lattice and calculates its moments for the one dimensional case. Thereafter he can show that the symmetric correlated random walk is recurrent in dimensions 1 and 2 and transient for all larger even dimensions. His conjecture that the random walk is transient for any $d \geq 3$ was proved by Domb and Fisher (1958) also for a certain class of non-symmetric correlated random walks. A more recent proof of Gillis' conjecture for the class considered by Domb and Fisher can be found in Chen and Renshaw (1992). Renshaw and Henderson (1981) give exact expressions for the n-step transition probabilities for a one-dimensional symmetric correlated random walk on the integers. Lal and Bhat (1989) extend this result to a multidimensional setting which allows for asymmetries as well.

Many authors focus on first-passage problems. Such investigations started with the work of Mohan (1955), at the same time when Gillis (1955) introduced his correlated random walk. Mohan modified the gambler's ruin problem to allow for a correlation between the results of two successive games, and then calculated the probability of ruin and expected duration of play in such a model. Lal and Bhat (1989) consider restricted correlated random walks, which are reflected if they hit certain boundaries, and determine equilibrium distributions and first-passage times for such random walks. Böhm (2000) discusses the distribution of a correlated random walk with absorbing boundary after n steps given that no absorption took place up to time n. Allaart (2004a,b) finds optimal stopping rules for correlated random walks.

In this chapter, we will give general conditions under which one-dimensional (in)homogeneous correlated random walks which are scaled like the classical random walk in Donsker's theorem converge towards a diffusion process. Thus, we are especially interested in earlier convergence results for correlated random walks. The oldest convergence result goes back to the very early stages of correlated random walks, namely to Goldstein (1951). However, he does not use Donsker's scaling, and thus produces a limit result which is not comparable with our results. He scales a symmetric homogeneous correlated random walk $\{X_k\}_{k\in\mathbb{N}}$ by the same order (say $O(n^{-1})$) in time and space, and chooses the transition probabilities of the increments $\{Z_k\}$ in such a way that the correlation between the direction of two successive steps converges to 1 with order $O(n^{-1})$ as $n \to \infty$ as well. Then he shows that for large $n \in \mathbb{N}$ the scaled correlated random walk can be approximated by a continuous-time process, for which the transition probability density solves the telegraph equation (see also Section 3.4d in Weiss (1994) on this point).

In later papers authors scaled the homogeneous correlated random walk by a factor $\delta = n^{-\frac{1}{2}}$ in the space direction and by a factor of order $O(\delta^2)$ in the time direction, as it is done in Donsker's invariance principle. Scaling a one-dimensional symmetric correlated random walk in such a way, Renshaw and Henderson (1981) take the limit in the Kolmogorov forward equations to conclude that a symmetric correlated random walk can be approximated by a Brownian motion with a variance depending on the correlation of the two successive increments of the random walk. In a two-dimensional setting similar results have been obtained by Henderson, Renshaw and Ford (1984).

Szász and Tóth (1984) and Tóth (1986) consider correlated random walks in a random environment. Though they introduce their model in a rather general way, such that the transition matrix $P = \{p_{ij}\}_{i,j\in I}$ of (1.1), which is now stochastic, may depend on the actual position x of the random walk (i.e. $P = P(x)$), all their results are only shown under the assumption that the probability matrices $\{P(x)\}_{x\in\mathbb{Z}}$ are either i.i.d. (as in Szász and Tóth (1984)) or stationary and ergodic (as in Szász and Tóth (1984)). Thus, the intersection of their model with models with a nonrandom environment consists just of the homogeneous random walks given in Definition 5.1. In a one-dimensional setting Szász and Tóth (1984) prove an invariance principle of Donsker type (in random environment) if the correlated random walk is symmetric: For almost all realizations of the environment the scaled correlated random walk converges weakly to a Brownian motion with some suitably chosen variance. This generalizes the pointwise convergence result of Renshaw and Henderson (1981) to convergence in distribution on the path space, even in a nonstochastic environment. If the transition matrix P is not symmetric but implies a positive drift for the correlated random walk, Szász and Tóth (1984) still derive a similar invariance principle. However, they can only state the limiting drift explicitly, but not the limiting volatility. Tóth (1986) generalizes the result of Szász and Tóth (1984) to the d-dimensional case, and proves that the finite-dimensional distributions

of correlated random walks, for which the corresponding transition matrix P is bistochastic and satisfies a uniform Döblin condition, converge to those of a Brownian motion with a positive definite covariance matrix, if the random walk is scaled as in Donsker's invariance principle. If the matrix P is also symmetric, then it especially is self-adjoint. In such a situation tightness, and hence weak convergence, of the scaled correlated random walks can be obtained.
Opitz (1999) employs homogeneous correlated random walks to prove convergence in distribution of an extension of Boyle and Vorst's (1992) transaction cost model. Though he does not recognize the logarithmic stock price in his model as a homogeneous random walk, the converge result which he has obtained for a sequence of those processes shows the convergence of a large family of homogeneous correlated random walks.

A class of related random walks are the *directionally reinforced random walks* introduced by Mauldin, Monticino and Weizsäcker (1996). The probability that a directionally reinforced random walk moves in a certain direction only depends on the number of steps which have been taken in that direction since the last change of direction. If the reinforcement does not depend on the number of steps, the directionally reinforced random walk reduces to a homogeneous symmetric correlated random walk. Mauldin et al. (1996) use directionally reinforced random walks to model certain aspects of ocean surface wave fields. They investigate recurrence and transience of such walks in different dimensions and then go on to prove the convergence of a one-dimensional directionally reinforced random walk, which is again scaled in the usual Donsker-type manner, to a Brownian motion with variance determined by the quotient of variance and expectation of the number of steps between two adjacent changes in direction. Using a constant reinforcement their method especially provides another proof of the convergence result of Renshaw and Henderson (1981) (see Example 4.2 in Mauldin et al. (1996)). Horváth and Shao (1998) extend the convergence result of Mauldin et al. (1996) to a multidimensional directionally reinforced random walk and show further limiting properties for such random walks. Finally, Allaart and Monticino (2001) consider optimal stopping rules for one-dimensional directionally reinforced random walks without and with transaction costs.

5.2 Our Results for General Correlated Random Walks

After our preliminary remarks about homogeneous correlated random walks, we can now go on and define general correlated random walks, which may be inhomogeneous in time and in space, in that the transition probabilities of (1.1) depend on the step k and the position x of the random walk in the kth step. To the best of our knowledge such random walks have not been considered so far in the literature.
We then scale a sequence of general correlated random walks like in Donsker's theorem. If the conditional probabilities for the direction of the random walk's next move can be approximated by a function of time, the random walk's current position and the direction of the random walk's previous move, then we show that the sequence of general correlated random walks converges in distribution to the solution of a stochastic differential equation. The coefficients of the SDE can be explicitly stated. As two corollaries to this main convergence theorem, we prove Theorem 4.1 of Section 4.1 and show that general correlated random walks can be used to approximate general diffusion processes by a recombining binomial tree.

Let us now formally introduce general correlated random walks. For simplicity, we restrict our considerations from the beginning to the one-dimensional case. Since we will later define random walks on a finite index set, we state their definition for the index set $I = \mathbb{N}_0$ and finite index sets of the form $I = \{0, 1, 2, \ldots, m\}$ for some $m \in \mathbb{N}$.

Definition 5.2. Let $\sigma \geq 0$ and $\mu \in I\!R$ be fixed real numbers, Y some real-valued random variable, and $\{Z_k\}_{k\in I}$ a sequence of $\{\pm 1\}$-valued random variables. Then the discrete stochastic process $X = \{X_k\}_{k\in I}$, which is defined by $X_k = Y + \sum_{j=1}^{k}(\mu + \sigma Z_j)$ for all $k \in I$, is called a **(general) correlated random walk** if $\big\{(X_k, Z_k)\big\}_{k\in I}$ is a two-dimensional Markov process. In this case the random variable Z_k is called the **tilt** of X in the kth step.

We emphasize that in this whole chapter μ always denotes the drift of a correlated random walk and not a price-determining measure as in the preceding chapters.

Remark. Note that due to the relation $X_{k+1} = X_k + \mu + \sigma Z_{k+1}$ the process $\{(X_k, Z_k)\}_{k\in I}$ is Markov if and only if

$$\mathbf{P}\big(X_{k+1} = X_k + \mu + \sigma,\, Z_{k+1} = 1 \,\big|\, X_j,\, Z_j;\, 0 \leq j \leq k\big) = \mathbf{P}\big(Z_{k+1} = 1 \big| k, X_k,\, Z_k\big)$$

for all $k \in I$ for which $k+1 \in I$ as well. In contrast to (1.1) this conditional probability may now depend not only on the tilt Z_k, but also on the actual position X_k of the correlated random walk at step k, and on the step number itself. Thus, the general correlated random walk can be inhomogeneous in time and space. $\Box$

From the previous remark we conclude that the distribution of a general correlated random walk $\{X_k\}_{k\in I}$ is fully determined by σ, μ, the distribution of (X_0, Z_0), and by the conditional probabilities

$$\hat{p}(k, x, z) := \mathbf{P}\big(Z_{k+1} = 1 \,\big|\, X_k = x,\, Z_k = z\big) \quad \text{for all } (k, x, z) \in \hat{I} \times I\!R \times \{\pm 1\}, \tag{2.1}$$

where $\hat{I} = \{k \in I \,|\, k+1 \in I\}$. Namely, starting with (X_0, Z_0), we can recursively define the random variables Z_k and X_k by $\mathbf{P}(Z_{k+1} = 1 \,|\, X_k,\, Z_k) = p(k, X_k, Z_k)$ and the relation $X_{k+1} = X_k + \mu + \sigma Z_{k+1}$ for all $k \in \hat{I}$. We call σ and μ the **volatility** and **drift parameter** of the random walk and the function $\hat{p} : \hat{I} \times I\!R \times \{\pm 1\} \to [0, 1]$ defined by (2.1) the **transition function** of the correlated random walk.

Remark. In contrast to the definition of homogeneous correlated random walks, Definition 5.2 utilizes the tilt Z_0 *before* the first step as well. However, it is easy to see that our Definition 5.2 includes (the one-dimensional version of) Chen and Renshaw's (1994) definition, as we introduced it in Definition 5.1. Namely, the process $X = \{X_k\}_{k\in I\!N_0}$ is a homogeneous correlated random walk in the sense of Definition 5.1 with transition matrix $P = \left(\begin{smallmatrix} r & 1-r \\ 1-s & s \end{smallmatrix}\right)$ and initial distribution $p^{(1)} = (p, 1-p)^{tr}$ of the increments $\{Z_k\}_{k\in I\!N}$ if and only if it is a general correlated random walk with constant starting value $X_0 = Y = x_0 \in \mathbb{Z}$, volatility and drift parameters $\sigma = 1$ and $\mu = 0$, and with the transition function $\hat{p} : I\!N_0 \times I\!R \times \{\pm 1\} \to [0, 1]$ and the distribution of Z_0 satisfying

$$\mathbf{P}(Z_0 = 1)\hat{p}(0, x, 1) + \mathbf{P}(Z_0 = -1)\hat{p}(0, x, -1) = p \quad \text{for all } x \in I\!R \tag{2.2}$$

as well as $\hat{p}(k, x, 1) = r$ and $\hat{p}(k, x, -1) = 1 - s$ for all $(k, x) \in I\!N \times I\!R$. If $r = s$, then X is a symmetric homogeneous correlated random walk.
The condition (2.2) guarantees that the initial condition $\mathbf{P}(Z_1 = 1) = p$ for the sequence $\{Z_k\}_{n\in I\!N}$, which only starts with Z_1, holds. Note that for example in the case $p > r \geq 1 - s$ condition (2.2) cannot be satisfied if we also take $\hat{p}(0, x, 1) = r$ and $\hat{p}(0, x, -1) = 1 - s$, such that the transition function $\hat{p} : I\!N_0 \times I\!R \times \{\pm 1\} \to [0, 1]$ does not depend on the first component. Of course (2.2) holds if we take for example $Z_0 = 1$ and $\hat{p}(0, x, 1) = p$ for all $x \in I\!R$. $\Box$

We will now give conditions such that a family $\{X^n\}_{n\in I\!N}$ of correlated random walks converges to some continuous diffusion process on any compact time interval $[0, T]$. Therefore, we divide the time interval $[0, T]$ as in Section 1.3.2 into $\lceil nT \rceil$ equidistant subintervals

$\{[t_{k-1}^n, t_k^n]\}_{1\le k\le\lceil nT\rceil}$ by setting $t_k^n = kT\lceil nT\rceil^{-1}$ for all $0 \le k \le \lceil nT\rceil$. Then we take a sequence $\{X^n\}_{n\in\mathbb{N}}$ of correlated random walks $X^n = \{X_k^n\}_{0\le k\le\lceil nT\rceil}$ living on some probability space $(\Omega^n, \mathcal{F}^n, \mathbf{P}^n)$, with volatility and drift parameters σ_n and μ_n, respectively, and transition function $\hat{p}_n : \{0, 1, \dots, \lceil nT\rceil - 1\} \times \mathbb{R} \times \{\pm 1\} \to [0, 1]$ for all $n \in \mathbb{N}$. Let us also introduce for each $n \in \mathbb{N}$ the process $Z^n = \{Z_k^n\}_{0\le k\le\lceil nT\rceil}$ of associated tilts, such that

$$X_k^n = X_0^n + \sum_{j=1}^{k}\bigl(\mu_n + \sigma_n Z_j^n\bigr) \quad \text{for } 0 \le k \le \lceil nT\rceil. \tag{2.3}$$

In order to make a rigorous convergence statement in the space $D[0, T]$ of càdlàg functions on $[0, T]$, we introduce for each $n \in \mathbb{N}$ the stochastic processes $U^n = \{U^n(t)\}_{t\in[0,T]}$ and $V^n = \{V^n(t)\}_{t\in[0,T]}$, which are constant on each of the intervals $[t_{k-1}^n, t_k^n)$ for all $1 \le k \le \lceil nT\rceil$, by

$$U^n(t) := X_{\lfloor \tilde{n}t\rfloor}^n \quad \text{and} \quad V^n(t) := Z_{\lfloor \tilde{n}t\rfloor}^n \quad \text{for all } 0 \le t \le T \text{ and all } n \in \mathbb{N}, \tag{2.4}$$

where $\tilde{n}$ is for all $n \in \mathbb{N}$ given by $\tilde{n} := T^{-1}\lceil nT\rceil$. We will call U^n the **continuous-time correlated random walk** and V^n the **continuous-time tilt process**, and we frequently drop the term "continuous-time" if it is clear whether we consider U^n or X^n, and V^n or Z^n, respectively. Of course, for each $n \in \mathbb{N}$ the processes U^n and V^n live on the same probability space as X^n and Z^n, namely on $(\Omega^n, \mathcal{F}^n, \mathbf{P}^n)$.

Remark. In Chapter 4 we have denoted continuous time processes U^n by $\{U_t^n\}_{t\in[0,T]}$. We slightly changed the notation, in order to prevent triple sub-indices in the proof of our main convergence theorem. □

In the same way as we introduced U^n and V^n, we can also introduce the **continuous-time transition function** $p_n : [0, T) \times \mathbb{R} \times \{\pm 1\} \to [0, 1]$ by

$$p_n(t, x, z) = \hat{p}_n\bigl(\lfloor \tilde{n}t\rfloor, x, z\bigr) \quad \text{for all } (t, x, z) \in [0, T) \times \mathbb{R} \times \{\pm 1\}, \tag{2.5}$$

which is constant on each interval of the form $[t_{k-1}^n, t_k^n)$ for $1 \le k \le \lceil nT\rceil$.

Remark. The definition of p_n is only for notational convenience. Note that $\hat{p}_n$ and p_n are equivalent, since $\hat{p}_n(k, u, z) = p_n\bigl(t_k^n, u, z\bigr)$ for all $0 \le k \le \lceil nT\rceil - 1$ and $(u, z) \in \mathbb{R} \times \{\pm 1\}$. □

We are interested in the behavior of $\{U^n\}_{n\in\mathbb{N}}$ as $n \to \infty$. For this reason, we have to assume some sort of convergence of the parameters μ_n and σ_n, and of the transition functions $p_n : [0, T) \times \mathbb{R} \times \{\pm 1\} \to [0, 1]$. We will require:

Assumption N. *For $\delta = \delta_n = n^{-\frac{1}{2}}$ there exist some constants $\beta \in (0, 1)$, $\sigma \ge 0$, and $\mu \in \mathbb{R}$ such that the volatility and drift parameters $\{\sigma_n\}_{n\in\mathbb{N}}$ and $\{\mu_n\}_{n\in\mathbb{N}}$ satisfy*

$$\sigma_n = \sigma\delta + O\bigl(\delta^{1+\beta}\bigr) \quad \textit{and} \quad \mu_n = \mu\delta^2 + O\bigl(\delta^{2+\beta}\bigr) \quad \textit{as } n \to \infty.$$

Moreover, there exist some functions $a : [0, T] \times \mathbb{R} \to \mathbb{R}$ and $b : [0, T] \times \mathbb{R} \to \mathbb{R}$ such that uniformly for all $(t, x) \in [0, T) \times \mathbb{R}$

$$p_n(t, x, \pm 1) = \frac{1}{2}\bigl(1 \pm a(t, x) + \delta b(t, x)\bigr) + O\bigl(\delta^{1+\beta}\bigr) \quad \textit{as } n \to \infty. \tag{2.6}$$

Remark. Note that by its definition U^n is scaled by order $O(\delta^2)$ in time. The assumptions on σ_n now ensure that the stochastic part of U^n is scaled by order $O(\delta)$ in space, as in Donsker's theorem. The deterministic drift μ_n has to be scaled by order $O(\delta^2)$ as $n \to \infty$ to avoid explosion. Since the $\delta b(t, x)$-term disappears in the limit, in view of the remark after (2.1) the convergence in (2.6) requires the correlated random walks $\{X_k^n\}_{0\le k\le\lceil nT\rceil}$ to be

at least asymptotically symmetric. However, it will turn out that the remaining first-order asymmetry represented by the term $\delta b(t,x)$ in the expansion of the transition functions still affects the behavior of the limit of the sequence $\{U^n\}_{n\in\mathbb{N}}$.
It is just for notational convenience that we have introduced the functions a and b on the closed domain $[0,T]\times\mathbb{R}$, and not only on $[0,T)\times\mathbb{R}$. □

The next example shows that we have to specify the limit functions $a:[0,T]\times\mathbb{R}\to\mathbb{R}$ and $b:[0,T]\times\mathbb{R}\to\mathbb{R}$ a little bit further.

Example 5.1. Let us consider the sequence $\{X^n\}_{n\in\mathbb{N}}$ of homogeneous correlated random walks given by $(X_0^n, Z_0^n)=(0,1)$, $\sigma_n=\delta=n^{-\frac{1}{2}}$, $\mu_n=0$ and $p_n(t,x,z)=\mathbf{1}_{\{z=1\}}$ for all $(t,x,z)\in[0,T)\times\mathbb{R}\times\{\pm1\}$ and all $n\in\mathbb{N}$. Obviously Assumption N is satisfied, but due to $\hat{p}_n(k,x,z)=p_n\big(t_k^n,x,z\big)=\mathbf{1}_{\{z=1\}}$ for all $0\le k\le\lceil nT\rceil-1$ and $(x,z)\in\mathbb{R}$ it follows that for each $n\in\mathbb{N}$ all the Z_k^n's are completely correlated, such that $Z_0^n=1$ implies $Z_k^n=1$ for all $1\le k\le\lceil nT\rceil$ and hence $X_k^n=\sum_{j=1}^k\delta Z_j=k\delta$. For all $t\in(0,T]$ this yields $U^n(t):=X^n_{\lfloor\tilde{n}t\rfloor}=\lfloor\tilde{n}t\rfloor\delta\to\infty$ as $n\to\infty$. □

However, we will see that the conditions on the limit functions a and b are not very severe. Recalling our convention that $\|f\|=\|f\|_{[0,T]\times\mathbb{R}}=\sup_{(t,x)\in[0,T]\times\mathbb{R}}|f(t,x)|$ if $f:[0,T]\times\mathbb{R}\to\mathbb{R}$ and $\|f\|=\|f\|_{\mathbb{R}}=\sup_{x\in\mathbb{R}}|f(x)|$ if $f:\mathbb{R}\to\mathbb{R}$, for our main convergence theorem we will suppose

Assumption O. *The functions $a:[0,T]\times\mathbb{R}\to\mathbb{R}$ and $b:[0,T]\times\mathbb{R}\to\mathbb{R}$ satisfy:*

(*i*) *There exists some $\mathfrak{a}\in(0,1)$ such that $\|a\|<\mathfrak{a}$. Moreover, $\|b\|<\infty$.*

(*ii*) *The spatial derivative $a':[0,T]\times\mathbb{R}\to\mathbb{R}$ given by $a'(t,x)=\frac{d}{dx}a(t,x)$ is uniformly bounded and continuous with respect to x for all $(t,x)\in[0,T]\times\mathbb{R}$.*

(*iii*) *$a':[0,T]\times\mathbb{R}\to\mathbb{R}$ satisfies a global Hölder condition, uniformly in $t\in[0,T]$, i.e. there exists some $K_0\in\mathbb{R}_+$ and some $\beta\in(0,1)$ such that*

$$\big|a'(t,x)-a'(t,y)\big|\le K_0|x-y|^\beta \quad \text{for all } x,y\in\mathbb{R} \text{ and all } t\in[0,T].$$

(*iv*) *The function $b:[0,T]\times\mathbb{R}\to\mathbb{R}$ is globally Hölder continuous in x with exponent β as well, i.e. there exists some $K_1\in\mathbb{R}_+$ such that*

$$\big|b(t,x)-b(t,y)\big|\le K_1|x-y|^\beta \quad \text{for all } x,y\in\mathbb{R} \text{ and all } t\in[0,T].$$

(*v*) *$a:[0,T]\times\mathbb{R}\to\mathbb{R}$, $a':[0,T]\times\mathbb{R}\to\mathbb{R}$, and $b:[0,T]\times\mathbb{R}\to\mathbb{R}$ are continuous in t.*

Remark. In this chapter we denote the spatial derivative by $a'(t,x)$, and not by $a_x(t,x)$ as we did in the previous chapters, in order to avoid confusions which sub-indices n that will frequently occur. □

We will subsequently employ the conditions of Assumption O as we need them. Before we can state our general convergence result for the sequence $\{U^n\}_{n\in\mathbb{N}}$, we have to introduce the generator $\boldsymbol{L}:C_b^2(\mathbb{R})\to C_b\big([0,T]\times\mathbb{R}\big)$ in terms of the limiting functions a and b. Our most general result will be stated in terms of the martingale problem for $\boldsymbol{L}$.

Definition 5.3. Suppose that the Assumptions O(i), (ii) and (v) hold. Then the generator $\boldsymbol{L}:C_b^2(\mathbb{R})\to C_b\big([0,T]\times\mathbb{R}\big)$ is for all $(t,u)\in[0,T]\times\mathbb{R}$ and $f\in C_b^2(\mathbb{R})$ given by

$$\big(\boldsymbol{L}\,f\big)(t,u)=\frac{1}{2}\sigma^2\frac{1+a(t,u)}{1-a(t,u)}\frac{d^2}{du^2}f(u)+\left(\mu+\frac{\sigma b(t,u)}{1-a(t,u)}+\frac{\sigma^2 a'(t,u)}{\big(1-a(t,u)\big)^2}\right)\frac{d}{du}f(u). \tag{2.7}$$

Now we have everything to state our main convergence theorem for correlated random walks.

Theorem 5.4 (Convergence of General Correlated Random Walks). *Suppose that the Assumptions N and O hold, and that $U^n(0) \Rightarrow U(0)$ for some random variable $U(0)$ with distribution ν on $\big(\mathbb{R}, \mathcal{B}(\mathbb{R})\big)$. If the martingale problem for $(\boldsymbol{L}, \nu)$ has a unique solution on $[0,T]$, then the sequence $\{U^n\}_{n\in\mathbb{N}}$ of continuous-time correlated random walks converges weakly to the process $U = \{U(t)\}_{t\in[0,T]}$ given by $U(0)$ and*

$$dU(t) = \left(\mu + \frac{\sigma b\big(t,U(t)\big)}{1-a\big(t,U(t)\big)} + \frac{\sigma^2 a'\big(t,U(t)\big)}{\big(1-a\big(t,U(t)\big)\big)^2}\right) dt + \sigma\sqrt{\frac{1+a\big(t,U(t)\big)}{1-a\big(t,U(t)\big)}}\, dW(t). \qquad (2.8)$$

Even if the martingale problem has no unique solution, any subsequence of $\{U^n\}_{n\in\mathbb{N}}$ contains a further subsequence which converges weakly to some *solution of* (2.8)*, which in this case need not be unique.*

Remark. The existence of a solution to the martingale problem is assured by Theorem 6.1.7 in Stroock and Varadhan (1979), since Assumption O implies that both coefficients of $\boldsymbol{L}$ are bounded, and for each $t \in [0,T]$ continuous in the space variable. □

The stochastic differential equation (2.8) shows the nontrivial influence of the correlation of two successive increments of the correlated random walk U^n, or, for $\sigma > 0$ equivalently, of two successive tilts. In order to convince ourselves that the diffusion limit is plausible, let us first consider the case $\sigma_n = \delta = n^{-\frac{1}{2}}$, $\mu_n = 0$ and $p_n \equiv \frac{1}{2}$ for all $n \in \mathbb{N}$. Then we are in the setting of Donsker's theorem, and Assumption N is satisfied with $a \equiv b \equiv 0$; the limiting diffusion given by (2.8) reduces to the standard Brownian motion, as in Donsker's theorem. The generalization to general σ_n's and μ_n's as in Assumption N is obvious.
In a next step, let us assume that we are given a sequence $\{U^n\}_{n\in\mathbb{N}}$ of general correlated random walks where for which each of the transition functions $p_n : [0,T) \times \mathbb{R} \times \{\pm 1\} \to [0,1]$ satisfies (2.6), but none of the p_n's depend on its last component. Then $a \equiv 0$, and the sequence $\{U^n\}_{n\in\mathbb{N}}$ is a sequence of simple Markovian binomial processes. It is easy to see that in this case $\mathbf{E}\big[U^n(t+\delta^2) - U^n(t) \,\big|\, U^n(t) = u\big] = \mu_n + \sigma_n \mathbf{E}\big[Z^n(t+\delta^2) \,\big|\, U^n(t) = u\big]$ and $\mathbf{E}\big[Z^n(t+\delta^2) \,\big|\, U^n(t) = u\big] = 2p_n(t,u,1) - 1$; hence Assumption N implies

$$\delta^{-2}\mathbf{E}\Big[\ U^n(t+\delta^2) - U^n(t) \quad \Big|\, U^n(t) = u\Big] \to \mu + b(t,u)$$

and similarly

$$\delta^{-2}\mathbf{E}\Big[\big(U^n(t+\delta^2) - U^n(t)\big)^2 \,\Big|\, U^n(t) = u\Big] \to \sigma \qquad\qquad \text{as } n \to \infty$$

uniformly for all $(t,u) \in [0,T) \times \mathbb{R}$. In this situation classical convergence results for simple binomial processes as in the papers of Nelson (1990) and Nelson and Ramaswamy (1990) lead to the same results as our Theorem 5.4: Not only the drift parameter μ, but also the asymptotic bias $\delta b(t,x)$ in the transition probabilities affects the drift of the limiting diffusion process.
Thirdly, let us assume $\mu_n = 0$, $\sigma_n = \sigma\delta$, and $p_n(t,x,\pm 1) = \frac{1}{2}(1 \pm a)$ for some constant $a \in (-1,1)$ and all $n \in \mathbb{N}$. Then $\{U^n\}_{n\in\mathbb{N}}$ is a sequence of continuous-time versions of homogeneous correlated random walks as considered by Renshaw and Henderson (1981). They have already proved that the correlation between two successive increments of the random walk introduced by a nonzero a only changes the volatility of the limiting diffusion, which in this cases is just a scaled Brownian motion. An increase in a, which indicates a higher probability that two successive steps of the correlated random walk move into the

same direction, increases the volatility. The reader might find this counterintuitive at the first sight, since for a given a close to 1 an observer, who looks at just a few successive steps of U^n will most likely not observe any change in direction at all. However, if at some point in time a change in direction occurs, the random walk will run in the new direction for quite a large number of steps, before it again changes the direction, such that the overall fluctuations of the correlated random walk with a high value for a are larger than for a walk with a lower a. Thus, the volatility of the limiting diffusion process indeed increases in the parameter a. The extreme case $a = -1$, which we excluded, would mean that the direction of the increments of length $\sigma\delta$ are changed at every step back and forth, such that in the limit, the random walk just stays constant. In contrast, we have shown above in Example 5.1 for the opposite extreme $a = 1$ that the correlated random walk need not converge. For $a \in (-1, 1)$ our reasoning depicts a key aspect of convergence of correlated random walks: We have to be careful about looking at the random walk with the right magnifier, since the behavior of the correlated random walk on a time scale of order $O(\delta^2)$ turns out to differ from its behavior on a time scale of order $O(\delta)$ as $n \to \infty$; in the case $a > 0$ the random walk seems to be inert if we look at it through $O(\delta^2)$-lenses, but using $O(\delta)$-lenses, we find it rather vivid.
The most demanding part of the proof of Theorem 5.4 consists of identifying the drift of the limiting diffusion process for general limiting functions $b : [0, T] \times I\!\!R \to I\!\!R$ and $a : [0, T] \times I\!\!R \to I\!\!R$ in (2.6). However, the qualitative properties of the limiting drift can be reasoned more easily: Let us take once again $\mu_n = 0$, $\sigma_n = \sigma\delta$, and let us now set $p_n(t, x, \pm 1) = \frac{1}{2}(1 \pm a + \delta b)$ for all $(t, x) \in [0, T) \times I\!\!R$ and some fixed positive constants a and b. We have seen for $a = 0$ that $b > 0$, which reflects the asymptotic asymmetry of the transition probabilities of the random walk, leads to a drift term in the stochastic differential equation of the limiting process U: It is always (slightly) more likely to move upwards than downwards. A positive a reinforces such a drift: though it is always more likely to move in the direction of the last move, it is even (slightly) more likely to move in this direction if the last move was an up-move. Thus, the drift-effect introduced by b is increased. The same reasoning holds when $b : [0, T] \times I\!\!R \to I\!\!R$ is not constant and σ_n and μ_n satisfy only the conditions of Assumption N. Note that the drift of the limiting diffusion process, which is due to the drift parameters $\{\mu_n\}_{n \in I\!\!N}$, is not affected by the choice of a.
Last but not least, for a general limiting function $a : [0, T] \times I\!\!R \to I\!\!R$, the drift of the limiting diffusion will also depend on the derivative of a.

The proof of Theorem 5.4 will take up the remainder of this chapter. Before we come to it, let us shortly state two corollaries, which immediately follow from Theorem 5.4.

Corollary 5.5. *Theorem 4.1 holds.*

Proof. In the special case where we have $\sigma = 0$, we obtain the deterministic convergence $U_t \to u_0 + \int_0^t \mu_0 ds = u_0 + \mu_0 t$ as $n \to \infty$, uniformly for all $t \in [0, T]$. Hence, for this case the weak convergence of $\{U^n\}_{n \in I\!\!N}$ is trivial, and we may suppose without loss of generality $\sigma > 0$. It is clear that $a \in H^{\frac{1}{2}\beta, 1+\beta}([0, T] \times I\!\!R)$ with $\|a\| < 1$ and $b \in H^{\frac{1}{2}\beta, \beta}([0, T] \times I\!\!R)$ satisfy Assumption O. Moreover, since $\sigma > 0$ and $\|a\| < 1$ we can bound the variance coefficient of $\boldsymbol{L}$ away from 0 by bounding $\sigma \frac{1 + a(t,x)}{1 - a(t,x)} \geq \sigma \frac{1 - \|a\|}{1 + \|a\|} > 0$ for all $(t, x) \in [0, T] \times I\!\!R$. Hence Theorem 3.2.1 of Stroock and Varadhan (1979) states that for each $t \in [0, T]$ and each function f with compact support which possesses bounded continuous derivatives of all orders, the final value problem

$$f_t(s, u) + (\boldsymbol{L} f)(s, u) = 0 \text{ for all } (s, u) \in [0, t) \times I\!\!R \quad \text{and} \quad f(t, u) = h(u) \text{ for all } u \in I\!\!R$$

which is induced by the operator $\boldsymbol{L} : C_b^2(I\!\!R) \to C_b(I\!\!R)$ of (2.7) has a unique solution $f \in C_b^{1,2}([0, T] \times I\!\!R)$. Thus, recalling the previous remark we can apply Theorems 6.1.7

and 6.3.2(i) in Stroock and Varadhan (1979) to conclude that the martingale problem for $\boldsymbol{L}$ is well-posed; especially, there exists a unique solution to the martingale problem starting at time 0 in $u_0 \in I\!R$. **q.e.d.**

Another immediate consequence of Theorem 5.4 is the following result:

Corollary 5.6. *Suppose* $\hat{\sigma} \in H^{\frac{1}{2}\beta,1+\beta}\big([0,T]\times I\!R\big)$ *and* $\hat{\mu} \in H^{\frac{1}{2}\beta,\beta}\big([0,T]\times I\!R\big)$. *If the function* $\hat{\sigma} : [0,T]\times I\!R \to I\!R$ *is bounded away from* 0 *then there exists a sequence* $\{X^n\}_{n\in I\!N}$ *of correlated random walks, such that the associated continuous time processes* $\{U^n\}_{n\in I\!N}$ *of* (2.4) *converge weakly to the process* U *given by*

$$dU(t) = \hat{\mu}\big(t, U(t)\big)dt + \hat{\sigma}\big(t, U(t)\big)dW(t), \quad \textit{and} \quad U(0) = u_0. \tag{2.9}$$

Proof. Let us define the functions $a : [0,T] \times I\!R \to I\!R$ and $b : [0,T] \times I\!R \to I\!R$ by

$$a(t,u) = \frac{\hat{\sigma}^2(t,u) - 1}{\hat{\sigma}^2(t,u) + 1} \quad \text{and} \quad b(t,u) = 2\frac{\hat{\mu}(t,u) - \hat{\sigma}(t,u)\hat{\sigma}'(t,u)}{\hat{\sigma}^2(t,u) + 1} \quad \text{for all } (t,u) \in [0,T] \times I\!R.$$

Then a straightforward calculation shows that

$$\frac{1 + a(t,u)}{1 - a(t,u)} = \hat{\sigma}^2(t,u) \text{ and } \frac{b(t,u)}{1 - a(t,u)} + \frac{a'(t,u)}{(1 - a(t,u))^2} = \hat{\mu}(t,u) \quad \text{for all } (t,u) \in [0,T] \times I\!R.$$

Hence we just have to define for each $n \in I\!N$ the random walk X^n by the initial condition $(X_0^n, Z_0^n) = (u_0, 1)$ (or $= (u_0, -1)$), by the volatility and drift parameters $\sigma_n = \delta = n^{-\frac{1}{2}}$, $\mu_n = 0$, and by the continuous-time transition function $p_n : [0,T) \times I\!R \times \{\pm 1\} \to [0,1]$ given by $p_n(t,u,\pm 1) = \frac{1}{2}\big(1 \pm a(t,u) + \delta b(t,u)\big)$ for all $(t,u) \in [0,T) \times I\!R$. Then U^n converges weakly to U due to Theorem 4.1. **q.e.d.**

Remark. Corollary 5.6 can be used to construct approximations of the diffusion U given by (2.9) via a recombining homogeneous tree. If the drift $\hat{\mu}$ of U depends on the space variable or the volatility $\hat{\sigma}$ depends on the time or space variable and if we construct a binomial process $X^n = \{X_k^n\}_{0 \le k \le \lceil nT \rceil}$, which approximates U, in the most straightforward manner by setting $\mathbf{P}\big(X_k^n = X_{k-1}^n + \hat{\mu}\big(t_{k-1}^n, X_{k-1}^n\big)\delta^2 \pm \hat{\sigma}\big(t_{k-1}^n, X_{k-1}^n\big)\delta \,\big|X_{k-1}^n\big) = \frac{1}{2}$ for $1 \le k \le \lceil nT \rceil$, then the possible realizations of $\{(t_k^n, X_k^n)\}_{0 \le k \le n}$ lead to a non-recombining tree, since the value of X^n after an up move followed by a down move need not coincide with the value obtained by performing the steps in reversed order. The common method to construct a binomial approximation of the diffusion given by (2.9) on a recombining tree is due to Nelson and Ramaswamy (1990). They construct a suitable transformation $g(U)$ of U with constant volatility, develop an approximation for $g(U)$ by a simple binomial process, which lives on a recombining tree, and then apply the inverse of g in order to construct a binomial approximation X^n of U. The paths of the approximation X^n are still recombining, but the tree on which X^n lives is compressed and stretched in space in a patchy way.

The proof of Corollary 5.6 now shows that we can construct a (discrete) approximation process X^n (namely a correlated random walk) which satisfies $\big|X_k^n - X_{k-1}^n\big| = \delta$ for all $1 \le k \le \lceil nT \rceil$. This means that the tree implied by the possible realizations of $\big\{(t_k^n, X_k^n)\big\}_{0 \le k \le n}$ is recombining and homogeneous. However, such a simple tree structure is attained by a more complicated structure of the probability weights, as the probability that the correlated random walk X^n moves from a certain vertex v, say $v = \big(t_k^n, X_k^n\big)$ in a certain direction, depends not only on the vertex v, but also on the tilt Z_k^n of the random walk at time t_k^n. Since $\sigma_n Z_k^n = X_k^n - X_{k-1}^n - \mu_n$ for $1 \le k \le \lceil nT \rceil$, the tilt can be interpreted as the direction from which the walk reached the vertex v. □

5.3 Proof of the Main Convergence Theorem

In this section we present the proof of the Theorem 5.4. We first show that the sequence $\{U^n\}_{n\in\mathbb{N}}$ of continuous-time correlated random walks is tight in the Skorohod space $D[0,T]$ by deriving precise bounds on product moments of the increments of the random walk and then employing techniques of Billingsley (1968) for the fluctuation of partial sums of not necessarily independent or identically distributed random variables. Then we show that the distribution of the limit of each converging subsequence of $\{U^n\}_{n\in\mathbb{N}}$ solves the martingale problem for $(\boldsymbol{L},\nu)$. Since the correlation between two successive increments of U^n does not vanish as $n\to\infty$, this requires a careful consideration of conditional moments on a time scale of order $O(\delta)$.

Before we give a detailed outline of the proof, let us agree on one simplification which does not lead to any loss of generality, and also introduce some more notation which will be used frequently in the sequel.

For ease of notation, we will assume from now on that $T=1$. Then we especially have $\tilde{n}=n$. Due to the Markov property of $\big\{\big(X_k^n,Z_k^n\big)\big\}_{0\le k\le n}$ we will frequently condition on events of the form $\big\{X_k^n=x,\,Z_k^n=z\big\}$ or also on $\big\{U^n(t)=u,\,V^n(t)=v\big\}$. In order to keep our notation compact, we introduce for all $(t,u,v)\in[0,T]\times\mathbb{R}\times\{\pm1\}$ the notation

$$\mathbf{P}_t^{u,v}\big(\,\cdot\,\big):=\mathbf{P}^n\big(\,\cdot\,\big|\,U^n(t)\!=\!u,\,V^n(t)\!=\!v\big)\ \text{ and }\ \mathbf{E}_t^{u,v}\big[\,\cdot\,\big]:=\mathbf{E}^n\big[\,\cdot\,\big|\,U^n(t)\!=\!u,\,V^n(t)\!=\!v\big]. \quad (3.1)$$

By taking $t=t_k^n$ and employing the definitions of U^n and V^n, our shorthands can be used to represent conditional expectations with respect to $\big(X_k^n,Z_k^n\big)$, namely we have

$$\mathbf{P}^n\big(\,\cdot\,\big|\,X_k^n=x,\,Z_k^n=z\big)=\mathbf{P}_{t_k^n}^{x,z}\big(\,\cdot\,\big)\quad\text{and}\quad\mathbf{E}^n\big[\,\cdot\,\big|\,X_k^n=x,\,Z_k^n=z\big]=\mathbf{E}_{t_k^n}^{x,z}\big[\,\cdot\,\big] \qquad (3.2)$$

for all $0\le k\le n$ and $(x,z)\in\mathbb{R}\times\{\pm1\}$. For our calculations it proves useful to write the continuous-time transition function $p_n:[0,T)\times\mathbb{R}\times\{\pm1\}\to[0,1]$ of (2.5) for all $n\in\mathbb{N}$ in a form which resembles the asymptotic representation of (2.6), namely we write it as

$$p_n(t,x,\pm1)=\frac{1}{2}\Big(1\pm a_n(t,x)+\delta b_n(t,x)\Big)\quad\text{for all } (t,x)\in[0,T)\times\mathbb{R} \qquad (3.3)$$

for some suitable functions $a_n:[0,T)\times\mathbb{R}\to\mathbb{R}$ and $b_n:[0,T)\times\mathbb{R}\to\mathbb{R}$. For each fixed $(t,x)\in[0,T)\times\mathbb{R}$ the solutions $a_n(t,x)$ and $b_n(t,x)$ of these two equations with two unknowns are easily found, and hence under Assumption N the functions a_n and b_n satisfy uniformly for all $(t,x)\in[0,T)\times\mathbb{R}$:

$$a_n(t,x)=p_n\big(t,x,1\big)-p_n\big(t,x,-1\big)=a(t,x)+O\big(\delta^{1+\beta}\big) \qquad (3.4)$$

and

$$b_n(t,x)=\frac{1}{\delta}\Big(p_n\big(t,x,1\big)+p_n\big(t,x,-1\big)-1\Big)=b(t,x)+O\big(\delta^{\beta}\big)\quad\text{as } n\to\infty. \qquad (3.5)$$

For each $n\in\mathbb{N}$ the function a_n measures the influence of the correlation between two successive tilts Z_{k-1}^n and Z_k^n for all $1\le k\le\lceil nT\rceil$ (respectively of $V^n(t-\delta^2)$ and $V^n(t)$ for $\delta^2\le t\le T$), while b^n describes the asymmetry of the tilt process $\{Z_k^n\}_{0\le k\le n}$ (respectively of $\{V^n(t)\}_{t\in[0,T]}$). Since p_n is a probability function it is clear from (3.4) and (3.5) that $|a_n(t,x)|\le1$ and $\delta|b_n(t,x)|\le1$ for all $(t,x)\in[0,T)\times\mathbb{R}$.

For the rest of this chapter we always assume that $n\in\mathbb{N}$ and $i,j,k,l,m\in\mathbb{N}_0$, even if we do not state it explicitly.

Steps in the Proof of the Main Convergence Theorem

The proof of Theorem 5.4 will be presented step by step. At first, we will calculate in Section 5.3.1 for each fixed $n \in \mathbb{N}$ the first and second conditional moment of the correlated random walk $X^n = \{X^n_k\}_{0 \le k \le n}$. These moments are given in terms of two auxiliary functions, which depend on the volatility and drift parameters σ_n and μ_n, and on the functions $a_n : [0,T] \times \mathbb{R} \to \mathbb{R}$ and $b_n : [0,T] \times \mathbb{R} \to \mathbb{R}$. In Section 5.3.2 we will derive approximations of various degrees of sharpness for those auxiliary functions by employing the Assumptions N and O. A set of weaker approximations, which can be derived by using Assumptions N and O(i) and (ii) only, will be used in Section 5.3.3 to bound the conditional moments of increments of X^n in terms of the number of steps, for all sufficiently large $n \in \mathbb{N}$ and a sufficiently small number of incremental steps. These bounds will allow us to prove by standard techniques that the sequence $\{U^n\}_{n\in\mathbb{N}}$ of associated continuous-time random walks given by (2.4) is tight.

We would like to conclude in analogy to Section 11.2 in Stroock and Varadhan (1979) that the distribution of the limit of any converging subsequence of $\{U^n\}_{n\in\mathbb{N}}$ is a solution to the martingale problem for $(\boldsymbol{L}, \nu)$. However, this is considerably more difficult than in Stroock and Varadhan's setting, since the correlation between two successive increments of U^n does not vanish as $n \to \infty$. A key step in showing that nevertheless the limit distribution does solve the martingale problem consists in determining the limit of the first two conditional local moments of U^n over a certain time interval, when normalized by the length of the interval. They should coincide with the coefficients of the generator $\boldsymbol{L}$ given by (2.7). As already mentioned in the reasoning of Theorem 5.4, in order to obtain the limit we will have to take care about selecting the proper lenses to look at our random walk. The right lens to prove convergence of the conditional local moments will be seen to be the $O(\delta)$-lens, i.e. we will consider the conditional moments of the increments of U^n during a time interval of the form $[\tau, \tau+\delta]$.

Since we do not require the functions $a : [0,T] \times \mathbb{R} \to \mathbb{R}$ and $b : [0,T] \times \mathbb{R} \to \mathbb{R}$ to be Hölder continuous in the time arguments of a and b, we will demonstrate the convergence of the conditional local moments of $\{U^n\}_{n\in\mathbb{N}}$ in two steps. In Section 5.3.4 first approximations of the conditional local moments will be derived by exploiting sharper approximations of our auxiliary functions than in the tightness proof. These approximations are also derived in Section 5.3.2, without assuming any continuity in the time variable for $a : [0,T] \times \mathbb{R} \to \mathbb{R}$ or $b : [0,T] \times \mathbb{R} \to \mathbb{R}$. The approximations for the second and third moment will only depend on the position $U^n(\tau)$ of the correlated random walk at the beginning of the time interval $[\tau, \tau+\delta]$, while the approximation of the first moment will still depend on the tilt $Z^n(\tau)$ of U^n at the beginning of that interval. However, we will see that the tilt's influence is manageable.

In Section 5.3.5 we will employ the time continuity of Assumption O(v) to show that the approximations for the first two local moments as obtained in Section 5.3.4 converge to the coefficents of the generator $\boldsymbol{L}$, uniformly on compact intervals. These results will be applied in Section 5.3.6 to show that for all sufficiently smooth functions $f : \mathbb{R} \to \mathbb{R}$ and all sufficiently large $n \in \mathbb{N}$ the process $\big\{f\big(U^n(t)\big) - \int_0^t \big(\boldsymbol{L}f\big)\big(U(s)\big)\,ds\big\}_{t\in[0,T]}$ is at least approximately a martingale.

In the concluding Section 5.3.6 it will be seen that this approximate martingale property suffices to apply the same arguments as in the setting of Stroock and Varadhan (1979). Especially, we will show that indeed the distribution of the limit of any converging subsequence of $\{U^n\}_{n\in\mathbb{N}}$ is a solution to the martingale problem for $(\boldsymbol{L}, \nu)$, which will complete the proof of Theorem 5.4.

Due to the generality of the convergence theorem, many bricks of the proof are very technical. Therefore the proofs of the following lemmas should be omitted on first reading.

5.3.1 Conditional Moments of the Correlated Random Walk

We start with calculating the conditional moments of the general correlated random walk $X^n = \{X_l^n\}_{0\le l\le n}$ given by (2.3) with $T = 1$. More exactly, for this section we fix $n \in \mathbb{N}$, $(x,z) \in \mathbb{R}\times\{\pm1\}$ and $0 \le i \le n$ and calculate the moments of the increments $X_l^n - X_i^n$ given the correlated random walk is at the ith step in x with tilt $Z_i^n = z$. Since the distribution of the correlated random walk is rather involved, we determine its moments in terms of some auxiliary functions $A_{n,F}^k$ and $B_{n,F}^k$.

Definition 5.7. Let us denote for all $n \in \mathbb{N}$ and $0 \le k \le n-1$ the set of time points $\{t_j^n\}_{0\le j\le n-k}$ by $\mathcal{T}_k^n$. Then for any function $F : \mathbb{R} \to \mathbb{R}$ we define the family $\{A_{n,F}^k\}_{0\le k\le n-1}$ of functions $A_{n,F}^k : \mathcal{T}_k^n \times \mathbb{R} \to \mathbb{R}$ recursively in terms of the function $a_n : [0,T)\times\mathbb{R} \to \mathbb{R}$ of (3.4) by

$$A_{n,F}^0\big(t_j^n, x\big) = F(x) \tag{3.6}$$

for all $0 \le j \le n$ and all $x \in \mathbb{R}$ and

$$A_{n,F}^k\big(t_j^n, x\big) = \frac{1}{2}\Big(A_{n,F}^{k-1}\big(t_{j+1}^n, x+\mu_n+\sigma_n\big) + A_{n,F}^{k-1}\big(t_{j+1}^n, x+\mu_n-\sigma_n\big)\Big)a_n\big(t_j^n, x\big) \tag{3.7}$$

for all $0 \le j \le n-k$, $x \in \mathbb{R}$, and $1 \le k \le n-1$. Moreover, we define the family $\{B_{n,F}^k\}_{1\le k\le n-1}$ of functions $B_{n,F}^k : \mathcal{T}_k^n \times \mathbb{R} \to \mathbb{R}$ by setting for all $1 \le k \le n-1$, $0 \le j \le n-k$, and $x \in \mathbb{R}$

$$\begin{aligned} B_{n,F}^k\big(t_j^n, x\big) = {} & \frac{1}{2}\Big(A_{n,F}^{k-1}(t_{j+1}^n, x+\mu_n+\sigma_n) - A_{n,F}^{k-1}(t_{j+1}^n, x+\mu_n-\sigma_n)\Big) \\ & + \frac{1}{2}\Big(A_{n,F}^{k-1}(t_{j+1}^n, x+\mu_n+\sigma_n) + A_{n,F}^{k-1}(t_{j+1}^n, x+\mu_n-\sigma_n)\Big)\delta b_n\big(t_j^n, x\big), \end{aligned} \tag{3.8}$$

where $b_n : [0,T)\times\mathbb{R} \to \mathbb{R}$ was defined in (3.5). If $F \equiv 1$, then we write A_n^k instead of $A_{n,1}^k$, and B_n^k instead of $B_{n,1}^k$.

Remark. It is easily seen from Definition 5.7 and from the representation (3.3) for the continuous-time transition function of X^n that the functions $A_{n,F}^k$ and $B_{n,F}^k$ satisfy

$$\mathbf{E}_{t_j^n}^{x,z}\Big[Z_{j+1}^n A_{n,F}^{k-1}\big(t_{j+1}^n, X_{j+1}^n\big)\Big] = zA_{n,F}^k\big(t_j^n, x\big) + B_{n,F}^k\big(t_j^n, x\big) \tag{3.9}$$

for all $(x,z) \in \mathbb{R}\times\{\pm1\}$, $0 \le j \le n-k$, and $1 \le k \le n-1$. Note that for all $0 \le k \le n-1$ and $t \in \mathcal{T}_k^n$ we have $A_n^k(t,x) = a^k$ if $a_n(t,x) = a$ for all $(t,x) \in \mathcal{T}_1^n \times \mathbb{R}$ and some constant $a \in \mathbb{R}$. Even if this does not hold for general functions $a_n : [0,T)\times\mathbb{R} \to \mathbb{R}$, the intuitive approximation $A_n^k(t,x) \approx a_n^k(t,x)$ for all $(t,x) \in \mathcal{T}_k^n \times \mathbb{R}$ is quite good, as we will see in Lemma 5.13. □

The auxiliary functions A_n^k and B_n^k are used frequently in the remainder of this chapter, the more general functions $A_{n,F}^k$ and $B_{n,F}^k$ are only needed in Lemma 5.28. Before we can calculate the moments of $X_l^n - X_i^n$, we have to state two lemmas:

Lemma 5.8. *We have for all $F \in C_b^1(\mathbb{R})$, the space of bounded functions on $\mathbb{R}$ with continuous and bounded derivative, and all $0 \le i \le l \le n-m$:*

$$\mathbf{E}_{t_i^n}^{x,z}\Big[Z_l^n A_{n,F}^m\big(t_l^n, X_l^n\big)\Big] = zA_{n,F}^{m+l-i}\big(t_i^n, x\big) + \sum_{j=i}^{l-1}\mathbf{E}_{t_i^n}^{x,z}\Big[B_{n,F}^{m+l-j}\big(t_j^n, X_j^n\big)\Big]. \tag{3.10}$$

Especially, the conditional mean of the tilt Z_l^n at step $l \geq i$ given the random walk has been in x with tilt z at the ith step, is given by

$$\mathbf{E}_{t_i^n}^{x,z}\big[Z_l^n\big] = zA_n^{l-i}\big(t_i^n, x\big) + \sum_{j=i}^{l-1} \mathbf{E}_{t_i^n}^{x,z}\Big[B_n^{l-j}\big(t_j^n, X_j^n\big)\Big]. \tag{3.11}$$

Proof. Let us fix $F \in C_b^1(I\!R)$. We prove the first assertion by induction over l. For $l = i$, the statement (3.10) is trivially satisfied for all $0 \leq m \leq n-l$. For the induction step let us suppose that we have already proved (3.10) for some $l \geq i$ and all $0 \leq m \leq n-l$. Then for all $0 \leq m \leq n-l-1$ we get from the Markov property of $\big\{(X_k^n, Z_k^n)\big\}_{0 \leq k \leq n}$, from (3.9) and from the induction hypothesis (3.10) (with m replaced by $m+1$):

$$\begin{aligned}
&\mathbf{E}_{t_i^n}^{x,z}\Big[Z_{l+1}^n A_{n,F}^m\big(t_{l+1}^n, X_{l+1}^n\big)\Big] \\
&\qquad = \mathbf{E}_{t_i^n}^{x,z}\Big[\mathbf{E}_{t_l^n}^{X_l^n, Z_l^n}\big[Z_{l+1}^n A_{n,F}^m\big(t_{l+1}^n, X_{l+1}^n\big)\big]\Big] \\
&\qquad = \mathbf{E}_{t_i^n}^{x,z}\Big[Z_l^n A_{n,F}^{m+1}\big(t_l^n, X_l^n\big)\Big] + \mathbf{E}_{t_i^n}^{x,z}\Big[B_{n,F}^{m+1}\big(t_l^n, X_l^n\big)\Big] \\
&\qquad = zA_{n,F}^{m+l-i+1}(i,x) + \sum_{j=i}^{l-1} \mathbf{E}_{t_i^n}^{x,z}\Big[B_{n,F}^{m+l-j+1}\big(t_j^n, X_j^n\big)\Big] + \mathbf{E}_{t_i^n}^{x,z}\Big[B_{n,F}^{m+1}\big(t_l^n, X_l^n\big)\Big],
\end{aligned}$$

which proves (3.10) for l replaced by $l+1$ and all $0 \leq m \leq n-(l+1)$. Hence the expression (3.10) holds for all $i \leq l \leq n-m$. Due to $A_n^0 \equiv 1$ the second statement (3.11) follows from (3.10) for $F \equiv 1$ and $m = 0$. **q.e.d.**

Remark. Since $Z_l^n \in \{\pm 1\}$, we have $\mathbf{P}_{t_i^n}^{x,z}\big(Z_l^n = 1\big) = \mathbf{E}_{t_i^n}^{x,z}\big[1_{\{Z_l^n = 1\}}\big] = \frac{1}{2}\mathbf{E}_{t_i^n}^{x,z}\big[1 + Z_l^n\big]$ for all $i \leq l \leq n$, and hence the probability that the correlated random walk has a positive tilt at the lth step given it starts at the ith step in x with tilt z is given by

$$\mathbf{P}_{t_i^n}^{x,z}\big(Z_l^n = 1\big) = \frac{1}{2}\left(1 + zA_n^{l-i}\big(t_i^n, x\big) + \sum_{j=i}^{l-1} \mathbf{E}_{t_i^n}^{x,z}\Big[B_n^{l-j}\big(t_j^n, X_j^n\big)\Big]\right) \quad \text{for all } i \leq l \leq n.$$

In particular, if the transition function $p_n : [0,T) \times I\!R \times \{\pm 1\} \to [0,1]$ is symmetric and only depends on the tilt of the random walk, but not on its position in space, such that we have $p_n\big(t, x, \pm 1\big) = \frac{1}{2}\big(1 \pm \tilde{a}_n(t)\big)$ for all $(t,x) \in [0,T) \times I\!R$ and some function $\tilde{a}_n : [0,T) \to [-1,1]$, then it follows from Definition 5.7 that $A_n^k\big(t_j^n, x\big) = \prod_{r=j}^{j+k-1} \tilde{a}_n\big(t_r^n\big)$ and $B_n^k\big(t_j^n, x\big) = 0$ for all $0 \leq k \leq n-1$, $0 \leq j \leq n-k$, and $x \in I\!R$. Hence the conditional probability that the correlated random walk has a positive tilt at the lth step simplifies to

$$\mathbf{P}_{t_i^n}^{x,z}\big(Z_l^n = 1\big) = \frac{1}{2}\left(1 + z\prod_{r=i}^{l-1} \tilde{a}_n\big(t_r^n\big)\right)$$

for all $i \leq l \leq n$. □

In the following Proposition 5.10 we will inductively calculate the second conditional moment of $X_l^n - X_i^n$ for $0 \leq i \leq l \leq n$. In the induction step, we will encounter the expression $\mathbf{E}_{t_i^n}^{x,z}\big[Z_{l+1}^n\big(X_l^n - X_i^n\big)\big]$. Our second lemma shows how we can deal with this term:

Lemma 5.9. *For all $i \leq l \leq n-1$ we can determine:*

$$\mathbf{E}_{t_i^n}^{x,z}\big[Z_{l+1}^n\big(X_l^n - X_i^n\big)\big] = \sigma_n \sum_{j=i+1}^{l} \mathbf{E}_{t_i^n}^{x,z}\Big[A_n^{l-j+1}\big(t_j^n, X_j^n\big)\Big] + \sum_{j=i+1}^{l} \mathbf{E}_{t_i^n}^{x,z}\Big[\big(X_j^n - X_i^n\big)B_n^{l-j+1}\big(t_j^n, X_j^n\big)\Big]$$

$$+ \mu_n \left(z(l-i)A_n^{l-i+1}\left(t_i^n, x\right) + \sum_{j=i}^{l-1}(l-j)\mathbf{E}_{t_i^n}^{x,z}\left[B_n^{l-j+1}\left(t_j^n, X_j^n\right)\right]\right).$$

Proof. Let us again fix $0 \leq i \leq n$ and choose $i \leq l \leq n-1$. By the Markov property, we can write $\mathbf{E}_{t_i^n}^{x,z}\left[Z_{l+1}^n\left(X_l^n - X_i^n\right)\right] = \mathbf{E}_{t_i^n}^{x,z}\left[\left(X_l^n - X_i^n\right)\mathbf{E}_{t_l^n}^{X_l^n, Z_l^n}\left[Z_{l+1}^n\right]\right]$, and applying (3.11) we obtain:

$$\mathbf{E}_{t_i^n}^{x,z}\left[Z_{l+1}^n\left(X_l^n - X_i^n\right)\right] = \mathbf{E}_{t_i^n}^{x,z}\left[\left(X_l^n - X_i^n\right)Z_l^n A_n^1\left(t_l^n, X_l^n\right)\right] + \mathbf{E}_{t_i^n}^{x,z}\left[\left(X_l^n - X_i^n\right)B_n^1\left(t_l^n, X_l^n\right)\right]. \quad (3.12)$$

In order to calculate $\mathbf{E}_{t_i^n}^{x,z}\left[\left(X_l^n - X_i^n\right)Z_l^n A_n^1\left(t_l^n, X_l^n\right)\right]$ we use an induction over l, and show that for all $i \leq l \leq n-m$ we have:

$$\begin{aligned}
&\mathbf{E}_{t_i^n}^{x,z}\left[\left(X_l^n - X_i^n\right)Z_l^n A_n^m\left(t_l^n, X_l^n\right)\right] \\
&\quad = \sigma_n \sum_{j=i+1}^{l} \mathbf{E}_{t_i^n}^{x,z}\left[A_n^{m+l-j}\left(t_j^n, X_j^n\right)\right] + \sum_{j=i+1}^{l-1} \mathbf{E}_{t_i^n}^{x,z}\left[\left(X_j^n - X_i^n\right)B_n^{m+l-j}\left(t_j^n, X_j^n\right)\right] \\
&\qquad + \mu_n\left(z(l-i)A_n^{m+l-i}\left(t_i^n, x\right) + \sum_{j=i}^{l-1}(l-j)\mathbf{E}_{t_i^n}^{x,z}\left[B_n^{m+l-j}\left(t_j^n, X_j^n\right)\right]\right).
\end{aligned} \quad (3.13)$$

With (3.13) being shown for all $i \leq l \leq n-m$ we can plug (3.13) with $m=1$ into (3.12) and incorporate the term $\mathbf{E}_{t_i^n}^{x,z}\left[\left(X_l^n - X_i^n\right)B_n^1\left(t_l^n, X_l^n\right)\right]$ in the second sum to validate the lemma's assertion. It remains to accomplish the induction to show (3.13). For $l = i$, both sides of (3.13) are 0. Now suppose that for some $l \geq i+1$ the expression (3.13) holds for $\tilde{l} = l-1$ and all $0 \leq m \leq n-l+1$. Then we can write $X_l^n - X_i^n = X_{l-1}^n - X_i^n + \left(\mu_n + \sigma_n Z_l^n\right)$ and apply the Markov property of $\left\{\left(X_k^n, Z_k^n\right)\right\}_{0 \leq k \leq n}$, (3.9), and $\left|Z_l^n\right| = 1$ to conclude:

$$\begin{aligned}
&\mathbf{E}_{t_i^n}^{x,z}\left[\left(X_l^n - X_i^n\right)Z_l^n A_n^m\left(t_l^n, X_l^n\right)\right] \\
&\quad = \mathbf{E}_{t_i^n}^{x,z}\left[\left(X_{l-1}^n - X_i^n\right)\mathbf{E}_{t_{l-1}^n}^{X_{l-1}^n, Z_{l-1}^n}\left[Z_l^n A_n^m\left(t_l^n, X_{l-1}^n + Z_l^n\right)\right]\right] + \mathbf{E}_{t_i^n}^{x,z}\left[\left(\mu_n + \sigma_n Z_l^n\right)Z_l^n A_n^m\left(t_l^n, X_l^n\right)\right] \\
&\quad = \mathbf{E}_{t_i^n}^{x,z}\left[\left(X_{l-1}^n - X_i^n\right)Z_{l-1}^n A_n^{m+1}\left(t_{l-1}^n, X_{l-1}^n\right)\right] + \mathbf{E}_{t_i^n}^{x,z}\left[\left(X_{l-1}^n - X_i^n\right)B_n^{m+1}\left(t_{l-1}^n, X_{l-1}^n\right)\right] \\
&\qquad + \mu_n \mathbf{E}_{t_i^n}^{x,z}\left[Z_l^n A_n^m\left(t_l^n, X_l^n\right)\right] + \sigma_n \mathbf{E}_{t_i^n}^{x,z}\left[A_n^m\left(t_l^n, X_l^n\right)\right].
\end{aligned}$$

Now by the induction hypothesis (3.13) with (l, m) replaced by $(l-1, m+1)$ holds, and if in addition we employ Lemma 5.8 we obtain by collecting terms

$$\begin{aligned}
&\mathbf{E}_{t_i^n}^{x,z}\left[Z_l^n\left(X_l^n - X_i^n\right)A_n^m\left(t_l^n, X_l^n\right)\right] \\
&\quad = \sigma_n \sum_{j=i+1}^{l-1} \mathbf{E}_{t_i^n}^{x,z}\left[A_n^{m+l-j}\left(t_j^n, X_j^n\right)\right] + \sum_{j=i+1}^{l-2} \mathbf{E}_{t_i^n}^{x,z}\left[\left(X_j^n - X_i^n\right)B_n^{m+l-j}\left(t_j^n, X_j^n\right)\right] \\
&\qquad + \mu_n\left(z(l-i-1)A_n^{m+l-i}\left(t_i^n, x\right) + \sum_{j=i}^{l-2}(l-1-j)\mathbf{E}_{t_i^n}^{x,z}\left[B_n^{m+l-j}\left(t_j^n, X_j^n\right)\right]\right) \\
&\qquad + \mathbf{E}_{t_i^n}^{x,z}\left[\left(X_{l-1}^n - X_i^n\right)B_n^{m+1}\left(t_{l-1}^n, X_{l-1}^n\right)\right] \\
&\qquad + \mu_n\left(zA_n^{m+l-i}\left(t_i^n, x\right) + \sum_{j=i}^{l-1} \mathbf{E}_{t_i^n}^{x,z}\left[B_n^{m+l-j}\left(t_j^n, X_j^n\right)\right]\right) + \sigma_n \mathbf{E}_{t_i^n}^{x,z}\left[A_n^m\left(t_l^n, X_l^n\right)\right]
\end{aligned}$$

$$= \sigma_n \sum_{j=i+1}^{l} \mathbf{E}_{t_i^n}^{x,z}\Big[A_n^{m+l-j}\big(t_j^n, X_j^n\big)\Big] + \sum_{j=i+1}^{l-1} \mathbf{E}_{t_i^n}^{x,z}\Big[\big(X_j^n - X_i^n\big)B_n^{m+l-j}\big(t_j^n, X_j^n\big)\Big]$$

$$+ \mu_n\bigg(z(l-i)A_n^{m+l-i}\big(t_i^n, x\big) + \sum_{j=i}^{l-1}(l-j)\mathbf{E}_{t_i^n}^{x,z}\Big[B_n^{m+l-j}\big(t_j^n, X_j^n\big)\Big]\bigg),$$

which shows that (3.13) holds for l and all $0 \le m \le n-l$ as well, hence it holds for all $i \le l \le n$ and all $0 \le m \le n-l$. This completes the proof of the lemma. **q.e.d.**

Now we can calculate the first two conditional moments of $X_l^n - X_i^n$:

Proposition 5.10. *For all $0 \le i \le l \le n$ and $(x,z) \in I\!R \times \{\pm 1\}$, the conditional expectation of $X_l^n - X_i^n$ given $\{X_i^n = x,\ Z_i^n = z\}$ can be expressed by*

$$\mathbf{E}_{t_i^n}^{x,z}\big[X_l^n - X_i^n\big] = \mu_n(l-i) + \sigma_n\left(z\sum_{k=1}^{l-i} A_n^k\big(t_i^n, x\big) + \sum_{j=i}^{l-1}\sum_{k=1}^{l-j} \mathbf{E}_{t_i^n}^{x,z}\big[B_n^k(t_j^n, X_j^n)\big]\right). \tag{3.14}$$

Moreover, the conditional second moment of $X_l^n - X_i^n$ becomes

$$\begin{aligned}\mathbf{E}_{t_i^n}^{x,z}\Big[\big(X_l^n - X_i^n\big)^2\Big] &= \sigma_n^2\left(l-i+2\sum_{j=i+1}^{l-1}\sum_{k=1}^{l-j}\mathbf{E}_{t_i^n}^{x,z}\Big[A_n^k\big(t_j^n, X_j^n\big)\Big]\right) + \mu_n^2(l-i)^2\\ &\quad + 2\mu_n\sigma_n\left(z(l-i)\sum_{k=1}^{l-i}A_n^k\big(t_i^n,x\big) + \sum_{j=i}^{l-1}\sum_{k=1}^{l-j}(l-j)\mathbf{E}_{t_i^n}^{x,z}\Big[B_n^k\big(t_j^n, X_j^n\big)\Big]\right)\\ &\quad + 2\sigma_n\sum_{j=i+1}^{l-1}\sum_{k=1}^{l-j}\mathbf{E}_{t_i^n}^{x,z}\Big[\big(X_j^n - X_i^n\big)B_n^k\big(t_j^n, X_j^n\big)\Big],\end{aligned} \tag{3.15}$$

and last but not least the conditional variance of $X_l^n - X_i^n$ is for all those $l \ge i$ given by

$$\begin{aligned}\mathbf{Var}_{t_i^n}^{x,z}\big[X_l^n - X_i^n\big] &= \sigma_n^2\Bigg(l-i+2\sum_{j=i+1}^{l-1}\sum_{k=1}^{l-j}\mathbf{E}_{t_i^n}^{x,z}\Big[A_n^k\big(t_j^n, X_j^n\big)\Big]\\ &\qquad - \left(z\sum_{k=1}^{l-i}A_n^k\big(t_i^n, x\big) + \sum_{j=i}^{l-1}\sum_{k=1}^{l-j}\mathbf{E}_{t_i^n}^{x,z}\Big[B_n^k\big(t_j^n, X_j^n\big)\Big]\right)^2\Bigg)\\ &\quad + 2\sigma_n\sum_{j=i+1}^{l-1}\sum_{k=1}^{l-j}\mathbf{E}_{t_i^n}^{x,z}\Big[\Big(X_j^n - X_i^n - (j-i)\mu_n\Big)B_n^k\big(t_j^n, X_j^n\big)\Big].\end{aligned} \tag{3.16}$$

Proof. We fix $0 \le i \le n$ and use again an induction argument. All three equations are trivially true for $l = i$. Thus, we only need to show the induction step. Let us first consider the mean. If (3.14) holds for $l-1$ for some $i \le l \le n$, we can use the definition of the correlated random walk $\{X_k^n\}_{0 \le k \le n}$ in (2.3), the induction hypothesis, and (3.11) of Lemma 5.8 to get:

$$\begin{aligned}\mathbf{E}_{t_i^n}^{x,z}\big[X_l^n - X_i^n\big] &= \mathbf{E}_{t_i^n}^{x,z}\big[X_{l-1}^n - X_i^n\big] + \mu_n + \sigma_n\mathbf{E}_{t_i^n}^{x,z}\big[Z_l^n\big]\\ &= \mu_n(l-i-1) + \sigma_n\left(z\sum_{k=1}^{l-i-1}A_n^k\big(t_i^n, x\big) + \sum_{j=i}^{l-2}\sum_{k=1}^{l-j-1}\mathbf{E}_{t_i^n}^{x,z}\big[B_n^k(t_j^n, X_j^n)\big]\right)\\ &\qquad + \mu_n + \sigma_n\left(zA_n^{l-i}\big(t_i^n, x\big) + \sum_{j=i}^{l-1}\mathbf{E}_{t_i^n}^{x,z}\big[B_n^{l-j}(t_j^n, X_j^n)\big]\right)\end{aligned}$$

$$= \mu_n(l-i) + \sigma_n \left(z \sum_{k=1}^{l-i} A_n^k(t_i^n, x) + \sum_{j=i}^{l-1} \sum_{k=1}^{l-j} \mathbf{E}_{t_i^n}^{x,z} \left[B_n^k(t_j^n, X_j^n) \right] \right),$$

and (3.14) holds for l as well.
Let us now turn to the second equation, and suppose that for some $i \le l \le n-1$ the second conditional moment of $X_l^n - X_i^n$ is given by (3.15). We write again $X_{l+1}^n - X_i^n = \left(\mu_n + \sigma_n Z_{l+1}^n\right) + \left(X_l^n - X_i^n\right)$ to obtain

$$\begin{aligned} \mathbf{E}_{t_i^n}^{x,z}\left[\left(X_{l+1}^n - X_i^n\right)^2\right] = \mathbf{E}_{t_i^n}^{x,z}\left[\left(\mu_n + \sigma_n Z_{l+1}^n\right)^2\right] + 2\mathbf{E}_{t_i^n}^{x,z}\left[\left(\mu_n + \sigma_n Z_{l+1}^n\right)\left(X_l^n - X_i^n\right)\right] \\ + \mathbf{E}_{t_i^n}^{x,z}\left[\left(X_l^n - X_i^n\right)^2\right]. \end{aligned} \tag{3.17}$$

For the first term we immediately get from $\left|Z_{l+1}^n\right| = 1$ and (3.11):

$$\begin{aligned} \mathbf{E}_{t_i^n}^{x,z}\left[\left(\mu_n + \sigma_n Z_{l+1}^n\right)^2\right] &= \mu_n^2 + \sigma_n^2 + 2\mu_n\sigma_n \mathbf{E}_{t_i^n}^{x,z}\left[Z_{l+1}^n\right] \\ &= \mu_n^2 + \sigma_n^2 + 2\mu_n\sigma_n\left(z A_n^{l-i+1}(t_i^n, x) + \sum_{j=i}^{l} \mathbf{E}_{t_i^n}^{x,z}\left[B_n^{l-j+1}(t_j^n, X_j^n)\right]\right). \end{aligned} \tag{3.18}$$

For the second term, we can draw on (3.14) and Lemma 5.9 to conclude

$$\begin{aligned} &\mathbf{E}_{t_i^n}^{x,z}\left[\left(\mu_n + \sigma_n Z_{l+1}^n\right)\left(X_l^n - X_i^n\right)\right] \\ &\quad = \mu_n \left\{ \mu_n(l-i) + \sigma_n \left(z \sum_{k=1}^{l-i} A_n^k(t_i^n, x) + \sum_{j=i}^{l-1} \sum_{k=1}^{l-j} \mathbf{E}_{t_i^n}^{x,z}\left[B_n^k(t_j^n, X_j^n)\right] \right) \right\} \\ &\qquad + \sigma_n \left\{ \sigma_n \sum_{j=i+1}^{l} \mathbf{E}_{t_i^n}^{x,z}\left[A_n^{l-j+1}(t_j^n, X_j^n)\right] + \sum_{j=i+1}^{l} \mathbf{E}_{t_i^n}^{x,z}\left[\left(X_j^n - X_i^n\right) B_n^{l-j+1}(t_j^n, X_j^n)\right] \right. \\ &\qquad\qquad \left. + \mu_n \left(z(l-i) A_n^{l-i+1}(t_i^n, x) + \sum_{j=i}^{l-1} (l-j) \mathbf{E}_{t_i^n}^{x,z}\left[B_n^{l-j+1}(t_j^n, X_j^n)\right] \right) \right\} \\ &\quad = \mu_n^2(l-i) + \sigma_n^2 \sum_{j=i+1}^{l} \mathbf{E}_{t_i^n}^{x,z}\left[A_n^{l-j+1}(t_j^n, X_j^n)\right] + \sigma_n \sum_{j=i+1}^{l} \mathbf{E}_{t_i^n}^{x,z}\left[\left(X_j^n - X_i^n\right) B_n^{l-j+1}(t_j^n, X_j^n)\right] \\ &\qquad + \mu_n\sigma_n \left(z \left((l-i) A_n^{l-i+1}(t_i^n, x) + \sum_{k=1}^{l-i} A_n^k(t_i^n, x) \right) \right. \\ &\qquad\qquad \left. + \sum_{j=i}^{l-1} \sum_{k=1}^{l-j} \mathbf{E}_{t_i^n}^{x,z}\left[B_n^k(t_j^n, X_j^n)\right] + \sum_{j=i}^{l-1} (l-j) \mathbf{E}_{t_i^n}^{x,z}\left[B_n^{l-j+1}(t_j^n, X_j^n)\right] \right). \end{aligned}$$

Now (3.15) with l replaced by $l+1$ follows from substituting in (3.17) the previous equation, (3.18), and the induction hypothesis (3.15), and then collecting terms. Thus, the induction step holds, and therefore (3.15) holds for all $i \le l \le n$. Finally, (3.16) directly follows from

$$\mathbf{Var}_{t_i^n}^{x,z}\left[X_l^n - X_i^n\right] = \mathbf{E}_{t_i^n}^{x,z}\left[\left(X_l^n - X_i^n\right)^2\right] - \left(\mathbf{E}_{t_i^n}^{x,z}\left[X_l^n - X_i^n\right]\right)^2 \tag{3.19}$$

and the two previous equations, (3.14) and (3.15). **q.e.d.**

5.3.2 Approximations for the Auxiliary Functions

Since we are mainly interested in the limiting behavior as $n \to \infty$ of our correlated random walk X^n, respectively of the continuous-time version U^n given by (2.4), we do not spend any more time on simplifying the exact formulæ for the conditional moments of Proposition 5.10, but rather look for approximations of these moments as $n \to \infty$. For that reason we subsequently invoke the conditions of Assumption N and Assumption O to obtain suitable approximations of the auxiliary functions $\{A^k_{n,F}\}_{0\le k\le n}$ and $\{B^k_{n,F}\}_{1\le k\le n}$.
Our first Lemma is a simple application of the definitions of Lipschitz and Hölder continuity, but it proves to be useful in the proofs of some of the following approximations for the two families of auxiliary functions.

Lemma 5.11. *Let us suppose that $F : \mathbb{R} \to \mathbb{R}$ is some bounded function satisfying a global Lipschitz condition, i.e. there exists some $L_F \in \mathbb{R}_+$ such that*

$$\big|F(x) - F(y)\big| \le L_F|x-y| \qquad \text{for all } x, y \in \mathbb{R}. \tag{3.20}$$

Then under Assumption O(i) and (ii) there exists some $K_2 = K_{2,F} \in \mathbb{R}_+$ such that for all $x, y \in \mathbb{R}$, for all $0 \le j \le n-k+1$, and for all functions $f, g : \mathbb{N}_0 \to \mathbb{R}$, which may depend on j and n,

$$\begin{aligned}\left|F(y)\prod_{l=j}^{j+k-1} a\big(t^n_l, y+g(l)\big) - F(x)\prod_{l=j}^{j+k-1} a\big(t^n_l, x+f(l)\big)\right|& \\ \le K_2(k+1)\mathfrak{a}^k\bigg(|y-x| + \sup_{j\le l\le j+k-1}|g(l)-f(l)|\bigg)&.\end{aligned} \tag{3.21}$$

If in addition Assumption O(iii) holds then there also exists some $K_3 \in \mathbb{R}_+$ such that for all $x, y \in \mathbb{R}$ and all $0 \le j \le n-k+1$

$$\begin{aligned}&\left|\sum_{l=j}^{j+k-1} a'\big(t^n_l, y{+}g(l)\big)\prod_{\substack{r=j\\ r\ne l}}^{j+k-1} a\big(t^n_r, y{+}g(r)\big) - \sum_{l=j}^{j+k-1} a'\big(t^n_l, x{+}f(l)\big)\prod_{\substack{r=j\\ r\ne l}}^{j+k-1} a\big(t^n_r, x{+}f(r)\big)\right| \\ &\le K_3(k+1)k\mathfrak{a}^{k-1}\bigg(|y{-}x|^\beta \vee |y{-}x| + \sup_{j\le r\le j+k-1}\Big(|g(r){-}f(r)|^\beta \vee |g(r){-}f(r)|\Big)\bigg).\end{aligned} \tag{3.22}$$

and last but not least we then get for all $x, z \in \mathbb{R}$ and all $0 \le j \le n-k+1$

$$\begin{aligned}\left|\frac{1}{2}\bigg(\prod_{l=j}^{j+k-1} a\big(t^n_l, x{+}z{+}f(l)\big) + \prod_{l=j}^{j+k-1} a\big(t^n_l, x{-}z{+}f(l)\big)\bigg) - \prod_{l=j}^{j+k-1} a\big(t^n_l, x{+}f(l)\big)\right|& \\ \le K_0 k\mathfrak{a}^{k-1}|z|^{1+\beta}&.\end{aligned} \tag{3.23}$$

Proof. Let us fix $x, y \in \mathbb{R}$ and $0 \le j \le n-k+1$. The first part of the Lemma is proved easily. We can use the telescoping sum

$$\begin{aligned}&\prod_{l=j}^{j+k-1} a\big(t^n_l, y+g(l)\big) - \prod_{l=j}^{j+k-1} a\big(t^n_l, x+f(l)\big) \\ &= \sum_{i=j}^{j+k-1}\bigg(\prod_{l=j}^{i} a\big(t^n_l, y+g(l)\big)\prod_{l=i+1}^{j+k-1} a\big(t^n_l, x+f(l)\big) - \prod_{l=j}^{i-1} a\big(t^n_l, y+g(l)\big)\prod_{l=i}^{j+k-1} a\big(t^n_l, x+f(l)\big)\bigg)\end{aligned}$$

and thereafter the triangular inequality to bound

$$
\begin{aligned}
D_1 &:= \left| F(y) \prod_{l=j}^{j+k-1} a\big(t_l^n, y+g(l)\big) - F(x) \prod_{l=j}^{j+k-1} a\big(t_l^n, x+f(l)\big) \right| \\
&\leq |F(y)| \sum_{i=j}^{j+k-1} \prod_{l=j}^{i-1} \big|a\big(t_l^n, y+g(l)\big)\big| \prod_{l=i+1}^{j+k-1} \big|a\big(t_l^n, x+f(l)\big)\big| \Big|a\big(t_i^n, y+g(i)\big) - a\big(t_i^n, x+f(i)\big)\Big| \\
&\quad + \big|F(y) - F(x)\big| \prod_{l=j}^{j+k-1} \big|a\big(t_l^n, x+f(l)\big)\big|.
\end{aligned}
$$

By the boundedness of $a : [0,T] \times I\!R \to I\!R$ and the Lipschitz continuity of $F : I\!R \to I\!R$ the second summand can be bounded by $L_F|y-x|\mathfrak{a}^k$. To bound the first term as well, we note that by Assumption O(ii) we have $\big|a\big(t_i^n, y+g(i)\big) - a\big(t_i^n, x+f(i)\big)\big| \leq \|a'\|\big|y+g(i)-x-f(i)\big|$ for all $j \leq i \leq j+k-1$. The latter term is bounded by $\|a'\|\big(|y-x| + \sup_{j \leq r \leq j+k-1} |g(r)-f(r)|\big)$, hence by the boundedness of $\|F\|$ we overall get

$$
\begin{aligned}
D_1 &\leq L_F|y-x|\mathfrak{a}^k + \|F\| \sum_{i=j}^{j+k-1} \mathfrak{a}^{k-1}\|a'\| \Bigg(|y-x| + \sup_{j \leq r \leq j+k-1} |g(r)-f(r)| \Bigg) \\
&= L_F|y-x|\mathfrak{a}^k + \|F\|\|a'\|k\mathfrak{a}^{k-1} \Bigg(|y-x| + \sup_{j \leq i \leq j+k-1} |g(i)-f(i)| \Bigg).
\end{aligned}
$$

If we now set $K_2 = \max\big\{L_F, \frac{1}{\mathfrak{a}}\|F\|\|a'\|\big\}$, the desired statement (3.21) follows. In order to prove (3.22) we use again a triangular inequality to bound:

$$
\begin{aligned}
D_2 &:= \left| \sum_{l=j}^{j+k-1} a'\big(t_l^n, y+g(l)\big) \prod_{\substack{r=j \\ r\neq l}}^{j+k-1} a\big(t_r^n, y+g(r)\big) - \sum_{l=j}^{j+k-1} a'\big(t_l^n, x+f(l)\big) \prod_{\substack{r=j \\ r\neq l}}^{j+k-1} a\big(t_r^n, x+f(r)\big) \right| \\
&\leq \sum_{l=j}^{j+k-1} \big|a'\big(t_l^n, y+g(l)\big) - a'\big(t_l^n, x+f(l)\big)\big| \prod_{\substack{r=j \\ r\neq l}}^{j+k-1} \big|a\big(t_r^n, y+g(r)\big)\big| \\
&\quad + \sum_{l=j}^{j+k-1} \big|a'\big(t_l^n, x+f(l)\big)\big| \left| \prod_{\substack{r=j \\ r\neq l}}^{j+k-1} a\big(t_r^n, y+g(r)\big) - \prod_{\substack{r=j \\ r\neq l}}^{j+k-1} a\big(t_r^n, x+f(r)\big) \right|.
\end{aligned}
$$

Due to the Hölder continuity of $a' : [0,T] \times I\!R \to I\!R$ we have for all $j \leq l \leq j+k-1$:

$$
\begin{aligned}
&\big|a'\big(t_l^n, y+g(l)\big) - a'\big(t_l^n, x+f(l)\big)\big| \\
&\qquad \leq \big|a'\big(t_l^n, y+g(l)\big) - a'\big(t_l^n, x+g(l)\big)\big| + \big|a'\big(t_l^n, x+g(l)\big) - a'\big(t_l^n, x+f(l)\big)\big| \\
&\qquad \leq K_0|x-y|^\beta + K_0\big|g(l)-f(l)\big|^\beta \leq K_0\Big(|y-x|^\beta + \sup_{j \leq r \leq j+k-1} |g(r)-f(r)|^\beta\Big),
\end{aligned}
$$

and by $\|a\| < \mathfrak{a}$ this gives us a bound on the first sum. Moreover, it is easy to see from the inequality (3.21) that the difference in the second sum can be bounded by the expression $K_{2,1}k\mathfrak{a}^{k-1}\big(|y-x| + \sup_{j \leq r \leq j+k-1} |g(r)-f(r)|\big)$. Thus we get

$$
D_2 \leq \sum_{l=j}^{j+k-1} K_0\Big(|y-x|^\beta + \sup_{j \leq r \leq j+k-1} |g(r)-f(r)|^\beta\Big)\mathfrak{a}^{k-1}
$$

$$+ \sum_{l=j}^{j+k-1} \|a'\| K_{2,1} k \mathfrak{a}^{k-1}\Big(|y-x| + \sup_{j\le r\le j+k-1} |g(r)-f(r)|\Big)$$
$$\le K_0 k \mathfrak{a}^{k-1}\Big(|y-x|^\beta + \sup_{j\le r\le j+k-1} |g(r)-f(r)|^\beta\Big)$$
$$+ \|a'\| K_{2,1} k^2 \mathfrak{a}^{k-1}\bigg(|y-x| + \sup_{j\le r\le j+k-1} |g(r)-f(r)|\bigg).$$

Setting $K_3 = \max\big\{K_0, \|a'\|K_{2,1}\big\}$ gives the desired claim.
Finally, the proof of (3.23) follows as the proof of (3.21) by adding and subtracting the terms $\frac{1}{2}\Big(\prod_{l=j}^{i} a\big(t_l^n, x{+}z{+}f(l)\big) + \prod_{l=j}^{i} a\big(t_l^n, x{-}z{+}f(l)\big)\Big) \prod_{l=i+1}^{j+k-1} a\big(t_l^n, x{+}f(l)\big)$ for all $j \le i \le j+k-2$. Instead of bounding the terms $\big|a(t_i^n, y{+}g(i)) - a(t_i^n, x{+}f(i))\big|$ we now have to bound the terms of the form $\big|\frac{1}{2}\big(a(t_i^n, x{+}z{+}f(l)) + a(t, x{-}z{+}f(l))\big) - a(t_i^n, x{+}f(l))\big|$ for all $j \le i \le j+k-1$. However, by the mean value theorem we find for all $t \in [0,T]$ and $y, z \in I\!R$ some $\theta_1, \theta_2 \in [0,1]$ such that $\big|\frac{1}{2}\big(a(t,y+z) + a(t,y-z)\big) - a(t,y)\big| = \frac{1}{2}|z|\big|a'(t, y+\theta_1 z) - a'(t, y-\theta_2 z)\big|$, and by the Hölder continuity of a' as required by Assumption O(iii) we can bound this term uniformly by $\frac{1}{2}|z|K_0|(\theta_1-\theta_2)z|^\beta \le K_0|z|^{1+\beta}$. Since we do not have the extra F-factor as in the proof of (3.21), we obtain (3.23). **q.e.d.**

Similarly, continuity in the time variable $t \in [0,T]$, implies that we can approximate expressions like the one in the previous lemma by much simpler expressions, which are evaluated only at one point in time. We state here a lemma, which is sufficient for our application in Lemma 5.30.

Lemma 5.12. *Let us assume that $\delta^2\kappa_n = n^{-1}\kappa_n \to 0$ as $n \to \infty$ and $F : [0,T] \times I\!R \to I\!R$ is continuous. Then under the Assumptions O(i), (ii), and (v) for any $R > 0$ and $\varepsilon > 0$ there exists some $N = N_F(R,\varepsilon) \in I\!N$ such that for all $n \ge N$, all $0 \le i \le j \le j+k-1 \le i+\kappa_n \le n$, and all $|x| \le R$ we have*

$$\left| F\big(t_j^n, x\big) \prod_{l=j}^{j+k-1} a\big(t_l^n, x\big) - F\big(t_i^n, x\big) a^k\big(t_i^n, x\big) \right| \le (k+1)\mathfrak{a}^k \varepsilon \tag{3.24}$$

and

$$\left| \sum_{l=j}^{j+k-1} a'\big(t_l^n, x\big) \prod_{\substack{r=j\\ r\ne l}}^{j+k-1} a\big(t_r^n, x\big) - a'\big(t_i^n, x\big) k a_n^{k-1}(i,x) \right| \le (k+1)k\mathfrak{a}^{k-1}\varepsilon. \tag{3.25}$$

Proof. Without loss of generality we may assume $\|F\| > 0$. Let us fix $R > 0$, $\varepsilon > 0$, and set $\hat\varepsilon = \varepsilon \min\big\{\mathfrak{a}\|F\|^{-1}, 1\big\}$. Since a and F are continuous on $[0,T] \times I\!R$, they are uniformly continuous on the compact set $[0,T] \times [-R,R]$, and hence there exists some $\eta = \eta(\hat\varepsilon) > 0$ such that

$$\big|F(s,x) - F(t,x)\big| \le \hat\varepsilon \quad \text{and} \quad \big|a(s,x) - a(t,x)\big| \le \hat\varepsilon \quad \text{for all } |s-t| \le \eta \text{ and } |x| \le R.$$

Since $\delta^2\kappa_n \to 0$, there exists some $N \in I\!N$ such that $\delta^2\kappa_n \le \eta$ for all $n \ge N$. Fix some $n \ge N$. Then it follows from the definition of t_i^n in Definition 5.7 that $\big|t_l^n - t_i^n\big| = \frac{l-i}{n} \le \delta^2\kappa_n \le \eta$ for all $0 \le l-i \le \kappa_n$. With $\hat\varepsilon$ playing the role of $L_F|y-x|$ and $\big\|a'\big\|\big|y-x\big|$, we now obtain by the same means as in the proof of (3.21) in Lemma 5.11 that for all $0 \le i \le j \le j+k-1 \le i+\kappa_n \le n$ we have

$$\left| F\big(t_j^n, x\big) \prod_{l=j}^{j+k-1} a\big(t_l^n, x\big) - F\big(t_i^n, x\big) a^k\big(t_i^n, x\big) \right| \le \hat\varepsilon\mathfrak{a}^k + \|F\|\hat\varepsilon k \mathfrak{a}^{k-1} \le \varepsilon(k+1)\mathfrak{a}^k.$$

In analogy, the proof of (3.25) follows by the means of the proof to (3.22). **q.e.d.**

Now we can start with stating approximations for the auxiliary functions $A^k_{n,F}$ and $B^k_{n,F}$ of Definition 5.7. At first, we give bounds and $O(\delta)$-approximations as $n \to \infty$ for these auxiliary functions, which only take advantage of the boundedness $\|a\| < 1$ and $\|b\| < \infty$ of the limiting functions $a : [0,T] \times I\!R \to I\!R$ and $b : [0,T] \times I\!R \to I\!R$ from Assumption N, and of the fact that $a : [0,T] \times I\!R \to I\!R$ is globally Lipschitz continuous in the space variable. These conditions are sufficient to prove tightness of the sequence of continuous time processes $\{U^n\}_{n\in I\!N}$ in Section 5.3.3. For the sake of clarity, we state the lemma and our tightness result under the slightly stronger conditions implied by Assumption O(ii).

Lemma 5.13. *Let us assume that $F : I\!R \to I\!R$ is uniformly bounded. Then under Assumption O(i) we have for all sufficiently large $n \in I\!N$,*

$$\sup_{(t,x)\in\mathcal{T}^n_k\times I\!R} \big|A^k_{n,F}(t,x)\big| \le \|F\|\mathfrak{a}^k \qquad \textit{for all } 0 \le k \le n-1. \tag{3.26}$$

If in addition Assumption O(ii) holds and $F : I\!R \to I\!R$ satisfies the global Lipschitz condition (3.20)*, then there exist some K_4, $K_5 \in I\!R^+$, which do not depend on k, such that for all sufficiently large $n \in I\!N$ we have for all $0 \le k \le n-1$:*

$$\sup_{0\le j\le n-k,\, |y-x|\le 2\sigma_n} \big|A^k_{n,F}(t,y) - A^k_{n,F}(t,x)\big| \le K_4(k+1)\mathfrak{a}^k\delta \tag{3.27}$$

and

$$\sup_{\substack{0\le j\le n-k\\ x\in I\!R}} \left|A^k_{n,F}\big(t^n_j,x\big) - F\big(x+k\mu_n\big)\prod_{l=j}^{j+k-1} a\big(t^n_l, x+(l-j)\mu_n\big)\right| \le K_5(k+1)k\mathfrak{a}^{k-1}\delta, \tag{3.28}$$

and for all $1 \le k \le n-1$ we also have

$$\sup_{(t,x)\in\mathcal{T}^n_k\times I\!R} \big|B^k_{n,F}\big(t,x\big)\big| \le K_6 k\mathfrak{a}^{k-1}\delta. \tag{3.29}$$

Proof. We prove the first three statements by an induction over k. Let us start with (3.26). Due to (3.4) and Assumption O(i) there exists some $N_1 \in I\!N$ such that $\|a_n\|_{[0,T)\times I\!R} \le \mathfrak{a}$ for all $n \ge N_1$. We show inductively that (3.26) holds for all $n \ge N_1$ as well. For this purpose fix such an $n \ge N_1$. Recalling the definition of $A^0_{n,F}\big(t^n_j,x\big)$ in (3.6) the bound (3.26) trivially holds for $k = 0$, hence our induction can be anchored at $k = 0$. For the induction step let us assume that $\sup_{(t,x)\in\mathcal{T}^n_{k-1}\times I\!R}\big|A^{k-1}_{n,F}(t,x)\big| \le \|F\|\mathfrak{a}^{k-1}$ for some $1 \le k \le n-1$. Using the shorthands $A^\pm = A^{k-1}_{n,F}(t^n_{j+1}, x+\mu_n \pm \sigma_n)$ we then get from the definition of $A^k_{n,F}\big(t^n_j,x\big)$ in (3.7) and the definition of N_1 that for all $0 \le j \le n-k$ and $x \in I\!R$

$$\big|A^k_{n,F}\big(t^n_j,x\big)\big| = \left|\frac{1}{2}\big(A^+ + A^-\big)a_n\big(t^n_j,x\big)\right| \le \frac{1}{2}\big(|A^+| + |A^-|\big)\big|a_n\big(t^n_j,x\big)\big| \le \|F\|\mathfrak{a}^{k-1}\mathfrak{a} = \|F\|\mathfrak{a}^k.$$

This proves (3.26) for all $n \ge N_1$. Let us now turn to the next estimate, and choose some $\hat\sigma > \sigma$ and $\hat K > \|a'\|$. We will then show that (3.27) with $K_4 := 2\hat\sigma \max\big\{L_F, \frac{1}{\mathfrak{a}}\hat K\|F\|\big\}$ holds for all $0 \le k \le n-1$ and all sufficiently large $n \in I\!N$. In order to specify "sufficiently large" we notice that due to Assumption N the exists some $N_3 \in I\!N$ such that $0 \le \sigma_n \le \hat\sigma\delta$ for all $n \ge N_3$. Hence (3.4) and $\big|a(t,y) - a(t,x)\big| \le \big\|a'\big\||y-x| \le 2\big\|a'\big\||\sigma_n|$ for all $t \in [0,T]$ and $x, y \in I\!R$ with $|y-x| \le 2\sigma_n$ imply that there exists some $N_4 \ge \max\big\{N_1, N_3\big\}$ such that

$$\sup_{t\in[0,T],\, |y-x|\le\sigma_n} \big|a_n(t,y) - a_n(t,x)\big| \le 2\hat\sigma\hat K\delta \quad \text{for all } n \ge N_4. \tag{3.30}$$

Then we will show that for all $n \geq N_4$ and $0 \leq k \leq n-1$ the bound (3.27) holds. Therefore, let us fix $n \geq N_4$ and apply again an induction over k. To anchor our induction, we notice that by the definition of $A^0_{n,F}$ and the Lipschitz continuity of F we have

$$\big|A^0_{n,F}(t^n_j, y) - A^0_{n,F}\big(t^n_j, x\big)\big| = \big|F(y) - F(x)\big| \leq L_F|y-x| \leq 2L_F|\sigma_n| \leq 2L_F\hat{\sigma}\delta \leq K_4\delta$$

for all $0 \leq j \leq n$ and $|y-x| \leq 2\sigma_n$, which leads to (3.27) for $k=0$. Now assume that (3.27) holds for some $0 \leq k \leq n-1$ with k replaced by $\tilde{k} = k-1$, and fix $0 \leq j \leq n-k$ and $x, y \in I\!R$ such that $|x-y| \leq 2\sigma_n$. By the definition of $A^k_{n,F}$ in (3.7) we have:

$$A^k_{n,F}\big(t^n_j, y\big) - A^k_{n,F}\big(t^n_j, x\big) = \frac{1}{2}\big(A^+(y) + A^-(y)\big)a_n\big(t^n_j, y\big) - \frac{1}{2}\big(A^+(x) + A^-(x)\big)a_n\big(t^n_j, x\big),$$

where we use the shorthands $A^\pm(z) := A^{k-1}_{n,F}\big(t^n_{j+1}, z + \mu_n \pm \sigma_n\big)$ for $z \in \{x, y\}$. Then an application of the equality $ab - cd = \frac{1}{2}(a-c)(b+d) + \frac{1}{2}(a+c)(b-d)$ for all $a, b, c, d \in I\!R$ gives us

$$\begin{aligned}
&A^k_{n,F}\big(t^n_j, y\big) - A^k_{n,F}\big(t^n_j, x\big) \\
&\quad = \frac{1}{4}\Big(A^+(y) + A^-(y) - A^+(x) - A^-(x)\Big)\Big(a_n\big(t^n_j, y\big) + a_n\big(t^n_j, x\big)\Big) \\
&\qquad + \frac{1}{4}\Big(A^+(y) + A^-(y) + A^+(x) + A^-(x)\Big)\Big(a_n\big(t^n_j, y\big) - a_n\big(t^n_j, x\big)\Big).
\end{aligned} \tag{3.31}$$

Due to the induction hypothesis for $\tilde{k} = k-1$ and the definitions of our shorthands $A^\pm$, we can bound

$$\frac{1}{2}\Big|A^+(y) + A^-(y) - A^+(x) - A^-(x)\Big| \leq \frac{1}{2}\Big(\big|A^+(y) - A^+(x)\big| + \big|A^-(y) - A^-(x)\big|\Big) \leq K_4 k \mathfrak{a}^{k-1}\delta,$$

and because of the first part of this lemma and $N_4 \geq N_1$ we obtain

$$\frac{1}{4}\Big|A^+(y) + A^-(y) + A^+(x) + A^-(x)\Big| \leq \frac{1}{4}\Big(\big|A^+(y)\big| + \big|A^+(x)\big| + \big|A^-(y)\big| + \big|A^-(x)\big|\Big) \leq \|F\|\mathfrak{a}^{k-1}.$$

Hence, inserting absolute values in (3.31) and using the bounds $\|a_n\|_{[0,T)\times I\!R} \leq \mathfrak{a}$ of the first part and (3.30) as well, we get:

$$\big|A^k_{n,F}(t^n_j, y) - A^k_{n,F}\big(t^n_j, x\big)\big| \leq K_4 k\mathfrak{a}^{k-1}\delta\mathfrak{a} + \|F\|\mathfrak{a}^{k-1}2\hat{\sigma}\hat{K}\delta \leq K_4(k+1)\mathfrak{a}^k\delta,$$

where the last inequality stems from the definition $K_4 = 2\hat{\sigma}\max\big\{L_F, \frac{1}{\mathfrak{a}}\hat{K}\|F\|\big\}$. Therefore, the statement (3.27) holds for k as well, and thus for all $k \in I\!N_0$.

The third statement follows similarly to the second one if we set for example $K_5 = \mathfrak{a}\hat{\sigma}K_2$, and choose $N_5 \geq N_0$ such that $\|a_n - a\|_{[0,T)\times I\!R} \leq \delta K_5$ for all $n \geq N_5$. In order to prove it, let us fix $n \geq N_5$. Then, first of all, we have by definition of $A^0_{n,F}$ for all $x \in I\!R$ the equality $\big|A^0_{n,F}\big(t^n_j, x\big) - F(x)\big| = \big|F(x) - F(x)\big| = 0$, which leads to (3.28) for $k=0$. Assume now that (3.28) holds for $\tilde{k} = k-1$ with $1 \leq k \leq n-1$, and fix $0 \leq j \leq n-k$ and $x \in I\!R$. Defining the function $\tilde{F} : I\!R \to I\!R$ by $\tilde{F}(y) = F\big(y + k\mu_n\big)$ for all $y \in I\!R$ and setting $x^\pm = x + \mu_n \pm \sigma_n$ we can use $F\big(x^\pm + (k-1)\mu_n\big) = \tilde{F}(x \pm \sigma_n)$, the induction hypothesis, and (3.21) to conclude:

$$\begin{aligned}
D_\pm &:= \left|A^{k-1}_{n,F}\big(t^n_{j+1}, x^\pm\big) - F\big(x + k\mu_n\big)\prod_{l=j+1}^{j+k-1} a\big(t^n_l, x + (l-j)\mu_n\big)\right| \\
&\leq \left|A^{k-1}_{n,F}\big(t^n_{j+1}, x^\pm\big) - F\big(x^\pm + (k-1)\mu_n\big)\prod_{l=j+1}^{j+k-1} a\big(t^n_l, x^\pm + (l-j-1)\mu_n\big)\right|
\end{aligned}$$

$$
\begin{aligned}
&+\left|\tilde{F}(x\pm\sigma_n)\prod_{l=j+1}^{j+k-1} a\big(t_l^n, x+\sigma_n+(l-j)\mu_n\big)-\tilde{F}(x)\prod_{l=j+1}^{j+k-1} a\big(t_l^n, x+(l-j)\mu_n\big)\right|\\
&\le K_5k(k-1)\mathfrak{a}^{k-2}\delta+K_2k\mathfrak{a}^{k-1}\big(|\sigma_n|+0\big)\\
&\le K_5k(k-1)\mathfrak{a}^{k-2}\delta+\hat{\sigma}K_2k\mathfrak{a}^{k-1}=K_5k^2\mathfrak{a}^{k-2}\delta.
\end{aligned}
$$

Since by (3.26) and by the definition of N_5 we also have $\big|A_{n,F}^{k-1}\big(t_{j+1}^n, x^{\pm}\big)\big|\le\|F\|\mathfrak{a}^{k-1}$ for all $n\ge N_5\ge N_0$, we see that for all $n\ge N_5$

$$
\begin{aligned}
&\left|A_{n,F}^k\big(t_j^n,x\big)-F\left(x+k\mu_n\right)\prod_{l=j}^{j+k-1} a\big(t_l^n, x+(l-j)\mu_n\big)\right|\\
&=\left|\frac{1}{2}\Big(A_{n,F}^{k-1}\big(t_{j+1}^n,x^+\big)+A_{n,F}^{k-1}\big(t_{j+1}^n,x^-\big)\Big)a_n\big(t_j^n,x\big)-F\left(x+k\mu_n\right)\prod_{l=j}^{j+k-1} a\big(t_l^n, x+(l-j)\mu_n\big)\right|\\
&\le\frac{1}{2}\Big|A_{n,F}^{k-1}\big(t_{j+1}^n,x^+\big)+A_{n,F}^{k-1}\big(t_{j+1}^n,x^-\big)\Big|\cdot\Big|a_n\big(t_j^n,x\big)-a\big(t_j^n,x\big)\Big|+\big|a\big(t_j^n,x\big)\big|\frac{1}{2}\left(D_++D_-\right)\\
&\le\mathfrak{a}^{k-1}K_5\delta+\mathfrak{a}K_5k^2\mathfrak{a}^{k-2}\delta=(k+1)k\mathfrak{a}^{k-1}K_5\delta.
\end{aligned}
$$

This proves the induction step, and hence (3.28) for all $0\le k\le n-1$ and $n\ge N_5$. Finally, let us recall from Assumptions N and O(i), and from (3.5), that $\|b\|<\infty$, and set $K_6:=\frac{1}{2}K_4+\|b\|\cdot\|F\|$. Then for all fixed $n\ge N_4$, $1\le k\le n-1$, $0\le j\le n-k$ and $x\in I\!R$ we may once again use our shorthands $A^{\pm}(x)=A_{n,F}^{k-1}\big(t_{j+1}^n, x+\mu_n\pm\sigma_n\big)$ to convince the reader that (3.29) follows from the definition of $B_{n,F}^k\big(t_j^n,x\big)$ in (3.8), $\frac{1}{2}|A^+(x)-A^-(x)|\le\frac{1}{2}K_4k\mathfrak{a}^{k-1}\delta$ by (3.27), and due to $\frac{1}{2}|b_n(t,x)|\big(|A^+(x)|+|A^-(x)|\big)\le\|b\|\cdot\|F\|\mathfrak{a}^{k-1}$. **q.e.d.**

The approximations of $A_n^k\big(t_j^n,x\big)$ and $B_n^k\big(t_j^n,x\big)$ of Lemma 5.13 are good enough to determine the volatility coefficient of the limiting diffusion process, but unfortunately they are not good enough to determine the drift of the diffusion as well. For this reason, we need stronger bounds, and as a preparatory lemma to find such bounds, we provide the reader with an improved version of the approximations for $A_n^k\big(t_j^n,x\big)$.

Lemma 5.14. *Under the Assumptions N and O(i) to (iii) we also have the bound:*

$$
\begin{aligned}
\sup_{\substack{0\le j\le n-k\\ |y-x|\le\sigma_n}}&\left|A_n^k\big(t_j^n,y\big)-\prod_{l=j}^{j+k-1} a\big(t_l^n, x+(l-j)\mu_n\big)\right.\\
&\left.-(y-x)\sum_{l=j}^{j+k-1} a'\big(t_l^n, x+(l-j)\mu_n\big)\prod_{\substack{r=j\\ r\ne l}}^{j+k-1} a\big(t_r^n, x+(r-j)\mu_n\big)\right| \qquad (3.32)\\
&\le K_7(k+2)(k+1)k\mathfrak{a}^{k-1}\delta^{1+\beta}.
\end{aligned}
$$

for all sufficiently large $n\in I\!N$ and all $0\le k\le n-1$.

Proof. Let us set $F\equiv 1$ and fix some $K_2,K_3\in I\!R_+$ as in Lemma 5.11, some $K_4,K_5\in I\!R_+$ as described in Lemma 5.13, and some $\hat{\sigma}>\sigma$. Due to (3.4) there exists some $L\in I\!R^+$ and $N_7\in I\!N$ such that for all $n\ge N_7$ we have $\|a_n-a\|_{[0,T)\times I\!R}\le L\delta^{1+\beta}$, and by Assumption N and Lemma 5.13 we can suppose without loss of generality that $0\le\sigma_n\le\hat{\sigma}\delta\le 1$, and that the bounds (3.26), (3.27), and (3.28) hold for all $0\le k\le n$ and $n\ge N_7$. Finally, recalling the Hölder constant K_0 of Assumption O(iii), we will show that for all $n \geq N_7$ the inequality (3.32) holds for example with

$$K_7 := \hat{\sigma} \max\Big\{\hat{\sigma}^{-1}L, K_0\hat{\sigma}^{\beta}, 2K_3\hat{\sigma}^{\beta}, K_4\big\|a'\big\|, 2\mathfrak{a}^{-1}\big\|a'\big\|K_5\Big\}.$$

Of course, we use again an induction over k. For that reason let us fix $n \geq N_7$. The starting assumption (3.32) for $k = 0$ is easily verified, since $A_n^0(t_j^n, y) = 1$ for all $0 \leq j \leq n$ by the definition in (3.6), $\prod_{l=j}^{j-1} a_n\big(t_l^n, x + (l-j)\mu_n\big) = 1$ by the definition of the empty product, and since for $k = 0$ the sum in (3.32) is also empty. Now assume that (3.32) is true for $k-1$; we want to show that it even holds for k as long as $1 \leq k \leq n-1$. Therefore, let us fix $x, y \in I\!R$ with $|x - y| \leq \sigma_n$ and $0 \leq j \leq n-k$, and set

$$\begin{aligned} D_0 :&= A_n^k(t_j^n, y) - \prod_{l=j}^{j+k-1} a\big(t_l^n, x + (l-j)\mu_n\big) \\ &\quad - (y-x) \sum_{l=j}^{j+k-1} a'\big(t_l^n, x + (l-j)\mu_n\big) \prod_{\substack{r=j\\ r\neq l}}^{j+k-1} a\big(t_r^n, x + (r-j)\mu_n\big). \end{aligned}$$

We separate those terms of $a : [0,T] \times I\!R \to I\!R$ and $a' : [0,T] \times I\!R \to I\!R$ which are evaluated at $\big(t_j^n, x\big)$, and rewrite D_0 in terms of nine differences $D_i^{(\pm)}$ as

$$\begin{aligned} D_0 = D_1 + a\big(t_j^n, x\big)\bigg(\frac{1}{2}\big(D_2^+ + D_2^-\big) + D_3 + (y-x)\frac{1}{2}\big(D_4^+ + D_4^-\big)\bigg) \\ + (y-x)a'\big(t_j^n, x\big)\bigg(\frac{1}{2}\big(D_5^+ + D_5^-\big) + D_6\bigg). \end{aligned} \tag{3.33}$$

Here the appearing differences are with $\tilde{y} = y + \mu_n$ given by

$$D_1 := A_n^k\big(t_j^n, y\big) - \frac{1}{2}\Big(A_n^{k-1}\big(t_{j+1}^n, \tilde{y}+\sigma_n\big) + A_n^{k-1}\big(t_{j+1}^n, \tilde{y}-\sigma_n\big)\Big)\Big(a\big(t_j^n, x\big) + (y-x)a'\big(t_j^n, x\big)\Big),$$

$$\begin{aligned} D_2^{\pm} := A_n^{k-1}\big(t_{j+1}^n, y + \mu_n \pm \sigma_n\big) - \prod_{l=j+1}^{j+k-1} a\big(t_l^n, x \pm \sigma_n + (l-j)\mu_n\big) \\ - (y-x) \sum_{l=j+1}^{j+k-1} a'\big(t_l^n, x \pm \sigma_n + (l-j)\mu_n\big) \prod_{\substack{r=j+1\\ r\neq l}}^{j+k-1} a\big(t_r^n, x \pm \sigma_n + (r-j)\mu_n\big), \end{aligned}$$

$$\begin{aligned} D_3 := \frac{1}{2}\Bigg(\prod_{l=j+1}^{j+k-1} a\big(t_l^n, x + \sigma_n + (l-j)\mu_n\big) + \prod_{l=j+1}^{j+k-1} a\big(t_l^n, x - \sigma_n + (l-j)\mu_n\big)\Bigg) \\ - \prod_{l=j+1}^{j+k-1} a\big(t_l^n, x + (l-j)\mu_n\big), \end{aligned}$$

$$\begin{aligned} D_4^{\pm} := \sum_{l=j+1}^{j+k-1} a'\big(t_l^n, x \pm \sigma_n + (l-j)\mu_n\big) \prod_{\substack{r=j+1\\ r\neq l}}^{j+k-1} a\big(t_r^n, x \pm \sigma_n + (r-j)\mu_n\big) \\ - \sum_{l=j+1}^{j+k-1} a'\big(t_l^n, x + (l-j)\mu_n\big) \prod_{\substack{r=j+1\\ r\neq l}}^{j+k-1} a\big(t_r^n, x + (r-j)\mu_n\big), \end{aligned}$$

$$D_5^{\pm} := A_n^{k-1}\big(t_{j+1}^n, y+\mu_n \pm \sigma_n\big) - A_n^{k-1}\big(t_{j+1}^n, x+\mu_n\big),$$

and

$$D_6 := A_n^{k-1}\big(t_{j+1}^n, x+\mu_n\big) - \prod_{r=j+1}^{j+k-1} a\big(t_r^n, x+(r-j)\mu_n\big).$$

We now bound these differences separately, and start with D_1. Due to the mean value theorem there exists some $\theta \in [0,1]$ such that $a\big(t_j^n, y\big) = a\big(t_j^n, x\big) + (y-x)a'\big(t_j^n, x+\theta(y-x)\big)$, and by the Hölder continuity of a' which has been stated in Assumption O(iii) we obtain $\big|a'\big(t_j^n, x+\xi(y-x)\big) - a'\big(t_j^n, x\big)\big| \le K_0|\theta(y-x)|^\beta \le K_0|y-x|^\beta$. Now we conclude from the inequalities $|y-x| \le \sigma_n \le \hat{\sigma}\delta$ and the definition of the constant K_7 that we can bound $|y-x|\big|a'\big(t_j^n, x+\theta(y-x)\big) - a'\big(t_j^n, x\big)\big| \le K_0\hat{\sigma}^{1+\beta}\delta^{1+\beta} \le K_7\delta^{1+\beta}$. By the definition of $L \le K_7$ and N_7 we also have $\big|a_n\big(t_j^n, y\big) - a\big(t_j^n, y\big)\big| \le K_7\delta^\beta$, hence we get by adding and subtracting $a\big(t_j^n, y\big)$ and $(y-x)a'\big(t_j^n, x+\theta(y-x)\big)$:

$$\begin{aligned}&\big|a_n\big(t_j^n, y\big) - a\big(t_j^n, x\big) - (y-x)a'\big(t_j^n, x\big)\big| \\ &\quad \le \big|a_n\big(t_j^n, y\big) - a\big(t_j^n, y\big)\big| + 0 + |y-x|\big|a'\big(t_j^n, x+\theta(y-x)\big) - a'\big(t_j^n, x\big)\big| \le 2K_7\delta^{1+\beta}.\end{aligned}$$

Additionally, we have by (3.26) that $\big|A_n^{k-1}(t_{j+1}^n, \tilde{y} \pm \sigma_n)\big| \le \mathfrak{a}^{k-1}$, and hence it follows by the definitions of $A_{n,F}^k$ in (3.7) and $\tilde{y} = y + \mu_n$, and by $\frac{1}{2}(k+1) \ge 1$, that

$$\begin{aligned}|D_1| &= \left|\frac{1}{2}\Big(A_n^{k-1}\big(t_{j+1}^n, \tilde{y}+\sigma_n\big) + A_n^{k-1}\big(t_{j+1}^n, \tilde{y}-\sigma_n\big)\Big)\right|\Big|a_n\big(t_j^n, y\big) - a\big(t_j^n, x\big) - (y-x)a'\big(t_j^n, x\big)\Big| \\ &\le \mathfrak{a}^{k-1}2K_7\delta^{1+\beta} \le K_7(k+1)k\mathfrak{a}^{k-1}\delta^{1+\beta}.\end{aligned}$$

In order to bound $\big|D_2^{\pm}\big|$ we can apply the induction hypothesis (3.32) with (k,j,x,y) replaced by $\big(k-1, j+1, x+\mu_n \pm \sigma, y+\mu_n \pm \sigma\big)$ to obtain

$$\big|D_2^{\pm}\big| \le K_7(k+1)k(k-1)\mathfrak{a}^{k-2}\delta^{1+\beta}.$$

Next, a bound on D_3 can be derived by (3.23) of Lemma 5.11, and since the definition of N_7 implies $\sigma_n \le \hat{\sigma}\delta$; we get

$$|D_3| \le K_0(k-1)\mathfrak{a}^{k-2}|\sigma_n|^{1+\beta} \le \hat{\sigma}^{1+\beta}K_0(k-1)\mathfrak{a}^{k-2}\delta^{1+\beta} \le \frac{1}{2}K_7k(k+1)\mathfrak{a}^{k-2}\delta^{1+\beta},$$

using once again $\frac{1}{2}(k+1) \ge 1$ for the last step. Analogously, we see by means of (3.22) in the same lemma, with $g(l) = f(l) = (l-j)\mu_n$ for all $l \in I\!N_0$, and due to $0 \le \sigma_n \le \hat{\sigma}\delta \le 1$ that

$$\big|D_4^{\pm}\big| \le K_3k(k-1)\mathfrak{a}^{k-2}\Big(|\sigma_n|^\beta \vee |\sigma_n| + 0\Big) \le \hat{\sigma}^\beta K_3k(k-1)\mathfrak{a}^{k-2}\delta^\beta \le \frac{1}{2\hat{\sigma}}K_7(k+1)k\mathfrak{a}^{k-2}\delta^\beta.$$

Now we consider the term before last and look for a bound for $D_5^{\pm}$. Because of the inequality $\big|y+\mu_n \pm \sigma_n - (x+\mu_n)\big| \le |y-x| + |\sigma_n| \le 2\sigma_n$, we can apply (3.27), and multiplying $\big|D_5^{\pm}\big|$ by $\big\|a'\big\|$ we get due to $\delta \le 1$ the bound

$$\big\|a'\big\|\big|D_5^{\pm}\big| \le \big\|a'\big\|K_4k\mathfrak{a}^{k-1}\delta \le \frac{1}{2\hat{\sigma}}K_7(k+1)k\mathfrak{a}^{k-1}\delta^\beta,$$

and finally we draw on (3.28) to bound $\big\|a'\big\||D_6|$ by

$$\big\|a'\big\||D_6| \le \big\|a'\big\|K_5k(k-1)\mathfrak{a}^{k-2}\delta \le \frac{1}{2\hat{\sigma}}K_7(k+1)k\mathfrak{a}^{k-1}\delta^\beta.$$

Hence we get from (3.33) and $\|a\| < \mathfrak{a}$ by collecting the above bounds:

$$\begin{aligned}
|D_0| &\leq |D_1| + \|a\| \Big(\frac{1}{2}\Big(\big|D_2^+\big| + \big|D_2^-\big|\Big) + |D_3| + |y-x|\frac{1}{2}\Big(\big|D_4^+\big| + \big|D_4^-\big|\Big)\Big) \\
&\quad + |y-x|\|a'\|\Big(\frac{1}{2}\Big(\big|D_5^+\big| + \big|D_5^-\big|\Big) + |D_6|\Big) \\
&\leq K_7(k+1)k\mathfrak{a}^{k-1}\delta^{1+\beta} + \mathfrak{a}K_7(k+1)k(k-1)\mathfrak{a}^{k-2}\delta^{1+\beta} \\
&\quad + \mathfrak{a}\frac{1}{2}K_7k(k+1)\mathfrak{a}^{k-2}\Big(\delta^{1+\beta} + |y-x|\hat{\sigma}^{-1}\delta^{\beta}\Big) + |y-x|\frac{1}{2\hat{\sigma}}K_7(k+1)k\mathfrak{a}^{k-1}\Big(\delta^{\beta}+\delta^{\beta}\Big).
\end{aligned}$$

If we now recall that $|x-y| \leq \sigma_n$ and N_7 was chosen such that $\sigma_n \leq \hat{\sigma}\delta$ for all $n \geq N_7$, we see that $|x-y|\hat{\sigma}^{-1} \leq \delta$, and we obtain

$$|D_0| \leq K_7\mathfrak{a}^{k-1}\big((k+1)k(k-1) + 3(k+1)k\big)\delta^{1+\beta} = K_7(k+2)(k+1)k\mathfrak{a}^{k-1}\delta^{1+\beta}.$$

This proves the induction step, and hence the assertion. **q.e.d.**

Now it is only a small step to give bounds on $\sigma_n A_n^k$, $\sigma_n^2 A_n^k$, and $\sigma_n B_n^k$ which are sufficiently accurate for our applications.

Lemma 5.15. *Under the Assumptions N and O(i) to (iii) there exist some constants K_8 and K_9 such that for all sufficiently large $n \in I\!N$, all $x, y \in I\!R$, $0 \leq k \leq n-1$, and $r \in \{1,2\}$*

$$\sup_{0\leq j\leq n-k}\left|\sigma_n^r A_n^k\big(t_j^n, y\big) - \delta^r\sigma^r\prod_{l=j}^{j+k-1}a\big(t_l^n, x\big)\right| \leq K_8(k+2)(k+1)\mathfrak{a}^k\delta^{r+\beta} + K_9(k+1)\mathfrak{a}^k\delta^r|y-x|. \quad (3.34)$$

Moreover, if in addition Assumption O(iv) holds there exist some constants K_{10} and K_{11} such that for all sufficiently large $n \in I\!N$ the following bound holds for all $x, y \in I\!R$ and $1 \leq k \leq n-1$:

$$\begin{aligned}
&\sup_{0\leq j\leq n-k}\left|\sigma_n B_n^k\big(t_j^n, y\big) - \delta^2\sigma b\big(t_j^n, x\big)\prod_{l=j+1}^{j+k-1}a\big(t_l^n, x\big) - \delta^2\sigma^2\sum_{l=j+1}^{j+k-1}a'\big(t_l^n, x\big)\prod_{\substack{r=j+1\\ r\neq l}}^{j+k-1}a\big(t_r^n, x\big)\right| \\
&\qquad \leq K_{10}(k+1)k\mathfrak{a}^{k-1}\delta^2\Big(|y-x|^{\beta} + |y-x|\Big) + K_{11}(k+2)(k+1)k\mathfrak{a}^{k-1}\delta^{2+\beta}.
\end{aligned} \quad (3.35)$$

Proof. The bound (3.34) follows easily from Lemma 5.13 and Lemma 5.11. Let us fix $\varepsilon > 0$ and set $F \equiv 1$. By the above two lemmas and Assumption N there exist some $N_8 \in I\!N$ and some $L \in I\!R_+$ such that for all $n \geq N_8$ the bounds (3.26), (3.28), and (3.21) hold, and additionally both $\big|\sigma_n^r - (\sigma\delta)^r\big| \leq L\delta^{r+\beta}$ for $r \in \{1,2\}$ and $|\mu_n| \leq \varepsilon\delta^{\beta}$ for all $n \geq N_8$. Hence, fixing some $n \geq N_8$, $0 \leq k \leq n-1$, and $x, y \in I\!R$, by (3.26) the bound $D_1 := \sup_{0\leq j\leq n-k}\big|A_n^k\big(t_j^n, y\big)\big| \leq \mathfrak{a}^k \leq \frac{1}{2}2(k+1)\mathfrak{a}^k$ holds, and due to (3.28) we bound

$$D_2 := \sup_{0\leq j\leq n-k}\left|A_n^k\big(t_j^n, y\big) - \prod_{l=j}^{j+k-1}a\big(t_l^n, y + (l-j)\mu_n\big)\right| \leq K_5(k+1)k\mathfrak{a}^{k-1}\delta.$$

Last but not least, let us notice that with $g(l) = (l-j)\mu_n$, and $f(l) = 0$ for all $l \in N_0$ we have $|g(l) - f(l)| \leq |l-j||\mu_n| \leq k\varepsilon\delta^{\beta}$ for all $j \leq l \leq j+k-1$. Therefore, we get by (3.21)

$$D_3 := \sup_{0\leq j\leq n-k}\left|\prod_{l=j}^{j+k-1}a\big(t_l^n, y+(l-j)\mu_n\big) - \prod_{l=j}^{j+k-1}a\big(t_l^n, x\big)\right| \leq K_2(k+1)\mathfrak{a}^k\Big(|y-x| + k\varepsilon\delta^{\beta}\Big).$$

If we now add and subtract the terms $\delta\sigma A_n^k\big(t_j^n,y\big)$ and $\delta\sigma\prod_{l=j}^{j+k-1} a\big(t_l^n,y\big)$ within the absolute value on the left-hand side of (3.34), we get for $r \in \{1,2\}$

$$\sup_{0\le j\le n-k}\left|\sigma_n^r A_n^k\big(t_j^n,y\big) - (\delta\sigma)^r \prod_{l=j}^{j+k-1} a\big(t_l^n,x\big)\right| \le \big|\sigma_n^r - (\delta\sigma)^r\big| D_1 + (\delta\sigma)^r\big(D_2 + D_3\big).$$

Employing $\big|\sigma_n^r - (\delta\sigma)^r\big| \le L\delta^{r+\beta}$ and the above bounds on D_1 to D_3, we obtain (3.34) for example with $K_8 := \max\big\{\sigma\big(\varepsilon K_2 + \frac{1}{\mathfrak{a}}K_5\big), \frac{1}{2}L_1\big\}$ and $K_9 := \sigma K_2$.
Now let us turn to (3.35). By Lemma 5.13 and Lemma 5.14 there exist some $K_4, K_5, K_7 > 0$ and $N_9 \in I\!N$ such that with $F \equiv 1$ the bounds (3.27), (3.28), and (3.32) hold for all $n \ge N_9$ and all $0 \le k \le n-1$. Due to Assumptions N and O(i) and (3.5) we can assume without loss of generality that for all $n \ge N_9$ we have $\delta\|b_n\|_{[0,T)\times I\!R} \le \varepsilon\delta^\beta$, for some $0 < \varepsilon \le \min\big\{1, \frac{2K_7}{\mathfrak{a}K_4}, 2\frac{K_7}{K_5}\big\}$, since δb_n is uniformly of order $O(\delta)$ as $n \to \infty$.
Let us now fix $n \ge N_9$, $1 \le k \le n-1$, $0 \le j \le n-k$, and $x, y \in I\!R$, and abbreviate products and sums of the type as they appear in (3.32) and (3.35) by

$$\Pi(z,\mu) := \prod_{l=j+1}^{j+k-1} a\big(t_l^n, z + (l-j)\mu\big)$$

and

$$\Sigma(z,\mu) := \sum_{l=j+1}^{j+k-1} a'\big(t_l^n, z + (l-j)\mu\big) \prod_{\substack{r=j+1\\ r\ne l}}^{j+k-1} a\big(t_r^n, z + (r-j)\mu\big) \quad \text{for all } z,\mu \in I\!R.$$

As a first step to prove (3.35) we will show that

$$\Big|B_n^k\big(t_j^n,y\big) - \delta b_n\big(t_j^n,y\big)\Pi(y,\mu_n) - \sigma_n\Sigma(y,\mu_n)\Big| \le K_7(k+2)(k+1)k\mathfrak{a}^{k-2}\delta^{1+\beta}. \tag{3.36}$$

In order to do so, we once again use the shorthands $A^\pm(y) := A_n^{k-1}(t_{j+1}^n, y + \mu_n \pm \sigma_n)$ and recall from the definition of $B_n^k\big(t_j^n,y\big)$ in (3.8) that

$$B_n^k\big(t_j^n,y\big) = \frac{1}{2}\big(A^+(y) - A^-(y)\big) + \frac{1}{2}\big(A^+(y) + A^-(y)\big)\delta b_n\big(t_j^n,y\big).$$

Hence the left-hand side of (3.36) can be written as $\big|D_1 + \delta b_n\big(t_j^n,y\big)D_2\big|$ where we have set $D_1 := \frac{1}{2}\big(A^+(y) - A^-(y)\big) - \sigma_n\Sigma(y,\mu_n)$ and $D_2 := \frac{1}{2}\big(A^+(y) + A^-(y)\big) - \Pi(y,\mu_n)$. We will bound D_1 and D_2 separately. Let us start with D_1. If we add and subtract $\frac{1}{2}\Pi(y,\mu_n)$ to D_1, and then apply (3.32) with (j,k,y,x) replaced by $(j+1,k-1,y+\mu_n\pm\sigma,y+\mu_n)$ we conclude

$$\begin{aligned}\big|D_1\big| &\le \frac{1}{2}\big|A^+(y) - \Pi(y,\mu_n) - \sigma_n\Sigma(y,\mu_n)\big| + \frac{1}{2}\big|\Pi(y,\mu_n) + (-\sigma_n)\Sigma(y,\mu_n) - A^-(y)\big| \\ &\le K_7(k+1)k(k-1)\mathfrak{a}^{k-2}\delta^{1+\beta}.\end{aligned}$$

In order to bound D_2, we employ (3.27) and (3.28) for $x = y + \mu_n$, and get with the abridged notation $\bar{A}(y) := A_n^{k-1}\big(t_{j+1}^n, y + \mu_n\big)$ and due to $\frac{1}{2}(k+1) \ge 1$ on the one hand the bound $\big|A^\pm(y) - \bar{A}(y)\big| \le K_4 k\mathfrak{a}^{k-1}\delta \le \frac{1}{2}\mathfrak{a}K_4(k+1)k\mathfrak{a}^{k-2}\delta$ and on the other hand also the bound $\big|\bar{A}(y) - \Pi(y,\mu_n)\big| \le K_5 k(k-1)\mathfrak{a}^{k-2}\delta \le K_5(k+1)k\mathfrak{a}^{k-2}\delta$. If we now notice that by the choice of ε we have $\mathfrak{a}K_4 \le \frac{2}{\varepsilon}K_7$ and $K_5 \le \frac{2}{\varepsilon}K_7$, we obtain by adding and subtracting $\bar{A}(y)$:

$$\big|D_2\big| \le \frac{1}{2}\big|A^+(y) - \bar{A}(y)\big| + \frac{1}{2}\big|A^-(y) - \bar{A}(y)\big| + \big|\bar{A} - \Pi(y,\mu_n)\big| \le \frac{3}{\varepsilon}K_7(k+1)k\mathfrak{a}^{k-1}\delta,$$

Since $\left|\delta b_n\big(t_j^n,y\big)\right| \le \varepsilon\delta^\beta$, we arrive at (3.36) by the triangular inequality:

$$\begin{aligned}\left|D_1+\delta b_n\big(t_j^n,y\big)D_2\right| &\le \left|D_1\right|+\left|\delta b_n\big(t_j^n,y\big)\right|\left|D_2\right|\\ &\le K_7(k+1)k(k-1)\mathfrak{a}^{k-2}\delta^{1+\beta}+\varepsilon\delta^\beta\frac{3}{\varepsilon}K_7(k+1)k\mathfrak{a}^{k-2}\delta\\ &= K_7(k+2)(k+1)k\mathfrak{a}^{k-2}\delta^{1+\beta}.\end{aligned}$$

If we multiply (3.36) by σ_n, it basically remains to bound the error, which is involved by replacing the term $\sigma_n b\big(t_j^n,y\big)\Pi(y,\mu_n)$ by $\delta\sigma b\big(t_j^n,y\big)\Pi(x,0)$, and by replacing $\sigma_n^2\Sigma(y,\mu_n)$ by $(\delta\sigma)^2\Sigma(x,0)$. By Assumption N and by (3.5) there exist some constants $\hat\sigma, L_1, L_2 \in I\!R_+$ and some $N_{10}\ge N_9$ such that for all $n\ge N_{10}$ we have

$$\sigma_n\le\hat\sigma\delta,\ \ \left|\sigma_n^2-(\delta\sigma)^2\right|\le L_1\delta^{2+\beta},\ \ \left\|\sigma_nb_n-\delta\sigma b\right\|_{[0,T)\times I\!R}\le L_2\delta^{1+\beta},\ \text{ and } |\mu_n|\le\varepsilon\delta. \quad (3.37)$$

Let us again fix some $n\ge N_{10}$, $1\le k\le n$, $0\le j\le n-k$ and $x,y\in I\!R$, and set $K_{10}:=\sigma^2\max\big\{K_1,\|b\|K_2,\frac{2}{\mathfrak{a}}K_3\big\}$. Combining the definition of L_2 with Assumption O(iv) we see from adding and subtracting $\delta\sigma b(t,y)$ that $\left|\sigma_nb_n(t,y)-\delta\sigma b(t,x)\right|\le L_2\delta^{1+\beta}+\sigma K_1\delta|y-x|^\beta$. Last but not least, let us notice that with $g(l)=(l-j)\mu_n$, and $f(l)=0$ for all $l\in I\!N_0$ we have $|g(l)-f(l)|\le|l-j||\mu_n|\le|k-1|\varepsilon\delta$ for all $j+1\le l\le j+k-1$. Thus, we can bound $D_3:=\sigma_nb\big(t_j^n,y\big)\Pi(y,\mu_n)-\delta\sigma b\big(t_j^n,x\big)\Pi(x,0)$ with the help of (3.21) and the bound $|\Pi(y,\mu_n)|\le\prod_{l=j+1}^{j+k-1}\mathfrak{a}=\mathfrak{a}^{k-1}$, which itself immediately follows from Assumption O(i), by

$$\begin{aligned}|D_3| &\le \left|\sigma_nb\big(t_j^n,y\big)-\delta\sigma b\big(t_j^n,x\big)\right|\left|\Pi(y,\mu_n)\right|+\delta\sigma\left|b\big(t_j^n,x\big)\right|\left|\Pi(y,\mu_n)-\Pi(x,0)\right|\\ &\le \Big(L_2\delta^{1+\beta}+\sigma K_1\delta|y-x|^\beta\Big)\mathfrak{a}^{k-1}+\delta\sigma\|b\|K_2k\mathfrak{a}^{k-1}\big(|y-x|+(k-1)\varepsilon\delta\big)\\ &\le \frac12K_{10}(k+1)k\mathfrak{a}^{k-1}\delta\Big(|y-x|^\beta+|y-x|\Big)+\Big(\frac16L_2+\frac12\varepsilon\sigma\|b\|K_2\Big)(k+2)(k+1)k\mathfrak{a}^{k-1}\delta^{1+\beta},\end{aligned}$$

where we also used $k\ge1$. Since $k\in I\!N$, we have $|k-1|^\beta\le|k-1|$, and on the other hand $\varepsilon\le1$ implies $|\mu_n|\le\varepsilon\delta\le1$ and hence $|\mu_n|\le|\mu_n|^\beta$. Employing the bounds $\left|\sigma_n^2-(\delta\sigma)^2\right|\le L_1\delta^{2+\beta}$, $|\Sigma(y,\mu_n)|\le\sum_{l=j+1}^{j+k-1}\left\|a'\right\|\mathfrak{a}^{k-2}=\left\|a'\right\|(k-1)\mathfrak{a}^{k-2}$, and (3.22) as well, we also find a bound on $D_4:=\sigma_n^2\Sigma(y,\mu_n)-(\delta\sigma)^2\Sigma(x,0)$, namely

$$\begin{aligned}|D_4| &\le \left|\sigma_n^2-(\delta\sigma)^2\right|\left|\Sigma(y,\mu_n)\right|+(\delta\sigma)^2\left|\Sigma(y,\mu_n)-\Sigma(x,0)\right|\\ &\le L_1\delta^{2+\beta}\left\|a'\right\|(k-1)\mathfrak{a}^{k-2}+(\delta\sigma)^2K_3k(k-1)\mathfrak{a}^{k-2}\Big(|y-x|^\beta\vee|y-x|+(k-1)\varepsilon^\beta\delta^\beta\Big)\\ &\le \frac12K_{10}(k+1)k\mathfrak{a}^{k-1}\delta^2\Big(|y-x|^\beta+|y-x|\Big)\\ &\quad+\Big(\frac16L_1\left\|a'\right\|+\sigma^2K_3\varepsilon^\beta\Big)(k+2)(k+1)k\mathfrak{a}^{k-2}\delta^{2+\beta},\end{aligned}$$

where we used $k\ge1$ and $|u|^\beta\vee|u|\le|u|^\beta+|u|$ for all $u\in I\!R$ in the last step. Hence we get by adding and subtracting $\sigma_n\delta b\big(t_j^n,x\big)\Pi(y,\mu_n)$ and $\sigma_n^2\Sigma(y,\mu_n)$

$$\begin{aligned}&\left|\sigma_nB_n^k\big(t_j^n,y\big)-\delta^2\sigma b\big(t_j^n,x\big)\Pi(x,0)-(\delta\sigma)^2\Sigma(x,0)\right|\\ &\quad=\left|\sigma_n\big(D_1+\delta b\big(t_j^n,y\big)D_2\big)+\delta D_3+D_4\right|\le\delta\hat\sigma\left|D_1+\delta b\big(t_j^n,y\big)D_2\right|+\delta|D_3|+|D_4|\\ &\quad\le K_{10}(k+1)k\mathfrak{a}^{k-1}\delta^2\Big(|y-x|^\beta+|y-x|\Big)+K_{11}(k+2)(k+1)k\mathfrak{a}^{k-1}\delta^{2+\beta},\end{aligned}$$

where K_{11} may be chosen as $K_{11}=\frac{\hat\sigma}{\mathfrak{a}}K_7+\frac16L_2+\frac12\varepsilon\sigma\|b\|K_2+\frac{1}{6\mathfrak{a}}\left\|a'\right\|L_1+\frac{1}{\mathfrak{a}}\sigma^2K_3\varepsilon^\beta$. **q.e.d.**

5.3.3 Tightness

Our next aim is to prove that the sequence $\{U^n\}_{n\in I\!N}$ of continuous time correlated random walks given by (2.4) is tight in the Skorohod space $D[0,T]$. This basically reduces to controlling the distribution of the maximum of partial sums of X^n. For the fluctuation of partial sums of a sequence of not necessarily independent or identically distributed random variables, we can invoke general techniques presented in Section 12 in Billingsley (1968). In order to apply these techniques, we first have to find suitable bounds on certain product moments. Such bounds can be derived by combining the formulæ for the conditional moments which we stated in Proposition 5.10 with the bounds on the auxiliary functions of Section 5.3.2. We give here a proposition, which only uses the weak bounds of Lemma 5.13.

Proposition 5.16. *Under the Assumptions N and O(i) and (ii) there exists some constant $M > 0$ such that for all sufficiently large $n \in I\!N$, for all $(x,z) \in I\!R \times \{\pm 1\}$ and all $0 \le i \le k \le l \le n$ with $l - i \le \lfloor (\delta^2 M)^{-1} \rfloor$ we have the following bounds on the conditional moments of the increments $X_l^n - X_i^n$ of the correlated random walk X^n given that it is in x with tilt z at the ith step:*

$$\mathbf{Var}^{x,z}_{t_i^n}\left[X_l^n - X_i^n\right] \le \delta^2 M(l-i), \tag{3.38}$$

$$\mathbf{E}^{x,z}_{t_i^n}\left[\left(X_l^n - X_i^n\right)^2\right] \le \delta^2 M(l-i), \tag{3.39}$$

and

$$\mathbf{E}^{x,z}_{t_i^n}\left[|X_k^n - X_i^n|^2 |X_l^n - X_k^n|^2\right] \le \delta^4 M^2 (l-i)^2. \tag{3.40}$$

Proof. Of course, we have the implications (3.40) $\Rightarrow$ (3.39) $\Rightarrow$ (3.38). But our argument below goes the other way round: We first prove (3.38), use this to prove (3.39), and use the latter plus conditioning to get (3.40).
Let us take some $\hat\sigma > \sigma$ and $\hat\mu > |\mu|$. By Assumption N and Lemma 5.13 there exist some $K \in I\!R_+$ and some $N \in I\!N$ such that for all $n \ge N$ we have $\sigma_n \le \hat\sigma\delta$, $|\mu_n| \le \hat\mu\delta^2$,

$$\left\|A_n^k\right\|_{\mathcal{T}_k^n \times I\!R} \le \mathfrak{a}^k \qquad \text{for all } 0 \le k \le n-1, \text{ and} \tag{3.41}$$

$$\left\|B_n^k\right\|_{\mathcal{T}_k^n \times I\!R} \le Kk\mathfrak{a}^{k-1}\delta \quad \text{for all } 1 \le k \le n-1. \tag{3.42}$$

Now let us fix some $n \ge N$, $(x,z) \in I\!R \times \{\pm 1\}$, and $0 \le i \le n$. In order to show (3.38) we will consider the two summands of expression (3.16) separately. The first summand will be easily bounded by a term $\delta^2 C_1 (l-i)$ for some $C_1 > 0$, without further limitations on $i \le l \le n$; for the second term we have to work harder, and we have to limit the size of $l - i$. But let us start with the first term, and notice that due to (3.26) and $0 \le \sigma_n \le \hat\sigma\delta$ we have for all $i \le l \le n$:

$$\begin{aligned} T_1(l) &:= \sigma_n^2 \Bigg(l - i + 2 \sum_{j=i+1}^{l-1} \sum_{k=1}^{l-j} \mathbf{E}^{x,z}_{t_i^n}\left[A_n^k(t_j^n, X_j^n)\right] \\ &\qquad - \left(z \sum_{k=1}^{l-i} A_n^k(t_i^n, x) + \sum_{j=i}^{l-1} \sum_{k=1}^{l-j} \mathbf{E}^{x,z}_{t_i^n}\left[B_n^k(t_j^n, X_j^n)\right] \right)^2 \Bigg) \\ &\le \sigma_n^2 \left(l-i + 2\sum_{j=i+1}^{l-1} \sum_{k=1}^{l-j} \mathbf{E}^{x,z}_{t_i^n}\Big[\left|A_n^k(t_j^n, X_j^n)\right|\Big] \right) \le \delta^2\hat\sigma^2 \left(l-i+2\sum_{j=i+1}^{l-1}\sum_{k=1}^{l-j} \mathfrak{a}^k \right) \le \delta^2 C_1 (l-i), \end{aligned}$$

where $C_1 := \hat{\sigma}^2\big(1 + 2\sum_{k=1}^{\infty} \mathfrak{a}^k\big) = \hat{\sigma}^2 \frac{1+\mathfrak{a}}{1-\mathfrak{a}}$. If $B_n^k(t,y) = 0$ for all $(t,y) \in \mathcal{T}_k^n \times I\!R$ and $1 \le k \le n-1$, this proves (3.38). However, in general B_n^k will not vanish, and we have to find a similar bound for the remainder term of (3.16), namely for

$$T_2(l) := 2\sigma_n \sum_{j=i+1}^{l-1} \sum_{k=1}^{l-j} \mathbf{E}_{t_i^n}^{x,z}\Big[\Big(X_j^n - X_i^n - (j-i)\mu_n\Big) B_n^k\big(t_j^n, X_j^n\big)\Big] = 2\sigma_n \sum_{j=i+1}^{l-1} \sum_{k=1}^{l-j} E(j,k) \quad (3.43)$$

for all $i \le l \le n$, where we have set $E(j,k) := \mathbf{E}_{t_i^n}^{x,z}\big[\big(X_j^n - X_i^n - (j-i)\mu_n\big) B_n^k\big(t_j^n, X_j^n\big)\big]$ for all $1 \le k \le n-1$ and $i \le j \le n-k$. This is more difficult, since we have to squeeze one factor δ out of each of the expressions $E(j,k)$ and we do not want to use a more complicated approximation of B_n^k than (3.29). For this purpose let us define

$$C_2 := \hat{\sigma} K \sum_{k=1}^{\infty} k \mathfrak{a}^{k-1} = \frac{\hat{\sigma} K}{(1-\mathfrak{a})^2}, \qquad C_3 := \hat{\sigma} \sum_{k=1}^{\infty} \mathfrak{a}^k = \frac{\hat{\sigma}\mathfrak{a}}{1-\mathfrak{a}}, \quad (3.44)$$

and

$$M_1 := C_1 + C_2\bigg(2(1+C_3) + \frac{1}{1+C_3}\bigg). \quad (3.45)$$

Then we are going to show via an induction over l that if we take $M = M_1$, the bound (3.38) holds for all $i \le l \le n$ which satisfy $0 \le l - i \le \lfloor(\delta^2 M_1)^{-1}\rfloor$. To start with, we remark that the statement is trivially true for $l = i$. Assume now that for some $i < l \le n$ with $l - i \le \lfloor(\delta^2 M_1)^{-1}\rfloor$ we have $\mathbf{Var}_{t_i^n}^{x,z}\big[X_j^n - X_i^n\big] \le \delta^2 M_1 (j-i)$ for all $i \le j \le l-1$. Since we have already shown outside the induction that we can bound $T_1(l)$ by $\delta^2 C_1(l-i)$, it remains to show that $|T_2(l)| \le \delta^2 C_2\big(2(1+C_3) + (1+C_3)^{-1}\big)$. In order to accomplish such an upper bound, we add and subtract $\mathbf{E}_{t_i^n}^{x,z}\big[X_j^n - X_i^n\big]\mathbf{E}_{t_i^n}^{x,z}\big[B_n^k\big(t_j^n, X_j^n\big)\big]$ to $E(j,k)$ and split $E(j,k)$ into two summands by writing $E(j,k) = E_1(j,k) + E_2(j,k)$ with

$$E_1(j,k) = \mathbf{E}_{t_i^n}^{x,z}\Big[\Big(X_j^n - X_i^n - \mathbf{E}_{t_i^n}^{x,z}\big[X_j^n - X_i^n\big]\Big) B_n^k\big(t_j^n, X_j^n\big)\Big]$$

and

$$E_2(j,k) = \mathbf{E}_{t_i^n}^{x,z}\big[B_n^k\big(t_j^n, X_j^n\big)\big]\Big(\mathbf{E}_{t_i^n}^{x,z}\big[X_j^n - X_i^n\big] - (j-i)\mu_n\Big)$$

for all $i \le j \le l-1$ and $1 \le k \le n-1$. At first, let us look for a bound on $E_1(j,k)$ and therefore apply (3.42) to get rid of the $B_n^k\big(t_j^n, X_j^n\big)$ term, since it implies for all $i \le j \le l-1$ and all $1 \le k \le n-1$:

$$\mathbf{E}_{t_i^n}^{x,z}\Big[\big|X_j^n - X_i^n - \mathbf{E}_{t_i^n}^{x,z}[X_j^n - X_i^n]\big|\big|B_n^k(t_j^n, X_j^n)\big|\Big] \le K k \mathfrak{a}^{k-1} \delta \mathbf{E}_{t_i^n}^{x,z}\Big[\big|X_j^n - X_i^n - \mathbf{E}_{t_i^n}^{x,z}[X_j^n - X_i^n]\big|\Big].$$

Now the remaining expectation can be bounded by 1, since Lyapunov's inequality and the induction hypothesis lead to the sequence of inequalities

$$\begin{aligned} \mathbf{E}_{t_i^n}^{x,z}\Big[\big|X_j^n - X_i^n - \mathbf{E}_{t_i^n}^{x,z}[X_j^n - X_i^n]\big|\Big] &\le \Big(\mathbf{E}_{t_i^n}^{x,z}\Big[\big(X_j^n - X_i^n - \mathbf{E}_{t_i^n}^{x,z}[X_j^n - X_i^n]\big)^2\Big]\Big)^{\frac{1}{2}} \\ &= \Big(\mathbf{Var}_{t_i^n}^{x,z}\big[X_j^n - X_i^n\big]\Big)^{\frac{1}{2}} \\ &\le \big(\delta^2 M_1 (j-i)\big)^{\frac{1}{2}} \le \big(\delta^2 M_1 \lfloor(\delta^2 M_1)^{-1}\rfloor\big)^{\frac{1}{2}} \le 1, \end{aligned}$$

and hence we can conclude for all $i \le j \le l-1$ and $1 \le k \le n-1$:

$$\big|E_1(j,k)\big| \le \mathbf{E}^{x,z}_{t^n_i}\Big[\big|X^n_j - X^n_i - \mathbf{E}^{x,z}_{t^n_i}[X^n_j - X^n_i]\big|\big|B^k_n(t^n_j, X^n_j)\big|\Big] \le Kk\mathfrak{a}^{k-1}\delta. \tag{3.46}$$

In order to bound $E_2(j,k)$ we note that (3.14), $z \in \{\pm 1\}$, $\sigma_n \le \hat{\sigma}\delta$, the bounds (3.41) and (3.42), and the definitions of C_2 and C_3 in (3.44) imply

$$\begin{aligned}\Big|\mathbf{E}^{x,z}_{t^n_i}\big[X^n_j - X^n_i\big] - (j-i)\mu_n\Big| &\le \sigma_n \sum_{r=1}^{j-i}\big|A^r_n(t^n_i, x)\big| + \sigma_n \sum_{s=i}^{j-1}\sum_{r=1}^{j-s}\mathbf{E}^{x,z}_{t^n_i}\Big[\big|B^r_n(s, X^n_s)\big|\Big] \\ &\le \hat{\sigma}\delta\sum_{r=1}^{j-i}\mathfrak{a}^r + \hat{\sigma}\delta\sum_{s=i}^{j-1}\sum_{r=1}^{j-s}Kr\mathfrak{a}^{r-1}\delta \le \delta C_3 + \delta^2\sum_{s=i}^{l-2}C_2\end{aligned}$$

for all $i \le j \le l-1$ and all $1 \le k \le n-j$, i.e. we have

$$\Big|\mathbf{E}^{x,z}_{t^n_i}[X^n_j - X^n_i] - (j-i)\mu_n\Big| \le \delta C_3 + \delta^2(l-i-1)C_2. \tag{3.47}$$

Now notice that $l-i-1 \le \lfloor(\delta^2 M_1)^{-1}\rfloor \le \frac{1}{\delta^2 M_1} \le \frac{1}{2\delta^2 C_2(1+C_3)}$, and using the bound $\delta \le 1$ for the first term in (3.47), we obtain $\big|\mathbf{E}^{x,z}_{t^n_i}\big[X^n_j - X^n_i\big] - (j-i)\mu_n\big| \le C_3 + \frac{1}{2(1+C_3)}$. Applying once again (3.42) we can write for all $i \le j \le l-1$ and $1 \le k \le n-1$:

$$\big|E_2(j,k)\big| \le \Big|\mathbf{E}^{x,z}_{t^n_i}\big[X^n_j - X^n_i\big] - (j-i)\mu_n\Big|\mathbf{E}^{x,z}_{t^n_i}\Big[\big|B^k_n(t^n_j, X^n_j)\big|\Big] \le \left(C_3 + \frac{1}{2(1+C_3)}\right)Kk\mathfrak{a}^{k-1}\delta.$$

Thus we get from the bounds on $E_1(j,k)$ and $E_2(j,k)$ that for all $1 \le k \le n-1$ we have $\big|E(j,k)\big| \le \big(1 + C_3 + (2(1+C_3))^{-1}\big)Kk\mathfrak{a}^{k-1}\delta$, and due to $\sigma_n \le \hat{\sigma}\delta$ and the definition of C_2 in (3.44) we can bound the remainder term $T_2(l)$ of (3.43) by

$$\big|T_2(l)\big| \le 2\sigma_n \sum_{j=i+1}^{l-1}\sum_{k=1}^{l-j}\big|E(j,k)\big| \le \delta^2 C_2\big(2(1+C_3) + (1+C_3)^{-1}\big)(l-i).$$

Hence, we indeed obtain $\mathbf{Var}^{x,z}_i\big[X^n_l - X^n_i\big] = T_1(l) + T_2(l) \le \delta^2 M_1(l-i)$ by the definition of M_1 in (3.45). Therefore the induction step holds, and we have proved

$$\mathbf{Var}^{x,z}_i\big[X^n_l - X^n_i\big] \le \delta^2 M_1(l-i) \quad \text{for all } i \le l \le n \text{ with } 0 \le l-i \le \big\lfloor(\delta^2 M_1)^{-1}\big\rfloor. \tag{3.48}$$

Now let us set $M_2 := \big(C_3 + M_1^{-\frac{1}{2}}(\hat{\mu} + C_2)\big)^2$ and write $M := M_1 + M_2$. Since $M \ge M_1$ we have $\lfloor(\delta^2 M)^{-1}\rfloor \le \lfloor(\delta^2 M_1)^{-1}\rfloor$ and (3.38) for $0 \le l-i \le \lfloor(\delta^2 M)^{-1}\rfloor$ follows from (3.48).
To prove the second inequality we still keep $n \ge N$, $(x,z) \in I\!R \times \{\pm 1\}$, and $0 \le i \le n$ fixed and note that the inductive proof of the first part has shown that (3.47) holds for all $1 \le l-i \le \lfloor(\delta^2 M_1)^{-1}\rfloor$ and $i \le j \le l-1$. Due to $\lfloor(\delta^2 M)^{-1}\rfloor \le \lfloor(\delta^2 M_1)^{-1}\rfloor$ we especially get for all $1 \le l-i \le \lfloor(\delta^2 M)^{-1}\rfloor$:

$$\begin{aligned}\big|\mathbf{E}^{x,z}_{t^n_i}\big[X^n_l - X^n_i\big]\big| &\le (l-i)|\mu_n| + \big|\mathbf{E}^{x,z}_{t^n_i}\big[X^n_l - X^n_i\big] - (l-i)\mu_n\big| \\ &\le (l-i)|\mu_n| + \delta C_3 + \delta^2(l-i-1)C_2 \le \delta C_3 + \delta^2(l-i)(\hat{\mu} + C_2),\end{aligned} \tag{3.49}$$

since the definition of N and $n \ge N$ imply $|\mu_n| \le \hat{\mu}\delta^2$. As opposed to the bound on $E_2(j,k)$ we now use the bound

$$(l-i) = \sqrt{l-i}\sqrt{l-i} \le \sqrt{l-i}\sqrt{\lfloor(\delta^2 M_1)^{-1}\rfloor} \le \sqrt{l-i}\,\frac{1}{\delta\sqrt{M_1}},$$

and spending an additional factor $\sqrt{l-i} \geq 1$ for the first term on the right-hand side of (3.49), we get due to $\delta \leq 1$:

$$\left|\mathbf{E}^{x,z}_{t^n_i}\left[X^n_l - X^n_i\right]\right| \leq \delta\big(C_3 + M_1^{-\frac{1}{2}}(\hat{\mu} + C_2)\big)\sqrt{l-i} \leq \delta\sqrt{M_2}\sqrt{l-i}.$$

Since this bound trivially holds for $l = i$ as well, we can write for all $0 \leq l-i \leq \lfloor(\delta^2 M)^{-1}\rfloor$:

$$\begin{aligned}\mathbf{E}^{x,z}_{t^n_i}\left[\left(X^n_l - X^n_i\right)^2\right] &= \mathbf{Var}^{x,z}_{t^n_i}\left[X^n_l - X^n_i\right] + \left(\mathbf{E}^{x,z}_{t^n_i}\left[X^n_l - X^n_i\right]\right)^2 \\ &\leq \delta^2 M_1(l-i) + \delta^2 M_2(l-i) = \delta^2 M(l-i).\end{aligned}$$

This proves the second statement. Now (3.40) follows immediately from (3.39) by conditioning on $\sigma\big(X^n_j,\, Z^n_j;\, 0 \leq j \leq k\big)$ and then using the Markov property of $\big\{\big(X^n_j, Z^n_j\big)\big\}_{0\leq j\leq n}$, since a first application of (3.39) with $(\tilde{i}, \tilde{l}) = (k, l)$ and a second one with $(\tilde{i}, \tilde{l}) = (i, k)$ yields

$$\begin{aligned}\mathbf{E}^{x,z}_{t^n_i}\left[\left|X^n_k - X^n_i\right|^2\left|X^n_l - X^n_k\right|^2\right] &= \mathbf{E}^{x,z}_{t^n_i}\left[\left|X^n_k - X^n_i\right|^2\mathbf{E}^{X^n_k, Z^n_k}_{t^n_k}\left[|X^n_l - X^n_k|^2\right]\right] \\ &\leq \mathbf{E}^{x,z}_{t^n_i}\left[\left|X^n_k - X^n_i\right|^2\delta^2 M(l-k)\right] \\ &\leq \delta^4 M^2(l-k)(k-i) \leq \delta^4 M^2(l-i)^2\end{aligned}$$

for all $0 \leq i \leq k \leq l \leq n$ with $l - i \leq \lfloor(\delta^2 M)^{-1}\rfloor$. **q.e.d.**

As announced, we can now apply the techniques of Section 12 in Billingsley to prove tightness of the sequence $\{U^n\}_{n\in\mathbb{N}}$. For this, it suffices to prove the next corollary for $\varepsilon_n = \varepsilon$ for all $n \in \mathbb{N}$. However, in Lemma 5.27 we will need a slightly more general formulation. Thus, we state for the constant $M > 0$ chosen as in Proposition 5.16:

Corollary 5.17. *Suppose that Assumptions N and O(i) and (ii) hold. Then for any $\hat{\sigma} > \sigma$ there exist some $N \in \mathbb{N}$ and some $K \in \mathbb{R}_+$ such that for any $n \in \mathbb{N}$ and $\varepsilon_n > 0$ we have: If $n \geq \max\big\{N, \big(4\frac{\hat{\sigma}}{\varepsilon_n}\big)^2\big\}$, $(x,z) \in \mathbb{R} \times \{\pm 1\}$, $0 \leq i \leq j \leq n$, and $m \leq \min\big\{n-j, \lfloor(\delta^2 M)^{-1}\rfloor\big\}$, then*

$$\mathbf{P}^{x,z}_{t^n_i}\left(\max_{j\leq l\leq j+m}\left|X^n_l - X^n_j\right| \geq \varepsilon_n\right) \leq \delta^4\frac{Km^2}{\varepsilon_n^4}. \tag{3.50}$$

Remark. The condition $m \leq n-j$ guarantees that all appearing X^n_j's are well defined. $\square$

Proof. Let us take some $\hat{\sigma} > \sigma$. By Assumption N there exists some N such that for all $n \geq N$ we have $|\mu_n| + \sigma_n \leq \hat{\sigma}\delta$. Without loss of generality we can assume that for all $n \geq N$ the statement of Proposition 5.16 holds as well.
Let us fix $n \in \mathbb{N}$ and $\varepsilon_n > 0$ such that $n \geq \max\big\{N, \big(4\frac{\hat{\sigma}}{\varepsilon_n}\big)^2\big\}$ holds. Then for any fixed $(x,z) \in \mathbb{R} \times \{\pm 1\}$, $0 \leq i \leq j \leq n$ and $0 \leq m \leq \min\big\{n-j, \lfloor(\delta^2 M)^{-1}\rfloor\big\}$ we want to apply results of Section 12 in Billingsley (1968). In order to align our notation with Billingsley's we set $S_k := X^n_{j+k} - X^n_j$, and note that by the definition of the correlated random walk $\big\{X^n_k\big\}_{0\leq k\leq n}$ in (2.3) we can write $S_k = \sum_{l=1}^k \xi_l$ for $0 \leq k \leq m$, where $\xi_l = \mu_n + \sigma_n Z^n_{j+l}$ for all $1 \leq l \leq m$. Last but not least, as in Billingsley (1968) we define the maxima

$$M_m = \max_{0\leq k\leq m}\big\{|S_k|\big\} \quad \text{and} \quad M'_m = \max_{0\leq k\leq m}\min\big\{|S_k|, |S_m - S_k|\big\}.$$

On page 88 in Billingsley (1968) it is shown that $M_m \leq 3M'_m + \max_{1\leq k\leq m}|\xi_k|$, and since $\delta = n^{-\frac{1}{2}}$ and $n \geq \max\big\{N, \big(4\frac{\hat{\sigma}}{\varepsilon_n}\big)^2\big\}$ imply $|\xi_k| \leq |\mu_n| + \sigma_n \leq \delta\hat{\sigma} \leq \frac{1}{4}\varepsilon_n$ for all $1 \leq k \leq m$, we conclude $M_m \le 3M'_m + \frac{1}{4}\varepsilon_n$. Moreover, $X_l^n - X_j^n = S_{l-j}$ for all $j \le l \le j+m$, and hence we get

$$\mathbf{P}_{t_i^n}^{x,z}\Big(\max_{j\le l\le j+m}|X_l^n - X_j^n| \ge \varepsilon_n\Big) = \mathbf{P}_{t_i^n}^{x,z}\big(M_m \ge \varepsilon_n\big) \\ \le \mathbf{P}_{t_i^n}^{x,z}\Big(3M'_m + \frac{1}{4}\varepsilon_n \ge \varepsilon_n\Big) = \mathbf{P}_{t_i^n}^{x,z}\Big(M'_m \ge \frac{1}{4}\varepsilon_n\Big).$$

Thus, in order to derive (3.50), it suffices to find an appropriate bound on $\mathbf{P}_{t_i^n}^{x,z}\big(M'_m \ge \frac{1}{4}\varepsilon_n\big)$. Seeking to apply Theorem 12.1 in Billingsley (1968), we notice that by our choice of m Proposition 5.16 implies for all $0 \le l_1 \le l_2 \le l_3 \le m$

$$\mathbf{E}_{t_i^n}^{x,z}\Big[\big|S_{l_2} - S_{l_1}\big|^2\big|S_{l_3} - S_{l_2}\big|^2\Big] = \mathbf{E}_{t_i^n}^{x,z}\Big[\mathbf{E}_{t_{j+l_1}^n}^{X_{j+l_1}^n, Z_{j+l_1}^n}\Big[\big|X_{j+l_2}^n - X_{j+l_1}^n\big|^2\big|X_{j+l_3}^n - X_{j+l_2}^n\big|^2\Big]\Big] \\ \le \delta^4 M^2 (l_3 - l_1)^2 = \Big(\sum_{l=l_1+1}^{l_3} u_l\Big)^2,$$

where $u_l = \delta^2 M$ for all $1 \le l \le m$, and hence an application of Billingsley's Theorem 12.1 yields

$$\mathbf{P}_{t_i^n}^{x,z}\Big(M'_m \ge \frac{1}{4}\varepsilon_n\Big) \le K_{2,1}\Big(\frac{4}{\varepsilon_n}\Big)^4\Big(\sum_{l=1}^{m} u_l\Big)^2 = K_{2,1}\Big(\frac{4}{\varepsilon_n}\Big)^4 m^2\delta^4 M^2,$$

where $K_{2,1}$ may be taken as $K_{2,1} = 4\big(2^{1/5} - 1\big)^{-5} \approx 55021$. This leads to (3.50) with $K = 4^4 K_{2,1} M^2$. **q.e.d.**

Let us now recall from the definition in (2.4) the continuous-time versions $U^n = \{U^n(t)\}_{t\in[0,T]}$ and $V^n = \{V^n(t)\}_{t\in[0,T]}$ of the correlated random walk $X^n = X^n_{0\le k\le\lceil nT\rceil}$ and the associated tilt process $Z^n = Z^n_{0\le k\le\lceil nT\rceil}$, respectively. In order to conclude the proof of tightness of the sequence $\{U^n\}_{n\in\mathbb{N}}$ in the Skorohod space $D[0,T]$, we recall the modulus of continuity:

Definition 5.18. For $s \in [0,T]$ the **modulus of continuity** of a function $f : [s,T] \to \mathbb{R}$ is defined by

$$w_s(f,\eta) = \sup_{t_1,t_2\in[s,T],\,|t_2-t_1|<\eta} |f(t_2) - f(t_1)| \quad \text{for all } \eta > 0. \tag{3.51}$$

Now standard methods yield the tightness of $\{U^n\}_{n\in\mathbb{N}}$. We single out the essential step and prove it also for conditioned processes, which start in a certain point $u \in \mathbb{R}$ with tilt $z \in \{\pm1\}$ at a certain time $s \in (0,T]$, since we are going to use this general form in Lemma 5.31.

Proposition 5.19. *Under the Assumptions N and O(i) and (ii), for any $\rho > 0$ and $\varepsilon > 0$ there exists some $\eta > 0$ such that $\mathbf{P}_s^{u,z}\big(w_s(X^n,\eta) \ge \rho\big) \le \varepsilon$ for all sufficiently large n and all $(s,u,z) \in [0,T] \times \mathbb{R} \times \{\pm1\}$.*

Proof. Let us fix $(s,u,z) \in [0,T] \times \mathbb{R} \times \{\pm1\}$. For any $\eta > 0$ let us consider the time points $\tau_i(\eta) := (s + i\eta) \wedge T$ for all $0 \le i \le r(\eta) := \min\{k \in \mathbb{N} : s + k\eta \ge T\}$, which divide the interval $[s,T]$ into the $r(\eta) - 1$ intervals $[\tau_{i-1}(\eta), \tau_i(\eta)]$ for $1 \le i \le r(\eta) - 1$ of length η and one possibly shorter residual interval $\big[\tau_{r(\eta)-1}(\eta), \tau_{r(\eta)}\big]$. Then by the corollary to Theorem 8.3 in Billingsley (1968) we have

$$\mathbf{P}_s^{u,z}\big(w_s(U^n,\eta) \ge 3\rho\big) \le \sum_{i=1}^{r(\eta)} \mathbf{P}_s^{u,z}\Big(\sup_{\tau_{i-1}(\eta)\le t\le\tau_i(\eta)} \big|U^n(t) - U^n\big(\tau_{i-1}(\eta)\big)\big| \ge \rho\Big). \tag{3.52}$$

In order to bound $\mathbf{P}_s^{u,z}\big(\sup_{\tau_{i-1}(\eta)\le t\le\tau_i(\eta)}\big|U^n(t)-U^n\big(\tau_{i-1}(\eta)\big)\big|\ge\rho\big)$ for all sufficiently large n and sufficiently small $\eta>0$, we take some $M>0$ for which the statement of Proposition 5.16 holds, fix $\rho>0$, and choose some $\hat\sigma>\sigma$ and some N as in Corollary 5.17. Without loss of generality we may take $N\ge 2M$. Then we have for all $0<\eta\le\frac{1}{2M}$, $1\le i\le r(\eta)$, and $n\ge N$:

$$\big\lfloor n\tau_i(\eta)\big\rfloor-\big\lfloor n\tau_{i-1}(\eta)\big\rfloor<(s+i\eta)n-\big(s+(i-1)\eta\big)n+1=n\eta+1\le\frac{n}{2M}+\frac{n}{2M}\le\frac{n}{M}.$$

Since the left-hand side is an integer, we even can conclude $\big\lfloor n\tau_i(\eta)\big\rfloor-\big\lfloor n\tau_{i-1}(\eta)\big\rfloor\le\big\lfloor\frac{n}{M}\big\rfloor$. Due to the definition of U^n in (2.4) we have

$$\sup_{\tau_{i-1}(\eta)\le t\le\tau_i(\eta)}\big|U^n(t)-U^n\big(\tau_i(\eta)\big)\big|=\max_{\lfloor n\tau_{i-1}(\eta)\rfloor\le l\le\lfloor n\tau_i(\eta)\rfloor}\big|X_l^n-X_{\lfloor n\tau_i(\eta)\rfloor}^n\big|,$$

and thus it follows by Corollary 5.17 that there exists some $K\in\mathbb{R}_+$, which does not depend on (s,u,z), such that for all $n\ge\max\big\{N,\big(4\frac{\hat\sigma}{\rho}\big)^2\big\}$, $0<\eta\le\frac{1}{2M}$, and all $1\le i\le r(\eta)$ we have

$$\begin{aligned}\mathbf{P}_s^{u,z}\bigg(\sup_{\tau_{i-1}(\eta)\le t\le\tau_i(\eta)}\big|U^n(t)-U^n\big(\tau_i(\eta)\big)\big|\ge\rho\bigg)&\le\delta^4\frac{K}{\rho^4}\Big(\big\lfloor n\tau_i(\eta)\big\rfloor-\big\lfloor n\tau_{i-1}(\eta)\big\rfloor\Big)^2\\&\le\delta^4\frac{K}{\rho^4}(n\eta+1)^2=\frac{K}{\rho^4}\big(\eta+\delta^2\big)^2,\end{aligned}$$

since $n\delta^2=1$. Hence we get from (3.52) for all $0<\eta\le\frac{1}{2M}$:

$$\mathbf{P}_s^{u,z}\big(w_s(U^n,\eta)\ge 3\rho\big)\le\frac{K}{\rho^4}\sum_{i=1}^{r(\eta)}\big(\eta+\delta^2\big)^2=\frac{K}{\rho^4}r(\eta)\big(\eta+\delta^2\big)^2\to\frac{K}{\rho^4}r(\eta)\eta^2\quad\text{as } n\to\infty.$$

Since $(s,u,z)\in[0,T]\times\mathbb{R}\times\{\pm1\}$ were chosen arbitrarily and since the definition of $r(\eta)$ implies $0\le r(\eta)<\frac{T-s}{\eta}+1$, we conclude

$$\lim_{\eta\searrow 0}\limsup_{n\to\infty}\sup_{(s,u,z)\in[0,T]\times\mathbb{R}\times\{\pm1\}}\mathbf{P}_s^{u,z}\big(w_s(X^n,\eta)\ge\rho\big)=0.$$

This gives us the assertion. **q.e.d.**

Now the tightness of $\{U^n\}_{n\in\mathbb{N}}$ in $D[0,T]$ follows straightforwardly:

Corollary 5.20. *In addition to Assumptions N and O(i) and (ii), suppose that* $U^n(0)\Rightarrow U(0)$ *as* $n\to\infty$. *Then the sequence* $\{U^n\}_{n\in\mathbb{N}}$ *is tight on* $D[0,T]$ *and every weak limit* U *is continuous a.s.*

Proof. Since $U^n(0)\Rightarrow U(0)$, the sequence of distributions $\big\{\mathbf{P}\big(U^n(0)\big)^{-1}\big\}_{n\in\mathbb{N}}$ on $\mathbb{R}$ is relatively compact, and since $\mathbb{R}$ is separable and complete, it follows from Prohorov's theorem that $\{U^n(0)\}_{n\in\mathbb{N}}$ is tight on $\mathbb{R}$, and hence for each $\eta>0$ there exists some $a>0$ such that $\mathbf{P}\big(|U^n(0)|>a\big)\le\eta$ for all $n\ge1$. Due to Proposition 5.19 we can now apply Theorem 15.5 in Billingsley (1968) and conclude that the sequence $\{U^n\}_{n\in\mathbb{N}}$ is tight on $D[0,T]$, and that every $U\in D[0,T]$ which is the weak limit of some subsequence $\big\{U^{n_k}\big\}_{k\in\mathbb{N}}$ must be continuous a.s. **q.e.d.**

5.3.4 Convergence of the Conditional Local Moments

With tightness having been demonstrated it remains to identify the limits of converging subsequences of $\{U^n\}_{n\in\mathbb{N}}$. In the standard results for the convergence of binomial models (as in Nelson and Ramaswamy (1990)) or more general Markov chains (as in Stroock and Varadhan (1979) or Nelson (1990)) it is assumed that the $O(\delta^2)$-local conditional moments converge, when time is scaled by order $O(\delta^2)$ as well, and these limits will then be identified as the instantaneous drift and the instantaneous variance of the limiting diffusion process. However, for the correlated random walk the corresponding $O(\delta^2)$-local drift $\lim_{n\to\infty}\delta^{-2}\mathbf{E}_t^{x,z}\big[U^n\big(t+\delta^2\big)-U^n(t)\big]$, given that the process U^n is at time t in $U^n(t)=x$ with tilt $V^n(t)=z$ will in general not exist, as the next lemma shows:

Lemma 5.21. *Suppose that Assumption N holds, and that the limit $a:[0,T]\times\mathbb{R}\to\mathbb{R}$ is continuous. Then for all $(t,x,z)\in[0,T)\times\mathbb{R}\times\{\pm1\}$ we have:*

$$\lim_{n\to\infty}\delta^{-2}\mathbf{E}_t^{x,z}\big[U^n\big(t+\delta^2\big)-U^n(t)\big]\quad \textit{exists if and only if}\quad a(t,x)=0.$$

Proof. Let us fix $(t,x,z)\in[0,T)\times\mathbb{R}\times\{\pm1\}$ and write $t_-^n=t_{\lfloor nt\rfloor}^n$. Recall from the end of Section 5.2 that we have assumed for simplicity $T=1$. Due to the definition of U^n in (2.4) and the definition of X^n in (2.3) we then have $U^n(t+\delta^2)-U^n(t)=X^n_{\lfloor nt\rfloor+1}-X^n_{\lfloor nt\rfloor}=\mu_n+\sigma_n Z^n_{\lfloor nt\rfloor+1}$ for all sufficiently large $n\in\mathbb{N}$, such that $\lfloor nt\rfloor+1\le n$. Moreover, since $U^n(t)=X^n_{\lfloor nt\rfloor}$ and $V^n(t)=Z^n_{\lfloor nt\rfloor}$, we see that the conditional expectations $\mathbf{E}_t^{x,z}\big[U^n\big(t+\delta^2\big)-U^n(t)\big]$ and $\mathbf{E}_{t_-^n}^{x,z}\big[U^n\big(t+\delta^2\big)-U^n(t)\big]$ coincide due the definition of the conditional expectation $\mathbf{E}_t^{x,z}$ in (3.1), and recalling that the continuous-time transition function $p_n:[0,T)\times\mathbb{R}\times\{\pm1\}\to\mathbb{R}$ was defined in terms of the discrete one given by (2.1), we can calculate

$$\mathbf{E}_t^{x,z}\big[U^n\big(t+\delta^2\big)-U^n(t)\big]=\mu_n+\sigma_n\mathbf{E}_{t_-^n}^{x,z}\Big[Z^n_{\lfloor nt\rfloor+1}\Big]=\mu_n+\sigma_n\big(2p_n(t_-^n,x,z)-1\big).$$

Thus, under Assumption N we get

$$\mathbf{E}_t^{x,z}\big[U^n\big(t+\delta^2\big)-U^n(t)\big]=\mu\delta^2+z\sigma a(t,x)\delta+\sigma b(t,x)\delta^2+O\big(\delta^{2+\beta}\big)\quad\text{as } n\to\infty,$$

and the statement of the lemma follows by dividing both sides by δ^2. **q.e.d.**

Remark 1. As opposed to the $O\big(\delta^2\big)$-local drift, it is easy to see that under Assumption N the $O\big(\delta^2\big)$-local variance $\lim_{n\to\infty}\delta^{-2}\mathbf{E}_t^{x,z}\big[(U^n(t+\delta^2)-U^n(t))^2\big]$ exists, and it is for all $(t,x,z)\in[0,T)\times\mathbb{R}\times\{\pm1\}$ given by σ^2. However, we have already discussed in the reasoning after the statement of Theorem 5.4 that the influence of the tilt on the transition probabilities for the discrete random walks should affect the variance and the drift of the limiting diffusion process U, and indeed, we will see that the $O\big(\delta^2\big)$-local variance does in general not coincide with the variance of any limit process U. □

Remark 2. Let us fix $t\in[0,T)$. In their general setting for the convergence of Markov processes Stroock and Varadhan (1979) and Nelson (1990) do not require the existence of the $O\big(\delta^2\big)$-local conditional moments in the form we introduced them either; they only require that truncated versions exist. Their generalization is of no use for our problem, since it is easily seen that for our binomial process U^n for fixed $t\in[0,T)$ and sufficiently large $n\in\mathbb{N}$ their truncation will not change U^n at all. □

Due to Lemma 5.21 we have to spend considerably more time than in the standard models on identifying the coefficients of the limiting diffusion process, though a large part of our calculations has already been carried out in the lemmas of Section 5.3.2. The basic idea behind our approach is quite simple: We do not look at the conditional local moments of

increments of U^n over time intervals with length of order $O(\delta^2)$, which would be sufficient for a non-correlated random walk, but we consider the conditional moments of increments over time intervals with length of order $O(\delta)$. Over intervals of this length the influence of the tilt levels out such that the $O(\delta)$-local drift $\lim_{n\to\infty} \delta^{-1}\mathbf{E}_t^{x,z}\left[U^n(t+\delta)-U^n(t)\right]$ and the $O(\delta)$-local variance $\lim_{n\to\infty} \delta^{-1}\mathbf{E}_t^{x,z}\left[(U^n(t+\delta)-U^n(t))^2\right]$ exist for all $(t,x,z) \in [0,T) \times I\!R \times \{\pm 1\}$.
In order to get precise, we have to introduce some additional notation, which permits to decompose the time interval $[0,T]$ in intervals of length $O(\delta)$, as the definition of the time points $t_k^n = kT\lceil nT\rceil^{-1}$ for $0 \le k \le \lceil nT\rceil$ allowed us to split $[0,T]$ in intervals of length $O(\delta^2)$:

Definition 5.22. For all $n \in I\!N$ and all $i \in I\!N_0$ the **time points on the** $O(\delta)$**-scale** are given by $\tau_i^n := \delta^2\lfloor i\sqrt{n}\rfloor$. For each $n \in I\!N$ and all $t \in [0,T]$ we then define $j_t^n := \sup\{i \in I\!N_0 \mid \tau_i^n \le t\}$ and $i_t^n := \inf\{i \in I\!N_0 \mid \tau_i^n \ge t\}$.

Remark. Recall our general restriction to the case $T = 1$. For a general $T > 0$, we would set $\tau_i^n := T\lceil nT\rceil^{-1}\lfloor i\sqrt{n}\rfloor$ for all $n \in I\!N$ and $i \in I\!N_0$. This is the reason why we do not simply define $\tau_i^n = i\delta$: Our definition assures that each time point $\tau_i^n \in [0,T]$ on the $O(\delta)$-scale is also a time point on the $O(\delta^2)$-scale, namely by taking $k(i) = \lfloor i\sqrt{n}\rfloor$ we have $\tau_i^n = t_{k(i)}^n$ for all $0 \le i \le j_T^n$ and $n \in I\!N$.
Moreover, at first sight, our notation of j_t^n and i_t^n is counterintuitive, since $j_t^n \le i_t^n$ for all $t \in [0,T]$. However, in the following, we will evaluate both expressions at different values of t: For fixed $0 \le s < t \le T$ we will see that for all sufficiently large $n \in I\!N$ we have $i_s^n \le j_t^n$. For those $n \in I\!N$ we then consider all integers between i_s^n and j_t^n. In order to achieve the canonical ordering of i and j in such a situation, we have to accept $j_t^n \le i_t^n$. □

In order to illustrate the two different partitions of $[0,T]$ which are induced by the $O(\delta^2)$-time points $\{t_k^n\}_{0\le k\le\lceil nT\rceil}$ and by the $O(\delta)$-time points $\{\tau_i^n\}_{0\le i\le j_T^n}$ we provide an example with $n = 40$ in Figure 5.1. Only the scaling on the time axis employs our general restriction to the case $T = 1$.

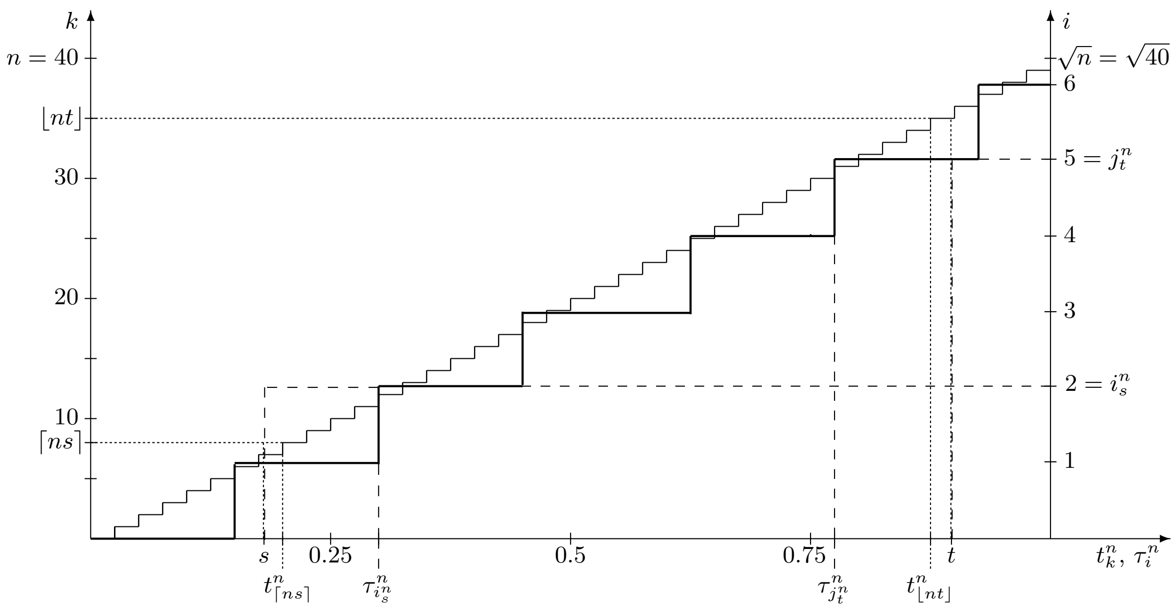

Figure 5.1: The definitions of τ_i^n, i_s^n and j_t^n

For each integer-valued $0 \le k \le n$ on the left-hand side, the flight of stairs with the small steps gives us the associated value of t_k^n on the horizontal axis, and similarly, starting with

some $i \in \mathbb{N}_0$ with $0 \le i \le \sqrt{n}$ on the right-hand axis, the flight of stairs with the large steps gives the value of τ_i^n on the horizontal axis. In addition, the figure explains in an illustrative way how to derive $t^n_{\lfloor nt \rfloor}$ and $\tau^n_{i_t^n}$: If we are given some $t \in [0,T]$ and look for the largest integer $0 \le k \le n$ with $t_k^n \le t$, we start with t on the x-axis, draw a vertical line up to the small-sized stairs, and then turn left to read off the corresponding k-value, which is of course given by $\lfloor nt \rfloor$. Evaluating $t^n_\cdot$ at $\lfloor nt \rfloor$ gives us the value $t^n_{\lfloor nt \rfloor} \le t$. In order to determine the largest integer $0 \le i \le j_T^n$ for which $\tau_i^n \le t$, we once again start at t, but now follow the horizontal up to the stairs with the large steps, and then turn right. This gives us the value j_t^n. Starting with this point we now arrive at $\tau^n_{j_t^n} \le t$. Since the range of values of $\{\tau_i^n\}_{0 \le i \le j_T^n}$ is contained in $\{t_k^n\}_{0 \le k \le n}$, we even have $\tau^n_{j_t^n} \le t^n_{\lfloor nt \rfloor} \le t$ for all $0 \le t \le T$.
Similarly, we can start with some $s \in [0,T]$ to derive $s \le t^n_{\lceil ns \rceil} \le \tau^n_{i_s^n}$. In this case, starting with s on the horizontal line, we have to move on one step beyond the point where the vertical line through s crosses the small or large stairs and then move to the left or right to read off $\lceil ns \rceil$ or i_s^n, respectively.
By the definition of $\lfloor \cdot \rfloor$ we get $x - (y+1) < \lfloor x \rfloor - \lfloor y \rfloor < x - (y-1)$, and since the difference $\lfloor x \rfloor - \lfloor y \rfloor$ is of course again integer-valued, even the stronger inequalities $\lfloor x - y \rfloor \le \lfloor x \rfloor - \lfloor y \rfloor$ and $\lfloor x \rfloor - \lfloor y \rfloor \le \lceil x - y \rceil$ hold for all $x, y \in \mathbb{R}$. Setting $x = (i+1)\sqrt{n}$ and $y = i\sqrt{n}$, and then multiplying the two inequalities by $\delta^2 = n^{-1}$ we obtain

$$\delta^2 \lfloor \sqrt{n} \rfloor \le \tau_{i+1}^n - \tau_i^n \le \delta^2 \lceil \sqrt{n} \rceil, \quad \text{for all } i \in \mathbb{N}_0 \text{ and } n \in \mathbb{N}. \tag{3.53}$$

This shows that the length of the intervals $[\tau_i^n, \tau_{i+1}^n]$ is indeed approximately δ. In our example, the interval lengths of the intervals $[\tau_i^n, \tau_{i+1}^n]$ with $0 \le i \le j_T^n - 1 = 5$ are given by $\delta^2 \lfloor \sqrt{n} \rfloor = 0.15$ for $i \ne 3$ and $\delta^2 \lceil \sqrt{n} \rceil = 0.175$ for $i = 3$.
Since our identification of the limiting distribution(s) of the converging subsequences of U^n only relies on the fact that $0 < \liminf_{i \in \mathbb{N}_0} \delta^{-1}(\tau_{i+1}^n - \tau_i^n)$ and $\limsup_{i \in \mathbb{N}_0} \delta^{-1}(\tau_{i+1}^n - \tau_i^n) < \infty$, we may derive from (3.53) the rather crude bounds

$$\frac{1}{4}\delta < \tau_{i+1}^n - \tau_i^n < 2\delta \quad \text{for all } i \in \mathbb{N}_0 \text{ and all } n \in \mathbb{N}. \tag{3.54}$$

For the lower bound note that it obviously holds for $n = 1$, and for $n \ge 2$ it follows from $\delta^2 \lfloor \sqrt{n} \rfloor > \delta^2(\sqrt{n} - 1) = \delta(1 - \delta) \ge \delta(1 - 2^{-\frac{1}{2}}) > \frac{1}{4}\delta$. The upper bound follows similarly from (3.53) and $1 \le \sqrt{n}$ for all $n \in \mathbb{N}$. Since $\tau^n_{j_t^n} \le t \le \tau^n_{i_t^n}$ and $\tau^n_{i_t^n - 1} < t < \tau^n_{j_t^n + 1}$ by Definition 5.22 we also obtain from the right-hand bound in (3.54) that

$$t - 2\delta < \tau^n_{j_t^n} \le t \le \tau^n_{i_t^n} < t + 2\delta \quad \text{for all } t \in [0,T] \text{ and } n \in \mathbb{N}. \tag{3.55}$$

As a preparation to find approximations for the conditional local moments on the $O(\delta)$-time scale as $n \to \infty$, we derive a lemma which helps us to bound increments of the continuous-time random walk during time intervals with length of order $O(\delta)$ as $n \to \infty$. In this formulation it will be used in Lemma 5.33. In order to be able to cite it directly in the proofs of Lemma 5.25 and Lemma 5.26, where we use it frequently, we provide a second, slightly weaker formulation.

Lemma 5.23. *Under the Assumptions N and O(i)-(ii) we have*

$$\sup_{(u,v) \in \mathbb{R} \times \{\pm 1\}} \sup_{\substack{0 \le s \le t \le T \\ t - s \le K\delta}} \mathbf{E}_s^{u,v}\Big[\big|U^n(t) - U^n(s)\big|^\alpha\Big] = O\Big(\delta^{\frac{1}{2}\alpha}\Big) \quad \text{as } n \to \infty \tag{3.56}$$

for all fixed $K \in \mathbb{R}_+$ and all $\alpha \in [0,2]$. Especially, we have for all $\alpha \in [\beta, 2]$:

$$\sup_{(i,j,x,z) \in I^n} \mathbf{E}^{x,z}_{\tau_i^n}\Big[\big|X_j^n - x\big|^\alpha\Big] = O\Big(\delta^{\frac{1}{2}\beta}\Big) \quad \text{as } n \to \infty, \tag{3.57}$$

where $I^n = \big\{(i,j,x,z) \in \mathbb{N}_0 \times \mathbb{N}_0 \times \mathbb{R} \times \{\pm 1\} \,\big|\, 0 \le i \le j_T^n - 1 \text{ and } n\tau_i^n \le j \le n\tau_{i+1}^n - 1\big\}$.

Proof. Let us start with (3.56), fix $K \in I\!R_+$ and $\alpha \in [0,2]$, and define $s_-^n := t_{\lfloor ns \rfloor}^n$ for all $s \in [0,T]$ and $n \in I\!N$. By the definition of t_k^n we have $s_-^n = \delta^2 \lfloor ns \rfloor \le s$. Then the definition of U^n in (2.4) implies for all $n \in I\!N$ and $0 \le s \le T$ that $U^n(s) = X_{\lfloor ns \rfloor}^n = X_{ns_-^n}^n$ and $V^n(s) = V_{ns_-^n}^n$, hence $\big(U^n(s), V^n(s)\big)$ can be calculated from $\big(X_{ns_-^n}^n, V_{ns_-^n}^n\big)$ and vice versa. Thus, by the definition of the conditional expectations in (3.2) and (3.1) we have for all $0 \le s \le t \le T$ and $(u,v) \in I\!R \times \{\pm 1\}$ the equality

$$\mathbf{E}_s^{u,v}\Big[\big|U^n(t) - U^n(s)\big|^\alpha\Big] = \mathbf{E}_{s_-^n}^{u,v}\Big[\big|X_{nt_-^n}^n - X_{ns_-^n}^n\big|^\alpha\Big].$$

If we now apply Lyapunov's inequality we get

$$\mathbf{E}_s^{u,v}\Big[\big|U^n(t) - U^n(s)\big|^\alpha\Big] \le \Big(\mathbf{E}_{s_-^n}^{u,v}\Big[\big|X_{nt_-^n}^n - X_{ns_-^n}^n\big|^2\Big]\Big)^{\frac{\alpha}{2}}. \tag{3.58}$$

Attempting to apply the bound (3.39) of Proposition 5.16 for the second conditional moment of increments of X^n we choose $M > 0$ such that for all sufficiently large $n \in I\!N$ the assertion of Proposition 5.16 holds. Since $\liminf_{n\to\infty} \delta \lfloor (\delta^2 M)^{-1} \rfloor = \infty$, there exists some $N \in I\!N$ such that for all $n \ge N$ both $\delta^{-1}(K+1) \le \lfloor (\delta^2 M)^{-1} \rfloor$ and the bound (3.39) hold, i.e.

$$\mathbf{E}_{t_i^n}^{u,v}\Big[\big(X_l^n - X_i^n\big)^2\Big] \le \delta^2 M(l-i) \quad \text{for all } 0 \le i \le k \le l \le n \text{ with } l - i \le \big\lfloor (\delta^2 M)^{-1} \big\rfloor \tag{3.59}$$

and all $(u,v) \in I\!R \times \{\pm 1\}$. Let us now take $n \ge N$, $(u,v) \in I\!R \times \{\pm 1\}$, and $0 \le s \le t \le T$ with $t - s \le K\delta$. Due to the definitions of s_-^n and N, and due to $n = \delta^{-2} \ge 1$ we have $nt_-^n - ns_-^n = \lfloor nt \rfloor - \lfloor ns \rfloor \le n(t-s) + 1 \le nK\delta + 1 \le \delta^{-1}(K+1) \le \big\lfloor (\delta^2 M)^{-1} \big\rfloor$, hence we can indeed apply (3.59) with $l = nt_-^n$ and $i = ns_-^n$ to conclude from (3.58)

$$\mathbf{E}_s^{u,v}\Big[\big|U^n(t) - U^n(s)\big|^\alpha\Big] \le \big(\delta^2 M(nt_-^n - ns_-^n)\big)^{\frac{\alpha}{2}} \le \big(\delta^2 M \delta^{-1}(K+1)\big)^{\frac{\alpha}{2}} = \big(M(K+1)\big)^{\frac{\alpha}{2}} \delta^{\frac{\alpha}{2}}.$$

Since this bound does not depend on $(u,v) \in I\!R \times \{\pm 1\}$, but only on $t - s \le K\delta$, this proves the first statement. For the second statement we recall again the definition of U^n in (2.4) and remark that the definition of the conditional probability $\mathbf{P}_{\cdot}^{x,z}$ in (3.2) and the equality $t_{n\tau_i^n}^n = \tau_i^n$ imply that under $\mathbf{P}_{\tau_i^n}^{x,z}$ we have $U^n(\tau_i^n) = X_{n\tau_i^n}^n = x$ for all $n \in I\!N$ and all $(i,j,x,z) \in I^n$. Moreover by dividing $n\tau_i^n \le j \le n\tau_{i+1}^n - 1$ by n and using the definition of t_j^n we obtain $\tau_i^n \le t_j^n \le \tau_{i+1}^n - \frac{1}{n}$, and due to (3.53) it follows that $t_j^n - \tau_i^n \le \tau_{i+1}^n - \tau_i^n - \frac{1}{n} \le \delta$. Hence we can indeed apply (3.56) with $K = 1$ to conclude that uniformly for all $(i,j,x,z) \in I_n$

$$\mathbf{E}_{\tau_i^n}^{x,z}\Big[\big|X_j^n - x\big|^\alpha\Big] = \mathbf{E}_{\tau_i^n}^{x,z}\Big[\big|U^n(t_j^n) - U^n(\tau_i^n)\big|^\alpha\Big] = O\Big(\delta^{\frac{1}{2}\alpha}\Big) \quad \text{as } n \to \infty$$

for all $\alpha \in [0,2]$. If $\alpha \in [\beta, 2]$ this immediately leads to the statement (3.57). **q.e.d.**

Instead of calculating the limits as $n \to \infty$ of the first two conditional local moments $\delta^{-1}\mathbf{E}_t^{u,v}\big[U^n(t+\delta) - U^n(t)\big]$ and $\delta^{-1}\mathbf{E}_t^{u,v}\big[(U^n(t+\delta) - U^n(t))^2\big]$ on the $O(\delta)$-scale for general $(t,u,v) \in [0,T) \times I\!R \times \{\pm 1\}$, it suffices to consider the asymptotic behavior as $n \to \infty$ of the corresponding conditional $O(\delta)$-local moments $\big(\tau_{i+1}^n - \tau_i^n\big)^{-1}\mathbf{E}_t^{u,v}\big[U^n(\tau_{i+1}^n) - U^n(\tau_i^n)\big]$ and $\big(\tau_{i+1}^n - \tau_i^n\big)^{-1}\mathbf{E}_t^{u,v}\big[(U^n(\tau_{i+1}^n) - U^n(\tau_i^n))^2\big]$. At first, we derive a pair of approximations for these $O(\delta)$-local moments, which hold uniformly for all $0 \le i \le j_T^n - 1$, without assuming any continuity in time of the limiting functions $a : [0,T] \times I\!R \to I\!R$ and $b : [0,T] \times I\!R \to I\!R$ of Assumption N. Therefore, the approximations are still rather complicated. In Section 5.3.5 we will employ the continuity in the time variable to replace the approximations of this section by the coefficients of the claimed limiting diffusion process U given in (2.8).

Now let us define for each $n \in I\!N$ the three functions which we will need in order to approximate the first two conditional moments:

Definition 5.24. For each $n \in I\!N$ let us introduce the set $\widetilde{\mathcal{T}}_n := \{\tau_i^n : 0 \le i \le j_T^n - 1\}$. Then we define for all $n \in I\!N$ the $O(\delta)$**-local drift approximation function** $\tilde{\mu}_n : \widetilde{\mathcal{T}}_n \times I\!R \to I\!R$ by setting for all $(\tau_i^n, x) \in \widetilde{\mathcal{T}}_n \times I\!R$

$$\tilde{\mu}_n(\tau_i^n, x) := \mu + \frac{\sigma\delta^2}{\tau_{i+1}^n - \tau_i^n} \sum_{j=n\tau_i^n}^{n\tau_{i+1}^n - 1} \sum_{k=1}^{n\tau_{i+1}^n - j} \left(b(t_j^n, x) \prod_{l=j+1}^{j+k-1} a(t_l^n, x) + \sum_{l=j+1}^{j+k-1} \sigma a'(t_l^n, x) \prod_{\substack{r=j+1 \\ r \ne l}}^{j+k-1} a(t_r^n, x) \right).$$

Moreover, we define the $O(\delta)$**-local drift correction function** $\tilde{d}_n : \widetilde{\mathcal{T}}_n \times I\!R \to I\!R$ by

$$\tilde{d}_n(\tau_i^n, x) := \frac{\sigma\delta}{\tau_{i+1}^n - \tau_i^n} \sum_{k=1}^{n(\tau_{i+1}^n - \tau_i^n)} \prod_{l=n\tau_i^n}^{n\tau_i^n + k - 1} a(t_l^n, x) \quad \text{for all } (\tau_i^n, x) \in \widetilde{\mathcal{T}}_n \times I\!R,$$

and the $O(\delta)$**-local volatility approximation function** $\tilde{\sigma}_n : \widetilde{\mathcal{T}}_n \times I\!R \to I\!R_+$ by

$$\tilde{\sigma}_n^2(\tau_i^n, x) := \sigma^2 + \frac{2\sigma^2\delta^2}{\tau_{i+1}^n - \tau_i^n} \sum_{j=n\tau_i^n + 1}^{n\tau_{i+1}^n - 1} \sum_{k=1}^{n\tau_{i+1}^n - j} \prod_{l=j}^{j+k-1} a(t_l^n, x) \quad \text{for all } (\tau_i^n, x) \in \widetilde{\mathcal{T}}_n \times I\!R.$$

Remark. Note that the three expressions simplify considerably if the transition function $p_n : [0, T) \times I\!R \times \{\pm 1\} \to [0, 1]$ of (2.5) does not depend on the tilt. In this case U^n is a simple binomial process as considered by Nelson and Ramaswamy (1990), and Assumption N implies $a \equiv 0$. Thus for all $n \in I\!N$ the $O(\delta)$-local drift approximation simplifies to $\tilde{\mu}_n(\tau_i^n, x) = \mu + \frac{\sigma\delta^2}{\tau_{i+1}^n - \tau_i^n} \sum_{j=n\tau_i^n}^{n\tau_{i+1}^n - 1} b(\tau_j^n, x)$ for all $(\tau_i^n, x) \in \widetilde{\mathcal{T}}_n \times I\!R$, the $O(\delta)$-local drift correction function $\tilde{d}_n \equiv 0$ vanishes, and the $O(\delta)$-local volatility approximation functions becomes constant, namely $\tilde{\sigma}_n^2 \equiv \sigma^2$. If in addition the limiting function $b : [0, T] \times I\!R \to I\!R$ is constant, then the $O(\delta)$-local drift approximation function $\tilde{\mu}_n$ is constant as well and for all $n \in I\!N$ given by $\tilde{\mu}_n \equiv \mu + \sigma b$, since $n\delta^2 = 1$.
For a non-vanishing $a : [0, T] \times I\!R \to I\!R$ the following Lemma 5.25 will show that for any $0 \le i \le j_T^n - 1$ some of the highest order terms in the approximation of the conditional local first moment of U^n over the time interval $[\tau_i^n, \tau_{i+1}^n]$ for $n \to \infty$ do not only depend on the position $U^n(\tau_i^n)$ of the random walk U^n at time τ_i^n, but also on the associated initial tilt $V^n(\tau_i^n)$. Depending on the sign of the tilt we have to add or subtract the local drift correction term $\tilde{d}_n(\tau_i^n, x)$ to the local drift approximation term $\tilde{\mu}_n(\tau_i^n, x)$ to obtain a suitable approximation of the conditional local first moment. We will however see in Lemma 5.28 that the influence of the initial tilt on such an $O(\delta)$-subinterval $[\tau_i^n, \tau_{i+1}^n] \subset [0, T]$ diminishes over a longer time horizon. □

After all the preparatory work and definitions, we are finally ready to state the first approximation for the term $(\tau_{i+1}^n - \tau_i^n)^{-1} \mathbf{E}_{\tau_i^n}^{x,z} [U^n(\tau_{i+1}^n) - U^n(\tau_i^n)]$ as $n \to \infty$, uniformly for all $0 \le i \le j_T^n - 1$:

Lemma 5.25. *Under the Assumptions N and O(i)-(iii) the normalized conditional expectation of the increment of U^n over the time interval $[\tau_i^n, \tau_{i+1}^n]$, given that U^n is at time τ_i^n in $U^n(\tau_i^n) = x$ with tilt $V^n(\tau_i^n) = z$, and normalized by the length of the time interval, can be approximated in terms of $\tilde{\mu}_n(\tau_i^n, x)$ and $\tilde{d}_n(\tau_i^n, x)$ as follows:*

$$\sup_{0 \le i \le j_T^n - 1} \left| \mathbf{E}_{\tau_i^n}^{x,z} \left[\frac{U^n(\tau_{i+1}^n) - U^n(\tau_i^n)}{\tau_{i+1}^n - \tau_i^n} \right] - \tilde{\mu}_n(\tau_i^n, x) - z\tilde{d}_n(\tau_i^n, x) \right| = O\left(\delta^{\frac{1}{2}\beta}\right) \quad \text{as } n \to \infty,$$

uniformly for all $(x, z) \in I\!R \times \{\pm 1\}$.

Proof. Let us fix $(x,z) \in I\!R \times \{\pm 1\}$, $n \in I\!N$, and $0 \le i \le j_T^n - 1$. Since $t^n_{n\tau_i^n} = \tau_i^n$ we get from the definition of U^n in (2.4) and from the formula for the conditional expectation for the increments of X^n in (3.14)

$$\begin{aligned}\mathbf{E}^{x,z}_{\tau_i^n}\big[U^n(\tau_{i+1}^n) - U^n(\tau_i^n)\big] &= \mathbf{E}^{x,z}_{\tau_i^n}\Big[X^n_{n\tau_{i+1}^n} - X^n_{n\tau_i^n}\Big] \\ &= \mu_n\big(n\tau_{i+1}^n - n\tau_i^n\big) + \sigma_n z \sum_{k=1}^{n\tau_{i+1}^n - n\tau_i^n} A_n^k\big(\tau_i^n, x\big) + \sigma_n \sum_{j=n\tau_i^n}^{n\tau_{i+1}^n - 1} \sum_{k=1}^{n\tau_{i+1}^n - j} \mathbf{E}^{x,z}_{\tau_i^n}\Big[B_n^k\big(t_j^n, X_j^n\big)\Big].\end{aligned}$$

Dividing this expression by $\tau_{i+1}^n - \tau_i^n$ and subtracting $\tilde{\mu}_n\big(\tau_i^n, x\big)$ and $z\tilde{d}_n\big(\tau_i^n, x\big)$ we get a representation of the form

$$\mathbf{E}^{x,z}_{\tau_i^n}\left[\frac{U^n(\tau_{i+1}^n) - U^n(\tau_i^n)}{\tau_{i+1}^n - \tau_i^n}\right] - \tilde{\mu}_n\big(\tau_i^n, x\big) - z\tilde{d}_n\big(\tau_i^n, x\big) = D_1^n + \frac{zD_2^n(i,x) + D_3^n(i,x)}{\tau_{i+1}^n - \tau_i^n}.$$

Here, for fixed $z \in \{\pm 1\}$, the differences D_1^n and $D_2^n(i,x)$ are for all $n \in I\!N$, $0 \le i \le j_T^n - 1$, and $x \in I\!R$ given by $D_1^n := \mu_n n - \mu$ and

$$D_2^n(i,x) := \sum_{k=1}^{n(\tau_{i+1}^n - \tau_i^n)} \left(\sigma_n A_n^k\big(\tau_i^n, x\big) - \delta\sigma \prod_{l=n\tau_i^n}^{n\tau_i^n + k - 1} a\big(t_l^n, x\big)\right), \tag{3.60}$$

and with us defining for all $n \in I\!N$, $0 \le i \le j_T^n - 1$, $n\tau_i^n \le j \le n\tau_{i+1}^n - 1$, $1 \le k \le n\tau_{i+1}^n - j$, and $x \in I\!R$ the expectations

$$E_i^n(j,k,x) := \mathbf{E}^{x,z}_{\tau_i^n}\left[\sigma_n B_n^k\big(t_j^n, X_j^n\big) - \delta^2\sigma b\big(t_j^n, x\big) \prod_{l=j+1}^{j+k-1} a\big(t_l^n, x\big) - \delta^2\sigma^2 \sum_{l=j+1}^{j+k-1} a'\big(t_l^n, x\big) \prod_{\substack{r=j+1 \\ r \ne l}}^{j+k-1} a\big(t_r^n, x\big)\right]$$

the difference $D_3^n(i,x)$ is for all $n \in I\!N$, $0 \le i \le j_T^n - 1$, and $x \in I\!R$ given as the double sum

$$D_3^n(i,x) := \sum_{j=n\tau_i^n}^{n\tau_{i+1}^n - 1} \sum_{k=1}^{n\tau_{i+1}^n - j} E_i^n(j,k,x). \tag{3.61}$$

Due to Assumption N it is obvious that $D_1^n = \mu_n n - \mu = O\big(\delta^\beta\big)$ as $n \to \infty$. Moreover, by (3.54) we have $\tau_{i+1}^n - \tau_i^n \ge \frac{1}{4}\delta$ for all $n \in I\!N$ and all $0 \le i \le j_T^n - 1$, hence it suffices to show that $\sup_{0 \le i \le j_T^n - 1}\big|D_2^n(i,x)\big| = O\big(\delta^{1+\frac{1}{2}\beta}\big)$ and $\sup_{0 \le i \le j_T^n - 1}\big|D_3^n(i,x)\big| = O\big(\delta^{1+\frac{1}{2}\beta}\big)$ uniformly for all $x \in I\!R$. Let us start with a bound on $\big|D_2^n(i,x)\big|$. By Lemma 5.15 there exists some $K_8 \in I\!R_+$ such that for all sufficiently large $n \in I\!N$ the bound (3.34) holds with $x = y$ and $r = 1$. For such $n \in I\!N$ we have for all $0 \le i \le j_T^n - 1$ and all $x \in I\!R$

$$\begin{aligned}\big|D_2^n(i,x)\big| &\le \sum_{k=1}^{n(\tau_{i+1}^n - \tau_i^n)} \left|\sigma_n A_n^k\big(\tau_i^n, x\big) - \delta\sigma \prod_{l=n\tau_i^n}^{n\tau_i^n + k - 1} a_n\big(t_l^n, x\big)\right| \\ &\le K_8 \sum_{k=1}^{n(\tau_{i+1}^n - \tau_i^n)} (k+2)(k+1)\mathfrak{a}^k \delta^{1+\beta} \le \frac{2K_8}{(1-\mathfrak{a})^3}\delta^{1+\beta},\end{aligned}$$

since $\sum_{k=0}^\infty (k+2)(k+1)\mathfrak{a}^k = \frac{d^2}{d\mathfrak{a}^2}\sum_{k=0}^\infty \mathfrak{a}^k = \frac{d^2}{d\mathfrak{a}^2}\frac{1}{1-\mathfrak{a}} = \frac{2}{(1-\mathfrak{a})^3}$. Hence we even have $\sup_{0 \le i \le j_T^n - 1}\big|D_2^n(i,x)\big| = O\big(\delta^{1+\beta}\big)$ as $n \to \infty$, uniformly for all $x \in I\!R$, and due to $\beta > 0$ this implies that the expression is of order $O\big(\delta^{1+\frac{1}{2}\beta}\big)$ as well.

We are now going to show that $\left|D_3^n(i,x)\right| = O\left(\delta^{1+\frac{1}{2}\beta}\right)$ as well. By a twofold application of (3.57) in Lemma 5.23 there exists some $M>0$ such that for all sufficiently large $n \in I\!N$ we have

$$\sup_{(i,j,x,z)\in I^n} \mathbf{E}^{x,z}_{\tau_i^n}\Big[\left|X_j^n - x\right|^\beta + \left|X_j^n - x\right|\Big] \le M\delta^{\frac{1}{2}\beta}, \tag{3.62}$$

with the set I^n given in Lemma 5.23. Fix $n \ge N$ sufficiently large, such that for some constants K_{10} and K_{11} as in Lemma 5.15 the bound (3.35) holds for all $1 \le k \le n$ and such that the bound (3.62) is in effect. Then taking the absolute value inside the expectation we can bound the term $\left|E_i^n(j,k,x)\right|$ by means of (3.35) and (3.62) for all $0 \le i \le j_T^n - 1$, $n\tau_i^n \le j \le n\tau_{i+1}^n - 1$, $1 \le k \le n\tau_{i+1}^n - j$ and $x \in I\!R$ by

$$\begin{aligned}\left|E_i^n(j,k,x)\right| &\le K_{10}(k{+}1)k\mathfrak{a}^{k-1}\delta^2\mathbf{E}^{x,z}_{\tau_i^n}\Big[\left|X_j^n{-}x\right|^\beta + \left|X_j^n{-}x\right|\Big] + K_{11}(k{+}2)(k{+}1)k\mathfrak{a}^{k-1}\delta^{2+\beta}\\ &\le MK_{10}(k{+}1)k\mathfrak{a}^{k-1}\delta^{2+\frac{1}{2}\beta} + K_{11}(k{+}2)(k{+}1)k\mathfrak{a}^{k-1}\delta^{2+\beta}.\end{aligned}$$

Hence it follows from $\sum_{k=1}^\infty (k+1)k\mathfrak{a}^{k-1} = \frac{2}{(1-\mathfrak{a})^3}$ and $\sum_{k=1}^\infty (k+2)(k+1)k\mathfrak{a}^{k-1} = \frac{6}{(1-\mathfrak{a})^4}$ that

$$\sum_{k=1}^{n\tau_{i+1}^n - j} \left|E_i^n(j,k,x)\right| \le \frac{2MK_{10}}{(1-\mathfrak{a})^3}\delta^{2+\frac{1}{2}\beta} + \frac{6K_{11}}{(1-\mathfrak{a})^4}\delta^{2+\beta} \quad \text{for all } n\tau_i^n \le j \le n\tau_{i+1}^n - 1.$$

Finally, if we take the sum over j from $n\tau_i^n$ to $n\tau_{i+1}^n - 1$ on both sides, then we obtain $n\tau_{i+1}^n - n\tau_i^n - 1 \le \delta^{-1}$ times the bound on the right hand side, and thus we get for all $x \in I\!R$ and $0 \le i \le j_T^n - 1$:

$$\left|D_3^n(i,x)\right| \le \sum_{j=n\tau_i^n}^{n\tau_{i+1}^n-1}\ \sum_{k=1}^{n\tau_{i+1}^n-j} \left|E_i^n(j,k,x)\right| \le \frac{2MK_{10}}{(1-\mathfrak{a})^3}\delta^{1+\frac{1}{2}\beta} + \frac{6K_{11}}{(1-\mathfrak{a})^4}\delta^{1+\beta}.$$

Since the bound on the right-hand side does not depend on i or x, we can indeed conclude that $\sup_{0\le i\le j_T^n-1}\left|D_3^n(i,x)\right| = O\left(\delta^{1+\frac{1}{2}\beta}\right)$ as $n \to \infty$, uniformly for all $x \in I\!R$. Thus, the lemma's assertion is proved. **q.e.d.**

A similar result to Lemma 5.25 can be derived for the normalized conditional second moments of the increments of U^n, namely for $\left(\tau_{i+1}^n - \tau_i^n\right)^{-1}\mathbf{E}^{x,z}_{\tau_i^n}\left[(U^n(\tau_{i+1}^n) - U^n(\tau_i^n))^2\right]$, uniformly for all $0 \le i \le j_T^n - 1$. However, the second moments converge more nicely than the first ones, in the sense that the $O\left(\delta^{\frac{1}{2}\beta}\right)$-approximation does not depend on the tilt z at time τ_i^n.

Lemma 5.26. *Under the Assumptions N and O(i)-(iv) the normalized second conditional moments of the increments of U^n over the time interval $\left[\tau_i^n, \tau_{i+1}^n\right]$, given that the correlated random walk U^n is at time τ_i^n in $U^n(\tau_i^n) = x$ with tilt $V^n(\tau_i^n) = z$, can be approximated in terms of $\tilde\sigma_n^2(\tau_i^n, x)$ as follows:*

$$\sup_{0\le i\le j_T^n-1}\left|\mathbf{E}^{x,z}_{\tau_i^n}\left[\frac{\left(U^n(\tau_{i+1}^n) - U^n(\tau_i^n)\right)^2}{\tau_{i+1}^n - \tau_i^n}\right] - \tilde\sigma_n^2\left(\tau_i^n, x\right)\right| = O\left(\delta^{\frac{1}{2}\beta}\right) \quad \text{as } n \to \infty, \tag{3.63}$$

uniformly for all $(x,z) \in I\!R \times \{\pm 1\}$.

Proof. Basically, the proof of Lemma 5.26 follows the lines of the proof to Lemma 5.25, where we now apply the long formula (3.15) for the second conditional moment of the increments of the correlated random walk instead of the equality (3.14). Because of the complexity it appears useful to introduce some notation for terms which will appear if we calculate the expression within the absolute value of (3.63). Thus, let us fix $z \in \{\pm 1\}$ and define for all $n \in I\!N$, $0 \le i \le j_T^n - 1$, and $x \in I\!R$ the differences D_1^n and D_2^n by $D_1^n := \sigma_n^2 n - \sigma^2$ and

$$D_2^n(i,x) := \frac{2}{\tau_{i+1}^n - \tau_i^n} \sum_{j=n\tau_i^n+1}^{n\tau_{i+1}^n - 1} \sum_{k=1}^{n\tau_{i+1}^n - j} \left(\sigma_n^2 \mathbf{E}_{\tau_i^n}^{x,z} \left[A_n^k\big(t_j^n, X_j^n\big)\right] - \delta^2\sigma^2 \prod_{l=j}^{j+k-1} a\big(t_l^n, x\big)\right).$$

Moreover, for the same values of n, i, and x let us define the four remainder terms R_1^n to R_4^n by

$$R_1^n(i,x) := \mu_n^2 n^2 \big(\tau_{i+1}^n - \tau_i^n\big),$$

$$R_2^n(i,x) := 2z\mu_n\sigma_n n \sum_{k=1}^{n(\tau_{i+1}^n - \tau_i^n)} A_n^k\big(\tau_i^n, x\big),$$

$$R_3^n(i,x) := 2\mu_n\sigma_n n \sum_{j=n\tau_i^n}^{n\tau_{i+1}^n - 1} \sum_{k=1}^{n\tau_{i+1}^n - j} \frac{\tau_{i+1}^n - j/n}{\tau_{i+1}^n - \tau_i^n} \mathbf{E}_{\tau_i^n}^{x,z}\left[B_n^k\big(t_j^n, X_j^n\big)\right],$$

and

$$R_4^n(i,x) := \frac{2\sigma_n}{\tau_{i+1}^n - \tau_i^n} \sum_{j=n\tau_i^n+1}^{n\tau_{i+1}^n - 1} \sum_{k=1}^{n\tau_{i+1}^n - j} \mathbf{E}_{\tau_i^n}^{x,z}\left[\big(X_j^n - X_i^n\big) B_n^k\big(t_j^n, X_j^n\big)\right].$$

By the definition of U^n in (2.4) we have $U^n(\tau_{i+1}^n) - U^n(\tau_i^n) = X_{n\tau_{i+1}^n}^n - X_{n\tau_i^n}^n$. Then the representation (3.15) with $l = n\tau_{i+1}^n$ and $\tilde{i} = n\tau_i^n$ and the definition of $\tilde{\sigma}_n$ in Definition 5.24 yield

$$\mathbf{E}_{\tau_i^n}^{x,z}\left[\frac{\big(U^n(\tau_{i+1}^n) - U^n(\tau_i^n)\big)^2}{\tau_{i+1}^n - \tau_i^n}\right] - \tilde{\sigma}_n^2(\tau_i^n, x) = D_1^n + D_2^n(i,x) + \sum_{m=1}^{4} R_m^n(n,i), \tag{3.64}$$

as a simple comparison shows. We will now prove that the differences D_1^n and $D_2^n(i,x)$ and the remainder terms $R_1^n(i,x)$ to $R_4^n(i,x)$ are all terms at least of order $O\big(\delta^{\frac{1}{2}\beta}\big)$ for $n \to \infty$, uniformly for all $0 \le i \le j_T^n - 1$ and $x \in I\!R$. For the first term D_1^n, this follows directly from Assumption N since it even implies that $\sigma_n^2 n - \sigma^2 = O\big(\delta^\beta\big)$. The bound on the second difference $D_2^n(i,x)$ follows by similar means to the bound on the difference (3.60): Taking the absolute value within the expectation and then applying (3.34) of Lemma 5.15 with $r = 2$ we see that there exist some $K_8, K_9 \in I\!R_+$ such that for all sufficiently large $n \in I\!N$ we can bound

$$\begin{aligned} \left| \sigma_n^2 \mathbf{E}_{\tau_i^n}^{x,z}\left[A_n^k\big(t_j^n, X_j^n\big)\right] - \delta^2\sigma^2 \prod_{l=j}^{j+k-1} a\big(t_l^n, x\big)\right| &\le \mathbf{E}_{\tau_i^n}^{x,z}\left| \sigma_n^2 A_n^k\big(t_j^n, X_j^n\big) - \delta^2\sigma^2 \prod_{l=j}^{j+k-1} a\big(t_l^n, x\big)\right| \\ &\le K_8(k+2)(k+1)\mathfrak{a}^k\delta^{2+\beta} + K_9(k+1)\mathfrak{a}^k\delta^2\mathbf{E}_{\tau_i^n}^{x,z}\left[\big|X_j^n - x\big|\right] \end{aligned}$$

for all $(x,z) \in I\!R \times \{\pm 1\}$, $0 \le i \le j_T^n - 1$, $n\tau_i^n + 1 \le j \le n\tau_{i+1}^n - 1$, and $1 \le k \le n\tau_{i+1}^n - j$. We can now apply (3.57) of Lemma 5.23 with $\alpha = 1$ to conclude that there exists some $M > 0$ such that for all sufficiently large $n \in I\!N$ not only the previous bound holds, but in addition the remaining conditional expectation can be bounded by $\mathbf{E}_{\tau_i^n}^{x,z}\big[|X_j^n - x|\big] \le M\delta^{\frac{1}{2}\beta}$ for all $n\tau_i^n + 1 \le j \le n\tau_{i+1}^n - 1$ and $x \in I\!R$. Since we can bound $\sum_{k=1}^{n\tau_{i+1}^n - j}(k+2)(k+1)\mathfrak{a}^k$

by $\sum_{k=0}^{\infty}(k+2)(k+1)\mathfrak{a}^k = \frac{2}{(1-\mathfrak{a})^3}$ and since on the other hand $\sum_{k=1}^{n\tau_{i+1}^n - j}(k+1)\mathfrak{a}^k \leq \frac{1}{(1-\mathfrak{a})^2}$ it follows for all those $n, j \in I\!N$ and all $x \in I\!R$

$$\big|D_2^n(i,x)\big| \leq \frac{2\delta^2}{\tau_{i+1}^n - \tau_i^n} \sum_{j=n\tau_i^n+1}^{n\tau_{i+1}^n - 1} \left(\frac{2K_8}{(1-\mathfrak{a})^3}\delta^\beta + \frac{K_9 M}{(1-\mathfrak{a})^2}\delta^{\frac{1}{2}\beta} \right) \leq \frac{4K_8}{(1-\mathfrak{a})^3}\delta^\beta + \frac{2K_9 M}{(1-\mathfrak{a})^2}\delta^{\frac{1}{2}\beta}.$$

Since this last bound does not depend on i or x we conclude $\sup_{0\leq i\leq j_T^n-1}\big|D_2^n(i,x)\big| = O\big(\delta^{\frac{1}{2}\beta}\big)$ as $n \to \infty$, uniformly for all $x \in I\!R$.
Now we come to the remainder terms R_1^n to R_4^n. At first, we immediately get from Assumption N and (3.53) that $\sup_{0\leq i\leq j_T^n-1}\big|R_1^n(i,x)\big| = O(\delta)$ as $n \to \infty$, and since $R_1^n(i,x)$ does not depend on x, this convergence holds of course uniformly for all $x \in I\!R$. In order to bound the second remainder term $R_2^n(i,x)$, we employ the inequality (3.26) to bound $\sup_{0\leq i\leq j_T^n-1}\sum_{k=1}^{n(\tau_{i+1}^n-\tau_i^n)}\big|A_n^k\big(\tau_i^n,x\big)\big| \leq \sum_{k=1}^{\infty}\mathfrak{a}^k < \infty$ for all sufficiently large $n \in I\!N$ and all $x \in I\!R$, and use $\mu_n\sigma_n n = O(\delta)$ by Assumption N to conclude $\sup_{0\leq i\leq j_T^n-1}\big|R_2^n(i,x)\big| = O(\delta)$ as $n \to \infty$, uniformly for all $x \in I\!R$. Thirdly, for all sufficiently large $n \in I\!N$, $0 \leq i \leq j_T^n - 1$ and $n\tau_i^n \leq j \leq n\tau_{i+1}^n - 1$ the fraction $\frac{\tau_{i+1}^n - j/n}{\tau_{i+1}^n - \tau_i^n}$ stays in the interval $[0,1]$, and bounding $\big|B_n^k\big(t_j^n, X_j^n\big)\big|$ by (3.29) for a suitable chosen $K_6 > 0$, we see that the inner sum in $R_3^n(i,x)$ can be bounded by

$$\sum_{k=1}^{n\tau_{i+1}^n - j}\left|\frac{\tau_{i+1}^n - j/n}{\tau_{i+1}^n - \tau_i^n}\mathbf{E}_{\tau_i^n}^{x,z}\big[B_n^k\big(t_j^n, X_j^n\big)\big]\right| \leq \sum_{k=1}^{\infty} K_6 k\mathfrak{a}^{k-1}\delta = \frac{K_6}{(1-\mathfrak{a})^2}\delta.$$

Therefore, the expression $\sum_{j=n\tau_i^n}^{n\tau_{i+1}^n-1}\sum_{k=1}^{n\tau_{i+1}^n-j}\frac{\tau_{i+1}^n-j/n}{\tau_{i+1}^n-\tau_i^n}\mathbf{E}_{\tau_i^n}^{x,z}\big[B_n^k\big(t_j^n, X_j^n\big)\big]$ can be bounded by $\frac{K_6}{(1-\mathfrak{a})^2}$ for all those sufficiently large $n \in I\!N$ and all $0 \leq i \leq j_T^n - 1$. Using again the bound $\mu_n\sigma_n n = O(\delta)$ it follows that $\sup_{0\leq i\leq j_T^n-1}\big|R_3^n(i,x)\big| = O(\delta)$ as $n \to \infty$ uniformly for all $x \in I\!R$ as well. For the fourth remainder term $R_4^n(i,x)$ there is slightly more work to do. At first, we once again take the absolute value within the expectation and use (3.29) to bound $\big|B_n^k\big(t_j^n, X_j^n\big)\big|$. This gives us for all sufficiently large $n \in I\!N$, and all $(i,j,x,z) \in I^n$ as defined in Lemma 5.23 the bound $\big|\mathbf{E}_{\tau_i^n}^{x,z}\big[\big(X_j^n - X_i^n\big)B_n^k\big(t_j^n, X_j^n\big)\big]\big| \leq \mathbf{E}_{\tau_i^n}^{x,z}\big|X_j^n - X_i^n\big|K_6 k\mathfrak{a}^{k-1}\delta$ for all $1 \leq k \leq n\tau_{i+1}^n - j$. For the remaining expectation we once again draw on Lemma 5.23, which guarantees the existence of some $M > 0$ such that $\sup_{(i,j,x,z)\in I^n}\mathbf{E}_{\tau_i^n}^{x,z}\big[\big|X_j^n - X_i^n\big|\big] \leq \delta^{\frac{1}{2}\beta}M$ for all sufficiently large $n \in I\!N$. Thus we get uniformly for all $0 \leq i \leq j_T^n - 1$, all $n\tau_i^n + 1 \leq j \leq n\tau_{i+1}^n - 1$, and all $x \in I\!R$ the bound

$$\sum_{k=1}^{n\tau_{i+1}^n - j}\big|\mathbf{E}_{\tau_i^n}^{x,z}\big[\big(X_j^n - X_i^n\big)B_n^k\big(t_j^n, X_j^n\big)\big]\big| \leq K_6 M\sum_{k=1}^{\infty} k\mathfrak{a}^{k-1}\delta^{1+\frac{1}{2}\beta},$$

and with the help of Assumption N we obtain $\sup_{0\leq i\leq j_T^n-1}\big|R_4^n(i,x)\big| = O\big(\delta^{\frac{1}{2}\beta}\big)$, uniformly for all $x \in I\!R$. Thus, since $z \in \{\pm 1\}$ can be chosen arbitrarily, we indeed conclude from the representation (3.64) and our order approximations of D_1^n, D_2^n, and R_1^n to R_4^n that the lemma's assertion holds. **q.e.d.**

In the next section, we employ the continuity of Assumption O(v) and replace the approximations of Definition 5.24 by their limits as $n \to \infty$. Before we come to that point, however, let us consider the normalized conditional local third moments of the increments of U^n over the $O(\delta)$-time intervals $\big[\tau_i^n, \tau_{i+1}^n\big]$, and show that they converge to 0 uniformly for all $0 \leq i \leq j_T^n - 1$. This turns out to be important later when we have to show that any limiting distribution of $\{U^n\}_{n\in\mathbb{N}}$ solves a martingale problem for the generator $\boldsymbol{L}$ from (2.7), since then we will need that for all sufficiently smooth and bounded functions $f:\mathbb{R}\to\mathbb{R}$ the remainder terms in the third-order Taylor expansions of $\mathbf{E}^{x,z}_{\tau_i^n}\left[f\big(U^n(\tau_{i+1}^n)\big)-f\big(U^n(\tau_i^n)\big)\right]$ vanish uniformly for all $0\le i\le j_T^n-1$ as $n\to\infty$.

Lemma 5.27. *Under Assumptions N and O(i) and (ii) we have*

$$\sup_{(x,z)\in\mathbb{R}\times\{\pm1\}}\ \sup_{0\le i\le j_T^n-1}\mathbf{E}^{x,z}_{\tau_i^n}\left[\frac{\left|U^n(\tau_{i+1}^n)-U^n(\tau_i^n)\right|^3}{\tau_{i+1}^n-\tau_i^n}\right]=O\Big(\delta^{\frac{5}{37}}\Big)\quad \textit{as } n\to\infty. \tag{3.65}$$

Proof. For some fixed $\hat\sigma>\sigma$ let us choose $M>0$, $K\in\mathbb{R}_+$ and $N\in\mathbb{N}$ such that the assertion of Corollary 5.17 holds. Due to Assumption N we can assume without loss of generality that $|\mu_n|+|\sigma_n|\le\hat\sigma\delta$ for all $n\ge N$. Now define

$$\tilde N:=\max\Big\{N,4M^2,(4\hat\sigma)^{\frac{74}{23}}\Big\}, \tag{3.66}$$

and let us fix $n\ge\tilde N$, $0\le i\le j_T^n-1$, and $(x,z)\in\mathbb{R}\times\{\pm1\}$. Since $n\ge\tilde N$ especially implies $n\ge 4M^2$, we obtain by (3.54) and $n=\delta^{-2}$:

$$n\tau_{i+1}^n-n\tau_i^n\le 2n\delta=2\delta^{-1}\le 2\delta^{-1}\sqrt{\frac{n}{4M^2}}=(\delta M)^{-2}. \tag{3.67}$$

Since the definition of τ_i^n in Definition 5.22 implies that the left-hand side of (3.67) is integer-valued, we even have $n\tau_{i+1}^n-n\tau_i^n\le\lfloor(\delta M)^{-2}\rfloor$. Moreover, due to the definitions of U^n in (2.44) and $n\tau_i^n,n\tau_{i+1}^n\in\mathbb{N}_0$ we get $U^n(\tau_{i+1}^n)-U^n(\tau_i^n)=X^n_{n\tau_{i+1}^n}-X^n_{n\tau_i^n}$, hence we can apply Corollary 5.17 and once again the left inequality in (3.67) to conclude that for any $\varepsilon_n>0$ with $n\ge(4\hat\sigma)^2\varepsilon_n^{-2}$ we have:

$$\mathbf{P}^{u,z}_{\tau_i^n}\Big(\left|U^n(\tau_{i+1}^n)-U^n(\tau_i^n)\right|\ge\varepsilon\Big)\le\delta^4\frac{K}{\varepsilon^4}\big(n\tau_{i+1}^n-n\tau_i^n\big)^2\le 4\delta^2K\varepsilon^{-4}. \tag{3.68}$$

We want to apply (3.68) for $\varepsilon_n=\delta^{\frac{14}{37}}$ and $\varepsilon_n=\delta^{\frac{8}{37}}$. In order to see that for these choices the condition $n\ge(4\hat\sigma)^2\varepsilon_n^{-2}$ is satisfied, we write at first $n=n^{\frac{23}{37}}n^{\frac{14}{37}}$ and then use $n\ge\tilde N\ge(4\hat\sigma)^{\frac{74}{23}}$ and $n=\delta^{-2}$ to obtain $n\ge(4\hat\sigma)^2\delta^{-\frac{28}{37}}$. This shows that (3.68) holds for $\varepsilon_n=\delta^{\frac{14}{37}}$. Secondly we notice that either $1\ge(4\hat\sigma)^{\frac{74}{29}}$ or $(4\hat\sigma)^{\frac{74}{23}}\ge(4\hat\sigma)^{\frac{74}{29}}$. Thus, in both cases we can conclude that $n\ge(4\hat\sigma)^{\frac{74}{29}}$ as well, and writing $n=n^{\frac{29}{37}}n^{\frac{8}{37}}$ we obtain $n\ge(4\hat\sigma)^2\delta^{-\frac{16}{37}}$, hence (3.68) also holds for $\varepsilon_n=\delta^{\frac{8}{37}}$.

Let us now split the set Ω^n in three disjoint sets by defining

$$\begin{aligned}K_1^n(i)&:=\Big\{\left|U^n(\tau_{i+1}^n)-U^n(\tau_i^n)\right|\le\delta^{\frac{14}{37}}\Big\}\\ K_2^n(i)&:=\Big\{\delta^{\frac{14}{37}}<\left|U^n(\tau_{i+1}^n)-U^n(\tau_i^n)\right|\le\delta^{\frac{8}{37}}\Big\},\end{aligned}$$

and

$$K_3^n(i):=\Big\{\left|U^n(\tau_{i+1}^n)-U^n(\tau_i^n)\right|>\delta^{\frac{8}{37}}\Big\}.$$

On $K_1^n(i)$ we can bound $\left|U^n(\tau_{i+1}^n)-U^n(\tau_i^n)\right|\le\delta^{\frac{14}{37}}$, and upon using the trivial estimate $\mathbf{P}^{u,z}_{\tau_i^n}\big(K_1^n(i)\big)\le 1$ we obtain:

$$\mathbf{E}^{x,z}_{\tau_i^n}\Big[\left|U^n(\tau_{i+1}^n)-U^n(\tau_i^n)\right|^3\mathbf{1}_{K_1^n(i)}\Big]\le\delta^{3\cdot\frac{14}{37}}\mathbf{P}^{u,z}_{\tau_i^n}\big(K_1^n(i)\big)\le\delta^{\frac{42}{37}}. \tag{3.69}$$

Similarly, we can bound $\left|U^n(\tau_{i+1}^n) - U^n(\tau_i^n)\right| \le \delta^{\frac{8}{37}}$ on $K_2^n(i)$. However, the trivial estimate $\mathbf{P}_{\tau_i^n}^{u,z}\big(K_2^n(i)\big) \le 1$ is not sharp enough for our desired convergence result. Instead, we will bound the remaining probability by $\mathbf{P}_{\tau_i^n}^{u,z}\big(K_2^n(i)\big) \le \mathbf{P}_{\tau_i^n}^{u,z}\big(\big|X_{n\tau_{i+1}^n}^n - X_{n\tau_i^n}^n\big| \ge \delta^{\frac{14}{37}}\big)$ and then use (3.68) with $\varepsilon_n = \delta^{\frac{14}{37}}$ to bound

$$\mathbf{E}_{\tau_i^n}^{x,z}\Big[\big|U^n(\tau_{i+1}^n) - U^n(\tau_i^n)\big|^3 \mathbf{1}_{K_2^n(i)}\Big] \le \delta^{\frac{24}{37}}\mathbf{P}_{\tau_i^n}^{u,z}\big(K_2^n(i)\big) \le \delta^{\frac{24}{37}} 4\delta^2 K \delta^{-4\cdot\frac{14}{37}} = 4K\delta^{\frac{42}{37}}. \quad (3.70)$$

Last but not least, on $K_3^n(i)$ we once again write

$$U^n\big(\tau_{i+1}^n\big) - U^n\big(\tau_i^n\big) = X_{n\tau_{i+1}^n}^n - X_{n\tau_i^n}^n = \sum_{j=n\tau_i^n+1}^{n\tau_{i+1}^n}\big(\mu_n + \sigma_n Z_j^n\big),$$

where the second equation stems from the definition of the correlated random walk X^n in (2.3), and then employ $|Z_j^n| = 1$ for all $0 \le j \le n$ and our assumption that $|\mu_n| + |\sigma_n| \le \hat{\sigma}\delta$ for all $n \ge N$ to conclude

$$\Big|U^n\big(\tau_{i+1}^n\big) - U^n\big(\tau_i^n\big)\Big| \le \sum_{k=n\tau_i^n+1}^{n\tau_{i+1}^n}\big(|\mu_n| + |\sigma_n|\big) \le \big(n\tau_{i+1}^n - n\tau_i^n\big)\hat{\sigma}\delta \le 2\hat{\sigma},$$

using (3.54) for the last inequality. If we then apply (3.68) with $\varepsilon_n = \delta^{\frac{8}{37}}$ we obtain

$$\mathbf{E}_{\tau_i^n}^{x,z}\Big[\big|U^n(\tau_{i+1}^n) - U^n(\tau_i^n)\big|^3 \mathbf{1}_{K_3^n(i)}\Big] \le (2\hat{\sigma})^3\mathbf{P}_{\tau_i^n}^{u,z}\big(K_2^n(i)\big) \le 4(2\hat{\sigma})^3 K\delta^{\frac{42}{37}}. \quad (3.71)$$

Adding up the three inequalities (3.69), (3.70) and (3.71) we get

$$\mathbf{E}_{\tau_i^n}^{x,z}\Big[\big|U^n(\tau_{i+1}^n) - U^n(\tau_i^n)\big|^3\Big] \le \Big(1 + 4K\big(1 + (2\hat{\sigma})^3\big)\Big)\delta^{\frac{42}{37}},$$

and since $\big(\tau_{i+1}^n - \tau_i^n\big)^{-1} \le 4\delta^{-1}$ by the lower bound in (3.54), the claimed rate of convergence for $\big(\tau_{i+1}^n - \tau_i^n\big)^{-1}\mathbf{E}_{\tau_i^n}^{u,z}\big[\big|U^n(\tau_{i+1}^n) - U^n(\tau_i^n)\big|^3\big]$, namely $O\big(\delta^{\frac{5}{37}}\big)$ as $n \to \infty$, follows. **q.e.d.**

Note that we could achieve some sharper bounds by dividing directly (3.68) by $\tau_{i+1}^n - \tau_i^n$. We did not care about this better bound since the rate of convergence in (3.65) that we have shown in the previous lemma is not the best that one could prove, anyhow. For example, one can easily find a better rate by splitting the conditional expectation $\mathbf{E}_{\tau_i^n}^{u,z}\big[\big|U^n(\tau_{i+1}^n) - U^n(\tau_i^n)\big|^3\big]$ not only into three, but into four or more terms. However, as it turns out in Lemma 5.32, the obtained rate of convergence is sufficient for our needs.
For our further calculations the drift correction term $z\tilde{d}_n\big(\tau_i^n, U^n(\tau_i^n)\big)$, which we had to employ in order to find an $o(1)$-approximation as $n \to \infty$ for the normalized first conditional moments over the time interval $\big[\tau_i^n, \tau_{i+1}^n\big]$ for any $0 \le i \le j_T^n - 1$, is a nuisance, since it still depends on the initial tilt $V^n(\tau_i^n) = z$ at the beginning of that interval. If U is a diffusion limit of some subsequence of $\{U^n\}_{n\in\mathbb{N}}$, then U cannot depend on a continuous limit of the tilt process V^n, since such a limit would have to be 1 if the limit U increases in t and -1 if U decreases in t, and with the Brownian path having no points of increase and no points of decrease, such a process cannot exist. Fortunately, the influence of the tilt is negligible if we just intersperse one more time interval with length of order $O(\delta)$ as $n \to \infty$, as the next lemma indicates. This is essential to circumvent dealing with some limit of the tilt processes $\{V^n\}_{n\in\mathbb{N}}$ associated to $\{U^n\}_{n\in\mathbb{N}}$.

Lemma 5.28. *Under the Assumptions O(i) and (ii) we have for any $f \in C_b^1(I\!R)$:*

$$\sup_{1\le i\le j_T^n-1}\left|\mathbf{E}^{x,z}_{\tau_{i-1}^n}\left[f\big(U^n(\tau_i^n)\big)\tilde{d}_n\big(\tau_i^n,U^n(\tau_i^n)\big)V^n(\tau_i^n)\right]\right| = O(\delta) \quad \text{as } n\to\infty, \tag{3.72}$$

uniformly for all $(x,z)\in I\!R\times\{\pm1\}$.

Proof. For all $n\in I\!N$ and $1\le i\le j_T^n-1$ let us introduce the function $F_i^n : I\!R\to I\!R$ by $F_i^n(x) := f(x)\tilde{d}_n\big(\tau_i^n,x\big)$ for all $x\in I\!R$, such that the definitions of the continuous-time tilt in (2.4) and the $O(\delta)$-time steps in Definition 5.22 imply the simplified representation $\mathbf{E}^{x,z}_{\tau_{i-1}^n}\left[f\big(U^n(\tau_i^n)\big)\tilde{d}_n\big(\tau_i^n,U^n(\tau_i^n)\big)V^n(\tau_i^n)\right] = \mathbf{E}^{x,z}_{\tau_{i-1}^n}\left[Z^n_{n\tau_i^n}F_i^n\big(X^n_{n\tau_i^n}\big)\right]$ for all $(x,z)\in I\!R\times\{\pm1\}$. Recall now the general definition of the auxiliary functions $A^k_{n,F}$ and $B^k_{n,F}$ of Definition 5.7. By the definition of A^0_{n,F_i^n} we have $F_i^n(x) = A^0_{n,F_i^n}\big(\tau_i^n,x\big)$ for all $x\in I\!R$, hence we can apply the definition of $U^n\big(\tau_i^n\big) = X^n_{n\tau_i^n}$ in (2.4), Lemma 5.8, and the equality $t^n_{n\tau_{i-1}^n} = \tau_{i-1}^n$ to rewrite the expectation within (3.72) as

$$\mathbf{E}^{x,z}_{\tau_{i-1}^n}\left[Z^n_{n\tau_i^n}F_i^n\big(X^n_{n\tau_i^n}\big)\right] = zA^{n(\tau_i^n-\tau_{i-1}^n)}_{n,F_i^n}\big(\tau_{i-1}^n,x\big) + \sum_{j=n\tau_{i-1}^n}^{n\tau_i^n-1}\mathbf{E}^{x,z}_{\tau_{i-1}^n}\left[B^{n\tau_i^n-j}_{n,F_i^n}\big(t_j^n,X_j^n\big)\right]. \tag{3.73}$$

In order to bound (3.73), we want to use Lemma 5.13. Hence we have to show that F_i^n is globally bounded and satisfies a global Lipschitz condition on $I\!R$. Let us fix $n\in I\!N$ and $1\le i\le j_T^n-1$. By (3.54) we have $\big(\tau_{i+1}^n-\tau_i^n\big)^{-1}\delta\le 4$, and due to Assumption O(i) it follows that $\prod_{l=n\tau_i^n}^{n\tau_i^n+k-1}\big|a\big(t_l^n,x\big)\big|\le\mathfrak{a}^k$ for all $n\tau_i^n\le k\le n\tau_{i+1}^n-1$, hence we get from the definition of $\tilde{d}_n\big(\tau_i^n,x\big)$ in Definition 5.24 that

$$\|F_i^n\| = \sup_{x\in I\!R}\left|f(x)\frac{\sigma\delta}{\tau_{i+1}^n-\tau_i^n}\sum_{k=1}^{n(\tau_{i+1}^n-\tau_i^n)}\prod_{l=n\tau_i^n}^{n\tau_i^n+k-1}a\big(t_l^n,x\big)\right| \le 4\sigma\|f\|\sum_{k=1}^{\infty}\mathfrak{a}^k\le 4\sigma\|f\|\frac{\mathfrak{a}}{1-\mathfrak{a}} \tag{3.74}$$

is finite. Now let us go on to show the Lipschitz continuity of $F_i^n : I\!R\to I\!R$. Since we assume $f\in C_b^1(I\!R)$, we especially have $|f(x)-f(y)|\le\|f'\||x-y|$ for all $x,y\in I\!R$. Hence we can apply (3.21) of Lemma 5.11 with $F\equiv f$ to conclude that there exists some $K_f\in I\!R_+$ such that

$$\left|f(x)\prod_{l=n\tau_i^n}^{n\tau_i^n+k-1}a\big(t_l^n,x\big) - f(y)\prod_{l=n\tau_i^n}^{n\tau_i^n+k-1}a\big(t_l^n,y\big)\right|\le K_f(k+1)\mathfrak{a}^k|y-x|,$$

and thus it easily follows from the definition of F_i^n and $\tilde{d}_n$ that $F_i^n : I\!R\to I\!R$ satisfies the Lipschitz condition

$$\big|F_i^n(x)-F_i^n(y)\big|\le\frac{\delta\sigma}{\tau_{i+1}^n-\tau_i^n}\sum_{k=1}^{n(\tau_{i+1}^n-\tau_i^n)}K_f(k+1)\mathfrak{a}^k|y-x|\le K_{F_i^n}|y-x|,$$

where $K_{F_i^n} = 4\sigma K_f\sum_{k=0}^{\infty}(k+1)\mathfrak{a}^k = \frac{4\sigma K_f}{(1-\mathfrak{a})^2}$. Hence the Assumptions of Lemma 5.13 are satisfied, and we can conclude from the bound in (3.26), the bound on $\|F_i^n\|$ in (3.74), $0<\mathfrak{a}<1$, and the lower bound in (3.53) that for all sufficiently large $n\in I\!N$ and all $1\le i\le j_T^n-1$ and $x\in I\!R$ we have

$$\left|A^{n(\tau_i^n-\tau_{i-1}^n)}_{n,F_i^n}\big(\tau_{i-1}^n,x\big)\right|\le\|F_i^n\|\mathfrak{a}^{n(\tau_i^n-\tau_{i-1}^n)}\le\frac{4\sigma}{1-\mathfrak{a}}\|f\|\mathfrak{a}^{\sqrt{n}} = O\big(\delta^\alpha\big)\quad\text{as } n\to\infty,$$

for any $\alpha > 0$, i.e. especially for $\alpha = 1$. Taking the absolute value inside the expectation and then using the bound (3.29) we also see that there exists some $K \in \mathbb{R}_+$ such that $\left|\mathbf{E}^{x,z}_{\tau^n_{i-1}}\left[B^{n\tau^n_i-j}_{n,F^n_i}\left(t^n_j, X^n_j\right)\right]\right| \leq K\left(n\tau^n_i - j\right)\mathfrak{a}^{n\tau^n_i-j-1}\delta$ for all sufficiently large $n \in \mathbb{N}$, all $1 \leq i \leq j^n_T - 1$, $n\tau^n_{i-1} \leq j \leq n\tau^n_i - 1$, and any $(x,z) \in \mathbb{R} \times \{\pm 1\}$. Thus, the substitution $k = n\tau^n_i - j - 1$ yields

$$\sum_{j=n\tau^n_{i-1}}^{n\tau^n_i-1}\left|\mathbf{E}^{x,z}_{\tau^n_{i-1}}\left[B^{n\tau^n_i-j}_{n,F^n_i}\left(t^n_j, X^n_j\right)\right]\right| \leq K \sum_{k=0}^{n(\tau^n_i-\tau^n_{i-1})-1}(k+1)\mathfrak{a}^k\delta \leq \frac{K}{(1-\mathfrak{a})^2}\delta = O(\delta) \quad \text{as } n \to \infty.$$

Since the bounds hold uniformly for all $1 \leq i \leq j^n_T - 1$ and $(x,z) \in \mathbb{R} \times \{\pm 1\}$, in view of (3.73) this completes the proof of Lemma 5.28. **q.e.d.**

5.3.5 Employing the Continuity in the Time Variable

In order to achieve a diffusion limit, we have to replace in our approximations of Lemma 5.25 and Lemma 5.26 the discrete functions $\tilde{\mu}_n$, $\tilde{d}_n$ and $\tilde{\sigma}^2_n$ of Definition 5.24, which still depend on n, by suitable limits as $n \to \infty$. The last lemma has indicated how we can eliminate the function $\tilde{d}_n$. Hence, it suffices to obtain limits for the two functions $\tilde{\mu}_n : \widetilde{\mathcal{T}}_n \times \mathbb{R} \to \mathbb{R}$ and $\tilde{\sigma}_n : \widetilde{\mathcal{T}}_n \times \mathbb{R} \to \mathbb{R}_+$. In order to do this, we have to employ the continuity of a, a' and b in the time variable t as stated in Assumption O(v).
More precisely, we will show that the functions $\tilde{\mu}_n$ and $\tilde{\sigma}^2_n$ converge on bounded subsets of their domains $\widetilde{\mathcal{T}}_n \times \mathbb{R}$ to certain functions $\check{\mu} : [0,T] \times \mathbb{R} \to \mathbb{R}$ and $\check{\sigma} : [0,T] \times \mathbb{R} \to \mathbb{R}_+$ introduced in terms of the limit functions $a : [0,T] \times \mathbb{R} \to \mathbb{R}$ and $b : [0,T] \times \mathbb{R} \to \mathbb{R}$, which satisfy (3.4) and (3.5), respectively:

Definition 5.29. The **(local) drift function** $\check{\mu} : [0,T] \times \mathbb{R} \to \mathbb{R}$ is given by

$$\check{\mu}(t,x) := \mu + \frac{\sigma b(t,x)}{1-a(t,x)} + \frac{\sigma^2 a'(t,x)}{\big(1-a(t,x)\big)^2} \quad \text{for all } (t,x) \in [0,T] \times \mathbb{R} \tag{3.75}$$

and the **(local) volatility function** $\check{\sigma} : [0,T] \times \mathbb{R} \to \mathbb{R}_+$ by

$$\check{\sigma}^2(t,x) := \sigma^2 \frac{1+a(t,x)}{1-a(t,x)} \quad \text{for all } (t,x) \in [0,T] \times \mathbb{R}. \tag{3.76}$$

Remark. If we compare the definitions of $\check{\mu}$ and $\check{\sigma}$ with the definition of the generator $\boldsymbol{L} : C^2_b(\mathbb{R}) \to C_b\big([0,T] \times \mathbb{R}\big)$ in (2.7), which is in Theorem 5.4 claimed to determine the limiting process U, we see that for all $f \in C^2_b(\mathbb{R})$ the generator can be rewritten as

$$(\boldsymbol{L}f)(t,x) = \frac{1}{2}\check{\sigma}^2(t,x)\frac{d^2}{dx^2}f(x) + \check{\mu}(t,x)\frac{d}{dx}f(x) \quad \text{for all } (t,x) \in [0,T] \times \mathbb{R}, \tag{3.77}$$

i.e. the local drift and local volatility functions are just the coefficients of $\boldsymbol{L}$. □

Now Lemma 5.12 gives us the tool to replace the tilded approximation functions $\tilde{\mu}_n$ and $\tilde{\sigma}_n$ of Definition 5.24 by the checked functions of Definition 5.29, namely it guarantees:

Lemma 5.30. *Under Assumptions O(i), (ii), and (v) we have for any fixed $R \in \mathbb{R}_+$:*

$$\sup_{(t,x)\in\widetilde{\mathcal{T}}_n\times[-R,R]}\left|\check{\mu}\left(\tau^n_i, x\right) - \tilde{\mu}_n\left(\tau^n_i, x\right)\right| \to 0 \quad \text{as } n \to \infty \tag{3.78}$$

and

$$\sup_{(t,x)\in\widetilde{\mathcal{T}}_n\times[-R,R]}\left|\check{\sigma}^2\left(\tau^n_i, x\right) - \tilde{\sigma}^2_n\left(i, x\right)\right| \to 0 \quad \text{as } n \to \infty. \tag{3.79}$$

Proof. Comparing the definition of $\tilde{\mu}_n : \widetilde{\mathcal{T}}_n \times I\!R \to I\!R$ in Definition 5.24 with the definition of $\check{\mu} : [0,T] \times I\!R \to I\!R$ in Definition 5.29, we easily see that in order to show (3.78) it suffices to show that the differences D_1^n and D_2^n, defined for all $0 \le i \le j_T^n - 1$ and $x \in I\!R$ by

$$D_1^n(i,x) := \frac{b\big(\tau_i^n, x\big)}{1 - a\big(\tau_i^n, x\big)} - \frac{\delta^2}{\tau_{i+1}^n - \tau_i^n} \sum_{j=n\tau_i^n}^{n\tau_{i+1}^n - 1} \sum_{k=1}^{n\tau_{i+1}^n - j} b\big(t_j^n, x\big) \prod_{l=j+1}^{j+k-1} a\big(t_l^n, x\big)$$

and

$$D_2^n(i,x) := \frac{a'\big(\tau_i^n, x\big)}{\big(1 - a(\tau_i^n, x)\big)^2} - \frac{\delta^2}{\tau_{i+1}^n - \tau_i^n} \sum_{j=n\tau_i^n}^{n\tau_{i+1}^n - 1} \sum_{k=1}^{n\tau_{i+1}^n - j} \sum_{l=j+1}^{j+k-1} a'\big(t_l^n, x\big) \prod_{\substack{r=j+1 \\ r \neq l}}^{j+k-1} a\big(t_r^n, x\big),$$

satisfy $\sup_{|x| \le R} \sup_{0 \le i \le j_T^n - 1} \big|D_r^n(i,x)\big| \to 0$ as $n \to \infty$ for $r \in \{1,2\}$. Let us first consider D_1^n and fix $n \in I\!N$ and $i \in I\!N_0$ with $0 \le i \le j_T^n - 1$. By adding and subtracting

$$\frac{\delta^2 b(\tau_i^n, x)}{\tau_{i+1}^n - \tau_i^n} \sum_{j=n\tau_i^n}^{n\tau_{i+1}^n - 1} \sum_{k=1}^{n\tau_{i+1}^n - j} a^{k-1}\big(\tau_i^n, x\big)$$

we can rewrite the difference $D_1^n(i,x)$ as $D_1^n(i,x) = b(\tau_i^n, x) D_{1,1}^n(i,x) + D_{1,2}^n(i,x)$ with the differences $D_{1,1}^n(i,x)$ and $D_{1,2}^n(i,x)$ given by

$$D_{1,1}^n(i,x) := \frac{1}{1 - a(\tau_i^n, x)} - \frac{\delta^2}{\tau_{i+1}^n - \tau_i^n} \sum_{j=n\tau_i^n}^{n\tau_{i+1}^n - 1} \sum_{k=1}^{n\tau_{i+1}^n - j} a^{k-1}\big(\tau_i^n, x\big)$$

and

$$D_{1,2}^n(i,x) := \frac{\delta^2}{\tau_{i+1}^n - \tau_i^n} \sum_{j=n\tau_i^n}^{n\tau_{i+1}^n - 1} \sum_{k=1}^{n\tau_{i+1}^n - j} \left(b(\tau_i^n, x) a^{k-1}\big(\tau_i^n, x\big) - b(t_j^n, x) \prod_{l=j+1}^{j+k-1} a\big(t_l^n, x\big) \right).$$

We will now bound the two differences $D_{1,1}^n(i,x)$ and $D_{1,2}^n(i,x)$ independently of i and x. For the first term $D_{1,1}^n(i,x)$, let us note that by the formula for the geometric sum we have

$$\frac{1}{1 - a(\tau_i^n, x)} - \sum_{k=1}^{n\tau_{i+1}^n - j} a^{k-1}\big(\tau_i^n, x\big) = \sum_{k=0}^{\infty} a^k\big(\tau_i^n, x\big) - \sum_{k=0}^{n\tau_{i+1}^n - j - 1} a^k\big(\tau_i^n, x\big) = \sum_{k=n\tau_{i+1}^n - j}^{\infty} a^k\big(\tau_i^n, x\big),$$

and hence we get due to $\sum_{j=n\tau_i^n}^{n\tau_{i+1}^n - 1} 1 = n\big(\tau_{i+1}^n - \tau_i^n\big)$ and $n = \delta^{-2}$ that $D_{1,1}^n(i,x)$ satisfies $D_{1,1}^n(i,x) = \delta^2 (\tau_{i+1}^n - \tau_i^n)^{-1} \sum_{j=n\tau_i^n}^{n\tau_{i+1}^n - 1} \sum_{k=n\tau_{i+1}^n - j}^{\infty} a^k\big(\tau_i^n, x\big)$. Moreover, the lower bound in (3.54) shows $\delta(\tau_{i+1}^n - \tau_i^n)^{-1} \le 4$. If we also use $\sum_{j=n\tau_i^n}^{n\tau_{i+1}^n - 1} 1 = n\big(\tau_{i+1}^n - \tau_i^n\big)$, apply Assumption O($i$) and then change the order of summation we obtain

$$\big|D_{1,1}^n(i,x)\big| \le 4\delta \sum_{j=1}^{n(\tau_{i+1}^n - \tau_i^n)} \sum_{k=j}^{\infty} \mathfrak{a}^k = 4\delta \sum_{k=1}^{\infty} \sum_{j=1}^{\min\{k, n(\tau_{i+1}^n - \tau_i^n)\}} \mathfrak{a}^k \le 4\delta \sum_{k=0}^{\infty} (k+1) \mathfrak{a}^k = \frac{4\delta}{(1 - \mathfrak{a})^2},$$

and therefore $\sup_{x \in I\!R} \sup_{0 \le i \le j_T^n - 1} \big|D_{1,1}^n(i,x)\big| \to 0$. Next we show the convergence of $D_{1,2}^n$. Since we have from (3.53) that $0 \le n\tau_{i+1}^n - n\tau_i^n - 1 \le \sqrt{n} = \delta^{-1}$, and since $t_{n\tau_i^n}^n = \tau_i^n$ we can apply (3.24) of Lemma 5.12 with $F \equiv b$, $(\tilde{i}, \tilde{j}, \tilde{k}) = (n\tau_i^n, j+1, k-1)$, and $\kappa_n = \sqrt{n}$ to conclude that for any $R > 0$ and any $\varepsilon > 0$ there exists some $N \in \mathbb{N}$ such that for all $n \geq N$, $0 \leq i \leq j_T^n - 1$, $n\tau_i^n \leq j \leq n\tau_{i+1}^n - 1$, and $|x| \leq R$ we obtain

$$\sum_{k=1}^{n\tau_{i+1}^n - j} \left| b(\tau_i^n, x) a^{k-1}(\tau_i^n, x) - b(t_j^n, x) \prod_{l=j+1}^{j+k-1} a(t_l^n, x) \right| \leq \sum_{k=1}^{n\tau_{i+1}^n - j} k\mathfrak{a}^{k-1}\varepsilon \leq \frac{\varepsilon}{(1-\mathfrak{a})^2}.$$

Now a summation of these bounds for all $n\tau_i^n \leq j \leq n\tau_{i+1}^n - 1$ and a division by $n(\tau_{i+1}^n - \tau_i^n)$ yields due to $n = \delta^{-2}$

$$\sup_{|x|\leq R} \sup_{0\leq i\leq j_T^n - 1} \left| D_{1,2}^n(i,x) \right| \leq \frac{1}{n(\tau_{i+1}^n - \tau_i^n)} \sum_{j=n\tau_i^n}^{n\tau_{i+1}^n - 1} \frac{\varepsilon}{(1-\mathfrak{a})^2} = \frac{\varepsilon}{(1-\mathfrak{a})^2},$$

and letting ε tend to 0 we see that the term on the left-hand side converges to 0 as $n \to \infty$ as well. Since $\|b\| < \infty$ by Assumption O(i), the convergence of $D_{1,1}^n$ and $D_{1,2}^n$ implies $\sup_{|x|\leq R} \sup_{0\leq i\leq j_T^n - 1} \left| D_1^n(i,x) \right| \to 0$ as $n \to \infty$.
Similarly we can write $D_2^n(i,x) = a'(\tau_i^n, x) D_{2,1}^n(i,x) + D_{2,2}^n(i,x)$ with

$$D_{2,1}^n(i,x) := \frac{1}{\left(1 - a(\tau_i^n, x)\right)^2} - \frac{1}{n(\tau_{i+1}^n - \tau_i^n)} \sum_{j=n\tau_i^n}^{n\tau_{i+1}^n - 1} \sum_{k=1}^{n\tau_{i+1}^n - j} (k-1) a^{k-2}(\tau_i^n, x)$$

and

$$D_{2,2}^n(i,x) := \frac{\delta^2}{\tau_{i+1}^n - \tau_i^n} \sum_{j=n\tau_i^n}^{n\tau_{i+1}^n - 1} \sum_{k=1}^{n\tau_{i+1}^n - j} \left(a'(\tau_i^n, x)(k-1) a^{k-2}(\tau_i^n, x) - \sum_{l=j+1}^{j+k-1} a'(t_l^n, x) \prod_{\substack{r=j+1 \\ r\neq l}}^{j+k-1} a(t_r^n, x) \right)$$

for all $n \in \mathbb{N}$, $0 \leq i \leq j_T^n - 1$ and $x \in \mathbb{R}$. In analogy to the treatment of the differences $D_{1,1}^n$ and $D_{1,2}^n$ we can draw on the equality $\left(1 - a(\tau_i^n, x)\right)^{-2} = \sum_{k=0}^\infty (k+1) a^k(\tau_i^n, x)$ to show the convergence of $\sup_{x\in\mathbb{R}} \sup_{0\leq i\leq j_T^n - 1} \left| D_{2,1}^n(i,x) \right|$ and (3.25) of Lemma 5.12 to show the convergence of the term $\sup_{|x|\leq R} \sup_{0\leq i\leq j_T^n - 1} \left| D_{2,2}^n(i,x) \right|$. Thus the boundedness of $a' : [0,T] \times \mathbb{R} \to \mathbb{R}$ implies $\sup_{|x|\leq R} \sup_{0\leq i\leq j_T^n - 1} \left| D_2^n(i,x) \right| \to 0$ as $n \to \infty$. By our definitions of D_1^n and D_2^n this completes the proof of (3.78).
The proof of (3.79) goes along the same lines as the proof of (3.78). Writing the fraction $\frac{1+a(t,x)}{1-a(t,x)}$ as $1 + 2\frac{a(t,x)}{1-a(t,x)}$ we only have to show that $\sup_{|x|\leq R} \sup_{0\leq i\leq j_T^n - 1} \left| D_3^n(i,x) \right| \to 0$ as $n \to \infty$, where $D_3^n(i,x)$ is for all $0 \leq i \leq j_T^n - 1$ and $x \in \mathbb{R}$ defined by

$$D_3^n(i,x) := \frac{a(\tau_i^n, x)}{1 - a(\tau_i^n, x)} - \frac{\delta^2}{\tau_{i+1}^n - \tau_i^n} \sum_{j=n\tau_i^n + 1}^{n\tau_{i+1}^n - 1} \sum_{k=1}^{n\tau_{i+1}^n - j} \prod_{l=j}^{j+k-1} a(t_l^n, x).$$

Now we can add and subtract the expression

$$\frac{\delta^2}{\tau_{i+1}^n - \tau_i^n} \sum_{j=n\tau_i^n + 1}^{n\tau_{i+1}^n} \sum_{k=1}^{n\tau_{i+1}^n - j} a^k(\tau_i^n, x) = \frac{\delta^2}{\tau_{i+1}^n - \tau_i^n} \sum_{j=n\tau_i^n + 1}^{n\tau_{i+1}^n - 1} \sum_{k=1}^{n\tau_{i+1}^n - j} a^k(\tau_i^n, x)$$

and apply $\sum_{k=1}^\infty a^k(\tau_i^n, x) = \frac{a(\tau_i^n, x)}{1 - a(\tau_i^n, x)}$ in order to show that if we define $D_{3,1}^n(i,x)$ for all $0 \leq i \leq j_T^n - 1$ and $x \in \mathbb{R}$ as

$$D_{3,1}^n(i,x) := \frac{a(\tau_i^n, x)}{1 - a(\tau_i^n, x)} - \frac{\delta^2}{\tau_{i+1}^n - \tau_i^n} \sum_{j=n\tau_i^n + 1}^{n\tau_{i+1}^n} \sum_{k=1}^{n\tau_{i+1}^n - j} a^k(\tau_i^n, x),$$

we get $\sup_{x\in\mathbb{R}}\sup_{0\le i\le j_T^n-1}\big|D_{3,1}^n(i,x)\big| = O(\delta)$ as $n\to\infty$, and using (3.24) of Lemma 5.12 with $F\equiv 1$ we obtain $\sup_{|x|\le R}\sup_{0\le i\le j_T^n-1}\big|D_{3,2}^n(i,x)\big|\to 0$ as $n\to\infty$, where

$$D_{3,2}^n(i,x) := \frac{\delta^2}{\tau_{i+1}^n-\tau_i^n}\sum_{j=n\tau_i^n+1}^{n\tau_{i+1}^n-1}\sum_{k=1}^{n\tau_{i+1}^n-j}\left(a^k\big(\tau_i^n,x\big)-\prod_{l=j}^{j+k-1}a\big(t_l^n,x\big)\right)$$

for all $0\le i\le j_T^n-1$ and $x\in\mathbb{R}$. Since $D_3^n(i,x)=D_{3,1}^n(i,x)+D_{3,2}^n(i,x)$ for all $0\le i\le j_T^n-1$ and $x\in\mathbb{R}$ this concludes the proof of (3.79). **q.e.d.**

Due to Proposition 5.19, we need not care too much about unbounded paths of U^n. In the next lemma we show that if we start at a deterministic time $s\in[0,T]$ in a deterministic point $U^n(s)=u\in\mathbb{R}$ with a deterministic tilt $V^n(s)=v\in\{\pm1\}$, the conditional expectation of the difference $\big|\tilde\mu\big(\tau_i^n,U^n(\tau_i^n)\big)-\check\mu\big(\tau_i^n,U^n(\tau_i^n)\big)\big|$ still converges. This convergence is seen to hold uniformly in the deterministic starting parameters (s,u,v), as long as they are taken from a compact subset of $[0,T]\times\mathbb{R}\times\{\pm1\}$.

Lemma 5.31. *Under the Assumptions N and O(i), (ii), and (v) we have for all $R\in\mathbb{R}_+$:*

$$\sup_{(s,u,v)\in I_R}\ \sup_{i_s^n\le i\le j_T^n-1}\mathbf{E}_s^{u,v}\Big[\big|\tilde\mu_n\big(\tau_i^n,U^n(\tau_i^n)\big)-\check\mu\big(\tau_i^n,U^n(\tau_i^n)\big)\big|\Big]\to 0\quad \text{as } n\to\infty \tag{3.80}$$

and

$$\sup_{(s,u,v)\in I_R}\ \sup_{i_s^n\le i\le j_T^n-1}\mathbf{E}_s^{u,v}\Big[\big|\tilde\sigma_n^2\big(\tau_i^n,U^n(\tau_i^n)\big)-\check\sigma^2\big(\tau_i^n,U^n(\tau_i^n)\big)\big|\Big]\to 0\quad \text{as } n\to\infty, \tag{3.81}$$

where $I_R:=[0,T]\times[-R,R]\times\{\pm1\}$.

Proof. Let us set $L:=\frac{\sigma^2\|b\|}{1-\|a\|}+\frac{\sigma^2\|a'\|}{(1-\|a\|)^2}$. By Proposition 5.19 there exist for any $\varepsilon>0$ some $\eta=\eta(\varepsilon)>0$ and $N=N(\varepsilon)\in\mathbb{N}$ such that for all $n\ge N$ and all $(s,u,v)\in I_R$ we have $\mathbf{P}_s^{u,v}\big(w_s(U^n,\eta)\ge1\big)\le\frac{1}{1+2L}\varepsilon$. Fixing $\varepsilon>0$ and $(s,u,v)\in I_R$, let us for all $n\ge N$ define the set

$$K_n:=K_n(\varepsilon,s,u,v):=\big\{\omega\in\Omega_n\,:\,U^n(s)=u,\ V^n(s)=v,\ \text{and } w_s\big(U^n,\eta(\varepsilon)\big)\le1\big\}. \tag{3.82}$$

We will only prove (3.80), since (3.81) follows by an analogous argument. Splitting the expectation in (3.80) into two parts we can rewrite it for all $n\ge N$ and $i_s^n\le i\le j_T^n-1$ as

$$\mathbf{E}_s^{u,v}\Big[\big|\tilde\mu_n\big(\tau_i^n,U^n(\tau_i^n)\big)-\check\mu\big(\tau_i^n,U^n(\tau_i^n)\big)\big|\Big]=E_s^{u,v}\big(n,i,K_n\big)+E_s^{u,v}\big(n,i,K_n^c\big)$$

where $E_s^{u,v}\big(n,i,D\big):=\mathbf{E}_s^{u,v}\big[\mathbf{1}_D\big|\tilde\mu_n\big(\tau_i^n,U^n(\tau_i^n)\big)-\check\mu\big(\tau_i^n,U^n(\tau_i^n)\big)\big|\big]$ for $D\in\{K_n,K_n^c\}$. We are going to show that the first summand converges to 0 as $n\to\infty$, uniformly for all $(s,u,v)\in I_R$ and $i_s^n\le i\le j_T^n-1$, hence we will have $\sup_{(s,u,v)\in I_R}\sup_{i_s^n\le i\le j_T^n-1}E_s^{u,v}(n,i,K_n)\le\frac{1}{1+2L}\varepsilon$ for all sufficiently large $n\in\mathbb{N}$. Moreover, we prove that the second summand can be uniformly bounded by $\frac{2L}{1+2L}\varepsilon$ for all $n\ge N(\varepsilon)$. If these two claims are shown, we obtain (3.80) as well, since $\varepsilon>0$ can be chosen arbitrarily small.

In order to show the convergence of the first summand, $E_s^{u,v}\big(n,i,K_n\big)$, we will start with bounding $U^n(t)$ on K_n for all $t\in[s,T]$. For this reason, let us pick some $t\in[s,T]$ and decompose the interval $[s,t]$ into $m(t):=\lfloor\frac{t-s}{\eta}\rfloor+1$ subintervals $\big\{\big[s+(k-1)\tilde\eta,s+k\tilde\eta\big]\big\}_{1\le k\le m(t)}$ of equal length $\tilde\eta:=\frac{t-s}{m(t)}<\eta$. Then writing $U^n(t)-U^n(s)$ as a telescoping sum we obtain

$$\big|U^n(t)\big|\le\big|U^n(s)\big|+\sum_{k=1}^{m(t)}\Big|U^n\big(s+k\tilde\eta\big)-U^n\big(s+(k-1)\tilde\eta\big)\Big|.$$

Since $\tilde{\eta} < \eta$, the increments can be bounded by means of the modulus of continuity defined in (3.51), and thus we get $\left|U^n(t)\right| \le \left|U^n(s)\right| + m(t) w_s\big(U^n, \eta\big)$. Since $t \in [s,T]$ was chosen arbitrarily, and since $m(t) \le \frac{T}{\eta(\varepsilon)}+1$, we obtain $\sup_{t\in[s,T]}\left|U^n(t)\right| \le R + \frac{T}{\eta(\varepsilon)}+1 =: \tilde{R}(\varepsilon)$ on K_n. Now $\tilde{R}(\varepsilon)$ does not depend on our choice of $(s,u,v) \in I_R$, hence the convergence statement (3.78) of Lemma 5.30 implies for any fixed $\varepsilon > 0$ and uniformly for all $(s,u,v) \in I_R$:

$$\begin{aligned} E_s^{u,v}\big(n,i,K_n(\varepsilon,s,u,v)\big) &\le \mathbf{E}_s^{u,v}\Big[\mathbf{1}_{\{|U^n(\tau_i^n)|\le \tilde{R}(\varepsilon)\}}\big|\tilde{\mu}_n\big(i,U^n(\tau_i^n)\big) - \check{\mu}\big(\tau_i^n, U^n(\tau_i^n)\big)\big|\Big] \\ &\le \sup_{|x|\le\tilde{R}(\varepsilon)}\ \sup_{0\le i\le j_T^n-1}\Big|\tilde{\mu}_n\big(\tau_i^n,x\big) - \check{\mu}\big(\tau_i^n,x\big)\Big| \to 0 \qquad \text{as } n\to\infty. \end{aligned}$$

On the complementary set $K_n^c = K_n^c(\varepsilon,s,u,v)$ we cannot bound $\sup_{t\in[s,T]}\left|U^n(t)\right|$, but because $\sup_{(s,u,v)\in I_R}\mathbf{P}_s^{u,v}\big(K_n^c\big) \le \frac{1}{1+2L}\varepsilon$ for all $n \ge N$ by the definition of K_n and $N = N(\varepsilon)$, it suffices to bound for example $\sup_{x\in I\!R}\sup_{0\le i\le j_T^n-1}\left|\tilde{\mu}_n\big(\tau_i^n,x\big) - \check{\mu}\big(\tau_i^n,x\big)\right| \le 2L$, since we then have uniformly for all $n \ge N(\varepsilon)$

$$\sup_{(s,u,v)\in I_R}\ \sup_{i_s^n\le i\le j_T^n-1} E_s^{u,v}\big(n,i,K_n^c\big) \le 2L \sup_{(s,u,v)\in I_R}\mathbf{P}_s^{u,v}\big(K_n^c\big) \le \frac{2L}{1+2L}\varepsilon,$$

as desired. In order to prove that $2L$ really gives us a uniform bound on the difference of the approximate mean local expectation $\tilde{\mu}_n\big(\tau_i^n,x\big)$ and the local drift parameter $\check{\mu}\big(\tau_i^n,x\big)$, let us fix $x \in I\!R$ and $0 \le i \le j_T^n - 1$. Adding and subtracting μ and then taking absolute values, we have $\left|\tilde{\mu}_n\big(\tau_i^n,x\big) - \check{\mu}\big(\tau_i^n,x\big)\right| \le \left|\tilde{\mu}_n\big(\tau_i^n,x\big) - \mu\right| + \left|\check{\mu}\big(\tau_i^n,x\big) - \mu\right|$. The last two terms can be easily bounded: On the one hand, we have by Definition 5.24 for the approximate mean local expectation $\tilde{\mu}_n\big(\tau_i^n,x\big)$ and due to $\delta^2 = \frac{1}{n}$, Assumptions O(i) and (ii), and the definition of L:

$$\begin{aligned} \left|\tilde{\mu}_n\big(\tau_i^n,x\big) - \mu\right| &\le \frac{\sigma^2}{n(\tau_{i+1}^n - \tau_i^n)}\sum_{j=n\tau_i^n}^{n\tau_{i+1}^n-1}\ \sum_{k=1}^{n\tau_{i+1}^n-j}\left(\|b\|\prod_{l=j+1}^{j+k-1}\|a\| + \sum_{l=j+1}^{j+k-1}\|a'\|\prod_{\substack{r=j+1\\ r\ne l}}^{j+k-1}\|a\|\right) \\ &\le \sigma^2\sum_{k=1}^{\infty}\Big(\|b\|\mathfrak{a}^{k-1} + \|a'\|(k-1)\mathfrak{a}^{k-2}\Big) = \frac{\sigma^2\|b\|}{1-\|a\|} + \frac{\sigma^2\|a'\|}{(1-\|a\|)^2} = L. \end{aligned}$$

On the other hand, it directly follows from the definition of the local drift in Definition 5.29 that $\left|\check{\mu}\big(\tau_i^n,x\big) - \mu\right| \le L$ as well. Hence we have for all $x \in I\!R$ and all $0 \le i \le j_T^n - 1$ the bound $\left|\tilde{\mu}_n\big(\tau_i^n,x\big) - \check{\mu}\big(\tau_i^n,x\big)\right| \le 2L$, which was left to prove. **q.e.d.**

5.3.6 Final Preparatory Steps

Let us recall from the outline of the proof for Theorem 5.4 given at the end of Section 5.2 that we want to show that the distribution of the limit U of any converging subsequence of $\{U^n\}_{n\in I\!N}$ is a solution of the martingale problem for $(\boldsymbol{L},\nu)$, where ν is the limiting initial distribution. Especially, we have to show for all sufficiently smooth and bounded functions $f : I\!R \to I\!R$ that $M = \{M(t)\}$, given by $M(t) = f\big(U(t)\big) - \int_0^t\big(\boldsymbol{L}\,f\big)\big(s,U(s)\big)ds$ for all $t\in[0,T]$, is a martingale. For Markov chains Stroock and Varadhan (1979) construct a sequence of discrete generators such that they can approximate M by a sequence of discrete martingales. Due to the influence of the correlation between successive increments of U^n, their approach does not completely carry over to our setting of correlated random walks. However, the idea stays the same: It suffices to construct a family $\{M^n\}_{n\in I\!N}$ of processes which converge to M and which are approximately martingales, in the sense that they force M to be a true martingale.

For that reason, we will equip the probability space $(\Omega^n, \mathcal{F}^n, \mathbf{P}^n)$ for each $n \in I\!N$ with the filtration $\{\mathcal{F}_t^n\}_{t\in[0,T]}$, which is generated by the continuous-time correlated random walk U^n and its continuous-time tilt process V^n. From the definition of these two processes in (2.4) and due to our standing restriction to the case $T = 1$ it is clear that

$$\mathcal{F}_t^n := \sigma\big(U^n(s),\, V^n(s);\, 0 \le s \le t\big) = \sigma\big(X_j^n,\, Z_j^n;\, 0 \le j \le \lfloor nt \rfloor\big) \quad \text{for all } t \in [0,T].$$

Due to the correlation between the increments of U^n we cannot prove that the conditional expectation of $f\big(U^n(\tau_{i+1}^n)\big) - f\big(U^n(\tau_i^n)\big) - \big(\tau_{i+1}^n - \tau_i^n\big)(\boldsymbol{L}f)\big(\tau_i^n, U^n(\tau_i^n)\big)$ with respect to $\mathcal{F}_{\tau_i^n}^n$ converges to 0. However, we can choose $s < \tau_i^n$ in such a way that $\mathcal{F}_s^n$ contains most of the information in $\mathcal{F}_{\tau_i^n}^n$, and show that the conditional expectation with respect to this slightly smaller σ-field is approximately zero. This serves us as a first estimate for the conditional expectation $\mathbf{E}_{\tau_i^n}^{u,v}\big[M(\tau_{i+1}^n) - M(\tau_i^n)\big]$ of the increments of M during the time interval $\big[\tau_i^n, \tau_{i+1}^n\big]$. Applying the Lemmas 5.25 to 5.28 and Lemma 5.31 we will make our reasoning precise in:

Proposition 5.32. *Under Assumptions N and O we have for all $f \in C_b^3(I\!R)$:*

$$\sup_{\substack{(s,u,v)\in I_R \\ i_s^n+1 \le i \le j_T^n - 1}} \left| \mathbf{E}_s^{u,v}\left[\frac{f\big(U^n(\tau_{i+1}^n)\big) - f\big(U^n(\tau_i^n)\big)}{\tau_{i+1}^n - \tau_i^n} - (\boldsymbol{L}f)\big(\tau_i^n, U^n(\tau_i^n)\big) \right] \right| \to 0 \quad \text{as } n \to \infty \tag{3.83}$$

for every fixed $R > 0$.

Proof. Let us fix $f \in C_b^3(I\!R)$ and set $K := \frac{1}{6}\|f'''\| < \infty$. By Taylor's theorem we have for all $x, y \in I\!R$

$$\left| f(y) - f(x) - f'(x)(y-x) - \frac{1}{2} f''(x)(y-x)^2 \right| \le K\big|y-x\big|^3. \tag{3.84}$$

Recalling the remark to Definition 5.29, we know that (3.77) holds, hence we obtain

$$\begin{aligned} &\left| \mathbf{E}_s^{u,v}\left[\frac{f\big(U^n(\tau_{i+1}^n)\big) - f\big(U^n(\tau_i^n)\big)}{\tau_{i+1}^n - \tau_i^n} - (\boldsymbol{L}f)\big(\tau_i^n, U^n(\tau_i^n)\big) \right] \right| \\ &\qquad\qquad \le \left| \mathbf{E}_s^{u,v}\left[f'\big(U^n(\tau_i^n)\big) D_1^n(i) + \frac{1}{2} f''\big(U^n(\tau_i^n)\big) D_2^n(i) \right] \right| + K\mathbf{E}_s^{u,v}\Big[\big|R^n(i)\big|\Big] \end{aligned}$$

for all $n \in I\!N$, $(s,u,v) \in I_R$ and $0 \le i \le j_T^n - 1$, where the two differences $D_1^n(i)$ and $D_2^n(i)$ and the remainder term $R^n(i)$ are given by

$$D_1^n(i) = \frac{U^n(\tau_{i+1}^n) - U^n(\tau_i^n)}{\tau_{i+1}^n - \tau_i^n} - \check{\mu}\big(\tau_i^n, U^n(\tau_i^n)\big),$$

$$D_2^n(i) = \frac{\big(U^n(\tau_{i+1}^n) - U^n(\tau_i^n)\big)^2}{\tau_{i+1}^n - \tau_i^n} - \check{\sigma}^2\big(\tau_i^n, U^n(\tau_i^n)\big),$$

and

$$R^n(i) = \frac{\big(U^n(\tau_{i+1}^n) - U^n(\tau_i^n)\big)^3}{\tau_{i+1}^n - \tau_i^n} \qquad \text{for all } n \in I\!N \text{ and } 0 \le i \le j_T^n - 1.$$

Thus, in order to show the assertion, it suffices to show that for every fixed $R > 0$ the three conditional expectations $\mathbf{E}_s^{u,v}\big[f'\big(U^n(\tau_i^n)\big)D_1^n(i)\big]$, $\mathbf{E}_s^{u,v}\big[f''\big(U^n(\tau_i^n)\big)D_2^n(i)\big]$, and $\mathbf{E}_s^{u,v}\big[|R^n(i)|\big]$ converge to 0 as $n \to \infty$, uniformly for all $(s,u,v) \in I_R$ and $i_s^n + 1 \le i \le j_T^n - 1$. We start with the first term and split the expression $f'\big(U^n(\tau_i^n)\big)D_1^n(i)$ into three parts by writing $f'\big(U^n(\tau_i^n)\big)D_1^n(i) = f'\big(U^n(\tau_i^n)\big)\big(D_{1,1}^n(i) + D_{1,2}^n(i)\big) + R_1^n(i)$, where

$$D_{1,1}^n(i) := \frac{U^n(\tau_{i+1}^n) - U^n(\tau_i^n)}{\tau_{i+1}^n - \tau_i^n} - \tilde{\mu}_n\big(\tau_i^n, U^n(\tau_i^n)\big) - V^n\big(\tau_i^n\big)\tilde{d}_n\big(i, U^n(\tau_i^n)\big)$$
$$D_{1,2}^n(i) := \tilde{\mu}_n\big(\tau_i^n, U^n(\tau_i^n)\big) - \check{\mu}\big(\tau_i^n, U^n(\tau_i^n)\big),$$

and

$$R_1^n(i) := V^n\big(\tau_i^n\big)f'\big(U^n(\tau_i^n)\big)\tilde{d}_n\big(i, U^n(\tau_i^n)\big)$$

for all $n \in I\!N$ and $0 \le i \le j_T^n - 1$. Then we have for all $n \in I\!N$, $(s,u,v) \in I_R$, and $i_s^n \le i \le j_T^n - 1$:

$$\big|\mathbf{E}_s^{u,v}\big[f'\big(U^n(\tau_i^n)\big)D_1^n(i)\big]\big| \le \|f'\|\Big(\big|\mathbf{E}_s^{u,v}\big[D_{1,1}^n(i)\big]\big| + \big|\mathbf{E}_s^{u,v}\big[D_{1,2}^n(i)\big]\big|\Big) + \big|\mathbf{E}_s^{u,v}\big[R_1^n(i)\big]\big|.$$

Now the bounds on $\big|\mathbf{E}_s^{u,v}\big[D_{1,1}^n(i)\big]\big|$ and $\big|\mathbf{E}_s^{u,v}\big[D_{1,2}^n(i)\big]\big|$ follow from the previous lemmas. Namely, by conditioning on $\mathcal{F}_{\tau_i^n}$ and by the definition of $\mathbf{P}_t^{u,v}$ in (3.1) we have for all $n \in I\!N$

$$\mathbf{E}_s^{u,v}\big[D_{1,1}^n(i)\big] = \mathbf{E}_s^{u,v}\Big[\mathbf{E}_{\tau_i^n}^{U^n(\tau_i^n),V^n(\tau_i^n)}\big[D_{1,1}^n(i)\big]\Big] \quad \text{for all } (s,u,v) \in I_R, \text{ and } i_s^n \le i \le j_T^n - 1,$$

and hence by taking the absolute value inside the outer expectation and then applying Lemma 5.25 we obtain $\sup_{(s,u,v)\in I_R} \sup_{i_s^n \le i \le j_T^n - 1}\big|\mathbf{E}_s^{u,v}\big[D_{1,1}^n(i)\big]\big| = O\big(\delta^{\frac{1}{2}\beta}\big)$ as $n \to \infty$. Moreover, Lemma 5.31 implies that $\sup_{(s,u,v)\in I_R} \sup_{i_s^n \le i \le j_T^n - 1}\big|\mathbf{E}_s^{u,v}\big[D_{1,2}^n(i)\big]\big| \to 0$ as $n \to \infty$. The reason why we can state the convergence result in our assertion only for all $i_s^n + 1 \le i \le j_T^n - 1$ rests on the remainder term $R_1^n(i)$. For $i_s^n + 1 \le i \le j_T^n - 1$ we can condition on $\mathcal{F}_{\tau_{i-1}^n} \supset \mathcal{F}_s^n$ and bound

$$\big|\mathbf{E}_s^{u,v}\big[R_1^n(i)\big]\big| \le \mathbf{E}_s^{u,v}\Big[\big|\mathbf{E}_{\tau_{i-1}^n}^{U^n(\tau_{i-1}^n),V^n(\tau_{i-1}^n)}\big[R_1^n(i)\big]\big|\Big]$$

to conclude in connection with Lemma 5.28 that $\sup_{i_s^n+1 \le i \le j_T^n - 1}\big|\mathbf{E}_s^{u,v}\big[R_1^n(i)\big]\big| = O(\delta)$ as $n \to \infty$, uniformly for all $(s,u,v) \in I_R$. If we combine the convergence of $D_{1,1}^n$, $D_{1,2}^n$, and R_1^n, we indeed get from the definitions of these three quantities that

$$\sup_{(s,u,v)\in I_R} \sup_{i_s^n+1 \le i \le j_T^n - 1}\big|\mathbf{E}_s^{u,v}\big[f'\big(U^n(\tau_i^n)\big)D_1^n(i)\big]\big| \to 0 \quad \text{as } n \to \infty.$$

Similarly, by writing $D_2^n(i) = D_{2,1}^n(i) + D_{2,2}^n(i)$ with

$$D_{2,1}^n(i) := \frac{(U^n(\tau_{i+1}^n) - U^n(\tau_i^n))^2}{\tau_{i+1}^n - \tau_i^n} - \tilde{\sigma}_n^2\big(\tau_i^n, U^n(\tau_i^n)\big)$$

and

$$D_{2,2}^n(i) := \tilde{\sigma}_n^2\big(\tau_i^n, U^n(\tau_i^n)\big) - \check{\sigma}^2\big(\tau_i^n, U^n(\tau_i^n)\big) \quad \text{for all } n \in I\!N \text{ and } 0 \le i \le j_T^n - 1$$

we obtain $\sup_{(s,u,v)\in I_R} \sup_{i_s^n \le i \le j_T^n - 1}\big|\mathbf{E}_s^{u,v}\big[f''\big(U^n(\tau_i^n)\big)D_2^n(i)\big]\big| \to 0$ if we use now Lemma 5.26 instead of Lemma 5.25. Last but not least, the uniform convergence of $\mathbf{E}_s^{u,v}\big[|R^n(i)|\big]$ follows from Lemma 5.27, since the lemma implies for all $(s,u,v) \in [0,T] \times I\!R \times \{\pm 1\} \supset I_R$ and all $i_s^n \le i \le j_T^n - 1$:

$$\begin{aligned}\mathbf{E}_s^{u,v}\big[|R^n(i)|\big] &\le \mathbf{E}_s^{u,v}\Big[\mathbf{E}_{\tau_i^n}^{U^n(\tau_i^n),V^n(\tau_i^n)}\Big[\big|R^n(i)\big|\Big]\Big] \\ &\le \sup_{(x,z)\in I\!R\times\{\pm 1\}} \sup_{0 \le j \le j_T^n - 1} \mathbf{E}_{\tau_j^n}^{x,z}\Big[\big|R^n(j)\big|\Big] = O\Big(\delta^{\frac{5}{37}}\Big) \quad \text{as } n \to \infty.\end{aligned}$$

This completes the proof of the proposition. **q.e.d.**

Remark. In order to show the convergence of the expectations $\mathbf{E}_s^{u,v}\big[D_{1,1}^n(i)\big]$, $\mathbf{E}_s^{u,v}\big[D_{2,1}^n(i)\big]$, and $\mathbf{E}_s^{u,v}\big[R_1^n(i)\big]$ it is important that the convergence statements of Lemmas 5.25 to 5.28 hold uniformly for all $(x,z) \in I\!R \times \{\pm 1\}$, and not only for $|x| \le R$: In contrast to the differences in Lemma 5.31 we cannot bound $(\tau_{i+1}^n - \tau_i^n)^{-1}\big(U^n(\tau_{i+1}^n) - U^n(\tau_i^n)\big)$ on the complement of the set K_n given in (3.82), and thus the proof of Lemma 5.31 cannot be modified to prove the convergence of the three expectations under the less rigid assumption that the statements of the Lemmas 5.25 to 5.28 only hold uniformly for all $(x,z) \in [-R,R] \times \{\pm 1\}$ and every fixed $R > 0$. □

If we iterate Proposition 5.32, we arrive at:

Corollary 5.33. *Under the Assumptions N and O we have for all $f \in C_b^3(I\!R)$, all $R \in I\!R_+$, and $0 \le s \le t \le T$*

$$\sup_{\substack{|u|\le R\\ |v|=1}} \left| \mathbf{E}_s^{u,v}\left[f\big(U^n(t)\big) - f\big(U^n(s)\big) - \sum_{i=i_s^n+1}^{j_t^n-1} \big(\tau_{i+1}^n - \tau_i^n\big)\big(\boldsymbol{L}f\big)\big(\tau_i^n, U^n(\tau_i^n)\big)\right]\right| \to 0 \quad \textit{as } n \to \infty.$$

Proof. Since the statement trivially holds for $s = t$, let us fix $f \in C_b^3(I\!R)$ and $0 \le s < t \le T$. Recall the definitions of τ_i^n, i_s^n and j_t^n in Definition 5.22. Due to the inequalities (3.55) and (3.54) we have

$$0 \le t - \tau_{j_t^n}^n \le 2\delta \quad \text{and} \quad 0 \le \tau_{i_s^n+1}^n - s \le \tau_{i_s^n+1}^n - \tau_{i_s^n}^n + \tau_{i_s^n}^n - s \le 4\delta \quad \text{for all } n \in I\!N. \quad (3.85)$$

This shows that $\tau_{j_t^n}^n \to t$ and $\tau_{i_s^n+1}^n \to s$ as $n \to \infty$. Hence we have $\tau_{i_s^n+1}^n \le \tau_{j_t^n}^n$ for all sufficiently large $n \in I\!N$, and for those n by employing the telescoping sum

$$\begin{aligned} f\big(U^n(t)\big) - f\big(U^n(s)\big) = f\big(U^n(t)\big) - f\big(U^n\big(\tau_{j_t^n}^n\big)\big) + \sum_{i=i_s^n+1}^{j_t^n-1}\Big(f\big(U^n(\tau_{i+1}^n)\big) - f\big(U^n(\tau_i^n)\big)\Big) \\ + f\big(U^n\big(\tau_{i_s^n+1}^n\big)\big) - f\big(U^n(s)\big) \end{aligned}$$

we see that it suffices to show that the three expressions

$$\begin{aligned} D_1^n(u,v) &:= \mathbf{E}_s^{u,v}\Big[f\big(U^n(t)\big) - f\big(U^n\big(\tau_{j_t^n}^n\big)\big)\Big], \\ D_2^n(u,v) &:= \mathbf{E}_s^{u,v}\Big[f\big(U^n\big(\tau_{i_s^n+1}^n\big)\big) - f\big(U^n(s)\big)\Big], \end{aligned}$$

and

$$D_3^n(u,v) := \sum_{i=i_s^n+1}^{j_t^n-1} \big(\tau_{i+1}^n - \tau_i^n\big)\mathbf{E}_s^{u,v}\left[\frac{f\big(U^n(\tau_{i+1}^n)\big) - f\big(U^n(\tau_i^n)\big)}{\tau_{i+1}^n - \tau_i^n} - \big(\boldsymbol{L}f\big)\big(\tau_i^n, U^n(\tau_i^n)\big)\right]$$

satisfy $\sup_{(u,v)\in[-R,R]\times\{\pm1\}}\big|D_i^n(u,v)\big| \to 0$ as $n \to \infty$ for $i \in \{1,2,3\}$. For the first two expressions, such a convergence result follows easily from Lemma 5.23. Namely, the bounds in (3.85) enable us to conclude with Lemma 5.23 that uniformly for all $(u,v) \in I\!R \times \{\pm 1\}$:

$$\big|D_1^n(u,v)\big| \le \mathbf{E}_s^{u,v}\Big[\big|f\big(U^n(t)\big) - f\big(U^n\big(\tau_{j_t^n}^n\big)\big)\big|\Big] \le \|f'\|\mathbf{E}_s^{u,v}\Big[\big|U^n(t) - U^n\big(\tau_{j_t^n}^n\big)\big|\Big] = O\Big(\delta^{\frac12}\Big)$$

and

$$\big|D_2^n(u,v)\big| \le \|f'\|\mathbf{E}_s^{u,v}\Big[\big|U^n\big(\tau_{i_s^n+1}^n\big) - U^n(s)\big|\Big] = O\Big(\delta^{\frac12}\Big) \quad \text{as } n \to \infty.$$

The convergence of $D_3^n(u,v)$ is not more difficult to prove. By Proposition 5.32 we have for all fixed $R \in I\!R_+$:

$$S^n := \sup_{\substack{(s,u,v)\in I_R \\ i_s^n+1\le i\le j_t^n-1}} \left| \mathbf{E}_s^{u,v}\left[\frac{f\big(U^n(\tau_{i+1}^n)\big) - f\big(U^n(\tau_i^n)\big)}{\tau_{i+1}^n - \tau_i^n} - (\boldsymbol{L}f)\big(\tau_i^n, U^n(\tau_i^n)\big)\right]\right| \to 0 \quad \text{as } n \to \infty.$$

Then it follows from $I_R = [0,T]\times[-R,R]\times\{\pm 1\}$ that uniformly for all $(u,v) \in [-R,R]\times\{\pm 1\}$

$$\big|D_3^n(s,t,u,v)\big| \le \sum_{i=i_s^n+1}^{j_t^n-1} \big(\tau_{i+1}^n - \tau_i^n\big) S^n = \Big(\tau_{j_t^n}^n - \tau_{i_s^n+1}^n\Big) S^n \le T S^n \to 0 \quad \text{as } n \to \infty.$$

Since $0 \le s \le t \le T$ and $R \in I\!R_+$ were chosen arbitrarily, our claim follows. **q.e.d.**

We now want to show that for all sufficiently large $n \in I\!N$ the process $M^n = \{M^n(t)\}$ given by $M^n(t) = f\big(U^n(t)\big) - \int_0^t \big(\boldsymbol{L}\, f\big)\big(s, U^n(s)\big)ds$ for all $t \in [0,T]$ is approximately a martingale, which approximates the continuous process M from the beginning of the section. Therefore, we have to show that we can replace the sum in Corollary 5.33 by the integral $\int_s^t \big(\boldsymbol{L}\, f\big)\big(s, U^n(s)\big)ds$. This is the aim of the last lemma which we need before concluding the proof of the main convergence theorem.

Lemma 5.34. *Let us assume that for each $n \in I\!N$ we are given some stochastic process S^n in $D[0,T]$ defined on some probability space $\big(\Omega^n, \mathcal{F}^n, \mathbf{P}^n\big)$. If $S^n \Rightarrow S$ as $n \to \infty$ for some a.s. continuous process S, then for any bounded continuous function $L : I\!R \to I\!R$ and all $0 \le s \le t \le T$ we have*

$$\mathbf{E}^n\left[\left|\int_s^t L\big(\tau, S^n(\tau)\big)d\tau - \sum_{i=i_s^n+1}^{j_t^n-1} \big(\tau_{i+1}^n - \tau_i^n\big) L\big(\tau_i^n, S^n(\tau_i^n)\big)\right|\right] \to 0 \quad \text{as } n \to \infty.$$

Proof. We want to apply the bounded convergence theorem. However, since the probability spaces $\big(\Omega^n, \mathcal{F}^n, \mathbf{P}^n\big)$, on which the S^n are defined, may differ for each $n \in I\!N$, we first have to find a common probability space for all $\{S^n\}_{n\in I\!N}$ and S, or at least for stochastic processes in $D[0,T]$ which have the same distributions. For this purpose we recall from Theorem 3.3.1 in Ethier and Kurtz (1986) that the weak convergence $(S^n \,|\, \mathbf{P}^n) \Rightarrow (S \,|\, \mathbf{P})$ as $n \to \infty$ is equivalent to the convergence of the corresponding distributions in the Prohorov metric, since $D[0,T]$ is separable. Thus, by the Skorohod representation (Theorem 3.1.8 in Ethier and Kurtz (1986)), there exists some probability space $\big(\Omega', \mathcal{F}', \mathbf{P}'\big)$ on which are defined some stochastic processes $\{Y^n\}_{n\in I\!N}$ and Y such that $\mathbf{P}'(Y^n)^{-1} = \mathbf{P}^n(S^n)^{-1}$, $\mathbf{P}'Y^{-1} = \mathbf{P}S^{-1}$, and such that under $\mathbf{P}'$ we have $\lim_{n\to\infty} Y_n = Y$ a.s.

Let us now fix $0 \le s \le t \le T$. If a sequence $\{z_n\}_{n\in I\!N}$ in $D[0,T]$ converges to some *continuous* function z in the Skorohod topology, it also converges in the uniform topology (see p.112 in Billingsley (1968)). In such a situation, we conclude from adding and subtracting $z(t_n)$ that $\lim_{n\to\infty} z_n(t_n) = z(t)$ for any sequence $\{t_n\}_{n\in I\!N}$ in $[0,T]$ with $\lim_{n\to\infty} t_n = t$. Moreover, we find some $a < \min_{t\in[0,T]} z(t)$ and $b > \max_{t\in[0,T]} z(t)$ such that $z_n(t) \in [a,b]$ for all $t \in [0,T]$ and $n \in I\!N$. On $[0,T]\times[a,b]$, the function $L : [0,T]\times I\!R \to I\!R$ is uniformly continuous, and since (3.54) yields $\tau_{i+1}^n - \tau_i^n < 2\delta$ for all $i_s^n + 1 \le i \le j_t^n - 1$ the uniform convergence of $\{z_n\}_{n\in I\!N}$ also implies

$$\sum_{i=i_s^n+1}^{j_t^n-1} \Big(L\big(\tau, z(\tau)\big) - L\big(\tau_i^n, z_n(\tau_i^n)\big)\Big)\mathbf{1}_{[\tau_i^n, \tau_{i+1}^n)}(\tau) \to 0 \quad \text{as } n \to \infty.$$

Since Y is continuous on $[0,T]$, we get by adding and subtracting $L(\tau, Y(\tau))$ and noting that (3.54) and (3.55) imply $s < \tau^n_{i^n_s+1} \to s$ and $t \geq \tau^n_{j^n_t} \to t$ the convergence

$$L\big(\tau, Y^n(\tau)\big) - \sum_{i=i^n_s+1}^{j^n_t-1} L\big(\tau^n_i, Y^n(\tau^n_i)\big)\mathbf{1}_{[\tau^n_i,\tau^n_{i+1})}(\tau) \to 0 \quad \text{as } n\to\infty,\ \lambda_{[s,t]} \times \mathbf{P}'\text{-a.e.},$$

where $\lambda_{[s,t]}$ is the Lebesgue measure on $[s,t]$. Since $\|L\| < \infty$ we can bound the last line by $2\|L\|$, and then apply the bounded convergence theorem to conclude

$$\begin{aligned}
&\mathbf{E}'\left[\left|\int_s^t L\big(\tau, Y^n(\tau)\big)\,d\tau - \sum_{i=i^n_s+1}^{j^n_t-1}\big(\tau^n_{i+1}-\tau^n_i\big)L\big(\tau^n_i, Y^n(\tau^n_i)\big)\right|\right] \\
&\quad = \mathbf{E}'\left[\left|\int_s^t \left(L\big(\tau, Y^n(\tau)\big) - \sum_{i=i^n_s+1}^{j^n_t-1} L\big(\tau^n_i, Y^n(\tau^n_i)\big)\mathbf{1}_{[\tau^n_i,\tau^n_{i+1})}(\tau)\right)d\tau\right|\right] \to 0 \quad \text{as } n\to\infty.
\end{aligned}$$

But for each $n \in I\!N$ the process Y^n has under $\mathbf{P}'$ the same distribution as S^n under $\mathbf{P}^n$, and hence the statement of the lemma follows. **q.e.d.**

5.3.7 Proof of the Main Convergence Theorem

Now we are finally in a position to prove the main convergence theorem for our correlated random walks, as it was stated in Theorem 5.4. As pointed out before, we want to imitate the proof of an analogous statement for the convergence of Markov chains as given in Section 11.2 of Stroock and Varadhan (1979). However, we have to adjust it because of the special features of correlated random walks, especially because of the dependences within the tilt process.

Proof of Theorem 5.4. We know by Corollary 5.20 that the sequence $\{U^n\}_{n\in I\!N}$ is tight, hence it is relatively compact by Prohorov's theorem (Theorem 6.1 in Billingsley (1968)), and, in addition, Corollary 5.20 also yields that any limiting process U is continuous. Thus, every subsequence of $\{U^n\}_{n\in I\!N}$ contains a further subsequence $\{U^{n_k}\}_{k\in I\!N}$ which converges weakly to *some* continuous process U on $D[0,T]$. If we can show that *all* converging subsequences weakly converge to the same process U, an application of Theorem 2.3 in Billingsley (1968) yields that then the whole sequence $\{U^n\}_{n\in I\!N}$ converges weakly to U.
To that end, let us take some converging subsequence $\{U^{n_k}\}_{k\in I\!N}$ of $\{U^n\}_{n\in I\!N}$ such that there exists some process U in $\big(D[0,T],\mathcal{B}(D[0,T])\big)$, which is defined on some probability space $\big(\widetilde{\Omega},\widetilde{\mathcal{F}},\widetilde{\mathbf{P}}\big)$ such that $U^{n_k} \Rightarrow U$ as $k\to\infty$. By considering the distribution $\mathbf{P} := \widetilde{\mathbf{P}}U^{-1}$ of U, we can without loss of generality assume that $\big(\widetilde{\Omega},\widetilde{\mathcal{F}},\widetilde{\mathbf{P}}\big) = \big(D[0,T],\mathcal{B}(D[0,T]),\mathbf{P}\big)$ and that U is the canonical process given by $U(t,\omega)=\omega(t)$ for all $t\in[0,T]$ and $\omega\in D[0,T]$. Let us then introduce the filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$ on the probability space $\big(D[0,T],\mathcal{B}(D[0,T]),\mathbf{P}\big)$ by setting $\mathcal{F}_t := \sigma\big(U(s);\,0\le s\le t\big)$ for all $t\in[0,T]$. We now want to show that $\mathbf{P}$ is a solution of the martingale problem for $(\boldsymbol{L},\nu)$. Since our assumptions imply $\mathbf{P}\big(U(0)\in A\big)=\nu(A)$ for all $A\in\mathcal{B}(I\!R)$, we only have to show that for all functions $f: I\!R\to I\!R$ with compact support and which possess bounded derivatives of all orders,

$$\left(f\big(U(t)\big) - \int_0^t (\boldsymbol{L}f)\big(\tau, U(\tau)\big)d\tau, \mathcal{F}_t;\ 0\le t\le T\right) \quad \text{is a } \mathbf{P}\text{-martingale.} \tag{3.86}$$

Slightly more than that, we will show that (3.86) holds for all $f\in C^3_b(I\!R)$. For this purpose, let us fix $f\in C^3_b(I\!R)$ and $0\le s\le t\le T$, and note that by the definition of $\boldsymbol{L}$ in (2.7) the function $(\boldsymbol{L}f): [0,T]\times I\!R\to I\!R$ is bounded and continuous, due to Assumption O. Thus, the functional $\Psi : D[0,T] \to I\!R$ given by $\Psi(g) := f\big(g(t)\big) - f\big(g(s)\big) - \int_s^t \big(\boldsymbol{L}\, f\big)\big(\tau, g(\tau)\big)\, d\tau$ for all $g \in D[0,T]$ is continuous as well, and it can be bounded by

$$\big|\Psi(g)\big| \le 2\|f\| + \int_s^t \big\|(\boldsymbol{L}\, f)\big\| du \le 2\|f\| + T\|\boldsymbol{L}\, f\| =: M < \infty \quad \text{for all } g \in D[0,T]. \tag{3.87}$$

By the definition of weak convergence, $U^{n_k} \Rightarrow U$ as $k \to \infty$ implies for all bounded and continuous functionals $\Phi : D[0,T] \to I\!R$

$$\begin{aligned}&\mathbf{E}\bigg[\Phi(U)\bigg(f\big(U(t)\big) - f\big(U(s)\big) - \int_s^t (\boldsymbol{L}f)\big(\tau, U(\tau)\big)d\tau\bigg)\bigg] \\ &\quad = \lim_{k\to\infty} \mathbf{E}^{n_k}\bigg[\Phi\big(U^{n_k}\big)\bigg(f\big(U^{n_k}(t)\big) - f\big(U^{n_k}(s)\big) - \int_s^t (\boldsymbol{L}f)\big(\tau, U^{n_k}(\tau)\big)d\tau\bigg)\bigg].\end{aligned} \tag{3.88}$$

If we can show that the limit on the right hand side vanishes for all $0 \le s \le t \le T$ and all bounded and continuous $\mathcal{F}_s$-measurable functionals $\Phi : D[0,T] \to I\!R$ then a monotone class argument and the definition of the conditional expectation imply that

$$\mathbf{E}\bigg[f\big(U(t)\big) - f\big(U(s)\big) - \int_s^t (\boldsymbol{L}f)(\tau, X(\tau))d\tau\bigg|\mathcal{F}_s\bigg] = 0 \quad \text{for all } 0 \le s \le t \le T, \tag{3.89}$$

and thus indeed (3.86) holds for the chosen function $f \in C_b^3(I\!R)$.
If (3.89) even holds for all functions $f \in C_b^3(I\!R)$, then $\mathbf{P}$ solves the martingale problem for $(\boldsymbol{L}, \nu)$. In this case we can apply Theorem 5.3.3 in Ethier and Kurtz (1986) to conclude that there exists an extended probability space on which U solves (2.8) with $\mathbf{P}\big(U(0) \in A\big) = \nu(A)$ for all $A \in \mathcal{B}(I\!R)$. Now the uniqueness of the martingale problem for $(\boldsymbol{L}, \nu)$ is equivalent to the uniqueness in the sense of probability law of solutions to the stochastic integral equation (2.8) with $\mathbf{P}\big(U(0) \in A\big) = \nu(A)$ for all $A \in \mathcal{B}(I\!R)$: If the martingale problem has a unique solution all weak limits of subsequences of $\{U^n\}_{n\in I\!N}$ have to coincide. Hence, $\{U^n\}_{n\in I\!N}$ converges to the diffusion process given by (2.8) and $\mathbf{P}\big(U(0) \in A\big) = \nu(A)$.

It remains to prove that for any $f \in C_b^3(I\!R)$, any $0 \le s \le t \le T$, any bounded and continuous $\mathcal{F}_s$-measurable function $\Phi : D[0,T] \times I\!R \to I\!R$, and any weakly convergent subsequence $\big\{U^{n_k}\big\}_{k\in I\!N}$ of $\{U^n\}_{n\in I\!N}$ the limit on the right-hand side of (3.88) is 0. Without loss of generality let us assume that the whole sequence converges, i.e. $U^n \Rightarrow U$ as $n \to \infty$ for some continuous process U. Then we have to show

$$\lim_{n\to\infty} \mathbf{E}^n\bigg[\Phi(U^n)\bigg(f\big(U^n(t)\big) - f\big(U^n(s)\big) - \int_s^t (\boldsymbol{L}f)\big(\tau, U^n(\tau)\big)d\tau\bigg)\bigg] = 0. \tag{3.90}$$

For that purpose keep $f \in C_b^3(I\!R)$, $0 \le s \le t \le T$, and some bounded and continuous $\big(\mathcal{F}_s, \mathcal{B}(I\!R)\big)$-measurable mapping $\Phi : D[0,T] \to I\!R$ fixed for the rest of the proof.
As a first step, we want to show that $\Phi(U^n) : \Omega^n \to I\!R$ is $\big(\mathcal{F}_s^n, \mathcal{B}(I\!R)\big)$-measurable for each fixed $n \in I\!N$. Since $\Phi : D[0,T] \to I\!R$ is $\big(\mathcal{F}_s, \mathcal{B}(I\!R)\big)$-measurable, it suffices to show that the mapping $U^n : \Omega^n \to D[0,T]$ is $\big(\mathcal{F}_s^n, \mathcal{F}_s\big)$-measurable. For this purpose define the projection $\pi : D[0,T] \to D[0,s]$ by $\pi(\omega, r) = \omega(r)$ for all $r \in [0,s]$. Then it is clear that the σ-field generated by all single observations of $U^n : \Omega^n \to D[0,T]$ up to time s satisfies $\sigma\big(U^n(r);\, 0 \le r \le s\big) = \sigma\big((\pi U^n)(r);\, 0 \le r \le s\big)$, and since $D[0,s]$ is separable it follows that $\sigma\big((\pi U^n)(r);\, 0 \le r \le s\big) = \big(\pi U^n\big)^{-1}\mathcal{B}\big(D[0,s]\big)$. Hence we conclude that

$$\sigma\big(U^n(r);\, 0 \le r \le s\big) = \big(\pi U^n\big)^{-1}\mathcal{B}\big(D[0,s]\big) = (U^n)^{-1}\pi^{-1}\mathcal{B}\big(D[0,s]\big). \tag{3.91}$$

Replacing $U^n : \Omega^n \to D[0,T]$ by the canonical process $U : D[0,T] \to D[0,T]$ the same arguments enable us to rewrite $\mathcal{F}_s$ as $\mathcal{F}_s = \sigma\big(U(r);\, 0 \le r \le s\big) = \pi^{-1}\mathcal{B}\big(D[0,s]\big)$ and hence, we may substitute this equality in (3.91) and apply the definition of $\mathcal{F}_s^n$ on Ω^n to conclude

$$\mathcal{F}_s^n = \sigma\Big(U^n(r),\, V^n(r);\, 0 \le r \le s\Big) \supset \sigma\big(U^n(r);\, 0 \le r \le s\big) = (U^n)^{-1}\mathcal{F}_s. \tag{3.92}$$

This shows that U^n is indeed $\big(\mathcal{F}_s^n, \mathcal{F}_s\big)$-measurable, and consequently $\Phi(U^n)$ is an $\big(\mathcal{F}_s^n, \mathcal{B}(I\!R)\big)$-measurable random variable for all $n \in I\!N$. Therefore we get for all $n \in I\!N$

$$\begin{aligned}&\mathbf{E}^n\left[\Phi(U^n)\Big(f\big(U^n(t)\big) - f\big(U^n(s)\big) - \int_s^t \big(\boldsymbol{L}\, f\big)\big(\tau, U^n(\tau)\big)d\tau\Big)\right]\\&\quad= \mathbf{E}^n\left[\Phi(U^n)\mathbf{E}^n\left[f\big(U^n(t)\big) - f\big(U^n(s)\big) - \int_s^t \big(\boldsymbol{L}\, f\big)\big(\tau, U^n(\tau)\big)d\tau\Big)\Big|\mathcal{F}_s^n\right]\right],\end{aligned}$$

and by (3.1) we can rewrite the inner expectation as $E^n\big(U^n(s), V^n(s)\big)$, where

$$E^n(u,v) := \mathbf{E}_s^{u,v}\left[f\big(U^n(t)\big) - f\big(U^n(s)\big) - \int_s^t \big(\boldsymbol{L}\, f\big)\big(\tau, U^n(\tau)\big)d\tau\right] \quad \text{for all } (u,v) \in I\!R \times \{\pm 1\}.$$

Taking now absolute values and using $|\!|\!|\Phi|\!|\!| := \|\Phi\|_{D[0,T]} < \infty$, we thus can bound

$$\left|\mathbf{E}^n\left[\Phi(U^n)\Big(f\big(U^n(s)\big) - f\big(U^n(s)\big) - \int_s^t (\boldsymbol{L}f)\big(\tau, U^n(\tau)\big)d\tau\Big)\right]\right| \le |\!|\!|\Phi|\!|\!|\mathbf{E}^n\Big[\big|E^n\big(U^n(s), V^n(s)\big)\big|\Big].$$

Since for all $(u,v) \in I\!R \times \{\pm 1\}$ we can rewrite $E^n(u,v)$ as $E^n(u,v) = D_1^n(u,v) - D_2^n(u,v)$ with $D_1^n(u,v)$ and $D_2^n(u,v)$ being defined by

$$D_1^n(u,v) = \mathbf{E}_s^{u,v}\left[f\big(U^n(s)\big) - f\big(U^n(s)\big) - \sum_{i=i_s^n+1}^{j_t^n-1} \big(\tau_{i+1}^n - \tau_i^n\big)(\boldsymbol{L}f)\big(\tau_i^n, U^n(\tau_i^n)\big)\right] \tag{3.93}$$

and

$$D_2^n(u,v) = \mathbf{E}_s^{u,v}\left[\int_s^t (\boldsymbol{L}f)\big(\tau, U^n(\tau)\big)d\tau - \sum_{i=i_s^n+1}^{j_t^n-1} \big(\tau_{i+1}^n - \tau_i^n\big)(\boldsymbol{L}f)\big(\tau_i^n, U^n(\tau_i^n)\big)\right], \tag{3.94}$$

respectively, we can apply the triangular inequality to conclude

$$\mathbf{E}^n\Big[\big|E^n\big(U^n(s), V^n(s)\big)\big|\Big] \le \mathbf{E}\Big[\big|D_1^n\big(U^n(s), V^n(s)\big)\big|\Big] + \mathbf{E}^n\Big[\big|D_2^n\big(U^n(s), V^n(s)\big)\big|\Big].$$

Hence it suffices to show that both $\mathbf{E}^n\big[\big|D_1^n\big(U^n(s), V^n(s)\big)\big|\big]$ and $\mathbf{E}^n\big[\big|D_2^n\big(U^n(s), V^n(s)\big)\big|\big]$ converge to 0 as $n \to \infty$.

In order to show the convergence of the first term, we employ Corollary 5.20, which gave us the tightness of $\{U^n\}$, and conclude that for any $\varepsilon > 0$ there exists a compact set $K_\varepsilon \subset D[0,T]$ such that $\mathbf{P}^n\big(U^n \in K_\varepsilon\big) \ge 1 - \varepsilon$ for all $n \in I\!N$. Let us fix $\varepsilon > 0$. On K_ε we have $R_\varepsilon := \sup_{g \in K_\varepsilon} \sup_{t \in [0,T]} \big|g(s)\big| < \infty$, and hence we get from the definition of D_1^n and Corollary 5.33:

$$\mathbf{E}^n\Big[\big|D_1^n\big(U^n(s), V^n(s)\big)\big|\mathbf{1}_{\{U^n \in K_\varepsilon\}}\Big] \le \sup_{(u,v) \in [-R_\varepsilon, R_\varepsilon] \times \{\pm 1\}} \big|D_1^n(u,v)\big| \to 0 \quad \text{as } n \to \infty.$$

On the other hand, by exactly the same arguments as in (3.87), we can bound the term within the expectation in (3.93) by M as well, and hence we also get $\big|D_1^n\big(U^n(s),V^n(s)\big)\big| \le M$, i.e.

$$\mathbf{E}^n\Big[\big|D_1^n\big(U^n(s),V^n(s)\big)\big|\mathbf{1}_{\{U^n\notin K_\varepsilon\}}\Big] \le M\mathbf{P}^n\big(U^n\notin K_\varepsilon\big) \le M\varepsilon.$$

Together, the last two bounds show that $\mathbf{E}^n\big[\big|D_1^n\big(U^n(s),V^n(s)\big)\big|\big] \to 0$ as $n\to\infty$. Last but not least, we plug in the definition of $D_2^n(u,v)$ from (3.94) and move the absolute value inside the conditional expectation to obtain

$$\mathbf{E}^n\Big[\big|D_2^n\big(U^n(s),V^n(s)\big)\big|\Big] \le \mathbf{E}^n\Bigg[\bigg|\int_s^t (\boldsymbol{L}f)\big(\tau,U^n(\tau)\big)\,d\tau - \sum_{i=i_s^n+1}^{j_t^n-1}\big(\tau_{i+1}^n-\tau_i^n\big)(\boldsymbol{L}f)\big(\tau_i^n,U^n(\tau_i^n)\big)\bigg|\Bigg].$$

Since the term on the right-hand side converges to 0 as $n\to\infty$ by Lemma 5.34, this shows $\lim_{n\to\infty}\mathbf{E}^n\Big[\big|D_2^n\big(U^n(s),V^n(s)\big)\big|\Big] = 0$. Thus, (3.90) holds, and the proof of Theorem 5.4 is complete. **q.e.d.**

Collection of Stated Assumptions

Assumption A (p. 78). *The associated small investor price function* $\overline{\psi} : [0,T] \times I\!R \to I\!R$ *satisfies*

$$\overline{\psi}\big(t_k^n, U_{k-1}^n - \delta\big) < \overline{\psi}\big(t_{k-1}^n, U_{k-1}^n\big) < \overline{\psi}\big(t_k^n, U_{k-1}^n + \delta\big) \quad \text{for all } 1 \le k \le n. \tag{2.1.5}$$

Assumption B (Multiplicative structure of ψ, p. 96). *There exists a locally bounded function* $f : I\!R \to (0,\infty)$ *which is continuous a.e. (with respect to the Lebesgue measure on* $I\!R$*) such that the equilibrium price function* $\psi : [0,T] \times I\!R^2 \to I\!R$ *can be written as*

$$\psi(t,u,\xi) = \overline{\psi}(t,u) f(\xi) \quad \text{for all } (t,u,\xi) \in [0,T] \times I\!R \times I\!R. \tag{2.4.1}$$

Assumption C (p. 129). *The small investor price function* $\overline{\psi} : [0,T] \times I\!R \to I\!R$*, mapping* $(t,u) \mapsto \overline{\psi}(t,u)$*, is continuously differentiable with respect to* t *and twice continuously differentiable with respect to* u*. The function* $\overline{\psi}$ *itself and its spacial derivative* $\overline{\psi}_u$ *are strictly positive, and the function satisfies satisfies* $\big\|\frac{\overline{\psi}_t}{\overline{\psi}_u}\big\| < \infty$ *and* $L_0 := \big\|\frac{\overline{\psi}}{\overline{\psi}_u}\big\| < \infty$*. For the function* $f : I\!R \to (0,\infty)$ *we assume that it is at least twice continuously differentiable. Finally we assume that the price determining measure* $\mu \in \mathcal{M}(f)$ *has a finite first moment.*

Assumption D (p. 129). *The price system* (ψ, μ) *excludes any immediate transaction gains, i.e. the local transaction loss rate function* $k_\mu : I\!R^2 \to I\!R$ *of Definition 2.13 is nonnegative.*

Assumption E (p. 159). *The transformed loss function* $\kappa : g(I\!R) \to I\!R$ *of (3.3.1) vanishes, i.e. we have* $d(\mu) f'(\xi) = 0$ *for all* $\xi \in I\!R$*. Moreover, suppose that the two components* $\overline{\psi}$ *and* f *of* ψ *belong to the Hölder spaces* $\overline{\psi} \in \widehat{H}^{1+\frac{1}{2}\beta, 3+\beta}\big([0,T] \times I\!R\big)$ *and* $H_{loc}^{1+\beta}(I\!R)$*, respectively. The function* $\zeta : I\!R \to I\!R$ *for the final condition is bounded and can be written as* $\zeta = \zeta^{ac} + \zeta^d$*, where* ζ^{ac} *is absolutely continuous with respect to the Lebesgue measure on* $I\!R$ *and* ζ^d *consists only of (left and right) jumps. The parameter* $\alpha \in I\!R$ *is some arbitrary real number.*

Assumption E (p. 160). *The transaction loss function* $\kappa : g(I\!R) \to I\!R$ *of (3.3.1) is nonnegative, i.e.* $d(\mu) f'(\xi) \ge 0$ *for all* $\xi \in I\!R$*. Moreover, suppose that the two components* $\overline{\psi}$ *and* f *of* ψ *belong to the Hölder spaces* $\overline{\psi} \in \widehat{H}^{1+\frac{1}{2}\beta, 3+\beta}\big([0,T] \times I\!R\big)$ *and* $H_{loc}^{3+\beta}(I\!R)$*, respectively. The function* $\zeta : I\!R \to I\!R$ *for the final condition belongs to the class* $C_b^1\big(I\!R\big)$*. The parameter* $\alpha \in I\!R$ *is sufficiently close to* 0 *so that a solution* $\varphi : [0,T] \times I\!R \to I\!R$ *to the final problem (3.4.3), (3.4.4) exists and so that this solution satisfies the constraint*

$$1 + 2d(\mu) \frac{\psi_\xi\big(t,u,\varphi(t,u)\big)}{\psi_u\big(t,u,\varphi(t,u)\big)} \varphi_u(t,u) \ge \varepsilon \quad \text{for all } (t,u) \in [0,T] \times I\!R \text{ and some } \varepsilon > 0.$$

Assumption G (On the price system (ψ, μ), p. 172). *There exist some strictly positive functions* $\overline{\psi} \in \widehat{H}^{2+\frac{1}{2}\beta, 4+\beta}\big([0,T] \times I\!R\big)$ *and* $f \in H_{loc}^{4+\beta}(I\!R)$ *such that the equilibrium price function* $\psi : [0,T] \times I\!R^2 \to I\!R$ *satisfies the multiplicative structure*

$$\psi(t,u,\xi) = \overline{\psi}(t,u) f(\xi) \quad \text{for all } (t,u,\xi) \in [0,T] \times I\!R^2.$$

We also have $L_0 := \left\|\frac{\bar{\psi}}{\bar{\psi}_u}\right\| < \infty$. *For the measure* μ *there exists some* $\eta > 0$ *such that* $\int e^{\eta|\theta|}\mu(d\theta) < \infty$. *The price system* (ψ, μ) *excludes any immediate transaction gains, i.e. by the remark following Definition 3.17 we have in particular* $d(\mu)f'(\xi) \geq 0$ *for all* $\xi \in I\!R$.

Assumption H (Solvability of the non-linear PDE for φ, p. 173). *For* $\zeta \in H^{4+\beta}(I\!R)$ *the parameter* $\alpha \in I\!R$ *is chosen so close to* 0 *that there exists some* $\varphi \in H^{2+\frac{1}{2}\beta,4+\beta}\big([0,T]\times I\!R\big)$ *which solves the final value problem (3.4.3), (3.4.4), which satisfies*

$$\alpha \inf_{z\in I\!R} \zeta(z) \leq \varphi(t,u) \leq \alpha \sup_{z\in I\!R} \zeta(z) \quad \textit{for all } (t,u) \in [0,T] \times I\!R, \tag{4.2.3}$$

and for which there exists some $\varepsilon > 0$ *such that*

$$2d(\mu)\frac{\psi_\xi\big(t,u,\varphi(t,u)\big)}{\psi_u\big(t,u,\varphi(t,u)\big)}\varphi_u(t,u) \geq -1+\varepsilon \quad \textit{for all } (t,u) \in [0,T] \times I\!R. \tag{4.2.4}$$

Assumption I (The p-martingale measures are well-defined, p. 173). *The scaling parameter* $\alpha \in I\!R$ *from Assumption H is also chosen so close to* 0 *that*

$$\frac{\psi_\xi\big(t,u,\varphi(t,u)\big)}{\psi_u\big(t,u,\varphi(t,u)\big)}\varphi_u(t,u) \geq -1+\varepsilon \quad \textit{for all } (t,u) \in [0,T] \times I\!R. \tag{4.2.5}$$

Assumption J (Stock holdings converge close to maturity, p. 173). *Immediately before and at maturity, the large investor's stock holdings converge in the sense that*

$$\max_{k\in\{n-1,n\}} \big\|\xi^n\big(t_k^n,\,\cdot\,\big) - \varphi\big(t_k^n,\,\cdot\,\big)\big\|_{\mathcal{U}_k^n} = O\big(\delta^{4+\beta}\big) \quad \textit{as } n \to \infty. \tag{4.2.7}$$

Assumption K (Cash holdings converge at maturity, p. 174). *For some* $b_0^\alpha \in I\!R$ *the function* $b^\alpha : I\!R \to I\!R$ *is given by*

$$b^\alpha(u) = b_0^\alpha - \alpha\int_{u_0}^{u} \psi\big(T,\tilde{u},\alpha\zeta(\tilde{u})\big)\zeta_u(\tilde{u})\,d\tilde{u},$$

and $\bar{v} : [0,T]\times I\!R \to I\!R$ *is the solution to the linear final value problem (3.4.6), (3.4.7) which corresponds to the continuous strategy function* $\varphi : [0,T]\times I\!R \to I\!R$ *via* $\gamma(t,u) = \int_0^{\varphi(t,u)} f(z)\,dz$ *for all* $(t,u) \in [0,T] \times I\!R$. *The cash holdings at maturity satisfy*

$$\big\|b^n(t_n^n,\,\cdot\,) - b^\alpha(\,\cdot\,)\big\|_{\mathcal{U}_n^n} = O\big(\delta^{2+\beta}\big) \quad \textit{as } n \to \infty. \tag{4.2.8}$$

Assumption L (Pre-trading behavior of the stock holdings, p. 174). *One of the two following conditions holds:*

(*i*) *The price determining measure* μ *is (or can be chosen as) the Dirac measure* δ_1 *concentrated in* 1, *so that for all* $(t,u,\xi_1,\xi_2) \in [0,T] \times I\!R^3$ *the large investor stock price equals the equilibrium stock price directly after his trades, i.e. the large investor price function* $S_\mu : [0,T]\times I\!R^3 \to I\!R$ *is for all* $(t,u,\xi_1,\xi_2) \in [0,T]\times I\!R^3$ *given by the equation* $S_\mu(t,u,\xi_1,\xi_2) = \psi(t,u,\xi_2)$.

(*ii*) *For each* $n \in I\!N$ *there is some* $Z_0^n \in \{\pm 1\}$ *such that the large investor's stock holdings* ξ_{-1}^n *immediately before time* $t_0^n = 0$ *satisfy*

$$\xi_{-1}^n = \xi^n(0,u_0) - \delta Z_0^n\varphi_u(0,u_0) + \delta^2\left(\frac{1}{2}\varphi_{uu}(0,u_0) - \varphi_t(0,u_0)\right) + O\big(\delta^{2+\beta}\big) \;\; \textit{as } n \to \infty.$$

Assumption M (p. 187). *Let φ be the solution of the final value problem (3.4.3), (3.4.4). For each fixed $t \in [0,T]$ the function $X_\varphi : I\!R \to (0,\infty)$, $u \mapsto \psi\big(t,u,\varphi(t,u)\big)$, is strictly increasing.*

Assumption N (p. 218). *For $\delta = \delta_n = n^{-\frac{1}{2}}$ there exist some constants $\beta \in (0,1)$, $\sigma \geq 0$, and $\mu \in I\!R$ such that the volatility and drift parameters $\{\sigma_n\}_{n\in I\!N}$ and $\{\mu_n\}_{n\in I\!N}$ satisfy*

$$\sigma_n = \sigma\delta + O\big(\delta^{1+\beta}\big) \quad \text{and} \quad \mu_n = \mu\delta^2 + O\big(\delta^{2+\beta}\big) \quad \text{as } n \to \infty.$$

Moreover, there exist some functions $a : [0,T] \times I\!R \to I\!R$ and $b : [0,T] \times I\!R \to I\!R$ such that uniformly for all $(t,x) \in [0,T) \times I\!R$

$$p_n(t,x,\pm 1) = \frac{1}{2}\big(1 \pm a(t,x) + \delta b(t,x)\big) + O\big(\delta^{1+\beta}\big) \quad \text{as } n \to \infty. \tag{4.2.6}$$

Assumption O (p. 219). *The functions $a : [0,T] \times I\!R \to I\!R$ and $b : [0,T] \times I\!R \to I\!R$ satisfy:*

(*i*) *There exists some $\mathfrak{a} \in (0,1)$ such that $\|a\| < \mathfrak{a}$. Moreover, $\|b\| < \infty$.*

(*ii*) *The spatial derivative $a' : [0,T] \times I\!R \to I\!R$ given by $a'(t,x) = \frac{d}{dx}a(t,x)$ is uniformly bounded and continuous with respect to x for all $(t,x) \in [0,T] \times I\!R$.*

(*iii*) *$a' : [0,T] \times I\!R \to I\!R$ satisfies a global Hölder condition, uniformly in $t \in [0,T]$, i.e. there exists some $K_0 \in I\!R_+$ and some $\beta \in (0,1)$ such that*

$$\big|a'(t,x) - a'(t,y)\big| \leq K_0|x-y|^\beta \quad \text{for all } x,y \in I\!R \text{ and all } t \in [0,T].$$

(*iv*) *The function $b : [0,T] \times I\!R \to I\!R$ is globally Hölder continuous in x with exponent β as well, i.e. there exists some $K_1 \in I\!R_+$ such that*

$$\big|b(t,x) - b(t,y)\big| \leq K_1|x-y|^\beta \quad \text{for all } x,y \in I\!R \text{ and all } t \in [0,T].$$

(*v*) *$a : [0,T] \times I\!R \to I\!R$, $a' : [0,T] \times I\!R \to I\!R$, and $b : [0,T] \times I\!R \to I\!R$ are continuous in t.*

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