Subspace Concentration of Geometric Measures

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Zusammenfassung

In dieser Arbeit untersuchen wir geometrische Maße in zwei verschiedenen Erweiterungen der Brunn-Minkowski-Theorie.

Der erste Teil dieser Arbeit befasst sich mit Problemen in der L_p-Brunn-Minkowski-Theorie, die auf dem Konzept der p-Addition konvexer Körper basiert, die zunächst von Firey für $p \geq 1$ eingeführt und später von Lutwak et al. für alle reellen p betrachtet wurde. Von besonderem Interesse ist das Zusammenspiel des Volumens und anderer Funktionale mit der p-Addition. Bedeutsame offene Probleme in diesem Setting sind die Gültigkeit von Verallgemeinerungen der berühmten Brunn-Minkowski-Ungleichung und der Minkowski-Ungleichung, insbesondere für $0 \leq p < 1$, da die Ungleichungen für kleinere p stärker werden. Die Verallgemeinerung der Minkowski-Ungleichung auf $p = 0$ wird als logarithmische Minkowski-Ungleichung bezeichnet, die wir hier für vereinzelte polytopale Fälle beweisen werden. Das Studium des Kegelvolumenmaßes konvexer Körper ist ein weiteres zentrales Thema in der L_p-Brunn-Minkowski-Theorie, das eine starke Verbindung zur logarithmischen Minkowski-Ungleichung aufweist. In diesem Zusammenhang stellen sich die grundlegenden Fragen nach einer Charakterisierung dieser Maße und wann ein konvexer Körper durch sein Kegelvolumenmaß eindeutig bestimmt ist. Letzteres ist für symmetrische konvexe Körper unbekannt, während das erstere Problem in diesem Fall gelöst wurde. Die Schlüsseleigenschaft in der Lösung ist eine Konzentrationsgrenze eines gegebenen Kegelvolumenmaßes eingeschränkt auf lineare Unterräume. Wir werden eine Charakterisierung von Kegelvolumenmaßen von Trapezen herleiten und neue Beispiele konvexer Körper mit nicht-eindeutigem Kegelvolumenmaß präsentieren. Dabei werden wir diskutieren, wie das Vorhandensein einer Schranke an die Konzentration auf Unterräumen die oben genannten Fragen beeinflusst.

Im zweiten Teil betrachten wir eine erst kürzlich entdeckte Familie geometrischer Maße, die in der dualen Brunn-Minkowski-Theorie vorkommt. Die sogenannten dualen Krümmungsmaße von konvexen Körpern fungieren als Gegenstücke zu Krümmungsmaßen in der klassischen Brunn-Minkowski-Theorie und schließen das Kegelvolumenmaß als Sonderfall ein. Duale Krümmungsmaße haben in den letzten Jahren großes Interesse geweckt. Die Aufgabe, die Resultate, die für Kegelvolumenmaße erzielt wurden, auf die allgemeineren dualen Krümmungsmaße auszudehnen, erfordert neuartige Abschätzungen der Unterraumkonzentration. Den Ideen von Kneser und Süss folgend, beweisen wir Varianten der Brunn-Minkowski-Ungleichung unter gewissen Symmetrievoraussetzungen, mit deren Hilfe wir scharfe Schranken an die Unterraumkonzentration für nahezu alle dualen Krümmungsmaße symmetrischer konvexer Körper folgern.

Abstract

In this work we study geometric measures in two different extensions of the Brunn-Minkowski theory.

The first part of this thesis is concerned with problems in L_p Brunn-Minkowski theory, that is based on the concept of p-addition of convex bodies, which was first introduced by Firey for $p \geq 1$ and later considered for all real p by Lutwak et al. The interplay of the volume and other functionals with the p-addition is of particular interest. Considerable open problems in this setting include the validity of extensions of the celebrated Brunn-Minkowski inequality and Minkowski's inequality, particularly for $0 \leq p < 1$ as the inequalities become stronger for smaller p. The generalization of Minkowski's inequality to $p = 0$ is called logarithmic Minkowski inequality, which we will prove here for some particular polytopal instances. The study of the cone-volume measure of convex bodies is another central subject in L_p Brunn-Minkowski theory, which exhibits a strong connection to the logarithmic Minkowski inequality. Fundamental questions in this context ask for a characterization of these measures and when a convex body is uniquely determined by its cone-volume measure. The latter is unknown even for symmetric convex bodies whereas the former problem was solved in this case. The key property in the solution is a concentration bound of a given cone-volume measure restricted to linear subspaces. We will establish a characterization of cone-volume measures of trapezoids and present new examples of convex bodies with non-unique cone-volume measure. Thereby we will discuss how the presence of a subspace concentration bound affects the aforementioned questions.

In the second part we consider an only recently discovered family of geometric measures arising in dual Brunn-Minkowski theory. The so-called dual curvature measures of convex bodies act as counterparts of curvature measures in the classical Brunn-Minkowski theory and include the cone-volume measure as a special case. Dual curvature measures gained much interest in the last few years. The task of extending the results obtained for cone-volume measures to the more general dual curvature measures requires novel subspace concentration inequalities. Following the ideas of Kneser and Süss we establish variants of the Brunn-Minkowski inequality under some symmetry assumptions with the aid of which we prove sharp subspace concentration bounds on nearly all dual curvature measures of symmetric convex bodies.

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Contents

Introduction

The Brunn-Minkowski theory combines the concept of convex sets, which find applications in many mathematical subfields due to their structural richness, with the volume and other functionals. The foundation of this theory is the famous (and name-giving) Brunn-Minkowski inequality linking the volume functional and the Minkowski addition of convex sets. It states that for two convex bodies K, M in Euclidean n-space $\mathbb{R}^n$ and a parameter $\lambda \in [0,1]$

$$\mathrm{vol}((1-\lambda)K + \lambda M)^{\frac{1}{n}} \geq (1-\lambda)\,\mathrm{vol}(K)^{\frac{1}{n}} + \lambda\,\mathrm{vol}(M)^{\frac{1}{n}}.$$

An important aspect is the study of differentials of the volume and other functionals with respect to Minkowski addition. Variational formulas for these functionals exhibit a connection between certain measures and the convex body under consideration. Those measures induced by convex bodies are called geometric measures. One of the most well-known examples is the (classical) surface area measure of a convex body K, that for a set $\mathcal{U}$ of unit outer normal vectors measures the area of the part of boundary of K which is associated to $\mathcal{U}$. The understanding of geometric measures has been proved to be an important ingredient for establishing sharp inequalities in convex geometric analysis. The task of characterizing geometric measures, i.e., to find necessary and sufficient conditions such that a given measure appears, for instance, as the surface area measure of a convex body, is called Minkowski problem. Although Minkowski problems have been worked on for decades, for many important geometric measures a full characterization is missing. There are two far-reaching extensions of the classical Brunn-Minkowski theory, both arising basically by replacing the classical Minkowski addition by another additive operation. The first one is the p-addition introduced by Firey [30] for $p \geq 1$ and extended by Lutwak [61, 62] to $p < 1$, which leads to the rich and emerging L_p Brunn-Minkowski theory. Therein a central object is the cone-volume measure which stands out due to its $\mathrm{SL}(n)$-invariance, whereas most other geometric measures are only $\mathrm{SO}(n)$-invariant. The second extension, introduced by Lutwak [58], is called dual Brunn-Minkowski theory and essentially emerges by replacing convex bodies and support functions by star bodies and radial functions, respectively. The word "dual" here refers to similarities of concepts in both theories rather than duality in a strict mathematical sense. The geometric measures of interest in dual Brunn-Minkowski theory are the dual curvature measures recently introduced by Huang, Lutwak, Yang and Zhang [48]. These are the long missing counterparts of curvature measures in classical Brunn-Minkowski theory and

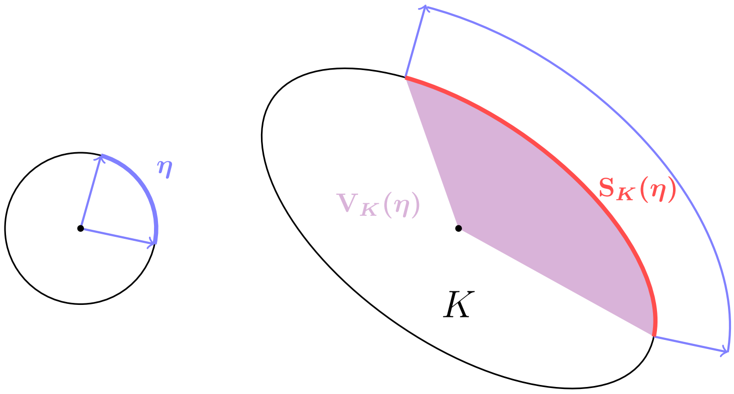

Figure 1: Surface area measure and cone-volume measure of a smooth convex body

they have already been studied thoroughly since their discovery. This thesis is mainly concerned with Minkowski problems regarding cone-volume measures and dual curvature measures.

We will provide notions and definitions that will be used later on in Chapter 1. In particular we will properly introduce the concept of area measures, including the classical surface area measure

$$\mathrm{S}_K(\eta) = \mathcal{H}^{n-1}(\boldsymbol{\nu}_K^{-1}(\eta)),$$

where $K \subseteq \mathbb{R}^n$ is a convex body, $\eta \subseteq \mathbb{S}^{n-1}$ a Borel set and $\mathcal{H}^{n-1}$ the $(n-1)$-dimensional Hausdorff measure. Here $\mathbb{S}^{n-1}$ denotes the unit sphere in Euclidean n-space and ν_K denotes the Gauss map (see p. 7 for the definition). The cone-volume measure of a convex body $K \subseteq \mathbb{R}^n$ with the origin in its interior is given by

$$\mathrm{V}_K(\eta) = \frac{1}{n} \int\limits_{\boldsymbol{\nu}_K^{-1}(\eta)} \langle \boldsymbol{x}, \boldsymbol{\nu}_K(\boldsymbol{x}) \rangle \,\mathrm{d}\mathcal{H}^{n-1}(\boldsymbol{x})$$

for every Borel set $\eta \subseteq \mathbb{S}^{n-1}$. It is closely related to the classical surface area measure (see Fig. 1). We will state the Minkowski problems associated to the respective measures and present (partial) solutions to them.

Chapter 2 focuses on some problems in L_p Brunn-Minkowski theory that are related to the cone-volume measure of convex bodies. The task of characterizing, when a given measure is the cone-volume measure of a convex body, is also known as the logarithmic Minkowski problem. Symmetric convex bodies are the largest class this goal has been achieved for. It was proved in [19] that a non-zero finite even Borel measure μ on $\mathbb{S}^{n-1}$ is the cone-volume measure of a symmetric convex body if and only if the subspace concentration inequality

$$\mu(L \cap \mathbb{S}^{n-1}) \leq \frac{\dim(L)}{n} \mu(\mathbb{S}^{n-1})$$

is satisfied for every proper subspace $L \subseteq \mathbb{R}^n$, and equality is attained for some L, if and only if there is a subspace L' complementary to L such that

μ is concentrated on $\mathbb{S}^{n-1} \cap (L \cup L')$. In the general setting the gap between known necessary and sufficient conditions is quite large. A little more is known for the two-dimensional case. Here we will examine the cone-volume measure of trapezoids explicitly to highlight the presence of conditions which are – in contrast to the subspace concentration inequality – non-linear in terms of cone-volumes. Moreover, we will discuss the question when a polygon is uniquely determined by its cone-volume measure. This has been answered by Stancu [77] for symmetric polygons, but for symmetric polytopes in higher dimensions it is an open problem. Here we will present examples of non-symmetric polygons with few vertices and non-unique cone-volume measure. In addition, the cone-volume measure appears naturally in problems of L_p Brunn-Minkowski theory aiming for results stronger than their counterpart in classical Brunn-Minkowski theory. For two convex bodies $K, M \subseteq \mathbb{R}^n$ with the origin in their interiors their log-combination with respect to $\lambda \in [0,1]$ is given by

$$\begin{aligned}(1-\lambda)K +_0 \lambda M& \\ &= \left\{\boldsymbol{x} \in \mathbb{R}^n : \langle \boldsymbol{u}, \boldsymbol{x}\rangle \leq \mathrm{h}_K(\boldsymbol{u})^{1-\lambda}\mathrm{h}_M(\boldsymbol{u})^{\lambda} \text{ for all } \boldsymbol{u} \in \mathbb{S}^{n-1}\right\},\end{aligned}$$

where h_K and h_M denote the support functions of K and M, respectively. The logarithmic Brunn-Minkowski inequality reads as

$$\mathrm{vol}((1-\lambda)K +_0 \lambda M) \geq \mathrm{vol}(K)^{1-\lambda}\,\mathrm{vol}(M)^{\lambda}.$$

As a consequence of the inequality of arithmetic and geometric means one can easily check that the logarithmic Brunn-Minkowski inequality, if it holds true, is in fact stronger than the classical Brunn-Minkowski inequality. It was proved (among other cases) for pairs of symmetric convex bodies in the plane in [18] and if both K and M are unconditional convex bodies in arbitrary dimension in [74]. It is conjectured that the logarithmic Brunn-Minkowski inequality holds whenever K and M are symmetric. It was shown by Böröczky, Lutwak, Yang and Zhang [18] that the logarithmic Brunn-Minkowski inequality is equivalent to the logarithmic Minkowski inequality

$$\int\limits_{\mathbb{S}^{n-1}} \log\left(\frac{\mathrm{h}_M(\boldsymbol{u})}{\mathrm{h}_K(\boldsymbol{u})}\right) \mathrm{dV}_K(\boldsymbol{u}) \geq \frac{\mathrm{vol}(K)}{n} \log\left(\frac{\mathrm{vol}(M)}{\mathrm{vol}(K)}\right)$$

in the sense that if one holds for all pairs of symmetric convex bodies K and M, the other one follows. The logarithmic Minkowski inequality does not hold for arbitrary pairs of convex bodies with the origin as an interior point. For centered convex bodies, i.e., convex bodies whose centroid (or center of mass) is the origin, there is no example known that violates the logarithmic Minkowski inequality. Here we will verify the logarithmic Minkowski inequality for some particular instances where one of the convex bodies is centered and the other one is either a simplex or a parallelepiped. The presented results are based on joint work with Martin Henk [43].

Chapter 3 revolves around Minkowski problems in dual Brunn-Minkowski theory, namely regarding the dual curvature measures. For some $q \in \mathbb{R}$ and a convex body $K \subseteq \mathbb{R}^n$ with the origin in its interior one may define the qth dual curvature measure of K via

$$\widetilde{\mathrm{C}}_q(K,\eta) = \frac{1}{n} \int\limits_{\boldsymbol{\nu}_K(\eta)^{-1}} |\boldsymbol{x}|^{q-n} \langle \boldsymbol{\nu}_K(\boldsymbol{x}), \boldsymbol{x}\rangle \,\mathrm{d}\mathcal{H}^{n-1}(\boldsymbol{x})$$

for every Borel set $\eta \subseteq \mathbb{S}^{n-1}$. They not only play an important role in dual Brunn-Minkowski theory, they also include well-known measures from classical Brunn-Minkowski theory, e.g., the cone-volume measure in case $q = n$. Regarding the dual Minkowski problem, i.e., finding necessary and sufficient conditions on dual curvature measures, much progress has been made during the past couple of years. In [48] it was shown that, given $q \in (0, n)$ and a non-zero finite even Borel measure μ on the sphere, a certain subspace concentration inequality is sufficent for the existence of a symmetric convex body K with $\mu = \widetilde{\mathrm{C}}_q(K, \cdot)$. In this work we will establish tight subspace concentration bounds of dual curvature measures of symmetric convex bodies for the parameter ranges $q \in (0, n)$ and $q \in [n + 1, \infty)$. The former supplements a result obtained by Böröczky, Lutwak, Yang, Zhang and Zhao [17] which lead to a solution of the even dual Minkowski problem for parameters $q \in (0, n)$: A non-zero finite even Borel measure μ on $\mathbb{S}^{n-1}$ is the qth dual curvature measure of a symmetric convex body if and only if the subspace concentration inequality

$$\mu(L \cap \mathbb{S}^{n-1}) < \min\left\{\frac{\dim(L)}{q}, 1\right\} \mu(\mathbb{S}^{n-1})$$

is satisfied for every proper subspace $L \subseteq \mathbb{R}^n$. Our proofs of the necessity of the subspace concentration inequality rely heavily on the symmetry of the involved convex bodies and variants of the Brunn-Minkowski inequality where the volume functional is replaced by a measure having a power of the Euclidean norm as Lebesgue density function. Moreover, we will discuss which of these inequalities can be easily extended to other norms. The remaining range $q \in (n, n + 1)$ in the symmetric setting is completely open as neither necessary nor sufficient conditions are known at this moment. At least in the planar case $n = 2$ we also provide tight subspace concentration bounds by extending the abovementioned inequalities. The results presented in this chapter are based on joint works with Károly Böröczky, Jr., and Martin Henk [14, 42].

1 Preliminaries

The aim of this chapter is to provide the essential definitions and concepts used throughout the thesis. We recommend the books by Gardner [32], Gruber [36] and Schneider [75] as excellent references on convex geometry.

Foundations

We consider the Euclidean n-space $\mathbb{R}^n = \{\boldsymbol{x} = (x_1, \ldots, x_n)^T : x_1, \ldots, x_n \in \mathbb{R}\}$ equipped with standard inner product $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \sum_{i=1}^n x_i y_i$ for $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$. The Euclidean norm will be denoted by $|\boldsymbol{x}| = \sqrt{\langle \boldsymbol{x}, \boldsymbol{x} \rangle}$ for $\boldsymbol{x} \in \mathbb{R}^n$ and if $\boldsymbol{x} \neq \boldsymbol{0}$ its normalization is $\overline{\boldsymbol{x}} = \frac{\boldsymbol{x}}{|\boldsymbol{x}|}$. We write $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n$ for the standard basis vectors in $\mathbb{R}^n$. For any set $X \subseteq \mathbb{R}^n$ we write $\operatorname{int}(X)$ and ∂X for its interior and its boundary points, respectively. We denote by B_n the n-dimensional Euclidean unit ball, i.e., $B_n = \{\boldsymbol{x} \in \mathbb{R}^n : |\boldsymbol{x}| \leq 1\}$, and by $\mathbb{S}^{n-1} = \partial B_n = \{\boldsymbol{x} \in \mathbb{R}^n : |\boldsymbol{x}| = 1\}$ its boundary.

For a linear subspace $L \subseteq \mathbb{R}^n$, $L^\perp$ is its **orthogonal complement** and the **orthogonal projection** onto L is denoted by $\cdot|L$. For a non-empty set $X \subseteq \mathbb{R}^n$ we define its **linear hull** by

$$\operatorname{lin}(X) = \left\{ \sum_{i=1}^m \lambda_i \boldsymbol{x}_i : m \in \mathbb{N}, \lambda_i \in \mathbb{R}, \boldsymbol{x}_i \in X \text{ for } i = 1, \ldots, m \right\},$$

its **affine hull** by

$$\operatorname{aff}(X) = \left\{ \sum_{i=1}^m \lambda_i \boldsymbol{x}_i : m \in \mathbb{N}, \lambda_i \in \mathbb{R}, \boldsymbol{x}_i \in X \text{ for } i = 1, \ldots, m, \sum_{i=1}^m \lambda_i = 1 \right\},$$

its **positive hull**

$$\operatorname{pos}(X) = \left\{ \sum_{i=1}^m \lambda_i \boldsymbol{x}_i : m \in \mathbb{N}, \lambda_i \geq 0, \boldsymbol{x}_i \in X \text{ for } i = 1, \ldots, m \right\}$$

and its **convex hull** by

$$\operatorname{conv}(X) = \left\{ \sum_{i=1}^{m} \lambda_i \boldsymbol{x}_i : m \in \mathbb{N}, \lambda_i \geq 0, \boldsymbol{x}_i \in X \text{ for } i = 1, \dots, m, \sum_{i=1}^{m} \lambda_i = 1 \right\}.$$

The convex hull $\operatorname{conv}\{\boldsymbol{x}, \boldsymbol{y}\}$ of two points $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$ will be abbreviated by $[\boldsymbol{x}, \boldsymbol{y}]$. A set is called **convex** if it equals its convex hull. A set of points $X \subseteq \mathbb{R}^n$ is called **affinely independent** if $\operatorname{aff}(X \setminus \{\boldsymbol{x}\}) \neq \operatorname{aff}(X)$ for every $\boldsymbol{x} \in X$. Its **dimension** is the maximal number of affinely independent points contained in it minus 1 and will be denoted by $\dim(X)$. For convenience we also define $\dim(\emptyset) = -1$. The **relative interior** of a set X is the interior of X with respect to its affine hull.

Convex bodies

A convex and compact set with non-empty interior is called a **convex body**. We write $\mathcal{K}^n$ for the set of all convex bodies in $\mathbb{R}^n$ and $\mathcal{K}_o^n$ for convex bodies containing the origin in the interior, i.e., $\mathcal{K}_o^n = \{K \in \mathcal{K}^n : \mathbf{0} \in \operatorname{int} K\}$.

As usual, **Minkowski addition** of subsets of $\mathbb{R}^n$ and multiplication with scalars are defined pointwise so that for $X, Y \subseteq \mathbb{R}^n$ and $\alpha, \beta \in \mathbb{R}$ we write

$$\alpha X + \beta Y = \{\alpha \boldsymbol{x} + \beta \boldsymbol{y} : \boldsymbol{x} \in X, \boldsymbol{y} \in Y\}. \tag{1.1}$$

In case $X, Y \in \mathcal{K}^n$ and $\alpha, \beta \geq 0$ or $0 \leq \beta = 1 - \alpha \leq 1$, we also speak of Minkowski combination and convex combination between convex bodies, respectively. For $A \subseteq \mathbb{R}$ we also define $A \cdot X = \{\alpha \boldsymbol{x} : \alpha \in A, \boldsymbol{x} \in X\}$.

The normalized k-dimensional Hausdorff measure will be denoted by $\mathcal{H}^k$ and instead of it we will sometimes write vol_k for the volume or just vol if the dimension is apparent from the context. The k-dimensional Hausdorff measure particularly coincides with the k-dimensional Lebesgue measure in affine subspaces and k-dimensional spherical Lebesgue measure on subspheres. In integrals we will often abbreviate $\mathrm{d}\mathcal{H}^n(\boldsymbol{x})$ by $\mathrm{d}\boldsymbol{x}$, when integrating with respect to $\mathcal{H}^n$, and $\mathrm{d}\mathcal{H}^{n-1}(\boldsymbol{u})$ by $\mathrm{d}\boldsymbol{u}$, when integrating with respect to $\mathcal{H}^{n-1}$. According to this, the **centroid** $\mathbf{c}(X)$ of a Lebesgue measurable set $X \subseteq \mathbb{R}^n$ with positive volume is defined by

$$\mathbf{c}(X) = \frac{1}{\operatorname{vol}(X)} \int_X \boldsymbol{x} \, \mathrm{d}\boldsymbol{x}.$$

If $\mathbf{c}(X) = \mathbf{0}$, the set X is called centered. In addition, we will denote the set of centered convex bodies by $\mathcal{K}_c^n$ and the subclass of symmetric convex bodies by $\mathcal{K}_s^n$, i.e., convex bodies K with $K = -K$. An even stronger notion of symmetry is held by **unconditional** convex bodies. These are symmetric about every coordinate hyperplane, i.e., if $K \in \mathcal{K}^n$ is an unconditional convex body, then $(x_1, \dots, x_n)^T \in K$ implies $(\pm x_1, \dots, \pm x_n)^T \in K$.

By the well-known Brunn-Minkowski inequality we know that the nth root of the volume of a Minkowski combination is a concave function.

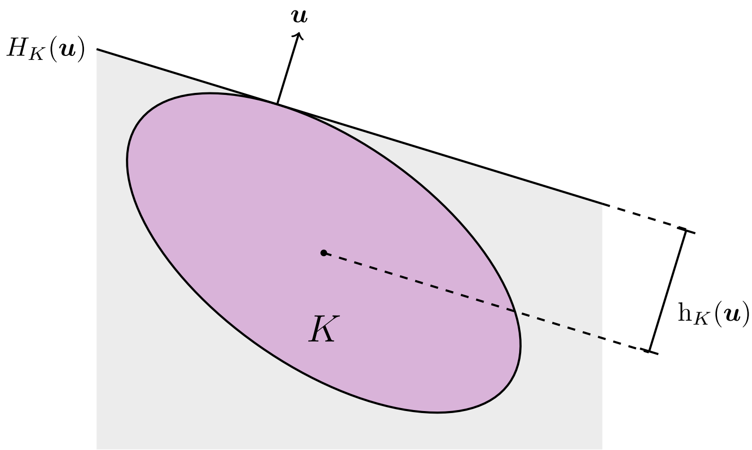

Figure 1.1: Smooth convex body and supporting hyperplane

Theorem 1.1 (Brunn-Minkowski inequality, see, e.g., [75, Thm. 7.1.1])**.** *If* $K, L \in \mathcal{K}^n$ *and* $0 < \lambda < 1$*, then*

$$\operatorname{vol}((1-\lambda)K + \lambda L)^{\frac{1}{n}} \geq (1-\lambda)\operatorname{vol}(K)^{\frac{1}{n}} + \lambda \operatorname{vol}(L)^{\frac{1}{n}} \tag{1.2}$$

and equality holds if and only if K *and* L *are homothetic, i.e., they are equal up to translation and scaling.*

For fixed $\boldsymbol{a} \in \mathbb{R}^n \setminus \{\boldsymbol{0}\}$ and $\alpha \in \mathbb{R}$ the hyperplane given by the equation $\langle \boldsymbol{a}, \boldsymbol{x} \rangle = \alpha$ will be denoted by $H(\boldsymbol{a}, \alpha)$. We will sometimes use $\boldsymbol{a}^\perp$ instead of $H(\boldsymbol{a}, 0)$. We write $H^-(\boldsymbol{a}, \alpha)$ and $H^+(\boldsymbol{a}, \alpha)$ for the halfspaces defined by $\langle \boldsymbol{a}, \boldsymbol{x} \rangle \leq \alpha$ and $\langle \boldsymbol{a}, \boldsymbol{x} \rangle \geq \alpha$, respectively. For a convex and compact set $K \subseteq \mathbb{R}^n$ the **support function** $\mathrm{h}_K \colon \mathbb{R}^n \to \mathbb{R}$ is defined by (see Fig. 1.1)

$$\mathrm{h}_K(\boldsymbol{x}) = \max_{\boldsymbol{y} \in K} \langle \boldsymbol{x}, \boldsymbol{y} \rangle .$$

It is worth noting that convex bodies are uniquely determined by their support functions since for given $K \in \mathcal{K}^n$ the support function h_K is the pointwise minimal function with

$$K = \{\boldsymbol{x} \in \mathbb{R}^n : \langle \boldsymbol{u}, \boldsymbol{x} \rangle \leq \mathrm{h}_K(\boldsymbol{u}) \text{ for all } \boldsymbol{u} \in \mathbb{S}^{n-1}\}.$$

On the other hand, each convex, positively 1-homogeneous function $\mathbb{R}^n \to \mathbb{R}$ is the support function of a convex and compact set. A boundary point $\boldsymbol{x} \in \partial K$ is said to have a (not necessarily unique) unit **outer normal vector** $\boldsymbol{u} \in \mathbb{S}^{n-1}$ if $\langle \boldsymbol{u}, \boldsymbol{x} \rangle = \mathrm{h}_K(\boldsymbol{u})$, i.e., $\boldsymbol{x} \in H(\boldsymbol{u}, \mathrm{h}_K(\boldsymbol{u}))$. The corresponding supporting hyperplane $H(\boldsymbol{u}, \mathrm{h}_K(\boldsymbol{u}))$ will be denoted by $H_K(\boldsymbol{u})$. We write $\partial' K$ for the set of boundary points of K with unique outer normal vector and define the **Gauss map** $\boldsymbol{\nu}_K \colon \partial' K \to \mathbb{S}^{n-1}$ such that $\boldsymbol{\nu}_K(\boldsymbol{x})$ is the unique unit outer normal vector of $\boldsymbol{x} \in \partial' K$. Almost every boundary point of convex body has a unique outer normal vector in the sense that $(\partial K) \setminus (\partial' K)$ has $(n-1)$-dimensional Hausdorff measure 0 (see, e.g., [75, Thm. 2.2.5]). If $\partial K = \partial' K$, K is called **smooth**.

Polytopes

A **polytope** is the convex hull of finitely many points. If a polytope is k-dimensional, we also call it k-polytope. Fulldimensional polytopes are also convex bodies and we will use the symbols $\mathcal{P}^n, \mathcal{P}^n_c, \mathcal{P}^n_s, \mathcal{P}^n_o \subseteq \mathcal{K}^n$ to denote the sets of n-polytopes, centered n-polytopes, symmetric n-polytopes and polytopes containing the origin in the interior, respectively. Polytopes in the plane $\mathbb{R}^2$ will also be called **polygons**. Each polytope can be written as the convex hull of a unique minimal set of points, called **vertices**. In addition to that, it is well known (c.f., e.g., [75, Sect. 2.4]) that a polytope can be equivalently described as a bounded intersection of finitely many halfspaces, i.e., a bounded set of the form

$$\{\boldsymbol{x} \in \mathbb{R}^n : A\boldsymbol{x} \leq \boldsymbol{b}\},$$

where $m \in \mathbb{N}$, $A \in \mathbb{R}^{m\times n}$ and $\boldsymbol{b} \in \mathbb{R}^m$. For an n-polytope $P \in \mathcal{P}^n$ we say that $\boldsymbol{u} \in \mathbb{S}^{n-1}$ is a unit outer normal vector of P if $\mathrm{vol}_{n-1}(P \cap H_P(\boldsymbol{u})) > 0$. The outer normal vectors of P are precisely the irredundant row vectors of A scaled to unit length. In particular, there are only finitely many of them. The set of all unit outer normal vectors of a polytope P is denoted by $U(P)$. For a given $\boldsymbol{u} \in U(P)$ we associate a set of points in the boundary of P arising from the intersection of P with the supporting hyperplane $H_P(\boldsymbol{u})$. This point set $F(P, \boldsymbol{u}) = P \cap H_P(\boldsymbol{u})$ is called **facet** of P normal to $\boldsymbol{u}$. Moreover, a given finite set $U \subseteq \mathbb{S}^{n-1}$ appears as the set of outer normal vectors of some fulldimensional polytope in $\mathbb{R}^n$ if and only if $\operatorname{pos} U = \mathbb{R}^n$, i.e., we say that U is not concentrated on a closed hemisphere.

There are two notable instances of polytopes in this work. A k**-simplex** or just simplex is the convex hull of $k+1$ affinely independent points. Volume and centroid of a simplex can be easily computed from their vertices, i.e., if an n-simplex is given as $S = \mathrm{conv}\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{n+1}\}$ for some affinely independent points $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{n+1} \in \mathbb{R}^n$, then

$$\begin{aligned} \mathrm{vol}(S) &= \left|\det(\boldsymbol{v}_{n+1} - \boldsymbol{v}_1, \boldsymbol{v}_{n+1} - \boldsymbol{v}_2, \ldots, \boldsymbol{v}_{n+1} - \boldsymbol{v}_n)\right|, \\ \mathbf{c}(S) &= \frac{1}{n+1}\sum_{i=1}^{n+1} \boldsymbol{v}_i. \end{aligned}$$

Moreover, n-simplices are the only polytopes in $\mathbb{R}^n$ having exactly $n+1$ outer normal vectors which is also minimal among all n-polytopes. Restricting to symmetric polytopes $\mathcal{P}^n_s$ the latter role is taken by linear images of the cube $[-1,1]^n$. These are called **parallelepipeds** and are precisely the polytopes given as solution sets of systems of the form

$$|\langle \boldsymbol{u}_i, \boldsymbol{x}\rangle| \leq h_i, \qquad i = 1, \ldots, n,$$

for some linearly independent $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_n \in \mathbb{S}^{n-1}$ and $h_1, \ldots, h_n > 0$. In this case the volume of the parallelepiped can be computed by

$$\frac{2^n \prod_{i=1}^n h_i}{|\det(\boldsymbol{u}_1, \ldots, \boldsymbol{u}_n)|} \tag{1.3}$$

Both simplices and parallelepipeds often appear as extremal cases of problems in convex geometry. Three-dimensionals instances of a simplex and a parallelepiped are depicted in Figure 1.2.

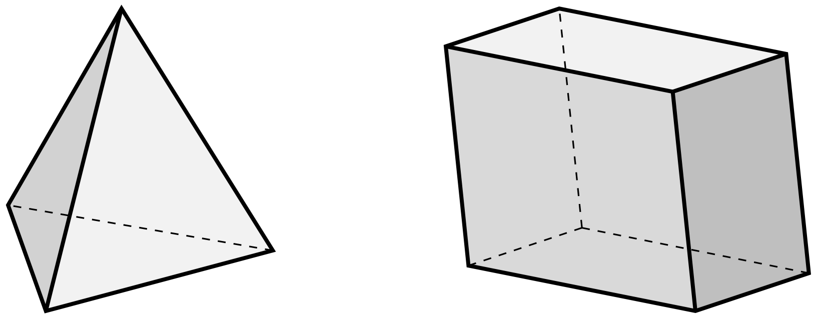

Figure 1.2: 3-simplex and 3-dimensional parallelepiped

Geometric measures

The combination of the volume functional and Minkowski addition is fundamental in the Brunn-Minkowki theory. The Minkowski sum $K+M$, $K, M \in \mathcal{K}^n$, for "small" M may be interpreted as a pertubation of K. Intriguingly, since support functions are linear with respect to Minkowski addition of convex bodies, the support function of the sum $K+M$ is represented by the sum of support functions $\mathrm{h}_K + \mathrm{h}_M$. When h_M is replaced by an arbitrary function $f\colon \mathbb{S}^{n-1} \to \mathbb{R}$, the convex body defined by

$$[\mathrm{h}_K + f] = \{\boldsymbol{x} \in \mathbb{R}^n : \langle \boldsymbol{u}, \boldsymbol{x} \rangle \leq \mathrm{h}_K(\boldsymbol{u}) + f(\boldsymbol{u}) \text{ for all } \boldsymbol{u} \in \mathbb{S}^{n-1}\},$$

is called the **Wulff shape** of $\mathrm{h}_K + f$. The support function of the Wulff shape $[\mathrm{h}_K + f]$ does not necessarily agree with $\mathrm{h}_K + f$, but we always have $[\mathrm{h}_K] = K$ for $K \in \mathcal{K}^n$. The variational formula

$$\lim_{\epsilon \to 0} \frac{\mathrm{vol}([\mathrm{h}_K + \varepsilon f]) - \mathrm{vol}([\mathrm{h}_K])}{\varepsilon} = \int\limits_{\mathbb{S}^{n-1}} f(\boldsymbol{u}) \,\mathrm{d}\mathrm{S}_K(\boldsymbol{u}), \tag{1.4}$$

for every continuous function $f\colon \mathbb{S}^{n-1} \to \mathbb{R}$, was originally established by Aleksandrov ([2], see also [75, Lem. 7.5.3]). Here S_K is the Borel measure on $\mathbb{S}^{n-1}$ known as the **surface area measure** of the convex body K and defined by

$$\mathrm{S}_K(\eta) = \mathcal{H}^{n-1}(\boldsymbol{\nu}_K^{-1}(\eta))$$

for each Borel set $\eta \subseteq \mathbb{S}^{n-1}$. The notion of surface area measures goes back to Lebesgue and Minkowski. If $K = P$ is a polytope, the surface area measure is discrete and concentrated on the outer normal vectors. Moreover, for each outer normal vector it assigns the area of the facet. To be more precise, it holds that

$$\mathrm{S}_P(\eta) = \sum_{\boldsymbol{u} \in U(P) \cap \eta} \mathrm{vol}_{n-1}(F(P, \boldsymbol{u}))$$

for every Borel set $\eta \subseteq \mathbb{S}^{n-1}$ (see Fig. 1.3). A common extension of the volume functional are the **quermassintegrals** $\mathrm{W}_i(K)$ of a convex body $K \in \mathcal{K}^n$, which may be defined via the classical Steiner formula, expressing the volume of the Minkowski sum of $K \in \mathcal{K}^n$ and λB_n, i.e., the volume of the parallel body of K at distance $\lambda \geq 0$, as a polynomial in λ (cf., e.g., [75, Sect. 4.2])

$$\mathrm{vol}(K + \lambda B_n) = \sum_{i=0}^{n} \lambda^i \binom{n}{i} \mathrm{W}_i(K). \tag{1.5}$$

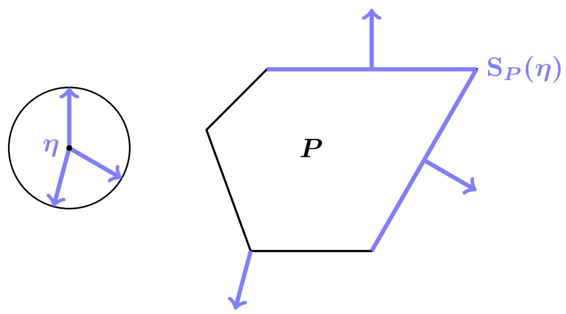

Figure 1.3: Surface area measure of a polygon

In particular, $\mathrm{W}_0(K) = \mathrm{vol}(K)$. The quermassintegrals of a convex body K may be interpreted as – up to normalization – the average volume of projections of K, i.e., Kubota's integral formula (cf., e.g., [75, Subsect. 5.3.2]) states that

$$\mathrm{W}_{n-i}(K) = \frac{\mathrm{vol}(B_n)}{\mathrm{vol}_i(B_i)} \int\limits_{\mathcal{G}(i,n)} \mathrm{vol}_i(K|L)\,\mathrm{d}L, \tag{1.6}$$

$i = 0, \ldots, n$, where integration is taken with respect to the rotation-invariant probability measure on the Grassmannian $\mathcal{G}(i,n)$ of all i-dimensional linear subspaces. Aleksandrov [2] and Fenchel and Jessen [28] established a variatonal formula similar to (1.4) for quermassintegrals in case f is a support function and only right limits are considered:

$$\lim_{\varepsilon \searrow 0} \frac{\mathrm{W}_{n-1-i}(K+\varepsilon M) - \mathrm{W}_{n-1-i}(K)}{\varepsilon} = \int\limits_{\mathbb{S}^{n-1}} \mathrm{h}_M(\boldsymbol{u})\,\mathrm{dS}_i(K,\boldsymbol{u}) \tag{1.7}$$

for $K, M \in \mathcal{K}^n$, $i = 0, \ldots, n-1$, where $\mathrm{S}_i(K,\cdot)$ are Borel measures on the sphere called **area measures** of K. The $(n-1)$th area measure $\mathrm{S}_{n-1}(K,\cdot)$ is just the surface area measure of K. Interestingly, the area measures of a convex body admit a local Steiner-type formula like (1.5) in the following sense. For a convex body $K \in \mathcal{K}^n$ and a point $\boldsymbol{x} \in \mathbb{R}^n$ the **metric projection** $\mathbf{p}_K(\boldsymbol{x})$ is the unique point in K that is closest to $\boldsymbol{x}$. For $\boldsymbol{x} \in \mathbb{R}^n \setminus K$ we also define $\boldsymbol{v}_K(\boldsymbol{x}) = \overline{\boldsymbol{x} - \mathbf{p}_K(\boldsymbol{x})}$, i.e., $\boldsymbol{v}_K(\boldsymbol{x})$ is a (not necessarily unique) unit outer normal vector of $\mathbf{p}_K(\boldsymbol{x})$ and for every $x \in \mathbb{R}^n$ its distance to K by

$$\mathrm{d}(K,\boldsymbol{x}) = \begin{cases} |\boldsymbol{x} - \mathbf{p}_K(\boldsymbol{x})|\,, & \text{if } \boldsymbol{x} \notin K, \\ 0, & \text{if } \boldsymbol{x} \in K. \end{cases}$$

Then for a Borel set $\eta \subseteq \mathbb{S}^{n-1}$ and $\lambda > 0$ we consider the local parallel body (see Fig. 1.4)

$$B_K(\lambda,\eta) = \{\boldsymbol{x} \in \mathbb{R}^n : 0 < \mathrm{d}(K,\boldsymbol{x}) \le \lambda \text{ and } \boldsymbol{v}_K(\boldsymbol{x}) \in \eta\}\,.$$

The local Steiner formula expresses the volume of $B_K(\lambda,\eta)$ as a polynomial in λ. Its coefficients are – up to constants depending on i and n – the area measures (cf., e.g., [75, Sect. 4.2])

$$\mathrm{vol}(B_K(\lambda,\eta)) = \frac{1}{n}\sum_{i=1}^{n} \lambda^i \binom{n}{i} \mathrm{S}_{n-i}(K,\eta). \tag{1.8}$$

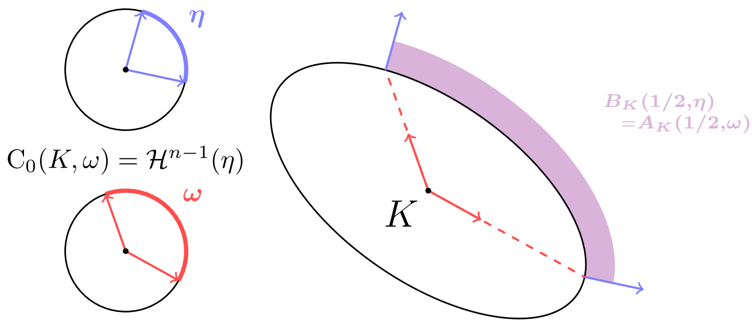

Figure 1.4: Local parallel bodies $B_K(\lambda,\eta)$ and $A_K(\lambda,\omega)$

In particular, the total of the area measures give the quermassintegrals, i.e., $n\,\mathrm{W}_i(K)=\mathrm{S}_{n-i}(K,\mathbb{S}^{n-1})$ for $i=1,\ldots,n$ (cf. (1.5) and (1.8)). Other notions of parallel bodies give rise to different sets of measures. For $K\in\mathcal{K}_o^n$, $\lambda>0$ and a Borel set $\omega\subseteq\mathbb{S}^{n-1}$ we set (see Fig. 1.4)

$$A_K(\lambda,\omega)=\left\{\boldsymbol{x}\in\mathbb{R}^n : 0<\mathrm{d}(K,\boldsymbol{x})\leq\lambda \text{ and } \overline{\mathbf{p}_K(\boldsymbol{x})}\in\omega\right\},$$

whose volume admits the Steiner-type formula

$$\mathrm{vol}(A_K(\lambda,\omega))=\frac{1}{n}\sum_{i=1}^{n}\lambda^i\binom{n}{i}\mathrm{C}_{n-i}(K,\omega), \tag{1.9}$$

where the Borel measures $\mathrm{C}_i(K,\cdot)$ are called **curvature measures** of the convex body K. Note that this is the definition of curvature measures given in [48] and other authors define the curvature measures on ∂K instead of $\mathbb{S}^{n-1}$. Both notions are related by an appropriate scaling of the points in ω. The 0th curvature measure was introduced by Aleksandrov [2] and called integral curvature. The name of the curvature measures stems from the relation

$$\mathrm{C}_0(K,\omega)=\mathcal{H}^{n-1}\left(\left\{\boldsymbol{u}\in\mathbb{S}^{n-1} : \boldsymbol{x}\in K\cap H_K(\boldsymbol{u}) \text{ and } \overline{\boldsymbol{x}}\in\omega\right\}\right),$$

i.e., if K is smooth, then every direction $\boldsymbol{v}\in\omega$ points towards a unique boundary point $\lambda\boldsymbol{v}\in\partial K$, which has a unique outer normal vector, and in that case $\mathrm{C}_0(K,\omega)$ measures the subset of points in $\mathbb{S}^{n-1}$, that are normal to such points in ∂K. Again, from (1.9) it can be seen that $n\,\mathrm{W}_i(K)=\mathrm{C}_{n-i}(K,\mathbb{S}^{n-1})$ for $i=1,\ldots,n$. An exhaustive treatment of surface area and curvature measures can be found in [75, Chapter 4] (see also [27]).

The **cone-volume measure** of a convex body $K\in\mathcal{K}_o^n$ is the Borel measure V_K on the sphere defined by

$$\mathrm{V}_K(\eta)=\frac{1}{n}\int_\eta \mathrm{h}_K(\boldsymbol{u})\,\mathrm{dS}_K(\boldsymbol{u})$$

for every Borel set $\eta\subseteq\mathbb{S}^{n-1}$. In case of a polytope $P\in\mathcal{P}_o^n$ the cone-volume measure takes on a simple form. Just as the surface area measure the cone-volume measure of a polytope is concentrated on its outer normal vectors $U(P)$,

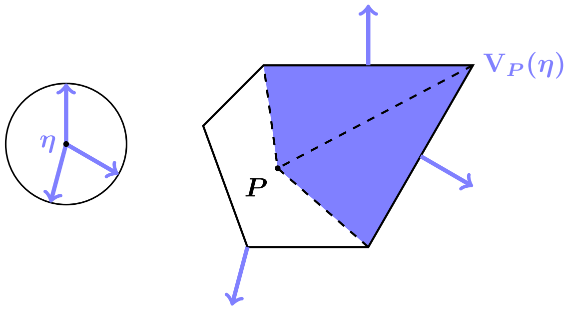

Figure 1.5: Cone-volume measure of a polygon

and for every $\boldsymbol{u} \in U(P)$ the cone-volume measure at $\boldsymbol{u}$ is the volume of the cone with apex at the origin and as base the facet of P with outer normal $\boldsymbol{u}$ (see Fig. 1.5), i.e.,

$$\mathrm{V}_P(\eta) = \sum_{\boldsymbol{u} \in U(P) \cap \eta} \mathrm{vol}(\mathrm{conv}(F(P, \boldsymbol{u}) \cup \{\boldsymbol{0}\})). \tag{1.10}$$

The cone-volume measure of convex bodies has been studied extensively over the last few years in many different contexts like the geometry of l_p^n balls ([5, 70, 71]), classification of $\mathrm{SL}(n)$-invariant valuations ([39, 57]), centro-affine surface area and curvature ([56, 73, 76, 85]), Schneider's projection problem ([40]), Ehrhart polynomials ([44]), L_p John ellipsoids ([45, 65]), isotropic measures ([12, 16]) and Orlicz-Brunn-Minkowski theory ([34, 66, 67]). One very important property of the cone-volume measure – and which makes it so essential – is its $\mathrm{SL}(n)$-invariance, or simply called affine invariance, i.e., for $K \in \mathcal{K}_o^n$, $A \in \mathbb{R}^{n \times n}$ with $|\det(A)| = 1$ it holds that $\mathrm{V}_{AK}(\eta) = \mathrm{V}_K(\overline{A^{-T}\eta})$ for every Borel set $\eta \subseteq \mathbb{S}^{n-1}$. The definition of the cone-volume measure also stems from L_p Brunn-Minkowski theory. This will be elaborated in Chapter 2.

Minkowski problems

A cornerstone of the Brunn-Minkowski theory is to characterize geometric measures. The problem originates from the characterization of surface area measures of convex bodies. Minkowski himself posed and solved the problem for surface area measures of polytopes. Later Aleksandrov [2] as well as Fenchel and Jessen [28] independently established the following solution for arbitrary convex bodies by using the variational formula (1.4) to transform the Minkowski problem into a minimization problem among support functions. This technique is still widely used to solve Minkowski problems related to other measures.

Theorem 1.2 (Aleksandrov [2], Fenchel, Jessen [28])**.** *Let μ be a non-zero finite Borel measure on $\mathbb{S}^{n-1}$ that is not concentrated on a closed hemisphere. Then there exists a convex body $K \in \mathcal{K}^n$ with $\mathrm{S}_K = \mu$ if and only if*

$$\int_{\mathbb{S}^{n-1}} \boldsymbol{u}\, \mathrm{d}\mu(\boldsymbol{u}) = \boldsymbol{0}. \tag{1.11}$$

If such K exists, it is unique up to translation.

Minkowski first solved the problem for measures which are either discrete or continuous, thus referred to as (classical) **Minkowski problem**. In case of a polytope P the equation (1.11) becomes

$$\sum_{\boldsymbol{u}\in U(P)} \mathrm{vol}_{n-1}(F(P,\boldsymbol{u}))\boldsymbol{u} = \boldsymbol{0}. \tag{1.12}$$

Today the characterization of area measures $\mathrm{S}_i(K,\cdot)$ of convex bodies $K \in \mathcal{K}^n$, $i \in \{1,\dots,n-1\}$, among the finite Borel measures on the sphere is known as the **Minkowski–Christoffel problem**, since for $i = n-1$ it is the classical Minkowski problem and for $i = 1$ it is known as Christoffel problem. For $i = 1$ Firey [29] and Berg [6] solved the problem independently and derived a necessary and sufficient condition which looks rather technical (see also [75, Thm. 8.3.8]). In the case $1 < i < n-1$ the characterization of area measures $\mathrm{S}_i(K,\cdot)$ still is a major open problem. If one considers the curvature measures we speak of **Aleksandrov problems** since for C_0 the problem has been solved by Aleksandrov [2]. We refer to [75, Chapter 8] and [48, p. 4] for more information and references on the Minkowski-Christoffel problem and also characterization of curvature measures.

The Minkowski problem associated to cone-volume measures of convex bodies in $\mathcal{K}^n_o$ is called **logarithmic Minkowski problem**. The discrete planar even logarithmic Minkowski problem was completely solved by Stancu [77, 78] via crystalline flows. She also proved that cone-volume measures of polytopes in $\mathcal{P}^2_s$ are unique with parallelograms being the only exception. The latter result was generalized to $\mathcal{K}^2_s$ by Böröczky, Lutwak, Yang and Zhang in [18]. Later, the same authors used a variational approach to extend Stancu's solution of the even logarithmic Minkowski problem for $\mathcal{P}^2_s$ to arbitrary dimensions and even measures on the sphere. One of the striking properties of cone-volume measures of symmetric convex bodies is an upper bound on the concentration on subspaces.

Theorem 1.3 (Böröczky, Lutwak, Yang, Zhang [19])**.** *Let μ be a non-zero finite even Borel measure on $\mathbb{S}^{n-1}$. Then there exists a symmetric convex body $K \in \mathcal{K}^n_s$ with $\mathrm{V}_K = \mu$ if and only if*

$$\mu(\mathbb{S}^{n-1}\cap L) \le \frac{\dim L}{n}\mu(\mathbb{S}^{n-1}) \tag{1.13}$$

for every proper subspace L of $\mathbb{R}^n$, and equality in (1.13) *is attained for some L, if and only if there is a subspace L' complementary to L such that μ is concentrated on $\mathbb{S}^{n-1}\cap(L\cup L')$.*

The inequality (1.13) along with its equality condition is known and will be referred to as **subspace concentration condition**. The above result settles the logarithmic Minkowski problem for symmetric convex bodies. However, the question of uniqueness of cone-volume measures of symmetric convex bodies remains an open problem. The general setting is more challenging. The validity of the subspace concentration condition for cone-volume measures of centered polytopes was established by Henk and Linke [41] and extended to centered convex bodies by Böröczky and Henk [13]. In the latter paper another remarkable result states the existence of lower bounds on concentration of cone-volume measures of centered convex bodies on open hemispheres. Stability in (1.13) for centered convex bodies is thematized in [12]. Regarding the general case, even less is known. Zhu [86] proved that in the discrete case there are no additional conditions on the measure if its support is in general position and not concentrated on a closed hemisphere. A set of vectors is said to be in **general position** if each n-element subset is linearly independent. In other words, Zhu showed that every measure on $\mathbb{S}^{n-1}$, that is concentrated on a finite subset of $\mathbb{S}^{n-1}$ in general position, but not on a closed hemisphere, can be realized as the cone-volume measure of a polytope. A refinement of this result can be found in [11], where it was proved that the validity of (1.13) (for a certain subset of subspaces) is a sufficient condition when the given measure is discrete but not necessarily symmetric. Chen, Li and Zhu [22] then used a sophisticated approximation argument to conclude that the strict inequality in (1.13) is also sufficient in case of non-even measures. Moreover, they gave the first examples of non-unique cone-volume measures not coming from parallelepipeds. On the other hand, necessary conditions on arbitrary cone-volume measures are widely missing. In [10], Böröczky and Zhu established a sharp upper bound on subspace concentration on 1-dimensional subspaces. Nevertheless, as they point out, their condition is non-sufficient even in the planar case.

2 The logarithmic Minkowski inequality and the planar cone-volume measure

2.1 An introduction to L_p Brunn-Minkowski theory

The starting point of this chapter is the study of the volume of certain sums of convex bodies other than the Minkowski addition defined by (1.1). Recall that the support function of a Minkowski combination $\lambda K + (1-\lambda)M$, $K, M \in \mathcal{K}^n$, $\lambda \in [0,1]$, is given by $\lambda \mathrm{h}_K + (1-\lambda)\mathrm{h}_M$ which represents the weighted arithmetic mean of h_K and h_M. It seems natural to consider other means of the respective support functions as well, where **p-means** (also called Hölder or genererealized means) are the most well-known examples. For $p \in \mathbb{R} \setminus \{0\}$, positive numbers $s, t \in \mathbb{R}_{>0}$ and a weighting parameter $\lambda \in [0,1]$ the p-mean of s and t is given by

$$M_p(s,t,\lambda) = [(1-\lambda)s^p + \lambda t^p]^{\frac{1}{p}},$$

where we may extend the definition to $p = 0$ via taking the limit

$$M_0(s,t,\lambda) = \lim_{p\to 0}[(1-\lambda)s^p + \lambda t^p]^{\frac{1}{p}} = s^{1-\lambda}t^{\lambda},$$

which is the weighted geometric mean of s and t. The family of p-means include the arithmetic mean as special case when $p = 1$. Moreover, the means are monotone with respect to p, i.e., for $p \leq p'$ it holds that

$$M_p(s,t,\lambda) \leq M_{p'}(s,t,\lambda).$$

In order to consider the p-mean of support functions we have to assure positivity. That is why here only the class $\mathcal{K}^n_o$ is considered. Now for $p \neq 0$, $K, M \in \mathcal{K}^n_o$, and scalars $s, t \geq 0$ the **p-combination** of K and M with respect to s and t is (see Fig. 2.1)

$$s \cdot_{\mathrm{p}} K +_{\mathrm{p}} t \cdot_{\mathrm{p}} M \\ = \left\{ \boldsymbol{x} \in \mathbb{R}^n : \langle \boldsymbol{u}, \boldsymbol{x} \rangle \leq [s\mathrm{h}_K(\boldsymbol{u})^p + t\mathrm{h}_M(\boldsymbol{u})^p]^{\frac{1}{p}} \text{ for all } \boldsymbol{u} \in \mathbb{S}^{n-1} \right\},$$

or in case $s = \lambda$, $t = 1-\lambda$,

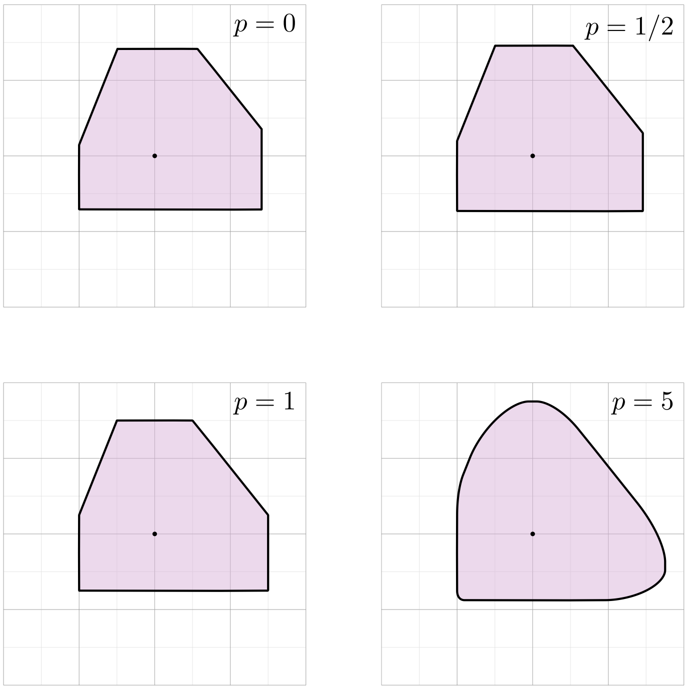

Figure 2.1: p-combination $\frac{1}{2}K +_{\mathrm{p}} \frac{1}{2}M$ for $K = \operatorname{conv}\left\{\binom{0}{2}, \binom{-1}{-1/2}, \binom{2}{-1/2}\right\}$ and $M = [-1,1]^2$

$$(1-\lambda) \cdot_{\mathrm{p}} K +_{\mathrm{p}} \lambda \cdot_{\mathrm{p}} M \\ = \{\boldsymbol{x} \in \mathbb{R}^n : \langle \boldsymbol{u}, \boldsymbol{x}\rangle \leq M_p(\mathrm{h}_K(\boldsymbol{u}), \mathrm{h}_M(\boldsymbol{u}), \lambda) \text{ for all } \boldsymbol{u} \in \mathbb{S}^{n-1}\}. \quad (2.1)$$

In the context of p-combinations we will usually just write $\cdot$ instead of $\cdot_{\mathrm{p}}$.

We want to point out that one may study p-combinations in a slightly more general way: The notion of p-means can be extended to $p \in \{\pm\infty\}$ via taking the limit, and if $p > 0$, the p-mean can be defined on the set of nonnegative numbers so that also convex bodies K with $\mathbf{0} \in \partial K$ may be considered in p-combinations. However, for the sake of simplicity we restrict ourselves to $p \in \mathbb{R}$ and p-combinations within $\mathcal{K}_o^n$.

The p-combination of convex bodies was first considered by Firey [30] for $p \geq 1$. He also established an extension of the Brunn-Minkowski inequality (1.2) to p-combinations for $p > 1$. More precisely, he proved that for $p > 1$, $K, M \in \mathcal{K}_o^n$ and $\lambda \in [0,1]$

$$\operatorname{vol}((1-\lambda)K +_{\mathrm{p}} \lambda M)^{\frac{p}{n}} \geq (1-\lambda)\operatorname{vol}(K)^{\frac{p}{n}} + \lambda \operatorname{vol}(M)^{\frac{p}{n}}. \quad (2.2)$$

Very recently Kolesnikov and Milman [54] established (2.2) under some smoothness assumptions on K and M, if $p < 1$ and p is sufficiently close to 1. Due to Lutwak [61] is the following variational formula

$$\lim_{\varepsilon \searrow 0} \frac{\operatorname{vol}(K +_{\mathrm{p}} \varepsilon M) - \operatorname{vol}(K)}{\varepsilon} = \frac{1}{p} \int_{\mathbb{S}^{n-1}} \mathrm{h}_M(\boldsymbol{u})^p \mathrm{h}_K(\boldsymbol{u})^{1-p} \,\mathrm{dS}_K(\boldsymbol{u}). \quad (2.3)$$

for $p > 1$ and $K, M \in \mathcal{K}_o^n$. The measure $\mathrm{S}_K^{(p)}$ given by

$$\mathrm{dS}_K^{(p)} = \mathrm{h}_K^{1-p}\,\mathrm{dS}_K$$

is called L_p **surface area measure** of K and can be defined this way for every $p \in \mathbb{R}$. In particular, the L_p surface area measure coincides with the classical surface area measure and – up to a factor – with the cone-volume measure in the cases $p = 1$ and $p = 0$, respectively. The task of characterizing L_p surface area measures is known as L_p **Minkowski problem** and was first addressed by Lutwak [61]. The L_p Brunn Minkowski theory has gained much interest throughout the years and has developed rapidly since 1990's. We refer to [75, Chapter 9] for a comprehensive overview of L_p Brunn-Minkowski theory. Recent progress on the L_p Minkowski problem was made in [7, 15, 21, 23, 24, 46, 49, 50, 55, 63, 84, 85, 87, 88].

As Schneider points out, Lutwaks solution to the L_p Minkowski problem for $p > 1$ in [61] actually only requires $p > 0$. This was the dawn of the L_p Brunn-Minkowski theory for $0 < p < 1$, which is both fascinating, since inequalities like (2.2) and (2.3) become stronger for smaller p, and challenging, since in general the p-mean of support functions is not the support function of the respective p-combination for $p < 1$. By taking the limit $p \to 0+$ the inequality (2.2) becomes the logarithmic Brunn-Minkowski inequality

$$\mathrm{vol}((1-\lambda)\cdot K +_0 \lambda \cdot M) \geq \mathrm{vol}(K)^{1-\lambda}\,\mathrm{vol}(M)^{\lambda}, \tag{2.4}$$

where we define by

$$\begin{aligned}(1-\lambda)K +_0 \lambda M \\ &= \{\boldsymbol{x} \in \mathbb{R}^n : \langle \boldsymbol{u}, \boldsymbol{x}\rangle \leq M_0(\mathrm{h}_K(\boldsymbol{u}), \mathrm{h}_M(\boldsymbol{u}), \lambda) \text{ for all } \boldsymbol{u} \in \mathbb{S}^{n-1}\}\end{aligned}$$

the **log-combination** of K and L with respect to λ (cf. (2.1)). It can be seen from the arithmetic-geometric mean inequality that the logarithmic Brunn-Minkowski inequality (2.4), if it holds true, is in fact stronger than (1.2). Böröczky, Lutwak, Yang and Zhang [18] conjectured that (2.4) holds for all pairs of symmetric convex bodies $K, M \in \mathcal{K}_s^n$, but so far it has been verified only in particular instances. In [18], the planar case $n = 2$ was established and Saroglou [74] showed that (2.4) holds for pairs of unconditional convex bodies in arbitrary dimension, i.e., for convex bodies that are symmetric about every coordinate hyperplane (see also [8, 25] for preliminary work). Both results were given alongside a characterization of the equality cases.

It is well-known that the classical Brunn-Minkowski inequality (1.2) is equivalent to Minkowski's mixed volume inequality, which, for two convex bodies $K, M \in \mathcal{K}_o^n$, can be stated as

$$\int_{\mathbb{S}^{n-1}} \frac{\mathrm{h}_M(\boldsymbol{u})}{\mathrm{h}_K(\boldsymbol{u})}\,\mathrm{dV}_K(\boldsymbol{u}) \geq \mathrm{vol}(K)^{\frac{n-1}{n}}\,\mathrm{vol}(M)^{\frac{1}{n}}. \tag{2.5}$$

Just as for the Brunn-Minkowski inequality, the Minkowski inequality (2.5) can be seen as a particular case of a family of L_p Minkowski inequalities in the L_p Brunn-Minkowski theory. The L_p Minkowski inequality states that for $K, M \in \mathcal{K}_o^n$

$$\int_{\mathbb{S}^{n-1}} \left(\frac{\mathrm{h}_M(\boldsymbol{u})}{\mathrm{h}_K(\boldsymbol{u})}\right)^p \mathrm{dV}_K(\boldsymbol{u}) \geq \mathrm{vol}(K)^{\frac{n-p}{n}} \mathrm{vol}(M)^{\frac{p}{n}} \tag{2.6}$$

and it was proved by Lutwak [61] for $p > 1$. So far, however, it is an open problem if (2.6) holds even for pairs of symmetric convex bodies when $0 < p < 1$. In a fundamental paper, Böröczky, Lutwak, Yang and Zhang [18] established the L_p Minkowski inequality in the plane in the case where $K, M \in \mathcal{K}_s^2$ and $0 < p < 1$. Moreover, they showed that in any dimension the L_p Brunn-Minkowski inequality (2.2) and the L_p Minkowski inequality (2.6) are equivalent in the class of symmetric convex bodies.

Theorem 2.1 (Böröczky, Lutwak, Yang, Zhang [18])**.** *Let $p > 0$. The L_p Brunn-Minkowski inequality* (2.2) *holds for all $K, M \in \mathcal{K}_s^n$ and $\lambda \in [0, 1]$ if and only if the L_p Minkowski inequality* (2.6) *holds true all $K, M \in \mathcal{K}_s^n$.*

The results extend to the limit case $p \to 0+$ which is as follows.

Theorem 2.2 (Böröczky, Lutwak, Yang, Zhang [18])**.** *The logarithmic Brunn-Minkowski inequality* (2.4) *holds for all $K, M \in \mathcal{K}_s^n$ and $\lambda \in [0, 1]$ if and only if the logarithmic Minkowski inequality*

$$\int_{\mathbb{S}^{n-1}} \log \frac{\mathrm{h}_M(\boldsymbol{u})}{\mathrm{h}_K(\boldsymbol{u})} \mathrm{dV}_K(\boldsymbol{u}) \geq \frac{\mathrm{vol}(K)}{n} \log \frac{\mathrm{vol}(M)}{\mathrm{vol}(K)} \tag{2.7}$$

holds true all $K, M \in \mathcal{K}_s^n$.

The inequality (2.7) is called the logarithmic Minkowski inequality and it was proved to hold in the plane ([18], see [68] for a different proof) and for pairs of unconditional convex bodies in any dimension ([74]).

In the general setting almost nothing is known. There are counterexamples showing that neither (2.4) nor (2.7) hold for pairs of arbitrary convex bodies containing the origin in the interior, e.g., a cube and a suitable translate of it. Instead one considers certain classes of convex bodies granting control over the location of the origin. Xi and Leng [80] proved that (2.4) holds if the two bodies are in so-called dilation position, which includes the class $\mathcal{K}_s^n$. Guan and Li [38] established among others the logarithmic Minkowski inequality when K is the Euclidean unit ball and the Santálo point of M is the origin. The Santálo point of a convex body K is the point $\boldsymbol{x} \in \operatorname{int} K$ minimizing $\mathrm{vol}([K - \boldsymbol{x}]^*)$ where $[K - \boldsymbol{x}]^*$ is the polar body of $K - \boldsymbol{x}$ (cf. (3.1)). Further results were obtained by Stancu [79], e.g., she proved among others that the logarithmic Minkowski inequality (2.7) holds true for convex bodies $K, M \in \mathcal{K}_o^n$, when K is a polytope and each facet of K touches the boundary of L. Stancu also established versions of (2.7) where in place of M an affine image of M is used; for instance when K is a centered simplex with constant edge length and M a given convex body, then there is an affine image $\tilde{M}$ of M such that (2.7) holds for K and $\tilde{M}$.

In the Sections 2.2 and 2.3 we will study the logarithmic Minkowski (2.7) inequality in the context of centered convex bodies. This particularly includes the class of symmetric convex bodies. Here we will prove the logarithmic Minkowski inequality in some very particular instances. More precisely, we establish the logarithmic Minkowski inequality (2.7) when the gauge body is a centered simplex and the other one is centered, and also when the gauge body is a parallelepiped and the other one is symmetric. The main tool in the proofs are sharp volume bounds on intersections of centered convex bodies with hyperplanes and halfspaces like Grünbaum's inequality (2.9). Furthermore, we extend the logarithmic Minkowski inequality for parallelepipeds to the setting where the second body is only centered rather than symmetric, but only in small dimensions. The results of the Sections 2.2 and 2.3 appeared as joint work with Martin Henk [43].

The Section 2.4 treats the logarithmic Minkowski problem in the plane for trapezoids. Our main result is a full characterization of cone-volume measures of trapezoids including the explicit computation of a trapezoid from a given cone-volume measure. Together with results stated in [78] this settles the logarithmic Minkowski problem for quadrilaterals. We discuss uniqueness of cone-volume measures in Section 2.5. Moreover, from the aforementioned explicit description we may derive a family of examples confirming non-uniqueness of cone-volume measures of quadrilaterals. So far, the only examples, which are not parallelepids, were given in [22] including fivegons in the planar case. The logarithmic Minkowski problem for polygons with at least five vertices is still open.

2.2 The logarithmic Minkowski inequality for simplices

We will now prove the logarithmic Minkowski inequality (2.7) for some special cases. They all have in common that one of the convex bodies under consideration is assumed to be centered and the other one is either a centered simplex or, in Section 2.3, a parallelepiped. One of the used tools is a reformulation of Minkowski's characterization theorem (Theorem 1.2). For a polytope $P \in \mathcal{P}_o^n$ and a unit outer normal vector $\boldsymbol{u} \in U(P)$ the volume of the corresponding cone can be computed by (cf. (1.10))

$$\mathrm{V}_P(\boldsymbol{u}) = \mathrm{vol}(\mathrm{conv}(F(P,\boldsymbol{u}) \cup \{\boldsymbol{0}\})) = \frac{\mathrm{h}_P(\boldsymbol{u})}{n}\,\mathrm{vol}_{n-1}(F(P,\boldsymbol{u})),$$

which is the well-known pyramid formula. Substituting into (1.12) yields the equivalent equation

$$\sum_{\boldsymbol{u}\in U(P)} \frac{\mathrm{V}_P(\boldsymbol{u})}{\mathrm{h}_P(\boldsymbol{u})}\boldsymbol{u} = \boldsymbol{0}, \tag{2.8}$$

so that Minkowski's characterization becomes a statement formulated entirely in terms of support functions and cone-volumes. The second important ingredient in the proofs is Grünbaum's inequality. Each symmetric convex body divided by a hyperplane through the origin will have its volume split into equal parts. Due to Grünbaum is the following sharp bound for centered convex bodies (see Fig. 2.2).

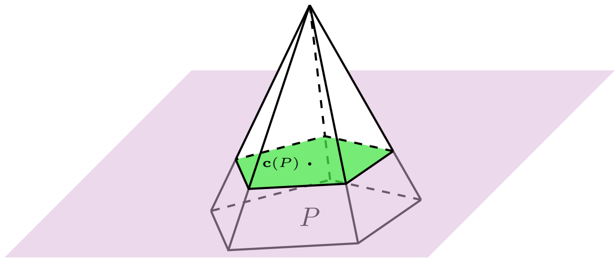

Figure 2.2: Extremal case of Grünbaum's inequality

Theorem 2.3 (Grünbaum [37])**.** *Let $K \in \mathcal{K}^n_c$ and $\boldsymbol{u} \in \mathbb{S}^{n-1}$. Then*

$$\operatorname{vol}(K \cap H^-(\boldsymbol{u}, 0)) \geq \operatorname{vol}(K) \left(\frac{n}{n+1}\right)^n \tag{2.9}$$

with equality if and only if K is a cone whose base is parallel to $H(\boldsymbol{u}, 0)$ and contained in $H^+(\boldsymbol{u}, 0)$.

We are now ready to prove the logarithmic Minkowski inequality for centered simplices and centered convex bodies. As an introductory remark we want to point out that for $P \in \mathcal{P}^n_o$ and $M \in \mathcal{K}^n_o$ with $\operatorname{vol}(P) = \operatorname{vol}(M)$ – which we can always assume because the logarithmic Minkowski inequality is scaling-invariant – the inequality (2.7) can be restated as

$$\prod_{\boldsymbol{u} \in U(P)} \left(\frac{\mathrm{h}_M(\boldsymbol{u})}{\mathrm{h}_P(\boldsymbol{u})}\right)^{\mathrm{V}_P(\boldsymbol{u})} \geq 1. \tag{2.10}$$

Theorem 2.4 (Henk, P. [43])**.** *Let $S, M \in \mathcal{K}^n_c$, where S is a simplex. Then*

$$\int\limits_{\mathbb{S}^{n-1}} \log \frac{\mathrm{h}_M(\boldsymbol{u})}{\mathrm{h}_S(\boldsymbol{u})} \, \mathrm{dV}_S(\boldsymbol{u}) \geq \frac{\operatorname{vol}(S)}{n} \log \frac{\operatorname{vol}(M)}{\operatorname{vol}(S)}. \tag{2.11}$$

Equality holds if and only if S and M are dilates.

Proof. The inequality (2.11) is homogeneous in both S and M, so without loss of generality we assume that $\operatorname{vol}(S) = \operatorname{vol}(M) = 1$. Let $U(S) = \{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_{n+1}\}$. First we note that since S is centered all cone-volumes $\mathrm{V}_S(\boldsymbol{u}_i)$ coincide. Indeed, if $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{n+1} \in \mathbb{R}^n$ are the vertices of S, then each facet $F(S, \boldsymbol{u}_i)$ of S contains exactly n of its vertices, say $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{i-1}, \boldsymbol{v}_{i+1}, \ldots, \boldsymbol{v}_{n+1}$, and

$$\begin{aligned}
\mathrm{V}_S(\boldsymbol{u}_i) &= \operatorname{vol}(\operatorname{conv}(F(S, \boldsymbol{u}_i) \cup \{\boldsymbol{0}\})) \\
&= \frac{1}{n} \left|\det(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{i-1}, \boldsymbol{v}_{i+1}, \ldots, \boldsymbol{v}_n)\right| \\
&= \frac{1}{n} \left|\det(\boldsymbol{v}_1 - (n+1)\,\mathbf{c}(S), \boldsymbol{v}_2, \ldots, \boldsymbol{v}_{i-1}, \boldsymbol{v}_{i+1}, \ldots, \boldsymbol{v}_n)\right| \\
&= \frac{1}{n} \left|\det(-(\boldsymbol{v}_2 + \ldots + \boldsymbol{v}_n + \boldsymbol{v}_{n+1}), \boldsymbol{v}_2, \ldots, \boldsymbol{v}_{i-1}, \boldsymbol{v}_{i+1}, \ldots, \boldsymbol{v}_n)\right| \\
&= \frac{1}{n} \left|\det(\boldsymbol{v}_i, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_{i-1}, \boldsymbol{v}_{i+1}, \ldots, \boldsymbol{v}_n)\right|
\end{aligned}$$

$$= \mathrm{V}_S(\boldsymbol{u}_1).$$

Thus, for each $i \in \{1, \dots, n+1\}$ we have $\mathrm{V}_S(\boldsymbol{u}_i) = \frac{1}{n+1}$. Hence in view of (2.10) we just have to verify

$$\prod_{i=1}^{n+1} \frac{\mathrm{h}_M(\boldsymbol{u}_i)}{\mathrm{h}_S(\boldsymbol{u}_i)} \geq 1 \tag{2.12}$$

and by (2.8) we also know

$$\sum_{i=1}^{n+1} \frac{1}{\mathrm{h}_S(\boldsymbol{u}_i)} \boldsymbol{u}_i = \boldsymbol{0}. \tag{2.13}$$

Grünbaum's centroid inequality (2.9) applied to the centered convex body M gives for $1 \leq i \leq n+1$

$$\mathrm{vol}(M \cap H^-(\boldsymbol{u}_i, 0)) \geq \mathrm{vol}(M) \left(\frac{n}{n+1}\right)^n \tag{2.14}$$

and for the simplex S we have the equality

$$\mathrm{vol}(S) = \mathrm{vol}(S \cap H^-(\boldsymbol{u}_i, 0)) \left(\frac{n+1}{n}\right)^n \tag{2.15}$$

for $i = 1, \dots, n+1$. Now suppose (2.12) does not hold. Then there exist also n indices $i = 1, \dots, n$, say, with

$$\prod_{i=1}^{n} \mathrm{h}_M(\boldsymbol{u}_i) < \prod_{i=1}^{n} \mathrm{h}_S(\boldsymbol{u}_i). \tag{2.16}$$

Since $M \subseteq \{\boldsymbol{x} \in \mathbb{R}^n : \langle \boldsymbol{u}_i, \boldsymbol{x} \rangle \leq \mathrm{h}_M(\boldsymbol{u}_i), 1 \leq i \leq n\}$, we conclude in view of (2.14)

$$\begin{aligned}\left(\frac{n}{n+1}\right)^n \mathrm{vol}(M) &\leq \mathrm{vol}(M \cap H^-(\boldsymbol{u}_{n+1}, 0)) \\ &\leq \mathrm{vol}(\{\boldsymbol{x} \in \mathbb{R}^n : \langle \boldsymbol{u}_i, \boldsymbol{x} \rangle \leq \mathrm{h}_M(\boldsymbol{u}_i), 1 \leq i \leq n, \langle \boldsymbol{u}_{n+1}, \boldsymbol{x} \rangle \leq 0\}).\end{aligned}$$

Let $T = \{\boldsymbol{x} \in \mathbb{R}^n : \langle \boldsymbol{e}_i, \boldsymbol{x} \rangle \leq 1, 1 \leq i \leq n, \langle \sum_{i=1}^n \boldsymbol{e}_i, \boldsymbol{x} \rangle \geq 0\}$ and A be the $(n \times n)$-matrix with columns $\frac{\boldsymbol{u}_i}{\mathrm{h}_M(\boldsymbol{u}_i)}$, $i = 1, \dots, n$. Then

$$A^{-T} T = \{\boldsymbol{x} \in \mathbb{R}^n : \langle \boldsymbol{u}_i, \boldsymbol{x} \rangle \leq \mathrm{h}_M(\boldsymbol{u}_i), 1 \leq i \leq n, \langle \boldsymbol{u}_{n+1}, \boldsymbol{x} \rangle \leq 0\}$$

and thus, by (2.16) and (2.15),

$$\begin{aligned}\left(\frac{n}{n+1}\right)^n \mathrm{vol}(M) &\leq \left|\det\left(\frac{\boldsymbol{u}_1}{\mathrm{h}_M(\boldsymbol{u}_1)}, \dots, \frac{\boldsymbol{u}_n}{\mathrm{h}_M(\boldsymbol{u}_n)}\right)\right|^{-1} \mathrm{vol}(T) \\ &= \frac{\prod_{i=1}^n \mathrm{h}_M(\boldsymbol{u}_i)}{|\det(\boldsymbol{u}_1, \dots, \boldsymbol{u}_n)|} \mathrm{vol}(T) \\ &< \frac{\prod_{i=1}^n \mathrm{h}_S(\boldsymbol{u}_i)}{|\det(\boldsymbol{u}_1, \dots, \boldsymbol{u}_n)|} \mathrm{vol}(T) \\ &= \mathrm{vol}(\{\boldsymbol{x} \in \mathbb{R}^n : \langle \boldsymbol{u}_i, \boldsymbol{x} \rangle \leq \mathrm{h}_S(\boldsymbol{u}_i), 1 \leq i \leq n, \langle \boldsymbol{u}_{n+1}, \boldsymbol{x} \rangle \leq 0\})\end{aligned}$$

$$= \left(\frac{n}{n+1}\right)^n \mathrm{vol}(S).$$

This contradicts, however, the assumption that S and M have the same volume.

Suppose now we have equality in (2.12). Then we must also have equality for each n-element subset of $\{1,\dots,n+1\}$, since in view of (2.16)

$$\prod_{\substack{i=1\\ i\neq j}}^{n+1} \mathrm{h}_M(\boldsymbol{u}_i) \geq \prod_{\substack{i=1\\ i\neq j}}^{n} \mathrm{h}_S(\boldsymbol{u}_i)$$

must hold for every j and

$$\begin{aligned}(\mathrm{h}_S(\boldsymbol{u}_1)\dots\mathrm{h}_S(\boldsymbol{u}_{n+1}))^n &= \prod_{j=1}^{n+1}\prod_{\substack{i=1\\ i\neq j}}^{n} \mathrm{h}_S(\boldsymbol{u}_i)\\ &\leq \prod_{j=1}^{n+1}\prod_{\substack{i=1\\ i\neq j}}^{n} \mathrm{h}_M(\boldsymbol{u}_i) = (\mathrm{h}_M(\boldsymbol{u}_1)\dots\mathrm{h}_M(\boldsymbol{u}_{n+1}))^n.\end{aligned}$$

Moreover, for every choice of n indices, say $\{1,\dots,n\}$, by repeating the steps above we have

$$\begin{aligned}\left(\frac{n}{n+1}\right)^n \mathrm{vol}(M) &= \left(\frac{n}{n+1}\right)^n \mathrm{vol}(S)\\ &= \frac{\prod_{i=1}^n \mathrm{h}_S(\boldsymbol{u}_i)}{|\det(\boldsymbol{u}_1,\dots,\boldsymbol{u}_n)|}\mathrm{vol}(T)\\ &= \frac{\prod_{i=1}^n \mathrm{h}_M(\boldsymbol{u}_i)}{|\det(\boldsymbol{u}_1,\dots,\boldsymbol{u}_n)|}\mathrm{vol}(T)\\ &= \mathrm{vol}(M\cap H^-(\boldsymbol{u}_{n+1},0))\end{aligned}$$

and by the characterization of the equality case in Theorem 2.3 we know that M is a simplex with outer normals $\boldsymbol{u}_i$ and centroid at the origin. Hence Minkowski's characterization formula (2.13) also holds with $\mathrm{h}_S(\boldsymbol{u}_i)$ replaced by $\mathrm{h}_M(\boldsymbol{u}_i)$ which shows that S and M must be equal. $\square$

It is not hard to see that the logarithmic Minkowski inequality for centered simplices (Theorem 2.4) implies the uniqueness of cone-volume measures concentrated on $n+1$ affinely independent directions. However, this is also a consequence of Minkowski's characterization theorem (cf. Theorem 1.2). For if, let $\boldsymbol{u}_1,\dots,\boldsymbol{u}_{n+1}$ be the outer unit normals of two simplices $S,T\in\mathcal{P}_o^n$ and the two simplices have equal cone-volumes $\gamma_i = \mathrm{V}_S(\boldsymbol{u}_i) = \mathrm{V}_T(\boldsymbol{u}_i)$, $1\leq i\leq n+1$. Let $\alpha_i(S),\alpha_i(T)$, $1\leq i\leq n+1$, be the $(n-1)$-dimensional volume of the facet with outer normal vector $\boldsymbol{u}_i$ of S and T, respectively. By (1.12) and the affine independence of the normal vectors we get that $(\alpha_1(S),\dots,\alpha_{n+1}(S))^T$ and $(\alpha_1(T),\dots,\alpha_{n+1}(T))^T$ are contained in a one-dimensional subspace, so $\alpha_i(S)=\mu\alpha_i(T)$ for $1\leq i\leq n+1$ and a positive scalar μ. Hence,

$$\frac{h_S(\boldsymbol{u}_i)}{h_T(\boldsymbol{u}_i)} = \frac{\left(\frac{n\gamma_i}{\alpha_i(S)}\right)}{\left(\frac{n\gamma_i}{\alpha_i(T)}\right)} = \frac{\alpha_i(S)}{\alpha_i(T)} = \mu,$$

and so $S = \mu T$. Since $\mathrm{vol}(S) = \mathrm{vol}(T)$ it follows that $S = T$.

2.3 The logarithmic Minkowski inequality for parallelepipeds

Using the same arguments as in the foregoing proof of Theorem 2.4 we may establish the logarithmic Minkowski inequality when both bodies are o-symmetric and one is a parallelepiped. Instead of Grünbaum's inequality (2.9) we use the fact that hyperplanes through the origin cut symmetric sets into equal halves.

Proposition 2.5 (Henk, P. [43]). *Let $Q, M \in \mathcal{K}_s^n$, where Q is a parallelepiped. Then*

$$\int\limits_{\mathbb{S}^{n-1}} \log \frac{\mathrm{h}_M(\boldsymbol{u})}{\mathrm{h}_Q(\boldsymbol{u})} \, \mathrm{dV}_Q(\boldsymbol{u}) \geq \frac{\mathrm{vol}(Q)}{n} \log \frac{\mathrm{vol}(M)}{\mathrm{vol}(Q)}.$$

Equality holds if and only if M is a parallelepiped with $U(M) = U(Q)$.

Proof. Without loss of generality we assume that $\mathrm{vol}(Q) = \mathrm{vol}(M)$. Let $U(Q) = \{\pm\boldsymbol{u}_1, \ldots, \pm\boldsymbol{u}_n\}$. First we note that all cone-volumes $\mathrm{V}_Q(\pm\boldsymbol{u}_i)$ coincide as Q is the linear image of a cube, so each cone of Q is the linear image of a cone of the cube. Therefore and since M is symmetric we just have to verify

$$\prod_{i=1}^{n} \frac{\mathrm{h}_M(\boldsymbol{u}_i)}{\mathrm{h}_Q(\boldsymbol{u}_i)} \geq 1. \tag{2.17}$$

Now suppose (2.17) does not hold. Since $M \subseteq \{\boldsymbol{x} \in \mathbb{R}^n : |\langle \boldsymbol{u}_i, \boldsymbol{x}\rangle| \leq \mathrm{h}_M(\boldsymbol{u}_i), 1 \leq i \leq n\}$, we get

$$\begin{aligned} \mathrm{vol}(M) &\leq \mathrm{vol}(\{\boldsymbol{x} \in \mathbb{R}^n : |\langle \boldsymbol{u}_i, \boldsymbol{x}\rangle| \leq \mathrm{h}_M(\boldsymbol{u}_i), 1 \leq i \leq n\}) \qquad (2.18)\\ &= \frac{2^n \prod_{i=1}^n \mathrm{h}_M(\boldsymbol{u}_i)}{|\det(\boldsymbol{u}_1, \ldots, \boldsymbol{u}_n)|} \\ &< \frac{2^n \prod_{i=1}^n \mathrm{h}_Q(\boldsymbol{u}_i)}{|\det(\boldsymbol{u}_1, \ldots, \boldsymbol{u}_n)|} \\ &= \mathrm{vol}(Q). \end{aligned}$$

This contradicts, however, the assumption of having the same volume.

Suppose now we have equality in (2.17). Then by the same arguments as above

$$\begin{aligned} \mathrm{vol}(Q) = \mathrm{vol}(M) &\leq \mathrm{vol}(\{\boldsymbol{x} \in \mathbb{R}^n : |\langle \boldsymbol{u}_i, \boldsymbol{x}\rangle| \leq \mathrm{h}_M(\boldsymbol{u}_i), 1 \leq i \leq n\}) \\ &= \frac{2^n \prod_{i=1}^n \mathrm{h}_M(\boldsymbol{u}_i)}{|\det(\boldsymbol{u}_1, \ldots, \boldsymbol{u}_n)|} \\ &= \frac{2^n \prod_{i=1}^n \mathrm{h}_Q(\boldsymbol{u}_i)}{|\det(\boldsymbol{u}_1, \ldots, \boldsymbol{u}_n)|} \\ &= \mathrm{vol}(Q), \end{aligned}$$

i.e., $M = \{\boldsymbol{x} \in \mathbb{R}^n : |\langle \boldsymbol{u}_i, \boldsymbol{x}\rangle| \leq \mathrm{h}_M(\boldsymbol{u}_i), i = 1, \ldots, n\}$. Thus, M is parallelepiped with outer normals $\pm\boldsymbol{u}_1, \ldots, \pm\boldsymbol{u}_n$. On the other hand, if Q and M are parallelepipeds with the same volume and sets of outer normal vectors, then (2.17) is satisfied with equality since the volumes are proportional to the product of the supporting distances. □

We conjecture that a similar result holds when the symmetry of M is replaced by the weaker assumption $M \in \mathcal{K}_c^n$. In this case, however, we can prove it only in dimensions $n \leq 4$.

Proposition 2.6 (Henk, P. [43])**.** *Let $n \in \{2, 3, 4\}$ and $Q \in \mathcal{K}_s^n$, $M \in \mathcal{K}_c^n$, where Q is a parallelepiped. Then*

$$\int_{\mathbb{S}^{n-1}} \log \frac{\mathrm{h}_M(\boldsymbol{u})}{\mathrm{h}_Q(\boldsymbol{u})} \, \mathrm{dV}_Q(\boldsymbol{u}) \geq \frac{\mathrm{vol}(Q)}{n} \log \frac{\mathrm{vol}(M)}{\mathrm{vol}(Q)}.$$

Equality holds if and only if M is a parallelepiped with $U(M) = U(Q)$.

This case seems to be more intricate and the proof of the above statement needs some preparation. Since the logarithmic Minkowski inequality only depends on the supporting distances in directions $\boldsymbol{u} \in U(Q)$, similar to (2.18) the problem becomes to find volume bounds for a certain M enclosing shifted parallelepiped of the form

$$\{\boldsymbol{x} \in \mathbb{R}^n : -\alpha_i^{-1} \leq \langle \boldsymbol{e}_i, \boldsymbol{x}\rangle \leq \alpha_i, 1 \leq i \leq n\}, \tag{2.19}$$

with some constants $\alpha_1, \ldots, \alpha_n \geq 1$. More precisely, establishing the logarithmic Minkowski inequality in this setting is equivalent to proving that a centered convex body contained in (2.19) has a volume smaller than 2^n unless $\alpha_i = 1$ for all $i = 1, \ldots, n$ (see Fig. 2.3).

For the proof of Proposition 2.6 we will need the following result by Milman and Pajor [69] (see also [4]).

Theorem 2.7 (Milman, Pajor [69])**.** *Let $K \in \mathcal{K}_c^n$. Then*

$$\mathrm{vol}(K \cap (-K)) \geq 2^{-n} \, \mathrm{vol}(K).$$

From Theorem 2.7 one can easily deduce a volume bound for a centered convex body with circumscribed axis-aligned parallelepiped.

Corollary 2.8. *Let $K \in \mathcal{K}_c^n$. Suppose there are numbers $\alpha_1, \ldots, \alpha_n \geq 1$ such that*

$$K \subseteq \{\boldsymbol{x} \in \mathbb{R}^n : -\alpha_i^{-1} \leq \langle \boldsymbol{e}_i, \boldsymbol{x}\rangle \leq \alpha_i, \, 1 \leq i \leq n\}$$

Then $\alpha_1 \cdot \ldots \cdot \alpha_n \leq \frac{4^n}{\mathrm{vol}(K)}$.

Proof. By Theorem 2.7 and since

$$K \cap (-K) \subseteq \{\boldsymbol{x} \in \mathbb{R}^n : -\alpha_i^{-1} \leq \langle \boldsymbol{e}_i, \boldsymbol{x}\rangle \leq \alpha_i^{-1}, 1 \leq i \leq n\}$$

we get

$$2^{-n} \, \mathrm{vol}(K) \leq \mathrm{vol}(K \cap (-K)) \leq 2^n \alpha_1^{-1} \cdot \ldots \cdot \alpha_n^{-1}.$$ □

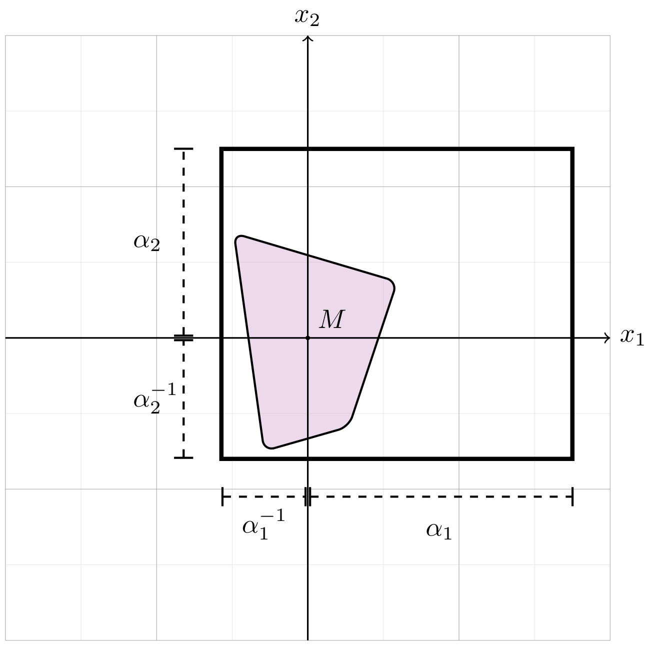

Figure 2.3: A centered convex body M enclosed by the shifted parallelepiped (2.19)

A more sophisticated estimate will be needed. We will establish next an upper bound on the volume of centered convex bodies with respect to volume sections and supporting distances. The intuition behind the following lemma is that if the mass of slices of a convex body inside a bounding box is shifted with respect to a fixed direction, then the centroid of the body moves into the same direction (see Fig. 2.4).

Lemma 2.9 (Henk, P. [43]). *Let $K \in \mathcal{K}_c^n$, $\boldsymbol{u} \in \mathbb{S}^{n-1}$ and for $t \in \mathbb{R}$ define $f(t) = \mathrm{vol}_{n-1}(K \cap H(\boldsymbol{u}, t))$. Then*

$$\mathrm{vol}(K) \leq 2\mathrm{h}_K(\boldsymbol{u}) \max_{t\in\mathbb{R}} f(t), \tag{2.20}$$

with equality if and only if K is a centered prism over a base parallel to $\boldsymbol{u}^\perp$.

Proof. Write $\|f\|_\infty = \max_{t\in\mathbb{R}} f(t)$. By Fubini's theorem we have

$$\mathrm{vol}(K) = \int_{-\mathrm{h}_K(-\boldsymbol{u})}^{\mathrm{h}_K(\boldsymbol{u})} f(t)\,\mathrm{d}t \tag{2.21}$$

and, since K is centered,

$$0 = \langle \boldsymbol{u}, \mathrm{vol}(K)\,\mathbf{c}(K)\rangle = \int_K \langle \boldsymbol{u}, \boldsymbol{x}\rangle\,\mathrm{d}\boldsymbol{x} = \int_{-\mathrm{h}_K(-\boldsymbol{u})}^{\mathrm{h}_K(\boldsymbol{u})} t f(t)\,\mathrm{d}t. \tag{2.22}$$

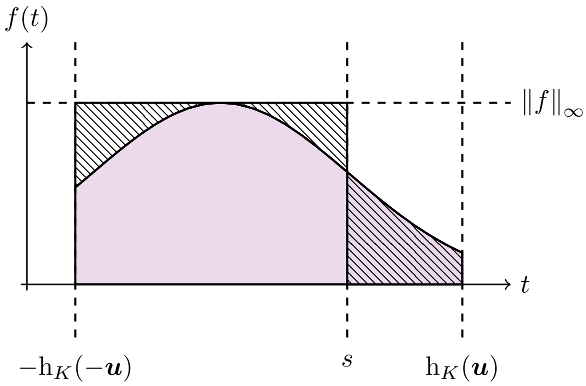

Figure 2.4: Mass distribution of a convex body shifted as in the proof of Lemma 2.9

By (2.21) we get

$$\mathrm{vol}(K) \leq (\mathrm{h}_K(\boldsymbol{u}) + \mathrm{h}_K(-\boldsymbol{u}))\, \|f\|_\infty .$$

Thus, for $s = \frac{\mathrm{vol}(K)}{\|f\|_\infty}$ we have $-\mathrm{h}_K(-\boldsymbol{u}) \leq \mathrm{h}_K(\boldsymbol{u}) - s \leq \mathrm{h}_K(\boldsymbol{u})$ and (2.22) yields

$$\begin{aligned} 0 &= \int\limits_{-\mathrm{h}_K(-\boldsymbol{u})}^{\mathrm{h}_K(\boldsymbol{u})-s} tf(t)\,\mathrm{d}t + \int\limits_{\mathrm{h}_K(\boldsymbol{u})-s}^{\mathrm{h}_K(\boldsymbol{u})} tf(t)\,\mathrm{d}t \\ &\leq (\mathrm{h}_K(\boldsymbol{u}) - s) \int\limits_{-\mathrm{h}_K(-\boldsymbol{u})}^{\mathrm{h}_K(\boldsymbol{u})-s} f(t)\,\mathrm{d}t + \int\limits_{\mathrm{h}_K(\boldsymbol{u})-s}^{\mathrm{h}_K(\boldsymbol{u})} tf(t)\,\mathrm{d}t. \end{aligned} \tag{2.23}$$

With (2.21) it follows that

$$\begin{aligned} 0 &\leq (\mathrm{h}_K(\boldsymbol{u}) - s) \left(\mathrm{vol}(K) - \int\limits_{\mathrm{h}_K(\boldsymbol{u})-s}^{\mathrm{h}_K(\boldsymbol{u})} f(t)\,\mathrm{d}t \right) + \int\limits_{\mathrm{h}_K(\boldsymbol{u})-s}^{\mathrm{h}_K(\boldsymbol{u})} tf(t)\,\mathrm{d}t \\ &= (\mathrm{h}_K(\boldsymbol{u}) - s) \left(\int\limits_{\mathrm{h}_K(\boldsymbol{u})-s}^{\mathrm{h}_K(\boldsymbol{u})} \big(\|f\|_\infty - f(t) \big)\,\mathrm{d}t \right) + \int\limits_{\mathrm{h}_K(\boldsymbol{u})-s}^{\mathrm{h}_K(\boldsymbol{u})} tf(t)\,\mathrm{d}t \\ &\leq \int\limits_{\mathrm{h}_K(\boldsymbol{u})-s}^{\mathrm{h}_K(\boldsymbol{u})} t\big(\|f\|_\infty - f(t) \big)\,\mathrm{d}t + \int\limits_{\mathrm{h}_K(\boldsymbol{u})-s}^{\mathrm{h}_K(\boldsymbol{u})} tf(t)\,\mathrm{d}t \\ &= \int\limits_{\mathrm{h}_K(\boldsymbol{u})-s}^{\mathrm{h}_K(\boldsymbol{u})} t\,\|f\|_\infty\,\mathrm{d}t \\ &= \frac{\|f\|_\infty}{2} \big(\mathrm{h}_K(\boldsymbol{u})^2 - (\mathrm{h}_K(\boldsymbol{u}) - s)^2 \big) \\ &= \frac{\|f\|_\infty}{2} \big(2s\mathrm{h}_K(\boldsymbol{u}) - s^2 \big) \end{aligned}$$

$$= \frac{s\,\|f\|_\infty}{2}\left(2\mathrm{h}_K(\boldsymbol{u}) - \frac{\mathrm{vol}(K)}{\|f\|_\infty}\right).$$

From the latter inequality we obtain (2.20).

Suppose now we have equality in (2.20). Then we also have equality in (2.23), and since f is positive on the interval $(-\mathrm{h}_K(-\boldsymbol{u}), \mathrm{h}_K(\boldsymbol{u}))$ it follows that $-\mathrm{h}_K(-\boldsymbol{u}) = \mathrm{h}_K(\boldsymbol{u}) - s$. Hence $\mathrm{vol}(K) = \Big(\mathrm{h}_K(\boldsymbol{u}) + \mathrm{h}_K(-\boldsymbol{u})\Big)\,\|f\|_\infty$, which by (2.21) shows that the volume sections $f(t)$ are constant. Then (2.22) yields $\mathrm{h}_K(\boldsymbol{u}) = \mathrm{h}_K(-\boldsymbol{u})$. Moreover, the equality conditions of the Brunn-Minkowski inequality (1.2) assert that the sections $K \cap H(\boldsymbol{u}, t)$, $t \in [-\mathrm{h}_K(\boldsymbol{u}), \mathrm{h}_K(\boldsymbol{u})]$, are translates. Thus K is a prism. □

As above we apply the foregoing lemma to the parallelepiped (2.19).

Corollary 2.10. *Let $K \in \mathcal{K}_c^n$ with $\mathrm{vol}_n(K) = 2^n$. Suppose there are numbers $\alpha_1, \ldots, \alpha_n \geq 1$ such that*

$$K \subseteq \{\boldsymbol{x} \in \mathbb{R}^n : -\alpha_i^{-1} \leq x_i \leq \alpha_i, 1 \leq i \leq n\}.$$

Then

$$\prod_{\substack{j=1\\ j\neq i}}^{n} \left(\frac{\alpha_j^{-1} + \alpha_j}{2}\right) \geq \alpha_i.$$

Proof. Let $i \in \{1, \ldots, n\}$. By applying Lemma 2.9 with $\boldsymbol{u} = -\boldsymbol{e}_i$ we obtain

$$\begin{aligned} 2^n = \mathrm{vol}(K) &\leq 2\mathrm{h}_K(-\boldsymbol{e}_i) \max_{t\in\mathbb{R}} \mathrm{vol}_{n-1}(K \cap H(\boldsymbol{e}_i, t)) \\ &\leq 2\alpha_i^{-1} \prod_{\substack{j=1\\ j\neq i}}^{n} (\alpha_j^{-1} + \alpha_j). \end{aligned}$$ □

As our next step we show that the inequalities obtained in the Corollaries 2.8 and 2.10 admit only a trivial solution $\alpha_1, \ldots, \alpha_n$ if the dimension n is particularly small.

Lemma 2.11. *Let $n \in \{2, 3, 4\}$. Then the system of inequalities*

$$\alpha_i \geq 1 \qquad \text{for each } i = 1, \ldots, n, \tag{2.24}$$

$$\alpha_1 \cdot \ldots \cdot \alpha_n \leq 2^n, \tag{2.25}$$

$$\prod_{\substack{j=1\\ j\neq i}}^{n} \left(\frac{\alpha_j^{-1} + \alpha_j}{2}\right) \geq \alpha_i, \qquad \text{for each } i = 1, \ldots, n, \tag{2.26}$$

has the only solution $\alpha_1 = \ldots = \alpha_n = 1$.

Proof. Without loss of generality we assume there is a solution to the above system with $\alpha_1 \geq \ldots \geq \alpha_n$.

Case $n = 2$: By (2.26) we readily have

$$\alpha_2^{-1} + \alpha_2 \geq 2\alpha_1 \geq 2\alpha_2,$$

which with (2.24) gives $\alpha_1 = \alpha_2 = 1$.

Case $n = 3$: Inequality (2.26) for $i = 1$ gives

$$(\alpha_2^2 + 1)(\alpha_3^2 + 1) \geq 4\alpha_1\alpha_2\alpha_3 \geq 4\alpha_2^2\alpha_3. \tag{2.27}$$

Rearranging the terms yields

$$\alpha_3^2 + 1 - (4\alpha_3 - \alpha_3^2 - 1)\alpha_2^2 \geq 0.$$

Note that (2.25) implies $\alpha_3 \leq 2$ which in turn shows

$$4\alpha_3 - \alpha_3^2 - 1 \geq 4\alpha_3 - 2\alpha_3 - 1 > 0.$$

Thus

$$\frac{\alpha_3^2 + 1}{4\alpha_3 - \alpha_3^2 - 1} \geq \alpha_2^2 \geq \alpha_3^2,$$

which again can be rearranged to the polynomial inequality

$$0 \leq \alpha_3^4 - 4\alpha_3^3 + 2\alpha_3^2 + 1 = (\alpha_3 - 1)(\alpha_3^3 - 3\alpha_3^2 - \alpha_3 - 1).$$

Since for the latter factor we have

$$\alpha_3^3 - 3\alpha_3^2 - \alpha_3 - 1 \leq 2\alpha_3^2 - 3\alpha_3^2 - \alpha_3 - 1 = -\alpha_3^2 - \alpha_3 - 1 < 0,$$

by (2.24) we find $\alpha_3 = 1$. By (2.27) then it follows that

$$2(\alpha_2^2 + 1) \geq 4\alpha_1\alpha_2 \geq 4\alpha_2^2.$$

Thus $\alpha_1 = \alpha_2 = 1$.

Case $n = 4$: From (2.26) for $i = 1$ we get

$$(\alpha_2^2 + 1)(\alpha_3^2 + 1)(\alpha_4^2 + 1) \geq 8\alpha_1\alpha_2\alpha_3\alpha_4 \geq 8\alpha_2^2\alpha_3\alpha_4$$

or

$$(\alpha_3^2 + 1)(\alpha_4^2 + 1) - \Big(8\alpha_3\alpha_4 - (\alpha_3^2 + 1)(\alpha_4^2 + 1)\Big)\alpha_2^2 \geq 0. \tag{2.28}$$

We will eliminate successively the variables α_2 and α_3 to obtain a polynomial inequality in the single variable α_4.

Note that from (2.24) and (2.25) we have

$$\begin{aligned} \alpha_4^4 \leq \alpha_1\alpha_2\alpha_3\alpha_4 \leq 16, \\ \alpha_3^3 \leq \alpha_3^3\alpha_4 \leq \alpha_1\alpha_2\alpha_3\alpha_4 \leq 16. \end{aligned} \tag{2.29}$$

Hence $\alpha_4 \leq 2$ and $\alpha_3 \leq 2^{4/3} \leq 13/5$. We aim to show that then

$$8\alpha_3\alpha_4 - (\alpha_3^2 + 1)(\alpha_4^2 + 1) > 0. \tag{2.30}$$

To this end, define $D = [1, \frac{13}{5}] \times [1, 2]$ and $f\colon D \to \mathbb{R}$ with $f(x, y) = 8xy - (x^2+1)(y^2+1)$. As a polynomial, f attains a minimum on D. Also f is differentiable and its gradient and Hessian are given by

$$\nabla f(x, y) = \begin{pmatrix} 8y - 2x(y^2+1) \\ 8x - 2y(x^2+1) \end{pmatrix},$$
$$\nabla^2 f(x, y) = \begin{pmatrix} -2(y^2+1) & 8 - 4xy \\ 8 - 4xy & -2(x^2+1) \end{pmatrix},$$

respectively. If $(x, y) \in D$ is a solution of $\nabla f(x, y) = \mathbf{0}$ we see from the first coordinate of ∇f that $x = \frac{4y}{y^2+1}$. Then from the second coordinate we find

$$\begin{aligned} 0 &= 8\left(\frac{4y}{y^2+1}\right) - 2y\left(\left(\frac{4y}{y^2+1}\right)^2 + 1\right) \\ &= (y^2+1)^{-2} y\Big(32(y^2+1) - 32y^2 - 2(y^2+1)^2\Big) \\ &= (y^2+1)^{-2} y\Big(-2(y^2-3)(y^2+5)\Big). \end{aligned}$$

Since $y \geq 1$, f has its only stationary point at $y = x = \sqrt{3}$ which is a not a minimum since

$$\nabla^2 f(\sqrt{3}, \sqrt{3}) = \begin{pmatrix} -8 & -4 \\ -4 & -8 \end{pmatrix}$$

is indefinite. Thus f attains its minimum at a point in ∂D, i.e., at some point $(1, y)$, $(\frac{13}{5}, y)$, $(x, 1)$ or $(x, 2)$, where $x \in [1, \frac{13}{5}]$ and $y \in [1, 2]$. Since

$$\begin{aligned} f(1, y) &= 8y - 2(y^2+1) \geq 8y - 2(2y+1) = 4y - 2 > 0, \\ f(13/5, y) &= \frac{104}{5}y - \frac{194}{25}y^2 - \frac{194}{25} = \frac{194}{25}\left(\frac{260}{97}y - y^2 - 1\right) \\ &> \frac{194}{25}\left(\frac{5}{2}y - y^2 - 1\right) = \frac{194}{25}(2-y)(y-1/2) \geq 0, \\ f(x, 1) &= 8x - 2(x^2+1) \geq 8x - 2\left(\frac{13}{5}x + 1\right) = \frac{2}{5}(7x - 5) > 0 \\ f(x, 2) &= 16x - 5x^2 - 5 = 5\left(\frac{16}{5}x - x^2 - 1\right) \\ &> 5\left(\frac{194}{65}x - x^2 - 1\right) = 5(13/5 - x)(x - 5/13) \geq 0, \end{aligned}$$

are positive in the given range, we have proved (2.30).

From (2.28) it then follows that

$$\frac{(\alpha_3^2+1)(\alpha_4^2+1)}{8\alpha_3\alpha_4 - (\alpha_3^2+1)(\alpha_4^2+1)} \geq \alpha_2^2 \geq \alpha_3^2$$

which can be rewritten as

$$\frac{\alpha_4^2+1}{\alpha_4} \geq \frac{8\alpha_3^3}{(\alpha_3^2+1)^2}. \tag{2.31}$$

The left hand side is increasing in α_4 when $1 \le \alpha_4$. We distinguish now two cases. First, if $\alpha_3 < 2$, we use $\alpha_4 \le \alpha_3$ to conclude from (2.31) that

$$\frac{\alpha_3^2+1}{\alpha_3} \ge \frac{8\alpha_3^3}{(\alpha_3^2+1)^2}$$

and so

$$\begin{aligned}0 \le (\alpha_3^2+1)^3 - 8\alpha_3^4 &= \alpha_3^6 - 5\alpha_3^4 + 3\alpha_3^2 + 1\\ &= (\alpha_3^2-1)(\alpha_3^4-4\alpha_3^2-1) \le (\alpha_3^2-1)(\alpha_3^4-4\alpha_3^2)\\ &= \alpha_3^2(\alpha_3^2-1)(\alpha_3^2-4).\end{aligned}$$

The only solution to this inequality in $[1,2)$ is $\alpha_3 = 1$.

Now suppose $\alpha_3 \ge 2$. From (2.29) we know $\alpha_4 \le \frac{16}{\alpha_3^3}$, which we may combine with (2.31) to obtain

$$\frac{8\alpha_3^3}{(\alpha_3^2+1)^2} \le \frac{\left(\frac{16}{\alpha_3^3}\right)^2+1}{\frac{16}{\alpha_3^3}} = \frac{256+\alpha_3^6}{16\alpha_3^3}$$

or equivalently

$$\begin{aligned}0 &\le (256+\alpha_3^6)(\alpha_3^2+1)^2 - 128\alpha_3^6\\ &= \alpha_3^{10} + 2\alpha_3^8 - 127\alpha_3^6 + 256\alpha_3^4 + 512\alpha_3^2 + 256.\end{aligned}$$

Clearly, in the range $\alpha_3 \in [2, \frac{13}{5}]$ we have

$$192 + 1744(\alpha_3^2-4) + 16(\alpha_3^2-4)^2 > 0$$

and therefore

$$\begin{aligned}0 &< \alpha_3^{10} + 2\alpha_3^8 - 127\alpha_3^6 + 256\alpha_3^4 + 512\alpha_3^2 + 256\\ &\quad + \left(192 + 1744(\alpha_3^2-4) + 16(\alpha_3^2-4)^2\right)\\ &= \alpha_3^{10} + 2\alpha_3^8 - 127\alpha_3^6 + 272\alpha_3^4 + 2128\alpha_3^2 - 6272\\ &= (\alpha_3^2-4)^2(\alpha_3^2-7)(\alpha_3^4+17\alpha_3^2+56).\end{aligned}$$

But this is false since $\alpha_3^2 < 7$. Thus, we have proved $\alpha_3 = 1$ and subsequently $\alpha_4 = 1$. But in this case (2.26) becomes $\alpha_2^{-1} + \alpha_2 \ge 2\alpha_1$ and as above this implies $\alpha_1 = \alpha_2 = 1$. $\square$

If $n \ge 5$, the system of inequalities (2.24), (2.25), (2.26) admits more solutions, e.g., for $\alpha_1 = \ldots = \alpha_n = 2$ the inequalities (2.26) become $(5/4)^{n-1} \ge 2$ which is true for $n \ge 5$.

We are now ready to give the proof of Proposition 2.6.

Proof of Proposition 2.6. Since Q is a parallelepiped, there exist linearly independent $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_n \in \mathbb{S}^{n-1}$ such that

$$Q = \{\boldsymbol{x} \in \mathbb{R}^n : |\langle \boldsymbol{u}_i, \boldsymbol{x}\rangle| \le \mathrm{h}_Q(\boldsymbol{u}_i)\}$$

Let A be the matrix with columns $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_n$. Without loss of generality we assume that $\mathrm{vol}(Q) = \mathrm{vol}(M) = \frac{2^n}{|\det(A)|}$. As in the proof of Proposition 2.5 we note that all cone-volumes $\mathrm{V}_Q(\pm\boldsymbol{u}_i)$ coincide. Hence we just have to verify

$$\prod_{i=1}^{n} \frac{\mathrm{h}_M(\boldsymbol{u}_i)\mathrm{h}_M(-\boldsymbol{u}_i)}{\mathrm{h}_Q(\boldsymbol{u}_i)^2} \geq 1. \tag{2.32}$$

Note that under these assumptions it follows from (1.3) that

$$\prod_{i=1}^{n} \mathrm{h}_Q(\boldsymbol{u}_i) = \frac{|\det(A)|}{2^n} \mathrm{vol}(Q) = 1.$$

Hence, with $K = A^T M$, i.e., $\mathrm{h}_K(\pm\boldsymbol{e}_i) = \mathrm{h}_M(\pm\boldsymbol{u}_i)$, we only have to show

$$\prod_{i=1}^{n} \Big(\mathrm{h}_K(\boldsymbol{e}_i)\mathrm{h}_K(-\boldsymbol{e}_i)\Big) \geq 1, \tag{2.33}$$

where $\mathrm{vol}(K) = 2^n$ and the centroid of K is at the origin. Moreover, after scaling by a suitable diagonal matrix of determinant 1 we may also assume $\mathrm{h}_K(\boldsymbol{e}_i)\mathrm{h}_K(-\boldsymbol{e}_i) = \gamma$, $1 \leq i \leq n$, for a suitable constant $\gamma > 0$.

Suppose (2.33) does not hold. Then $\gamma = \mathrm{h}_K(\boldsymbol{e}_i)\mathrm{h}_K(-\boldsymbol{e}_i) < 1$ for $1 \leq i \leq n$, and without loss of generality assume $\mathrm{h}_K(\boldsymbol{e}_i) \geq \mathrm{h}_K(-\boldsymbol{e}_i)$, $i = 1, \ldots, n$. Then for every $\boldsymbol{x} \in K$ and $i = 1, \ldots, n$,

$$-\mathrm{h}_K(\boldsymbol{e}_i)^{-1} < -\mathrm{h}_K(-\boldsymbol{e}_i) \leq \langle \boldsymbol{e}_i, \boldsymbol{x} \rangle \leq \mathrm{h}_K(\boldsymbol{e}_i).$$

Hence, K is strictly contained in a shifted parallelepiped

$$P = \{\boldsymbol{x} \in \mathbb{R}^n : -\alpha_i^{-1} \leq \langle \boldsymbol{e}_i, \boldsymbol{x} \rangle \leq \alpha_i\}$$

for some $\alpha_1, \ldots, \alpha_n \geq 1$. Now Corollaries 2.8 and 2.10 yield $\alpha_1 \cdot \ldots \cdot \alpha_n \leq 2^n$ and

$$\prod_{\substack{j=1 \\ j \neq i}}^{n} \left(\frac{\alpha_j^{-1} + \alpha_j}{2}\right) \geq \alpha_i$$

for each $i = 1, \ldots, n$. By Lemma 2.11 it follows that $\alpha_1 = \ldots = \alpha_n = 1$. In particular, $\mathrm{vol}(P) = 2^n$ which contradicts $\mathrm{vol}(P) > \mathrm{vol}(K) = 2^n$.

Suppose now we have equality in (2.32). Then we also have equality in (2.33). Repeating the steps above we find that K is contained in the cube $[-1, 1]^n$. Since $\mathrm{vol}(K) = 2^n$ it follows that $K = [-1, 1]^n$. Thus, M is a parallelepiped with outer normals $\pm\boldsymbol{u}_1, \ldots, \pm\boldsymbol{u}_n$. □

We conjecture that Proposition 2.6 also holds in arbitrary dimension which could possibly be proved by establishing an analogue of Lemma 2.9 for lower-dimensional sections. It is, however, not clear how such an extension should look like.

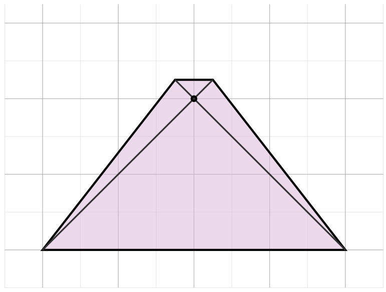

Figure 2.5: Trapezoid violating the subspace concentration inequality (2.34)

2.4 The logarithmic Minkowski problem for trapezoids

It was mentioned earlier that the even logarithmic Minkowski problem, i.e., the task of characterizing the set of cone-volume measures of bodies in $\mathcal{K}_s^n$, was solved in [19] (see Theorem 1.3). As a starting point of the search for the solution in the general setting we will consider the cone-volume measures of polygons with few vertices. Restricting to the plane allows the use of tools that may not have a direct extension to higher dimensions. For instance, the equality cases of even cone-volume measures in the plane were fully characterized by Böröczky, Lutwak, Yang and Zhang [18], but in higher dimensions this is considered a significant open problem. Another example is the following result proved by Stancu [78]. Here, for a discrete measure μ on the sphere, its **support** $\operatorname{supp}(\mu)$ is the set of points $\boldsymbol{u}$ with $\mu(\boldsymbol{u}) > 0$.

Theorem 2.12 (Stancu [78])**.** *Let μ be a non-zero, finite, discrete Borel measure on $\mathbb{S}^1$, that is not concentrated on a closed hemisphere, and let $m = \#(\operatorname{supp}(\mu)) \geq 4$. If either $m > 4$ and*

$$\mu(\mathbb{S}^1 \cap L) < \frac{1}{2}\mu(\mathbb{S}^1) \tag{2.34}$$

for every proper subspace L of $\mathbb{R}^2$ with $\#(L \cap \operatorname{supp}(\mu)) = 2$, or $\operatorname{supp}(\mu)$ is in general position, then there exists a polygon $P \in \mathcal{P}_o^2$ with $\mathrm{V}_P = \mu$.

An extension of the preceding theorem to higher dimensions was discovered only after a decade by Böröczky, Hegedűs and Zhu [11]. The subspace concentration inequality (2.34) is not necessary as can be easily seen from the trapezoid with vertices $\binom{\pm\varepsilon}{\varepsilon}, \binom{\pm 1}{-1}$, for small $\varepsilon > 0$ (see Fig. 2.5). Still, Böröczky and Hegedűs [10] proved a necessary condition for cone-volume measures evaluated at antipodal points, which in the plane reads as follows.

Theorem 2.13 (Böröczky, Hegedűs [10])**.** *Let $K \in \mathcal{K}_o^2$ and $\boldsymbol{u} \in \mathbb{S}^1$. Then*

$$\mathrm{V}_K(\{\pm\boldsymbol{u}\}) \leq \operatorname{vol}(K) - 2\sqrt{\mathrm{V}_K(\boldsymbol{u})\mathrm{V}_K(-\boldsymbol{u})} \tag{2.35}$$

and equality is attained if and only if K is trapezoid with two sides parallel to $\boldsymbol{u}^\perp$ and the intersection point of the diagonals is contained in $\boldsymbol{u}^\perp$.

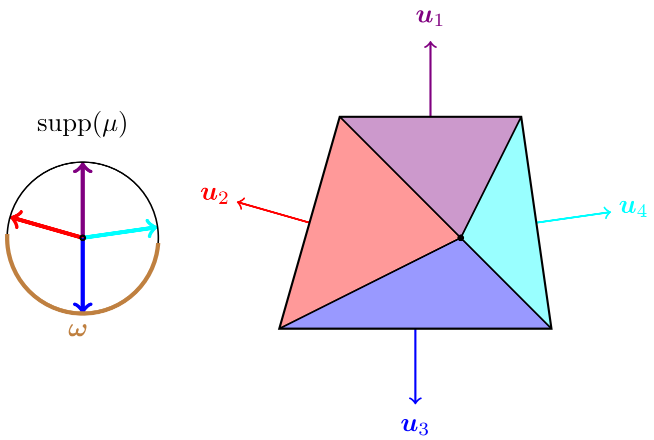

Figure 2.6: Exemplary starting point (and solution) in Theorem 2.14 and Corollary 2.18

To this day no sufficient or necessary condition other than (2.34) and (2.35) is known for the unrestricted planar logarithmic Minkowski problem.

Since cone-volumes of parallelograms associated to antipodal outer normal vectors sum up to half of the parallelograms area (which characterizes their cone-volume measures), and quadrilaterals without parallel sides are covered by Theorem 2.12, trapezoids are the vertex-minimal open case in the discrete logarithmic Minkowski problem. It is the main goal of this section to prove a characterization of cone-volume measures of trapezoids which together with Theorem 2.12 solves the logarithmic Minkowski problem for quadrilaterals.

Theorem 2.14. *Let μ be a non-zero, finite Borel measure on $\mathbb{S}^1$ supported on pairwise distinct and counterclockwise ordered unit vectors $\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3, \boldsymbol{u}_4 \in \mathbb{S}^1$. Suppose* $\operatorname{supp}(\mu)$ *contains a single pair of antipodal points, say $\boldsymbol{u}_1 = -\boldsymbol{u}_3$, and there is an open hemisphere $\omega \subsetneq \mathbb{S}^1$ such that* $\operatorname{supp}(\mu) \cap \omega = \{\boldsymbol{u}_3\}$. *Then there exists a polygon $P \in \mathcal{P}_o^2$ with $\mathrm{V}_P = \mu$ if and only if either*

(i) $\mu(\boldsymbol{u}_1) + \mu(\boldsymbol{u}_3) < \mu(\boldsymbol{u}_2) + \mu(\boldsymbol{u}_4)$, *or*

(ii) $\mu(\boldsymbol{u}_1) + \mu(\boldsymbol{u}_3) \geq \mu(\boldsymbol{u}_2) + \mu(\boldsymbol{u}_4) \geq 2\sqrt{\mu(\boldsymbol{u}_1)\mu(\boldsymbol{u}_3)}$ *and* $\mu(\boldsymbol{u}_1) < \mu(\boldsymbol{u}_3)$.

Moreover, P is uniquely determined unless $\mu(\boldsymbol{u}_1) + \mu(\boldsymbol{u}_3) \geq \mu(\boldsymbol{u}_2) + \mu(\boldsymbol{u}_4) > 2\sqrt{\mu(\boldsymbol{u}_1)\mu(\boldsymbol{u}_3)}$ *and* $\mu(\boldsymbol{u}_1) < \mu(\boldsymbol{u}_3)$, *and in the latter case there are exactly two such polygons.*

See Fig. 2.6 for a graphic representation of the notation.

It is remarkable that condition (i) is the subspace concentration inequality (2.34) whereas condition (ii) is mainly a rearrangement of (2.35). This shows that the inequality (2.35) is almost a sufficient condition. The peculiar thing about this characterization is the appearance of an extra condition if the subspace concentration inequality is not satisfied.

The proof of Theorem 2.14 is based on an explicit computation of P for a given μ. Thereby a polynomial system arises which is stated and solved in the following lemma.

Lemma 2.15. *Let $a, b \in \mathbb{R}$ with $a < b$, and $\gamma_i \in \mathbb{R}_{>0}$, $1 \le i \le 4$. Write $l_a = \sqrt{1+a^2}$, $l_b = \sqrt{1+b^2}$ and $D = (\gamma_2+\gamma_4)^2 - 4\gamma_1\gamma_3$. Consider the system*

$$\left.\begin{aligned} &2\gamma_1 = h_1(-(b-a)h_1 + l_a h_2 + l_b h_4),\\ &2\gamma_2 = l_a h_2(h_1+h_3),\\ &2\gamma_3 = h_3((b-a)h_3 + l_a h_2 + l_b h_4),\\ &2\gamma_4 = l_b h_4(h_1+h_3),\\ &l_a h_2 + l_b h_4 > (b-a)h_1,\\ &h_1, h_2, h_3, h_4 > 0. \end{aligned}\right\} \tag{2.36}$$

(i) If either

(a) $\gamma_1+\gamma_3 \ge \gamma_2+\gamma_4$ and $\gamma_1 \ge \gamma_3$, or

(b) $\gamma_2+\gamma_4 < 2\sqrt{\gamma_1\gamma_3}$,

then (2.36) *admits no solutions.*

(ii) If either

(a) $\gamma_1+\gamma_3 < \gamma_2+\gamma_4$,

(b) $\gamma_1+\gamma_3 = \gamma_2+\gamma_4$ and $\gamma_1 < \gamma_3$, or

(c) $\gamma_1+\gamma_3 > \gamma_2+\gamma_4 = 2\sqrt{\gamma_1\gamma_3}$ and $\gamma_1 < \gamma_3$,

then (2.36) *has a unique solution given by*

$$\begin{aligned} h_1 &= \frac{1}{\sqrt{2(b-a)}}\frac{-2\gamma_1+\gamma_2+\gamma_4+\sqrt{D}}{\sqrt{\gamma_3-\gamma_1+\sqrt{D}}},\\ h_2 &= \frac{\sqrt{2(b-a)}}{l_a}\frac{\gamma_2}{\sqrt{\gamma_3-\gamma_1+\sqrt{D}}},\\ h_3 &= \frac{1}{\sqrt{2(b-a)}}\frac{2\gamma_3-\gamma_2-\gamma_4+\sqrt{D}}{\sqrt{\gamma_3-\gamma_1+\sqrt{D}}},\\ h_4 &= \frac{\sqrt{2(b-a)}}{l_b}\frac{\gamma_4}{\sqrt{\gamma_3-\gamma_1+\sqrt{D}}}. \end{aligned}$$

(iii) If $\gamma_1+\gamma_3 > \gamma_2+\gamma_4 > 2\sqrt{\gamma_1\gamma_3}$ and $\gamma_1 < \gamma_3$, then (2.36) *has exactly two*

different solutions given by

$$\begin{aligned}
h_1 &= \frac{1}{\sqrt{2(b-a)}}\frac{-2\gamma_1+\gamma_2+\gamma_4\pm\sqrt{D}}{\sqrt{\gamma_3-\gamma_1\pm\sqrt{D}}},\\
h_2 &= \frac{\sqrt{2(b-a)}}{l_a}\frac{\gamma_2}{\sqrt{\gamma_3-\gamma_1\pm\sqrt{D}}},\\
h_3 &= \frac{1}{\sqrt{2(b-a)}}\frac{2\gamma_3-\gamma_2-\gamma_4\pm\sqrt{D}}{\sqrt{\gamma_3-\gamma_1\pm\sqrt{D}}},\\
h_4 &= \frac{\sqrt{2(b-a)}}{l_b}\frac{\gamma_4}{\sqrt{\gamma_3-\gamma_1\pm\sqrt{D}}}.
\end{aligned}$$

We shall prove Theorem 2.14 first.

Proof of Theorem 2.14. After carrying out a suitable rotation and reflection we may assume that $\boldsymbol{u}_1=(0,1)^T$, $\boldsymbol{u}_2=\frac{1}{l_a}(-1,-a)^T$, $\boldsymbol{u}_3=(0,-1)^T$ and $\boldsymbol{u}_4=\frac{1}{l_b}(1,b)^T$, where $a,b\in\mathbb{R}$, $b>a$ and $l_a=\left|(-1,-a)^T\right|$, $l_b=\left|(1,b)^T\right|$. Suppose $h_1,h_2,h_3,h_4>0$. A polygon

$$P=\bigcap_{i=1}^{4}H^-(\boldsymbol{u}_i,h_i)$$

is a proper trapezoid if and only if the intersection point of $H(\boldsymbol{u}_2,h_2)$ and $H(\boldsymbol{u}_4,h_4)$ given by

$$\frac{1}{b-a}\begin{pmatrix}-(l_abh_2+l_bah_4)\\ l_ah_2+l_bh_4\end{pmatrix}$$

is contained in $\operatorname{int}H^+(\boldsymbol{u}_1,h_1)$, i.e., $l_ah_2+l_bh_4>(b-a)h_1$, since otherwise P is a triangle. Moreover, in this case the vertices of P are given as intersections of index-adjacent supporting hyperplanes, namely

$$\begin{aligned}
\boldsymbol{v}_1 &= \begin{pmatrix}-bh_1+l_bh_4\\ h_1\end{pmatrix}, & \boldsymbol{v}_3 &= \begin{pmatrix}ah_3-l_ah_2\\ -h_3\end{pmatrix},\\
\boldsymbol{v}_2 &= \begin{pmatrix}-ah_1-l_ah_2\\ h_1\end{pmatrix}, & \boldsymbol{v}_4 &= \begin{pmatrix}bh_3+l_bh_4\\ -h_3\end{pmatrix},
\end{aligned}$$

so that

$$\begin{aligned}
F(P,\boldsymbol{u}_1) &= [\boldsymbol{v}_1,\boldsymbol{v}_2], & F(P,\boldsymbol{u}_3) &= [\boldsymbol{v}_3,\boldsymbol{v}_4]\\
F(P,\boldsymbol{u}_2) &= [\boldsymbol{v}_2,\boldsymbol{v}_3], & F(P,\boldsymbol{u}_4) &= [\boldsymbol{v}_4,\boldsymbol{v}_1].
\end{aligned}$$

Moreover, the cone-volumes of P (dependent on h_i) are given by

$$\begin{aligned}
\mathrm{V}_P(\boldsymbol{u}_1) &= \frac{1}{2}h_1\cdot(-(b-a)h_1+l_ah_2+l_bh_4),\\
\mathrm{V}_P(\boldsymbol{u}_2) &= \frac{1}{2}h_2\cdot l_a(h_1+h_3),
\end{aligned}$$

$$\mathrm{V}_P(\boldsymbol{u}_3) = \frac{1}{2}h_3 \cdot ((b-a)h_3 + l_a h_2 + l_b h_4),$$
$$\mathrm{V}_P(\boldsymbol{u}_4) = \frac{1}{2}h_4 \cdot l_b(h_1 + h_3).$$

Applying Lemma 2.15 to the system $\mathrm{V}_P(\boldsymbol{u}_i) = \mu(\boldsymbol{u}_i)$, $1 \le i \le 4$, yields the desired assertion. □

We finish this section with the proof of Lemma 2.15 where we reduce solving (2.36) to finding positive solutions of a biquadratic equation. To this end, we recall that for $p, q \in \mathbb{R}$, and denoting $D = p^2 - q$, the biquadratic equation $x^4 - 2px^2 + q = 0$ in the variable $x \in \mathbb{R}$ has

- no positive solution, if either $D < 0$ or $p \le -\sqrt{|D|}$,
- exactly one positive solution, if either $D \ge 0$ and $p \in (-\sqrt{D}, \sqrt{D}]$, or $D = 0$ and $p > 0$, given by $x = \sqrt{p + \sqrt{D}}$,
- exactly two positive solutions, if $D > 0$ and $p > \sqrt{D}$, given by $x = \sqrt{p \pm \sqrt{D}}$.

Proof of Lemma 2.15. Suppose $h_1, h_2, h_3, h_4 > 0$ solve the system (2.36). Both h_2 and h_4 can be computed from h_1 and h_3 and the given data, so that

$$h_2 = \frac{2l_a^{-1}\gamma_2}{h_1 + h_3}, \tag{2.37}$$
$$h_4 = \frac{2l_b^{-1}\gamma_4}{h_1 + h_3}, \tag{2.38}$$
$$2\gamma_1 = h_1\left(-(b-a)h_1 + \frac{2(\gamma_2+\gamma_4)}{h_1+h_3}\right), \tag{2.39}$$
$$2\gamma_3 = h_3\left((b-a)h_3 + \frac{2(\gamma_2+\gamma_4)}{h_1+h_3}\right). \tag{2.40}$$

By considering the sum and difference of (2.39) and (2.40) and substituting the variables $h_1, h_3 > 0$ by $x > 0$ and $y \in \mathbb{R}$ via

$$\begin{aligned} h_1 &= \frac{x^2 - y}{x\sqrt{2(b-a)}}, \\ h_3 &= \frac{x^2 + y}{x\sqrt{2(b-a)}}, \end{aligned} \tag{2.41}$$

we find that the variables $x = \sqrt{\frac{b-a}{2}}(h_1 + h_3)$ and $y = \frac{b-a}{2}(h_3^2 - h_1^2)$ satisfy

$$\begin{aligned} 2(\gamma_1+\gamma_3) &= (b-a)(h_3^2 - h_1^2) + 2(\gamma_2+\gamma_4) \\ &= 2y + 2(\gamma_2+\gamma_4), \end{aligned} \tag{2.42}$$
$$\begin{aligned} 2(\gamma_3-\gamma_1) &= (b-a)(h_1^2 + h_3^2) + 2(\gamma_2+\gamma_4)\frac{h_3 - h_1}{h_1 + h_3} \\ &= (b-a)\left(\frac{x^2}{b-a} + \frac{y^2}{(b-a)x^2}\right) + 2(\gamma_2+\gamma_4)\frac{y}{x^2} \end{aligned}$$

$$= x^{-2}\Big(x^4 + y^2 + 2(\gamma_2 + \gamma_4)y\Big). \tag{2.43}$$

As a final step we normalize (2.43) and substitute y by means of (2.42). Summed up, we find that each solution of (2.36) yields a positive root of

$$x^4 - 2(\gamma_3 - \gamma_1)x^2 + (\gamma_1 + \gamma_3)^2 - (\gamma_2 + \gamma_4)^2. \tag{2.44}$$

Let $p = \gamma_3 - \gamma_1$ and $D = (\gamma_2 + \gamma_4)^2 - 4\gamma_1\gamma_3$.

(i) If $\gamma_2 + \gamma_4 < 2\sqrt{\gamma_1\gamma_3}$, then $D < 0$. Also, if $\gamma_1 + \gamma_3 \geq \gamma_2 + \gamma_4$ and $\gamma_1 \geq \gamma_3$, then $p \leq 0$ and

$$p^2 = (\gamma_3 - \gamma_1)^2 = (\gamma_1 + \gamma_3)^2 - 4\gamma_1\gamma_3 \geq D.$$

Hence (2.44) has no positive root.

(ii) Suppose that $D \geq 0$ and $p \in (-\sqrt{D}, \sqrt{D}]$, so that $x = \sqrt{p + \sqrt{D}} > 0$ solves (2.44). Then backsubstituting via (2.42), (2.41), (2.37) and (2.38) yields

$$\begin{aligned}
y &= (\gamma_1 + \gamma_3) - (\gamma_2 + \gamma_4), \\
h_1 &= \frac{p + \sqrt{D} - (\gamma_1 + \gamma_3) + (\gamma_2 + \gamma_4)}{\sqrt{2(b-a)}\sqrt{p + \sqrt{D}}} \\
&= \frac{1}{\sqrt{2(b-a)}} \frac{-2\gamma_1 + \gamma_2 + \gamma_4 + \sqrt{D}}{\sqrt{\gamma_3 - \gamma_1 + \sqrt{D}}}, \\
h_3 &= \frac{p + \sqrt{D} + (\gamma_1 + \gamma_3) - (\gamma_2 + \gamma_4)}{\sqrt{2(b-a)}\sqrt{p + \sqrt{D}}} \\
&= \frac{1}{\sqrt{2(b-a)}} \frac{2\gamma_3 - \gamma_2 - \gamma_4 + \sqrt{D}}{\sqrt{\gamma_3 - \gamma_1 + \sqrt{D}}}, \\
h_2 &= \frac{2\gamma_2}{l_a} \frac{\sqrt{2(b-a)}\sqrt{\gamma_3 - \gamma_1 + \sqrt{D}}}{2(\gamma_3 - \gamma_1 + \sqrt{D})} \\
&= \frac{\sqrt{2(b-a)}}{l_a} \frac{\gamma_2}{\sqrt{\gamma_3 - \gamma_1 + \sqrt{D}}}, \\
h_4 &= \frac{2\gamma_4}{l_b} \frac{\sqrt{2(b-a)}\sqrt{\gamma_3 - \gamma_1 + \sqrt{D}}}{2(\gamma_3 - \gamma_1 + \sqrt{D})} \\
&= \frac{\sqrt{2(b-a)}}{l_b} \frac{\gamma_4}{\sqrt{\gamma_3 - \gamma_1 + \sqrt{D}}}.
\end{aligned}$$

In any case, $\gamma_2, \gamma_4 > 0$ implies $h_2, h_4 > 0$. Moreover,

$$0 < 4\gamma_1(\gamma_1 + \gamma_2 + \gamma_3 + \gamma_4) = (2\gamma_1 + \gamma_2 + \gamma_4)^2 - (\gamma_2 + \gamma_4)^2 + 4\gamma_1\gamma_3$$

and so $2\gamma_1 + \gamma_2 + \gamma_4 > \sqrt{D}$. This gives

$$(b-a)h_1 = \sqrt{\frac{b-a}{2}} \frac{-2\gamma_1 + \gamma_2 + \gamma_4 + \sqrt{D}}{\sqrt{\gamma_3 - \gamma_1 + \sqrt{D}}}$$

$$
\begin{aligned}
&< \sqrt{\frac{b-a}{2}}\frac{2\gamma_2+2\gamma_4}{\sqrt{\gamma_3-\gamma_1+\sqrt{D}}}\\
&= l_a h_2 + l_b h_4.
\end{aligned}
$$

It remains to show, that in either of the cases $D \geq 0$, $p \in (-\sqrt{D}, \sqrt{D}]$, or $D = 0$ and $p > 0$, and that consequently $h_1, h_3 > 0$ holds. The latter inequalities are equivalent to $\sqrt{D} > \max(2\gamma_1 - \gamma_2 - \gamma_4, \gamma_2 + \gamma_4 - 2\gamma_3)$.

Case $\gamma_1 + \gamma_3 < \gamma_2 + \gamma_4$**.** The assumption yields

$$
\begin{aligned}
D &= (\gamma_2+\gamma_4)^2 - 4\gamma_1\gamma_3 > (\gamma_1+\gamma_3)^2 - 4\gamma_1\gamma_3 = (\gamma_3-\gamma_1)^2 \geq 0,\\
p^2 &= (\gamma_3-\gamma_1)^2 = (\gamma_1+\gamma_3)^2 - 4\gamma_1\gamma_3 < (\gamma_2+\gamma_4)^2 - 4\gamma_1\gamma_3 = D,
\end{aligned}
$$

and

$$
\begin{aligned}
\sqrt{D} &= \sqrt{(\gamma_2+\gamma_4)^2 - 4\gamma_1\gamma_3}\\
&= \sqrt{(\gamma_2+\gamma_4)^2 + 4\gamma_3^2 - 4\gamma_3(\gamma_1+\gamma_3)}\\
&> \sqrt{(\gamma_2+\gamma_4)^2 + 4\gamma_3^2 - 4\gamma_3(\gamma_2+\gamma_4)}\\
&= \sqrt{(\gamma_2+\gamma_4-2\gamma_3)^2}\\
&\geq \gamma_2+\gamma_4-2\gamma_3,\\
&> \gamma_2+\gamma_4-2\gamma_3 + 2(\gamma_1+\gamma_3-\gamma_2-\gamma_4),\\
&= 2\gamma_1-\gamma_2-\gamma_4.
\end{aligned}
$$

Case $\gamma_1 + \gamma_3 = \gamma_2 + \gamma_4$**,** $\gamma_1 < \gamma_3$**.** We have

$$D = (\gamma_2+\gamma_4)^2 - 4\gamma_1\gamma_3 = (\gamma_1+\gamma_3)^2 - 4\gamma_1\gamma_3 = (\gamma_3-\gamma_1)^2 = p^2 > 0.$$

Hence $0 < p = \sqrt{D}$. Therefore, from the assumptions we also find

$$2\gamma_1 - \gamma_2 - \gamma_4 = \gamma_2 + \gamma_4 - 2\gamma_3 = \gamma_1 - \gamma_3 < 0 < \sqrt{D}.$$

Case $\gamma_1 + \gamma_3 > \gamma_2 + \gamma_4 = 2\sqrt{\gamma_1\gamma_3}$**,** $\gamma_1 < \gamma_3$**.** We have

$$D = (\gamma_2+\gamma_4)^2 - 4\gamma_1\gamma_3 = 0$$

and $p = \gamma_3 - \gamma_1 > 0$. Furthermore,

$$2\gamma_3 > \gamma_1 + \gamma_3 > \gamma_2 + \gamma_4 = 2\sqrt{\gamma_1\gamma_3} > 2\gamma_1,$$

so that $2\gamma_1 - \gamma_2 - \gamma_4 < 0 = \sqrt{D}$ and $\gamma_2 + \gamma_4 - 2\gamma_3 < 0 = \sqrt{D}$.

(iii) From $\gamma_2 + \gamma_4 > 2\sqrt{\gamma_1\gamma_3}$ it follows that

$$D = (\gamma_2+\gamma_4)^2 - 4\gamma_1\gamma_3 > 0.$$

Furthermore we readily have $p = \gamma_3 - \gamma_1 > 0$. Then $\gamma_1 + \gamma_3 > \gamma_2 + \gamma_4$ yields

$$p^2 = (\gamma_3-\gamma_1)^2 = (\gamma_3+\gamma_1)^2 - 4\gamma_1\gamma_3 > (\gamma_2+\gamma_4)^2 - 4\gamma_1\gamma_3 = D,$$

which shows that $x = \sqrt{p \pm \sqrt{D}}$ are the positive roots of (2.44). It remains to show that both roots lead to distinct solutions of (2.36). In full analogy to (ii), backsubstituting gives

$$\begin{aligned}
y &= (\gamma_1 + \gamma_3) - (\gamma_2 + \gamma_4), \\
h_1 &= \frac{1}{\sqrt{2(b-a)}} \frac{-2\gamma_1 + \gamma_2 + \gamma_4 \pm \sqrt{D}}{\sqrt{\gamma_3 - \gamma_1 \pm \sqrt{D}}}, \\
h_3 &= \frac{1}{\sqrt{2(b-a)}} \frac{2\gamma_3 - \gamma_2 - \gamma_4 \pm \sqrt{D}}{\sqrt{\gamma_3 - \gamma_1 \pm \sqrt{D}}}, \\
h_2 &= \frac{\sqrt{2(b-a)}}{l_a} \frac{\gamma_2}{\sqrt{\gamma_3 - \gamma_1 \pm \sqrt{D}}}, \\
h_4 &= \frac{\sqrt{2(b-a)}}{l_b} \frac{\gamma_4}{\sqrt{\gamma_3 - \gamma_1 \pm \sqrt{D}}}.
\end{aligned}$$

Again, $\gamma_2, \gamma_4 > 0$ implies $h_2, h_4 > 0$ and from h_2 and h_4 it can be seen that both solutions are distinct. Moreover,

$$0 < 4\gamma_1(\gamma_1 + \gamma_2 + \gamma_3 + \gamma_4) = (2\gamma_1 + \gamma_2 + \gamma_4)^2 - (\gamma_2 + \gamma_4)^2 + 4\gamma_1\gamma_3$$

and so $2\gamma_1 + \gamma_2 + \gamma_4 > \sqrt{D} > -\sqrt{D}$. This gives

$$\begin{aligned}
(b-a)h_1 &= \sqrt{\frac{b-a}{2}} \frac{-2\gamma_1 + \gamma_2 + \gamma_4 \pm \sqrt{D}}{\sqrt{\gamma_3 - \gamma_1 \pm \sqrt{D}}} \\
&< \sqrt{\frac{b-a}{2}} \frac{2\gamma_2 + 2\gamma_4}{\sqrt{\gamma_3 - \gamma_1 \pm \sqrt{D}}} \\
&= l_a h_2 + l_b h_4.
\end{aligned}$$

It remains to show that in both cases $h_1, h_3 > 0$. From $\gamma_1 + \gamma_3 > \gamma_2 + \gamma_4$ we find

$$\begin{aligned}
\sqrt{D} &= \sqrt{(\gamma_2 + \gamma_4)^2 - 4\gamma_1\gamma_3} \\
&= \sqrt{(\gamma_2 + \gamma_4)^2 + 4\gamma_1^2 - 4\gamma_1(\gamma_1 + \gamma_3)} \\
&< \sqrt{(\gamma_2 + \gamma_4)^2 + 4\gamma_1^2 - 4\gamma_1(\gamma_2 + \gamma_4)} \\
&= \sqrt{(\gamma_2 + \gamma_4 - 2\gamma_1)^2}.
\end{aligned}$$

Since $\gamma_2 + \gamma_4 > 2\sqrt{\gamma_1\gamma_3} > 2\gamma_1$ it follows that

$$\begin{aligned}
-\sqrt{D} < \sqrt{D} &< \gamma_2 + \gamma_4 - 2\gamma_1 \\
&< \gamma_2 + \gamma_4 - 2\gamma_1 + 2(\gamma_1 + \gamma_3 - \gamma_2 - \gamma_4) \\
&= 2\gamma_3 - \gamma_2 - \gamma_4.
\end{aligned}$$

Thus, $h_1, h_3 > 0$. $\square$

2.5 Subspace concentration and uniqueness of the cone-volume measure

This section is concerned with non-uniqueness of cone-volume measures, i.e., the question when it is possible that two distinct convex bodies $K, M \in \mathcal{K}_o^n$ satisfy $\mathrm{V}_K = \mathrm{V}_M$. We particularly discuss the correlation between uniqueness of cone-volume measures and the validity of the subspace concentration inequality (1.13). This problem remarkably connects to the logarithmic Minkowski inequality (2.7) – at least to some extent – as we will demonstrate. Suppose $K \in \mathcal{K}_s^n$ strictly satisfies the subspace concentration inequality (1.13). Böröczky, Lutwak, Yang and Zhang [19] transformed the even logarithmic Minkowski problem into the optimization problem

$$\inf \left\{ \int_{\mathbb{S}^{n-1}} \log \mathrm{h}_Q(\boldsymbol{u}) \,\mathrm{d}\mu(\boldsymbol{u}) : Q \in \mathcal{K}_s^n, \mathrm{vol}(Q) = \mu(\mathbb{S}^{n-1}) \right\}, \tag{2.45}$$

where μ is a given even measure on the sphere satisfying the strict subspace concentration inequality. In particular, Lemma 4.1 and Theorem 6.3 in [19] assert that (2.45) has a global minimum and every minimizer Q_0 satisfies $\mathrm{V}_{Q_0} = \mu$. Hence, by setting $\mu = \mathrm{V}_K$ in (2.45) we find that V_K is unique if and only if for every $M \in \mathcal{K}_s^n$ and $r = (\mathrm{vol}(K)/\mathrm{vol}(M))^{1/n}$ it holds that

$$\int\limits_{\mathbb{S}^{n-1}} \log \mathrm{h}_{rM}(\boldsymbol{u}) \,\mathrm{dV}_K(\boldsymbol{u}) > \int\limits_{\mathbb{S}^{n-1}} \log \mathrm{h}_K(\boldsymbol{u}) \,\mathrm{dV}_K(\boldsymbol{u}),$$

which can be equivalently written as

$$\int\limits_{\mathbb{S}^{n-1}} \log \frac{\mathrm{h}_M(\boldsymbol{u})}{\mathrm{h}_K(\boldsymbol{u})} \,\mathrm{dV}_K(\boldsymbol{u}) > -\mathrm{vol}(K) \log r = \frac{\mathrm{vol}(K)}{n} \log \frac{\mathrm{vol}(M)}{\mathrm{vol}(K)},$$

i.e., V_K is unique if and only if the strict logarithmic Minkowski inequality (2.7) is satisfied for K and every $M \in \mathcal{K}_s^n$.

A significant aspect of Theorem 2.14 is that it asserts the non-uniqueness of cone-volume measures of trapezoids. Only recently, Chen, Li and Zhu [22] gave the first non-trivial examples for non-uniqueness of cone-volume measures. Their construction is based on a family of truncated cross-polytopes, which in the planar case become fivegons (see Fig. 2.7).

The explicit computation in Lemma 2.15 produces every example of non-unique cone-volume measures of trapezoids from prescribed unit outer normals and cone-volumes. The solutions corresponding to the given data

$$\begin{aligned}
\boldsymbol{u}_1 &= \begin{pmatrix} 0 \\ 1 \end{pmatrix}, & \mu(\boldsymbol{u}_1) &= 4, \\
\boldsymbol{u}_2 &= \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \end{pmatrix}, & \mu(\boldsymbol{u}_2) &= 24, \\
\boldsymbol{u}_3 &= \begin{pmatrix} 0 \\ -1 \end{pmatrix}, & \mu(\boldsymbol{u}_3) &= 56, \\
\boldsymbol{u}_4 &= \begin{pmatrix} 1 \\ 0 \end{pmatrix}, & \mu(\boldsymbol{u}_4) &= 12,
\end{aligned}$$

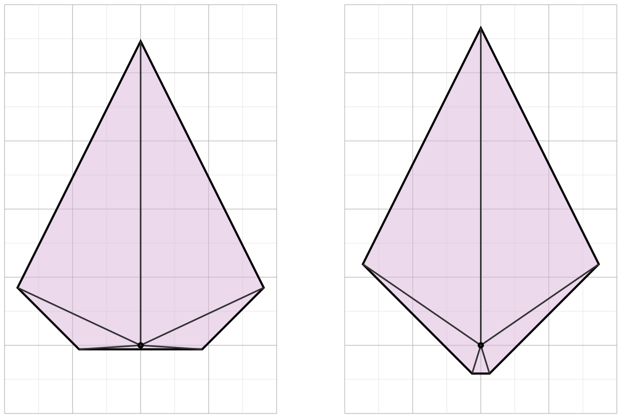

Figure 2.7: Two fivegons with equal cone-volume measure found by Chen, Li and Zhu [22, Thm. 6.1]

are illustrated in Fig. 2.8. Overall, on the one hand the uniqueness question for cone-volume measures in the class $\mathcal{K}_s^2$ is settled and the logarithmic Minkowski inequality (2.7) is known to hold. On the other hand both assertions are false for the class $\mathcal{K}_o^2$. For centered convex bodies in the plane they are both open problems.

From the characterization result in Theorem 2.14 we may draw conclusions about cone-volume measures of centered trapezoids. In this case the subspace concentration inequality (1.13) holds, which is stronger than (2.35).

Theorem 2.16 (Böröczky, Henk [13]). *Let $K \in \mathcal{K}_c^n$. Then*

$$\mathrm{V}_K(\mathbb{S}^{n-1} \cap L) \leq \frac{\dim L}{n} \operatorname{vol}(K) \tag{2.46}$$

for every proper subspace L of $\mathbb{R}^n$, and equality in (2.46) is attained for some L, if and only if there is a subspace L' complementary to L such that μ is concentrated on $\mathbb{S}^{n-1} \cap (L \cup L')$.

We will use the following relation which is probably known, but no reference is known to the author.

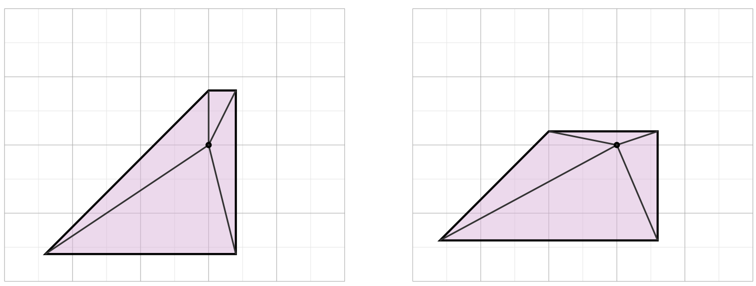

Figure 2.8: Two trapezoids with equal cone-volume measure

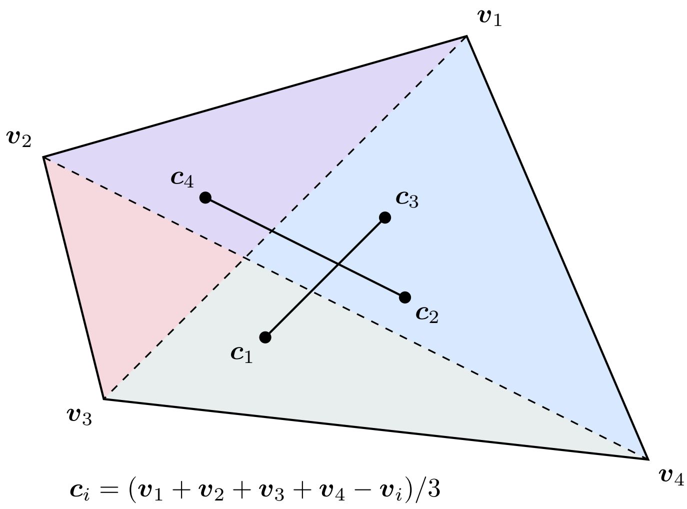

Figure 2.9: Determining the centroid of a quadrilateral from triangulations as in the proof of Proposition 2.17

Proposition 2.17 (Xue[1])**.** *Let $Q \in \mathcal{P}_o^2$ be a quadrilateral with counterclockwise ordered vertices $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4 \in \mathbb{S}^1$. Then Q is centered if and only if*

$$\begin{aligned} \det(\boldsymbol{v}_2, \boldsymbol{v}_4) &= -\det(\boldsymbol{v}_2, \boldsymbol{v}_1) + \det(\boldsymbol{v}_3, \boldsymbol{v}_2) + \det(\boldsymbol{v}_4, \boldsymbol{v}_3) - \det(\boldsymbol{v}_1, \boldsymbol{v}_4), \\ \det(\boldsymbol{v}_3, \boldsymbol{v}_1) &= -\det(\boldsymbol{v}_2, \boldsymbol{v}_1) - \det(\boldsymbol{v}_3, \boldsymbol{v}_2) + \det(\boldsymbol{v}_4, \boldsymbol{v}_3) + \det(\boldsymbol{v}_1, \boldsymbol{v}_4). \end{aligned} \tag{2.47}$$

In particular, if Q is centered with ordered unit outer normals $\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3, \boldsymbol{u}_4 \in \mathbb{S}^1$, then

$$\begin{aligned} &\mathrm{V}_Q(\boldsymbol{u}_1)^2 - \mathrm{V}_Q(\boldsymbol{u}_1)\mathrm{V}_Q(\boldsymbol{u}_3) + \mathrm{V}_Q(\boldsymbol{u}_3)^2 \\ &\qquad\qquad = \mathrm{V}_Q(\boldsymbol{u}_2)^2 - \mathrm{V}_Q(\boldsymbol{u}_2)\mathrm{V}_Q(\boldsymbol{u}_4) + \mathrm{V}_Q(\boldsymbol{u}_4)^2. \end{aligned} \tag{2.48}$$

Proof. Let $Q \in \mathcal{P}_o^2$ with vertices $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4 \in \mathbb{R}^2$ and ordered outer normals $\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3, \boldsymbol{u}_4 \in \mathbb{S}^1$ such that

$$\begin{aligned} F(Q, \boldsymbol{u}_1) &= [\boldsymbol{v}_1, \boldsymbol{v}_2], & F(Q, \boldsymbol{u}_3) &= [\boldsymbol{v}_3, \boldsymbol{v}_4] \\ F(Q, \boldsymbol{u}_2) &= [\boldsymbol{v}_2, \boldsymbol{v}_3], & F(Q, \boldsymbol{u}_4) &= [\boldsymbol{v}_4, \boldsymbol{v}_1]. \end{aligned}$$

The centroid of any measurable set $X \subseteq \mathbb{R}^n$ with $\mathrm{vol}(X) > 0$ satisfies for any partition into measurable sets $X_1 \cup X_2 = X$, $\mathrm{vol}(X_1 \cap X_2) = 0$, the relation

$$\mathbf{c}(X) = \frac{\mathrm{vol}(X_1)}{\mathrm{vol}(X)}\,\mathbf{c}(X_1) + \frac{\mathrm{vol}(X_2)}{\mathrm{vol}(X)}\,\mathbf{c}(X_2),$$

i.e., $\mathbf{c}(X)$ lies on the segment $[\mathbf{c}(X_1), \mathbf{c}(X_2)]$. Consider the partitions of Q coming from its diagonals, i.e.,

$$Q = \mathrm{conv}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\} \cup \mathrm{conv}\{\boldsymbol{v}_1, \boldsymbol{v}_3, \boldsymbol{v}_4\} = \mathrm{conv}\{\boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4\} \cup \mathrm{conv}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_4\}.$$

[1] Fei Xue, private communication.

The centroid is the unique intersection point of the segments joining the centroids of the cells (see Fig. 2.9), i.e.,

$$\mathbf{c}(Q) \in \left[\frac{1}{3}(\boldsymbol{v}_1+\boldsymbol{v}_2+\boldsymbol{v}_3), \frac{1}{3}(\boldsymbol{v}_1+\boldsymbol{v}_3+\boldsymbol{v}_4)\right] \cap \left[\frac{1}{3}(\boldsymbol{v}_2+\boldsymbol{v}_3+\boldsymbol{v}_4), \frac{1}{3}(\boldsymbol{v}_1+\boldsymbol{v}_2+\boldsymbol{v}_4)\right].$$

The origin lies on each of both segments if and only if

$$\begin{aligned}\det(\boldsymbol{v}_1+\boldsymbol{v}_2+\boldsymbol{v}_3, \boldsymbol{v}_1+\boldsymbol{v}_3+\boldsymbol{v}_4) = 0,\\ \det(\boldsymbol{v}_2+\boldsymbol{v}_3+\boldsymbol{v}_4, \boldsymbol{v}_1+\boldsymbol{v}_2+\boldsymbol{v}_4) = 0.\end{aligned}$$

Rearranging these equations by using the properties of the determinant yields the equivalent equations

$$\begin{aligned}\det(\boldsymbol{v}_2,\boldsymbol{v}_4) &= -\det(\boldsymbol{v}_2,\boldsymbol{v}_1)+\det(\boldsymbol{v}_3,\boldsymbol{v}_2)+\det(\boldsymbol{v}_4,\boldsymbol{v}_3)-\det(\boldsymbol{v}_1,\boldsymbol{v}_4),\\ \det(\boldsymbol{v}_3,\boldsymbol{v}_1) &= -\det(\boldsymbol{v}_2,\boldsymbol{v}_1)-\det(\boldsymbol{v}_3,\boldsymbol{v}_2)+\det(\boldsymbol{v}_4,\boldsymbol{v}_3)+\det(\boldsymbol{v}_1,\boldsymbol{v}_4).\end{aligned}$$

This is (2.47). Now if Q is centered, by multiplying the latter equations we find

$$\begin{aligned}&\det(\boldsymbol{v}_2,\boldsymbol{v}_4)\det(\boldsymbol{v}_3,\boldsymbol{v}_1)\\ &\quad= \Big(\det(\boldsymbol{v}_4,\boldsymbol{v}_3)-\det(\boldsymbol{v}_2,\boldsymbol{v}_1)\Big)^2 - \Big(\det(\boldsymbol{v}_3,\boldsymbol{v}_2)-\det(\boldsymbol{v}_1,\boldsymbol{v}_4)\Big)^2\\ &\quad= 4\left(\Big(\mathrm{V}_Q(\boldsymbol{u}_3)-\mathrm{V}_Q(\boldsymbol{u}_1)\Big)^2 - \Big(\mathrm{V}_Q(\boldsymbol{u}_2)-\mathrm{V}_Q(\boldsymbol{u}_4)\Big)^2\right).\end{aligned} \tag{2.49}$$

On the other hand, for any vectors $\boldsymbol{a},\boldsymbol{b},\boldsymbol{c},\boldsymbol{d}\in\mathbb{R}^2$ it holds that

$$\begin{aligned}\det(\boldsymbol{a},\boldsymbol{b})\det(\boldsymbol{c},\boldsymbol{d}) &= \det((\boldsymbol{a},\boldsymbol{b})^T(\boldsymbol{c},\boldsymbol{d}))\\ &= \det\begin{pmatrix}\langle\boldsymbol{a},\boldsymbol{c}\rangle & \langle\boldsymbol{a},\boldsymbol{d}\rangle\\ \langle\boldsymbol{b},\boldsymbol{c}\rangle & \langle\boldsymbol{b},\boldsymbol{d}\rangle\end{pmatrix} = \langle\boldsymbol{a},\boldsymbol{c}\rangle\langle\boldsymbol{b},\boldsymbol{d}\rangle - \langle\boldsymbol{a},\boldsymbol{d}\rangle\langle\boldsymbol{b},\boldsymbol{c}\rangle.\end{aligned} \tag{2.50}$$

In particular,

$$\begin{aligned}&\det(\boldsymbol{v}_2,\boldsymbol{v}_4)\det(\boldsymbol{v}_3,\boldsymbol{v}_1)\\ &= \langle\boldsymbol{v}_2,\boldsymbol{v}_3\rangle\langle\boldsymbol{v}_4,\boldsymbol{v}_1\rangle - \langle\boldsymbol{v}_2,\boldsymbol{v}_1\rangle\langle\boldsymbol{v}_4,\boldsymbol{v}_3\rangle\\ &= \langle\boldsymbol{v}_2,\boldsymbol{v}_3\rangle\langle\boldsymbol{v}_4,\boldsymbol{v}_1\rangle - \langle\boldsymbol{v}_3,\boldsymbol{v}_1\rangle\langle\boldsymbol{v}_4,\boldsymbol{v}_2\rangle + \langle\boldsymbol{v}_3,\boldsymbol{v}_1\rangle\langle\boldsymbol{v}_4,\boldsymbol{v}_2\rangle - \langle\boldsymbol{v}_2,\boldsymbol{v}_1\rangle\langle\boldsymbol{v}_4,\boldsymbol{v}_3\rangle\\ &= \langle\boldsymbol{v}_3,\boldsymbol{v}_1\rangle\langle\boldsymbol{v}_2,\boldsymbol{v}_4\rangle - \langle\boldsymbol{v}_3,\boldsymbol{v}_4\rangle\langle\boldsymbol{v}_2,\boldsymbol{v}_1\rangle - \langle\boldsymbol{v}_2,\boldsymbol{v}_4\rangle\langle\boldsymbol{v}_1,\boldsymbol{v}_3\rangle + \langle\boldsymbol{v}_2,\boldsymbol{v}_3\rangle\langle\boldsymbol{v}_1,\boldsymbol{v}_4\rangle\\ &= \det(\boldsymbol{v}_3,\boldsymbol{v}_2)\det(\boldsymbol{v}_1,\boldsymbol{v}_4) - \det(\boldsymbol{v}_2,\boldsymbol{v}_1)\det(\boldsymbol{v}_4,\boldsymbol{v}_3)\\ &= 4\Big(\mathrm{V}_Q(\boldsymbol{u}_2)\mathrm{V}_Q(\boldsymbol{u}_4) - \mathrm{V}_Q(\boldsymbol{u}_1)\mathrm{V}_Q(\boldsymbol{u}_3)\Big).\end{aligned}$$

Together with (2.49) it follows that

$$\begin{aligned}&\mathrm{V}_Q(\boldsymbol{u}_2)\mathrm{V}_Q(\boldsymbol{u}_4) - \mathrm{V}_Q(\boldsymbol{u}_1)\mathrm{V}_Q(\boldsymbol{u}_3)\\ &\qquad= \Big(\mathrm{V}_Q(\boldsymbol{u}_3)-\mathrm{V}_Q(\boldsymbol{u}_1)\Big)^2 - \Big(\mathrm{V}_Q(\boldsymbol{u}_2)-\mathrm{V}_Q(\boldsymbol{u}_4)\Big)^2,\end{aligned}$$

which gives (2.48). □

The conditions of Theorem 2.16 and Proposition 2.17 together are almost sufficient for centered trapezoids. In fact, we may deduce a characterization of cone-volume measures of centered trapezoids from Theorem 2.14 (see Fig. 2.6).

Corollary 2.18. *Let μ be a non-zero, finite Borel measure on $\mathbb{S}^1$ supported on pairwise distinct and counterclockwise ordered unit vectors $\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3, \boldsymbol{u}_4 \in \mathbb{S}^1$. Suppose* $\operatorname{supp}(\mu)$ *contains a single pair of antipodal points, say $\boldsymbol{u}_1 = -\boldsymbol{u}_3$, and there is open hemisphere $\omega \subseteq \mathbb{S}^1$ such that* $\operatorname{supp}(\mu) \cap \omega = \{\boldsymbol{u}_3\}$. *Then there exists a polygon $P \in \mathcal{P}_c^2$ with $\mathrm{V}_P = \mu$ if and only if*

(i) $\mu(\boldsymbol{u}_1) + \mu(\boldsymbol{u}_3) < \mu(\boldsymbol{u}_2) + \mu(\boldsymbol{u}_4)$,

(ii) $\mu(\boldsymbol{u}_1)^2 - \mu(\boldsymbol{u}_1)\mu(\boldsymbol{u}_3) + \mu(\boldsymbol{u}_3)^2 = \mu(\boldsymbol{u}_2)^2 - \mu(\boldsymbol{u}_2)\mu(\boldsymbol{u}_4) + \mu(\boldsymbol{u}_4)^2$,

(iii) $\mu(\boldsymbol{u}_1) < \mu(\boldsymbol{u}_3)$,

(iv) $\mu(\boldsymbol{u}_2) = \mu(\boldsymbol{u}_4)$.

Moreover, P is uniquely determined.

Proof. We start by proving the necessity of all the conditions. Let $T \in \mathcal{P}_c^n$ be a centered trapezoid with order outer normal vectors $\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3, \boldsymbol{u}_4 \in \mathbb{S}^1$, $\boldsymbol{u}_1 = -\boldsymbol{u}_3$, such that $\boldsymbol{u}_3 = U(T) \cap \omega$ for a suitable open hemisphere $\omega \subseteq \mathbb{S}^1$. Condition (i) is just the strict subspace concentration inequality for centered convex polygons that are not parallelograms (see Theorem 2.16). The assertion (2.48) of Proposition 2.17 is condition (ii). For condition (iv) consider the segment S connecting the midpoints of $F(T, \boldsymbol{u}_1)$ and $F(T, \boldsymbol{u}_3)$. It divides T into two parts of equal area such that exactly half of each segment $T \cap H(\boldsymbol{u}_1, t)$, $t \in \mathbb{R}$, lies on either side of S. This shows that $\mathbf{c}(T)$ is contained in S. Moreover, S also divides the cones $\operatorname{conv}(F(T, \boldsymbol{u}_i) \cup \{\mathbf{0}\})$, $i \in \{1, 3\}$ into equal parts. Thus, $\mathrm{V}_T(\boldsymbol{u}_2) = \mathrm{V}_T(\boldsymbol{u}_4)$. The condition (iii) can be seen as follows. We abbreviate

$$\begin{aligned} h_i &= \mathrm{h}_T(\boldsymbol{u}_i), \\ f_i &= \mathrm{vol}_1(F(T, \boldsymbol{u}_i)), \end{aligned}$$

for $i = 1, 2, 3, 4$. Since T is centered, Fubini's theorem yields

$$\begin{aligned} 0 &= \int_{-h_3}^{h_1} t \operatorname{vol}_1(T \cap H(\boldsymbol{u}_1, t)) \,\mathrm{d}t \\ &= \int_{-h_3}^{h_1} t \left(\frac{t + h_3}{h_1 + h_3}(f_1 - f_3) + f_3 \right) \mathrm{d}t \\ &= \frac{h_1 + h_3}{6} \Big((2f_1 + f_3)h_1 - (f_1 + 2f_3)h_3 \Big) \\ &= \frac{2(h_1 + h_3)f_1 f_3}{3} \Big((2f_1 + f_3) f_3 \mathrm{V}_T(\boldsymbol{u}_1) - (f_1 + 2f_3) f_1 \mathrm{V}_T(\boldsymbol{u}_3) \Big), \end{aligned}$$

and so

$$\mathrm{V}_T(\boldsymbol{u}_1) = \mathrm{V}_T(\boldsymbol{u}_3) \cdot \frac{(f_1 + 2f_3)f_1}{(2f_1 + f_3)f_3}.$$

From Minkowski's characterization (2.8) and the fact that $U(T) \cap \omega = \{\boldsymbol{u}_3\}$ it follows that $f_1 < f_3$ and thus $(f_1 + 2f_3)f_1 < (2f_1 + f_3)f_3$, which shows (iii).

On the other hand, suppose μ is given as above such that (i), (ii), (iii) and (iv) hold. Theorem 2.14 asserts that there exists a unique trapezoid $T \in \mathcal{P}_o^2$ with $\mathrm{V}_T = \mu$. It remains to show that T is centered. To this end, let $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4 \in \mathbb{R}^2$ be the vertices of T and sorted such that

$$\begin{aligned} F(T, \boldsymbol{u}_1) &= [\boldsymbol{v}_1, \boldsymbol{v}_2], & F(T, \boldsymbol{u}_3) &= [\boldsymbol{v}_3, \boldsymbol{v}_4] \\ F(T, \boldsymbol{u}_2) &= [\boldsymbol{v}_2, \boldsymbol{v}_3], & F(T, \boldsymbol{u}_4) &= [\boldsymbol{v}_4, \boldsymbol{v}_1]. \end{aligned}$$

We start by proving that $\det(\boldsymbol{v}_3, \boldsymbol{v}_1) = \det(\boldsymbol{v}_2, \boldsymbol{v}_4)$. Indeed, since $F(T, \boldsymbol{u}_1)$ and $F(T, \boldsymbol{u}_3)$ are parallel we find that

$$0 = \det(\boldsymbol{v}_2 - \boldsymbol{v}_1, \boldsymbol{v}_4 - \boldsymbol{v}_3) = \det(\boldsymbol{v}_2, \boldsymbol{v}_4) - \det(\boldsymbol{v}_2, \boldsymbol{v}_3) - \det(\boldsymbol{v}_1, \boldsymbol{v}_4) + \det(\boldsymbol{v}_1, \boldsymbol{v}_3).$$

Now (iv) is equivalent to $\det(\boldsymbol{v}_3, \boldsymbol{v}_2) = \det(\boldsymbol{v}_1, \boldsymbol{v}_4)$. Hence $\det(\boldsymbol{v}_3, \boldsymbol{v}_1) = \det(\boldsymbol{v}_2, \boldsymbol{v}_4)$. As above, by (2.50) we have

$$\begin{aligned} &\det(\boldsymbol{v}_2, \boldsymbol{v}_4) \det(\boldsymbol{v}_3, \boldsymbol{v}_1) \\ &= \langle \boldsymbol{v}_2, \boldsymbol{v}_3 \rangle \langle \boldsymbol{v}_4, \boldsymbol{v}_1 \rangle - \langle \boldsymbol{v}_2, \boldsymbol{v}_1 \rangle \langle \boldsymbol{v}_4, \boldsymbol{v}_3 \rangle \\ &= \langle \boldsymbol{v}_2, \boldsymbol{v}_3 \rangle \langle \boldsymbol{v}_4, \boldsymbol{v}_1 \rangle - \langle \boldsymbol{v}_3, \boldsymbol{v}_1 \rangle \langle \boldsymbol{v}_4, \boldsymbol{v}_2 \rangle + \langle \boldsymbol{v}_3, \boldsymbol{v}_1 \rangle \langle \boldsymbol{v}_4, \boldsymbol{v}_2 \rangle - \langle \boldsymbol{v}_2, \boldsymbol{v}_1 \rangle \langle \boldsymbol{v}_4, \boldsymbol{v}_3 \rangle \\ &= \langle \boldsymbol{v}_3, \boldsymbol{v}_1 \rangle \langle \boldsymbol{v}_2, \boldsymbol{v}_4 \rangle - \langle \boldsymbol{v}_3, \boldsymbol{v}_4 \rangle \langle \boldsymbol{v}_2, \boldsymbol{v}_1 \rangle - \langle \boldsymbol{v}_2, \boldsymbol{v}_4 \rangle \langle \boldsymbol{v}_1, \boldsymbol{v}_3 \rangle + \langle \boldsymbol{v}_2, \boldsymbol{v}_3 \rangle \langle \boldsymbol{v}_1, \boldsymbol{v}_4 \rangle \\ &= \det(\boldsymbol{v}_3, \boldsymbol{v}_2) \det(\boldsymbol{v}_1, \boldsymbol{v}_4) - \det(\boldsymbol{v}_2, \boldsymbol{v}_1) \det(\boldsymbol{v}_4, \boldsymbol{v}_3) \\ &= 4\Big(\mathrm{V}_T(\boldsymbol{u}_2)\mathrm{V}_T(\boldsymbol{u}_4) - \mathrm{V}_T(\boldsymbol{u}_1)\mathrm{V}_T(\boldsymbol{u}_3)\Big) \\ &= 4\Big(\mathrm{V}_T(\boldsymbol{u}_2)^2 - \mathrm{V}_T(\boldsymbol{u}_1)\mathrm{V}_T(\boldsymbol{u}_3)\Big) \\ &= 4\Big(\mathrm{V}_T(\boldsymbol{u}_1)^2 + \mathrm{V}_T(\boldsymbol{u}_3)^2 - 2\mathrm{V}_T(\boldsymbol{u}_1)\mathrm{V}_T(\boldsymbol{u}_3)\Big) \\ &= 4(\mathrm{V}_T(\boldsymbol{u}_3) - \mathrm{V}_T(\boldsymbol{u}_1))^2, \end{aligned}$$

where we used (ii) and (iv). Since $\det(\boldsymbol{v}_2, \boldsymbol{v}_4) = \det(\boldsymbol{v}_3, \boldsymbol{v}_1)$ and because of (iii) and (iv) it follows that

$$\begin{aligned} \det(\boldsymbol{v}_2, \boldsymbol{v}_4) &= 2(\mathrm{V}_T(\boldsymbol{u}_3) - \mathrm{V}_T(\boldsymbol{u}_1)) \\ &= 2(\mathrm{V}_T(\boldsymbol{u}_3) - \mathrm{V}_T(\boldsymbol{u}_1) + \mathrm{V}_T(\boldsymbol{u}_2) - \mathrm{V}_T(\boldsymbol{u}_4)) \\ &= \det(\boldsymbol{v}_4, \boldsymbol{v}_3) - \det(\boldsymbol{v}_2, \boldsymbol{v}_1) + \det(\boldsymbol{v}_3, \boldsymbol{v}_2) - \det(\boldsymbol{v}_1, \boldsymbol{v}_4) \end{aligned}$$

and

$$\begin{aligned} \det(\boldsymbol{v}_3, \boldsymbol{v}_1) &= 2(\mathrm{V}_T(\boldsymbol{u}_3) - \mathrm{V}_T(\boldsymbol{u}_1)) \\ &= 2(\mathrm{V}_T(\boldsymbol{u}_3) - \mathrm{V}_T(\boldsymbol{u}_1) - \mathrm{V}_T(\boldsymbol{u}_2) + \mathrm{V}_T(\boldsymbol{u}_4)) \\ &= \det(\boldsymbol{v}_4, \boldsymbol{v}_3) - \det(\boldsymbol{v}_2, \boldsymbol{v}_1) - \det(\boldsymbol{v}_3, \boldsymbol{v}_2) + \det(\boldsymbol{v}_1, \boldsymbol{v}_4). \end{aligned}$$

Since (2.47) is satisfied, Proposition 2.17 asserts that T is centered. □

The presented examples confirming non-uniqueness of non-trivial cone-volume measure all have in common that either the subspace concentration inequality is violated for a pair of antipodal outer normals, or the considered polygon has

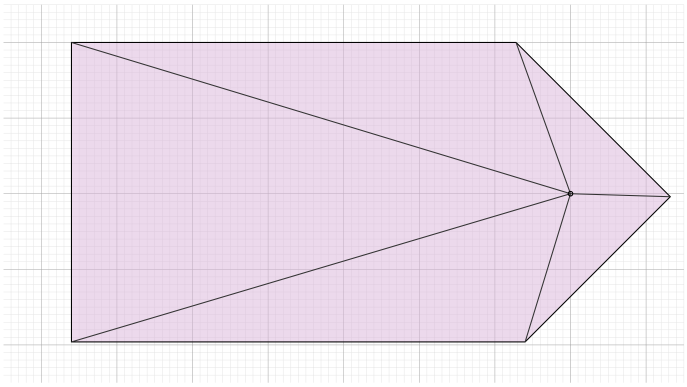

Figure 2.10: Fivegon in Proposition 2.19

no parallel sides. In case of quadrilaterals, Theorem 2.14 assures that this is necessary. However, Malikiosis discovered the existence of a fivegon satisfying the strict subspace concentration inequality with non-unqiue cone-volume measures.

Proposition 2.19 (Malikiosis[2])**.** *There are two fivegons satisfying the strict subspace concentration inequality that have the same cone-volume measure.*

Proof. The fivegon given as (see Fig. 2.10)

$$P = \operatorname{conv}\left\{ \begin{pmatrix} -36 \\ 100 \end{pmatrix}, \begin{pmatrix} -330 \\ 100 \end{pmatrix}, \begin{pmatrix} -330 \\ -98 \end{pmatrix}, \begin{pmatrix} -30 \\ -98 \end{pmatrix}, \begin{pmatrix} 66 \\ -2 \end{pmatrix} \right\},$$

has cone-volume data

$$\begin{aligned}
\boldsymbol{u}_1 &= \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}, & \mathrm{V}_P(\boldsymbol{u}_1) &= 3264, \\
\boldsymbol{u}_2 &= \begin{pmatrix} 0 \\ 1 \end{pmatrix}, & \mathrm{V}_P(\boldsymbol{u}_2) &= 14700, \\
\boldsymbol{u}_3 &= \begin{pmatrix} -1 \\ 0 \end{pmatrix}, & \mathrm{V}_P(\boldsymbol{u}_3) &= 32670, \\
\boldsymbol{u}_4 &= \begin{pmatrix} 0 \\ -1 \end{pmatrix}, & \mathrm{V}_P(\boldsymbol{u}_4) &= 14700, \\
\boldsymbol{u}_5 &= \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}, & \mathrm{V}_P(\boldsymbol{u}_5) &= 3264.
\end{aligned}$$

In particular, P satisfies the subspace concentration inequality (2.34). Moreover, the cone-volume measure of P is invariant under reflection about $\boldsymbol{e}_2^\perp$, but P is not. □

The results of this section raise the questions whether cone-volume measures of centered polygons are unique and if the the number of distinct polygons realizing given cone-volume data can be bounded.

[2]Romanos Diogenes Malikiosis, private communication.

3 Dual curvature measures

3.1 An introduction to dual Brunn-Minkowski theory

Duality of concepts generally describes the act of translating aspects of one theory into another by an (often involutive) operation, e.g., dual spaces of vector spaces or primal and dual linear programs in optimization. A duality within the space of convex bodies is described by the following operation. For a non-empty set $X \subseteq \mathbb{R}^n$ we define its **polar set** by

$$X^* = \{\boldsymbol{y} \in \mathbb{R}^n : \langle \boldsymbol{x}, \boldsymbol{y} \rangle \leq 1 \text{ for all } \boldsymbol{x} \in X\}. \tag{3.1}$$

The polar set is always a closed convex set containing the origin. Moreover, for $K \in \mathcal{K}_o^n$ its polar K^* is a convex body in $\mathcal{K}_o^n$ and $(K^*)^* = K$. There is no relation between the support functions h_K and h_{K^*} of the same simplicity, but h_{K^*} exhibits a geometric interpretation regarding K, which is

$$(\mathrm{h}_{K^*}(\boldsymbol{u}))^{-1} = \max\{\lambda > 0 : \lambda \boldsymbol{u} \in K\}, \tag{3.2}$$

for $\boldsymbol{u} \in \mathbb{S}^{n-1}$. Its (-1)-homogeneous extension defined by $\rho_K(\boldsymbol{x}) = (\mathrm{h}_{K^*}(\boldsymbol{x}))^{-1}$ for $\boldsymbol{x} \in \mathbb{R}^n \setminus \{\boldsymbol{0}\}$ is called **radial function** of K. The geometric intuition behind this duality is that points contained in K^* interchange roles with halfspaces containing K, i.e., $\boldsymbol{x} \in K^*$ if and only if $K \subseteq H^-(\boldsymbol{x}, 1)$, for $\boldsymbol{x} \neq \boldsymbol{0}$, and vice versa.

The dual Brunn-Minkowski theory is a conceptually dual to the classical Brunn-Minkowski theory which arises by replacing the Minkowski addition (1.1), i.e., the addition of support functions, by addition of radial functions in the following way. For $K, M \in \mathcal{K}_o^n$, and scalars $s, t \geq 0$ the **radial combination** of K and M with respect to s and t is

$$sK \,\widetilde{+}\, tM = \{\boldsymbol{x} \in \mathbb{R}^n : s\rho_K(\boldsymbol{x}) + t\rho_M(\boldsymbol{x}) \geq 1\}. \tag{3.3}$$

In general, the radial combination of convex bodies is not necessarily a convex set, but it is always a star-shaped set, i.e., a non-empty set $S \subseteq \mathbb{R}^n$ such that for every $\boldsymbol{x} \in S$ the segment $[\boldsymbol{0}, \boldsymbol{x}]$ is contained in S (see Fig. 3.1). The

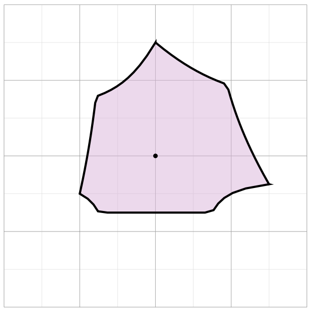

Figure 3.1: Radial combination $\frac{1}{2}K \mathbin{\tilde{+}} \frac{1}{2}M$ for $K = \operatorname{conv}\left\{\binom{0}{2}, \binom{-1}{-1/2}, \binom{2}{-1/2}\right\}$ and $M = [-1,1]^2$

definition (3.2) may be extended to compact star-shaped sets $S \subseteq \mathbb{R}^n$ via

$$\rho_S(\boldsymbol{x}) = \max\{\lambda \geq 0 : \lambda\boldsymbol{x} \in S\},$$

for $\boldsymbol{x} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$. If ρ_S is positive and continuous, we call S a **star body**. The set of star bodies in $\mathbb{R}^n$ will be denoted by $\mathcal{S}_o^n$. Then the left hand side of (3.3) can be characterized as the unique star body with

$$\rho_{sK \mathbin{\tilde{+}} tM} = s\rho_K + t\rho_M.$$

The dual Brunn-Minkowski theory was introduced by Lutwak [58]. His contributions include among others an analogue of the Minkowski inequality (2.5) from which the dual Brunn-Minkowski inequality

$$\operatorname{vol}((1-\lambda)K \mathbin{\tilde{+}} \lambda M)^{\frac{1}{n}} \leq (1-\lambda)\operatorname{vol}(K)^{\frac{1}{n}} + \lambda \operatorname{vol}(M)^{\frac{1}{n}},$$

$K, M \in \mathcal{S}_o^n$ and $\lambda \in [0,1]$, may be derived (see also [59]). Moreover, there exist functionals similar to the quermassintegrals defined in (1.5). Lutwak [58] found that for a star body $K \in \mathcal{S}_o^n$ its dual parallel body $K \mathbin{\tilde{+}} \lambda B_n$ also admits a polynomial expansion given by

$$\operatorname{vol}(K \mathbin{\tilde{+}} \lambda B_n) = \sum_{i=0}^{n} \lambda^i \binom{n}{i} \widetilde{\mathrm{W}}_i(K),$$

where the functionals $\widetilde{\mathrm{W}}_i(K)$ defined this way are called **dual quermassintegrals** of K. In analogy to Kubota's formula (1.6) the dual quermassintegrals $\widetilde{\mathrm{W}}_i(K)$ exhibit the following integral geometric representation as the means of the volumes of sections (see [60])

$$\widetilde{\mathrm{W}}_{n-i}(K) = \frac{\operatorname{vol}(B_n)}{\operatorname{vol}_i(B_i)} \int\limits_{\mathcal{G}(i,n)} \operatorname{vol}_i(K \cap L)\,\mathrm{d}L,$$

for $i = 0, \ldots, n$. The celebrated solution of the Busemann-Petty problem is amongst the recent successes of the dual Brunn-Minkowski theory (cf. [31, 35, 81]) and it also has connections and applications to integral geometry, Minkowski geometry and the local theory of Banach spaces. For more information about the study of central sections of convex bodies we recommend the

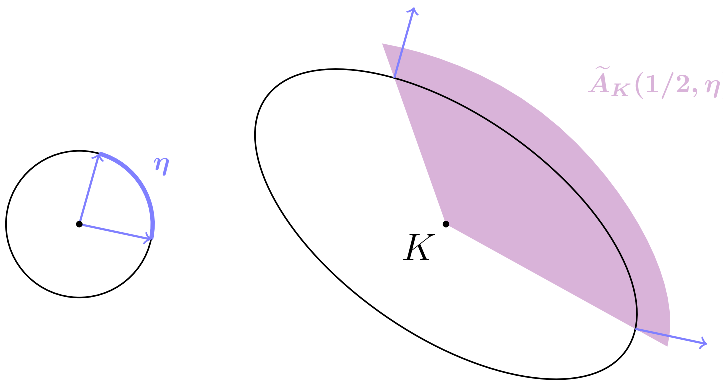

Figure 3.2: Local dual parallel body $\widetilde{A}_K(\lambda,\eta)$

books of Gardner [32], Koldobsky [53] and Schneider [75, Sect. 9.3] and the references therein.

For a long time an analogue of the variational formula (1.7) for dual quermassintegral has been unknown. This gap between classical and dual Brunn-Minkowski theory was recently closed in a groundbreaking paper by Huang, Lutwak, Yang and Zhang [48]. Define the **logarithmic Wulff shape** of $K \in \mathcal{K}_o^n$ with respect to a function $f\colon \mathbb{S}^{n-1} \to \mathbb{R}$, as the convex body

$$[\mathrm{h}_K \exp(f)] = \{\boldsymbol{x} \in \mathbb{R}^n : \langle \boldsymbol{u}, \boldsymbol{x}\rangle \le \mathrm{h}_K(\boldsymbol{u}) \exp(f(\boldsymbol{u})) \text{ for all } \boldsymbol{u} \in \mathbb{S}^{n-1}\}.$$

Huang, Lutwak, Yang and Zhang proved for $K \in \mathcal{K}_o^n$ and every continuous function $f\colon \mathbb{S}^{n-1} \to \mathbb{R}$ the variational formula

$$\lim_{\epsilon\to 0} \frac{\widetilde{\mathrm{W}}_{n-i}([\mathrm{h}_K \exp(\varepsilon f + o(\varepsilon,\cdot))]) - \widetilde{\mathrm{W}}_{n-i}([\mathrm{h}_K])}{\varepsilon} = i \int_{\mathbb{S}^{n-1}} f(\boldsymbol{u})\, \mathrm{d}\widetilde{\mathrm{C}}_i(K,\boldsymbol{u}),$$

$i = 1,\dots,n$, where $o(\varepsilon,\cdot)$ is an arbitrary family of real functions on $\mathbb{S}^{n-1}$ with $\lim_{\epsilon\to 0} \frac{o(\varepsilon,\boldsymbol{u})}{\varepsilon} = 0$, uniformly in $\boldsymbol{u}$, and $\widetilde{\mathrm{C}}_i(K,\cdot)$ are Borel measures on $\mathbb{S}^{n-1}$ (see [48, Thm. 4.5]). Amazingly, the measures $\widetilde{\mathrm{C}}_i(K,\cdot)$ admit a local Steiner-type formula comparable to (1.8). For a convex body $K \in \mathcal{K}_o^n$ and a point $\boldsymbol{x} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ we define by $\widetilde{\mathbf{p}}_K(\boldsymbol{x}) = \rho_K(\boldsymbol{x})\boldsymbol{x} \in \partial K$ its **radial projection** onto K and by

$$\widetilde{\mathrm{d}}(K,\boldsymbol{x}) = \begin{cases} |\boldsymbol{x} - \widetilde{\mathbf{p}}_K(\boldsymbol{x})|\,, & \text{if } \boldsymbol{x} \notin K,\\ 0, & \text{if } \boldsymbol{x} \in K, \end{cases}$$

its radial distance to K. Then for a Borel set $\eta \subseteq \mathbb{S}^{n-1}$ and $\lambda > 0$ we consider the local dual parallel body (see Fig. 3.2)

$$\widetilde{A}_K(\lambda,\eta) = \Big\{\boldsymbol{x} \in \mathbb{R}^n : 0 \le \widetilde{\mathrm{d}}(K,\boldsymbol{x}) \le \lambda \\ \text{and, if } \boldsymbol{x} \neq \mathbf{0},\ \widetilde{\mathbf{p}}_K(\boldsymbol{x}) \in H_K(\boldsymbol{u}) \text{ for some } \boldsymbol{u} \in \eta\Big\}.$$

Huang, Lutwak, Yang and Zhang established the polynomial expansion

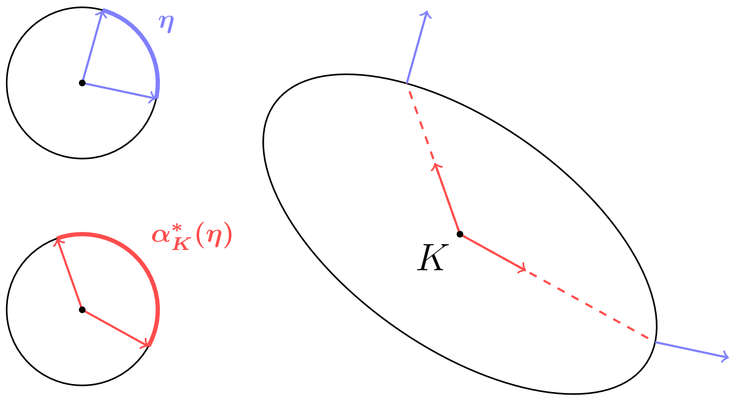

Figure 3.3: Reverse radial Gauss image of a smooth convex body

$$\mathrm{vol}(\widetilde{A}_K(\lambda,\eta)) = \sum_{i=0}^{n} \binom{n}{i} \lambda^i \widetilde{\mathrm{C}}_{n-i}(K,\eta) \tag{3.4}$$

for every Borel set $\eta \subseteq \mathbb{S}^{n-1}$ (see [48, Thm. 3.1]) and used (3.4) as the definition for the **dual curvature measures** $\widetilde{\mathrm{C}}_i(K,\cdot)$ of K, $0 \le i \le n$, which act as conceptual counterparts of the curvature measures defined in (1.9) in the dual Brunn-Minkowski theory. Moreover, they gave an explicit integral representation of the dual curvature measures which is as follows. For a Borel set $\eta \subseteq \mathbb{S}^{n-1}$ we define the **reverse radial Gauss image** by (see Fig. 3.3)

$$\boldsymbol{\alpha}^*_K(\eta) = \{\boldsymbol{u} \in \mathbb{S}^{n-1} : \widetilde{\mathbf{p}}_K(\boldsymbol{u}) \in H_K(\boldsymbol{v}) \text{ for some } \boldsymbol{v} \in \eta\}.$$

Then the dual curvature measures satisfy

$$\widetilde{\mathrm{C}}_i(K,\eta) = \frac{1}{n} \int\limits_{\boldsymbol{\alpha}^*_K(\eta)} \rho_K(\boldsymbol{u})^i \,\mathrm{d}\boldsymbol{u}$$

for $i = 0, \ldots, n$. This representation justifies to define for any $q \in \mathbb{R}$ the **qth dual curvature measure** of $K \in \mathcal{K}^n_o$ by

$$\widetilde{\mathrm{C}}_q(K,\eta) = \frac{1}{n} \int\limits_{\boldsymbol{\alpha}^*_K(\eta)} \rho_K(\boldsymbol{u})^q \,\mathrm{d}\boldsymbol{u}. \tag{3.5}$$

If η is a Borel set, then $\boldsymbol{\alpha}^*_K(\eta)$ is $\mathcal{H}^{n-1}$-measurable (see [75, Lemma 2.2.11.]) and so the qth dual curvature measure given in (3.5) is well defined. An astonishing feature of these dual curvature measures is that they link two other fundamental geometric measures of a convex body. When $q = 0$ the dual curvature measure is – up to a factor of n – Aleksandrov's integral curvature of the polar body of K, i.e., $\widetilde{\mathrm{C}}_0(K,\cdot) = \frac{1}{n}\mathrm{C}_0(K^*,\cdot)$, and for $q = n$ the dual curvature measure coincides with the cone-volume measure of K (see [48, Lem. 3.8]).

We want to point out that there are also **dual area measures** corresponding to the area measures defined in (1.8) given by

$$\widetilde{\mathrm{S}}_q(K,\omega) = \frac{1}{n} \int\limits_{\omega} \rho_K(\boldsymbol{u})^q \,\mathrm{d}\boldsymbol{u}$$

for $q \in \mathbb{R}$ and Borel sets $\omega \subseteq \mathbb{S}^{n-1}$. An extensive description of the duality between classical and dual area measures and curvature measures is contained in [48].

The analogue to the Minkowski-Christoffel problem in the dual Brunn-Minkowski theory is the **dual Minkowski problem**. The task is, given $q \in \mathbb{R}$ and a finite Borel measure μ on $\mathbb{S}^{n-1}$, to find necessary and sufficient conditions for the existence of a convex body $K \in \mathcal{K}_o^n$ such that $\widetilde{\mathrm{C}}_q(K, \cdot) = \mu$. It was introduced in [48] and therein a sufficient condition for even measures was established.

Theorem 3.1 (Huang, Lutwak, Yang, Zhang [48])**.** *Let $q \in (0, n]$ and μ be a non-zero finite even Borel measure on $\mathbb{S}^{n-1}$. If*

$$\mu(\mathbb{S}^{n-1} \cap L) < \min \left\{ 1 - \frac{q-1}{q} \frac{n - \dim L}{n-1}, 1 \right\} \mu(\mathbb{S}^{n-1}) \tag{3.6}$$

for every proper subspace L of $\mathbb{R}^n$, then there exists a symmetric convex body $K \in \mathcal{K}_s^n$ with $\widetilde{\mathrm{C}}_q(K, \cdot) = \mu$.

There are two remarkable aspects of Theorem 3.1. As for $q = n$ the qth dual curvature measure is equal to the cone-volume measure, also (3.6) becomes the strict form of the subspace concentration inequality (1.13). Furthermore, for $0 < q \leq 1$ the inequality (3.6) only asserts that μ is not concentrated on a great subsphere. This is a necessary condition that can be seen from (3.5). In particular, Theorem 3.1 solves the even dual Minkowski problem in the range $q \in (0, 1]$. Examples of convex bodies showing that for $q \in (1, n)$ the inequality (3.6) is not a necessary condition were given in [14, Prop. 1.5] and [82, Prop. A.1]. A necessary subspace concentration inequality for qth dual curvature measures of symmetric convex bodies was proved by Böröczky, Henk and the author. It was also shown that the inequality is sharp since the aforementioned examples get arbitrarily close to the upper bound.

Theorem 3.2 (Böröczky, Henk, P. [14])**.** *Let $q \in (1, n)$ and $K \in \mathcal{K}_s^n$. Then*

$$\widetilde{\mathrm{C}}_q(K, \mathbb{S}^{n-1} \cap L) < \min \left\{ \frac{\dim L}{q}, 1 \right\} \widetilde{\mathrm{C}}_q(K, \mathbb{S}^{n-1}) \tag{3.7}$$

for every proper subspace L of $\mathbb{R}^n$.

We remark that since the bounds (3.6) and (3.7) coincide for $\dim L = 1$, the Theorems 3.1 and 3.2 solved the even dual Minkowski problem in the plane for $q \in (1, 2)$. The sufficiency of (3.7) was first proved by Zhao [82], when q is an integer, and very recently in full generality by Böröczky, Lutwak, Yang, Zhang and Zhao [17].

Theorem 3.3 (Böröczky, Lutwak, Yang, Zhang, Zhao [17])**.** *Let $q \in (0, n)$ and μ be a non-zero finite even Borel measure on $\mathbb{S}^{n-1}$. If*

$$\mu(\mathbb{S}^{n-1} \cap L) < \min \left\{ \frac{\dim L}{q}, 1 \right\} \mu(\mathbb{S}^{n-1})$$

for every proper subspace L of $\mathbb{R}^n$, then there exists a symmetric convex body $K \in \mathcal{K}_s^n$ with $\widetilde{\mathrm{C}}_q(K, \cdot) = \mu$.

The Theorems 3.2 and 3.3 combined solve the dual Minkowski problem for symmetric convex bodies in the range $q \in (0,n)$. The case $q < 0$ was treated by Zhao [83]. He proved the absence of a non-trivial subspace concentration bound even without any symmetry assumptions on the given measure. Moreover, he established the uniqueness of dual curvature measures of convex bodies when $q < 0$.

Theorem 3.4 (Zhao [83])**.** *Let $q < 0$ and μ be a non-zero finite Borel measure on $\mathbb{S}^{n-1}$. Then there exists a convex body $K \in \mathcal{K}^n_o$ with $\widetilde{\mathrm{C}}_q(K,\cdot) = \mu$ if and only if μ is not concentrated on any closed hemisphere. Moreover, $K \in \mathcal{K}^n_o$ is uniquely determined.*

In the remaining paramater range $q > n$ no sufficient condition in the (even) dual Minkowski problem is known. A necessary subspace concentration inequality for dual curvature measures of symmetric convex bodies when $q \geq n+1$ was proved by Henk and the author.

Theorem 3.5 (Henk, P. [42])**.** *Let $q \geq n+1$ and $K \in \mathcal{K}^n_s$. Then*

$$\widetilde{\mathrm{C}}_q(K, \mathbb{S}^{n-1} \cap L) < \frac{q - n + \dim L}{q} \widetilde{\mathrm{C}}_q(K, \mathbb{S}^{n-1}) \tag{3.8}$$

for every proper subspace L of $\mathbb{R}^n$.

The main goal of this chapter is to present the proofs of the Theorems 3.2 and 3.5 in full detail. We use the representation (3.5) to conclude that – similar to the cone-volume measure – $\widetilde{\mathrm{C}}_q(K,\eta)$ measures the cones associated to the outer normals in η just by a measure different from the volume. We will follow the approach of Henk, Schürmann and Wills [44] who proved the necessity of (1.13) for cone-volume measures of symmetric polytopes using only the Brunn-Minkowski inequality (1.2). To this end, we establish Brunn-Minkowski type inequalities in the Sections 3.2 and 3.3, where the volume is replaced by a measure with a Lebesgue-density that satisfies a convexity principle.

In Section 3.4 we will prove Theorems 3.2 and 3.5 and also the sharpness of the given bounds. Moreover, in Section 3.5 we establish an extension of Theorem 3.5 to the range $q \in (n, n+1)$ in case $n = 2$ and discuss the occurring obstructions when $n > 2$.

Finally, we bring up a generalization of dual curvature measures proposed in [64] in Section 3.6 and survey extensions of Theorems 3.2 and 3.5 in this setting.

The results of this chapter originally appeared as joint works with Károly Böröczky, Jr., and Martin Henk [14, 42].

3.2 A generalization of Anderson's theorem on even quasiconcave functions

The integral representation of dual curvature measures (3.5) has a striking resemblance to the well-known volume formula of convex bodies $K \in \mathcal{K}_o^n$

$$\mathrm{vol}(K) = \frac{1}{n} \int_{\mathbb{S}^{n-1}} \rho_K(\boldsymbol{u})^n \,\mathrm{d}\boldsymbol{u},$$

which can be proved just by using spherical coordinates in $\mathbb{R}^n$. Using the convention $0^\alpha = \infty$, $\alpha < 0$, for convenience, which is consistent to measure theoretic considerations, the same method yields the following formula.

Lemma 3.6. *Suppose $q > 0$, $K \in \mathcal{K}_o^n$ and $Q \in \mathcal{S}_o^n$. Let $\eta \subseteq \mathbb{S}^{n-1}$ be a Borel set. Then*

$$\int_{\boldsymbol{\alpha}_K^*(\eta)} \rho_Q(\boldsymbol{u})^{n-q} \rho_K(\boldsymbol{u})^q \,\mathrm{d}\boldsymbol{u} = q \int_{K \cap \left(\mathbb{R}_{\geq 0} \cdot \boldsymbol{\alpha}_K^*(\eta)\right)} \rho_Q(\boldsymbol{x})^{n-q} \,\mathrm{d}\boldsymbol{x}. \tag{3.9}$$

Proof. Since $q > 0$, the integral $\int_0^a r^{q-1} \,\mathrm{d}r$ exists for any $a > 0$. By using spherical coordinates and (3.5)

$$\begin{aligned} q \int_{K \cap \left(\mathbb{R}_{\geq 0} \cdot \boldsymbol{\alpha}_K^*(\eta)\right)} \rho_Q(\boldsymbol{x})^{n-q} \,\mathrm{d}\boldsymbol{x} &= q \int_{\boldsymbol{\alpha}_K^*(\eta)} \left(\int_0^{\rho_K(\boldsymbol{u})} \rho_Q(r\boldsymbol{u})^{n-q} r^{n-1} \,\mathrm{d}r \right) \mathrm{d}\boldsymbol{u} \\ &= q \int_{\boldsymbol{\alpha}_K^*(\eta)} \rho_Q(\boldsymbol{u})^{n-q} \left(\int_0^{\rho_K(\boldsymbol{u})} r^{q-1} \,\mathrm{d}r \right) \mathrm{d}\boldsymbol{u} \\ &= \int_{\boldsymbol{\alpha}_K^*(\eta)} \rho_Q(\boldsymbol{u})^{n-q} \rho_K(\boldsymbol{u})^q \,\mathrm{d}\boldsymbol{u}. \end{aligned}$$

□

The nature of Lemma 3.6 is revealed when we consider (3.5) for a polytope $P \in \mathcal{P}_o^n$ rather than an arbitrary convex body $K \in \mathcal{K}_o^n$. In this case (3.9) with $Q = B_n$ gives (cf. (1.10))

$$\begin{aligned} \widetilde{\mathrm{C}}_q(P, \eta) &= \frac{1}{n} \int_{\boldsymbol{\alpha}_P^*(\eta)} \rho_P(\boldsymbol{u})^q \,\mathrm{d}\boldsymbol{u} \\ &= \frac{q}{n} \int_{P \cap \left(\mathbb{R}_{\geq 0} \cdot \boldsymbol{\alpha}_P^*(\eta)\right)} |\boldsymbol{x}|^{q-n} \,\mathrm{d}\boldsymbol{x} \\ &= \frac{q}{n} \sum_{\boldsymbol{u} \in U(P) \cap \eta} \left(\int_{\mathrm{conv}(F(P, \boldsymbol{u}) \cup \{\boldsymbol{0}\})} |\boldsymbol{x}|^{q-n} \,\mathrm{d}\boldsymbol{x} \right), \end{aligned} \tag{3.10}$$

i.e., the sum in (3.10) is taken over the same cones as the cone-volume measure, which are measured by $|\cdot|^\alpha \,\mathrm{d}\mathcal{H}^n(\cdot)$, for some $\alpha \in (-n, \infty)$, rather than the Lebesgue measure. Following the idea of Henk, Schürmann and Wills [44] we therefore seek for Brunn-Minkowski type inequalities for measures of this kind.

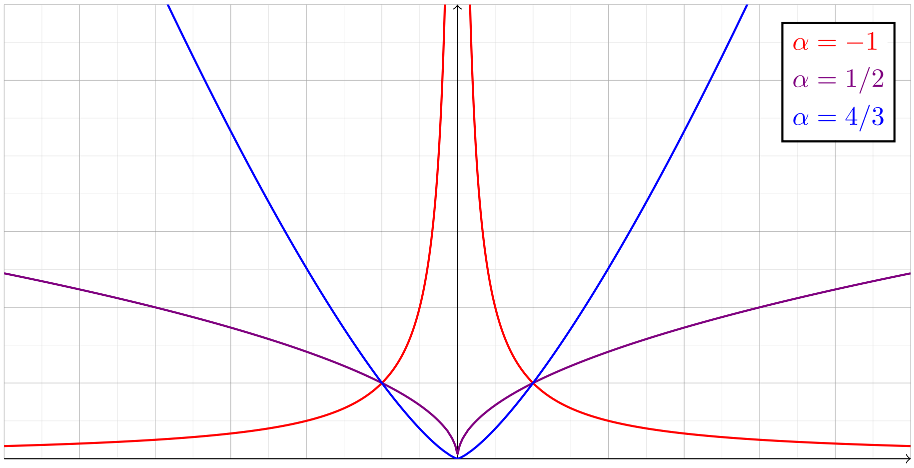

Figure 3.4: Quasiconcave (red), quasiconvex (violet) and convex (blue) instance of the family $\{|\cdot|^\alpha : \alpha \in \mathbb{R} \setminus \{0\}\}$

We point out that Lemma 3.6 is crucial for this approach which is why it is only viable when $q > 0$.

For each $\alpha \in \mathbb{R} \setminus \{0\}$ the function $|\cdot|^\alpha$ has a convexity property (see Fig. 3.4). If $\alpha \geq 1$, then it is convex. When $\alpha \in (0, 1)$, it is a quasiconvex function, but not convex. For $\alpha < 0$, the function is quasiconcave. We say that a nonnegative function $f \colon \mathbb{R}^n \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ is **quasiconvex** if for every $c \in \mathbb{R}_{\geq 0}$ the sublevel set

$$\{f \leq c\} = f^{-1}([0, c]) = \{\boldsymbol{x} \in \mathbb{R}^n : f(\boldsymbol{x}) \leq c\}$$

is convex, and **quasiconcave** if the superlevel sets

$$\{f \geq c\} = f^{-1}([c, \infty]) = \{\boldsymbol{x} \in \mathbb{R}^n : f(\boldsymbol{x}) \geq c\}$$

are convex. Since convex sets are measurable and measurability of sub-/superlevel sets implies measurability of functions, this definition yields that quasiconvex and quasiconcave functions are Hausdorff-measurable functions. In [3], Anderson proved that if $K \in \mathcal{K}_s^n$ and $f \colon \mathbb{R}^n \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ is quasiconcave and even, the minimum value of

$$\int_{\boldsymbol{t}+K} f(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x}$$

with respect to $\boldsymbol{t} \in \mathbb{R}^n$ is attained at $\boldsymbol{t} = \mathbf{0}$. This is obvious if restricted to the one-dimensional case, since the evenness of f together with the quasiconcavity implies that f is monotonically decreasing on any ray starting from the origin. The proof of Anderson relies only on the Brunn-Minkowski inequality (1.2), but the result can be strengthened to a Brunn-Minkowski type inequality for integrals of quasiconcave functions. Here we compare the integral of f over a convex body K with the integral over Minkowski combinations of the form $(1-\lambda)K + \lambda(-K)$. It can be observed, that on the one hand $(1-\lambda)K + \lambda(-K)$ is larger than K regarding volume, but also it lies somewhat closer to the origin where f attains its peak (see Fig. 3.5).

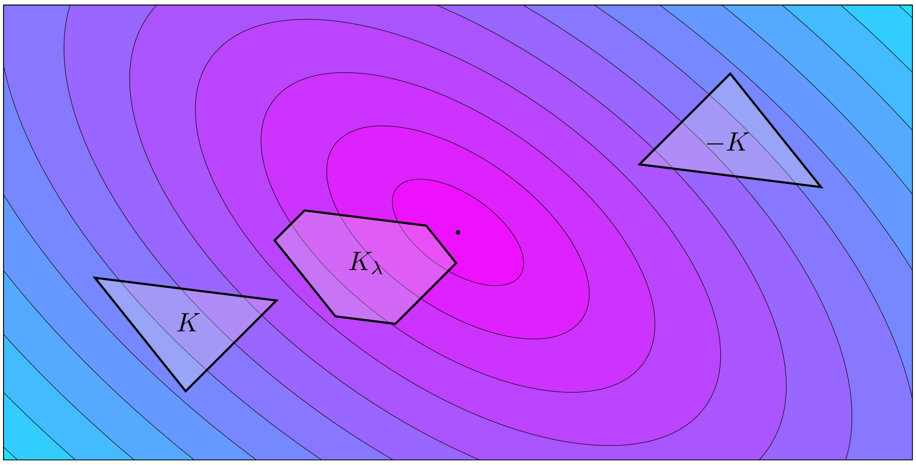

Figure 3.5: Level sets of a two-dimensinal quasiconcave function, and a Minkowski combination $(1-\lambda)K + \lambda(-K)$

Theorem 3.7 (Böröczky, Henk, P. [14])**.** *Let $f\colon \mathbb{R}^n \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ be an even quasiconcave function. Let $K \subset \mathbb{R}^n$ be a compact, convex set with* $\dim K = k$*. Then for $\lambda \in [0,1]$*

$$\int\limits_{(1-\lambda)K+\lambda(-K)} f(\boldsymbol{x})\, \mathrm{d}\mathcal{H}^k(\boldsymbol{x}) \geq \int\limits_K f(\boldsymbol{x})\, \mathrm{d}\mathcal{H}^k(\boldsymbol{x}). \tag{3.11}$$

If the superlevel sets $\{f \geq c\}$ are closed for every $c \geq 0$, then equality holds if and only if for every $c > 0$

$$\mathrm{vol}_k\left([\lambda K + (1-\lambda)(-K)] \cap \{f \geq c\}\right) = \mathrm{vol}_k(K \cap \{f \geq c\}).$$

Proof. Let $K_\lambda = (1-\lambda)K + \lambda(-K)$. By the convexity of the superlevel sets of f we have for every $c \geq 0$

$$K_\lambda \cap \{f \geq c\} \supseteq (1-\lambda)\Big(K \cap \{f \geq c\}\Big) + \lambda\Big((-K) \cap \{f \geq c\}\Big). \tag{3.12}$$

The Brunn-Minkowski inequality (1.2) applied to the set on right hand side of (3.12) gives

$$\begin{aligned} &\mathrm{vol}_k(K_\lambda \cap \{f \geq c\}) \\ &\quad\geq \mathrm{vol}_k\left((1-\lambda)(K \cap \{f \geq c\}) + \lambda((-K) \cap \{f \geq c\})\right) \\ &\quad\geq \Big((1-\lambda)\,\mathrm{vol}_k(K \cap \{f \geq c\})^{\frac{1}{k}} + \lambda\,\mathrm{vol}_k((-K) \cap \{f \geq c\})^{\frac{1}{k}}\Big)^k. \end{aligned} \tag{3.13}$$

Since f is even, the superlevel sets $\{f \geq c\}$ are symmetric. Hence,

$$\mathrm{vol}_k(K \cap \{f \geq c\}) = \mathrm{vol}_k((-K) \cap \{f \geq c\})$$

and so (3.13) becomes

$$\mathrm{vol}_k(K_\lambda \cap \{f \geq c\}) \geq \mathrm{vol}_k(K \cap \{f \geq c\})$$

for every $c \geq 0$. Fubini's theorem yields

$$\begin{aligned}\int_{K_\lambda} f(\boldsymbol{x})\,\mathrm{d}\mathcal{H}^k(\boldsymbol{x}) &= \int_{K_\lambda}\left(\int_0^{f(\boldsymbol{x})} 1\,\mathrm{d}c\right)\mathrm{d}\mathcal{H}^k(\boldsymbol{x}) \\ &= \int_0^\infty \mathrm{vol}_k(K_\lambda \cap \{f \geq c\})\,\mathrm{d}c \\ &\geq \int_0^\infty \mathrm{vol}_k(K \cap \{f \geq c\})\,\mathrm{d}c \\ &= \int_K f(\boldsymbol{x})\,\mathrm{d}\mathcal{H}^k(\boldsymbol{x}).\end{aligned}$$

Suppose $\{f \geq c\}$ is closed for every $c > 0$, and we have equality in (3.11). Then $\mathrm{vol}_k(K_\lambda \cap \{f \geq c\})$ is lower semi-continuous with respect to c, and for $\lambda \in [0, 1]$ we find that

$$\mathrm{vol}_k(K_\lambda \cap \{f \geq c\}) = \mathrm{vol}_k(K \cap \{f \geq c\})$$

for every $c > 0$. □

Theorem 3.7 includes the result of Anderson [3] as a special case when K is the translate of a symmetric set. If f is chosen to be the constant function $f(\boldsymbol{x}) = 1$, $\boldsymbol{x} \in \mathbb{R}^n$, then (3.11) becomes the Brunn-Minkowski inequality (1.2) for $M = -K$. Another interpretation is as follows. For a quasiconcave function $f\colon \mathbb{R}^n \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ we denote by Ψ_f the measure with Hausdorff-density function f, i.e., $\mathrm{d}\Psi_f = f\,\mathrm{d}\mathcal{H}^n$. The inequality (3.11) for $k = n$ states that

$$\Psi_f((1-\lambda)K + \lambda(-K)) \geq \Psi_f(K)$$

for $K \in \mathcal{K}^n$ and $\lambda \in [0, 1]$. The natural extension reads

$$\Psi_f((1-\lambda)K + \lambda M) \geq \min\{\Psi_f(K), \Psi_f(M)\} \tag{3.14}$$

for any $K, M \in \mathcal{K}^n$, which is known as quasiconcavity of the measure Ψ_f. However, the restriction to pairs $(K, -K) \in (\mathcal{K}^n)^2$ of Theorem 3.7 is crucial. Counterexamples to (3.14) can be easily found also when f is an even function. As a matter of fact, the famous Borell-Brascamp-Lieb inequality asserts that (3.14) holds if $f^{-\frac{1}{n}}$ is a convex function (which is stronger than f being quasiconcave). For more information on how convexity of measures and density functions are related we refer to [9], [26] and [33, Sect. 15].

3.3 A Brunn–Minkowski type inequality for moments of the Euclidean norm

Establishing a Brunn-Minkowski type inequality like (3.11) for convex functions f is significantly more challenging. In this case the sublevel sets $\{f \leq c\}$ are convex, but adapting the proof of Theorem 3.7 is not possible. In fact, for any

$M \in \mathcal{K}_s^n$, and $\boldsymbol{t} \in \mathbb{R}^n$ such that $M \cap (M + \boldsymbol{t}) = \emptyset$ we may choose f to be 0 on M and 1 otherwise, and $K = M + \boldsymbol{t}$. Then f is an even convex function, but (3.11) is wrong for any $\lambda \in [0,1]$ sufficently close to $\frac{1}{2}$. Nevertheless, in the special case when f is a power of the Euclidean norm we prove the following result.

Theorem 3.8 (Henk, P. [42])**.** *Let $\alpha \geq 1$ and $K \in \mathcal{K}^n$ with $\dim K = k \geq 1$. Then for $\lambda \in [0,1]$ we have*

$$\int\limits_{(1-\lambda)K+\lambda(-K)} |\boldsymbol{x}|^\alpha \,\mathrm{d}\mathcal{H}^k(\boldsymbol{x}) \geq |2\lambda - 1|^\alpha \int\limits_K |\boldsymbol{x}|^\alpha \,\mathrm{d}\mathcal{H}^k(\boldsymbol{x}), \tag{3.15}$$

with equality for $\lambda \in (0,1)$ if and only if $\alpha = 1$, there exists a $\boldsymbol{u} \in \mathbb{S}^{n-1}$ such that $K \subset \operatorname{lin} \boldsymbol{u}$, and the origin is not in the relative interior of the segment $(1-\lambda)K + \lambda(-K)$.

Note, that when $\alpha = 0$, also the inequality (3.15) becomes the Brunn-Minkowski inequality (1.2) for K and $-K$. The rest of this section is devoted to the proof of Theorem 3.8, but actually we prove a slightly more general statement.

Theorem 3.9 (Henk, P. [42])**.** *Let $\alpha \geq 1$ and $K_0, K_1 \in \mathcal{K}^n$ such that $\dim K_0 = \dim K_1 = k \geq 1$, $\mathrm{vol}_k(K_0) = \mathrm{vol}_k(K_1)$ and their affine hulls are parallel. For $\lambda \in [0,1]$ let $K_\lambda = (1-\lambda)K_0 + \lambda K_1$. Then we have*

$$\begin{aligned} &\int\limits_{K_\lambda} |\boldsymbol{x}|^\alpha \,\mathrm{d}\mathcal{H}^k(\boldsymbol{x}) + \int\limits_{K_{1-\lambda}} |\boldsymbol{x}|^\alpha \,\mathrm{d}\mathcal{H}^k(\boldsymbol{x}) \\ &\qquad \geq |2\lambda - 1|^\alpha \left(\int\limits_{K_0} |\boldsymbol{x}|^\alpha \,\mathrm{d}\mathcal{H}^k(\boldsymbol{x}) + \int\limits_{K_1} |\boldsymbol{x}|^\alpha \,\mathrm{d}\mathcal{H}^k(\boldsymbol{x}) \right) \end{aligned} \tag{3.16}$$

with equality for $\lambda \in (0,1)$ if and only if $\alpha = 1$, there exists a $\boldsymbol{u} \in \mathbb{S}^{n-1}$ such that $K_0, K_1 \subset \operatorname{lin} \boldsymbol{u}$, and the hyperplane $H(\boldsymbol{u}, 0)$ separates K_λ and $K_{1-\lambda}$.

Now Theorem 3.8 follows by setting $K = K_0 = -K_1$. We start by recalling a variant of the well-known Karamata inequality which often appears in the context of Schur-convex functions.

Theorem 3.10 (Karamata's inequality, see, e.g., [51, Theorem 1])**.** *Let $D \subseteq \mathbb{R}$ be convex and let $\varphi \colon D \to \mathbb{R}$ be a non-decreasing, convex function. Let $x_1, \ldots, x_k$, $y_1, \ldots, y_k \in D$ such that*

(i) $x_1 \geq x_2 \geq \ldots \geq x_k$,

(ii) $y_1 \geq y_2 \geq \ldots \geq y_k$, *and*

(iii) $x_1 + x_2 + \ldots + x_i \geq y_1 + y_2 + \ldots + y_i$ *for all* $i = 1, \ldots, k$.

Then

$$\varphi(x_1) + \varphi(x_2) + \ldots + \varphi(x_k) \geq \varphi(y_1) + \varphi(y_2) + \ldots + \varphi(y_k). \tag{3.17}$$

If φ is strictly convex, then equality in (3.17) *holds if and only if $x_i = y_i$, $i = 1, \dots, k$.*

As a consequence we obtain a 0-dimensional version of Theorem 3.9, i.e., when the convex sets are replaced by singletons.

Lemma 3.11 (Henk, P. [42])**.** *Let $\alpha \geq 1$ and $z_0, z_1 \in \mathbb{R}$. Then for $\lambda \in [0,1]$ we have*

$$|(1-\lambda)z_0 + \lambda z_1|^\alpha + |\lambda z_0 + (1-\lambda)z_1|^\alpha \geq |2\lambda - 1|^\alpha \left(|z_0|^\alpha + |z_1|^\alpha \right) \tag{3.18}$$

with equality for $\lambda \in (0,1)$ if and only if either

(i) $z_1 = -z_0$*, or*

(ii) $\alpha = 1$*,* $z_0 z_1 < 0$ *and* $\max\{\lambda, 1-\lambda\} \geq \frac{\max\{|z_0|, |z_1|\}}{|z_0| + |z_1|}$.

Proof. By symmetry we may assume that $\lambda \leq \frac{1}{2}$, $|z_0| \geq |z_1|$ and $z_0 \geq 0$. Write

$$\begin{aligned} x_1 &= |(1-\lambda)z_0 + \lambda z_1|, \\ x_2 &= |\lambda z_0 + (1-\lambda)z_1|, \\ y_1 &= (1-2\lambda)z_0, \\ y_2 &= (1-2\lambda)|z_1|. \end{aligned}$$

We want to apply Karamata's inequality with $D = \mathbb{R}_{\geq 0}$ and $f(t) = t^\alpha$. We readily have

$$\begin{gathered} y_1 \geq y_2, \\ x_1^2 - x_2^2 = (1-2\lambda)(z_0^2 - z_1^2) \geq 0, \end{gathered}$$

which gives $x_1 \geq x_2$. Since $z_1 \geq -z_0$ we also have

$$(1-\lambda)z_0 + \lambda z_1 \geq \lambda z_0 + \lambda z_1 = \lambda(z_0 + z_1) \geq 0, \tag{3.19}$$

from which we find

$$x_1 = (1-\lambda)z_0 + \lambda z_1 \geq (1-\lambda)z_0 - \lambda z_0 = y_1.$$

It remains to show that $x_1 + x_2 \geq y_1 + y_2$. The triangle inequality gives

$$|((1-\lambda)z_0 + \lambda z_1) \pm (\lambda z_0 + (1-\lambda)z_1)| \leq x_1 + x_2.$$

Hence

$$\begin{aligned} x_1 + x_2 &\geq \max\{|z_0 + z_1|, |(1-2\lambda)(z_0 - z_1)|\} \\ &\geq (1-2\lambda)\max\{|z_0 + z_1|, |z_0 - z_1|\} \\ &= (1-2\lambda)(z_0 + |z_1|) \\ &= y_1 + y_2 \end{aligned} \tag{3.20}$$

and Karamata's inequality (3.17) yields $x_1^\alpha + x_2^\alpha \geq y_1^\alpha + y_2^\alpha$.

Suppose now we have equality in (3.18) and as before we assume $0 < \lambda \leq \frac{1}{2}$, $|z_0| \geq |z_1|$ and $z_0 \geq 0$. In view of our assumptions we have (3.19). First let

$\alpha > 1$. In this case the equality condition of Karamata's inequality asserts $x_1 = y_1$ and from (3.19) we conclude $\lambda z_1 = -\lambda z_0$. So for $\alpha > 1$ equality in (3.18) is equivalent to $z_1 = -z_0$.

Now suppose $\alpha = 1$, and $z_1 \neq -z_0$. In this case (3.18) becomes $x_1 + x_2 = y_1 + y_2$ and the equality in (3.20) yields

$$(1-\lambda)z_0 + \lambda z_1 + |\lambda z_0 + (1-\lambda)z_1| = x_1 + x_2 = |(1-2\lambda)(z_0 - z_1)|\,,$$

i.e., we have equality in the triangle inequality

$$|((1-\lambda)z_0 + \lambda z_1) + (-\lambda z_0 - (1-\lambda)z_1)| \leq x_1 + x_2.$$

Thus, by (3.19) it follows that $\lambda z_0 + (1-\lambda)z_1 \leq 0$. Since $z_0 \geq 0$ and $z_1 \neq -z_0$ we must have $z_1 < 0$. Moreover, since $z_0 \geq |z_1|$ this also shows $z_0 > 0$. By rearranging $\lambda z_0 + (1-\lambda)z_1 \leq 0$ the inequality becomes

$$1 - \lambda \geq \frac{z_0}{z_0 - z_1}.$$

In consideration of the made assumptions this is condition (ii). On the other hand, following the same arguments condition (ii) implies $x_1 + x_2 = y_1 + y_2$. □

At the present day there is a vast number of different proofs of the Brunn-Minkowski inequality (1.2). Here we want to emphasize the version of Kneser and Süss [52] who used an inductive argument and Fubini's theorem to apply a lower-dimensional Brunn-Minkowski inequality onto sections of convex bodies with hyperplanes (see [75, Thm. 7.1.1] for an English version). An inductive approach in the setting of Theorem 3.9 comes with the task of considering the restriction of the function $|\cdot|^\alpha$ to sections of convex bodies. Instead, the next lemma will allow us to replace spheres appearing as level sets of the norm function by hyperplanes. It appeared first in Alesker [1] and for more explicit versions of it we refer to [72, Lemma 2.1] and [4, (10.4.2)].

Lemma 3.12. *Let $\alpha > -1$. There is a constant $c = c(n,\alpha)$ such that for every $\boldsymbol{x} \in \mathbb{R}^n$*

$$|\boldsymbol{x}|^\alpha = c \cdot \int_{\mathbb{S}^{n-1}} |\langle \boldsymbol{\theta}, \boldsymbol{x}\rangle|^\alpha \,\mathrm{d}\boldsymbol{\theta}.$$

Consequently, in the subsequent lemma we will prove a version of Theorem 3.9 with the function $|\langle \boldsymbol{\theta}, \cdot\rangle|^\alpha$, $\boldsymbol{\theta} \in \mathbb{S}^{n-1}$ fixed, in place of $|\cdot|^\alpha$.

Lemma 3.13 (Henk, P. [42])**.** *Let $\alpha \geq 1$ and $K_0, K_1 \in \mathcal{K}^n$ such that $\dim K_0 = \dim K_1 = k \geq 1$, $\mathrm{vol}_k(K_0) = \mathrm{vol}_k(K_1)$ and their affine hulls are parallel. Let $\boldsymbol{\theta} \in \mathbb{S}^{n-1}$ and for $\lambda \in [0,1]$ let $K_\lambda = (1-\lambda)K_0 + \lambda K_1$. Then we have*

$$\int_{K_\lambda} |\langle \boldsymbol{\theta}, \boldsymbol{x}\rangle|^\alpha \,\mathrm{d}\mathcal{H}^k(\boldsymbol{x}) + \int_{K_{1-\lambda}} |\langle \boldsymbol{\theta}, \boldsymbol{x}\rangle|^\alpha \,\mathrm{d}\mathcal{H}^k(\boldsymbol{x})$$
$$\geq |2\lambda - 1|^\alpha \left(\int_{K_0} |\langle \boldsymbol{\theta}, \boldsymbol{x}\rangle|^\alpha \,\mathrm{d}\mathcal{H}^k(\boldsymbol{x}) + \int_{K_1} |\langle \boldsymbol{\theta}, \boldsymbol{x}\rangle|^\alpha \,\mathrm{d}\mathcal{H}^k(\boldsymbol{x}) \right). \quad (3.21)$$

Moreover, if K_0 and K_1 are not contained in affine spaces parallel to $\boldsymbol{\theta}^\perp$ and equality holds in (3.21) for $\lambda \in (0,1)$, then $\alpha = 1$, K_0 and K_1 are translates and the hyperplane $H(\boldsymbol{\theta}, 0)$ separates K_λ and $K_{1-\lambda}$.

Proof. Without loss of generality we assume that $\mathrm{vol}_k(K_0) = \mathrm{vol}_k(K_1) = 1$. The Brunn-Minkowski-inequality (1.2) gives in this setting for any $\lambda \in [0,1]$

$$\mathrm{vol}_k(K_\lambda) \geq 1. \tag{3.22}$$

For $t \in \mathbb{R}$ denote

$$\begin{aligned} H(\boldsymbol{\theta}, t) &= \{\boldsymbol{x} \in \mathbb{R}^n : \langle \boldsymbol{\theta}, \boldsymbol{x}\rangle = t\}, \\ H^-(\boldsymbol{\theta}, t) &= \{\boldsymbol{x} \in \mathbb{R}^n : \langle \boldsymbol{\theta}, \boldsymbol{x}\rangle \leq t\}. \end{aligned}$$

First suppose that K_0 lies in a hyperplane parallel to $\boldsymbol{\theta}^\perp$ (and therefore K_λ for $\lambda \in [0,1]$), i.e., $K_i \subset H(\boldsymbol{\theta}, t_i)$ for some $t_i \in \mathbb{R}$, $i \in \{0,1\}$. By (3.22) and Lemma 3.11 we find

$$\begin{aligned} &\int_{K_\lambda} |\langle \boldsymbol{\theta}, \boldsymbol{x}\rangle|^\alpha \,\mathrm{d}\mathcal{H}^k(\boldsymbol{x}) + \int_{K_{1-\lambda}} |\langle \boldsymbol{\theta}, \boldsymbol{x}\rangle|^\alpha \,\mathrm{d}\mathcal{H}^k(\boldsymbol{x}) \\ &\quad = \mathrm{vol}_k(K_\lambda)\, |(1-\lambda)t_0 + \lambda t_1|^\alpha + \mathrm{vol}_k(K_{1-\lambda})\, |\lambda t_0 + (1-\lambda)t_1|^\alpha \\ &\quad \geq |(1-\lambda)t_0 + \lambda t_1|^\alpha + |\lambda t_0 + (1-\lambda)t_1|^\alpha \\ &\quad \geq |2\lambda - 1|^\alpha \left(|t_0|^\alpha + |t_1|^\alpha\right) \\ &\quad = |2\lambda - 1|^\alpha \left(\mathrm{vol}_k(K_0)\, |t_0|^\alpha + \mathrm{vol}_k(K_1)\, |t_1|^\alpha\right) \\ &\quad = |2\lambda - 1|^\alpha \left(\int_{K_0} |\langle \boldsymbol{\theta}, \boldsymbol{x}\rangle|^\alpha \,\mathrm{d}\mathcal{H}^k(\boldsymbol{x}) + \int_{K_1} |\langle \boldsymbol{\theta}, \boldsymbol{x}\rangle|^\alpha \,\mathrm{d}\mathcal{H}^k(\boldsymbol{x}) \right). \end{aligned}$$

Thus, in the following we may assume that $K_0 \not\subseteq \boldsymbol{v} + \boldsymbol{\theta}^\perp$ for all $\boldsymbol{v} \in \mathbb{R}^n$ and hence, $-\mathrm{h}_{K_i}(-\boldsymbol{\theta}) < \mathrm{h}_{K_i}(\boldsymbol{\theta})$ for $i = 0,1$. For $t \in \mathbb{R}$ and for $i = 0,1$ we set

$$\begin{aligned} v_i(t) &= \mathrm{vol}_{k-1}(K_i \cap H(\boldsymbol{\theta}, t)), \\ w_i(t) &= \mathrm{vol}_k(K_i \cap H^-(\boldsymbol{\theta}, t)), \end{aligned}$$

so that $w_i(t) = \int_{-\infty}^t v_i(\zeta)\,\mathrm{d}\zeta$, $i \in \{0,1\}$. On $(-\mathrm{h}_{K_i}(-\boldsymbol{\theta}), \mathrm{h}_{K_i}(\boldsymbol{\theta}))$ the function v_i is continuous and hence w_i is differentiable. For $i \in \{0,1\}$ let z_i be the inverse function of w_i. Then z_i is differentiable with

$$z_i'(\tau) = \frac{1}{w_i'(z_i(\tau))} = \frac{1}{v_i(z_i(\tau))} \tag{3.23}$$

for $\tau \in (0,1)$. Write $z_\nu(\tau) = (1-\nu)z_0(\tau) + \nu z_1(\tau)$ for $\nu \in [0,1]$. By Fubini's theorem and a change of variables via $t = z_\nu(\tau)$ we find

$$\begin{aligned} \int_{K_\nu} |\langle \boldsymbol{\theta}, \boldsymbol{x}\rangle|^\alpha \,\mathrm{d}\mathcal{H}^k(\boldsymbol{x}) &= \int_{-\infty}^{\infty} |t|^\alpha\, \mathrm{vol}_{k-1}(K_\nu \cap H(\boldsymbol{\theta}, t))\,\mathrm{d}t \\ &= \int_0^1 |z_\nu(\tau)|^\alpha\, \mathrm{vol}_{k-1}\left(K_\nu \cap H(\boldsymbol{\theta}, z_\nu(\tau))\right) z_\nu'(\tau)\,\mathrm{d}\tau. \end{aligned} \tag{3.24}$$

Since $K_\nu \cap H(\boldsymbol{\theta}, z_\nu(\tau)) \supseteq (1-\nu)(K_0 \cap H(\boldsymbol{\theta}, z_0(\tau))) + \nu(K_1 \cap H(\boldsymbol{\theta}, z_1(\tau)))$ we may apply the Brunn-Minkowski inequality (1.2) to the latter set and together with (3.23) we get

$$\begin{aligned}
&\mathrm{vol}_{k-1}\Big(K_\nu \cap H(\boldsymbol{\theta}, z_\nu(\tau))\Big) z_\nu'(\tau) \\
&\quad\geq \mathrm{vol}_{k-1}\Big((1-\nu)\big(K_0 \cap H(\boldsymbol{\theta}, z_0(\tau))\big) + \nu\big(K_1 \cap H(\boldsymbol{\theta}, z_1(\tau))\big)\Big) z_\nu'(\tau) \\
&\quad\geq \Big[(1-\nu) v_0(z_0(\tau))^{\frac{1}{k-1}} + \nu v_1(z_1(\tau))^{\frac{1}{k-1}}\Big]^{k-1} \left[\frac{1-\nu}{v_0(z_0(\tau))} + \frac{\nu}{v_1(z_1(\tau))}\right] \\
&\quad\geq \Big[v_0(z_0(\tau))^{\frac{1-\nu}{k-1}} v_1(z_1(\tau))^{\frac{\nu}{k-1}}\Big]^{k-1} \Big[v_0(z_0(\tau))^{-(1-\nu)} v_1(z_1(\tau))^{-\nu}\Big] \\
&\quad= 1,
\end{aligned} \tag{3.25}$$

where for the last inequality we used the weighted arithmetic-geometric mean inequality. Therefore, (3.24) and (3.25) yield

$$\begin{aligned}
&\int_{K_\lambda} |\langle \boldsymbol{\theta}, \boldsymbol{x}\rangle|^\alpha \, \mathrm{d}\mathcal{H}^k(\boldsymbol{x}) + \int_{K_{1-\lambda}} |\langle \boldsymbol{\theta}, \boldsymbol{x}\rangle|^\alpha \, \mathrm{d}\mathcal{H}^k(\boldsymbol{x}) \\
&\quad\geq \int_0^1 |z_\lambda(\tau)|^\alpha + |z_{1-\lambda}(\tau)|^\alpha \, \mathrm{d}\tau \\
&\quad= \int_0^1 |(1-\lambda) z_0(\tau) + \lambda z_1(\tau)|^\alpha + |\lambda z_0(\tau) + (1-\lambda) z_1(\tau)|^\alpha \, \mathrm{d}\tau.
\end{aligned}$$

Next in order to estimate the integrand we use Lemma 3.11 and then we substitute back via (3.23)

$$\begin{aligned}
&\int_{K_\lambda} |\langle \boldsymbol{\theta}, \boldsymbol{x}\rangle|^\alpha \, \mathrm{d}\mathcal{H}^k(\boldsymbol{x}) + \int_{K_{1-\lambda}} |\langle \boldsymbol{\theta}, \boldsymbol{x}\rangle|^\alpha \, \mathrm{d}\mathcal{H}^k(\boldsymbol{x}) \\
&\quad\geq \int_0^1 |(1-\lambda) z_0(\tau) + \lambda z_1(\tau)|^\alpha + |\lambda z_0(\tau) + (1-\lambda) z_1(\tau)|^\alpha \, \mathrm{d}\tau \\
&\quad\geq \int_0^1 |2\lambda - 1|^\alpha \Big(|z_0(\tau)|^\alpha + |z_1(\tau)|^\alpha \Big) \, \mathrm{d}\tau \qquad (3.26) \\
&\quad= |2\lambda - 1|^\alpha \left(\int_0^1 |z_0(\tau)|^\alpha \, \mathrm{d}\tau + \int_0^1 |z_1(\tau)|^\alpha \, \mathrm{d}\tau \right) \\
&\quad= |2\lambda - 1|^\alpha \left(\int_0^1 |z_0(\tau)|^\alpha \, v_0(z_0(\tau)) \cdot z_0'(\tau) \, \mathrm{d}\tau \right. \\
&\qquad\qquad \left. + \int_0^1 |z_1(\tau)|^\alpha \, v_1(z_1(\tau)) \cdot z_1'(\tau) \, \mathrm{d}\tau \right)
\end{aligned}$$

$$
\begin{aligned}
&= |2\lambda-1|^\alpha \left(\int_{-\infty}^{\infty} |t|^\alpha \operatorname{vol}_{k-1}(K_0 \cap H(\boldsymbol{\theta},t))\,\mathrm{d}t \right. \\
&\qquad \left. + \int_{-\infty}^{\infty} |t|^\alpha \operatorname{vol}_{k-1}(K_1 \cap H(\boldsymbol{\theta},t))\,\mathrm{d}t \right) \\
&= |2\lambda-1|^\alpha \left(\int_{K_0} |\langle \boldsymbol{\theta}, \boldsymbol{x}\rangle|^\alpha \,\mathrm{d}\mathcal{H}^k(\boldsymbol{x}) + \int_{K_1} |\langle \boldsymbol{\theta}, \boldsymbol{x}\rangle|^\alpha \,\mathrm{d}\mathcal{H}^k(\boldsymbol{x}) \right).
\end{aligned}
$$

Now suppose that equality holds in (3.21) for two k-dimensional convex bodies K_0, K_1 of k-dimensional volume 1 with $K_i \not\subseteq \boldsymbol{v} + \boldsymbol{\theta}^\perp$ for any $\boldsymbol{v} \in \mathbb{R}^n$, $i \in \{0,1\}$. Without loss of generality we may assume that $0 < \lambda \leq \frac{1}{2}$. Then the equality in (3.21) implies equality in (3.25) for $\nu = \lambda$ and thus

$$
\begin{aligned}
\operatorname{vol}_k(K_\lambda) &= \int_0^1 \operatorname{vol}_{k-1}\Big(K_\lambda \cap H(\boldsymbol{\theta}, z_\lambda(\tau))\Big) z_\lambda'(\tau)\,\mathrm{d}\tau \\
&= 1 = \lambda \operatorname{vol}_k(K_0) + (1-\lambda)\operatorname{vol}_k(K_1).
\end{aligned}
$$

Hence, by the equality conditions of the Brunn-Minkowski inequality (1.2), K_0 and K_1 are homothets and we conclude $K_0 = \boldsymbol{v} + K_1$ for some $\boldsymbol{v} \in \mathbb{R}^n$. Thus for $\tau \in [0,1]$ and $s = \langle \boldsymbol{\theta}, \boldsymbol{v}\rangle$

$$
z_0(\tau) = z_1(\tau) + \langle \boldsymbol{\theta}, \boldsymbol{v}\rangle = z_1(\tau) + s. \tag{3.27}
$$

Since we must also have equality in (3.26) the equality conditions (i) and (ii) of Lemma 3.11 can be applied to $z_i(\tau)$, $i = 0,1$. If $\alpha > 1$, then Lemma 3.11 (i) implies $z_0(\tau) = -z_1(\tau)$ for $\tau \in [0,1]$. Together with (3.27), however, we get that $z_0(\tau) = \frac{s}{2}$ is constant with respect to τ. Then $\frac{\boldsymbol{v}}{2} + \boldsymbol{\theta}^\perp$ contains K_0, which contradicts the assumption.

Thus we must have $\alpha = 1$ and in this case we get from Lemma 3.11 (ii) and (3.27)

$$
0 > z_0(\tau) z_1(\tau) = z_0(\tau)(z_0(\tau) - s), \tag{3.28}
$$

$$
1 - \lambda = \max\{\lambda, 1-\lambda\} \geq \frac{\max\{|z_0(\tau)|, |z_0(\tau) - s|\}}{|z_0(\tau)| + |z_0(\tau) - s|} \tag{3.29}
$$

for every $\tau \in [0,1]$ except when $z_1(\tau) = -z_0(\tau)$. The above considerations show that there is at most one element in $\tau \in [0,1]$ satisfying the latter equation.

By interchanging the roles of $z_0(\tau)$ and $z_1(\tau)$, if necessary, we may additionally assume that $s \geq 0$. In particular, by (3.28) and since z_0 is continuous and monotone, it follows that $s > z_0(\tau) > 0$ for all $\tau \in (0,1)$. Then (3.29) becomes

$$
1 - \lambda \geq \frac{\max\{z_0(\tau), s - z_0(\tau)\}}{s}
$$

and we obtain

$$
z_\lambda(\tau) = (1-\lambda) z_0(\tau) + \lambda(z_0(\tau) - s) = z_0(\tau) - \lambda s \geq 0,
$$

$$z_{1-\lambda}(\tau) = \lambda z_0(\tau) + (1-\lambda)(z_0(\tau) - s) = z_0(\tau) - (1-\lambda)s \le 0$$

for all $\tau \in (0,1)$. By the continuity of z_0 we have $-\mathrm{h}_{K_\lambda}(-\boldsymbol{\theta}) \ge 0 \ge \mathrm{h}_{K_{1-\lambda}}(\boldsymbol{\theta})$, i.e., $H(\boldsymbol{\theta},0)$ separates K_λ and $K_{1-\lambda}$. □

Now we are ready to prove Theorem 3.9.

Proof of Theorem 3.9. Without loss of generality we assume that $\mathrm{vol}_k(K_0) = \mathrm{vol}_k(K_1) = 1$. In order to prove the desired inequality (3.16) we first substitute the integrand via Lemma 3.12 which leads, after an application of Fubini's theorem, to the equivalent inequality

$$\begin{aligned}
&\int\limits_{\mathbb{S}^{n-1}} \left(\int\limits_{K_\lambda} |\langle \boldsymbol{\theta}, \boldsymbol{x}\rangle|^\alpha \, \mathrm{d}\mathcal{H}^k(\boldsymbol{x}) + \int\limits_{K_{1-\lambda}} |\langle \boldsymbol{\theta}, \boldsymbol{x}\rangle|^\alpha \, \mathrm{d}\mathcal{H}^k(\boldsymbol{x}) \right) \mathrm{d}\boldsymbol{\theta} \\
&\quad \ge |2\lambda - 1|^\alpha \times \\
&\qquad \int\limits_{\mathbb{S}^{n-1}} \left(\int\limits_{K_0} |\langle \boldsymbol{\theta}, \boldsymbol{x}\rangle|^\alpha \, \mathrm{d}\mathcal{H}^k(\boldsymbol{x}) + \int\limits_{K_1} |\langle \boldsymbol{\theta}, \boldsymbol{x}\rangle|^\alpha \, \mathrm{d}\mathcal{H}^k(\boldsymbol{x}) \right) \mathrm{d}\boldsymbol{\theta}.
\end{aligned} \tag{3.30}$$

By Lemma 3.13 this inequality holds pointwise for every $\boldsymbol{\theta} \in \mathbb{S}^{n-1}$. Hence we have shown (3.30) and thus (3.16).

Now suppose that $\lambda \in (0,1)$ and equality holds in (3.16) for two k-dimensional convex bodies K_0, K_1 of k-dimensional volume 1. Then we also have equality in (3.30) for any $\boldsymbol{\theta} \in \mathbb{S}^{n-1}$. In particular, since K_0 contains at least two distinct points $\boldsymbol{x}, \boldsymbol{y} \in K_0$, we may choose $\boldsymbol{\theta} = \overline{\boldsymbol{x} - \boldsymbol{y}}$, so that K_0 is not contained in an affine space parallel to $\boldsymbol{\theta}^\perp$. The necessary conditions for equality in Lemma 3.13 assert that $\alpha = 1$, K_0 and K_1 are translates, and $H(\boldsymbol{\theta}, 0)$ separates K_λ and $K_{1-\lambda}$. On the other hand, if $\boldsymbol{x}$ and $\boldsymbol{y}$ are linearly independent, we find $\boldsymbol{v} \in \boldsymbol{x}^\perp \setminus \boldsymbol{y}^\perp$ and $\iota \in \mathbb{R}$ such that

$$\begin{aligned}
\langle \boldsymbol{x} + \iota \boldsymbol{v}, \boldsymbol{x} \rangle &= |\boldsymbol{x}|^2 > 0, \\
\langle \boldsymbol{x} + \iota \boldsymbol{v}, \boldsymbol{y} \rangle &= \langle \boldsymbol{x}, \boldsymbol{y} \rangle + \iota \langle \boldsymbol{v}, \boldsymbol{y} \rangle < 0,
\end{aligned}$$

i.e., K_0 has a non-empty intersection with $H(\boldsymbol{x} + \iota \boldsymbol{v}, 0)$. Setting $\boldsymbol{\theta} = \overline{\boldsymbol{x} + \iota \boldsymbol{v}}$ then contradicts Lemma 3.13 as K_λ and $K_{1-\lambda}$ cannot be separated by $H(\boldsymbol{\theta}, 0)$. Thus every two points in K_0 are linearly dependent, which means that $K \subseteq \mathrm{lin}\, \boldsymbol{u}$ for some $\boldsymbol{u} \in \mathbb{S}^{n-1}$. Following the same line of reasoning for K_1 leads to the same conclusion and since the affine hulls of K_0 and K_1 are parallel we also have $K_1 \subseteq \mathrm{lin}\, \boldsymbol{u}$. In particular, the function $\langle \boldsymbol{u}, \cdot \rangle$ is not constant on K_0 or K_1 and by Lemma 3.13 the hyperplane $H(\boldsymbol{u}, 0)$ separates K_λ and $K_{1-\lambda}$.

It remains to show that this separating property also implies equality in (3.16). If there exists such a $\boldsymbol{u}$, then with an appropriate choice of coordinates we may assume that $K_0, K_1 \subseteq \mathbb{R}$, i.e., $K_0 = [\xi_0, \xi_0 + 1]$ and $K_1 = [-(\xi_1 + 1), -\xi_1]$ for some $\xi_0, \xi_1 \ge 0$. By symmetry we may assume $\lambda \le \frac{1}{2}$. The assumption that

$$K_\lambda = [(1-\lambda)\xi_0 - \lambda(\xi_1 + 1), (1-\lambda)(\xi_0 + 1) - \lambda \xi_1]$$

and

$$K_{1-\lambda} = [\lambda\xi_0 - (1-\lambda)(\xi_1+1), \lambda(\xi_0+1) - (1-\lambda)\xi_1]$$

are separated by the origin, means $\lambda(\xi_0+1)-(1-\lambda)\xi_1 \leq 0 \leq (1-\lambda)\xi_0-\lambda(\xi_1+1)$ and so

$$\begin{aligned}
&\int\limits_{K_\lambda} |\boldsymbol{x}| \,\mathrm{d}\boldsymbol{x} + \int\limits_{K_{1-\lambda}} |\boldsymbol{x}| \,\mathrm{d}\boldsymbol{x} \\
&\qquad = \int\limits_{(1-\lambda)\xi_0-\lambda(\xi_1+1)}^{(1-\lambda)(\xi_0+1)-\lambda\xi_1} t \,\mathrm{d}t + \int\limits_{\lambda\xi_0-(1-\lambda)(\xi_1+1)}^{\lambda(\xi_0+1)-(1-\lambda)\xi_1} -t \,\mathrm{d}t \\
&\qquad = \frac{1}{2}\Big[\Big((1-\lambda)(\xi_0+1)-\lambda\xi_1\Big)^2 - \Big((1-\lambda)\xi_0-\lambda(\xi_1+1)\Big)^2 \\
&\qquad\qquad - \Big(\lambda(\xi_0+1)-(1-\lambda)\xi_1\Big)^2 + \Big(\lambda\xi_0-(1-\lambda)(\xi_1+1)\Big)^2\Big] \\
&\qquad = (2\lambda-1)(\xi_0+\xi_1+1) \\
&\qquad = (2\lambda-1)\cdot\frac{1}{2}\Big[(\xi_0+1)^2-\xi_0^2+(\xi_1+1)^2-\xi_1^2\Big] \\
&\qquad = (2\lambda-1)\left(\int\limits_{\xi_0}^{\xi_0+1} t\,\mathrm{d}t + \int\limits_{\xi_1}^{\xi_1+1} t\,\mathrm{d}t\right) \\
&\qquad = (2\lambda-1)\left(\int\limits_{K_0} |\boldsymbol{x}|\,\mathrm{d}\boldsymbol{x} + \int\limits_{K_1} |\boldsymbol{x}|\,\mathrm{d}\boldsymbol{x}\right). \qquad \square
\end{aligned}$$

We remark that the factor $|2\lambda-1|^\alpha$ in the Theorems 3.8 and 3.9 cannot be replaced by a smaller one. This can be seen from the following example. Let $C \in \mathcal{K}_s^n$, $\boldsymbol{u} \in \mathbb{S}^{n-1}$, and for every $r \in \mathbb{R}_{>0}$ set $K_0^{(r)} = C + r\cdot\boldsymbol{u}$ and $K_1^{(r)} = -K_0^{(r)}$. Then, for every $\nu \in [0,1]$ we have $K_\nu^{(r)} = C - (2\nu-1)r\boldsymbol{u}$ and

$$\int\limits_{K_\nu^{(r)}} |\boldsymbol{x}|^\alpha \,\mathrm{d}\boldsymbol{x} = r^\alpha \int\limits_C \left|r^{-1}\boldsymbol{x} - (2\nu-1)\boldsymbol{u}\right|^\alpha \,\mathrm{d}\boldsymbol{x}.$$

Moreover, the symmetry of C gives $K_{1-\nu}^{(r)} = -K_\nu^{(r)}$. Therefore and since

$$\left|r^{-1}\boldsymbol{x} - (2\nu-1)\boldsymbol{u}\right|^\alpha \leq (|\boldsymbol{x}| + |(2\nu-1)\boldsymbol{u}|)^\alpha$$

for $r \geq 1$, for a given $\lambda \in [0,1]$, we conclude by the dominated convergence theorem

$$\lim_{r\to\infty} \frac{\int_{K_{1-\lambda}^{(r)}} |\boldsymbol{x}|^\alpha \,\mathrm{d}\boldsymbol{x}}{\int_{K_1^{(r)}} |\boldsymbol{x}|^\alpha \,\mathrm{d}\boldsymbol{x}} = \lim_{r\to\infty} \frac{\int_{K_\lambda^{(r)}} |\boldsymbol{x}|^\alpha \,\mathrm{d}\boldsymbol{x}}{\int_{K_0^{(r)}} |\boldsymbol{x}|^\alpha \,\mathrm{d}\boldsymbol{x}} = |2\lambda-1|^\alpha.$$

Although (3.15) also holds for $\alpha = 0$ we point out that an extension of Theorem 3.8 to the intermediate cases $\alpha \in (0,1)$, where $|\cdot|^\alpha$ is a non-convex function, is not possible.

Proposition 3.14 (Henk, P. [42])**.** *Let $\alpha \in (0,1)$. There is a closed interval $K \subseteq \mathbb{R}$ and $\lambda \in [0,1]$ such that for $K_\lambda = (1-\lambda)K + \lambda(-K)$ we have*

$$\int_{K_\lambda} |x|^\alpha \,\mathrm{d}x < |2\lambda - 1|^\alpha \int_K |x|^\alpha \,\mathrm{d}x.$$

Proof. Let $\varepsilon > 0$, $K = [\varepsilon, 1+\varepsilon]$ and $\lambda = \frac{\varepsilon}{1+2\varepsilon} < \frac{1}{2}$ so that $K_\lambda = [0,1]$. Then

$$(1+\alpha)\int_{K_\lambda} |x|^\alpha \,\mathrm{d}x = 1$$

and

$$\begin{aligned}(1+\alpha)(1-2\lambda)^\alpha \int_K |x|^\alpha \,\mathrm{d}x &= (1+\alpha)(1+2\varepsilon)^{-\alpha}\int_\varepsilon^{1+\varepsilon} x^\alpha \,\mathrm{d}x \\ &= \frac{(1+\varepsilon)^{1+\alpha} - \varepsilon^{1+\alpha}}{(1+2\varepsilon)^\alpha}.\end{aligned}$$

Let $f(t) = (1+t)^{1+\alpha} - t^{1+\alpha}$ and $g(t) = (1+2t)^\alpha$. Since

$$\begin{aligned}f(0) &= 1 = g(0),\\ f'(0) &= (1+\alpha) > 2\alpha = g'(0),\end{aligned}$$

we have

$$\frac{(1+\varepsilon)^{1+\alpha} - \varepsilon^{1+\alpha}}{(1+2\varepsilon)^\alpha} > 1$$

for small ε and hence,

$$\int_{K_\lambda} |x|^\alpha \,\mathrm{d}x < (1-2\lambda)^\alpha \int_K |x|^\alpha \,\mathrm{d}x. \qquad \square$$

3.4 Subspace concentration in the even dual Minkowski problem

In this section we will prove bounds on the subspace concentration of dual curvature measures of symmetric convex bodies. In the following lemma we evaluate an integral that repeatedly appears in the proofs of Theorems 3.2 and 3.5.

Lemma 3.15. *Let $q > 0$, $K \in \mathcal{K}_s^n$, $Q \in \mathcal{S}_o^n$ and L be a k-dimensional subspace of $\mathbb{R}^n$, where $1 \le k \le n-1$. Then for any $s < k$*

$$\int_{K|L} \left(\rho_{K|L}(\boldsymbol{y})^s \int_{K_{\bar{\boldsymbol{p}}_{K|L}(\boldsymbol{y})}} \rho_Q(\boldsymbol{z})^{n-q} \,\mathrm{d}\boldsymbol{z} \right) \mathrm{d}\boldsymbol{y} = \frac{c}{k-s}, \tag{3.31}$$

where for $\boldsymbol{v} \in K|L$ we write $K_{\boldsymbol{v}} = K \cap (\boldsymbol{v} + L^\perp)$ and $c = c(q, K, Q, L)$ is a constant independent of s.

Proof. Since $\widetilde{\mathbf{p}}_{K|L}(\boldsymbol{y})$ is homogeneous of degree 0 in $\boldsymbol{y}$ we have $\widetilde{\mathbf{p}}_{K|L}(\overline{\boldsymbol{y}}) = \widetilde{\mathbf{p}}_{K|L}(\boldsymbol{y})$ for every $\boldsymbol{y} \in K|L$, $\boldsymbol{y} \neq \mathbf{0}$. Using spherical coordinates in $K|L$ the left hand side of (3.31) becomes

$$\int\limits_{K|L} \left(\rho_{K|L}(\boldsymbol{y})^s \int\limits_{K_{\tilde{\mathbf{p}}_{K|L}(\boldsymbol{y})}} \rho_Q(\boldsymbol{z})^{n-q} \, \mathrm{d}\boldsymbol{z} \right) \mathrm{d}\boldsymbol{y}$$
$$= \int\limits_{\mathbb{S}^{n-1}\cap L} \int\limits_0^{\rho_{K|L}(\boldsymbol{u})} \rho_{K|L}(r\boldsymbol{u})^s g(\boldsymbol{u}) r^{k-1} \, \mathrm{d}r \, \mathrm{d}\boldsymbol{u},$$

where $g(\boldsymbol{u}) = \int_{K_{\tilde{\mathbf{p}}_{K|L}(\boldsymbol{u})}} \rho_Q(\boldsymbol{z})^{n-q} \, \mathrm{d}\boldsymbol{z}$. Homogeneity of the radial function ρ_Q and integration with respect to r yields

$$\int\limits_{K|L} \left(\rho_{K|L}(\boldsymbol{y})^s \int\limits_{K_{\tilde{\mathbf{p}}_{K|L}(\boldsymbol{y})}} \rho_Q(\boldsymbol{z})^{n-q} \, \mathrm{d}\boldsymbol{z} \right) \mathrm{d}\boldsymbol{y}$$
$$= \int\limits_{\mathbb{S}^{n-1}\cap L} \rho_{K|L}(\boldsymbol{u})^s g(\boldsymbol{u}) \int\limits_0^{\rho_{K|L}(\boldsymbol{u})} r^{k-s-1} \, \mathrm{d}r \, \mathrm{d}\boldsymbol{u}$$
$$= \int\limits_{\mathbb{S}^{n-1}\cap L} \rho_{K|L}(\boldsymbol{u})^s g(\boldsymbol{u}) \frac{\rho_{K|L}(\boldsymbol{u})^{k-s}}{k-s} \, \mathrm{d}\boldsymbol{u}$$
$$= \frac{1}{k-s} \int\limits_{\mathbb{S}^{n-1}\cap L} \rho_{K|L}(\boldsymbol{u})^k g(\boldsymbol{u}) \, \mathrm{d}\boldsymbol{u},$$

where we used that the involved integrals are finite since $s < k$. $\square$

Now we are ready to prove the Theorems 3.2 and 3.5. We use Fubini's theorem to decompose the dual curvature measure of $K \in \mathcal{K}_s^n$ into integrals over sections with affine planes orthogonal to the given subspace L. These sections contain Minkowski combinations of parallel sections which lie in ∂K (see Fig. 3.6), and by applying a Brunn-Minkowski type inequality we may compare these integrals with the corresponding integrals over sections associated to the dual curvature measure of the Borel set $\mathbb{S}^{n-1} \cap L$. We start with the case $q \in (0, n)$ where we prove a slightly more general statement.

Theorem 3.16 ([14, cf. Rem. 4.1])**.** *Let $q \in (0,n)$ and $K, Q \in \mathcal{K}_s^n$. Then*

$$\int\limits_{\boldsymbol{\alpha}_K^*(\mathbb{S}^{n-1}\cap L)} \rho_Q(\boldsymbol{u})^{n-q} \rho_K(\boldsymbol{u})^q \, \mathrm{d}\boldsymbol{u} < \frac{\dim L}{q} \int\limits_{\mathbb{S}^{n-1}} \rho_Q(\boldsymbol{u})^{n-q} \rho_K(\boldsymbol{u})^q \, \mathrm{d}\boldsymbol{u} \tag{3.32}$$

for every proper subspace L of $\mathbb{R}^n$.

Proof. Let $\dim L = k \in \{1, \ldots, n-1\}$. For $\boldsymbol{y} \in K|L$ and $\boldsymbol{u} \in \mathbb{S}^{n-1} \cap L$ we

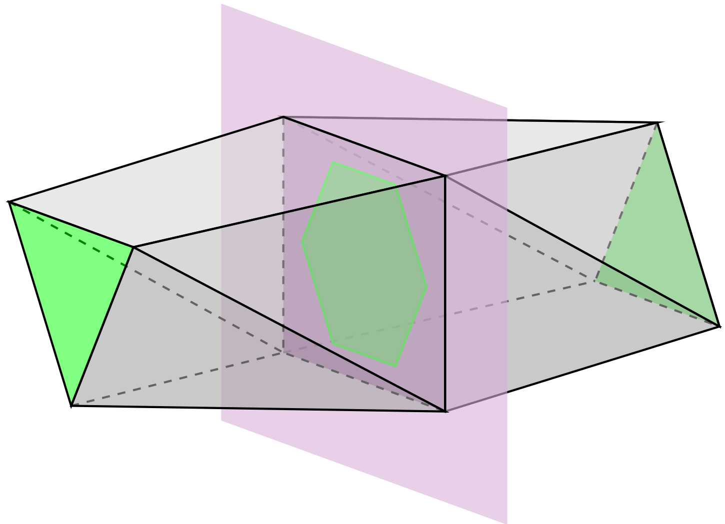

Figure 3.6: Sections of convex bodies contain suitable Minkowski combinations of parallel sections

denote

$$\begin{aligned} K_{\boldsymbol{y}} &= K \cap (\boldsymbol{y} + L^{\perp}), \\ M_{\boldsymbol{u}} &= \operatorname{conv}\left\{K_{\mathbf{0}}, K_{\tilde{\mathbf{p}}_{K|L}(\boldsymbol{u})}\right\}. \end{aligned}$$

By (3.5), Lemma 3.6, Fubini's theorem and the fact that $M_{\overline{\boldsymbol{y}}} \cap (\boldsymbol{y} + L^{\perp}) \subseteq K_{\boldsymbol{y}}$ we may write

$$\begin{aligned} \int_{\mathbb{S}^{n-1}} \rho_Q(\boldsymbol{u})^{n-q} \rho_K(\boldsymbol{u})^q \,\mathrm{d}\boldsymbol{u} &= q \int_K \rho_Q(\boldsymbol{x})^{n-q} \,\mathrm{d}\boldsymbol{x} \\ &= q \int_{K|L} \left(\int_{K_{\boldsymbol{y}}} \rho_Q(\boldsymbol{z})^{n-q} \,\mathrm{d}\boldsymbol{z} \right) \mathrm{d}\boldsymbol{y} \\ &\geq q \int_{K|L} \left(\int_{M_{\overline{\boldsymbol{y}}} \cap (\boldsymbol{y}+L^{\perp})} \rho_Q(\boldsymbol{z})^{n-q} \,\mathrm{d}\boldsymbol{z} \right) \mathrm{d}\boldsymbol{y}. \end{aligned} \tag{3.33}$$

In order to estimate the inner integral we fix a $\boldsymbol{y} \in K|L$, $\boldsymbol{y} \neq \mathbf{0}$, and for abbreviation we set $\tau = \rho_{K|L}(\boldsymbol{y})^{-1} \leq 1$. Then, by the symmetry of K we find

$$\begin{aligned} M_{\overline{\boldsymbol{y}}} \cap (\boldsymbol{y} + L^{\perp}) &\supseteq \tau K_{\tilde{\mathbf{p}}_{K|L}(\boldsymbol{y})} + (1-\tau) K_{\mathbf{0}} \\ &\supseteq \tau K_{\tilde{\mathbf{p}}_{K|L}(\boldsymbol{y})} + (1-\tau) \left(\frac{1}{2} K_{\tilde{\mathbf{p}}_{K|L}(\boldsymbol{y})} + \frac{1}{2} (-K_{\tilde{\mathbf{p}}_{K|L}(\boldsymbol{y})}) \right) \\ &= \frac{1+\tau}{2} K_{\tilde{\mathbf{p}}_{K|L}(\boldsymbol{y})} + \frac{1-\tau}{2} (-K_{\tilde{\mathbf{p}}_{K|L}(\boldsymbol{y})}). \end{aligned}$$

Hence, $M_{\overline{\boldsymbol{y}}} \cap (\boldsymbol{y} + L^{\perp})$ contains a convex combination of a set and its reflection at the origin. Note that the superlevel sets of $\rho_Q(\cdot)^{n-q}$ are dilates of Q. This allows us to apply Theorem 3.7 from which we obtain

$$\int_{M_{\overline{\boldsymbol{y}}}\cap(\boldsymbol{y}+L^\perp)} \rho_Q(\boldsymbol{z})^{n-q}\,\mathrm{d}\boldsymbol{z} \geq \int_{K_{\tilde{\mathbf{p}}_{K|L}(\boldsymbol{y})}} \rho_Q(\boldsymbol{z})^{n-q}\,\mathrm{d}\boldsymbol{z} \tag{3.34}$$

for every $\boldsymbol{y} \in K|L$, $\boldsymbol{y} \neq \mathbf{0}$. Together with (3.33) we find

$$\int_{\mathbb{S}^{n-1}} \rho_Q(\boldsymbol{u})^{n-q}\rho_K(\boldsymbol{u})^q\,\mathrm{d}\boldsymbol{u} \geq q\int_{K|L}\left(\int_{K_{\tilde{\mathbf{p}}_{K|L}(\boldsymbol{y})}} \rho_Q(\boldsymbol{z})^{n-q}\,\mathrm{d}\boldsymbol{z}\right)\mathrm{d}\boldsymbol{y}.$$

By using Lemma 3.15 with $s = 0$ we then obtain the lower bound

$$\int_{\mathbb{S}^{n-1}} \rho_Q(\boldsymbol{u})^{n-q}\rho_K(\boldsymbol{u})^q\,\mathrm{d}\boldsymbol{u} \geq \frac{qc}{k}, \tag{3.35}$$

where c is a constant independent of s.

In order to evaluate the left hand side of (3.32) we recall that for $\boldsymbol{x} \in K$ we have $\overline{\boldsymbol{x}} \in \boldsymbol{\alpha}_K^*(\mathbb{S}^{n-1}\cap L)$ if and only if the boundary point $\widetilde{\mathbf{p}}_K(\boldsymbol{x})$ has a unit outer normal in L. Hence,

$$\begin{aligned} &K \cap \left(\mathbb{R}_{\geq 0}\cdot\boldsymbol{\alpha}_K^*(\mathbb{S}^{n-1}\cap L)\right)\\ &\qquad = \bigcup_{\boldsymbol{v}\in\partial(K|L)} \operatorname{conv}(\{\mathbf{0}\}\cup K_{\boldsymbol{v}}) \end{aligned}$$

and in view of (3.5), Lemma 3.6 and Fubini's theorem we obtain

$$\begin{aligned} &\int_{\boldsymbol{\alpha}_K^*(\mathbb{S}^{n-1}\cap L)} \rho_Q(\boldsymbol{u})^{n-q}\rho_K(\boldsymbol{u})^q\,\mathrm{d}\boldsymbol{u}\\ &\qquad = q\int_{K\cap\left(\mathbb{R}_{\geq 0}\cdot\boldsymbol{\alpha}_K^*(\mathbb{S}^{n-1}\cap L)\right)} \rho_Q(\boldsymbol{x})^{n-q}\,\mathrm{d}\boldsymbol{x}\\ &\qquad = q\int_{K|L}\left(\int_{\operatorname{conv}\left(\{\mathbf{0}\}\cup K_{\tilde{\mathbf{p}}_{K|L}(\boldsymbol{y})}\right)\cap(\boldsymbol{y}+L^\perp)} \rho_Q(\boldsymbol{z})^{n-q}\,\mathrm{d}\boldsymbol{z}\right)\mathrm{d}\boldsymbol{y}\\ &\qquad = q\int_{K|L}\left(\int_{\rho_{K|L}(\boldsymbol{y})^{-1}K_{\tilde{\mathbf{p}}_{K|L}(\boldsymbol{y})}} \rho_Q(\boldsymbol{z})^{n-q}\,\mathrm{d}\boldsymbol{z}\right)\mathrm{d}\boldsymbol{y}\\ &\qquad = q\int_{K|L}\rho_{K|L}(\boldsymbol{y})^{k-q}\left(\int_{K_{\tilde{\mathbf{p}}_{K|L}(\boldsymbol{y})}} \rho_Q(\boldsymbol{z})^{n-q}\,\mathrm{d}\boldsymbol{z}\right)\mathrm{d}\boldsymbol{y}. \end{aligned}$$

Using Lemma 3.15 again with $s = k - q$ it follows from (3.35) that

$$\int_{\boldsymbol{\alpha}_K^*(\mathbb{S}^{n-1}\cap L)} \rho_Q(\boldsymbol{u})^{n-q}\rho_K(\boldsymbol{u})^q\,\mathrm{d}\boldsymbol{u} = c \leq \frac{k}{q}\int_{\mathbb{S}^{n-1}} \rho_Q(\boldsymbol{u})^{n-q}\rho_K(\boldsymbol{u})^q\,\mathrm{d}\boldsymbol{u}.$$

It remains to show that the inequality is strict. To this end, suppose that we have equality in (3.32) for a proper subspace L. Then we must have equality in (3.34) for almost every $\boldsymbol{y}$ in the relative interior of $K|L$. Hence, since $q < n$ and in view of the equality condition of Theorem 3.7, the equality in (3.34) implies that $M_{\overline{\boldsymbol{y}}} \cap (\boldsymbol{y} + L^{\perp}) \cap rQ$ and $K_{\tilde{\mathbf{p}}_{K|L}(\boldsymbol{y})} \cap rQ$ have the same $(n-k)$-dimensional volume for almost every $r > 0$. Moreover, the equality in (3.33) shows $K_{\boldsymbol{y}} = M_{\overline{\boldsymbol{y}}} \cap (\boldsymbol{y} + L^{\perp})$ and therefore

$$\mathrm{vol}_{n-k}(K_{\boldsymbol{y}} \cap rQ) = \mathrm{vol}_{n-k}(K_{\tilde{\mathbf{p}}_{K|L}(\boldsymbol{y})} \cap rQ)$$

for almost every $r > 0$. Let $\boldsymbol{\xi}$ be a point maximizing ρ_Q over $K_{\tilde{\mathbf{p}}_{K|L}(\boldsymbol{y})}$. In particular, $\rho_{K|L}(\boldsymbol{y})^{-1}\boldsymbol{\xi} \in K_{\boldsymbol{y}}$. Since $\boldsymbol{y}$ is a relative interior point of $K|L$, we have $\rho_{K|L}(\boldsymbol{y}) > 1$ and we may choose $r > 0$ such that

$$\rho_Q(\boldsymbol{\xi}) < r^{-1} < \rho_{K|L}(\boldsymbol{y})\rho_Q(\boldsymbol{\xi}) = \rho_Q\left(\rho_{K|L}(\boldsymbol{y})^{-1}\boldsymbol{\xi}\right).$$

Since $\boldsymbol{\xi}$ maximizes ρ_Q on $K_{\tilde{\mathbf{p}}_{K|L}(\boldsymbol{y})}$ the former inequality gives $K_{\tilde{\mathbf{p}}_{K|L}(\boldsymbol{y})} \cap rQ = \emptyset$, whereas the latter shows

$$\mathrm{vol}_{n-k}(K_{\boldsymbol{y}} \cap rQ) > 0. \qquad \square$$

In view of (3.5), Theorem 3.2 is an immediate consequence of Theorem 3.16. The proof of Theorem 3.5 is carried out analogously where instead of (3.11) we use (3.15).

Proof of Theorem 3.5. Let $\dim L = k \in \{1, \ldots, n-1\}$. For $\boldsymbol{y} \in K|L$ and $\boldsymbol{u} \in \mathbb{S}^{n-1} \cap L$ we denote

$$\begin{aligned} K_{\boldsymbol{y}} &= K \cap (\boldsymbol{y} + L^{\perp}), \\ M_{\boldsymbol{u}} &= \mathrm{conv}\left\{K_{\mathbf{0}}, K_{\tilde{\mathbf{p}}_{K|L}(\boldsymbol{u})}\right\}. \end{aligned}$$

By (3.5), Lemma 3.6, Fubini's theorem and the fact that $M_{\overline{\boldsymbol{y}}} \cap (\boldsymbol{y} + L^{\perp}) \subseteq K_{\boldsymbol{y}}$ we may write (cf. (3.33))

$$\begin{aligned} \widetilde{\mathrm{C}}_q(K, \mathbb{S}^{n-1}) &= \frac{1}{n} \int_{\mathbb{S}^{n-1}} \rho_K(\boldsymbol{u})^q \,\mathrm{d}\boldsymbol{u} \\ &\geq \frac{q}{n} \int_{K|L} \left(\int_{M_{\overline{\boldsymbol{y}}} \cap (\boldsymbol{y} + L^{\perp})} |\boldsymbol{z}|^{q-n} \,\mathrm{d}\boldsymbol{z} \right) \mathrm{d}\boldsymbol{y}. \end{aligned} \tag{3.36}$$

In order to estimate the inner integral we fix a $\boldsymbol{y} \in K|L$, $\boldsymbol{y} \neq \mathbf{0}$, and for abbreviation we set $\tau = \rho_{K|L}(\boldsymbol{y})^{-1} \leq 1$. Then, by the symmetry of K we find

$$\begin{aligned} M_{\overline{\boldsymbol{y}}} \cap (\boldsymbol{y} + L^{\perp}) &\supseteq \tau(K_{\tilde{\mathbf{p}}_{K|L}(\boldsymbol{y})}) + (1-\tau)(K_{\mathbf{0}}) \\ &\supseteq \tau(K_{\tilde{\mathbf{p}}_{K|L}(\boldsymbol{y})}) + (1-\tau)\left(\frac{1}{2}K_{\tilde{\mathbf{p}}_{K|L}(\boldsymbol{y})} + \frac{1}{2}(-K_{\tilde{\mathbf{p}}_{K|L}(\boldsymbol{y})})\right) \\ &= \frac{1+\tau}{2}K_{\tilde{\mathbf{p}}_{K|L}(\boldsymbol{y})} + \frac{1-\tau}{2}(-K_{\tilde{\mathbf{p}}_{K|L}(\boldsymbol{y})}). \end{aligned}$$

Hence, $M_{\overline{\boldsymbol{y}}} \cap (\boldsymbol{y} + L^\perp)$ contains a convex combination of a set and its reflection at the origin. This allows us to apply Theorem 3.8 from which we obtain

$$\int\limits_{M_{\overline{\boldsymbol{y}}} \cap (\boldsymbol{y}+L^\perp)} |\boldsymbol{z}|^{q-n} \,\mathrm{d}\boldsymbol{z} \geq \tau^{q-n} \int\limits_{K_{\tilde{\mathbf{p}}_{K|L}(\boldsymbol{y})}} |\boldsymbol{z}|^{q-n} \,\mathrm{d}\boldsymbol{z}$$

for every $\boldsymbol{y} \in K|L$, $\boldsymbol{y} \neq \mathbf{0}$. By the equality characterization of Theorem 3.8 the last inequality is strict whenever $\tau < 1$, i.e., $\boldsymbol{y}$ belongs to the relative interior of $K|L$. Together with (3.36) we find

$$\widetilde{\mathrm{C}}_q(K, \mathbb{S}^{n-1}) > \frac{q}{n} \int\limits_{K|L} \rho_{K|L}(\boldsymbol{y})^{n-q} \left(\int\limits_{K_{\tilde{\mathbf{p}}_{K|L}(\boldsymbol{y})}} |\boldsymbol{z}|^{q-n} \,\mathrm{d}\boldsymbol{z} \right) \mathrm{d}\boldsymbol{y}.$$

By using Lemma 3.15 with $Q = B_n$ and $s = n - q$ we then obtain the lower bound

$$\widetilde{\mathrm{C}}_q(K, \mathbb{S}^{n-1}) > \frac{q}{n} \frac{c}{k+q-n},$$

where c is a constant independent of s.

On the other hand, in the same manner as in the proof of Theorem 3.16 we find

$$\begin{aligned}
\widetilde{\mathrm{C}}_q(K, \mathbb{S}^{n-1} \cap L) &= \frac{1}{n} \int\limits_{\boldsymbol{\alpha}^*_K(\mathbb{S}^{n-1} \cap L)} \rho_K(\boldsymbol{u})^q \,\mathrm{d}\boldsymbol{u} \\
&= \frac{q}{n} \int\limits_{K|L} \rho_{K|L}(\boldsymbol{y})^{k-q} \left(\int\limits_{K_{\tilde{\mathbf{p}}_{K|L}(\boldsymbol{y})}} |\boldsymbol{z}|^{q-n} \,\mathrm{d}\boldsymbol{z} \right) \mathrm{d}\boldsymbol{y}.
\end{aligned}$$

Inequality (3.35) and Lemma 3.15, again, with $Q = B_n$ and $s = k - q$ yield

$$\widetilde{\mathrm{C}}_q(K, \mathbb{S}^{n-1} \cap L) = \frac{q}{n} \frac{c}{q} < \frac{k+q-n}{q} \widetilde{\mathrm{C}}_q(K, \mathbb{S}^{n-1}). \qquad \square$$

Next we show that the bounds given in Theorem 3.5 are tight for every choice of $q \geq n+1$.

Proposition 3.17 (Henk, P. [42])**.** *Let $q > n$ and $k \in \{1, \ldots, n-1\}$. There exists a sequence of convex bodies $K_l \in \mathcal{K}^n_s$, $l \in \mathbb{N}$, and a k-dimensional subspace $L \subseteq \mathbb{R}^n$ such that*

$$\lim_{l \to \infty} \frac{\widetilde{\mathrm{C}}_q(K_l, \mathbb{S}^{n-1} \cap L)}{\widetilde{\mathrm{C}}_q(K_l, \mathbb{S}^{n-1})} = \frac{q-n+k}{q}.$$

Proof. Let $k \in \{1, \ldots, n-1\}$ and for $l \in \mathbb{N}$, let K_l be the cylinder given as the cartesian product of two lower-dimensional balls

$$K_l = (lB_k) \times B_{n-k}.$$

Let $L = \mathrm{lin}\{\boldsymbol{e}_1, \dots, \boldsymbol{e}_k\}$ be the k-dimensional subspace generated by the first k canonical unit vectors $\boldsymbol{e}_i$. For $\boldsymbol{x} \in \mathbb{R}^n$ write $\boldsymbol{x} = (\boldsymbol{x}_1, \boldsymbol{x}_2)$, where $\boldsymbol{x}_1 \in \mathbb{R}^k$ and $\boldsymbol{x}_2 \in \mathbb{R}^{n-k}$. The supporting hyperplane of K_l with respect to a unit vector $\boldsymbol{v} = (\boldsymbol{v}_1, \boldsymbol{v}_2) \in \mathbb{S}^{n-1} \cap L$ is given by

$$H_{K_l}(\boldsymbol{v}) = \{\boldsymbol{x} \in \mathbb{R}^n : \langle \boldsymbol{v}_1, \boldsymbol{x}_1 \rangle = l\}.$$

Hence the part of the boundary of K_l covered by all these supporting hyperplanes is given by $l\mathbb{S}^{k-1} \times B_{n-k}$. In view of Lemma 3.6 and Fubini's theorem we conclude

$$\widetilde{\mathrm{C}}_q(K_l, \mathbb{S}^{n-1} \cap L) = \frac{q}{n} \int\limits_{lB_k} \left(\int\limits_{\frac{|\boldsymbol{x}_1|}{l} B_{n-k}} (|\boldsymbol{x}_1|^2 + |\boldsymbol{x}_2|^2)^{\frac{q-n}{2}} \, \mathrm{d}\boldsymbol{x}_2 \right) \mathrm{d}\boldsymbol{x}_1. \tag{3.37}$$

Denote the volume of B_n by ω_n. Recall that the surface area of B_n is given by $n\omega_n$ and for abbreviation we set

$$c = c(q, k, n) = \frac{q}{n} k\omega_k (n-k)\omega_{n-k}.$$

Switching to the cylindrical coordinates

$$\boldsymbol{x}_1 = s\boldsymbol{u}, \quad s \geq 0, \boldsymbol{u} \in \mathbb{S}^{k-1}, \qquad \boldsymbol{x}_2 = t\boldsymbol{v}, \quad t \geq 0, \boldsymbol{v} \in \mathbb{S}^{n-k-1},$$

transforms the right hand side of (3.37) to

$$\begin{aligned}
\widetilde{\mathrm{C}}_q(K_l, \mathbb{S}^{n-1} \cap L) &= c \int\limits_0^l \int\limits_0^{s/l} s^{k-1} t^{n-k-1} (s^2 + t^2)^{\frac{q-n}{2}} \, \mathrm{d}t \, \mathrm{d}s \\
&= c \int\limits_0^l \int\limits_0^1 s^{q-1} l^{k-n} t^{n-k-1} (1 + l^{-2} t^2)^{\frac{q-n}{2}} \, \mathrm{d}t \, \mathrm{d}s \\
&= c\, l^{q-n+k} \int\limits_0^1 \int\limits_0^1 s^{q-1} t^{n-k-1} (1 + l^{-2} t^2)^{\frac{q-n}{2}} \, \mathrm{d}t \, \mathrm{d}s.
\end{aligned}$$

Analogously we obtain

$$\begin{aligned}
\widetilde{\mathrm{C}}_q(K_l, \mathbb{S}^{n-1}) &= \frac{q}{n} \int\limits_{lB_k} \left(\int\limits_{B_{n-k}} (|\boldsymbol{x}_1|^2 + |\boldsymbol{x}_2|^2)^{\frac{q-n}{2}} \, \mathrm{d}\boldsymbol{x}_2 \right) \mathrm{d}\boldsymbol{x}_1 \\
&= c \int\limits_0^l \int\limits_0^1 s^{k-1} t^{n-k-1} (s^2 + t^2)^{\frac{q-n}{2}} \, \mathrm{d}t \, \mathrm{d}s \\
&= c l^k \int\limits_0^1 \int\limits_0^1 s^{k-1} t^{n-k-1} (l^2 s^2 + t^2)^{\frac{q-n}{2}} \, \mathrm{d}t \, \mathrm{d}s \\
&= c l^{q-n+k} \int\limits_0^1 \int\limits_0^1 s^{k-1} t^{n-k-1} (s^2 + l^{-2} t^2)^{\frac{q-n}{2}} \, \mathrm{d}t \, \mathrm{d}s.
\end{aligned}$$

The monotone convergence theorem gives

$$\begin{aligned}\lim_{l\to\infty}&(cl^{q-n+k})^{-1}\widetilde{\mathrm{C}}_q(K_l,\mathbb{S}^{n-1}\cap L)\\&=\lim_{l\to\infty}\int_0^1\int_0^1 s^{q-1}t^{n-k-1}(1+l^{-2}t^2)^{\frac{q-n}{2}}\,\mathrm{d}t\,\mathrm{d}s\\&=\int_0^1 s^{q-1}\,\mathrm{d}s\cdot\int_0^1 t^{n-k-1}\,\mathrm{d}t=\frac{1}{q(n-k)}\end{aligned}$$

and

$$\begin{aligned}\lim_{l\to\infty}&(cl^{q-n+k})^{-1}\widetilde{\mathrm{C}}_q(K_l,\mathbb{S}^{n-1})\\&=\lim_{l\to\infty}\int_0^1\int_0^1 s^{k-1}t^{n-k-1}(s^2+l^{-2}t^2)^{\frac{q-n}{2}}\,\mathrm{d}t\,\mathrm{d}s\\&=\int_0^1 s^{q-n+k-1}\,\mathrm{d}s\cdot\int_0^1 t^{n-k-1}\,\mathrm{d}t=\frac{1}{(q-n+k)(n-k)}.\end{aligned}$$

Hence,

$$\lim_{l\to\infty}\frac{\widetilde{\mathrm{C}}_q(K_l,\mathbb{S}^{n-1}\cap L)}{\widetilde{\mathrm{C}}_q(K_l,\mathbb{S}^{n-1})}=\frac{q-n+k}{q}.\qquad\square$$

The optimality of (3.7) can be established with a similar example where $K_l = B_k \times (lB_{n-k})$ as it was done in [14] (see also [82]). Obviously this is also a consequence of Theorem 3.3.

3.5 Further results

The Proposition 3.17 above particularly shows that the upper bound $\frac{q-n+\dim L}{q}$ would be also optimal in the range $q \in (n, n+1)$. Unfortunately, our approach to prove Theorem 3.5 can not cover this missing range. One reason is that the Brunn-Minkowski-type inequality (3.15) does not extend to $\alpha \in (0,1)$ as shown in Proposition 3.14. In the proof of Theorem 3.5 we applied (3.15) pointwise onto the inner integral on the right hand side of (3.36). This brings up the question, whether (3.15) is correct on a certain average, i.e., when both sides are integrated with respect to $\lambda \in [0,1]$. Note that in order to extend Theorem 3.5 to $\alpha \in (0,1)$ we would need to prove

$$\begin{aligned}\int_0^1 |2\lambda-1|^{m-1}&\int_{(1-\lambda)K+\lambda(-K)}|\boldsymbol{x}|^\alpha\,\mathrm{d}\mathcal{H}^k(\boldsymbol{x})\,\mathrm{d}\lambda\\&\geq\frac{1}{m+\alpha}\int_K|\boldsymbol{x}|^\alpha\,\mathrm{d}\mathcal{H}^k(\boldsymbol{x}),\end{aligned}\tag{3.38}$$

for $m \in \mathbb{N}$, $K \in \mathcal{K}^n$ with $\dim(K) = k \geq 1$ and $\alpha \in (0,1)$. But this is also false for some examples like $m = 2$, $\alpha = \frac{1}{2}$ and $K = [1,5]$. In this case one can compute

$$\begin{aligned}\int_0^1 |2\lambda - 1| \int_{(1-\lambda)K+\lambda(-K)} \sqrt{|x|}\, \mathrm{d}x\, \mathrm{d}\lambda &= \frac{4}{945}\Big(-19 + 32\sqrt{2} + 275\sqrt{5}\Big) \\ &\approx 2.713\ldots\end{aligned}$$

and

$$\frac{1}{2+\frac{1}{2}} \int_K \sqrt{|x|}\, \mathrm{d}x = \frac{4}{15}\Big(5\sqrt{5} - 1\Big) \approx 2.714\ldots$$

In the particular case when $m = 1$ we will prove (3.38). To this end, we start with an integral version of (3.18) for $\alpha \in (0,1)$.

Lemma 3.18. *Let $\alpha \in (0,1)$ and $z_0, z_1 \in \mathbb{R}$. Then*

$$\int_0^1 |(1-\lambda)z_0 + \lambda z_1|^\alpha \, \mathrm{d}\lambda \geq \frac{|z_0|^\alpha + |z_1|^\alpha}{2(1+\alpha)}$$

with equality if and only if $z_1 = -z_0$.

Proof. Without loss of generality we may assume that $|z_0| \geq |z_1|$ and, since the inequality holds for $z_0 = z_1 = 0$, also $z_0 \neq 0$. Then the reverse triangle inequality yields

$$\begin{aligned}&\int_0^1 |(1-\lambda)z_0 + \lambda z_1|^\alpha \, \mathrm{d}\lambda \\ &\quad\geq \int_0^1 \big|(1-\lambda)\,|z_0| - \lambda\,|z_1|\big|^\alpha \, \mathrm{d}\lambda \\ &\quad= \int_0^{\frac{|z_0|}{|z_0|+|z_1|}} \big(\,|z_0| - \lambda(|z_0| + |z_1|)\big)^\alpha \, \mathrm{d}\lambda \\ &\qquad+ \int_{\frac{|z_0|}{|z_0|+|z_1|}}^1 \big(\lambda(|z_0| + |z_1|) - |z_0|\,\big)^\alpha \, \mathrm{d}\lambda \\ &\quad= -(1+\alpha)^{-1}(|z_0| + |z_1|)^{-1}\Big[\,|z_0| - \lambda(|z_0| + |z_1|)\big)^{1+\alpha}\Big]_0^{\frac{|z_0|}{|z_0|+|z_1|}} \\ &\qquad+ (1+\alpha)^{-1}(|z_0| + |z_1|)^{-1}\Big[\lambda(|z_0| + |z_1|) - |z_0|\,\big)^{1+\alpha}\Big]_{\frac{|z_0|}{|z_0|+|z_1|}}^1 \\ &\quad= (1+\alpha)^{-1}(|z_0| + |z_1|)^{-1}\big(\,|z_0|^{1+\alpha} + |z_1|^{1+\alpha}\,\big)\end{aligned}$$

and equality holds in the first inequality if and only if $z_0 z_1 \leq 0$. It remains to show that $\frac{|z_0|^{1+\alpha}+|z_1|^{1+\alpha}}{|z_0|+|z_1|} \geq \frac{|z_0|^{\alpha}+|z_1|^{\alpha}}{2}$. Indeed, since

$$0 \leq (|z_0|-|z_1|)\Big(|z_0|^{\alpha}-|z_1|^{\alpha}\Big) = \Big(|z_0|^{1+\alpha}+|z_1|^{1+\alpha}\Big)-\Big(|z_0|\,|z_1|^{\alpha}+|z_1|\,|z_0|^{\alpha}\Big),$$

we find

$$\begin{aligned}\Big(|z_0|^{\alpha}+|z_1|^{\alpha}\Big)(|z_0|+|z_1|) &= |z_0|^{1+\alpha}+|z_1|^{1+\alpha}+|z_0|\,|z_1|^{\alpha}+|z_1|\,|z_0|^{\alpha}\\ &\leq 2\Big(|z_0|^{1+\alpha}+|z_1|^{1+\alpha}\Big)\end{aligned}$$

with equality if and only if $|z_0| = |z_1|$. □

Proposition 3.19. *Let $\alpha \in (0,1)$ and $K \in \mathcal{K}^n$ with $\dim K = k \geq 1$. Then*

$$\int_0^1 \int_{(1-\lambda)K+\lambda(-K)} |\boldsymbol{x}|^{\alpha}\, \mathrm{d}\mathcal{H}^k(\boldsymbol{x})\, \mathrm{d}\lambda > \frac{1}{1+\alpha}\int_K |\boldsymbol{x}|^{\alpha}\, \mathrm{d}\mathcal{H}^k(\boldsymbol{x}). \tag{3.39}$$

Proof. Without loss of generality we assume that $\mathrm{vol}_k(K) = 1$. Denote $K_\lambda = (1-\lambda)K+\lambda(-K)$. As in the proof of Theorem 3.8 the inequality (3.39) is equivalent to

$$\int_{\mathbb{S}^{n-1}}\int_0^1\int_{K_\lambda} |\langle\boldsymbol{\theta},\boldsymbol{x}\rangle|^{\alpha}\, \mathrm{d}\mathcal{H}^k(\boldsymbol{x})\, \mathrm{d}\lambda\, \mathrm{d}\boldsymbol{\theta} \geq \frac{1}{1+\alpha}\int_{\mathbb{S}^{n-1}}\int_K |\langle\boldsymbol{\theta},\boldsymbol{x}\rangle|^{\alpha}\, \mathrm{d}\mathcal{H}^k(\boldsymbol{x})\, \mathrm{d}\boldsymbol{\theta}.$$

by Lemma 3.12. In full analogy to the proof of Theorem 3.9 and using the same notation z_0, z_1 where $K_0 = K$ and $K_1 = -K$ we find

$$\int_{K_\lambda} |\langle\boldsymbol{\theta},\boldsymbol{x}\rangle|^{\alpha}\, \mathrm{d}\mathcal{H}^k(\boldsymbol{x}) \geq \int_0^1 |(1-\lambda)z_0(\tau)+\lambda z_1(\tau)|^{\alpha}\, \mathrm{d}\tau.$$

Now Lemma 3.18 and backsubstituting yield

$$\int_0^1\int_{K_\lambda} |\langle\boldsymbol{\theta},\boldsymbol{x}\rangle|^{\alpha}\, \mathrm{d}\mathcal{H}^k(\boldsymbol{x})\, \mathrm{d}\lambda > \frac{1}{1+\alpha}\int_K |\langle\boldsymbol{\theta},\boldsymbol{x}\rangle|^{\alpha}\, \mathrm{d}\mathcal{H}^k(\boldsymbol{x}),$$

where the inequality is strict since both z_0 and z_1 are strictly increasing functions whenever $\mathrm{h}_K(\boldsymbol{\theta}) \neq \mathrm{h}_K(-\boldsymbol{\theta})$ and therefore $z_0(\tau) = -z_1(\tau)$ cannot hold for every $\tau \in (0,1)$. This proves (3.39). □

As a consequence we establish (3.8) when $q \in (n, n+1)$ and L is a one-dimensional subspace.

Theorem 3.20. *Let $q \in (n, n+1)$, $K \in \mathcal{K}^n_s$ and L be a one-dimensional subspace of $\mathbb{R}^n$. Then*

$$\widetilde{\mathrm{C}}_q(K, \mathbb{S}^{n-1}\cap L) < \frac{q-n+1}{q}\widetilde{\mathrm{C}}_q(K, \mathbb{S}^{n-1}).$$

Proof. Following the lines of the proof of Theorem 3.5 and using the same notation we find

$$\widetilde{\mathrm{C}}_q(K,\mathbb{S}^{n-1}) \geq \frac{q}{n}\int\limits_{K|L}\left(\int\limits_{\frac{1+\tau}{2}K_{\tilde{\mathbf{p}}_{K|L}(\boldsymbol{y})}-\frac{1-\tau}{2}K_{\tilde{\mathbf{p}}_{K|L}(\boldsymbol{y})}}|\boldsymbol{z}|^{q-n}\,\mathrm{d}\boldsymbol{z}\right)\mathrm{d}\boldsymbol{y}$$

and

$$\widetilde{\mathrm{C}}_q(K,\mathbb{S}^{n-1}\cap L) = \frac{q}{n}\int\limits_{K|L}\tau^{q-k}\left(\int\limits_{K_{\tilde{\mathbf{p}}_{K|L}(\boldsymbol{y})}}|\boldsymbol{z}|^{q-n}\,\mathrm{d}\boldsymbol{z}\right)\mathrm{d}\boldsymbol{y},$$

where $\tau = \rho_{K|L}(\boldsymbol{y})^{-1}$. Using spherical coordinates in $K|L$, i.e., $\boldsymbol{y} = r\cdot\boldsymbol{u}$, $r \geq 0$, $\boldsymbol{u} \in \mathbb{S}^{n-1}$, and thereafter $r = (1-2\lambda)\rho_{K|L}(\boldsymbol{u})$ the upper inequality becomes

$$\begin{aligned}
&\widetilde{\mathrm{C}}_q(K,\mathbb{S}^{n-1})\\
&\quad\geq \frac{q}{n}\int\limits_{\mathbb{S}^{n-1}\cap L}\int\limits_0^{\rho_{K|L}(\boldsymbol{u})}\left(\int\limits_{\frac{1+\tau}{2}K_{\tilde{\mathbf{p}}_{K|L}(\boldsymbol{u})}-\frac{1-\tau}{2}K_{\tilde{\mathbf{p}}_{K|L}(\boldsymbol{u})}}|\boldsymbol{z}|^{q-n}\,\mathrm{d}\boldsymbol{z}\right)r^{k-1}\,\mathrm{d}r\,\mathrm{d}\boldsymbol{u}\\
&\quad= \frac{q}{n}\int\limits_{\mathbb{S}^{n-1}\cap L}\int\limits_0^{\frac{1}{2}}\left(\int\limits_{(1-\lambda)K_{\tilde{\mathbf{p}}_{K|L}(\boldsymbol{u})}-\lambda K_{\tilde{\mathbf{p}}_{K|L}(\boldsymbol{u})}}|\boldsymbol{z}|^{q-n}\,\mathrm{d}\boldsymbol{z}\right)\\
&\qquad\qquad\times\big((1-2\lambda)\rho_{K|L}(\boldsymbol{u})\big)^{k-1}(2\rho_{K|L}(\boldsymbol{u}))\,\mathrm{d}\lambda\,\mathrm{d}\boldsymbol{u}\\
&\quad= \frac{q}{n}\int\limits_{\mathbb{S}^{n-1}\cap L}\rho_{K|L}(\boldsymbol{u})^k\\
&\qquad\qquad\times\int\limits_0^1|2\lambda-1|^{k-1}\left(\int\limits_{(1-\lambda)K_{\tilde{\mathbf{p}}_{K|L}(\boldsymbol{u})}-\lambda K_{\tilde{\mathbf{p}}_{K|L}(\boldsymbol{u})}}|\boldsymbol{z}|^{q-n}\,\mathrm{d}\boldsymbol{z}\right)\mathrm{d}\lambda\,\mathrm{d}\boldsymbol{u}.
\end{aligned}$$

Since $k = 1$ we apply Proposition 3.19 to get

$$\begin{aligned}
&\widetilde{\mathrm{C}}_q(K,\mathbb{S}^{n-1})\\
&\quad> \frac{q}{n(q-n+1)}\int\limits_{\mathbb{S}^{n-1}\cap L}\rho_{K|L}(\boldsymbol{u})^k\int\limits_{K_{\tilde{\mathbf{p}}_{K|L}(\boldsymbol{u})}}|\boldsymbol{z}|^{q-n}\,\mathrm{d}\boldsymbol{z}\,\mathrm{d}\boldsymbol{u}\\
&\quad= \frac{q^2}{n(q-n+1)}\int\limits_{\mathbb{S}^{n-1}\cap L}\rho_{K|L}(\boldsymbol{u})^{k-q}\int\limits_0^{\rho_{K|L}(\boldsymbol{u})}r^{q-1}\int\limits_{K_{\tilde{\mathbf{p}}_{K|L}(\boldsymbol{u})}}|\boldsymbol{z}|^{q-n}\,\mathrm{d}\boldsymbol{z}\,\mathrm{d}r\,\mathrm{d}\boldsymbol{u}\\
&\quad= \frac{q^2}{n(q-n+1)}\int\limits_{\mathbb{S}^{n-1}\cap L}\int\limits_0^{\rho_{K|L}(\boldsymbol{u})}\rho_{K|L}(r\boldsymbol{u})^{k-q}r^{k-1}\int\limits_{K_{\tilde{\mathbf{p}}_{K|L}(\boldsymbol{u})}}|\boldsymbol{z}|^{q-n}\,\mathrm{d}\boldsymbol{z}\,\mathrm{d}r\,\mathrm{d}\boldsymbol{u}
\end{aligned}$$

$$= \frac{q^2}{n(q-n+1)} \int\limits_{K|L} \rho_{K|L}(\boldsymbol{y})^{k-q} \int\limits_{K_{\tilde{\mathbf{p}}_{K|L}(\boldsymbol{y})}} |\boldsymbol{z}|^{q-n} \,\mathrm{d}\boldsymbol{z}\,\mathrm{d}\boldsymbol{y}$$

$$= \frac{q}{q-n+1} \widetilde{\mathrm{C}}_q(K, \mathbb{S}^{n-1} \cap L). \qquad \square$$

This settles the extension of Theorem 3.5 to $\alpha \in (0,1)$ when $n = 2$.

Corollary 3.21 (Henk, P. [42])**.** *Let $q > 2$ and $K \in \mathcal{K}_s^2$. Then*

$$\widetilde{\mathrm{C}}_q(K, \mathbb{S}^1 \cap L) < \frac{q-1}{q} \widetilde{\mathrm{C}}_q(K, \mathbb{S}^1) \tag{3.40}$$

for every one-dimensional subspace L of $\mathbb{R}^2$.

The proof of Corollary 3.21 given in [42] is different and might be of some interest on its own. It is based on sharp subspace concentration inequalities for dual curvature measures of parallelepipeds. To this end, we need a variant of Theorem 3.7 for quasiconvex functions.

Lemma 3.22 (Henk, P. [42])**.** *Let $f\colon \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ be an even quasiconvex function. Let $K \subset \mathbb{R}^n$ be a compact, convex set with $\dim K = k$. Let $\lambda \in [0,1]$ and $\boldsymbol{v} \in \mathbb{R}^n$. Then*

$$\int\limits_{K+\lambda\boldsymbol{v}} f(\boldsymbol{x}) \,\mathrm{d}\mathcal{H}^k(\boldsymbol{x}) \leq \int\limits_{K+\boldsymbol{v}} f(\boldsymbol{x}) \,\mathrm{d}\mathcal{H}^k(\boldsymbol{x}).$$

If the sublevel sets $\{f \leq c\}$ are closed for every $c \geq 0$, then equality holds if and only if

$$(K + \boldsymbol{v}) \cap \{f \leq c\} = (K \cap \{f \leq c\}) + \boldsymbol{v}.$$

for every $c > 0$.

Proof. Apply Theorem 3.7 to the quasiconcave function $\tilde{f}(\boldsymbol{x}) = \max(S - f(\boldsymbol{x}), 0)$ where $S = \sup\{f(\boldsymbol{x}+\boldsymbol{v}) : \boldsymbol{x} \in K\}$ and the convex body $M = K + \boldsymbol{v}$ with $\mu = \frac{1-\lambda}{2}$, so that $(1-\mu)M + \mu(-M) = K + \lambda\boldsymbol{v}$. $\square$

Based on Lemma 3.22 we can easily get a lower bound on the subspace concentration of prisms.

Proposition 3.23. *Let $q > n$ and $K \in \mathcal{K}_s^n$ be a prism, i.e., there is a $\boldsymbol{u} \in \mathbb{S}^{n-1}$, and $Q \subset \boldsymbol{u}^\perp$ with $Q = -Q$ and $\dim Q = n-1$, such that*

$$K = \operatorname{conv}\big((Q - \boldsymbol{v}) \cup (Q + \boldsymbol{v})\big)$$

for some $\boldsymbol{v} \in \mathbb{R}^n \setminus \boldsymbol{u}^\perp$. Then

$$\widetilde{\mathrm{C}}_q(K, \{\pm\boldsymbol{u}\}) > \frac{1}{q} \widetilde{\mathrm{C}}_q(K, \mathbb{S}^{n-1}). \tag{3.41}$$

Proof. Let $L = \operatorname{lin}\{\boldsymbol{u}\}$. Since the dual curvature measure is homogeneous, we may assume that $|\langle \boldsymbol{u}, \boldsymbol{v}\rangle| = 1$. There exists an invertible matrix $A \in \mathbb{R}^{n\times n}$ with

$Av = u$ and $Aw = w$ for all $w \in u^\perp$, and our assumption yields $|\det(A)| = 1$. By Lemma 3.6 we may write

$$\begin{aligned}\widetilde{\mathrm{C}}_q(K, \mathbb{S}^{n-1}) &= \frac{q}{n} \int\limits_K |x|^{q-n} \, \mathrm{d}x \\ &= \frac{q}{n} \int\limits_{AK} \left|A^{-1} x\right|^{q-n} \, \mathrm{d}x.\end{aligned}$$

Clearly $AK = \operatorname{conv}\big((Q - u) \cup (Q + u)\big)$ and Fubini's theorem gives

$$\begin{aligned}\widetilde{\mathrm{C}}_q(K, \mathbb{S}^{n-1}) &= \frac{q}{n} \int\limits_{-1}^{1} \int\limits_{Q+\tau u} \left|A^{-1} z\right|^{q-n} \, \mathrm{d}z \, \mathrm{d}\tau \\ &= \frac{2q}{n} \int\limits_{0}^{1} \int\limits_{Q+\tau v} |z|^{q-n} \, \mathrm{d}z \, \mathrm{d}\tau.\end{aligned}$$

Applying Lemma 3.22 to the inner integral gives

$$\begin{aligned}\widetilde{\mathrm{C}}_q(K, \mathbb{S}^{n-1}) &\le \frac{2q}{n} \int\limits_{0}^{1} \int\limits_{Q+v} |z|^{q-n} \, \mathrm{d}z \, \mathrm{d}\tau \\ &= \frac{2q}{n} \int\limits_{Q+v} |z|^{q-n} \, \mathrm{d}z.\end{aligned}$$

On the other hand,

$$\begin{aligned}\widetilde{\mathrm{C}}_q(K, \{\pm u\}) &= \frac{2q}{n} \int\limits_{0}^{1} \int\limits_{\tau(Q+v)} |z|^{q-n} \, \mathrm{d}z \, \mathrm{d}\tau \\ &= \frac{2q}{n} \int\limits_{0}^{1} \int\limits_{Q+v} \tau^{q-n} |z|^{q-n} \cdot \tau^{n-1} \, \mathrm{d}z \, \mathrm{d}\tau \\ &= \frac{2q}{n} \int\limits_{Q+v} |z|^{q-n} \, \mathrm{d}z \int\limits_{0}^{1} \tau^{q-1} \, \mathrm{d}\tau \\ &= \frac{2}{n} \int\limits_{Q+v} |z|^{q-n} \, \mathrm{d}z.\end{aligned}$$

This gives (3.41) without strict inequality. Suppose we have equality. Then the equality characterization of Lemma 3.22 implies

$$(Q + v) \cap r B_n = (Q \cap r B_n) + v$$

for almost all $r > 0$. But for small r the left hand side is empty and hence equality in (3.41) cannot be attained. □

As a consequence we deduce an upper bound on the subspace concentration of dual curvature measures of parallelepipeds.

Corollary 3.24. *Let $q > n$ and $P \in \mathcal{P}_s^n$ be a parallelepiped. Then*

$$\widetilde{\mathrm{C}}_q(P, \mathbb{S}^{n-1} \cap L) < \frac{q - n + \dim L}{q} \widetilde{\mathrm{C}}_q(P, \mathbb{S}^{n-1})$$

for every proper subspace L of $\mathbb{R}^n$.

Proof. Let $\pm \boldsymbol{u}_1, \ldots, \pm \boldsymbol{u}_n \in \mathbb{S}^{n-1}$ be the outer normal vectors of P. In particular, $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_n$ are linearly independent. For a given k-dimensional subspace L of $\mathbb{R}^n$ we may assume that $\boldsymbol{u}_{k+1}, \ldots, \boldsymbol{u}_n \notin L$. By Proposition 3.23

$$\begin{aligned}
\widetilde{\mathrm{C}}_q(P, \mathbb{S}^{n-1} \cap L) &= \sum_{i=1}^{k} \widetilde{\mathrm{C}}_q(P, \{\boldsymbol{u}_i, -\boldsymbol{u}_i\}) \\
&= \widetilde{\mathrm{C}}_q(P, \mathbb{S}^{n-1}) - \sum_{i=k+1}^{n} \widetilde{\mathrm{C}}_q(P, \{\boldsymbol{u}_i, -\boldsymbol{u}_i\}) \\
&< \widetilde{\mathrm{C}}_q(P, \mathbb{S}^{n-1}) - \sum_{i=k+1}^{n} \frac{1}{q} \widetilde{\mathrm{C}}_q(P, \mathbb{S}^{n-1}) \\
&= \widetilde{\mathrm{C}}_q(P, \mathbb{S}^{n-1}) - \frac{n - \dim L}{q} \widetilde{\mathrm{C}}_q(P, \mathbb{S}^{n-1}) \\
&= \frac{q - n + \dim L}{q} \widetilde{\mathrm{C}}_q(P, \mathbb{S}^{n-1}).
\end{aligned}$$

□

In particular this gives yet another proof of Corollary 3.21 as it can be reduced to proving a subspace bound for parallelograms as follows. Let $K \in \mathcal{K}_s^2$ and L be a one-dimensional subspace of $\mathbb{R}^2$ spanned by $\boldsymbol{u} \in \mathbb{S}^1$. If $F = H_K(\boldsymbol{u})$ is a singleton, inequality (3.40) trivially holds. If $\dim F = 1$, by an inclusion argument it suffices to prove (3.40) for $P = \operatorname{conv}(F \cup (-F))$. Since F is a line segment, P is a parallelogram and Corollary 3.24 gives (3.40) for P.

3.6 L_p dual curvature measures

In their recent paper Lutwak, Yang and Zhang [64] unified some notions of L_p Brunn-Minkowski theory and dual Brunn-Minkowski theory. Their framework is build upon the **ith generalized dual curvature measure of K relative to Q**, i.e., for $K \in \mathcal{K}_o^n$ and $Q \in \mathcal{S}_o^n$ and $i \in \{0, \ldots, n\}$ it is a Borel measure on $\mathbb{S}^{n-1} \times \mathbb{S}^{n-1}$ given by

$$\widetilde{\Theta}_i(K, Q, \Xi) = \frac{1}{n} \int\limits_{\psi_K(\Xi)} \rho_K(\boldsymbol{u})^i \rho_Q(\boldsymbol{u})^{n-i} \, \mathrm{d}\boldsymbol{u}$$

for every Borel set $\Xi \subseteq \mathbb{S}^{n-1} \times \mathbb{S}^{n-1}$, where

$$\psi_K(\Xi) = \left\{ \boldsymbol{u} \in \mathbb{S}^{n-1} : \widetilde{\mathbf{p}}_K(\boldsymbol{u}) \in H_K(\boldsymbol{v}) \text{ for some } \boldsymbol{v} \in \mathbb{S}^{n-1} \text{ and } (\boldsymbol{u}, \boldsymbol{v}) \in \Xi \right\}.$$

The ith dual area measure and the ith dual curvature measure of a convex body K arise as marginal measures from Θ_i via

$$\widetilde{\mathrm{S}}_q(K,\omega) = \widetilde{\Theta}_i(K, B_n, \omega \times \mathbb{S}^{n-1}),$$
$$\widetilde{\mathrm{C}}_q(K,\eta) = \widetilde{\Theta}_i(K, B_n, \mathbb{S}^{n-1} \times \eta),$$

where $K \in \mathcal{K}_o^n$, $i \in \{0,\dots,n\}$ and $\omega, \eta \subseteq \mathbb{S}^{n-1}$ are Borel sets. Moreover, for each $p, q \in \mathbb{R}$ and $Q \in \mathcal{S}_o^n$ they define the L_p **dual curvature measure** or, more precisely, the (p,q)th dual curvature measure of a convex body $K \in \mathcal{K}_o^n$ relative to Q by

$$\widetilde{\mathrm{C}}_{p,q}(K,Q,\eta) = \frac{1}{n} \int\limits_{\boldsymbol{\alpha}_K^*(\eta)} \mathrm{h}_K(\boldsymbol{\alpha}_K(\boldsymbol{u}))^{-p} \rho_K(\boldsymbol{u})^q \rho_Q(\boldsymbol{u})^{n-q} \,\mathrm{d}\boldsymbol{u} \tag{3.42}$$

for Borel sets $\eta \subseteq \mathbb{S}^{n-1}$, where $\boldsymbol{\alpha}_K(\boldsymbol{u}) = \{\boldsymbol{v} \in \mathbb{S}^{n-1} : \boldsymbol{u} \in H_K(\boldsymbol{v})\}$ for $\boldsymbol{u} \in \mathbb{S}^{n-1}$. The L_p dual curvature measures include the L_p surface area measures and the dual curvature measures as special cases since

$$\widetilde{\mathrm{C}}_{p,q}(K,K,\cdot) = \widetilde{\mathrm{C}}_{p,n}(K,B_n,\cdot) = \frac{1}{n}\mathrm{S}_K^{(p)},$$
$$\widetilde{\mathrm{C}}_{0,q}(K,B_n,\cdot) = \widetilde{\mathrm{C}}_q(K,\cdot),$$

for $p,q \in \mathbb{R}$ and $K \in \mathcal{K}_o^n$ (see [64, Prop. 5.4]). Given $p, q \in \mathbb{R}$ and $Q \in \mathcal{S}_o^n$, the task of characterizing L_p dual curvature measures of convex bodies among Borel measures on the sphere is called L_p **dual Minkowski problem**. It was first posed in [64] where for $p \le q$ uniqueness of L_p dual curvature measures of polytopes was studied. Given a Borel measure $\mu \in \mathbb{S}^{n-1}$, Huang and Zhao [47] related existence of solutions of $\widetilde{\mathrm{C}}_{p,q}(K,Q,\cdot) = \mu$ when $Q = B_n$ to an optimization problem which led to a characterization of L_p dual curvature measures in the particular cases when $p > 0$ and $q < 0$, and when $p, q > 0$, $p \neq q$ and μ is even. In case $p, q < 0$, $p \neq q$ and μ is even, they also proved a sufficient condition. Böröczky and Fodor [20] established for arbitrary $Q \in \mathcal{S}_o^n$ a sufficent condition when $p > 1$, $q > 0$, $p \neq q$ and μ is discrete. In addition, regularity of the involved convex bodies is discussed. The main focus in this section is on the case $p = 0$, i.e., for fixed $Q \in \mathcal{S}_o^n$ we consider the $(0,q)$th dual curvature measures $\widetilde{\mathrm{C}}_{0,q}(K,Q,\cdot)$ for $K \in \mathcal{K}_o^n$. Here we discuss how the subspace concentration inequalities presented in Section 3.4 may be generalized to L_p dual curvature measures. For $q \in (0,n)$ and $Q \in \mathcal{K}_s^n$ we already proved a straightforward generalization of Theorem 3.2.

Theorem 3.25 (Böröczky, Henk, P. [14]). *Let $q \in (0,n)$ and $K, Q \in \mathcal{K}_s^n$. Then*

$$\widetilde{\mathrm{C}}_{0,q}(K,Q,\mathbb{S}^{n-1} \cap L) < \min\left\{\frac{\dim L}{q}, 1\right\} \widetilde{\mathrm{C}}_{0,q}(K,Q,\mathbb{S}^{n-1}) \tag{3.43}$$

for every proper subspace L of $\mathbb{R}^n$.

Proof. If $\dim L \ge q$, there is nothing to show. If $\dim L < q$, by (3.42) and

Theorem 3.16

$$\begin{aligned}\widetilde{\mathrm{C}}_{0,q}(K,Q,\mathbb{S}^{n-1}\cap L) &= \frac{1}{n}\int\limits_{\boldsymbol{\alpha}_K^*(\eta)} \rho_K(\boldsymbol{u})^q\rho_Q(\boldsymbol{u})^{n-q}\,\mathrm{d}\boldsymbol{u}\\ &< \frac{\dim L}{qn}\int\limits_{\mathbb{S}^{n-1}} \rho_K(\boldsymbol{u})^q\rho_Q(\boldsymbol{u})^{n-q}\,\mathrm{d}\boldsymbol{u}\\ &= \frac{\dim L}{q}\widetilde{\mathrm{C}}_{0,q}(K,Q,\mathbb{S}^{n-1}). \qquad \square\end{aligned}$$

This bound is also optimal as the next example shows.

Proposition 3.26. *Let $q\in(0,n)$ and $k\in\{1,\dots,n-1\}$. There exists a sequence of convex bodies $K_l\in\mathcal{K}_s^n$, $l\in\mathbb{N}$, and a k-dimensional subspace $L\subseteq\mathbb{R}^n$ such that*

$$\lim_{l\to\infty}\frac{\widetilde{\mathrm{C}}_{0,q}(K_l,Q,\mathbb{S}^{n-1}\cap L)}{\widetilde{\mathrm{C}}_{0,q}(K_l,Q,\mathbb{S}^{n-1})} = \begin{cases}\frac{k}{q}, & \text{if } k<q,\\ 1, & \text{if } k\ge q.\end{cases}$$

The proof will be omitted since it repeats the arguments of the proof of Proposition 3.17 with $K_l=B_k\times(lB_{n-k})$ and ρ_Q^{n-q} instead of $|\cdot|^{q-n}$.

The point about (3.43), which deserves particular attention, is that the given bound is independent of the body $Q\in\mathcal{K}_s^n$. In fact, one cannot prove a Q independent subspace concentration bound if either symmetry or convexity in this assumption is dropped. We will prove this in the following two propositions and begin with the case, when Q is not necessarily convex, but a star body.

Proposition 3.27. *Let $q\in(0,n)$. There is no non-trivial subspace concentration bound on the $(0,q)$th dual curvature measures of symmetric convex bodies that is uniform with respect to origin-symmetric star bodies, i.e., for every $k\in\{1,\dots,n-1\}$ and $\varepsilon>0$ there exist $K\in\mathcal{K}_s^n$, an origin-symmetric $Q\in\mathcal{S}_o^n$ and a k-dimensional subspace L of $\mathbb{R}^n$ such that*

$$\widetilde{\mathrm{C}}_{0,q}(K,Q,\mathbb{S}^{n-1}\cap L) > (1-\varepsilon)\widetilde{\mathrm{C}}_{0,q}(K,Q,\mathbb{S}^{n-1}).$$

Proof. Let $K=B_k\times B_{n-k}$ and $L=\mathrm{lin}\{\boldsymbol{e}_1,\dots,\boldsymbol{e}_k\}$ be the k-dimensional subspace generated by the first k canonical unit vectors $\boldsymbol{e}_i$. For $\boldsymbol{x}\in\mathbb{R}^n$ write $\boldsymbol{x}=(\boldsymbol{x}_1,\boldsymbol{x}_2)$, where $\boldsymbol{x}_1\in\mathbb{R}^k$ and $\boldsymbol{x}_2\in\mathbb{R}^{n-k}$. The supporting hyperplane of K with respect to a unit vector $\boldsymbol{v}\in\mathbb{S}^{n-1}\cap L$ is given by

$$H_K(\boldsymbol{v})=\{\boldsymbol{x}\in\mathbb{R}^n:\langle\boldsymbol{v}_1,\boldsymbol{x}_1\rangle=1\}.$$

Hence the part of the boundary of K covered by all these supporting hyperplanes is given by $\mathbb{S}^{k-1}\times B_{n-k}$. Let

$$Q=\{(\boldsymbol{x}_1,\boldsymbol{x}_2)\in\mathbb{R}^n:|\boldsymbol{x}_1|\le 1,|\boldsymbol{x}_2|\le|\boldsymbol{x}_1|\}.$$

Thus Q is star-shaped and for $(\boldsymbol{x}_1,\boldsymbol{x}_2)\in\mathbb{R}^n$ with $\boldsymbol{x}_1\neq\boldsymbol{0}$

$$\rho_Q(\boldsymbol{x}_1,\boldsymbol{x}_2)=\begin{cases}0, & \text{if } |\boldsymbol{x}_2|>|\boldsymbol{x}_1|,\\ |\boldsymbol{x}_1|^{-1}, & \text{otherwise.}\end{cases}$$

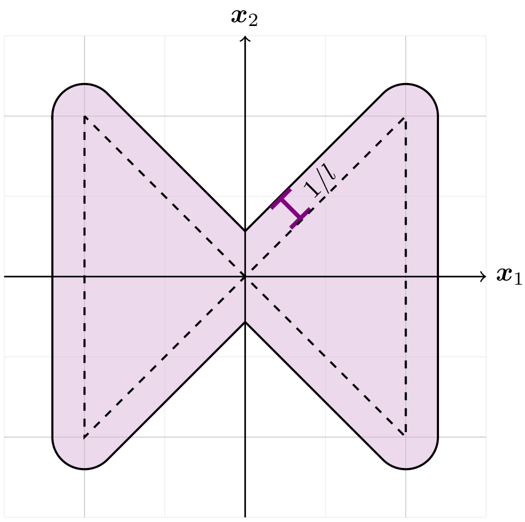

Figure 3.7: The star body Q_l in the proof of Prop. 3.27

For each $l \in \mathbb{N}$ we define a star body $Q_l = Q + \frac{1}{l} B_n$ (see Fig. 3.7). Then ρ_{Q_l} converges pointwise to ρ_Q. By (3.42), Lemma 3.6, Fubini's theorem and the monotone convergence theorem we conclude

$$\begin{aligned}
&\lim_{l\to\infty} \widetilde{\mathrm{C}}_{0,q}(K, Q_l, \mathbb{S}^{n-1} \cap L) \\
&\qquad = \lim_{l\to\infty} \frac{q}{n} \int\limits_{B_k} \left(\int\limits_{|\boldsymbol{x}_1| B_{n-k}} \rho_{Q_l}(\boldsymbol{x}_1, \boldsymbol{x}_2)^{n-q} \,\mathrm{d}\boldsymbol{x}_2 \right) \mathrm{d}\boldsymbol{x}_1 \\
&\qquad = \frac{q}{n} \int\limits_{B_k} \left(\int\limits_{|\boldsymbol{x}_1| B_{n-k}} \rho_{Q}(\boldsymbol{x}_1, \boldsymbol{x}_2)^{n-q} \,\mathrm{d}\boldsymbol{x}_2 \right) \mathrm{d}\boldsymbol{x}_1 \\
&\qquad = \frac{q}{n} \int\limits_{B_k} \left(\int\limits_{B_{n-k}} \rho_{Q}(\boldsymbol{x}_1, \boldsymbol{x}_2)^{n-q} \,\mathrm{d}\boldsymbol{x}_2 \right) \mathrm{d}\boldsymbol{x}_1 \\
&\qquad = \lim_{l\to\infty} \frac{q}{n} \int\limits_{B_k} \left(\int\limits_{B_{n-k}} \rho_{Q_l}(\boldsymbol{x}_1, \boldsymbol{x}_2)^{n-q} \,\mathrm{d}\boldsymbol{x}_2 \right) \mathrm{d}\boldsymbol{x}_1 \\
&\qquad = \lim_{l\to\infty} \widetilde{\mathrm{C}}_{0,q}(K, Q_l, \mathbb{S}^{n-1}),
\end{aligned}$$

where we used, that $\rho_Q(\boldsymbol{x}_1, \cdot)$ is zero on $B_{n-k} \setminus (|\boldsymbol{x}_1| B_{n-k})$. □

We use a similar construction when we consider $Q \in \mathcal{K}_o^n$.

Proposition 3.28. *Let $q \in (0, n)$. There is no non-trivial subspace concentration bound on the $(0, q)$th dual curvature measures of symmetric convex bodies that is uniform with respect to convex bodies in $\mathcal{K}_o^n$, i.e., for every $k \in$*

$\{1, \dots, n-1\}$ *and* $\varepsilon > 0$ *there exist* $K \in \mathcal{K}_s^n$, *an* $Q \in \mathcal{K}_o^n$ *and a* k*-dimensional subspace* L *of* $\mathbb{R}^n$ *such that*

$$\widetilde{\mathrm{C}}_{0,q}(K, Q, \mathbb{S}^{n-1} \cap L) > (1-\varepsilon)\widetilde{\mathrm{C}}_{0,q}(K, Q, \mathbb{S}^{n-1}).$$

Proof. Let $P = [-1,1]^n$ and $L = \operatorname{lin}\{\boldsymbol{e}_1, \dots, \boldsymbol{e}_k\}$ as above and define

$$Q = \operatorname{conv}(\{\mathbf{0}\} \cup F(P, \boldsymbol{e}_1)).$$

Thus Q is a convex body and for each $l \in \mathbb{N}$ we define $Q_l = Q + \frac{1}{l}B_n \in \mathcal{K}_o^n$. In particular, ρ_{Q_l} converges pointwise to ρ_Q. By (3.42), Lemma 3.6, Fubini's theorem and the monotone convergence theorem we conclude

$$\begin{aligned}
\lim_{l\to\infty} &\widetilde{\mathrm{C}}_{0,q}(P, Q_l, \mathbb{S}^{n-1} \cap L) \\
&= \lim_{l\to\infty} \frac{q}{n} \sum_{i=1}^{k} \int_{\operatorname{conv}(\{\mathbf{0}\}\cup F(P,\boldsymbol{e}_i))} \rho_{Q_l}(\boldsymbol{x})^{n-q} + \rho_{Q_l}(-\boldsymbol{x})^{n-q} \,\mathrm{d}\boldsymbol{x} \\
&= \frac{q}{n} \sum_{i=1}^{k} \int_{\operatorname{conv}(\{\mathbf{0}\}\cup F(P,\boldsymbol{e}_i))} \rho_{Q}(\boldsymbol{x})^{n-q} + \rho_{Q}(-\boldsymbol{x})^{n-q} \,\mathrm{d}\boldsymbol{x} \\
&= \frac{q}{n} \sum_{i=1}^{n} \int_{\operatorname{conv}(\{\mathbf{0}\}\cup F(P,\boldsymbol{e}_i))} \rho_{Q}(\boldsymbol{x})^{n-q} + \rho_{Q}(-\boldsymbol{x})^{n-q} \,\mathrm{d}\boldsymbol{x} \\
&= \lim_{l\to\infty} \frac{q}{n} \sum_{i=1}^{n} \int_{\operatorname{conv}(\{\mathbf{0}\}\cup F(P,\boldsymbol{e}_i))} \rho_{Q_l}(\boldsymbol{x})^{n-q} + \rho_{Q_l}(-\boldsymbol{x})^{n-q} \,\mathrm{d}\boldsymbol{x} \\
&= \lim_{l\to\infty} \widetilde{\mathrm{C}}_{0,q}(K, Q_l, \mathbb{S}^{n-1}).
\end{aligned}$$

□

The foregoing propositions suggest that if, for fixed $q \in \mathbb{R}$ and $Q \in \mathcal{S}_o^n$, there are any non-trivial subspace concentration bounds on $\widetilde{\mathrm{C}}_{0,q}(K, Q, \cdot)$, they depend on Q.

As a final remark we want to discuss if and how Theorem 3.5 may be extended to the $(0, q)$th dual curvature measure with respect to $Q \in \mathcal{K}_s^n$. The main ingredient in our proof was Theorem 3.8. In this setting we would need to prove (3.15) with

$$\|\cdot\|_Q^\alpha = \rho_Q(\cdot)^{-\alpha}$$

in place of $|\cdot|^\alpha$. Note that for symmetric star bodies Q the function

$$\|\boldsymbol{x}\|_Q = \begin{cases} \rho_Q(\boldsymbol{x})^{-1}, & \text{if } \boldsymbol{x} \neq \mathbf{0}, \\ 0, & \text{if } \boldsymbol{x} = \mathbf{0}, \end{cases}$$

defines a quasinorm on $\mathbb{R}^n$, which is a norm if and only if Q is convex. In view of Karamata's inequality, which implies the 0-dimensional version of (3.15), one could even consider $\varphi(\|\cdot\|_Q)$ for a convex non-decreasing function $\varphi\colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ instead of $|\cdot|^\alpha$. The major obstruction for generalizing our method is a missing variant of Lemma 3.12 for arbitrary norms. Unfortunately, this is only possible in some cases.

Lemma 3.29 ([53, Lem. 6.4]). *Let $Q \in \mathcal{S}_o^n$ be an origin-symmetric star body and $\alpha > 0$. There exists a finite Borel measure μ on $\mathbb{S}^{n-1}$ with*

$$\|\boldsymbol{x}\|_Q^\alpha = \int\limits_{\mathbb{S}^{n-1}} |\langle \boldsymbol{\theta}, \boldsymbol{x} \rangle|^\alpha \, \mathrm{d}\mu(\boldsymbol{\theta})$$

for all $\boldsymbol{x} \in \mathbb{R}^n$, if and only if $(\mathbb{R}^n, \|\cdot\|_Q)$ can be isometrically embedded into the space of functions whose αth power of the absolute value is Lebesgue integrable over $\mathbb{R}^n$.

Conclusion

The present dissertation addressed Minkowski problems in Brunn-Minkowski, L_p-Brunn-Minkowski and dual Brunn-Minkowski theory and related topics.

In Chapter 2 we investigated the cone-volume measure, that takes on a significant role in the study of the logarithmic Brunn-Minkowski inequality (2.4) and the logarithmic Minkowski inequality (2.7), which are conjectured to hold for pairs of symmetric convex bodies. Known examples violating (2.7) typically involve a convex body with boundary points close to the origin. We established (2.7) in the context of centered convex bodies when one of the convex bodies is a simplex, or a parallelepiped in dimensions up to four. It was pointed out that the logarithmic Minkowski inequality (2.7) is connected to the problem of determining uniqueness of cone-volume measures. In Section 2.4 we fully characterized cone-volume measures of trapezoids alongside with necessary and sufficient conditions for their uniqueness. On this basis, we constructed vertex-minimal polygons with non-unique cone-volume measure in Section 2.5 and presented a fivegon found by Malikiosis confirming that the strict subspace concentration inequality (2.34) is not sufficent for uniqueness. From the characterization of cone-volume measures of trapezoids (Theorem 2.14) we deduced necessary and sufficient conditions in the logarithmic Minkowski problem for centered trapezoids. We are the first to state sufficient conditions for cone-volume measures of centered bodies that are neither symmetric nor simplices. It remains an open problem if in any dimension the cone-volume measures of centered convex bodies are unique and if the logarithmic Minkowski inequality holds for pairs of centered convex bodies. Future studies could also target the logarithmic problem for polygons with five or more vertices.

Chapter 3 addressed subspace concentration of dual curvature measures of symmetric convex bodies, which are geometric measures recently introduced by Huang, Lutwak, Yang and Zhang. They fill a blank spot in the dual Brunn-Minkowski theory and consequently have been studied thoroughly since their discovery. In addition, they deserve attention since the family of dual curvature measures include cone-volume measures of convex bodies. While most published works on the dual Minkowski problem aim for establishing sufficient conditions, we stated and proved – in the symmetric setting – the only known non-trivial necessary conditions. The given proofs are based on sharp estimates of integrals of quasiconcave and convex density functions over certain regions within a convex body. To this end, we established new Brunn-Minkowski type inequalites for the measures defined by these density functions in the Sections 3.2 and 3.3 leading to subspace concentration bounds on the qth dual curvature measure of an n-dimensional symmetric convex body in the parameter ranges $0 < q < n$ and $q \geq n + 1$. The remaining range $n < q < n + 1$ cannot be covered by our approach as was pointed out in Section 3.5. A detailed analysis of the occurring obstructions enabled us to extend the subspace concentration bounds to $(n, n + 1)$ for one-dimensional subspaces or when the convex body is a parallelepiped. From each of these results we deduced a tight subspace concentration inequality for qth dual curvature measures of symmetric convex bodies in the plane when $q > 2$. In Section 3.6 we applied our methods to the more general theory of L_p dual curvature measures to prove another subspace concentration inequality in this setting at least for $0 < q < n$. Future studies could aim for an extension of Theorem 3.8 to arbitrary norms which seems to require a different approach than the Kneser-Süss proof of Theorem 3.9. Other noteworthy research directions include establishing sufficient conditions in the dual Minkowski problem for $q > n$ or considering, e.g., centered convex bodies to investigate if known results for the cone-volume measure have natural generalizations.

Bibliography

[1] S. Alesker. ψ_2-estimate for the Euclidean norm on a convex body in isotropic position. *Geometric aspects of functional analysis (Israel, 1992–1994)*. Vol. 77. Oper. Theory Adv. Appl. Birkhäuser, Basel, 1995, 1–4 (cit. on p. 59).

[2] A. D. Alexandrov. *Selected works. Part I.* Vol. 4. Classics of Soviet Mathematics. Selected scientific papers, Translated from the Russian by P. S. V. Naidu, Edited and with a preface by Yu. G. Reshetnyak and S. S. Kutateladze. Gordon and Breach Publishers, Amsterdam, 1996, x+322 (cit. on pp. 9–13).

[3] T. W. Anderson. The integral of a symmetric unimodal function over a symmetric convex set and some probability inequalities. *Proc. Amer. Math. Soc.* 6 (1955): 170–176 (cit. on pp. 54, 56).

[4] S. Artstein-Avidan, A. Giannopoulos, and V. D. Milman. *Asymptotic geometric analysis. Part I.* Vol. 202. Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 2015, xx+451 (cit. on pp. 24, 59).

[5] F. Barthe, O. Guédon, S. Mendelson, and A. Naor. A probabilistic approach to the geometry of the l_p^n-ball. *Ann. Probab.* 33 (2) (2005): 480–513 (cit. on p. 12).

[6] C. Berg. Corps convexes et potentiels sphériques. *Mat.-Fys. Medd. Danske Vid. Selsk.* 37 (6) (1969): 64 pp. (1969) (cit. on p. 13).

[7] G. Bianchi, K. J. Böröczky, A. Colesanti, and D. Yang. The L_p-Minkowski problem for $-n < p < 1$. *Adv. Math.* 341 (2019): 493–535 (cit. on p. 17).

[8] B. Bollobás and I. Leader. Products of unconditional bodies. *Geometric aspects of functional analysis (Israel, 1992–1994)*. Vol. 77. Oper. Theory Adv. Appl. Birkhäuser, Basel, 1995, 13–24 (cit. on p. 17).

[9] C. Borell. Convex set functions in d-space. *Period. Math. Hungar.* 6 (2) (1975): 111–136 (cit. on p. 56).

[10] K. J. Böröczky and P. Hegedűs. The cone volume measure of antipodal points. *Acta Math. Hungar.* 146 (2) (2015): 449–465 (cit. on pp. 14, 32).

[11] K. J. Böröczky, P. Hegedűs, and G. Zhu. On the discrete logarithmic Minkowski problem. *Int. Math. Res. Not. IMRN*, (6) (2016): 1807–1838 (cit. on pp. 14, 32).

[12] K. J. Böröczky and M. Henk. Cone-volume measure and stability. *Adv. Math.* 306 (2017): 24–50 (cit. on pp. 12, 14).

[13] K. J. Böröczky and M. Henk. Cone-volume measure of general centered convex bodies. *Adv. Math.* 286 (2016): 703–721 (cit. on pp. 13, 41).

[14] K. J. Böröczky, M. Henk, and H. Pollehn. Subspace concentration of dual curvature measures of symmetric convex bodies. *J. Differential Geom.* 109 (3) (2018): 411–429 (cit. on pp. 4, 51, 52, 55, 66, 72, 79).

[15] K. J. Böröczky and H. T. Trinh. The planar L_p-Minkowski problem for $0 < p < 1$. *Adv. in Appl. Math.* 87 (2017): 58–81 (cit. on p. 17).

[16] K. J. Böröczky, E. Lutwak, D. Yang, and G. Zhang. Affine images of isotropic measures. *J. Differential Geom.* 99 (3) (2015): 407–442 (cit. on p. 12).

[17] K. J. Böröczky, E. Lutwak, D. Yang, G. Zhang, and Y. Zhao. The dual Minkowski problem for symmetric convex bodies (2017). eprint: `https://arxiv.org/abs/1703.06259` (cit. on pp. 4, 51).

[18] K. J. Böröczky, E. Lutwak, D. Yang, and G. Zhang. The log-Brunn-Minkowski inequality. *Adv. Math.* 231 (3-4) (2012): 1974–1997 (cit. on pp. 3, 13, 17, 18, 32).

[19] K. J. Böröczky, E. Lutwak, D. Yang, and G. Zhang. The logarithmic Minkowski problem. *J. Amer. Math. Soc.* 26 (3) (2013): 831–852 (cit. on pp. 2, 13, 32, 40).

[20] K. J. Böröczky and F. Fodor. The L_p dual Minkowski problem for $p > 1$ and $q > 0$. 2018. eprint: `https://arxiv.org/abs/1802.00933` (cit. on p. 79).

[21] S. Chen, Q.-R. Li, and G. Zhu. On the L_p Monge-Ampère equation. *J. Differential Equations*, 263 (8) (2017): 4997–5011 (cit. on p. 17).

[22] S. Chen, Q.-R. Li, and G. Zhu. The logarithmic Minkowski problem for non-symmetric measures. *Trans. Amer. Math. Soc.* (2018). eprint: `https://doi.org/10.1090/tran/7499` (cit. on pp. 14, 19, 40, 41).

[23] W. Chen. L_p Minkowski problem with not necessarily positive data. *Adv. Math.* 201 (1) (2006): 77–89 (cit. on p. 17).

[24] K.-S. Chou and X.-J. Wang. The L_p-Minkowski problem and the Minkowski problem in centroaffine geometry. *Adv. Math.* 205 (1) (2006): 33–83 (cit. on p. 17).

[25] D. Cordero-Erausquin, M. Fradelizi, and B. Maurey. The (B) conjecture for the Gaussian measure of dilates of symmetric convex sets and related problems. *J. Funct. Anal.* 214 (2) (2004): 410–427 (cit. on p. 17).

[26] S. Dancs and B. Uhrin. On a class of integral inequalities and their measure-theoretic consequences. *J. Math. Anal. Appl.* 74 (2) (1980): 388–400 (cit. on p. 56).

[27] H. Federer. Curvature measures. *Trans. Amer. Math. Soc.* 93 (1959): 418–491 (cit. on p. 11).

[28] W. Fenchel and B. Jessen. *Mengenfunktionen und konvexe Körper*. Matematisk-fysiske meddelelser. Levin & Munksgaard, 1938, 31 (cit. on pp. 10, 12).

[29] W. J. Firey. Christoffel's problem for general convex bodies. *Mathematika*, 15 (1968): 7–21 (cit. on p. 13).

[30] W. J. Firey. p-means of convex bodies. *Math. Scand.* 10 (1962): 17–24 (cit. on pp. 1, 16).

[31] R. J. Gardner. A positive answer to the Busemann-Petty problem in three dimensions. *Ann. of Math. (2)*, 140 (2) (1994): 435–447 (cit. on p. 48).

[32] R. J. Gardner. *Geometric tomography.* Second. Vol. 58. Encyclopedia of Mathematics and its Applications. Cambridge University Press, Cambridge, 2006, xxii+492 (cit. on pp. 5, 49).

[33] R. J. Gardner. The Brunn-Minkowski inequality. *Bull. Amer. Math. Soc. (N.S.)* 39 (3) (2002): 355–405 (cit. on p. 56).

[34] R. J. Gardner, D. Hug, and W. Weil. The Orlicz-Brunn-Minkowski theory: a general framework, additions, and inequalities. *J. Differential Geom.* 97 (3) (2014): 427–476 (cit. on p. 12).

[35] R. J. Gardner, A. Koldobsky, and T. Schlumprecht. An analytic solution to the Busemann-Petty problem on sections of convex bodies. *Ann. of Math. (2)*, 149 (2) (1999): 691–703 (cit. on p. 48).

[36] P. M. Gruber. *Convex and discrete geometry.* Vol. 336. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Springer, Berlin, 2007, xiv+578 (cit. on p. 5).

[37] B. Grünbaum. Partitions of mass-distributions and of convex bodies by hyperplanes. *Pacific J. Math.* 10 (1960): 1257–1261 (cit. on p. 20).

[38] P. Guan and L. Ni. Entropy and a convergence theorem for Gauss curvature flow in high dimension. *J. Eur. Math. Soc. (JEMS)*, 19 (12) (2017): 3735–3761 (cit. on p. 18).

[39] C. Haberl and L. Parapatits. The centro-affine Hadwiger theorem. *J. Amer. Math. Soc.* 27 (3) (2014): 685–705 (cit. on p. 12).

[40] B. He, G. Leng, and K. Li. Projection problems for symmetric polytopes. *Adv. Math.* 207 (1) (2006): 73–90 (cit. on p. 12).

[41] M. Henk and E. Linke. Cone-volume measures of polytopes. *Adv. Math.* 253 (2014): 50–62 (cit. on p. 13).

[42] M. Henk and H. Pollehn. Necessary subspace concentration conditions for the even dual Minkowski problem. *Adv. Math.* 323 (2018): 114–141 (cit. on pp. 4, 52, 57–59, 65, 70, 76).

[43] M. Henk and H. Pollehn. On the log-Minkowski inequality for simplices and parallelepipeds. *Acta Math. Hungar.* 155 (1) (2018): 141–157 (cit. on pp. 3, 19, 20, 23–25).

[44] M. Henk, A. Schürmann, and J. M. Wills. Ehrhart polynomials and successive minima. *Mathematika*, 52 (1-2) (2005): 1–16 (2006) (cit. on pp. 12, 52, 53).

[45] J. Hu and G. Xiong. The logarithmic John ellipsoid. *Geom. Dedicata* (2018) (cit. on p. 12).

[46] Y. Huang and Q. Lu. On the regularity of the L_p Minkowski problem. *Adv. in Appl. Math.* 50 (2) (2013): 268–280 (cit. on p. 17).

[47] Y. Huang and Y. Zhao. On the L_p dual Minkowski problem. *Adv. Math.* 332 (2018): 57–84 (cit. on p. 79).

[48] Y. Huang, E. Lutwak, D. Yang, and G. Zhang. Geometric measures in the dual Brunn-Minkowski theory and their associated Minkowski problems. *Acta Math.* 216 (2) (2016): 325–388 (cit. on pp. 1, 4, 11, 13, 49–51).

[49] D. Hug, E. Lutwak, D. Yang, and G. Zhang. On the L_p Minkowski problem for polytopes. *Discrete Comput. Geom.* 33 (4) (2005): 699–715 (cit. on p. 17).

[50] H. Jian, J. Lu, and X.-J. Wang. Nonuniqueness of solutions to the L_p-Minkowski problem. *Adv. Math.* 281 (2015): 845–856 (cit. on p. 17).

[51] Z. Kadelburg, D. Đukić, M. Lukić, and I. Matić. Inequalities of Karamata, Schur and Muirhead, and some applications. *The Teaching of Mathematics*, 8 (1) (2005): 31–45 (cit. on p. 57).

[52] H. Kneser and W. Süss. Die Volumina in linearen Scharen konvexer Körper. *Mat. Tidsskr. B* (1932): 19–25 (cit. on p. 59).

[53] A. Koldobsky. *Fourier analysis in convex geometry.* Vol. 116. Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 2005, vi+170 (cit. on pp. 49, 83).

[54] A. Kolesnikov and E. Milman. Local L^p-Brunn-Minkowski inequalities for $p < 1$. 2018. eprint: `https://arxiv.org/abs/1711.01089` (cit. on p. 16).

[55] J. Lu and X.-J. Wang. Rotationally symmetric solutions to the L_p-Minkowski problem. *J. Differential Equations*, 254 (3) (2013): 983–1005 (cit. on p. 17).

[56] M. Ludwig. General affine surface areas. *Adv. Math.* 224 (6) (2010): 2346–2360 (cit. on p. 12).

[57] M. Ludwig and M. Reitzner. A classification of $\mathrm{SL}(n)$ invariant valuations. *Ann. of Math. (2)*, 172 (2) (2010): 1219–1267 (cit. on p. 12).

[58] E. Lutwak. Dual mixed volumes. *Pacific J. Math.* 58 (2) (1975): 531–538 (cit. on pp. 1, 48).

[59] E. Lutwak. Intersection bodies and dual mixed volumes. *Adv. in Math.* 71 (2) (1988): 232–261 (cit. on p. 48).

[60] E. Lutwak. Mean dual and harmonic cross-sectional measures. *Ann. Mat. Pura Appl. (4)*, 119 (1979): 139–148 (cit. on p. 48).

[61] E. Lutwak. The Brunn-Minkowski-Firey theory. I. Mixed volumes and the Minkowski problem. *J. Differential Geom.* 38 (1) (1993): 131–150 (cit. on pp. 1, 16–18).

[62] E. Lutwak. The Brunn-Minkowski-Firey theory. II. Affine and geominimal surface areas. *Adv. Math.* 118 (2) (1996): 244–294 (cit. on p. 1).

[63] E. Lutwak and V. Oliker. On the regularity of solutions to a generalization of the Minkowski problem. *J. Differential Geom.* 41 (1) (1995): 227–246 (cit. on p. 17).

[64] E. Lutwak, D. Yang, and G. Zhang. L_p dual curvature measures. *Adv. Math.* 329 (2018): 85–132 (cit. on pp. 52, 78, 79).

[65] E. Lutwak, D. Yang, and G. Zhang. L_p John ellipsoids. *Proc. London Math. Soc. (3)*, 90 (2) (2005): 497–520 (cit. on p. 12).

[66] E. Lutwak, D. Yang, and G. Zhang. Orlicz centroid bodies. *J. Differential Geom.* 84 (2) (2010): 365–387 (cit. on p. 12).

[67] E. Lutwak, D. Yang, and G. Zhang. Orlicz projection bodies. *Adv. Math.* 223 (1) (2010): 220–242 (cit. on p. 12).

[68] L. Ma. A new proof of the log-Brunn-Minkowski inequality. *Geom. Dedicata*, 177 (2015): 75–82 (cit. on p. 18).

[69] V. D. Milman and A. Pajor. Entropy and asymptotic geometry of non-symmetric convex bodies. *Adv. Math.* 152 (2) (2000): 314–335 (cit. on p. 24).

[70] A. Naor. The surface measure and cone measure on the sphere of l_p^n. *Trans. Amer. Math. Soc.* 359 (3) (2007): 1045–1079 (electronic) (cit. on p. 12).

[71] A. Naor and D. Romik. Projecting the surface measure of the sphere of l_p^n. *Ann. Inst. H. Poincaré Probab. Statist.* 39 (2) (2003): 241–261 (cit. on p. 12).

[72] G. Paouris. Ψ_2-estimates for linear functionals on zonoids. *Geometric aspects of functional analysis.* Vol. 1807. Lecture Notes in Math. Springer, Berlin, 2003, 211–222 (cit. on p. 59).

[73] G. Paouris and E. M. Werner. Relative entropy of cone measures and L_p centroid bodies. *Proc. Lond. Math. Soc. (3)*, 104 (2) (2012): 253–286 (cit. on p. 12).

[74] C. Saroglou. Remarks on the conjectured log-Brunn-Minkowski inequality. *Geom. Dedicata*, 177 (2015): 353–365 (cit. on pp. 3, 17, 18).

[75] R. Schneider. *Convex bodies: the Brunn-Minkowski theory.* expanded. Vol. 151. Encyclopedia of Mathematics and its Applications. Cambridge University Press, Cambridge, 2014, xxii+736 (cit. on pp. 5, 7–11, 13, 17, 49, 50, 59).

[76] A. Stancu. Centro-affine invariants for smooth convex bodies. *Int. Math. Res. Not. IMRN*, (10) (2012): 2289–2320 (cit. on p. 12).

[77] A. Stancu. On the number of solutions to the discrete two-dimensional L_0-Minkowski problem. *Adv. Math.* 180 (1) (2003): 290–323 (cit. on pp. 3, 13).

[78] A. Stancu. The discrete planar L_0-Minkowski problem. *Adv. Math.* 167 (1) (2002): 160–174 (cit. on pp. 13, 19, 32).

[79] A. Stancu. The logarithmic Minkowski inequality for non-symmetric convex bodies. *Adv. in Appl. Math.* 73 (2016): 43–58 (cit. on p. 18).

[80] D. Xi and G. Leng. Dar's conjecture and the log-Brunn-Minkowski inequality. *J. Differential Geom.* 103 (1) (2016): 145–189 (cit. on p. 18).

[81] G. Zhang. A positive solution to the Busemann-Petty problem in $\mathbf{R}^4$. *Ann. of Math. (2)*, 149 (2) (1999): 535–543 (cit. on p. 48).

[82] Y. Zhao. Existence of solutions to the even dual Minkowski problem. *J. Differential Geom.* 110 (3) (2018): 543–572 (cit. on pp. 51, 72).

[83] Y. Zhao. The dual Minkowski problem for negative indices. *Calc. Var. Partial Differential Equations*, 56 (2) (2017): Art. 18, 16 (cit. on p. 52).

[84] G. Zhu. Continuity of the solution to the L_p Minkowski problem. *Proc. Amer. Math. Soc.* 145 (1) (2017): 379–386 (cit. on p. 17).

[85] G. Zhu. The centro-affine Minkowski problem for polytopes. *J. Differential Geom.* 101 (1) (2015): 159–174 (cit. on pp. 12, 17).

[86] G. Zhu. The logarithmic Minkowski problem for polytopes. *Adv. Math.* 262 (2014): 909–931 (cit. on p. 14).

[87] G. Zhu. The L_p Minkowski problem for polytopes for $0 < p < 1$. *J. Funct. Anal.* 269 (4) (2015): 1070–1094 (cit. on p. 17).

[88] G. Zhu. The L_p Minkowski problem for polytopes for $p < 0$. *Indiana Univ. Math. J.* 66 (4) (2017): 1333–1350 (cit. on p. 17).

Index

List of Symbols

$[\boldsymbol{x}, \boldsymbol{y}]$	convex hull/segment between $\boldsymbol{x}$ and $\boldsymbol{y}$
$\#X$	number of elements of a finite set X
$\cdot\vert L$	orthogonal projection onto L
$\vert\cdot\vert$	(Euclidean) norm
$\overline{\boldsymbol{x}}$	normalization $\frac{\boldsymbol{x}}{\vert\boldsymbol{x}\vert}$
$\langle\cdot,\cdot\rangle$	standard inner product
$(1-\lambda)K +_{\mathrm{p}} \lambda M$	p-combination of the convex bodies K and M w.r.t. λ (for $p = 0$ called log-combination)
$(1-\lambda)K \mathbin{\widetilde{+}} \lambda M$	radial combination of the star bodies K and M w.r.t. λ
aff	affine hull
$\boldsymbol{\alpha}_K^*$	reverse radial Gauss image of a convex body K
B_n	Euclidean unit ball
$\mathbf{c}$	centroid
$\mathrm{C}_i(K,\cdot)$	ith curvature measure of the convex body K
$\widetilde{\mathrm{C}}_q(K,\cdot)$	qth dual curvature measure of the convex body K
$\widetilde{\mathrm{C}}_{p,q}(K,\cdot)$	(p,q)th dual curvature measure of the convex body K
conv	convex hull
∂X	boundary of the set X
$\partial' K$	boundary points of convex body K with unique outer normal vector
dim	dimension
$\boldsymbol{e}_i$	ith standard basis vector of $\mathbb{R}^n$
$\mathcal{G}(i,n)$	Grassmannian of i-dimensional subspaces of $\mathbb{R}^n$
$H(\boldsymbol{a},\alpha)$	hyperplane defined by $\langle\boldsymbol{a},\boldsymbol{x}\rangle = \alpha$

$H^+(\boldsymbol{a},\alpha)$	halfspace defined by $\langle \boldsymbol{a},\boldsymbol{x}\rangle \geq \alpha$
$H^-(\boldsymbol{a},\alpha)$	halfspace defined by $\langle \boldsymbol{a},\boldsymbol{x}\rangle \leq \alpha$
$\mathcal{H}^k$	normalized k-dimensional Hausdorff measure
h_K	support function of convex body K
$\operatorname{int} X$	interior points of the set X
$\mathcal{K}^n$	set of convex bodies in $\mathbb{R}^n$
$\mathcal{K}^n_c$	set of centered convex bodies in $\mathbb{R}^n$
$\mathcal{K}^n_o$	set of convex bodies in $\mathbb{R}^n$ containing $\mathbf{0}$ in the interior
$\mathcal{K}^n_s$	set of symmetric convex bodies in $\mathbb{R}^n$
lin	linear hull
$L^\perp$	orthogonal complement of the subspace L
$M_p(\cdot,\cdot,\lambda)$	p-mean with weighting parameter λ
$\mathbb{N}$	natural numbers $1,2,\ldots$
$\boldsymbol{\nu}_K$	Gauss map of K
$\mathbf{p}_K$	metric projection map onto a convex body K
$\widetilde{\mathbf{p}}_K$	radial projection map onto a convex body K
$\mathcal{P}^n$	set of n-polytopes in $\mathbb{R}^n$
$\mathcal{P}^n_c$	set of centered polytopes in $\mathbb{R}^n$
$\mathcal{P}^n_o$	set of polytopes in $\mathbb{R}^n$ containing $\mathbf{0}$ in the interior
$\mathcal{P}^n_s$	set of symmetric n-polytopes in $\mathbb{R}^n$
pos	positive hull
ρ_K	radial function of convex body K
$\mathbb{R}$	real numbers
$\mathbb{R}_{>0}$ ($\mathbb{R}_{\geq 0}$)	positive (nonnegative) real numbers
$\mathbb{R}^n$	Euclidean n-space
$\mathbb{R}^{n\times n}$	real n-by-n matrices
$\mathbb{S}^{n-1}$	$(n-1)$-dimensional sphere in Euclidean n-space
$\mathrm{S}_i(K,\cdot)$	ith area measure of the convex body K
$\widetilde{\mathrm{S}}_q(K,\cdot)$	qth dual area measure of the convex body K
S_K	surface area measure of the convex body K
$\mathrm{S}^{(p)}_K$	L_p surface area measure of the convex body K
$\mathcal{S}^n_o$	set of star bodies in $\mathbb{R}^n$
supp	support

$U(P)$	set of unit outer normal vectors of a polytope P
V_K	cone-volume measure of the convex body K
vol_k	k-dimensional volume
W_i	ith quermassintegral