Aspects of Volume of Convex Bodies

Discretization, Subspace Concentration and Polarity

vorgelegt von
M. Sc.
Ansgar Freyer

an der Fakultät II – Mathematik und Naturwissenschaften
der Technischen Universität Berlin
zur Erlangung des akademischen Grades

Doktor der Naturwissenschaften
Dr. rer. nat.

genehmigte Dissertation

Promotionsausschuss:

Vorsitzender: Prof. Dr. John M. Sullivan
Gutachter: Prof. Dr. Martin Henk
Gutachter: Prof. Dr. Paolo Gronchi
Gutachter: Prof. Dr. Achill Schürmann

Tag der wissenschaftlichen Aussprache: 29. September 2022

Berlin 2023

Zusammenfassung

Gegenstand dieser Dissertation ist das Volumen konvexer Körper $K \subseteq \mathbb{R}^n$. Konkret untersuchen wir das Volumen $\mathrm{vol}(K)$ vor dem Hintergrund von drei verschiedenen Teilgebieten der Konvexgeometrie.

Im ersten Teil der Arbeit vergleichen wir das Volumen von K mit der Anzahl der ganzzahligen Punkte $\mathrm{G}(K) = |K \cap \mathbb{Z}^n|$, die in K enthalten sind. $\mathrm{G}(K)$ darf dabei als ein diskreter Volumenbegriff verstanden werden. Zum einen stellen wir Bezüge zwischen der Anzahl der ganzzahligen Punkte in K zu den Gitterpunkten in Schnitten von K mit geeigneten Hyperebenen her, wie man sie in ähnlicher Form aus der klassischen Konvexgeometrie kennt. Zum anderen wollen wir die Diskrepanz zwischen dem diskreten Volumen $\mathrm{G}(K)$ und dem kontinuierlichen Volumen $\mathrm{vol}(K)$ mit Hilfe von Parametern aus der Geometrie der Zahlen, wie etwa den sukzessiven Minima, kontrollieren.

Der zweite Teil widmet sich einer sogenannten Affine Subspace Concentration Condition für zentrierte Polytope. Eine solche wurde zuerst von Wu für zentrierte Polytope, die zusätzlich reflexiv und glatt sind, bewiesen [Wu22]. Wus Resultat gibt neue Erkenntnisse über die Verteilung des Volumens innerhalb dieser Polytope. Wir zeigen, dass die Reflexivität und Glattheit für die Gültigkeit von Wus Ungleichung nicht notwendig sind, d.h. wir verallgemeinern die Affine Subspace Concentration Conditions auf beliebige zentrierte Polytope. Dabei stoßen wir auf einen Zusammenhang zu den klassichen Linear Subspace Concentration Conditions in hohen Dimensionen.

Im dritten Teil befassen wir uns mit dem Mahlervolumen auf zwei speziellen Klassen von Polytopen, deren Eckenzahl in jeder Dimension beschränkt ist. Unser Ziel ist es für diese Klassen obere Schranken an das Mahlervolumen anzugeben, die eine Verbesserung der Blaschke-Santalóschen Ungleichung darstellen. Zunächst betrachten wir dazu die Voronoizellen von Gittern im $\mathbb{R}^3$. Wir zeigen, dass das Gitter $A_3^\star$ ein lokales Maximum des Mahlervolumens der Voronoizelle darstellt und wir präsentieren einen Ansatz um mit Hilfe von sogenannten Shadow Systems zu zeigen, dass das Maximum in der Tat global ist. Die zweite Klasse, die wir untersuchen, sind die Matching Polytope von Wäldern mit fester Kantenzahl. Wir konstruieren Triangulierungen dieser Polytope und ihrer polaren Polytope, die es uns erlauben ihr Volumen mit Mitteln der Kombinatorik auszudrücken. So erhalten wir eine obere Schranke an ihr Mahlervolumen, welche die Blaschke-Santaló Ungleichung für Wälder mit hinreichend vielen Blättern verbessert.

Abstract

The subject of this thesis is the volume of convex bodies $K \subseteq \mathbb{R}^n$. Specifically, we investigate the volume $\mathrm{vol}(K)$ in the context of three different branches of convex geometry.

In the first part of the thesis, we compare the volume of K to the number of integer points $\mathrm{G}(K) = |K \cap \mathbb{Z}^n|$ contained in K. Here, one might think of $\mathrm{G}(K)$ as a discrete notion of volume. First we establish relations between the lattice points in K and the lattice points in suitable hyperplane sections of K, similar to those that are known for $\mathrm{vol}(K)$ from classical convex geometry. Moreover, we control the discrepancy between the discrete volume $\mathrm{G}(K)$ and the continuous volume $\mathrm{vol}(K)$ with the help of parameters from the geometry of numbers, such as the successive minima.

The second part is devoted to so-called affine subspace concentration conditions for centered polytopes. Such conditions have first been proven by Wu [Wu22] for centered polytopes that are, in addition, reflexive and smooth. Wu's result gives new insights on the distribution of volume within those polytopes. We prove that the conditions of reflexivity and smoothness are not necessary for Wu's inequalities to hold, i.e., we generalize the affine subspace concentration conditions to arbitrary centered polytopes. In doing so we find a connection to the classical linear subspace conditions in high dimensions.

In the third part, we consider the Mahler volume on two classes of polytopes, whose vertex numbers are bounded in each dimension. Our goal is to formulate upper bounds on the Mahler volume for these classes that improve on the Blaschke-Santaló inequality. First, we work with Voronoi cells of lattices in $\mathbb{R}^3$. We prove that the lattice $A_3^\star$ is a local maximizer of the Mahler volume of these Voronoi cells and we present an approach to showing that it is indeed the global maximizer with the help of so-called shadow systems. The second class that we investigate, are matching polytopes of forest with a fixed number of edges. We construct triangulations of these polytpes and their polars that allow us to express their volume in combinatorial terms. As a consequence, we obtain an upper bound on the Mahler volume which improves on the Blaschke-Santaló inequality for forests with sufficiently many leaves.

Acknowledgments

I want to express my sincere gratitude to my advisor Martin Henk for introducing me to the wonderful world of convex geometry, as well as for his guidance, knowledge and constant support. It was a pleasure to be a member of his research group. Moreover, I wish to thank Paolo Gronchi, Achill Schürmann and John M. Sullivan for co-refereeing this thesis.

Besides my advisor, I thank Christian Kipp, Eduardo Lucas and Raman Sanyal for joyous collaborations. I also want to thank Marcel Celaya, Christoph Hunkenschröder, Christian Kipp, Nicolas Klenert and Stefan Kuhlmann for proofreading parts of this thesis, as well as Katharina Eller for many inspiring discussions.

Thanks go to my flatmates Alex, Frederic, Hossen, Maggie and Melanie who where always open to some geometry over a cup of coffee.

Finally, I thank my family, in particular Astrid, Bernd and Nils Freyer, and my girlfriend Lilli Leifheit with all my heart for their lasting encouragement throughout my studies.

Contents

1 Introduction

The present thesis is devoted to the investigation of three different aspects of the volume functional on the class of convex bodies. Reduced to three keywords, these are *discretization*, *subspace concentration* and *polarity*. More precisely, the thesis consists of three parts. In the first part, we will compare the volume of a convex body to the number of integer points it contains. In the second part, we prove a so-called affine subspace concentration condition for centered polytopes. In the third part, we compare the volumes of lattice Voronoi cells and matching polytopes of forests to the volume of their respective polar polytopes. The purpose of this chapter is to give an overview of the results obtained in the three parts, assuming some of the terms and definitions that are to be introduced in the upcoming chapters.

In Part I, we study the connection of the volume $\mathrm{vol}(K)$ of a convex body K to the number of lattice points $\mathrm{G}(K) = |K \cap \mathbb{Z}^n|$ contained in K. The two questions that we investigate are

1) Which properties of the volume functional $\mathrm{vol}(\cdot)$ carry over to the lattice point enumerator $\mathrm{G}(\cdot)$?
2) Are there bounds on the deviation $\mathrm{G}(K)/\mathrm{vol}(K)$ of the volume and the lattice point enumerator that are invariant with respect to unimodular transformations and converge to 1 as we replace K by rK, where $r \to \infty$?

Chapter 3 addresses the first question. We strive for inequalities that relate $\mathrm{G}(K)$ to the number of lattice points of K in suitable hyperplane sections and projections. A key objective of this research is to find "fully discrete" relations – these are typically inequalities that depend solely on $K \cap \mathbb{Z}^n$. Among other results, we find for an origin-symmetric body $K \subseteq \mathbb{R}^n$ the following lower bound in terms of coordinate sections:

$$\mathrm{G}(K)^{\frac{n-1}{n}} > \frac{1}{4^{n-1}} \Big(\prod_{i=1}^{n} \mathrm{G}(K \cap e_i^{\perp}) \Big)^{\frac{1}{n}}.$$

This may be regarded as a discrete variant of Meyer's inequality for the volume [Mey88]

$$\mathrm{vol}(K)^{\frac{n-1}{n}} \geq \frac{n!^{\frac{1}{n}}}{n} \Big(\prod_{i=1}^{n} \mathrm{vol}_{n-1}(K \cap e_i^{\perp}) \Big)^{\frac{1}{n}}.$$

Moreover, we construct a sequence of convex bodies that shows that, in contrast to Meyer's inequality, our result for the symmetric case cannot be extended to arbitrary convex bodies.

In a similar spirit, we given an upper bound on $\mathrm{G}(K)$ in terms of hyperplane sections by showing that

$$\mathrm{G}(K)^{(n-1)/n} \leq c_n \mathrm{G}(K \cap H)$$

for a suitable hyperplane H depending on K and a constant $c_n > 0$ depending only on the dimension n. As we will see, the order of magnitude of c_n is between $\sqrt{n}$ and n^2.

In Chapter 4 we treat Question 2. It is a general intuition that the discrete measure $\mathrm{G}(K)$ behaves similarly to $\mathrm{vol}(K)$ once the convex body in question is large enough. Our goal is to specify the meaning of the words "similarly" and "large" in this context. The main result of the chapter is the following threefold relation between the volume, the lattice point enumerator and the so-called successive minima $\lambda_i(K)$, $1 \leq i \leq n$, of an n-dimensional convex body $K \subseteq \mathbb{R}^n$:

$$\mathrm{vol}(K) \prod_{i=1}^{n} \left(1 - \frac{n\lambda_i(K)}{2}\right) \leq \mathrm{G}(K) \leq \mathrm{vol}(K) \prod_{i=1}^{n} \left(1 + \frac{n\lambda_i(K)}{2}\right), \tag{1.1}$$

where for the lower bound $\lambda_n(K) \leq 2/n$ is necessary. We apply this result to obtain a weak resolution of a long-standing open problem by Betke, Henk and Wills who conjectured that the following discrete version of Minkowski's classical second theorem on successive minima holds [BHW93]:

$$\mathrm{G}(K) \leq \prod_{i=1}^{n} \left\lfloor \frac{2}{\lambda_i(K)} + 1 \right\rfloor.$$

We deduce the weaker inequality

$$\mathrm{G}(K) \leq \prod_{i=1}^{n} \left(\frac{2}{\lambda_i(K)} + n\right).$$

It shows that a discrete version of Minkowski's theorem, which is equivalent to the continuous original, exists. A key step in our proof is to control the successive minima during a geometrical process known as the *Blaschke shaking* of K.

Although the bounds in (1.1) meet the criteria that are formulated in Question 2, they lack the property of being "tight at every scale". By this we mean that there is most likely no convex body K for which an arbitrary dilate $m\,K$, where $m \in \mathbb{N}$, achieves equality. In the plane we can overcome this issue for the upper bound by showing

$$\mathrm{G}(K) \leq \mathrm{vol}(K) \left(1 + \frac{\lambda_1(K)}{2}\right)(1 + \lambda_2(K)).$$

Equality is obtained for any right triangle of the form $K = \mathrm{conv}\{0, me_1, me_2\}$, $m \in \mathbb{N}$.

In Part II we turn to subspace concentration conditions. A polytope P that contains the origin in its interior, given by an irredundant representation of the form $P = \{x \in \mathbb{R}^n : \langle a_i, x\rangle \leq 1,\ 1 \leq i \leq m\}$, is said to fulfill the (linear) *subspace concentration condition* with respect to the linear subspace L, if

$$\sum_{i\colon a_i \in L} \operatorname{vol}(C_i) \leq \frac{\dim L}{n} \operatorname{vol}(P),$$

where C_i denotes the pyramid obtained by taking the convex hull of the origin together with the facet of P that corresponds to a_i. Apart from their key role in the study of the log-Minkowski problem, the subspace concentration conditions are interesting in their on right, since they encode strong geometric information on P. It has been shown by Henk and Linke that the subspace concentration conditions hold true for any centered polytope P together with any linear subspace L [HL14]. In the special class of smooth reflexive centered polytopes, Wu proved an affine version of the subspace concentration conditions, using the special role of these polytopes in toric geometry [Wu22]:

$$\sum_{i\colon a_i \in A} \operatorname{vol}(C_i) \leq \frac{\dim A + 1}{n+1} \operatorname{vol}(P),$$

where A is an affine subspace of $\mathbb{R}^n$. The main result of Chapter 5 is an extension of Wu's inequality to arbitrary centered polytopes. The proof makes use of a surprising phenomenon in high dimensions, where the linear subspace concentration conditions, applied to multiple pyramids over P, yield the affine subspace concentration conditions for P.

Part III is about the relation between the volume of an origin-symmetric body K to the volume of its polar body $K^\star$. Since the polarity operation is inclusion reversing, one can see (e.g. by considering the John ellipsoid) that there are constants $0 < c_n \leq C_n$ depending only on the dimension such that

$$c_n \leq \operatorname{vol}(K)\operatorname{vol}(K^\star) \leq C_n.$$

Thus, $\operatorname{vol}(K)$ and $\operatorname{vol}(K^\star)$ are reciprocal to one another, up to a constant. Their product is called the Mahler volume. The quest for the largest possible lower bound c_n is known as the Mahler conjecture. Although it dates back to 1930s, it is still one of the major open problems in convex geometry. We briefly summarize the state of the conjecture in Chapter 2. For more exhaustive references, we also refer to [Sch14, Sec. 10.7].

The smallest possible upper bound C_n, however, is known to be $\operatorname{vol}(B^n)^2$, where B^n is the Euclidean unit ball [Sch14, Eq. 10.28]. This bound is called the Blaschke-Santaló inequality. An important aspect of this bound is that equality is achieved, if and only if K is an ellipsoid. This raises the natural question of upper bounding the Mahler volume of a polytope that is reasonably far from being an ellipsoid (for example, because it has a bounded number of vertices). In the plane, Meyer and Reisner proved that the polygons with m vertices that maximize the Mahler volume are linear images of the regular m-gon [MR11]. In the general setting, Böröczky proved a stability version of the Blaschke-Santaló inequality [Bör10].

Here we are aim for explicit improvements of the Blaschke-Santaló inequality for two classes of polytopes for which the reciprocity relation between $\operatorname{vol}(P)$ and $\operatorname{vol}(P^\star)$ becomes particularly apparent. The first of these two classes are the Voronoi cells V_Λ of lattices

Λ, which we study in Chapter 6. As we shall see, the polar $V_\Lambda^\star$ of the Voronoi cell is the convex hull of the inversion of $\Lambda \setminus \{0\}$ at the unit sphere (up to a factor 2).

In the planar case, we identify the hexagonal lattice as the lattice Λ, which uniquely (up to dilations and isometries) maximizes the Mahler volume of $\mathrm{vol}(V_\Lambda)\mathrm{vol}(V_\Lambda^\star)$. Even though this fact follows from the aforementioned theorem of Meyer and Reisner, we provide an independent proof that exploits the periodic structure of the Voronoi subdivision.

The fact that the hexagonal lattice maximizes the Mahler volume is accordant with the idea of the hexagonal lattice as the "roundest lattice" in the plane – it also minimizes the covering radius and maximizes the packing radius among all 2-dimensional lattices of a fixed density. In higher dimensions, these parameters are in general not extremal for the same lattice, which is why there is no natural candidate for a lattice that might maximize the Mahler volume of its Voronoi cell. In the three-dimensional case, we can show that $A_3^\star$, the dual lattice to the root lattice A_3, is at least a local maximizer of the Mahler volume, and we are able to give supporting evidence for the conjecture, that it is in fact the unique global maximizer.

The second class of polytopes, which we investigate in Chapter 7, are matching polytopes of forests. For a forest G, the matching polytope $\mathrm{M}(G)$ is the convex hull of all indicator vectors of matchings of G. It has been shown by Liu that the volume of $\mathrm{M}(G)$ is given by the number of certain permutations of the edges of G, the so-called standard labelings [Liu12]. We extend Liu's ideas to the polar $\mathrm{AM}(G)$ of G and we find a second class of permutations, which we will call co-standard labelings, that encode the volume of the polar. It will turn out that the standard labelings act in a "global" way on the edges, while the co-standard labelings permute the edges "locally". We use this characterization of the volume of $\mathrm{M}(G)$ and its polar to obtain upper bounds on the Mahler volume of $\mathrm{M}(G)$ that depend only on a combinatorial parameter of G and improve on the Blaschke-Santaló inequality for forests with many leaves.

In the course of our study of the Mahler volume of $\mathrm{M}(G)$ we construct two explicit pulling triangulations of the matching polytope and its polar, respectively. In particular, we will see that the triangulation of $\mathrm{AM}(G)$ has an interesting greedy structure that bears strong similarities with the Stanley's triangulation of the chain polytope of a poset.

Chapter 3 is published in [FH22], a joint work with Martin Henk, while the results of Chapter 4 can be found in the joint work with Eduardo Lucas [FL22]. Chapter 5 is a joint work with Martin Henk and Christian Kipp and Chapter 7 originates from an ongoing project together with Raman Sanyal.

2 Basics

The present thesis investigates problems in the fields of convex and discrete geometry, as well as the geometry of numbers. In this chapter, we recall the basic concepts in these areas in order to provide the mathematical groundwork of the thesis. In addition to the summary presented here, there is a multitude of books that provide an exhaustive overview, which goes far beyond this chapter. We refer to the books of Gruber [Gru07] and Schneider [Sch14] as resources for convex geometry. The books [GL87], by Gruber and Lekkerkerker, and [Cas71], by Cassels provide a solid background on the geometry of numbers. Moreover [CS99], by Conway and Sloane, is a good reference for specific lattices and their properties, and in [BR07], by Beck and Robins, the number of integer points in lattice polytopes is studied in detail. Finally, Ziegler's book [Zie12] gives a thorough introduction into the theory of polytopes.

Before we come to the geometric background of the thesis, we fix standard terms and notations.

For a non-zero vector $x \in \mathbb{R}^n$, we denote the orthogonal complement of $\operatorname{span}\{x\}$ by $x^\perp$ and its Euclidean norm by $|x|$. For a set $X \subseteq \mathbb{R}^n$, $\mathrm{bd}X$, $\mathrm{cl}X$ and $\mathrm{int}X$ denote the boundary, closure and interior of X within $\mathbb{R}^n$. We write $B^n = \{x \in \mathbb{R}^n : |x| \leq 1\}$ and $\mathbb{S}^{n-1} = \mathrm{bd}B^n$ for the Euclidean unit ball and sphere, respectively, as well as $C_n = [-1,1]^n$ for the symmetric cube. The *line segment* between $x, y \in \mathbb{R}^n$ is denoted by $[x,y] = \{\lambda x + (1-\lambda)y : \lambda \in [0,1]\}$. For two non-empty sets $A, B \subseteq \mathbb{R}^n$ the *Minkowski sum* is defined elementwise, i.e., $A + B = \{a + b : a \in A, b \in B\}$. Similarly, for a scalar $\lambda \in \mathbb{R}$, one defines $\lambda A = \{\lambda\, a : a \in A\}$ and we write $-A = (-1)A$. For an affine subspace $A \subseteq \mathbb{R}^n$ and a set $M \subseteq \mathbb{R}^n$ we denote by $M|A$ the image of the orthogonal projection of M on A. We write $\{x\}|A = x|A$. Moreover, for $n \in \mathbb{N}$, we write $[n] = \{1, ..., n\}$ and $\binom{[n]}{k} = \{I \subseteq [n] : |I| = k\}$. For $i \in [n]$, we denote by $e_i \in \mathbb{R}^n$ the i-th standard unit vector. The symbol $\mathbb{1}_n$ denotes the vector $(1, ..., 1)^T \in \mathbb{R}^n$. If n is clear from the context we simply write $\mathbb{1} = \mathbb{1}_n$.

2.1 Convex Geometry

Convex geometry refers to the investigation of convex sets in $\mathbb{R}^n$. This has been initiated by Hermann Brunn and Hermann Minkowski. Since convex sets arise in various areas, such as optimization, functional analysis, or probability, to name a few, convex geometry is a very colourful branch of mathematics in which several different disciplines play together.

Classes of convex bodies. A *convex body* is a compact convex set $K \subseteq \mathbb{R}^n$. We say that K is *origin symmetric* (short: *symmetric*), if $K = -K$. The set of all convex bodies in $\mathbb{R}^n$ is denoted by $\mathcal{K}^n$, the set of all n-dimensional convex bodies $K \in \mathcal{K}^n$ is denoted by $\mathcal{K}^n_n$ and the set of all origin symmetric convex bodies is denoted by $\mathcal{K}^n_{os}$. A convex body $K \in \mathcal{K}^n$ is called *unconditional*, if it is symmetric with respect to all the coordinate hyperplanes, i.e., $(\pm x_1, ..., \pm x_n) \in K$, for any $x \in K$.

Functions associated to a convex body. A structural advantage of convex bodies towards arbitrary compact sets is the fact that convex bodies may be described by several well-behaved functions. A classical example of such a function is the *support function*. The *support function* of a convex body K is defined for $x \in \mathbb{R}^n$ as $\mathrm{h}(K, x) = \sup_{y \in K} \langle x, y \rangle$.

If the origin is an interior point of K, the *polar body* of K is defined as

$$K^\star = \{y \in \mathbb{R}^n : \langle x, y \rangle \leq 1, \forall x \in K\} \in \mathcal{K}^n.$$

Moreover, for such K, the *gauge function* $|\cdot|_K : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is defined by $|x|_K = \min\{\mu \geq 0 : x \in \mu K\}$. The gauge function is related to the support function via the formula $|\cdot|_{K^\star} = \mathrm{h}(K, \cdot)$. If $K \in \mathcal{K}^n_{os}$, then $|\cdot|_K$ defines a norm on $\mathbb{R}^n$ whose unit ball is K.

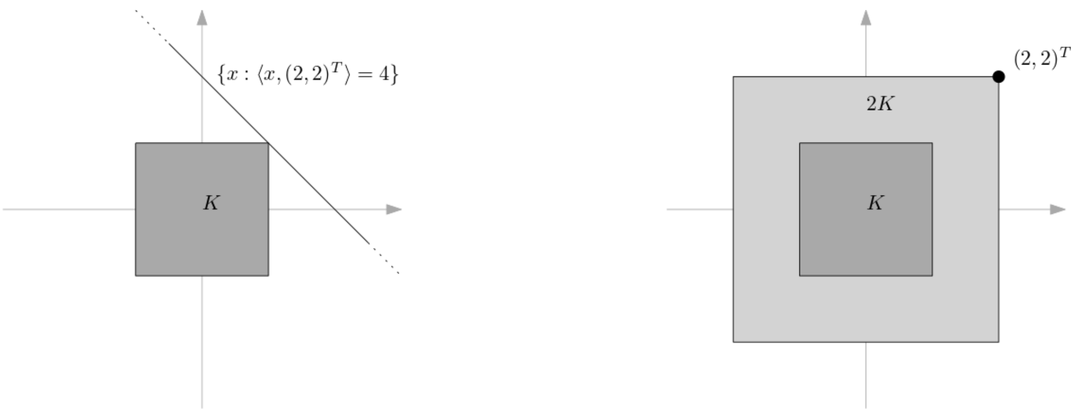


Figure 2.1: Illustration of the support and gauge functions of $K = C_2$ at the point $x = (2, 2)^T$. We have $\mathrm{h}(K, x) = 4$ and $|x|_K = 2$.

Volume. The *volume* $\mathrm{vol}(K)$ of a convex body K is its n-dimensional Lebesgue measure. If K is contained in a k-dimensional affine space F, we denote by $\mathrm{vol}_k(K)$ its k-dimensional Lebesgue measure in F.

One of the fundamental theorems concerning the volume of a convex body is the Brunn-Minkwoski inequality. For two convex bodies (or, more generally, compact sets) $K, L \subseteq \mathbb{R}^n$ and a scalar $\alpha \in [0, 1]$, it states that

$$\mathrm{vol}(\alpha K + (1-\alpha)L)^{\frac{1}{n}} \geq \alpha\, \mathrm{vol}(K)^{\frac{1}{n}} + (1-\alpha)\mathrm{vol}(L)^{\frac{1}{n}}. \tag{2.1}$$

In words, the volume functional to the power $1/n$ is concave with respect to Minkowski addition. In the convex case, equality is obtained in (2.1), if and only if $K = \beta L + t$, for certain $\beta \geq 0$ and $t \in \mathbb{R}^n$, or, if K and L lie in parallel hyperplanes.

An important consequence of (2.1) is the *concavity principle* of Brunn, which states that for any convex body $K \subseteq \mathbb{R}^n$ and any linear subspace $V \subseteq \mathbb{R}^n$ the *intersection function*

$$V \to \mathbb{R}, \quad x \mapsto \mathrm{vol}(K \cap (x + V^{\perp}))$$

is concave on its support, which is given by $K|V$. In particular, if $K \in \mathcal{K}^n_{os}$, the volume-maximal section with an affine space parallel to $V^{\perp}$ passes through the origin.

In convex geometry, there is a large variety of problems centered around the volume functional. One of them, which plays a particular role in this thesis, is the relation between the volume of K and its polar body $K^{\star}$. More precisely, the problem is to bound the product $\mathrm{vol}(K)\mathrm{vol}(K^{\star})$, for $K \in \mathcal{K}^n_{os}$.

As for the upper bound, it is known that

$$\mathrm{vol}(K)\mathrm{vol}(K^{\star}) \leq \mathrm{vol}(B^n)^2,$$

with equality, if and only if $K = AB^n$, for some $A \in \mathrm{GL}_n(\mathbb{R})$, i.e., K is an ellipsoid. This inequality is known as the *Blaschke-Santaló inequality*.

On the other end of the spectrum, Mahler conjectured that

$$\mathrm{vol}(K)\mathrm{vol}(K^{\star}) \geq \frac{4^n}{n!}. \tag{2.2}$$

This conjecture is known as the *Mahler conjecture* and the quantity $\mathrm{vol}(K)\mathrm{vol}(K^{\star})$ is also called the *Mahler volume* of K. It may be regarded as one of the central open problems in convex geometry.

Equality holds in (2.2) for the class of *Hanner polytopes*. A class that is defined by the following rules.

i) $C_1 = [-1, 1]$ is a Hanner polytope.

ii) $P \times Q$ is a Hanner polytope, if P and Q are Hanner polytopes.

iii) $P^{\star}$ is a Hanner polytope, if P is a Hanner polytope.

The Mahler Conjecture is confirmed in dimensions up to 3 [Mah39, IS20] and in general, by a result of Kuperberg [Kup09], it is known that

$$\mathrm{vol}(K)\mathrm{vol}(K^{\star}) \geq \frac{\pi^n}{n!}. \tag{2.3}$$

Moreover, it has been shown by Saint-Raymond that the Mahler conjecture holds true for unconditional bodies [Sai81].

Polytopes. For a set $X \subseteq \mathbb{R}^n$, we denote the *convex hull* of X by $\mathrm{conv}\, X$. If X is finite, $\mathrm{conv}\, X$ is called a *polytope*. On many occasions, polytopes act as the "discrete building blocks" of convex geometry.

By the Minkowski-Weyl theorem, any polytope $P \subseteq \mathbb{R}^n$ may be represented as an intersection of finitely many half-spaces, i.e.,

$$P = \{x \in \mathbb{R}^n : \langle x, a_i \rangle \leq b_i, \ \forall i \in [m]\}, \tag{2.4}$$

where $a_1, ..., a_m \in \mathbb{R}^n$ and $b_1, ..., b_m \in \mathbb{R}$. Conversely, any *bounded* intersection of finitely many half-spaces is a polytope. A constraint $\langle x, a_i \rangle \leq b_i$ in (2.4) is called *irredundant*, if it cannot be omitted without altering the polytope. In this case, the set $\{x \in P : \langle a_i, x \rangle = b_i\}$ is called a *facet* of P. More generally, a *face* F of a polytope P is an intersection of P with a supporting hyperplane, i.e., $F = \{x \in P : \langle x, a \rangle = \mathrm{h}(P, a)\}$ for some $a \neq 0$. We say that F is a k-face, if F is k-dimensional, and 0-dimensional faces are called *vertices* of P. The empty set and P itself are (-1)- resp. n-faces of P per convention.

If the origin is an interior point of P, one can normalize the inequalities on the right hand side of (2.4) such that $b_i = 1$ holds for all $i \in [m]$, i.e.,

$$P = \{x \in \mathbb{R}^n : \langle x, a_i \rangle \leq 1, \ \forall i \in [m]\}. \tag{2.5}$$

If this description is irredundant, the a_i's turn out to be the vertices of $P^\star$. In particular, the description is unique. There is a duality between the faces of P and $P^\star$: For a k-face $F \subseteq P$, $0 \leq k \leq n-1$ the set

$$F^\diamond = \{y \in P^\star : \langle y, x \rangle = 1, \ \forall x \in F\} \tag{2.6}$$

is an $(n-1-k)$-face of $P^\star$, which we call the *polar face* of F and every $(n-k-1)$-face of $P^\star$ arises as an $F^\diamond$, for some k-face $F \subseteq P$. Moreover, the diamond-operation is inclusion reversing, i.e., we have $F \subseteq G$, if and only if $G^\diamond \subseteq F^\diamond$.

A special class of polytopes in the context of integer points are *lattice polytopes*. These are polytopes of the form $P = \operatorname{conv} X$, where $X \subseteq \mathbb{Z}^n$. A lattice polytope P whose polar $P^\star$ is again a lattice polytope is called *reflexive*.

Decompositions. As in many other areas of mathematics, the "Divide-and-Conquer"-principle also finds its applications in the study of convex bodies. Most generally, a polyhedral *polyhedral decomposition* of a set X is a set of polytopes $\overline{P} = (P_i)_{i \in I}$ such that $X = \bigcup_{i \in I} P_i$ and any two distinct P_i's intersect in a set of measure zero.

Decompositions in this general sense are a convenient tool to compute the volume of a convex body. However, oftentimes it is helpful to impose more structure on a decomposition. This leads to what we shall call a *polyhedral subdivision* of X. By this, we mean a polyhedral decomposition $\overline{P}$ that satisfies the following properties:

i) If $P \in \overline{P}$ and $F \subseteq P$ is a face of P, then $F \in \overline{P}$.

ii) For any $P, P' \in \overline{P}$, the intersection $P \cap P'$ is a face of both P and P'.

The inclusion-maximal polytopes in $\overline{P}$ are called *facets* of $\overline{P}$. We notice that the facets of a polytope P coincide with the facets of its *boundary complex*, which is the polyhedral subdivision of $\operatorname{bd} P$ given by all k-faces of P, for $k < \dim P$. Moreover, a polyhedral subdivision is uniquely defined by its facets, which is why we shall oftentimes identify $\overline{P}$ with its facets.

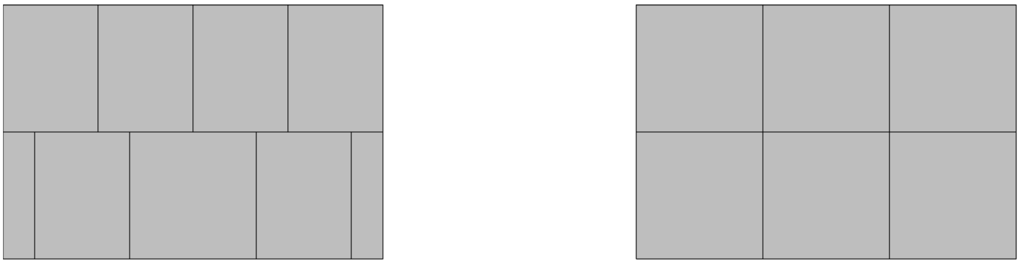


Figure 2.2: Two decompositions of a rectangle into smaller rectangles. The decomposition on the left is not a subdivision, but the decomposition on the right is.

Finally, a *triangulation* is a polyhedral subdivision $\overline{P}$, such that every $P \in \overline{P}$ is a *simplex*, i.e., the convex hull of $\dim(P) + 1$ affinely independent points.

Centroid. Intuitively, the *centroid* of a convex body is its center of gravity. Formally, one defines the centroid for a d-dimensional convex body $K \in \mathcal{K}^n$ as

$$\mathrm{c}(K) = \frac{1}{\mathrm{vol}_d(K)} \int_K x \,\mathrm{d}x \in K$$

where the integral is to be understood componentwise in the affine hull of K. K is called *centered*, if $\mathrm{c}(K) = 0$. Centered bodies are comparatively close to origin symmetric bodies in the sense that we have

$$-K \subseteq dK. \tag{2.7}$$

The factor d in the above inclusion cannot be improved as the centered simplex shows.

Moreover, the centroid may be computed from a decomposition. Let $\overline{P} = (P_1, ..., P_m)$ be a decomposition of a d-dimensional body K into d-dimensional polytopes (if any), then we have

$$\mathrm{c}(K) = \frac{\mathrm{vol}_d(P_1)}{\mathrm{vol}_d(K)} \mathrm{c}(P_1) + \cdots + \frac{\mathrm{vol}_d(P_m)}{\mathrm{vol}_d(K)} \mathrm{c}(P_m) \tag{2.8}$$

by the additivity of the integral.

Anti-blocking bodies. We finish this section by introducing the class of anti-blocking convex bodies, which play a major role at several points in this thesis. Anti-blocking bodies have been studied primarily in the context of optimization, their properties as listed below can be found in [Sch86, Sec. 9.3].

$K \in \mathcal{K}^n$ is called *anti-blocking*, if $K = L \cap \mathbb{R}^n_{\geq 0}$, for some unconditional body L. This is equivalent to the condition that $K \subseteq \mathbb{R}^n_{\geq 0}$ and for any $x \in K$, we have $(x'_1, ..., x'_n)^T \in K$, for all $x'_i \in [0, x_i]$. Both unconditional and anti-blocking bodies fulfill the equations $K \cap e_i^\perp = K | e_i^\perp$ for all $i \in [n]$. Indeed, anti-blocking bodies are characterized by this equation among convex bodies in $\mathbb{R}^n_{\geq 0}$.

Further, there is a one-to-one correspondence between anti-blocking and unconditional bodies: If K is unconditional, then $K \cap \mathbb{R}^n_{\geq 0}$ is anti-blocking by definition and if K is anti-blocking, then

$$\mathrm{U}K = \left\{ (\sigma_1 x_1, ..., \sigma_n x_n)^T : x \in K, \sigma_i \in \{-1, 1\} \right\}$$

is unconditional. This correspondence gives rise to a natural definition of polarity for anti-

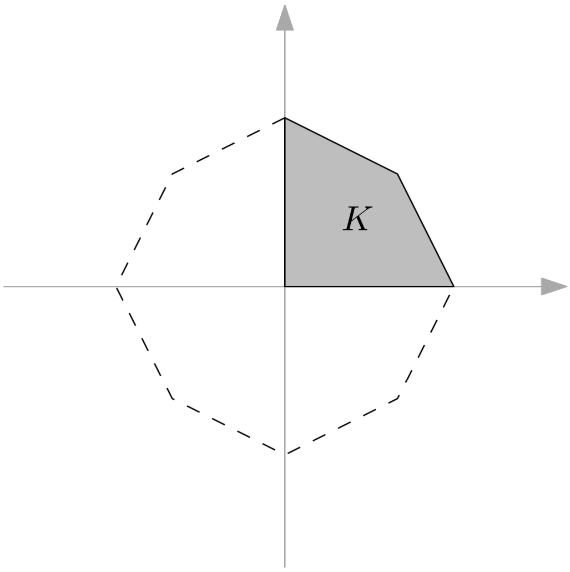


Figure 2.3: An example of an anti-blocking $K \subseteq \mathbb{R}^2$. Its associated unconditional $\mathrm{U}K$ is depicted by the dashed line.

blocking bodies: One defines the anti-blocking polar of an n-dimensional anti-blocking body K as $\mathrm{A}K = (\mathrm{U}K)^\star \cap \mathbb{R}^n_{\geq 0}$. This is equivalent to

$$\mathrm{A}K = \{ y \in \mathbb{R}^n_{\geq 0} \colon \langle x, y \rangle \leq 1, \ \forall x \in K \}.$$

If P is an anti-blocking polytope, there is a unique irredundant representation of P of the form (cf. (2.4))

$$P = \{ x \in \mathbb{R}^n_{\geq 0} : \langle x, a_i \rangle \leq 1, \ \forall i \in [m] \},$$

where $a_1, ..., a_m \in \mathbb{R}^n_{\geq 0}$. Unlike for polytopes that contain the origin in their interior, the vectors $a_1, ..., a_m$ are *not* the vertices of $\mathrm{A}P$. But we have $\mathrm{A}P = \{a_1, ..., a_m\}^\downarrow$, where for $X \subseteq \mathbb{R}^n_{\geq 0}$, one defines

$$X^\downarrow = \mathrm{conv}\{ x' \in \mathbb{R}^n_{\geq 0} : \exists x \in X \ \forall i \in [n] \ x'_i \leq x_i \}.$$

2.2 Geometry of Numbers

The geometry of numbers was historically introduced by Hermann Minkowski in 1910 in his fundamental work *Geometrie der Zahlen* [Min10] in order to treat problems that arise in number theory, such as the minima of quadratic forms, geometrically. Today, the theory has grown beyond its original purpose and serves as a powerful instrument for those who aim to count, estimate or compute the integer points in a convex set.

Lattices. At the heart of the theory are *(Euclidean) lattices*. These are discrete subgroups $\Lambda \subseteq \mathbb{R}^n$. Equivalently, a lattice may be represented as

$$\Lambda = \left\{ \sum_{i=1}^{k} \alpha_i b_i : \alpha_i \in \mathbb{Z},\ \forall i \in [k] \right\},$$

where $b_1, ..., b_k$ are linearly independent. The set on the right hand side is also referred to as the integral span of $b_1, ..., b_k$ and we denote it by $\operatorname{span}_{\mathbb{Z}}\{b_1, ..., b_k\}$. The set $\{b_1, \ldots, b_k\}$ is called a *(lattice) basis* of Λ and one defines $\det \Lambda = \operatorname{vol}_k([0, b_1] + \cdots + [0, b_k])$.

If $b_1, ..., b_n \in \mathbb{Z}^n$ form a basis of the lattice $\mathbb{Z}^n$, the corresponding matrix $U = [b_1, ..., b_n]$ is called *unimodular*. Unimodular matrices form a subgroup of $\mathrm{GL}_n(\mathbb{R})$, which is denoted by $\mathrm{GL}_n(\mathbb{Z})$. If A and B are (the matrices of) two bases of Λ, then there exists a unimodular matrix $U \in \mathrm{GL}_n(\mathbb{Z})$ such that $A = BU$. The converse is also true. With this characterization of lattice bases, one can define the topological *space of n-dimensional lattices* as the quotient $\mathcal{L}^n = \mathrm{GL}_n(\mathbb{R})/\mathrm{GL}_n(\mathbb{Z})$. This way, a sequence of lattices $(\Lambda^{(i)})_{i\in\mathbb{N}} \subseteq \mathcal{L}^n$ is convergent to a lattice $\Lambda \in \mathcal{L}^n$, if and only if for any basis B of Λ, there exists a sequence of bases $B^{(i)}$ of $\Lambda^{(i)}$, such that $B^{(i)}$ converges to B in $\mathbb{R}^{n,n}$.

A *sublattice* of Λ is a subgroup of Λ. Sublattices of Λ that arise as intersections $\Lambda \cap L$, where $L \subseteq \mathbb{R}^n$ is a linear subspace, are called *primitive*. If L fulfills $\dim L = \dim(\Lambda \cap L)$, L is called a *lattice subspace* of Λ. A point $v \in \Lambda \setminus \{0\}$ is called primitive, if $\mathbb{Z}v$ is a primitive sublattice of Λ. The *index* of a sublattice $\Lambda' \subseteq \Lambda$ is the number of different cosets $a + \Lambda'$, $a \in \Lambda$. If $\dim \Lambda' = \dim \Lambda$, this number is known to be finite and is given by $\det \Lambda' / \det \Lambda$.

The *polar lattice* of Λ is defined as

$$\Lambda^\star = \{a \in \operatorname{span} \Lambda : \langle b, a\rangle \in \mathbb{Z}, \forall b \in \Lambda\}.$$

There are several duality relations between Λ and $\Lambda^\star$ of which we recall a few here (cf. e.g. [Mar03, Prop. 1.3.4]). First of all the determinants of Λ and $\Lambda^\star$ are linked by the simple formula $\det \Lambda \det \Lambda^\star = 1$. Further, a k-dimensional subspace $L \subseteq \operatorname{span} \Lambda$ is a lattice subspace of Λ, if and only if its orthogonal complement $L^\perp$ in $\operatorname{span} \Lambda$ is a $(\dim \Lambda - k)$-dimensional lattice subspace of $\Lambda^\star$. In particular, every lattice hyperplane H of Λ possesses a primitive normal vector $v^\star \in \Lambda^\star$ and the determinant of $\Lambda \cap H$ is given by $|v^\star| \det \Lambda$. Moreover, the orthogonal projection $\Lambda^\star | L$ is a k-dimensional lattice and we have the following relation:

$$(\Lambda \cap L)^\star = \Lambda^\star | L. \tag{2.9}$$

The *lattice point enumerator* of a set $A \subseteq \mathbb{R}^n$ with respect to the lattice Λ is defined as $\mathrm{G}_\Lambda(A) = |A \cap \Lambda|$. In the case $\Lambda = \mathbb{Z}^n$ we write $\mathrm{G}(A) = \mathrm{G}_{\mathbb{Z}^n}(A)$

Relations between vol(·) and $\mathrm{G}_\Lambda(\cdot)$. The lattice point enumerator may be regarded as a discrete way to measure the size of a set M; while $\mathrm{vol}(M)$ counts every point in M via integration, $\mathrm{G}_\Lambda(M)$ forgets about all points in M except for the discrete set $M \cap \Lambda$. It is natural to ask under which conditions, these two notions of size may be related.

First of all, it follows from the properties of the Riemann integral that the volume and the lattice point enumerator agree asymptotically, i.e., one has

$$\lim_{r\to\infty} \frac{\mathrm{vol}_d(rM)}{\mathrm{G}_\Lambda(rM)} = \det \Lambda. \tag{2.10}$$

for any d-dimensional lattice Λ and any full-dimensional Jordan-measurable set $M \subseteq \mathrm{span}(\Lambda)$. Note that any convex body $K \subseteq \mathrm{span}(\Lambda)$ is Jordan-measurable. Moreover, (2.10) also holds, if one replaces $\mathrm{G}_\Lambda(rM)$ by $|M \cap (t + \Lambda)|$ for some $t \in \mathrm{span}(\Lambda)$.

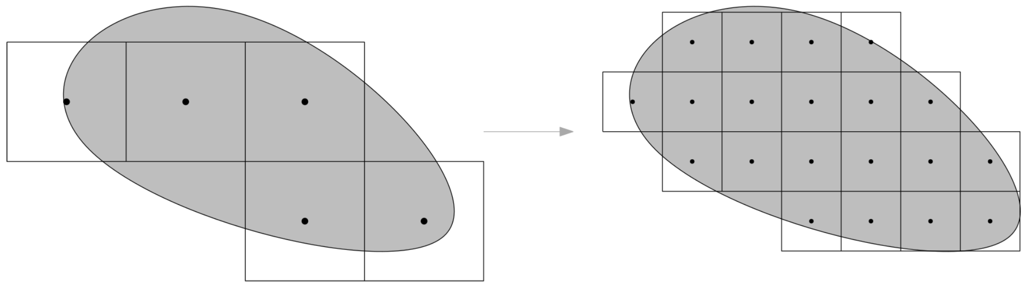


Figure 2.4: Illustration of (2.10); Counting the lattice points of Λ in $2M$ is equivalent to counting the points of $\frac{1}{2}\Lambda$ in M. If we replace Λ by $\varepsilon\,\Lambda$ as in the figure, the complex of boxes converges to M as $\varepsilon \searrow 0$. Note that each box in this complex has volume $\varepsilon^n \det \Lambda$.

The asymptotic relation (2.10) does not yield a direct insight on the number of lattice points of M itself in comparison to its volume. Indeed, if M is not convex, one can easily find examples for which $\mathrm{vol}(M)$ is arbitrarily small/large compared to $\mathrm{G}_\Lambda(M)$. For convex bodies, however, universal bounds are known. Let us consider a convex body $K \in \mathcal{K}^n$ and an n-dimensional lattice $\Lambda \subseteq \mathbb{R}^n$. By a a result of van der Corput [GL87, Ch. 2, Thm. 7.1] one has

$$\mathrm{vol}(K) \leq \big(2^{n-1}(\mathrm{G}_\Lambda(K) + 1)\big) \det \Lambda, \tag{2.11}$$

whenever $-K = K$. A lower bound on the volume has been obtained by Blichfeldt [Bli21], who showed for $K \in \mathcal{K}^n$ with $\dim(K \cap \Lambda) = n$ that

$$\mathrm{vol}(K) \geq \frac{1}{n!}\Big(\mathrm{G}_\Lambda(K) - n\Big) \det \Lambda. \tag{2.12}$$

Further, it is known that for any Lebesgue-measurable set $M \subseteq \mathbb{R}^n$, there exists a vector $t \in \mathbb{R}^n$ such that

$$\mathrm{vol}(M) \leq \mathrm{G}(M + t). \tag{2.13}$$

This result is attributed to van der Corput and Remak [GL87, Ch. 2, Sec. 6].

Finally, we want to mention here a classical identity of Pick that gives a precise connection between the volume and the number of lattice points in a planar lattice polygon $P \subseteq \mathbb{R}^2$:

$$\mathrm{vol}(P) = \mathrm{G}(P) - \frac{\mathrm{G}(\mathrm{bd}\, P)}{2} - 1. \tag{2.14}$$

Successive minima and covering radius. Consider an n-dimensional lattice Λ and an n-dimensional convex body $K \in \mathcal{K}^n_{os}$. Then, the *i-th successive minimum* is defined as

$$\lambda_i(K, \Lambda) = \min \left\{ \lambda > 0 : \dim(\lambda K \cap \Lambda) \geq i \right\},$$

for $i = 1, \ldots, n$. Roughly speaking, the successive minima of a K with respect to Λ measure the extend or "dimensionality" of K in the lattice Λ; For instance, if the first k successive minima $\lambda_i(K, \lambda)$ are significantly smaller than the remaining $n - k$, one might think of K as being concentrated around a k-dimensional lattice subspace of Λ.

If K is not necessarily symmetric, we define $\lambda_i(K, \Lambda) = \lambda_i(\mathrm{cs}(K), \Lambda)$, where $\mathrm{cs}(K) = \frac{1}{2}(K - K)$ is the *central symmetral* of K(cf. Figure 2.5). The successive minima of K can

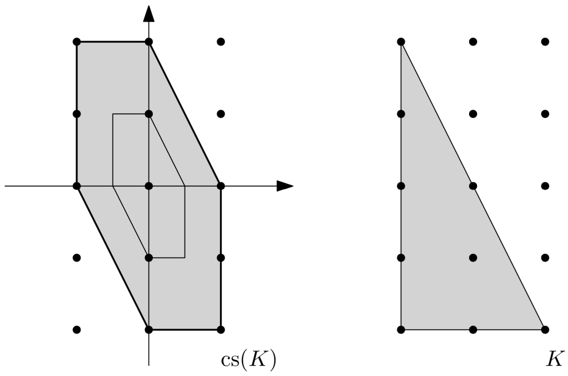


Figure 2.5: Illustration of the successive minima of a non-symmetric body. Since $\pm e_2$ are the only non-zero vectors in $\frac{1}{2}\mathrm{cs}(K)$, we have $\lambda_1(K) = 1/2$. Further, we have $\lambda_2(K) = 1$, which is attained by $\pm e_1$. In general, one may think of $\lambda_1(K)$ as the reciprocal of the maximum number of collinear lattice points in K, up to a factor 2 and rounding.

also be expressed with the help of the gauge function of $\mathrm{cs}(K)$; First, we have $\lambda_1(K, \Lambda) = \min |x|_{\mathrm{cs}(K)}$, where x ranges over $\Lambda \setminus \{0\}$, and we let $x_1 \in \Lambda \setminus \{0\}$ be a realizer of this minimum. For $1 < i \leq n$ we assume that we already found $i - 1$ independent vectors $x_1, ..., x_{i-1} \in \Lambda$ such that $\lambda_j(K, \Lambda) = |x_j|_{\mathrm{cs}(K)}$, $1 \leq j < i$. Then we have $\lambda_i(K, \Lambda) = \min |x|_{\mathrm{cs}(K)}$, where x ranges over $\Lambda \setminus \mathrm{span}\{x_1, ..., x_{i-1}\}$ and we let x_i be a realizer of this minimum.

For $K \in \mathcal{K}^n_n$, the linearly independent lattice points $x_i \in \Lambda$, $1 \leq i \leq n$, corresponding to the successive minima, i.e., $x_i \in \lambda_i(K, \Lambda)\, \mathrm{cs}(K)$, do not form a basis of Λ in general. It was shown by Mahler, however, that there exists a lattice basis $b_1, \ldots, b_n \in \Lambda$ such that

$$|b_i|_{\mathrm{cs}(K)} \leq \max \left\{ 1, \frac{i}{2} \right\} \lambda_i(K, \Lambda). \tag{2.15}$$

When considering the successive minima of the polar body $K^\star$ of $K \in \mathcal{K}^n_{os}$, the following useful reciprocity relation holds

$$\lambda_i(K^\star, \Lambda^\star)\lambda_{n+1-i}(K, \Lambda) \geq 1. \tag{2.16}$$

Moreover, the parameter $\lambda_1((K-K)^\star, \Lambda^\star)$ is called the *lattice width* of K with respect to Λ. The flatness theorem of Banaszczyk et al. states that [BLPS99]

$$\lambda_1((K-K)^\star, \Lambda^\star) \leq cn^{\frac{3}{2}} \tag{2.17}$$

for any convex body $K \in \mathcal{K}^n_n$ with $\mathrm{int} K \cap \mathbb{Z}^n = \emptyset$, where $c > 0$ is an absolute constant.

If $\Lambda = \mathbb{Z}^n$, we write $\lambda_i(K) = \lambda_i(K, \Lambda)$. Since one has $\lambda_i(K, A\mathbb{Z}^n) = \lambda(A^{-1}K, \mathbb{Z}^n)$, it is oftentimes enough to consider restrict to the case $\Lambda = \mathbb{Z}^n$.

As an "inhomogeneous counterpart" of the successive minima, the *covering radius* $\mu(K, \Lambda)$ is defined as the smallest number $\mu > 0$ such that $\mu K + \Lambda = \mathbb{R}^n$. Again, we write $\mu(K) = \mu(K, \mathbb{Z}^n)$. In terms of the gauge function, one has

$$\mu(K, \Lambda) = \max_{x \in \mathbb{R}^n} \min_{y \in \Lambda} |x - y|_{\mathrm{cs}(K)}.$$

For this reason, $\mu(K)$ is also referred to as the "inhomogeneous minimum" of K, while the $\lambda_i(K)$ is called the "i-th homogeneous minimum" of K. A distinction that dates back to Minkowski's studies of quadratic forms. The covering radius is related to the successive minima via the inequality [KL88, Lemma 2.4]

$$\frac{\lambda_n(K, \Lambda)}{2} \leq \mu(K, \Lambda) \leq \sum_{i=1}^{n} \frac{\lambda_i(K, \Lambda)}{2}. \tag{2.18}$$

Minkowski's theorems on successive minima. Minkowski's first theorem on successive states that a symmetric convex body $K \in \mathcal{K}^n_{os}$ with $K \cap \Lambda = \{0\}$, for some n-dimensional lattice Λ, has volume at most $2^n \det \Lambda$. Since $\lambda_1(K, \Lambda)$ contains no lattice points except for the origin, this theorem can be reformulated as

$$\lambda_1(K, \Lambda)^n \,\mathrm{vol}(K) \leq 2^n \det \Lambda.$$

Minkowski's first theorem has applications in various areas of number theory, ranging from Lagrange's four-square theorem to the study of number fields.

Minkowski also gave a generalization of his first theorem that takes all successive minima into account.

$$\frac{1}{n!} \prod_{i=1}^{n} \frac{2}{\lambda_i(K, \Lambda)} \leq \frac{\mathrm{vol}(K)}{\det \Lambda} \leq \prod_{i=1}^{n} \frac{2}{\lambda_i(K, \Lambda)}. \tag{2.19}$$

This classical result is known as Minkowski's second theorem on successive minima. For origin-symmetric K, this has been proven by Minkowski [GL87, Ch. 2, Thms. 9.1 and 9.2]. For general $K \in \mathcal{K}^n_n$, the upper bound follows directly from the inequality $\mathrm{vol}(K) \leq \mathrm{vol}(\mathrm{cs}(K))$, which in turn is a special case of the Brunn-Minkowski inequality (2.1). The lower bound can also be proved by an inclusion argument, similar to the symmetric case: One considers the convex hull of the n segments in K that realize the $\lambda_i(K, \Lambda)$ [HHH16, Rem. 1.1].

Betke, Henk and Wills studied the relation of the lattice point enumerator to the successive minima of K and conjectured for $K \in \mathcal{K}_n^n$ that [BHW93, Conj. 2.1]:

$$\mathrm{G}_\Lambda(K) \leq \prod_{i=1}^{n} \left\lfloor \frac{2}{\lambda_i(K, \Lambda)} + 1 \right\rfloor, \tag{2.20}$$

where for a real number $x \in \mathbb{R}$, $\lfloor x \rfloor = \max\{z \in \mathbb{Z} : z \leq x\}$ denotes the floor function. Malikiosis was able to show that [Mal12, Thm. 3.2.1]

$$\mathrm{G}_\Lambda(K) \leq \frac{4}{e}(\sqrt{3})^{n-1} \prod_{i=1}^{n} \left(\frac{2}{\lambda_i(K, \Lambda)} + 1 \right). \tag{2.21}$$

In the plane, the conjecture has been settled by Betke, Henk and Wills themselves [BHW93, Theorem 2.2] and in dimension 3 by Malikiosis [Mal12]. In arbitrary dimension, however, the factor $(4/e)\sqrt{3}^{n-1}$ in (2.21) is the best known. We will revisit the conjectured inequality (2.20) in more detail in Chapter 4.

Part I

Discretization

3 Bounds on the Lattice Point Enumerator via Slices and Projections

In this chapter, we formulate and prove inequalities that relate the lattice points inside a convex body K to the number of lattice points in hyperplane sections or projections of K. We obtain inequalities that may be regarded as "discrete analogues" of classical and modern theorems on the volume of convex bodies.

With the exceptions of Remark 3.1.6 and Theorem 3.2.6, the results in this chapter are published in [FH22], a joint work with Martin Henk.

3.1 A Discrete Meyer Inequality

One of the central questions in Geometric Tomography is to determine or to reconstruct a set K in the n-dimensional Euclidean space $\mathbb{R}^n$ by some of its lower dimensional "structures" (see [Gar06]). Usually, these are projections on and sections with lower dimensional subspaces of $\mathbb{R}^n$. A classical and very well-known example in this context is the famous Loomis-Whitney inequality [LW49], which compares the volume of a non-empty compact set K to the geometric mean of its projections onto the coordinate hyperplanes:

$$\mathrm{vol}(K)^{\frac{n-1}{n}} \leq \Big(\prod_{i=1}^{n} \mathrm{vol}_{n-1}(K|e_i^{\perp})\Big)^{\frac{1}{n}}. \tag{3.1}$$

Equality is attained, e.g., if $K = [a_1, b_1] \times \cdots \times [a_n, b_n]$, $a_i < b_i$, is a rectangular box. For various generalizations and extensions of this inequality we refer to [BGL18] and the references within.

Loomis and Whitney proved (3.1) by observing that it suffices to prove it when K is the non-overlapping union of equal cubes which is then a purely combinatorial problem. In particular, this combinatorial version implies (and is actually equivalent to) the following discrete variant of (3.1):

$$\mathrm{G}(K)^{\frac{n-1}{n}} \leq \Big(\prod_{i=1}^{n} \mathrm{G}(K|e_i^{\perp})\Big)^{\frac{1}{n}}. \tag{3.2}$$

So (3.1) and (3.2) are equivalent statements for compact sets. The discrete version (3.2) was also independently proven by Schwenk and Munro [SM83].

Due to the comparison of n- and $(n-1)$-dimensional volumes in (3.1), it is easy to see that there is no lower bound on the volume in terms of the geometric mean of $\mathrm{vol}_{n-1}(K|e_i^\perp)$. However, if we further assume that $K \in \mathcal{K}^n$, and if we replace projections by sections, then it was shown by M. Meyer [Mey88] that

$$\mathrm{vol}(K)^{\frac{n-1}{n}} \geq \frac{n!^{\frac{1}{n}}}{n}\Big(\prod_{i=1}^{n} \mathrm{vol}_{n-1}(K \cap e_i^\perp)\Big)^{\frac{1}{n}}, \tag{3.3}$$

where equality is attained, if and only if K is a generalized crosspolytope, i.e., $K = \mathrm{conv}\{a_1e_1, -b_1e_1, \dots, a_ne_n, -b_ne_n\}$, for some $a_i, b_i \geq 0$. Observe that $n!^{1/n}/n$ is asymptotically $1/e$. Meyer's inequality may be regarded as a dual inequality to (3.1) in the setting of polarity of convex bodies.

In [GGZ05], Gardner, Gronchi and Zong posed the question to find a discrete analogue of M. Meyer's inequality (3.3); more precisely, they asked

Question. *Let $n \in \mathbb{N}$. Is there a constant $c_n > 0$ such that for all $K \in \mathcal{K}^n$,*

$$\mathrm{G}(K)^{\frac{n-1}{n}} \geq c_n\Big(\prod_{i=1}^{n} \mathrm{G}(K \cap e_i^\perp)\Big)^{\frac{1}{n}} \ ? \tag{3.4}$$

As in the case of the Loomis-Whitney inequality, a discrete version (3.4) would imply the analogous inequality for the volume, and hence, by (3.3) we certainly have $c_n \leq n!^{1/n}/n$ (cf. (2.10)). In the plane, Gardner et al. [GGZ05] proved

$$\mathrm{G}(K)^{\frac{1}{2}} > \frac{1}{\sqrt{3}}\big(\mathrm{G}(K \cap e_1^\perp) \cdot \mathrm{G}(K \cap e_2^\perp)\big)^{\frac{1}{2}},$$

for any $K \in \mathcal{K}^2$. The elongated cross-polytope $K = \mathrm{conv}\{\pm e_1, \pm h e_2\}$ shows that $\frac{1}{\sqrt{3}}$ is asymptotically best possible, i.e., for $h \to \infty$. Hence, in contrast to the Loomis-Whitney inequality, (3.3) has no equivalent discrete version, since in the plane the constant in Meyer's inequality is $1/\sqrt{2}$.

Even more, in dimension $n \geq 3$ the answer to the above question of Gardner et al. is negative, as the following proposition shows.

Proposition 3.1.1. *Let $n \geq 3$ be fixed. There exists no positive number $c > 0$ such that for all $K \in \mathcal{K}^n$*

$$\mathrm{G}(K)^{\frac{n-1}{n}} \geq c\Big(\prod_{i=1}^{n} \mathrm{G}(K \cap e_i^\perp)\Big)^{\frac{1}{n}}. \tag{3.5}$$

Proof. We first prove it for $n = 3$. For an integer $k \in \mathbb{N}$, let T_k be the simplex with vertices $\{0, e_1, e_1 + k\, e_2, k\, e_3\}$ (see Figure 3.1). Then, $\mathrm{G}(T_k) = 2(k+1)$ and also $\mathrm{G}(T_k \cap e_1^\perp) = k+1$

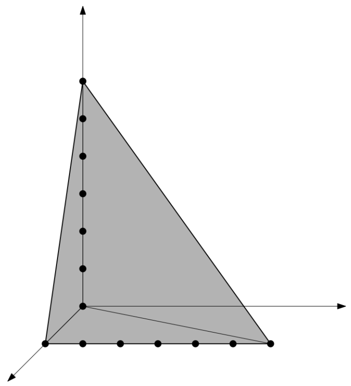


Figure 3.1: The simplex T_k.

and $\mathrm{G}(T_k \cap e_2^\perp) = \mathrm{G}(T_k \cap e_3^\perp) = k + 2$. Thus

$$\frac{(\mathrm{G}(T_k))^{\frac{2}{3}}}{\left(\prod_{i=1}^{3} \mathrm{G}(T_k \cap e_i^\perp)\right)^{\frac{1}{3}}} \leq 2^{\frac{2}{3}} \frac{(k+1)^{\frac{2}{3}}}{k+1} = 2^{\frac{2}{3}} (k+1)^{-\frac{1}{3}}$$

and so the left hand side tends to 0 as $k \to \infty$.

For $n \geq 4$ we just can consider, e.g., the simplices $\mathrm{conv}(T_k \cup \{e_4, \ldots, e_n\})$. $\square$

Roughly speaking, the simplex T_k from above falsifies (3.5) because the two skew segments $[0, ke_3]$ and $[e_1, e_1 + k\, e_2]$ are both "long", but do not generate any additional lattice points in T_k. Such a construction is not possible in the symmetric case. In fact, if $K \in \mathcal{K}_{os}^n$ possesses $2h + 1$ lattice points on the coordinate axis $\mathbb{R}e_n$, any interior lattice point $v \in K \cap e_n^\perp$ will contribute $O_{v,n}(h)$ lattice points to K. Here, $O_{v,n}$ hides a constant that only depends on v and n.

However, unlike the simplex above, a *symmetric* convex body always contains at least $\mathrm{G}(K)/3^n$ interior lattice points (see [GS16]). Motivated by this heuristic, we conjecture the following polytopes to be extremal in (3.4), when restricted to $\mathcal{K}_{os}^n$.

Example 3.1.2. For an integer $h \in \mathbb{N}$ let $K_h = \mathrm{conv}\big((C_{n-1} \times \{0\}) \cup \{\pm h\, e_n\}\big)$ be a double pyramid over the $(n-1)$-dimensional cube $C_{n-1} = [-1, 1]^{n-1}$ (see Figure 3.2). Then $\mathrm{G}(K_h) = 3^{n-1} + 2h$, $\mathrm{G}(K_h \cap e_i^\perp) = 3^{n-2} + 2h$, for $1 \leq i < n$, and $\mathrm{G}(K_h \cap e_n^\perp) = 3^{n-1}$. Thus,

$$\lim_{h \to \infty} \frac{\mathrm{G}(K_h)^{n-1}}{\prod_{i=1}^{n} \mathrm{G}(K_h \cap e_i^\perp)} = \frac{1}{3^{n-1}},$$

and we conjecture (3.4) to hold with $c_n = 3^{-\frac{n-1}{n}} \approx 1/3$, when restricted to symmetric convex bodies. $\diamond$

Indeed, there is strong computational evidence that the conjectured bound is correct in low dimensions (up to 5). We can prove the following weak version of the conjecture.

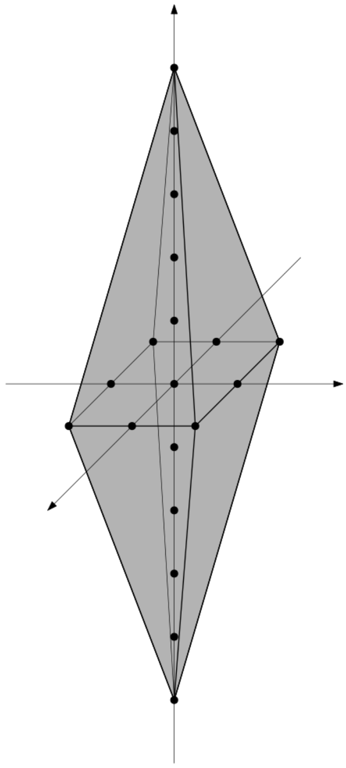


Figure 3.2: The double pyramid K_h.

Theorem 3.1.3. *Let $K \in \mathcal{K}^n_{os}$. Then*

$$\mathrm{G}(K)^{\frac{n-1}{n}} > \frac{1}{4^{n-1}}\Big(\prod_{i=1}^{n} \mathrm{G}(K \cap e_i^{\perp})\Big)^{\frac{1}{n}}.$$

For the proof, we need to understand the behaviour of the lattice point enumerator of a symmetric convex body regarding translations and dilations. These bounds will also be used extensively in the further sections of this chapter.

Lemma 3.1.4. *Let $K \in \mathcal{K}^n_{os}$, $t \in \mathbb{R}^n$ and $A \in \mathbb{Z}^{n,n}$ be an invertible matrix with integral entries. Then,*

$$\mathrm{G}\big(A\,K + t\big) \leq 2^{n-1}|\det A|(\mathrm{G}(K) + 1) \leq 2^n|\det A|\mathrm{G}(K).$$

In particular, we have

$$\mathrm{G}(K + t) \leq 2^{n-1}(\mathrm{G}(K) + 1) \leq 2^n\mathrm{G}(K), \tag{3.6}$$

which is best possible, and for $m \in \mathbb{N}$,

$$\mathrm{G}(m\,K) \leq 2^{n-1}\,m^n(\mathrm{G}(K) + 1) \leq (2m)^n\,\mathrm{G}(K), \tag{3.7}$$

which is best possible up to a factor $(1+\frac{1}{2m-1})^n$.

Proof. Let $\Lambda = 2A\mathbb{Z}^n \subseteq \mathbb{Z}^n$ and let $\Gamma_i \subseteq \mathbb{Z}^n$, $1 \le i \le 2^n|\det A|$, be the cosets of Λ in $\mathbb{Z}^n$. Consider two points $y_j = A\,x_j + t \in A\,K + t$, where $x_j \in K$, $j = 1,2$, that belong to a common Γ_i, say. For such points, we have $y_1 - y_2 \in \Lambda$. Thus, by the symmetry of K, we have

$$\frac{1}{2}(y_1 - y_2) = A\Big(\frac{1}{2}(x_1 - x_2)\Big) \in AK \cap A\mathbb{Z}^n = A(K \cap \mathbb{Z}^n).$$

That means

$$\big|(A\,K + t) \cap \Gamma_i - (A\,K + t) \cap \Gamma_i\big| \le \big|A(K \cap \mathbb{Z}^n)\big| = \mathrm{G}(K). \tag{3.8}$$

Since for any two finite sets $A, B \subseteq \mathbb{R}^n$ (cf. [TV06, Sec. 5.1])

$$|A + B| \ge |A| + |B| - 1,$$

we get from (3.8) $\big|(A\,K+t)\cap\Gamma_i\big| \le (\mathrm{G}(K)+1)/2$. Since the Γ_i's form a partition of $\mathbb{Z}^n$, the desired inequality follows.

In order to see that (3.6) is best-possible, consider the rectangular box $Q_k = \frac{1}{2}[-1,1]^{n-1} \times [-k+\frac{1}{2}, k-\frac{1}{2}]$, where $k \in \mathbb{N}$, and $t = (\frac{1}{2}, \dots, \frac{1}{2})^T \in \mathbb{R}^n$. Then, we have $\mathrm{G}(Q_k) = 2k-1$ and $\mathrm{G}(Q_k + t) = 2^{n-1}2k = 2^{n-1}(\mathrm{G}(Q_k)+1)$.

For (3.7), let $K = \big[-(1-\frac{1}{2m}), 1-\frac{1}{2m}\big]^n$; then $\mathrm{G}(m\,K) = (2m-1)^n\mathrm{G}(K)$. □

Remark 3.1.5.

i) In his thesis, Berg showed that for any $n \in \mathbb{N}$ there exists a $2K \in \mathcal{K}^n_{os}$ with $\mathrm{G}(2K) = 4^n - 2^n + 1$ such that any line passing through the origin contains at most 3 lattice points of $2K$ [Ber18, Prop. 5.13]. This implies that $\mathrm{G}(K) = 1$. Regarding (3.7) with $m = 2$, this example shows that the constant 4^n cannot be replaced by a constant of order c^n with $c < 4$.

ii) Restricted to the class of origin-symmetric lattice polytopes, (3.6) is best possible up to a factor 4, as the polytopes

$$K_h = \mathrm{conv}\big((C_{n-1} \times \{0\}) \cup \{\pm h\,e_n\}\big)$$

of Example 3.1.2 together with the vector $t = (1/2, \dots, 1/2, 0)^T$ show. On the one hand, one has $\mathrm{G}(K_h) = 2h + 3^{n-1}$, where $O(\cdot)$ describes the asymptotic behaviour for $h \to \infty$. On the other hand, the cube $[0,1]^{n-1}$ is contained in the relative interior of $K \cap e_n^\perp + t$. Even more, each of its vertices v are at half distance between t and one of the vertices of $K \cap e_n^\perp + t$. As a result of Brunn's concavity principle (cf. Section 2.1), the line $v + \mathbb{R}e_n$ is half as long as the line $t + \mathbb{R}e_n$. Consequently, it contributes $h + O(1)$ points to $(K_h + t) \cap \mathbb{Z}^n$. Since there are 2^{n-1} such lines, we obtain

$$\lim_{h\to\infty} \frac{\mathrm{G}(K_h + t)}{\mathrm{G}(K_h)} = \lim_{h\to\infty} \frac{2^{n-1}(h + O(1))}{2h + O(1)} = 2^{n-2}.$$

In fact, Wills [Wil73] showed that for any lattice polygon $P \subseteq \mathbb{R}^2$ and $t \in \mathbb{R}^2$ one has $\mathrm{G}(P+t) \leq \mathrm{G}(P)$. We conjecture that for any lattice polytope $P \in \mathcal{K}^n_{os}$,

$$\mathrm{G}(P+t) \leq 2^{n-2}\mathrm{G}(P).$$

iii) If K is not necessarily symmetric, (3.6) and (3.7) fail. In that case, counterexamples are given by the simplices T_k in the proof of Proposition 3.1.1. Basically, the reason for this is that T_k contains $O(k)$ lattice points, while in a translation or dilation of T_k one may find a right-angled triangle spanned by two orthogonal segments of length $O(k)$ each, lying in a hyperplane of the form $e_i^{\perp} + t$, $t \in \mathbb{Z}^3$. Such a triangle contributes $O(k^2)$ points. Again, letting $k \to \infty$ shows that the inequalities cannot be generalized to the non-symmetric case. This has also been observed independently by Lovett and Regev in [LR17]. ◇

Next we come to the proof of Theorem 3.1.3.

Proof of Theorem 3.1.3. For $i \in [n]$, let $K_i = K \cap e_i^{\perp}$. We consider the linear map

$$\phi : (K_1 \cap \mathbb{Z}^n) \times \cdots \times (K_n \cap \mathbb{Z}^n) \to ((K-K) \cap \mathbb{Z}^n)^{n-1} \text{ given by}$$
$$\phi\big((a_1, \ldots, a_n)\big) = \big((a_2 - a_1), (a_3 - a_1), \ldots, (a_n - a_1)\big).$$

Since this is an injective map, if follows that

$$\mathrm{G}(K-K)^{n-1} \geq \prod_{i=1}^{n} \mathrm{G}(K_i). \tag{3.9}$$

Since K is origin symmetric, we have $K - K = 2K$ and the bound follows from (3.7). □

Remark 3.1.6. At the expanse of a worse constant, one can extend the above proof to centered convex bodies. Indeed, for $K \in \mathcal{K}^n$ centered, we have

$$\begin{aligned}\mathrm{G}(K-K) &= \mathrm{G}\Big((n+1)\,\frac{1}{n+1}(K-K)\Big)\\ &\leq (2(n+1))^n\,\mathrm{G}\Big(\frac{1}{n+1}(K-K)\Big) \leq (2(n+1))^n\mathrm{G}(K),\end{aligned}$$

where we used Lemma 3.1.4, (3.7) and (2.7). Combining this with (3.9), we obtain

$$\mathrm{G}(K)^{\frac{n-1}{n}} \geq \frac{1}{(2(n+1))^{n-1}}\Big(\prod_{i=1}^{n}\mathrm{G}(K \cap e_i^{\perp})\Big)^{\frac{1}{n}}. \tag{3.10}$$

As the class of centered bodies is broader than the origin symmetric bodies, we do not expect $3^{-(n-1)/n}$ to be the best constant in (3.10). Indeed, by replacing the cube C_{n-1} in the construction of Example 3.1.2 by the centered simplex

$$S = \{x \in \mathbb{R}^{n-1} : x_i \geq -1,\, x_1 + \cdots + x_n \leq n-1\}$$

we see, by considering the centered bodies

$$P_h = \operatorname{conv}\big(S \times \{0\} \cup \{\pm e_n\}\big),$$

that the constant in (3.10) cannot be greater than $\mathrm{G}(S)^{-1/n} = \binom{2n-1}{n-1}^{-1/n} \approx 1/4$. $\diamond$

3.2 Reversing the Discrete Meyer Inequality

While it is not possible to bound the volume of a symmetric convex set K from above in terms of $\prod_{i=1}^{n} \mathrm{vol}_{n-1}(K \cap e_i^{\perp})$, Feng, Huang and Li proved in [FHL19] that there is a constant $\tilde{c}_n \leq (n-1)!$ such that for any $K \in \mathcal{K}_{os}^n$ there exists an orthogonal basis $u_1, \ldots, u_n \in \mathbb{R}^n$ such that:

$$\mathrm{vol}(K)^{\frac{n-1}{n}} \leq \tilde{c}_n \Big(\prod_{i=1}^{n} \mathrm{vol}_{n-1}(K \cap u_i^{\perp}) \Big)^{\frac{1}{n}}. \tag{3.11}$$

Alonso-Gutiérrez and Brazitikos [AB21] improved this result considerably: they showed that up to a universal constant the best possible $\tilde{c}_n$ is equal to the so called maximum isotropic constant in dimension n, which is bounded from above by $O((\log n)^4)$ (see [KL22]). For a definition of the isotropic constant and extensive background material thereunto, we refer to [BGVV14]. Moreover, they proved that this is valid for any centered convex body. Here we prove the following inequalities:

Theorem 3.2.1. *Let $K \in \mathcal{K}_{os}^n$. There exists a basis $b_1, \ldots, b_n$ of the lattice $\mathbb{Z}^n$ such that*

$$\mathrm{G}(K)^{\frac{n-1}{n}} < O(n^2\, 2^n) \Big(\prod_{i=1}^{n} \mathrm{G}(K \cap b_i^{\perp}) \Big)^{\frac{1}{n}}, \tag{3.12}$$

and there exists $t_i \in \mathbb{Z}^n$, $1 \leq i \leq n$, such that

$$\mathrm{G}(K)^{\frac{n-1}{n}} < O(n^2) \Big(\prod_{i=1}^{n} \mathrm{G}\big(K \cap (t_i + b_i^{\perp})\big) \Big)^{\frac{1}{n}}. \tag{3.13}$$

The idea of the proof is to choose a lattice basis that is short with respect to $|\cdot|_{K^\star}$ and then decompose $\mathbb{Z}^n = \bigcup_{j \in \mathbb{Z}} \{x \in \mathbb{Z}^n : \langle x, b_i \rangle = j\}$ and estimate the sections parallel to $b_i^{\perp}$ against the central one. Recall from Section 2.1 that for the volume, Brunn's concavity principle states that

$$\mathrm{vol}\big(K \cap (t + L)\big) \leq \mathrm{vol}(K \cap L),$$

for any k-dimensional linear subspace $L \subseteq \mathbb{R}^n$, $t \in \mathbb{R}^n$ and $K \in \mathcal{K}_{os}^n$. So the volume-maximal section of K parallel to L is indeed always the one containing the origin. Unfortunately, this is false in the discrete setting as the following example shows: Let $K = \operatorname{conv}\big(\pm([0,1]^{n-1} \times \{1\})\big)$ and for $1 \leq k \leq n-1$, let $L_k = \operatorname{span}\{e_1, ..., e_k\}$. Then $\mathrm{G}(K \cap L_k) = 1$, but $\mathrm{G}(K \cap (e_n + L_k)) = 2^k$.

Here we have the following kind of a discrete Brunn's concavity principle, which also appears in [Ber18, Sec. 5.2]:

Lemma 3.2.2. *Let* $K \in \mathcal{K}_{os}^n$ *and let* $L \subset \mathbb{R}^n$ *be a* k*-dimensional linear lattice subspace, i.e.,* $\dim(L \cap \mathbb{Z}^n) = k$, $k \in \{0, \dots, n-1\}$. *Then for any* $t \in \mathbb{R}^n$,

$$\mathrm{G}(K \cap (t + L)) \leq 2^k \mathrm{G}(K \cap L), \tag{3.14}$$

and the inequality is best possible.

For various discrete versions of the classical Brunn-Minkowski theorem, which in particular implies Brunn's concavity principle we refer to [GG01, HKS21, HIY18, IYZ20].

Proof. Without loss of generality, let $t \in \mathbb{Z}^n$. We denote $m = \mathrm{G}(K \cap L)$. Towards a contradiction, assume that $\mathrm{G}(K \cap (t + L)) > 2^k\, m$. Consider the k-dimensional lattice $\Lambda = \mathbb{Z}^n \cap L$. By the pigeon hole principle, there exist $(m+1)$ distinct lattice points $x_0, \dots, x_m \in K \cap (t + \Lambda)$ such that $x_0 - t, \dots, x_m - t$ are in the same coset of $\Lambda/2\Lambda$. This implies that

$$\frac{1}{2}(x_i - x_0) = \frac{1}{2}((x_i - t) - (x_0 - t)) \in K \cap \Lambda$$

are $m + 1$ distinct points in $K \cap \Lambda$, a contradiction to the choice of m. The tightness of the bound has already been verified in (3.14). □

Proof of Theorem 3.2.1. By induction on the dimension, we will show that for any n-dimensional convex body $K \in \mathcal{K}_{os}^n$ and any n-dimensional lattice Λ, there exists a basis $b_1, \dots, b_n$ of $\Lambda^\star$ and vectors $t_1, \dots, t_n \in \Lambda$ such that

$$\mathrm{G}_\Lambda(K)^{n-1} \leq (n!)^2 4^n \prod_{i=1}^{n} \mathrm{G}_\Lambda\big(K \cap (t_i + b_i^\perp)\big). \tag{3.15}$$

From this, (3.13) follows by considering $\Lambda = \mathbb{Z}^n$ and taking the nth root. Moreover, (3.12) follows immediately from (3.13) and Lemma 3.2.2.

First, we assume $\dim(K \cap \Lambda) = n$. For any $b \in \Lambda^\star$ we may write

$$\begin{aligned} \mathrm{G}_\Lambda(K) &= \sum_{i=-\lfloor \mathrm{h}(K,b) \rfloor}^{\lfloor \mathrm{h}(K,b) \rfloor} \mathrm{G}_\Lambda\big(K \cap \{x \in \mathbb{R}^n : \langle b, x \rangle = i\}\big) \\ &\leq (2\lfloor \mathrm{h}(K,b) \rfloor + 1)\mathrm{G}_\Lambda\big(K \cap (t_b + b^\perp)\big), \end{aligned} \tag{3.16}$$

where $t_b \in \Lambda$ is chosen to be the translation that maximizes the number of lattice points in a section parallel to $b^\perp$.

Now let $b_1, \dots, b_n \in \Lambda^\star$ be a basis of $\Lambda^\star$ obtained from (2.15) with respect to the polar body $K^\star$, i.e., we have $|b_i|_{K^\star} \leq i\lambda_i(K^\star, \Lambda^\star)$, $1 \leq i \leq n$. For the vectors b_i we denote the above translation vectors t_{b_i} by t_i. Then, on account of $\mathrm{h}(K, b_i) = |b_i|_{K^\star}$ we conclude from (3.16) that

$$\begin{aligned} \mathrm{G}_\Lambda(K)^n &\leq n! \prod_{i=1}^{n} \big(2\lambda_i(K^\star, \Lambda^\star) + 1\big) \prod_{i=1}^{n} \mathrm{G}_\Lambda\big(K \cap (t_i + b_i^\perp)\big) \\ &\leq n! 3^n \prod_{i=1}^{n} \lambda_i(K^\star, \Lambda^\star) \prod_{i=1}^{n} \mathrm{G}_\Lambda\big(K \cap (t_i + b_i^\perp)\big), \end{aligned} \tag{3.17}$$

where for the last inequality we used $\lambda_i(K^\star, \Lambda^\star) \geq 1$ which follows from the assumption $\dim(K \cap \Lambda) = n$ via (2.16).

Using the upper bound of Minkowski's theorem (2.19), the lower bound on the volume product (2.3) and van der Corput's inequality (2.11), we estimate

$$\prod_{i=1}^{n} \lambda_i(K^\star, \Lambda^\star) \leq \frac{2^n \det \Lambda^\star}{\mathrm{vol}(K^\star)} \leq n! \Big(\frac{2}{\pi}\Big)^n \mathrm{vol}(K) \det \Lambda^\star \leq n! \Big(\frac{4}{3}\Big)^n \mathrm{G}_\Lambda(K). \tag{3.18}$$

Substituting this into (3.17) yields the desired inequality (3.15) for this case.

It remains to consider the case $\dim(K \cap \Lambda) < n$, so let $K \cap \Lambda \subseteq H$ for some $(n-1)$-dimensional lattice subspace $H \subseteq \mathbb{R}^n$. Let $\Gamma = \Lambda \cap H$. We apply our induction hypothesis to Γ and $K \cap H$. Hence, we find a basis $y_1, ..., y_{n-1}$ of $\Gamma^\star$ and vectors $t_1, ..., t_{n-1} \in \Gamma$ such that

$$\mathrm{G}_\Gamma(K)^{n-2} \leq (n-1)!^2 4^{n-1} \prod_{i=1}^{n-1} \mathrm{G}_\Gamma\big(K \cap (y_i^\perp + t_i)\big),$$

which is equivalent to

$$\mathrm{G}_\Gamma(K)^{n-1} \leq (n-1)!^2 4^{n-1} \mathrm{G}_\Gamma(K \cap b_n^\perp) \prod_{i=1}^{n-1} \mathrm{G}_\Gamma\big(K \cap (y_i^\perp + t_i)\big), \tag{3.19}$$

where $b_n \in \Lambda^\star$ is a primitive normal vector of H. Unfortunately, the independent system $\{y_1, \ldots, y_{n-1}, b_n\}$ is in general not a basis of $\Lambda^\star$. In fact, the y_i's are not elements of $\Lambda^\star$ in the first place.

In view of (2.9), we have $\Gamma^\star = \Lambda^\star | H = \Lambda^\star | b_n^\perp$. So there are vectors $b_i \in (y_i + \mathbb{R} b_n) \cap \Lambda^\star$. For these vectors, one has $b_i^\perp \cap H = y_i^\perp \cap H$. By our assumption on K, this means

$$\mathrm{G}_\Gamma\big(K \cap (y_i^\perp + t_i)\big) = \mathrm{G}_\Lambda\big(K \cap (b_i^\perp + t_i)\big), \tag{3.20}$$

for all $1 \leq i \leq n-1$. Moreover, $\{b_1, \ldots, b_n\}$ is a $\Lambda^\star$-basis, since

$$\begin{aligned} \det(b_1, \ldots, b_n) &= \det(y_1, \ldots, y_{n-1}, b_n) \\ &= |b_n| \det \Gamma^\star = \frac{|b_n|}{\det \Gamma} = \frac{|b_n|}{|b_n| \det \Lambda} = \det \Lambda^\star. \end{aligned}$$

In view of (3.19) and (3.20), $\{b_1, \ldots, b_n\}$ is the desired basis and our proof is complete. □

If we just want to find linearly independent lattice points $a_i \in \mathbb{Z}^n$, $1 \le i \le n$, for the slices in Theorem 3.2.1 instead of a basis, then one can save one factor of n in the bounds of Theorem 3.2.1. To see this, we replace in the proof above the basis vectors b_i by linearly independent lattice points $a_i \in \lambda_i(K^\star)K^\star \cap \mathbb{Z}^n$, $1 \le i \le n$. In this case we do not need the estimate (2.15). In particular this leads to

$$\mathrm{G}(K)^{\frac{n-1}{n}} < O(n) \max_{t \in \mathbb{Z}^n, u \in \mathbb{Z}^n \setminus \{0\}} \mathrm{G}(K \cap (t + u^\perp)). \tag{3.21}$$

This may be regarded as a lattice version of the well-known *slicing problem* for volumes asking for the correct order of a constant c such that for all centered convex bodies $K \in \mathcal{K}^n$ there exists a $u \in \mathbb{R}^n \setminus \{0\}$ such that

$$\mathrm{vol}(K)^{\frac{n-1}{n}} \le c \,\mathrm{vol}(K \cap u^\perp). \tag{3.22}$$

To this day, the best known bound is of order $(\log n)^4$ (cf. [KL22]), improving an earlier result of Chen.

We obtain a discrete slicing inequality for non-symmetric convex bodies similar to (3.21).

Theorem 3.2.3. *Let $K \in \mathcal{K}^n$. Then*

$$\mathrm{G}(K)^{\frac{n-1}{n}} \le O(n^2) \max_{t \in \mathbb{Z}^n, u \in \mathbb{Z}^n \setminus \{0\}} \mathrm{G}(K \cap (t + u^\perp)). \tag{3.23}$$

If $K \in \mathcal{K}^n_{os}$, the constant can be replaced by $O(n)$.

Before we come to the proof, we note that if we consider (3.16), we can also estimate $\mathrm{G}(K)$, if K is not symmetric; in that case one replaces the factor $(2\lfloor \mathrm{h}(K, b)\rfloor + 1)$ on the right hand side by the number of hyperplanes parallel to $b^\perp$ that intersect K. As it will turn out, the challenge then is to compare the number of lattice points in $K - K$ to the number of points in K.

For volume, such a comparison is provided by the *Rogers–Shephard* inequality [RS58a], which asserts that

$$\mathrm{vol}(K - K) \le \binom{2n}{n} \mathrm{vol}(K).$$

The simplices T_k given in the proof of Proposition 3.1.1 show that there is no similar inequality for the lattice point enumerator: While we have $\mathrm{G}(T_k) = O(k)$, in $T_k - T_k$ we find the triangle $\mathrm{conv}\{0, k/2e_2, ke_3\}$ which contains $O(k^2)$ lattice points. Since k can be arbitrarily large, there is no constant $c_n > 0$ depending only on the dimension such that $\mathrm{G}(K - K) \le c_n \,\mathrm{G}(K)$.

However, the simplices T_k are extremely flat and therefore easily admit a large hyperplane section. Our strategy in order to prove Theorem 3.2.3 will be to argue that convex bodies K whose difference body $K - K$ contains disproportionately many lattice points are automatically flat. This reasoning is inspired by the proof of Theorem 4 in [AGP+17].

Proof of Theorem 3.2.3. The inequality for $K \in \mathcal{K}^n_{os}$ is already given by (3.21). So let $K \in \mathcal{K}^n$. We may assume $\dim(K \cap \mathbb{Z}^n) = n$, because otherwise, $K \cap \mathbb{Z}^n$ is contained in a hyperplane itself and the inequality follows directly. Therefore, $K - K$ contains n linearly independent lattice points and it follows $\lambda^\star = \lambda_1\big((K-K)^\star\big) \geq 1$ (cf. (2.16)). Let $y \in \lambda^\star (K-K)^\star \cap \mathbb{Z}^n \setminus \{0\}$. Then

$$\mathrm{h}(K, y) + \mathrm{h}(K, -y) = \mathrm{h}(K - K, y) = |y|_{(K-K)^\star} = \lambda^\star,$$

and similarly to (3.16), (3.17), we obtain for a certain $t \in \mathbb{Z}^n$

$$\mathrm{G}(K) \leq (2\lambda^\star + 1)\mathrm{G}\big(K \cap (t + y^\perp)\big) \leq 3\lambda^\star \mathrm{G}\big(K \cap (t + y^\perp)\big). \tag{3.24}$$

Let $c = \mathrm{c}(K)$ be the centroid of K. From (2.7) we deduce that

$$K - K \subset (n+1)(-c + K). \tag{3.25}$$

First we assume that $\frac{1}{2}(K + c) = c + \frac{1}{2}(-c + K)$ does not contain any integral lattice point. By the flatness theorem (2.17) we have

$$\lambda^* = 2\lambda_1\left(\big(\frac{1}{2}(c+K) - \frac{1}{2}(c+K)\big)^\star \right) = O(n^{3/2}).$$

In view of (3.24) we are done.

So we can assume that there is a lattice point $a \in \frac{1}{2}(c + K) \cap \mathbb{Z}^n$. By the choice of c (cf. (3.25)), we get

$$\begin{aligned} a + \frac{1}{2(n+1)}(K - K) &\subseteq \frac{1}{2}(c + K) + \frac{1}{2(n+1)}(K-K) \\ &= \frac{1}{2}K + \frac{1}{2}\big((c + \frac{1}{(n+1)}(K-K)\big) \\ &\subseteq \frac{1}{2}K + \frac{1}{2}K = K. \end{aligned} \tag{3.26}$$

Now in order to bound λ^* in this case we use (3.18) applied to $(K-K)^\star$ and $\mathbb{Z}^n$, which implies

$$(\lambda^*)^n \leq n! \left(\frac{4}{3}\right)^n \mathrm{G}(K - K).$$

Together with (3.24) we obtain

$$\mathrm{G}(K)^n \leq n! 4^n \mathrm{G}(K-K) \mathrm{G}\big(K \cap (t + y^\perp)\big)^n. \tag{3.27}$$

In order to estimate the number of lattice points in $K - K$ we may apply (3.7) and with (3.26) we get

$$\mathrm{G}(K-K) \leq 4^n (n+1)^n \mathrm{G}\Big(\frac{1}{2(n+1)}(K-K)\Big) \leq 4^n(n+1)^n \mathrm{G}(K).$$

By plugging this into (3.27), we obtain

$$\mathrm{G}(K)^{n-1} \le 16^n n!(n+1)^n \mathrm{G}\big(K \cap (t+y^\perp)\big)^n.$$

After taking the n-th root, we have

$$\mathrm{G}(K)^{\frac{n-1}{n}} \le O(n^2)\mathrm{G}\big(K \cap (t+y^\perp)\big),$$

as desired. □

Remark 3.2.4. The slicing problem (3.22) has also been extensively studied for other measures. For instance, Koldobsky [Kol14b] proved for origin-symmetric convex bodies that

$$\mu(K) \le O(\sqrt{n}) \max_{x \in \mathbb{R}^n \setminus \{0\}} \mu(K \cap x^\perp)\mathrm{vol}(K)^{1/n}$$

for measures μ that admit a continuous density and it was shown by Klartag and Livshyts [KL20] that the order of $\sqrt{n}$ is optimal (see also [KK18]). An extension to lower dimensional sections was given by Koldobsky in [Kol14a, Kol15], and Chasapis, Giannopoulos and Liakopoulos [CGL17] proved for general convex bodies that

$$\mu(K) \le O(k)^{(n-k)/2} \max_F \mu(K \cap F)\mathrm{vol}(K)^{\frac{n-k}{n}}, \tag{3.28}$$

where F ranges over all k–dimensional subspaces of $\mathbb{R}^n$ and μ is a measure with a locally integrable density function. In [AHZ17] the authors obtained an inequality similar to (3.28) for $K \in \mathcal{K}^n_{os}$ and the lattice point enumerator:

$$\mathrm{G}(K) \le O(1)^n \, n^{n-k} \max_F \mathrm{G}(K \cap F)\mathrm{vol}(K)^{\frac{n-k}{n}},$$

where F ranges over all k–dimensional linear subspaces with $\dim(F \cap \mathbb{Z}^n) = k$. In the case $k = n-1$ and convex bodies of "small" volume, Regev [Reg16] proved via a probabilistic approach such an inequality with the constant $O(n)$ instead of $O(n)^{n-1}$. ◇

Next, we will give an example that shows that all the constants in (3.12), (3.13), (3.21) and (3.23) must be at least of order $\sqrt{n}$.

Theorem 3.2.5. *For $n \in \mathbb{N}$ there exists a sequence of n-dimensional origin-symmetric convex bodies $(K_j)_{j \in \mathbb{N}}$ such that*

$$\limsup_{j \to \infty} \frac{\mathrm{G}(K_j)^{\frac{n-1}{n}}}{\sup_H \mathrm{G}(K_j \cap H)} \ge c\sqrt{n},$$

where H ranges over all affine hyperplanes in $\mathbb{R}^n$ and $c > 0$ is a universal constant.

Proof of Theorem 3.2.5. We consider a lattice $\Lambda \subseteq \mathbb{R}^n$ such that Λ is *self-polar* (i.e., $\Lambda = \Lambda^\star$), and $\lambda_1(B^n, \Lambda) = c\sqrt{n}$, where c is an absolute constant. Such lattices have been detected by Conway and Thompson [HM73, Thm. 9.5]. We will use a volume approximation argument for the Euclidean ball $rB^n = \{x \in \mathbb{R}^n : |x| \le r\}$, where $r \to \infty$.

For $x \in \mathbb{R}^n \setminus \{0\}$ and $\alpha \in \mathbb{R}$, let $H(x,\alpha) = \{y \in \mathbb{R}^n : \langle x, y\rangle = \alpha\}$ be the corresponding hyperplane. For $r > 0$ let $a_r \in \mathbb{R}^n$ and $\alpha_r \in \mathbb{R}_{\ge 0}$ be such that $\mathrm{G}_\Lambda(rB^n \cap H(a_r,\alpha_r))$ is maximal. Since Λ is self–polar, we may assume that $a_r \in \Lambda$ and $\alpha_r \in \mathbb{Z}$. In order to control the limit as $r \to \infty$ we want to find a sequence of radii $(r_j)_{j\in\mathbb{N}} \subseteq \mathbb{N}$ such that $r_j \to \infty$ and $H(a_{r_j}, \alpha_{r_j})$ is constant. To this end, fix a primitive vector $a_0 \in \Lambda$. Van der Corput's inequality (2.11) yields

$$\mathrm{G}_\Lambda\big(rB^n \cap H(a_r,\alpha_r)\big) \ge \mathrm{G}_\Lambda(rB^n \cap a_0^\perp) \ge 2^{-(n-1)} r^{n-1} \frac{\omega_{n-1}}{|a_0|}, \tag{3.29}$$

where ω_i denotes the volume of the i–dimensional Euclidean unit ball and we used that the determinant of $\Lambda \cap a_0^\perp$ is given by $|a_0| \det \Lambda = |a_0|$, since the determinant of any self-polar lattice is 1 (cf. Section 2.2).

On the other hand, if r is large enough, rB^n contains n linearly independent points of Λ. Thus, the maximal section $rB^n \cap H(a_r,\alpha_r)$ contains $(n-1)$ affinely independent points of Λ; otherwise, we might choose another point $x \in rB^n \cap \Lambda$ and replace $H(a_r,\alpha_r)$ by the affine hull of $rB^n \cap H(a_r,\alpha_r) \cap \Lambda$ and x. This yields a hyperplane that contains more lattice points of rB^n than $H(a_r,\alpha_r)$, contradicting the maximality. Hence, Blichfeldt's inequality (2.12) yields

$$\mathrm{G}_\Lambda\big(rB^n \cap H(a_r,\alpha_r)\big) \le n!\, r^{n-1} \frac{\omega_{n-1}}{|a_r|}.$$

Combining with (3.29), we obtain $|a_r| \le 2^{n-1}\, n!\, |a_0|$, for all but finitely many $r \in \mathbb{N}$. Since this bound is independent of r, we find a sequence $(r_j)_{j\in\mathbb{N}} \subseteq \mathbb{N}$ that tends to infinity such that $a_{r_j} = \overline{a}$, for all j and some primitive $\overline{a} \in \Lambda$ independent of j.

Since $\overline{a} \in \Lambda$, we have for any $\alpha > |\overline{a}|^2$,

$$-\overline{a} + (r_j B^n \cap H(\overline{a},\alpha) \cap \Lambda) \subseteq r_j B^n \cap H(\overline{a}, \alpha - |\overline{a}|^2) \cap \Lambda.$$

Hence, we may assume that $\alpha_{r_j} \le |\overline{a}|^2$. Since α_r is integral, we even find a sequence of radii $(r_j)_{j\in\mathbb{N}} \subseteq \mathbb{N}$ such that $H(a_{r_j}, \alpha_{r_j}) = H(\overline{a}, \overline{\alpha}) =: \overline{H}$ for all j and a fixed $\overline{\alpha} \in \mathbb{N}$.

We choose $K_j = r_j B^n$. In order to estimate the limit, we want to apply (2.10) to $r_j B^n$ and $r_j B^n \cap \overline{H}$. The latter body may be viewed as a ball of radius $r_j - o(r_j)$ that is embedded in an $(n-1)$-space together with a translation of $\Lambda \cap \overline{a}^\perp$. Thus, by (2.10),

$$\begin{aligned}
&\lim_{j\to\infty} \frac{\mathrm{G}_\Lambda(r_j B^n)^{n-1}}{\max \mathrm{G}_\Lambda(r_j B^n \cap H)^n} \\
&= \lim_{j\to\infty} \frac{\big(\mathrm{G}_\Lambda(r_j B^n)/r_j^n\big)^{n-1}}{\big(\mathrm{G}_\Lambda(r_j B^n \cap \overline{H})/(r_j - o(r_j))^{n-1}\big)^n} \\
&= |\overline{a}|^n \frac{\omega_n^{n-1}}{\omega_{n-1}^n} \ge (c\sqrt{n})^n e^{-c'n},
\end{aligned}$$

where $c' > 0$ is an absolute constant. In the last step we used the assumption that $\lambda_1(B^n, \Lambda) = c\sqrt{n}$ and Stirling's formula to estimate the volumes, together with the formula $\omega_n = \pi^{\frac{n}{2}}/\Gamma(\frac{n}{2}+1)$, where Γ is the Gamma-function. Taking the n-th root yields the claim. □

We conclude the section by proving an exact version of (3.13) in the plane.

Theorem 3.2.6. *Let $K \in \mathcal{K}^2$. Then there exist two non-parallel lattice lines $\ell_1, \ell_2 \subseteq \mathbb{R}^2$ such that*

$$\mathrm{G}(K) \leq \mathrm{G}(K \cap \ell_1) \cdot \mathrm{G}(K \cap \ell_2)$$

and the bound is tight.

Proof. Let ℓ_1 be the lattice line with the maximum number of lattice points in K. We choose ℓ_2 to be the lattice line that is non-parallel to ℓ_1 and contains the maximum number of lattice points in K among all such lines. After applying a unimodular transformation and a translation by an integral vector, we may assume that $\ell_1 = \mathbb{R}e_1$. Let $H_i = \ell_1 + ie_2$ be the horizontal line at height i and let $l_j = \mathrm{G}(K \cap \ell_j)$, $j = 1, 2$. Then we have

$$\mathrm{G}(K) = \sum_{i \in \mathbb{Z}} \mathrm{G}(K \cap H_i) = \sum_{i=0}^{l_2-1} \sum_{j \equiv i \bmod l_2} \mathrm{G}(K \cap H_j). \tag{3.30}$$

Let

$$M_i = \bigcup_{j \equiv i \bmod l_2} (K \cap H_j \cap \mathbb{Z}^2)$$

and $m_i = |M_i|$. Towards a contradiction, assume that $m_i > l_1$ for some $i \in \{0, ..., l_2 - 1\}$. As by construction $l_1 \geq l_2$, there exist two distinct points $x, y \in M_i$ with $x_1 \equiv y_1 \mod l_2$. By definition of M_i, we also have $x_2 \equiv y_2 \mod l_2$. Hence, by convexity, K contains the $l_2 + 1$ collinear lattice points $x + \frac{k}{l_2}(y - x)$, $k = 0, ..., l_2$ (cf. Figure 3.3)

It follows from the maximality of ℓ_2 that the segment $[x, y]$ must be parallel to ℓ_1, i.e., $x, y \in H_j$ for some $j \equiv i \mod l_2$. Also, by the maximality of ℓ_2, there cannot be a point $z \in M_i \setminus H_j$: By the pigeon hole principle, we could find a point in $z' \in [x, y]$ with $z_1 \equiv z'_1 \mod l_2$. Again, by definition of M_i, the points z and z' are congruent mod $l_2\mathbb{Z}^2$ and therefore, $[z, z']$ contains more lattice points than ℓ_2. But since $[z, z']$ is not parallel to ℓ_1, this is a contradiction.

Thus, it follows that $M_i \subseteq H_j$. In particular, the points in M_i are collinear. This contradicts the maximality of ℓ_1, since we assumed $|M_i| = m_i > l_1$. So we have $m_i \leq l_1$ and by (3.30) we obtain

$$\mathrm{G}(K) = \sum_{i=0}^{l_2-1} m_i \leq l_1 l_2$$

as desired. In order to see that the bound is tight, it suffices to consider rectangles of the form $[0, a] \times [0, b]$, where $a, b \in \mathbb{Z}$. □

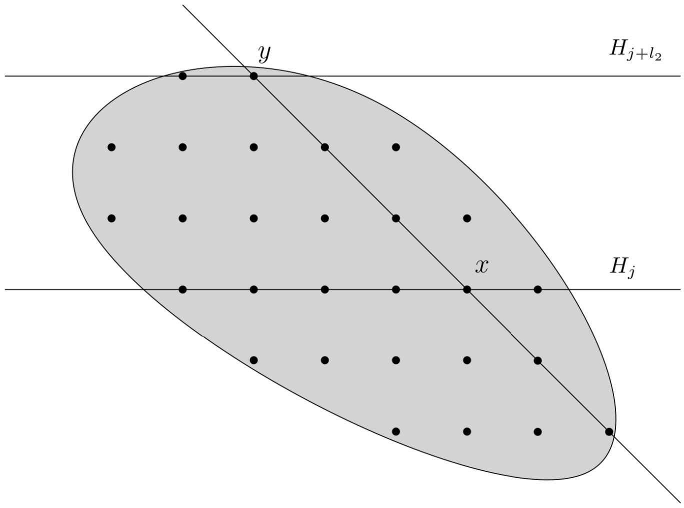


Figure 3.3: Two congruent points x and y lead to at most l_2+1 many collinear points.

Bounding the number of lattice points in a convex body by the maximal 1-dimensional affine section has been also done by Rabinowitz [Rab89] who obtained for $K \in \mathcal{K}^n$

$$\mathrm{G}(K) \leq \left(\max_{\ell} \mathrm{G}(K \cap \ell)\right)^n, \tag{3.31}$$

where the maximum ranges over all affine lattice lines ℓ. Berg proved a homogeneous version of (3.31) in [Ber18, Thm. 5.14]: For $K \in \mathcal{K}^n_{os}$, $n > 1$, there exists a 1-dimensional linear subspace $\ell \subseteq \mathbb{R}^n$ such that

$$\mathrm{G}(K) \leq \left(\frac{4}{3}\right)^n \mathrm{G}(K \cap \ell)^n.$$

3.3 Reversing the Discrete Loomis–Whitney Inequality

As for the projections, we discuss a reverse Loomis-Whitney inequality in the spirit of (3.11). Campi, Gritzmann, and Gronchi [CGG18] reversed the Loomis–Whitney inequality (3.1) by showing that there exists a constant $\tilde{d}_n \geq c/n$, where c is an absolute constant, such that

$$\mathrm{vol}(K)^{\frac{n-1}{n}} \geq \tilde{d}_n \left(\prod_{i=1}^{n} \mathrm{vol}(K|u_i^{\perp})\right)^{\frac{1}{n}}, \tag{3.32}$$

where again $u_1, \ldots, u_n$ form a suitable orthonormal basis. In [KSZ19], Koldobsky, Saroglou and Zvavitch showed that the optimal order of the constant $\tilde{d}_n$ is of size $n^{-1/2}$.

In order to get a meaningful discrete version of (3.32) we have to project so that $\mathbb{Z}^n|u_i^{\perp}$ is again a lattice, i.e., $u_i \in \mathbb{Z}^n$, and we have to count the lattice points of $K|u_i^{\perp}$ with respect to this lattice.

Theorem 3.3.1. *Let $K \in \mathcal{K}^n_{os}$ with $\dim(K \cap \mathbb{Z}^n) = n$. There exist linearly independent vectors $v_1, \dots, v_n \in \mathbb{Z}^n$ such that*

$$\mathrm{G}(K)^{\frac{n-1}{n}} \geq O(1)^{-n} \Big(\prod_{i=1}^{n} \mathrm{G}_{\mathbb{Z}^n|v_i^\perp}(K|v_i^\perp) \Big)^{\frac{1}{n}}.$$

Before we come to the proof, we have to estimate the number of points in the projection of $K|v^\perp$ with respect to the projected lattice $\mathbb{Z}^n|v^\perp$ against the number of points in $(K \cap \mathbb{Z}^n)|v^\perp$ for a given $v \in \mathbb{Z}^n$. To this end, we prove the following lemma.

Lemma 3.3.2. *Let $K \in \mathcal{K}^n_{os}$ and $v \in (K \cap \mathbb{Z}^n) \setminus \{0\}$. Then we have*

$$\mathrm{G}_{\mathbb{Z}^n|v^\perp}(K|v^\perp) \leq 4^{n-1} \cdot \big|(K \cap \mathbb{Z}^n)|v^\perp\big|.$$

Proof. Let $z \in (1/2)(K|v^\perp)$ be a lattice point in $\Lambda = \mathbb{Z}^n|v^\perp$. As the function $x \mapsto \mathrm{vol}_1(K \cap (x + \mathbb{R}v))$ is concave on $K|v^\perp$ by Brunn's concavity principle and we have, by assumption, $\mathrm{vol}_1(K \cap \mathbb{R}v) \geq 2|v|$, we deduce that the preimage of z has length at least $|v|$. Since it is located in a lattice line parallel to $\mathbb{R}v$, it contains a lattice point (cf. Figure 3.4).

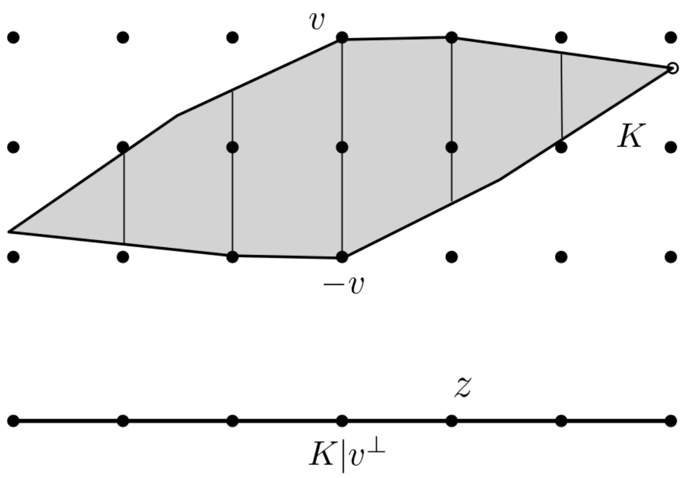


Figure 3.4: The point z is located deep inside of $K|v^\perp$. Therefore, its preimages is long enough to guarantee a lattice point.

So we have shown

$$\mathrm{G}_\Lambda\Big(\frac{1}{2}K|v^\perp\Big) \leq |(K \cap \mathbb{Z}^n)|v^\perp|.$$

Applying Lemma 3.1.4, (3.7), in the $(n-1)$-dimensional space $v^\perp$ with the lattice Λ gives $\mathrm{G}_\Lambda(K|v^\perp) \leq 4^{n-1}\mathrm{G}_\Lambda((1/2)(K|v^\perp))$, which concludes the proof. $\square$

Remark 3.3.3. The inequality of Lemma 3.3.2 is essentially best possible, in the sense that in any dimension, there is a convex body $K \in \mathcal{K}_{os}^n$ with $e_n \in K$, such that

$$\mathrm{G}_{\mathbb{Z}^n|e_n^\perp}(K|e_n^\perp) = 3^{n-1} \quad \text{and} \quad (K \cap \mathbb{Z}^n)|e_n^\perp = \{0\}.$$

To see this, let $u = (1, 2, 4, ..., 2^{n-1})^T \in \mathbb{R}^n$. We have $C_n \cap u^\perp \cap \mathbb{Z}^n = \{0\}$; Suppose there were a non-zero point $x \in C_n \cap u^\perp \cap \mathbb{Z}^n$. Let i be the largest index such that $x_i \neq 0$. By symmetry, we may assume that $x_i > 0$. It follows from

$$\sum_{j=0}^{i-1} 2^j = 2^i - 1, \tag{3.33}$$

that $\langle x, u\rangle \geq 1$, a contradiction to $x \in u^\perp$.

On the other hand, we have $(C_n \cap u^\perp)|e_n^\perp = C_{n-1}$. Let $x \in \{\pm 1\}^{n-1}$ be a vertex of C_{n-1}. Then, by (3.33), $\left|\sum_{i=1}^{n-1} x_i 2^{i-1}\right| \leq 2^{n-1}$. So there exists $x_n \in [-1, 1]$ such that $(x, x_n)^T \in C_n \cap u^\perp$.

Thus, the convex body $K = \mathrm{conv}\big((C_n \cap u^\perp) \cup \{\pm e_n\}\big)$ has the desired properties. $\diamond$

Now we are ready for the proof of Theorem 3.3.1.

Proof of Theorem 3.3.1. We abbreviate $\lambda_i = \lambda_i(K)$ and let $v_i \in \lambda_i\, K \cap \mathbb{Z}^n$, $1 \leq i \leq n$, be linearly independent. Due to our assumption we have $\lambda_n \leq 1$ and so $v_i \in K$. So we may apply Lemma 3.3.2 to obtain

$$\prod_{i=1}^n \mathrm{G}_{\mathbb{Z}^n|v_i^\perp}(K|v_i^\perp) \leq 4^{n(n-1)} \prod_{i=1}^n \left|Z|v_i^\perp\right|,$$

where $Z = K \cap \mathbb{Z}^n$. It is therefore enough to show that

$$\prod_{i=1}^n \left|Z|v_i^\perp\right| \leq O(1)^{n^2} |Z|^{n-1}. \tag{3.34}$$

To this end we set $S_i = Z \cap \mathbb{R}v_i$, $1 \leq i \leq n$. Then $|S_i| = 2\lfloor 1/\lambda_i \rfloor + 1$. Now we choose a subset $Z_i \subseteq Z$ such that the projection $Z_i \to Z|v_i^\perp$ is bijective. Clearly, $Z_i + S_i \subseteq Z + Z$ and so we have

$$\left|Z + Z\right| \geq \left|Z_i + S_i\right| = (2\lfloor 1/\lambda_i \rfloor + 1) \cdot \left|Z|v_i^\perp\right| \geq \frac{2}{3}((2/\lambda_i + 1)\left|Z|v_i^\perp\right|,$$

where for the last estimate we used once more that $\lambda_n \leq 1$. So we obtain

$$|Z + Z|^n \geq \left(\frac{2}{3}\right)^n \prod_{i=1}^n (2/\lambda_i + 1) \prod_{i=1}^n \left|Z|v_i^\perp\right|.$$

In view of Lemma 3.1.4, (3.7) we have $|Z+Z| \leq \mathrm{G}(2K) \leq 4^n \mathrm{G}(K) = 4^n |Z|$ and thus

$$4^{n^2}\left(\frac{3}{2}\right)^n |Z|^n \geq \prod_{i=1}^{n}(2/\lambda_i + 1) \prod_{i=1}^{n} |Z|v_i^{\perp}|.$$

Finally we use Malikiosis's discrete version of Minkowski's 2nd theorem (2.21), i.e., $\prod_{i=1}^{n}(2/\lambda_i+1) \geq \sqrt{3}^{-n}|Z|$ in order to get (3.34). □

Remark 3.3.4.

i) The above proof does not depend on the particular properties of the lattice $\mathbb{Z}^n$. So one obtains the same statement for an arbitrary n-dimensional lattice $\Lambda \subseteq \mathbb{R}^n$. More precisely, if $K \in \mathcal{K}^n_{os}$ fulfills $\dim(K \cap \Lambda) = n$, we have

$$\mathrm{G}_{\Lambda}(K)^{\frac{n-1}{n}} \geq c^{-n}\Big(\prod_{i=1}^{n} \mathrm{G}_{\Lambda|v_i^{\perp}}(K|v_i^{\perp})\Big)^{\frac{1}{n}},$$

where $v_i \in \lambda_i(K,\Lambda)K \cap \Lambda$ are linearly independent.

ii) Also, the above approach yields a reverse Loomis-Whitney-type inequality, if one aims for a lattice basis, instead of merely independent lattice vectors. However, in order to apply Lemma 3.3.2 one has to ensure that the basis is contained in K. For the basis $\{b_1, ..., b_n\}$ from equation (2.15), this means that one has to enlarge K by a factor n. So in this case, we obtain

$$\mathrm{G}(K)^{\frac{n-1}{n}} \geq (cn)^{-n}\Big(\prod_{i=1}^{n} \mathrm{G}_{\mathbb{Z}^n|v_i^{\perp}}(K|b_i^{\perp})\Big)^{\frac{1}{n}}.$$

◇

3.4 Unconditional Bodies

In this section, we improve some of our inequalities for unconditional bodies. We start with (3.7).

Lemma 3.4.1. *Let $K \in \mathcal{K}^n$ be unconditional and $m \in \mathbb{N}$. Then $\mathrm{G}(mK) \leq (2m-1)^n \mathrm{G}(K)$ and the inequality is sharp.*

Proof. First, we prove the claim for $\dim K = 1$, i.e., we may assume that $K = [-x, x] \subseteq \mathbb{R}$, $x \geq 0$. Then, $\mathrm{G}(K) = 2\lfloor x \rfloor + 1$. In case that $x \in \mathbb{Z}$ we have

$$\mathrm{G}(mK) = 2mx + 1 \leq m(2x+1) = m \cdot \mathrm{G}(K).$$

So let $x \notin \mathbb{Z}$. Then,

$$\mathrm{G}(mK) \leq 2mx + 1 < 2m(\lfloor x \rfloor + 1) + 1.$$

Since both sides of the inequality are odd integers, we obtain

$$\begin{aligned}\mathrm{G}(mK) &\le 2m(\lfloor x\rfloor+1)-1 = 2m\lfloor x\rfloor+2m-1\\ &= (2m-1)\Big(\frac{m}{2m-1}2\lfloor x\rfloor+1\Big) \le (2m-1)\mathrm{G}(K).\end{aligned}$$

Next, let $K\subseteq\mathbb{R}^n$ be an arbitrary unconditional convex body. Consider the unconditional body K' obtained by multiplying the first coordinates in K by m. The lattice points in K and K' can be partitioned into intervals parallel to e_1. The intervals that we see in K' are exactly the intervals of K, multiplied by m. So by the 1-dimensional case we have $\mathrm{G}(K')\le(2m-1)\mathrm{G}(K)$. If we repeat this argument for every coordinate, we end up with the desired inequality. The cubes $K=\big[-(1-\frac{1}{2m}),1-\frac{1}{2m}\big]^n$ show that the inequality is sharp. □

Lemma 3.4.1 yields a slightly improved version of Theorem 3.1.3 for the class of unconditional bodies.

Proposition 3.4.2. *Let $K\in\mathcal{K}^n_{os}$ be unconditional. Then,*

$$\mathrm{G}(K)^{\frac{n-1}{n}} \ge \frac{1}{3^{n-1}}\Big(\prod_{i=1}^{n}\mathrm{G}(K\cap e_i^\perp)\Big)^{\frac{1}{n}}.$$

Proof. Again, we write $K_i=K\cap e_i^\perp$ and $h_i=\mathrm{h}(K,e_i)$. Note that h_i is attained by a multiple of e_i, since K is unconditional. This implies that

$$K_i+[-h_i,h_i]e_i\subseteq 2K,$$

and we obtain

$$\mathrm{G}(2K)^{n-1}\ge\prod_{i=1}^{n-1}\big((2\lfloor h_i\rfloor+1)\mathrm{G}(K_i)\big)\ge\prod_{i=1}^{n}\mathrm{G}(K_i),$$

where the last inequality follows from $K_n\subseteq[-h_1,h_1]\times\cdots\times[-h_{n-1},h_{n-1}]$. The claim follows by applying Lemma 3.4.1 to the left-hand side above. □

Note that for an unconditional body $K\subseteq\mathbb{R}^n$ one has $K\cap e_i^\perp=K|e_i^\perp$, $1\le i\le n$. Therefore, Proposition 3.4.2 is also a sharpening of Theorem 3.3.1. In fact, following the lines of the proof of the discrete reverse Loomis-Whitney inequality in Section 3.3, the above proof is a simplification of the proof in Section 3.3.

Moreover, the inequalities of Theorem 3.2.1 and 3.2.3 hold with constant 1 for unconditional bodies, by the Loomis–Whitney inequality.

As for the discrete Brunn inequality, a constant 1 is obtained when intersecting an unconditional body K with a coordinate subspace L, since every slice $K\cap(L+t)$ is mapped into the central slice injectively by the orthogonal projection onto L. Moreover, for any hyperplane H there is a coordinate i such that the projection $H\to e_i^\perp$ is bijective and maps lattice points in H to lattice points in $e_i^\perp$. The index i can be chosen to be an index for which the normal vector v of H is non-zero. Therefore, the maximal hyperplane section with respect to $\mathrm{G}(\cdot)$ can always be chosen to be a coordinate section.

However, for general subspaces L we cannot hope for a constant 1 in the discrete Brunn inequality, as the next example illustrates.

Example 3.4.3. Consider the symmetric cube $C_n = [-1, 1]^n$ and the vector

$$u = (1, 2, \ldots, 2^{n-1})^T \in \mathbb{R}^n.$$

Then we have $\mathrm{G}(C_n \cap u^\perp) = 1$ (cf. Remark 3.3.3).

On the other hand, we have $\mathrm{G}(C_n \cap \{x \in \mathbb{R}^n : \langle x, u\rangle = 1\}) = n$, because, in view of (3.33), the points $x_k = e_k - \sum_{j=0}^{k-1} e_j$, $1 \leq k \leq n$ are contained in this section and as in Remark 3.3.3, by considering the maximal non-zero coordinate of a lattice point in this section, there are no further points. $\diamond$

4 Interpolating Between Volume and Lattice Point Enumerator with Successive Minima

This chapter aims at building a bridge between Minkowski's second theorem on volume and successive minima (4.2) and its conjectured discrete analogue (4.3). Apart from Proposition 4.4.5 and Section 4.5, the results presented here can be found in the joint work [FL22] with Eduardo Lucas.

4.1 The Conjectures of Betke, Henk and Wills

Minkowski's second theorem on volume and successive minima states for an n-dimensional convex body $K \in \mathcal{K}_n^n$ that (cf. (2.19) and the references thereafter)

$$\frac{1}{n!}\prod_{i=1}^{n}\frac{2}{\lambda_i(K)} \leq \mathrm{vol}(K) \leq \prod_{i=1}^{n}\frac{2}{\lambda_i(K)}. \tag{4.1}$$

The lower and upper bound in (4.1) are attained, e.g., by simplices and parallelepipeds respectively. Recall from Chapter 2 that for an n-dimensional convex body K, which is not necessarily symmetric, the successive minima are defined as $\lambda_i(K) = \lambda_i(\mathrm{cs}(K))$, where $\mathrm{cs}(K) = \frac{1}{2}(K - K)$ is the central symmetral of K.

Many alternatives to Minkowski's complicated original proof have been obtained. One of the first short proofs was given by Davenport [Dav39]. More analytic proofs were obtained by Weyl [Wey42] and Estermann [Est46]; and Bambah, Woods and Zassenhaus provided three new proofs in [BWZ65]. A more recent example was obtained by Henk [Hen02].

The result has been extended, for instance, to more general successive minima by Hlawka [GL87, Sec. 9.5]; to more general discrete sets, not necessarily lattices, by Woods [Woo66]; to intrinsic volumes by Henk [Hen90]; or to surface area measures by Henk, Henze and Hernández Cifre [HHH16].

Although (4.1) gives a deep insight on the lattice point structure of K, it may be regarded as a "continuous result", as both the volume and the successive minima vary continuously in K. In their paper [BHW93], Betke, Henk and Wills use the successive minima to infer estimates on the actual number of lattice points in K, a discrete parameter. They obtain for $K \in \mathcal{K}_{os}^n$ that

$$\frac{1}{n!}\prod_{i=1}^{n}\Big(\frac{1}{\lambda_i(K)}-1\Big) \leq \mathrm{G}(K) \leq \prod_{i=1}^{n}\Big(\frac{2i}{\lambda_i(K)}+1\Big), \tag{4.2}$$

where for the lower bound $\lambda_n(K) \leq 2$ is needed [BHW93, Prop. 2.1 and Cor. 2.1]. While the lower bound is best-possible, it is conjectured that the upper bound can be strengthened as follows [BHW93, Conj. 2.1]:

Conjecture 4.1.1 (Betke, Henk, Wills)**.** *Let $K \in \mathcal{K}_n^n$ and $\lambda_i = \lambda_i(K)$. Then one has*

$$\mathrm{G}(K) \leq \prod_{i=1}^{n}\Big\lfloor\frac{2}{\lambda_i}+1\Big\rfloor. \tag{4.3}$$

Equality would be attained, e.g., for boxes of the form $[-m_1, m_1] \times ... \times [-m_n, m_n]$, where $m_i \in \mathbb{Z}_{>0}$. In dimension 2 the conjecture has been confirmed by Betke, Henk and Wills themselves [BHW93, Thm. 2.2] and in dimension 3 it has been shown by Malikiosis [Mal12, Sec. 3.2], who also proved the benchmark result (2.21).

Betke, Henk and Wills pointed out in [BHW93, Prop. 2.2] that any inequality of the form

$$\mathrm{G}(K) \leq \prod_{i=1}^{n}\Big(\frac{2}{\lambda_i}+c_i\Big), \tag{4.4}$$

for some numbers c_i, $1 \leq i \leq n$, independent of K (but not necessarily of n), would imply the upper bound in Minkowski's second theorem (4.1). Indeed, one can asymptotically approximate the volume of K by the lattice point enumerator with respect to progressively finer lattices (using the properties of the Riemann integral), to which (4.4) could then be applied, and the resulting limit is precisely Minkowski's bound.

In this chapter, we use Minkowski's second theorem to show (4.4) with $c_i = n$ (cf. Corollary 4.1.4). In order to do so, we aim to express the deviation between $\mathrm{G}(K)$ and $\mathrm{vol}(K)$ in terms of the successive minima $\lambda_i(K)$, $i = 1, ..., n$. Our approach stems from another conjecture by Betke, Henk and Wills that relates the volume, the lattice point enumerator and the successive minima simultaneously.

Conjecture 4.1.2 (Betke, Henk, Wills)**.** *Let $K \in \mathcal{K}_n^n$ and $\lambda_i = \lambda_i(K)$. Then,*

$$\mathrm{G}(K) \leq \mathrm{vol}(K)\prod_{i=1}^{n}\Big(1+\frac{i\,\lambda_i}{2}\Big) \tag{4.5}$$

and, if $\lambda_n \leq \frac{2}{n}$,

$$\mathrm{G}(\mathrm{int}K) \geq \mathrm{vol}(K)\prod_{i=1}^{n}\Big(1-\frac{i\,\lambda_i}{2}\Big). \tag{4.6}$$

Moreover, if $K = -K$ and $\lambda_n \leq 2$, we have

$$\mathrm{G}(\mathrm{int}K) \geq \mathrm{vol}(K) \prod_{i=1}^{n} \Big(1 - \frac{\lambda_i}{2}\Big). \tag{4.7}$$

The bound (4.7) is stated as Conjecture 2.2 in [BHW93], where it is formulated for arbitrary n-dimensional lattices. However, there is no loss of generality in restricting to the integer lattice $\mathbb{Z}^n$. (4.5) and (4.6) have been communicated to the authors of [FL22] by Martin Henk personally. In the general case, we obtain the following weakenings of (4.5) and (4.6):

Theorem 4.1.3. *Let $K \in \mathcal{K}_n^n$ and $\lambda_i = \lambda_i(K)$, $i \in [n]$. Then we have*

$$\mathrm{G}(K) \leq \mathrm{vol}(K) \prod_{i=1}^{n} \left(1 + \frac{n\lambda_i}{2}\right). \tag{4.8}$$

Moreover, if $\lambda_n \leq \frac{2}{n}$, we have

$$\mathrm{G}(\mathrm{int}K) \geq \mathrm{vol}(K) \prod_{i=1}^{n} \left(1 - \frac{n\lambda_i}{2}\right). \tag{4.9}$$

From this we can deduce immediately, by applying the upper bound in (4.1) to the volume in (4.8), the following inequality:

Corollary 4.1.4. *Let $K \in \mathcal{K}_n^n$ and $\lambda_i = \lambda_i(K)$, $i \in [n]$. Then we have*

$$\mathrm{G}(K) \leq \prod_{i=1}^{n} \left(\frac{2}{\lambda_i} + n\right).$$

While our bound is tight for convex bodies rK, $r \to \infty$, it is weaker than Malikiosis's bound (2.21), if, e.g., $\lambda_i(K) = 1/c$ for some fixed number $c > 0$. Then our bound is of order n^n, while the bound in (2.21) is of order $\sqrt{3}^n$.

Apart from yielding discrete versions of Minkowski's second theorem, Conjecture 4.1.2 is interesting in its own right; on the one hand, one can deduce the well-known formula (cf. (4.10))

$$\lim_{r \to \infty} \frac{\mathrm{vol}(rK)}{\mathrm{G}(rK)} = 1 \tag{4.10}$$

from it, since $\lambda_i(rK)$ tends to 0 as $r \to \infty$. On the other hand, if K contains an n-dimensional set of lattice points, it follows that $\lambda_i(K) \leq 2$ holds, and, if $K = -K$, one has $\lambda_i(K) \leq 1$, $1 \leq i \leq n$. Therefore, we retrieve the universal bounds

$$\mathrm{G}(K) \leq (n+1)!\,\mathrm{vol}(K),$$

for K with $\dim(K \cap \mathbb{Z}^n) = n$, and

$$\mathrm{G}(\mathrm{int}K) \geq 2^{-n} \operatorname{vol}(K),$$

for $K = -K$ with $\dim(K \cap \mathbb{Z}^n) = n$, from Conjecture 4.1.2. These bounds essentially correspond to the classical results of Blichfeldt (2.12) and van der Corput (2.11).

In fact, all inequalities in Conjecture 4.1.2 have equality cases that are invariant with respect to integer scaling; (4.5) is tight, e.g., for integer multiples of the standard simplex $T_n = \operatorname{conv}\{0, e_1, ..., e_n\}$, since $\lambda_i(T_n) = 2$ and thus,

$$\operatorname{vol}(mT_n) \prod_{i=1}^{n} \left(1 + \frac{i\lambda_i(mT_n)}{2}\right) = \frac{1}{n!} \prod_{i=1}^{n} (m+i),$$

where the right hand side is exactly the Ehrhart polynomial of T_n [BR07, Thm. 2.2 (a)]. In view of [BR07, Thm. 2.2 (b)], we have

$$\mathrm{G}\big(\mathrm{int}(mT_n)\big) = \frac{1}{n!} \prod_{i=1}^{n} (m-i) = \operatorname{vol}(mT_n) \prod_{i=1}^{n} \left(1 - \frac{i\lambda_i(mT_n)}{2}\right)$$

and so (4.6) is tight for integer multiples of T_n as well. As it has been mentioned already in [BHW93], equality cases for (4.7) are given for example by boxes parallel to the coordinate axes with integral side lengths.

In dimension 2, we can confirm the upper bound in Conjecture 4.1.2. For the non-symmetric lower bound we obtain an asymptotic confirmation:

Theorem 4.1.5. *Let $K \in \mathcal{K}_2^2$ and $\lambda_i = \lambda_i(K)$. Then we have*

$$\mathrm{G}(K) \leq \operatorname{vol}(K) \left(1 + \frac{\lambda_1}{2}\right) (1 + \lambda_2) \tag{4.11}$$

and

$$\mathrm{G}(\mathrm{int}K) \geq \operatorname{vol}(K) \left(1 - \frac{\lambda_1}{2} - \lambda_2\right). \tag{4.12}$$

In particular, for any $\varepsilon > 0$, if $\lambda_1 \leq \frac{2\varepsilon}{1+\varepsilon}$, it follows from (4.12) that

$$\mathrm{G}(\mathrm{int}K) \geq \operatorname{vol}(K) \left(1 - \frac{\lambda_1}{2}\right) (1 - (1+\varepsilon)\lambda_2).$$

The remainder of the chapter is organized as follows: In the upcoming Section 4.2 we study the behaviour of the successive minima in a reduction process called “Blaschke's shaking procedure”. This will allow us to reduce the proofs of the Theorems 4.1.3 and 4.1.5 to anti-blocking bodies. Section 4.3 contains the proof of Theorems 4.1.3 as well as further results that connect the volume and the lattice point enumerator of n-dimensional convex bodies. In Section 4.4 we deal with the planar case and we prove Theorem 4.1.5 and (4.7) for symmetric lattice polygons and right triangles.

4.2 Anti-Blocking Convex Bodies

Recall that a convex body K is anti-blocking if $K \subseteq \mathbb{R}^n_{\geq 0}$ and for every $x = (x_1, \ldots, x_n) \in K$ the set $\{x' \in \mathbb{R}^n_{\geq 0} : x'_i \leq x_i, \forall i \in [n]\}$ is also contained in K. Given the convexity of K, the latter condition is equivalent to $K \cap e_i^\perp = K|e_i^\perp$, for all $i \in [n]$.

Anti-blocking bodies have been introduced in [Ful71]. Their volumes have been extensively studied in [AASS]. In the discrete setting, the set of lattice points $K \cap \mathbb{Z}^n$ inside of an anti-blocking body K is called a compressed set. Compressed sets have been considered in [GG01] and [GT05] (in the context of sum-set estimates) and in [VR12] (in the context of discrete isoperimetric inequalities).

The goal of this section is to prove the following statement:

Theorem 4.2.1. *For any convex body $K \in \mathcal{K}^n_n$, there exists an anti-blocking convex body $A \subseteq \mathbb{R}^n_{\geq 0}$ such that the following holds:*

i) $\mathrm{vol}(K) = \mathrm{vol}(A)$,

ii) $\mathrm{G}(K) \leq \mathrm{G}(A)$,

iii) $\mathrm{G}(\mathrm{int}K) \geq \mathrm{G}(\mathrm{int}A)$ *and*

iv) $\lambda_i(K) \geq \lambda_i(A)$, *for all* $i \in [n]$.

This shows that it is enough to prove (4.5) and (4.6) for the special class of anti-blocking bodies.

An important tool for the proof of Theorem 4.2.1 is the Blaschke shaking of a convex body $K \in \mathcal{K}^n_n$ with respect to an oriented hyperplane $u^\perp$, $u \neq 0$, which is defined as

$$\mathrm{sh}_u(K) = \bigcup_{x \in K|u^\perp} \left[x,\, x + \frac{f_{u,K}(x)}{|u|} \cdot u\right],$$

where $f_{u,K}(x)$ denotes the length of the preimage of x under the orthogonal projection $K \to u^\perp$ (cf. Figure 4.1). The Blaschke shaking has been introduced in [Bla17]. This process, which bares resemblance to Steiner's symmetrization, belongs to a wider class of transformations known as "shakings". These processes have been explored, for instance, to obtain discrete isoperimetric inequalities by Kleitman [Kle79], and more recently by Bollobás and Leader [BL91]. Stability results, akin to that of Gross for Steiner's symmetrization, have been obtained by Biehl [Bie23], Schöpf [Sch76], and more recently, Campi, Colesanti and Gronchi [CCG01], for example. Other applications were obtained in [Uhr94] and [CCG99].

The operator sh_u is known to preserve convexity [CCG01, Lemma 1.1] and we have the following lemma:

Lemma 4.2.2. *Let $K \in \mathcal{K}^n_n$ and $u \in \mathbb{R}^n \setminus \{0\}$. For the Blaschke shaking $\mathrm{sh}_u(K)$, the following relations hold:*

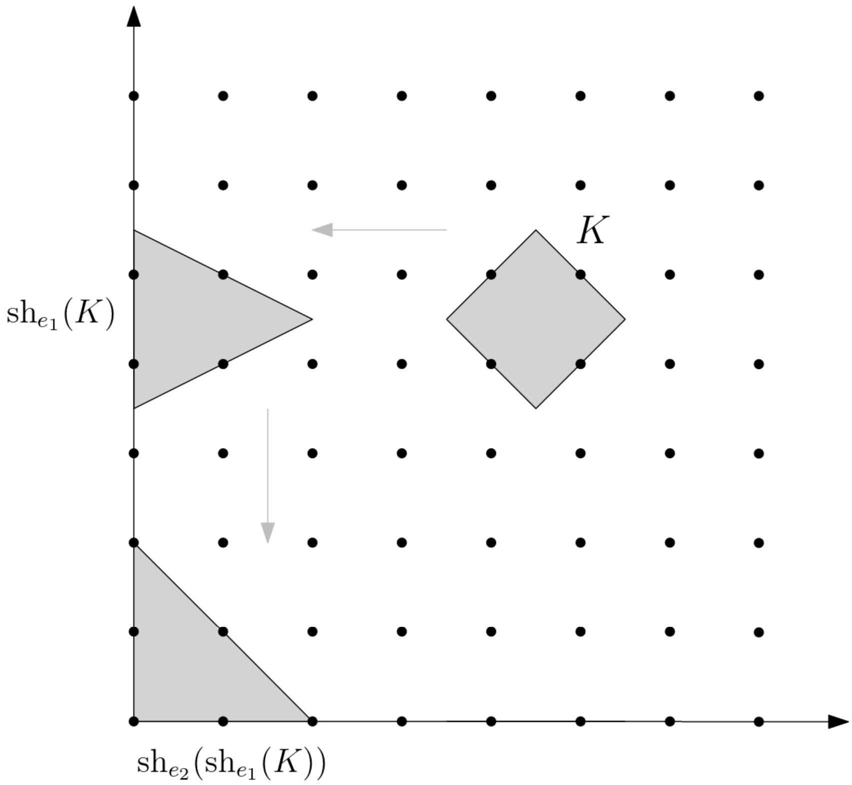


Figure 4.1: Illustration of two consecutive Blaschke shakings. The body $\mathrm{sh}_{e_2}(\mathrm{sh}_{e_1}(K))$ is anti-blocking.

i) $K|u^{\perp} \subseteq \mathrm{sh}_u(K)$

ii) $\mathrm{vol}(K) = \mathrm{vol}(\mathrm{sh}_u(K))$,

iii) $|u|_{\mathrm{cs}(K)} = |u|_{\mathrm{cs}(\mathrm{sh}_u(K))}$,

iv) $|x|_{\mathrm{cs}(K)} \geq \left|x|u^{\perp}\right|_{\mathrm{cs}(\mathrm{sh}_u(K))}$, *for all* $x \in \mathbb{R}^n$.

If $u = e_i$, *for some* $i \in [n]$, *we also have*

v) $\mathrm{G}(K) \leq \mathrm{G}(\mathrm{sh}_{e_i}(K))$,

vi) $\mathrm{G}(\mathrm{int}K) \geq \mathrm{G}(\mathrm{int}(\mathrm{sh}_{e_i}(K))$.

Proof. i) and ii) follow directly from the definition of $\mathrm{sh}_u(K)$. For iii), we note that

$$\begin{aligned}|u|_{\mathrm{cs}(K)} &= \min\left\{r \geq 0 : \exists x, y \in K.\, u = \frac{r}{2}(x-y)\right\}\\ &= \min\left\{\frac{2|u|}{|x-y|} : x, y \in K,\ [x,y] \text{ is parallel to } u\right\} = \frac{2|u|}{\max \mathrm{vol}_1(S)},\end{aligned}$$

where the maximum ranges over all segments $S \subseteq K$ that are parallel to u. Now iii) follows directly from the definition of $\mathrm{sh}_u(K)$.

For iv) let $r = |x|_{\mathrm{cs}(K)}^{-1}$. Then there are $a, b \in K$ such that $rx = \frac{1}{2}(a - b)$ and from i) it follows that

$$r \cdot x|u^\perp = \frac{1}{2}\big(a|u^\perp - b|u^\perp\big) \in \mathrm{cs}\big(\mathrm{sh}_u(K)\big).$$

Thus $\left|r \cdot x|u^\perp\right|_{\mathrm{cs}(\mathrm{sh}_u(K))} \le 1$, which implies iv) by the choice of r.

In order to prove v), we start with an interval $I = [a, b] \subseteq \mathbb{R}$ and show that the number of lattice points in an interval of length $b - a$ is maximized when $a \in \mathbb{Z}$. Otherwise, we could let $\delta = a - \lfloor a \rfloor$ and observe that

$$\begin{aligned} |(I - \delta) \cap \mathbb{Z}| &= \lfloor b - \delta \rfloor - \lceil a - \delta \rceil + 1 = \lfloor b - \delta \rfloor - \lfloor a \rfloor + 1 \\ &\ge \lfloor b \rfloor - \lfloor a \rfloor = \lfloor b \rfloor - \lceil a \rceil + 1 = |I \cap \mathbb{Z}| \end{aligned}$$

In order to obtain v) it is then enough to note that the lattice points in K and $\mathrm{sh}_{e_i}(K)$ are contained in intervals of the same lengths, while in $\mathrm{sh}_{e_i}(K)$, these intervals start at a lattice point and therefore contain at least as many lattice points as those in K (cf. Figure 4.1).

vi) is proved with the same argument, but since the intervals involved are open, translating them such that they start at a lattice point will potentially reduce, but never increase, their lattice point count. □

Proof of Theorem 4.2.1. Let $v_1, ..., v_n \in \mathbb{Z}^n$ be linearly independent such that $|v_i|_{\mathrm{cs}(K)} = \lambda_i(K)$. Since all the functionals involved are invariant with respect to unimodular transformations, we may assume that the matrix $[v_1, ..., v_n]$ is an upper triangular matrix (e.g., a Hermite-normal-form [Sch86, Sec. 4.1]). Let $K_0 = K$ and for $j \in [n]$, let $K_j = \mathrm{sh}_{e_j}(K_{j-1})$. We show that $A := K_n$ is the desired body. To this end, we prove the following statement inductively.

Claim 1. For $j \in \{0, ..., n\}$, there exist linearly independent vectors $u_1, ..., u_n \in \mathbb{Z}^n$ such that $|u_i|_{\mathrm{cs}(K_j)} \le \lambda_i(K)$ and the matrix $[u_1, ..., u_n]$ is of the form

$$\begin{pmatrix} D_j & 0 \\ 0 & T_{n-j} \end{pmatrix},$$

where D_j is a $j \times j$-diagonal matrix and T_{n-j} is an $(n-j) \times (n-j)$-upper triangular matrix.

For $j = 0$, Claim 1 is clearly true with $u_i = v_i$, $1 \le i \le n$. So we assume that the claim holds for some $j < n$. We choose $u_i' = u_i|e_{j+1}^\perp$, for $i \ne j+1$, and $u_{j+1}' = u_{j+1}$. In view of $K_{j+1} = \mathrm{sh}_{e_{j+1}}(K_j)$, Lemma 4.2.2 iv) and our induction hypothesis, we have

$$|u_i'|_{\mathrm{cs}(K_{j+1})} = \left|u_i|e_{j+1}^\perp\right|_{\mathrm{cs}(K_{j+1})} \le |u_i|_{\mathrm{cs}(K_j)} \le \lambda_i(K),$$

for all $i \ne j+1$. From Lemma 4.2.2 iii) it also follows that

$$|u_{j+1}'|_{\mathrm{cs}(K_{j+1})} = |u_{j+1}|_{\mathrm{cs}(K_j)} \le \lambda_{j+1}(K).$$

The matrix $[u'_1, ..., u'_n]$ differs from $[u_1, ..., u_n]$ only by the zeros in the $(j+1)$-th row after the diagonal entry. Therefore, the system $u'_1, ..., u'_n \in \mathbb{Z}^n$ is also linearly independent and it fulfills the requirements of Claim 1 for $j+1$.

Hence, $A = K_n$ satisfies $\lambda_i(A) \leq \lambda_i(K)$ for all $i \in [n]$. The other requirements i)–iii) on A follow from a repeated application of Lemma 4.2.2 ii),v) and vi). It remains to prove that A is indeed anti-blocking. To this end, we use induction again to prove:

Claim 2. For $j \in \{0, ..., n\}$ and $x \in K_j$, we have $x|e_i^\perp \in K_j$, for all $1 \leq i \leq j$.

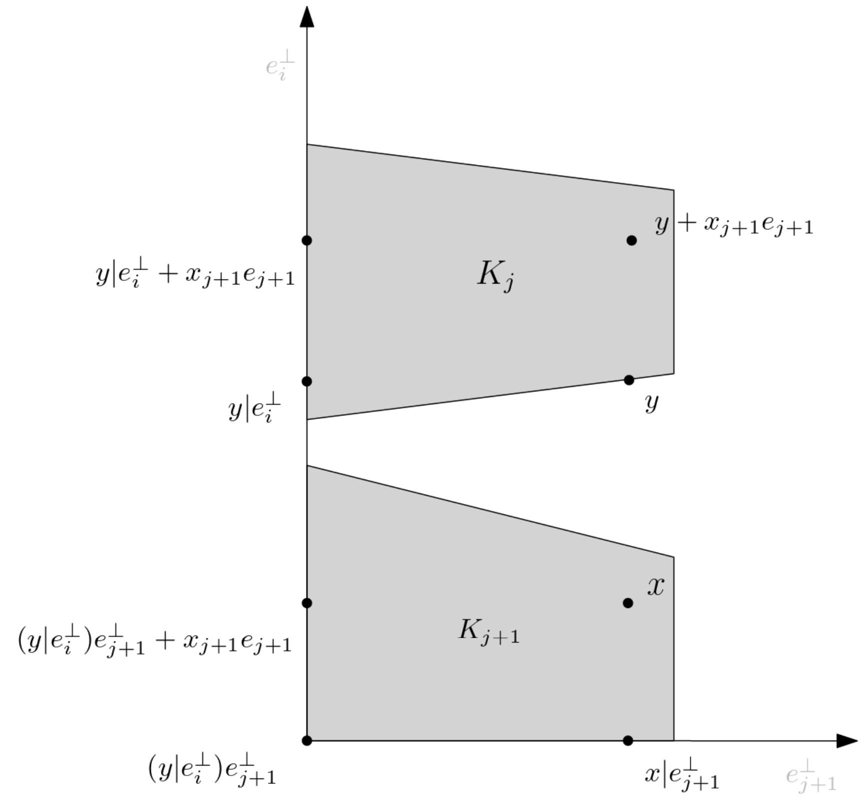


Figure 4.2: The construction for the proof of Claim 2.

For $j = 0$, the statement is trivial. So we assume Claim 2 holds for some $j \in [n]$. Let $x \in K_{j+1}$. By Lemma 4.2.2 i) it follows that $x|e_{j+1}^\perp \in K_{j+1}$. So we consider $i \in [j]$. Let x_{j+1} be the $(j+1)$-th entry of x and let $y \in K_j$ be the lowest (with respect to e_{j+1}) point in the preimage of $x|e_{j+1}^\perp$ under the orthogonal projection $K_j \to e_{j+1}^\perp$ (cf. Figure 4.2). Then, $[y, y + x_{j+1}e_{j+1}] \subseteq K_j$. By induction, it follows that $[y|e_i^\perp, y|e_i^\perp + x_{j+1}e_{j+1}] \subseteq K_j$. Since $(y|e_i^\perp)|e_{j+1}^\perp = (x|e_i^\perp)|e_{j+1}^\perp$, the interval $[(x|e_i^\perp)|e_{j+1}^\perp, (x|e_i^\perp)|e_{j+1}^\perp + x_{j+1}e_{j+1}]$ is contained in K_{j+1}. Since $(x|e_i^\perp)|e_{j+1}^\perp + x_{j+1}e_{j+1} = x|e_i^\perp$, Claim 2 holds for $j+1$.

For $j = n$, Claim 2 yields that A is anti-blocking. □

The proof of Claim 2 essentially corresponds to the argument given in the proof of Lemma 1.2 in [CCG01].

One of the reasons why anti-blocking bodies are beneficial when dealing with successive minima problems is that the successive minima are always realized by the standard basis directions of $\mathbb{Z}^n$. Even more, the segments that testify for the successive minima are of the form $[0, (2/\lambda_i)e_i]$, as the next lemma will show. In particular, all these segments share the origin as a common point, a property that will be crucial in the proof of Theorem 4.1.3.

Lemma 4.2.3. *Let $K \in \mathcal{K}_n^n$ be anti-blocking. Then the coordinates can be permuted in such a way that $|e_i|_{\mathrm{cs}(K)} = \lambda_i(K)$ holds. In this case, one also has $\frac{2}{\lambda_i(K)} e_i \in K$, $1 \le i \le n$.*

Proof. Let $v_1, ..., v_n \in \mathbb{Z}^n$ be linearly independent with $|v_i|_{\mathrm{cs}(K)} = \lambda_i(K)$. Then there exists a permutation σ of $[n]$ such that the σ_i-th entry of v_i is non-zero. Otherwise the determinant of $[v_1, ..., v_n]$ would be zero, a contradiction. For the sake of simplicity we assume that σ is the identity. Since K is anti-blocking, the projection w_i of v_i on $\mathrm{span}\{e_i\} = \bigcap_{j \ne i} e_j^\perp$ is contained in K and a repeated application of Lemma 4.2.2 iv) shows that $|w_i|_{\mathrm{cs}(K)} \le |v_i|_{\mathrm{cs}(K)} = \lambda_i(K)$. By the minimality of the λ_i's and the fact that $w_i \in \mathrm{span}\{e_i\} \cap \mathbb{Z}^n$, we obtain $w_i = e_i$ and $|e_i|_{\mathrm{cs}(K)} = \lambda_i(K)$.

For the second part we deduce from $|e_i|_{\mathrm{cs}(K)} = \lambda_i(K)$ that

$$\frac{1}{\lambda_i(K)} e_i = \frac{1}{2}(a - b),$$

for some $a, b \in K$. Since $1/\lambda_i(K)$ is the maximal number r such $re_i \in \mathrm{cs}(K)$, b_i must be zero. So b is a member of $e_i^\perp$ and since K is anti-blocking we obtain

$$\frac{2}{\lambda_i(K)} e_i = \Big(\frac{2}{\lambda_i(K)} e_i + b\Big)\Big|\mathrm{span}\{e_i\} = a|\mathrm{span}\{e_i\} \in K$$

as desired. $\square$

4.3 n-Dimensional Case

We start with the following bounds in terms of the covering radius $\mu(K)$ of $K \in \mathcal{K}_n^n$. Recall that $\mu(K) = \min\{\mu \ge 0 : \mu K + \mathbb{Z}^n = \mathbb{R}^n\}$.

Proposition 4.3.1. *Let $K \in \mathcal{K}_n^n$ and $\mu = \mu(K)$. Then we have*

$$\mathrm{G}(K) \le \mathrm{vol}(K)\big(1 + \mu\big)^n. \tag{4.13}$$

If $\mu \le 1$, i.e., $K + \mathbb{Z}^n = \mathbb{R}^n$, we also have

$$\mathrm{G}(\mathrm{int}K) \ge \mathrm{vol}(K)\big(1 - \mu\big)^n. \tag{4.14}$$

Both inequalities are tight.

The upper bound (4.13) has also been shown independently by Dadush in [Dad12, Lemma 7.4.1].

Proof of Proposition 4.3.1. For the upper bound, it is enough to show that μK contains a measurable tiling set, i.e., a set S with $\text{int}S \cap (z + \text{int}S) = \emptyset$, for all $z \in \mathbb{Z}^n \setminus \{0\}$ and $S + \mathbb{Z}^n = \mathbb{R}^n$. Given the former condition, the latter condition is equivalent to $\text{vol}(S) = 1$. For a tiling set S we have

$$\mathrm{G}(K) = \text{vol}((K \cap \mathbb{Z}^n) + S) \leq \text{vol}(K + \mu K) = (1 + \mu)^n \text{vol}(K).$$

In order to find S, let $P = [0, 1]^n$. There are finitely many translates $\mu K + x_i$, $x_i \in \mathbb{Z}^n$, $1 \leq i \leq m$, that cover P. We define inductively $P_1 = P \cap (\mu K + x_1)$ and

$$P_i = \big(P \setminus (\bigcup_{j<i} P_j)\big) \cap (\mu K + x_i).$$

Now, let $S_i = P_i - x_i \subseteq \mu K$ and $S = \bigcup_{i=1}^m S_i$. We claim that S is the desired set. To prove this, we show that S has volume 1 and that its $\mathbb{Z}^n$-translates do not overlap.

Clearly the P_i's are interiorly disjoint, i.e., $\text{int}(P_i) \cap \text{int}(P_j) = \emptyset$, and satisfy $\bigcup_{i=1}^m P_i = P$. The S_i's are interiorly disjoint too; suppose there are $i \neq j$ such that $\text{int}S_i$ intersects $\text{int}S_j$. Then, $\text{int}P_i$ intersects $\text{int}P_j + x_i - x_j$. Since the $\mathbb{Z}^n$ translates of P are interiorly disjoint, we must have $x_i = x_j$, a contradiction. Therefore the S_i's are interiorly disjoint and it follows that

$$\text{vol}(S) = \sum_{i=1}^m \text{vol}(S_i) = \sum_{i=1}^m \text{vol}(P_i) = \text{vol}(P) = 1.$$

Now assume that $\text{int}S$ intersects $\text{int}S + x$ for some $x \in \mathbb{Z}^n$. Then there exist $i, j \in [m]$ such that $\text{int}P_i - x_i$ intersects $\text{int}P_j - x_j + x$. Again, since the $\mathbb{Z}^n$-translates of P are interiorly disjoint, as well as the P_i's, we must have $i = j$ and $x = 0$. Hence, the $\mathbb{Z}^n$-translates of S are interiorly disjoint and so S is as desired. This finishes the proof of the upper bound.

For the lower bound, we apply (2.13) to $K' = (1 - \mu)\text{int}K$ and obtain a vector $t \in \mathbb{R}^n$ such that $\text{vol}(K') \leq \mathrm{G}(K' + t)$. Since $\mu K + \mathbb{Z}^n = \mathbb{R}^n$, we may assume that $t \in \mu K$ holds. Thus, for $\mu < 1$,

$$\text{vol}(K)(1 - \mu)^n = \text{vol}(K') \leq \mathrm{G}(K' + t) \leq \mathrm{G}\big((1 - \mu)\text{int}K + \mu K\big) = \mathrm{G}(\text{int}K).$$

For $\mu = 1$ the lower bound is trivially satisfied.

In order to see that the inequality is tight, consider $K = [0, m]^n$, where $m \in \mathbb{Z}_{>0}$. For such cubes one has $\text{vol}(K) = m^n$, $\mathrm{G}(K) = (m + 1)^n$, $\mathrm{G}(\text{int}K) = (m - 1)^n$ and $\mu(K) = 1/m$. So equality is achieved for both of the bounds. $\square$

Remark 4.3.2. i) The strategy of finding an appropriate tiling used in the proof of the upper bound above has also been applied, for instance, in the proof of Blichfeldt's classical variant of (2.13) [GL87, Ch. 2, Thm. 5.2]. Moreover, in [XZ14], the authors showed that convex tilings in these conditions need not exist.

ii) The disadvantage of (4.13) in comparison to the upper bound (4.8) is that it cannot profit from K being large in a lattice subspace. Consider the convex body $K = [-r, r]^{n-1} \times [-1/2, 1/2]$, where r is large. Then it holds that $\mu(K) = 1$, so the constant in (4.13) is 2^n. But the constant in (4.8) is of order $n + 1$, since $\lambda_i(K)$ tends to 0 for $i < n$ as $r \to \infty$.

   On the other hand, (4.14) is actually stronger than the lower bound (4.9) in Theorem 4.1.3. We will use (4.14) to prove (4.9) in Section 4.3.

iii) Applied to the special class of convex lattice tiles, i.e., convex bodies K with $K + \mathbb{Z}^n = \mathbb{R}^n$ and $\mathrm{int}K \cap (z + \mathrm{int}K) = \emptyset$, for all $z \in \mathbb{Z}^n \setminus \{0\}$, Proposition 4.3.1 yields for $r \geq 1$ that
$$(r-1)^n \leq \mathrm{G}(\mathrm{int}(rK)) \leq \mathrm{G}(rK) \leq (r+1)^n,$$
since $\mathrm{vol}(K) = \mu(K) = 1$ and $\mu(rK) = \frac{\mu(K)}{r}$, which is sharp for $K = [0, 1]^n$ and $r \in \mathbb{Z}_{>0}$. ◇

For the proof of Theorem 4.1.3, we also need an inequality of Davenport [Dav51], which states that for any convex body $K \in \mathcal{K}^n_n$ one has the bound
$$\mathrm{G}(K) \leq \sum_{k=1}^{n} \sum_{I \in \binom{[n]}{k}} \mathrm{vol}_{n-k}(K|L_I^\perp), \tag{4.15}$$
where $L_I = \mathrm{span}\{e_i : i \in I\}$. For anti-blocking bodies K, (4.15) can also be derived directly as follows:
$$\begin{aligned} \mathrm{G}(K) &= \mathrm{vol}\big((K \cap \mathbb{Z}^n) + [-1, 0]^n\big) \leq \mathrm{vol}(K + [-1, 0]^n) \\ &= \sum_{k=1}^{n} \sum_{I \in \binom{[n]}{k}} \mathrm{vol}_{n-k}(K|L_I^\perp), \end{aligned}$$
where the last equation follows, since for anti-blocking bodies, the Minkowski sum can be decomposed into a union of disjoint prisms:
$$K + [-1, 0]^n = \bigcup_{I \subseteq [n]} \left\{ x \in \mathbb{R}^n : x|L_I^\perp \in K|L_I^\perp \text{ and } x_i \in [-1, 0],\ \forall i \in I \right\}.$$

We are now ready for the proof of Theorem 4.1.3.

Proof of Theorem 4.1.3. In order to prove (4.8), we may assume that K is anti-blocking by Theorem 4.2.1. After renumbering the coordinates, we can also assume that $|e_i|_{\mathrm{cs}(K)} = \lambda_i$,

$1 \leq i \leq n$ holds (cf. Lemma 4.2.3). For a set $I \subseteq [n]$, let $L_I = \mathrm{span}\{e_i : i \in I\}$. An inequality of Rogers and Shephard [RS58a, Thm. 1] yields that

$$\mathrm{vol}_k(K \cap L_I)\mathrm{vol}_{n-k}(K|L_I^{\perp}) \leq \tbinom{[n]}{k}\mathrm{vol}(K), \tag{4.16}$$

for any $I \in \binom{[n]}{k}$. By Lemma 4.2.3 we have $\frac{2}{\lambda_i}e_i \in K$, so from (4.16), we deduce that

$$\begin{aligned}\mathrm{vol}_{n-k}(K|L_I^{\perp}) &\leq \binom{n}{k}\frac{\mathrm{vol}(K)}{\mathrm{vol}_k(K \cap L_I)} \leq \binom{n}{k}\frac{\mathrm{vol}(K)}{\mathrm{vol}_k(\mathrm{conv}\{(2/\lambda_i)e_i : i \in I\})}\\ &= k!\binom{n}{k}\mathrm{vol}(K)\prod_{i\in I}\frac{\lambda_i}{2} \leq \mathrm{vol}(K)\prod_{i\in I}\frac{n\lambda_i}{2}.\end{aligned}$$

Combining this with Davenport's inequality (4.15) yields

$$\mathrm{G}(K) \leq \mathrm{vol}(K)\sum_{I\subseteq[n]}\prod_{i\in I}\frac{n\lambda_i}{2} = \mathrm{vol}(K)\prod_{i=1}^{n}\left(1+\frac{n\lambda_i}{2}\right)$$

as desired. In order to prove (4.9), we use the lower bound (4.14) in terms of $\mu(K)$, as well as the relation (2.18), which states

$$\mu(K) \leq \sum_{i=1}^{n}\frac{\lambda_i}{2}. \tag{4.17}$$

Since $\lambda_n \leq 2/n$, (4.17) yields that $\mu(K) \leq 1$. Thus, we may apply Proposition 4.3.1 and obtain

$$\begin{aligned}\mathrm{G}(\mathrm{int}K) &\geq \mathrm{vol}(K)\big(1-\mu(K)\big)^n \geq \mathrm{vol}(K)\Big(1-\sum_{i=1}^{n}\frac{\lambda_i}{2}\Big)^n\\ &= \mathrm{vol}(K)\Big(\frac{1}{n}\sum_{i=1}^{n}\Big(1-\frac{n\lambda_i}{2}\Big)\Big)^n\\ &\geq \mathrm{vol}(K)\prod_{i=1}^{n}\Big(1-\frac{n\lambda_i}{2}\Big),\end{aligned}$$

where we used the inequality of arithmetic and geometric means in the last step. $\square$

Schymura generalized Davenport's inequality (4.15) and obtained for an arbitrary linearly independent set $\{b_1, ..., b_n\} \subseteq \mathbb{Z}^n$ that

$$\mathrm{G}(K) \leq \sum_{k=1}^{n}\sum_{I\in\binom{[n]}{k}}\mathrm{vol}_{n-k}(K|L_I^{\perp})\mathrm{vol}_k(P_I), \tag{4.18}$$

where $L_I = \mathrm{span}\{b_i : i \in I\}$ and $P_I = \sum_{i\in I}[0, b_i]$ [Hen12, Lemma 1.1]. We reverse (4.18) in the following way.

Theorem 4.3.3. *Let $K \in \mathcal{K}_n^n$ and let $b_1, ..., b_n \in \mathbb{Z}^n$ be linearly independent. Then*

$$\mathrm{vol}(K) \leq \sum_{I \subseteq [n]} \mathrm{G}_{\mathbb{Z}^n | L_I^\perp}(\mathrm{int}K | L_I^\perp) \tag{4.19}$$

holds, where $L_I = \mathrm{span}\{b_i : i \in I\}$ and $\mathrm{G}_{\mathbb{Z}^n|L_I^\perp}$ denotes the lattice point enumerator with respect to the projected lattice $\mathbb{Z}^n|L_I^\perp$. The inequality is tight.

The factor $\mathrm{vol}_k(P_I)$ in (4.18) is hidden in the correspondingly higher density of $\mathbb{Z}^n|L_I^\perp$. In fact, one has $\det(\mathbb{Z}^n|L_I^\perp) \geq \mathrm{vol}_k(P_I)^{-1}$.

Proof. For an ordered linearly independent set $B = \{b_1, ..., b_k\}$ and a vector $x = \sum_{i=1}^k \alpha_i b_i$ in $\mathrm{span}\, B$, we write $\mathrm{supp}_B(x) = \{i : \alpha_i \neq 0\}$. We will show the following statement:

Claim 3. Let Λ be an n-dimensional lattice and $B = \{b_1, ..., b_n\} \subseteq \Lambda$ be a linearly independent set. For any convex and bounded (but not necessarily closed) set $K \subseteq \mathrm{span}(\Lambda)$, and any $t \in \mathrm{span}(\Lambda)$, it holds that

$$\mathrm{G}_\Lambda(K + t) \leq \sum_{I \subseteq \mathrm{supp}_B(t)} \mathrm{G}_{\Lambda|L_I^\perp}(K|L_I^\perp), \tag{4.20}$$

where $L_I = \mathrm{span}\{b_i : i \in I\}$.

If $t = 0$ there is only one summand in (4.20) corresponding to $I = \emptyset$, and so (4.20) reads as $\mathrm{G}_\Lambda(K) \leq \mathrm{G}_\Lambda(K)$, a tautology. Thus, from now on we assume that $t \neq 0$.

First we note that if $n = 1$, then (4.20) states for non-zero t that

$$\mathrm{G}_\Lambda(K + t) \leq \mathrm{G}_\Lambda(K) + 1. \tag{4.21}$$

Since any bounded convex set $K \subseteq \mathbb{R}^1$ is a bounded interval, the statement is confirmed.

Now, for any $n > 1$, we will prove (4.20) by induction on $|\mathrm{supp}_B(t)|$. If $|\mathrm{supp}_B(t)| = 1$ then $t = \alpha_1 b_1$ for some $\alpha_1 \neq 0$, and thus

$$\mathrm{G}_\Lambda(K + t) = \sum_{x \in K|b_1^\perp \cap \Lambda|b_1^\perp} \mathrm{G}_\Lambda((K + t) \cap (x + \mathbb{R}b_1)).$$

Since the bodies on the right hand side are segments parallel to t, we can apply (4.21) and obtain

$$\begin{aligned} \mathrm{G}_\Lambda(K + t) &\leq \sum_{x \in K|b_1^\perp \cap \Lambda|b_1^\perp} \big(\mathrm{G}_\Lambda(K \cap (x + \mathbb{R}b_1)) + 1\big) \\ &= \mathrm{G}_\Lambda(K) + \mathrm{G}_{\Lambda|b_1^\perp}(K|b_1^\perp), \end{aligned} \tag{4.22}$$

which corresponds to (4.20) in this case.

Finally, let $t = \sum_{i=1}^n \alpha_i b_i$ be an arbitrary non-zero vector in $\operatorname{span}(\Lambda)$. Consider any $j \in \operatorname{supp}_B(t)$. We define $t' = t - \alpha_j b_j$ and $t'' = t'|b_j^\perp$ as well as $B' = B \setminus \{b_j\}$ and $B'' = B'|b_j^\perp$. Then, we observe that

$$\operatorname{supp}_B(t') = \operatorname{supp}_{B''}(t'') = \operatorname{supp}_B(t) \setminus \{j\}.$$

Therefore, we obtain with $\widetilde{L_I} = \operatorname{span}\{b_i|b_j^\perp : i \in I\}$ that

$$\begin{aligned} \mathrm{G}_\Lambda(K+t) =& \mathrm{G}_\Lambda(K+t'+\alpha_j b_j) \\ \leq& \mathrm{G}_\Lambda(K+t') + \mathrm{G}_{\Lambda|b_j^\perp}((K+t')|b_j^\perp) \\ =& \mathrm{G}_\Lambda(K+t') + \mathrm{G}_{\Lambda|b_j^\perp}(K|b_j^\perp + t'') \\ \leq& \sum_{I \subseteq \operatorname{supp}_B(t)\setminus j} \mathrm{G}_{\Lambda|L_I^\perp}(K|L_I^\perp) \\ &+ \sum_{I \subseteq \operatorname{supp}_B(t)\setminus j} \mathrm{G}_{(\Lambda|b_j^\perp)|\widetilde{L_I}^\perp}((K|b_j^\perp)|\widetilde{L_I}^\perp) \\ =& \sum_{I \subseteq \operatorname{supp}(t)\setminus j} \left(\mathrm{G}_{\Lambda|L_I^\perp}(K|L_I^\perp) + \mathrm{G}_{\Lambda|L_{I\cup j}^\perp}(K|L_{I\cup j}^\perp)\right) \\ =& \sum_{I \subseteq \operatorname{supp}_B(t)} \mathrm{G}_{\Lambda|L_I^\perp}(K|L_I^\perp). \end{aligned}$$

For the first inequality we used (4.22), and for the second inequality we used the induction hypothesis (4.20) applied to K, Λ, B and t', as well as to $K|b_j^\perp$, $\Lambda|b_j^\perp$, B'' and t''. This finishes the proof of Claim 3 and so we obtain

$$\mathrm{G}(\operatorname{int} K + t) \leq \sum_{I \subseteq [n]} \mathrm{G}_{\mathbb{Z}^n|L_I^\perp}(\operatorname{int} K|L_I^\perp)$$

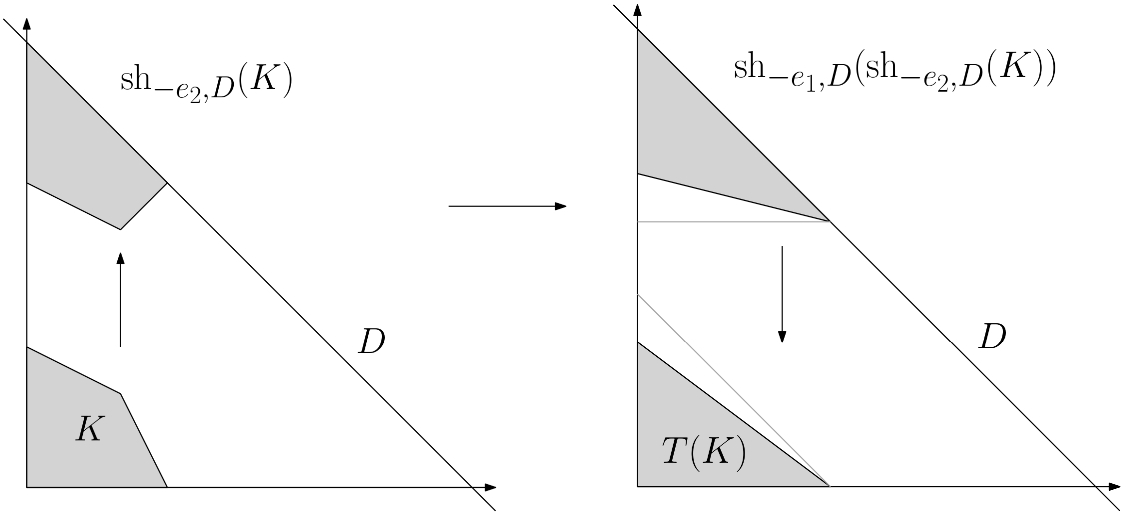
for any $t \in \mathbb{R}^n$. Now inequality (4.19) follows from (2.13).

To see that it is tight let $K = [0, k_i] \times \ldots \times [0, k_n]$, where $k_i \in \mathbb{Z}_{>0}$, and $b_i = e_i$. Then we have

$$\begin{aligned} \operatorname{vol}(K) &= \prod_{i=1}^n k_i = \prod_{i=1}^n \big((k_i - 1) + 1\big) \\ &= \sum_{I \subseteq [n]} \prod_{i \in I} (k_i - 1) = \sum_{I \subseteq [n]} \mathrm{G}_{\mathbb{Z}^n|L_I^\perp}(\operatorname{int} K|L_I^\perp). \end{aligned}$$ □

4.4 Two-Dimensional Case

For the proof of Theorem 4.1.5, we take the reduction from Section 4.2 a step further, by shaking K in such a way that it is anti-blocking and, in addition, located below the diagonal line passing through $(2/\lambda_1)e_1$ and $(2/\lambda_1)e_2$ (cf. Figure 4.3).



(a) First, K is pushed under the diagonal D from below, ...

(b) ...next from the right and then back down on $e_2^\perp$.

Figure 4.3: Illustration of the shaking process T.

To this end, we consider non-orthogonal shakings as a generalization of the Blaschke shakings in Section 4.2; For an affine line $\ell \subseteq \mathbb{R}^2$ and a vector $u \in \mathbb{R}^2 \setminus \{0\}$ which is not parallel to ℓ, let $\pi_{u,\ell}$ denote the projection on ℓ along u. For $K \in \mathcal{K}_2^2$, we then define

$$\mathrm{sh}_{u,\ell}(K) = \bigcup_{x \in \pi_{u,\ell}(K)} \left[x,\, x + \mathrm{vol}_1 \left(K \cap (x + \mathbb{R}u) \right) \frac{u}{|u|} \right]$$

as the Blaschke shaking of K with respect to u and ℓ. Note that in the setting of Section 4.2, we have $\mathrm{sh}_u = \mathrm{sh}_{u,u^\perp}$.

As we saw in Section 4.2, it is enough to prove Theorem 4.1.5 for $K \in \mathcal{K}_2^2$ anti-blocking. Starting with an anti-blocking body K that satisfies $|e_i|_{\mathrm{cs}(K)} = \lambda_i(K)$ (cf. Lemma 4.2.3), we construct a new body A by shaking K first vertically and then horizontally from below against a lattice diagonal $D = \{x \in \mathbb{R}^2 : x_1 + x_2 = m\}$, $m \in \mathbb{Z}$, and finally back down on $e_2^\perp$. (The value $m \in \mathbb{Z}$ may be chosen arbitrarily since lattice translations do not change the involved parameters.) Formally, we define $A = T(K)$, where

$$T = \mathrm{sh}_{e_2} \circ \mathrm{sh}_{-e_1,D} \circ \mathrm{sh}_{-e_2,D}.$$

We claim that A satisfies the following properties:

Lemma 4.4.1. *Let K and A be as above. Then the following statements hold true:*

i) *A is convex,*

ii) *A is anti-blocking,*

iii) $\mathrm{vol}(A) = \mathrm{vol}(K)$, *iv)* $\mathrm{G}(A) \geq \mathrm{G}(K)$,

v) $\mathrm{G}(\mathrm{int}A) \leq \mathrm{G}(\mathrm{int}K)$,

vi) $\lambda_1(A) \leq \lambda_1(K)$,

vii) $\lambda_2(A) = \lambda_2(K)$ *and*

viii) $A \subseteq \{x \in \mathbb{R}^2 : x_1 + x_2 \leq 2/\lambda_1(A)\}$.

For the proof of the lemma, we will use that non-orthogonal Blaschke shakings are monotonous:

Lemma 4.4.2. *Let $K, L \in \mathcal{K}_2^2$ with $K \subseteq L$. Also, let $\ell \subseteq \mathbb{R}^2$ be a line and $u \in \mathbb{R}^n \setminus \{0\}$ be not parallel to ℓ. Then we have*

$$\mathrm{sh}_{u,\ell}(K) \subseteq \mathrm{sh}_{u,\ell}(L).$$

This is a widely known fact in the context of classical Blaschke shakings (cf. [CCG01, Lemma 1.1 (iii)]).

Proof. Let $\mathrm{sh} = \mathrm{sh}_{u,\ell}$ and $\pi = \pi_{u,\ell}$. Consider a point $x \in \mathrm{sh}(K)$. Then we have $x \in \pi(K) \subseteq \pi(L)$. Also, by inclusion, we have

$$l_1 := \mathrm{vol}_1\big(K \cap (x + \mathbb{R}u)\big) \leq \mathrm{vol}_1\big(L \cap (x + \mathbb{R}u)\big) =: l_2.$$

Hence, since $x \in \mathrm{sh}(K)$,

$$x \in \Big[\pi(x), \pi(x) + l_1 \frac{u}{|u|}\Big] \subseteq \Big[\pi(x), \pi(x) + l_2 \frac{u}{|u|}\Big] \subseteq \mathrm{sh}(L). \qquad \square$$

Proof of Lemma 4.4.1. For i) we show that $\mathrm{sh}_{u,\ell}(K)$ is convex for all u, K and ℓ as in Lemma 4.4.2. To see this, we consider $x, y \in \mathrm{sh}_{u,\ell}(K)$. Let $\overline{x}$ and $\overline{y}$ denote the points in K on the lines $x + \mathbb{R}u$ and $y + \mathbb{R}u$ respectively that minimize $\langle \cdot, u\rangle$. Then, the points

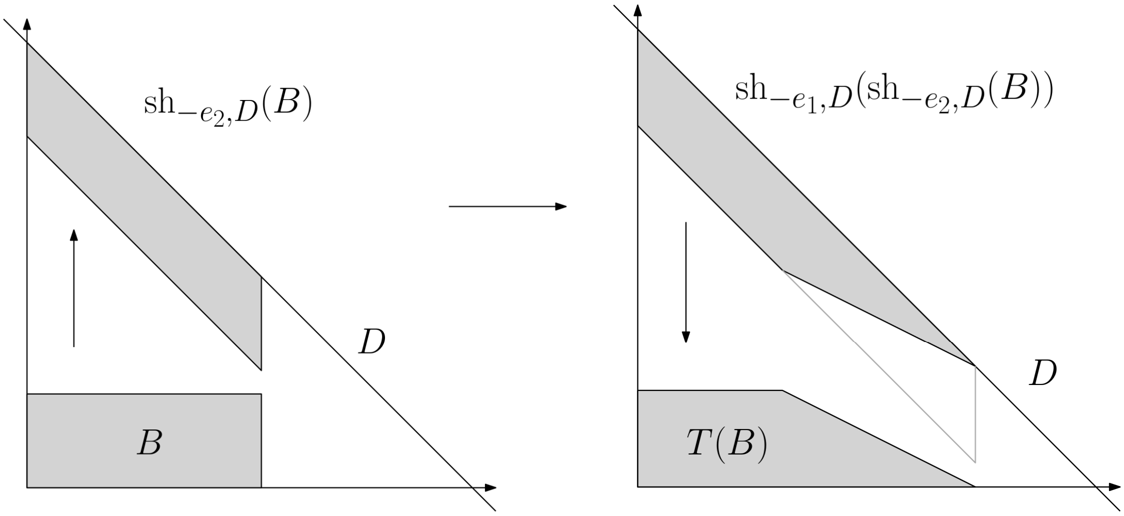
$$\widetilde{z} = \overline{z} + |z - \pi_{u,\ell}(z)| \cdot u/|u|, \quad z \in \{x, y\},$$

are contained in K. Lemma 4.4.2 then yields that

$$\mathrm{conv}\{\pi_{u,\ell}(x), \pi_{u,\ell}(y), x, y\} = \mathrm{sh}_{u,\ell}\big(\mathrm{conv}\{\overline{x}, \widetilde{x}, \overline{y}, \widetilde{y}\}\big) \subseteq \mathrm{sh}_{u,\ell}(K).$$

In particular, $[x, y] \subseteq \mathrm{sh}_{u,\ell}(K)$, which shows that $\mathrm{sh}_{u,\ell}(K)$ is convex. Thus, A is convex as well.

Next, we consider the box $B = [0, 2/\lambda_1(K)] \times [0, 2/\lambda_2(K)]$. Clearly, we have $K \subseteq B$ and by Lemma 4.4.2 it follows that $A \subseteq T(B)$. The vertical segments $\mathrm{sh}_{-e_2,D}$ are of length $2/\lambda_2(K)$. The vertical segments in $\mathrm{sh}_{-e_1,D}(\mathrm{sh}_{-e_2,D}(B))$ are also not longer than those in



(a) Applying $\mathrm{sh}_{-e_1,D}$ to the box B does not change the lengths of the vertical segments.

(b) Applying $\mathrm{sh}_{-e_1,D}$ does not lead to any vertical segments of length higher than $2/\lambda_2(K)$ as they are all contained in the grey area.

Figure 4.4: Behaviour of the vertical segments when passing from B to $T(B)$

$\mathrm{sh}_{-e_2,D}(K)$. Therefore, all vertical segments in $T(B)$ (and thus also in A) are of length at most $2/\lambda_2(K)$ (cf. Figure 4.4). On the other hand, by considering the triangle

$$\Delta = \mathrm{conv}\{0, 2/\lambda_1(K)e_1, 2/\lambda_2(K)e_2\} \subseteq K,$$

which fulfills $T(\Delta) = \Delta$ because of $\lambda_1(K) \leq \lambda_2(K)$, we see that the segment over the origin in A has length precisely $2/\lambda_2(K)$. Since by construction we have $A \cap e_2^\perp = A|e_2^\perp$, we obtain from this $A \cap e_1^\perp = A|e_1^\perp$ as well. Therefore, A is anti-blocking and fulfills $|e_2|_{\mathrm{cs}(A)} = \lambda_2(K)$, as well as $|e_1|_{\mathrm{cs}(A)} \leq |e_1|_{\mathrm{cs}(\Delta)} = \lambda_1$. So we obtained ii), vi) and vii).

iii) follows from Fubini's theorem applied to $\mathrm{span}\{u\}$ and $u^\perp$, since also for arbitrary non-orthogonal shakings one has that $\mathrm{sh}_{u,\ell}(K)|u^\perp = K|u^\perp$ and $\mathrm{vol}_1(\mathrm{sh}_{u,\ell}(K) \cap (x + \mathbb{R}u)) = \mathrm{vol}_1(K \cap (x + \mathbb{R}u))$, for any $x \in u^\perp$.

iv) and v) are proven in the same way as Lemma 4.2.2 v) and vi), since $\pi_{-e_i,D}(\mathbb{Z}^2) = \mathbb{Z}^2 \cap D$ and, thus, all the shaken lattice segments in e_i-direction of any $\mathrm{sh}_{-e_i,D}(K)$ contain a lattice point on D as an endpoint.

For viii), let p denote the lowest point on the diagonal (with respect to e_2) such that $p \in \mathrm{sh}_{-e_1,D}(\mathrm{sh}_{-e_2,D}(K))$. Then we have

$$\mathrm{sh}_{-e_1,D}(\mathrm{sh}_{-e_2,D}(K)) \subseteq \mathrm{conv}\{me_2, p|e_1^\perp, p\},$$

where m is the integer with the property that $me_2 \in D$ (cf. Figure 4.3). Applying sh_{e_2} to both sides of the inclusion yields that

$$A \subseteq \mathrm{conv}\{0, 2/\lambda_1(A)e_1, 2/\lambda_1(A)e_2\} \subseteq \{x \in \mathbb{R}^2 : x_1 + x_2 \leq 2/\lambda_1(A)\}. \qquad \square$$

In order to prove Theorem 4.1.5, we also need the following estimates, which follow from elementary properties of concave functions:

Lemma 4.4.3. *Let $f : [a, b] \to \mathbb{R}$ be a concave function, then we have*

$$\frac{1}{2}\big(f(a) + f(b)\big)(b - a) \leq \int_a^b f(t)\,\mathrm{d}t.$$

Moreover, if $f'(a)$ exists, we also have

$$\int_a^b f(t)\,\mathrm{d}t \leq (b - a)\Big(f(a) + (b - a)\frac{1}{2}f'(a)\Big).$$

Proof. For the upper bound, let g be the affine linear function given by $g(a) = f(a)$ and $g(b) = f(b)$, i.e., $g(t) = \frac{f(b)-f(a)}{b-a}(t - a) + f(a)$. By concavity, we have $f \geq g$ and therefore

$$\int_a^b f(t)\,\mathrm{d}t \geq \int_a^b g(t)\,\mathrm{d}t = \frac{1}{2}\big(f(a) + f(b)\big)(b - a).$$

For the upper bound let h be the tangent of f at a, i.e., $h(t) = f'(a)(t - a) + f(a)$. Again by concavity, we have $h \geq f$ and, thus,

$$\int_a^b f(t)\,\mathrm{d}t \leq \int_a^b h(t)\,\mathrm{d}t \leq (b - a)\Big(f(a) + (b - a)\frac{1}{2}f'(a)\Big)$$

□

Proof of Theorem 4.1.5. We write $\lambda_i = \lambda_i(K)$, $i = 1, 2$. In view of Theorem 4.2.1 and Lemma 4.4.1, we can assume that K is an anti-blocking body with $|e_i|_{\mathrm{cs}(K)} = \lambda_i$ and $K \subseteq \{x \in \mathbb{R}^2 : x_1 + x_2 \leq 2/\lambda_1\}$. We let $\ell_t = \{x \in \mathbb{R}^2 : x_2 = t\}$ denote the horizontal line at height $t \in \mathbb{R}$ and we consider

$$f : \Big[0, \frac{2}{\lambda_2}\Big] \to \mathbb{R},\ t \mapsto \mathrm{vol}_1(K \cap \ell_t).$$

We observe that $\mathrm{h}(K, e_2) = 2/\lambda_2$ holds. Since K is convex, this implies that f is concave. Moreover, since K is anti-blocking, f is decreasing. From the inclusion $K \subseteq \{x \in \mathbb{R}^2 : x_1 + x_2 \leq 2/\lambda_1\}$ it follows that

$$f(t) \leq f(0) - t = \frac{2}{\lambda_1} - t \tag{4.23}$$

holds for all $t \in [0, 2/\lambda_2]$. Exploiting the fact that the sections $K \cap \ell_t$ are 1-dimensional, we obtain that

$$\mathrm{G}(K) = \sum_{i=0}^{\lfloor 2/\lambda_2 \rfloor} \mathrm{G}(K \cap \ell_t) \leq \sum_{i=0}^{\lfloor 2/\lambda_2 \rfloor} \big(f(i) + 1\big)$$

$$= \frac{1}{2}\big(f(0) + f(\lfloor 2/\lambda_2 \rfloor) + \sum_{i=1}^{\lfloor 2/\lambda_2 \rfloor} \Big(\frac{1}{2}\big(f(i-1) + f(i)\big)\Big) + \lfloor 2/\lambda_2 \rfloor + 1.$$

We apply (4.23) to $f(\lfloor 2/\lambda_2 \rfloor)$ and the lower bound in Lemma 4.4.3 to the summands $\frac{1}{2}\big(f(i-1) + f(i)\big)$ and deduce

$$\begin{aligned}
\mathrm{G}(K) &\leq \frac{2}{\lambda_1} - \frac{1}{2}\lfloor 2/\lambda_2 \rfloor + \int_0^{\lfloor 2/\lambda_2 \rfloor} f(t)\,\mathrm{d}t + \lfloor 2/\lambda_2 \rfloor + 1 \\
&\leq \mathrm{vol}(K) + \frac{2}{\lambda_1} + \frac{1}{\lambda_2} + 1 \\
&\leq \mathrm{vol}(K)\Big(1 + \lambda_2 + \frac{\lambda_1}{2} + \frac{\lambda_1\lambda_2}{2}\Big) \\
&= \mathrm{vol}(K)\Big(1 + \frac{\lambda_1}{2}\Big)\big(1 + \lambda_2\big),
\end{aligned}$$

where we used the lower bound in Minkowski's second theorem (4.1), applied to the summands $2/\lambda_1$, $1/\lambda_2$ and 1 each, to obtain the third line. This shows the upper bound (4.11) of the theorem.

For the lower bound, we assume that f is differentiable. Else, we might approximate f with a linear spline φ from below. φ in turn can be approximated by a smooth concave function g from below by rounding its corners. This function satisfies $g(0) = f(0)$, and thus, the anti-blocking convex body

$$K' = \{(x, y) \in \mathbb{R}^2 : 0 \leq y \leq 2/\lambda_2,\ 0 \leq x \leq g(y)\} \subseteq K$$

is located underneath the diagonal $\{x \in \mathbb{R}^2 : x_1 + x_2 = 2/\lambda_2(K')\}$ as well.

We observe that $\lceil 2/\lambda_2 - 1 \rceil$ is the height of the highest horizontal integer line that intersects $\mathrm{int}K$. Therefore, we can estimate the number of interior lattice points of K as follows:

$$\begin{aligned}
\mathrm{G}(\mathrm{int}K) &= \sum_{i=1}^{\lceil 2/\lambda_2 - 1 \rceil} \mathrm{G}(\mathrm{int}K \cap \ell_i) \geq \sum_{i=1}^{\lceil 2/\lambda_2 - 1 \rceil} \big(f(i) - 1\big) \\
&= \sum_{i=1}^{\lceil 2/\lambda_2 - 1 \rceil} f(i) - \lceil 2/\lambda_2 - 1 \rceil.
\end{aligned}$$

We use the upper bound in Lemma 4.4.3 in order to estimate

$$f(i) \geq \int_i^{i+1} f(t)\,\mathrm{d}t - \frac{1}{2} f'(i),$$

for all $1 \leq i < \lceil 2/\lambda_2 - 1 \rceil$ and, since $2/\lambda_2 - \lceil 2/\lambda_2 - 1 \rceil \leq 1$,

$$f\big(\lceil 2/\lambda_2 - 1 \rceil\big) \geq \int_{\lceil 2/\lambda_2 - 1 \rceil}^{2/\lambda_2} f(t)\,\mathrm{d}t - (2/\lambda_2 - \lceil 2/\lambda_2 - 1 \rceil)\frac{1}{2} f'\big(\lceil 2/\lambda_2 - 1 \rceil\big).$$

Combining this, we obtain

$$
\begin{aligned}
\mathrm{G}(\mathrm{int}K) \geq & \sum_{i=1}^{\lceil 2/\lambda_2-1\rceil-1} \int_i^{i+1} f(t)\,\mathrm{d}t + \int_{\lceil 2/\lambda_2-1\rceil}^{\lambda_2/2} f(t)\,\mathrm{d}t \\
& + \frac{1}{2}\Big(\sum_{i=1}^{\lceil 2/\lambda_2-1\rceil-1} \big(-f'(i)\big) - (2/\lambda_2 - \lceil 2/\lambda_2-1\rceil) f'\big(\lceil 2/\lambda_2-1\rceil\big)\Big) \\
& - \lceil 2/\lambda_2 - 1\rceil \\
= & \operatorname{vol}(K) - \int_0^1 f(t)\,\mathrm{d}t \\
& + \frac{1}{2}\Big(\sum_{i=1}^{\lceil 2/\lambda_2-1\rceil-1} \big(-f'(i)\big) - (2/\lambda_2 - \lceil 2/\lambda_2-1\rceil) f'\big(\lceil 2/\lambda_2-1\rceil\big)\Big) \\
& - \lceil 2/\lambda_2 - 1\rceil
\end{aligned} \tag{4.24}
$$

Due to (4.23), we have

$$
\int_0^1 f(t)\,\mathrm{d}t \leq \frac{2}{\lambda_1} - \frac{1}{2}. \tag{4.25}
$$

Next, we estimate the bracket term in the last but one line of (4.24). To this end, we observe that $-f'$ is increasing, since f is concave, and that $-f'$ is non-negative, since K is anti-blocking. Therefore, we obtain that

$$
\begin{aligned}
& \sum_{i=1}^{\lceil 2/\lambda_2-1\rceil-1} \big(-f'(i)\big) - (2/\lambda_2 - \lceil 2/\lambda_2-1\rceil) f'\big(\lceil 2/\lambda_2-1\rceil\big) \\
\geq & \sum_{i=1}^{\lceil 2/\lambda_2-1\rceil-1} \int_{i-1}^{i} -f'(t)\,\mathrm{d}t \\
& + \big((2/\lambda_2-1) - (\lceil 2/\lambda_2-1\rceil - 1)\big)(-f'(2/\lambda_2-1)) \\
\geq & \int_0^{2/\lambda_2-1} -f'(t)\,\mathrm{d}t = f(0) - f(2/\lambda_2-1) \\
\geq & \frac{2}{\lambda_2} - 1,
\end{aligned}
$$

where we used (4.23) in the last step. Substituting this and (4.25) into (4.24) yields

$$
\begin{aligned}
\mathrm{G}(\mathrm{int}K) & \geq \operatorname{vol}(K) - \frac{2}{\lambda_1} + \frac{1}{2} + \frac{1}{2}\Big(\frac{2}{\lambda_2} - 1\Big) - \left\lceil \frac{2}{\lambda_2} - 1 \right\rceil \\
& \geq \operatorname{vol}(K) - \frac{2}{\lambda_1} - \frac{1}{\lambda_2} = \operatorname{vol}(K)\Big(1 - \frac{1}{\operatorname{vol}(K)}\Big(\frac{2}{\lambda_1} + \frac{1}{\lambda_2}\Big)\Big) \\
& \geq \operatorname{vol}(K)\Big(1 - \frac{\lambda_1}{2} - \lambda_2\Big),
\end{aligned}
$$

where we used $\operatorname{vol}(K) \geq \frac{2}{\lambda_1\lambda_2}$ in the last step. Therefore, the proof of (4.12) is finished. □

Next, we prove (4.7) in the special classes of symmetric lattice polygons and rectangular triangles.

Proposition 4.4.4. *Let $P \in \mathcal{K}_2^2$ be an origin-symmetric lattice polygon, i.e., we have $-P = P$ and P is the convex hull of finitely many integer points. Then we have*

$$\mathrm{G}(\mathrm{int}P) \geq \mathrm{vol}(P)\left(1 - \frac{\lambda_1}{2}\right)\left(1 - \frac{\lambda_2}{2}\right),$$

where $\lambda_i = \lambda_i(P)$.

Proof. Let $\lambda_i = \lambda_i(P)$, $i = 1, 2$. By Pick's theorem (2.14), we have for a lattice polygon P that

$$\mathrm{G}(\mathrm{int}P) = \mathrm{vol}(P) - \frac{\mathrm{G}(\mathrm{bd}P)}{2} + 1. \tag{4.26}$$

An inequality of Henk, Schürmann and Wills yields that [HSW05, Eq. 1.6]

$$\frac{\mathrm{G}(\mathrm{bd}P)}{2} \leq \mathrm{vol}(P)\Big(\frac{\lambda_1}{2} + \frac{\lambda_2}{2}\Big). \tag{4.27}$$

Combining (4.26) with (4.27) yields

$$\mathrm{G}(\mathrm{int}P) \geq \mathrm{vol}(P) - \mathrm{vol}(P)\Big(\frac{\lambda_1}{2} + \frac{\lambda_2}{2}\Big) + 1.$$

By the upper bound in (4.1), we have $1 \geq \mathrm{vol}(P)\lambda_1\lambda_2/4$. Hence,

$$\mathrm{G}(\mathrm{int}P) \geq \mathrm{vol}(P)\Big(1 - \frac{\lambda_1}{2} - \frac{\lambda_2}{2} + \frac{\lambda_1\lambda_2}{4}\Big) \geq \mathrm{vol}(P)\Big(1 - \frac{\lambda_1}{2}\Big)\Big(1 - \frac{\lambda_2}{2}\Big),$$

and the proof is finished. □

Now we give the proof of (4.6) for right triangles, where we can reformulate the inequality as follows.

Proposition 4.4.5. *Let $a, b \in \mathbb{R}$, where $2 \leq b \leq a$, and define $T = \mathrm{conv}\{0, ae_1, be_2\}$. Then,*

$$\mathrm{G}(\mathrm{int}T) \geq \frac{1}{2}(a-1)(b-2).$$

The proof builds on the classical argument for counting the number of lattice points in a right triangle with integer vertices. There one considers the rectangle $[0, a] \times [0, b]$ and uses the fact that this rectangle decomposes into two copies of T, namely T and $(a, b)^T - T$, that only intersect in the diagonal. For this argument, it is crucial that a and b are integers. Only then one can guarantee that both copies contain the same number of lattice points.

In the proof we will solve the issue by rounding a and b appropriately. For this, we need some notation. For $c \in \mathbb{R}$ we write $\lfloor c \rceil \in \mathbb{Z}$ to denote the unique integer with $c - \lfloor c \rceil \in (-1/2, 1/2]$. Furthermore, we define $\{c\} = c - \lfloor c \rceil$.

Proof. Let $t = (\lfloor a \rceil, \lfloor b \rceil)$. We consider the triangle

$$T' = t - T = \Big\{ y \le t : \frac{\lfloor a \rceil - y_1}{a} + \frac{\lfloor b \rceil - y_2}{b} \le 1 \Big\} = \Big\{ y \le t : \frac{y_1}{a} + \frac{y_2}{b} \ge \beta \Big\},$$

where the inequality $y \le t$ is to be understood componentwise and $\beta = 1 - \frac{\{a\}}{a} - \frac{\{b\}}{b}$. Observe that $\mathrm{G}(T') = \mathrm{G}(T)$. We distinguish the cases $\beta < 1$ and $\beta \ge 1$, i.e., T and T' overlap and T and T' do not overlap (cf. Figure 4.5).

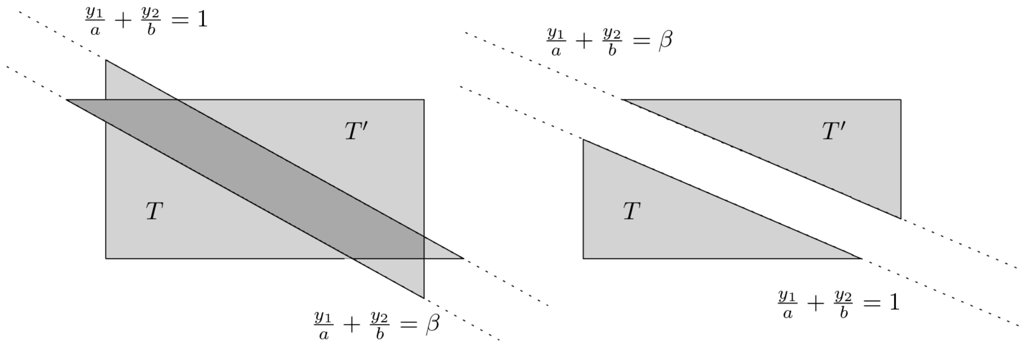


Figure 4.5: Illustration of the two main cases in the proof of Proposition 4.4.5. The situation on the left describes the case $\beta < 1$, in which the triangles T and T' overlap. On the right, we see the case $\beta > 1$, where T and T' are disjoint.

Case 1 $\beta < 1$.

Case 1.1 $\{a\} > 0$, $\{b\} > 0$.

This means $\lfloor a \rceil = \lfloor a \rfloor$ and $\lfloor b \rceil = \lfloor b \rfloor$. Since T and T' overlap, estimating the number of lattice points in $[0, \lfloor a \rfloor] \times [0, \lfloor b \rfloor]$ leads to

$$(\lfloor a \rfloor + 1)(\lfloor b \rfloor + 1) \le 2\lfloor a \rfloor + 2\lfloor b \rfloor + \mathrm{G}(\mathrm{int}T) + \mathrm{G}(\mathrm{int}T') = 2\lfloor a \rfloor + 2\lfloor b \rfloor + 2\mathrm{G}(\mathrm{int}T).$$

It follows that.

$$2\mathrm{G}(\mathrm{int}T) \ge (\lfloor a \rfloor - 1)(\lfloor b \rfloor - 1) \ge \Big(a - \frac{3}{2}\Big)\Big(b - \frac{3}{2}\Big) \ge (a-1)(b-2),$$

where in the second step we used the assumption of Case 1.1 and in the third step the fact that $a \ge b$. So the case is finished.

Case 1.2 $\{a\} > 0$, $\{b\} \le 0$.

Similarly to the previous case, we have

$$\begin{aligned} 2\mathrm{G}(\mathrm{int}T) &= (\lfloor a \rceil - 1)(\lfloor b \rceil - 1) = (a - \{a\} - 1)(b - \{b\} - 1) \\ &\ge \Big(a - \frac{3}{2}\Big)(b-1) = ab - b - 2a + \frac{3}{2} + \frac{1}{2}(a-b) + \frac{1}{2}a \\ &\ge ab - 2a - b + 2 = (a-1)(b-2). \end{aligned}$$

For the second inequality, we used the assumption of Case 1.2, for the third inequality we used $2 \le b \le a$.

Case 1.3 $\{a\} \leq 0$, $\{b\} > 0$.

As in Case 1.1, we obtain

$$\begin{aligned} 2\mathrm{G}(\mathrm{int}T) &\geq (a - \{a\} - 1)(b - \{b\} - 1) =\geq (a-1)\Big(b - \frac{3}{2}\Big) \\ &= ab - b - \frac{3}{2}a + \frac{3}{2} = ab - b - 2a + \frac{3}{2} + \frac{1}{2}a \geq (a-1)(b-2). \end{aligned}$$

For the last step we used that $a \geq 2$ and thus, $3/2 + a/2 \geq 2$. Since $\{a\}, \{b\} \leq 0$ is not compatible with $\beta < 1$, Case 1 is completed.

Case 2 $\beta \geq 1$.

Case 2.1 $\{a\} \leq 0$, $\{b\} \leq 0$.

We have

$$\beta = 1 - \frac{\{a\}}{a} - \frac{\{b\}}{b} \leq 1 + \frac{1}{2a} + \frac{1}{2b} \leq 1 + \frac{1}{b}.$$

This implies that a vertical segment in the strip $S = \{x \in \mathbb{R}^2 : 1 \leq x_1/a + x_2/b \leq \beta\}$ has length < 1 and, thus, contains at most one lattice point (cf. Figure 4.6).

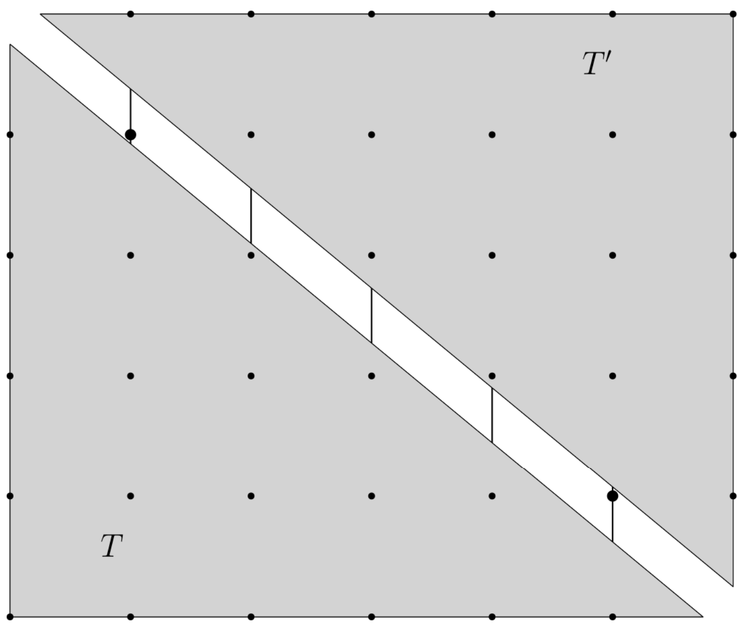


Figure 4.6: The vertical segments between T and T' cannot contain more than one lattice point each.

The assumption of Case 2.1 is equivalent to $\lfloor a \rceil = \lceil a \rceil$ and $\lfloor b \rceil = \lceil b \rceil$. Counting the lattice points in $B = [0, \lceil a \rceil] \times [0 \lceil b \rceil]$ leads to

$$\begin{aligned} (\lceil a \rceil + 1)(]\lceil b \rceil + 1) &= 2\lceil a \rceil + 2\lceil b \rceil + 2\mathrm{G}(\mathrm{int}T) + \mathrm{G}(\mathrm{int}B \cap S) \\ &\leq 2\lceil a \rceil + 2\lceil b \rceil + 2\mathrm{G}(\mathrm{int}T) + \lceil a \rceil - 1. \end{aligned}$$

Rearranging gives

$$2\mathrm{G}(\mathrm{int}T) \geq \lceil a \rceil \lceil b \rceil - 2\lceil a \rceil - \lceil b \rceil + 2 = (\lceil a \rceil - 2)(\lceil b \rceil - 1) \geq (a-2)(b-1).$$

The claim follows from $a \geq b$, since this implies $(a-2)(b-1) \geq (a-1)(b-2)$.

Case 2.2 $\{a\} < 0$ and $\{b\} \geq 0$.

This means $\lfloor a \rceil = \lceil a \rceil$ and $\lfloor b \rceil = \lfloor b \rfloor$. Since T and $(\lfloor a \rfloor, \lfloor b \rfloor)^T - T$ always intersect, we have

$$\begin{aligned} 2\mathrm{G}(\mathrm{int}T) &\geq \lfloor a \rfloor \lfloor b \rfloor - \lfloor a \rfloor - \lfloor b \rfloor + 1 \\ &= (a - \{a\} - 1)(b - \{b\}) - (a - \{a\} - 1) - (b - \{b\}) + 1 \\ &= ab - b\{a\} - a\{b\} + \{a\}\{b\} - b + \{b\} - a + \{a\} + 1 - b + \{b\} + 1. \end{aligned} \tag{4.28}$$

Observe that in general, we have

$$\beta \geq 1 \iff -b\{a\} - a\{b\} \geq 1. \tag{4.29}$$

So by Case 2 it follows that $-b\{a\} - a\{b\} \geq 1$ and we may continue as follows:

$$\begin{aligned} 2\mathrm{G}(\mathrm{int}T) &\geq ab - 2b - a + 2 + \{a\}\{b\} + 2\{b\} + \{a\} \\ &= ab - 2a - b + 2 + a - b + \{a\}\{b\} + 2\{b\} + \{a\} \\ &\geq (a-1)(b-2) + \frac{1}{2} + \{a\}\{b\} + \{b\} + \{a\} \\ &= (a-1)(b-2) + \left(\frac{1}{2} + \{a\}\right) + \{b\}(\{a\} + 1). \end{aligned}$$

For the second inequality we used that by $a \geq b$ and the assumption of Case 2.2 we must have $a \geq \lfloor b \rfloor + \frac{1}{2}$ and therefore $a - b \geq \frac{1}{2} - \{b\}$. It remains to observe that the last two bracket terms in the above equation are non-negative and the case is completed.

Case 2.3 $\{a\} \geq 0$, $\{b\} < 0$.

This is equivalent to $\lfloor a \rceil = \lfloor a \rfloor$ and $\lfloor b \rceil = \lceil b \rceil$. In Case 2.2, we did not use $a \geq b$ until (4.28). So we may exchange the roles of a and b and obtain

$$2\mathrm{G}(\mathrm{int}T) \geq ab - 2a - b + 2 - a\{b\} - b\{a\} + \{a\}\{b\} + 2\{a\} + \{b\}.$$

We let $\gamma = -a\{b\} - b\{a\} + \{a\}\{b\} + 2\{a\} + \{b\}$, so we have to show $\gamma \geq 0$. To this end, we distinguish two final cases.

Case 2.3.1 $\{a\} \geq \frac{-\{b\}}{2+\{b\}}$.

In view of (4.29), we have

$$\gamma \geq \{a\}\{b\} + 2\{a\} + \{b\} = \{a\}(\{b\} + 2) + \{b\} \geq -\{b\} + \{b\} = 0,$$

where we used the assumption of Case 2.3.1 for the last inequality.

Case 2.3.2 $\{a\} < \frac{-\{b\}}{2+\{b\}}$.

In view of $-b + \{b\} + 2 \leq \{b\} < 0$, we have

$$\gamma = \{a\}(-b + \{b\} + 2) + \{b\} - a\{b\} \geq \{b\}\frac{b}{2+\{b\}} - \{b\} + \{b\} - a\{b\}$$
$$= \{b\}\Big(\frac{b}{2+\{b\}} - a\Big) \geq \{b\}\Big(\frac{b}{2+\{b\}} - b\Big) = b\{b\}\Big(\frac{1}{2+\{b\}} - 1\Big) \geq 0.$$

As $\{a\}\{b\} > 0$ is impossible within Case 2, Case 2 is completed, and therefore, the proposition is shown. □

4.5 Further Applications of the Interpolation

In this subsection, we demonstrate two simple applications of the interpolating inequalities of Theorem 4.1.3 and Proposition 4.3.1.

Rectangular Simplices

For $a \in \mathbb{R}^n_{>0}$, let

$$S(a) = \operatorname{conv}\{0, a_1 e_1, ..., a_n e_n\}$$

be the rectangular simplex with side lengths $a_1, ..., a_n$ at the origin. For $a \in \mathbb{Z}^n$, the number of lattice points in $S(a)$ has been studied by various authors, see for instance [DR97] and [Pom93]. For arbitrary real parameters $a \in \mathbb{R}^n$, the number $\mathrm{G}(S(a))$ is not as well understood.

Since we have $\operatorname{vol}(S(a)) = \frac{1}{n!} a_1 \cdots a_n$ and (up to a renumbering) $\lambda_i(S(a))/2 = 1/a_i$, we obtain from Theorem 4.1.3 the following bound.

Proposition 4.5.1. *For real numbers $a_1 > \cdots > a_n$ we have*

$$\mathrm{G}(S(a)) \leq \frac{a_1 \cdots a_n}{n!} \prod_{i=1}^{n} \Big(1 + \frac{n}{a_i}\Big). \tag{4.30}$$

An upper bound of this form has also been proven by Yau and Zhang [YZ06], who obtained

$$\mathrm{G}(S(a)) \leq \frac{a_1 \cdots a_n}{n!} \prod_{i=1}^{n} \Big(1 + \sum_{j \neq i} \frac{1}{a_j}\Big) \tag{4.31}$$

for $n \geq 3$. In general, none of the two bounds (4.30) and (4.31) is stronger than the other. If all the a_i's are equal to a fixed number c, the product in (4.31) is $(1 + (n-1)c)^n$, while in (4.30), we have $(1 + nc)^n$. On the other hand, for $a_1 = c$ fixed and $a_i \to 0$, $i > 1$, the product in (4.30) is of order $(1 + n)$ and in (4.31) it is of order n^n (see also Remark 4.3.2).

Discrete Brunn-Minkowski Inequality

Let us recall the Brunn-Minkowski inequality (2.1) here. For two convex bodies $K, L \subseteq \mathbb{R}^n$ and $\alpha \in [0,1]$, we have

$$\mathrm{vol}(\alpha K + (1-\alpha)L)^{\frac{1}{n}} \geq \alpha\, \mathrm{vol}(K)^{\frac{1}{n}} + (1-\alpha)\mathrm{vol}(L)^{\frac{1}{n}}. \tag{4.32}$$

In fact, convexity of K and L is not necessary for (4.32) to hold, but compactness suffices. This inequality plays a role in several distinct disciplines of mathematics, and there is a large variety of reformulations and generalizations of this inequality [Gar01]. One kind of these generalizations are again discretizations of the inequality. There are two different approaches to this.

- Lower bounds on the cardinality of $|A+B|$, where $A, B \subseteq \mathbb{Z}^n$,
- Lower bounds on $\mathrm{G}(K+L)$, where $K, L \in \mathcal{K}^n$.

In the case of the discrete Meyer inequality and its reverse that we considered in Chapter 3, the distinction between the lattice point enumerator and the cardinality of a set in the integer lattice was not necessary. In case of Minkowski addition, however, one has $\mathrm{G}(K+L) \geq |(K \cap \mathbb{Z}^n) + (L \cap \mathbb{Z}^n)|$ and the inequality is strict, for instance, if $K = L = \frac{1}{2}C_n$.

In [GG01, HIY18] discrete variants of the Brunn-Minkowski inequality within the integer lattice are obtained, while in [HKS21, IYZ20, ILY22] one finds Brunn-Minkowski-type inequalities for the lattice point enumerator of compact sets. It is pointed out in [IYZ20] that no direct discretization of (4.32) exists, i.e., it is not true that

$$\mathrm{G}(\alpha K + (1-\alpha)L)^{\frac{1}{n}} \geq \alpha\, \mathrm{G}(K)^{\frac{1}{n}} + (1-\alpha)\mathrm{G}(L)^{\frac{1}{n}}.$$

for all $\lambda \in [0,1]$ and all compact sets (or even convex bodies) $K, L \subseteq \mathbb{R}^n$. Instead, they prove the elegant inequality

$$\mathrm{G}(\alpha K + (1-\alpha)L + (-1,1)^n)^{\frac{1}{n}} \geq \alpha\, \mathrm{G}(K)^{\frac{1}{n}} + (1-\alpha)\mathrm{G}(L)^{\frac{1}{n}}, \tag{4.33}$$

in which the convex combination of two compact sets K and L is enlarged by the open cube $(-1,1)^n$. Using the approximation of the volume by the lattice point enumerator, the classical Brunn-Minkowski inequality can be deduced from (4.33) and, moreover, the proof of (4.33) is independent of the Brunn-Minkowski inequality.

A disadvantage of the enlargement by the cube $(-1,1)^n$ in (4.33) is, however, that the inequality is not unimodularly invariant; If $\alpha K + (1-\alpha)L$ is the closed square C_2, then, $\mathrm{G}(C_2 + (-1,1)^2) = \mathrm{G}(C_2)$, while for a shearing $M_k = \mathrm{conv}\{\pm(2k-1,1)^T, \pm(2k+1,1)^T\}$ of C_2, we have $\mathrm{G}(M_k + (-1,1)^2) \to \infty$ as $k \to \infty$. Thus, applying a unimodular transformation to K and L might affect the sharpness of (4.33) drastically.

We can use the approximation of the lattice point enumerator by the volume and the covering radius (Proposition 4.3.1) in conjunction with the Brunn-Minkowski inequality to obtain a Brunn-Minkowski-type inequality for the lattice point enumerator of convex bodies that is equivalent to the continuous inequality and invariant with respect to unimodular transformations.

Proposition 4.5.2. *Let $K, L \in \mathcal{K}_n^n$ and $\alpha \in [0,1]$. If, for the convex combination $M = \alpha K + (1-\alpha)L$, we have $M + \mathbb{Z}^n = \mathbb{R}^n$, then,*

$$\mathrm{G}(\mathrm{int} M)^{\frac{1}{n}} \geq \frac{1-\mu(M)}{1+\mu(M)}\big(\alpha\, \mathrm{G}(K)^{\frac{1}{n}} + (1-\alpha)\mathrm{G}(L)^{\frac{1}{n}}\big).$$

Proof. As in the proof of Proposition 4.3.1, we find a tile $T \subseteq \mu(M)M$. Hence, we obtain

$$\begin{aligned}\alpha\, \mathrm{G}(K)^{\frac{1}{n}} + (1-\alpha)\mathrm{G}(L)^{\frac{1}{n}} &\leq \alpha\, \mathrm{vol}(K+T)^{\frac{1}{n}} + (1-\alpha)\mathrm{vol}(L+T)^{\frac{1}{n}}\\ &\leq \alpha\, \mathrm{vol}(K+\mu(M)M)^{\frac{1}{n}} + (1-\alpha)\mathrm{vol}(L+\mu(M)M)^{\frac{1}{n}}\\ &\leq \mathrm{vol}(\alpha(K+\mu(M)M) + (1-\alpha)(L+\mu(M)M))^{\frac{1}{n}}\\ &= (1+\mu(M))\mathrm{vol}(M)^{\frac{1}{n}},\end{aligned}$$

where for the last inequality, we applied the Brunn-Minkowski inequality to $K + \mu(M)M$ and $L + \mu(M)M$. Since by assumption $M + \mathbb{Z}^n = \mathbb{R}^n$, we have $\mu(M) \leq 1$. For $\mu(M) = 1$, there is nothing to show, so we assume $\mu(M) < 1$. Thus, we can apply the lower bound of Proposition 4.3.1 and obtain

$$\alpha\, \mathrm{G}(K)^{\frac{1}{n}} + (1-\alpha)\mathrm{G}(L)^{\frac{1}{n}} \leq \frac{1+\mu(M)}{1-\mu(M)}\mathrm{G}(M)^{\frac{1}{n}},$$

which finishes the proof. □

It is not clear to see how the quantity $\mu(M)$ depends on the convex bodies K and L. In order to gain a better understanding of the constant $(1-\mu(M))/(1+\mu(M))$ in Proposition 4.5.2 it would therefore be beneficial to estimate $\mu(M)$ from above by $\mu(K)$ and $\mu(L)$. Since the covering radius is (-1)-homogeneous one might expect that an inequality of the following form holds:

$$\mu(M) \leq \frac{1}{\frac{\alpha}{\mu(K)} + \frac{1-\alpha}{\mu(L)}}.$$

We are not aware of any counterexamples to this inequality. However, we can only confirm it in the special case of $n = 2$ and $L = -K$; Then, $\mu(K)K$ contains a *convex* tile P, which is then necessarily centrally symmetric with respect to some center $c \in \mathbb{R}^n$ [GL87, Ch. 3, Lemma 22.3]. Due to the translation invariance of $\mu(\cdot)$, we may assume $c = 0$. Thus, we have $P = \alpha P - (1-\alpha)P \subseteq \mu(K)M$. From this, it follows that

$$\mu(M) \leq \mu(K) = \frac{1}{\frac{\alpha}{\mu(K)} + \frac{1-\alpha}{\mu(-K)}}.$$

It is decisive for this argument that the tile P is symmetric, which is why it does not extend to higher dimensions.

Part II

Subspace Concentration

5 Affine Subspace Concentration Conditions for Centered Polytopes

In this chapter, we generalize the affine subspace concentration conditions that K.-Y. Wu showed for centered smooth reflexive polytopes to arbitrary centered polytopes. The results in this chapter stem from a work in progress with Martin Henk and Christian Kipp.

Before we dive into the affine setting, we shall recall the linear case and its significance in modern convex geometry.

5.1 Linear Subspace Concentration Conditions

Let P be an n-dimensional polytope in $\mathbb{R}^n$ that contains the origin as an interior point. We recall from Chapter 2 that P admits a unique representation as (cf. (2.5))

$$P = \{x \in \mathbb{R}^n : \langle a_i, x\rangle \leq 1,\ 1 \leq i \leq m\},$$

where the vectors $a_i \in \mathbb{R}^n \setminus \{0\}$ are pairwise different and $F_i = P \cap \{x \in \mathbb{R}^n : \langle a_i, x\rangle = 1\}$, $1 \leq i \leq m$, are the facets of P. Then the volume of P can be written as

$$\mathrm{vol}(P) = \frac{1}{n}\sum_{i=1}^{m} \mathrm{vol}_{n-1}(F_i)\frac{1}{|a_i|}.$$

This identity is also known as the pyramid formula, as it sums up the volumes of the pyramids (cones)

$$C_i = \mathrm{conv}(\{0\} \cup F_i),$$

which form a subdivision of P. Observe that

$$\mathrm{vol}(C_i) = \frac{1}{n}\frac{1}{|a_i|}\mathrm{vol}_{n-1}(F_i),\ 1 \leq i \leq m.$$

These cone volumes are the geometric base of the cone-volume measure of an arbitrary convex body, which is a finite positive Borel measure on the $(n-1)$-dimensional unit sphere $\mathbb{S}^{n-1} \subseteq \mathbb{R}^n$. The cone-volume measure is the subject of the well-known and important log-Minkowski problem in modern convex geometry, see for instance [BHZ16, BLYZ13].

In the discrete setting, i.e., the polytopal case, the cone-volume measure $\mathrm{V}_P(\cdot)$ associated to P is the discrete measure

$$\mathrm{V}_P(\eta) = \sum_{i=1}^{m} \mathrm{vol}(C_i)\, \delta_{u_i}(\eta),$$

where $\eta \subseteq \mathbb{S}^{n-1}$ is a Borel set, and $\delta_{u_i}(\cdot)$ denotes the delta measure concentrated on u_i. In analogy to the classical Minkowski-problem, the discrete log-Minkowski problem asks for sufficient and necessary conditions such that a discrete Borel measure $\mu = \sum_{i=1}^{m} \gamma_i\, \delta_{u_i}(\cdot)$, $\gamma_i \in \mathbb{R}_{>0}$, $u_i \in \mathbb{S}^{n-1}$, is the cone-volume measure of a polytope.

Böröczky, Lutwak, Yang and Zhang settled the general (i.e., not necessarily discrete) log-Minkowski problem for arbitrary finite even Borel measures. Here *even* means that $\mu(A) = \mu(-A)$ holds for all Borel sets $A \subseteq \mathbb{S}^{n-1}$. This assumption corresponds to the case of origin-symmetric convex bodies; reduced to the discrete setting their result may be stated as follows:

Theorem 5.1.1 (Böröczky, Lutwak, Yang and Zhang, [BLYZ13]). *A discrete even Borel measure $\mu : \mathbb{S}^{n-1} \to \mathbb{R}_{\geq 0}$ given by $\mu = \sum_{i=1}^{m} \gamma_i\, \delta_{u_i}$, $\gamma_i \in \mathbb{R}_{>0}$, $u_i \in \mathbb{S}^{n-1}$, is the cone-volume measure of an origin-symmetric n-polytope if and only if the* subspace concentration conditions *are fulfilled, i.e., for every linear subspace $L \subseteq \mathbb{R}^n$ it holds*

$$\mu(L \cap \mathbb{S}^{n-1}) = \sum_{i:\, u_i \in L} \gamma_i \leq \frac{\dim L}{n} \sum_{i=1}^{m} \gamma_i = \frac{\dim L}{n} \mu(\mathbb{S}^{n-1}) \tag{5.1}$$

and equality is achieved, if and only if there exists a complementary subspace L' such that μ is concentrated on $L \cup L'$.

In the non-even case, even in the discrete setting, a complete characterization is still missing, see [CLZ19]. The main problem here is the position of the origin.

Recall that an n-polytope $P \subseteq \mathbb{R}^n$ is called *centered* if its centroid $\mathrm{c}(P)$ is at the origin, i.e.,

$$\mathrm{c}(P) = \mathrm{vol}(P)^{-1} \int_P x \,\mathrm{d}x = 0.$$

It is known that centered polytopes satisfy the subspace concentration conditions.

Theorem 5.1.2 (Henk and Linke, [HL14]). *Let $P = \{x \in \mathbb{R}^n : \langle a_i, x\rangle \leq 1,\, 1 \leq i \leq m\}$ be a centered polytope and let $L \subseteq \mathbb{R}^n$ be a linear subspace. Then,* (5.1) *holds true, i.e.,*

$$\sum_{i:\, a_i \in L} \mathrm{vol}(C_i) \leq \frac{\dim L}{n} \mathrm{vol}(P).$$

Equality is obtained if and only if there exists a complementary linear subspace $L' \subseteq \mathbb{R}^n$ to L such that $\{a_i : 1 \leq i \leq m\} \subseteq L \cup L'$.

For a generalization to centered convex bodies we refer to [BH16].

Remark 5.1.3. i) The statement of Theorem 5.1.2 only depends on the direction of the a_i's but not on their lengths. I.e., if we renormalize the inequalities of P to obtain representation $P = \{x \in \mathbb{R}^n : \langle a'_i, x\rangle \leq b_i,\ 1 \leq i \leq m\}$, for certain $a'_i \in \mathbb{R}^n \setminus \{0\}$ and $b_i \in \mathbb{R}$, then it still holds that

$$\sum_{i\colon a'_i \in L} \mathrm{vol}(C_i) \leq \frac{\dim L}{n}\mathrm{vol}(P).$$

This circumstance distinguishes the linear case from the affine inequalities to come.

ii) The subspace concentration conditions are used in convex and discrete geometry beyond the context of the log-Minkowski problem, since they offer a deep insight into the distribution of volume in a centered body (cf. Figure 5.1). Indeed, the first time in recorded history that the subspace concentration conditions have been formulated and proven for origin-symmetric polytopes was in [HSW05, Lemma 3.1]. In that paper, the authors use these inequalities to derive a relation between the successive minima and the Ehrhart coefficients of a symmetric lattice polytope. We encountered their result in the proof of Proposition 4.4.4 in the previous chapter. $\diamond$

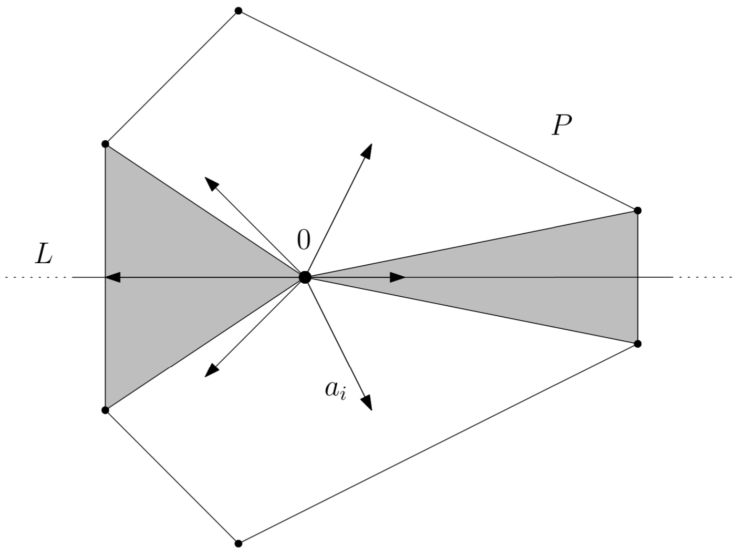


Figure 5.1: The linear situation in Theorem 5.1.2.

5.2 Affine Subspace Concentration Conditions

The subspace concentration conditions have been recently reinterpreted in the context of toric geometry in [HNS19], exploiting the deep connection between lattice polytopes and toric varieties. This lead K.-Y. Wu to prove an elegant variant to Theorem 5.1.2 in which the linear subspace concentration conditions (5.1) are replaced by an affine subspace concentration condition.

Theorem 5.2.1 (K.-Y. Wu, [Wu22]). *Let $P = \{x \in \mathbb{R}^n : \langle a_i, x\rangle \le 1,\ 1 \le i \le m\}$ be a centered reflexive smooth polytope and let $A \subseteq \mathbb{R}^n$ be a proper affine subspace. Then,*

$$\sum_{i\colon a_i \in A} \mathrm{vol}(C_i) \le \frac{\dim A + 1}{n+1} \mathrm{vol}(P).$$

Equality is obtained if and only if there exists a complementary affine subspace A', i.e., $A \cap A' = \emptyset$ and $\mathrm{aff}(A \cup A') = \mathbb{R}^n$, such that $\{a_i : 1 \le i \le m\} \subseteq A \cup A'$.

Recall from Chapter 2 that a lattice polytope P is reflexive if P and $P^\star$, the polar of P, are both lattice polytopes. In other words, the vectors a_i, $1 \le i \le m$, as well as the vertices of P are points of $\mathbb{Z}^n$. A lattice polytope P is said to be smooth if it is simple, i.e., each vertex of P is contained in exactly n facets $F_{j_1}, \dots, F_{j_n}$, say, and the corresponding normals $a_{j_1}, \dots, a_{j_n}$ form a lattice basis of $\mathbb{Z}^n$, i.e., $(a_{j_1}, \dots, a_{j_n})\mathbb{Z}^n = \mathbb{Z}^n$.

The purpose of this section is to generalize K.-Y. Wu's affine subspace concentration inequality to arbitrary centered polytopes.

Theorem 5.2.2. *Let $P = \{x \in \mathbb{R}^n : \langle x, a_i\rangle \le 1,\ 1 \le i \le m\}$ be a centered polytope and let $A \subseteq \mathbb{R}^n$ be an affine subspace. Then,*

$$\sum_{i\colon a_i \in A} \mathrm{vol}(C_i) \le \frac{\dim A + 1}{n+1} \mathrm{vol}(P). \tag{5.2}$$

Unlike Theorem 5.2.1, our proof does not give us insight into the characterization of the equality case. In Section 5.3, we will characterize the equality case of (5.2) in two special cases.

Remark 5.2.3. In contrast to the linear case, the affine subspace concentration conditions depend on the normalization of the inequalities in the description of P in Theorems 5.2.1 and 5.2.2. If we rewrite $P = \{x \in \mathbb{R}^n : \langle a_i', x\rangle \le b_i,\ 1 \le i \le m\}$ as in Remark 5.1.3, the affine dependencies between the new normals a_i' can change. Hence, there is no equivalent of the affine subspace concentration conditions on the level of Borel measures on the sphere, without knowing the support function of the underlying polytope (if any). However, the affine subspace concentration conditions supplement the geometric insight concerning the volume distribution within centered polytopes from Remark 5.1.3 (cf. Figure 5.2). ◇

For the proof of Theorem 5.2.2, we will lift the polytope P to successively higher dimensions using a pyramid construction. The following lemma allows us to determine the centroids of the pyramids that we construct this way.

Lemma 5.2.4. *Let $F \subseteq \mathbb{R}^n$ be an $(n-1)$-dimensional polytope and $v \notin \mathrm{aff}(F)$. Then,*

$$\mathrm{c}\big(\mathrm{conv}(F \cup \{v\})\big) = \frac{n}{n+1}\mathrm{c}(F) + \frac{1}{n+1}v. \tag{5.3}$$

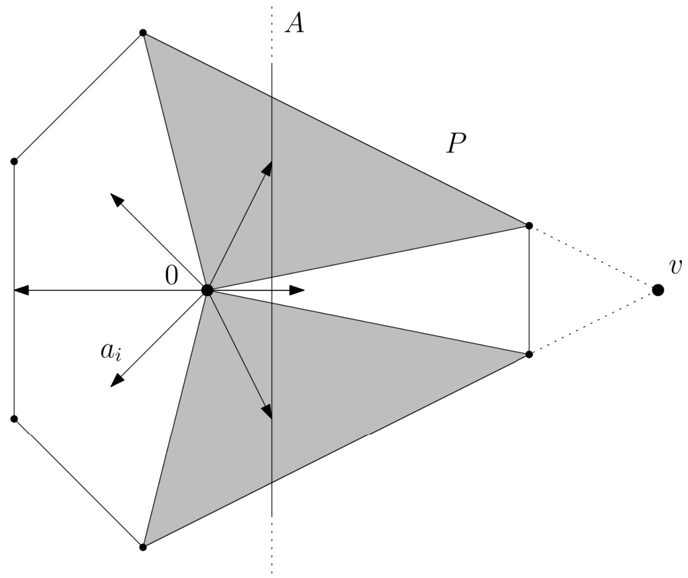


Figure 5.2: The affine situation in Theorem 5.2.2. Note that in general a subset of the a_i's affinely spans a k-subspace, if and only if the affine hulls of the corresponding facets intersect in an $(n-1-k)$-subspace. Here, this is a single point v.

Proof. As $\mathrm{c}(\cdot)$ is affinely equivariant, it is enough to consider the case where $F \subseteq \{x \in \mathbb{R}^n : x_n = 0\}$, $\mathrm{c}(F) = 0$ and $v = e_n$, where e_n denotes the n-th standard unit vector. Let $H_t = \{x \in \mathbb{R}^n : x_n = t\}$. Using Fubini's theorem, we have

$$\mathrm{c}(P) = \frac{1}{\mathrm{vol}(P)} \int_0^1 \mathrm{vol}_{n-1}(P \cap H_t)\, \mathrm{c}(P \cap H_t)\, \mathrm{d}t.$$

In our setting, we have $P \cap H_t = (1-t)F + te_n$. Thus, it follows that

$$\begin{aligned} \mathrm{c}(P) &= \frac{\mathrm{vol}_{n-1}(F)}{\mathrm{vol}(P)} \Big(\int_0^1 (1-t)^{n-1} t \, \mathrm{d}t \Big)\, e_n \\ &= \frac{\mathrm{vol}_{n-1}(F)}{\mathrm{vol}(P)} \frac{1}{n(n+1)} e_n = \frac{1}{n+1} e_n, \end{aligned}$$

where the last equality follows from the fact that P is a pyramid with height one over F and therefore $\mathrm{vol}(P) = \mathrm{vol}_{n-1}(F)/n$. Given our assumptions, the proof of the lemma is finished. □

We define for a k-dimensional polytope $Q \subseteq \mathbb{R}^k$ the pyramid $\mathrm{pyr}(Q)$ by

$$\mathrm{pyr}(Q) = \mathrm{conv}((Q \times \{1\}) \cup \{-(k+1)e_{k+1}\}) \subseteq \mathbb{R}^{k+1}.$$

We will need the following properties of this embedding.

Lemma 5.2.5. *Let $P \subseteq \mathbb{R}^n$ be given as in Theorem 5.2.2 and let $P^{(1)} = \mathrm{pyr}(P)$. Then the following holds:*

i)
$$P^{(1)} = \left\{ x \in \mathbb{R}^{n+1} : \left\langle \begin{pmatrix} \frac{n+2}{n+1} a_i \\ -\frac{1}{n+1} \end{pmatrix}, x \right\rangle \leq 1,\ 1 \leq i \leq m, x_{n+1} \leq 1 \right\},$$

ii)
$$\mathrm{vol}_{n+1}(P^{(1)}) = \frac{n+2}{n+1} \mathrm{vol}_n(P),$$

iii) $P^{(1)}$ is centered,

iv) Let $C_i^{(1)}$ be the cone given by the facet of $P^{(1)}$ corresponding to the outer normal vector $\left(\frac{n+2}{n+1} a_i, -\frac{1}{n+1}\right)^T$ and the origin. Then for $1 \leq i \leq m$ we have

$$\mathrm{vol}_{n+1}(C_i^{(1)}) = \mathrm{vol}_n(C_i).$$

Proof. i) and ii) follow directly from the fact that $P^{(1)}$ is indeed a pyramid; iii) is a consequence of Lemma 5.2.4. For iv), let $\overline{C}_i = C_i \times \{1\}$, $G_1 = \mathrm{conv}(\overline{C}_i \cup \{-(n+1)e_{n+1}\})$ and $G_2 = \mathrm{conv}(\overline{C}_i \cup \{0\}) \subseteq G_1$. Then we have $C_i^{(1)} = G_1 \setminus G_2$ and therefore

$$\begin{aligned} \mathrm{vol}_{n+1}(C_i^{(1)}) &= \mathrm{vol}_{n+1}(G_1) - \mathrm{vol}_{n+1}(G_2) \\ &= \frac{n+2}{n+1} \mathrm{vol}_n(C_i) - \frac{1}{n+1} \mathrm{vol}_n(C_i) \\ &= \mathrm{vol}_n(C_i). \end{aligned}$$
□

We finish the section by proving Theorem 5.2.2.

Proof of Theorem 5.2.2. Let $A \subseteq \mathbb{R}^n$ be a proper affine space, $d = \dim A$, $I = \{i \in [m] : a_i \in A\}$ and we may assume $\dim\{a_i : i \in I\} = d$. For any k, let

$$\varphi_k : \mathbb{R}^k \to \mathbb{R}^{k+1},\ x \mapsto \begin{pmatrix} \frac{k+2}{k+1} x \\ -\frac{1}{k+1} \end{pmatrix}.$$

For $j \geq 1$ and $i \in [m]$ we set

$$a_i^{(j)} = (\varphi_{n+j-1} \circ \cdots \circ \varphi_n)(a_i) \in \mathbb{R}^{n+j},$$

and let $L^{(j)} = \mathrm{span}\{a_i^{(j)} : i \in I\} \subseteq \mathbb{R}^{n+j}$. Observe that the vectors $a_i^{(j)}$ have the form

$$a_i^{(j)} = \begin{pmatrix} \frac{n+j+1}{n+1} a_i \\ c_{n+1} \\ \vdots \\ c_{n+j} \end{pmatrix},$$

where

$$c_{n+k} = -\frac{n+j+1}{(n+k)(n+k+1)}, \quad 1 \le k \le j.$$

The c_{n+k}'s only depend on n and j, but not on a_i. Therefore, $L^{(1)}$ is a $(d+1)$-dimensional linear space and since the matrix $(a_i^{(j+1)} : i \in I)$ differs from $(a_i^{(j)} : i \in I)$ only by an additional constant row and a multiplication of the first $n+j$ rows, we have $\dim L^{(j)} = d+1$ for all $j \ge 1$.

Now consider the pyramids $P^{(j)} = \mathrm{pyr}(P^{(j-1)})$ with $P^{(0)} = P$. A repeated application of Lemma 5.2.5 i) and iii) shows that each $P^{(j)}$ is a centered pyramid that has the vectors $\{a_i^{(j)} : i \in [m]\}$ among its normal vectors and from Lemma 5.2.5 ii) we get

$$\mathrm{vol}_{n+j}(P^{(j)}) = \left(\prod_{k=1}^{j} \frac{n+k+1}{n+k}\right) \mathrm{vol}_n(P) = \frac{n+j+1}{n+1} \mathrm{vol}_n(P).$$

Let $C_i^{(j)}$ be the cone of $P^{(j)}$ corresponding to $a_i^{(j)}$. Lemma 5.2.5 iv) shows that $\mathrm{vol}_{n+j}(C_i^{(j)}) = \mathrm{vol}_n(C_i)$, and so by Theorem 5.1.2 applied to $P^{(j)}$ and $L^{(j)}$ we obtain

$$\begin{aligned}\sum_{i \in I} \mathrm{vol}_n(C_i) &= \sum_{i \in I} \mathrm{vol}_{n+j}(C_i^{(j)}) \\ &\le \frac{d+1}{n+j} \mathrm{vol}_{n+j}(P^{(j)}) \\ &= \frac{\dim A + 1}{n+1} \frac{n+j+1}{n+j} \mathrm{vol}_n(P).\end{aligned}$$

The claim follows from letting $j \to \infty$. □

5.3 On the Equality Case in the Affine Subspace Concentration Conditions

In this section we discuss the equality case of the affine subspace concentration conditions. For a centered polytope $P = \{x \in \mathbb{R}^n : \langle a_i, x \rangle \le 1,\ 1 \le i \le m\}$, which is in addition smooth and reflexive, Wu proves that equality is achieved in (5.2), if and only if there exists an affine subspace A' complementary to A such that $\{a_1, ..., a_m\} \subseteq A \cup A'$. It is not hard to see that this condition is sufficient for the equality case, also in the general setting; it suffices to apply (5.2) to both A and A' and obtain

$$\mathrm{vol}(P) = \sum_{i: a_i \in A} \mathrm{vol}(C_i) + \sum_{i: a_i \in A'} \mathrm{vol}(C_i) \le \mathrm{vol}(P).$$

Thus, we have equality in (5.2) for A (and also for A'). However, due to the limiting process in our proof of Theorem 5.2.2, we cannot exclude further equality cases at this point.

The main result of this section is the following theorem, which characterizes the equality case in (5.2) for affine subspaces that either consist of a single point, or are given as the affine hull of a facet of $P^\star$, the polar of P.

Theorem 5.3.1. *Let $P = \{x \in \mathbb{R}^n : \langle x, a_i \rangle \leq 1, 1 \leq i \leq m\}$ be a centered n-dimensional polytope.*

i) If $A = \{a_i\}$ for some $1 \leq i \leq m$, then equality holds in (5.2) *if and only if P is a pyramid with base F_i.*

ii) If A is the hyperplane spanned by the a_i's corresponding to all the facets containing a vertex v of P, then equality holds in (5.2) *if and only if P is a pyramid with apex v.*

As a byproduct of the proof of Theorem 5.3.1, we will see alternative proofs of (5.2) in these special cases. The first case of Theorem 5.3.1 slightly generalizes a former result by Zhou and He [ZH17, Thm. 1.2]. There an additional technical assumption on P is made.

In contrast to the description of the equality case in Theorem 5.2.1, the descriptions of the equality cases in Theorem 5.3.1 do not explicitly refer to the normal vectors a_i. The following proposition gives an equivalent formulation of the equality case in Theorem 5.2.1; it shows that the two conditions in Theorem 5.3.1 are indeed special cases of the general description in terms of the a_i's:

Proposition 5.3.2. *Let $P = \{x \in \mathbb{R}^n : \langle x, a_i \rangle \leq 1, 1 \leq i \leq m\}$. Then there exist a proper affine subspace A and a complementary affine subspace A' such that $\{a_i : 1 \leq i \leq m\} \subseteq A \cup A'$ if and only if P can be written as*

$$P = \mathrm{conv}(Q_1 \cup Q_2),$$

where $Q_1, Q_2 \subseteq \mathbb{R}^n$ are polytopes with $\dim Q_1 + \dim Q_2 = n - 1$ and $\mathrm{aff} Q_1 \cap \mathrm{aff} Q_2 = \emptyset$.

A polytope that can be expressed as the convex hull of two polytopes Q_1 and Q_2 in complementary affine subspaces is also called the *join* of Q_1 and Q_2 [HRZ17, p. 390]. Proposition 5.3.2 essentially states that a polytope P is a join of two polytopes, if and only if its polar $P^\star$ can be expressed as a join of two polytopes. This fact seems to be well-known, but as we did not find a proof in the literature, we provide one here.

For the proof of Proposition 5.3.2, we recall from Chapter 2 that d-faces of P correspond one-to-one to $(n - d - 1)$-faces of $P^\star$ via the polarity operation (cf. (2.6))

$$F^\diamond = \{y \in P^\star : \langle y, x \rangle = 1, \ \forall x \in F\},$$

where F is a d-face of P.

Proof of Proposition 5.3.2. Since the a_i's are the vertices of $P^\star$, the condition that $\{a_1, \dots, a_m\}$ is contained in $A \cup A'$ is equivalent to $P^\star = \operatorname{conv}(P_1 \cup P_2)$, where P_1 is a d-polytope and P_2 is an $(n-d-1)$-polytope and $\operatorname{aff} P_1 = A$ and $\operatorname{aff} P_2 = A'$ are complementary affine spaces. By polarity, it is therefore enough to prove one direction of the equivalence.

So let $P = \operatorname{conv}(Q_1 \cup Q_2)$ with Q_1 and Q_2 as in the statement of the proposition. We will prove that $P^\star = \operatorname{conv}\{Q_1^\diamond \cup Q_2^\diamond\}$, where $Q_i^\diamond$ is the polar face of Q_i in $P^\star$. To this end, we need to show first that Q_1 and Q_2 are faces of P. Let $x_1 \in Q_1$, $x_2 \in Q_2$ and $L = \operatorname{span}((Q_1 - x_1) \cup (Q_2 - x_2))$. Since $\dim Q_1 + \dim Q_2 = n-1$, we have $\dim L \leq n-1$. Choosing a vector $u \in L^\perp \setminus \{0\}$, the linear functional $f \colon \mathbb{R}^n \to \mathbb{R}$, $x \mapsto \langle u, x \rangle$ satisfies $f(Q_1) = \{\alpha\}$ and $f(Q_2) = \{\beta\}$ for certain $\alpha, \beta \in \mathbb{R}$. Since P is n-dimensional and of the form $P = \operatorname{conv}(Q_1 \cup Q_2)$, we have $\alpha \neq \beta$, $f(P) = \operatorname{conv}\{\alpha, \beta\}$ and

$$f^{-1}(\{\alpha\}) \cap P = Q_1, \quad f^{-1}(\{\beta\}) \cap P = Q_2.$$

This shows that Q_1 and Q_2 are faces of P.

Now we consider the polar faces $P_i = Q_i^\diamond \subseteq P^\star$, $i \in \{1, 2\}$, of the two faces $Q_1, Q_2 \subset P$. Note that $\dim P_1 = n - d - 1$ and $\dim P_2 = d$. Clearly, we have $\operatorname{conv}(P_1 \cup P_2) \subseteq P^\star$. If the inclusion was strict, we find a vertex v of $P^\star$ which is neither a vertex of P_1, nor of P_2. Consider the corresponding facet $F = v^\diamond$ of P. Since v is not contained in $P_1 \cup P_2$, it follows by polarity that neither Q_1, nor Q_2 is contained in F. But $F_i = Q_i \cap F$ is a face of Q_i. Thus, we have $\dim F_1 \leq d-1$ and $\dim F_2 \leq n-d-2$. Due to the assumption $P = \operatorname{conv}(Q_1 \cup Q_2)$, the vertices of F are contained in $Q_1 \cup Q_2$, i.e., $F = \operatorname{conv}(F_1 \cup F_2)$. It follows that

$$\dim F \leq 1 + \dim F_1 + \dim F_2 = 1 + (d-1) + (n-d-2) = n-2,$$

a contradiction. So we have proven $P^\star = \operatorname{conv}(P_1 \cup P_2)$. Since $\dim(P^\star) = n$ and $P^\star \subseteq \operatorname{aff}(P_1 \cup P_2)$, we have $\operatorname{aff}(P_1 \cup P_2) = \mathbb{R}^n$, so the affine hulls of P_1 and P_2 are indeed complementary. □

In order to prove Theorem 5.3.1, we need a characterization of pyramids by their intersection function.

Lemma 5.3.3. *Let $P \subseteq \mathbb{R}^n$ be an n-dimensional polytope, $u \in \mathbb{S}^{n-1}$ and let $f \colon \mathbb{R} \to \mathbb{R}$, given by*

$$f(t) = \operatorname{vol}_{n-1}((tu + u^\perp) \cap P)^{\frac{1}{n-1}}.$$

Let $[\alpha, \beta] = \operatorname{supp}(f)$. If f is affine on $[\alpha, \beta]$ and $f(\beta) = 0$, then P is a pyramid with base $(\alpha u + u^\perp) \cap P$ and apex $(\beta u + u^\perp) \cap P$.

Proof. Let $S = (\alpha u + u^\perp) \cap P$ and $T = (\beta u + u^\perp) \cap P$. Since f is affine, $f(\beta) = 0$ and $\operatorname{vol}(P) = \int_{\mathbb{R}} f(x)^{n-1} dx > 0$, we know that $\operatorname{vol}_{n-1}(S) = f(\alpha) > 0$. Let $\lambda \in [0, 1]$. By the convexity of P, we have

$$\lambda T + (1-\lambda) S \subseteq \big((\lambda\beta + (1-\lambda)\alpha) u + u^\perp\big) \cap P =: P_\lambda.$$

Combining this with the Brunn-Minkowski inequality (cf. Section 2.1), we obtain

$$\begin{aligned} f(\lambda\beta + (1-\lambda)\alpha) &\geq \mathrm{vol}_{n-1}(\lambda T + (1-\lambda)S)^{\frac{1}{n-1}} \\ &\geq \lambda \mathrm{vol}_{n-1}(T)^{\frac{1}{n-1}} + (1-\lambda)\mathrm{vol}_{n-1}(S)^{\frac{1}{n-1}} \\ &= \lambda f(\beta) + (1-\lambda) f(\alpha) \end{aligned} \tag{5.4}$$

Since f is affine, both inequalities hold with equality. The equality in the Brunn-Minkowski inequality implies that S and T are homothetic (the other equality case being ruled out by the fact that $\mathrm{vol}_{n-1}(S) > 0$). Because $\mathrm{vol}_{n-1}(T) = 0$, this shows that T is a singleton. Finally, since the polytope P_λ contains the polytope $\lambda T + (1-\lambda)S$, the first equality in (5.4) implies that $P_\lambda = \lambda T + (1-\lambda)S$. Since $\lambda \in [0,1]$ was arbitrary, it follows that P is a pyramid with S as its base. $\square$

Now we ware ready to prove Theorem 5.3.1. We wil show the 0- and the $(n-1)$-dimensional case separately. Also, for the $(n-1)$-dimensional case, we will give two alternative proofs. One that uses elementary geometric arguments and another one that takes a more analytic perspective.

We start with case of A being a singleton. Our proof is inspired by the proof of Grünbaum's theorem on central sections of centered convex bodies [Grü60].

Proof of Theorem 5.3.1 i). Without loss of generality, we assume that $F_i = P \cap \{x \in \mathbb{R}^n : \langle e_1, x\rangle = -\alpha\}$ for an appropriately chosen $\alpha > 0$. Let $Q = \mathrm{conv}(F_i \cup \{\beta e_1\})$, where $\beta > -\alpha$ is chosen such that $\mathrm{vol}(Q) = \mathrm{vol}(P)$. We define two functions $\mathbb{R} \to \mathbb{R}$ via

$$f(t) = \mathrm{vol}_{n-1}((te_1 + e_1^\perp) \cap P)^{\frac{1}{n-1}}, \quad g(t) = \mathrm{vol}_{n-1}((te_1 + e_1^\perp) \cap Q)^{\frac{1}{n-1}}.$$

If $\langle e_1, c(Q)\rangle \geq \langle e_1, c(P)\rangle$, then by Lemma 5.2.4 it would follow that

$$\mathrm{vol}(C_i) \leq \mathrm{vol}(\mathrm{conv}(F_i \cup \{\mathrm{c}(Q)\})) = \frac{1}{n+1}\mathrm{vol}(Q),$$

as desired. Recalling that P is centered, we have to show for $\gamma = \langle e_1, \mathrm{c}(Q)\rangle$ that

$$\gamma = \langle e_1, \mathrm{c}(Q) - \mathrm{c}(P)\rangle = \int_{-\infty}^{\infty} t(g(t)^{n-1} - f(t)^{n-1})\mathrm{d}t \geq 0,$$

with equality if and only if P is a pyramid.

Since Q is a pyramid with base orthogonal to e_1, g is affine on $\mathrm{supp}(g) = [-\alpha, \beta]$. By Brunn's concavity principle [AGM15, Thm. 1.2.1], f is concave on $\mathrm{supp}(f)$. Hence, $g - f$ is convex on $\mathrm{supp}(f) \cap \mathrm{supp}(g)$. In fact, we have $\mathrm{supp}(f) \subseteq \mathrm{supp}(g)$: If there was a $t > \beta$ with $f(t) > 0$, then the concavity of f would imply $f > g$ on $\mathrm{supp}(g)$, in contradiction to $\mathrm{vol}(Q) = \mathrm{vol}(P)$. Hence, $g - f$ is convex on $\mathrm{supp}(f)$ and the sublevel set

$$\mathrm{supp}(f) \cap \{g - f \leq 0\} = \mathrm{supp}(f) \cap \{g^{n-1} - f^{n-1} \leq 0\}$$

is convex. Since $f(-\alpha) = g(-\alpha)$, it follows that $\operatorname{supp}(f) \cap \{g - f \leq 0\} = [-\alpha, \tau]$ for a $\tau \leq \beta$. On $[\tau, \beta]$ we have $g \geq f$, leading to the desired estimate

$$\begin{aligned}
\gamma &= \int_{-\alpha}^{\tau} t(g(t)^{n-1} - f(t)^{n-1})\mathrm{d}t + \int_{\tau}^{\beta} t(g(t)^{n-1} - f(t)^{n-1})\mathrm{d}t \\
&\geq \int_{-\alpha}^{\tau} \tau(g(t)^{n-1} - f(t)^{n-1})\mathrm{d}t + \int_{\tau}^{\beta} \tau(g(t)^{n-1} - f(t)^{n-1})\mathrm{d}t \\
&= \tau \left(\int_{-\alpha}^{\beta} (g(t)^{n-1} - f(t)^{n-1})\mathrm{d}t \right) = \tau \left(\operatorname{vol}(Q) - \operatorname{vol}(P) \right) = 0.
\end{aligned}$$

Equality holds if and only if $g = f$ on $[-\alpha, \beta]$. It is clear that this is the case if P is a pyramid with base F_i; the other direction follows from Lemma 5.3.3. □

Next, we give a geometric proof of Theorem 5.3.1 ii).

Geometric proof of Theorem 5.3.1 ii). Let $I \subseteq [m]$ be the set of indices such that $\langle v, a_i \rangle = 1$, i.e., $A = \operatorname{aff}\{a_i : i \in I\}$. Since P is centered, we have $-\frac{1}{n}v \in P$ (see (2.7)). For $i \in I$, we consider the cones $\overline{C}_i = \operatorname{conv}(F_i \cup \{-(1/n)v\}) \subseteq P$, where F_i is the facet of P with normal a_i. By the volume formula for pyramids, we have $\operatorname{vol}(\overline{C}_i) = \frac{n+1}{n}\operatorname{vol}(C_i)$. As the $\overline{C}_i$'s intersect in a set of measure zero, we obtain

$$\operatorname{vol}(P) \geq \sum_{i \in I} \operatorname{vol}(\overline{C}_i) = \frac{n+1}{n}\operatorname{vol}(C_i), \tag{5.5}$$

so we have reproven Theorem 5.2.2 in this case. In order to have equality in the above, we must have $P = \bigcup_{i \in I} \overline{C}_i$. Let $J = [m] \setminus I$. Then we have

$$\langle -(1/n)v, a_j \rangle = 1, \ \forall j \in J, \tag{5.6}$$

since otherwise, the cone C_j would have a positive volume and we could not achieve equality in (5.5).

For $j \in J$, let $K_j = \operatorname{conv}\{F_j, v\} \subseteq P$. Just like the $\overline{C}_i$'s, the K_j's subdivide P, i.e., $P = \bigcup_{j \in J} K_j$ and the pyramids intersect in sets of measure zero. By (5.3), we have $\mathrm{c}(K_j) = \frac{n}{n+1}\mathrm{c}(F_j) + \frac{1}{n}v$. It follows from (2.8) that

$$0 = \mathrm{c}(P) = \sum_{j \in J} \frac{\operatorname{vol}(K_j)}{\operatorname{vol}(P)} \left(\frac{n}{n+1}\mathrm{c}(F_j) + \frac{1}{n+1}v \right).$$

Multiplying with $(n+1)/n$ and rearranging yields

$$-\frac{1}{n}v = \sum_{j \in J} \frac{\operatorname{vol}(K_j)}{\operatorname{vol}(P)} \left(-\frac{1}{n}v \right) = \sum_{j \in J} \frac{\operatorname{vol}(K_j)}{\operatorname{vol}(P)}\mathrm{c}(F_j).$$

For any $j \in J$, we have by (5.6):

$$1 = \langle -\frac{1}{n}v, a_j\rangle = \sum_{k\in J} \frac{\mathrm{vol}(K_k)}{\mathrm{vol}(P)} \langle \mathrm{c}(F_k), a_j\rangle.$$

Towards a contradiction, assume that J contains more than one element. Then there is a $k \in J \setminus \{j\}$. Since $\mathrm{c}(F_k) \in \mathrm{relint} F_k$ we have $\langle \mathrm{c}(F_k), a_j\rangle < 1$. It follows that $1 < \sum_{k\in J} \mathrm{vol}(K_k)/\mathrm{vol}(P) = 1$. Therefore, J can contain only one element, which corresponds to the case that P is a pyramid with apex v. □

We give another proof of Theorem 5.3.1 ii) via a probabilistic approach.

Analytic proof of Theorem 5.3.1 ii). Again, we only show the "only if" part of the equality case. To this end, we assume that $\mathrm{vol}(P) = 1$, which is not a restriction as both sides of (5.2) are n-homogeneous. By definition, we have $\mathrm{c}(P) = \mathbb{E}[X]$, where X is a uniformly distributed random vector in P. We consider the functional

$$f : \mathbb{R}^n \to \mathbb{R},\ f(x) = \frac{1}{n} \sum_{i:a_i\in A} \mathrm{dist}(x, \mathrm{aff}(F_i))\, \mathrm{vol}_{n-1}(F_i),$$

where $\mathrm{dist}(x, \mathrm{aff}(F_i))$ is the signed Euclidean distance to $\mathrm{aff}(F_i)$, oriented such that it is non-negative inside P. Note that for $x \in P$ one has

$$f(x) = \sum_{i:a_i\in A} \mathrm{vol}\big(\mathrm{conv}(F_i \cup \{x\})\big).$$

As f is an affine map, we have

$$\sum_{i:a_i\in A} \mathrm{vol}(C_i) = \mathbb{E}[f(X)] = \int_0^1 \mathbb{P}_X(f \geq t)\mathrm{d}t = 1 - \int_0^1 \mathbb{P}_X(f < t)\mathrm{d}t. \tag{5.7}$$

We consider the function $p : [0, 1] \to [0, 1],\ t \mapsto \mathbb{P}_X(f < t)^{\frac{1}{n}}$. There holds $p(0) = 0$ and $p(t) = 1$, for $t \geq m = \max f(P) \leq 1$. Let $H(t) = \{x \in \mathbb{R}^n : f(x) \leq t\}$ be the half-space where $f \leq t$. Since the vertex v is the unique point that is contained in all facets F_i, where $a_i \in A$, we have $0 \in f(P)$ and $f(x) = 0$ for $x \in P$, if and only if $x = v$. Thus, $P \cap H(0) = \{v\}$. Using the inclusion

$$P \cap H(t) \supseteq \frac{t}{m}\big(P \cap H(m)\big) + \frac{m-t}{m}v,$$

we deduce that, for any $t \in [0, m]$,

$$\begin{aligned} p(t) = \mathrm{vol}(P \cap H(t))^{\frac{1}{n}} &\geq \mathrm{vol}\Big(\frac{t}{m}\big(P \cap H(m)\big) + \frac{m-t}{m}v\Big)^{\frac{1}{n}} \\ &= \frac{t}{m}\mathrm{vol}\big(P \cap H(m)\big)^{\frac{1}{n}} = \frac{t}{m}p(m) = \frac{t}{m}. \end{aligned} \tag{5.8}$$

Applying this to (5.7), we have

$$\sum_{i:a_i\in A} \mathrm{vol}(C_i) = 1 - \int_0^m p(t)^n \mathrm{d}t - (1-m)$$
$$\leq m - \int_0^m \left(\frac{t}{m}\right)^n \mathrm{d}t = \frac{mn}{n+1} \leq \frac{n}{n+1}.$$

By our assumption that $\mathrm{vol}(P) = 1$, this is (5.2). In order to have equality, we must have $m = 1$ and equality in (5.8), i.e., $\mathrm{vol}(P \cap H(t)) = t^n$ for $t \in [0,1]$. This is equivalent to $\mathrm{vol}_{n-1}(P \cap \{x \in \mathbb{R}^n : f(x) = t\}) = nt^{n-1}$ for $t \in [0,1]$. By Lemma 5.3.3, this implies that P is a pyramid with apex v. $\square$

Remark 5.3.4. It is natural to ask whether the assumption in Theorem 5.3.1 ii) that v is a vertex of P can be removed. In other words, is it possible to adapt our proofs to the situation where the hyperplanes $\{x \in \mathbb{R}^n : \langle a_i, x\rangle = 1\}$, $a_i \in A$, intersect in a single point v that is not necessarily contained in P (cf. Figure 5.2)? Both proofs of Theorem 5.3.1 ii) make use of the assumption that $v \in P$: In the first proof, we use it to derive $-\frac{1}{n}v \in P$; in the second proof, it ensures that p is concave on $[0, \max f(P)]$. It is not clear how the first proof could be modified to dispense with the assumption. In the second proof, a suitable upper bound on $\max f(P)$ in terms of $\min f(P)$ would be sufficient: The concavity of p on $[\min f(P), \max f(P)]$ leads to the desired estimate if we additionally assume that $\max f(P) \leq 1 - \frac{\min f(P)}{n}$. $\diamond$

Finally, let us take a closer look at the affine subspace concentration inequality in the case of simple polytopes. For a k-face of F of P, there are exactly $n-k-1$ vectors among the a_i's such that $F \subseteq F_i$. Without loss of generality, we assume that $a_1, ..., a_{n-k-1}$ are these vectors. In view of Theorem 5.3.1 i), we obtain

$$\mathrm{vol}(C_i) \leq \frac{1}{n+1}\mathrm{vol}(P), \text{ for all } 1 \leq i \leq n-k-1. \tag{5.9}$$

Summing up these inequalities gives the affine subspace concentration condition (5.2) for P and A and, by the characterization of the equality case in Theorem 5.3.1 i), equality holds, if and only if equality holds in each of the inequalities in (5.9). In particular, equality holds, only if P is a pyramid with base F_1. Since P is simple, this implies that P is a simplex. So we obtained the following corollary:

Corollary 5.3.5. *Let $P = \{x \in \mathbb{R}^n : \langle x, a_i\rangle \leq 1, 1 \leq i \leq m\}$ be a centered simple polytope. Let $A \subseteq \mathbb{R}^n$ be an affine subspace spanned by a_i's corresponding to all the facets containing a k-face of P with $1 \leq k \leq n-1$. Then we have equality in* (5.2) *if and only if P is a centered simplex.*

Remark 5.3.6. Suppose that we have equality in Theorem 5.3.6 for a proper affine d-subspace A and a centered reflexive smooth polytope $P = \{x \in \mathbb{R}^n : \langle x, a_i\rangle \leq 1, 1 \leq i \leq m\}$. Then, we obtain from Theorem 5.2.1 that $\{a_1, ..., a_m\} \subseteq A \cup A'$ holds for some affine subspace A' which is complementary to A and in particular A and A' are affinely spanned by the a_i's they contain. Hence, we have $P^\star = \mathrm{conv}(P_1 \cup P_2)$, for

$$P_1 = \mathrm{conv}(A \cap \{a_1, ..., a_m\}) \quad \text{ and } P_2 = \mathrm{conv}(A' \cap \{a_1, ..., a_m\}).$$

As we saw in the proof of Proposition 5.3.2, P_1 and P_2 are faces of $P^\star$. So A is the affine space spanned by the a_i's such that $P_1^\diamond \subseteq F_i$. Recalling that smooth polytopes are simple, it follows from Corollary 5.3.5 that P is a simplex. Simplices are therefore the only equality cases in Theorem 5.2.1. $\diamond$

Part III

Polarity

6 Mahler Volume of Low-Dimensional Voronoi Cells of Lattices

This chapter is devoted to upper bounds on the Mahler volume of lattice Voronoi cells. In a fixed dimension n, these convex bodies have at most $(n+1)!$ vertices and at most $2(2^n-1)$ facets. Therefore, we expect the maximum Mahler volume of a lattice Voronoi cell to be significantly smaller than the Blaschke-Santaló bound for origin-symmetric convex bodies $K \in \mathcal{K}^n_{os}$;

$$\mathrm{vol}(K)\mathrm{vol}(K^\star) \leq \mathrm{vol}(B^n)^2,$$

where one has equality, if and only if K is an ellipsoid.

6.1 Voronoi Cells and Delaunay Subdivisions

In this section, we briefly recall the Voronoi and Delaunay subdivisions associated to lattices. For a detailed introduction to these types of subdivisions we refer to [AKL13].

The *(Dirichlet-)Voronoi cell* of a lattice $\Lambda \subseteq \mathbb{R}^n$ (around the origin) is defined as

$$V_\Lambda = \{x \in \operatorname{span}\Lambda : |x| \leq |x-a|,\ \forall a \in \Lambda\}, \tag{6.1}$$

i.e., V_Λ is the collection of points that are at least as close to the origin as to any other lattice point. By squaring the norms in (6.1) and rearranging, we obtain:

$$V_\Lambda = \left\{x \in \operatorname{span}\Lambda : \langle x, a\rangle \leq |a|^2/2, \forall a \in \Lambda\right\}. \tag{6.2}$$

Indeed, V_Λ is a polytope, since in the representation above, only finitely many lattice vectors are relevant. These are called *Voronoi relevant vectors* and we denote them by $\mathrm{vor}(\Lambda)$. In the literature, one oftentimes distinguishes between strict and weak Voronoi relevant vectors, where weak Voronoi relevant vectors also include those lattice vectors such that the corresponding inequality in (6.2) is fulfilled with equality for a point $x \in V_\Lambda$ (but these points do not need to form a facet). In these terms, $\mathrm{vor}(\Lambda)$ denotes the set of strict Voronoi relevant vectors.

Since Λ is an origin-symmetric point set, V_Λ is an origin-symmetric polytope. Since $x + \Lambda = \Lambda$ holds for all $x \in \Lambda$, the Voronoi cell of Λ around any other $x \in \Lambda$ (not necessarily the origin) is given by $x + V_\Lambda$. In fact, the cells $(x+V_\Lambda)_{x\in\Lambda}$ form a subdivision of $\mathbb{R}^n$ [CS99, Ch. 2, Sec. 1.2] and thus, $\mathrm{vol}(V_\Lambda) = \det \Lambda$.

The dual concept to the Voronoi subdivision is the *Delaunay subdivision*. For the definition, let $c \in \mathbb{R}^n$ and $r > 0$ be such that $(c + \mathrm{int}(r\,B^n)) \cap \Lambda = \emptyset$, where B^n is the Euclidean unit ball. Then,

$$P_{c,r}(\Lambda) = \mathrm{conv}(\Lambda \cap (c + \mathrm{bd}(r\,B^n))$$

is called a *Delaunay polytope* of Λ, if it is non-empty. We write $P_{c,r} = P_{c,r}(\Lambda)$ if Λ is understood. The Delaunay subdivision is then defined as

$$\mathrm{Del}(\Lambda) = \{P \subseteq \mathbb{R}^n : P \text{ is a Delaunay polytope of } \Lambda\}.$$

We write $\mathrm{Del}_n(\Lambda)$ for the set of n-polytopes in $\mathrm{Del}(\Lambda)$ and we say that Λ is *generic*, if $\mathrm{Del}(\Lambda)$ is a triangulation.

The Delaunay subdivision is dual to the Voronoi subdivision in the following sense: Consider a k-face $F \subseteq V_\Lambda$ of the Voronoi cell. Then, there exists a Delaunay $(n-k)$-polytope P of Λ that contains the Voronoi relevant vectors whose corresponding facets contain F, as well as the origin, among its vertices. In particular, the affine hulls of P and F are orthogonal. If $F = \{v\}$ is a vertex, then P is an n-dimensional polytope and we have $P = P_{v,|v|}$ (cf. Figure 6.1).

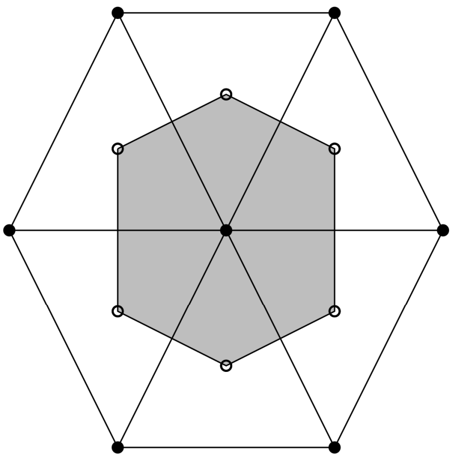


Figure 6.1: The Voronoi cell of $\Lambda = \mathrm{span}_{\mathbb{Z}}\{(2,0)^T, (1,2)^T\}$ (depicted in gray), together with the six Delaunay triangles of Λ at the origin. The vertices of the Voronoi cell are the centers of the circumcircles of the Delaunay triangles they are contained in.

Converserly, for every $(n-k)$-dimensional Delaunay polytope P containing the origin, there exists a k-face $F \subseteq V_\Lambda$ of Λ, which is orthogonal to P. If $P = P_{c,r}$ is n-dimensional, then $F = \{c\}$ is a vertex of P. We refer to [LD09] for a detailed account on these connections.

The following two examples will play a key role in our investigation.

Example 6.1.1. i) The Voronoi cell of $\mathbb{Z}^n$ is given by $[-\frac{1}{2}, \frac{1}{2}]^n$ and we have $\operatorname{vor}(\mathbb{Z}^n) = \{\pm e_1, ..., \pm e_n\}$. The Delaunay subdivision of $\mathbb{Z}^n$ is determined by the n-polytopes $(z + [0,1]^n)_{z \in \mathbb{Z}^n}$.

ii) Let $A_n = \mathbb{Z}^{n+1} \cap \mathbb{1}_{n+1}^{\perp}$ and let $A_n^\star$ be its dual lattice. The Voronoi cell of $A_n^\star$ is given as the convex hull of all permutations of the vector [CS99, Ch. 4, Sec. 6.6]

$$\frac{1}{2(n+1)}(-n, -n+2, ..., n-2, n)$$

This is a regular permutohedron (cf. [Zie12, Ex. 0.10]) and therefore a simple polytope. Hence, $A_n^\star$ is a generic lattice. $\diamond$

The geometry of V_Λ encodes a lot of information on the lattice Λ. Its inradius $r(V_\Lambda)$, i.e., the maximum radius of a Euclidean ball contained in V_Λ, is the packing radius $\frac{1}{2}\lambda_1(B_n, \Lambda)$ of Λ. Its circumradius $R(V_\Lambda)$, i.e., the minimum radius of a Euclidean ball containing V_Λ, equals the covering radius $\mu(B_n, \Lambda)$. And also the second moment

$$\overline{\mu}(V_\Lambda) = \int_{V_\Lambda} |x|^2 \mathrm{d}x$$

is known as the "lattice quantization error", a quantity that has applications to coding theory [CS99] and also to Minkowski's conjecture on inhomogeneous forms [RS17].

Gauss showed that the densest sphere packing by a lattice in 3-dimensional space is given by the lattice A_3 from Example 6.1.1. (If one considers arbitrary 3-dimensional packings, this problem is known as the Kepler Conjecture for which a computer assisted proof has been given in [Hal05, HAB$^+$17]). Gauss's result may be rephrased as an isoperimetric-type inequality for the Voronoi cells of 3-dimensional lattices, namely

$$\frac{r(V_\Lambda)^3}{\operatorname{vol}(V_\Lambda)} \leq \frac{r(V_{A_3})^3}{\operatorname{vol}(V_{A_3})},$$

for all 3-dimensional lattices Λ. In a similar spirit, one has

$$\frac{R(V_\Lambda)^n}{\operatorname{vol}(V_\Lambda)} \geq \frac{R(V_{A_n^\star})^n}{\operatorname{vol}(V_{A_n^\star})},$$

where $n \leq 5$ (cf. [CS99, Ch. 2, Sec. 1.6]). More recently, Lángi showed in [Lán22] that

$$\frac{\overline{w}(V_\Lambda)^3}{\operatorname{vol}(V_\Lambda)} \geq \frac{\overline{w}(V_{A_3^\star})^3}{\operatorname{vol}(V_{A_3^\star})},$$

where $\overline{w}(K) = \int_{\mathbb{S}^2} \mathrm{h}(K,u) - \mathrm{h}(K,-u)\, \mathrm{d}^2u$ denotes the *mean width* of $K \in \mathcal{K}^3$. If one would allow arbitrary convex bodies in the above inequalities, the extremal cases would be given by the Euclidean ball in all three of the inequalities. Of course, the ball is not among the lattice Voronoi cells. Nonetheless, these results give an insight on the "roundness" of the Voronoi cell.

Here we attempt to contribute to these results by studying upper bounds on the Mahler volume of two and three dimensional Voronoi cells. A functional which, by the Blaschke-Santaló inequality, is likewise maximized by the ball, but also by any other ellipsoid owing to the fact that the Mahler volume is invariant under linear transformations.

The chapter is organized as follows. In Section 6.2 we take a closer look at the polar body $V_\Lambda^\star$ of the Voronoi cell of Λ. For a generic lattice Λ, we develop a formula for $\mathrm{vol}(V_\Lambda^\star)$ depending on the Delaunay simplices of Λ that meet the origin. We use this formula in Sections 6.3 and 6.4 in order to study the Mahler volume of 2- resp. 3-dimensional lattice Voronoi cells. We will see that $A_2^\star$ is the strict (up to dilations and isometries) global maximum of the Mahler volume among all 2-dimensional lattices and that $A_3^\star$ is a strict local maximum among all 3-dimensional lattices. Section 6.5 contains a potential proof strategy for the conjecture that $A_3^\star$ is indeed the global maximum in dimension 3. We conclude with examples of lattice Voronoi cells and their Mahler volumes in higher dimensions in Section 6.6.

6.2 The Polar Body of a Lattice Voronoi Cell

In view of (6.2), we are able to describe the polar of V_Λ as a convex hull:

$$V_\Lambda^\star = \mathrm{conv}\Big\{\frac{2x}{|x|^2} : x \in \Lambda \setminus \{0\}\Big\}.$$

The vertices of $V_\Lambda^\star$ are precisely the points $2x/|x|^2$, where $x \in \mathrm{vor}(\Lambda)$. The body $\frac{1}{2}V_\Lambda^\star$ bears the remarkable property that it intersects any hyperplane of the form $\{v \in \mathbb{R}^n : \langle v, x\rangle = 1\}$, where $x \in \Lambda \setminus \{0\}$ – a point of intersection is the point $x/|x|^2$. Lower bounds on the volume of such bodies have been studied by Álvarez Paiva et al. in [APBT16] who phrased the term *unavoidable bodies* for general convex bodies $K \in \mathcal{K}^n$ with the property that $K \cap \{x \in \mathbb{R}^n : \langle v, x\rangle = 1\} \neq \emptyset$, for all $v \in \mathbb{Z}^n \setminus \{0\}$.

For a generic lattice Λ, the body $V_\Lambda^\star$ can be triangulated in a natural way.

Proposition 6.2.1. *Let Λ be generic. Then $V_\Lambda^\star$ is simplicial, and can be triangulated as follows:*

$$V_\Lambda^\star = \bigcup_T \mathrm{conv}\Big\{0, \frac{2x}{|x|^2} : x \in \mathrm{vert}(T)\Big\},$$

where T ranges over all full-dimensional Delaunay-simplices of Λ that include the origin. In particular, we have

$$\mathrm{vol}(V_\Lambda^\star) = 2^n \sum_{0 \in T \in \mathrm{Del}_n(\Lambda)} \mathrm{vol}(T) \prod_{x \in \mathrm{vert}(T)\setminus\{0\}} |x|^{-2}. \tag{6.3}$$

Proof. We show that $F_T = \mathrm{conv}\{2x/|x|^2 : x \in \mathrm{vert}(T) \setminus \{0\}\}$, $T \in \mathrm{Del}_n(\Lambda)$, are facets of $V_\Lambda^\star$. Let c be the center of the sphere S that testifies for T. Then, since T has the origin as a vertex, $|x - c|^2 \geq |c|^2$, for any $x \in \Lambda$. Equality holds, if and only if $x \in \mathrm{vert}(T)$.

Rearranging the inequality gives

$$\langle \frac{2x}{|x|^2}, c \rangle \leq 1,$$

for all $x \in \Lambda \setminus \{0\}$, with equality, if and only if $x \in \mathrm{vert}(T) \setminus \{0\}$. Thus, $F_T \subseteq V_\Lambda^\star$ is the polar facet $c^\diamond$ of the vertex c of V_Λ. The volume formula follows from expressing the simplices $\mathrm{conv}\{0, 2x/|x|^2 : x \in \mathrm{vert}(T)\}$ as a linear image of T. □

Note that the point $x/|x|^2$ is the inverse point of x with respect to inversion in the unit sphere. The above proof essentially recalls the fact that inversion in the unit sphere takes the sphere S to a hyperplane H. Thereby, a point is mapped to the open half-space defined by H which contains the origin, if and only if it is contained in the exterior of S.

Before we finish this section, let us take a second look at the Delaunay simplices of a generic lattice that meet the origin.

Remark 6.2.2. We define the star S_Λ of the Delaunay triangulation $\mathrm{Del}(\Lambda)$ as the union of all simplices in $\mathrm{Del}(\Lambda)$ that contain the origin, i.e.,

$$S_\Lambda = \bigcup \{T : 0 \in T \in \mathrm{Del}_n(\Lambda),\ \dim(T) = n\}.$$

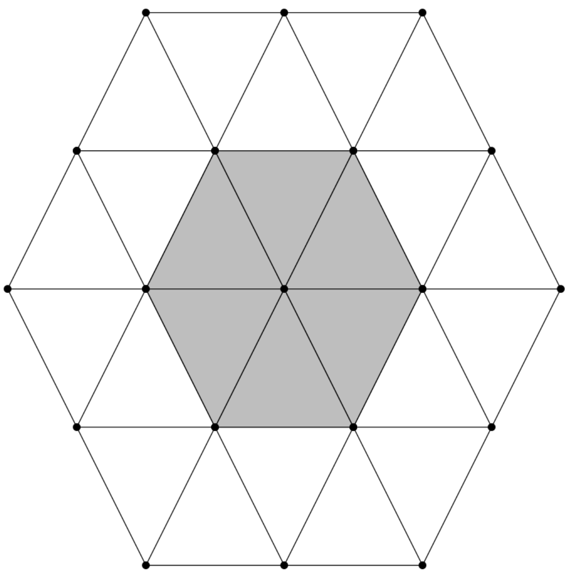


Figure 6.2: The Delaunay star S_Λ of a 2-dimensional lattice.

The translates $(x + S_\Lambda)_{x \in \Lambda}$ form an $(n+1)$-fold tiling of Λ, since any point $y \in \mathbb{R}^n$ (up to a set of measure zero) is contained in the interior of a some Delaunay n-simplex $T = \mathrm{conv}\{x_0, ..., x_n\}$ and therefore precisely in the translates $-x_i + S_\Lambda$, for $i \in \{0, ..., n\}$.

From that it follows that the volume of S_Λ is $(n+1)\det\Lambda$ and, thus,

$$
\begin{aligned}
(n+1)\det\Lambda = \mathrm{vol}(S_\Lambda) &= \frac{1}{n!}\sum_{0\in T\in \mathrm{Del}_n(\Lambda)} \big|\det \mathrm{span}_{\mathbb{Z}}(T)\big| \\
&\geq \frac{1}{n!}\det\Lambda\cdot\big|\{T\in\mathrm{Del}_n(\Lambda) : 0\in T\}\big|.
\end{aligned}
$$

As the simplices in $\mathrm{Del}_n(\Lambda)$ correspond to vertices of V_Λ, we see that a generic lattice Voronoi cell can have at most $(n+1)!$ vertices, which is attained, for instance, if $\Lambda = A_n^\star$ (cf. Example 6.1.1). This fact was already known to Voronoi who showed that the number of k-faces of a (not necessarily generic) lattice Voronoi cell is maximal for $A_n^\star$ [Vor08b, §63-§68, §101]. In particular, this result implies that in a given dimension, the Voronoi cells of lattices cannot be arbitrarily close to an ellipsoid. More precisely, the *Banach-Mazur* distance

$$
d_{\mathrm{BM}}(V_\Lambda, B^n) = \inf\big\{\lambda>0 : \exists\mathcal{E}\text{ ellipsoid}.\,\mathcal{E}\subseteq V_\Lambda\subseteq\lambda\mathcal{E}\big\}
$$

is bounded from below by a number $c_n > 1$, independent of Λ (see, for instance, [Bör00]). In view of Börčzky's stability version of the Blaschke-Santaló inequality [Bör10], this raises the hope for better upper bounds on the Mahler volume in this class of polytopes. $\diamond$

6.3 2-Dimensional Lattices

In this section we give a sharp upper bound on the Mahler volume of a 2-dimensional lattice. We will see that it is attained, if and only if Λ is isometric to a dilation of $A_2^\star$.

For a 2-dimensional lattice Λ, a Delaunay triangle is always a non-obtuse triangle T such that $T\cap\Lambda = \mathrm{vert}(T)$ [LD09, Eq. (13.2.9)]. Thus, we have by Pick's theorem (2.14) $\mathrm{vol}(T) = \frac{1}{2}\det\Lambda$. Our key observation is that up to translation and reflection, there exists only one Delaunay triangle in $\mathrm{Del}(\Lambda)$.

Lemma 6.3.1. *Let Λ be generic 2-dimensional lattice and $S, T\in\mathrm{Del}_2(\Lambda)$. Then there exist $x\in\Lambda$ and $s\in\{-1,1\}$ such that $S = x+sT$.*

Proof. Without loss of generality, we can assume that S contains the origin as a vertex. Let $T = \mathrm{conv}\{a,b,c\}$, for certain $a,b,c\in\Lambda$. Since we have $\Lambda = x\pm\Lambda$, for all $x\in\Lambda$, the simplices $T_{x,\pm} = \pm(T-x)$ are Delaunay simplices of Λ, for all $x\in\{a,b,c\}$. Indeed, every $T_{x,\pm}$ contains the origin and the $T_{x,\pm}$ are pairwise distinct. By Remark 6.2.2, there are at most 6 Delaunay triangles at the origin. Hence, one of the $T_{x,\pm}$ must be S (cf. Figure 6.3). $\square$

Again, we consider a 2-dimensional lattice Λ and let T be one if its Delaunay-triangles. Let $a,b,c\in\Lambda$ be the vertices of T and let F_x be the edge opposite to x, $x = a,b,c$. As a consequence of Lemma 6.3.1, (6.3) can be rewritten as follows:

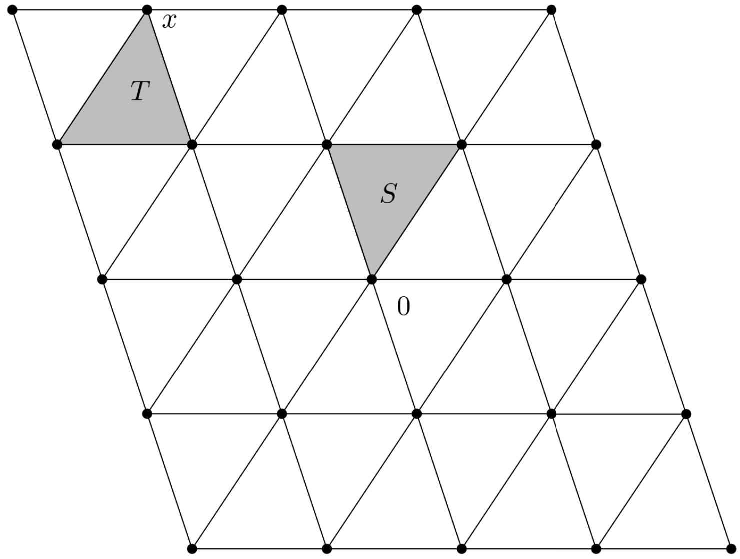


Figure 6.3: The triangle T is equivalent to S, since $S = -(T - x)$.

$$\mathrm{vol}(V_\Lambda^\star) = 8\Big(\frac{\mathrm{vol}(T)}{\mathrm{vol}(F_a)^2\mathrm{vol}(F_b)^2} + \frac{\mathrm{vol}(T)}{\mathrm{vol}(F_a)^2\mathrm{vol}(F_c)^2} + \frac{\mathrm{vol}(T)}{\mathrm{vol}(F_b)^2\mathrm{vol}(F_c)^2}\Big).$$

Each summand corresponds to the translation of T to the origin by $-a$, $-b$, or $-c$. The factor $8 = 2^2 \cdot 2$ comes from considering $-T$. Since $\mathrm{vol}(V_\Lambda) = \det \Lambda = 2\mathrm{vol}(T)$, we have

$$\mathrm{vol}(V_\Lambda)\mathrm{vol}(V_\Lambda^\star) = 16\Big(\frac{\mathrm{vol}(T)^2}{\mathrm{vol}(F_a)^2\mathrm{vol}(F_b)^2} + \frac{\mathrm{vol}(T)^2}{\mathrm{vol}(F_a)^2\mathrm{vol}(F_c)^2} + \frac{\mathrm{vol}(T)^2}{\mathrm{vol}(F_b)^2\mathrm{vol}(F_c)^2}\Big).$$

Let α be the angle measured in radians at a, and β and γ are the angles at b and c, respectively. By the law of sines, one has

$$\mathrm{vol}(T) = \frac{\mathrm{vol}(F_a)\mathrm{vol}(F_b)\sin\gamma}{2} = \frac{\mathrm{vol}(F_a)\mathrm{vol}(F_c)\sin\beta}{2} = \frac{\mathrm{vol}(F_b)\mathrm{vol}(F_c)\sin\alpha}{2}.$$

Hence,

$$\mathrm{vol}(V_\Lambda)\mathrm{vol}(V_\Lambda^\star) = 4(\sin^2\alpha + \sin^2\beta + \sin^2\gamma). \tag{6.4}$$

We are now ready to prove a sharp upper bound on the volume product of a 2-dimensional lattice Voronoi cell:

Proposition 6.3.2. *For a 2-dimensional lattice Λ, we have*

$$\mathrm{vol}(V_\Lambda)\mathrm{vol}(V_\Lambda^\star) \le 9.$$

Equality is attained, if and only if Λ is equivalent to $A_2^\star$ up to dilations and isometries.

Since the Voronoi cell of $A_2^\star$ is a regular hexagon, Proposition 6.3.2 is a special case of a more general theorem of Meyer and Reisner in [MR11], which states that among all polygons with m vertices the affine images of the regular m-gon maximize the Mahler volume.

Proof. As the map $\Lambda \mapsto V_\Lambda \mapsto \mathrm{vol}(V_\Lambda)\mathrm{vol}(V_\Lambda^\star)$ is continuous, it is enough to consider a generic lattice Λ. In view of (6.4) and since T contains no obtuse angles, it is enough to show $\sin^2\alpha + \sin^2\beta + \sin^2\gamma \leq 9/4$, for any choice of $0 \leq \alpha, \beta, \gamma \leq \frac{\pi}{2}$ with $\alpha + \beta + \gamma = \pi$. Suppose that there are two angle $< \pi/4$. Then it follows that the third angle must be $> \pi/2$, a contradiction. So we can assume that $\alpha, \beta \in [\pi/4, \pi/2]$. On this interval, the function $x \mapsto \sin^2 x$ is strictly concave. Therefore,

$$\begin{aligned}\sin^2\alpha + \sin^2\beta + \sin^2\gamma &\leq 2\sin^2\left(\frac{\alpha+\beta}{2}\right) + \sin^2\gamma \\ &= 1 - \cos(\alpha+\beta) + \sin^2(\pi - (\alpha+\beta))\end{aligned}$$

with equality, if and only if $\alpha = \beta$. We consider the function

$$f : [\pi/2, \pi] \to \mathbb{R}, \quad f(\phi) = 1 - \cos\phi + \sin^2(\pi - \phi).$$

A basic curve sketching shows that f attains its unique maximum at the point $2\pi/3$ and $f(2\pi/3) = 9/4$. Since $\alpha + \beta \in [\pi/2, \pi]$, it follows that

$$\sin^2\alpha + \sin^2\beta + \sin^2\gamma \leq \frac{9}{4},$$

as desired.

In order to have equality, we must have $\alpha = \beta$ and $2\pi/3 = \alpha + \beta = 2\alpha$. From this, it follows that $\alpha = \beta = \gamma = \pi/3$. By Lemma 6.3.1, this characterizes Λ up to a dilation and an isometry. It remains to note that $A_2^\star$ satisfies $\alpha = \beta = \gamma$, which can be computed from Example 6.1.1. $\square$

6.4 3-Dimensional Lattices

The goal of this section is to use an explicit description of the Vornoi cell of a 3-dimensional lattice in order to express its Mahler volume as a rational function in 6 variables. This will enable us to prove that $A_3^\star$ is a strict local maximum of the Mahler volume up to dilations and isometries.

The structure of 3-dimensional lattice Voronoi cells has been studied by Fedorov who proved that there exist exactly 5 different combinatorially non-equivalent 3-dimensional lattice Voronoi cells [Fed85, Fed91]. Only one of these types is a simple polytope, the permutohedron. This means, that any generic lattice in dimension 3 has a Voronoi cell that is combinatorially equivalent to a permutohedron. In [CS92], Conway and Sloane compute the coordinates of these permutohedra, using a parametrization of 3-dimensional lattices due to Selling [Sel74]. In the following we sum up the results of Conway and Sloane in order to obtain an explicit formula for the Mahler volume of a 3-dimensional lattice.

In general, an n-dimensional lattice Λ is said to be of the first kind, if there exists an *obtuse superbasis (OSB)* $v_0, ..., v_n$ of Λ, that is, we have

i) $\Lambda = \mathrm{span}_{\mathbb{Z}}\{v_1, ..., v_n\}$,

ii) $v_0 + \cdots + v_n = 0$ and

iii) $\langle v_i, v_j\rangle \leq 0$ for all $i \neq j$.

The numbers $p_{ij} = -\langle v_i, v_j\rangle \geq 0$, $0 \leq i \neq j \leq n$ are then called the *Selling parameters* of Λ. An OSB is called *strict*, if $p_{ij} > 0$ for all choices of i and j. It follows from the classification in [CS92] that a 3-dimensional lattice Λ is generic, if and only if it possesses a strict OSB. Since $p_{ij} = p_{ji}$, we may index the Selling parameters of Λ by 2-elementary subsets of $\{0, 1, 2, 3\}$. But for the sake of readability we abbreviate $p_{ij} = p_{\{i,j\}}$.

Similarly to the angles α, β and γ in Section 6.3, the Selling parameters characterize a lattice of the first kind Λ up to an isometry; to see this, consider the matrix $V = (v_1, ..., v_n)$. We note that

$$|v_i|^2 = \langle v_i, -\sum_{j\neq i} v_j\rangle = \sum_{j\neq i} p_{ij}.$$

Hence the Gram-Matrix $V^T V$ maybe expressed in terms of the Selling parameters via

$$V^T V = \begin{pmatrix} \sum_{j\neq 1} p_{1j} & -p_{12} & \cdots & -p_{1n} \\ -p_{12} & \sum_{j\neq 2} p_{2j} & \cdots & -p_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -p_{1n} & -p_{2n} & \cdots & \sum_{j\neq n} p_{jn} \end{pmatrix}. \tag{6.5}$$

From this we can reconstruct V uniquely up to a multiplication with an orthogonal matrix from the right.

Note that for any choice of positive p_{ij}'s the expression on the right hand side is a positive definite matrix, since it is a strictly diagonal dominant matrix. A Cholesky decomposition of this matrix gives a basis $\{v_1, v_2, v_3\}$ of a lattice Λ with Selling parameters (p_{ij}).

Example 6.4.1. i) In $\mathbb{Z}^n$, the system $\{-\mathbb{1}, e_1, ..., e_n\}$ constitutes an OSB, which is not strict; Selling parameters of $\mathbb{Z}^n$ are $p_{0i} = 1$ and $p_{ij} = 0$, for $1 \leq i < j \leq n$.

ii) In $A_n^\star$, the vectors of the form

$$\Big(\frac{-1}{n+1}, ..., \frac{-1}{n+1}, \frac{n}{n+1}, \frac{-1}{n+1}, ..., \frac{-1}{n+1}\Big)^T \in \mathbb{R}^{n+1} \cap \mathbb{1}_{n+1}^\perp$$

are a strict OSB with Selling parameters $p_{ij} = \frac{1}{n+1}$ for all $0 \leq i < j \leq n$. Observe that the Selling parameters of $A_n^\star$ do not depend on i and j. $\diamond$

Voronoi showed in his memoires [Vor08a, Vor08b, Vor09] that any 3-dimensional lattice is of the first kind. An algorithmic proof of this fact is given in [CS92, Sec. 7].

For the remainder of the chapter, we shall think of $\mathbb{R}^6$ as $\mathbb{R}^{\binom{4}{2}}$, the coordinates are indexed by 2-elementary subsets of $\{0,1,2,3\}$. As the Mahler volume is invariant with respect to orthogonal transformations, we may choose for any set of Selling parameters $p \in \mathbb{R}^6_{>0}$ a 3-dimensional generic lattice Λ_p with those parameters and restrict ourselves to bounding the function

$$f : \mathbb{R}^6_{>0} \to \mathbb{R},\ f(p) = \mathrm{vol}(V_{\Lambda_p})\mathrm{vol}(V^\star_{\Lambda_p}). \tag{6.6}$$

We will express f as a rational function in order to identify $A_3^\star$ as a local maximum of the Mahler volume.

As for the volume of the Voronoi cell, (6.5) yields

$$\mathrm{vol}(V_\Lambda)^2 = \det(\Lambda)^2 = \det(V^TV) = \sum p_{ij}p_{kl}p_{mn}, \tag{6.7}$$

where the sum ranges of all triples of pairs of indices $\{i,j\}$, $\{k,l\}$ and $\{m,n\}$ whose symmetric difference is non-empty (cf. [CS92, Eq. 15]). In order to express $\mathrm{vol}(V^\star_\Lambda)$ in terms of p, we recall the description of V_Λ due to Conway and Sloane in [CS92].

Voronoi cells of generic 3-dimensional lattices. Let Λ be a generic 3-dimensional lattice with OSB $\{v_0, v_1, v_2, v_3\}$ and Selling parameters $p = (p_{ij})_{0\le i<j\le 3} \in \mathbb{R}^6_{>0}$. Consider the linear isomorphism

$$\varphi : \mathbb{R}^3 \to \mathbb{1}_4^\perp,\ x \mapsto (\langle x, v_i\rangle)_{0\le i\le 3}.$$

It has been shown by Conway and Sloane [CS92, Sec. 8] that

$$\varphi(V_\Lambda) = Z(p), \tag{6.8}$$

where $Z(p) = \mathrm{conv}\{p_\sigma : \sigma \in S_4\}$. Here, S_4 denotes the group of permutations of $\{0,1,2,3\}$ and for $\sigma = (i,j,k,l)$ one has

$$\begin{aligned}(p_\sigma)_i &= \frac{1}{2}(p_{ij}+p_{ik}+p_{il}), & (p_\sigma)_j &= \frac{1}{2}(-p_{ij}+p_{jk}+p_{jl}),\\ (p_\sigma)_k &= \frac{1}{2}(-p_{ik}-p_{jk}+p_{kl}), & (p_\sigma)_l &= \frac{1}{2}(-p_{il}-p_{jl}-p_{kl}).\end{aligned} \tag{6.9}$$

Moreover, we have [CS92, Thm. 3]

$$\mathrm{vor}(\Lambda) = \Big\{v_S = \sum_{s\in S} v_s : S \subseteq \{0,...,3\},\ S \neq \emptyset,\ S \neq \{0,...,3\}\Big\}$$

The Voronoi relevant vectors whose inequalities in (6.2) are active at a vertex $\varphi^{-1}(p_\sigma)$ of V_Λ, $\sigma = (i,j,k,l) \in S_4$, are given by v_i, v_{ij} and v_{ijk} [CS92, Sec. 8]. Hence, these 3 vectors together with the origin form a Delaunay 3-simplex of Λ and conversely, every Delaunay 3-simplex of Λ at the origin is of this form. Since $\{v_1, v_2, v_3\}$ form a basis of Λ, so do the vectors $\{v_i, v_j, v_k\}$ for any $\{i,j,k\} \subseteq \{0,...,3\}$ and thus, also $\{v_i, v_{ij}, v_{ijk}\}$ form a basis. Finally, the norm of any v_S, where S is a proper subset of $\{0,...,3\}$ is given by [CS92, Thm. 4]

$$|v_S|^2 = \sum_{i\in S,\, j\in S^c} p_{ij}.$$

With this knowledge at hand, we express (6.3) in terms of p and obtain for a generic 3-dimensional lattice with Selling parameters $p \in \mathbb{R}^6_{>0}$:

$$\mathrm{vol}(V_\Lambda^\star) = \frac{4}{3}\mathrm{vol}(V_\Lambda) \sum_{(ijkl)\in S_4} \frac{1}{(p_{ij}+p_{ik}+p_{il})(p_{ik}+p_{il}+p_{jk}+p_{jl})(p_{il}+p_{jl}+p_{kl})}.$$

From (6.7) it follows that

$$\begin{aligned} f(p) = &\frac{4}{3}\Big(\sum p_{ij}p_{kl}p_{mn}\Big) \\ &\cdot\Big(\sum_{(ijkl)\in S_4} \frac{1}{(p_{ij}+p_{ik}+p_{il})(p_{ik}+p_{il}+p_{jk}+p_{jl})(p_{il}+p_{jl}+p_{kl})}\Big), \end{aligned} \tag{6.10}$$

for all $p \in \mathbb{R}^6_{>0}$, where again the first sum ranges over all triples of index sets whose symmetric difference is non-empty. This is indeed a rational function in $p \in \mathbb{R}^6_{>0}$. Using `sagemath` [SageMath], we obtain

Lemma 6.4.2. *For $p \in \mathbb{R}^6_{>0}$, let $f(p)$ be as in* (6.6)*. Then,*

$$\{p \in \mathbb{R}^6_{>0} : \langle p, \mathbb{1}_6\rangle = 6\} \to \mathbb{R},\ p \mapsto f(p)$$

attains a strict local maximum at $p = \mathbb{1}$.

Proof. In (6.10), We substitute p_{23} by $6 - p_{01} - \cdots - p_{13}$. This gives a function g in the five variables $p_{01}, ..., p_{13}$. In `sagemath`, we can compute the gradient and the Hessian of f and evaluate both in exact rational arithmetic at $\mathbb{1}_5$. This second derivative test shows that g has a strict local maximum at $\mathbb{1}_5$. $\square$

The symbolic expression given by (6.10) is provided as a sage-object under this URL:

`https://github.com/AnsgarFreyer/Dissertation_Data.git`

As p encodes a unique lattice up to isometry and f is invariant with respect to dilation, we deduce from Example 6.4.1 that $A_3^\star$ is a strict local maximum of the volume product up to scaling and isometries. More precisely,

Theorem 6.4.3. *For a 3-dimensional lattice Λ, let $[\Lambda]$ be the equivalence class of lattices obtained from Λ by applying dilations and isometries. Let*

$$\mathcal{L}^3_\sim = \{[\Lambda] : \Lambda \text{ 3-dimensional lattice}\}.$$

Then the map

$$F : \mathcal{L}^3_\sim \to \mathbb{R},\ [\Lambda] \mapsto \mathrm{vol}(V_\Lambda)\mathrm{vol}(V_\Lambda^\star)$$

attains a strict local maximum at $[A_3^\star]$.

Proof. First, we recall that by the invariance properties of the Mahler volume, the map F is indeed well-defined.

Suppose $[A_3^\star]$ was not a strict local maximum of F. Then we find a sequence $(\Lambda^{(n)})_{n\in\mathbb{N}}$ of 3-dimensional lattices in $\mathbb{1}_4^\perp$ that converges to $A_3^\star$ such that $F([\Lambda^{(n)}]) \geq F([A_3^\star])$, but no $\Lambda^{(n)}$ is equivalent to $A_3^\star$. For each $\Lambda^{(n)}$, we choose a superbasis $B^{(n)}$ in such a way that $B^{(n)}$ converges pointwise to the obtuse superbasis B of $A_3^\star$ that has been described in Example 6.4.1. Since B is a strict OSB, for large enough n, the superbases $B^{(n)}$ are also strict OSBs. Let $p^{(n)}$ be the Selling parameters of $\Lambda^{(n)}$ that we gain from $B^{(n)}$. By continuity, we have $\lim_{n\to\infty} p^{(n)} = \mathbb{1}_6$. But on the other hand, we have

$$f(p^{(n)}) = \mathrm{vol}(V_{\Lambda^{(n)}})\mathrm{vol}(V_{\Lambda^{(n)}}^\star) = F([\Lambda^{(n)}]) \geq F([A_3^\star]) = f(\mathbb{1}_6).$$

Since none of the $\Lambda^{(n)}$ is equivalent to $A_3^\star$, none of the $p^{(n)}$ is a multiple of $\mathbb{1}_6$. Since f is invariant with respect to scaling of p, we obtained a contradiction to Lemma 6.4.2. Hence, $[A_3]^\star$ is indeed a strict local maximum of F. ☐

We conjecture that $A_3^\star$ is indeed a global maximum.

Conjecture 6.4.4. *For any 3-dimensional lattice Λ, we have*

$$\mathrm{vol}(V_\Lambda)\mathrm{vol}(V_\Lambda^\star) \leq \mathrm{vol}(V_{A_3^\star})\mathrm{vol}((V_{A_3^\star})^\star) = \frac{128}{9}$$

with equality, if and only if Λ can be obtained from $A_3^\star$ by dilations and isometries.

6.5 Towards Quasi-Concavity of f

So far we reduced the function f from (6.6) to a rational function in the Selling parameters. In this section we want to use the underlying geometry in order to develop a strategy to prove Conjecture 6.4.4. In other words, f attains its maximum at $p \in \mathbb{R}_{>0}\mathbb{1}$. Unfortunately, a proof of this statement is out of reach at the moment, but nonetheless, the insights of this chapter may be regarded as supporting evidence of the conjecture.

Let us recall the representation of a generic lattice Voronoi cell as a linear image of $Z(p)$, $p \in \mathbb{R}^6_{>0}$ (cf. (6.8)). As the Mahler volume is invariant with respect to linear transformations, we have

$$f(p) = \mathrm{vol}(Z(p))\mathrm{vol}(Z(p)^\star),$$

where the polarity and the volume are computed within $\mathrm{span}(Z(p)) = \mathbb{1}_4^\perp$. We will use this view on f to show the following proposition.

Proposition 6.5.1. *For $0 \leq i < j \leq 3$, let e_{ij} denote the standard unit vector of $\mathbb{R}^6$ indexed by 2-element subsets of $\{0, ..., 3\}$. For $p \in \mathbb{R}^6_{>0}$ fixed, and $\{i, j\}$ arbitrary, the function*

$$g : \mathbb{R}_{>0} \to \mathbb{R},\ t \mapsto f(p + te_{ij})$$

is quasi-concave.

A function $\phi : C \to \mathbb{R}$ on a convex set C is called *quasi-concave*, if $\{x \in C : \phi(x) \geq s\}$ is convex for any $s \in \mathbb{R}$. If the function f itself was quasi-concave, then Conjecture 6.4.4 would follow, as Proposition 6.5.3 will show. However, this cannot be deduced from Proposition 6.5.1.

In order to prove Proposition 6.5.1, we need the notion of *shadow systems*. For a fixed vector $v \neq 0$, a bounded set $X \subseteq \mathbb{R}^n$, a bounded function $\alpha : X \to \mathbb{R}$ and an interval $I \subseteq \mathbb{R}$, the family of convex sets

$$K_t = \operatorname{conv}\{x + \alpha(x)tv : x \in X\}, \quad t \in I,$$

is called a shadow system. Shadow systems were introduced by Rogers and Shephard [RS58b, She64]. We call a shadow system *symmetric*, if $K_t \in \mathcal{K}^n_{os}$, for any $t \in I$. In [CG06], Campi and Gronchi prove that for a symmetric shadow system, the function $t \mapsto \operatorname{vol}(K_t^\star)^{-1}$ is convex. If in addition $t \mapsto \operatorname{vol}(K_t)$ is an affine function, the Mahler volume $t \mapsto \operatorname{vol}(K_t)\operatorname{vol}(K_t^\star)$ is quasi-concave in t [FMZ12, Cor. 2].

Proof of Proposition 6.5.1. For $t \in \mathbb{R}_{>0}$, consider the family of convex bodies

$$K_t = Z(p + te_{ij}).$$

We have $g(t) = \operatorname{vol}(K_t)\operatorname{vol}(K_t^\star)$ and indeed, $(K_t)_{t>0}$ is a shadow system, which follows from (6.9); we let $X = \{p_\sigma : \sigma \in S_4\}$, $v = e_i - e_j$ and $\alpha(p_\sigma) = 1$, if i precedes j in σ and $\alpha(p_\sigma) = -1$ otherwise. Moreover, we have by (6.8) that $\operatorname{vol}(Z(p)) = \det(\Lambda_p)^2$ for any $p \in \mathbb{R}^6_{>0}$ and therefore, by (6.7),

$$\operatorname{vol}(Z(p)) = \sum p_{ij}p_{kl}p_{mn},$$

where the sum ranges over all index sets whose symmetric difference is non-empty. From this, we obtain that the volume of K_t is an affine function in t, which finishes the proof. $\square$

Remark 6.5.2. From the coordinate representation (6.9), it follows that

$$Z(p) = \frac{1}{2} \sum_{0 \leq i < j \leq 3} p_{ij}[e_i - e_j, e_j - e_i],$$

where e_i denotes the i-th standard unit vector in $\mathbb{R}^4$ indexed by $\{0, ..., 3\}$. In particular $Z(p)$ is a zonotope with fixed edge directions. Thus, geometrically, the shadow system $(K_t)_{t \geq 0}$ describes the process of elongating the edge in direction $e_i - e_j$, while keeping the remaining edge lengths fixed. That way, we have an alternative argument why $\operatorname{vol}(K_t)$ is an affine function in t (cf. Figure 6.4). $\diamond$

We finish the section by proving that quasi-concavity of f would imply Conjecture 6.4.4.

Proposition 6.5.3. *Suppose that the function f as defined* (6.6) *is quasi-concave. Then, Conjecture 6.4.4 holds true.*

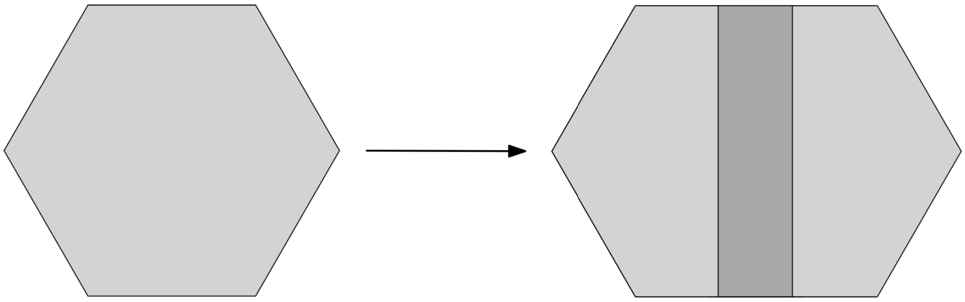


Figure 6.4: 2-dimensional illustration of the shadow system; If we elongate the edge in direction $e_i - e_j$ by t, the new zonotope differs from $Z(p)$ only by an "inserted" prism of height t over $Z(p)|(e_i - e_j)^{\perp}$. Thus, its volume is an affine function in t.

Proof. Suppose there was a lattice Λ which is not obtained from $A_3^\star$ by isometries and dilations, but fulfills

$$\mathrm{vol}(V_\Lambda)\mathrm{vol}(V_\Lambda^\star) \geq \mathrm{vol}(V_{A_3^\star})\mathrm{vol}((V_{A_3^\star})^\star).$$

Then there exists a set of Selling parameters $p \in \mathbb{R}^6_{>0}$, which is not a multiple of $\mathbb{1}$ but satisfies $f(p) \geq f(\mathbb{1})$. We consider the S_4-action on $\mathbb{R}^6_{>0}$ given by $(\sigma p)_{ij} = p_{\sigma_i \sigma_j}$. By Remark 6.5.2, we have $Z(\sigma p) = \sigma^{-1} Z(p)$, where the S_4-action on the right hand side is the usual permutation of the coordinates. Since $f(p)$ is the Mahler volume of $Z(p)$, we see that f is invariant with respect to the S_4-action on $\mathbb{R}^6_{>0}$. Hence, by quasi-concavity, $f(q) \geq f(\mathbb{1})$ for any $q \in O = \mathrm{conv}\{\sigma p : \sigma \in S_4\}$.

Now we prove that $\frac{1}{24}\sum_{\sigma \in S_4} \sigma p \in O \cap \mathbb{R}^6_{>0}\mathbb{1}$: Again, let e_{kl} denote the standard unit vector of $\mathbb{R}^6$ with respect to the indexing by 2-elementary sets. Then we have

$$\Big\langle \sum_{\sigma \in S_4} \sigma p, e_{kl} \Big\rangle = \sum_{0 \leq i < j \leq 3} p_{ij}\, s(kl; ij),$$

where $s(kl; ij) = |S(kl; ij)|$ and

$$S(kl; ij) = \big\{\sigma \in S_4 : \{i, j\} = \{\sigma_k, \sigma_l\}\big\}.$$

In order to prove the claim, it is enough to show $s(kl; ij) = s(ab; ij)$ for all 2-elementary subsets $\{a, b\}$ and $\{k, l\}$ of $\{0, ..., 3\}$. For this, consider any permutation $\pi \in S_4$ with $\pi_a = k$ and $\pi_b = l$. Then we have $S(kl; ij) \cdot \pi \subseteq S(ab; ij)$, which yields $s(kl; ij) \leq s(ab; ij)$. Reversing the roles $\{k, l\}$ and $\{i, j\}$ shows that we have equality. This proves the claim.

Since p is not a multiple of $\mathbb{1}$, the polytope O is not a segment in $\mathbb{R}^6_{>0}\mathbb{1}$. Thus, by projecting O stereografically with respect to the origin on the hyperplane $\{\langle \cdot, \mathbb{1}\rangle = 6\}$, we obtain a segment in that hyperplane that contains $\mathbb{1}$ on which f is at least $f(\mathbb{1})$. This contradicts the fact that $\mathbb{1}$ is a strict local maximum of f on that hyperplane (cf. Lemma 6.3.1). $\square$

Note that f is quasi-concave, if and only if it satisfies $f(\lambda p + (1-\lambda) q) \geq \min\{f(p), f(q)\}$ for all $p, q \in \mathbb{R}^6_{>0}$ and $\lambda \in [0, 1]$ [BV13, Sec. 3.4.2]. With this definition we can efficiently test the quasi-concavity of f at two points p and q with random coordinates in a large interval. No counterexamples have been obtained.

6.6 Examples in Higher Dimensions

With the help of the software `polymake` [AGH+17], we can compute the Voronoi cell of a low-dimensional lattice from one of its Gram matrices. Below, we list examples of lattices whose Voronoi cells have a potentially high Mahler volume.

The `polymake` script that we use has been implemented as part of a joint project with Martin Henk and Lilli Leifheit. Given a basis $B = \{b_1, ..., b_n\}$ of a lattice $\Lambda = B\mathbb{Z}^n$, we consider the parallelepiped $P = \{x \in \mathbb{R}^n : |\langle x, b_i \rangle| \leq |b_i|^2/2, \forall i \in [n]\}$. In view of (6.2), we have $V_\Lambda \subseteq P$. Moreover, one has $\mathrm{vor}(\Lambda) \subseteq 2V_\Lambda$; If there was a vector $v \in \mathrm{vor}(\Lambda)$, which is not contained in $2V_\Lambda$, then there exists $w \in \mathrm{vor}(\Lambda)$ with $\langle v, w \rangle > |w|^2$. As we have $\langle w, x \rangle \leq |w|^2/2$ and $\langle v - w, x \rangle \leq |v - w|^2/2$ for any $x \in V_\Lambda$, we deduce $\langle v, x \rangle < |v|^2/2$ for all $x \in V_\Lambda$. This contradicts $v \in \mathrm{vor}(\Lambda)$. Thus, in order to compute V_Λ, it suffices to compute the lattice points in $2P$.

As the examples we computed were well-behaved in the sense that the basis vectors were not too long, we could use this approach without any preprocessing of the basis. In general, one might apply an LLL-reduction to obtain a basis whose vectors are not too long compared to the successive minima of Λ with respect to the Euclidean ball [LLL82]. In [HRS20], the authors prove upper and lower bounds on the size of parallelepipeds that contain $\mathrm{vor}(\Lambda)$. Our polymake script can be found under the following URL:

`https://github.com/AnsgarFreyer/Dissertation_Data.git`

As it is oftentimes more convenient to encode a lattice in terms of its Gram matrix $Q = B^T B$ rather than its basis B, our algorithm computes for a given positive definite matrix Q the Voronoi cell

$$V(Q) = \{x \in \mathbb{R}^n : x^T Q z \leq \frac{1}{2} z^t Q z, \forall z \in \mathbb{Z}^n\},$$

i.e., the Voronoi cell of $\mathbb{Z}^n$ with respect to the scalar product induced by Q. This is done in order to avoid irrational numbers that might occur in a Cholesky decomposition of Q. In fact, $V(Q)$ may be computed with the same algorithm as described above for V_Λ, by replacing the Euclidean scalar product with the one induced by Q. Moreover, we have $V(Q) = B^{-1}V_\Lambda$ and therefore $\mathrm{vol}(V(Q))\mathrm{vol}(V(Q)^\star) = \mathrm{vol}(V_\Lambda)\mathrm{vol}(V_\Lambda^\star)$, which allows us to compute the Mahler volume of V_Λ in rational arithmetic, whenever a rational Gram matrix Q of Λ is given.

Root lattices. The root lattices A_n (cf. Example 6.1.1), $D_n = \{x \in \mathbb{Z}^n : x_1 + \cdots + x_n \in 2\mathbb{Z}\}$ and $\mathbb{Z}^n$, as well as their polar lattices have a particularly rich symmetry group and are therefore natural candidates for extrema of geometric functionals. The following table shows their Mahler Volumes. Note that $\mathbb{Z}^n$ is self-polar and that the Voronoi cell of $\mathbb{Z}^n$ is the unit cube, which is conjectured to attain the minimum Mahler volume among all origin-symmetric convex bodies. Also, D_3 is is equivalent to A_3 and D_4 is self-polar. In the last column we also list the Mahler volume of the Euclidean ball in the respective dimension for the sake of comparison. All values are rounded up to the third decimal place.

n	$\mathbb{Z}^n$	A_n	$A_n^\star$	D_n	$D_n^\star$	ball
3	10.667	13.334	14.223	13.334	14.223	17.546
4	10.667	14.584	17.361	16	16	24.352
5	8.533	12.6	17.28	14.4	16.384	27.707
6	5.689	8.984	14.525	10.311	12.642	26.705

Generic lattices in dimension 4. Our investigation of the 3-dimensional case relied heavily on the fact that any generic 3-dimensional lattice Voronoi cell is combinatorially equivalent to the permutohedron (cf. (6.9) and Remark 6.5.2). In higher dimensions, the number of combinatorially distinct Delaunay lattice triangulations grows rapidly. While there are 3 of them in dimension 4 [Vor08b, Vor09], there are already 222 in dimension 5 [EG02, SV06, DG09] and at least 567.613.632 in dimension 6 [BE13].

Interestingly, the matrices

$$D_4^1 = \begin{pmatrix} 4 & -1 & -1 & -1 \\ -1 & 4 & -1 & -1 \\ -1 & -1 & 4 & -1 \\ -1 & -1 & -1 & 4 \end{pmatrix}, \quad Di_{4a} = \begin{pmatrix} 2 & \alpha & -1 & -1 \\ \alpha & 2 & -1 & -1 \\ -1 & -1 & 2 & 1-\alpha \\ -1 & -1 & 2 & 1-\alpha \end{pmatrix},$$

and

$$Di_{4b} = \begin{pmatrix} 3-\gamma & \gamma & -1 & -1 \\ \gamma & 3-\gamma & -1 & -1 \\ -1 & -1 & 2+2\beta & \beta \\ -1 & -1 & \beta & 2+2\beta \end{pmatrix},$$

where $\alpha \approx 0.697$, $\beta \approx 0.544$ and $\gamma \approx 0.499$ are algebraic numbers, are not only representatives of the three different types of generic 4-dimensional lattices, but are also the only local minimizers of the covering radius, i.e., of the function $\Lambda \mapsto \mu(B_n, \Lambda)$ (cf. [CS99, Ch. 2, Sec. 1.3]). In fact, we can check with `polymake` that the Voronoi cells that belong to the above matrices are combinatorially distinct and, thus, serve as "round" representatives of the three different types of generic Voronoi cells in dimension 4. Note that D_4^1 is the Gram matrix of $A_4^\star$. Let us denote the lattice corresponding to Di_{4a} and Di_{4b} by Λ_{4a} and Λ_{4b}. We obtain the following values for the Mahler volume of these lattices. The Mahler volume in the table below is rounded up to the third decimal place.

Lattice	Voronoi cell	Mahler volume
$\mathbb{Z}^4$	unit cube	10.667
$A_4^\star$	regular permutohedron	17.361
Λ_{4a}	$\sim$	17.218
Λ_{4b}	$\sim$	16.268
$\sim$	4-dimensional ball	24.352

Note that the Mahler volumes of Λ_{4a} and $A_4^\star$ are remarkably close to one another.

5-dimensional lattices. In [DGSW16], the authors classify the combinatorial types of all (not necessarily generic) lattice Voronoi cells and provide a list of representatives for each of the 110244 combinatorial types. Although there are infinitely many affine types of lattice Voronoi cells, the examples in [DGSW16] serve as an interesting database for our purposes. Among them, we find a lattice Λ_5 whose volume product is exceptionally close to the one of $A_5^\star$; we have $\mathrm{vol}(V_{\Lambda_5})\mathrm{vol}((V_{\Lambda_5})^\star) \approx 17.273$, while the volume product of the Voronoi cell of $A_5^\star$ is 17.28. The Gram matrix of Λ_5 is

$$\begin{pmatrix} 7 & 3 & -3 & -3 & -2 \\ 3 & 7 & -2 & -2 & -3 \\ -3 & -2 & 7 & 2 & -2 \\ -3 & -2 & 2 & 7 & -2 \\ -2 & -3 & -2 & -2 & 7 \end{pmatrix}.$$

It is the 75774th element in the list computed in [DGSW16].

7 Two Polytopes Associated to Forests

In this chapter we consider two anti-blocking polytopes that can be associated to a forest; the *matching polytope* and the *substar polytope*. These polytopes are dual to one another and the volume of both polytopes can be described combinatorially in terms of the underlying forest. We will use the combinatorial view on the volume to obtain upper bounds on the Mahler volume of this pair of polytopes. The results in this chapter are part of an ongoing joint work with Raman Sanyal.

7.1 Basics From Combinatorics

Before we begin, let us briefly recall the basic terms and definitions from combinatorics that are necessary for this chapter.

Graphs. We consider a *graph* as a pair $G = (V, E)$, where V is finite and $E \subseteq \binom{V}{2}$ is a set of 2-elementary subsets of V. The elements of V are called *vertices* of G and the elements of E are called *edges* of G. If we are dealing with graphs and polytopes at the same time, we might also call the elements of V *nodes*, in order to avoid confusion with the vertices of polytopes. The *degree* of a vertex v in G is the number of edges of G that contain v. An *orientation* $\mathcal{O}$ of a graph G is a binary relation on V such that $\{x, y\} \in E$ holds, if and only if either $(x, y) \in \mathcal{O}$, or $(y, x) \in \mathcal{O}$ holds. If $(x, y) \in \mathcal{O}$ for an edge $e = \{x, y\} \in E$, we call x the *source* and y the *target* vertex of e in $\mathcal{O}$, and we say that e is oriented from x to y.

We introduce shorthand notations for particular graphs. We denote by P_n the path graph with n edges and by C_n the cycle graph with n edges. A *matching* of G is a set $M \subseteq E$ such that no two edges of M intersect. By M_n, we shall denote the matching graph with n edges, i.e., M_n contains exactly n disjoint edges and $2n$ vertices. Moreover, for a vertex v we call the vertices of G that form an edge with v the *neighbors* of v. The set of edges S that contain v is called the *star* around v and v is the *center* of S. An edge that contains a vertex is also said to *cover* this vertex. We let S_n denote the *star graph* with n edges, by which we mean the graph that contains $n + 1$ vertices, and n edges such that one vertex is covered by all edges.

An important concept in the study of polytopes that are associated to graphs are *cliques* and *stable sets*. A *clique* C of $G = (V, E)$ is a subset of V such that any two vertices in C are joined by an edge. Conversely, a stable set is a subset of V in which no two vertices are joined by an edge. We say that G is *bipartite*, if there exist two stable sets $S_1, S_2 \subseteq V$ such that $V = S_1 \cup S_2$. It is important for the upcoming sections to note that

i) A stable set of G is a clique in the *complement graph* $\overline{G} = (V, \binom{V}{2} \setminus E)$,

ii) A matching is a stable set in the *line graph* $\mathsf{L}G = (E, F)$, where we have $\{e, f\} \in F$, if and only if $|e \cap f| = 1$.

For a graph $G = (V, E)$, another graph $H = (V', E')$ is called a *subgraph* of G, if $V' \subseteq V$ and $E' \subseteq E$. H is called an *induced* subgraph, if $E' = E \cap \binom{V'}{2}$. A *forest* is a graph that does not contain a C_k as a subgraph, for any $k \in \mathbb{N}$. A *tree* is a connected forest, i.e., any two vertices are connected by a path.

A graph G is called *perfect*, if every induced subgraph $H = (V', E')$ of G fulfills

$$\max\{|C| : C \subseteq V' \text{ is a clique}\} = \min\{k : \exists S_1, ..., S_k \subseteq V' \text{ stable. } V' = S_1 \cup ... \cup S_k\}.$$

Perfect graphs are characterized as those graphs G that do not contain C_{2k+1}, or $\overline{\mathsf{C}_{2k+1}}$ as an induced subgraph for $k \geq 2$. This result is known as the *strong perfect graph theorem*, which has been proven in 2006 by Chudnovsky et al. [CRST06]. It implies in particular that G is perfect, if and only if $\overline{G}$ is perfect, a result that is known as the (weak) *perfect graph theorem*, which has been proven by Lovász in 1972 [Lov72a, Lov72b].

Examples of graph classes that are perfect are

i) bipartite graphs,

ii) line graphs of bipartite graphs [Lov74],

iii) (in)comparability graphs of posets (cf. Section 7.3),

and many more "natural classes". We refer to [BC84] for a more comprehensive overview.

Posets. Finally, we recall the basic terms concerning posets. For a detailed introduction, we refer to [Sta11, Ch. 3]. A *partially ordered set (poset)* is a pair $\mathcal{P} = (X, \leq)$, where X is a set and $\leq$ is a relation on X such that

i) $x \leq x$, for all $x \in X$,

ii) $x \leq y$ and $y \leq x$ implies $x = y$, for all $x, y \in X$,

iii) $x \leq y$ and $y \leq z$ implies $x \leq z$, for all $x, y, z \in X$.

A *chain* of $\mathcal{P}$ is a subset $C \subseteq X$ in which any two elements are comparable. Dual to this, an *anti-chain* of $\mathcal{P}$ is a subset $A \subseteq X$ such that any two distinct elements of A are incomparable. We will be concerned with finite posets only. For such posets, a *linear extension* is an order-preserving bijection $\sigma : X \to [n]$. That is, we have $\sigma(x) \leq \sigma(y)$, whenever $x \leq y$.

7.2 Combinatorial Anti-Blocking Polytopes

In this section we introduce the class of stable set polytopes, which may be regarded as the most general class of anti-blocking polytopes that are associated to a graph. Afterwards, we recall the definition of Stanley's chain polytope and consider it in the framework of stable set polytopes.

For a graph $G = (V, E)$, the *stable set polytope* is defined as

$$\mathrm{Stab}(G) = \mathrm{conv}\{\mathbb{1}[S] : S \subseteq V \text{ is stable in } G\} \subseteq \mathbb{R}^V,$$

where $\mathbb{1}[S]$ is the indicator vector of S in $\mathbb{R}^V$, the real $|V|$-dimensional vector space whose coordinates are indexed by V. Note that $\mathrm{Stab}(G)$ is an anti-blocking polytope, since any subset of a stable set is stable.

We say that an anti-blocking lattice polytope P is *reflexive*, if its anti-blocking polar $\mathrm{A}P$ is again a lattice polytope. Equivalently, P is reflexive, if and only if the associated unconditional $\mathrm{U}P$ is a reflexive polytope (cf. Section 2.1).

Kohl, Olsen and Sanyal showed that reflexive anti-blocking polytopes are exactly the stable set polytopes of perfect graphs [KOS20, Thms. 4.6 and 4.9].

Theorem 7.2.1 (Kohl, Olsen and Sanyal, [KOS20])**.** *Let $P \subseteq \mathbb{R}^n_{\geq 0}$ be an anti-blocking polytope. Then, P is reflexive, if and only if there exists a unique (up to the labeling of the nodes) perfect graph G with $P = \mathrm{Stab}(G)$.*

As a byproduct of Matoušek's proof of the weak perfect graph theorem [Mat02, Thm. 12.1.2], one obtains for a perfect graph G that its anti-blocking polar is given by $\mathrm{A}P = \mathrm{Stab}(\overline{G})$. In other words, we have

$$\mathrm{Stab}(G) = \{x \in \mathbb{R}^V_{\geq 0} : \langle x, \mathbb{1}[C]\rangle \leq 1,\ \forall C \subseteq G \text{ clique}\}$$

and the irredundant inequalities in the description are given by the maximal cliques of G.

Theorem 7.2.1 raises the question whether geometric quantities of a reflexive anti-blocking polytope P may be expressed in terms of its underlying perfect graph G. As for the volume, an explicit connection to the combinatorial properties of G has only been established for the class of chain polytopes.

Chain Polytopes. Consider a finite poset $\mathcal{P} = (X, \leq)$. The *chain polytope* of $\mathcal{P}$ has been defined by Stanley in [Sta86] as

$$\mathrm{C}(\mathcal{P}) = \{x \in \mathbb{R}^X_{\geq 0} : \langle x, \mathbb{1}[C]\rangle \leq 1,\ \forall C \subset \mathcal{P} \text{ chain}\}.$$

The terms "chain polytope" and "stable set polytope" are somewhat inconsistent, since the chain polytope is not the convex hull of all indicator vectors of chains of $\mathcal{P}$. However, we shall stick to this terminology, since it is widely used in the literature.

The irredundant constraints of the chain polytopes are given by the maximal chains and Stanley proved that the vertices of $\mathrm{C}(\mathcal{P})$ are precisely the points $\mathbb{1}[A]$, where $A \subseteq \mathcal{P}$ is an anti-chain [Sta86, Thm. 2.2]. This shows that $\mathrm{C}(\mathcal{P})$ is a reflexive anti-blocking polytope and as such, there exists a perfect graph G such that $\mathrm{C}(\mathcal{P}) = \mathrm{Stab}(G)$. Indeed, if we consider the comparability graph

$$\mathsf{Comp}(\mathcal{P}) = \big(X, \{\{x,y\} : x < y \text{ or } y < x\}\big),$$

we find that $\mathrm{C}(\mathcal{P}) = \mathrm{Stab}(\mathsf{Comp}(\mathcal{P}))$, since chains and anti-chains of $\mathcal{P}$ correspond to cliques and stable sets of $\mathsf{Comp}(\mathcal{P})$, respectively. The fact that comparability graphs of posets are perfect can also be regarded as a reformulation of Mirsky's theorem, which states that the size of the largest chain in a poset $\mathcal{P}$ equals the minimum number of anti-chains needed to partition $\mathcal{P}$ [Mir71].

Unlike for general stable set polytopes, the volume of $\mathrm{C}(\mathcal{P})$ is known to have a strong combinatorial meaning for $\mathcal{P}$ [Sta86, Cor. 4.2].

Theorem 7.2.2 (Stanley)**.** *For a poset $\mathcal{P}$ with n elements, we have*

$$\mathrm{vol}(\mathrm{C}(\mathcal{P})) = \frac{1}{n!} e(\mathcal{P}),$$

where $e(\mathcal{P})$ denotes the number of linear extensions of $\mathcal{P}$.

Stanley's result allows for geometric interpretations of combinatorial results involving the number of linear extensions of a poset. For instance, Sidorenko proved that for any poset $\mathcal{P}$ with the property that its incomparability graph $\overline{\mathsf{Comp}(\mathcal{P})}$ is the comparability graph of another poset $\overline{\mathcal{P}}$ one has [Sid91]

$$e(\mathcal{P})e(\overline{\mathcal{P}}) \geq n!. \tag{7.1}$$

In view of Theorem 7.2.2, Sidorenko's inequality becomes [BBS99]

$$\mathrm{vol}(\mathrm{C}(\mathcal{P}))\mathrm{vol}(\mathrm{AC}(\mathcal{P})) = \mathrm{vol}(\mathrm{C}(\mathcal{P}))\mathrm{vol}(\mathrm{C}(\overline{\mathcal{P}})) \geq \frac{1}{n!}. \tag{7.2}$$

If we consider the corresponding unconditional body $K = \mathrm{UC}(\mathcal{P})$, (7.2) yields $\mathrm{vol}(K)\mathrm{vol}(K^\star) \geq 4^n/n!$, i.e., the Mahler conjecture (cf. (2.3)) holds true for these bodies. In general, for an anti-blocking body K we shall refer to the volume product $\mathrm{vol}(K)\mathrm{vol}(\mathrm{A}K)$ as the *Mahler volume* of K.

Indeed, it has been shown by Saint-Raymond that the Mahler conjecture holds for an arbitrary unconditional convex body [Sai81]. For an anti-blocking body K, this means that the inequality

$$\mathrm{vol}(K)\mathrm{vol}(\mathrm{A}K) \geq \frac{1}{n!} \tag{7.3}$$

holds. While the proof of Saint-Raymond uses methods from harmonic analysis, Meyer gave a more elementary proof of the statement [Mey86]. In [BB00], remarkable parallels between Meyer's proof and the combinatorial argument of Sidorenko are established.

Inspired by the history of Sidorenko's inequality (7.1) and its interpretation (7.2), we study the (Mahler) volume of two particular stable set polytopes, the matching and the substar polytope of forests.

7.3 Matching and Substar Polytopes

In the following, we consider a forest $G = (V, E)$ with n edges and we define the *matching polytope* of G as

$$\mathrm{M}(G) = \operatorname{conv}\{\mathbb{1}[M] : M \subseteq E \text{ matching in } G\} \subseteq \mathbb{R}^E_{\geq 0}. \tag{7.4}$$

We have $\mathrm{M}(G) = \mathrm{Stab}(\mathsf{L}G)$ and, since a forest is bipartite, $\mathsf{L}G$ is perfect. Note that a clique $S \subseteq \mathsf{L}G$ is a set of edges of G that intersect in common vertex of G, because G contains no triangles. We call such a set S a *substar* of G. If S is a maximal clique, it is called a *star* of G. With these terms, we have

$$\mathrm{M}(G) = \{x \in \mathbb{R}^E_{\geq 0} : \langle x, \mathbb{1}[S]\rangle \leq 1,\ \forall S \subseteq E \text{ star}\} \tag{7.5}$$

and the description is irredundant. This insight is indeed a special case of *Edmond's matching polytope theorem*, which generalizes the description of the matching polytope to the non-bipartite case [LP86, Sec. 7.3]. Note that in particular, a matching and a substar intersect in at most one edge.

Remark 7.3.1. There are two types of stars in a forest G; either, the star consists of all edges incident to a non-leaf vertex of G, or it consists of a single isolated edge. ◇

The anti-blocking polar of $\mathrm{M}(G)$ is called the *substar polytope* of G. In view of (7.4) and (7.5), we have

$$\begin{aligned}\mathrm{AM}(G) &= \operatorname{conv}\{\mathbb{1}[S] : S \subseteq E \text{ substar of } G\} \\ &= \{x \in \mathbb{R}^E_{\geq 0} : \langle x, \mathbb{1}[M]\rangle \leq 1,\ \forall M \subseteq E \text{ maximal matching}\},\end{aligned} \tag{7.6}$$

and the descriptions are irredundant. Before we continue, let us consider some examples.

Example 7.3.2. i) If $G = \mathsf{M}_n$ is a single matching of size n, then every subset of E is again a matching and the only (sub)stars are the n isolated edges of M_n. Thus, $\mathrm{M}(\mathsf{M}_n) = [0,1]^E$ is the cube of side length 1 and $\mathrm{AM}(\mathsf{M}_n) = \operatorname{conv}\{0, \mathbb{1}[e] : e \in E\}$ is the standard orthogonal simplex.

Conversely for the star graph S_n, that is, the graph that consists of n edges, all of which meet at a common vertex, a matching can contain at most one edge. Moreover, any subset of the edges of S_n forms a substar. Hence, $\mathrm{M}(\mathsf{S}_n) = \operatorname{conv}\{0, \mathbb{1}[e] : e \in E\}$ and $\mathrm{AM}(\mathsf{S}_n) = [0,1]^E$.

Note that both $\mathrm{M}(\mathsf{M}_n)$ and $\mathrm{M}(\mathsf{S}_n)$ satisfy the Saint-Raymond inequality (7.3) with equality.

ii) Consider the path graph P_n with n edges. For convenience, we label the edges of P_n with numbers from $[n]$ such that any two consecutive edges intersect. That way, a matching corresponds to a 0-1-string without consecutive 1s, while a star is a 0-1-string with exactly two consecutive 1s. Consequently,

$$\mathrm{M}(\mathsf{P}_n) = \{x \in \mathbb{R}^n_{\geq 0} : x_i + x_{i+1} \leq 1, \ \forall i \in [n-1]\}$$

and the number of vertices (i.e., the number of matchings of P_n) is the $(n+2)$nd Fibonacci number F_{n+2}, since the number of 0-1-strings of length n obeys the same recursion as the Fibonacci numbers. For this reason $\mathrm{M}(\mathsf{P}_n)$ is also called "Fibonacci-polytope". It has been studied in a general context in [Ris05]. We will revisit this polytope in Section 7.5.

iii) Consider the tree $T = (V, \{e_1, e_2, e_3, f_1, f_2, f_3\})$ as drawn in Figure 7.1.

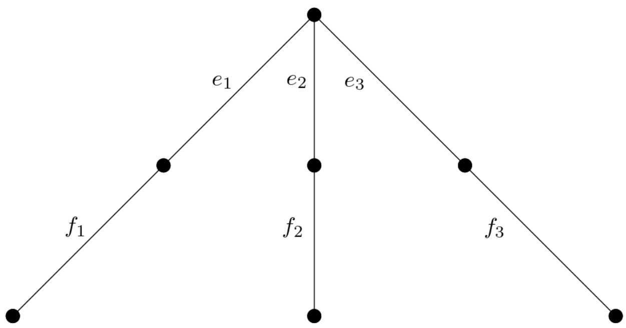


Figure 7.1: A tree such that neither its matching polytope, nor its substar polytope can be expressed as the chain polytope of some poset.

Suppose $\mathrm{M}(T)$ is the chain polytope of some poset $\mathcal{P}$. Since $\mathrm{M}(T) = \mathrm{Stab}(\mathsf{L}T)$ and $\mathrm{C}(\mathcal{P}) = \mathrm{Stab}(\mathsf{Comp}(\mathcal{P}))$, it follows from Theorem 7.2.1 that $\mathsf{L}T = \mathsf{Comp}(\mathcal{P})$. So we may assume that the ground set of $\mathcal{P}$ is E and we have that e_1, e_2, e_3 are a chain in $\mathcal{P}$, since they form a clique in $\mathsf{L}T$. Without loss of generality let $e_1 < e_2 < e_3$ in $\mathcal{P}$. Since e_2 and f_2 intersect in T, they are comparable in $\mathcal{P}$. If, for instance, $f_2 < e_2$ it follows that $f_2 < e_3$. But since f_2 and e_3 do not intersect in T they need to be incomparable in $\mathcal{P}$, a contradiction. The case $e_2 < f_2$ is treated analogously.

Moreover $\mathrm{AM}(T) = \mathrm{Stab}(\overline{\mathsf{L}T})$ is not a chain polytope either. If it was, we would have by the same argument that we applied above $\mathsf{L}T = \overline{\mathsf{Comp}(\mathcal{P})}$ for some poset $\mathcal{P}$. In words, the line graph of T is the incomparability graph of $\mathcal{P}$. Again, we identify the ground set of $\mathcal{P}$ with E. Then a matching in T is a chain in $\mathcal{P}$. So without loss of generality, we may assume $f_1 < f_2 < f_3$. Also $\{e_1, f_2, f_3\}$ is a matching in T and therefore a chain in $\mathcal{P}$. Since e_1 is incomparable to f_1 and $f_2 < f_3$, we must have $e_1 < f_2 < f_3$. In the same way, we obtain $f_1 < f_2 < e_3$. From this, we deduce $e_1 < e_3$, a contradiction, since e_1 and e_3 meet and are therefore incomparable.

This example shows that neither the class of matching polytopes, nor the class of substar polytopes is contained in the class of chain polytopes. $\diamond$

In order to compute the volume of $\mathrm{M}(G)$, we follow an approach by Liu in [Liu12]. A matching M of a forest G is called *almost perfect*, if it covers any non-leaf vertex of G and also contains all isolated edges. In the next section, we will construct an almost perfect matching for a forest explicitly. However, in general, a bipartite graph does not need to contain an almost perfect matching as the complete bipartite graph of size $(2, 3)$ shows.

For a forest G, a map m that assigns to each subforest H of G a matching $m(H)$ which is almost perfect for H is called a *matching choice function (MCF)* of G. Given a forest G with n edges and a matching choice function m of G, Liu defines a *standard labeling* of G as a bijection $\sigma : E \to [n]$ with the property that

$$\sigma^{-1}(i) \in m(G \setminus \{\sigma^{-1}(1), ..., \sigma^{-1}(i-1)\}).$$

The set of all standard labelings of G with respect to m is denoted by $\mathrm{St}(G, m)$ and we write $\mathrm{st}(G, m) = |\mathrm{St}(G, m)|$ for the number of standard labelings of G with respect to m.

It is convenient to denote a standard labeling in inline notation, i.e., we write $\sigma = (e_1, ..., e_n)$, where $e_i = \sigma^{-1}(i)$. We think of σ as a process of deleting the edges of G, where at each step, where are only allowed to choose an edge from the current almost perfect matching $m(G \setminus \{e_1, ..., e_{i-1}\})$ (see also Figure 7.2).

The following result is due to Liu [Liu12].

Lemma 7.3.3 (Liu). *Let G be a forest with n edges and let m be an MCF of G. Then,*

$$\mathrm{vol}(\mathrm{M}(G)) = \frac{1}{n!}\mathrm{st}(G, m).$$

In particular, it follows from the lemma that $\mathrm{st}(G, m)$ does not depend on the MCF m. Therefore, we may write $\mathrm{st}(G) = \mathrm{st}(G, m) = n!\mathrm{vol}(\mathrm{M}(G))$.

Liu's proof of Lemma 7.3.3 uses a recursive argument in order to characterize the volume of $\mathrm{M}(G)$ as a functional of G. Here, we shall give a proof of the lemma that highlights the special geometric properties of $\mathrm{M}(G)$ using an alternative recursion.

Geometric proof of Lemma 7.3.3. We prove the claim by induction. For $n = 1$, the statement is clearly true. Let $n > 1$ and consider $M = m(G)$, the almost perfect matching of G chosen by the MCF m. Then, $\mathbb{1}[M]$ is a vertex of $\mathrm{M}(G)$. Since M covers any non-leaf vertex of G and contains any isolated edge, we have $|M \cap S| = 1$, for all stars $S \subseteq E$. In view of (7.5), this means that the vertex $\mathbb{1}[M]$ is contained in all facets of $\mathrm{M}(G)$ except for the coordinate facets of the form $F_e = \{x \in \mathrm{M}(G) : x_e = 0\}$ for $e \in M$. The distance of $\mathbb{1}[M]$ to F_e is 1. We observe that $F_e = \mathrm{M}(G \setminus \{e\})$. Therefore, by the pyramid formula and our induction hypothesis, we have

$$\mathrm{vol}(\mathrm{M}(G)) = \frac{1}{n} \sum_{e \in M} \mathrm{vol}(\mathrm{M}(G \setminus \{e\})) = \frac{1}{n!} \sum_{e \in M} \mathrm{st}(G \setminus \{e\}, m). \tag{7.7}$$

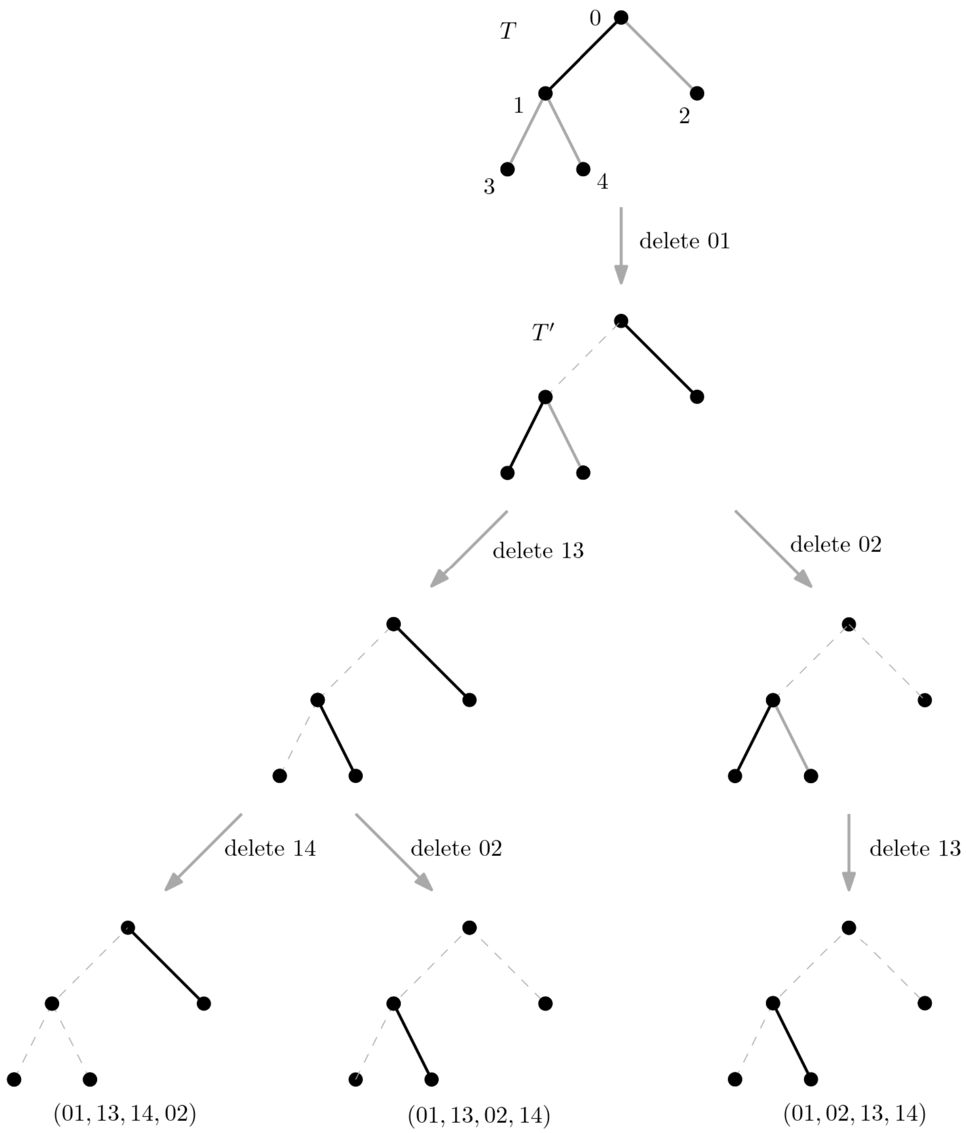


Figure 7.2: Illustrations of the three standard labelings of the given tree. At the top, we see a tree T with its vertices labeled from 0 to 4. The almost perfect matching is depicted in black, while the remaining edges are grey. The tree T' below is obtained from T by deleting the only black edge in its almost perfect matching. In the third row, we have two different trees obtained from T' by deleting one of the edges in its almost perfect matching. Continuing in this manner, we can delete all edges of T in three different ways. These ways are the three standard labelings of T, they are listed at the bottom of the figure.

But on the other hand, any $\sigma = (e_1, ..., e_n) \in \mathrm{St}(G, m)$ has to start with an edge $e_1 \in M$. By definition, the sequence $(e_2, ..., e_n)$ is then an element of $\mathrm{St}(G \setminus \{e_1\}, m)$. Thus,

$$\mathrm{st}(G, m) = \sum_{e \in M} \mathrm{st}(G \setminus \{e\}, m).$$

We obtain, by rearrangig (7.7),

$$n! \operatorname{vol}(\mathrm{M}(G)) = \sum_{e \in M} \mathrm{st}(G \setminus \{e\}, m) = \mathrm{st}(G, m)$$

and the proof is finished. □

The recursive application of the pyramid formula in the above proof corresponds to a decomposition of $\mathrm{M}(G)$ into simplices. In fact, for any standard labeling $\sigma = (e_1, ..., e_n) \in \mathrm{St}(G, m)$, there exists a lattice n-simplex

$$S_\sigma = \mathrm{conv}\{\mathbb{1}[m(G \setminus \{e_1, ..., e_i\})] : 0 \leq i \leq n\}. \tag{7.8}$$

Moreover, S_σ has integral vertices and its volume is $1/n!$. We call such simplices *unimodular*.

If we choose m appropriately, the simplices $(S_\sigma)_{\sigma \in \mathrm{St}(G,m)}$ form a triangulation of $\mathrm{M}(G)$. We will explain these triangulations in detail in Section 7.4. Before that, let us introduce a dual concept to standard labelings for the substar polytope.

A star $S \subseteq E$ is called *almost perfect*, if it covers a leaf of G. A *star choice function (SCF)* of G is a map s that assigns to each subgraph H of G an almost perfect star $s(H)$ and a *co-standard labeling* of G with respect to s is a bijection $\pi : E \to [n]$ with the property that

$$\pi^{-1}(i) \in s(G \setminus \{\pi^{-1}(1), ..., \pi^{-1}(i-1)\}).$$

The set of co-standard labelings of G with respect to s is denoted by $\mathrm{Co}(G, s)$ and we write $\mathrm{co}(G, s) = |\mathrm{Co}(G, s)|$. As for the standard labelings we may represent $\pi \in \mathrm{Co}(G, s)$ as $\pi = (e_1, ..., e_n)$, where $e_i = \pi^{-1}(i)$, and we think of π as a process of deleting the edges of G, where at each step, we may only delete an edge from the substar $s(G \setminus \{\pi^{-1}(1), ..., \pi^{-1}(i-1)\})$ (see also Figure 7.3).

An almost perfect star $S \subseteq E$ has the property that it intersects every maximal matching M of G; If a matching M does not intersect S, we could just add the edge to the leaf covered by S to M, so M was not maximal. In view of (7.6), we see that $\mathbb{1}[S]$ is contained in every facet of $\mathrm{AM}(G)$ except for $\{x \in \mathrm{AM}(G) : x_e = 0\} = \mathrm{AM}(G \setminus \{e\})$, $e \in S$. Thus, we may follow the lines of the proof of Lemma 7.3.3 and obtain a combinatorial description of $\operatorname{vol}(\mathrm{AM}(G))$.

Lemma 7.3.4. *Let G be a forest with n edges let s be an SCF of G. Then,*

$$\operatorname{vol}(\mathrm{AM}(G)) = \frac{1}{n!}\mathrm{co}(G, s).$$

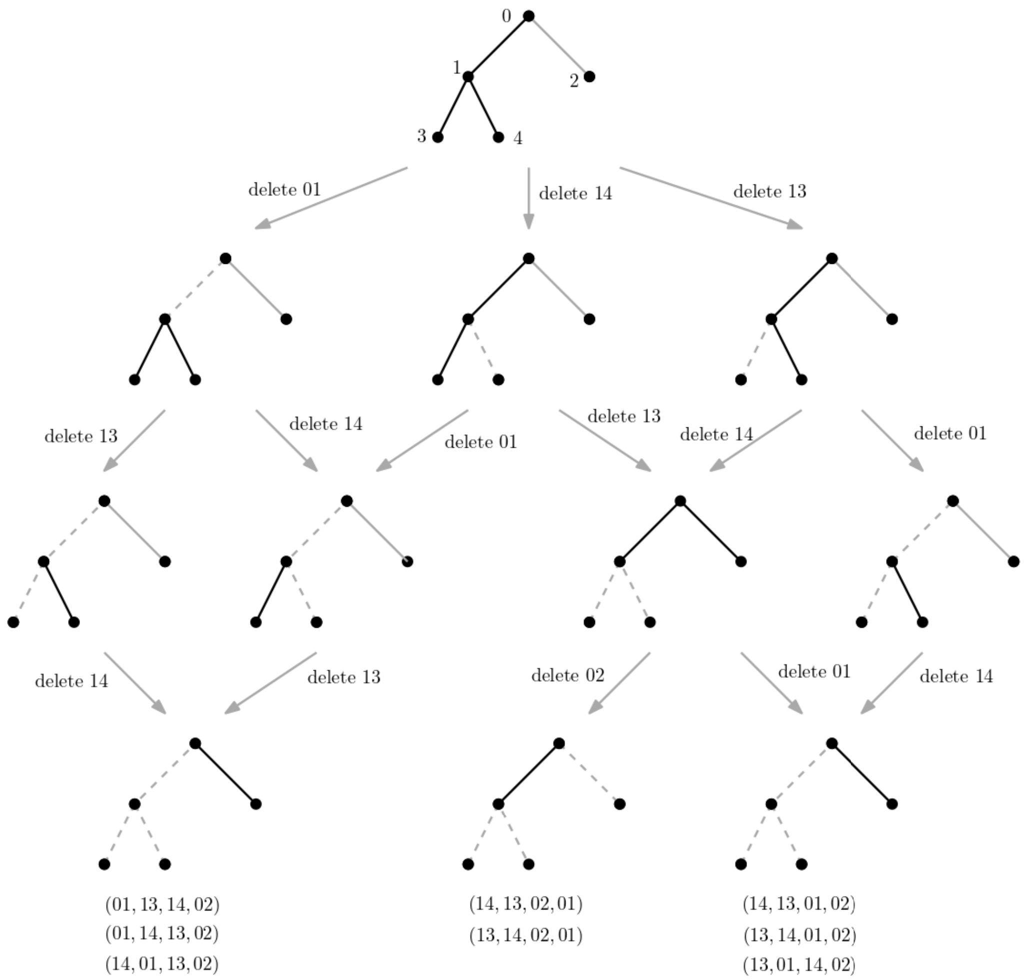


Figure 7.3: Illustrations of the eight co-standard labelings of the tree from Figure 7.2.

Again, we find that $\mathrm{co}(G, s)$ is independent of the SCF s, which justifies the notation $\mathrm{co}(G) = \mathrm{co}(G, s) = n!\mathrm{vol}(\mathrm{AM}(G))$. Similarly to (7.8), we obtain for any $\pi = (e_1, ..., e_n) \in \mathrm{Co}(G, s)$ a unimodular simplex

$$T_\pi = \mathrm{conv}\{\mathbb{1}[s(G \setminus \{e_1, ..., e_i\})] : 0 \leq i \leq n\} \tag{7.9}$$

and the simplices $(T_\pi)_{\pi \in \mathrm{Co}(G,s)}$ form a decomposition of $\mathrm{AM}(G)$.

7.4 Explicit Pulling Triangulations of M(G) and AM(G)

In this section, we will construct a matching choice function m and a star choice function s such that the decompostions $(S_\sigma)_{\sigma \in \mathrm{St}(G,m)}$ and $(T_\pi)_{\pi \in \mathrm{Co}(G,s)}$ as defined in (7.8) and (7.9) are in fact triangulations of $\mathrm{M}(G)$ and $\mathrm{AM}(G)$, respectively.

For this we need the notion of a pulling triangulation that goes back to Stanley [Sta80].

Let us recall the definition; In general, we consider a polytope $P \subseteq \mathbb{R}^n$ and we fix an enumeration $\mathrm{vert}(P) = \{v_1, ..., v_m\}$ of its vertex set. For a non-empty face $F \subseteq P$, let $\max(F) = \mathrm{argmax}\{i : v_i \in F\}$. A chain of faces

$$\mathcal{F}\colon F_0 \subsetneq ... \subsetneq F_n = P$$

is called a *flag* of P and $\mathcal{F}$ is called a *full flag*, if $\max(F_i) \notin F_{i-1}$, for all $i \in [n]$. For a full flag $\mathcal{F}$, one defines the simplex

$$S_{\mathcal{F}} = \mathrm{conv}\{\max(F_i) : 0 \leq i \leq n\}.$$

It has been observed in [Sta80, Lemma 1.1] that the simplices $(S_{\mathcal{F}})_{\mathcal{F}}$, where $\mathcal{F}$ ranges over all full flags of P, form a triangulation of P. This triangulation is referred to as the *pulling triangulation* of P with respect to the vertex enumeration $(v_1, ..., v_m)$.

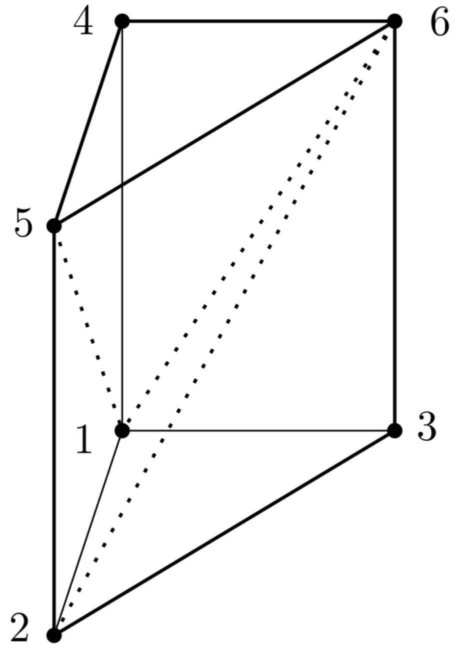


Figure 7.4: Pulling Triangulation of a prism. The 3-simplices in the triangulation are given by the convex hulls of the vertices labeled by $\{6, 3, 2, 1\}$, $\{6, 5, 2, 1\}$ and $\{6, 5, 4, 1\}$.

Remark 7.4.1. Intuitively, the decompositions $(S_\sigma)_{\sigma \in \mathrm{St}(G,m)}$ and $(T_\pi)_{\pi \in \mathrm{Co}(G,s)}$ of $\mathrm{M}(G)$ and $\mathrm{AM}(G)$ are closely related to a pulling triangulation: A simplex S_σ, where $\sigma = (e_1, ..., e_n)$, as defined in (7.8) corresponds to the flag

$$\{0\} \subsetneq \mathrm{M}(G \setminus \{e_1, ..., e_{n-1}\}) \subsetneq ... \subsetneq \mathrm{M}(G \setminus \{e_1\}) \subsetneq \mathrm{M}(G).$$

Out of each face $\mathrm{M}(H)$ in this flag, we pick the vertex $\mathbb{1}[m(H)]$. What keeps $(S_\sigma)_{\sigma \in \mathrm{St}(G,m)}$ from being a pulling triangulation in general, is the circumstance that the rule by which the vertex $\mathbb{1}[m(H)]$ is chosen from $\mathrm{M}(H)$ might not be "consistent". For instance, if $H' \subseteq H \subseteq G$ are subforests of G and $m(H') \neq m(H)$, but $m(H') \subseteq H$ (i.e., it could have been chosen as the almost perfect matching of H), it is possible that there are simplices in the decomposition $(S_\sigma)_{\sigma\in\mathrm{St}(G,m)}$ that do not intersect in a common face (cf. Figure 7.5). $\diamond$

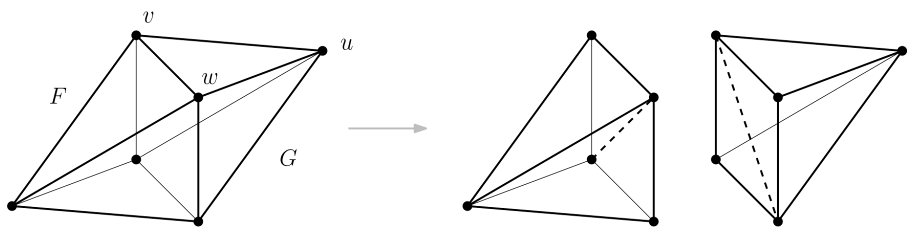


Figure 7.5: Let us consider F and G as facets of a hypothetical 4-dimensional matching polytope $\mathrm{M}(G)$ such that $F = \mathrm{M}(H)$ and $G = M(H')$ for two subforests $H, H' \subseteq G$. Suppose our MCF m is such that $\mathbb{1}[m(H)] = w$, $\mathbb{1}[m(H')] = u$ and $\mathbb{1}[m(H \cap H')] = v$. Then the facets F and G are decomposed as it is drawn on the right, so they do not intersect in a common face.

Our goal is to identify the decompositions of the previous section as pulling triangulations for suitable MCFs and SCFs. Before we do so, we need one more piece of terminology. A *shelling* of a forest $G = (V, E)$ is an injective map $\omega : V \to \mathbb{N}$ with the property that

$$\left|\{x \in N(v) : \omega(x) < \omega(v)\}\right| \leq 1, \quad \forall v \in V,$$

i.e., every vertex has at most one neighbor with a smaller number. By applying for instance a breadth first search on each connected component, one can see that any forest admits a shelling. Shellings of trees have been studied in the context of anti-matroids [LKS12, Ch. III]. The next lemma contains two important observations.

Lemma 7.4.2. *Let $G = (V, E)$ be a forest with a shelling ω. For $e = \{u, v\} \in E$ with $\omega(u) < \omega(v)$, let $c_e = 2^{-\omega(v)}$ and consider $c = (c_e)_{e\in E} \in \mathbb{R}^E_{\geq 0}$, as well as $c^{-1} = (c_e^{-1})_{e\in E} \in \mathbb{R}^E_{\geq 0}$. Then,*

i) For every $v \in V$, there is at most one edge $e = \{v, u\} \in E$ with $\omega(v) > \omega(u)$.

ii) The maps $\{0,1\}^E \to \mathbb{R},\ x \mapsto \langle x, c\rangle$ and $\{0,1\}^E \to \mathbb{R},\ x \mapsto \langle x, c^{-1}\rangle$ are injective.

Proof. i) is a direct consequence of the definition of a shelling. For ii), we note that

$$\langle x, c\rangle = \sum_{e\colon x_e=1} 2^{-\max\omega(e)}. \tag{7.10}$$

Since the exponents on the right hand side are all distinct by i), this expression is the unique binary representation of some number. The statement for c^{-1} is proven in the same way. $\square$

The critical step for our pulling triangulations is the following theorem.

Theorem 7.4.3. *Let $G = (V, E)$ be a forest with n edges and a shelling ω. For $e = \{u, v\} \in E$ with $\omega(u) < \omega(v)$, let $c_e = 2^{-\omega(v)}$ and consider $c = (c_e)_{e \in E} \in \mathbb{R}^E_{\geq 0}$, as well as $c^{-1} = (c_e^{-1})_{e \in E} \in \mathbb{R}^E_{\geq 0}$. Then,*

i) There exists a unique vertex $\mathbb{1}[M]$ of $\mathrm{M}(G)$ that maximizes $\langle \cdot, c\rangle$. M is an almost perfect matching.

ii) There exists a unique vertex $\mathbb{1}[S]$ of $\mathrm{AM}(G)$ that maximizes $\langle \cdot, c^{-1}\rangle$. S is an almost perfect star.

Proof of Theorem 7.4.3. The uniqueness follows in both cases from Lemma 7.4.2.

i): Let M be a matching such that $\langle \mathbb{1}[M], c\rangle$ is maximal. Clearly, M is a maximal matching and as such it contains every isolated edge. Towards a contradiction, let us assume that there exists a non-leaf vertex $v_0 \in V$ which is not covered by M. Then, v_0 has a neighbor v_1 with $\omega(v_1) > \omega(v_0)$. Since M is maximal, v_1 must be covered by M, i.e., there exists an edge $\{v_1, v_2\} \in M$, $v_2 \neq v_0$. Since ω is a shelling, we have $\omega(v_2) > \omega(v_1)$. Now we can replace the edge $\{v_1, v_2\}$ by $\{v_0, v_1\}$ and obtain a matching M' with $\langle c, \mathbb{1}[M']\rangle > \langle c, \mathbb{1}[M]\rangle$.

ii): Clearly, S is a maximal substar, i.e., a star. Let v be the vertex of G for which ω is maximal. It follows from the shelling property that v is a leaf. In general, we have

$$\langle \mathbb{1}[S], c^{-1}\rangle = \sum_{e \in S} 2^{\max \omega(e)}.$$

Thus, the product is maximal, if and only if v is covered by S. Since v is a leaf, S is almost perfect. $\square$

In view of Lemma 7.4.2, we may order the vertices of $\mathrm{M}(G)$ and $\mathrm{AM}(G)$ by their products with c resp. c^{-1}. These orderings yield pulling triangulations $\mathcal{S}$ and $\mathcal{T}$, and we have:

Corollary 7.4.4. *Let G, c, c^{-1}, $\mathcal{S}$ and $\mathcal{T}$ be as above. For a subforest $H \subseteq G$, let*

$$\begin{aligned} m(H) &= \operatorname{argmax}\{\langle c, \mathbb{1}[M]\rangle : M \subseteq E \text{ matching in } H\} \quad \text{and} \\ s(H) &= \operatorname{argmax}\{\langle c^{-1}, \mathbb{1}[S]\rangle : S \subseteq E \text{ substar in } H\}. \end{aligned} \tag{7.11}$$

These are well-defined MCFs resp. SCFs and the simplices $(S_\sigma)_{\sigma \in \mathrm{St}(G,m)}$ and $(T_\pi)_{\pi \in \mathrm{Co}(G,s)}$ are the facets of $\mathcal{S}$ and $\mathcal{T}$.

Proof. We prove the statement for the matching polytope, the substar case follows the same lines. In order to see that m is well-defined it suffices to apply Theorem 7.4.3 to the subforest H with the shelling induced by ω. Moreover, for any $\sigma \in \mathrm{St}(G, m)$, the simplex S_σ arises as $S_\mathcal{F}$, where $\mathcal{F}$ is the flag defined in Remark 7.4.1. By our choice of m, $\mathcal{F}$ is indeed a full flag with respect to the ordering given by c. Therefore, S_σ is a facet of $\mathcal{S}$. Since we already know that the S_σ's decompose $\mathrm{M}(G)$, there cannot be any further facets in $\mathcal{S}$. $\square$

In the remainder of the chapter, unless otherwise stated, we will assume that our MCFs and SCFs are of the form (7.11). The following characterization of these choice functions follows directly from the interpretation of $\langle x, c\rangle$ as a binary number (cf. (7.10)).

Proposition 7.4.5. *In the setting of Corollary 7.4.4, we have*

i) $m(H)$ is the matching in H, obtained by greedily adding edges in order to maximize the cost function given by c.

ii) $s(H)$ is the star in H that covers the leaf v, which is maximal with respect to the shelling ω.

The matching and star choice functions of the trees in Figures 7.2 and 7.3 are the ones given by (7.11). In the remainder of the chapter, we will consider *shelled forests*, by which we mean pairs (G, ω), where G is a forest and ω is a shelling of G. Since the actual choice of ω does not affect our arguments, we will oftentimes omit it in the notation.

Remark 7.4.6. i) Reflexive anti-blocking polytopes P are *compressed*, i.e., any pulling triangulation of P is unimodular [Sul05], where a triangulation is called unimodular, if the edges at any n-simplex in the triangulation form a basis of Z^n. Apart from matching and substar polytopes, an explicit description of a pulling triangulation is known for chain polytopes [Sta86]. In Section 7.6, we will compare the triangulation of the chain polytope to the triangulation of the substar polytope.

ii) The particular structure of the forest G and its line graph has enabled us to get a good grip on a pulling triangulation of $\mathrm{M}(G)$. For the construction it was crucial that G possesses an almost perfect matching, i.e., a matching M that intersects every star. On the level of the line graph $\mathsf{L}G$, this is equivalent to saying that there exists a stable set $M \subseteq \mathsf{L}G$, which meets every maximal clique of $\mathsf{L}G$. In general, a graph G in which every vertex induced subgraph satisfies this property is called *strongly perfect*. Every strongly perfect graph is perfect, but the converse is not true, as can be seen by considering the complement graph of a cycle $\overline{\mathsf{C}_{2n}}$, where $n \geq 3$. It is also important to note that C_{2n} itself is strongly perfect and therefore the class of all strongly perfect graphs is not closed under complements.

Strongly perfect graphs have been considered in [BD84]. In [Rav99], many examples of subclasses of strongly perfect graphs are given. Among them, one rediscovers the line graphs of forests [Rav99, Fact 8] and their complements [Rav99, Fact 18]. Indeed, there are further examples of graph classes such that both G and $\overline{G}$ are strongly perfect, such as the class of *triangulated (or chordal) graphs* [Rav99, Facts 3 and 16]. It is an interesting direction for future research to investigate the (Mahler) volume of the stable set polytopes of those graphs.

7.5 Combinatorial Bounds on the Mahler Volume

In this section, we use the combinatorial interpretation of the volume that we obtained from Lemma 7.3.3 and 7.3.4 to derive an upper bound on the Mahler volume of $\mathrm{M}(G)$. Let us note that due to the correspondence between unconditional and anti-blocking bodies, the Blaschke-Santaló inequality translates to the realm of anti-blocking bodies as

$$\mathrm{vol}(K)\mathrm{vol}(\mathrm{A}K) \leq \mathrm{vol}(B^n \cap \mathbb{R}^n_{\geq 0})^2 = \left(\frac{\pi}{2} + o(1)\right)^n \frac{1}{n!}, \tag{7.12}$$

for any anti-blocking body $K \subseteq \mathbb{R}^n_{\geq 0}$, as $\mathrm{vol}(B^n) = \pi^{n/2}/\Gamma(n/2+1)$, where Γ denotes Euler's Gamma-function. The main result of this section is Theorem 7.5.8, which improves the Blaschke-Santaló inequality for forests with sufficiently many leaves. Before we formulate and prove the theorem, let us consider the following example, which illustrates the interplay of standard and co-standard labelings.

Example 7.5.1. Consider the "forest of stars" G as drawn in Figure 7.6. In a subgraph

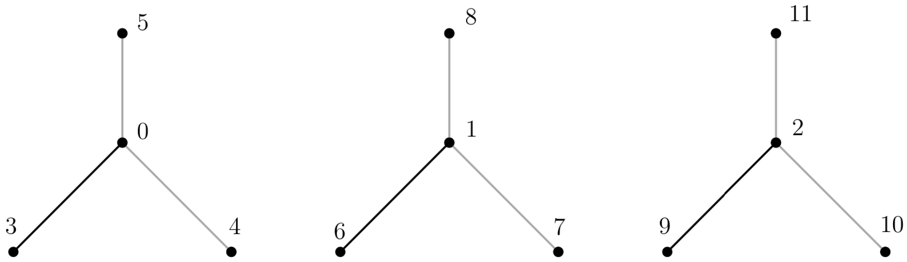


Figure 7.6: A forest consisting of three disjoint stars. Its almost perfect matching is drawn in black, while the remaining edges are grey. Its almost perfect star is the one on the right, since it covers the leaf with the highest label.

$H \subseteq G$, exactly one edge for each of the three stars is present in $m(H)$, except if we deleted all edges of one star already. A standard labeling may therefore be identified with a word of length 9 over the alphabet $\{0, 1, 2\}$ in which every letter appears exactly 3 times. The i-th letter in the word signifies the star out of which the i-th edge in the standard labeling is deleted. We obtain that there are $\frac{9!}{3!3!3!}$ standard labelings of G.

On the other hand, given the way the leaves are labeled in Figure 7.6, a co-standard labeling will first delete all edges at 2 in an arbitrary order, then all edges at 1 and finally it deletes the edges at 0. Thus, it is made up of 3 independent permutations, one for the edges of each star. So there are 3!3!3! co-standard labelings. Thus,

$$\mathrm{vol}(\mathrm{M}(G))\mathrm{vol}(\mathrm{AM}(G)) = \frac{\mathrm{st}(G)\mathrm{co}(G)}{(9!)^2} = \frac{1}{9!}.$$

This implies that the unconditional body $\mathrm{UM}(G)$ satisfies $\mathrm{vol}(\mathrm{UM}(G))\mathrm{vol}((\mathrm{UM}(G))^\star) = 4^9/9!$, i.e., the Mahler conjecture holds with equality for $\mathrm{UM}(G)$. This is not a coincidence; Since G is a disjoint union of three stars, its matching polytope $\mathrm{M}(G)$ is the product of three $\mathrm{M}(\mathsf{S}_3)$'s, i.e., three 3-simplices (cf. Example 7.3.2). This translates to the unconditional body $\mathrm{UM}(G)$; it is a product of three 3-dimensional cross-polytopes and therefore a Hanner-polytope. $\diamond$

In this example, the co-standard labelings acted as "local" permutations at each non-leaf vertex, while the standard labelings where unable to distinguish between the leaves, but only between the stars. They contributed the "global" part.

In the following we want to extend this idea to general forests. For this we need the notion of a local edge ordering: Let $\sigma = (e_1, ..., e_n)$ be a permutation of the edges of a forest G and let v be a vertex of V. Then σ induces an ordering of the edges incident to v in a natural way; we read them off from left to right in σ. This ordering of the edges at v is called the *local edge ordering (LEO)* at v given by σ and we denote it by $\sigma^{(v)}$.

Our goal is to establish the correspondence between a co-standard labeling σ of a forest G and its LEOs $(\sigma^{(v)})_v$, where v ranges over all non-leaf vertices of G. Unfortunately, the map $\sigma \mapsto (\sigma^{(v)})_v$ is in general not surjective, as the following example shows.

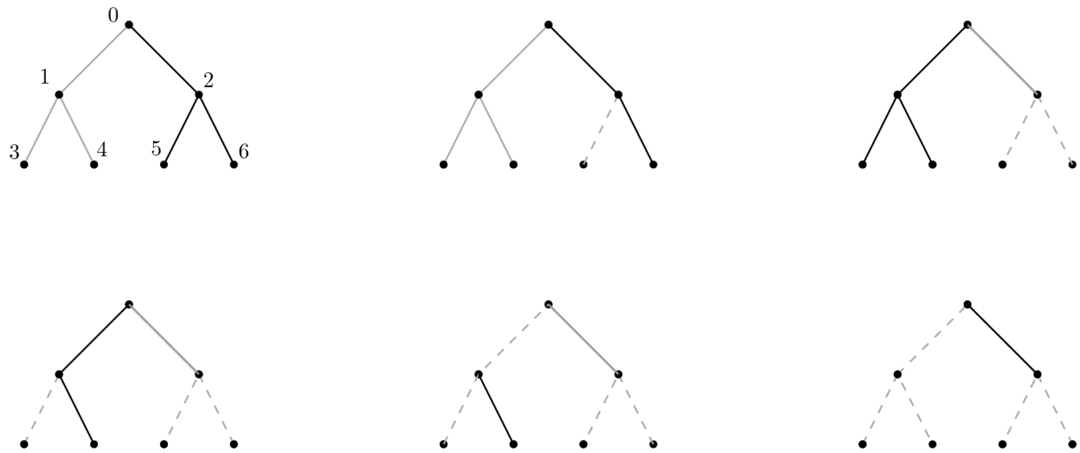
Example 7.5.2. Consider the co-standard labeling $\pi = (01, 13, 14, 02)$ of the tree T in Figure 7.3. It induces the LEOs $\pi^{(0)} = (01, 02)$ and $\pi^{(1)} = (01, 13, 14)$. But in contrast to the forest of stars in Example 7.5.1, not every pair of LEOs can be realized by a co-standard labeling π. For instance, the pair $(\pi^{(0)}, \pi^{(1)})$, where $\pi^{(0)} = (02, 01)$ and $\pi^{(1)} = (01, 13, 14)$, cannot be realized. If there was a co-standard labeling π with these LEOs, the first edge of π must be 01, since the first edge in π is chosen from the star around 1 (cf. Figure 7.3). But then the first edge in $\pi^{(0)}$ is 01 as well. $\diamond$

In order to fully understand the co-standard labelings of G, we will work with orientations of G. First, for a forest $G = (V, E)$ with a shelling ω, we fix the canonical orientation

$$\mathcal{O}_c = \big\{(x, y) : \{x, y\} \in E \text{ and } \omega(x) > \omega(y)\big\}.$$

We say that a vertex v is a *root* of G if it is minimal with respect to ω among all vertices of its connected component in G. From now on, we call a vertex v a *leaf* of G if it has $d_{\text{in}}(v, \mathcal{O}_c) = 0$, where $d_{\text{in}}(v, \mathcal{O}_c)$ is the in-degree of v with respect to $\mathcal{O}_c$. Note that these are precisely the leaves in the traditional sense except for the roots if they happen to be of degree 1. If a vertex v is not a root, there exists a unique neighbor w of v with $\omega(w) < \omega(v)$. This vertex w is called the *parent* of v and we denote it by $w = \mathrm{parent}(v)$. Neighbors of v that are not the parent of v are referred to as *children* of v. In the literature, an orientation of a tree such that every edge is oriented away from a root, is also called an *arborescence*.

Next, we introduce a second orientation of G that arises from a co-standard labeling $\sigma : E \to [n]$. For an edge $e = \{x, y\} \in E$, let $i \in [n]$ be such that $\sigma(e) = i$. We orient the edge e from x to y, if $S = s(G \setminus \sigma^{-1}([i-1]))$ is a star around y (we also say that y is the *center* of S). And we orient e from y to x if S is a star around x. Note that if S contains only a single edge, S is technically a star around both x and y. In this case we shall consider S as a star around y, if and only if $\omega(y) < \omega(x)$. The orientation that results this way is denoted by $\mathcal{O}_\sigma$.



$\sigma = (25, 26, 13, 01, 14, 02)$

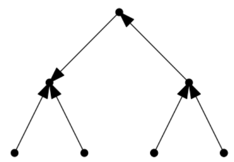
$\Longrightarrow \quad \mathcal{O} =$ 

Figure 7.7: The figure illustrates another co-standard labeling of a tree. Note that the edge 01 is chosen at a time when the almost perfect star of the subgraph is $\{01, 14\}$. This star is centered at 1, so 01 is oriented towards 1. On the other hand, 02 is chosen as the last edge. The almost perfect star of the subgraph at that time is just $\{02\}$ itself, which is centered at 0 per convention. Thus, 02 is directed towards 0.

Remark 7.5.3. Recall that the star $s(G \setminus \sigma^{-1}([i-1]))$ is the star whose center is the the parent w of the largest leaf v in $G \setminus \sigma^{-1}([i-1])$. This parent may itself have a parent x and the edge $e = \{w, x\}$ could be chosen for the i-th place of σ. We then say that e *is chosen as an edge at* w. If this happens, the orientation of e in $\mathcal{O}_\sigma$ is reverse to the canonical orientation $\mathcal{O}_c$. An example is given in Figure 7.7. $\diamond$

The orientations $\mathcal{O}_\sigma$, $\sigma \in \mathrm{Co}(G, s)$, satisfy the following properties.

Lemma 7.5.4. *Let G be a shelled forest and let σ be a co-standard labeling of G. Consider the orientation $\mathcal{O}_\sigma$ associated to σ.*

i) A leaf of G has no ingoing edge in $\mathcal{O}_\sigma$.

ii) If an edge $e = (x, y) \in \mathcal{O}_c$ is reversed in $\mathcal{O}_\sigma$, then e was not the last edge incident to x in σ.

iii) Let v be neither a leaf nor a root. Let w be its parent. Then, $(w, v) \in \mathcal{O}_\sigma$ implies $(x, v) \in \mathcal{O}_\sigma$ for some $x \neq v$.

Proof. i): Let v be a leaf and let $e = \{v, w\}$ be the edge that is incident to v. Suppose $(w, v) \in \mathcal{O}_\sigma$. Then e is chosen as an edge at v. In particular, there is a subgraph H of G such that $s(H)$ is a star whose center is v. Such a star can contain no other edge than e. But by our convention it is then a star with center w, since $\omega(w) < \omega(v)$.

ii): Towards a contradiction, suppose that e was the last edge in σ that is incident to x. Let $i \in [n]$ with $\sigma(e) = i$. Then, x is a leaf of $G \setminus \sigma^{-1}([i-1])$ and we have $e \in S = s(G \setminus \sigma^{-1}([i-1]))$. This implies that S is a substar with center y. So e is chosen as an edge at y. But then it is not reversed in $\mathcal{O}_\sigma$.

iii): Assume that such a vertex x does not exist. Let $i = \sigma(\{v, w\})$. Then, $s(G \setminus \sigma^{-1}[i-1])$ is a star with center v. Since v is not a leaf, there exist edges $e_j = \{v, y_i\} \neq \{v, w\}$, $j \in [k]$ for some $k > 0$, in $G \setminus \sigma^{-1}[i-1]$. Since ω is a shelling and w is the parent of v, we have $\omega(y_j) > \omega(v)$, for all j. By our assumption, we have $(v, y_j) \in \mathcal{O}_\sigma$, for all j, so e_j is reversed in $\mathcal{O}_\sigma$. But since all e_j's are chosen after $\{v, w\}$, one of them will be the last edge incident to x which is deleted by σ. This contradicts ii). □

Let G be a forest with an orientation $\mathcal{O}$ and let σ be a permutation of its edges. Similar to the undirected case we define a *directed local edge ordering (directed LEO)* of σ at v as the ordering of the ingoing edges at v as they appear in σ. This ordering is denoted by $\sigma^{(v,\mathcal{O})}$. If the orientation is understood, we simply write $\sigma^{(v)} = \sigma^{(v,\mathcal{O})}$. As from now on we are only concerned with directed LEOs, this notation will cause no misconceptions with the undirected case.

For a forest $G = (V, E)$, let V° denote its non-leaf vertices. Consider an orientation $\mathcal{O}$ of G and a set of permutations $(\sigma^{(v)})_{v \in V^\circ}$ of the ingoing edges of G at v. We say that the tuple $(\mathcal{O}, (\sigma^{(v)})_{v \in V^\circ})$ is *admissible*, if for any $v \in V$ such that $(\mathrm{parent}(v), v) \in \mathcal{O}$, we have that $\{\mathrm{parent}(v), v\}$ is not the last edge in $\sigma^{(v)}$. Note that this implies in particular that v is not a leaf.

Theorem 7.5.5. *Let G be a shelled forest and let V° be the set of non-leaf vertices of G. Then the map*

$$\begin{aligned} F\colon \mathrm{Co}(G, s) &\to \left\{(\mathcal{O}, (\pi^{(v)})_{v \in V^\circ}) \text{ admissible tuple for } G\right\} \\ \sigma &\mapsto (\mathcal{O}_\sigma, (\sigma^{(v)})_{v \in V^\circ}) \end{aligned}$$

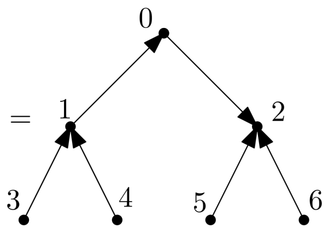
is a bijection.

Proof. Lemma 7.5.4 yields that for any co-standard labeling σ, the tuple $(\mathcal{O}_\sigma, (\sigma^{(v)})_{v \in V^\circ})$ is admissible. So the map is well-defined. Conversely, let $(\mathcal{O}, (\pi^{(v)})_{v \in V^\circ})$ be an arbitrary admissible tuple. We will construct a unique co-standard labeling from that tuple.

Let e_1 be the first edge in $\pi^{(v)}$, where v is the parent of the largest leaf w of G. By the definition of an admissible tuple, the edge $\{v, w\}$ must be oriented towards v, so $\pi^{(v)}$ is non-empty and e_1 indeed exists. We delete e_1 from $\pi^{(v)}$ and G and continue. Suppose the first $i-1$ edges of σ are already fixed and deleted from G and their respective $\pi^{(v)}$'s. Let H be the subgraph of G given by the remaining edges and let $v \in H$ be the largest vertex therein. Since the vertices are labeled with respect to a shelling, v is a leaf of H. Let $w = \mathrm{parent}(v)$ be its parent. Then we must have $(v, w) \in \mathcal{O}$. Otherwise, $e = \{v, w\}$ was ingoing at v and since v is a leaf, e would be the last edge in $\pi^{(v)}$, a contradiction to the fact that $(\mathcal{O}, (\pi^{(v)})_{v \in V^\circ})$ is admissible. Thus, we did not delete all edges of $\pi^{(w)}$ yet and we may choose the first of its remaining edges as e_i. The resulting labeling $\sigma = (e_1, ..., e_n)$ is co-standard since in every step we pick an edge from the almost perfect star of the current subgraph. By construction we have $F(\sigma) = (\mathcal{O}, (\pi^{(v)})_{v \in V^\circ})$ and we can reconstruct σ uniquely from $(\mathcal{O}_\sigma, (\sigma^{(v)})_{v \in V^\circ})$ with the above construction. Thus, we have found an inverse map to F. $\square$

Theorem 7.5.5 may be regarded as an abstraction of the idea developed for the forest of stars in Example 7.5.1. Its benefit is that we are no longer faced with the situation that one edge in a co-standard labeling influences two LEOs (cf. Example 7.5.2), since in the directed case, an edge only contributes to one of the directed LEOs. Before we come to the Mahler volume of $\mathrm{M}(G)$, let us illustrate the construction in the proof of Theorem 7.5.5.

Example 7.5.6. Consider the tree T from Figure 7.7 along with the admissible tuple presented in Figure 7.8. Let σ be a co-standard labeling that yields this tuple. The almost

$\mathcal{O} =$ 

$\sigma^{(0)} = (01)$

$\sigma^{(1)} = (13, 14)$

$\sigma^{(2)} = (26, 02, 25)$

Figure 7.8: Another admissible tuple for the tree in Figure 7.7.

perfect star of T is $\{02, 25, 26\}$, it is centered around 2.. Thus, the first edge of σ must be chosen from among those edges. We take a look into $\sigma^{(2)}$ and see that 26 is the first edge in $\sigma^{(2)}$. So it must be the first edge in σ as well. We delete this edge from T and $\sigma^{(2)}$. In order to find the next edge in σ we consider the subtree $T' = T \setminus \{26\}$. Its almost perfect star is still centered around 2, since 25 is the edge to the largest leaf. The next edge in $\sigma^{(2)}$ is 02 and so it must be the second edge in σ. Continuing in this manner, we see that the only co-standard labeling that can be reconstructed from the given admissible tuple is $\sigma = (26, 02, 25, 13, 14, 01)$. $\diamond$

Remark 7.5.7. If we draw a forest in such a way that each vertex v is drawn below its parent, an orientation $\mathcal{O}$ with the property

$$\forall v \in V. \quad (\mathrm{parent}(v), v) \in \mathcal{O} \implies \exists x \neq \mathrm{parent}(v).\ (x, v) \in \mathcal{O} \qquad \text{(NDF)}$$

can be thought of as an orientation, such that we do **n**ot see a **d**ownward **f**low as in Figure 7.9. We call such an orientation NDF.

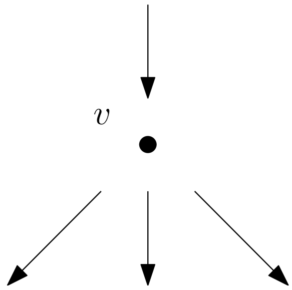


Figure 7.9: This situation is excluded by the property (NDF).

In particular, (NDF) implies that a leaf has no ingoing edge in $\mathcal{O}$. NDF orientations will play a key role in our estimate of the Mahler volume. $\diamond$

With the understanding of co-standard labelings from Theorem 7.5.5, we can prove an upper bound on the Mahler volume of $\mathrm{M}(G)$.

Theorem 7.5.8. *Let G be a shelled forest with n edges. Define $\Omega(G)$ as the number of NDF orientations $\mathcal{O}$ of G. Then,*

$$\mathrm{vol}(\mathrm{M}(G))\mathrm{vol}(\mathrm{AM}(G)) \leq \frac{\Omega(G)}{n!}.$$

Proof. Again, let V° denote the set of non-leaf vertices of G. Note that by Theorem 7.5.5 and Lemma 7.5.4, the orientation $\mathcal{O}$ in an admissible tuple of G is an NDF orientation. It, thus, follows from Theorem 7.5.5 that

$$\mathrm{co}(G) \leq \sum_{\mathcal{O}} \prod_{v\in V^\circ} d_{\mathrm{in}}(v,\mathcal{O})!, \tag{7.13}$$

where $\mathcal{O}$ ranges over all NDF orientations of G and $d_{\mathrm{in}}(v,\mathcal{O})$ denotes the in-degree of v with respect to $\mathcal{O}$.

For a fixed NDF orientation $\mathcal{O}$ and an edge $e \in E$, let $t(e)$ be the target vertex of e in $\mathcal{O}$. For a standard labeling $\sigma = (e_1, ..., e_n)$ of G, consider the sequence $(t(e_1), ..., t(e_n))$. This is a word over the alphabet V° in which every $v \in V^\circ$ appears exactly $d_{\mathrm{in}}(v,\mathcal{O})$ times. Since in a matching, a vertex is covered at most once, we can uniquely reconstruct $(e_1, ..., e_n)$ from $(t(e_1), ..., t(e_n))$. Thus, for any NDF orientation $\mathcal{O}$, we have

$$\begin{aligned}\mathrm{st}(G) &\leq \left|\{s \in (V^\circ)^n : \text{Every } v \in V^\circ \text{ appears } d_{\mathrm{in}}(v_i,\mathcal{O}) \text{ times in } s\}\right| \\ &= \frac{n!}{\prod_{v\in V^\circ} d_{\mathrm{in}}(v,\mathcal{O})!}\end{aligned}$$

Combining this with (7.13) gives

$$\mathrm{st}(G)\mathrm{co}(G) \leq \sum_{\mathcal{O}} \mathrm{st}(G) \prod_{v\in V^\circ} d_{\mathrm{in}}(v,\mathcal{O})! \leq \Omega(G)\, n!.$$

The claim follows from the Lemmas 7.3.3 and 7.3.4 that relate the (co-)standard labelings to the volume of $\mathrm{M}(G)$ resp. $\mathrm{AM}(G)$. $\square$

In an NDF orientation, the edges incident to a leaf have their leaf as a source (cf. Remark 7.5.7), so their orientation is fixed. If G has k connected components, then

$$\Omega(G) \leq 2^{|V^\circ|-k}. \tag{7.14}$$

For the forest of stars in Example 7.5.1 we thus have $\Omega(G) = 1$ and we have equality in Theorem 7.5.8. In general, the bound in Theorem 7.5.8 is particularly strong, if G has many leaves. For instance, we have

Corollary 7.5.9. *Let G be a forest with n edges such that any non-leaf vertex is the parent of at least 2 distinct vertices. Then,*

$$\mathrm{vol}(\mathrm{M}(G))\mathrm{vol}(\mathrm{AM}(G)) \leq \frac{\sqrt{2}^n}{n!}.$$

This improves (7.12), since $\sqrt{2} < \pi/2$.

Proof. For such a forest, one has $|V^\circ| \leq |V|/2 = \frac{n+k}{2}$, where k is the number of connected components of G. So the claim follows from combining Theorem 7.5.8 with (7.14). $\square$

But what is the maximum number of NDF orientations that an arbitrary forest with n edges can have? (7.14) gives an upper bound of 2^{n-2}, but this is not sharp. Instead we have the following bound.

Proposition 7.5.10. *For a shelled forest G with n edges, we have*

$$\Omega(G) \leq \Omega(\mathsf{P}_n) = F_{n+1},$$

where F_n is the n-th Fibonacci number and the shelling of P_n is given by the natural ordering of its vertices.

Proof. Let $\mathcal{O}$ be an NDF orientation of G. Consider a non-leaf vertex w of G such that all its children are leaves. If w is a root, then G contains a star as a connected component and there is only one NDF for this component, so the claim follows from induction. Let us therefore assume that $x = \mathrm{parent}(w)$ exists.

Case 1 $(w, x) \in \mathcal{O}$.

Then it follows directly that for any child v of w, the orientation $\mathcal{O} \setminus \{(v, w)\}$ is an NDF orientation of $G \setminus \{\{v, w\}\}$, since no configurations as in Figure 7.9 can arise by deleting $\{v, w\}$.

Case 2 $(x, w) \in \mathcal{O}$.

Then, for any child v of w, $\mathcal{O}' = \mathcal{O} \setminus \{(v,x),(x,w)\}$ is an NDF orientation for $G' = G \setminus \{\{v,x\},\{x,w\}\}$. To see this, we check that the defining property (NDF) is satisfied at any vertex of G'. The only vertices for which (NDF) might fail are v, w and x because the orientations at the remaining vertices are unaffected. But v was a leaf in G and is now isolated in G', so we do not need to take it into account. w has lost its parent edge, so (NDF) is trivially fulfilled for w in G'. As for x, let us assume without loss of generality that x has a parent y such that $(y,x) \in \mathcal{O}'$. It follows that $(y,x) \in \mathcal{O}$. Since $\mathcal{O}$ is an NDF orientation, there is a child w' of x with $(w',x) \in \mathcal{O}$. By assumption, we have $w' \neq w$. Thus, $(w',x) \in \mathcal{O}'$ and (NDF) is fulfilled.

From the two cases, we obtain

$$\Omega(G) \leq \Omega(G \setminus \{\{v,w\}\}) + \Omega(G \setminus \{\{v,w\},\{w,x\}\}).$$

Checking the maximal values of Ω for forests with 1 and 2 edges yields $\Omega(G) \leq F_{n+1}$ by induction.

In order to see that $F_{n+1} = \Omega(\mathsf{P}_n)$, we label the vertices of P_n by the numbers $0,...,n$ such that $\{i-1,i\}$ forms an edge for every $i \in [n]$. This labeling is a shelling of P_n. We observe that the NDF orientations of P_n are precisely the orientations for which we do not have two consecutive "increasing edges", i.e., we do not have $(i-1,i),(i,i+1) \in \mathcal{O}$ for any $i \in [n-1]$. It can be checked with a simple recursion that the number of such orientations satisfies a Fibonacci recurrence and by checking the initial values, the claim follows. □

Since $F_{n+1}^{1/n} \approx \frac{1+\sqrt{5}}{2} > \frac{\pi}{2}$, the inequality

$$\mathrm{vol}(\mathrm{M}(G)\mathrm{vol}(\mathrm{AM}(G)) \leq \frac{F_{n+1}}{n!}$$

obtained by combining Theorem 7.5.8 and Proposition 7.5.10 is no improvement of the Blaschke-Santaló inequality (7.12). For this, we believe, a more detailed understanding of the standard labelings would be necessary.

We conclude this section by using the (co-)standard labelings in order to determine the Mahler volume of the Fibonacci polytope that we encountered in Example 7.3.2. Again, let F_n be the n-th Fibonacci number and let A_n denote the number of *alternating permutations* of $[n]$, i.e., $\sigma \in S_n$ such that $\sigma(1) < \sigma(2) > \sigma(3) < \sigma(4) > ....$ The number A_n occurs in various places within combinatorics, but also in the context of analytic functions. It is sequence A000111 in the `Online Encyclopedia of Integer Sequences`. We refer to [Sta09] for a survey on this sequence of numbers.

Proposition 7.5.11. *Consider the path graph P_n with n edges. We have*

i) $\mathrm{M}(\mathsf{P}_n) = \frac{A_n}{n!}$ *and*

ii) $\mathrm{AM}(\mathsf{P}_n) = \frac{F_{n+1}}{n!}$.

Proof. We equip P_n with the natural shelling, as we did in Proposition 7.5.10.

i): Here we consider a matching choice function that is different to the one that we found in Section 7.3. Let $M_0 = \{\{0,1\},\{2,3\},...\}$. This is an almost perfect matching of P_n, so we set $m(\mathsf{P}_n) = M_0$. For a subforest $H \subseteq \mathsf{P}_n$, let $m(H)$ be the union of $H \cap M_0$ together with all isolated edges of H. Since H is subgraph of a path, this is indeed an almost perfect matching; A non-leaf vertex v of H is covered by exactly two edges e and f of H. These are also the only edges in P_n that cover v. Thus, one of them is contained in M_0 and therefore also in $m(H)$.

Now we identify an edge $\{i-1,i\}$ of P_n with i. That way, a standard labeling $\sigma \in \mathrm{St}(\mathsf{P}_n, m)$ is a permutation of $[n]$. We claim that the following are equivalent:

(a) σ is a standard labeling,

(b) σ is an alternating permutation.

Suppose first that $\sigma \in \mathrm{St}(\mathsf{P}_n, m)$ and let $i \in [n]$ be even. Then, $\{i-1,i\} \notin M_0$. Hence, by our choice of m, $\{i-1,i\}$ is not available for σ until $\{i-2,i-1\}$ and $\{i,i+1\}$ are deleted. That means, we have $\sigma(i-1) < \sigma(i) > \sigma(i+1)$ as desired.

Conversely, let σ be an alternating permutation. We have to show $\sigma^{-1}(i) \in m(\mathsf{P}_n \setminus \sigma^{-1}([i-1])$ for any i. Let j be the edge such that $\sigma(j) = i$, i.e., $j = \sigma^{-1}(i)$.

Case 1 j is even.

Since σ is alternating, we have $\sigma(j-1) < \sigma(j) > \sigma(j+1)$. Thus, j is isolated in the subgraph $\mathsf{P}_n \setminus \sigma^{-1}([i-1])$ and by definition of m it follows that $j \in m(\mathsf{P}_n \setminus \sigma^{-1}([i-1]))$.

Case 2 j is odd.

Then we readily have $j \in M_0 \setminus \sigma^{-1}([i-1]) \subseteq m(\mathsf{P}_n \setminus \sigma^{-1}([i-1]))$.

ii): This is a very similar argument to the proof of Proposition 7.5.10; At any point, we can choose the last or the second to last edge of the current subgraph, provided that the last edge is not isolated. $\square$

For trees T with $n \leq 10$ edges, a `sagemath` [SageMath] computation shows that

$$\mathrm{st}(T)\mathrm{co}(T) \leq \mathrm{st}(\mathsf{P}_n)\mathrm{co}(\mathsf{P}_n).$$

So P_n is a potential maximizer of the Mahler volume among all forests with n edges. Recall that $A_n^{1/n} = \frac{2}{\pi} + o(1)$ [Sta09, Eq. 1.10] and $F_n^{1/n} = \frac{1+\sqrt{5}}{2}$. So we have

$$\mathrm{vol}(\mathrm{M}(\mathsf{P}_n))\mathrm{vol}(\mathrm{AM}(\mathsf{P}_n)) = \frac{1}{n!^2} A_n F_{n+1} = \frac{1}{n!} \left(\frac{1+\sqrt{5}}{\pi} + o(1) \right)^n.$$

Note that $\frac{1+\sqrt{5}}{\pi} \approx 1.03$, which is significantly smaller than $\pi/2$, the base of the exponential factor in the Blaschke-Santaló inequality for anti-blocking bodies (7.12). The sage script that computes the (co-)standard labelings of a given forest is available here:

https://github.com/AnsgarFreyer/Dissertation_Data.git

The volume of $\mathsf{M}(\mathsf{P}_n)$ has first been determined in [Sta86], where the matching polytope is considered as a chain polytope of the so-called fence poset.

7.6 Further Properties of the Substar Polytope

In this section, we investigate the substar polytope of a forest more closely. The main result is an alternative characterization of its triangulation that we constructed in Section 7.4. The facets of this triangulation were given by the simplices

$$T_\sigma = \operatorname{conv}\{\mathbb{1}[s(G \setminus \{e_1, ..., e_i\})] : 0 \le i \le n\},$$

where $\sigma = (e_1, ..., e_n) \in \operatorname{Co}(G, s)$ and the star choice function s is chosen with respect to a shelling ω of G (cf. Corollary 7.4.4).

For a graph $G = (V, E)$ and a permutation $\sigma = (e_1, ..., e_n)$, we can construct a matching M recursively as follows: We start with $M_0 = \{e_n\}$. For $i < n$ we define $M_i = M_{i-1} \cup \{e_{n-i}\}$ if this yields a matching and $M_i = M_{i-1}$ otherwise. We set $M = M_{n-1}$. This is the matching obtained by greedily picking the highest edge in σ. Therefore, we call it the *greedy matching of* G with respect to σ. Moreover, for a subset $F \subseteq E$, we write $G|_F = (V, F)$. With these notions, we can describe the substar polytope $\operatorname{AM}(G)$ of a forest G as follows:

Theorem 7.6.1. *Let G be a shelled forest with n edges and let $\sigma = (e_1, ..., e_n)$ be a co-standard labeling. We denote the greedy matching of $G|_{\sigma^{-1}([i])}$ with respect to $(e_1, ..., e_i)$ by M_i^σ. For the corresponding facet T_σ in the triangulation of $\operatorname{AM}(G)$, we have*

$$T_\sigma = \big\{x \in \mathbb{R}^E : 0 \le \langle x, \mathbb{1}[M_1^\sigma]\rangle \le \cdots \le \langle x, \mathbb{1}[M_n^\sigma]\rangle \le 1\big\}. \tag{7.15}$$

Example 7.6.2. Once again, let us consider the tree T from Figure 7.3. We identify an edge ij of T where $i < j$ with its larger vertex j. Consider the co-standard labeling $\sigma = (13, 14, 02, 01) = (3, 4, 2, 1)$. Then we can write T_σ as follows:

$$\begin{aligned} T_\sigma &= \operatorname{conv}\left\{ \begin{pmatrix}1\\0\\1\\1\end{pmatrix}, \begin{pmatrix}1\\0\\0\\1\end{pmatrix}, \begin{pmatrix}1\\1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\0\end{pmatrix} \right\} \\ &= \{x \in \mathbb{R}^4 : 0 \le x_3 \le x_4 \le x_2 + x_4 \le x_1 \le 1\}. \end{aligned}$$

Note that $\{3\}$, $\{4\}$, $\{2, 4\}$, $\{1\}$ are indeed the greedy matchings of the respective $T|_{\sigma^{-1}([i])}$ and that the i-th vertex in the convex hull description satisfies the i-th inequality with strict inequality and the remaining ones with equality. ◇

Proof of Theorem 7.6.1. Let $S_i = s(G \setminus \sigma^{-1}([i]))$, for $0 \leq i \leq n$ with the convention $[0] = \emptyset$. For $i < n$, let (i) be the inequality

$$(i): \quad \langle x, \mathbb{1}[M_i^\sigma] \rangle \leq \langle x, \mathbb{1}[M_{i+1}^\sigma] \rangle,$$

where $M_0^\sigma = \emptyset$. Moreover let (n) be the inequality

$$(n): \quad \langle x, \mathbb{1}[M_n^\sigma] \rangle \leq 1.$$

We show that $\mathbb{1}[S_i]$ fulfills (i) with strict inequality and (j) with equality for all $j \neq i$. For this, we first notice that $\langle \mathbb{1}[S_i], \mathbb{1}[M_j^\sigma] \rangle = |S_i \cap M_i^\sigma| \in \{0, 1\}$.

We begin with $S_0 = s(G)$. Suppose that $S_0 \cap M_i^\sigma = \emptyset$ for some $i > 0$. By definition, S_0 is the star around v, the parent of the largest leaf w of G. $S_0 \cap M_i^\sigma = \emptyset$ means that v is not covered by M_i^σ. It follows that $\sigma(\{v, w\}) > i$, otherwise the greedy algorithm would not have overlooked the edge $\{v, w\}$. This means that $\{v, w\}$ is still present in any $G_j = G \setminus \sigma^{-1}([j])$ with $j \leq i$. As w has the largest label, any $s(G_j)$ is a substar centered at $v = \text{parent}(w)$, for $j \leq i$. But then, M_i^σ consists only of edges that cover v, a contradiction. Hence, M_i^σ and S_0 intersect for all $i > 0$, which gives equality in (1) to (n) and strict inequality in (0).

Next, consider an arbitrary S_i, for $i < n$. For $j \leq i$ we have $M_j^\sigma \subseteq \sigma^{-1}([j]) \subseteq \sigma^{-1}([i])$, while on the other hand, $S_i \subseteq E \setminus \sigma^{-1}([i])$. Consequently, we have $S_i \cap M_i^\sigma = \emptyset$ for all $j \leq i$ and therefore $\langle \mathbb{1}[S_i], \mathbb{1}[M_j^\sigma] \rangle = 0$.

For $j > i$, we let $H = G \setminus \sigma^{-1}([i])$. Then

$$\tau : E \setminus \sigma^{-1}([i]) \to [n-i], \quad \tau(e) = \sigma(e) - i$$

is a co-standard labeling of H by definition and we have $S_i = s(H)$. Moreover, we have $M_{j-i}^\tau \subseteq M_j^\sigma$, where M_{j-i}^τ is the greedy matching of $H|_{\sigma^{-1}([j-i])}$ w.r.t. τ. Applying the discussion of the case $i = 0$ from above to $s(H)$ in H, we find that $S_i \cap M_{j-i}^\tau \neq \emptyset$ and therefore $S_i \cap M_j^\sigma \neq \emptyset$. It follows that $\langle \mathbb{1}[S_i], \mathbb{1}[M_j^\sigma] \rangle = 1$ for all $j > i$. Thus, $\mathbb{1}[S_i]$ satisfies (i) with strict inequality and all the other inequalities with equality.

Since $S_n = \emptyset$, we have $\mathbb{1}[S_n] = 0$, which satisfies (i) with equality for $i < n$ and (n) with strict inequality. Our proof is now finished. □

Remark 7.6.3. There is no direct analogue of Theorem 7.6.1 for the matching polytope $\mathrm{M}(G)$, i.e., it is not true that the simplices in (7.8) may be written as

$$S_\sigma = \{x \in \mathbb{R}^E : 0 \leq \langle x, \mathbb{1}[S_1^\sigma] \leq \cdots \leq \langle x, \mathbb{1}[S_n^\sigma] \rangle \leq 1\},$$

where σ is a standard labeling of G and the S_i^σ's are substars that are constructed in from σ in a certain way. As a counterexample, let us consider the path P_4 with 4 edges and the natural shelling, as well as the MCF m defined by this shelling. Then, $\sigma =$

$(01, 34, 12, 23) \in \mathrm{St}(G, m)$ is a standard labeling and one quickly verifies that

$$\begin{aligned} S_\sigma &= \mathrm{conv}\{\mathbb{1}[01, 23], \mathbb{1}[12, 34], \mathbb{1}[12], \mathbb{1}[34], 0\} \\ &= \{x \in \mathbb{R}^E : 0 \leq x_{01} \leq x_{01} + x_{34} \leq x_{01} + x_{12} \leq x_{12} + x_{23} \leq 1\}, \end{aligned}$$

but the set $\{01, 34\}$ is not a substar. $\diamond$

The chain polytope revisited. Let us compare the triangulation of $\mathrm{AM}(G)$ to the triangulation of the chain polytope $\mathrm{C}(\mathcal{P})$ from Section 7.3 as it has been described by Stanley in [Sta86, Sec. 5]. For a poset $\mathcal{P}$, Stanley constructs a triangulation of $\mathrm{C}(\mathcal{P})$ as follows. For a linear extension $\sigma = (x_1, ..., x_n)$ of $\mathcal{P}$ and $i \in [n]$, consider the chain

$$K_i^\sigma = \{z_j < \cdots < z_0\},$$

where

i) $z_0 = x_i$,

ii) z_i is the last element in σ that is smaller than z_{i-1} for any $i \in [j]$,

iii) $z_j \in \min(\mathcal{P})$.

Then, the n-simplices

$$\Delta_\sigma = \{x \in \mathbb{R}^X : 0 \leq \langle x, \mathbb{1}[K_1^\sigma]\rangle \leq \cdots \leq \langle x, \mathbb{1}[K_n^\sigma]\rangle \leq 1\}$$

form a triangulation of $\mathrm{C}(\mathcal{P})$. These simplices appear to be very similar to the facets $(T_\pi)_{\pi \in \mathrm{Co}(G,s)}$ of the triangulation of the substar polytope of a forest, as presented in (7.15). Indeed, the chain K_i^σ is the one obtained by greedily adding elements from $(x_1, ..., x_i)$, starting with x_i. Thus, it is in fact the "poset-analogue" of M_i^σ.

This connection goes further. For a linear extension $\sigma = (x_1, ..., x_n)$ of $\mathcal{P}$, we have

$$\Delta_\sigma = \mathrm{conv}\{\mathbb{1}[\min(P \setminus \{x_1, ..., x_i\})] \colon 0 \leq i \leq n\}$$

as can be checked similarly to the proof of Theorem 7.6.1, since $\min(P \setminus \{x_1, ..., x_i\}$ is an anti-chain and therefore meets K_j in at most one point. From the convex hull representation of Δ_σ we see that Stanley's triangulation is in fact a pulling triangulation; We can order the anti-chains of $\mathcal{P}$ by their scalar product with

$$c = (2^{-\pi(x)})_{x \in X},$$

where $\pi : X \to [n]$ is a fixed linear extension of $\mathcal{P}$. $\diamond$

We finish the investigation by proving that the path graph P_n achieves the minimum volume among all substar polytopes of trees with n edges.

Proposition 7.6.4. *Let G be a tree with n edges. Then,*

$$\mathrm{vol}(\mathrm{AM}(G)) \geq \mathrm{vol}(\mathrm{AM}(\mathsf{P}_n)) = \frac{F_{n+1}}{n!},$$

where F_n is the n-th Fibonacci number.

Note that among all *forests* with n edges, the minimum is attained for the matching graph M_n (cf. Example 7.3.2).

Proof. It suffices to prove $\operatorname{co}(G) \geq F_{n+1}$. We consider a vertex v of G whose children are all leaves. We choose the shelling of G in such a way that these leaves have the highest labels. This does not affect the number of co-standard labelings.

Case 1 v has at least two children w_1 and w_2.

Let $e_i = \{v, w_i\}$, $\in \{1, 2\}$. Then we have

$$\begin{aligned}\operatorname{co}(G) &\geq |\{\sigma \in \operatorname{Co}(G, s) : \sigma_1 = e_1\}| + |\{\sigma \in \operatorname{Co}(G, s) : \sigma_1 = e_2\}| \\ &= \operatorname{co}(G \setminus \{e_1\}) + \operatorname{co}(G \setminus \{e_2\}) \geq F_n + F_n > F_{n+1}.\end{aligned}$$

Case 2 v has exactly one child w.

Since G is a tree, v has a parent, whenever $n > 1$. The first edge in a co-standard labeling σ of G is either $e = \{v, w\}$ or $f = \{v, \operatorname{parent}(v)\}$. If f is the first edge, then v has no parent in $G \setminus \{f\}$, so the second edge in σ is e. We obtain,

$$\operatorname{co}(G) = \operatorname{co}(G \setminus \{e\}) + \operatorname{co}(G \setminus \{e, f\}) \geq F_n + F_{n-1} = F_{n+1}. \qquad \square$$

Conclusion

We investigated three aspects of the volume of convex bodies. To conclude, let us reflect on the main results and open questions of the three parts of the thesis.

In the first part we contributed to the study of the lattice point enumerator by showing discrete analogues of Meyer's inequality and its reverse, as well as the slicing inequality and the reverse Loomis-Whitney inequality. In doing so, we also gained new insights on the behaviour of $\mathrm{G}(K)$ under operations such as translations and dilations. We learned that although the lattice point enumerator behaves similarly to the volume when K extends far in every dimension, many geometric inequalities of the volume fail, or must be weakened, when translated to the the integral setting. What remains open is the question for the best-possible constants in many of these results.

We were able to use the successive minima in order to prove bounds on the lattice point enumerator $\mathrm{G}(K)$ in terms of $\mathrm{vol}(K)$ that account for the above mentioned extension of K in each dimension. As a Corollary, we obtained a discrete version of Minkowski's second theorem which is equivalent to the original version for the volume. A key element of this research was to unify the concepts of compressions and Blaschke shakings within the geometry of numbers. The open questions here are of course the sharp versions of the conjectures of Betke Henk and Wills, but also the study of other parameters in the geometry of numbers, such as the lattice width, under Blaschke shaking and similar processes.

In contrast to the discrete arrangement of lattice points in a convex body K, in Part II we considered the volume of K as a continuous distribution of mass inside of K. By extending Wu's affine subspace concentration conditions to arbitrary centered polytopes, we saw new properties of this distribution. However, we are lacking a full characterization of the equality case in the affine subspace concentration conditions.

In the first two parts, the polar body $K^\star$ of K was a helpful tool, while in the third part it was the subject of our studies itself. We investigated the polar bodies of lattice Voronoi cells and matching polytopes of forests in detail, with the aim of bounding their Mahler volume. As for the lattice Voronoi cells, we saw that the lattice $A_3^\star$ constitutes a strict local maximum of the Mahler volume of its Voronoi cell among all 3-dimensional lattices. Despite strong computational and theoretical evidence, the question whether $A_3^\star$ is indeed the global maximizer of the Mahler volume is still open. For the matching polytopes $\mathrm{M}(G)$, we could exploit the underlying combinatorics of the forest G to give an upper bound on its Mahler volume. In order to strengthen our bound, we believe that a deeper understanding of the standard labelings of a forest would be necessary.

Bibliography

[AASS] S. Artstein-Avidan, S. Sadovsky, and R. Sanyal. Geometric inequalities for anti-blocking bodies. *Commun. Contemp. Math.*, to appear.

[AB21] D. Alonso-Gutiérrez and S. Brazitikos. Reverse Loomis-Whitney inequalities via isotropicity. *Proc. Am. Math. Soc.*, 149:817–828, 2021.

[AGH+17] B. Assarf, E. Gawrilow, K. Herr, M. Joswig, B. Lorenz, A. Paffenholz, and T. Rehn. Computing convex hulls and counting integer points with `polymake`. *Math. Program. Comput.*, 9(1):1–38, 2017.

[AGM15] S. Artstein-Avidan, A. Giannopoulos, and V.-D. Milman. *Asymptotic Geometric Analysis. Part I.* AMS, Providence, RI, 2015.

[AGP+17] G. Averkov, B. González Merino, I. Paschke, M. Schymura, and S. Weltge. Tight bounds on discrete quantitative Helly numbers. *Adv. Math.*, 89:76–101, 2017.

[AHZ17] M. Alexander, M. Henk, and A. Zvavitch. A discrete version of Koldobsky's slicing inequality. *Israel J. Math.*, 222(1):261–278, 2017.

[AKL13] F. Aurenhammer, R. Klein, and D.-T. Lee. *Voronoi diagrams and Delaunay triangulations.* World Scientific, Singapore u.a., 2013.

[APBT16] J. C. Álvarez Paiva, F. Balacheff, and K. Tzanev. Isosystolic inequalities for optical hypersurfaces. *Advances in Mathematics*, 301:934–972, 2016.

[BB00] B. Bollobás and G. R. Brightwell. Convex bodies, graphs and partial orders. *Proc. London Math. Soc.*, 80(2):415–450, 2000.

[BBS99] B. Bollobás, G. Brightwell, and A. Sidorenko. Geometrical techniques for estimating numbers of linear extensions. *Eur. J. Comb.*, 20(5):329–35, 1999.

[BC84] C. Berge and V. Chvátal. *Topics on Perfect Graphs.* ISSN. Elsevier Science, 1984.

[BD84] C. Berge and P. Duchet. Strongly perfect graphs. *Ann. Discrete Math.*, 21:57–61, 1984.

[BE13] I. Baburin and P. Engel. On the enumeration of the combinatorial types of primitive parallelohedra in E^d, $2 \leq d \leq 6$. *Acta Crystallogr. A*, 69:510–516, 09 2013.

[Ber18] S. L. Berg. *Lattice points in convex bodies: Counting and approximating.* TU Berlin, Berlin, 2018.

[BGL18] S. Brazitikos, A. Giannopoulos, and D.-M. Liakopoulos. Uniform cover inequalities for the volume of coordinate sections and projections of convex bodies. *Adv. Geom.*, 18(3):345–354, 2018.

[BGVV14] S. Brazitikos, A. Giannopoulos, P. Valettas, and B.-H. Vritsiou. *Geometry of isotropic convex bodies.* AMS, Providence, RI, 2014.

[BH16] K. J. Böröczky and M. Henk. Cone-volume measures of general centered convex bodies. *Adv. Math.*, 286:703–721, 2016.

[BHW93] U. Betke, M. Henk, and J. M. Wills. Successive-minima-type inequalities. *Discrete Comput. Geom.*, 9:165–175, 1993.

[BHZ16] K. J. Böröczky, P. Hegedus, and G. Zhu. On the discrete logarithmic Minkowski problem. *IMRN. International Mathematics Research Notices*, 2016(6):1807–1838, 2016.

[Bie23] T. Biehl. Über affine Geometrie XXXVIII. Über die Schüttelung von Eikörpern. *Abh. Math. Semin. Hambg. Univ.*, 2:69–70, 1923.

[BL91] B. Bollobás and I. Leader. Compressions and isoperimetric inequalites. *J. Comb. Theory*, A56:47–62, 1991.

[Bla17] W. Blaschke. Über affine Geometrie XI: Lösung des "Vierpunktproblems" von Sylvester aus der geometrischen Wahrscheinlichkeitstheorie. *Leipziger Berichte*, 69:436–453, 1917.

[Bli21] H. F. Blichfeldt. Notes on Geometry of Numbers, The october meeting of the San Francisco section of the AMS. *Bull. Am. Math. Soc.*, 27(4):150–153, 1921.

[BLPS99] W. Banaszczyk, A. E. Litvak, A. Pajor, and J. Stanislaw. The flatness theorem for nonsymmetric convex bodies via the local symmetry of Banach spaces. *Math. Oper. Res.*, 24(3):728–750, 1999.

[BLYZ13] K. J. Böröczky, E. Lutwak, D. Yang, and G. Zhang. The logarithmic Minkowski problem. *J. Am. Math. Soc.*, 26:831–852, 2013.

[Bör00] K. J. Böröczky. Approximation of general smooth convex bodies. *Adv. Math.*, 153:325–341, 2000.

[Bör10] K. Böröczky. Stability of the Blaschke–Santaló and the affine isoperimetric inequality. *Adv. Math.*, 225:1914–1928, 2010.

[BR07] M. Beck and S. Robins. *Computing the continuum discretely.* Undergraduate Texts in Mathematics, Springer, New York, 2007.

[BV13] S. Boyd and L. Vandenberghe. *Convex optimization.* Cambridge University Press, seventh printing with corrections edition, 2013.

[BWZ65] R. P. Bambah, A. C. Woods, and H. Zassenhaus. Three proofs of Minkowski's second inequality in the geometry of numbers. *J. Austral. Math. Soc.*, 5:453–462, 1965.

[Cas71] J. W. S. Cassels. *An introduction to the Geometry of Numbers.* Springer, 1971.

[CCG99] S. Campi, A. Colesanti, and P. Gronchi. A note on Sylvester's problem for random polytopes in a convex body. *Rend. Istit. Mat. Univ. Trieste*, 31:79–94, 1999.

[CCG01] S. Campi, A. Colesanti, and P. Gronchi. Shaking compact sets. *Beitr. Algebra Geom.*, 42:123–136, 2001.

[CG06] S. Campi and P. Gronchi. On volume product inequalities for convex sets. *Proc. Am. Math. Soc.*, 134(8):2393–2402, 2006.

[CGG18] S. Campi, P. Gritzmann, and P. Gronchi. On the reverse Loomis-Whitney inequality. *Discrete Comput. Geom.*, 60(1):115–144, 2018.

[CGL17] G. Chasapis, A. Giannopoulos, and D.-M. Liakopoulos. Estimates for measures of lower dimensional sections of convex bodies. *Adv. Math.*, 306:880–904, 2017.

[CLZ19] S. Chen, Q. Li, and G. Zhu. The logarithmic Minkowski problem for non-symmetric measures. *Trans. Am. Math. Soc.*, 371:2623–2641, 2019.

[CRST06] M. Chudnovsky, N. Robertson, P. Seymour, and R. Thomas. The strong perfect graph theorem. *Ann. Math.*, 164:51–229, 2006.

[CS92] J. H. Conway and N. J. A. Sloane. Low-dimensional lattices IV. Voronoi reduction of three-dimensional lattices. *Proc. R. Soc. Lond. A*, 436:55–68, 1992.

[CS99] J. H. Conway and N. J. A. Sloane. *Sphere Packings, Lattices and Groups*, volume 290 of *Grundlehren der mathematischen Wissenschaften, A Series of Comprehensive Studies in Mathematics.* Springer, New York, NY, third edition, 1999.

[Dad12] D. Dadush. *Integer programming, lattice algorithms, and deterministic volume Estimation.* Georgia Institute of Technology, USA, 2012.

[Dav39] H. Davenport. Minkowski's inequality for the minima associated with a convex body. *Quaterl J. Math.*, 10:119–121, 1939.

[Dav51] H. Davenport. On a principle of Lipschitz. *J. London Math. Soc.*, 26(2):179–183, 1951.

[DG09] M. Dutour Sikirić and V. P. Grishukhin. The decomposition of the hypermetric cone into L-domains. *Eur. J. Comb.*, 30:853–865, 2009.

[DGSW16] M. Dutour Sikirić, A. Garber, A. Schürmann, and C. Waldmann. The complete classification of five-dimensional Dirichlet-Voronoi polyhedra of translational lattices. *Acta Crystallogr. A*, 72 Pt 6:673–683, 2016.

[DR97] R. Diaz and S. Robins. The Ehrhart polynomial of a lattice polytope. *Ann. of Math.*, 145:503–518, 1997.

[EG02] P. Engel and V. Grishukhin. There are exactly 222 L-types of primitive five-dimensional lattices. *Eur. J. Comb.*, 23:275–279, 04 2002.

[Est46] T. Estermann. Note on a theorem of Minkowski. *J. London Math. Soc.*, 21:179–182, 1946.

[Fed85] E. S. Fedorov. Elements of the study of figures. *Zap. Miner. Obshch.*, 21(2):1–279, 1885. Reprinted in *Izv. Akad. Nauk SSSR* (1953).

[Fed91] E. S. Fedorov. The symmetry of regular systems of figures. *Zap. Miner. Obshch.*, 28(2):1–146, 1891.

[FH22] A. Freyer and M. Henk. Bounds on the lattice point enumerator via slices and projections. *Discrete Comput. Geom.*, 67:895–918, 2022.

[FHL19] Y.-R. Feng, Q. Huang, and A.-J. Li. On the reverse dual Loomis-Whitney inequality. *Results Math.*, 74(3):Art. 106, 15, 2019.

[FL22] A. Freyer and E. Lucas. Interpolating between volume and lattice point enumerator with successive minima. *Monatsh. Math.*, 198:717–740, 2022.

[FMZ12] M. Fradelizi, M. Meyer, and A. Zvavitch. An application of shadow systems to Mahler's conjecture. *Discrete Comput. Geom.*, 48:721–734, 2012.

[Ful71] D. R. Fulkerson. Blocking and anti-blocking pairs of polyhedra. *Math. Program. 1*, no.1:168–194, 1971.

[Gar01] R. J. Gardner. The Brunn-Minkowski inequality. *Bull. Am. Math. Soc.*, 39(5):355–405, 11 2001.

[Gar06] R. J. Gardner. *Geometric Tomography.* Cambridge University Press, New York, second edition, 2006.

[GG01] R. Gardner and P. Gronchi. A Brunn-Minkowski inequality for the integer lattice. *Trans. Math. Soc.*, 353(10):3995–4024, 2001.

[GGZ05] R. Gardner, P. Gronchi, and C. Zong. Sums, projections, and sections of lattice sets, and the discrete covariogram. *Discrete Comput. Geom.*, 34(2):391–409, 2005.

[GL87] P. M. Gruber and C. G. Lekkerkerker. *Geometry of Numbers.* North-Holland, second edition, 1987.

[Grü60] B. Grünbaum. Partitions of mass-distribution and of convex bodies by hyperplanes. *Pac. J. Math.*, 10(4):1257–1261, 1960.

[Gru07] P. M. Gruber. *Convex and Discrete Geometry*, volume 336 of *A Series of Comprehensive Studies in Mathematics*. Springer Berlin Heidelberg, 2007.

[GS16] B. González Merino and M. Schymura. A generalization of the discrete version of Minkowski's fundamental theorem. *Mathematika*, 62:637–652, 2016.

[GT05] B. Green and T. Tao. Compressions, convex geometry and the Freiman-Bilu theorem. *Q. J. Math.*, 57:495–504, 2005.

[HAB+17] T. Hales, M. Adams, G. Bauer, T. D. Dang, J. Harrison, L. T. Hoang, C. Kaliszyk, V. Magron, S. McLaughlin, T. Nguyen, et al. A formal proof of the Kepler conjecture. *Forum of Mathematics, Pi*, 5:e2, 2017.

[Hal05] T. Hales. A proof of the Kepler conjecture. *Ann. Math. (2)*, 162(3):1065–1185, 2005.

[Hen90] M. Henk. Inequalities between successive minima and intrinsic volumes of a convex body. *Monatsh. Math.*, 110:279–282, 1990.

[Hen02] M. Henk. Successive minima and lattice points. *Rend. Circ. Mat. Palermo (2) Suppl.*, 70, part I:377–384, 2002.

[Hen12] M. Henze. A Blichfeldt-type inequality for centrally symmetric convex bodies. *Monatsh. Math.*, 170(3-4):371–379, 2012.

[HHH16] M. Henk, M. Henze, and M. Hernandez Cifre. On extensions of Minkowski's second theorem on successive minima. *Forum Math.*, 28(2):311–326, 2016.

[HIY18] M. Hernández Cifre, D. Iglesias, and J. Yepes Nicolás. On a discrete Brunn-Minkowski type inequality. *SIAM J. Discrete Math.*, 32:1840–1856, 2018.

[HKS21] D. Halikias, B. Klartag, and B. A. Slomka. Discrete variants of Brunn-Minkowski type inequalities. *Ann. Fac. Sci. Toulouse Math., Serie 6*, 30(2):267–279, 2021.

[HL14] M. Henk and E. Linke. Cone-volume measures of polytopes. *Adv. Math.*, 253:50–62, 2014.

[HM73] D. Husemoller and J. W. Milnor. *Symmetric Bilinear Forms*. Springer, 1973.

[HNS19] M. Hering, B. Nill, and H. Süß. Stability of tangent bundles on smooth toric Picard-rank-2 varieties and surfaces. Preprint arXiv:1910.08848, 2019.

[HRS20] C. Hunkenschröder, G. Reuland, and M. Schymura. On compact representations of Voronoi cells of lattices. *Math. Program.*, 183:337–358, 2020.

[HRZ17] M. Henk, J. Richter-Gebert, and G. M. Ziegler. Basic properties of convex polytopes. In J. E. Goodman, J. O'Rourke, and C. D. Tóth, editors, *Handbook of Discrete and Computational Geometry*, pages 383–413. CRC Press, 3rd edition, 2017.

[HSW05] M. Henk, A. Schürmann, and J. M. Wills. Ehrhart polynomials and successive minima. *Mathematatika*, 52(1-2):1–16, 2005.

[ILY22] D. Iglesias, E. Lucas, and J. Yepes Nicolás. On discrete Brunn-Minkowski and isoperimetric type inequalities. *Discrete Math.*, 345(1):112640, 2022.

[IS20] H. Iriyeh and M. Shibata. Symmetric Mahler's conjecture for the volume product in the 3-dimensional case. *Duke Math. J.*, 169(6):1077 – 1134, 2020.

[IYZ20] D. Iglesias, J. Yepes Nicolás, and A. Zvavitch. Brunn-Minkowski type inequalities for the lattice point enumerator. *Adv. Math.*, 370:107193, 2020.

[KK18] B. Klartag and A. Koldobsky. An example related to the slicing inequality for general measures. *J. Funct. Anal.*, 274(7):2089–2112, 2018.

[KL88] R. Kannan and L. Lovasz. Covering minima and lattice-point-free convex bodies. *Ann. Math. (2)*, 128(3):577–602, 1988.

[KL20] B. Klartag and G. V. Livshyts. The lower bound for Koldobsky's slicing inequality via random rounding. In *Geometric aspects of functional analysis, Lecture notes in Math.*, volume 2266, pages 43–63, Cham, 2020. Springer.

[KL22] B. Klartag and J. Lehec. Bourgain's slicing problem and kls isoperimetry up to polylog. Preprint arXiv:2203.15551, 2022.

[Kle79] D. J. Kleitman. Extremal hypergraph problems. In *Surveys in combinatorics*, pages 44–65. Cambridge University Press, 1979.

[Kol14a] A. Koldobsky. Estimates for measures of sections of convex bodies. In *Geometric aspects of functional analysis, Lecture notes in Math.*, volume 2116, pages 261–271, Cham, 2014. Springer.

[Kol14b] A. Koldobsky. A $\sqrt{n}$ estimate for measures of hyperplane sections of convex bodies. *Adv. Math.*, 254:33–40, 2014.

[Kol15] A. Koldobsky. Slicing inequalities for measures of convex bodies. *Adv. Math.*, 283:473–488, 2015.

[KOS20] F. Kohl, M. Olsen, and R. Sanyal. Unconditional reflexive polytopes. *Discrete Comput. Geom.*, 64:427–452, 2020.

[KSZ19] A. Koldobsky, C. Saroglou, and A. Zvavitch. Estimating volume and surface area of a convex body via its projections or sections. *Studia Math.*, 244(3):245–246, 2019.

[Kup09] G. Kuperberg. From the Mahler conjecture to Gauss linking integrals. *Geom. Funct. Anal.*, 18(3):870–892, 2009.

[Lán22] Z. Lángi. An isoperimetric problem for three-dimensional parallelohedra. *Pac. J. Math.*, 316(1):169–181, 2022.

[LD09] M. Laurent and M. M. Deza. *Geometry of Cuts and Metrics*, volume 15 of *Algorithms and Combinatorics*. Springer, 2009.

[Liu12] R. I. Liu. Matching polytopes and Specht modules. *Trans. Am. Math. Soc.*, 364(2):1089–1107, 2012.

[LKS12] L. Lovász, B. Korte, and R. Schrader. *Greedoids*, volume 4 of *Algorithms and Combinatorics*. Springer, 2012.

[LLL82] A. K. Lenstra, H. W. Lenstra, and L. Lovász. Factoring polynomials with rational coefficients. *Math. Ann.*, 261:515–534, 1982.

[Lov72a] L. Lovász. A characterization of perfect graphs. *J. Comb. Theory. Ser. B*, 13(2):95–98, 1972.

[Lov72b] L. Lovász. Normal hypergraphs and the perfect graph conjecture. *Discrete Math.*, 2(3):253–267, 1972.

[Lov74] L. Lovász. Minimax theorems for hypergraphs. In *Hypergraph Seminar*, page 111–126, Berlin, Heidelberg, 1974. Springer Berlin Heidelberg.

[LP86] L. Lovász and M. Plummer. *Matching Theory*, volume 121 of *North-Holland Mathematics Studies*. North-Holland, 1986.

[LR17] S. Lovett and O. Regev. A counterexample to a strong variant of the Freiman–Rusza conjecture in Euclidean space. *Discrete Anal.*, 8, 2017.

[LW49] L. H. Loomis and H. Whitney. An inequality related to the isoperimetric inequality. *Bull. Am. Math. Soc.*, 55(10):961–962, 1949.

[Mah39] K. Mahler. Ein Minimalproblem für konvexe Polygone. *Mathematica (Zutphen)*, B7:118–127, 1939.

[Mal12] R. Malikiosis. A discrete analogue of Minkowski's second theorem on successive minima. *Adv. Geom.*, 12(2):365–380, 2012.

[Mar03] J. Martinet. *Perfect lattices in Euclidean spaces*. Springer, Berlin, 2003.

[Mat02] J. Matoušek. *Lectures on Discrete Geometry*. Springer, New York, NY, first edition, 2002.

[Mey86] M. Meyer. Une caractérisation volumique de certains espaces normés de dimension finie. *Israel J. Math.*, 55(3):317–326, 1986.

[Mey88] M. Meyer. A volume inequality concerning sections of convex sets. *Bull. London Math. Soc.*, 20(2):151–155, 1988.

[Min10] H. Minkowski. *Geometrie der Zahlen*. B.G. Teubner, 1910.

[Mir71] L. Mirsky. A dual of Dilworth's decomposition theorem. *Amer. Math. Monthly*, 78(8):876–877, 1971.

[MR11] M. Meyer and S. Reisner. On the volume product of polygons. *Abh. Math. Semin. Univ. Hambg.*, 81(1):93–100, 2011.

[Pom93] J. E. Pommersheim. Toric varieties, lattice points and Dedekind sums. *Math. Annalen*, 295:1–24, 1993.

[Rab89] S. Rabinowitz. A theorem about collinear lattice points. *Utilitas Math.*, 36:93–95, 1989.

[Rav99] G. Ravindra. Some classes of strongly perfect graphs. *Discrete Math.*, 206(1):197–203, 1999.

[Reg16] O. Regev. A note on Koldobsky's lattice slicing inequality. Preprint arXiv: 1608.04945, 2016.

[Ris05] F. Rispoli. Fibonacci polytopes and their applications. *Fibonacci Q.*, 43:227–233, 2005.

[RS58a] C. A. Rogers and G. C. Shephard. Convex bodies associated with a given convex body. *J. London Math. Soc.*, 33:270–281, 1958.

[RS58b] C. A. Rogers and G. C. Shephard. Some extremal problems for convex bodies. *Mathematika*, 5(2):93–102, 1958.

[RS17] O. Regev and N. Stephens-Davidowitz. A reverse Minkowski theorem. In *Proceedings of the 49th Annual ACM SIGACT Symposium on Theory of Computing*, STOC 2017, pages 941–953, Montreal, Canada, 2017. Association for Computing Machinery.

[SageMath] The Sage Developers. *SageMath, the Sage Mathematics Software System (Version 9.4)*, 2022. `https://www.sagemath.org`.

[Sai81] J. Saint-Raymond. Sur le volume des corps convexes symétriques. In *Séminaire d'Initiation à l'Analyse*, volume 11, page 25pp. Université Paris VI, 1980/81.

[Sch76] R. Schöpf. Schüttelung konvexer Mengen. *Ber. Math.-Statist.*, 68, 1976.

[Sch86] A. Schrijver. *Theory of linear and integer programming.* John Wiley & Sons Inc., USA, 1986.

[Sch14] R. Schneider. *Convex bodies: The Brunn-Minkowski Theory.* Cambridge University press, second expanded edition, 2014.

[Sel74] E. Selling. Ueber die binären und ternären quadratischen formen. *J. reine angew. Math.*, 77:143–229, 1874.

[She64] G. C. Shephard. Shadow systems of convex sets. *Israel J. Math.*, 2:229–236, 1964.

[Sid91] A. Sidorenko. Inequalities for the number of linear extensions. *Order*, 8(4):331–340, 1991.

[SM83] A. J. Schwenk and J. I. Munro. How small can the mean shadow of a set be? *Amer. Math. Monthly*, 90(5):325–329, 1983.

[Sta80] R. P. Stanley. Decompositions of rational convex polytopes. *Ann. Discrete Math.*, 6:333–342, 1980.

[Sta86] R. P. Stanley. Two poset polytopes. *Discrete Comput. Geom.*, 1:9–23, 1986.

[Sta09] R. P. Stanley. A survey of alternating permutations. *Contemp. Math.*, 531:165–196, 2009.

[Sta11] R. P. Stanley. *Enumerative Combinatorics: Volume 1*. Cambridge University Press, USA, 2nd edition, 2011.

[Sul05] S. Sullivant. Compressed polytopes and statistical disclosure limitation. *Tohoku Math. J.*, 58(3):433–445, 2005.

[SV06] A. Schürmann and F. Vallentin. Computational approaches to lattice packing and covering problems. *Discrete Comp. Geom.*, 35:73–116, 10 2006.

[TV06] T. Tao and V. H. Vu. *Additive combinatorics*. Cambridge University Press, 2006.

[Uhr94] B. Uhrin. Curvlinear extensions of the Brunn-Minkowski-Lusternik inequality. *Adv. Math.*, 109:288–312, 1994.

[Vor08a] G. Voronoi. Nouvelles applications des paramètres continus à la théorie des formes quadratiques. *J. reine angew. Math.*, 133:97–178, 1908.

[Vor08b] G. Voronoi. *J. reine angew. Math.*, 134:198–287, 1908.

[Vor09] G. Voronoi. *J. reine angew. Math.*, 136:67–181, 1909.

[VR12] E. Veomett and A. J. Radcliffe. Vertex isoperimetric inequalities for a family of graphs on $\mathbb{Z}^k$. *Electron. J. Comb.*, 19:10.37236/2426, 2012.

[Wey42] H. Weyl. On geometry of numbers. *Proc. London Math. Soc.*, 47(2):268–289, 1942.

[Wil73] J. M. Wills. Über konvexe Gitterpolygone. *Comment. Math. Helv.*, 48:188–194, 1973.

[Woo66] A. C. Woods. A generalization of Minkowski's second inequality in the geometry of numbers. *J. Austral. Math. Soc.*, 6:148–152, 1966.

[Wu22] K.-Y. Wu. Affine subspace concentration conditions. Preprint arXiv: 2201.06062, 2022.

[XZ14] F. Xue and C. Zong. A note on lattice coverings. Preprint arXiv:1402.4965, 2014.

[YZ06] S. T. Yau and L. Zhang. An upper estimate of integral points in real simplices with an application to singularity theory. *Math. Res. Lett.*, 13(6):911–921, 2006.

[ZH17] Y. Zhou and B. He. On LYZ's conjecture for the U-functional. *Adv. in Appl. Math.*, 87:43–57, 2017.

[Zie12] G. M. Ziegler. *Lectures on Polytopes.* Graduate Texts in Mathematics. Springer, 2012.

List of Symbols

$\mathrm{A}K$ Anti-blocking polar of an anti-blocking body K

$\mathrm{AM}(G)$ Substar polytope of G

$\mathrm{bd}K$ Boundary of K

B^n Euclidean unit ball

$\mathrm{c}(K)$ Centroid of K

$\mathrm{cl}K$ Closure of K

C_n n-dimensional origin symmetric cube with side length 2

$\mathrm{co}(G)$ Number of co-standard labelings of G

$\mathrm{C}(\mathcal{P})$ Chain polytope of $\mathcal{P}$

$\mathrm{cs}(K)$ Central symmetral of K

$\mathrm{Del}(\Lambda)$ Delaunay decomposition associated to Λ

$F^\diamond$ Polar face in $P^\star$ of a face $F \subseteq P$

$\mathrm{G}(\cdot)$ Lattice Point enumerator of $\mathbb{Z}^n$

$\overline{G}$ Complement graph of G

$\mathrm{GL}_n(\mathbb{Z})$ n-dimensional unimodular group

$\mathrm{h}(K,\cdot)$ Support function of K

$\mathrm{int}K$ Interior of K

$\mathcal{K}^n$ Convex compact sets in $\mathbb{R}^n$

$\mathcal{K}^n_n$ n-dimensional convex compact sets in $\mathbb{R}^n$

$\mathcal{K}^n_{os}$ origin-symmetric convex compact sets in $\mathbb{R}^n$

$K^\star$	Polar body of K
$K\|L$	Orthogonal projection of K on L
$\lambda_i(K)$	i-th successive minimum of K
$\Lambda^\star$	Polar lattice of Λ
$\mathsf{L}G$	Line graph of G
$\mathrm{M}(G)$	Matching polytope of G
$\mu(K)$	Covering radius of K
$\|x\|$	Euclidean norm of x
$\|x\|_K$	Gauge function of K at x
$O(\cdot)$	Landau O-notation
$\mathbb{1}_n$	All-ones vector of length n
$\mathbb{1}[X]$	Indicator vector/function of a set X
p_{ij}	Selling parameters of Λ
$\mathrm{sh}_u(K)$	Orthogonal Blasche shaking of K on $u^\perp$
$\mathrm{sh}_{u,\ell}(K)$	Blaschke shaking of K on ℓ along u
S_n	Group of permutations of n elements
$\mathrm{Stab}(G)$	Stable set polytope of G
$\mathrm{st}(G)$	Number of standard labelings of G
$\mathrm{U}K$	Unconditional body obtained from an anti-blocking body K
V_Λ	Voronoi cell of Λ
vol	Lesbesgue measure in $\mathbb{R}^n$
vol_k	Lesbesgue measure in a k-dimensional subspace

Index