Dynamical Properties of Rough Delay Equations

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Zusammenfassung

In dieser Monografie untersuchen wir das Langzeitverhalten von stochastischen Verzögerungsgleichungen. Unser Ansatz sind *zufällige dynamische Systeme*, und wir lösen unsere Gleichung unter dem Gesichtspunkt der Theorie der *rough paths.* Wir befassen uns vor allem mit dem singulären Fall, in dem die Verzögerungsterme auch im Diffusionsteil vorkommen.

Obwohl wir die Gleichung mit den klassischen Werkzeugen der stochastischen Analysis lösen können, ist das Haupthindernis das Fehlen der Flusseigenschaft. Genauer gesagt hängt die Lösung nicht kontinuierlich vom Anfangswert ab. Um dieses Problem zu lösen, definieren wir diese Eigenschaft anders. Wir werden zeigen, wie wir eine Flusseigenschaft auf Feldern von Banach-Räumen mithilfe der rough path Theorie erzeugen können. Infolgedessen beweisen wir die Kozykel-Eigenschaft und stellen ein Wong-Zakai-Theorem auf. Da wir die rough path Theorie verwenden, können wir unsere Ergebnisse auf den Fall anwenden, dass das Rauschen aus Brownschen Bewegungen oder fraktionalen Brownschen Bewegungen mit $\frac{1}{3} < H < \frac{1}{2}$ besteht.

Das wichtigste Theorem in zufälligen dynamischen Systemen ist der berühmte *Multiplikative Ergodensatz* (MET). Angeregt durch unseren Rahmen beweisen wir eine Version dieses Theorems auf Feldern von Banachräumen. Außerdem zeigen wir unter der Annahme der Invertierbarkeit der Basis das Oseledets Splitting. Anschließend wenden wir dieses Theorem auf die stochastischen linearen Verzögerungsgleichungen an und zeigen, dass die linearen Verzögerungsgleichungen ein Lyapunov-Spektrum besitzen. Dieses Ergebnis ist bemerkenswert, denn es liefert eine umfassende Erklärung für die Stabilität und das chaotische Verhalten der stochastischen Verzögerungsgleichung.

Das Vorhandensein von invarianten Mannigfaltigkeiten ist eine Anwendung des MET. Mithilfe des MET beweisen wir dieses Theorem für nichtlineare Kozykeln, die auf messbaren Feldern von Banach-Räumen wirken. Insbesondere beweisen wir lokale stabile und instabile Mannigfaltigkeit für nichtlineare, singuläre stochastische Verzögerungsgleichungen um die stationären Punkte.

Diese Monografie enthält auch ein eigenständiges Kapitel über das Konzept der metrischen Entropie für die stochastischen Flüsse, die in endlich vielen Richtungen invariant sind. Nach der Definition der Entropie für diese Klasse von Flüssen, beweisen wir die Ruelle'sche Ungleichung entsprechend. Diese Ungleichung besagt dass, die metrische Entropie durch die Summe der positiven Lyapunov-Exponenten begrenzt ist.

Abstract

In this monograph, we investigate the long-time behavior of stochastic delay equations. Our approach is *random dynamical systems*, and we solve our equation in the *rough path* point of view. Namely, we deal with the singular case, i.e., when the delay terms also are appearing in the diffusion part.

Although we can solve the equation using the classical tools of stochastic analysis, the main obstacle is the lack of flow property. More precisely, the solution does not depend continuously on the initial value. To solve this problem, we define this property differently. We will show how we can generate a flow property on fields of Banach spaces using rough path theory. As a consequence, we prove the cocycle property and establish a Wong-Zakai theorem. Since we use rough path theory, we can apply our results to the case where the noise consists of Brownian motions or fractional Brownian motions with $\frac{1}{3} < H < \frac{1}{2}$.

The main theorem in random dynamical systems is the celebrated *multiplicative ergodic theorem* (MET). Inspired by our framework, we prove a version of this theorem on fields of Banach spaces. Moreover, assuming the invertibility of the basis, we show Oseledets splitting. We then apply this theorem to stochastic linear delay equations and demonstrate linear delay equations possess a Lyapunov spectrum. This result is remarkable, as it provides a comprehensive explanation for the stability and chaotic behavior of the stochastic delay flows.

The existence of invariant manifolds is an application of the MET. Using the MET, we prove this theorem for nonlinear cocycles acting on measurable fields of Banach spaces. In particular, we prove local stable and unstable manifold theorems for nonlinear, singular stochastic delay differential equations around the stationary points.

This monograph also contains a separate chapter on the concept of the metric entropy for the stochastic flows, which are invariant in finitely many directions. Having defined the entropy for this class of flows, we prove Ruelle's inequality accordingly. This inequality states that metric entropy is bounded by the sum of the positive Lyapunov exponents.

Dedicated to my family

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1

Introduction

Abstract

The aim of this monograph is the study of the long-time behavior of stochastic delay equations. Our approach to this is pathwise. Namely, we will consider equations that can be solved using the rough paths method developed by T. Lyons. The main tools used in this work are the theory of *rough paths* and the *random dynamical system* method. In the following, we briefly review these two subjects, and at the end, we outline our contributions in this thesis.

Rough Path Theory

Let us start with the rough path theory first. Deterministic systems do not capture the essence of fluctuations in the real situation. In fact, for many physical phenomena, our system is subject to white noise. In the simple case, our system is controlled by the following *controlled* ODE

$$\dot{Y}_t = f_0(Y_t) + f_1(Y_t)\dot{X}_t. \tag{1.0.1}$$

Here Y is our output which takes values in $\mathbb{R}^d$. Typically, X is a process that does not have differentiable sample paths like the Brownian motion. The main challenge here is to make sense of this equation. Indeed, if we reformulate equation (1.0.1) in integral form, then

$$Y_t = Y_0 + \int_0^t f_0(Y_s)ds + \int_0^t f_1(Y_s)dX_s. \tag{1.0.2}$$

The question now is how we can define the second integral. When X is a Brownian motion or a martingale, this integral is typically defined in the Itô sense. In this case, the Itô integral can be defined like the Riemann–Stieltjes integral, that is, as a limit in probability of Riemann sums; such a limit does not necessarily exist pathwise.

On the other hand, the rough path theory provides an alternative approach to solve stochastic differential equations. It is even more general in many respects since it is not based on the classical martingale framework. The main difference with the classical probabilistic approach is, it depends on some algebraic machinery.

Returning to (1.0.2), we fix the path X and do not assume that it is a random process; what the theory of rough path is providing is to make sense of this equation in terms of additional information of X. For simplicity, let us assume that X is a α- Hölder path such that $\frac{1}{3} < \alpha < \frac{1}{2}$, and we can give meaning to the second iterated integral $\Gamma_{s,t} := \int_s^t X_{s,\tau} dX_\tau$, moreover assume Γ has 2α-regularity. Then under some regularity assumptions on f_0 and f_1, we can give meaning to this equation in terms of X, $\int X dX$, and initial value.

In the case of Brownian motion, the natural way to define the second iterated integrals is the usual Itô or Stratonovich integral:

$$\int_s^t B_{s,\tau} dB_\tau, \qquad \int_s^t B_{s,\tau} d^\circ B_\tau.$$

The rough path theory is robust to consider other equations subject to the irregular driving signal. Moreover, ideas from the rough path have been shown conclusive, especially for equations in infinite-dimensional spaces. To wit, Martin Hairer used this idea to invent the theory of regularity structures. He then applied this theory to solve celebrated ill-posed stochastic partial differential equations, including Burgers type and the KPZ equation.

Random Dynamical System

Let us now briefly talk about the second tool. A random dynamical system (RDS) is a dynamical system in which the equation of motion contains an element of randomness. This approach goes back to L. Arnold [1].

The main components of a RDS are

- A *measurable dynamical system* in the sense of *ergodic* theory.
- A smooth (topological) dynamical system, typically generated by a differential equation.

To be more specific

- Let $(\Omega, \mathcal{F})$ and $(X, \mathcal{B})$ be measurable spaces. Let $\mathbb{T}$ be either $\mathbb{R}$ or $\mathbb{Z}$, equipped with a σ-algebra $\mathcal{I}$ given by the Borel σ-algebra $\mathcal{B}(\mathbb{R})$ in the case of $\mathbb{T} = \mathbb{R}$ and by $\mathcal{P}(\mathbb{Z})$ in the case of $\mathbb{T} = \mathbb{Z}$. A family $\theta = (\theta_t)_{t\in\mathbb{T}}$ of maps from Ω to itself is called a *measurable dynamical system* if

  (i) $(\omega, t) \mapsto \theta_t \omega$ is $\mathcal{F} \otimes \mathcal{I}/\mathcal{F}$-measurable,

  (ii) $\theta_0 = \mathrm{Id}$,

  (iii) $\theta_{s+t} = \theta_s \circ \theta_t$, for all $s, t \in \mathbb{T}$.

  If $\mathbb{T} = \mathbb{Z}$, we will also use the notation $\theta := \theta_1$, $\theta^n := \theta_n$ and $\theta^{-n} := \theta_{-n}$ for $n \geq 1$. If $\mathbb{P}$ is furthermore a probability on $(\Omega, \mathcal{F})$ that is invariant under any of the elements of θ,

  $$\mathbb{P} \circ \theta_t^{-1} = \mathbb{P}$$

  for every $t \in \mathbb{T}$, we call the tuple $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$ a *measurable metric dynamical system.*

- Let $\mathbb{T}^+ := \{t \in \mathbb{T} : t \geq 0\}$, equipped with the trace σ-algebra. A measurable random dynamical system on $(X, \mathcal{B})$ is a measurable metric dynamical system $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$ with a measurable map
$$\varphi\colon \mathbb{T}^+ \times \Omega \times X \to X$$
that enjoys the *cocycle property*, i.e. $\varphi(0, \omega, \cdot) = \mathrm{Id}_X$, for all $\omega \in \Omega$, and
$$\varphi(t + s, \omega, \cdot) = \varphi(t, \theta_s \omega, \cdot) \circ \varphi(s, \omega, \cdot)$$

for all $s, t \in \mathbb{T}^+$ and $\omega \in \Omega$. The map φ is called *cocycle*.

This theory allows us to describe not only whether a solution is stable or unstable, but also to identify the *directions* of stability, using the concept of stable or unstable invariant manifolds [2, 3]. Furthermore, domains of attraction can be identified using random attractors [4, 5, 6], and stochastic bifurcation can be studied [1, Chapter 9]. The concept of random dynamical systems was successfully applied to stochastic differential equations (SDEs) in finite and infinite dimensions. It is a natural approach to study the long-time behaviour of stochastic delay equations.

The results of this thesis

As we stated earlier, the primary motivation of this thesis is to investigate the long-time behavior of *stochastic delay differential equations* (SDDE). In particular, we are interested in the application of the RDS approach. Let us now summarize the results of this monograph.

Chapter 2. The celebrated *multiplicative ergodic theorem* (MET) is the main theorem in RDS. This theorem is describing the generic asymptotic behavior of a *stationary* product of linear operators $A^n = A_n \circ A_{n-1} \circ ... \circ A_1$. Remarkably, in many cases of natural interest, this body of ergodic-theoretical tools ensures that generic compositions of these operators exhibit defined asymptotic exponential growth rates in various directions in the underlying vector space. Originally proved by Oseledets in the late 1960s for compositions of $d \times d$ matrices, the MET has been extended and refined in the ensuing years in various follow-up works. The MET forms the theoretical foundation for many areas of dynamical systems research, notably smooth ergodic theory and the theory of SRB measures for both finite-dimensional systems (ODE and SDE) and infinite-dimensional systems (PDE and SPDE). To fix ideas while retaining some informality, let us consider compositions of a stationary sequence $A_1, A_2, ...$ of $d \times d$ real matrices. Indeed, A_i's are random matrices drawn from the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$, $A_i : \Omega \to M^{d\times d}(\mathbb{R})$, and that the law of A_i for $\mathbb{P}$ does not depend on i. Let us agree to write $A^i = A_i \circ A_{i-1} \circ ... \circ A_1$ for i-th random composition. The MET itself has two parts. The first is the "one-sided" MET for compositions of operators drawn from a "one-sided" stationary sequence of linear operators. The one-sided version says that, under suitable ergodicity and integrability assumptions, the following holds: there exist (deterministic) constants $1 \leqslant r \leqslant d$ and $\lambda_1 > \lambda_2 > ... > \lambda_r$, as well as a random filtration

$$\{0\} = F_{r+1} \subset F_r \subset ... \subset F_2 \subset F_1 = \mathbb{R}^d,$$

of $\mathbb{R}^d$ with the property that with probability 1,

$$\lim_{n\to\infty} \frac{1}{n} \log \|A^n v\| = \lambda_i,$$

for all $v \in F_i \setminus F_{i+1}$. Note, of course, that the sequence (A_i) is correlated with the filtration of subspace (F_i). The second aspect of the MET is its "two-sided" version, which in this context is stated for bi-infinite stationary sequences $(A_i)_{i\in\mathbb{Z}}$ of $d \times d$ matrices, $A_i : \Omega \to M_{d\times d}$. Define $A^{-n} = (A_0 \circ ... \circ A_{-(n-1)})^{-1}$ for $n > 0$, and for now assume that the A_i are almost-surely invertible. Then, under suitable ergodicity and integrability assumptions, with $\lambda_1, ..., \lambda_r$ as in the one-sided MET, there exists a random splitting

$$\mathbb{R}^d = E_1 \oplus ... \oplus E_r$$

of $\mathbb{R}^d$ into random subspaces E_i so that with probability 1, we have

$$\lim_{n\to\infty} \frac{1}{n} \log \|A^{-n} v\| = -\lambda_i,$$

for all $v \in E_i \setminus \{0\}$, $1 \leqslant i \leqslant d$. Various extensions of the scope and assumptions of these theorems have been made over the years, e.g.: extending to stationary compositions of compact or quasi-compact linear operators on Hilbert and Banach spaces, as well as to compositions of linear operators which are not invertible (or not even necessarily injective), the latter sometimes being referred to as "semi-invertible" MET's. Motivated by our model we made following contributions on the MET state-of-the-art:

(i) an MET for stationary compositions on a (possibly random) field of (potentially distinct) Banach spaces, depending on the random sample.

(ii) weakening the measurability requirements for the stationary sequence.

Our main theorems in this chapter are Theorem 2.2.16 which is obtained in collaboration with Prof. Michael Scheutzow and Dr. Sebastian Riedel and Theorem 2.3.20, brought in partnership with Dr. Sebastian Riedel.

Chapter 3. Invariant manifolds are topological manifolds that are invariant under the action of the dynamical system. In ODE's, these invariant sets play an essential role in questions of stability and bifurcations near the equilibrium points. To illustrate the idea of this chapter, we begin with the following simple example from ODE

$$\begin{cases} \dot{X} = -\lambda X + F(X,Y), & \lambda > 0 \ \text{ and } \ F(0,0) = 0 \\ \dot{Y} = \mu Y + G(X,Y), & \mu > 0 \ \text{ and } \ G(0,0) = 0. \end{cases}$$

The invariant manifold theorem states that the solutions of this system near $(0,0)$ look like the following picture

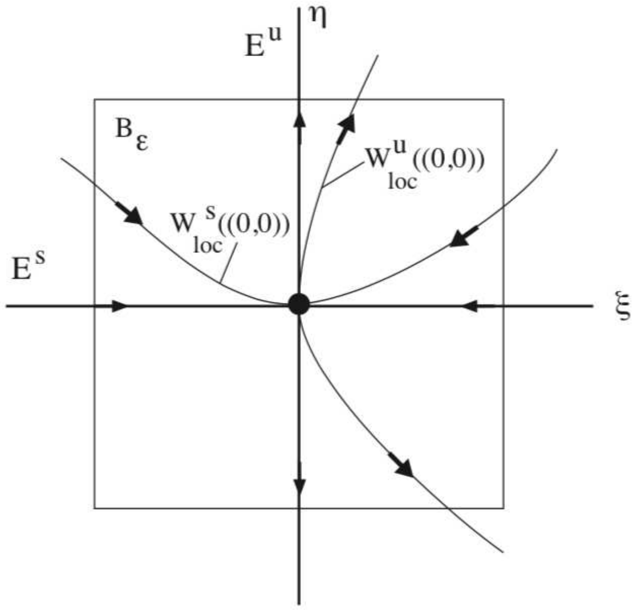

Figure 1.1: Stable and Unstable manifolds

the Indeed, we have two curves, so if our initial value starts on either curve, the corresponding trajectory stays in the same curve. In the stable curve, the solution converges exponentially toward the equilibrium point. In contrast, in the unstable curve, the solution escapes away from the equilibrium. However, in the negative times (if we are allowed to go back), the trajectory converges to the equilibrium point. We should also point out that there is another invariant curve called the center manifold; the behavior of the solution in this set can be either stable or unstable.

The key ingredient for these theorems is the linearized equation at the equilibrium point. Returning to RDS, the MET shows that linear and linearised random dynamical systems possess a *Lyapunov spectrum* which can be interpreted as an analog to the spectrum of eigenvalues of a matrix. Here, positive Lyapunov exponents lead to the existence of an unstable manifold, and similarly, the negative Lyapunov exponents generate the stable manifold.

The main theorems of this chapter are Theorem 3.2.9 and Theorem 3.3.6, both of these theorems are obtained with Dr. Sebastian Riedel.

Chapter 4. Stochastic delay equations are the type of stochastic equations, in which the derivative of the function is given in terms of the values of the function at present and in previous times. The simple case of this type of equation is in the form of

$$dy_t = b(y_t, y_{t-r})\,dt + \sigma(y_t, y_{t-r})\,dB_t(\omega). \tag{1.0.3}$$

Here B can be a Brownian motion or a fractional Brownian motion with $\frac{1}{3} < H < \frac{1}{2}$.

Although we can solve the equation by assuming standard assumptions on b and σ, the absence of the flow property is a severe obstacle to a dynamical theory. As a result, it was long believed that a dynamical approach to this type of equation was impossible. Our strategy to debunk this problem is to solve the equations using the theory of rough paths and then generate a flow property in the fibers of Banach spaces. More precisely, we do not fix the space of initial values and let this space also depends on our random object; this space is updated as we evolve in time.

It turns out that we can define a fiber-like dynamical system. This new setting indeed coincides with our framework in Chapter 2 and Chapter 3. Recall, in these chapters, we developed a version of MET and applied this theorem to prove the existence of invariant manifolds.

The main contributions of this chapter are

- We extend the theory of rough delay differential equations introduced by Deya, Neuenkirch, and Tindel [7]. The new results are an a priori bound for linear equations (Theorem 4.2.11), a semi-flow property (Theorem 4.2.13), a Wong-Zakai theorem (Theorem 4.2.28), and the existence of the random dynamical system (Theorem 4.3.7).

- We show that SDDE induces an RDS on a field of Banach spaces where the fibers are (essentially) the spaces of controlled paths (Theorem 4.3.14). En passant, we prove that the spaces of controlled paths form a measurable field of Banach spaces, which we believe is interesting in its own right since it sheds new light on the geometry of the spaces of controlled paths.

- We apply the MET to linear SDDE and prove the existence of a Lyapunov spectrum (Theorem 4.4.1 and Corollary 4.4.2). In the case of the (simple) SDDE (4.1.3), we show that the largest Lyapunov exponent coincides with the exponential growth rate, which was studied in [8] (Theorem 4.5.1).

I obtained the results of this chapter in collaboration with Prof. Michael Scheutzow and Dr. Sebastian Riedel.

Chapter 5. This chapter is the sequel to the previous chapter. Indeed, we notably address the nonlinear equations and harvest the fruits of our previous results. We accept our framework in Chapter 4. The main idea is to apply our Multiplicative ergodic theorem and our theorems on invariant manifolds from Chapter 2 and Chapter 3.

We first solve our equations when the drift components can be unbounded (but linear), and the nonlinear diffusion coefficient satisfies certain smoothness assumptions. The primary technique here is to decompose the flows. After solving the equation, we prove the regularity (In Fréchet sense) and estimate the growth of the solution. Based on these estimates and earlier results in other chapters, we show the existence of local stable/unstable manifolds around a stationary trajectory for delay equations. We then give two examples: a rough delay equation having 0 as a stationary solution and an Itô delay equation with an exponentially stable linear part.

The main results of this chapter are Theorem 5.4.4 and Theorem 5.4.5 where we prove the existence of invariant manifolds for nonlinear stochastic delay equations. Both of these theorems are based on joint work with Dr. Sebastian Riedel.

Chapter 6. This chapter is independent of the other chapters. This chapter aims to define and explore metric entropy for specific flows that are invariant under a finite of family vectors. Metric entropy is an essential concept in ergodic theory that measures the chaoticity of the system. In fact, the positive Lyapunov exponents are responsible for the chaotic behavior of the systems. After defining this concept for our model (while there is not an invariant measure) and generating the Lyapunov exponents, we prove in this chapter that our entropy is less than and equal to the sum of the Lyapunov exponents. This result is known as Ruelle's inequality.

The main results of this chapter are Theorems 6.3.5 and 6.3.7. And these results are obtained with collaboration with Prof. Michael Scheutzow, Prof. Marc Keßeböhmer and Dr. Vitalii Senin.

2

Multiplicative Ergodic Theorem

2.1 Introduction

The Multiplicative Ergodic Theorem (MET) is a powerful tool with various applications in different fields of mathematics, including analysis, probability theory, and geometry, and a cornerstone in smooth ergodic theory. It was first proved by Oseledets [9] for matrix cocycles. Since then, the theorem attracted many researchers to provide new proofs and formulations with increasing generality [10, 11, 12, 13, 14, 15, 16, 17, 18, 19]. In this section we give a proof for an MET for cocycles acting on *measurable fields of Banach spaces.* We first prove the MET and then apply it to prove the Oseledets splitting. Let us quickly recall the setting here: If $(\Omega, \mathcal{F}, \mathbb{P})$ denotes a probability space, we call a family of Banach spaces $\{E_\omega\}_{\omega\in\Omega}$ a measurable field if there exists a linear subspace Δ of all sections $\Pi_{\omega\in\Omega}E_\omega$ and a countable subset $\Delta_0 \subset \Delta$ such that $\{g(\omega) \,:\, g \in \Delta_0\}$ is dense in E_ω for every $\omega \in \Omega$ and $\omega \mapsto \|g(\omega)\|_{E_\omega}$ is measurable for every $g \in \Delta$. Note that this definition implies that every Banach space E_ω is separable. On the other hand, every separable Banach space defines a field of Banach spaces by simply setting $E_\omega = E$. This structure is similar to a measurable version of a Banach bundle with base Ω and total space $\Pi_{\omega\in\Omega}E_\omega$ in which every space E_ω is a fiber. However, the fundamental difference is that we do *not* put any measurable (or topological) structure on the bundle $\Pi_{\omega\in\Omega}E_\omega$ itself! In fact, the existence of the set Δ is a substitute for the measurable structure and will help to prove measurability for functionals defined on $\Pi_{\omega\in\Omega}E_\omega$ as we will see many times in this chapter. Remember for a measure preserving dynamical systems $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$, a *cocycle* acting on the field $\{E_\omega\}_{\omega\in\Omega}$ consists of a family of maps $\psi_\omega \colon E_\omega \to E_{\theta\omega}$. Setting $\psi^n_\omega := \psi_{\theta^{n-1}\omega} \circ \cdots \circ \psi_\omega$, we furthermore assume that $\omega \mapsto \|\psi^n_\omega(g(\omega))\|_{E_{\theta^n\omega}}$ is measurable for every $g \in \Delta$ and every $n \in \mathbb{N}$. Our first main result in this chapter is a MET on a measurable field of Banach spaces. We state a simplified version here, the full statement can be found in Theorem 2.2.16 below.

Theorem 2.1.1. *Let $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$ be an ergodic measurable metric dynamical system and ψ be a compact linear cocycle acting on a measurable field of Banach spaces $\{E_\omega\}_{\omega\in\Omega}$. For*

$\mu \in \mathbb{R} \cup \{-\infty\}$ and $\omega \in \Omega$, define

$$F_\mu(\omega) := \{x \in E_\omega \, : \, \limsup_{n\to\infty} \frac{1}{n} \log \|\psi_\omega^n(x)\| \leqslant \mu\}.$$

Assume that

$$\log^+ \|\psi_\omega\| \in L^1(\Omega).$$

Then there is a measurable forward invariant set $\tilde{\Omega} \subset \Omega$ of full measure and a decreasing sequence $\{\mu_i\}_{i\geq 1}$, $\mu_i \in [-\infty, \infty)$ with the properties that $\lim_{n\to\infty} \mu_n = -\infty$ and either $\mu_i > \mu_{i+1}$ or $\mu_i = \mu_{i+1} = -\infty$ such that for every $\omega \in \tilde{\Omega}$,

$$x \in F_{\mu_i}(\omega) \setminus F_{\mu_{i+1}}(\omega) \quad \text{if and only if} \quad \lim_{n\to\infty} \frac{1}{n} \log \|\psi_\omega^n(x)\| = \mu_i. \tag{2.1.1}$$

Moreover, there are numbers $m_1, m_2, \ldots$ such that $\operatorname{codim} F_{\mu_j}(\omega) = m_1 + \ldots + m_{j-1}$ for every $\omega \in \tilde{\Omega}$.

Let us mention here that, motivated by our example of a stochastic delay equation, we proved this theorem for compact cocycles only, but it is straightforward to generalize it to the quasi-compact case as Thieullen did in [14]. Consequently, we believe that all our results in this work will hold for quasi-compact cocycles, too.

The numbers $\{\mu_i\}$ are the *Lyapunov exponents*, the subspaces $F_\mu(\omega)$ are sometimes called *slow-growing subspaces* and the resulting filtration

$$E_\omega = F_{\mu_1}(\omega) \supset F_{\mu_2}(\omega) \supset \ldots$$

is called *Oseledets filtration*. Is is easily seen that the slow-growing spaces are *equivariant*, meaning that $\psi_\omega(F_{\mu_i}(\omega)) \subset F_{\mu_i}(\theta\omega)$. In the proof of this theorem, no invertibility of θ or ψ is assumed, in which case a filtration of slow-growing subspaces is the best one can hope for. However, things change when we assume that the base θ is invertible. In this case, it is possible to deduce a *splitting* of the spaces E_ω consisting of *fast-growing subspaces* which are invariant under ψ. Such a splitting is called *Oseledets splitting*, and the corresponding theorem is called *semi-invertible MET*. Let us emphasize that we only need to assume invertibility of the base θ and no invertibility of the cocyle ψ. In the context of SPDE or stochastic delay equations, these assumptions are quite natural: θ usually denotes the shift of a random trajectory (which can be shifted forward and backward in time) and the cocycle denotes the solution map, which is not injective if the equation can be solved forward in time only.

Our second main result is a semi-invertible MET on a measurable field of Banach spaces. The full statement can be found in Theorem 2.3.20 below.

Theorem 2.1.2. *In addition to the assumptions made in Theorem 2.2.16, assume that θ is invertible with measurable inverse $\sigma := \theta^{-1}$ and that Assumption 2.3.1 holds. Then there is a θ-invariant set $\tilde{\Omega}$ of full measure such that for every $i \geq 1$ with $\mu_i > \mu_{i+1}$ and $\omega \in \tilde{\Omega}$, there is an m_i-dimensional subspace H_ω^i with the following properties:*

(i) (Invariance) $\psi_\omega^k(H_\omega^i) = H_{\theta^k\omega}^i$ for every $k \geq 0$.

(ii) (Splitting) $H^i_\omega \oplus F_{\mu_{i+1}}(\omega) = F_{\mu_i}(\omega)$. *In particular,*

$$E_\omega = H^1_\omega \oplus \cdots \oplus H^i_\omega \oplus F_{\mu_{i+1}}(\omega).$$

(iii) ('Fast-growing' subspace) *For each* $h_\omega \in H^i_\omega \setminus \{0\}$,

$$\lim_{n\to\infty} \frac{1}{n} \log \|\psi^n_\omega(h_\omega)\| = \mu_j$$

and

$$\lim_{n\to\infty} \frac{1}{n} \log \|(\psi^n_{\sigma^n\omega})^{-1}(h_\omega)\| = -\mu_j.$$

Moreover, the spaces are uniquely determined by properties (i), (ii) and (iii).

Clearly, the Oseledets splitting provides much more information about the cocycle than the filtration.

Let us discuss some important preceeding results. In the finite dimensional case, an MET for cocycles acting on measurable bundles can be found in the monograph [1, 4.2.6 Theorem] by L. Arnold. In [13], Mañé proved an MET with Oseledets splitting on a Banach bundle, assuming a topological structure on Ω and continuity of the map $\omega \mapsto \psi_\omega$. He also assumed injectivity of ψ. Besides these results, we are not aware of any METs for cocycles acting on a bundle-type structure. Lian and Lu [16] proved an MET for cocycles acting on a fixed Banach space, assuming only a measurable structure on Ω, but injectivity of the cocycle. This assumption was later removed by Doan in [17] without giving an Oseledets splitting, however. In [20], González-Tokman and Quas used this result as a "black-box" and proved that an Oseledets splitting holds in this case, too.

Let us mention that our result is not only the first which provides MET and a splitting on a bundle structure of Banach spaces without using a topological structure on Ω, it also weakens the measurability assumption on ψ significantly in case we are dealing with a single Banach space E only. In fact, the standard measurability assumption, for instance in [19], is *strong measurability* of ψ, meaning that for fixed $x \in E$, the map

$$\Omega \ni \omega \mapsto \psi_\omega(x) \in E \tag{2.1.2}$$

should be measurable. In contrast, our assumption means that the maps

$$\Omega \ni \omega \mapsto \|\psi^{k+n}_\omega(x) - \psi^k_{\theta^n\omega}(\tilde{x})\|_E \in \mathbb{R}$$

should be measurable for every $n, k \in \mathbb{N}_0$ and $x, \tilde{x} \in S$ where S is a countable and dense subset of E. This assumption is clearly implied by (2.1.2).

The proof of Theorem 2.1.2 pushes forward the volume growth-approach advocated by Blumenthal [18] and González-Tokman, Quas [19] which provides a clear growth interpretation of the Lyapunov exponents. In a way, our result covers and complements these two works in case of a single Banach space E. In particular, we are not imposing any further assumptions on E like reflexivity or separability of the dual as in [19].

The structure of this chapter is as follows. In Section 2.2, we prove a MET for cocycles acting on measurable fields of Banach spaces. In Section 2.3 we assume in addition, the base θ is invertible and then prove a semi-invertible MET again for cocycles acting on measurable fields of Banach spaces.

Notation

- For Banach spaces $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$, $L(X,Y)$ denotes the space of bounded linear functions from X to Y equipped with usual operator norm. We will often not explicitly write a subindex for Banach space norms and use the symbol $\|\cdot\|$ instead. *Differentiability* of a function $f\colon X \to Y$ will always mean Fréchet-differentiability. A C^m *function* denotes an m-times Fréchet-differentiable function such that the m-th order derivatives are continuous. If $A, B \subseteq X$, we denote by $d(A,B) := \inf_{a\in A, b\in B} \|a-b\|$ the distance between two sets A and B. We also set $d(x,B) := d(B,x) := d(\{x\}, B)$ for $x \in X$, $B \subseteq X$.

- Let E be a vector space. If we can write E as a direct sum $E = F \oplus H$ of vector spaces, we have an *algebraic splitting*. We also say that F *is a complement of* H and vice versa. The projection operator $\Pi_{F\|H}(e) = f$ with $e = f + h$, $f \in F$, $h \in H$, is called the *projection operator onto* F *parallel to* H. If E is a normed space and $\Pi_{F\|H}$ is bounded linear, i.e.

$$\|\Pi_{F\|H}\| = \sup_{f\in F, h\in H, f+h\neq 0} \frac{\|f\|}{\|f+h\|} < \infty,$$

  we call $E = F \oplus H$ a *topological splitting*. For normed spaces, a *splitting* will always mean a topological splitting.

- Let $(\Omega, \mathcal{F})$ be a measurable space. We call a family of Banach spaces $\{E_\omega\}_{\omega\in\Omega}$ a *measurable field of Banach spaces* if there is a set of sections

$$\Delta \subset \prod_{\omega\in\Omega} E_\omega$$

  with the following properties:

  (i) Δ is a linear subspace of $\prod_{\omega\in\Omega} E_\omega$.

  (ii) There is a countable subset $\Delta_0 \subset \Delta$ such that for every $\omega \in \Omega$, the set $\{g(\omega) : g \in \Delta_0\}$ is dense in E_ω.

  (iii) For every $g \in \Delta$, the map $\omega \mapsto \|g(\omega)\|_{E_\omega}$ is measurable.

- Let $(\Omega, \mathcal{F})$ be a measurable space. If there exists a measurable map $\theta\colon \Omega \to \Omega$, $\omega \mapsto \theta\omega$, we call $(\Omega, \mathcal{F}, \theta)$ a *measurable dynamical system*. We will use the notation $\theta^n\omega$ for n-times applying θ to an element $\omega \in \Omega$. We also set $\theta^0 := \mathrm{Id}_\Omega$. If $\mathbb{P}$ is a probability measure on $(\Omega, \mathcal{F})$ that is invariant under θ, i.e. $\mathbb{P}(A) = \mathbb{P}(\theta^{-1}A)$ for every $A \in \mathcal{F}$, we call the tuple $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$ a *measure-preserving dynamical system*. The system is called *ergodic* if every θ-invariant set has probability 0 or 1.

- When we say θ is invertible then we also assume, θ^{-1} is measurable and we set $\theta^{-n} := (\theta^n)^{-1}$. In this case we call the tuple $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$ a *measure-preserving dynamical system* if for every $A \in \mathcal{F}$, $\mathbb{P}(A) = \mathbb{P}(\theta A) = \mathbb{P}(\theta^{-1}A)$.
- Let $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$ be a measure-preserving dynamical system and $(\{E_\omega\}_{\omega\in\Omega}, \Delta, \Delta_0)$ a measurable field of Banach spaces. A *continuous cocycle on* $\{E_\omega\}_{\omega\in\Omega}$ consists of a family of continuous maps

$$\psi_\omega \colon E_\omega \to E_{\theta\omega}. \tag{2.1.3}$$

If ψ is a continuous cocycle, we define $\psi^n_\omega \colon E_\omega \to E_{\theta^n\omega}$ as

$$\psi^n_\omega := \psi_{\theta^{n-1}\omega} \circ \cdots \circ \psi_\omega.$$

We also set $\psi^0_\omega := \mathrm{Id}_{E_\omega}$. We say that ψ *acts on* $\{E_\omega\}_{\omega\in\Omega}$ if the maps

$$\omega \mapsto \|\psi^n_\omega(g(\omega))\|_{E_{\theta^n\omega}}, \quad n \in \mathbb{N} \tag{2.1.4}$$

are measurable for every $g \in \Delta$. In this case, we will speak of a *continuous random dynamical system on a field of Banach spaces.* If the map (2.1.3) is bounded linear/compact, we call ψ a bounded linear/compact cocycle.

2.2 MET on fields of Banach spaces

Throughout this chapter, we assume $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$ is a measurable metric dynamical system and $(\{E_\omega\}_{\omega\in\Omega}, \Delta, \Delta_0)$ is a measurable field of Banach spaces and also that ψ is a bounded and linear cocycle acting on $\{E_\omega\}_{\omega\in\Omega}$.

We start with an easy observation.

Lemma 2.2.1. *For every $n \in \mathbb{N}$, the map*

$$\omega \mapsto \|\psi^n_\omega(.)\|_{L(E_\omega, E_{\theta^n\omega})}$$

is measurable.

Proof. Using properties of Δ and continuity of ψ,

$$\|\psi^n_\omega(.)\|_{L(E_\omega, E_{\theta^n\omega})} = \sup_{\xi\in E_\omega\setminus\{0\}} \frac{\|\psi^n_\omega(\xi)\|}{\|\xi\|} = \sup_{g\in\Delta_0} \frac{\|\psi^n_\omega(g(\omega))\|}{\|g(\omega)\|} \chi_{\{\|g\|>0\}}(\omega)$$

with the convention $\infty \cdot 0 = 0$. Since the fraction on the right hand side is a quotient of measurable functions and the supremum runs over a countable set, measurability follows. $\square$

The next lemma proves a further measurability result. The assumptions will be justified in the sequel.

Lemma 2.2.2. *For $\omega \in \Omega$ and $\mu \in \mathbb{R}$, define the subspace*

$$F_\mu(\omega) := \left\{ \xi \in E_\omega \, : \, \limsup_{n\to\infty} \frac{1}{n} \log \|\psi^n_\omega(\xi)\| \leq \mu \right\}.$$

Assume that there is a strictly decreasing sequence $(\mu_j)_{1\leqslant j\leqslant N}$, $N \leqslant \infty$, and a θ-invariant, measurable set $\Omega_0 \subset \Omega$ of full measure with the following properties:

(i) $F_{\mu_1}(\omega) = E_\omega$ *for every* $\omega \in \Omega_0$.

(ii) For every $j < N$, *there is a number* $m_j \in \mathbb{N}$ *such that* $F_{\mu_{j+1}}(\omega)$ *is closed and* m_j*-codimensional in* $F_{\mu_j}(\omega)$ *for every* $\omega \in \Omega_0$.

(iii) For every $j < N$,

$$\lim_{n\to\infty} \frac{1}{n} \log \|\psi^n_\omega(\cdot)|_{F_{\mu_j}(\omega)}\| = \mu_j \tag{2.2.1}$$

for every $\omega \in \Omega_0$.

(iv) For every $j < N$, *if* H^j_ω *is any complement of* $F_{\mu_{j+1}}(\omega)$ *in* $F_{\mu_j}(\omega)$,

$$\lim_{n\to\infty} \frac{1}{n} \log \inf_{h\in H^j_\omega\setminus\{0\}} \frac{\|\psi^n_\omega(h)\|}{\|h\|} = \mu_j \tag{2.2.2}$$

for every $\omega \in \Omega_0$.

(v)

$$\limsup_{n\to\infty} \frac{1}{n} \log \|\psi^n_\omega(\cdot)\ |_{F_{\mu_N}(\omega)}\ \| \leq \mu_N \tag{2.2.3}$$

for every $\omega \in \Omega_0$.

Then for every $n \in \mathbb{N}$ *and* $j \leqslant N$, *the map*

$$\omega \mapsto \|\psi^n_\omega(\cdot)\ |_{F_{\mu_j}(\omega)}\ \|\chi_{\Omega_0}(\omega) \tag{2.2.4}$$

is measurable.

Proof. First we claim that for every $g \in \Delta$ and $j \leqslant N$ the map

$$\omega \mapsto d\big(g(\omega), F_{\mu_j}(\omega)\big) \tag{2.2.5}$$

is measurable. To see this, it suffices to show measurability of the function

$$d\big(g(\omega), S_{F_{\mu_j}(\omega)}\big) := \inf_{\substack{\xi\in F_{\mu_j}(\omega)\\ \|\xi\|=1}} \|g(\omega) - \xi\|$$

where $S_{F_{\mu_j}(\omega)}$ is the unit sphere in $F_{\mu_j}(\omega)$. We use induction to prove the claim. The statement is clear for $j = 1$, so let $j \geqslant 2$. For every $1 \leqslant i < j$, since $\dim \big[\frac{F_{\mu_i}(\omega)}{F_{\mu_{i+1}}(\omega)}\big] < \infty$, we can find a finite-dimensional subspace $H_i(\omega)$ such that for a constant[1] M,

$$F_{\mu_i}(\omega) = H_i(\omega) \oplus F_{\mu_{i+1}}(\omega) \quad \text{and} \quad \|\pi_{H_i(\omega)||F_{\mu_{i+1}}(\omega)}\| < M. \tag{2.2.6}$$

[1] The existence of this complement with the given bound for the projection is a classical result and follows e.g. from [21, III.B.11], cf. also [18, Lemma 2.3].

For $\mu_0 := \mu_1$ and $l, k \geq 1$ set

$$B_\omega^{l,k}(\mu_j) = \Big\{ \xi \in E_\omega : \ \|\xi\| = 1, \ \|\psi_\omega^k(\xi)\| < \exp\big(k(\mu_j + \frac{1}{l})\big) \text{ and } \\ d(\xi, F_{\mu_i}(\omega)) < \exp\big(k(\mu_j - \mu_{i-1})\big), 1 \leqslant i < j \Big\}.$$

We claim that

$$d(g(\omega), S_{F_{\mu_j}(\omega)}) = \lim_{k\to\infty} \liminf_{l\to\infty} d(g(\omega), B_\omega^{l,k}(\mu_j)). \tag{2.2.7}$$

Set the right side equal to A. By definition, it is straightforward to show that $d(g(\omega), S_{F_{\mu_j}(\omega)}) \geqslant A$. For the opposite direction, let $\epsilon > 0$. For large k, l we can find $\xi^{l,k} \in B_\omega^{l,k}(\mu_j)$ such that $\|g(\omega) - \xi^{l,k}\| \leqslant A + \epsilon$. By our assumptions on $B_\omega^{k,l}(\mu_j)$, we have a decomposition of the form

$$\xi^{l,k} = \sum_{1\leqslant i<j-1} h_i^{l,k} + h_{j-1}^{l,k} + f^{l,k}$$

such that for $1 \leqslant i < j$, $h_i^{l,k} \in H_i(\omega)$ and $f^{l,k} \in F_{\mu_j}(\omega)$. Moreover, there is a constant $\tilde{M}$ such that for $1 \leqslant i < j-1$,

$$\|h_i^{l,k}\| < \tilde{M}\, d(\xi^{l,k}, F_{\mu_{i+1}}(\omega)) \quad \text{and} \quad \|f^{l,k}\| < \tilde{M}.$$

From (2.2.2), choosing k larger if necessary, we obtain that for a given $\delta > 0$,

$$\exp\big(k(\mu_{j-1} - \delta)\big)\|h_{j-1}^{l,k}\| \leqslant \|\psi_\omega^k(h_{j-1}^{l,k})\| \leqslant \|\psi_\omega^k(\xi^{l,k})\| + \\ \sum_{1\leqslant i<j-1} \|\psi_\omega^k(h_i^{l,k})\| + \tilde{M}\, \|\psi_\omega^k(\cdot)|_{F_{\mu_j}(\omega)}\|.$$

Consequently, from our assumptions on $B_\omega^{l,k}(\mu_j)$ and (2.2.1), we obtain for large l, k

$$\|h_{j-1}^{l,k}\| \leqslant \tilde{M}_0 \exp\big(k(\mu_j - \mu_{j-1} + 2\delta)\big)$$

for a constant $\tilde{M}_0$. Now for large l, k,

$$\|\sum_{1\leqslant i<j} h_i^{l,k}\| < \epsilon\,, \qquad 1 - \epsilon \leqslant \|f^{l,k}\| \leqslant 1 + \epsilon.$$

Consequently, $d(g(\omega), S_{F_{\mu_j(\omega)}}(\omega)) \leqslant A$ and (2.2.7) is proved. The rest of the proof is straightforward: For $\tilde{g} \in \Delta$ we set

$$C^{l,k,j}(\tilde{g}) := \{\omega \ : \ \frac{\tilde{g}(\omega)}{\|\tilde{g}(\omega)\|} \in B_\omega^{l,k}(\mu_j)\}$$

From the definition of $B_\omega^{l,k}(\mu_j)$ and the induction hypothesis, $C^{l,k,j}(\tilde{g})$ is measurable for every $k, l \geq 1$. Note that

$$d(g(\omega), S_{F_{\mu_j}(\omega)}) = \inf_{\tilde{g}\in\Delta_0} J_{\tilde{g}}(\omega)$$

where

$$J_{\tilde{g}}(\omega) = \begin{cases} \infty & \text{if } \omega \notin C^{l,k,j}(\tilde{g}) \\ \|g(\omega) - \frac{\tilde{g}(\omega)}{\|\tilde{g}(\omega)\|}\| & \text{otherwise.} \end{cases} \tag{2.2.8}$$

Since $J_{\tilde{g}}(\omega)$ is measurable, this proves the claim. Therefore, we have also shown measurability of $C^{l,k,j}(g)$ for every $j,k,l \geq 1$ and $g \in \Delta$. Next, with the same argument as above, we can show that

$$\|(\psi_\omega^n(\cdot) \mid_{F_{\mu_j}(\omega)}\|\chi_{\Omega_0}(\omega) = \lim_{l\to\infty} \liminf_{k\to\infty} \left[\sup_{\xi \in B_\omega^{l,k}(\mu_j)} \|\psi_\omega^k(\xi)\| \right] \chi_{\Omega_0}(\omega)$$

for every $j \geq 2$. Since

$$\sup_{\xi \in B_\omega^{l,k}(\mu_j)} \|\psi_\omega^k(\xi)\| = \sup_{g \in \Delta_0} \frac{\|\psi_\omega^k(g(\omega))\|}{\|g(\omega)\|} \chi_{C^{l,k,j}(g)}(\omega),$$

measurability of (2.2.4) follows. □

We also have the following lemma.

Lemma 2.2.3. *Let the same assumptions as in Lemma 2.2.2 be satisfied. Then there exists a θ-invariant, measurable set $\Omega_1 \subset \Omega$ of full measure such that for every $\omega \in \Omega_1$, if H_ω is a complement of $F_{\mu_2}(\omega)$ in E_ω, we have*

$$\lim_{n\to\infty} \frac{1}{n} \log \|\Pi_{\psi_\omega^n(H_\omega)\|F_{\mu_2}(\theta^n\omega)}\| = 0. \tag{2.2.9}$$

Proof. It is enough to show that

$$\limsup_{n\to\infty} \log \|\Pi_{\psi_\omega^n(H_\omega)\|F_{\mu_2}(\theta^n\omega)}\| \leqslant 0. \tag{2.2.10}$$

Define

$$\phi_1(\omega) = \sup_{p\geqslant 0} \exp\left(-p(\mu_1+\delta)\right) \|\psi_\omega^p(\cdot)\|$$

$$\phi_2(\omega) = \sup_{p\geqslant 0} \exp\left(-p(\mu_2+\delta)\right) \|\psi_\omega^p(\cdot)|_{F_{\mu_2}(\omega)}\|$$

From Lemma 2.2.2, ϕ_1 and ϕ_2 are measurable functions and bounded on a set of full measure Ω_0. So from [13, Lemma III.8], there exists a measurable subset Ω_1 of full measure such that for any $\omega \in \Omega_1$,

$$\lim_{n\to\infty} \frac{1}{n} \log^+ \phi(\theta^n\omega) = 0 \tag{2.2.11}$$

where $\phi(\omega) = \max\{\phi_1(\omega), \phi_2(\omega)\}$. Note that we can assume that Ω_1 is also θ-invariant, otherwise we can replace it by $\bigcap_{j\in\mathbb{Z}} (\theta^j)^{-1}(\Omega_1)$. Fix $\omega \in \Omega_1$ and assume that $H_\omega \oplus F_{\mu_2}(\omega) = E_\omega$.

Let $\epsilon > 0$. From (2.2.1) and (2.2.2), we can find an $N \in \mathbb{N}$ such that for $n \geqslant N$,

$$\begin{gathered}\|\psi_\omega^n(\cdot)\| \leqslant \exp\left(n(\mu_1+\delta)\right)\ ,\quad \inf_{h\in H_\omega\setminus\{0\}} \frac{\|\psi_\omega^n(h)\|}{\|h\|} \geqslant \exp\left(n(\mu_1-\delta)\right) \\ \phi(\theta^n\omega) \leqslant \exp(n\epsilon).\end{gathered} \tag{2.2.12}$$

We prove (2.2.10) by contradiction. Assume there is a $\gamma > 0$ and a sequence $(n_k, h_k, f_k) \in (\mathbb{N}, H_\omega, F_{\mu_2}(\theta^{n_k}\omega))$ such that

$$n_k \to \infty\ ,\ \|\ h_k\| = 1 \text{ and } \frac{\|\psi_\omega^{n_k}(h_k)\|}{\|\psi_\omega^{n_k}(h_k) - f_k\|} \geqslant \frac{1}{2}\exp(n_k\gamma) \text{ for all } k \geq 1. \tag{2.2.13}$$

For $p \geq 0$,

$$\begin{aligned}\|\psi_\omega^{n_k+p}(h_k)\| &= \|\psi_{\theta^{n_k}\omega}^p(\psi_\omega^{n_k}(h_k))\| \\ &\leqslant \|\psi_{\theta^{n_k}\omega}^p(.)\|\|\psi_\omega^{n_k}(h_k) - f_k\| + \|\psi_{\theta^{n_k}\omega}^p(\cdot)|_{F_{\theta^{n_k}\omega}}\|\|f_k\|\end{aligned} \tag{2.2.14}$$

From (2.2.13), it follows that $\|f_k\| \leqslant 3\|\psi_\omega^{n_k}(h_k)\|$. Now for large n_k, from (2.2.12) and (2.2.14),

$$\begin{aligned}\exp\left((n_k+p)(\mu_1-\delta)\right) \leqslant{}& 2\exp\Big(n_k\epsilon + p(\mu_1+\delta) + n_k(\mu_1+\delta) - n_k\gamma\Big) \\ &+ 3\exp\Big(p(\mu_2+\delta) + n_k\epsilon + n_k(\mu_1+\delta)\Big).\end{aligned}$$

Choosing $p = n_k$ and δ, ϵ small, we will have a contradiction. □

We need the following definition.

Definition 2.2.4. *Let X, Y be Banach spaces. For $x_1, ..., x_k \in X$, we define*

$$\mathrm{Vol}(x_1, x_2, ..., x_k) := \|x_1\| \prod_{i=2}^k d(x_i, \langle x_j\rangle_{1\leqslant j<i}) \tag{2.2.15}$$

where d denotes the usual distance between a point and a subset in X. For a given bounded linear function $T : X \to Y$ and $k \geq 1$, set

$$D_k(T) := \sup_{\|x_i\|=1; i=1,\ldots,k} \mathrm{Vol}\left(T(x_1), T(x_2), ..., T(x_k)\right)$$

We summarize some basic properties of D_k in the next lemma.

Lemma 2.2.5. *Let X, Y, Z be Banach spaces and $T : X \to Y$, $S : Y \to Z$ bounded linear maps.*

(i) $D_1(T) = \|T\|$ and $D_k(T) \leqslant \|T\|^k$ for $k \geq 1$.

(ii) $D_k(S \circ T) \leqslant D_k(S)D_k(T)$ for $k \geq 1$.

Proof. The proof of (i) is straightforward, (ii) is proven in [19, Lemma 1]. □

Lemma 2.2.6. *Let $T : X \to Y$ be a bounded linear map between two Banach spaces, $x \in \langle x_i \rangle_{1\leqslant i\leqslant k}$ and $\|x_i\| = 1$. Then there exists a constant α_k which only depends on k such that*

$$\mathrm{Vol}\left(T(x_1), T(x_2), ..., T(x_k)\right) \leqslant \alpha_k \|T\|^{k-1} \frac{\|Tx\|}{\|x\|}$$

Proof. Assume $\frac{x}{\|x\|} = \sum_{1\leqslant j\leqslant k} \beta_j x_j$. Consequently, there exists $1 \leqslant t \leqslant k$ such that $\beta_t \geqslant \frac{1}{k}$. Define $y = (y_1, \dots, y_k)$ as

$$y_i = \begin{cases} x_i & \text{for } i \neq t, n, \\ x_n & \text{for } i = t, \\ x_t & \text{for } i = n. \end{cases}$$

By definition,

$$\begin{aligned} \mathrm{Vol}\left(T(y_1), T(y_2), ..., T(y_n)\right) &\leqslant \|T\|^{k-1} d\big(T(y_n), \langle T(y_i)\rangle_{1\leqslant i\leqslant n-1}\big) \\ &\leqslant k\|T\|^{k-1}\frac{\|Tx\|}{\|x\|}. \end{aligned} \tag{2.2.16}$$

From [18, Proposition 2.14], there is an inner product $(\cdot,\cdot)_V$ on $V = \langle T(x_i)\rangle_{1\leqslant i\leqslant k}$ such that

$$\frac{1}{\sqrt{k}} \leqslant \frac{\|T(x)\|_V}{\|T(x)\|} \leqslant \sqrt{k} \qquad \forall x \in \langle x_i\rangle_{1\leqslant i\leqslant k}.$$

It is not hard to see that this implies that

$$\frac{1}{\sqrt{k}} \leqslant \frac{d_V\big(T(x_j), \langle T(x_i)\rangle_{1\leqslant i<j}\big)}{d\big(T(x_j), \langle T(x_i)\rangle_{1\leqslant i<j}\big)} \leqslant \sqrt{k}$$

and, consequently,

$$(\frac{1}{\sqrt{k}})^k \leqslant \frac{\mathrm{Vol}_V\left(T(x_1), ..., T(x_k)\right)}{\mathrm{Vol}\left(T(x_1), ..., T(x_k)\right)} \leqslant (\sqrt{k})^k. \tag{2.2.17}$$

Note that $\mathrm{Vol}_V\left(T(x_1), ..., T(x_k)\right) = \mathrm{Vol}_V\left(T(y_1), T(y_2), ..., T(y_k)\right)$ so our claim follows from (2.2.16) and (2.2.17). □

Lemma 2.2.7. *Assume that X, Y are Banach spaces and that $T : X \to Y$ is a linear map. Let $V \subset X$ be a closed subspace of codimension m. Then for $k > m$, there exists a constant C which only depends on k and m such that*

$$D_k(T) \leqslant C D_m(T) D_{k-m}(T|_V) \tag{2.2.18}$$

Proof. [19, Lemma 8]. □

Proposition 2.2.8. *Let ψ be a bounded linear cocycle acting on a measurable field of Banach spaces $(\{E_\omega\}_{\omega\in\Omega}, \Delta, \Delta_0)$. Then for every $n, k \geqslant 1$, the map*

$$\begin{aligned}\Psi_n^k \colon \Omega &\to \mathbb{R}\\ \omega &\mapsto D_k(\psi_\omega^n(\cdot))\end{aligned}$$

is measurable.

Proof. For $k = 1$, the claim follows from Lemma 2.2.5 and Lemma 2.2.1. Note that for $\omega \in \Omega$,

$$\Psi_n^k(\omega) = \sup_{g_1,\dots,g_k\in\Delta_0} \operatorname{Vol}\left(\psi_\omega^n(\tilde{g}_1(\omega)), \dots, \psi_\omega^n(\tilde{g}_k(\omega))\right)\chi_{\{\|g_1\|>0,\dots,\|g_k\|>0\}}(\omega)$$

where we used the notation $\tilde{g}_i(\omega) = g_i(\omega)/\|g_i(\omega)\|$, $i = 1, \dots, k$. It is therefore sufficient to prove that for fixed $g_1, \dots, g_k \in \Delta$,

$$\omega \mapsto \operatorname{Vol}\left(\psi_\omega^n(\tilde{g}_1(\omega)), \dots, \psi_\omega^n(\tilde{g}_k(\omega))\right)\chi_{\{\|g_1\|>0,\dots,\|g_k\|>0\}}(\omega)$$

is measurable. For $i \geqslant 2$, we have

$$\begin{aligned}
d\big(\psi_\omega^n(\tilde{g}_i(\omega)), &\langle\psi_\omega^n(\tilde{g}_t(\omega))\rangle_{1\leqslant t<i}\big)\\
&= \inf_{q_1,\dots,q_{i-1}\in\mathbb{Q}} \left\|\psi_\omega^n(\tilde{g}_i(\omega)) - \Sigma_{1\leqslant t<i} q_t \psi_\omega^n(\tilde{g}_t(\omega))\right\|\\
&= \frac{1}{\|g_i(\omega)\|} \inf_{q_1,\dots,q_{i-1}\in\mathbb{Q}} \left\|\psi_\omega^n(\tilde{g}_i(\omega)) - \Sigma_{1\leqslant t<i} q_t \psi_\omega^n(\tilde{g}_t(\omega))\right\|\\
&= \frac{1}{\|g_i(\omega)\|} \inf_{q_1,\dots,q_{i-1}\in\mathbb{Q}} \left\|\psi_\omega^n(\tilde{g}_i(\omega) - \Sigma_{1\leqslant t<i} q_t \tilde{g}_t(\omega))\right\|.
\end{aligned}$$

The claim follows by definition of Vol.

□

Lemma 2.2.9. *Under the same setting as in Proposition 2.2.8, let $\chi_n^k(\omega) = \log(\Psi_n^k(\omega))$. Assume that*

$$\log^+ \|\psi_\omega^1(.)\| \in L^1(\Omega).$$

Then there exists a measurable forward invariant set $\Omega_1 \subset \Omega$ of full measure such that the limit

$$\Lambda_k(\omega) := \lim_{n\to\infty} \frac{\chi_n^k(\omega)}{n} \in [-\infty, \infty) \tag{2.2.19}$$

exists for every $\omega \in \Omega_1$ and $k \geq 1$. Furthermore, $\Lambda_k(\theta\omega) = \Lambda_k(\omega)$ for every $k \geq 1$, $\omega \in \Omega_1$ and $\Lambda_k(\omega)$ is constant on Ω_1 in case the underlying metric dynamical system is ergodic.

Proof. From Lemma 2.2.5 and the cocycle property,

$$\chi_{n+m}^k(\omega) \leqslant \chi_n^k(\theta^m\omega) + \chi_m^k(\omega). \tag{2.2.20}$$

By assumption and Lemma 2.2.5, it follows that $\chi_1^{k;+} \in L^1(\Omega)$. Therefore, we can directly apply Kingman's Subadditive Ergodic Theorem [1, 3.3.2 Theorem] to conclude. □

Remark 2.2.10. *(i) From Birkhoff's Ergodic Theorem, we can furthermore assume that*

$$\lim_{n\to\infty} \frac{\log^+ \|\psi^1_{\theta^n\omega}(\cdot))\|}{n} = 0 \tag{2.2.21}$$

for all $\omega \in \Omega_1$*.*

(ii) From Lemma 2.2.7, it follows that

$$\Lambda_k \leq \Lambda_m + \Lambda_{k-m}$$

for every $k > m$*. In particular, if* $\Lambda_m = -\infty$*, it follows that* $\Lambda_k = -\infty$ *for every* $k > m$*.*

Definition 2.2.11. *If the assumptions of Lemma 2.2.9 are satisfied, we define*

$$\lambda_k(\omega) := \begin{cases} \Lambda_k(\omega) - \Lambda_{k-1}(\omega) & \textit{if } \Lambda_k(\omega), \in \mathbb{R} \\ -\infty & \textit{if } \Lambda_k(\omega) = -\infty \end{cases}$$

for $k \geq 1$*, where we set* $\Lambda_0(\omega) := 0$*. We call* λ_k *the* k-th Lyapunov exponent *of* ψ*. Note that they are deterministic almost surely in case the underlying system is ergodic.*

Remark 2.2.12. *One can easily show that* $(\lambda_k)_{k\geq 1}$ *is a decreasing sequence.*

The next lemma shows that the sequence (λ_k) does not have real cluster points in case the cocycle is compact.

Lemma 2.2.13. *Let* ψ *be as in Lemma 2.2.9. Furthermore, assume that it is compact. Then there is a measurable forward invariant subset* $\tilde{\Omega} \subset \Omega$ *with full measure such that for any* $\omega \in \tilde{\Omega}$ *and* $\rho \in \mathbb{R}$*, there are only finitely many exponents* $\lambda_k(\omega)$ *that exceed* ρ*.*

Proof. Let Ω_1 be the set provided in Lemma 2.2.9. For $\omega \in \Omega$, let B_ω be the unit ball in E_ω. Set

$$G(\vartheta, \nu) := \left\{ \omega \in \Omega_1 \,:\, \psi^1_\omega(B_\omega) \text{ can be covered by } e^\vartheta \text{ balls with sizes less than } e^\nu \right\}. \tag{2.2.22}$$

We claim that $G(\vartheta, \nu)$ is a measurable subset. To see this, define

$$S(\omega) := \left\{ s \in B_\omega \,:\, s = r \frac{g(\omega)}{\|g(\omega)\|} \chi_{\{\|g\|>0\}}(\omega),\, g \in \Delta_0,\, r \in \mathbb{Q} \cap [0,1] \right\}.$$

One can easily check that $S(\omega)$ is dense in B_ω. Let $p = e^\vartheta$ and define

$$H(\omega) = \inf_{s_1,\ldots,s_p \in S(\omega)} \left(\sup_{s \in S(\omega)} \min_{1 \leqslant i \leqslant p} \left(\|\psi^1_\omega(s) - \psi^1_\omega(s_i)\| \right) \right).$$

It is not hard to see that

$$G(\vartheta, \nu) = \{\omega \in \Omega_1 \,:\, H(\omega) < e^\nu\}$$

and consequently $G(\vartheta, \nu)$ is indeed measurable. Since ψ is compact, for any $\nu \in \mathbb{R}$,

$$\lim_{\vartheta \to \infty} \mathbb{P}(G(\vartheta, \nu)) = 1.$$

Let $\omega \in \Omega_1$. We can prove that $\psi_\omega^m(B_\omega)$ can be covered by $N_m = e^{m\vartheta}$ balls of size $R_m^{\vartheta,\nu} = e^{m\gamma_m^{\vartheta,\nu}}$ where

$$\gamma_m^{\vartheta,\nu}(\omega) = \frac{1}{m}\Big[\nu \sum_{0\leqslant j\leqslant m} \chi_{G(\vartheta,\nu)}(\theta^j\omega) + \sum_{0\leqslant j\leqslant m} \chi_{G(\vartheta,\nu)^c} \log^+ \|\psi^1_{\theta^j\omega}(\cdot)\|\Big] =: \nu A_m^{\vartheta,\nu}(\omega) + B_m^{\vartheta,\nu}(\omega).$$

Let $\lambda_k(\omega) > \rho$. For large m, we must have $k(\rho - \gamma_m^{\vartheta,\nu}) \leqslant \vartheta$. If we can show that $\rho - \gamma_m^{\vartheta,\nu} > 0$ for some m, ϑ, μ, the proof is finished since in that case, $k < \frac{\vartheta}{\rho - \gamma_m^{\vartheta,\nu}}$.
Let $\epsilon > 0$ and choose $\nu < 0$ such that $\nu < \frac{\rho-\epsilon}{\epsilon}$. From integrability of $\log^+ \|\psi^1_\omega(\cdot)\|$, there exists a $\delta > 0$ such that for $\mathbb{P}(E) < \delta$,

$$\int_E \log^+ \|\psi^1_\omega(\cdot)\| \, d\mathbb{P} \leqslant \epsilon^2. \tag{2.2.23}$$

Now we choose $\vartheta > 0$ such that

$$\mathbb{P}\big(G(\vartheta,\nu)^c\big) \leqslant \epsilon \wedge \delta. \tag{2.2.24}$$

Since $0 \leqslant A_m^{\vartheta,\nu}(\omega) \leqslant 1$,

$$\int_\Omega A_m^{\vartheta,\nu} \, d\mathbb{P} \leqslant \mathbb{P}(A_m^{\nu,r} > \epsilon) + \epsilon \quad \text{and}$$
$$\mathbb{P}(B_m^{\vartheta,\nu} > \epsilon) \leqslant \frac{1}{\epsilon} \int_\Omega B_m^{\vartheta,\nu} \, d\mathbb{P}.$$

Now from (2.2.23), (2.2.24) and Birkhoff's Ergodic theorem, for large m,

$$\mathbb{P}(A_m^{\vartheta,\nu} > \epsilon) \geqslant 1 - 3\epsilon \quad \text{and} \quad \mathbb{P}(B_m^{\vartheta,\nu} > \epsilon) \leqslant 2\epsilon.$$

Set $A_1 := \{A_m^{\vartheta,\nu} > \epsilon\}$ and $B_1 := \{B_m^{\vartheta,\nu} \leqslant \epsilon\}$ and note that $\mathbb{P}(A_1 \cap B_1) \geqslant 1 - 5\epsilon$. For $\omega \in A_1 \cap B_1$,

$$\gamma_m^{\vartheta,\nu} < \rho.$$

Since ϵ is arbitrary, we can find a set $\Omega_2 \subset \Omega_1$ of full measure with the desired property. Finally we put $\Omega_3 := \bigcap_{j=0}^\infty (\theta^j)^{-1}\Omega_2$. □

The following proposition, a trajectory-wise version of the Multiplicative Ergodic Theorem, will play a central role in the proof of our main result. It is a slight reformulation of [18, Proposition 3.4]. The proof is very similar to Blumenthal's original proof, but because of its importance, we decided to sketch it in the appendix, cf. page 135.

Proposition 2.2.14. *Let $\{V_j\}_{j\geqslant 0}$ be a sequence of Banach spaces and $T_i : V_i \to V_{i+1}$ a sequence of bounded linear operators. Set $T^n = T_{n-1} \circ \ldots \circ T_0$. Assume that:*

(i) $\limsup_{n\to\infty} \frac{1}{n} \log^+ \|T_n\| = 0$.

(ii) For any $k \geq 1$, the following limits exists:

$$L_k = \lim_{n\to\infty} \frac{1}{n} \log D_k(T^n).$$

(iii) *Setting* $L_0 := 0$ *and* $l_k := L_k - L_{k-1}$ *for* $k \geq 1$, *assume that there is a number* $m < \infty$ *for which* $\bar{l} := l_1 = \ldots = l_m > l_{m+1} =: \underline{l}$.

Then the subspace

$$F := \{v \in V_0 : \limsup_{n\to\infty} \frac{1}{n} \log \|T^n v\| \leqslant \underline{l}\}$$

is closed and m*-codimensional. Also, for* $v \in V_0 \setminus F$,

$$\lim_{n\to\infty} \frac{1}{n} \log \|T^n v\| = \bar{l}. \tag{2.2.25}$$

Furthermore, for any complement H *of* F,

$$\lim_{n\to\infty} \frac{1}{n} \log \inf_{v\in H\setminus\{0\}} \frac{\|T^n v\|}{\|v\|} = \bar{l}. \tag{2.2.26}$$

Finally, if $h_1, \ldots, h_m \in V_0$ *are linearly independent and* $H = \langle h_1, ..., h_m\rangle$,

$$\lim_{n\to\infty} \frac{1}{n} \log \mathrm{Vol}\,(T^n h_1, T^n h_2, ..., T^n h_m) = m\bar{l}. \tag{2.2.27}$$

Remark 2.2.15. *In the proof of the proposition above, we will also see that*

$$\limsup_{n\to\infty} \frac{1}{n} \log \|T^n|_F\| \leqslant \underline{l} \tag{2.2.28}$$

holds.

We finally state the main result of this section, a Multiplicative Ergodic Theorem for cocycles acting on measurable fields of Banach spaces.

Theorem 2.2.16. *Let* $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$ *be an ergodic measurable metric dynamical system and* ψ *be a compact linear cocycle acting on a measurable field of Banach spaces* $(\{E_\omega\}_{\omega\in\Omega}, \Delta, \Delta_0)$. *For* $\mu \in \mathbb{R} \cup \{-\infty\}$ *and* $\omega \in \Omega$, *remember*

$$F_\mu(\omega) := \{x \in E_\omega \,:\, \limsup_{n\to\infty} \frac{1}{n} \log \|\psi^n_\omega(x)\| \leqslant \mu\}.$$

Assume that

$$\log^+ \|\psi^1_\omega(\cdot)\| \in L^1(\Omega).$$

Then there is a measurable forward invariant set $\tilde{\Omega} \subset \Omega$ *of full measure such that:*

(i) *For any* $\omega \in \tilde{\Omega}$ *and* $k \geqslant 1$, *the limit*

$$\Lambda_k := \lim_{n\to\infty} \frac{1}{n} \log D_k(\psi^n_\omega(\cdot)) \in [-\infty, \infty) \tag{2.2.29}$$

exists and is independent of ω.

(ii) *Setting* $\Lambda_0 := 0$ *and* $\lambda_k := \Lambda_k - \Lambda_{k-1}$ *with* $\lambda_k = -\infty$ *if* $\Lambda_k = -\infty$, *the sequence* (λ_k) *is decreasing. If the number of distinct values of this sequence is infinite, then*

$\lim_{k\to\infty}\lambda_k = -\infty$. *We denote the decreasing subsequence of distinct values by* $(\mu_j)_{j\geqslant 1}$, *which can be a finite or an infinite sequence, and* m_j *will denote the multiplicity of* μ_j *in the sequence* (λ_j). *If* $\mu_j \in \mathbb{R}$, m_j *is finite.*

(iii) For $\mu_i \neq -\infty$ *and* $\omega \in \tilde{\Omega}$,

$$x \in F_{\mu_i}(\omega) \setminus F_{\mu_{i+1}}(\omega) \quad \textit{if and only if} \quad \lim_{n\to\infty}\frac{1}{n}\log\|\psi^n_\omega(x)\| = \mu_i. \tag{2.2.30}$$

(iv) For any μ_j, $\operatorname{codim} F_{\mu_j}(\omega) = m_1 + \ldots + m_{j-1}$ *for every* $\omega \in \tilde{\Omega}$.

(v) For $\omega \in \tilde{\Omega}$, *if* $h^1,\ldots,h^k \in E_\omega$ *are linearly independent and* $H_\omega = \langle h^1,...,h^k\rangle$ *is a complement subspace for* $F_{\mu_j}(\omega)$ *in* E_ω, *then*

$$\lim_{n\to\infty}\frac{1}{n}\log\operatorname{Vol}\left(\psi^n_\omega(h^1),...,\psi^n_\omega(h^k)\right) = \sum_{1\leqslant i\leqslant j} m_i\mu_i. \tag{2.2.31}$$

Remark 2.2.17. *The sequence* (μ_j) *is called the* Lyapunov spectrum, *the filtration of spaces*

$$F_{\mu_1}(\omega) \supset F_{\mu_2}(\omega) \supset \cdots$$

is called Oseledets filtration.

Proof. Note that (i) and (ii) are direct consequences of Lemma 2.2.9 and Lemma 2.2.13, hence we only have to prove (iii), (iv) and (v). The idea is to prove the consecutive statements for each Lyapunov exponent by induction, where Proposition 2.2.14 will play a central role. We will only give the proof in case that the Lyapunov spectrum is infinite, the case of a finite Lyapunov spectrum is similar.

Let us start to formulate a result for the first Lyapunov exponent μ_1. Consider $\Omega_1 \subset \Omega$ as in Lemma 2.2.9. We may assume that (2.2.21) is also satisfied for every $\omega \in \Omega_1$. Fix some $\omega \in \Omega_1$ and define $V_j := E_{\theta^j\omega}$ and $T_j := \psi^1_{\theta^j\omega}(\cdot)$. Note that, by definition, $\mu_1 = \lambda_1 = \ldots = \lambda_{m_1} > \lambda_{m_1+1} = \mu_2$ and $\mu_1 = \Lambda_1$, therefore $F_{\mu_1}(\omega) = E_\omega = V_0$. Proposition 2.2.14 now implies that for $x \in F_{\mu_1}(\omega)\setminus F_{\mu_2}(\omega)$, we have $\lim_{n\to\infty}\frac{1}{n}\log\|\psi^n_\omega(x)\| = \mu_1$ and that $F_{\mu_2}(\omega)$ is m_1-codimensional. Furthermore, if $H_\omega = \langle h^1,\ldots,h^{m_1}\rangle$ is a complement for $F_{\mu_2}(\omega)$,

$$\lim_{n\to\infty}\frac{1}{n}\log\operatorname{Vol}\left(\psi^n_\omega(h^1),...,\psi^n_\omega(h^k)\right) = m_1\mu_1. \tag{2.2.32}$$

For the next step, we set $V_j := F_{\mu_2}(\theta^j\omega)$ and $T_j := \psi^1_{\theta^j\omega}(\cdot)\mid_{F_{\mu_2}(\theta^j\omega)}$. Note that from the cocycle property, $T_j\colon V_j \to V_{j+1}$. We claim that there is a measurable and θ-invariant subset $\Omega_2 \subset \Omega_1$ with full measure such that for any $\omega \in \Omega_2$ and $k \geqslant 1$,

$$\lim_{n\to\infty}\frac{1}{n}\log D_k\left[\psi^n_\omega(\cdot)\mid_{F_{\mu_2}(\omega)}\right] = \Lambda_{k+m} - \Lambda_m \tag{2.2.33}$$

where we set $m := m_1$ for simplicity. Let $\Omega_2 \subset \Omega_1$ be a measurable subset with the properties stated in Lemma 2.2.3. Fix some $\omega \in \Omega_2$. As a consequence of Lemma 2.2.7,

$$\Lambda_{k+m} \leqslant \Lambda_m + \liminf_{n\to\infty}\frac{1}{n}\log D_k\left[\psi^n_\omega(\cdot)\mid_{F_{\mu_2}(\omega)}\right]. \tag{2.2.34}$$

For $n \in \mathbb{N}$ to be specified later, let $\{f^i\}_{1 \leqslant i \leqslant k} \subset F_{\mu_2}(\omega)$ be chosen such that $\|f^i\| = 1$ for every i and

$$\mathrm{Vol}\left(\psi_\omega^n(f^1), ..., \psi_\omega^n(f^k)\right) \geqslant \frac{1}{2} D_k \left[\psi(n, \omega, \cdot) \,|_{F_{\mu_2}(\omega)}\,\right]. \tag{2.2.35}$$

Let $H_\omega = \langle h^1, h^2, ..., h^m \rangle$ be a complement subspace for $F_{\mu_2}(\omega)$. We can assume that $\|h^i\| = 1$ for all i. By definition,

$$\begin{aligned} &D_{k+m}(\psi_\omega^n(\cdot)) \geqslant \mathrm{Vol}\left(\psi_\omega^n(h^1), ..., \psi_\omega^n(h^m), \psi_\omega^n(f^1), ..., \psi_\omega^n(f^k)\right) \\ &= \mathrm{Vol}\left(\psi_\omega^n(h^1), ..., \psi_\omega^n(h^m)\right) \prod_{j=1}^m d\Big(\psi_\omega^n(f_\omega^j), \langle \psi_\omega^n(h^1), ..., \psi_\omega^n(h^m), \psi_\omega^n(f^1), ..., \psi_\omega^n(f^{j-1})\rangle\Big). \end{aligned} \tag{2.2.36}$$

It is not hard to see that

$$\frac{d\Big(\psi_\omega^n(f^j), \langle \psi_\omega^n(f^1), ..., \psi_\omega^n(f^{j-1})\rangle\Big)}{d\Big(\psi_\omega^n(f^j), \langle \psi_\omega^n(h^1), ..., \psi_\omega^n(h^m), \psi_\omega^n(f^1), ..., \psi_\omega^n(f^{j-1})\rangle\Big)} \leqslant \|\Pi_{F_{\mu_2}(\theta^n\omega) || \psi(n,\omega,H_\omega)}\|.$$

Consequently, by (2.2.35) and (2.2.36),

$$\begin{aligned} D_{k+m}(\psi_\omega^n(\cdot)) &\geqslant \|\Pi_{F_{\mu_2}(\theta^n\omega) || \psi(n,\omega,H_\omega)}\|^{-m} \,\mathrm{Vol}\left(\psi_\omega^n(h^1), ..., \psi_\omega^n(h^m)\right) \mathrm{Vol}(\psi_\omega^n(f^1), ..., \psi_\omega^n(f^n)) \\ &\geqslant \frac{1}{2} \|\Pi_{F_{\mu_2}(\theta^n\omega) || \psi(n,\omega,H_\omega)}\|^{-m} \,\mathrm{Vol}\left(\psi_\omega^n(h^1), ..., \psi_\omega^n(h^m)\right) D_k \left[\psi_\omega^n(\cdot) \,|_{F_{\mu_2}(\omega)}\,\right]. \end{aligned}$$

Note that, by definition of the projection operator,

$$1 \leqslant \|\Pi_{F_{\mu_2}(\theta^n\omega) || \psi_\omega^n(H_\omega)}\| \leqslant \|\Pi_{\psi_\omega^n(H_\omega) || F_{\mu_2}(\theta^n\omega)}\| + 1.$$

Choosing n large, using (2.2.32) and Lemma 2.2.3, we see that

$$\limsup_{n\to\infty} \frac{1}{n} \log D_k \left[\psi_\omega^n(\cdot \,|_{F_{\mu_2}(\omega)}\,\right] + \Lambda_m \leqslant \Lambda_{k+m} \tag{2.2.37}$$

and (2.2.33) is shown. We can now use Proposition 2.2.14 again with $\bar{l} = \mu_2$, $\underline{l} = \mu_3$ and $m = m_2$ which proves that for $\omega \in \Omega_2$ and $x \in F_{\mu_2}(\omega) \setminus F_{\mu_3}(\omega)$,

$$\lim_{n\to\infty} \frac{1}{n} \log \|\psi_\omega^n(x)\| = \mu_2.$$

Moreover, $F_{\mu_3}(\omega)$ is m_2-codimensional in $F_{\mu_2}(\omega)$. Using that $F_{\mu_2}(\omega)$ is m_1-codimensional in E_ω implies that $F_{\mu_3}(\omega)$ has codimension $m_1 + m_2$ in E_ω.

It remains to prove (v). Let $\langle h^1, ..., h^{m_1+m_2} \rangle$ be a complement subspace for $F_{\mu_3}(\omega)$. Note that $\mathrm{Vol}\left(\psi_\omega^n(h^1), ..., \psi_\omega^n(h^{m_1+m_2})\right)$ is not invariant under permutation, but all permutations are equivalent up to a constant which only depends on $m_1 + m_2$, cf. the proof of Lemma 2.2.6. We may assume that $H_\omega = \langle h^1, ..., h^{m_1} \rangle$ is a complement subspace for $F_{\mu_2}(\omega)$ and that for $m_1 + 1 \leqslant j \leqslant m_1 + m_2$, we have $h^j = g^{j-m_1} + f^{j-m_1}$ where $g^{j-m_1} \in F_{\mu_2}(\omega)$ and $f^{j-m_1} \in H_\omega$. It is not hard to see that $G_\omega := \langle g^1, ..., g^{m_2} \rangle$ is a complement subspace for $F_{\mu_3}(\omega)$ in $F_{\mu_2}(\omega)$.

By definition,

$$\begin{aligned}&\mathrm{Vol}\left(\psi_\omega^n(g^1),...,\psi_\omega^n(g^{m_2}),\psi_\omega^n(h^1),...,\psi_\omega^n(h^{m_1})\right)\\ =\ &\mathrm{Vol}\left(\psi_\omega^n(g^1),...,\psi_\omega^n(g^{m_2})\right)\prod_{j=1}^{m_1} d\big(\psi_\omega^n(h^j),\langle\psi_\omega^n(g^1),...,\psi_\omega^n(g^{m_2}),\psi_\omega^n(h^1),...,\psi_\omega^n(h^{j-1})\rangle\big).\end{aligned}$$

Note that

$$1\leqslant\frac{d\big(\psi_\omega^n(h^j),\langle\psi_\omega^n(h^1),...,\psi_\omega^n(h^{j-1})\rangle\big)}{d\big(\psi_\omega^n(h^j),\langle\psi_\omega^n(g^1),...,\psi_\omega^n(g^{m_2}),\psi_\omega^n(h^1),..,\psi_\omega^n(h^{j-1})\rangle\big)}\leqslant\|\Pi_{\psi_\omega^n(H_\omega)||F_{\mu_2}(\theta^n\omega)}\|.$$

Together with Lemma 2.2.3 and (2.2.27) in Proposition 2.2.14, this implies that

$$\begin{aligned}&\lim_{n\to\infty}\frac{1}{n}\log\mathrm{Vol}\left(\psi_\omega^n(h^1),...,\psi_\omega^n(h^{m_1}),\psi_\omega^n(g^1),...,\psi_\omega^n(g^{m_2})\right)\\ =\ &\lim_{n\to\infty}\frac{1}{n}\log\mathrm{Vol}\left(\psi_\omega^n(g^1),...,\psi_\omega^n(g^{m_2}),\psi_\omega^n(h^1),...,\psi_\omega^n(h^{m_1})\right)=m_1\mu_1+m_2\mu_2.\end{aligned}\tag{2.2.38}$$

Since $f^k\in H_\omega$ for $1\leqslant j\leqslant m_1$,

$$\begin{aligned}&d\big(\psi_\omega^n(g^j),\langle\psi_\omega^n(h^1),...,\psi_\omega^n(h^{m_1}),\psi_\omega^n(g^1),...,\psi_\omega^n(g^{j-1})\rangle\big)\\ =\ &d\big(\psi_\omega^n(h^{m_1+j}),\langle\psi_\omega^n(h^1),...,\psi_\omega^n(h^{m_1}),\psi_\omega^n(h^{m_1+1}),..,\psi_\omega^n(h^{m_1+j-1})\rangle\big).\end{aligned}$$

Consequently, by (2.2.38),

$$\begin{aligned}&\lim_{n\to\infty}\frac{1}{n}\log\mathrm{Vol}\left(\psi_\omega^n(h^1),...,\psi_\omega^n(h^{m_1}),\psi_\omega^n(h^{m_1+1}),...,\psi_\omega^n(h^{m_1+m_2})\right)\\ =\ &\lim_{n\to\infty}\frac{1}{n}\log\mathrm{Vol}\left(\psi_\omega^n(h^{m_1+1}),...,\psi_\omega^n(h^{m_1+m_2}),\psi_\omega^n(h^1),...,\psi_\omega^n(h^{m_1})\right)]=m_1\mu_1+m_2\mu_2.\end{aligned}$$

This finishes step 2. We can now iterate the procedure and the general result follows by induction.

□

2.3 Semi-invertible MET on fields of Banach spaces

In this section, we assume θ is invertible and $(\Omega,\mathcal{F},\mathbb{P},\theta)$ will denote an ergodic measure-preserving dynamical system. We set $\sigma:=\theta^{-1}$. Let $(\{E_\omega\}_{\omega\in\Omega},\Delta,\Delta_0)$ be a measurable field of Banach space and let $\psi_\omega\colon E_\omega\to E_{\theta\omega}$ be a compact linear cocycle acting on it. In the sequel, we will furthermore assume that the following assumption is satisfied:

Assumption 2.3.1. *For each $g,\tilde g\in\Delta$ and $n,k\geq 0$,*

$$\omega\to\|\psi_{\theta^n\omega}^k[\psi_\omega^n(g(\omega))-\tilde g(\theta^n\omega)]\|_{E_{\theta^{n+k}\omega}}$$

is measurable.

We will always assume that

$$\log^+\|\psi_\omega\|\in L^1(\Omega).$$

Under this condition, the Multiplicative Ergodic Theorem 2.2.16 applies and yields the existence of Lyapunov exponents $\{\mu_1 > \mu_2 > \ldots\} \subset [-\infty, \infty)$ on a θ-invariant set of full measure $\tilde{\Omega} \subset \Omega$. Remember for all $\omega \in \tilde{\Omega}$

$$\Lambda_k = \lim_{n\to\infty} \frac{1}{n} \log D_k(\psi_\omega^n), \quad k \geq 1$$

and $\lambda_k = \Lambda_k - \Lambda_{k-1}$ also, the sequence (μ_k) is the subsequence of (λ_k) defined by removing all multiple elements. Note that ψ is invariant on these spaces in the sense that

$$\psi_\omega^n|_{F_\mu(\omega)} \colon F_\mu(\omega) \to F_\mu(\theta^n\omega).$$

We also saw in the last section that there are numbers $m_i \in \mathbb{N}$ such that $m_i = \dim\left(F_{\mu_i}(\omega)/F_{\mu_{i+1}}(\omega)\right)$ for every $\omega \in \tilde{\Omega}$.

If not otherwise stated, $\tilde{\Omega} \subset \Omega$ will always denote a θ-invariant set of full measure. Note that we can always assume w.l.o.g. that a given set of full measure $\Omega_0 \subset \Omega$ is θ-invariant, otherwise we can consider

$$\bigcap_{k\in\mathbb{Z}} \theta^k(\Omega_0)$$

instead.

Next, we collect some basic Lemmas. Recall the definition of Vol and D_k.

Lemma 2.3.2. *Let X, Y be Banach spaces and $T : X \to Y$ a linear operator. For $k \in \mathbb{N}$, there exist positive constants c_k, C_k depending only on k such that*

$$c_k D_k(T) \leqslant D_k(T^*) \leqslant C_k D_k(T) \tag{2.3.1}$$

where by $T^* : Y^* \to X^*$ we mean the dual map of T.

Proof. [19, Lemma 3]. ☐

Lemma 2.3.3. *For a Banach space X and $k \geqslant 1$, the map*

$$\begin{aligned} \mathrm{Vol} : X^k &\longrightarrow \mathbb{R} \\ (x_1, x_2, ..., x_k) &\mapsto \|x_1\| \prod_{i=2}^{k} d(x_i, \langle x_j \rangle_{1\leqslant j<i}) \end{aligned} \tag{2.3.2}$$

is continuous.

Proof. [16, Lemma 4.2]. ☐

For a Banach space X and a closed subspace $U \subset X$, the quotient space X/U is again a Banach space with norm

$$\|[x]\|_{X/U} = \inf_{u\in U} \|x - u\|.$$

For an element $x \in E_\omega$, we denote by $[x]_\mu$ its equivalence class in the quotient space $E_\omega/F_\mu(\omega)$. From the invariance property of ψ, the map

$$[\psi^n_\omega]_{\mu_{j+1}} : \frac{F_{\mu_j}(\omega)}{F_{\mu_{j+1}}(\omega)} \longrightarrow \frac{F_{\mu_j}(\theta^n\omega)}{F_{\mu_{j+1}}(\theta^n\omega)}, \quad [\psi^n_\omega]_{\mu_{j+1}}([x]) := [\psi^n_\omega(x)]_{\mu_{j+1}}$$

is well-defined for every $j \geq 1$ and $n \in \mathbb{N}$. Note also that $[\psi^n_\omega]_{\mu_{j+1}}$ is bijective for $\omega \in \tilde{\Omega}$. Indeed, injectivity is straightforward and surjectivity follows from the fact that $F_{\mu_j}(\omega)/F_{\mu_{j+1}}(\omega)$ and $F_{\mu_j}(\theta^n\omega)/F_{\mu_{j+1}}(\theta^n\omega)$ are finite-dimensional with the same dimension m_i.

Lemma 2.3.4. *For $m, n \in \mathbb{N}$, the maps*

$$f_1(\omega) := D_m(\psi^n_\omega\,|_{F_{\mu_2}(\omega)}) \quad \textit{and} \quad f_2(\omega) := D_m([\psi^n_\omega]_{\mu_2})$$

are measurable.

Proof. It is not hard to see that

$$f_1(\omega) = \lim_{l\to\infty} \liminf_{k\to\infty} \Big[\sup_{\{\xi^t_\omega\}_{1\leqslant t\leqslant m} \subset B^{l,k}_\omega(\mu_2)} \operatorname{Vol}\left(\psi^n_\omega(\xi^1_\omega), ..., \psi^n_\omega(\xi^m_\omega)\right)\Big] \tag{2.3.3}$$

where

$$B^{l,k}_\omega(\mu_2) = \{\xi \in F_{\mu_1}(\omega) :\ \|\xi\| = 1,\ \|\psi^k_\omega(\xi)\| < \exp\left(k(\mu_2 + \frac{1}{l})\right)\ \},$$

cf. the proof of Lemma 2.2.2. Let $\{g_t\}_{1\leqslant t\leqslant m} \subset \Delta_0$ and $C(g_t) := \{\omega\ :\ g_t(\omega) \in B^{l,k}_\omega(\mu_2)\}$. As a consequence of the proof of Lemma 2.2.2, these sets are measurable and we have

$$\sup_{\{\xi^t_\omega\}_{1\leqslant t\leqslant m} \subset B^{l,k}_\omega(\mu_2)} \operatorname{Vol}\left(\psi^n_\omega(\xi^1_\omega), ..., \psi^n_\omega(\xi^m_\omega)\right) =$$

$$\sup_{\{g_t\}_{1\leqslant t\leqslant m} \subset \Delta_0} \operatorname{Vol}\left(\psi^n_\omega\Big(\frac{g_1(\omega)}{\|g_1(\omega)\|}\Big), ..., \psi^n_\omega\Big(\frac{g_m(\omega)}{\|g_m(\omega)\|}\Big)\right) \prod_{1\leqslant t\leqslant m} \chi_{C(g_t)}(\omega)$$

which implies measurability of f_1. For f_2, note first that

$$f_2(\omega) = \lim_{l\to\infty} \liminf_{k\to\infty} \Big[\sup_{\{\xi^t_\omega\}_{1\leqslant t\leqslant m} \subset F_{\mu_1}(\omega)} \frac{\operatorname{Vol}\left([\psi^n_\omega(\xi^1_\omega)]_{\mu_2}, ..., [\psi^n_\omega(\xi^m_\omega)]_{\mu_2}\right)}{\prod_{1\leqslant t\leqslant m} \|[\xi^t_\omega]_{\mu_2}\|}\Big]$$

where we set $\frac{0}{0} := 0$. Again as before

$$\sup_{\{\xi^t_\omega\}_{1\leqslant t\leqslant m} \subset F_{\mu_1}(\omega)} \frac{\operatorname{Vol}\left([\psi^n_\omega(\xi^1_\omega)]_{\mu_2}, ..., [\psi^n_\omega(\xi^m_\omega)]_{\mu_2}\right)}{\prod_{1\leqslant t\leqslant m} \|[\xi^t_\omega]_{\mu_2}\|} =$$

$$\sup_{\{g_t\}_{1\leqslant t\leqslant m} \subset \Delta_0} \frac{\operatorname{Vol}\left([\psi^n_\omega(g_1(\omega))]_{\mu_2}, ..., [\psi^n_\omega(g_k(\omega))]_{\mu_2}\right)}{\prod_{1\leqslant t\leqslant m} d(g_t(\omega), F_{\mu_2}(\omega))}.$$

It remains to show that for $g \in \Delta$, $d\big(\psi^n_\omega(g(\omega)), F_{\mu_2}(\theta^n\omega)\big)$ is measurable, which can be achieved using Assumption 2.3.1 with a proof similar the proof of Lemma 2.2.2. $\square$

Lemma 2.3.5. *For every $i \geq 0$, there is a constant $M_i > 0$ such that*

$$\|[\psi^1_\omega]_{\mu_{i+1}}\| < M_i \|\psi^1_\omega\|$$

for every $\omega \in \tilde{\Omega}$.

Proof. Since $\dim[\frac{F_{\mu_i}(\omega)}{F_{\mu_{i+1}}(\omega)}] = m_i$, we can choose $H_\omega \subset F_{\mu_i}(\omega)$ such that

$$H_\omega \oplus F_{\mu_{i+1}}(\omega) = F_{\mu_i}(\omega) \quad \text{and} \quad \|\Pi_{H_\omega \| F_{\mu_{i+1}}(\omega)}\| \leq \sqrt{m_i} + 2 =: M_i, \tag{2.3.4}$$

cf. [18, Lemma 2.3]. Let $\xi_\omega \in F_{\mu_i}(\omega) \setminus F_{\mu_{i+1}}(\omega)$ with corresponding decomposition $\xi_\omega = h_\omega + f_\omega \in H_\omega \oplus F_{\mu_{i+1}}(\omega)$. From (2.3.4), we know that $\frac{\|h_\omega\|}{\|[\xi_\omega]_{\mu_{i+1}}\|} \leqslant M_i$ and consequently

$$\frac{\|[\psi^1_\omega(\xi_\omega)]_{\mu_{i+1}}\|}{\|[\xi_\omega]_{\mu_{i+1}}\|} \leq M_i \frac{\|[\psi^1_\omega(h_\omega)]_{\mu_{i+1}}\|}{\|h_\omega\|} \leq M_i \frac{\|\psi^1_\omega(h_\omega)\|}{\|h_\omega\|} \leq M_i \|\psi^1_\omega\|.$$

The claim follows. ☐

Lemma 2.3.6. *Assume that $\{f_n(\omega)\}_{n\geqslant 1}$ is a subadditive sequence with respect to θ and set $g_n(\omega) := f_n(\sigma^n \omega)$. Assume $f_1^+(\omega) \in L^1(\Omega)$. Then there is a θ-invariant set $\tilde{\Omega} \in \mathcal{F}$ with full measure such that for every $\omega \in \tilde{\Omega}$,*

$$\lim_{n\to\infty} \frac{1}{n} f_n(\omega) = \lim_{n\to\infty} \frac{1}{n} g_n(\omega) \in [-\infty, \infty)$$

where the limit does not depend on ω.

Proof. We can easily check that $\{g_n(\omega)\}_{n\geqslant 1}$ is a subadditive sequence with respect to σ. Since $f_n(\omega)$ and $g_n(\omega)$ have same law, the result follows from Kingman's Subadditive Ergodic Theorem. ☐

As a consequence, we obtain the following:

Lemma 2.3.7. *There is a θ-invariant set of full measure $\tilde{\Omega} \in \mathcal{F}$ such that*

$$\lim_{n\to\infty} \frac{1}{n} \log D_k(\psi^n_\omega) = \lim_{n\to\infty} \frac{1}{n} \log D_k(\psi^n_{\sigma^n\omega}) = \lim_{n\to\infty} \frac{1}{n} \log D_k((\psi^n_{\sigma^n\omega})^*) = \Lambda_k \tag{2.3.5}$$

and

$$\begin{aligned} &\lim_{n\to\infty} \frac{1}{n} \log D_k(\psi^n_\omega |_{F_{\mu_2}(\omega)}) = \lim_{n\to\infty} \frac{1}{n} \log D_k(\psi^n_{\sigma^n\omega} |_{F_{\mu_2}(\sigma^n\omega)}) \\ &= \lim_{n\to\infty} \frac{1}{n} \log D_k((\psi^n_{\sigma^n\omega})^* |_{(F_{\mu_2}(\sigma^n\omega))^*}) = \Lambda_{k+m_1} - \Lambda_{m_1} \end{aligned} \tag{2.3.6}$$

Proof. We already noted that $\lim_{n\to\infty} \frac{1}{n} \log D_k(\psi^n_\omega) = \Lambda_k$. The equality

$$\lim_{n\to\infty} \frac{1}{n} \log D_k(\psi^n_\omega |_{F_{\mu_2}(\omega)}) = \Lambda_{k+m_1} - \Lambda_{m_1} \tag{2.3.7}$$

was a partial result in the proof of Theorem 2.2.16. The remaining inequalities follow by a combination of Lemmas 2.3.2 - 2.3.6. ☐

From now on, we will assume that $\tilde{\Omega}$ is the set provided in Lemma 2.3.7.

Lemma 2.3.8. *Fix* $\omega \in \tilde{\Omega}$ *and let* $(\xi_{\sigma^n\omega})_n$ *be a sequence such that* $\xi_{\sigma^n\omega} \in F_{\mu_1}(\sigma^n\omega) \setminus F_{\mu_2}(\sigma^n\omega)$ *and* $\|[\xi_{\sigma^n\omega}]_{\mu_2}\| = 1$ *for every* $n \in \mathbb{N}$. *Then*

$$\lim_{n\to\infty} \frac{1}{n} \log \|[\psi^n_{\sigma^n\omega}(\xi_{\sigma^n\omega})]_{\mu_2}\| = \mu_1. \tag{2.3.8}$$

Proof. By applying Lemma 2.3.4, Lemma 2.3.5 and Lemma 2.3.6, Kingman's Subadditive Ergodic Theorem shows that

$$\lim_{n\to\infty} \frac{1}{n} \log D_k\big([\psi^n_\omega]_{\mu_2}\big) = \lim_{n\to\infty} \frac{1}{n} \log D_k\big([\psi^n_{\sigma^n\omega}]_{\mu_2}\big)$$

exist for every $k \geq 1$. Let H_ω be a complement subspace for $F_{\mu_2}(\omega)$ in $F_{\mu_1}(\omega)$. Using a slight generalization of Lemma 2.2.3, we have that

$$\lim_{n\to\infty} \frac{1}{n} \log \|\Pi_{\psi^n_\omega(H_\omega)||F_{\mu_2}(\theta^n\omega)}\| = 0.$$

For $\xi_\omega \in F_{\mu_1}(\omega) \setminus F_{\mu_2}(\omega)$, since

$$\frac{\|\psi^n_\omega(\Pi_{H_\omega||F_{\mu_2}(\omega)}(\xi_\omega))\|}{\|[\psi^n_\omega(\xi_\omega)]_{\mu_2}\|} \leqslant \|\Pi_{\psi^n_\omega(H_\omega)||F_{\mu_2}(\theta^n\omega)}\|$$

it follows that

$$\lim_{n\to\infty} \frac{1}{n} \log \|[\psi^n_\omega(\xi_\omega)]_{\mu_2}\| = \mu_1. \tag{2.3.9}$$

Let

$$k := \max\{m : \lim_{n\to\infty} \frac{1}{n} \log D_m\big([\psi^n_\omega]_{\mu_2}\big) = m\mu_1\}.$$

We claim $k = m_1$. Indeed, otherwise from Proposition 2.2.14, there exists a subspace $F_\omega \subset \frac{F_{\mu_1}(\omega)}{F_{\mu_2}(\omega)}$ with codimension k such that for every $\xi_\omega \in F_\omega$

$$\limsup_{n\to\infty} \frac{1}{n} \log \|[\psi^n_\omega(\xi_\omega)]_{\mu_2}\| < \mu_1.$$

Since $\dim[\frac{F_{\mu_1}(\omega)}{F_{\mu_2}(\omega)}] = m_1$, we can find a non-zero element in F_ω which contradicts (2.3.9). Hence we have shown that

$$\lim_{n\to\infty} \frac{1}{n} \log D_{m_1}\big([\psi^n_\omega]_{\mu_2}\big) = m_1\mu_1.$$

Therefore, for every $n \in \mathbb{N}$, we can find $\{\xi^j_{\sigma^n\omega}\}_{1\leqslant j\leqslant m_1} \subset F_{\mu_1}(\sigma^n\omega)$ such that $\|[\xi^j_\omega]_{\mu_2}\| = 1$ and

$$\lim_{n\to\infty} \frac{1}{n} \mathrm{Vol}\,\big([\psi^n_{\sigma^n\omega}(\xi^1_{\sigma^n\omega})]_{\mu_2}, \dots, [\psi^n_{\sigma^n\omega}(\xi^{m_1}_{\sigma^n\omega})]_{\mu_2})\big] = m_1\mu_1. \tag{2.3.10}$$

Using the definition of Vol, it follows that for every $2 \leqslant t \leqslant m_1$,

$$\lim_{n\to\infty} \frac{1}{n} \log d\big([\psi^n_{\sigma^n\omega}(\xi^t_\omega)]_{\mu_2}, \langle[\psi^n_{\sigma^n\omega}(\xi^j_{\sigma^n\omega})]_{\mu_2}\rangle_{1\leqslant j\leqslant t-1}\big) = \mu_1. \tag{2.3.11}$$

We have $\xi_{\sigma^n\omega} = \sum_{1\leqslant j\leqslant m_1} \alpha_j \xi^j_{\sigma^n\omega} \mod F_{\mu_2}(\sigma^n\omega)$. In the proof of Lemma 2.2.6, we already saw that the the Vol-function is symmetric up to a constant. By our assumption on $\xi_{\sigma^n\omega}$, we can therefore assume that $\alpha_{m_1} \geqslant \frac{1}{m_1}$. Finally from (2.3.11)

$$\lim_{n\to\infty} \frac{1}{n} \log \|[\psi^n_{\sigma^n\omega}(\xi_{\sigma^n\omega})]_{\mu_2}\| = \lim_{n\to\infty} \frac{1}{n} [d([\psi(\xi^{m_i}_{\sigma^n\omega})]_{\mu_2}, \langle[\psi^n_{\sigma^n\omega}(\xi^j_{\sigma^n\omega})]_{\mu_2}\rangle_{1\leqslant j\leqslant m_1-1}) = \mu_1.$$

□

Definition 2.3.9. *Let X be a Banach space. We define $G(X)$ to be the Grassmanian of closed subspaces of X equipped with the Hausdorff distance*

$$d_H(A,B) := \max\{\sup_{a\in S_A} d(a,S_B), \sup_{b\in S_B} d(b,S_A)\}$$

where $S_A = \{a \in A \ : \ \|a\| = 1\}$. Set

$$G_k(X) = \{A \in G(X) \ : \ \dim[A] = k\} \quad \textit{and} \quad G^k(X) = \{A \in G(X) \ : \ \dim[X/A] = k\}.$$

It can be shown that $(G(X), d_H)$ is a complete metric space and that $G_k(X)$ and $G^k(X)$ are closed subsets [22, Chapter IV]. The following lemma will be useful.

Lemma 2.3.10. *For $A, B \in G(X)$ set*

$$\delta(A,B) := \sup_{a\in S_A} d(a,B).$$

Then the following holds:

(i) $d_H(A,B) \leqslant 2\max\{\delta(A,B), \delta(B,A)\}$.

(ii) If $A, B \in G_k(X)$ with $d(A,B) < \frac{1}{k}$ for some $k \in \mathbb{N}$, we have

$$\delta(B,A) \leqslant \frac{k\delta(A,B)}{1-k\delta(A,B)}.$$

Proof. [18, Lemma 2.6]. □

Proposition 2.3.11. *Assume $\mu_1 > -\infty$. Fix $\omega \in \tilde{\Omega}$. For every $n \in \mathbb{Z}$, let $H^n_{\sigma^n\omega} \subset F_{\mu_1}(\sigma^n\omega)$ be a complementary subspace for $F_{\mu_2}(\omega)$ satisfying* (2.3.4)*. Set $\tilde{H}^n_\omega := \psi^n_{\sigma^n\omega}(H^n_{\sigma^n\omega})$. Then the sequence $\{\tilde{H}^n_\omega\}_{n\geqslant 1}$ is Cauchy in $(G_{m_1}(F_{\mu_1}(\omega)), d_H)$.*

Proof. From (2.3.4), we can deduce that for every $n \in \mathbb{N}$ and $\xi_{\sigma^n\omega} \in S_{H^n_{\sigma^n\omega}}$,

$$\frac{1}{M_1} < \|[\xi_{\sigma^n\omega}]_{\mu_2}\| \leq 1. \tag{2.3.12}$$

Note that $\psi^k_{\sigma^n\omega}|_{H^n_{\sigma^n\omega}}$ is injective for any $k \geq 1$, therefore $\dim(\tilde{H}^n_\omega) = \dim(H^n_{\sigma^n\omega}) = m_1$. Since $\mu_2 < \mu_1$, we know that $\tilde{H}^n_\omega \cap F_{\mu_2}(\omega) = \{0\}$ and since $\dim[\frac{F_{\mu_1}(\omega)}{F_{\mu_2}(\omega)}] = m_1$, we obtain that

$$\tilde{H}^n_\omega \oplus F_{\mu_2}(\omega) = F_{\mu_1}(\omega)$$

for any $n \in \mathbb{N}$. Let $\{\xi^j_{\sigma^n\omega}\}_{1\leqslant j\leqslant m_1} \subset S_{F_{\mu_1}(\sigma^n\omega)}$ be a base for $H^n_{\sigma^n\omega}$. Then for $\xi_{\sigma^{n+1}\omega} \in S_{F_{\mu_1}(\sigma^{n+1}\omega)} \cap H^{n+1}_{\sigma^{n+1}\omega}$, there exist $\{\beta_j\}_{1\leqslant j\leqslant m_1} \subset \mathbb{R}$ such that

$$Z^n_\omega := \frac{\psi^{n+1}_{\sigma^{n+1}\omega}(\xi_{\sigma^{n+1}\omega})}{\|\psi^{n+1}_{\sigma^{n+1}\omega}(\xi_{\sigma^{n+1}\omega})\|} - \sum_{1\leqslant j\leqslant m_1} \beta_j \frac{\psi^n_{\sigma^n\omega}(\xi^j_{\sigma^n\omega})}{\|\psi^n_{\sigma^n\omega}(\xi^j_{\sigma^n\omega})\|} \in F_{\mu_2}(\omega).$$

It follows that

$$Y^n_{\sigma^n\omega} := \frac{\psi^1_{\sigma^{n+1}\omega}(\xi_{\sigma^{n+1}\omega})}{\|\psi^{n+1}_{\sigma^{n+1}\omega}(\xi_{\sigma^{n+1}\omega})\|} - \sum_{1\leqslant j\leqslant m_1} \beta_j \frac{\xi^j_{\sigma^n\omega}}{\|\psi^n_{\sigma^n\omega}(\xi^j_{\sigma^n\omega})\|} \in F_{\mu_2}(\sigma^n\omega),$$

thus

$$\begin{aligned}\Big\| \sum_{1\leqslant j\leqslant m_1} \beta_j \frac{\xi^j_{\sigma^n\omega}}{\|\psi^n_{\sigma^n\omega}(\xi^j_{\sigma^n\omega})\|}\Big\| &\leqslant \|\Pi_{H^n_{\sigma^n\omega}\|F_{\mu_2}(\sigma^n\omega)}\| \frac{\|\psi^1_{\sigma^{n+1}\omega}\|}{\|\psi^{n+1}_{\sigma^{n+1}\omega}(\xi_{\sigma^{n+1}\omega})\|} \\ &\leqslant M_1 \frac{\|\psi^1_{\sigma^{n+1}}\|}{\|\psi^{n+1}_{\sigma^{n+1}\omega}(\xi_{\sigma^{n+1}\omega})\|}\end{aligned}$$

and so

$$d\Big(\frac{\psi^{n+1}_{\sigma^{n+1}\omega}(\xi_{\sigma^{n+1}\omega})}{\|\psi^{n+1}_{\sigma^{n+1}\omega}(\xi_{\sigma^{n+1}\omega})\|}, \tilde{H}^n_\omega\Big) \leqslant \|Z^n_\omega\| = \|\psi^n_{\sigma^n\omega}(Y^n_{\sigma^n\omega})\| \leqslant \ (M_1+1)\frac{\|\psi^n_{\sigma^n\omega}|_{F_{\mu_2}(\sigma^n\omega)}\|\|\psi^1_{\sigma^{n+1}\omega}\|}{\|\psi^{n+1}_{\sigma^{n+1}\omega}(\xi_{\sigma^{n+1}\omega})\|}. \tag{2.3.13}$$

Note that $\lim_{n\to\infty} \frac{1}{n} \log \|\psi^1_{\sigma^n\omega}\| = 0$ from Birkhoff's Ergodic Theorem. Using Lemma 2.3.6 and (2.3.7) for $k=1$, we have

$$\limsup_{n\to\infty} \frac{1}{n} \log \|\psi^n_{\sigma^n\omega}|_{F_{\mu_2}(\sigma^n\omega)}\| \le \mu_2.$$

From Lemma 2.3.8 the estimate 2.3.12 and Lemma 2.3.10, (2.3.13) implies that for $\epsilon > 0$ small and large n,

$$d_H(\tilde{H}^n_\omega, \tilde{H}^{n+1}_\omega) < M \exp\big(n(\mu_2 - \mu_1 + \epsilon)\big)$$

for a constant $M > 0$. The claim is proved. □

Next, we collect some facts about the limit of the sequence above.

Lemma 2.3.12. *Assume $\tilde{H}^n_\omega \xrightarrow{d_H} \tilde{H}_\omega$. Then the following holds:*

(i) $\tilde{H}_\omega$ is invariant, i.e. $\psi^k_\omega(\tilde{H}_\omega) = \tilde{H}_{\theta^k\omega}$ for any $k \geq 0$.

(ii) $\tilde{H}_\omega \cap F_{\mu_2}(\omega) = \{0\}$.

(iii) $\tilde{H}_\omega$ only depends on ω. In particular, it does not depend on the choice of the sequence $\{\tilde{H}^n_\omega\}_{n\geq 1}$.

Proof. By construction, $\tilde{H}_\omega$ is invariant. We proceed with (ii). Consider the dual map

$$(\psi^n_{\sigma^n\omega})^*_{\mu_1} : (F_{\mu_1}(\omega))^* \to (F_{\mu_1}(\sigma^n\omega))^*.$$

It is straightforward to see that $\left(\psi^n_{\sigma^n\omega}\right)^*_{\mu_1}$ enjoys the cocycle property. From (2.3.5) and Proposition 2.2.14, we can find a closed subspace $G^*_{\mu_2}(\omega) \subset \left(F_{\mu_1}(\omega)\right)^*$ such that $\dim[(F_{\mu_1}(\omega))^*/G^*_{\mu_2}(\omega)] = m_1$ and for $\xi^*_\omega \in G^*_{\mu_2}(\omega)$, $\limsup_{n\to\infty} \frac{1}{n} \log \left\| \left(\psi^n_{\sigma^n\omega}\right)^*_{\mu_1}(\xi^*_\omega)\right\| \leqslant \mu_2$. Set

$$\left(F_{\mu_2}(\omega)\right)^{\perp}_{\mu_1} = \{\xi^*_\omega \in \left(F_{\mu_1}(\omega)\right)^* \ : \ \xi^*_\omega|_{F_{\mu_2}(\omega)} = 0\}.$$

By Hahn-Banach separation theorem,

$$\dim\left[\left(F_{\mu_2}(\omega)\right)^{\perp}_{\mu_1}\right] = \dim\left[F_{\mu_1}(\omega)/F_{\mu_2}(\omega)\right] = m_1.$$

Let $\xi^*_\omega \in \left(F_{\mu_2}(\omega)\right)^{\perp}_{\mu_1} \cap G^*_{\mu_2}(\omega)$ and assume that $\xi^*_\omega \neq 0$. Then for some $\xi_\omega \notin F_{\mu_1}(\omega) \setminus F_{\mu_2}(\omega)$, $\langle \xi^*_\omega, \xi_\omega\rangle = 1$. Using surjectivity of $[\psi^n_{\sigma^n\omega}]_{\mu_2}$, for every $n \in \mathbb{N}$, we can find $\xi_{\sigma^n\omega} \in H^n_{\sigma^n\omega}$ such that

$$\psi^n_{\sigma^n\omega}(\xi_{\sigma^n\omega}) = \xi_\omega \mod F_{\mu_2}(\omega).$$

Consequently, $\langle (\psi^n_{\sigma^n\omega})^*_{\mu_1}(\xi^*_\omega), \xi_{\sigma^n\omega}\rangle = 1$. From Lemma 2.3.8 ,

$$\lim_{n\to\infty} \frac{1}{n} \log \left\| [\psi^n_{\sigma^n\omega}(\frac{\xi_{\sigma^n\omega}}{\|[\xi_{\sigma^n\omega}]_{\mu_2}\|})]_{\mu_2} \right\| = \lim_{n\to\infty} \frac{1}{n} \log \left\| \frac{\|[\xi_\omega]_{\mu_2}\|}{\|[\xi_{\sigma^n\omega}]_{\mu_2}\|} \right\| = \mu_1. \tag{2.3.14}$$

Hence for $\epsilon > 0$ and large n,

$$\|[\xi_{\sigma^n\omega}]_{\mu_2}\| < \exp(-n(\mu_1 - \epsilon))$$

which is a contradiction since $\|(\psi^n_{\sigma^n\omega})^*_{\mu_1}(\xi^*_\omega)\| \leqslant \exp\left(n(\mu_2 + \epsilon)\right)$. Thus we have shown that

$$\left(F_{\mu_1}(\omega)\right)^* = \left(F_{\mu_2}(\omega)\right)^{\perp}_{\mu_1} \oplus G^*_{\mu_2}(\omega). \tag{2.3.15}$$

Now let $\xi_\omega \in \tilde{H}_\omega \cap F_{\mu_2}(\omega)$ and assume that $\|\xi_\omega\| = 1$. From 2.3.15, we can find $\xi^*_\omega \in G^*_{\mu_2}(\omega)$ such that $\langle \xi^*_\omega, \xi_\omega\rangle = 1$. By definition of $\tilde{H}_\omega$, there exist $\xi^n_{\sigma^n\omega} \in S_{H^n_{\sigma^n\omega}}$ such that $\frac{\psi^n_{\sigma^n\omega}(\xi^n_{\sigma^n\omega})}{\|\psi^n_{\sigma^n\omega}(\xi^n_{\sigma^n\omega})\|} \to \xi_\omega$ as $n \to \infty$, and consequently

$$\langle \xi^*_\omega, \frac{\psi^n_{\sigma^n\omega}(\xi^n_{\sigma^n\omega})}{\|\psi^n_{\sigma^n\omega}(\xi^n_{\sigma^n\omega})\|}\rangle = \langle (\psi^n_{\sigma^n\omega})^*(\xi^*_\omega), \frac{\xi^n_{\sigma^n\omega}}{\|\psi^n_{\sigma^n\omega}(\xi^n_{\sigma^n\omega})\|}\rangle \to 1$$

as $n \to \infty$. With Lemma 2.3.8 and a similar argument as above, this is again a contradiction and we have shown (ii). It remains to prove (iii). For $\xi_\omega \in \tilde{H}_\omega \subset (F_{\mu_1}(\omega))^{**}$, $\xi^*_\omega \in G^*_{\mu_2}(\omega)$ and a sequence $\xi^n_{\sigma^n\omega}$ chosen as above,

$$\langle \frac{\psi^n_{\sigma^n\omega}(\xi^n_{\sigma^n\omega})}{\|\psi^n_{\sigma^n\omega}(\xi^n_{\sigma^n\omega})\|}, \xi^*_\omega\rangle \to 0$$

as $n \to \infty$. Therefore, $\tilde{H}_\omega \subset \left(G^*_{\mu_2}(\omega)\right)^{\perp}_{\mu_1} = \{\xi^{**}_\omega \in \left(F_{\mu_1}(\omega)\right)^{**} \ : \ \xi^{**}_\omega|_{G^*_{\mu_2}(\omega)} = 0\}$ and since $\dim\left[\left(G^*_{\mu_2}(\omega)\right)^{\perp}_{\mu_1}\right] = m_1$, we obtain $\tilde{H}_\omega = \left(G^*_{\mu_2}(\omega)\right)^{\perp}_{\mu_1}$ which proves (iii). $\square$

Combining Proposition 2.3.11 and Lemma 2.3.12, we see that if $\mu_1 > -\infty$, there is a θ-invariant set $\tilde{\Omega} \subset \Omega$ of full measure such that for every $\omega \in \tilde{\Omega}$, there is an m_1-dimensional subspace H^1_ω with the properties

- $\psi^k_\omega(H^1_\omega) = H^1_{\theta^k\omega}$ for every $k \geq 0$ and
- $H^1_\omega \oplus F_{\mu_2}(\omega) = F_{\mu_1}(\omega)$.

Thanks to the following lemma, we can invoke an induction argument to deduce the existence of a sequence of invariant spaces H^i_ω, $i \geq 1$.

Lemma 2.3.13. *The family of Banach spaces $\{F_{\mu_2}(\omega)\}_{\omega\in\tilde{\Omega}}$ is a measurable field of Banach spaces with*

$$\tilde{\Delta} = \{\tilde{g} := \Pi_{F_{\mu_2}||H^1} \circ g,\ g \in \Delta\} \quad \textit{and} \quad \tilde{\Delta}_0 = \{\tilde{g} := \Pi_{F_{\mu_2}||H^1} \circ g,\ g \in \Delta_0\}.$$

In addition, $\psi_\omega|_{F_{\mu_2}(\omega)} : F_{\mu_2}(\omega) \to F_{\mu_2}(\theta\omega)$ is a linear compact cocycle satisfying Assumption 2.3.1 with Δ replaced by $\tilde{\Delta}$. Moreover, the maps

$$f_1(\omega) := \|\Pi_{H^1_\omega||F_{\mu_2}(\omega)}\| \quad \textit{and} \quad f_2(\omega) := \|\Pi_{F_{\mu_2}(\omega)||H^1_\omega}\|$$

are measurable.

Proof. The only non-trivial part in proving that $\{F_{\mu_2}(\omega)\}_{\omega\in\tilde{\Omega}}$ is a measurable field of Banach spaces is to show that

$$\omega \mapsto \|\Pi_{F_{\mu_2}(\omega)||H^1_\omega}(g(\omega))\| \tag{2.3.16}$$

is measurable for every $g \in \Delta$. Let

$$\{g_i\ :\ i \in \mathbb{N}\} = \Delta_0 \quad \text{and} \quad \{(g_{k_1}, \dots, g_{k_{m_1}})\ :\ k \in \mathbb{N}\} = \Delta_0^{m_1}.$$

Fix $n \in \mathbb{N}$ and $\omega \in \tilde{\Omega}$. We define $\{U^k_{\sigma^n\omega}\}_{k\geqslant 1}$ to be the family of subspaces of $E_{\sigma^n\omega}$ given by $U^k_{\sigma^n\omega} = \langle g_{k_i}(\sigma^n\omega)\rangle_{1\leqslant i\leqslant m_1, g_{k_i}\in\Delta_0}$. Using the same technique as in Lemma 2.3.4, one can show that the map

$$\omega \mapsto G_k(\sigma^n\omega) = \begin{cases} \|\Pi_{U^k_{\sigma^n\omega}||F_{\mu_2}(\sigma^n\omega)}\| & U^k_{\sigma^n\omega} \oplus F_{\mu_2}(\sigma^n\omega) = F_{\mu_1}(\sigma^n\omega) \\ \infty & \text{otherwise} \end{cases}$$

is measurable. Set $\psi_n(\omega) := \inf\{k : G_k(\sigma^n\omega) < M_1\}$ with M_1 as in Lemma 2.3.5. This map is clearly measurable. By Proposition 2.3.11, $\tilde{H}^n_\omega := \psi^n_{\sigma^n\omega}(U^{\psi_n(\omega)}_{\sigma^n\omega}) \xrightarrow{d_H} H^1_\omega$ and consequently

$$\Pi_{\tilde{H}^n_\omega||F_{\mu_2}(\omega)} \to \Pi_{H^1_\omega||F_{\mu_2}(\omega)} \qquad \text{as } n \to \infty. \tag{2.3.17}$$

Let $g \in \Delta$. Then we have a decomposition of the form

$$\Pi_{\tilde{H}^n_\omega||F_{\mu_2}(\omega)} g(\omega) = \sum_{1\leqslant t\leqslant m_1} \alpha_t(\omega)\psi^n_{\sigma^n\omega}(g_{\iota_t(\omega)}(\sigma^n\omega))$$

where $\iota_1, \ldots, \iota_{m_1} \colon \Omega \to \mathbb{N}$ are measurable. We assume $m_1 = 1$ first. To ease notation, set $\iota := \iota_1$. Since $g(\omega) - \alpha_1(\omega)\psi^n_{\sigma^n\omega}(g_{\iota(\omega)}(\sigma^n\omega)) \in F_{\mu_2}(\omega)$, we have $\|[g(\omega)]_{\mu_2}\| = |\alpha_1(\omega)|\|[\psi^n_{\sigma^n\omega}(g_{\iota(\omega)}(\sigma^n\omega))]\|$ and therefore

$$|\alpha_1(\omega)| = \frac{d\big(g(\omega), F_{\mu_2}(\omega)\big)}{d\big(\psi_{\sigma^n\omega}(g_{\iota(\omega)}(\sigma^n\omega)), F_{\mu_2}(\omega)\big)}.$$

Set

$$d_0(\omega) := d\big(g(\omega), F_{\mu_2}(\omega)\big) \quad \text{and} \quad d_1(\omega) := d\big(\psi_{\sigma^n\omega}(g_{\iota(\omega)}(\sigma^n\omega)), F_{\mu_2}(\omega)\big).$$

From the proof of Lemma 2.2.2, we know that d_0 is measurable. Furthermore, a slight adaptation of the proof yields the measurability of $\omega \mapsto d\big(\psi_{\sigma^n\omega}(g_k(\sigma^n\omega)), F_{\mu_2}(\omega)\big)$ for any fixed $k \in \mathbb{N}$. Since ι is measurable, this implies the measurability of d_1, too. We have

$$\Pi_{\tilde{H}^n_\omega || F_{\mu_2}(\omega)} g(\omega) = G(\omega)\frac{d_0(\omega)}{d_1(\omega)}\psi^n_{\sigma^n\omega}(g_{\iota(\omega)}(\sigma^n\omega))$$

where $G(\omega)$ takes values in $\{-1, 0, 1\}$. Set $h_0(\omega) := g(\omega) - \frac{d_0(\omega)}{d_1(\omega)}\psi^n_{\sigma^n\omega}(g_{\iota(\omega)}(\sigma^n\omega))$ and $h_1(\omega) := g(\omega) + \frac{d_0(\omega)}{d_1(\omega)}\psi^n_{\sigma^n\omega}(g_{\iota(\omega)}(\sigma^n\omega))$ and define

$$J_0(\omega) := \lim_{m\to\infty}\frac{1}{m}\log\|\psi^m_\omega(h_0(\omega))\|, \qquad J_1(\omega) := \lim_{m\to\infty}\frac{1}{m}\log\|\psi^m_\omega(h_1(\omega))\|.$$

It follows that J_0 and J_1 are measurable and that

$$\Pi_{\tilde{H}^n_\omega || F_{\mu_2}(\omega)} g(\omega) = (1 - \chi_{\{g(\omega)\in F_{\mu_2}(\omega)\}})\left[g(\omega) - \chi_{\mu_2}(J_0(\omega))h_0(\omega) - \chi_{\mu_2}(J_1(\omega))h_1(\omega)\right]. \tag{2.3.18}$$

Then (2.3.18) and (2.3.17) prove the measurability of (2.3.16) for every $g \in \Delta$ in the case $m_1 = 1$. Furthermore, measurability of f_1 and f_2 and Assumption 2.3.1 for $\tilde{\Delta}$ can also be deduced. It remains to consider the case $m_1 > 1$ for which we invoke the same technique: Let

$$d_0(\omega) = d\big(g(\omega), F_{\mu_2}(\omega) \oplus \langle\psi^n_{\sigma^n\omega}(g_{\iota_t(\omega)}(\sigma^n\omega))\rangle_{2\leqslant t\leqslant m_1}\big),$$
$$d_1(\omega) = d\big(\psi^n_{\sigma^n\omega}(g_{\iota_1(\omega)}(\sigma^n\omega)), F_{\mu_2}(\omega) \oplus \langle\psi^n_{\sigma^n\omega}(g_{\iota_t(\omega)}(\sigma^n\omega))\rangle_{2\leqslant t\leqslant m_1}\big).$$

For $h_0(\omega) = g(\omega) - \frac{d_0(\omega)}{d_1(\omega)}\psi^n_{\sigma^n\omega}(g_{\iota_1(\omega)}(\sigma^n\omega))$ and $h_1(\omega) = g(\omega) + \frac{d_0(\omega)}{d_1(\omega)}\psi^n_{\sigma^n\omega}(g_{\iota_1(\omega)}(\sigma^n\omega))$ let

$$d_{i0}(\omega) := d\big(h_i(\omega), F_{\mu_2}(\omega) \oplus \langle\psi^n_{\sigma^n\omega}(g_{\iota_t(\omega)}(\sigma^n\omega))\rangle_{3\leqslant t\leqslant m_1}\big), \ i \in \{0,1\}$$
$$d_{01}(\omega) = d_{11}(\omega) = d\big(\psi^n_{\sigma^n\omega}(g_{\iota_2(\omega)}(\sigma^n\omega)), F_{\mu_2}(\omega) \oplus \langle\psi^n_{\sigma^n\omega}(g_{\iota_t(\omega)}(\sigma^n\omega))\rangle_{3\leqslant t\leqslant m_1}\big).$$

For $i \in \{0, 1\}$ define

$$h_{0,i} = h_0(\omega) + (-1)^{i+1}\frac{d_{00}(\omega)}{d_{01}(\omega)}\psi^n_{\sigma^n\omega}(g_{\iota_2(\omega)}(\sigma^n\omega))$$
$$h_{1,i} = h_1(\omega) + (-1)^{i+1}\frac{d_{10}(\omega)}{d_{11}(\omega)}\psi^n_{\sigma^n\omega}(g_{\iota_2(\omega)}(\sigma^n\omega)).$$

We repeat the same procedure with our four new functions. Iterating this, we end up with 2^{m_1} functions $\{I_t(\omega)\}_{1\leqslant t\leqslant 2^{m_1}}$ for which we define $J_t(\omega) := \lim_{m\to\infty}\frac{1}{m}\log\|\psi_\omega^m(I_t(\omega))\|$. Since

$$\Pi_{\tilde{H}^n_\omega||F_{\mu_2}(\omega)}g(\omega) = (1-\chi_{\{g(\omega)\in F_{\mu_2}(\omega)\}})\left[g(\omega) - \sum_{0\leqslant t\leqslant 2^{m_1}}\chi_{\mu_2}(J_t(\omega))I_t(\omega)\right],$$

our claim follows for arbitrary m_1. $\square$

Proposition 2.3.14. *Let $i\in\mathbb{N}$ and assume $\mu_i>\infty$. Then there is a θ-invariant set of full measure $\tilde{\Omega}$ such that for every $\omega\in\tilde{\Omega}$, there is an m_i-dimensional space H^i_ω with the properties*

1. *$\psi^k_\omega(H^i_\omega) = H^i_{\theta^k\omega}$ for every $k\geq 0$ and*
2. *$H^i_\omega\oplus F_{\mu_{i+1}}(\omega) = F_{\mu_i}(\omega)$.*

Proof. For $i=1$, the statement follows from Proposition 2.3.11 and Lemma 2.3.12. For $i=2$, we consider the restricted cocycle $\psi^k_\omega|_{F_{\mu_2}(\omega)}$. From Lemma 2.3.13, we know that this cocycle acts on the measurable field of Banach spaces $\{F_{\mu_2}(\omega)\}_{\omega\in\Omega}$ and we can thus apply Proposition 2.3.11 and Lemma 2.3.12 to this cocycle again. It remains to make sure that the top Lyapunov exponent of the restricted cocycle coincides with μ_2. This, however, was deduced in Lemma 2.3.7. We can now repeat the argument until we reach i. $\square$

From now on, H^i_ω will always denote the spaces deduced in Proposition 2.3.14.

Remark 2.3.15. *Using identities of the form*

$$\Pi_{F_{\mu_j}(\omega)||\oplus_{l\leqslant i<j}H^i_\omega} = \Pi_{F_{\mu_j}(\omega)||H^{j-1}_\omega}\circ\Pi_{F_{\mu_{j-1}}(\omega)||H^{j-2}_\omega}\circ\ldots\circ\Pi_{F_{\mu_{l+1}}(\omega)||H^l_\omega},$$

we can use the same strategy as in Lemma 2.3.13 to see that for each $1\leq l\leq j$ and $k\geq 0$,

$$f_1(\omega) := \|\Pi_{\oplus_{l\leqslant i<j}H^i_\omega\oplus F_{\mu_j}(\omega)}\|,\ f_2(\omega) := \|\Pi_{F_{\mu_j}(\omega)||\oplus_{l\leqslant i<j}H^i_\omega}\| \text{ and } f_3(\omega) := \|\psi^k_\omega|_{\oplus_{l\leqslant i<j}H^i_\omega}\|$$

are measurable.

Lemma 2.3.16. *For a measurable and non-negative function $f:\Omega\to\mathbb{R}$*

$$\lim_{n\to\infty}\frac{1}{n}f(\theta^n\omega)=0 \ a.s. \quad \text{if and only if} \quad \lim_{n\to\infty}\frac{1}{n}f(\sigma^n\omega)=0 \ a.s.$$

Proof. The main idea is due to Jack Feldman, cf. [23, Lemma 7.2]. Assume that $\lim_{n\to\infty}\frac{1}{n}f(\theta^n\omega)=0$ on a set of full measure Ω^0. Let $\epsilon>0$ and set

$$\Omega_n := \{\omega\in\Omega^0\ :\ \forall i\geqslant n\ \ \frac{f(\theta^i\omega)}{i}\leqslant\epsilon\}.$$

From our assumptions, for some $n_0\in\mathbb{N}$,

$$\mathbb{P}(\Omega_{n_0}) > \frac{9}{10}.$$

From Birkhoff's ergodic theorem, there is a set of full measure Ω^1 such that for every $\omega \in \Omega^1$, we can find $m_0 = m_\omega$ such that for $m \geqslant m_0$,

$$\frac{1}{m} \sum_{0 \leqslant j \leqslant m} \chi_{\Omega_{n_0}}(\sigma^j \omega) > \frac{9}{10}. \tag{2.3.19}$$

W.l.o.g., we may assume that $\Omega^0 = \Omega^1$. Now for $k \geqslant \max\{3n_0, m_0\}$, set $m = \lfloor \frac{5}{3} k \rfloor + 1$. Then from (2.3.19)

$$\frac{1}{m} \Big[\sum_{0 \leqslant j \leqslant \frac{4m}{5}} \chi_{\Omega_{n_0}}(\sigma^j \omega) + \sum_{\frac{4m}{5} < j \leqslant m} \chi_{\Omega_{n_0}}(\sigma^j \omega) \Big] > \frac{9}{10}.$$

Consequently, there exists $\frac{4m}{5} < j \leqslant m$ such that $\sigma^j \omega \in \Omega_{n_0}$. Set $i := j - k > n_0$. Then by the definition of Ω_{n_0},

$$\frac{f(\theta^i \sigma^j \omega)}{i} = \frac{f(\sigma^k \omega)}{j-k} \leqslant \epsilon.$$

Since $j - k \leq \frac{2}{3} k + 1$ and ϵ is arbitrary, our claim is shown. The other direction can be proved similarly. $\square$

As a consequence, we obtain the following:

Lemma 2.3.17. *For each $1 \leq l \leq j$ and $\omega \in \tilde{\Omega}$,*

$$\lim_{n\to\infty} \frac{1}{n} \log \|\Pi_{\oplus_{l \leqslant i < j} H^i_{\theta^n \omega} \| F_{\mu_j}(\theta^n \omega)}\| = \lim_{n\to\infty} \frac{1}{n} \log \|\Pi_{\oplus_{l \leqslant i < j} H^i_{\sigma^n \omega} \| F_{\mu_j}(\sigma^n \omega)}\| = 0. \tag{2.3.20}$$

Proof. Follows from a straightforward generalization of Lemma 2.2.3 and Lemma 2.3.16. $\square$

The following lemma characterizes the spaces H^i_ω as 'fast-growing' subspaces.

Proposition 2.3.18. *For $\omega \in \tilde{\Omega}$, every $i \geqslant N$ and $\xi_\omega \in H^i_\omega \setminus \{0\}$,*

$$\lim_{n\to\infty} \frac{1}{n} \log \|\psi^n_\omega(\xi_\omega)\| = \lim_{n\to\infty} \frac{1}{n} \log \|\psi^n_\omega|_{H^i_\omega}\| = \mu_i \tag{2.3.21}$$

and

$$\lim_{n\to\infty} \frac{1}{n} \log \|(\psi^n_{\sigma^n \omega})^{-1}(\xi_\omega)\| = \lim_{n\to\infty} \frac{1}{n} \log \|(\psi^n_{\sigma^n \omega}|_{H^i_\omega})^{-1}\| = -\mu_i. \tag{2.3.22}$$

Proof. The equalities (2.3.21) follow by applying the Multiplicative Ergodic Theorem 2.2.16 to the map $\psi^n_\omega|_{H^i_\omega} : H^i_\omega \to H^i_{\theta^n \omega}$. It remains to prove (2.3.22). By definition, for every $\xi_\omega \in H^i_\omega$,

$$\frac{\|(\psi^n_{\sigma^n \omega})^{-1}(\xi_\omega)\|}{\|[\xi_\omega]_{\mu_{i+1}}\|} \times \frac{\|[\psi^n_{\sigma^n \omega}((\psi^n_{\sigma^n \omega})^{-1}(\xi_\omega))]_{\mu_{i+1}}\|}{\|[(\psi^n_{\sigma^n \omega})^{-1}(\xi_\omega)]_{\mu_{i+1}}\|} =$$

$$\frac{\|(\psi^n_{\sigma^n \omega})^{-1}(\xi_\omega)\|}{\|[(\psi^n_{\sigma^n \omega})^{-1}(\xi_\omega)]_{\mu_{i+1}}\|} \leqslant \|\Pi_{H^i_{\sigma^n \omega} \| F_{\mu_{i+1}}(\sigma^n \omega)}\|.$$

From Lemma 2.3.8,

$$\lim_{n\to\infty}\frac{1}{n}\inf_{\bar{\xi}_{\sigma^n\omega}\in H^i_{\sigma^n\omega}}\frac{\|[\psi^n_{\sigma^n\omega}(\bar{\xi}_{\sigma^n\omega})]_{\mu_{i+1}}\|}{\|[\bar{\xi}_{\sigma^n\omega}]_{\mu_{i+1}}\|}=\lim_{n\to\infty}\frac{1}{n}\frac{\|[\psi^n_{\sigma^n\omega}(\hat{\xi}_{\sigma^n\omega})]_{\mu_{i+1}}\|}{\|[\hat{\xi}_{\sigma^n\omega}]_{\mu_{i+1}}\|}=\mu_i$$

where $\hat{\xi}_{\sigma^n\omega}\in H^i_{\sigma^n\omega}$ is chosen such that

$$\frac{\|[\psi^n_{\sigma^n\omega}(\hat{\xi}_{\sigma^n\omega})]_{\mu_{i+1}}\|}{\|[\hat{\xi}_{\sigma^n\omega}]_{\mu_{i+1}}\|}=\min_{\bar{\xi}_{\sigma^n\omega}\in H^i_{\sigma^n\omega}}\frac{\|[\psi^n_{\sigma^n\omega}(\bar{\xi}_{\sigma^n\omega})]_{\mu_{i+1}}\|}{\|[\bar{\xi}_{\sigma^n\omega}]_{\mu_{i+1}}\|}.$$

Consequently, from (2.3.20),

$$\limsup_{n\to\infty}\frac{1}{n}\log\|(\psi^n_{\sigma^n\omega}|_{H^i_\omega})^{-1}\|\leqslant-\mu_i$$

Finally, from inequality $\|\xi_\omega\|\leqslant\|\psi^n_{\sigma^n\omega}|_{H^i_{\sigma^n\omega}}\|\|(\psi^n_{\sigma^n\omega})^{-1}(\xi_\omega)\|$, Lemma 2.3.6 and (2.3.21), the equalities (2.3.22) can be deduced. □

Lemma 2.3.19. *Let $\omega\in\tilde{\Omega}$ and $i<k$. For every $i\le j<k$, let $\{\xi^t_\omega\}_{t\in I_j}$ be a basis of H^j_ω. Set $I:=\cup_{i\le j<k}I_j$ and assume $\xi^t_\omega\in H^j_\omega$. Then*

$$\lim_{n\to\infty}\frac{1}{n}\log d(\psi^n_\omega(\xi^t_\omega),\langle\psi^n_\omega(\xi^{t'}_\omega)\rangle_{t'\in I\setminus\{t\}})=\mu_j \tag{2.3.23}$$

and

$$\lim_{n\to\infty}\frac{1}{n}\log d((\psi^n_{\sigma^n\omega})^{-1}(\xi^t_\omega),\langle(\psi^n_{\sigma^n\omega})^{-1}(\xi^{t'}_\omega)\rangle_{t'\in I\setminus\{t\}})=-\mu_j. \tag{2.3.24}$$

Proof. We will prove (2.3.24) only, the proof for (2.3.23) is completely analogous. First, we claim that the statement is true for $j=i$ and $k=i+1$. Indeed, in this case we have the inequalities

$$\frac{1}{\|\psi^n_{\sigma^n\omega}|_{H^i_{\sigma^n\omega}}\|}\leqslant\frac{d\big((\psi^n_{\sigma^n\omega})^{-1}(\xi^t_\omega),\langle(\psi^n_{\sigma^n\omega})^{-1}(\xi^{t'}_\omega)\rangle_{t'\in I\setminus\{t\}}\big)}{d\big(\xi^t_\omega,\langle\xi^{t'}_\omega\rangle_{t'\in I\setminus\{t\}}\big)}\leqslant\|(\psi^n_{\sigma^n\omega})^{-1}|_{H^i_\omega}\|$$

and we can conclude with Proposition 2.3.18. For arbitrary k and $j=i$, we can use the inequalities

$$1\leqslant\frac{d\big((\psi^n_{\sigma^n\omega})^{-1}(\xi^t_\omega),\langle(\psi^n_{\sigma^n\omega})^{-1}(\xi^{t'}_\omega)\rangle_{t'\in I_i\setminus\{t\}}\big)}{d\big((\psi^n_{\sigma^n\omega})^{-1}(\xi^t_\omega),\langle(\psi^n_{\sigma^n\omega})^{-1}(\xi^{t'}_\omega)\rangle_{t'\in I\setminus\{t\}}\big)}\leqslant\|\Pi_{H^i_{\sigma^n\omega}\|F_{\mu_{i+1}}(\sigma^n\omega)}\|,$$

Lemma 2.3.17 and our previous result above. The definition of Vol allows to deduce that

$$\lim_{n\to\infty}\frac{1}{n}\log\mathrm{Vol}\left(\big((\psi^n_{\sigma^n\omega})^{-1}(\xi^t_\omega)\big)_{t\in I_{k-1}},\ldots,\big((\psi^n_{\sigma^n\omega})^{-1}(\xi^t_\omega)\big)_{t\in I_i}\right)=\sum_{i\leqslant j<k}-\mu_j|I_j|. \tag{2.3.25}$$

Since Vol is symmetric up to a constant, the claim (2.3.24) follows for arbitrary j.

□

The following theorem is the announced semi-invertible Oseledets theorem on fields of Banach spaces. It summarizes the main result of this section.

Theorem 2.3.20. *There is a θ-invariant set of full measure $\tilde{\Omega}$ such that for every $i \geq 1$ with $\mu_i > \mu_{i+1}$ and $\omega \in \tilde{\Omega}$, there is an m_i-dimensional subspace H^i_ω with the following properties:*

(i) (Invariance) $\psi^k_\omega(H^i_\omega) = H^i_{\theta^k \omega}$ *for every* $k \geq 0$.

(ii) (Splitting) $H^i_\omega \oplus F_{\mu_{i+1}}(\omega) = F_{\mu_i}(\omega)$. *In particular,*

$$E_\omega = H^1_\omega \oplus \cdots \oplus H^i_\omega \oplus F_{\mu_{i+1}}(\omega).$$

(iii) ('Fast-growing' subspace I) For each $h_\omega \in H^i_\omega \setminus \{0\}$,

$$\lim_{n\to\infty} \frac{1}{n} \log \|\psi^n_\omega(h_\omega)\| = \mu_i.$$

(iv) ('Fast-growing' subspace II) For each $h_\omega \in H^i_\omega \setminus \{0\}$,

$$\lim_{n\to\infty} \frac{1}{n} \log \|(\psi^n_{\sigma^n\omega})^{-1}(h_\omega)\| = -\mu_i.$$

(v) If $\{\xi^t_\omega\}_{1\leqslant t\leqslant m}$ *is a basis of* $\oplus_{1\leqslant i\leqslant j} H^i_\omega$, *then*

$$\begin{aligned}\lim_{n\to\infty} \frac{1}{n} \log \operatorname{Vol}\left(\psi^n_\omega(\xi^1_\omega), ..., \psi^n_\omega(\xi^m_\omega)\right) &= \sum_{1\leqslant i\leqslant j} m_i\mu_i \quad \textit{and} \\ \lim_{n\to\infty} \frac{1}{n} \log \operatorname{Vol}\left((\psi^n_{\sigma^n\omega})^{-1}(\xi^1_\omega), ..., (\psi^n_{\sigma^n\omega})^{-1}(\xi^m_\omega)\right) &= \sum_{1\leqslant i\leqslant j} -m_i\mu_i.\end{aligned} \tag{2.3.26}$$

Moreover, the properties (i) - (iv) uniquely determine the spaces H^i_ω.

Proof. Properties (i) and (ii) are proven in Proposition 2.3.14. (iii) and (iv) are shown in Proposition 2.3.18 and (v) can be deduced from Lemma 2.3.19, using the definition of Vol and symmetry modulo a constant of this function. It remains to prove the uniqueness statement. Fix $i \geq 1$ and assume $\mu_i > \mu_{i+1}$. We define $G^*_{\mu_{i+1}}(\omega)$ and $\left(G^*_{\mu_{i+1}}(\omega)\right)^\perp_{\mu_i}$ as in Lemma 2.3.12 and claim that

$$H^i_\omega = \left(G^*_{\mu_{i+1}}(\omega)\right)^\perp_{\mu_i}. \tag{2.3.27}$$

Let $h_\omega \in H^i_\omega$, $h^*_\omega \in G^*_{\mu_{i+1}}(\omega)$ and set $h_{\sigma^n\omega} := (\psi^n_{\sigma^n\omega})^{-1}(h_\omega)$. Property (iv) implies that there is an $\epsilon > 0$ sufficiently small such that

$$\langle h_\omega, h^*_\omega\rangle = \langle \psi^n_{\sigma^n\omega}(h_{\sigma^n\omega}), h^*_\omega\rangle = \langle h_{\sigma^n\omega}, (\psi^n_{\sigma^n\omega})^*(h^*_\omega)\rangle \leqslant \exp\left(-n(\mu_i - \mu_{i+1} - \epsilon)\right) \to 0$$

as $n \to \infty$ which reveals $H^i_\omega \subset \left(G^*_{\mu_{i+1}}(\omega)\right)^\perp_{\mu_i}$. Finally, since these spaces have the same dimension, (2.3.27) follows.

$\square$

3

Invariant Manifolds

3.1 Introduction

A typical application for MET is the construction of stable and unstable manifolds, cf. [11, 12, 13]. Here, the existence of the Oseledets splitting is crucial. In this chapter, we prove an invariant manifold theorem for nonlinear cocycles acting on fields of Banach spaces. We state an informal version here; the precise statements are formulated in Theorem 3.2.9 and Theorem 3.3.6.

Theorem 3.1.1. *Let φ be a nonlinear, differentiable cocycle acting on a measurable field of Banach spaces $\{E_\omega\}_{\omega\in\Omega}$. Assume that Y_ω is a random fixed point of φ, in particular $\varphi_\omega(Y_\omega) = Y_{\theta\omega}$. Then, under the same measurability and integrability assumptions as in Theorem 2.1.2, the linearized cocycle $D_{Y_\omega}\varphi_\omega$ has a Lyapunov spectrum $\{\mu_n\}_{n\geq 1}$. Under further assumptions on φ and Y, there is a θ-invariant set $\tilde{\Omega}$ of full measure, closed subspaces S_ω and U_ω of E_ω and immersed submanifolds $S_{loc}(\omega)$ and $U_{loc}(\omega)$ of E_ω such that for every $\omega \in \tilde{\Omega}$,*

$$T_{Y(\omega)}S_{loc}(\omega) = S_\omega \qquad \text{and} \qquad T_{Y(\omega)}U_{loc}(\omega) = U_\omega$$

and the properties that for every $Z_\omega \in S_{loc}(\omega)$,

$$\limsup_{n\to\infty} \frac{1}{n} \log \|\varphi^n_\omega(Z_\omega) - Y_{\theta^n\omega}\| \leqslant \mu_{j_0} < 0$$

and for every $Z_\omega \in U_{loc}(\omega)$ one has $\varphi^n_{\sigma^n\omega}(Z_{\sigma^n\omega}) = Z_\omega$ and

$$\limsup_{n\to\infty} \frac{1}{n} \log \|Z_{\sigma^n\omega} - Y_{\sigma^n\omega}\| \leqslant -\mu_{k_0} < 0.$$

Here we have set $\mu_{j_0} = \max\{\mu_j \,:\, \mu_j < 0\}$ and $\mu_{k_0} = \min\{\mu_k \,:\, \mu_k > 0\}$. In the hyperbolic case, i.e. if all Lyapunov exponents are non-zero, the submanifolds $S^{\upsilon}_{loc}(\omega)$ and $U^{\upsilon}_{loc}(\omega)$ are transversal, i.e.

$$E_\omega = T_{Y_\omega}U^{\upsilon}_{loc}(\omega) \oplus T_{Y_\omega}S^{\upsilon}_{loc}(\omega).$$

Notation

In the whole of this chapter we assume θ is invertible and $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$ will denote an ergodic measure-preserving dynamical system also we set $\sigma := \theta^{-1}$. We assume $(\{E_\omega\}_{\omega\in\Omega}, \Delta, \Delta_0)$ is a measurable field of Banach space and φ^n_ω is a nonlinear cocycle acting on it, i.e.

$$\begin{aligned}
&\varphi^n_\omega \colon E_\omega \to E_{\theta^n\omega} \\
&\varphi^{n+m}_\omega(.) = \varphi^n_{\theta^m\omega}(\varphi^m_\omega(.)).
\end{aligned}$$

We also need following definition.

Definition 3.1.2. *We say that φ^n_ω admits a* stationary solution *if there exists a map $Y : \Omega \longrightarrow \prod_{\omega\in\Omega} E_\omega$ such that*

(i) $Y_\omega \in E_\omega$,

(ii) $\varphi^n_\omega(Y_\omega) = Y_{\theta^n\omega}$ *and*

(iii) $\omega \to \|Y_\omega\|$ *is measurable.*

Stationary solutions should be thought of random analogues to fixed points in (deterministic) dynamical systems. If φ^n_ω is Fréchet differentiable, one can easily check that the derivative around a stationary solution also enjoys the cocycle property, i.e for $\psi^n_\omega(.) = D_{Y_\omega}\varphi^n_\omega(.)$, one has

$$\psi^{n+m}_\omega(.) = \psi^n_{\theta^m\omega}(\psi^m_\omega(.)).$$

In the following, we will assume that φ is Fréchet differentiable, that there exists a stationary solution Y and that the linearized cocycle ψ around Y is compact and satisfies Assumption 2.3.1. Furthermore, we will assume that

$$\log^+ \|\psi_\omega\| \in L^1(\Omega).$$

Therefore, we can apply the MET to ψ. In the following, we will use the same notation as in the previous chapter.

3.2 Stable manifolds

Definition 3.2.1. *Let Y be a stationary solution, let $\{... < \mu_j < \mu_{j-1} < ... < \mu_1\} \in [-\infty, \infty)$ be the corresponding Lyapunov spectrum and $\tilde{\Omega}$ the θ-invariant set on which the MET holds. Set $\mu_{j_0} = \max\{\mu_j : \mu_j < 0\}$ and $\mu_{j_0} = -\infty$ if all finite μ_j are nonnegative. We define the* stable subspace

$$S_\omega := F_{\mu_{j_0}}(\omega).$$

By the unstable subspace *we mean*

$$U_\omega := \oplus_{1\leqslant i<j_0} H^i_\omega.$$

Note that $\dim[E_\omega/S_\omega] = \dim[U_\omega] =: k < \infty$ *for every* $\omega \in \tilde{\Omega}$.

Lemma 3.2.2. *For* $\omega \in \tilde{\Omega}$ *and* $\epsilon \in (0, -\mu_{j_0})$, *set*

$$F(\omega) := \sup_{p \geqslant 0} \exp[-p(\mu_{j_0} + \epsilon)] \|\psi^p_\omega|_{S_\omega}\|.$$

Then

$$\lim_{n\to\infty} \frac{1}{n} \log^+ [F(\theta^n\omega)] = 0. \tag{3.2.1}$$

Proof. Follows from (2.3.7). $\square$

Lemma 3.2.3. *Let* $\omega \in \tilde{\Omega}$, $U_\omega = \langle \xi^t_\omega \rangle_{1 \leqslant t \leqslant k}$ *and* $n, p \geq 0$. *Then*

$$\|[\psi^n_{\theta^p\omega}]^{-1}\|_{L[U_{\theta^{n+p}\omega}, U_{\theta^p\omega}]} \leqslant \sum_{1 \leqslant t \leqslant k} \frac{\|\psi^p_\omega(\xi^t_\omega)\|}{\|\psi^{n+p}_\omega(\xi^t_\omega)\|} \times \frac{\|\psi^{n+p}_\omega(\xi^t_\omega)\|}{d(\psi^{n+p}_\omega(\xi^t_\omega), \langle \psi^{n+p}_\omega(\xi^{t'}_\omega)\rangle_{t' \neq t})} \tag{3.2.2}$$

and

$$\begin{aligned} &\|[\psi^p_{\sigma^n\omega}]^{-1}\|_{L[U_{\sigma^{n-p}\omega}, U_{\sigma^n\omega}]} \leqslant \\ &\quad \sum_{1 \leqslant t \leqslant k} \frac{\|(\psi^n_{\sigma^n\omega})^{-1}(\xi^t_\omega)\|}{\|(\psi^{n-p}_{\sigma^{n-p}\omega})^{-1}(\xi^t_\omega)\|} \times \frac{\|(\psi^{n-p}_{\sigma^{n-p}\omega})^{-1}(\xi^t_\omega)\|}{d((\psi^{n-p}_{\sigma^{n-p}\omega})^{-1}(\xi^t_\omega), \langle (\psi^{n-p}_{\sigma^{n-p}(\omega)})^{-1}(\xi^{t'}_\omega)\rangle_{t' \neq t})}. \end{aligned} \tag{3.2.3}$$

Proof. Choose $u \in U_{\theta^p\omega}$ and assume that $u = \sum_{1 \leqslant t \leqslant k} u^t \frac{\psi^p_\omega(\xi^t_\omega)}{\|\psi^p_\omega(\xi^t_\omega)\|}$. Then

$$\frac{|u^t|}{\|u\|} \leqslant \frac{\|\psi^p_\omega(\xi^t_\omega)\|}{d(\psi^p_\omega(\xi^t_\omega), \langle \psi^p_\omega(\xi^{t'}_\omega)\rangle_{t' \neq t})}. \tag{3.2.4}$$

From $\psi^n_{\theta^p\omega} u = \sum_{1 \leqslant t \leqslant k} u^t \frac{\|\psi^{n+p}_\omega(\xi^t_\omega)\|}{\|\psi^p_\omega(\xi^t_\omega)\|} \frac{\psi^{n+p}_\omega(\xi^t_\omega)}{\|\psi^{n+p}_\omega(\xi^t_\omega)\|}$ and (3.2.4),

$$\frac{|u^t|}{\|\psi^n_{\theta^p\omega} u\|} \leqslant \frac{\|\psi^p_\omega(\xi^t_\omega)\|}{\|\psi^{n+p}_\omega(\xi^t_\omega)\|} \times \frac{\|\psi^{n+p}_\omega(\xi^t_\omega)\|}{d(\psi^{n+p}_\omega(\xi^t_\omega), \langle \psi^{n+p}_\omega(\xi^{t'}_\omega)\rangle_{t' \neq t})}$$

and (3.2.2) follows. The estimate (3.2.3) is proven similarly.

$\square$

Definition 3.2.4. *For* $\omega \in \Omega$ *set* $\Sigma_\omega := \prod_{j \geqslant 0} E_{\theta^j\omega}$. *For* $\upsilon > 0$ *we define*

$$\Sigma^\upsilon_\omega := \left\{ \Gamma \in \Sigma_\omega : \|\Gamma\| = \sup_{j \geqslant 0} \left[\|\Pi^j_\omega \Gamma\| \exp(\upsilon j) \right] < \infty \right\}$$

where $\Pi^j_\omega : \prod_{i \geqslant 0} E_{\theta^i\omega} \to E_{\theta^j\omega}$ *denotes the projection map.*

One can check that Σ^υ_ω is a Banach space.

Lemma 3.2.5. *Let* $\omega \in \Omega$ *and* $0 < \upsilon < -\mu_{j_0}$. *Define*

$$\begin{aligned} P_\omega &: E_\omega \to E_{\theta\omega} \\ &\quad \xi_\omega \mapsto \varphi^1_\omega(Y_\omega + \xi_\omega) - \varphi^1_\omega(Y_\omega) - \psi^1_\omega(\xi_\omega). \end{aligned}$$

Let $\rho\colon \Omega \to \mathbb{R}^+$ be a random variable with the property that

$$\liminf_{n\to\infty} \frac{1}{n} \log \rho(\theta^n\omega) \geq 0$$

almost surely. Assume that for $\|\xi_\omega\|, \|\tilde{\xi}_\omega\| < \rho(\omega)$,

$$\|P_\omega(\xi_\omega) - P_\omega(\tilde{\xi}_\omega)\| \leqslant \|\xi_\omega - \tilde{\xi}_\omega\| f(\omega) h(\|\xi_\omega\| + \|\tilde{\xi}_\omega\|) \tag{3.2.5}$$

almost surely where $f:\Omega \to \mathbb{R}^+$ is a measurable function such that $\lim_{n\to\infty} \frac{1}{n}\log^+ f(\theta^n\omega) = 0$ almost surely and $h(x) = x^r g(x)$ for some $r > 0$ where $g:\mathbb{R}\to\mathbb{R}^+$ is an increasing C^1 function. Set

$$\tilde{\rho}(\omega) := \inf_{n\geqslant 0} \exp(n\upsilon)\rho(\theta^n\omega). \tag{3.2.6}$$

Then the map

$$I_\omega\colon S_\omega \times \Sigma_\omega^\upsilon \cap B(0,\tilde{\rho}(\omega)) \to \Sigma_\omega^\upsilon,$$

$$\Pi_\omega^n[I_\omega(v_\omega,\Gamma)] = \begin{cases} \psi_\omega^n(v_\omega) + \sum_{0\leqslant j\leqslant n-1} [\psi_{\theta^{1+j}\omega}^{n-1-j} \circ \Pi_{S_{\theta^{1+j}\omega}\|U_{\theta^{1+j}\omega}}] P_{\theta^j\omega}(\Pi_\omega^j[\Gamma]) & \\ \qquad - \sum_{j\geqslant n} [[\psi_{\theta^n\omega}^{j-n+1}]^{-1} \circ \Pi_{U_{\theta^{1+j}\omega}\|S_{\theta^{1+j}\omega}}] P_{\theta^j\omega}(\Pi_\omega^j[\Gamma]) & \text{for } n \geq 1, \\ v_\omega - \sum_{j\geqslant 0} [[\psi_\omega^{j+1}]^{-1} \circ \Pi_{U_{\theta^{1+j}\omega}\|S_{\theta^{1+j}\omega}}] P_{\theta^j\omega}(\Pi_\omega^j[\Gamma]) & \text{for } n = 0. \end{cases}$$

is well-defined on a θ-invariant set of full measure $\tilde{\Omega}$.

Proof. We collect some estimates first. Let $\epsilon \in (0, -\mu_{j_0})$. From (2.3.20), we can find a random variable $R(\omega) > 1$ such that for $j \geqslant 0$,

$$\|\Pi_{U_{\theta^j\omega}\|S_{\theta^j\omega}}\| \leqslant R(\omega)\exp(\epsilon j)\ , \qquad \|\Pi_{S_{\theta^j\omega}\|U_{\theta^j\omega}}\| \leqslant R(\omega)\exp(\epsilon j). \tag{3.2.7}$$

Also from (3.2.1), for $n, p \geqslant 0$,

$$\|\psi_{\theta^n\omega}^p|_{S_{\theta^n\omega}}\| \leqslant R(\omega)\exp\left(p\mu_{j_0} + \epsilon(n+p)\right). \tag{3.2.8}$$

In addition, from (2.3.23) and (3.2.2) for $n, p \geqslant 0$,

$$\|[\psi_{\theta^p\omega}^n]^{-1}\|_{L[U_{\theta^{n+p}\omega}, U_{\theta^p\omega}]} \leqslant R(\omega)\exp\left(\epsilon(n+p)\right)\exp(-n\mu_{j_0-1}). \tag{3.2.9}$$

From our assumptions,

$$\|P_{\theta^j\omega}(\Pi_\omega^j[\Gamma])\| \leqslant \|\Pi_\omega^j[\Gamma]\|^{1+r}[f(\theta^j\omega)g(\|\Pi_\omega^j[\Gamma]\|)].$$

So for $j \geqslant 0$ and a random variable $\tilde{R}(\omega) > 1$,

$$\|P_{\theta^j\omega}(\Pi_\omega^j[\Gamma])\| \leqslant \tilde{R}(\omega)\|\Pi_\omega^j[\Gamma]\|^{1+r} g(\|\Pi_\omega^j[\Gamma]\|)\exp(\epsilon j). \tag{3.2.10}$$

Now from (3.2.7), (3.2.8), (3.2.9) and (3.2.10), we obtain

$$\|\Pi_\omega^n[I_\omega(v_\omega,\Gamma)]\| \leqslant R(\omega)\Big[\exp((\mu_{j_0}+\epsilon)n)\|v_\omega\|+ \\ \sum_{0\leqslant j\leqslant n-1} R(\omega)\tilde{R}(\omega)\exp\left(\epsilon n+2\epsilon(1+j)+(n-1-j)\mu_{j_0}\right)\|\Pi_\omega^j(\Gamma)\|^{1+r}g(\|\Pi_\omega^j[\Gamma]\|)+ \\ \sum_{j\geqslant n} R(\omega)\tilde{R}(\omega)\exp\left(3\epsilon(1+j)-(j-n+1)\mu_{j_0-1}\right)\|\Pi_\omega^j(\Gamma)\|^{1+r}g(\|\Pi_\omega^j[\Gamma]\|)\Big].$$

Since g is increasing,

$$\|\Pi_\omega^n[I_\omega(v_\omega,\Gamma)]\| \leqslant R(\omega)\Big[\exp\left((\mu_{j_0}+\epsilon)n\right).\|v_\omega\|+ \\ R(\omega)\tilde{R}(\omega)\|\Gamma\|_{\Sigma_\omega^\upsilon}^{1+r}g(\|\Gamma\|_{\Sigma_\omega^\upsilon})\exp\left(\epsilon n+2\epsilon+(n-1)\mu_{j_0}\right)\sum_{0\leqslant j\leqslant n-1}\exp\left(j(2\epsilon-\mu_{j_0}-(1+r)\upsilon)\right)+ \\ R(\omega)\tilde{R}(\omega)\|\Gamma\|_{\Sigma_\omega^\upsilon}^{1+r}g(\|\Gamma\|_{\Sigma_\omega^\upsilon})\exp\left(3\epsilon+(n-1)\mu_{j_0-1}\right)\sum_{j\geqslant n}\exp\left(j(3\epsilon-\mu_{j_0-1}-(1+r)\upsilon)\right)\Big].$$

Since $\mu_{j_0-1}\geqslant 0$ and $0<\upsilon<-\mu_{j_0}$, we can choose $\epsilon>0$ smaller if necessary to see that

$$\sup_{n\geqslant 0}\Big[\|\Pi_\omega^n[I_\omega(v_\omega,\Gamma)]\|\exp(\upsilon n)\Big]<\infty.$$

As a result, I_ω is well-defined . □

Lemma 3.2.6. *With the same setting as in Lemma 3.2.5, for* $\Gamma\in\Sigma_\omega^\upsilon\cap B(0,\tilde{\rho}(\omega))$,

$$I_\omega[v_\omega,\Gamma]=\Gamma \quad\Longleftrightarrow\quad \forall j\geqslant 0:\Pi_\omega^j[\Gamma]=\varphi_\omega^j(Y_\omega+\xi_\omega)-\varphi_\omega^j(Y_\omega) \tag{3.2.11}$$

where

$$\xi_\omega=v_\omega-\sum_{j\geqslant 0}\left[[\psi_\omega^{j+1}]^{-1}\circ\Pi_{U_{\theta^{1+j}\omega}\|S_{\theta^{1+j}\omega}}\right]P_{\theta^j\omega}(\Pi_\omega^j[\Gamma]). \tag{3.2.12}$$

Proof. The strategy of the proof is similar to [13, Lemma VI.5]. Let $I_\omega[v_\omega,\Gamma]=\Gamma$. Then $\xi_\omega=\Pi_\omega^0[\Gamma]$ and the claim is shown for $j=0$. We proceed by induction. Assume that $\Pi_\omega^n[\Gamma]=\varphi_\omega^n(Y_\omega+\xi_\omega)-\varphi_\omega^n(Y_\omega)$. By definition,

$$\varphi_\omega^{n+1}(Y_\omega+\xi_\omega)-\varphi_\omega^{n+1}(Y_\omega)=\varphi_{\theta^n\omega}^1(\varphi_\omega^n(Y_\omega+\xi_\omega))-\varphi_{\theta^n\omega}^1(Y_{\theta^n\omega})= \\ P_{\theta^n\omega}(\varphi_\omega^n(Y_\omega+\xi_\omega)-Y_{\theta^n\omega})+\psi_{\theta^n\omega}^1(\varphi_\omega^n(Y_\omega+\xi_\omega)-Y_{\theta^n\omega})=P_{\theta^n\omega}(\Pi_\omega^n[\Gamma])+\psi_{\theta^n\omega}^1(\Pi_\omega^n[I_\omega(v_\omega,\Gamma)]).$$

Note that for $j\geqslant n$,

$$\psi_{\theta^n\omega}^1\circ[\psi_{\theta^n\omega}^{j-n+1}]^{-1}=[\psi_{\theta^{n+1}\omega}^{j-n}]^{-1}:U_{\theta^{1+j}\omega}\to U_{\theta^{1+n}\omega}.$$

By definition

$$\psi^1_{\theta^n\omega}(\Pi^n_\omega[I_\omega(v_\omega,\Gamma)]) = \psi^{n+1}_\omega(v_\omega) + \sum_{0\leqslant j\leqslant n-1} [\psi^{n-j}_{\theta^{1+j}\omega} \circ \Pi_{S_{\theta^{1+j}\omega}\|U_{\theta^{1+j}\omega}}] P_{\theta^j\omega}(\Pi^j_\omega[\Gamma]) -$$
$$\sum_{j\geqslant n} [[\psi^{j-n}_{\theta^n\omega}]^{-1} \circ \Pi_{U_{\theta^{1+j}\omega}\|S_{\theta^{1+j}\omega}}] P_{\theta^j\omega}(\Pi^j_\omega[\Gamma]).$$

Consequently, $\Pi^{n+1}_\omega[\Gamma] = \varphi^{n+1}_\omega(Y_\omega+\xi_\omega) - \varphi^{n+1}_\omega(Y_\omega)$ which finishes the induction step. Conversely, for $\xi_\omega \in E_\omega$ and $\Gamma \in \Sigma^\upsilon_\omega \cap B(0,\tilde\rho(\omega))$, assume that for every $j \geqslant 0$, $\Pi^j_\omega[\Gamma] = \varphi^j_\omega(Y_\omega+\xi_\omega) - \varphi^j_\omega(Y_\omega)$. Set

$$v_\omega := \xi_\omega + \sum_{j\geqslant 0} [[\psi^{j+1}_\omega]^{-1} \circ \Pi_{U_{\theta^{1+j}\omega}\|S_{\theta^{1+j}\omega}}] P_{\theta^j\omega}(\Pi^j_\omega[\Gamma]).$$

Similar to Lemma 3.2.5, we can see that v_ω is well-defined. Morever,

$$\begin{aligned}\Pi^n_\omega[I_\omega(v_\omega,\Gamma)] &= \psi^n_\omega(\xi_\omega) + \sum_{0\leqslant j\leqslant n-1} \psi^{n-1-j}_{\theta^{1+j}\omega} P_{\theta^j\omega}(\Pi^j_\omega[\Gamma]) \\ &= \varphi^j_\omega(Y_\omega+\xi_\omega) - \varphi^j_\omega(Y_\omega) = \Pi^j_\omega[\Gamma]\end{aligned}$$

which proves the claim. □

Lemma 3.2.7. *Under the same assumptions as in Lemma 3.2.6, set*

$$\begin{aligned}&h^\upsilon_1(\omega) := \sup_{n\geqslant 0}\big[\exp(n\upsilon)\|\psi^n_\omega|_{S_\omega}\|\big] \quad \textit{and} \\ &h^\upsilon_2(\omega) := \sup_{n\geqslant 0}\big[\exp(n\upsilon)\sum_{0\leqslant j\leqslant n-1} \exp(-j\upsilon(1+r)) f(\theta^j\omega)\|\psi^{n-j}_{\theta^{j+1}\omega}|_{S_{\theta^{j+1}\omega}}\|\|\Pi_{S_{\theta^{j+1}\omega}\|U_{\theta^{j+1}\omega}}\| \\ &\qquad + \exp(n\upsilon)\sum_{j\geqslant n}\exp(-j\upsilon(1+r))f(\theta^j\omega)\|(\psi^{j-n+1}_{\theta^n\omega}|_{U_{\theta^{j+1}}})^{-1}\|\|\Pi_{U_{\theta^{j+1}\omega}\|S_{\theta^{j+1}\omega}}\|\big].\end{aligned}$$

Then h^υ_1 and h^υ_2 are measurable and finite on a θ-invariant set of full measure $\tilde\Omega$. In addition,

$$\lim_{n\to\infty}\frac{1}{n}\log^+ h^\upsilon_1(\theta^n\omega) = \lim_{n\to\infty}\frac{1}{n}\log^+ h^\upsilon_2(\theta^n\omega) = 0$$

for every $\omega\in\tilde\Omega$. Furthermore, the estimates

$$\begin{aligned}\|I_\omega(v_\omega,\Gamma)\| &\leqslant h^\upsilon_1(\omega)\|v_\omega\| + h^\upsilon_2(\omega)\|\Gamma\|^{1+r}g(\|\Gamma\|) \quad \textit{and} \\ \|I_\omega(v_\omega,\Gamma) - I_\omega(v_\omega,\tilde\Gamma)\| &\leqslant h^\upsilon_2(\omega)h(\|\Gamma\|+\|\tilde\Gamma\|)\ \|\Gamma-\tilde\Gamma\|\end{aligned}$$

hold for every $\omega\in\tilde\Omega$, $\Gamma,\tilde\Gamma\in\Sigma^\upsilon_\omega\cap B(0,\tilde\rho(\omega))$ and $v_\omega\in S_\omega$.

Proof. The statements about h^υ_1 and h^υ_2 follow from our assumption on f, (2.3.7), Lemma 2.3.7 and Proposition 2.3.18. The claimed estimates follow by definition of I_ω. □

Recall that $h(x) = x^r g(x)$. In particular, h is invertible and h and h^{-1} are strictly increasing.

Lemma 3.2.8. *Assume that for $v_\omega \in S_\omega$,*

$$\|v_\omega\| \leqslant \frac{1}{2h_1^\upsilon(\omega)} \min\{\frac{1}{2}h^{-1}(\frac{1}{2h_2^\upsilon(\omega)}), \tilde{\rho}(\omega)\}.$$

Then the equation

$$I_\omega(v_\omega, \Gamma) = \Gamma$$

admits a unique solution $\Gamma = \Gamma(v_\omega)$ and the bound

$$\|\Gamma(v_\omega)\| \leqslant \min\{\frac{1}{2}h^{-1}(\frac{1}{2h_2^\upsilon(\omega)}), \tilde{\rho}(\omega)\} =: H_1^\upsilon(\omega) \tag{3.2.13}$$

holds true.

Proof. We can use the estimates provided in Lemma 3.2.7 to conclude that $I(v_\omega, \cdot)$ is a contraction on the closed ball with radius $\min\{\frac{1}{2}h^{-1}(\frac{1}{2h_2^\upsilon(\omega)}), \tilde{\rho}(\omega)\}$. □

Now we can formulate the main theorem about the existence of local stable manifolds.

Theorem 3.2.9. *Let $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$ be an ergodic measure-preserving dynamical systems and φ a Fréchet-differentiable cocycle acting on a measurable field of Banach spaces $\{E_\omega\}_{\omega\in\Omega}$. Assume that φ admits a stationary solution Y and that the linearized cocycle ψ around Y is compact, satisfies Assumption 2.3.1 and the integrability condition*

$$\log^+ \|\psi_\omega\| \in L^1(\omega).$$

Moreover, assume that (3.2.5) *holds for φ and ψ. Let $\mu_{j_0} < 0$ and S_ω be defined as in Definition 3.2.1. For $0 < \upsilon < -\mu_{j_0}$, $\omega \in \Omega$ and $R^\upsilon(\omega) := \frac{1}{2h_1^\upsilon(\omega)} \min\{\frac{1}{2}h^{-1}(\frac{1}{2h_2^\upsilon(\omega)}), \tilde{\rho}(\omega)\}$ with $\tilde{\rho}$ defined as in* (3.2.6)*, let*

$$S_{loc}^\upsilon(\omega) := \{Y_\omega + \Pi_\omega^0[\Gamma(v_\omega)], \quad \|v_\omega\| < R^\upsilon(\omega)\}. \tag{3.2.14}$$

Then there is a θ-invariant set of full measure $\tilde{\Omega}$ on which the following properties are satisfied for every $\omega \in \tilde{\Omega}$:

(i) There are random variables $\rho_1^\upsilon(\omega), \rho_2^\upsilon(\omega)$, positive and finite on $\tilde{\Omega}$, for which

$$\liminf_{p\to\infty} \frac{1}{p} \log \rho_i^\upsilon(\theta^p\omega) \geq 0, \quad i = 1, 2 \tag{3.2.15}$$

and such that

$$\{Z_\omega \in E_\omega \,:\, \sup_{n\geqslant 0} \exp(n\upsilon)\|\varphi_\omega^n(Z_\omega) - Y_{\theta^n\omega}\| < \rho_1^\upsilon(\omega)\} \subseteq S_{loc}^\upsilon(\omega)$$
$$\subseteq \{Z_\omega \in E_\omega \,:\, \sup_{n\geqslant 0} \exp(n\upsilon)\|\varphi_\omega^n(Z_\omega) - Y_{\theta^n\omega}\| < \rho_2^\upsilon(\omega)\}.$$

(ii) $S_{loc}^\upsilon(\omega)$ is an immersed submanifold of E_ω and

$$T_{Y_\omega} S_{loc}^\upsilon(\omega) = S_\omega.$$

(iii) *For $n \geqslant N(\omega)$,*

$$\varphi^n_\omega(S^{\upsilon}_{loc}(\omega)) \subseteq S^{\upsilon}_{loc}(\theta^n\omega).$$

(iv) *For $0 < \upsilon_1 \leqslant \upsilon_2 < -\mu_{j_0}$,*

$$S^{\upsilon_2}_{loc}(\omega) \subseteq S^{\upsilon_1}_{loc}(\omega).$$

Also for $n \geqslant N(\omega)$,

$$\varphi^n_\omega(S^{\upsilon_1}_{loc}(\omega)) \subseteq S^{\upsilon_2}_{loc}(\theta^n(\omega))$$

and consequently for $Z_\omega \in S^{\upsilon}_{loc}(\omega)$,

$$\limsup_{n\to\infty} \frac{1}{n} \log \|\varphi^n_\omega(Z_\omega) - Y_{\theta^n\omega}\| \leqslant \mu_{j_0}. \tag{3.2.16}$$

(v)

$$\limsup_{n\to\infty} \frac{1}{n} \log \left[\sup \left\{ \frac{\|\varphi^n_\omega(Z_\omega) - \varphi^n_\omega(\tilde{Z}_\omega)\|}{\|Z_\omega - \tilde{Z}_\omega\|}, \quad Z_\omega \neq \tilde{Z}_\omega,\ Z_\omega, \tilde{Z}_\omega \in S^{\upsilon}_{loc}(\omega) \right\} \right] \leqslant \mu_{j_0}.$$

Proof. We start with (i). For the first inclusion, note that we can find a random variable $\rho^{\upsilon}_1(\omega)$ satisfying

$$\liminf_{p\to\infty} \frac{1}{p} \log \rho^{\upsilon}_1(\theta^p\omega) \geqslant 0 \tag{3.2.17}$$

and such that whenever $\|\Gamma\| \leqslant \rho^{\upsilon}_1(\omega)$,

$$\|\Gamma\| + h^{\upsilon}_2(\omega)\|\Gamma\|^{r+1} g(\|\Gamma\|) \leqslant \frac{1}{2h^{\upsilon}_1(\omega)} \min\{\frac{1}{2} h^{-1}(\frac{1}{2h^{\upsilon}_2(\omega)}), \tilde{\rho}(\omega)\} =: H^{\upsilon}_2(\omega).$$

For example, we can define

$$\rho^{\upsilon}_1(\omega) := \min\{h^{-1}(\frac{1}{h^{\upsilon}_2(\omega)}), H^{\upsilon}_2(\omega)/2, H^{\upsilon}_1(\omega)\}$$

with H^{υ}_1 defined as in (3.2.13). Assume that $Z_\omega \in E_\omega$ has the property that

$$\sup_{n\geqslant 0} \exp(n\upsilon)\|\varphi^n_\omega(Z_\omega) - Y_{\theta^n\omega}\| < \rho^{\upsilon}_1(\omega).$$

Setting

$$\tilde{v}_\omega := Z_\omega - Y_\omega + \sum_{j\geqslant 0} [[\psi^{j+1}_\omega]^{-1} \circ \Pi_{U_{\theta^{1+j}\omega} \| S_{\theta^{1+j}\omega}}] P_{\theta^j\omega}(\Pi^j_\omega[\tilde{\Gamma}]),$$

it follows that $\|\tilde{v}_\omega\| < R^{\upsilon}(\omega)$. From Lemma 3.2.6, we conclude that $I_\omega[\tilde{v}_\omega, \tilde{\Gamma}] = \tilde{\Gamma}$. By uniqueness of the fixed point map, we have $\tilde{\Gamma} = \Gamma(\tilde{v}_\omega)$, therefore $Z_\omega = Y_\omega + \Pi^0_\omega(\Gamma(\tilde{v}_\omega)) \in S^{\upsilon}_{loc}(\omega)$. Next, let $Z_\omega \in S^{\upsilon}_{loc}(\omega)$, i.e. $Z_\omega = Y_\omega + \Pi^0_\omega(\Gamma(v_\omega))$ for some $\|v_\omega\| < R^{\upsilon}(\omega)$. From Lemma 3.2.6

and Lemma 3.2.8,

$$\|\Gamma(v_\omega)\| = \sup_{n\geqslant 0} \exp(n\upsilon)\|\varphi_\omega^n(Z_\omega) - Y_{\theta^n\omega}\| \leqslant R^\upsilon(\omega).$$

We can therefore choose $\rho_2^\upsilon(\omega) = R^\upsilon(\omega)$ and the second inclusion is shown.

The second item immediately follows from our definition for $S_{loc}^\upsilon(\omega)$.

For item (iii), by (3.2.15), we can find $N(\omega)$ such that for $n \geqslant N(\omega)$,

$$\exp(-n\upsilon)\rho_2^\upsilon(\omega) \leqslant \rho_1^\upsilon(\theta^n\omega).$$

Now the claim follows from item (i).

For item (iv), note first that $R^{\upsilon_2}(\omega) \leqslant R^{\upsilon_1}(\omega)$. By definition of $\Gamma_\omega^\upsilon(v_\omega)$, it immediately follows that

$$S_{loc}^{\upsilon_2}(\omega) \subseteq S_{loc}^{\upsilon_1}(\omega).$$

Now take $Z_\omega \in S_{loc}^{\upsilon_1}(\omega)$. From Lemma 2.3.17 and (i), we can find $N(\omega)$ such that for $n \geqslant N(\omega)$,

$$\|\Pi_{S_{\theta^n\omega}\|U_{\theta^n\omega}}(\varphi_\omega^n(Z_\omega) - Y_{\theta^n\omega})\| < R^{\upsilon_2}(\theta^n\omega).$$

We may also assume that $\exp(-n\upsilon_1)\rho_2^{\upsilon_1}(\omega) \leqslant \rho_1^{\upsilon_1}(\theta^n\omega)$ for $n \geq N(\omega)$. For

$$v_{\theta^n\omega} := \Pi_{S_{\theta^n\omega}\|U_{\theta^n\omega}}(\varphi_\omega^n(Z_\omega) - Y_{\theta^n\omega})$$

let

$$Z_{\theta^n\omega} := \Pi_{\theta^n\omega}^0(\Gamma(v_{\theta^n\omega})) + Y_{\theta^n\omega} \in S_{loc}^{\upsilon_2}(\theta^n\omega) \subset S_{loc}^{\upsilon_1}(\theta^n\omega).$$

We claim that $Z_{\theta^n\omega} = \varphi_\omega^n(Z_\omega)$. Since $Z_\omega \in S_{loc}^{\upsilon_1}(\omega)$,

$$\sup_{j\geqslant 0} \exp(j\upsilon_1)\|\varphi_{\theta^n\omega}^j(\varphi_\omega^n(Z_\omega)) - Y_{\theta^j\theta^n\omega}\| \leqslant \exp(-n\upsilon_1)\rho_2^{\upsilon_1}(\omega) \leqslant \rho_1^{\upsilon_1}(\theta^n\omega).$$

So from item (i), $\varphi_\omega^n(Z_\omega) \in S_{loc}^{\upsilon_1}(\theta^n\omega)$. Remember $Z_{\theta^n\omega} \in S_{loc}^{\upsilon_1}(\theta^n\omega) \cap S_{loc}^{\upsilon_2}(\theta^n\omega)$ and

$$\Pi_{S^{\theta^n\omega}||U^{\theta^n\omega}}(Z_{\theta^n\omega} - Y_{\theta^n\omega}) = \Pi_{S^{\theta^n\omega}||U^{\theta^n\omega}}(\varphi_\omega^n(Z_\omega) - Y_{\theta^n\omega}).$$

So by uniqueness of the fixed point, we indeed have

$$\varphi_\omega^n(Z_\omega) = Z_{\theta^n\omega} \in S_{loc}^{\upsilon_2}(\theta^n\omega).$$

To prove (3.2.16), let $\upsilon \leq \upsilon_2 < -\mu_0$ and take $Z_\omega \in S_{loc}^\upsilon(\omega)$. Then we know that for large enough N, $\varphi_\omega^N(Z_\omega) \in S_{loc}^{\upsilon_2}(\theta^N\omega)$, therefore

$$\sup_{j\geq 0} \exp(j\upsilon_2)\|\varphi_\omega^{j+N}(Z_\omega) - Y_{\theta^{j+N}\omega}\| < \infty$$

and it follows that

$$\limsup_{n\to\infty} \frac{1}{n} \log \|\varphi_\omega^n(Z_\omega) - Y_{\theta^n\omega}\| \leqslant -\upsilon_2.$$

We can choose υ_2 arbitrarily close to $-\mu_0$, therefore the claim follows and item (iv) is proved.

For item (v), first by definition,

$$\begin{aligned}
\|\Gamma(v_\omega) - \Gamma(\tilde{v}_\omega)\| &= \|I_\omega(v_\omega, \Gamma(v_\omega)) - I_\omega(\tilde{v}_\omega, \Gamma(\tilde{v}_\omega))\| \\
&\leqslant \|I_\omega(v_\omega, \Gamma(v_\omega)) - I_\omega(\tilde{v}_\omega, \Gamma(v_\omega))\| + \|I_\omega(\tilde{v}_\omega, \Gamma(v_\omega)) - I_\omega(\tilde{v}_\omega, \Gamma(\tilde{v}_\omega))\| \\
&\leqslant h_1^\upsilon(\omega)\|v_\omega - \tilde{v}_\omega\| + \frac{1}{2}\|\Gamma(v_\omega) - \Gamma(\tilde{v}_\omega)\|
\end{aligned}$$

for every $v_\omega, \tilde{v}_\omega \in S_\omega$ with $\|v_\omega\|, \|\tilde{v}_\omega\| \leq R^\upsilon(\omega)$. Consequently,

$$\|\Gamma(v_\omega) - \Gamma(\tilde{v}_\omega)\| \leqslant 2h_1^\upsilon(\omega)\|v_\omega - \tilde{v}_\omega\|. \tag{3.2.18}$$

Also by definition, cf. (3.2.12),

$$\|\Pi_\omega^0(\Gamma(v_\omega)) - \Pi_\omega^0(\Gamma(\tilde{v}_\omega))\| \geqslant \|v_\omega - \tilde{v}_\omega\| - h_2^\upsilon(\omega)\, \|\Gamma(v_\omega) - \Gamma_\omega(\tilde{v}_\omega)\|\, h(\|\Gamma(v_\omega)\| + \|\Gamma_\omega(\tilde{v}_\omega)\|).$$

So from (3.2.18)

$$\|\Pi_\omega^0(\Gamma(v_\omega)) - \Pi_\omega^0(\Gamma(\tilde{v}_\omega))\| \geqslant \|v_\omega - \tilde{v}_\omega\| \big[1 - 2h_1^\upsilon(\omega)h_2^\upsilon(\omega)h(\|\Gamma(v_\omega)\| + \|\Gamma_\omega(\tilde{v}_\omega)\|)\big]. \tag{3.2.19}$$

First assume that

$$\max\{\|\Gamma(v_\omega), \Gamma(\tilde{v}_\omega)\|\} \leqslant \frac{1}{2} h^{-1}(\frac{1}{4h_1^\upsilon(\omega)h_2^\upsilon(\omega)}).$$

Then from (3.2.18) and (3.2.19),

$$\frac{\|\Gamma(v_\omega) - \Gamma(\tilde{v}_\omega)\|}{\|\Pi_\omega^0(\Gamma(v_\omega)) - \Pi_\omega^0(\Gamma(\tilde{v}_\omega))\|} \leqslant 4h_1^\upsilon(\omega). \tag{3.2.20}$$

Thus if $Z_\omega = Y_\omega + \Pi_\omega^0[\Gamma(v_\omega)]$ and $\tilde{Z}_\omega = Y_\omega + \Pi_\omega^0[\Gamma(v_\omega)]$, it follows that

$$\frac{\|\varphi_\omega^n(Z_\omega) - \varphi_\omega^n(\tilde{Z}_\omega)\|}{\|Z_\omega - \tilde{Z}_\omega\|} \leqslant 4\exp(-n\upsilon)h_1^\upsilon(\omega)$$

for every $n \geq 1$. In the general case, we can use item (i) and that $h^{-1}(\frac{1}{4h_1^\upsilon(\omega)h_2^\upsilon(\omega)})$ satisfies (3.2.15) to see that for some $N = N(\omega)$,

$$\sup_{j\geqslant 0} \exp(j\upsilon)\|\varphi_{\theta^N\omega}^j(\varphi_\omega^N(Z_\omega)) - Y_{\theta^j\theta^N\omega}\| \leqslant \exp(-N\upsilon)\rho_2^\upsilon(\omega) \leqslant \frac{1}{2}h^{-1}(\frac{1}{4h_1^\upsilon(\theta^N\omega)h_2^\upsilon(\theta^N\omega)}).$$

Consequently, from (3.2.20),

$$\sup_{j\geqslant 0} \frac{\exp(j\upsilon)\|\varphi_\omega^{j+N}(Z_\omega) - \varphi_\omega^{j+N}(\tilde{Z}_\omega)\|}{\|\varphi_\omega^N(Z_\omega) - \varphi_\omega^N(\tilde{Z}_\omega)\|} \leqslant 4h_1^\upsilon(\theta^N\omega)$$

and hence for every $n \geqslant N$,

$$\frac{\|\varphi_\omega^n(Z_\omega) - \varphi_\omega^n(\tilde{Z}_\omega)\|}{\|Z_\omega - \tilde{Z}_\omega\|} \leqslant 4\exp((-n-N)\upsilon)h_1^\upsilon(\theta^N\omega)H_N^\upsilon(\omega) \tag{3.2.21}$$

where

$$H_N^\upsilon(\omega) = \sup\left\{\frac{\|\varphi_\omega^N(Z_\omega) - \varphi_\omega^N(\tilde{Z}_\omega)\|}{\|Z_\omega - \tilde{Z}_\omega\|},\quad Z_\omega \neq \tilde{Z}_\omega,\ Z_\omega, \tilde{Z}_\omega \in S_{loc}^\upsilon(\omega)\right\}.$$

We claim that $H_N^\upsilon(\omega)$ is finite. Indeed, by assumption (3.2.5),

$$\begin{aligned}\|\varphi_\omega^N(Z_\omega) - \varphi_\omega^N(\tilde{Z}_\omega)\| &\leqslant \|\psi^1_{\theta^{N-1}\omega}\|\ \|\varphi_\omega^{N-1}(Z_\omega) - \varphi_\omega^{N-1}(\tilde{Z}_\omega)\| \\ &+ f(\theta^N\omega)\ \|\varphi_\omega^{N-1}(Z_\omega) - \varphi_\omega^{N-1}(\tilde{Z}_\omega)\| h(\|\varphi_\omega^{N-1}(Z_\omega) - Y_{\theta^{N-1}\omega}\| + \|\varphi_\omega^{N-1}(\tilde{Z}_\omega) - Y_{\theta^{N-1}\omega}\|)\end{aligned}$$

and we can proceed by induction to conclude. Finally, from (3.2.21) and item (iv), our claim is proved. □

Remark 3.2.10. *Assume that for $\omega \in \tilde{\Omega}$ the function φ_ω is C^m. Then, since*

$$I_\omega(0,0) = \frac{\partial}{\partial\Gamma}I_\omega(0,0) = 0,$$

we can deduce from the Implicit function theorem that $S_{loc}^\upsilon(\omega)$ is locally C^{m-1}.

3.3 Unstable manifolds

We invoke the same strategy for proving the existence of unstable manifolds. Since the arguments are very similar, we will only sketch them briefly. In this section, we will assume that the largest Lyapunov exponent is strictly positive, i.e. that $\mu_1 > 0$.

Definition 3.3.1. *Set $k_0 := \min\{k : \mu_k > 0\}$, $\tilde{S}_\omega := F_{\mu_{k_0+1}}(\omega)$ and $\tilde{U}_\omega = \oplus_{1\leqslant i\leqslant k_0} H_\omega^i$ for $\omega \in \tilde{\Omega}$. For $\tilde{\Sigma}_\omega := \prod_{j\geqslant 0} E_{\sigma^j\omega}$ and $\upsilon > 0$, we define the Banach space*

$$\tilde{\Sigma}_\omega^\upsilon := \left\{\Gamma \in \tilde{\Sigma}_\omega \ :\ \|\Gamma\| = \sup_{k\geqslant 0}\,[\|\tilde{\Pi}_\omega^k\Gamma\|\exp(k\upsilon)] < \infty\right\}$$

where $\tilde{\Pi}_\omega^k : \prod_{i\geqslant 0} E_{\sigma^i\omega} \to E_{\sigma^k\omega}$ is the projection map. Similar to last section, we also set

$$\begin{aligned}\tilde{h}_1^\upsilon(\omega) &:= \sup_{n\geqslant 0}\left[\exp(n\upsilon)\|(\psi^n_{\sigma^n\omega}|_{\tilde{U}_\omega})^{-1}\|\right] \quad \text{and} \\ \tilde{h}_2^\upsilon(\omega) &:= \sup_{n\geqslant 0}\Big[\exp(n\upsilon)\sum_{0\leqslant k\leqslant n-1}\exp\big(-\upsilon(n-k)(1+r)\big)f(\sigma^{n-k}\omega)\|(\psi^{k+1}_{\sigma^n\omega}|_{\tilde{U}_{\sigma^{n-1-k}\omega}})^{-1}\| \\ &\quad\times\|\Pi_{\tilde{U}_{\sigma^{n-1-k}\omega}\|\tilde{S}_{\sigma^{n-1-k}\omega}}\| \\ &\quad+\exp(n\upsilon)\sum_{k\geqslant n}\exp(-\upsilon(k+1)(1+r))f(\sigma^{k+1}\omega)\|\psi^{k-n}_{\sigma^k\omega}|_{\tilde{S}_{\sigma^k\omega}}\|\|\Pi_{\tilde{S}_{\sigma^k\omega}\|\tilde{U}_{\sigma^k\omega}}\|\Big].\end{aligned}$$

Lemma 3.3.2. *Let $\omega \in \Omega$, $0 < \upsilon < \mu_{k_0}$ and assume that $\rho\colon \Omega \to \mathbb{R}^+$ satisfies*

$$\liminf_{n\to\infty}\frac{1}{n}\log\rho(\sigma^n\omega) \geq 0 \tag{3.3.1}$$

almost surely. Define P as in Lemma 3.2.5 and assume that (3.2.5) *holds for a random variable $f\colon \Omega \to \mathbb{R}^+$ which satisfies $\lim_{n\to\infty} f(\sigma^n\omega) = 0$ almost surely. Set*

$$\tilde{\rho}(\omega) := \inf_{n\geq 0} \exp(n\upsilon)\rho(\sigma^n\omega). \tag{3.3.2}$$

Then the map

$$\tilde{I}_\omega : \tilde{U}_\omega \times \tilde{\Sigma}^\upsilon_\omega \cap B(0,\tilde{\rho}(\omega)) \to \tilde{\Sigma}^\upsilon_\omega,$$

$$\tilde{\Pi}^n_\omega[\tilde{I}_\omega(u_\omega,\Gamma)] = \begin{cases} [\psi^n_{\sigma^n\omega}]^{-1}(u_\omega) & \\ \qquad -\sum_{0\leqslant k\leqslant n-1} \big[[\psi^{k+1}_{\sigma^n\omega}]^{-1}\circ \Pi_{\tilde{U}_{\sigma^{n-1-k}\omega}\|\tilde{S}_{\sigma^{n-1-k}\omega}}\big] P_{\sigma^{n-k}\omega}(\tilde{\Pi}^{n-k}_\omega[\Gamma]) & \\ \qquad +\sum_{k\geqslant n} \big[\psi^{k-n}_{\sigma^k\omega}\circ \Pi_{\tilde{S}_{\sigma^k\omega}\|\tilde{U}_{\sigma^k\omega}}\big] P_{\sigma^{k+1}\omega}(\tilde{\Pi}^{k+1}_\omega[\Gamma]) & \text{for } n\geq 1, \\ u_\omega + \sum_{k\geqslant 0}\big[\psi^k_{\sigma^k\omega}\circ \Pi_{\tilde{S}_{\sigma^k\omega}\|\tilde{U}_{\sigma^k\omega}}\big] P_{\sigma^{k+1}\omega}(\tilde{\Pi}^{k+1}_\omega[\Gamma]) & \text{for } n = 0. \end{cases}$$

is well-defined on a θ-invariant set of full measure $\tilde{\Omega}$.

Proof. We can use Lemma 2.3.16 to obtain a version of Lemma 3.2.2 where we replace θ by σ. The rest of the proof is similar to Lemma 3.2.5. ☐

Lemma 3.3.3. *For $0<\upsilon<\mu_{k_0}$, $\omega\in\tilde{\Omega}$ and $\Gamma\in\Sigma^\upsilon_\omega\cap B(0,\tilde{\rho}(\omega))$,*

$$\tilde{I}_\omega(u_\omega,\Gamma) = \Gamma \quad\iff\quad \forall\ 0\leq k\leqslant n:\quad \tilde{\Pi}^{n-k}_\omega\Gamma = \varphi^k_{\sigma^n\omega}(\tilde{\Pi}^n_\omega\Gamma + Y_{\sigma^n\omega}) - Y_{\sigma^{n-k}\omega}. \tag{3.3.3}$$

Proof. Similar to Lemma 3.2.6. ☐

Lemma 3.3.4. *For $0<\upsilon<\mu_{k_0}$, $\tilde{h}^\upsilon_1$ and $\tilde{h}^\upsilon_2$ are measurable and finite on a θ-invariant set of full measure $\tilde{\Omega}$. Moreover,*

$$\lim_{p\to\infty}\frac{1}{p}\log^+\tilde{h}^\upsilon_1(\sigma^p\omega) = \lim_{p\to\infty}\frac{1}{p}\log^+\tilde{h}^\upsilon_2(\sigma^p\omega) = 0 \tag{3.3.4}$$

and

$$\begin{aligned} \|\tilde{I}_\omega(u_\omega,\Gamma)\| &\leqslant \tilde{h}^\upsilon_1(\omega)\|u_\omega\| + \tilde{h}^\upsilon_2(\omega)\|\Gamma\|^{r+1}g(\|\Gamma\|) \\ \|\tilde{I}_\omega(u_\omega,\Gamma) - \tilde{I}_\omega(u_\omega,\tilde{\Gamma})\| &\leqslant \tilde{h}^\upsilon_2(\omega)h(\|\Gamma\|+\|\tilde{\Gamma}\|)\ \|\Gamma-\tilde{\Gamma}\| \end{aligned}$$

hold for every $\omega\in\tilde{\Omega}$, $\Gamma,\tilde{\Gamma}\in\tilde{\Sigma}^\upsilon_\omega\cap B(0,\tilde{\rho}(\omega))$ and $u_\omega\in\tilde{U}_\omega$.

Proof. As in Lemma 3.2.7. ☐

Lemma 3.3.5. *Assume that for $u_\omega\in\tilde{U}_\omega$,*

$$\|u_\omega\| \leqslant \frac{1}{2\tilde{h}^\upsilon_1(\omega)}\min\Big\{\frac{1}{2}h^{-1}\Big(\frac{1}{2\tilde{h}^\upsilon_2(\omega)}\Big),\tilde{\rho}(\omega)\Big\}.$$

Then the equation

$$\tilde{I}_\omega(u_\omega,\Gamma) = \Gamma$$

admits a uniques solution $\Gamma = \Gamma(u_\omega)$ *and the bound*

$$\|\Gamma(u_\omega)\| \leqslant \min\{\frac{1}{2}h^{-1}(\frac{1}{2\tilde{h}_2^{\upsilon}(\omega)}), \tilde{\rho}(\omega)\}$$

holds true.

Proof. We can show that $\tilde{I}(u_\omega, \cdot)$ is a contraction using Lemma 3.3.4. $\square$

Finally we can formulate our main results about the existence of local unstable manifolds.

Theorem 3.3.6. *Let* $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$ *be an ergodic measure-preserving dynamical systems,* $\sigma := \theta^{-1}$ *and* φ *a Fréchet-differentiable cocycle acting on a measurable field of Banach spaces* $\{E_\omega\}_{\omega\in\Omega}$*. Assume that* φ *admits a stationary solution* Y *and that the linearized cocycle* ψ *around* Y *is compact, satisfies Assumption 2.3.1 and the integrability condition*

$$\log^+ \|\psi_\omega\| \in L^1(\omega).$$

Moreover, assume that (3.2.5) *holds for* φ *and* ψ *and a random variable* $\rho\colon \Omega \to \mathbb{R}^+$ *satisfying* (3.3.1)*. Assume that* $\mu_1 > 0$ *and let* $\mu_{k_0} > 0$ *and* $\tilde{U}_\omega$ *be defined as in Definition 3.3.1. For* $0 < \upsilon < \mu_{k_0}$, $\omega \in \Omega$ *and* $R^\upsilon(\omega) := \frac{1}{2\tilde{h}_1^{\upsilon}(\omega)} \min\{\frac{1}{2}h^{-1}(\frac{1}{2\tilde{h}_2^{\upsilon}(\omega)}), \tilde{\rho}(\omega)\}$ *with* $\tilde{\rho}$ *defined as in* (3.3.2)*, let*

$$U^\upsilon_{loc}(\omega) := \{Y_\omega + \tilde{\Pi}^0_\omega[\Gamma(u_\omega)], \quad \|u_\omega\| < \tilde{R}^\upsilon(\omega)\}. \tag{3.3.5}$$

Then there is a θ*-invariant set of full measure* $\tilde{\Omega}$ *on which the following properties are satisfied for every* $\omega \in \tilde{\Omega}$*:*

(i) There are random variables $\tilde{\rho}_1^\upsilon(\omega), \tilde{\rho}_2^\upsilon(\omega)$*, positive and finite on* $\tilde{\Omega}$*, for which*

$$\liminf_{p\to\infty} \frac{1}{p} \log \tilde{\rho}_i^\upsilon(\sigma^p\omega) \geq 0, \quad i = 1, 2$$

and such that

$$\Big\{Z_\omega \in E_\omega \,:\, \exists\{Z_{\sigma^n\omega}\}_{n\geqslant 1} \text{ s.t. } \varphi^m_{\sigma^n\omega}(Z_{\sigma^n\omega}) = Z_{\sigma^{n-m}\omega} \text{ for all } 0 \leq m \leq n \text{ and}$$
$$\sup_{n\geqslant 0} \exp(n\upsilon)\|Z_{\sigma^n\omega} - Y_{\sigma^n\omega}\| < \tilde{\rho}_1^\upsilon(\omega)\Big\} \subseteq U^\upsilon_{loc}(\omega) \subseteq \Big\{Z_\omega \in E_\omega \,:\, \exists\{Z_{\sigma^n\omega}\}_{n\geqslant 1} \text{ s.t.}$$
$$\varphi^m_{\sigma^n\omega}(Z_{\sigma^n\omega}) = Z_{\sigma^{n-m}\omega} \text{ for all } 0 \leq m \leq n \text{ and } \sup_{n\geqslant 0} \exp(n\upsilon)\|Z_{\sigma^n\omega} - Y_{\sigma^n\omega}\| < \tilde{\rho}_2^\upsilon(\omega)\Big\}.$$

(ii) $U^\upsilon_{loc}(\omega)$ *is an immersed submanifold of* E_ω *and*

$$T_{Y_\omega} U^\upsilon_{loc}(\omega) = \tilde{U}_\omega.$$

(iii) For $n \geqslant N(\omega)$*,*

$$U^\upsilon_{loc}(\omega) \subseteq \varphi^n_{\sigma^n\omega}(U^\upsilon_{loc}(\sigma^n\omega)).$$

(iv) For $0 < \upsilon_1 \leqslant \upsilon_2 < \mu_{k_0}$,

$$U_{loc}^{\upsilon_2}(\omega) \subseteq U_{loc}^{\upsilon_1}(\omega).$$

Also for $n \geqslant N(\omega)$,

$$U_{loc}^{\upsilon_1}(\omega) \subseteq \varphi_{\sigma^n\omega}^{n}(U_{loc}^{\upsilon_2}(\sigma^n(\omega))$$

and consequently for $Z_\omega \in U_{loc}^{\upsilon}(\omega)$,

$$\limsup_{n\to\infty} \frac{1}{n} \log \|Z_{\sigma^n\omega} - Y_{\sigma^n\omega}\| \leqslant -\mu_{k_0}.$$

(v)

$$\limsup_{n\to\infty} \frac{1}{n} \log \left[\sup \left\{ \frac{\|Z_{\sigma^n\omega} - \tilde{Z}_{\sigma^n\omega}\|}{\|Z_\omega - \tilde{Z}_\omega\|}, \quad Z_\omega \neq \tilde{Z}_\omega,\ Z_\omega, \tilde{Z}_\omega \in U_{loc}^{\upsilon}(\omega) \right\} \right] \leqslant -\mu_{k_0}.$$

Proof. One uses the same arguments as in the proof of Theorem 3.2.9. □

Remark 3.3.7. *We have:*

(i) As in the stable case, if φ_ω *is* C^m *for every* $\omega \in \tilde{\Omega}$, *one can deduce that* $U_{loc}^{\upsilon}(\omega)$ *is locally* C^{m-1}.

(ii) In the hyperbolic *case, i.e. if all Lyapunov exponents are non-zero, if the assumptions of Theorem 3.2.9 and 3.3.6 are satisfied, we have* $S_\omega = \tilde{S}_\omega$ *and* $U_\omega = \tilde{U}_\omega$. *In particular, the submanifolds* $S_{loc}^{\upsilon}(\omega)$ *and* $U_{loc}^{\upsilon}(\omega)$ *are* transversal, *i.e.*

$$E_\omega = T_{Y_\omega} U_{loc}^{\upsilon}(\omega) \oplus T_{Y_\omega} S_{loc}^{\upsilon}(\omega).$$

4

Rough Delay Equations I

4.1 Introduction

Stochastic delay differential equations (SDDEs) describe stochastic processes for which the dynamics do not only depend on the present state, but may depend on the whole past of the process. In its simplest formulation, an SDDE takes the form

$$dy_t = b(y_t, y_{t-r})\, dt + \sigma(y_t, y_{t-r})\, dB_t^H(\omega) \tag{4.1.1}$$

for some delay $r > 0$ where B^H is a fractional Brownian motion, b is the drift and σ the diffusion coefficient, both depending on the present and a delayed state of the system. In this case, we speak of a (single) *discrete time delay.* SDDEs appear frequently in practice. For instance, they can be used to model cell population growth and neural control mechanisms, cf. [24] and the references therein, they are applied in financial modeling [25], for climate models [26] and for models of the formation of blood cellular components, called *hematopoiesis* [27]. To be able to solve (4.1.1) uniquely, an initial condition has to be given which is a path or, more generally, a stochastic process. This means that we are led to solve an equation on an infinite dimensional (path) space. Popular choices for spaces of initial conditions are continuous paths or L^2 paths. Standard Itō theory can be applied without too much effort to solve (4.1.1) for such initial conditions, cf. [28, 29].

Due to its numerous applications, the study of the long-time behaviour for SDDEs of the type (4.1.1) is an important issue. However, it turns out that this is a challenging problem. Consequently, there is only a relatively small number of works devoted to this topic. One of the few articles dealing with this problem is [27]. There, the authors study the moment stability of a stochastically perturbed model of the hematopoietic stem cell (HSC) regulation system to model different diseases like leukemia or anemia. They show that stability domains for the perturbed and unperturbed system differ if the equation is perturbed by a multiplicative noise. In fact, the multiplicative noise poses a major technical problem, and its stability domain is not described completely. The theoretical and numerical results in this paper show that the HSC regulation system is sensitive to perturbations in certain parameters and insensitive in others which then gives a hint on the origin of the above mentioned diseases, cf. [27, Section 5].

Studying the moment stability of a stochastic system is a first step, but a rather coarse one to describe its long-time behaviour. Indeed, much more information could be deduced if the equation induces a *Random dynamical system* (RDS), an approach going back to L. Arnold [1]. The concept of random dynamical systems was successfully applied to stochastic differential equations (SDEs) in both finite and infinite dimensions, and it is a natural approach to study the long-time behaviour for SDDEs, too. Unfortunately, it turns out that there are serious obstacles. In fact, for a long time, it was believed that it is impossible to use the RDS approach to study SDDEs of the form (4.1.1). Here we claim, that indeed, it is possible.

Let us explain in more detail the difficulty one faces when applying the RDS approach to SDDE. A necessary condition for the existence of an RDS is that the equation generates a continuous *stochastic semi-flow*. Recall that given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, a continuous stochastic semi-flow on a topological space E is a measurable map

$$\phi\colon \{(s,t) \in [0,\infty)^2 \,|\, s \leq t\} \times \Omega \times E \to E$$

such that on a set of full measure $\tilde{\Omega}$, we have $\phi(t,t,\omega,x) = x$ and $\phi(s,u,\omega,x) = \phi(t,u,\omega,\phi(s,t,\omega,x))$ for every $s,t,u \in [0,\infty)$, $s \leq t \leq u$, every $x \in E$ and every $\omega \in \tilde{\Omega}$ and $x \mapsto \phi(s,t,\omega,x)$ is assumed to be continuous for every choice of $s,t \in [0,\infty)$, $s \leq t$, and every $\omega \in \tilde{\Omega}$. Consider the linear stochastic delay equation

$$\begin{aligned} dy_t &= y_{t-1}\, dB_t(\omega); \quad t \geq 0 \\ y_t &= \xi_t; \quad t \in [-1,0] \end{aligned} \tag{4.1.2}$$

interpreted as an Itō integral equation. It is clear that the solution on the time interval $[0,1]$ should be given by

$$y_t = \xi_0 + \int_0^t \xi_{s-1}\, dB_s(\omega) \tag{4.1.3}$$

whenever the stochastic integral makes sense. However, Mohammed proved in [30] that there is no modification of the process y which depends continuously on ξ in the supremum norm. This rules out the choice of $E = C([-1,0],\mathbb{R})$ on which a possible semi-flow ϕ induced by (4.1.2) could be defined. At this stage, one might still hope that another choice of E could be a possible state space for our semi-flow. We will prove now that there is in fact no such choice. Similar as in [31, Section 1.5.1], we make the following definition:

Definition 4.1.1. *Let E be a Banach space of functions mapping from $[-1,0]$ to $\mathbb{R}$. We say that E* carries the Wiener measure *if the functions $t \mapsto \sin[(n-1/2)\pi t]$ are contained in E for every $n \geq 1$ and if the series*

$$\sum_{n=1}^{\infty} Z_n(\omega) \frac{\sin[(n-1/2)\pi t]}{(n-1/2)\pi}, \quad t \in [-1,0]$$

converges in E almost surely for every sequence (Z_n) of independent, $\mathcal{N}(0,1)$-distributed random variables.

Note that *carrying the Wiener measure* is indeed a minimum requirement for the state space E of a possible semi-flow induced by (4.1.2), otherwise we would not even be able to choose constant paths as initial conditions. This assumption already rules out the possibility of the existence of a continuous semi-flow, as the following theorem shows.

Theorem 4.1.2. *There is no space E carrying the Wiener measure for which the equation* (4.1.2) *induces a continuous mapping $I\colon E \to \mathbb{R}$, $I(\xi) = y_1$, on a set of full measure, which extends the pathwise defined mapping for smooth initial conditions.*

Proof. The proof is inspired by [31, Proposition 1.29]. Let (Z_n) be a sequence of independent standard normal random variables. Set

$$B_t^N(\omega) = \sum_{n=1}^{N} Z_n(\omega) \frac{\sin[(n-1/2)\pi t]}{(n-1/2)\pi}.$$

Then $B^N \to B$ as $N \to \infty$ in α-Hölder norm, $\alpha < 1/2$, on a set of full measure Ω_1, cf. (4.1.5) where we recall the definition of the Hölder norm and [32, 3.5.1. Theorem] for a general result about Gaussian sequences from which the convergence above follows. Assume that E carries the Wiener measure. Then there is a set of full measure Ω_2 such that the limit

$$\sum_{n=1}^{\infty} \tilde{Z}_n(\omega) \frac{\sin[(n-1/2)\pi t]}{(n-1/2)\pi} =: \lim_{N\to\infty} \tilde{B}_t^N(\omega) =: \tilde{B}_t(\omega)$$

exists in E for every $\omega \in \Omega_2$ where $\tilde{Z}_n := (-1)^n Z_n$. The theory of Young integration [33] implies that

$$\int_0^1 \tilde{B}_t^N(\omega)\, dB_t^M(\omega) \to \int_0^1 \tilde{B}_t^N(\omega)\, dB_t(\omega)$$

as $M \to \infty$ for every $\omega \in \Omega_1 \cap \Omega_2$. Noting that $\tilde{Z}_n \sin[(n-1/2)\pi t] = Z_n \cos[(n-1/2)\pi(1+t)]$, we obtain that

$$\int_0^1 \tilde{B}_t^N(\omega)\, dB_t^M(\omega) = \sum_{n=1}^{N} \frac{Z_n^2(\omega)}{(n-1/2)\pi}$$

for all $M \geq N$. Therefore,

$$\int_0^1 \tilde{B}_t^N(\omega)\, dB_t(\omega) = \sum_{n=1}^{N} \frac{Z_n^2(\omega)}{(n-1/2)\pi} \to \infty$$

as $N \to \infty$ on a set of full measure $\Omega_3 \subset \Omega_1 \cap \Omega_2$. Now we can argue by contradiction. Assume that there is a set of full measure Ω_4 such that for every $\omega \in \Omega_4$, the map

$$E \ni \xi \mapsto \xi_0 + \int_0^1 \xi_{t-1}\, dB_t(\omega)$$

is continuous. Since $\Omega_3 \cap \Omega_4$ has full measure, the set is nonempty and we can choose $\omega \in \Omega_3 \cap \Omega_4$. Set $\xi_n := \tilde{B}^n(\omega)$ and $\xi := \tilde{B}(\omega)$. Then we have $\xi_n \to \xi$ in E as $n \to \infty$, but $\int_0^1 \xi_{t-1}^n\, dB_t(\omega)$ diverges as $n \to \infty$ which leads to a contradiction.

□

This theorem shows that *there is no reasonable space of functions on which the SDDE* (4.1.2) *induces a continuous semi-flow*, and using RDS to study such equations seems indeed hopeless. Let us mention here that only a delay in the diffusion part causes the trouble, a delay in a possible drift part would be harmless. For this reason, we will discard the drift here and study equations of the form (4.1.1) with $b = 0$ only. We also remark that studying delay equations where the diffusion coefficient may depend on a whole path segment of the solution, so-called *continuous delay*, can lead to easier equations since in that case, the diffusion coefficient might have a smoothing effect. Such equations are called *regular* stochastic delay differential equations, and they can indeed be studied using RDS, cf. [34] and [35]. The equation (4.1.2) is an example of *singular* stochastic delay differential equation.

Let us now explain the idea of this chapter. We have seen that there is no space of paths E on which $E \ni \xi \mapsto \int_0^1 \xi_s \, dB_s(\omega)$ is a continuous map on a set of full measure. However, in rough path theory, one knows that there is a family of Banach spaces $\{E_\omega\}_{\omega \in \Omega}$ and a set of full measure $\tilde{\Omega}$ such that the maps

$$E_\omega \ni \xi \mapsto \int_0^1 \xi_s \, d\mathbf{B}_s(\omega)$$

are continuous for every $\omega \in \tilde{\Omega}$ where the integral has to be interpreted as a *rough paths integral*. Indeed, the spaces E_ω are nothing but the usual spaces of *controlled paths* introduced by Gubinelli in [36]. Therefore, we can hope to establish a semi-flow property for solutions to (4.1.2) (and even more general equations) if we allow the state spaces to be random and by interpreting the equation as a delay differential equation driven by a random rough path. Fortunately, Neuenkirch, Nourdin, and Tindel already studied delay equations driven by rough paths in [37], and we can build on their results. Having established such a semi-flow property, the corresponding RDS will involve random spaces as well. We then argue this family of random spaces, constitutes a *measurable field of Banach spaces.* One of our main theorems in this work is to prove that SDDE induce RDS on a field of Banach spaces, cf. Theorem 4.3.14.

An obvious question is whether this structure is indeed useful for our actual goal which is to study the long-time behaviour of SDDE. This is not obvious at all since there is no example yet in the literature where an RDS was defined on a field of Banach spaces. As we pointed out earlier, the crucial result on which the theory of RDS is built is a *Multiplicative Ergodic Theorem* (MET). With the MET proved in Chapter 2, we can indeed deduce the existence of a Lyapunov spectrum for linear SDDE. Our main result in this chapter can loosely be formulated as follows:

Theorem 4.1.3. *Linear stochastic delay differential equations of the form*

$$dy_t = \sigma(y_t, y_{t-r}) \, dB_t(\omega) \tag{4.1.4}$$

induce linear RDS on measurable fields of Banach spaces given by the spaces of controlled paths defined by $B(\omega)$. Furthermore, an MET applies and provides the existence of a Lyapunov spectrum for the linear RDS.

Let us remark that stochastic differential equations on infinite dimensional spaces frequently lack the semi-flow property. For instance, this is often the case for stochastic partial differential equations (SPDEs), too, cf. e.g. [38] and the references therein. We believe that the approach we present here can be applied also in the context of SPDEs to provide a dynamical systems approach to equations for which the semi-flow property is known not to hold.

We finally remark that, although we focused on the (seemingly) simple SDDE (4.1.1) here, all of the presented results can be easily generalised to SDDE of the form

$$dy_t = \sigma\big(y_t, \int_{-r}^{0} y_{t+\tau}\mu(d\tau)\big)\, dB_t(\omega),$$

where μ is a finite signed measure on $[-r, 0]$. However, for the sake of simplicity, we prefer to work with a discrete delay term. We will later briefly state the necessary required modifications for this form of equations, cf. Remark 4.4.3.

This chapter is structured as follows. In Section 4.2, we introduce the techniques to study delay equations driven by rough paths and prove some basic properties. The content of Section 4.2 is to show that the fractional Brownian motion can drive rough delay equations and to prove a Wong-Zakai theorem. In Section 4.3, we establish the connection to Arnold's theory and define RDS on measurable fields of Banach spaces. The main results of this chapter and a discussion of them are contained in Section 4.4. Finally, we come back to the example (4.1.2) and discuss it in more detail in Section 4.5.

Preliminaries and notation

In this section we collect some notations which will be used throughout the chapter.

- If not stated differently, U, V, W and $\bar{W}$ will always denote finite-dimensional, normed vector spaces over the real numbers, with norm denoted by $|\cdot|$. By $L(U, W)$ we mean the set of linear and continuous functions from U to W equipped with usual operator norm.
- Let I be an interval in $\mathbb{R}$. A map $m : I \to U$ will also be called a *path*. For a path m, we denote its increment by $m_{s,t} = m_t - m_s$ where by m_t we mean $m(t)$. We set

  $$\|m\|_{\infty;I} := \sup_{s\in I} |m_s|$$

  and define the γ-Hölder seminorm, $\gamma \in (0, 1]$, by

  $$\|m\|_{\gamma;I} := \sup_{s,t\in I; s\neq t} \frac{|m_{s,t}|}{|t-s|^{\gamma}}.$$

  For a general 2-parameter function $m^{\#} : I \times I \to U$, the same notation is used. We will sometimes omit I as subindex if the domain is clear from the context. The space $C^0(I, U)$ consists of all continuous paths $m : I \to U$ equipped with the uniform norm, $C^{\gamma}(I, U)$ denotes the space of all γ-Hölder continuous functions equipped with the norm

  $$\|\cdot\|_{C^{\gamma};I} := \|\cdot\|_{\infty;I} + \|\cdot\|_{\gamma;I}. \tag{4.1.5}$$

  $C^{\infty}(I, U)$ is the space of all arbitrarily often differentiable functions. If $0 \in I$, using 0 as subindex such as for $C_0^{\gamma}(I, U)$ denotes the subspace of functions for which $x_0 = 0$.

An upper index such as $C^{0,\gamma}(I,U)$ means taking the closure of smooth functions in the corresponding norms.

Next, we introduce some basic objects from rough paths theory needed in this chapter. We refer the reader to [39] for a general overview.

- Let $X\colon \mathbb{R} \to U$ be a locally γ-Hölder path, $\gamma \in (0,1]$. A *Lévy area* for X is a continuous function

$$\mathbb{X}\colon \mathbb{R} \times \mathbb{R} \to U \otimes U$$

for which the algebraic identity

$$\mathbb{X}_{s,t} = \mathbb{X}_{s,u} + \mathbb{X}_{u,t} + X_{s,u} \otimes X_{u,t}$$

is true for every $s,u,t \in \mathbb{R}$ and for which $\|\mathbb{X}\|_{2\gamma;I} < \infty$ holds on every compact interval $I \subset \mathbb{R}$. If $\gamma \in (1/3, 1/2]$ and X admits Lévy area $\mathbb{X}$, we call $\mathbf{X} = (X, \mathbb{X})$ a *γ-rough path*. If $\mathbf{X}$ and $\mathbf{Y}$ are γ-rough paths, one defines

$$\varrho_{\gamma;I}(\mathbf{X},\mathbf{Y}) := \sup_{s,t\in I; s\neq t} \frac{|X_{s,t} - Y_{s,t}|}{|t-s|^{\gamma}} + \sup_{s,t\in I; s\neq t} \frac{|\mathbb{X}_{s,t} - \mathbb{Y}_{s,t}|}{|t-s|^{2\gamma}}.$$

- Let $I = [a,b]$ be a compact interval. A path $m\colon I \to \bar{W}$ is a *controlled path* based on X on the interval I if there exists a γ-Hölder path $m'\colon I \to L(U, \bar{W})$ such that

$$m_{s,t} = m'_s X_{s,t} + m^{\#}_{s,t}$$

for all $s,t \in I$ where $m^{\#}\colon I \times I \to \bar{W}$ satisfies $\|m^{\#}\|_{2\gamma;I} < \infty$. The path m' is called a *Gubinelli derivative* of m. We use $\mathscr{D}^{\gamma}_X(I,\bar{W})$ to denote the space of controlled paths based on X on the interval I. It can be shown that this space is a Banach space with norm

$$\|m\|_{\mathscr{D}^{\gamma}_X} := \|(m,m')\|_{\mathscr{D}^{\gamma}_X} := |m_a| + |m'_a| + \|m'\|_{\gamma;I} + \|m^{\#}\|_{2\gamma;I}.$$

If X and $\tilde{X}$ are γ-Hölder paths, $(m,m') \in \mathscr{D}^{\gamma}_X(I,\bar{W})$ and $(\tilde{m},\tilde{m}') \in \mathscr{D}^{\gamma}_{\tilde{X}}(I,\bar{W})$, we set

$$d_{2\gamma;I}((m,m'),(\tilde{m},\tilde{m}')) := \|m' - \tilde{m}'\|_{\gamma;I} + \|m^{\#} - \tilde{m}^{\#}\|_{2\gamma;I}.$$

If $\bar{W} = \mathbb{R}$, we will also use the notation $\mathscr{D}^{\gamma}_X(I)$ instead of $\mathscr{D}^{\gamma}_X(I,\mathbb{R})$.

We finally again recall the definition of a random dynamical system introduced in Chapter 2.

- Let $(\Omega,\mathcal{F})$ and $(X,\mathcal{B})$ be measurable spaces. Let $\mathbb{T}$ be either $\mathbb{R}$ or $\mathbb{Z}$, equipped with a σ-algebra $\mathcal{I}$ given by the Borel σ-algebra $\mathcal{B}(\mathbb{R})$ in the case of $\mathbb{T} = \mathbb{R}$ and by $\mathcal{P}(\mathbb{Z})$ in the case of $\mathbb{T} = \mathbb{Z}$. A family $\theta = (\theta_t)_{t\in\mathbb{T}}$ of maps from Ω to itself is called a *measurable dynamical system* if

  (i) $(\omega,t) \mapsto \theta_t\omega$ is $\mathcal{F} \otimes \mathcal{I}/\mathcal{F}$-measurable,

(ii) $\theta_0 = \mathrm{Id}$,

(iii) $\theta_{s+t} = \theta_s \circ \theta_t$, for all $s, t \in \mathbb{T}$.

If $\mathbb{T} = \mathbb{Z}$, we will also use the notation $\theta := \theta_1$, $\theta^n := \theta_n$ and $\theta^{-n} := \theta_{-n}$ for $n \geq 1$. If $\mathbb{P}$ is furthermore a probability on $(\Omega, \mathcal{F})$ that is invariant under any of the elements of θ,

$$\mathbb{P} \circ \theta_t^{-1} = \mathbb{P}$$

for every $t \in \mathbb{T}$, we call the tuple $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$ a *measurable metric dynamical system.* The system is called *ergodic* if every θ-invariant set has probability 0 or 1.

- Let $\mathbb{T}^+ := \{t \in \mathbb{T} : t \geq 0\}$, equipped with the trace σ-algebra. An *(ergodic) measurable random dynamical system* on $(X, \mathcal{B})$ is an (ergodic) measurable metric dynamical system $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$ with a measurable map

$$\varphi \colon \mathbb{T}^+ \times \Omega \times X \to X$$

that enjoys the *cocycle property*, i.e. $\varphi(0, \omega, \cdot) = \mathrm{Id}_X$, for all $\omega \in \Omega$, and

$$\varphi(t + s, \omega, \cdot) = \varphi(t, \theta_s \omega, \cdot) \circ \varphi(s, \omega, \cdot)$$

for all $s, t \in \mathbb{T}^+$ and $\omega \in \Omega$. The map φ is called *cocycle.* If X is a topological space with $\mathcal{B}$ being the Borel σ-algebra and the map $\varphi_\cdot(\omega, \cdot) \colon \mathbb{T}^+ \times X \to X$ is continuous for every $\omega \in \Omega$, it is called a *continuous (ergodic) random dynamical system.* In general, we say that φ *has property P* if and only if $\varphi(t, \omega, \cdot) \colon X \to X$ has property P for every $t \in \mathbb{T}^+$ and $\omega \in \Omega$ whenever the latter statement makes sense.

4.2 Basic properties of rough delay equations

In this section, we show how to solve rough delay differential equations and present some basic properties of the solution.

Basic objects, existence, uniqueness and stability

This section basically summarizes the concepts and results from [37]. We start by introducing "delayed" versions of rough paths and controlled paths. Note that, as already mentioned in the introduction, we restrict ourselves to the case of one time delay only. We refer to [37] for corresponding definitions for a finite number of delays.

Definition 4.2.1. *Let $X \colon \mathbb{R} \to U$ be a locally γ-Hölder path and $r > 0$. A* delayed Lévy area *for X is a continuous function*

$$\mathbb{X}(-r) \colon \mathbb{R} \times \mathbb{R} \to U \otimes U$$

for which the algebraic identity

$$\mathbb{X}_{s,t}(-r) = \mathbb{X}_{s,u}(-r) + \mathbb{X}_{u,t}(-r) + X_{s-r,u-r} \otimes X_{u,t}$$

is true for every $s, u, t \in \mathbb{R}$ *and for which* $\|\mathbb{X}(-r)\|_{2\gamma;I} < \infty$ *holds on every compact interval* $I \subset \mathbb{R}$. *If* $\gamma \in (1/3, 1/2]$ *and* X *admits Lévy- and delayed Lévy area* $\mathbb{X}$ *and* $\mathbb{X}(-r)$, *we call* $\mathbf{X} = \big(X, \mathbb{X}, \mathbb{X}(-r)\big)$ *a* delayed γ-rough path with delay $r > 0$. *If* $\mathbf{X}$ *and* $\mathbf{Y}$ *are delayed* γ*-rough paths, we set*

$$\varrho_{\gamma;I}(\mathbf{X}, \mathbf{Y}) := \sup_{s,t \in I; s \neq t} \frac{|X_{s,t} - Y_{s,t}|}{|t-s|^{\gamma}} + \sup_{s,t \in I; s \neq t} \frac{|\mathbb{X}_{s,t} - \mathbb{Y}_{s,t}|}{|t-s|^{2\gamma}} + \sup_{s,t \in I; s \neq t} \frac{|\mathbb{X}(-r)_{s,t} - \mathbb{Y}(-r)_{s,t}|}{|t-s|^{2\gamma}}.$$

Remark 4.2.2. *For* X *as in the former definition, set*

$$Z := (X, X_{\cdot - r}) \in U \oplus U.$$

If X *admits a Lévy- and delayed Lévy area, also* Z *admits a Lévy area* $\mathbb{Z}$ *given by*

$$\mathbb{Z} = \begin{pmatrix} \mathbb{X} & \bar{\mathbb{X}}(-r) \\ \mathbb{X}(-r) & \mathbb{X}_{\cdot - r, \cdot - r} \end{pmatrix}$$

where $\bar{\mathbb{X}}^{ij}(-r) := X^i_{s,t} X^j_{s-r,t-r} - \mathbb{X}^{ji}_{s,t}(-r)$. *Conversely, if* Z *admits a Lévy area, the path* X *admits both Lévy- and delayed Lévy area. The delayed Lévy area can therefore be understood as the usual Lévy area of a path enriched with its delayed path.*

Next, we recall what is a delayed controlled path.

Definition 4.2.3. *Let* $I = [a, b]$ *be a compact interval. A path* $m \colon I \to \bar{W}$ *is a* delayed controlled path *based on* X *on the interval* I *if there exist* γ*-Hölder paths* $\zeta^0, \zeta^1 \colon I \to L(U, \bar{W})$ *such that*

$$m_{s,t} = \zeta^0_s X_{s,t} + \zeta^1_s X_{s-r,t-r} + m^{\#}_{s,t} \tag{4.2.1}$$

for all $s, t \in I$ *where* $m^{\#} \colon I \times I \to \bar{W}$ *satisfies* $\|m^{\#}\|_{2\gamma;I} < \infty$. *The path* (ζ^0, ζ^1) *will again be called* Gubinelli derivative *of* m. *We use* $\mathcal{D}^{\gamma}_X(I, \bar{W})$ *to denote the space of delayed controlled paths based on* X *on the interval* I. *A norm on this space can be defined by*

$$\|m\|_{\mathcal{D}^{\gamma}_X} := \|(m, \zeta^0, \zeta^1)\|_{\mathcal{D}^{\gamma}_X} := |m_a| + |\zeta^0_a| + |\zeta^1_a| + \|\zeta^0\|_{\gamma;I} + \|\zeta^1\|_{\gamma;I} + \|m^{\#}\|_{2\gamma;I}. \tag{4.2.2}$$

Remark 4.2.4. *Note that any controlled path is also a delayed controlled path (by the choice* $\zeta^1 = 0$*), but the converse might not be true. However, considering again the enhanced path*

$$Z = (X, X_{\cdot - r}) \in U \oplus U,$$

the identity (4.2.1) *shows that* m *is a usual* $\bar{W}$*-valued controlled path based on* Z *with Gubinelli derivative* $\bar{\zeta} \colon I \to L(U \oplus U, \bar{W})$ *given by* $\bar{\zeta}_t(v, w) := \zeta^0_t v + \zeta^1_t w$.

With these objects, we can define an integral as follows.

Theorem 4.2.5. *Let* $\mathbf{X} = \big(X, \mathbb{X}, \mathbb{X}(-r)\big)$ *be a delayed* γ*-rough path and* m *an* $L(U, W)$*-valued delayed controlled path based on* X *with decomposition as in* (4.2.1) *on the interval* $[a, b]$. *Then* *the limit*

$$\int_a^b m_s \, d\mathbf{X}_s := \lim_{|\Pi| \to 0} \sum_{t_j \in \Pi} m_{t_j} X_{t_j, t_{j+1}} + \zeta^0_{t_j} \mathbb{X}_{t_j, t_{j+1}} + \zeta^1_{t_j} \mathbb{X}_{t_j, t_{j+1}}(-r) \tag{4.2.3}$$

exists where Π *denotes a partition of* $[a,b]$*. Moreover, there is a constant* C *depending on* γ *and* $(b-a)$ *only such that for all* $s < t \in [a,b]$*, the estimate*

$$\begin{aligned} &\left| \int_s^t m_u \, d\mathbf{X}_u - m_s X_{s,t} - \zeta^0_s \mathbb{X}_{s,t} - \zeta^1_s \mathbb{X}_{s,t}(-r) \right| \\ &\quad \leq C \left(\|m^\#\|_{2\gamma} \|X\|_\gamma + \|\zeta^0\|_\gamma \|\mathbb{X}\|_{2\gamma} + \|\zeta^1\|_\gamma \|\mathbb{X}(-r)\|_{2\gamma} \right) |t-s|^{3\gamma} \end{aligned}$$

holds. In particular,

$$t \mapsto \int_s^t m_u \, d\mathbf{X}_u$$

is controlled by X *with Gubinelli derivative* m*.*

Proof. This is just an application of the Sewing lemma, cf. e.g. [39, Lemma 4.2], applied to

$$\Xi_{s,t} = m_s X_{s,t} + \zeta^0_s \mathbb{X}_{s,t} + \zeta^1_s \mathbb{X}_{s,t}(-r).$$

□

Example 4.2.6. *Let* $U = W = \mathbb{R}$ *and* $\mathbf{X} = \big(X, \mathbb{X}, \mathbb{X}(-1)\big)$ *be a delayed* γ*-rough path. We aim to solve the equation*

$$\begin{aligned} dy_t &= y_{t-1} \, d\mathbf{X}_t; \quad t \geq 0 \\ y_t &= \xi_t; \quad t \in [-1, 0]. \end{aligned} \tag{4.2.4}$$

If $\xi \in \mathscr{D}^\gamma_X([-1,0])$*, the path* $[0,1] \ni t \mapsto \xi_{t-1}$ *is a delayed controlled path, thus the integral*

$$[0,1] \ni t \mapsto \int_0^t \xi_{s-1} \, d\mathbf{X}_s$$

exists. Therefore, the path

$$y_t := \begin{cases} \xi_t & \text{if } t \in [-1,0] \\ \int_0^t \xi_{s-1} \, d\mathbf{X}_s + \xi_0 & \text{if } t \in [0,1] \end{cases}$$

is the unique continuous solution to (4.2.4) *on* $[-1,1]$*. Since the integral is again an element in* $\mathscr{D}^\gamma_X([0,1])$*, we can iterate the procedure to solve* (4.2.4) *on the whole positive real line.*

We will need the following class of vector fields:

Definition 4.2.7. *By* $C^3_b(W^2, L(U,W))$*, we denote the space of bounded functions* $\sigma \colon W \oplus W \to L(U,W)$ *possessing 3 bounded derivatives.*

We can now state the first existence and uniqueness result for rough delay equations.

Theorem 4.2.8 (Neuenkirch, Nourdin, Tindel)**.** *For $r > 0$, let $\mathbf{X}$ be a delayed γ-rough path for $\gamma \in (1/3, 1/2]$, $\sigma \in C_b^3(W^2, L(U,W))$ and $(\xi, \xi') \in \mathscr{D}_X^\beta([-r,0], W)$ for some $\beta \in (1/3, \gamma)$. Then the equation*

$$\begin{aligned} y_t &= \xi_0 + \int_0^t \sigma(y_s, y_{s-r})\, d\mathbf{X}_s; \quad t \in [0,r] \\ y_t &= \xi_t; \quad t \in [-r, 0] \end{aligned} \tag{4.2.5}$$

has a unique solution $(y, y') \in \mathscr{D}_X^\beta([0,T], W)$ with Gubinelli derivative given by $y'_t = \sigma(y_t, y_{t-r})$.

Proof. The theorem was proved in [37, Theorem 4.2], we quickly sketch the idea here: First, it can be shown that for an element $\zeta \in \mathscr{D}_X^\beta([0,r], W)$, the path $\sigma(\zeta_\cdot, \xi_{\cdot - r})$ is a delayed controlled path. Therefore, one can consider the map

$$\zeta \mapsto \xi_0 + \int_0^\cdot \sigma(\zeta_u, \xi_{u-r})\, d\mathbf{X}_u$$

and prove that it has a fixed point in the space $\mathscr{D}_X^\beta([0,r], W)$ to obtain a solution on $[0,r]$. The claimed Gubinelli derivative can be deduced using the estimate provided in Theorem 4.2.5. □

We proceed with a theorem which shows that the solution map induced by (4.2.5) is continuous. Unfortunately, the corresponding result stated in [37, Theorem 4.2] is not correct, therefore we can not cite it directly. We will first formulate the correct statement and then discuss the difference compared to [37, Theorem 4.2].

Theorem 4.2.9. *Let $\mathbf{X}$ and $\tilde{\mathbf{X}}$ be a delayed γ-rough paths with $\gamma \in (1/3, 1/2]$, $\sigma \in C_b^3(W^2, L(U,W))$ and choose $(\xi, \xi') \in \mathscr{D}_X^\beta([-r,0], W)$ and $(\tilde{\xi}, \tilde{\xi}') \in \mathscr{D}_{\tilde{X}}^\beta([-r,0], W)$ for some $\beta \in (1/3, \gamma)$. Consider the solutions (y, y') and $(\tilde{y}, \tilde{y}')$ to*

$$\begin{aligned} dy_t &= \sigma(y_t, y_{t-r})\, d\mathbf{X}; \quad t \in [0,r] \\ y_t &= \xi_t; \quad t \in [-r, 0] \end{aligned}$$

resp.

$$\begin{aligned} d\tilde{y}_t &= \sigma(\tilde{y}_t, \tilde{y}_{t-r})\, d\tilde{\mathbf{X}}; \quad t \in [0,r] \\ \tilde{y}_t &= \tilde{\xi}_t; \quad t \in [-r, 0]. \end{aligned}$$

Then

$$\begin{aligned} &d_{2\beta;[0,r]}((y, y'), (\tilde{y}, \tilde{y}')) \\ &\leq C \left(|\xi_{-r} - \tilde{\xi}_{-r}| + |\xi'_{-r} - \tilde{\xi}'_{-r}| + d_{2\beta;[-r,0]}((\xi, \xi'), (\tilde{\xi}, \tilde{\xi}')) + \varrho_{\gamma;[0,r]}(\mathbf{X}, \tilde{\mathbf{X}}) \right) \end{aligned} \tag{4.2.6}$$

holds for some constant $C > 0$ depending on r, γ, β and M, where M is chosen such that

$$\begin{aligned} M \geq &\|\xi\|_{\mathscr{D}_X^\beta} + \|\tilde{\xi}\|_{\mathscr{D}_{\tilde{X}}^\beta} + \|X\|_\gamma + \|\mathbb{X}\|_{2\gamma} + \|\mathbb{X}(-r)\|_{2\gamma} \\ &+ \|\tilde{X}\|_\gamma + \|\tilde{\mathbb{X}}\|_{2\gamma} + \|\tilde{\mathbb{X}}(-r)\|_{2\gamma}. \end{aligned}$$

Remark 4.2.10. *In [37, Theorem 4.2], the authors state that the estimate*

$$\|y - \tilde{y}\|_{\beta;[0,r]} \leq C(|\xi_{-r} - \tilde{\xi}_{-r}| + \|\xi - \tilde{\xi}\|_{\beta;[-r,0]} + \rho_\gamma(\mathbf{X}, \tilde{\mathbf{X}})) \tag{4.2.7}$$

holds for the usual Hölder norm. However, this estimate can not be true in general. To see this, assume $\mathbf{X}^1 = \mathbf{X}^2 =: \mathbf{X}$ *and consider the equation in Example 4.2.6. If* (4.2.7) *was true, the map*

$$\xi \mapsto \int \xi \, d\mathbf{X}$$

would be continuous in the β*-Hölder norm, which is clearly not the case for a genuine rough path* $\mathbf{X}$.

The proof of Theorem 4.2.9 is a bit lengthy, but mostly straightforward. We sketch it in the appendix, cf. page 136.

Linear equations

In this section, we consider the case where σ is linear, i.e. $\sigma \in L(W^2, L(U,W))$. Note that in this case, there are $\sigma_1, \sigma_2 \in L(W, L(U,W))$ such that $\sigma(y_1,y_2) = \sigma_1(y_1) + \sigma_2(y_2)$ for all $y_1, y_2 \in W$. Since linear vector fields are unbounded, we cannot directly apply Theorem 4.2.8. However, we can prove an a priori bound for any solution of the equation and then deduce existence, uniqueness and stability for linear equations from Theorem 4.2.8 and 4.2.9 by truncating the vector field σ.

Theorem 4.2.11. *Let* $\mathbf{X}$ *be a delayed* γ*-rough path over* X *with* $\gamma \in (1/3, 1/2]$ *and* $\sigma \in L(W^2, L(U,W))$. *Then any solution* $y \colon [0,r] \to W$ *of*

$$\begin{aligned} dy_t &= \sigma(y_t, y_{t-r}) \, d\mathbf{X}; \quad t \in [0,r] \\ y_t &= \xi_t; \quad t \in [-r, 0] \end{aligned} \tag{4.2.8}$$

satisfies, for $(y, y') = (y, \sigma(y, \xi_{\cdot - r}))$, *the bound*

$$\begin{aligned} &\|y\|_{\mathscr{D}^\beta_X([0,r],W)} \leqslant \\ &C(1 + r^{\gamma-\beta}\|X\|_{\gamma;[0,r]})\|\xi\|_{\mathscr{D}^\beta_X([-r,0],W)} \exp\left\{C(\|X\|_{\gamma;[0,r]} + \|\mathbb{X}\|_{2\gamma;[0,r]} + \|\mathbb{X}(-r)\|_{2\gamma;[0,r]})^{\frac{1}{\gamma-\beta}}\right\} \end{aligned} \tag{4.2.9}$$

where C *depends on* r, $\|\sigma\|$, γ *and* β.

Proof. For $s,t \in [0,r]$, we have

$$y_{s,t} = y'_s X_{s,t} + y^{\#}_{s,t}$$

where

$$y'_s = \sigma(y_s, \xi_{s-r}) \tag{4.2.10}$$

and

$$\begin{aligned} y^{\#}_{s,t} &= \int_s^t \sigma(y_u, y_{u-r})\, d\mathbf{X}_u - \sigma(y_s, \xi_{s-r}) X_{s,t} \\ &= \tilde{\rho}_{s,t} + \sigma_1 y'_s \mathbb{X}_{s,t} + \sigma_2 \xi'_{s-r} \mathbb{X}_{s,t}(-r) \end{aligned}$$

with $\tilde{\rho}$ given by

$$\tilde{\rho}_{s,t} = \int_s^t \sigma(y_u, y_{u-r})\, d\mathbf{X}_u - \sigma(y_s, \xi_{s-r}) X_{s,t} - \sigma_1 y'_s \mathbb{X}_{s,t} - \sigma_2 \xi'_{s-r} \mathbb{X}_{s,t}(-r).$$

Note that $u \mapsto \sigma(y_u, \xi_{u-r})$ is a delayed controlled path with Gubinelli derivative $u \mapsto (\sigma_1 y'_u, \sigma_2 \xi'_{u-r})$. Therefore, we can use the estimate provided in Theorem 4.2.5 to see that for a constant $M = M(\beta, r)$ and $I = [a, b] \subset [0, r]$:

$$\begin{aligned} \|y^{\#}\|_{2\beta;I} \leqslant{}& \|\sigma\| \big(\|y'\|_{\infty;I} \|\mathbb{X}\|_{2\gamma;I} + \|\xi'\|_{\infty;[-r,0]} \|\mathbb{X}(-r)\|_{2\gamma;I} \big) (b-a)^{2\gamma-2\beta} \\ &+ M\|\sigma\| \big(\|y^{\#}\|_{2\beta;I} \|X\|_{\gamma;I} + \|\xi^{\#}\|_{2\beta;[-r,0]} \|X\|_{\gamma;I} \big) (b-a)^{\gamma} \\ &+ M\|\sigma\| \big(\|y'\|_{\beta;I} \|\mathbb{X}\|_{2\gamma;I} + \|\xi'\|_{\beta;[-r,0]} \|\mathbb{X}(-r)\|_{2\gamma;I} \big) (b-a)^{2\gamma-\beta} \end{aligned} \tag{4.2.11}$$

and by relation (4.2.10) :

$$\begin{aligned} &\|y\|_{\beta;I} \leqslant \|\sigma\| \big(\|y\|_{\infty,I} + \|\xi\|_{\infty,[-r,0]} \big) \|X\|_{\gamma;I} (b-a)^{\gamma-\beta} + \|y^{\#}\|_{2\beta;I} (b-a)^{\beta} \quad \text{and} \\ &\|y'\|_{\beta;I} \leqslant \|\sigma\| \big(\|y\|_{\beta;I} + \|\xi\|_{\beta;[-r,0]} \big). \end{aligned}$$

Now assume that $b - a = \theta < 1 \wedge r$ for a given θ and set

$$A := 1 + \|X\|_{\gamma;[0,r]} + \|\mathbb{X}\|_{2\gamma;[0,r]} + \|\mathbb{X}(-r)\|_{2\gamma;[0,r]}.$$

Our former estimates imply that there are constants $\tilde{M}, \tilde{N}$ depending on $\|\sigma\|$ such that

$$\begin{aligned} &\|y\|_{\beta;I} + \|y\|_{\infty;I} + \|y'\|_{\beta;I} + \|y^{\#}\|_{2\beta;I} + \|y'\|_{\infty;I} \leqslant \\ &\tilde{M} A \theta^{\gamma-\beta} \big(\|y\|_{\beta;I} + \|y\|_{\infty;I} + \|y'\|_{\beta;I} + \|y^{\#}\|_{2\beta;I} + \|y'\|_{\infty;I} \big) + \\ &\tilde{N} A \big(\|\xi\|_{\beta;[-r,0]} + \|\xi\|_{\infty;[-r,0]} + \|\xi'\|_{\infty;[-r,0]} + \|\xi^{\#}\|_{2\beta;[-r,0]} \big) + (1 + \|\sigma\|) \|y\|_{\infty;I}. \end{aligned} \tag{4.2.12}$$

Choose θ small enough such that

$$\tilde{M} A \theta^{\gamma-\beta} \leqslant \frac{1}{4} \quad \text{and} \quad \theta^{\beta} (1 + \|\sigma\|) \leqslant \frac{1}{4}. \tag{4.2.13}$$

For $n \geqslant 1$ and $n\theta \leqslant r$, set $I_n := [(n-1)\theta, n\theta]$ and

$$\begin{aligned} B_n &= \|y\|_{\beta;I_n} + \|y\|_{\infty;I_n} + \|y'\|_{\beta;I_n} + \|y^{\#}\|_{2\beta;I_n} + \|y'\|_{\infty;I_n} \\ B_0 &= \|\xi\|_{\beta;[-r,0]} + \|\xi\|_{\infty;[-r,0]} + \|\xi'\|_{\infty;[-r,0]} + \|\xi^{\#}\|_{2\beta;[-r,0]}. \end{aligned}$$

Note that $\|y\|_{\infty;I_n} \leqslant B_{n-1} + \theta^{\beta} B_n$. By (4.2.12) and (4.2.13),

$$B_n \leqslant 2\tilde{N} A B_0 + 2(1 + \|\sigma\|) B_{n-1}.$$

Set $C = 2\tilde{N}A$ and $\tilde{C} = 2(1 + \|\sigma\|)$. By a simple induction argument, it is not hard to verify that for $k \leq n$,

$$B_n \leqslant C(1 + \tilde{C} + \tilde{C}^2 + ... + \tilde{C}^{k-1})B_0 + \tilde{C}^k B_{n-k}$$

which implies

$$B_n \leqslant \tilde{C}^n(1 + C)B_0$$

for $k = n$. Note that since $y^{\#}_{s,t} = y^{\#}_{s,u} + y^{\#}_{u,t} + y'_{s,u}X_{u,t}$,

$$\|y^{\#}\|_{2\beta;[0,r]} \leqslant \sum_{1 \leqslant n \leqslant m} \|y^{\#}\|_{2\beta;I_n} + r^{\gamma-\beta}\|X\|_{\gamma;[0,r]} \sum_{1 \leqslant n \leqslant m} \|y'\|_{\beta;I_n} \tag{4.2.14}$$

Now set $m = [\frac{r}{\theta}] + 1$. By (4.2.14) and subadditivity of the Hölder norm,

$$\begin{aligned}\|y\|_{\beta;[0,r]} + \|y\|_{\infty;[0,r]} + \|y'\|_{\beta;[0,r]} + \|y^{\#}\|_{2\beta;[0,r]} + \|y'\|_{\infty;[0,r]} \\ \leqslant (1 + r^{\gamma-\beta}\|X\|_{\gamma;[0,r]}) \sum_{1 \leqslant n \leqslant m} B_n \leqslant (1 + r^{\gamma-\beta}\|X\|_{\gamma;[0,r]})\tilde{C}^{m+1}(1 + C)B_0.\end{aligned}$$

Note that an appropriate choice for θ is

$$\theta = \frac{1}{(4\tilde{M}A)^{\frac{1}{\gamma-\beta}} + (4(1 + \|\sigma\|))^{\frac{1}{\beta}} + 1 + \frac{1}{r}} \tag{4.2.15}$$

which implies the claimed bound. $\square$

From Theorem 4.2.11, it follows that in the case of linear vector fields σ, the solution map induced by (4.2.8) is a bounded linear map. We now prove that it is even compact.

Proposition 4.2.12. *Under the same assumptions as in Theorem 4.2.11, the solution map induced by* (4.2.8) *is a compact linear map for every* $1/3 < \beta < \gamma$.

Proof. Fix $\beta < \gamma$. Let $\{\xi^{(n)}\}_{n \geqslant 1}$ be a bounded sequence in $\mathscr{D}^{\beta}_X([-r, 0], W)$, i.e.

$$\xi^{(n)}_{u,v} = (\xi^{(n)})'_u X_{u,v} + (\xi^{(n)})^{\#}_{u,v}$$

with uniformly bounded β-Hölder norm of $\xi^{(n)}$ and $(\xi^{(n)})'$ and uniformly bounded 2β-Hölder norm of $(\xi^{(n)})^{\#}$. From the Arzelà-Ascoli theorem, there are continuous functions ξ and ξ' such that

$$(\xi^{(n)}, (\xi^{(n)})') \to (\xi, \xi')$$

uniformly along a subsequence, which we will henceforth denote by $(\xi^{(n)}, (\xi^{(n)})')_n$ itself. It follows that $(\xi^{(n)}, (\xi^{(n)})') \to (\xi, \xi')$ in δ-Hölder norm for every $\delta < \beta$. Define $\xi^{\#}_{u,v} := \xi_{u,v} - \xi'_u X_{u,v}$. Clearly, $(\xi^{(n)})^{\#} \to \xi^{\#}$ uniformly, and since

$$|\xi^{\#}_{u,v}| \leq \sup_n \|(\xi^{(n)})^{\#}\|_{2\beta;[-r,0]}|v - u|^{2\beta}$$

for every $-r \leq u \leq v \leq 0$, it follows that $(\xi^{(n)})^{\#} \to \xi^{\#}$ in 2δ-Hölder norm for every $\delta < \beta$. This implies that $(\xi^{(n)}, (\xi^{(n)})') \to (\xi, \xi')$ in the space $\mathscr{D}_X^{\delta}([-r,0],W)$ for every $\delta < \beta$. Let $(y^n, (y^n)')$ denotes the solutions to (4.2.8) for the initial conditions $(\xi^n, (\xi^n)')$. Fix some $1/3 < \delta < \beta$. From continuity, the solutions $(y^n, (y^n)')$ converge in the space $\mathscr{D}_X^{\delta}([0,r],W)$, too. Choose $\beta < \beta' < \gamma$. Using a similar estimate as (4.2.11) in Theorem 4.2.11 where we apply the estimate in Theorem 4.2.5 for δ shows that we can bound $\|(y^n, (y^n)')\|_{\mathscr{D}_X^{\beta'}([0,r],W)}$ uniformly over n where the bound depends, in particular, on $\sup_n \|(y^n, (y^n)')\|_{\mathscr{D}_X^{\delta}([0,r],W)}$. This implies convergence also in the space $\mathscr{D}_X^{\beta}([0,r],W)$ and therefore proves compactness. □

A semi-flow property

In this section, we discuss the flow property induced by a rough delay equation. Recall that a *flow* on some set M is a mapping

$$\phi\colon [0,\infty) \times [0,\infty) \times M \to M$$

such that $\phi(t,t,\xi) = \xi$ and

$$\phi(s,t,\xi) = \phi(u,t,\phi(s,u,\xi)) \tag{4.2.16}$$

hold for every $\xi \in M$ and $s,t,u \in [0,\infty)$. Our prime example of a flow is a differential equation in which case $\xi \in M$ denotes an initial condition at time point s and $\phi(s,t,\xi)$ denotes the solution at time t. In the setting of a delay equation, we can only expect to solve the equation forward in time, i.e. $\phi(s,t,\xi)$ will only be defined for $s \leq t$. If (4.2.16) is assumed to hold only for $s \leq u \leq t$, we will speak of a *semi-flow*. In case of a rough delay equation, we will give up the idea of choosing a common set of admissible initial conditions M which will work for all time instances. Instead, our semi-flow will actually consist of a family of maps

$$\phi(s,t,\cdot)\colon M_s \to M_t$$

where $(M_t)_{t\geq 0}$ are sets (later: spaces) indexed by time. Note that the semi-flow property (4.2.16) still makes perfect sense in this setting, and this is what we are going to prove. Note also that the phenomenon of time-varying spaces is already visible in Example 4.2.6: admissible initial conditions are controlled paths defined on intervals depending on the time when we start to solve the equation.

Theorem 4.2.13. *Let $\mathbf{X}$ be a delayed γ-rough path over X with $\gamma \in (1/3, 1/2]$ and $\sigma \in C_b^3(W^2, L(U,W))$. Consider the equation*

$$\begin{aligned} dy_t &= \sigma(y_t, y_{t-r})\, d\mathbf{X}; \quad t \in [s,\infty) \\ y_t &= \xi_t; \quad t \in [s-r, s] \end{aligned} \tag{4.2.17}$$

for $s \in \mathbb{R}$. Let $\beta \in (1/3, \gamma)$. If $\xi \in \mathscr{D}_X^{\beta}([s-r,s],W)$, the equation (4.2.17) has a unique solution $y\colon [s,\infty) \to W$ and for $t \geq s$, we denote by $\phi(s,t,\xi)$ the solution path segment

$$\phi(s,t,\xi) = (y_u)_{t-r \leq u \leq t}.$$

If $r \leqslant t-s$, *we have that* $\phi(s,t,\xi) \in \mathscr{D}_X^{\beta}([t-r,t],W)$ *with Gubinelli derivative* $\phi'(s,t,\xi) = (\sigma(y_u, y_{u-r}))_{t-r\leq u\leq t}$ *and*

$$\begin{aligned} \phi(s,t,\cdot)\colon \mathscr{D}_X^{\beta}([s-r,s],W) &\to \mathscr{D}_X^{\beta}([t-r,t],W) \\ \xi &\mapsto \phi(s,t,\xi) \end{aligned} \tag{4.2.18}$$

is a continuous map. In case that $\xi'_s = \sigma(\xi_s, \xi_{s-r})$, *we have* $\phi(s,t,\xi) \in \mathscr{D}_X^{\beta}([-r+t,t],W)$ *for all* $s \leq t$ *with Gubinelli derivative given by*

$$\phi(s,t,\xi)(u) = \begin{cases} \xi'_u & \text{for } t-r \leq u \leq s \\ \sigma(y_u, y_{u-r}) & \text{for } s \leq u \leq t \end{cases}$$

for $r > t-s$. *For* $s \leq u \leq t$ *and* $r \leqslant u-s$, *we have the semi-flow property*

$$\begin{aligned} &\phi(s,s,\cdot) = \mathrm{Id}_{\mathscr{D}_X^{p}([-r+s,s],W)} \qquad \text{and} \\ &\phi(u,t,\cdot) \circ \phi(s,u,\xi) = \phi(s,t,\xi). \end{aligned} \tag{4.2.19}$$

Again, if $\xi'_s = \sigma(\xi_s, \xi_{-r+s})$, (4.2.19) *is true for all* $s \leq u \leq t$.

Proof. As in Theorem 4.2.8, we can first solve (4.2.17) on the time interval $[s, s+r]$. This can now be iterated to obtain a solution on $[s,\infty)$. The claimed Gubinelli derivatives on every interval $[s+kr, s+(k+1)r]$, $k \in \mathbb{N}_0$, are a consequence of Theorem 4.2.5. Since the derivatives agree on the boundary points of the intervals, we can "glue them together" to obtain a controlled path on arbitrary intervals $[u,v] \subset [s,\infty)$. If the assumption $\xi'_s = \sigma(\xi_s, \xi_{s-r})$ holds, this can even be done for every interval $[u,v] \subset [s-r,\infty)$. Continuity of the map (4.2.18) is a consequence of Theorem 4.2.9. The semi-flow property (4.2.19) is a consequence of existence and uniqueness of solutions: Let $y_\tau^{s,\xi}$ be the solution of (4.2.17) for $\tau \geq s-r$ where $y_\tau^{s,\xi} = \xi_\tau$ for $s-r \leqslant \tau \leqslant s$. Let $s \leq u \leq t$ and assume either $r \leqslant u-s$ or $\xi'_s = \sigma(\xi_s, \xi_{s-r})$. For $\tau < u$, it is not hard to verify that $y_\tau^{s,\xi} = y_\tau^{u,\phi(s,u,\xi)}$. If $u \leqslant \tau$ by definition:

$$\begin{aligned} y_\tau^{s,\xi} &= \xi_s + \int_s^\tau \sigma(y_z^{s,\xi}, y_{z-r}^{s,\xi})\,d\mathbf{X}_z = y_u^{s,\xi} + \int_u^\tau \sigma(y_z^{s,\xi}, y_{z-r}^{s,\xi})\,d\mathbf{X}_z \quad \text{and} \\ y_\tau^{u,\phi(s,u,\xi)} &= y_u^{s,\xi} + \int_u^\tau \sigma(y_z^{u,\phi(s,u,\xi)}, y_{z-r}^{u,\phi(s,u,\xi)})\,d\mathbf{X}_z. \end{aligned}$$

Given the uniqueness of the solution, $y_\tau^{s,\xi} = y_\tau^{u,\phi(s,u,\xi)}$ which indeed implies (4.2.19). □

Existence of delayed Lévy areas for the fractional Brownian motion and a Wong-Zakai theorem

In order to apply the results from Section 4.2 to stochastic delay differential equations, we need to make sure that the fractional Brownian motion can be "lifted" to a process taking values in the space of delayed rough paths. In this section, $B = (B^1, \ldots, B^d)\colon \mathbb{R} \to \mathbb{R}^d$ will always denote an $\mathbb{R}^d$-valued two-sided fractional Brownian motion with $\frac{1}{3} < H \leqslant \frac{1}{2}$, defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. When $H = \frac{1}{2}$, we mean the usual Brownian motion adapted to some two-parameter filtration $(\mathcal{F}_s^t)_{s\leq t}$, i.e. $(B_{t+s}-B_s)_{t\geq 0}$ is a usual $(\mathcal{F}_s^{t+s})_{t\geq 0}$-Brownian motion for every $s\in\mathbb{R}$.

Also $B_0=0$ almost surely (cf. [1, Section 2.3.2] for a more detailed discussion about two-sided stochastic processes).

Brownian Motion

We will first start with the easy case when $H=\frac{1}{2}$.

Definition 4.2.14. *For $r>0$ and $H=\frac{1}{2}$ set*

$$\mathbf{B}_{s,t}^{It\bar{o}} := \Big(B_{s,t}, \mathbb{B}_{s,t}^{It\bar{o}}, \mathbb{B}_{s,t}^{It\bar{o}}(-r)\Big) := \Big(B_t - B_s, \int_s^t (B_u - B_s)\otimes dB_u, \int_s^t (B_{u-r}-B_{s-r})\otimes dB_u\Big)$$

for $s\leq t\in\mathbb{R}$ where the stochastic integrals are understood in Itō-sense. We furthermore define

$$\mathbf{B}_{s,t}^{Strat} := \Big(B_{s,t}, \mathbb{B}_{s,t}^{It\bar{o}} + \frac{1}{2}(t-s)I_d, \mathbb{B}_{s,t}^{It\bar{o}}(-r)\Big)$$

where I_d denotes the identity matrix in $\mathbb{R}^d$.

Proposition 4.2.15. *Both processes $\mathbf{B}^{It\bar{o}}$ and $\mathbf{B}^{Strat}$ have modifications, henceforth denoted with the same symbols, with sample paths being delayed γ-rough paths for every $\gamma<1/2$ almost surely. Moreover, the γ-Hölder norms of both processes have finite p-th moment for every $p>0$ on any compact interval.*

Proof. The assertion follows by considering the usual Itō- and Stratonovich lifts of the enhanced process $(B, B_{\cdot-r})$ as in [39, Section 3.2 and 3.], using the Kolmogorov criterion for rough paths stated in [39, Theorem 3.1] (cf. also Remark 4.2.2). $\square$

The next proposition justifies the names of the processes defined above.

Proposition 4.2.16. *Let $(m(\omega),\zeta^0(\omega),\zeta^1(\omega))\in\mathcal{D}_{B(\omega)}^{\gamma}$ almost surely. Furthermore, assume that the process $(m_t,\zeta_t^0,\zeta_t^1)_{t\geq 0}$ is $(\mathcal{F}_0^t)_{t\geq 0}$-adapted. Then*

$$\int_0^T m_s\, dB_s = \int_0^T m_s\, d\mathbf{B}_s^{It\bar{o}} \quad \textit{and} \quad \int_0^T m_s \circ dB_s = \int_0^T m_s\, d\mathbf{B}^{Strat}$$

almost surely for every $T>0$.

Proof. We will first consider the Itō-case which is similar to [39, Proposition 5.1]. Set $\mathcal{F}_t := \mathcal{F}_0^t$. To simplify notation, assume $W=\mathbb{R}$. Let (τ_j) be a partition of $[0,T]$. We first prove that

$$\mathbb{E}\Big[\big(\zeta_{\tau_j}^0\mathbb{B}_{\tau_j,\tau_{j+1}} + \zeta_{\tau_j}^1\mathbb{B}_{\tau_j,\tau_{j+1}}(-r)\big)\big(\zeta_{\tau_k}^0\mathbb{B}_{\tau_k,\tau_{k+1}} + \zeta_{\tau_k}^1\mathbb{B}_{\tau_k,\tau_{k+1}}(-r)\big)\Big] = 0 \tag{4.2.20}$$

for $j<k$. To see this, note that

$$\mathbb{E}[(\zeta_{\tau_j}^1\mathbb{B}_{\tau_j,\tau_{j+1}}(-r))(\zeta_{\tau_k}^1\mathbb{B}_{\tau_k,\tau_{k+1}}(-r))] = \mathbb{E}\Big[\mathbb{E}[(\zeta_{\tau_j}^1\mathbb{B}_{\tau_j,\tau_{j+1}}(-r))(\zeta_{\tau_k}^1\mathbb{B}_{\tau_k,\tau_{k+1}}(-r))|\mathcal{F}_{\tau_k}]\Big] =$$

$$\mathbb{E}\Big[\zeta_{\tau_j}^1\mathbb{B}_{\tau_j,\tau_{j+1}}(-r)\zeta_{\tau_k}^1\mathbb{E}[\mathbb{B}_{\tau_k,\tau_{k+1}}(-r)|\mathcal{F}_{\tau_k}]\Big].$$

We show that $\mathbb{E}[\mathbb{B}_{s,u}(-r)|\mathcal{F}_s] = 0$ for $s \leq u$. By definition,

$$\mathbb{B}_{s,u}(-r) = \lim_{|\Pi|\to 0} \sum_{t_k \in \Pi} B_{s-r,t_k-r} \otimes B_{t_k,t_{k+1}}$$

where Π is a partition for $[s,u]$ and the limit is understood in $L^2(\Omega)$-sense. Consequently,

$$\mathbb{E}[\mathbb{B}_{s,u}(-r)|\mathcal{F}_s] = \lim_{|\Pi|\to 0} \sum_{t_k \in \Pi} \mathbb{E}[B_{s-r,t_k-r} \otimes B_{t_k,t_{k+1}}|\mathcal{F}_s]$$

again in L^2. Note that

$$\begin{aligned} &\mathbb{E}[B_{s-r,t_k-r} \otimes B_{t_k,t_{k+1}}|\mathcal{F}_s] = \\ &\qquad \begin{cases} B_{s-r,t_k-r} \otimes \mathbb{E}[B_{t_k,t_{k+1}}|\mathcal{F}_s] = 0, & \text{if } t_k - r \leqslant s \\ B_{s-r,s} \otimes \mathbb{E}[B_{t_k,t_{k+1}}|\mathcal{F}_s] + \mathbb{E}[B_{s,t_k-r} \otimes B_{t_k,t_{k+1}}|\mathcal{F}_s] = 0, & \text{if } s < t_k - r. \end{cases} \end{aligned}$$

Other cases are similar and (4.2.20) can be deduced. Using a stopping argument, we may assume that there is a deterministic $M > 0$ such that

$$\sup_{t\in[0,T]} \|\zeta_t^0(\omega)\| \vee \|\zeta_t^1(\omega)\| \leq M$$

almost surely. Then,

$$\mathbb{E}\Big[\big(\sum_j \zeta_{t_j}^0 \mathbb{B}_{\tau_j,\tau_{j+1}} + \zeta_{\tau_j}^1 \mathbb{B}_{\tau_j,\tau_{j+1}}(-r)\big)^2\Big] \leqslant M \sum_j (\tau_{j+1} - \tau_j)^2 \leq MT \max_j |\tau_{j+1} - \tau_j|$$

which converges to 0 when the mesh size of the partition gets small. The claim now follows using the definition of the Itō integral as a limit of Riemann sums. The proof for the Stratonovich integral is similar to [39, Corollary 5.2]. □

The following corollary is immediate.

Corollary 4.2.17. *The solution to the Itō equation*

$$dY_t = \sigma(Y_t, Y_{t-r})\, dB_t$$

is almost surely equal to the solution to the random rough delay equation

$$dY_t = \sigma(Y_t, Y_{t-r})\, d\mathbf{B}_t^{It\bar{o}}$$

if the initial condition is $\mathcal{F}_{-1}^0$-measurable and almost surely controlled by B. The same statement holds in the Stratonovich case.

Next, we prove an approximation result.

Definition 4.2.18. *Let $\rho : \mathbb{R} \to [0,2]$ be a smooth function such that* $\operatorname{supp}(\rho) \subset [0,1]$ *and which integrates to* 1*. We set*

$$B_t^\varepsilon := \int_{\mathbb{R}} B_{-\varepsilon z, t-\varepsilon z} \rho(z) dz, \quad \varepsilon \in (0,1].$$

It is not hard to see that

$$\mathbb{E}|B^{\varepsilon}_{s,t}|^2 \leqslant M(t-s) \quad \text{and} \quad \lim_{\varepsilon\to 0}\mathbb{E}|B^{\varepsilon}_{s,t}-B_{s,t}|^2 = 0 \tag{4.2.21}$$

where M is independent of ε.

Lemma 4.2.19. *We have the following pathwise identity:*

$$\int_s^t B^{\varepsilon}_{s-r,u-r}\otimes dB^{\varepsilon}_u = \int_{\mathbb{R}}\rho(z)\int_{s-\varepsilon z}^{t-\varepsilon z} B^{\varepsilon}_{s-r,u+\varepsilon z-r}\otimes dB_u dz. \tag{4.2.22}$$

Proof. Note that both integrals in (4.2.22) are indeed pathwise defined since B^{ε} is smooth and B is Hölder continuous. Using integration by parts, for $i,j\in\{1,\dots,d\}$,

$$\int_{s-\varepsilon z}^{t-\varepsilon z}(B^{\varepsilon})^i_{s-r,u+\varepsilon z-r}dB^j_u = (B^{\varepsilon})^i_{s-r,t-r}B^j_{t-\varepsilon z} - \int_{s-r}^{t-r}B^j_{u+r-\varepsilon z}d(B^{\varepsilon})^i_u.$$

Consequently,

$$\begin{aligned}&\int_{\mathbb{R}}\rho(z)\int_{s-\varepsilon z}^{t-\varepsilon z}(B^{\varepsilon})^i_{s-r,u+\varepsilon z-r}dB^j_u dz = \\ &\qquad \int_{\mathbb{R}}(B^{\varepsilon})^i_{s-r,t-r}\rho(z)B^j_{t-\varepsilon z}dz - \int_{\mathbb{R}}\int_{s-r}^{t-r}\rho(z)B^j_{u+r-\varepsilon z}d(B^{\varepsilon})^i_u dz = \\ &\qquad (B^{\varepsilon})^i_{s-r,t-r}(B^{\varepsilon})^j_t - \int_{s-r}^{t-r}(B^{\varepsilon})^j_{u+r}d(B^{\varepsilon})^i_u.\end{aligned}$$

Using integration by parts again, we have

$$(B^{\varepsilon})^i_{s-r,t-r}(B^{\varepsilon})^j_t - \int_{s-r}^{t-r}(B^{\varepsilon})^j_{u+r}d(B^{\varepsilon})^i_u = \int_s^t(B^{\varepsilon})^i_{s-r,u-r}d(B^{\varepsilon})^j_u$$

which implies the claim. □

Lemma 4.2.20. *For $\mathbb{B}_{s,t}(-r)=\int_s^t B_{s-r,u-r}\otimes dB_u$ and $\mathbb{B}^{\varepsilon}_{s,t}(-r)=\int_s^t B^{\varepsilon}_{s-r,u-r}\otimes dB^{\varepsilon}_u$,*

$$\mathbb{E}\big|\mathbb{B}_{s,t}(-r)\big|^2 \le M(t-s)^2 \quad \textit{and} \quad \mathbb{E}\big|\mathbb{B}^{\varepsilon}_{s,t}(-r)\big|^2 \le M(t-s)^2 \tag{4.2.23}$$

for a constant $M>0$ independent of s,t and ε.

Proof. An easy consequence of the Cauchy-Schwarz inequality and Lemma 4.2.19. □

Lemma 4.2.21. *We have*

$$\lim_{\varepsilon\to 0}\mathbb{E}\big|\mathbb{B}_{s,t}(-r)-\mathbb{B}^{\varepsilon}_{s,t}(-r)\big|^2 = 0. \tag{4.2.24}$$

Proof. A direct consequence of Lemma 4.2.19 and (4.2.21). □

Theorem 4.2.22. *Setting*

$$\mathbf{B}^{\varepsilon}_{s,t} := \Big(B^{\varepsilon}_{s,t},\mathbb{B}^{\varepsilon}_{s,t},\mathbb{B}^{\varepsilon}_{s,t}(-r)\Big) := \left(B^{\varepsilon}_{s,t},\int_s^t B^{\varepsilon}_{s,u}\otimes dB^{\varepsilon}_u,\int_s^t B^{\varepsilon}_{s-r,u-r}\otimes dB^{\varepsilon}_u\right),$$

we have

$$\lim_{\varepsilon\to\infty}\sup_{q\geqslant 1}\frac{\left\|d_{\gamma;I}(\mathbf{B}^\varepsilon,\mathbf{B}^{Strat})\right\|_{L^q}}{\sqrt{q}}=0$$

for every $\gamma < 1/2$ *and every compact interval* $I\subset\mathbb{R}$ *where* $d_{\gamma;I}$ *denotes the homogeneous metric*

$$d_{\gamma;I}(\mathbf{X},\mathbf{Y})=\sup_{s,t\in I;s\neq t}\frac{|X_{s,t}-Y_{s,t}|}{|t-s|^\gamma}+\sqrt{\sup_{s,t\in I;s\neq t}\frac{|\mathbb{X}_{s,t}-\mathbb{Y}_{s,t}|}{|t-s|^{2\gamma}}}+\sqrt{\sup_{s,t\in I;s\neq t}\frac{|\mathbb{X}_{s,t}(-r)-\mathbb{Y}_{s,t}(-r)|}{|t-s|^{2\gamma}}}.$$

Proof. The strategy of the proof is standard, cf. [40, Chapter 15], we only sketch the main arguments. First, the uniform bounds (4.2.23) and the convergence (4.2.24) hold for $\mathbb{B}^\varepsilon$ and $\mathbb{B}^{\text{Strat}}$, too, cf. [40, Theorem 15.33 and Theorem 15.37]. Since all objects are elements in the second Wiener chaos, the results even hold in the L^q-norm for any $q\geq 1$. We can now argue as in the proof of [40, Proposition 15.24] to conclude. □

Fractional Brownian Motion

In this subsection, we state similar results for the fractional Brownian motion when $\frac{1}{3}<H<\frac{1}{2}$.

Definition 4.2.23. *For* $r>0$ *and* $\frac{1}{3}<H<\frac{1}{2}$, *set*

$$\mathbf{B}_{s,t}=(B_{s,t},\mathbb{B}_{s,t},\mathbb{B}_{s,t}(-r)):=\left(B_t-B_s,\int_s^t(B_u-B_s)\otimes d^\circ B_u,\int_s^t(B_{u-r}-B_{s-r})\otimes d^\circ B_u\right) \tag{4.2.25}$$

for $s\leq t\in\mathbb{R}$, *here the integrals are defined in the symmetric sense(cf. Definition A.3.1 for more details).*

Similar to Proposition 4.2.15, we have:

Proposition 4.2.24. *Process* $\mathbf{B}$ *have a modification with sample path being* γ*-rough path for every* $\gamma<H$ *almost surly. In addition the* γ*-Hölder norms of both processes have finite p-th moment for every* $p>0$ *on any compact interval.*

Proof. [37, Proposition 5.2] □

Similar to Lemma 4.2.19 we have:

Lemma 4.2.25. *We have the following pathwise identity:*

$$\int_s^t B^\varepsilon_{s-r_p,u-r_p}\otimes dB^\varepsilon_u=\int_{\mathbb{R}}\rho(z)\int_{s-\varepsilon z}^{t-\varepsilon z}B^\varepsilon_{s-r_p,u+\varepsilon z-r_p}\otimes d^\circ B_u dz. \tag{4.2.26}$$

Where $r_p\in\{0,r\}$.

Proof. Enough to show

$$\lim_{\delta\to 0}E\left(\int_{\mathbb{R}}\rho(z)\int_{s-\epsilon z}^{t-\epsilon z}F(u+\epsilon z)\otimes\frac{B_{u+\delta z}-B_{u-\delta z}}{2\delta}dudz-\int_s^t F(u)\otimes dB^\epsilon_u\right)^2=0.$$

Where $F(u)=B^\epsilon_{s-r_p,u-r_p}$. □

To prove similar results like Theorem 4.2.22 for the fractional Brownian motion, we need the following auxiliary lemma.

Lemma 4.2.26. *For $\tau < r$*

$$E(\int_s^t B_{u-r,u-\tau} \otimes d^\circ B_u)^2 \lesssim (r-\tau)^2(t-s)^2 + \sum_{4\leqslant i\leqslant 7} (t-s)^{\frac{i}{2}H}(r-\tau)^{(4-\frac{i}{2})H}. \tag{4.2.27}$$

Proof. For $\Phi_u := B^i_{u-r,u-\tau}\chi_{[s,t]}(u)$, by Proposition A.3.2

$$\int_0^t \Phi_u d^\circ B^j_u = \delta^{B^j}_{[s,t]}(\Phi) + Tr_{[s,t]}(D^{B^j}\Phi). \tag{4.2.28}$$

It can be shown $Tr_{[s,t]}D^{B^j}\Phi = H\delta_{i=j}\big[sgn(r)|r|^{2H-1} - sgn(\tau)|\tau|^{2H-1}\big](t-s)$, also

$$D^{B^j}_u(\Phi_x) = \delta_{i=j}\ \chi_{[s,t]}(x)\ \chi_{[x-\tau,x-r]}(u).$$

Consequently, for $t-s \leqslant r-\tau$

$$\begin{aligned}
&\delta^{B^j}_{[s,t]}(D^{B^j}_u\Phi_.) = \\
&\delta_{i=j}\ \delta^{B^j}_{[s,t]}\big(\chi_{[s-\tau,t-\tau]}(u)\chi_{[u+\tau,t]}(.) + \chi_{[t-r,s-\tau]}(u)\chi_{[s,t]}(.) + \chi_{[s-r,t-r]}(u)\chi_{[s,u+r]}(.)\big) = \\
&\delta_{i=j}\big[\chi_{[s-\tau,t-\tau]}(u)B_{u+\tau,t} + \chi_{[t-r,s-\tau]}(u)B_{s,t} + \chi_{[s-r,t-r]}(u)B_{s,u+r}\big].
\end{aligned} \tag{4.2.29}$$

For $r-\tau \leqslant t-s$

$$\begin{aligned}
&\delta^{B^j}_{[s,t]}(D^{B^j}_u\Phi_.) = \\
&\delta_{i=j}\delta^{B^j}_{[s,t]}\big(\chi_{[t-r,t-\tau]}(u)\chi_{[u+\tau,t]}(.) + \chi_{[s-r,t-r]}(u)\chi_{[s,\sigma+r]}(.) - \chi_{[s-\tau,t-r]}(u)\chi_{[s,u+\tau]}(.)\big) = \\
&\delta_{i=j}\big[\chi_{[t-r,t-\tau]}(u)B_{u+\tau,t} + \chi_{[s-r,t-r]}(u)B_{s,u+r} - \chi_{[s-\tau,t-r]}(u)B_{s,u+\tau}\big].
\end{aligned} \tag{4.2.30}$$

In addition, from (A.3.2) and (A.3.3)

$$\begin{aligned}
E[(\delta^{B^j}_{[s,t]}(\Phi))^2] &= E(\|\Phi\|^2_{\mathcal{H}_{[s,t]}}) + E(< \delta^{B^j}_{[s,t]}(D^{B^j}\Phi), \Phi >_{\mathcal{H}_{[s,t]}}) \\
&\lesssim E(\|\Phi\|^2_{\mathcal{H}_{[s,t]}}) + E(\|\delta^{B^j}_{[s,t]}(D^{B^j}\Phi)\|^2_{\mathcal{H}_{[s,t]}})^{\frac{1}{2}}E(\|\Phi\|^2_{\mathcal{H}_{[s,t]}})^{\frac{1}{2}}.
\end{aligned} \tag{4.2.31}$$

By (A.3.1)

$$\begin{aligned}
&E(\|\Phi\|^2_{\mathcal{H}_{[s,t]}}) \lesssim \int_s^t (r-\tau)^{2H}(t-s)^{2H-1}du + \int_{-\infty}^s (\int_s^t \frac{(r-\tau)^H}{(x-u)^{3/2-H}}dx)^2du + \\
&\int_s^t \Big(\int_u^t \frac{(E(|\Phi_\sigma - \Phi_u|^2))^{\frac{1}{2}}}{(\sigma-u)^{\frac{3}{2}-H}}d\sigma\Big)^2 du \lesssim (r-\tau)^{2H}(t-s)^{2H} + (r-\tau)^H(t-s)^{3H}.
\end{aligned} \tag{4.2.32}$$

Note that, we used the following inequality

$$0 \leqslant 2x^{2H} + 2y^{2H} - |x+y|^{2H} - |x-y|^{2H} \leqslant 4x^H y^H, \qquad x,y \geqslant 0.$$

Now (4.2.27) can be deduced from (4.2.28), (4.2.29), (4.2.30), (4.2.31) and (4.2.32). $\square$

Finally we have:

Theorem 4.2.27. *For*

$$\mathbf{B}^{\varepsilon}_{s,t} := \Big(B^{\varepsilon}_{s,t}, \mathbb{B}^{\varepsilon}_{s,t}, \mathbb{B}^{\varepsilon}_{s,t}(-r)\Big) := \Big(B^{\varepsilon}_{s,t}, \int_s^t B^{\varepsilon}_{s,u} \otimes dB^{\varepsilon}_u, \int_s^t B^{\varepsilon}_{s-r,u-r} \otimes dB^{\varepsilon}_u\Big),$$

we have

$$\lim_{\varepsilon\to\infty} \sup_{q\geqslant 1} \frac{\left\| d_{\gamma;I}(\mathbf{B}^{\epsilon}, \mathbf{B})\right\|_{L^q}}{\sqrt{q}} = 0$$

for every $\gamma < H$ and every compact interval I.

Proof. We claim

$$E\Big(\int_s^t B^{\epsilon}_{s-r_p,u-r_p} \otimes dB^{\epsilon}_u - \int_s^t B_{s-r_p,u-r_p} d^{\circ} B_u\Big)^2 \lesssim \sum_{4\leqslant l\leqslant 7} (t-s)^{\frac{l}{2}H} \epsilon^{(4-\frac{l}{2})H}. \tag{4.2.33}$$

From Lemma (4.2.19)

$$\begin{aligned}
&E\Big(\int_s^t B^{\epsilon}_{s-r_p,u-r_p} \otimes dB^{\epsilon}_u - \int_s^t B_{s-r_p,u-r_p} \otimes d^{\circ} B_u\Big)^2 \lesssim \\
&E\Big(\int_{\mathbb{R}} \rho(z)\big[\int_{s-\epsilon z}^{t-\epsilon z} (B^{\epsilon}_{s-r_p,u+\epsilon z-r_p} - B_{s-r_p,u+\epsilon z-r_p}) \otimes d^{\circ}\ B_u\big] dz\Big)^2 + \\
&E\Big(\int_{\mathbb{R}} \rho(z)\big[\int_{s-\epsilon z}^{t-\epsilon z} B_{s-r_p,u+\epsilon z-r_p} \otimes d^{\circ} B_u - \int_s^t B_{s-r_p,u-r_p} \otimes d^{\circ} B_u\big] dz\Big)^2 = I + II.
\end{aligned}$$

It is not hard to verify the following integration by parts property

$$\begin{aligned}
&\int_{s-\frac{1}{n}z}^{t-\frac{1}{n}z} B^i_{s-r_p,u+\frac{1}{n}z-r_p} d^{\circ} B^j_u - \int_s^t B^i_{s-r_p,u-r_p} d^{\circ} B^j_u = \\
&\qquad - B^i_{s-r_p,t-r_p} B^j_{t-\frac{1}{n}z,t} + \int_{s-r_p}^{t-r_p} B^j_{u+r_p-\frac{1}{n}z,u+r_p} d^{\circ} B^i_u.
\end{aligned}$$

By lemma 4.2.26 :

$$\begin{aligned}
II &\lesssim \int_{\mathbb{R}} \phi(z) E\Big(- B^i_{s-r_p,t-r_p} B^j_{t-\frac{1}{n}z,t} + \int_{s-r_p}^{t-r_p} B^j_{u+r-\frac{1}{n}z,u+r_p} d^{\circ} B^i_u\Big)^2 dz \\
&\lesssim \sum_{4\leqslant l\leqslant 7} (t-s)^{\frac{l}{2}H} (\frac{1}{n})^{(4-\frac{l}{2})H}.
\end{aligned} \tag{4.2.34}$$

Also

$$I \lesssim \int_{\mathbb{R}} \phi(z) E\Big(\int_{s-\frac{1}{n}z}^{t-\frac{1}{n}z} ((B^i)^n_{s-r_p,u+\frac{1}{n}z-r_p} - B^i_{s-r_p,u+\frac{1}{n}z-r_p}) d^{\circ} B^j_u\Big)^2 dz.$$

Set $s_1 = s - \frac{1}{n}z$ and $t_1 = t - \frac{1}{n}z$ then :

$$
\begin{aligned}
&E\Big(\int_{s_1}^{t_1} [(B^i)^n_{s_1,u+\frac{1}{n}z-r_p} - B^i_{s_1,u+\frac{1}{n}z-r_p}] d^\circ B^j_u\Big)^2 \\
&\lesssim \int_{\mathbb{R}} \phi(y) E\Big(\int_{s_1}^{t_1} (B^i_{s_1-\frac{1}{n}y,u+\frac{1}{n}z-r_p-\frac{1}{n}y} - B^i_{s_1,u+\frac{1}{n}z-r_p}) d^\circ B^j_u\Big)^2 dy \lesssim \sum_{4\leqslant l\leqslant 7} (t-s)^{\frac{l}{2}H} (\frac{1}{n})^{(4-\frac{l}{2})H},
\end{aligned}
$$

consequently

$$
I \lesssim \sum_{4\leqslant l\leqslant 7} (t-s)^{\frac{l}{2}H} (\frac{1}{n})^{(4-\frac{l}{2})H}. \tag{4.2.35}
$$

Now (4.2.34) and (4.2.35) yield (4.2.33).

Following with $\sup_n E\big(\int_s^t (B^i)^n_{s-r,u-r} d(B^j)^n_u)\big)^2, E\big(\int_s^t B^i_{s-r,u-r} d^\circ B^j_u\big)^2 \lesssim (t-s)^{4H}$, interpolation technique and second chaos property, for every $\gamma < H$ and every compact interval I

$$
\lim_{\varepsilon\to\infty} \sup_{q\geqslant 1} \frac{\|d_{\gamma;I}(\mathbf{B}^\epsilon, \mathbf{B})\|_{L^q}}{\sqrt{q}} = 0.
$$

Cf. [40, Chapter 15] for more details. $\square$

As an application, we can prove a Wong-Zakai theorem for stochastic delay equations.

Theorem 4.2.28. *Let $\sigma \in C_b^3(W^2, L(U,W))$ and B^ε be defined as above. Assume that there is a set of full measure $\tilde{\Omega} \subset \Omega$ such that*

$$
(\xi(\omega), \xi'(\omega)) \in \mathscr{D}_{B^\varepsilon(\omega)}([-r,0], W) \cap \mathscr{D}_{B(\omega)}([-r,0], W) \tag{4.2.36}
$$

holds for every $\varepsilon \in (0,1]$ and every $\omega \in \tilde{\Omega}$. Then the solutions to random delay ordinary differential equations

$$
\begin{aligned}
dY_t^\varepsilon &= \sigma(Y_t^\varepsilon, Y_{t-r}^\varepsilon)\, dB_t^\varepsilon; \quad t \geq 0 \\
Y_t^\varepsilon &= \xi_t; \qquad t \in [-r, 0]
\end{aligned}
$$

converge in probability as $\varepsilon \to 0$ in γ-Hölder norm on compact sets for every $\gamma < H$ to the solution Y of

$$
\begin{aligned}
dY_t &= \sigma(Y_t, Y_{t-r})\, d\mathbf{B}_t; \quad t \geq 0 \\
Y_t &= \xi_t; \qquad t \in [-r, 0].
\end{aligned}
$$

Moreover when $H = \frac{1}{2}$, if (ξ_t, ξ_t') is $\mathcal{F}^0_{-1}$-measurable for every $t \in [-r, 0]$, the solution Y coincides almost surely with the solution of the Stratonovich delay equation

$$
\begin{aligned}
dY_t &= \sigma(Y_t, Y_{t-r}) \circ dB_t; \quad t \geq 0 \\
Y_t &= \xi_t; \qquad t \in [-r, 0].
\end{aligned}
$$

Proof. A combination of the stability result in Theorem 4.2.9, Theorem 4.2.22 and Theorem 4.2.27. $\square$

Remark 4.2.29. *Note that* (4.2.36) *is satisfied, for instance, if ξ has almost surely differentiable sample paths, in which case we can choose $\xi' \equiv 0$.*

4.3 Random Dynamical Systems induced by stochastic delay equations

This section establishes the connection between stochastic delay equations and Arnold's concept of a random dynamical system.

Delayed rough path cocycles

We start by describing the object which will drive our equation. The following definition is an analogue of a *rough paths cocycle* defined in [41] for delay equations.

Definition 4.3.1. *Let $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t\in\mathbb{R}})$ be a measurable metric dynamical system and $r > 0$. A* delayed γ-rough path cocycle $\mathbf{X}$ (with delay $r > 0$) *is a delayed γ-rough path valued stochastic process $\mathbf{X}(\omega) = (X(\omega), \mathbb{X}(\omega), \mathbb{X}(-r)(\omega))$ such that*

$$\mathbf{X}_{s,s+t}(\omega) = \mathbf{X}_{0,t}(\theta_s \omega) \tag{4.3.1}$$

holds for every $\omega \in \Omega$ and every $s, t \in \mathbb{R}$.

Our goal is to prove that Brownian motion together with Lévy- and delayed Lévy area can be understood as delayed rough path cocycles.

Definition 4.3.2. *For a finite-dimensional vector space U, set*

$$\tilde{T}^2(U) := \big\{ (1 \oplus (\alpha, \beta) \oplus (\gamma, \theta)) \mid \alpha, \beta \in U \text{ and } \gamma, \theta \in U \otimes U \big\}.$$

We define projections Π_i^j by

$$\Pi_i^j(1 \oplus (\alpha, \beta) \oplus (\gamma, \theta)) := \begin{cases} \alpha & \text{if } i = 1,\ j = 1 \\ \beta & \text{if } i = 1,\ j = 2 \\ \gamma & \text{if } i = 2,\ j = 1 \\ \theta & \text{if } i = 2,\ j = 2. \end{cases}$$

Furthermore, we set

$$\begin{aligned}(1 \oplus (\alpha_1, \beta_1) \oplus (\gamma_1, \theta_1)) \circledast (1 \oplus (\alpha_2, \beta_2) \oplus (\gamma_2, \theta_2)) :=& \\ (1 \oplus (\alpha_1 + \alpha_2, \beta_1 + \beta_2) \oplus (\gamma_1 + \gamma_2 + \alpha_1 \otimes \alpha_2, \theta_1 + \theta_2 + \beta_1 \otimes \alpha_2))&\end{aligned}$$

and $\mathbf{1} := (1, (0, 0), (0, 0))$.

It is not hard to verify that $(\tilde{T}^2(U), \circledast)$ is a topological group with identity $\mathbf{1}$. For a continuous U-valued path of bounded variation x, we can define the following natural lifting map

$$\tilde{S}_2(x)_{u,v} := \Big(1 \oplus (x_{u,v}, x_{u-r,v-r}) \oplus \big(\int_u^v x_{u,\tau} \otimes dx_\tau, \int_u^v x_{u-r,\tau-r} \otimes dx_\tau \big)\Big) \in \tilde{T}^2(U).$$

Definition 4.3.3. *Assume $I \subset \mathbb{R}$ and $0 \in I$. We define $C_0^{0,1-var}(I,U)$ as the closure of the set of arbitrarily often differentiable paths x from I to U with $x_0 = 0$ with respect to the 1-variation norm. Furthermore, $C_0^{0,p-var}(I, \tilde{T}^2(U))$ is defined as the set of continuous maps $\mathbf{x}\colon I \to \tilde{T}^2(U)$ such that $\mathbf{x}_0 = \mathbf{1}$ and for which there exists a sequence $x_n \in C_0^{0,1-var}(I,U)$ with*

$$d_{p-var}(\mathbf{x}, \tilde{S}_2(x_n)) := \sup_{i,j\in\{1,2\}} \Big(\sup_{\mathcal{P}\subset I} \sum_{t_k\in\mathcal{P}} |\Pi_i^j(\mathbf{x}_{t_k,t_{k+1}} - \tilde{S}_2(x_n)_{t_k,t_{k+1}})|^{\frac{p}{i}} \Big)^{\frac{1}{p}} \longrightarrow 0$$

as $n \to \infty$. We use the notation $\mathbf{x}_{s,t} := \mathbf{x}_s^{-1} \circledast \mathbf{x}_t$ here. The space $C_0^{0,p-var}(\mathbb{R}, \tilde{T}^2(U))$ consists of all continuous paths $\mathbf{x}\colon \mathbb{R} \to \tilde{T}^2(U)$ for which $\mathbf{x}|_I \in C_0^{0,p-var}(I, \tilde{T}^2(U))$ for every I as above.

We can now state the following results:

Theorem 4.3.4. *Let $p \geqslant 1$ and let $\overline{\mathbf{X}}$ be an $C_0^{0,p-var}(\mathbb{R}, \tilde{T}^2(U))$-valued random variable on a probability space $(\overline{\Omega}, \overline{\mathcal{F}}, \overline{\mathbb{P}})$. Assume that $\mathbf{X}$ has stationary increments, i.e. the law of the process $(\overline{\mathbf{X}}_{t_0,t_0+h})_{h\in\mathbb{R}}$ does not depend on $t_0 \in \mathbb{R}$. Then we can define a metric dynamical system $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$ and a $C_0^{0,p-var}(\mathbb{R}, \tilde{T}^2(U))$-valued random variable $\mathbf{X}$ on Ω with the same law as $\overline{\mathbf{X}}$ which satisfies the cocycle property* (4.3.1)*.*

Proof. The proof in all lines is similar to Theorem 5 in [41] by setting $\Omega = C_0^{0,p-var}(\mathbb{R}, \tilde{T}^2(U))$, $\mathcal{F}$ being the Borel σ-algebra, $\mathbb{P}$ the law of $\bar{\mathbb{X}}$ and for $\omega \in \Omega$, we define

$$(\theta_s\omega)(t) := \omega(s)^{-1} \circledast \omega(t+s) \ , \qquad \mathbf{X}_t(\omega) = \omega(t).$$

□

Remark 4.3.5. *Note that the cocycle property* (4.3.1) *is equivalent to $\mathbf{X}_t(\theta_s(\omega)) = \mathbf{X}_s^{-1}(\omega) \circledast \mathbf{X}_{t+s}(\omega)$ for every $s,t \in \mathbb{R}$ and every $\omega \in \Omega$.*

We will also ask for ergodicity of rough cocycles. The following lemma will be useful.

Lemma 4.3.6. *Let $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t\in\mathbb{R}})$ and $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}, (\tilde{\theta}_t)_{t\in\mathbb{R}})$ be two measurable metric dynamical systems and let $\Phi\colon \Omega \to \tilde{\Omega}$ be a measurable map such that $\tilde{\mathbb{P}} = \mathbb{P} \circ \Phi^{-1}$. Assume that for every $t \in \mathbb{R}$, there is a set of full $\mathbb{P}$-measure $\Omega_t \subset \Omega$ on which $\Phi \circ \theta_t = \tilde{\theta}_t \circ \Phi$ holds. Then, if $\mathbb{P}$ is ergodic, $\tilde{\mathbb{P}}$ is ergodic, too.*

Proof. The reader will have no difficulties to check that the assertion is just a slight generalization of [42, Lemma 3]. □

Theorem 4.3.7. *Consider the processes $\mathbf{B}$ and $\mathbf{B}^{It\bar{o}}$ (when $H = \frac{1}{2}$) defined in Section 4.2. Then for each process, we can find an ergodic metric dynamical system $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$ on which we can define a new process with the same law, satisfying the cocycle property* (4.3.1)*, i.e. both processes are delayed γ-rough path cocycles for every $\gamma \in (1/3, H)$.*

Proof. We will first consider $\mathbf{B}$. From the approximation result in Theorem 4.2.22 and Theorem 4.2.27, we see that $\mathbf{B}$ takes values in $C_0^{0,p-var}(\mathbb{R}, \tilde{T}^2(\mathbb{R}^d))$ for every $p \in (\frac{1}{H}, 3)$. It is easy to check that the process has stationary increments, therefore we can apply Theorem 4.3.4. It remains to show ergodicity. By construction, $\Omega = C_0^{0,p-var}(\mathbb{R}, \tilde{T}^2(\mathbb{R}^d))$, $\mathcal{F}$ is the Borel σ-algebra and $\mathbb{P} = \hat{\mathbb{P}} \circ S^{-1}$ where $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{P}}, \hat{\theta})$ is the measurable metric dynamical system given by $\hat{\Omega} = C_0^0(\mathbb{R}, \mathbb{R}^d)$, $\hat{\mathcal{F}}$ the corresponding Borel σ-algebra, $\hat{\mathbb{P}}$ the Wiener measure and $\hat{\theta} = (\hat{\theta}_t)_{t \in \mathbb{R}}$ the Wiener shift. The map $S \colon \hat{\Omega} \to \Omega$ is defined as follows: For $x \in \hat{\Omega}$, set

$$S(x) = \Big(1 \oplus (x_{s,t}, x_{s-r,t-r}) \oplus \big(\int_s^t x_{s,\tau} \otimes dx_\tau, \int_s^t x_{s-r,\tau-r} \otimes dx_\tau\big)\Big)_{s \leq t}$$

if the integrals exist as limits of Riemann sums, in Statonovich sense, on compact sets for the sequence of partitions given by $\Pi_n = \{k/2^n \,:\, k \in \mathbb{Z}\}$ as $n \to \infty$, and $S(x) = (1, 0, 0)$ otherwise. It is not hard to see that there is a set of full $\hat{\mathbb{P}}$-measure on which the limits do exist. It follows that for every $t \in \mathbb{R}$, there is a set of full measure $\hat{\Omega}_t$ such that for every $x \in \hat{\Omega}_t$,

$$S(\hat{\theta}_t x) = \theta(S(x)).$$

Since $\hat{\mathbb{P}}$ is ergodic, ergodicity of $\mathbb{P}$ follows by Lemma 4.3.6 which completes the proof .

When $H = \frac{1}{2}$, for the Itō-case, we can argue analogously: First, we define a map

$$\hat{S}_2(x)_{s,t} := \Big(1 \oplus (x_{s,t}, x_{s-r,t-r}) \oplus \big(\int_s^t x_{s,\tau} \otimes dx_\tau - \frac{1}{2}(t-s)I_d, \int_s^t x_{s-r,\tau-r} \otimes dx_\tau\big)\Big) \in \tilde{T}^2(U)$$

for smooth paths and a corresponding (separable!) space $\hat{C}_0^{0,p-var}(\mathbb{R}, \tilde{T}^2(\mathbb{R}^d))$ in which, using again the approximation result for the Stratonovich lift, the random variable $\mathbf{B}^{\text{Itō}}$ takes its values. Then a version of [41, Theorem 5] applies and shows the claim. Ergodicity is proven analogously . $\square$

Cocycle property of the solution map

Let $I \subset \mathbb{R}$ be a compact interval and $X \colon I \to U$ a γ-Hölder continuous path. It is easy to see that for $\alpha \leq \beta \leq \gamma$,

$$\begin{aligned} i_{\alpha,\beta} : \mathscr{D}_X^\beta(I, W) &\to \mathscr{D}_X^\alpha(I, W), \\ (\xi, \xi') &\mapsto (\xi, \xi') \end{aligned}$$

is a continuous embedding. We make the following definition:

Definition 4.3.8. *We define $\mathscr{D}_X^{\alpha,\beta}(I, W)$ as the closure of $\mathscr{D}_X^\beta(I, W)$ in the space $\mathscr{D}_X^\alpha(I, W)$.*

The reason why we introduce these spaces is their separability, which we will prove in the next lemma.

Lemma 4.3.9. *For all $\alpha < \beta$, the spaces $\mathscr{D}_X^{\alpha,\beta}(I, W)$ are separable.*

Proof. The space $\mathscr{D}_X^{\alpha,\beta}(I, W))$ can be viewed as a subset of

$$C^{\alpha,\beta}(I, W) \times C^{\alpha,\beta}(I, L(U, W)) \times C^{2\alpha,2\beta}(I, W)$$

where $C^{\alpha,\beta}$ again means taking the closure of β-Hölder functions in the α-Hölder norm. Since all spaces above are separable, the result follows.

$\square$

For $\alpha < \beta \leqslant \gamma$, we can find a very explicit dense subset. This is the content of the next theorem, which has far reaching consequences, as we will see.

Theorem 4.3.10. *Let $\alpha < \beta \leqslant \gamma \leqslant H$. Then the set*

$$\left\{(\psi,\psi') \mid \psi_{s,t} = \int_s^t f(\tau)\,dX_\tau + R_{s,t},\ \psi'_s = f(s) \text{ where } f \in C^\infty(I, L(U,W)) \text{ and } R \in C^\infty(I,W)\right\}$$

is dense in $\mathscr{D}_X^{\alpha,\beta}(I,W)$, the integral being understood as a Young-integral here. In particular, $\mathscr{D}_X^{\alpha,\beta}(I,W)$ does not depend on α.

Proof. Take $(\xi,\xi') \in \mathscr{D}_X^{\alpha,\beta}(I,W)$, i.e. $\xi_{s,t} = \xi'_s X_{s,t} + \xi^{\#}_{s,t}$. Since $\mathscr{D}_X^{\alpha,\beta}(I,W)$ is the closure of the space $\mathscr{D}_X^{\beta}(I,W)$ in $\mathscr{D}_X^{\alpha}(I,W)$, it suffices to prove that set is dense $\mathscr{D}_X^{\beta}(I,W)$. Thus we assume without loss of generality that $\|\xi'\|_{\beta;I}, \|\xi^{\#}\|_{2\beta;I} < \infty$. For a partition $\Pi^n = \{t_i\}_{1\leqslant i\leqslant n}$ with $\Delta t_j = \theta$, we define a function $\bar{\xi}' : I \to L(U,W)$ by setting

$$\bar{\xi}'_\tau := \xi'_{t_i} + \frac{(\tau - t_i)}{\Delta t_i}\xi'_{t_i,t_{i+1}}, \qquad t_i \leqslant \tau < t_{i+1}.$$

Our goal is to find a function $\tilde{R}$ with $\tilde{R}_0 = 0$ and such that for

$$\bar{\xi}_{s,t} := \int_s^t \bar{\xi}'_\tau\,dX_\tau + \tilde{R}_{s,t}, \quad \bar{\xi}_a = \xi_a, \tag{4.3.2}$$

we have $\|(\xi,\xi') - (\bar{\xi},\bar{\xi}')\|_{\mathscr{D}_X^{\alpha}(I,W))} < \varepsilon$ for any given $\varepsilon > 0$ when choosing θ small enough, i.e.

$$\|\xi' - \bar{\xi}'\|_\alpha \to 0 \quad \text{and} \quad \|\xi^{\#} - \bar{\xi}^{\#}\|_{2\alpha} \to 0$$

as $\theta \to 0$ where $\bar{\xi}^{\#} := \bar{\xi}_{s,t} - \bar{\xi}' X_{s,t}$. Set $\eta_{s,t} = \bar{\xi}'_{s,t} - \xi'_{s,t}$ and note that by construction the map vanishes on the subdivision, i.e. $\eta_{t_i,t_i+1} = 0$ for all $1 \le i \le n-1$. Inserting subdivision points, we obtain the (global) estimate,

$$\|\eta\|_\alpha \leqslant 4\theta^{\beta-\alpha}\|\xi'\|_\beta. \tag{4.3.3}$$

First note that, by our construction, at the mesh points, $\bar{\xi}$ and $\bar{\xi}$ coincide, now for arbitrary s,t which not belong to same interval, we can assume $t_k \leqslant s \leqslant t_{k+1} \leqslant \ldots \leqslant t_j \leqslant t \leqslant t_{j+1}$ since η is an increment path

$$\begin{aligned} n_{s,t} &= (\bar{\xi}'_{s,t_{k+1}} - \xi'_{s,t_{k+1}}) + (\bar{\xi}'_{t_{k+1},t_j} - \xi'_{t_{k+1},t_j}) + (\bar{\xi}'_{t_j,t} - \xi'_{t_j,t}) \\ &= \frac{t_{k+1}-s}{t_{k+1}-t_k}\xi'_{t_k,t_{k+1}} - \xi'_{s,t_{k+1}} + \frac{t-t_j}{t_{j+1}-t_j}\xi'_{t_j,t_{j+1}} - \xi'_{t_j,t} \end{aligned}$$

Consequently

$$\frac{|\eta_{s,t}|}{(t-s)^\alpha} \leqslant \frac{t_{k+1}-s}{(t-s)^\alpha}\frac{|\xi'_{t_k,t_{k+1}}|}{(t_{k+1}-t_k)} + \frac{|\xi'_{s,t_{k+1}}|}{(t-s)^\alpha} + \frac{t-t_j}{(t-s)^\alpha}\frac{|\xi'_{t_j,t_{j+1}}|}{t_{j+1}-t_j} + \frac{|\xi'_{t_j,t}|}{(t-s)^\alpha}$$

Note that $t_{k+1} - s \leqslant t - s$ and $t - t_j \leqslant t - s$. Now, (4.3.3), immediately follows, note that here the main task is to get ride of the $\bar{\xi}'_{t_{k+1},t_j} - \xi'_{t_{k+1},j}$, then the remaining terms can be controlled.

Define $\rho_{s,t} := \int_s^t \bar{\xi}'_\tau \, dX_\tau - \bar{\xi}'_s X_{s,t}$. For $s, t \in [t_i, t_{i+1}]$, we can use integration by parts to see that

$$\rho_{s,t} = \frac{\xi'_{t_i,t_{i+1}}}{\Delta t_i} \int_s^t (\tau - s) \; dX_\tau = \frac{t-s}{\Delta t_i} \xi'_{t_i,t_{i+1}} X_t - \frac{\xi'_{t_i,t_{i+1}}}{\Delta t_j} \int_s^t X_\tau \; d\tau = \frac{\xi'_{t_i,t_{i+1}}}{\Delta t_i} \int_s^t X_{\tau,t} \; d\tau.$$

Consequently, for $I_i = [t_i, t_{i+1}]$,

$$\|\rho\|_{2\alpha;I_i} \leqslant \frac{\|\xi'\|_\beta \; \|X\|_\gamma}{(\gamma+1)} \; \theta^{\gamma+\beta-2\alpha}. \tag{4.3.4}$$

Note that we can deduce the same results without integration by parts, using the estimate

$$\left| \int_s^t (\tau - s) \; dX_\tau \right| \leq C_\gamma |t-s|^{1+\gamma} \|X\|_\gamma$$

for the Young integral instead. Next,

$$\begin{aligned} \rho_{t_k,t_j} &= \sum_{k \leqslant i < j} \left[\int_{t_i}^{t_{i+1}} \bar{\xi}'_{t_i,\tau} dX_\tau + \bar{\xi}'_{t_k,t_i} X_{t_i,t_{i+1}} \right] \\ &= \sum_{k \leqslant i < j} \Big[\rho_{t_i,t_{i+1}} - \xi^{\#}_{t_i,t_{i+1}} + \xi_{t_i,t_{i+1}} - \xi'_k X_{t_i,t_{i+1}} \Big] \\ &= \sum_{k \leqslant i < j} \Big[\rho_{t_i,t_{i+1}} - \xi^{\#}_{t_i,t_{i+1}} \Big] + \xi^{\#}_{t_k,t_j}. \end{aligned} \tag{4.3.5}$$

Set $\tilde{\rho}_{s,t} := \xi^{\#}_{s,t} - \rho_{s,t}$. (4.3.4) implies that

$$\|\tilde{\rho}\|_{2\alpha;I_i} \leqslant \|\xi^{\#}\|_{2\beta} \theta^{2(\beta-\alpha)} + \frac{\|\xi'\|_\beta \; \|X\|_\gamma}{(\gamma+1)} \; \theta^{\gamma+\beta-2\alpha} \tag{4.3.6}$$

and from (4.3.5),

$$\tilde{\rho}_{t_k,t_j} = \sum_{k \leqslant i < j} \Big[\xi^{\#}_{t_i,t_{i+1}} - \rho_{t_i,t_{i+1}} \Big]. \tag{4.3.7}$$

It is easy to verify that

$$\tilde{\rho}_{s,t} = \tilde{\rho}_{s,u} + \tilde{\rho}_{u,t} + \eta_{s,u} X_{u,t}. \tag{4.3.8}$$

Let $\tilde{R}$ be the piecewise linear function defined by

$$\tilde{R}_{s,t} := \frac{t-s}{\Delta t_i} \left(\xi^{\#}_{t_i,t_{i+1}} - \rho_{t_i,t_{i+1}} \right), \qquad s, t \in [t_i, t_{i+1}]$$

on I_i.

For $s,t \in I$ with $t_k \leqslant s \leqslant t_{k+1} \leqslant ... \leqslant t_j \leqslant t \leqslant t_{j+1}$, we have

$$\begin{aligned}\tilde{R}_{s,t} &= \tilde{R}_{s,t_{k+1}} + \tilde{R}_{t_{k+1},t_{k+2}} + ... + \tilde{R}_{t_j,t} = \tilde{R}_{s,t_{k+1}} + \tilde{R}_{t_j,t} + \tilde{\rho}_{t_{k+1},t_j}, \\ \rho_{s,t} &= \rho_{s,t_{k+1}} + \rho_{t_{k+1},t_j} + \rho_{t_j,t} + \bar{\xi}'_{s,t_{k+1}} X_{t_{k+1},t_j} + \bar{\xi}'_{s,t_j} X_{t_j,t}.\end{aligned} \tag{4.3.9}$$

Note that

$$\|\tilde{R}\|_{2\alpha;I_i} \leqslant \|\xi^{\#}\|_{2\beta}\theta^{2(\beta-\alpha)} + \frac{\|\xi'\|_{\beta}\|X\|_{\gamma}\theta^{\gamma+\beta-2\alpha}}{\gamma+1}. \tag{4.3.10}$$

Also,

$$\tilde{\rho}_{s,t} = \tilde{\rho}_{s,t_{k+1}} + \tilde{\rho}_{t_{k+1},t_j} + \tilde{\rho}_{t_j,t} + \eta_{s,t_{k+1}} X_{t_{k+1},t} + \eta_{t_{k+1},t_j} X_{t_j,t}. \tag{4.3.11}$$

So from (4.3.9) and (4.3.11),

$$\bar{\xi}^{\#}_{s,t} - \xi^{\#}_{s,t} = \tilde{R}_{s,t} - \tilde{\rho}_{s,t} = \tilde{R}_{s,t_{k+1}} + \tilde{R}_{t_j,t} - \eta_{s,t_{k+1}} X_{t_{k+1},t} - \eta_{t_{k+1},t_j} X_{t_j,t} - \tilde{\rho}_{s,t_{k+1}} - \tilde{\rho}_{t_j,t}.$$

From (4.3.3), (4.3.6) and (4.3.10), we deduce the following:

$$\frac{|\bar{\xi}^{\#}_{s,t} - \xi^{\#}_{s,t}|}{(t-s)^{2\alpha}} \leqslant 2\|\xi^{\#}\|_{2\beta}\theta^{2(\beta-\alpha)} + \frac{4\|\xi'\|_{\beta}\|X\|_{\gamma}\theta^{\gamma+\beta-2\alpha}}{\gamma+1} + 8(b-a)^{\gamma-\alpha}\|X\|_{\gamma}\|\xi'\|_{\beta}\theta^{\beta-\alpha}. \tag{4.3.12}$$

We can now pass to the supremum over all $s < t$ on the left hand side and send $\theta \to 0$ which proves the claim. ☐

Theorem 4.3.11. *Let* $\mathbf{X}$ *be a delayed* γ*-rough path cocycle for some* $\gamma \in (1/3, 1/2]$. *Under the assumptions of Theorem 4.2.13, the map*

$$\varphi(n,\omega,\cdot) := \phi(0, nr, \omega, \cdot) \tag{4.3.13}$$

is a continuous map

$$\varphi(n,\omega,\cdot)\colon \mathscr{D}^{\beta}_{X(\omega)}([-r,0],W) \to \mathscr{D}^{\beta}_{X(\theta_{nr}\omega)}([-r,0],W)$$

and the cocycle property

$$\varphi(n+m,\omega,\cdot) = \varphi(n,\theta_{mr}\omega,\cdot) \circ \varphi(m,\omega,\cdot) \tag{4.3.14}$$

holds for every $s,t \in [0,\infty)$. *If* σ *is linear, the cocycle is compact linear. Furthermore, all assertions remain true if we replace the spaces* $\mathscr{D}^{\beta}$ *by* $\mathscr{D}^{\alpha,\beta}$ *for* $1/3 < \alpha < \beta < \gamma$.

Proof. Note that $\mathscr{D}^{\beta}_{X(\omega)}([-r+nr,nr],W) \cong \mathscr{D}^{\beta}_{X(\theta_{nr}\omega)}([-r,0],W)$ by the natural linear map

$$\begin{aligned}\Psi\colon \mathscr{D}^{\beta}_{X(\omega)}([-r+nr,nr],V) &\longrightarrow \mathscr{D}^{\beta}_{X(\theta_{nr}\omega)}([-r,0],V) \\ (\xi_{\tau})_{-r+nr\leqslant\tau\leqslant nr} &\mapsto (\tilde{\xi}_{\tau} = \xi_{\tau+nr})_{-r\leqslant\tau\leqslant 0}.\end{aligned}$$

Continuity of φ is a consequence of Theorem 4.2.13. Regarding the cocycle property, by the semi-flow property (4.2.19) it is enough to show that

$$\phi(0, nr, \theta_{mr}\omega, \cdot) = \phi(mr, (m+n)r, \omega, \cdot).$$

Using again the semi-flow property (4.2.19), it is enough to show the equality for $n = 1$ only. Finally, by the definition of the integral in (4.2.3) and the cocycle property of a rough cocycle, this can easily be verified. The statements about linearity and compactness are a consequence of 4.2.11 and Proposition 4.2.12. The claim that all spaces $\mathscr{D}^{\beta}$ can be replaced by $\mathscr{D}^{\alpha,\beta}$ follows from the invariance

$$\varphi(n, \omega, \mathscr{D}^{\alpha,\beta}_{X(\omega)}([-r,0], W)) \subset \mathscr{D}^{\alpha,\beta}_{X(\theta_{nr}\omega)}([-r,0], W) \tag{4.3.15}$$

which is a consequence of the continuity of φ. □

Note that so far, we worked with delayed rough path cocycles $\mathbf{X}$ which are defined on a continuous-time metric dynamical system $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t\in\mathbb{R}})$. In Theorem 4.3.11, we saw that stochastic delay equations a priori induce discrete-time RDS only. The reason is that we cannot expect that the semi-flow property (4.2.16) holds in full generality for all times, cf. Theorem 4.2.13. Therefore, in what follows, we will continue working with discrete time only. From now on, whenever we consider cocycles induced by delay equations with delay $r > 0$, our underlying discrete-time metric dynamical system is given by $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$ with $\theta := \theta_r$. We also use the notation $\varphi(\omega, \cdot) := \varphi(1, \omega, \cdot)$ for the cocycle φ defined in (4.3.13).

We recall the definition of measurable field of Banach spaces again.

Definition 4.3.12. *Let $(\Omega, \mathcal{F})$ be a measurable space. A family of Banach spaces $\{E_\omega\}_{\omega\in\Omega}$ is called a* measurable field of Banach spaces *if there is a set of sections*

$$\Delta \subset \prod_{\omega\in\Omega} E_\omega$$

with the following properties:

(i) Δ is a linear subspace of $\prod_{\omega\in\Omega} E_\omega$.

(ii) There is a countable subset $\Delta_0 \subset \Delta$ such that for every $\omega \in \Omega$, the set $\{g(\omega) : g \in \Delta_0\}$ is dense in E_ω.

(iii) For every $g \in \Delta$, the map $\omega \mapsto \|g(\omega)\|_{E_\omega}$ is measurable.

Proposition 4.3.13. *Let $X \colon \Omega \to C^\gamma(I, U)$ be a stochastic process. Assume that there are $\alpha < \beta < \gamma$. Then $\{\mathscr{D}^{\alpha,\beta}_{X(\omega)}(I, W)\}_{\omega\in\Omega}$ is a measurable field of Banach spaces.*

Proof. For $s = (v, f, R) \in \mathbb{R} \times C^\infty(I, L(U, W)) \times C^\infty_0(I, W)$, define

$$g_s(\omega) := \left(v + \int_{-r}^{\cdot} f(\tau)\, dX_\tau(\omega) + R, f \right) \in \mathscr{D}^{\alpha,\beta}_{X(\omega)}(I, W)$$

and set

$$\Delta := \{ g_s : s \in \mathbb{R} \times C^\infty(I, L(U, W)) \times C^\infty_0(I, W) \}. \tag{4.3.16}$$

It is clear that (i) holds for Δ. Let S be a countable and dense subset of $\mathbb{R} \times C^\infty(I, L(U,W)) \times C_0^\infty(I,W)$ and define $\Delta_0 := \{g_s : s \in S\}$. By definition, Δ_0 is countable, and $\{g_s(\omega) : s \in S\}$ is dense in $\mathscr{D}_{X(\omega)}^{\alpha,\beta}(I,W)$ for fixed $\omega \in \Omega$ by Theorem 4.3.10. It remains to prove (iii). Let $I = [a,b]$ and choose $s = (v, f, R)$. Then

$$\begin{aligned} \|g_s(\omega)\| = |v| + |f(a)| &+ \sup_{s,t \in I \cap \mathbb{Q}, s<t} \frac{|f(t) - f(s)|}{(t-s)^\alpha} \\ &+ \sup_{s,t \in I \cap \mathbb{Q}, s<t} \frac{|R_{s,t} + \int_s^t f(\tau)\, dX_\tau(\omega) - f(s) X_{s,t}(\omega)|}{(t-s)^{2\alpha}}. \end{aligned}$$

The integral is measurable since it is a limit of measurable Riemann sums. Measurability of $\omega \mapsto \|g_s(\omega)\|$ thus follows which finishes the proof.

$\square$

Theorem 4.3.14. *The continuous cocycle*

$$\varphi(\omega, \cdot) \colon \mathscr{D}_{X(\omega)}^{\alpha,\beta}([-r,0],W) \to \mathscr{D}_{X(\theta_r\omega)}^{\alpha,\beta}([-r,0],W)$$

defined in Theorem 4.3.11 induces a random dynamical system on the field of Banach spaces $\{\mathscr{D}_{X(\omega)}^{\alpha,\beta}([-r,0],W)\}_{\omega \in \Omega}$.

Proof. Let Δ be defined as (4.3.16) and take $g \in \Delta$. Consider the solution y to

$$\begin{aligned} y_t(\omega) &= g_0(\omega) + \int_0^t \sigma(y_\tau(\omega), y_{\tau-r}(\omega))\, d\mathbf{X}_\tau(\omega), \quad t \geq 0; \\ y_t(\omega) &= g_t(\omega), \quad t \in [-r, 0]. \end{aligned}$$

To simplify notation, set $\|\cdot\|_{\mathscr{D}_{X(\omega)}([0,r])} := \|\cdot\|_{\mathscr{D}_{X(\omega)}^{\alpha,\beta}([0,r],W)}$. We will prove that $\omega \mapsto \|y(\omega)\|_{\mathscr{D}_{X(\omega)}([0,r])}$ is measurable. Define

$$y_t^1(\omega) := g_0(\omega) + \int_0^t \sigma\big(g_0(\omega), g_{\tau-r}(\omega)\big)\, d\mathbf{X}_\tau(\omega)$$

and recursively for $n \geqslant 1$

$$y_t^{n+1}(\omega) := g_0(\omega) + \int_0^t \sigma\big(y_\tau^n(\omega), g_{\tau-r}(\omega)\big)\, d\mathbf{X}_\tau(\omega).$$

By induction, one can show that $\omega \mapsto y_t^n(\omega)$ is measurable for every $t \in [0,r]$ and $n \geq 1$. By a similar strategy for proving continuity of the Itō-Lyons map, one can show that $y^n(\omega) \to y(\omega)$ in the space $\mathscr{D}_{X(\omega)}^{\alpha,\beta}([0, T(A(\omega))], W)$ as $n \to \infty$ where

$$A(\omega) = \|X(\omega)\|_{\gamma;[0,r]} + \|\mathbb{X}(\omega)\|_{2\gamma;[0,r]} + \|\mathbb{X}(\omega)(-r)\|_{2\gamma;[0,r]}$$

and $T \colon [0,\infty) \to (0, r]$ is a decreasing function. Define

$$\Omega_m := \left\{ \omega \in \Omega \,:\, T(A(\omega)) \leq \frac{r}{m} \right\}.$$

Then Ω_m is a measurable subset and $\Omega = \bigcup_{m\geqslant 1} \Omega_m$. Fix $m \in \mathbb{N}$ and choose $\omega \in \Omega_m$. Then $(y^n(\omega))_n$ is a Cauchy sequence in the space $\mathscr{D}^{\alpha,\beta}_{X(\omega)}([0,r/m],W)$ and, consequently, converges to some element $\tilde{y}^0(\omega)$ for which we can conclude that $\omega \mapsto \tilde{y}^0_t(\omega)$ is measurable for every $t \in [0,r/m]$. Now we can repeat this argument in $[\frac{jr}{m}, \frac{(j+1)r}{m}]$ for $j = 0,\dots,m-1$ and obtain a sequence of elements $\tilde{y}^j(\omega) \in \mathscr{D}^{\alpha,\beta}_{X(\omega)}([jr/m,(j+1)r/m],W)$ with the properties that $\omega \mapsto \tilde{y}^j_t(\omega)$ is measurable for every $t \in [jr/m,(j+1)r/m]$ and

$$y_t(\omega) = \sum_{j=0}^{m-1} \tilde{y}^j_t(\omega)\chi_{[\frac{jr}{m},\frac{(j+1)r}{m})}(t).$$

This implies that $\omega \mapsto y_t(\omega)$ is measurable for every $t \in [0,r]$ on the subspace Ω_m. Since m was arbitrary, measurability follows also on the space Ω. Note that $y'_t(\omega) = \sigma(y_t(\omega), g_{t-r}(\omega))$, thus

$$\begin{aligned}\|y(\omega)\|_{\mathscr{D}_{X(\omega)}([0,r])} &= |y_0(\omega)| + |y'_0(\omega)| + \sup_{s<t\in[0,r]\cap\mathbb{Q}} \frac{|y'_{s,t}|}{|t-s|^\alpha} \\ &\quad + \sup_{s<t\in[0,r]\cap\mathbb{Q}} \frac{\left|\int_s^t \sigma(y_\tau(\omega), g_{\tau-r}(\omega))\,d\mathbf{X}_\tau(\omega) - \sigma(y_s(\omega), g_{s-r}(\omega))\right|}{|t-s|^{2\alpha}}\end{aligned}$$

and measurability of $\omega \mapsto \|y(\omega)\|_{\mathscr{D}_{X(\omega)}([0,r])}$ follows. We can now repeat this argument to see that $\omega \mapsto \|y(\omega)\|_{\mathscr{D}_{X(\omega)}([nr,(n+1)r])}$ is measurable for every $n \geq 0$ which proves the theorem. $\square$

4.4 The Lyapunov spectrum for linear equations

In this section, we formulate the main results of the chapter.

Theorem 4.4.1. *Let $(\Omega,\mathcal{F},\mathbb{P},(\theta)_{t\in\mathbb{R}})$ be an ergodic measurable metric dynamical system and $\mathbf{X}$ a delayed γ-rough path cocycle for some $\gamma \in (1/3,1/2]$ and some delay $r > 0$. Assume that $\alpha < \beta < \gamma$. In addition, we assume that*

$$\|X\|_{\gamma;[0,r]} + \|\mathbb{X}\|_{2\gamma;[0,r]} + \|\mathbb{X}(-r)\|_{2\gamma;[0,r]} \in L^{\frac{1}{\gamma-\beta}}(\Omega). \tag{4.4.1}$$

Let $\sigma \in L(W^2, L(U,W))$. Then we have the following:

(i) The equation

$$\begin{aligned} dy_t &= \sigma(y_t, y_{t-r})\,d\mathbf{X}_t(\omega); \quad t \geq 0 \\ y_t &= \xi_t; \quad t \in [-r,0] \end{aligned} \tag{4.4.2}$$

has a unique solution $y\colon [0,\infty) \to W$ for every initial condition $(\xi,\xi') \in \mathscr{D}^{\alpha,\beta}_{X(\omega)}([-r,0],W)$ with

$$(y_{t+n}(\omega), y'_{t+n}(\omega))_{t\in[-r,0]} \in \mathscr{D}^{\alpha,\beta}_{X(\theta_{nr}\omega)}([-r,0],W)$$

for every $n \geq 0$ where

$$y'_t(\omega) = \begin{cases} \sigma(y_t(\omega), y_{t-r}(\omega)) & \text{for } t \geq 0 \\ \xi'_t & \text{for } t \in [-r, 0]. \end{cases}$$

(ii) Set $\varphi(n,\omega,\xi) := (y_{t+n}(\omega), y'_{t+n}(\omega))_{t\in[-r,0]}$ and $E_\omega := \mathscr{D}^{\alpha,\beta}_{X(\omega)}([-r,0],W)$. Then φ is a compact linear cocycle defined on the discrete ergodic measurable metric dynamical system $(\Omega, \mathcal{F}, \mathbb{P}, \theta_r)$ acting on the measurable field of Banach spaces $\{E_\omega\}_{\omega\in\Omega}$ and all statements of the Multiplicative Ergodic Theorem 2.2.16 hold. In particular, a deterministic Lyapunov spectrum $(\mu_j)_{j\geq 0}$ exists and induces an Oseledets filtration of the space of admissible initial conditions $\mathscr{D}^{\alpha,\beta}_{X(\omega)}([-r,0],W)$ on a set of full measure.

Proof. Theorem 4.3.11 together with Theorem 4.3.14 show that (4.4.2) induces a cocycle acting on a measurable field of Banach spaces given by the spaces of controlled paths. The estimate in Theorem 4.2.11 together with our assumption (4.4.1) show that the moment condition of the MET 2.2.16 is satisfied and the theorem follows. □

Finally, we apply our results for the fractional Brownian motion.

Corollary 4.4.2. *Theorem 4.4.1 can be applied for X being a two-sided fractional Brownian motion B. In case $H = \frac{1}{2}$ and $\mathbf{B} = \mathbf{B}^{It\bar{o}}$, the solution to* (4.4.2) *coincides with the usual Itō-solution almost surely in case the initial condition is $\mathcal{F}^0_{-1}$-measurable.*

Proof. The fact that $\mathbf{B}$ and $\mathbf{B}^{\text{Itō}}$ (when $H = \frac{1}{2}$) are delayed γ-rough path cocycles on an ergodic measurable metric dynamical system for every $\gamma \in (1/3, H)$ was shown in Theorem 4.3.7. In Proposition 4.2.15, we saw that the integrability condition (4.4.1) is satisfied in the Brownian case, and we can indeed apply Theorem 4.4.1. The fact that the solution to (4.4.2) coincides with the usual Itō resp. Stratonovich solution (when $H = \frac{1}{2}$) was shown in Corollary 4.2.17. □

Remark 4.4.3. *As we pointed out earlier, our results are applicable to equations where the dynamics depend on the past in a much more general way, namely to those of the form*

$$dy_t = \sigma\big(y_t, \int_{-r}^0 y_{t+\tau}\mu(d\tau)\big)\, d\mathbf{X}_t(\omega).$$

Note that this is the most general form of delay equations in the linear case (without drift). In addition, for the non-linear case, this is the most common form. To give a meaning to this integral, we invoke the following ansatz:

$$\int_s^t \sigma\big(y_t, \int_{-r}^0 y_{t+\tau}\mu(d\tau)\big)\, d\mathbf{X}_t(\omega) \approx \sum \big[\sigma\big(y_{s_j}, \int_{-r}^0 y_{s_j+\tau}\mu(d\tau)\big)X_{s_j,s_{j+1}} + \\ \sigma^1(-_{s_j})y'_{s_j}\mathbb{X}_{s_j,s_{j+1}} + \sigma^2(-_{s_j})\int_{-r}^0 y'_{s_j+\tau}\mathbb{X}_{s_j,s_{j+1}}(\tau)\mu(d\tau)\big].$$

It is not hard to see that the above increment is satisfying the assumptions of the sewing lemma. All the results concerning existence, uniqueness, and our estimates remain valid with *some straightforward modifications. Theorem 4.3.4 has to be modified slightly by taking* $\overline{\mathbf{X}}$ *as a* $C([-r,0), C_0^{0,p-var}(\mathbb{R}, \tilde{T}^2(U)))$*-valued random variable. We take* $\Omega = C([-r,0), C_0^{0,p-var}(\mathbb{R}, \tilde{T}^2(U)))$ *and for* $\omega \in \Omega$*, we define*

$$(\theta_s\omega)(\tau,t) := \omega(\tau,s)^{-1} \circledast \omega(\tau,t+s), \qquad \tau \in [-r,0), \quad s,t \in \mathbb{R}.$$

It follows that also in the case of this general SDDE, the solution induced a cocycle, and we can apply the MET.

Remark 4.4.4. *It is possible to use the language of Hairer's Regularity Structures [43] to reformulate our results. In that case, the space of controlled paths has to be replaced by the space of* modelled distributions*. We decided to use the language of rough paths here because less theory is needed and we can directly rely on prior work such as [37]. However, it might be useful to use regularity structures in the future.*

4.5 An example

In view of our main results obtained in the former section, we now come back to the previous example already discussed in the introduction: we consider the stochastic delay equation

$$\begin{aligned} dy_t &= y_{t-1}\, d\mathbf{B}_t^{\text{Itō}}; \quad t \geq 0 \\ y_t &= \xi_t; \quad t \in [-1,0]. \end{aligned} \tag{4.5.1}$$

This equation can be considered as the prototype of a singular stochastic delay equation. In its classical Itō formulation, it was studied in [8]. In that work, it was shown that there exists a deterministic real number Λ such that

$$\Lambda = \lim_{t\to\infty} \frac{1}{t} \log \|\varphi(t,\omega,\xi)\| \tag{4.5.2}$$

almost surely for any initial condition $\xi \in C([-1,0],\mathbb{R}) \setminus \{0\}$ (the exceptional set depends on ξ). In (4.5.2), the norm $\|\cdot\|$ may denote the uniform norm or the M_2-norm which we will define below. It is a natural question to ask whether Λ coincides with the top Lyapunov exponent provided by the Multiplicative Ergodic Theorem 2.2.16. We will give an affirmative answer in this section. We point out that the proof of (4.5.2) in [8] was quite long. It relied on the uniqueness of the invariant measure of the Markov process obtained by projecting the solution process onto the unit sphere of the state space M_2 introduced below and then applying a suitable version of the Furstenberg-Hasminskii formula. To establish uniqueness, the author constructed a tailor-made generalized (asymptotic) coupling.

Set $E_\omega = \mathscr{D}^{\alpha,\beta}_{B(\omega)}([-1,0])$ with $\alpha < \beta$. Take $(\xi,\xi') \in E_\omega$. On the time interval $[-1,1]$, the unique solution to (4.5.1) is given by

$$(y_t, y'_t) = \begin{cases} (\xi_t, \xi'_t) & \text{if } t \in [-1,0] \\ \left(\int_0^t \xi_{s-1}\, d\mathbf{B}^{\text{Itō}} + \xi_0, \xi_{t-1}\right) & \text{if } t \in [0,1]. \end{cases} \tag{4.5.3}$$

Note that $C^1([-1,0],\mathbb{R}) \subset E_\omega$ for every $\omega \in \Omega$ by the embedding $\eta \mapsto (\eta, 0)$. Let us introduce the Hilbert space $M_2 := \mathbb{R} \times L^2([-1,0],\mathbb{R})$ furnished with the norm

$$\|(\nu,\eta)\|_{M_2} := \Big(|\nu|^2 + \|\eta\|_{L^2}^2\Big)^{\frac{1}{2}}$$

for $(\nu,\eta) \in M_2$. Note that $C([-1,0],\mathbb{R}) \subset M_2$ using the embedding $\eta \mapsto (\eta_0, \eta)$. Recall the definition of Vol given in Definition 2.2.4. Our main result in this section is the following.

Theorem 4.5.1. *For every $\eta_1, ..., \eta_k \in C^1([-1,0],\mathbb{R}) \setminus \{0\}$, the limit*

$$\lim_{n\to\infty} \frac{1}{n} \log \operatorname{Vol}\left(\varphi(n,\omega,\eta_1), ..., \varphi(n,\omega,\eta_k)\right) \tag{4.5.4}$$

exists almost surely in $[-\infty,\infty)$. Moreover, the limit is independent of the choice of the norm when we take $\|\cdot\|_{E_{\theta^n\omega}}$, $\|\cdot\|_{C^\alpha}$, $\|\cdot\|_\infty$ or $\|\cdot\|_{M_2}$ in the definition of Vol. *For $k = 1$, if $\|\cdot\|$ denotes any of the norms above, the limit*

$$\lim_{n\to\infty} \frac{1}{n} \log \|\varphi(n,\omega,\eta)\|$$

is independent of the choice of $\eta \in C^1([-1,0],\mathbb{R}) \setminus \{0\}$ and coincides with the largest Lyapunov exponent provided by the Multiplicative Ergodic Theorem 2.2.16.

Before proving Theorem 4.5.1, we need two classical inequalities:

Lemma 4.5.2. *Let $\alpha < \frac{1}{2}$, $p > 2$ and let $\xi \colon [-1,0] \to \mathbb{R}$ be an α-Hölder path. Then there is a constant A_p such that*

$$\|\xi\|_\alpha = \sup_{-1\leqslant s<t\leqslant 0} \frac{|\xi_{s,t}|}{(t-s)^\alpha} \leqslant A_p \bigg(\iint_{[-1,0]^2} \frac{|\xi_u - \xi_v|^p}{|u-v|^{p\alpha+2}}\, du\, dv \bigg)^{\frac{1}{p}}. \tag{4.5.5}$$

If X is α-Hölder and $(\xi,\xi') \in \mathscr{D}_X^\alpha([-1,0],\mathbb{R})$,

$$\sup_{-1\leqslant s<t\leqslant 0} \frac{|\xi^{\#}_{s,t}|}{(t-s)^{2\alpha}} \leqslant A_p \bigg[\bigg(\iint_{-1\leqslant u<v\leqslant 0} \frac{|\xi^{\#}_{u,v}|^p}{|u-v|^{2\alpha p+2}}\, du\, dv \bigg)^{\frac{1}{p}} + \|\xi'\|_\alpha \|X\|_\alpha \bigg]. \tag{4.5.6}$$

Proof. Cf. [36, Corollary 4]. □

Proof of Theorem 4.5.1. First, we claim that the limit (4.5.4) exists for any choice of $\eta_1, \ldots, \eta_k$ for the norm $\|\cdot\|_{E_{\theta^n\omega}}$. Indeed, if $\eta_1, \ldots, \eta_k$ are linearly dependent, the limit (4.5.4) clearly exists and equals $-\infty$. Also if for every $j \geqslant 1$ we have $\langle \eta_1, ..., \eta_k \rangle \cap F_{\mu_j}(\omega) \neq \{0\}$, since $\mu_j \to -\infty$, Lemma 2.2.6 implies that (4.5.4) exists and equals $-\infty$. So we can assume that for some $j \geqslant 1$, $\langle \eta_1, ..., \eta_k \rangle \cap F_{\mu_{j+1}}(\omega) = \{0\}$. For $i \leqslant j$ we can find a finite-dimensional subspace $H_i(\omega)$ such that $H_i(\omega) \bigoplus F_{\mu_{i+1}}(\omega) = F_{\mu_i}(\omega)$. Furthermore, for each $i \leqslant j$, there is a subspace $\tilde{H}_i(\omega) \subset H_i(\omega)$ with $\dim \big[\tilde{H}_i(\omega)\big] = n_i$ such that

$$\frac{\langle \eta_1, ..., \eta_k \rangle}{F_{\mu_{j+1}}(\omega)} = \frac{\bigoplus_{1\leqslant i\leqslant j} \tilde{H}_i(\omega)}{F_{\mu_{j+1}}(\omega)}.$$

Now as a consequence of item (v) in the Multiplicative Ergodic Theorem 2.2.16,

$$\lim_{n\to\infty} \frac{1}{n} \log \operatorname{Vol}\left(\varphi(n,\omega,\eta_1),...,\varphi(n,\omega,\eta_k)\right) = \sum_{1\leqslant i\leqslant j} n_i\mu_i$$

which shows the claim.

The strategy of the proof now is to compare all norms against one another. For $-1 \leqslant t \leqslant 0$, $\xi, \eta \in C^1([-1,0],\mathbb{R}) \setminus \{0\}$ and $n \in \mathbb{N}_0$ set $\xi^n_t = y^{\xi}_{n+t}$ and $\eta^n_t = y^{\eta}_{n+t}$ where y^{ξ} and y^{η} are solutions to (4.5.1) starting from ξ, η respectively. By definition,

$$(\xi^n)'_t = \xi^{n-1}_t, \qquad (\xi^n)^{\#}_{s,t} = \int_s^t \xi^{n-1}_{s,u} dB_{n+u} \tag{4.5.7}$$

for $-1 \leq s \leq t \leq 0$ and $n \geq 0$ where we define $\xi^{-1} \equiv 0$. Set $\mathcal{F}_t := \mathcal{F}^t_0$. From Lemma 4.5.2, for any $C > 0$,

$$\mathbb{P}(\|\xi^n\|_\alpha > C \,|\, \mathcal{F}_{n-1}) \leqslant \mathbb{P}\left(\iint_{[-1,0]^2} \frac{|\xi^n_{v,u}|^p}{|u-v|^{2+p\alpha}}\, du\, dv \geqslant \frac{C^p}{(A_p)^p} |\mathcal{F}_{n-1}\right) =$$
$$\mathbb{P}\left(\iint_{[-1,0]^2} \frac{|\int_{[u,v]} \xi^{n-1}_\tau \, dB_{n+\tau}|^p}{|u-v|^{p\alpha+2}}\, du\, dv \geqslant \frac{C^p}{(A_p)^p} |\mathcal{F}_{n-1}\right)$$

almost surely. Similarly,

$$\mathbb{P}\left(\inf_{\beta\in\mathbb{Q}} \|\eta^n - \beta\xi^n\|_\alpha > C |\mathcal{F}_{n-1}\right) \leqslant$$
$$\inf_{\beta\in\mathbb{Q}} \mathbb{P}\left(\iint_{[-1,0]^2} \frac{|\int_{[u,v]} \eta^{n-1}_\tau - \beta\xi^{n-1}_\tau \, dB_{n+\tau}|^p}{|u-v|^{p\alpha+2}}\, du\, dv \geqslant \frac{C^p}{(A_p)^p} |\mathcal{F}_{n-1}\right)$$

almost surely. Set $p = 2m$ for m chosen such that $m(1-2\alpha) > 1$. From the Burkholder-Davis-Gundy inequality, it follows that

$$\mathbb{E}\left(\left|\int_{[u,v]} \xi^{n-1}_\tau \, dB_{n+\tau}\right|^{2m} |\mathcal{F}_{n-1}\right) \leqslant B_{2m}|u-v|^m \|\xi^{n-1}\|^{2m}_\infty$$

almost surely for some constant $B_{2m} > 0$. Consequently,

$$\mathbb{P}\left(\|\xi^n\|_\alpha > C|\mathcal{F}_{n-1}\right) \leqslant \tilde{A}_{2m} \frac{\|\xi^{n-1}\|^{2m}_\infty}{C^{2m}} \quad \text{and} \tag{4.5.8}$$
$$\mathbb{P}\left(\inf_{\beta\in\mathbb{Q}} \|\eta^n - \beta\xi^n\|_\alpha > C|\mathcal{F}_{n-1}\right) \leqslant \tilde{A}_{2m} \frac{\inf_{\beta\in\mathbb{Q}} \|\eta^{n-1} - \beta\xi^{n-1}\|^{2m}_\infty}{C^{2m}} \tag{4.5.9}$$

for a general constant $\tilde{A}_{2m}$. Now for any $\varepsilon > 0$, (4.5.8) implies that

$$\begin{aligned}\mathbb{P}\left(\frac{1}{n} \log \|\xi^n\|_\alpha \geqslant \varepsilon + \frac{1}{n-1} \log \|\xi^{n-1}\|_\infty]\right) &\leqslant \mathbb{P}\left(\|\xi^{n-1}\|_\alpha \geqslant \|\xi^{n-1}\|_\infty \exp[\varepsilon(n-1)]\right) \\ &\leqslant \frac{\tilde{A}_{2m}}{\exp\left[2m\varepsilon(n-1)\right]} \longrightarrow 0\end{aligned} \tag{4.5.10}$$

as $n \to \infty$. Similarly,

$$\mathbb{P}\left(\frac{1}{n}\log\inf_{\beta\in\mathbb{Q}}\|\eta^n-\beta\xi^n\|_\alpha \geqslant \varepsilon + \frac{1}{n-1}\log\inf_{\beta\in\mathbb{Q}}\|\eta^{n-1}-\beta\xi^{n-1}\|_\infty\right) \longrightarrow 0 \tag{4.5.11}$$

as $n \to \infty$. Now from (4.5.6) and (4.5.7),

$$\begin{aligned}
&\mathbb{P}\left(\sup_{-1\leqslant s<t\leqslant 0}\frac{|(\xi^n)^{\#}_{s,t}|}{(t-s)^{2\alpha}} > C \,|\, \mathcal{F}_{n-1}\right) \leqslant \\
&\mathbb{P}\left(A_p\left[\left(\iint_{-1\leqslant u<v\leqslant 0}\frac{|\int_{u,v}\xi^{n-1}_{u,\tau}\,dB_{n+\tau}|^p}{(v-u)^{2p\alpha+2}}\,du\,dv\right)^{\frac{1}{p}} + \|\xi^{n-1}\|_\alpha\|B^n\|_\alpha\right] > C \,|\, \mathcal{F}_{n-1}\right) \leqslant \\
&\mathbb{P}\left(\iint_{-1\leqslant u<v\leqslant 0}\frac{|\int_{u,v}\xi^{n-1}_{u,\tau}\,dB_{n+\tau}|^p}{(v-u)^{2p\alpha+2}}\,du\,dv + \|\xi^{n-1}\|^p_\alpha\|B^n\|^p_\alpha > \frac{C^p}{(2A_p)^p} \,|\, \mathcal{F}_{n-1}\right)
\end{aligned}$$

almost surely. Similarly,

$$\begin{aligned}
&\mathbb{P}\left(\inf_{\beta\in\mathbb{Q}}\left[\sup_{-1\leqslant s<t\leqslant 0}\frac{|(\eta^n-\beta\xi^n)^{\#}_{s,t}|}{(t-s)^{2\alpha}}\right] > C \,|\, \mathcal{F}_{n-1}\right) \leqslant \\
&\inf_{\beta\in\mathbb{Q}}\mathbb{P}\left(\iint_{-1\leqslant u<v\leqslant 0}\frac{|\int_{u,v}(\eta^n_{u,\tau}-\beta\xi^n_{u,\tau})\,dB_{n+\tau}|^p}{(v-u)^{2p\alpha+2}}\,du\,dv + \|\eta^{n-1}-\beta\xi^{n-1}\|^p_\alpha\|B^n\|^p_\alpha > \frac{C^p}{(2A_p)^p} \,|\, \mathcal{F}_{n-1}\right)
\end{aligned}$$

almost surely. Set $p = 2m$ such that $m(1-2\alpha) > 1$. Then

$$\mathbb{E}\left(\left|\int_{[u,v]}\xi^{n-1}_{u,\tau}\,dB_{n+\tau}\right|^{2m} \,|\, \mathcal{F}_{n-1}\right) \leqslant B_{2m}(v-u)^{m(2\alpha+1)}\|\xi^{n-1}\|^{2m}_\alpha$$

almost surely. Consequently, for general constants M and $\tilde{M}$,

$$\begin{aligned}
\mathbb{P}\left(\sup_{-1\leqslant s<t\leqslant 0}\frac{|(\xi^n)^{\#}_{s,t}|}{(t-s)^{2\alpha}} > C \,|\, \mathcal{F}_{n-1}\right) &\leqslant \mathbb{P}\left(M\|\xi^{n-1}\|^{2m}_\alpha(1+\|B^n\|^{2m}_\alpha) > C^{2m} \,|\, \mathcal{F}_{n-1}\right) \\
&\leqslant \frac{\tilde{M}}{C^{2m}}\|\xi^{n-1}\|^{2m}_\alpha
\end{aligned}$$

almost surely and

$$\mathbb{P}\left(\inf_{\beta\in\mathbb{Q}}\sup_{-1\leqslant s<t\leqslant 0}\frac{|(\eta^n-\beta\xi^n)^{\#}_{s,t}|}{(t-s)^{2\alpha}} > C \,|\, \mathcal{F}_{n-1}\right) \leqslant \frac{\tilde{M}}{C^{2m}}\inf_{\beta\in\mathbb{Q}}\|\eta^{n-1}-\beta\xi^{n-1}\|^{2m}_\alpha$$

almost surely. Similarly to (4.5.10), for any $\varepsilon > 0$,

$$\begin{aligned}
&\mathbb{P}\left(\frac{1}{n}\log\|(\xi^n)^{\#}\|_{2\alpha} \geqslant \varepsilon + \frac{1}{n-1}\log\|\xi^{n-1}\|_\alpha]\right) \longrightarrow 0 \qquad \text{and} \\
&\mathbb{P}\left(\frac{1}{n}\log\inf_{\beta\in\mathbb{Q}}\|(\eta^n-\beta\xi^n)^{\#}\|_{2\alpha} \geqslant \varepsilon + \frac{1}{n-1}\log\inf_{\beta\in\mathbb{Q}}\|\eta^{n-1}-\beta\xi^{n-1}\|_\alpha]\right) \longrightarrow 0
\end{aligned} \tag{4.5.12}$$

as $n \to \infty$. Remember $\|\xi^n\|_{M_2}^2 = |\xi_{-1}^n|^2 + \int_{-1}^0 (\xi_t^n)^2\,dt$. From Doob's submartingale inequality, for a general constant M,

$$\begin{aligned}\mathbb{P}(\|\xi^n\|_\infty > C \mid \mathcal{F}_{n-1}) &\leqslant \mathbb{P}\left(|\xi_{-1}^n| + \sup_{-1\leqslant t\leqslant 0} |\xi_{-1,t}^n| > C \mid \mathcal{F}_{n-1}\right)\\ &\leqslant \frac{4|\xi_{-1}^n|^2 + 4\mathbb{E}|\xi_{-1,0}^n|^2}{C^2} \leqslant \frac{M}{C^2}\|\xi^{n-1}\|_{M_2}^2\end{aligned}$$

almost surely. Also,

$$\mathbb{P}\left(\inf_{\beta\in\mathbb{Q}} \|\eta^n - \beta\xi^n\|_\infty > C \mid \mathcal{F}_{n-1}\right) \leqslant \frac{M}{C^2}\inf_{\beta\in\mathbb{Q}} \|\eta^{n-1} - \beta\xi^{n-1}\|_{M_2}^2$$

almost surely. Again as in (4.5.10), for any $\varepsilon > 0$,

$$\begin{aligned}&\mathbb{P}\left(\frac{1}{n}\log\inf_{\beta\in\mathbb{Q}} \|\xi^n\|_\infty \geqslant \varepsilon + \frac{1}{n-1}\log\inf_{\beta\in\mathbb{Q}}\|\xi^{n-1}\|_{M_2}\right) \longrightarrow 0 \qquad \text{and}\\ &\mathbb{P}\left(\frac{1}{n}\log\inf_{\beta\in\mathbb{Q}} \|\eta^n - \beta\xi^n\|_\infty \geqslant \varepsilon + \frac{1}{n-1}\log\inf_{\beta\in\mathbb{Q}}\|\eta^{n-1} - \beta\xi^{n-1}\|_{M_2}\right) \longrightarrow 0\end{aligned} \tag{4.5.13}$$

as $n \to \infty$. Now from the Multiplicative Ergodic Theorem 2.2.16, (4.5.10), (4.5.11), (4.5.12) and (4.5.13), the following limits exist

$$\begin{aligned}&\lim_{n\to\infty}\frac{1}{n}\log\|\varphi(n,\omega,\xi)\|\\ &\lim_{n\to\infty}\frac{1}{n}\log\operatorname{Vol}\left(\varphi(n,\omega,\xi),\varphi(n,\omega,\eta)\right)\end{aligned} \tag{4.5.14}$$

as $n \to \infty$ where $\|\cdot\|$ could be any of the proposed norms, used also in the definition of Vol, and the limit is independent of the choice of the norm. From the definition of Vol, the above argument together with a simple induction generalizes to every $k \geqslant 1$ which proves the first claim.

To prove the second claim, let $\eta \in C^1([-1,0],\mathbb{R}) \setminus \{0\}$. Then the limit $\mu := \lim_{n\to\infty}\frac{1}{n}\log\|\varphi(n,\omega,\eta)\|$ is independent from η, cf. [8, Theorem 1.1]. Therefore, from the Multiplicative Ergodic Theorem 2.2.16, $C^1([-1,0],\mathbb{R}) \setminus \{0\} \subset F_{\mu_j}(\omega) \setminus F_{\mu_{j+1}}(\omega)$ for some $j \geq 1$. Let $\xi, \eta \in C^\infty([-1,0],\mathbb{R}) \setminus \{0\}$, $a \in \mathbb{R}$ and set $\tilde{\xi}_t := \int_{-1}^t \xi_\tau\,dB_\tau$. Using (4.5.3), we have

$$\gamma_t := \tilde{\xi}_t + \eta_t + a = \varphi(1,\theta^{-1}\omega,\xi)[t] + \eta_t + a - \xi_0 \tag{4.5.15}$$

and (4.5.14) implies that $\lim_{n\to\infty}\frac{1}{n}\log\|\varphi(n,\omega,\gamma)\| \leqslant \mu$. From Theorem 4.3.10, we know that elements of the form γ are dense in E_ω. Choose $\xi_\omega \in F_{\mu_1}(\omega) \setminus F_{\mu_2}(\omega)$. Since $F_{\mu_2}(\omega)$ is a closed subspace, we can find a neighborhood $B(\xi_\omega,\delta) \subset F_{\mu_1}(\omega) \setminus F_{\mu_2}(\omega)$ and an element $\gamma \in B(\xi_\omega,\delta)$ of the form (4.5.15). Therefore, $\mu_1 \leq \mu$, thus $\mu = \mu_1$. □

Remark 4.5.3. *Taking the Hilbert space norm* $\|\cdot\|_{M_2}$ *in the definition of* Vol*, we actually have*

$$\operatorname{Vol}(X_1,...,X_k) = \|X_1 \wedge X_2 \wedge ... \wedge X_k\|_{M_2}.$$

We conjecture that the limit

$$\lim_{n\to\infty} \frac{1}{n} \log \|\varphi(n,\omega,\eta_1) \wedge \cdots \wedge \varphi(n,\omega,\eta_k)\|_{M_2}$$

is independent of the choice of $\eta_1,\dots,\eta_k$ *whenever these vectors are linearly independent, and that the limit coincides with* Λ_k *almost surely. This would be in good accordance with the classical definition of Lyapunov exponents in the finite dimensional case, cf. [1, Chapter 3].*

5

Rough Delay Equations II

The following chapter is the sequel to the previous chapter. We developed an appropriate setting to investigate the dynamics of stochastic delay equations. In this chapter, we mainly deal with non-linear equations. We aim to study stochastic delay differential equations (SDDEs) of the form

$$dy_t = b(y_t, y_{t-r})\, dt + \sigma(y_t, y_{t-r})\, dB_t(\omega) \tag{5.0.1}$$

from a dynamical systems point of view. Remember in (5.0.1), $r > 0$ denotes a time delay, B is a multidimensional fractional Brownian motion, b is the drift and σ the diffusion coefficient. The goal in this chapter is to prove the existence of *random invariant manifolds* for (5.0.1). Invariant manifolds are key objects in the theory of dynamical systems, both deterministic and random, and play a central role, for instance, in stochastic bifurcation theory [44, 1, 45] and model reduction for stochastic differential equations [46, 47, 48, 49].

One of our main results in the previous chapter was that (5.0.1) does indeed induce a cocycle. However, one has to pay a price: the spaces on which the cocycle map is defined will depend on the trajectory of the driving path $B(\omega)$. More precisely, if $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$ is a random dynamical system, the cocycle φ is a continuous map

$$\varphi(n, \omega, \cdot) \colon E_\omega \to E_{\theta^n \omega}$$

where $\{E_\omega\}_{\omega \in \Omega}$ is a family of Banach spaces. One key idea in the previous chapter was to interpret (5.0.1) as a random rough differential equation in the sense of Lyons [50, 37, 39]. Doing this, we showed that Gubinelli's spaces of controlled paths [36] are possible choices for E_ω when studying (5.0.1).

In Chapter 2, we proved a version of *Multiplicative Ergodic Theorem* (MET) in this framework. We then applied this theorem in the previous chapter. We showed that cocycles induced by linear equations of the form (5.0.1) possess a *Lyapunov spectrum*, an analogue to the set of eigenvalues of a matrix. In Chapter 2, in addition, we proved in a more abstract framework that an *Oseledets splitting*, i.e., a decomposition of E_ω into a direct sum of φ-

invariant spaces. This decomposition was indeed the basis to prove the existence of local stable and unstable manifolds in Chapter 3 in the same framework.

In this chapter, we harvest the fruit of our former works. In our main results, Theorem 5.4.4 and Theorem 5.4.5, we formulate sufficient conditions under which we can deduce the existence of local stable and unstable manifolds for equation (5.0.1). Let us mention that one difficulty in the unstable case is that the cocycle induced by (5.0.1) is not invertible, which is natural for delay equations: solutions exist only forward in time. Therefore, we can not just apply the stable manifold theorem to the inverse cocycle as, for instance, in [2]. To overcome this difficulty, in Chapter 3, we used our semi-invertible MET (Theorem (2.3.20)) to obtain the existence of unstable manifolds. We formulated both theorems ((3.2.9) and (3.3.6)) in a generality that allows them to be applied to equations that are driven by other noise than fractional Brownian motion, e.g., by semimartingales with stationary increments.

There are many invariant manifold theorems for stochastic differential equations. In the case of a finite dimensional state space, let us mention [51, 52, 53, 2, 54]. For infinite dimensional state spaces, invariant manifold theorems were proved by Mohammed and Scheutzow for a class SDDEs in [3] and for different classes of stochastic partial differential equations in [55, 56, 57, 58, 59, 60, 61, 62, 63, 64].

The structure of the chapter is as follows: In Section 5.1, we study properties of rough delay differential equations. In particular, we prove their differentiability and provide bounds for the derivative. We furthermore study equations with a linear drift term. Section 5.3 contains our main results. We introduce random fixed points for cocycles (*stationary trajectories*) around which the invariant manifolds exist. The main results are formulated in Theorem 5.4.4 and Theorem 5.4.5. Subsection 5.4 contains examples of equations for which our theorems apply.

Preliminaries and notation

We use the same notations as previous chapters; here, we collect some additional notations.

- *Differentiable* will always mean differentiability in Fréchet-sense.
- If not stated differently, U, V, W and $\bar{W}$ will always denote finite-dimensional, normed vector spaces over the real numbers, with norm denoted by $|\cdot|$. The space $L(U,W)$ consists of all bounded linear functions from U to W equipped with usual operator norm.
- By $C_b^n(W^2,\bar{W})$, we denote the space of bounded functions $\sigma\colon W\oplus W\to\bar{W}$ having n bounded derivatives such that the n-the order derivatives are continuous, where $n\geq 0$. Often, we will omit domain and codomain and just write C_b^n. We set $\sigma_{x^n,y^m} := \frac{\partial^{n+m}}{\partial x^n\partial y^m}\sigma(x,y)$ for $n,m\geq 0$ and $\sigma_x := \sigma_{x^1,y^0}$, $\sigma_y := \sigma_{x^0,y^1}$. Dropping the subindex b means dropping the boundedness assumption.

5.1 Properties of nonlinear rough delay equations

In this section, we study different aspects of nonlinear rough delay differential equations. For simplicity, we will study equations without a drift coefficient first. Fix a delay $r>0$ and consider

$$\begin{aligned} y_t &= \xi_0 + \int_0^t \sigma(y_s, y_{s-r})\, d\mathbf{X}_s; \quad t \in [0,r] \\ y_t &= \xi_t; \quad t \in [-r, 0] \end{aligned} \tag{5.1.1}$$

where $\mathbf{X} = (X, \mathbb{X}, \mathbb{X}(-r))$ is a delayed γ-rough path, $\gamma \in (1/3, 1/2]$, and $X \colon \mathbb{R} \to U$ is locally γ-Hölder continuous. We recall the following result:

Theorem 5.1.1. *Assume $\sigma \in C_b^3(W^2, L(U,W))$, $1/3 < \alpha \leq \beta < \gamma \leq 1/2$ and either $\xi \in \mathscr{D}_X^{\beta}([-r,0],W)$ or $\xi \in \mathscr{D}_X^{\alpha,\beta}([-r,0],W)$. Then the equation* (5.1.1) *has a unique solution $y \in \mathscr{D}_X^{\beta}([0,T],W)$ resp. $y \in \mathscr{D}_X^{\alpha,\beta}([0,T],W)$ for any $T > 0$. In both cases, $y'_t = \sigma(y_t, y_{t-r})$.*

Proof. The case $\xi \in \mathscr{D}_X^{\beta}([-r,0],W)$ was shown in Theorem 4.2.8 and the case $\xi \in \mathscr{D}_X^{\alpha,\beta}([-r,0],W)$ follows from continuity of the solution map, cf. Theroem 4.2.9. □

Regularity

In this subsection, we will study the regularity of the solution map induced by (5.1.1). More precisely, we will give sufficient conditions under which this map is differentiable in the initial condition, which means differentiability in Fréchet-sense on the space of controlled paths. To prove our result, we will follow a similar strategy as in [65] and [66].

Definition 5.1.2. *For $m \in \mathbb{N}$ and $0 < \kappa \leqslant 1$, we say that $f \colon V^2 \to W$ belongs to $\mathscr{C}^{m+\kappa}(V^2, W)$ if its derivatives up to order m are bounded and continuous and if $D^m f$ is κ- Hölder continuous. The space is equipped by the norm*

$$\|f\|_{\mathscr{C}^{m+\kappa}} = \max_{j=0,\dots,m} \{\|D^j f\|_\infty, \|D^m f\|_\kappa\}.$$

Next, we give a more general definition of a delayed controlled path.

Definition 5.1.3. *Let $I = [a,b]$. We say that $m \colon I \to W$ is a* delayed (α,β,θ)-controlled path based on X on the interval I *if there exist paths $\zeta^0, \zeta^1 \colon I \to L(U, \bar{W})$ such that*

$$m_{s,t} = \zeta_s^0 X_{s,t} + \zeta_s^1 X_{s-r,t-r} + m_{s,t}^{\#}$$

holds for all $s,t \in I$ where

$$\|m\|_{\alpha;I}, \|\zeta^0\|_{\beta;I}, \|\zeta^1\|_{\beta;I} \text{ and } \|m^{\#}\|_{\theta;I} < \infty.$$

We denote the corresponding space by $\mathcal{D}_X^{\alpha,\beta,\theta}(I, \bar{W})$ where the norm on this space is defined as

$$\|m\|_{\mathcal{D}_X^{\gamma}} := \|(m, \zeta^0, \zeta^1)\|_{\mathcal{D}_X^{\gamma}} := |m_a| + |\zeta_a^0| + |\zeta_a^1| + \|m\|_{\alpha;I} + \|\zeta^0\|_{\beta;I} + \|\zeta^1\|_{\beta;I} + \|m^{\#}\|_{\theta;I}. \tag{5.1.2}$$

Remark 5.1.4. *Clearly, $\mathcal{D}_X^{\beta,\beta,2\beta}(I,\bar{W}) = \mathcal{D}_X^{\beta}(I,\bar{W})$. Using the sewing lemma [39, Lemma 4.2], it is easy to check that we can define an integral of the form*

$$\int m\, d\mathbf{X}$$

as in Theorem 4.2.5 for delayed γ-rough paths $\mathbf{X}$ and delayed (α,β,θ)-controlled paths m provided $\theta+\gamma>1$ and $\beta+2\gamma>1$. Furthermore, the (linear) map

$$\begin{aligned} \mathcal{D}_X^{\alpha,\beta,\theta}(I, L(U,W)) &\to \mathcal{D}_X^{\gamma,\alpha,2\gamma}(I,W) \\ m &\mapsto \int m\, d\mathbf{X} \end{aligned}$$

is well defined and continuous .

The next theorem is a version of the Omega lemma [66, Proposition 5] for delayed controlled paths.

Theorem 5.1.5. *(Delayed Omega lemma) Let $n\in\mathbb{N}$ and $0<\kappa\leqslant 1$ for $G\in\mathscr{C}^{n+1+\kappa}(V^2,W)$, $\eta\in(0,1)$ and $r>0$. Then the map*

$$\begin{aligned} \mathfrak{D}G : \mathscr{D}_X^{\beta}([0,r],V)\times\mathscr{D}_X^{\beta}([-r,0],V) &\to \mathcal{D}_X^{\beta,\beta\eta\kappa,\beta(1+\eta\kappa)\wedge 2\beta}([0,r],W) \\ (y_t,\xi_{t-r})_{t\in[0,r]} &\mapsto (G(\xi_0+y_t,\xi_{t-r}))_{t\in[0,r]} \end{aligned}$$

is locally of class $\mathscr{C}^{n+\kappa(1-\eta)}$.

Proof. We noted in Remark 4.2.4 that every delayed controlled path based on X can be seen as a usual controlled path based on $(X, X_{\cdot-r})$ and vice versa. Using this identification, the assertion just follows from [66, Proposition 5]. $\square$

Thanks to the delayed Omega lemma, we can state the following theorem:

Theorem 5.1.6. *Let $0<\kappa\leqslant 1$, $2\leqslant n+\kappa$ and $\sigma\in\mathscr{C}^{n+1+\kappa}(W^2, L(U,W))$. For a delayed γ-rough path $\mathbf{X}$, consider equation (5.1.1). Then, under the same assumptions as in Theorem 5.1.1, the solution map induced by (5.1.1) is locally of class $\mathscr{C}^{n+\kappa(1-\eta)}$ for any $\eta\in(0,1)$ provided $\beta(2+\kappa\eta)>1$.*

Proof. Fix $\hat{\xi}\in\mathscr{D}_X^{\beta}([-r,0],W)$. We aim to prove the claimed regularity in a neighbourhood around $\hat{\xi}$. Choose $M>0$ such that

$$\hat{\xi}\in B := \{\xi\in\mathscr{D}_X^{\beta}([-r,0],W),\ \|\xi\|_{\mathscr{D}_X^{\beta}([-r,0],W)} < M\}.$$

Let $\mathscr{D}_{X,0}^{\beta}([a,b],W)$ be the set of functions in $\mathscr{D}_X^{\beta}([a,b],W)$ starting from 0. Let $0<t_0\leqslant r$ and define

$$\begin{aligned} \Gamma : B\times\mathscr{D}_{X,0}^{\beta}([0,t_0],W) &\to \mathscr{D}_{X,0}^{\beta}([0,t_0],W) \\ (\xi_{t-r},y_t)_{0\leqslant t\leqslant t_0} &\mapsto \left(\int_0^t \sigma(y_\tau+\xi_0,\xi_{\tau-r})\,d\mathbf{X}_\tau\right)_{0\leqslant t\leqslant t_0}. \end{aligned} \tag{5.1.3}$$

Note that by Remark 5.1.4 and Theorem 5.1.5, this map is locally of class $\mathscr{C}^{n+\kappa(1-\eta)}$. Using the estimates (59) and (61) in [37], we see that

$$\begin{aligned}\|\Gamma(\xi,y)\|_{\mathscr{D}_X^\beta[0,t_0]} &\leqslant C_1A^3(1+\|\xi\|^2_{\mathscr{D}_X^\beta[-r,0]})(1+t_0^{\gamma-\beta}\|y\|^2_{\mathscr{D}_X^\beta[0,t_0]})\\ \|\Gamma(\xi,y)-\Gamma(\xi,\tilde y)\|_{\mathscr{D}_X^\beta[0,t_0]} &\leqslant C_1A^3\big(1+\|y\|_{\mathscr{D}_X^\beta[0,t_0]}+\|\tilde y\|_{\mathscr{D}_X^\beta[0,t_0]}+\|\xi\|_{\mathscr{D}_X^\beta[-r,0]}\big)^2\|y-\tilde y\|_{\mathscr{D}_X^\beta[0,t_0]}t_0^{\gamma-\beta}\end{aligned} \tag{5.1.4}$$

where C_1 only depends on σ. Let $C := C_1A^3(1+M^2)$ and set $\tau_1 := (8C^2)^{\frac{-1}{\gamma-\beta}}$. From [37, Lemma 4.1],

$$\sup\{u\in\mathbb{R}^+ \,:\, C(1+\tau_1^{\gamma-\beta}u^2)\leqslant u\}\leqslant(4+2\sqrt{2})C =: M_1. \tag{5.1.5}$$

Choose τ_2 such that

$$C_1A^3(1+2M_1+M)^2\tau_2^{\gamma-\beta}\le\frac{1}{2}.$$

Set $\tau_3 := \min\{\tau_1,\tau_2,r\}$. Choosing τ_3 smaller if necessary, we can assume that $N := \frac{r}{\tau_3}\in\mathbb{N}$. Set

$$B_1 := \Big\{y\in\mathscr{D}^\beta_{X,0}([0,\tau_3],W) \,:\, \|y\|_{\mathscr{D}^\beta_{X,0}([0,\tau_3],W)}\leqslant M_1\Big\}.$$

With this choice, the map

$$\Gamma_1 := \Gamma|_{B\times B_1}\colon B\times B_1\to B_1$$

is well defined. Moreover, for fixed $\hat\xi\in B$,

$$\begin{aligned}\Lambda_1 : B_1 &\to B_1\\ (y_s)_{0\leqslant s\leqslant\tau_3} &\mapsto \Big(\int_0^s\sigma(\hat\xi_0+y_\tau,\hat\xi_{\tau-r})\,d\mathbf{X}_\tau\Big)_{0\leqslant s\leqslant\tau_3}\end{aligned}$$

is a contraction, so it admits a unique fixed point which we denote by $(z_s^{1,\hat\xi})_{0\leqslant s\leqslant\tau_3}$. This shows that we can use the implicit function theorem on Banach spaces (cf. [67, 2.5.7 Implicit Function Theorem] or [66, Theorem 1]) to see that there is a neighbourhood U around $\hat\xi$ such that for every $\xi\in U$, there are functions $(z_s^{1,\xi})_{0\leqslant s\leqslant\tau_3}$ with the property that $\Lambda_1(z^{1,\xi})=z^{1,\xi}$ and the map $\xi\mapsto z^{1,\xi}$ is of class $\mathscr{C}^{n+\kappa(1-\eta)}$. Therefore, $\xi\mapsto(y_s^{1,\xi}=\xi_0+z_s^{1,\xi})_{0\leqslant s\leqslant\tau_3}$, which is the solution of equation (5.1.1) in $[0,\tau_3]$, is also locally of class $\mathscr{C}^{n+(1-\eta)\kappa}$. Moreover,

$$\|z^{1,\xi}\|_{\mathscr{D}_X^\beta([0,\tau_3])}\leqslant(4+2\sqrt{2})C \tag{5.1.6}$$

holds for every $\xi\in U$. Now we proceed inductively. For $2\leqslant j\leqslant N$, define

$$B_j = \Big\{y\in\mathscr{D}^\beta_{X,0}([(j-1)\tau_3,j\tau_3],W) \,:\, \|y\|_{\mathscr{D}^\beta_{X,0}[(j-1)\tau_3,j\tau_3]}\leqslant M_1\Big\}$$

and

$$\begin{aligned}\Lambda_j : B_j &\to B_j\\ (y_s)_{(j-1)\tau_3\leqslant s\leqslant j\tau_3} &\mapsto \left(\int_{(j-1)\tau_3}^s \sigma(y^{j-1,\hat\xi}_{(k-1)\tau_3}+y_\tau,\hat\xi_{\tau-r})d\mathbf{X}_\tau\right)_{(j-1)\tau_3\leqslant s\leqslant j\tau_3}.\end{aligned}$$

Again, this map is contraction and admits a unique fixed point, namely $(z^{j,\hat\xi}_s)_{(j-1)\tau_3\leqslant s\leqslant j\tau_3}$, and a locally defined map $\xi\mapsto (z^{j,\xi}_s)_{(j-1)\tau_3\leqslant s\leqslant j\tau_3}$ which is of class $\mathscr{C}^{n+\kappa(1-\eta)}$. Again,

$$\|z^{j,\xi}\|_{\mathscr{D}^\beta_X([(j-1)\tau_3,j\tau_3])}\leqslant (4+2\sqrt{2})C \tag{5.1.7}$$

holds for all ξ in a neighbourhood around $\hat\xi$. This shows that $(y^{j,\xi}_s = y^{j-1,\xi}_{(j-1)\tau_3}+z^{j,\xi}_s)_{(j-1)\tau_3\leqslant s\leqslant j\tau_3}$, the solution of (5.1.1) in $[(j-1)\tau_3, j\tau_3]$, has the same local regularity. Finally, the following map is locally of class $\mathscr{C}^{n+\kappa(1-\eta)}$:

$$\begin{aligned}\Lambda : B &\to \prod_{1\leqslant j\leqslant N}\mathscr{D}^\beta_X[(j-1)\tau_3,j\tau_3]\\ \xi &\mapsto \prod_{1\leqslant j\leqslant N}(y^{j,\xi}_s)_{(j-1)\tau_3\leqslant s\leqslant j\tau_3}.\end{aligned}$$

Since we can consider $\mathscr{D}^\beta_X[0,r]$ as a closed subspace of $\prod_{1\leqslant j\leqslant N}\mathscr{D}^\beta_X[(j-1)\tau_3,j\tau_3]$, the regularity claim is proved. ☐

Remark 5.1.7. *Since $C^3_b\subset\mathscr{C}^3$, Theorem 5.1.6 implies that the solution of* (5.1.1) *is Fréchet differentiable in the initial condition.*

The proof of Theorem 5.1.6 also reveals a bound for the solution to (5.1.1) which we record in the next theorem.

Theorem 5.1.8. *Under the same assumptions as in Theorem 5.1.1, there exists a polynomial $P:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ such that its coefficients depend on σ, β and γ and if y^ξ denotes the solution to* (5.1.1) *with initial condition ξ, we have*

$$\|y^\xi\|_{\mathscr{D}^\beta_X([0,r])}\leqslant P(A,\|\xi\|_{\mathscr{D}^\beta_X([-r,0])}) \tag{5.1.8}$$

where $A = 1+\|\mathbf{X}\|_{\gamma,[0,r]}$.

Proof. With the same notation as in the proof of Theorem 5.1.6,

$$\|(y^\xi)^\#\|_{2\beta,[0,r]}\leqslant \sum_{1\leqslant k\leqslant N}\|(z^{k,\xi})^\#\|_{2\beta,[(k-1)\tau_3,k\tau_3]} + r^{\gamma-\beta}\|X\|_{\gamma,[0,r]}\sum_{1\leqslant k\leqslant N}\|(z^{k,\xi})'\|_{\beta,[(k-1)\tau_3,k\tau_3]}. \tag{5.1.9}$$

The estimate (5.1.8) now follows from (5.1.7), (5.1.9), subadditivity of the Hölder norm and our choice for τ_3. ☐

It is possible to show that all derivatives solve linear, non-autonomous rough delay equations obtained by formally taking the derivatives of (5.1.1). We give a proof of this result for the first derivative in the next proposition. Higher order derivatives can be treated similarly.

Proposition 5.1.9. *For $\xi \in \mathscr{D}_X^{\beta}([-r,0],W)$, let $(y_t^{\xi})_{0\le t\le r}$ be the solution to* (5.1.1). *The derivative of the solution at ξ in the direction of $\tilde{\xi}$ exists and satisfies the following equation:*

$$\begin{aligned} Dy^{\xi}[\tilde{\xi}](t) - \tilde{\xi}_0 &= \int_0^t [\sigma_x(y_\tau^{\xi}, \xi_{\tau-r}) Dy^{\xi}[\tilde{\xi}](\tau) + \sigma_y(y_\tau^{\xi}, \xi_{\tau-r})\tilde{\xi}_{\tau-r}]\, d\mathbf{X}_\tau; \quad t \in [0,r] \\ Dy^{\xi}[\tilde{\xi}](t) &= \tilde{\xi}_t; \quad t \in [-r,0]. \end{aligned} \tag{5.1.10}$$

Proof. By definition,

$$\begin{aligned} &\frac{y_{s,t}^{\xi+z\tilde{\xi}} - y_{s,t}^{\xi}}{z} - \int_s^t [\sigma_x(y_\tau^{\xi}, \xi_{\tau-r}) Dy^{\xi}[\tilde{\xi}](\tau) + \sigma_y(y_\tau^{\xi}, \tau_{\tau-r})\tilde{\xi}_{\tau-r}]\, d\mathbf{X}_\tau \\ &= \int_s^t \left[\frac{\sigma(y_\tau^{\xi+z\tilde{\xi}}, \xi_\tau + z\tilde{\xi}_{\tau-r}) - \sigma(y_\tau^{\xi}, \xi_{\tau-r})}{z} - [\sigma_x(y_\tau^{\xi}, \xi_{\tau-r}) Dy^{\xi}[\tilde{\xi}](\tau) + \sigma_y(y_\tau^{\xi}, \xi_{\tau-r})\tilde{\xi}_{\tau-r}] \right] d\mathbf{X}_\tau \\ &= \int_s^t \Big[[A_\tau^z M_\tau^z + B_\tau^z] - [A_\tau M_\tau + B_\tau] \Big] d\mathbf{X}_\tau \end{aligned}$$

where

$$A_\tau^z = \int_0^1 \sigma_x(\eta y_\tau^{\xi+z\tilde{\xi}} + (1-\eta) y_\tau^z, \xi_{\tau-r}, +\eta z \tilde{\xi}_{\tau-r})\, d\eta \quad , \quad M_\tau^z = \frac{y_\tau^{\xi+z\tilde{\xi}} - y_\tau^{\xi}}{z}$$

$$B_\tau^z = \int_0^1 \sigma_y(\eta y_\tau^{\xi+z\tilde{\xi}} + (1-\eta) y_\tau^{\xi}, \xi_{\tau-r}, +\eta z \tilde{\xi}_{\tau-r})\tilde{\xi}_{\tau-r}\, d\eta$$

and

$$A_\tau = \sigma_x(y_\tau^{\xi}, \xi_{\tau-r}), \quad M_\tau = Dy^{\xi}[\tilde{\xi}](\tau), \quad B_\tau = \sigma_y(y_\tau^{\xi}, \xi_{\tau-r})\tilde{\xi}_{\tau-r}.$$

Note that by Theorem (5.1.6), $\lim_{z\to 0} \|M_\cdot^z - M_\cdot\|_{\mathscr{D}_X^{\beta}[0,r]} = 0$. From continuity in the initial condition, we furthermore see that $\lim_{z\to 0} \|y^{\xi+z\tilde{\xi}} - y^{\xi}\|_{\mathscr{D}_X^{\beta}[0,r]} = 0$. Consequently, thanks to our assumptions on σ, it is not hard too see that

$$\lim_{z\to 0} \left[\big\| [A_\cdot^z M_\cdot^z + B_\cdot^z] - [A_\cdot M_\cdot + B_\cdot] \big\|_{\mathcal{D}_X^{\beta}[0,r]} \right] = 0.$$

Using remark (5.1.4), equality (5.1.10) can be verified. □

5.2 Rough delay equations with a linear drift

Our next goal is to generalize the theory in order to include a drift term in the equation. More precisely, we aim to solve the equation

$$\begin{aligned} dy_t &= B(y_t, y_{t-r})dt + \sigma(y_t, y_{t-r})d\mathbf{X}_t \\ y_s &= \xi_s, \quad -r \leqslant s \leqslant 0 \end{aligned} \tag{5.2.1}$$

with initial condition $\xi \in \mathscr{D}_X^{\beta}([-r,0],W)$ for a linear drift $B : W^2 \to W$ and to give a bound for the solution map. We believe that we could even include a nonlinear drift satisfying suitable growth assumptions as in [68], but we restrict ourselves to a linear drift here for the sake of simplicity. The next theorem is the main result of this section.

Theorem 5.2.1. *Let $\sigma \in C_b^4$. Then the equation* (5.2.1) *has a unique solution $y \in \mathscr{D}_X^\beta([0,r],W)$. Moreover, there is a polynomial Q depending on B, σ, γ and β such that*

$$\|y\|_{\mathscr{D}_X^\beta([0,r])} \leq Q(A, \|\xi\|_{\mathscr{D}_X^\beta([-r,0])})$$

where $A = 1 + \|\mathbf{X}\|_{\gamma,[0,r]}$.

Proof. The idea is to give a representation of the solution to (5.2.1) using the flow map of the respective equation omitting the drift term. Let $\xi \in \mathscr{D}_X^\beta([-r,0],W)$ be fixed and consider the equation

$$\begin{aligned} dy_t &= \sigma(y_t, \xi_{t-r})\, d\mathbf{X}_t \\ y_s &= x, \quad 0 \leqslant s \leqslant t \leqslant r. \end{aligned} \tag{5.2.2}$$

Existence and uniqueness of this equation can be shown similarly to the usual delay case. We use $\bar{\varphi}(s,t,x)$ to denote the solution of (5.2.2) at time t with initial condition $y_s = x$. From uniqueness of the solution, we have for every $\tau \leqslant s \leqslant t$,

$$\bar{\varphi}(\tau,t,x) = \bar{\varphi}(s,t,\bar{\varphi}(\tau,s,x)).$$

As for usual rough differential equations [40, Theorem 10.14], one can show that there is a polynomial P_1 such that

$$\sup_{x\in W, 0\leqslant s\leqslant t\leqslant r} \|\bar{\varphi}(s,t,x) - x\| \leqslant (t-s)^\beta P_1(A, \|\xi\|_{\mathscr{D}_X^\beta([-r,0])}). \tag{5.2.3}$$

In addition, one can check that the solution is differentiable with respect to initial value and that its derivative is the matrix solution of the equation

$$D\bar{\varphi}(s,t,x) - I = \int_s^t \sigma_x(\bar{\varphi}(s,\tau,x), \xi_{\tau-r}) D\bar{\varphi}(s,\tau,x) d\mathbf{X}_\tau.$$

Let $0 < t_0 < r$ be fixed. For $0 \leqslant \tau < \varsigma \leqslant t_0$, we define

$$\tilde{X}_\tau := X_{t_0-\tau}, \quad \tilde{\mathbb{X}}_{\tau,\varsigma} := -\mathbb{X}_{t_0-\varsigma,t_0-\tau}, \quad \tilde{\mathbb{X}}_{\tau,\varsigma}(-r) := -\mathbb{X}_{t_0-\varsigma,t_0-\tau}(-r).$$

We say that $\eta \in \tilde{\mathscr{D}}_{\tilde{X}}^\beta([a,b],W)$ if we have a decomposition of the form

$$\eta_{s,t} = \eta'_t \tilde{X}_{s,t} + \eta^\#_{s,t}$$

where

$$\|\eta'\|_{\beta;[a,b]} < \infty \quad \text{and} \quad \sup_{s<t} \frac{|\eta^\#_{s,t}|}{(t-s)^{2\beta}} < \infty.$$

Using the sewing lemma [39, Lemma 4.2] we can also define

$$\int_{[a,b]} \eta_\tau d\tilde{\mathbf{X}}_\tau := \lim_{|\Pi|\to 0} \sum_{\Pi} [\eta_{\tau_{j+1}} \tilde{X}_{\tau_j,\tau_{j+1}} + \eta'_{\tau_{j+1}} \tilde{\mathbb{X}}_{\tau_j,\tau_{j+1}}]$$
$$\int_{[a,b]} \eta_{\tau-r} d\tilde{\mathbf{X}}_\tau := \lim_{|\Pi|\to 0} \sum_{\Pi} [\eta_{\tau_{j+1}-r} \tilde{X}_{\tau_j,\tau_{j+1}} + \eta'_{\tau_{j+1}-r} \tilde{\mathbb{X}}_{\tau_j,\tau_{j+1}}(-r)].$$

For $\xi \in \mathscr{D}_X^\beta([a,b],W)$, it is straightforward to check that $\tilde{\xi}_\cdot := \xi_{t_0-\cdot} \in \tilde{\mathscr{D}}_{\tilde{X}}^\beta([t_0-b,t_0-a],W)$ and that

$$\int_{[a,b]} \xi_\tau d\mathbf{X}_\tau = \int_{[t_0-b,t_0-a]} \tilde{\xi}_\tau d\tilde{\mathbf{X}}_\tau.$$

For $s_0 \leqslant t_0 \leqslant r$ and $\tilde{\varphi}(s_0,t,x) := \bar{\varphi}(s_0,t_0-t,x)$ we consider the equation

$$\begin{aligned} dZ_t &= \sigma_x(\tilde{\varphi}(s_0,t,x_0),\tilde{\xi}_{t-r}) Z_t d\tilde{\mathbf{X}}_t \\ Z_0 &= I, \quad 0 \leqslant t \leqslant t_0 - s_0. \end{aligned} \tag{5.2.4}$$

Then

$$Z_{t_0-s_0} = [D\bar{\varphi}(s_0,t_0,x)]^{-1}.$$

Thus by standard estimates for linear equations [40, Theorem 10.53], we have a bound of the form

$$\begin{aligned} &\sup_{s\leqslant t\leqslant r, x\in W} \|[D\bar{\varphi}(s,t,x)]^{-1} - I\| \leqslant \\ &\qquad M(t-s)^\beta P_2(A, \|\xi\|_{\mathscr{D}_X^\beta([-r,0])}) \exp\left((t-s)P_2(A, \|\xi\|_{\mathscr{D}_X^\beta([-r,0])})\right) \end{aligned} \tag{5.2.5}$$

where M is just a general constant and P_2 is a polynomial. Now we consider the ODE

$$\begin{aligned} d\eta_t &= [D\bar{\varphi}(0,t,\eta_t)]^{-1} B(\bar{\varphi}(0,t,\eta_t),\xi_{t-r}) dt \\ \eta_0 &= \xi_0. \end{aligned}$$

Using the chain rule, it is straightforward to see that $\bar{\varphi}(0,t,\eta_t)$ solves (5.2.1). Next, we choose $\tau > 0$ sufficiently small such that

$$M\tau^\beta P_2(A, \|\xi\|_{\mathscr{D}_X^\beta([-r,0])}) \exp(\tau P_2(A, \|\xi\|_{\mathscr{D}_X^\beta([-r,0])})) \leqslant 1$$

holds. Using some basic calculations, we can check that there is a polynomial P_3 such that

$$\frac{r}{\tau} = P_3(A, \|\xi\|_{\mathscr{D}_X^\beta([-r,0])}). \tag{5.2.6}$$

Choosing τ smaller if necessary, we can assume that there is some $n \in \mathbb{N}$ such that $n\tau = r$. Define $I_m := [(m-1)\tau, m\tau]$ for $1 \leq m \leqslant n$ and $\eta_0^0 := \xi_0$. Inductively, we define the equations

$$\begin{aligned} d\eta_t^m &= [D\bar{\varphi}_x((m-1)\tau, t, \eta_t^m)]^{-1} B(\bar{\varphi}((m-1)\tau, t, \eta_t^m), \xi_{t-r}) dt, \quad t \in [(m-1)\tau, m\tau] \\ \eta_{(m-1)\tau}^m &= \bar{\varphi}((m-1)\tau, \eta_{(m-1)\tau}^{m-1}). \end{aligned} \tag{5.2.7}$$

Again, it is not hard to see that

$$y_t = \bar{\varphi}((m-1)\tau, t, \eta_t^m), \quad t \in [(m-1)\tau, m\tau]$$

solves (5.2.1). From (5.2.5),

$$\|\eta_t^m\| - \|\eta_{(m-1)\tau}^m\| \leqslant 2\|B\| \int_{(m-1)\tau}^t [\|\bar{\varphi}((m-1)\tau, \varsigma, \eta_\varsigma^m)\| + \|\xi_{\varsigma-r}\|] d\varsigma.$$

By Grönwall's lemma and (5.2.3), we can deduce that there is for a constant M and polynomial P_4 such that

$$\|\eta^m\|_{\infty;I_m} \leqslant \exp(2\|B\|\tau)\|\eta^m\|_{\infty;I_{m-1}} + M\big[\exp(2\|B\|\tau) - 1\big]\big[\|\xi\|_\infty + P_4(A, \|\xi\|_{\mathscr{D}_X^\beta([-r,0])})\big].$$

Finally, from (5.2.3) and (5.2.6), for a polynomial P_5,

$$\|y\|_{\infty;[0,r]} \leqslant P_5(A, \|\xi\|_{\mathscr{D}_X^\beta([-r,0])}). \tag{5.2.8}$$

Remember that

$$y_{s,t} = \int_s^t B(y_\varsigma, \xi_{\varsigma-r})\, d\varsigma + \int_s^t \sigma(y_\varsigma, \xi_{\varsigma-r})\, d\mathbf{X}_\varsigma.$$

Using the standard estimate for the rough integral [39, Theorem 4.10] and (5.2.8), we obtain for $0 \leqslant s < t \leqslant r$

$$\begin{aligned} &\|y\|_{\beta;[s,t]} + \|y^\#\|_{2\beta;[s,t]} \leqslant \\ &\qquad P_6(A, \|\xi\|_{\mathscr{D}_X^\beta([-r,0])}) + (t-s)^{\gamma-\beta} P_7(A, \|\xi\|_{\mathscr{D}_X^\beta([-r,0])})[\|y\|_{\beta;[s,t]} + \|y^\#\|_{2\beta;[s,t]}] \end{aligned} \tag{5.2.9}$$

where P_6 and P_7 are polynomials. Again, we can find a polynomial P_8 and $\tau > 0$ such that

$$\frac{r}{\tau} = P_8(A, \|\xi\|_{\mathscr{D}_X^\beta([-r,0])}) \quad \text{and} \quad \tau^{\gamma-\beta} P_7(A, \|\xi\|_{\mathscr{D}_X^\beta([-r,0])}) \leqslant \frac{1}{2}.$$

Finally, from (5.2.9) and subadditivity of the Hölder norm, we can deduce the existence of a polynomial Q such that

$$\|y\|_{\mathscr{D}_X^\beta([0,r])} \leqslant Q(A, \|\xi\|_{\mathscr{D}_X^\beta([-r,0])}). \tag{5.2.10}$$

□

Corollary 5.2.2. *Under the same assumptions as in Theorem 5.2.1, the results of Theorem 5.1.6 and Proposition 5.1.9 hold for equation* (5.2.1)*, too.*

Proof. We can rewrite the equation (5.2.1) as

$$\begin{aligned} dy_t &= \tilde{\sigma}(y_t, y_{t-r})d\tilde{\mathbf{X}}_t \\ y_s &= \xi_s, \quad -r \leqslant s \leqslant 0 \end{aligned} \tag{5.2.11}$$

where $\tilde{\sigma} := (B, \sigma)$ and $\tilde{\mathbf{X}}$ is the delayed rough path obtained from $\mathbf{X}$ by including $t \mapsto t$ as a smooth component, cf. [40, Section 9.4]. Note that $\tilde{\sigma}$ has the same smoothness as σ. Fixing an initial condition ξ and a neighbourhood around it, we can assume that $\tilde{\sigma}$ is bounded for these initial conditions by replacing the unbounded $\tilde{\sigma}$ by a version which is compactly supported in the region where the respective solutions take their values. Therefore, we can directly apply Theorem 5.1.6 and Proposition 5.1.9 to (5.2.11). □

We finally give some bounds for the solution to the linearized equation. Since the proofs are a bit technical, we decided to put them in the appendix.

We finally give some bounds for the solution to the linearized equation. Since the proofs are a bit technical, we decided to put them in the appendix.

Theorem 5.2.3. *Assume $\sigma \in C_b^3$. Then the solution of* (5.1.1) *is differentiable and if $Dy^\xi[\tilde{\xi}]$ denotes the derivative at ξ in the direction $\tilde{\xi}$, we have the bound*

$$\|Dy^\xi[\tilde{\xi}]\|_{\mathscr{D}_X^\beta[0,r]} \leqslant \|\tilde{\xi}\|_{\mathscr{D}_X^\beta[-r,0]} \exp[Q(A, \|\xi\|_{\mathscr{D}_X^\beta[-r,0]})] \tag{5.2.12}$$

where Q is a polynomial and $A = 1 + \|\mathbf{X}\|_{\gamma,[0,r]}$. If $\sigma \in C_b^4$, we have the same result for equation (5.2.1)*.*

Proof. Cf. appendix. □

Theorem 5.2.4. *Under the same assumptions as in Theorem 5.2.3,*

$$\|Dy^\xi[\eta] - Dy^{\tilde{\xi}}[\eta]\|_{\mathscr{D}_X^\beta[0,r]} \leqslant \|\xi - \tilde{\xi}\|_{\mathscr{D}_X^\beta[-r,0]} \|\eta\|_{\mathscr{D}_X^\beta[-r,0]} \exp\left[P(A, \|\xi\|_{\mathscr{D}_X^\beta[-r,0]}, \|\xi - \tilde{\xi}\|_{\mathscr{D}_X^\beta[-r,0]})\right] \tag{5.2.13}$$

for a polynomial P.

Proof. Cf. appendix. □

Remark 5.2.5. *Note that since P is a polynomial, we can find a polynomial $\tilde{P}$ and an increasing function $\tilde{Q}$ such that also*

$$\begin{aligned} \|Dy^\xi[\eta] - Dy^{\tilde{\xi}}[\eta]\|_{\mathscr{D}_X^\beta[0,r]} \leqslant \|\xi - \tilde{\xi}\|_{\mathscr{D}_X^\beta[-r,0]} \|\eta\|_{\mathscr{D}_X^\beta[-r,0]} \exp\left[\tilde{P}(A, \|\xi\|_{\mathscr{D}_X^\beta[-r,0]})\right] \\ \times \exp\left[\tilde{Q}(\|\xi - \tilde{\xi}\|_{\mathscr{D}_X^\beta[-r,0]})\right] \end{aligned} \tag{5.2.14}$$

holds.

Remark 5.2.6. *If $f\colon W^2 \to W$ has the same smoothness as σ and is bounded with bounded derivatives, the equation*

$$\begin{aligned} dy_t &= B(y_t, y_{t-r})\,dt + f(y_t, y_{t-r})\,dt + \sigma(y_t, y_{t-r})\,d\mathbf{X}_t \\ y_s &= \xi_s, \quad -r \leqslant s \leqslant 0 \end{aligned} \tag{5.2.15}$$

with initial condition $\xi \in \mathscr{D}_X^{\beta}([-r,0],W)$ has a unique solution and all results in this section hold for (5.2.15)*, too, where the constants will now depend on f as well. As in the proof of Corollary 5.2.2, this just follows by including $t \mapsto t$ as a smooth component of $\mathbf{X}$ and viewing (f,σ) as an element in $C_b^4(W^2, L(\mathbb{R}\oplus U, W))$.*

5.3 Invariant manifolds for random rough delay equations

Let $B\colon W^2 \to W$ be a linear map and $\sigma \in C_b^3$ resp. $\sigma \in C_b^4$ in the case when $C \neq 0$. Our goal is to study invariant manifolds for the solution to stochastic delay differential equations of the form

$$dy_t = B(y_t, y_{t-r})\,dt + \sigma(y_t, y_{t-r}) \star dB_t(\omega) \tag{5.3.1}$$

where $\star dB(\omega)$ can be either the Itō- (when $H = \frac{1}{2}$) or the Stratonovich (symmetric integral) differential. As already pointed out in the previous chapter, it is equivalent to study the random rough delay equation

$$dy_t = B(y_t, y_{t-r})\,dt + \sigma(y_t, y_{t-r})\,d\mathbf{X}_t(\omega) \tag{5.3.2}$$

where $\mathbf{X}$ is either $\mathbf{B}^{\text{Itō}}$ (when $H = \frac{1}{2}$) or $\mathbf{B}$.

Recall that we could also add a smooth drift term to (5.3.2) as explained in Remark 5.2.6, but we will not do so in the sequel for the sake of clarity.

Using the same cut-off argument as in the proof to Corollary 5.2.2, we can deduce from theorem 4.2.13 that the solution to (5.3.2) induces a semi-flow ϕ on the spaces of controlled paths. From Theorem 4.3.7 , we can assume that there is an ergodic metric dynamical system $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t\in\mathbb{R}})$ on which $\mathbf{B}^{\text{Itō}}$ (when $H = \frac{1}{2}$) and $\mathbf{B}$ are defined and satisfy the cocycle property. More generally, from now on, we will consider an arbitrary *delayed γ-rough path cocycle* $\mathbf{X}$ which drives the equation (5.3.2), cf. Definition 4.3.1. With Theorem 4.3.11, we can deduce that $\varphi(n,\omega,\cdot) := \phi(0, nr, \omega, \cdot)$ is a continuous map

$$\varphi(n,\omega,\cdot)\colon \mathscr{D}_{X(\omega)}^{\alpha,\beta}([-r,0],W) \to \mathscr{D}_{X(\theta_{nr}\omega)}^{\alpha,\beta}([-r,0],W)$$

satisfying the cocycle property

$$\varphi(n+m,\omega,\cdot) = \varphi(n, \theta_{mr}\omega, \cdot) \circ \varphi(m,\omega,\cdot) \tag{5.3.3}$$

for every $n, m \in \mathbb{N}_0$ with parameters $\frac{1}{3} < \alpha < \beta < H$. From Corollary 5.2.2, the cocycle is differentiable. Set $\theta^n := \theta_{nr}$, $\theta := \theta^1$, then by Proposition 4.3.10, $\{\mathscr{D}_{X(\omega)}^{\alpha,\beta}([-r,0],W)\}_{\omega\in\Omega}$

constitutes a measurable field of Banach spaces, and the cocycle φ defined on the discrete metric dynamical system $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$ acts on it, cf. Theorem 4.3.14.

5.4 Random fixed points and formulation of the main theorems

In order to deduce the existence of invariant manifolds, we aim to linearize the equation (5.3.2) around random fixed points which we define now.

Definition 5.4.1. *Let φ be a cocycle defined on a metric dynamical system $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$ acting on a measurable field of Banach spaces $\{E_\omega\}_{\omega\in\Omega}$. A map $Y : \Omega \longrightarrow \prod_{\omega\in\Omega} E_\omega$ is called* stationary trajectory *if the following properties are satisfied:*

(i) $Y_\omega \in E_\omega$,

(ii) $\varphi(n, \omega, Y_\omega) = Y_{\theta^n\omega}$ *and*

(iii) $\omega \to \|Y_\omega\|_{E_\omega}$ *is measurable.*

For the given random fixed point, we first linearize (5.3.2) around it. We aim to apply our Multiplicative Ergodic Theorem to the linearized equation. The following Lemma gives a sufficient condition for this goal.

Lemma 5.4.2. *Assume that the cocycle induced by* (5.3.2) *admits a stationary trajectory Y and that*

$$Q(A_\omega, \|Y_\omega\|) \in L^1(\Omega)$$

holds for the polynomial Q obtained in Theorem 5.2.3 where $A_\omega = 1 + \|\mathbf{X}(\omega)\|_{\gamma,[0,r]}$. Then $\psi^n_\omega := D_{Y_\omega}\varphi(n, \omega, \cdot)$ defines a compact linear cocycle acting on the measurable field of Banach spaces $\{\mathscr{D}^{\alpha,\beta}_{X(\omega)}([-r, 0], W))\}_{\omega\in\Omega}$ and the semi-invertible Mutliplicative Ergodic Theorem 2.3.20 holds true.

Proof. It is straightforward to check that ψ satisfies the cocycle property. We need to verify Assumption 2.3.1 which also implies the measurability condition (2.1.4). The proof of Assumption 2.3.1 is very similar to the proof of Theorem 4.3.14 using that ψ solves a (non-autnonomous) linear delay equation, cf. Proposition 5.1.9 resp. Corollary 5.2.2, so we decided to omit it here. Compactness follows as in the proof of Proposition 4.2.12. From our assumption and Theorem 5.2.3, it follows that $\log^+ \|\psi^1\|$ is integrable. Therefore, all conditions of Theorem 2.3.20 are indeed satisfied. □

From now on, we assume that the conditions of Lemma 5.4.2 are satisfied. Let $\tilde{\Omega}$ denote the θ-invariant set of full measure provided in Theorem 2.3.20.

Definition 5.4.3. *Let $\{... < \mu_j < \mu_{j-1} < ... < \mu_1\} \in [-\infty, \infty)$ be the Lyapounov spectrum of ψ provided by the MET (Theorem 2.2.16) and let $\{H^i_\omega\}_{i\in\mathbb{N}}$ be the fast growing subspaces provided by the semi-invertible MET (Theorem 2.3.20). Recall the splitting*

$$\mathscr{D}^{\alpha,\beta}_{X(\omega)}([-r, 0], W)) = H^1_\omega \oplus \cdots \oplus H^n_\omega \oplus F_{\mu_{n+1}}(\omega)$$

for every $n \in \mathbb{N}_0$ and $\omega \in \tilde{\Omega}$ with $F_\mu(\omega)$ defined as inTheorem 2.2.16. Set $\mu_{j_0} := \max\{\mu_j : \mu_j < 0\}$ and $\mu_{j_0} := -\infty$ if all μ_j for which $\mu_j \neq -\infty$ are nonnegative. We define the stable subspace

$$S_\omega := F_{\mu_{j_0}}(\omega)$$

for $\omega \in \tilde{\Omega}$. Similarly, if $\mu_1 > 0$, set $k_0 := \min\{k : \mu_k > 0\}$ and define the unstable subspace

$$U_\omega := \oplus_{1 \leqslant i \leqslant k_0} H_\omega^i$$

for $\omega \in \tilde{\Omega}$. If $\mu_1 \leq 0$, we set $U_\omega := \{0\}$.

From both METs Theorem 2.2.16 and Theorem 2.3.20, we know that

$$\dim[\mathscr{D}_{X(\omega)}^{\alpha,\beta}([-r,0],W))/S_\omega] < \infty \quad \text{and} \quad \dim[U_\omega] < \infty$$

for every $\omega \in \tilde{\Omega}$ and that the dimension does not depend on ω. Note also that

$$\mathscr{D}_{X(\omega)}^{\alpha,\beta}([-r,0],W)) = U_\omega \oplus S_\omega$$

in the case where all Lyapounov exponents are nonzero.

Now we are ready to state our main results of this section. Note that they are basically reformulations of the abstract stable and unstable manifold theorems in Chapter 3, but we decided to give a full statement here for the readers convenience. We start with the stable case.

Theorem 5.4.4 (Local stable manifolds)**.** *Let $\mathbf{X}$ be a delayed γ-rough path cocycle defined on an ergodic metric dynamical system $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t\in\mathbb{R}})$ and let $\frac{1}{3} < \alpha < \beta < \gamma < \frac{1}{2}$. Assume $\sigma \in C_b^3$ resp. $\sigma \in C_b^4$ in the case $B \neq 0$. Assume also that the cocycle φ induced by* (5.3.2) *admits a stationary trajectory Y for which*

$$\tilde{P}(A_\omega, \|Y_\omega\|) \in L^1(\Omega) \quad \textit{and} \quad Q(A_\omega, \|Y_\omega\|) \in L^1(\Omega) \tag{5.4.1}$$

where $A_\omega = 1 + \|\mathbf{X}(\omega)\|_{\gamma,[0,r]}$, $\tilde{P}$ is the polynomial in (5.2.14) *and Q is the polynomial in* (5.2.12)*. Then there is a θ-invariant set of full measure $\tilde{\Omega}$ and a family of immersed submanifolds $S_{loc}^\upsilon(\omega)$ of $\mathscr{D}_{X(\omega)}^{\alpha,\beta}([-r,0],W))$, $0 < \upsilon < -\mu_{j_0}$ and $\omega \in \tilde{\Omega}$, satisfying in the following properties for every $\omega \in \tilde{\Omega}$:*

(i) There are random variables $\rho_1^\upsilon(\omega), \rho_2^\upsilon(\omega)$, positive and finite on $\tilde{\Omega}$, for which

$$\liminf_{p\to\infty} \frac{1}{p} \log \rho_i^\upsilon(\theta^p\omega) \geq 0, \quad i = 1,2 \tag{5.4.2}$$

and such that

$$\{\xi \in \mathscr{D}_{X(\omega)}^{\alpha,\beta} : \sup_{n\geqslant 0} \exp(n\upsilon)\|\varphi(n,\omega,\xi) - Y_{\theta^n\omega}\| < \rho_1^\upsilon(\omega)\} \subseteq S_{loc}^\upsilon(\omega)$$
$$\subseteq \{\xi \in \mathscr{D}_{X(\omega)}^{\alpha,\beta} : \sup_{n\geqslant 0} \exp(n\upsilon)\|\varphi(n,\omega,\xi) - Y_{\theta^n\omega}\| < \rho_2^\upsilon(\omega)\}.$$

(ii)

$$T_{Y_\omega} S^{\upsilon}_{loc}(\omega) = S_\omega.$$

(iii) For $n \geqslant N(\omega)$,

$$\varphi(n,\omega,S^{\upsilon}_{loc}(\omega)) \subseteq S^{\upsilon}_{loc}(\theta^n\omega).$$

(iv) For $0 < \upsilon_1 \leqslant \upsilon_2 < -\mu_{j_0}$,

$$S^{\upsilon_2}_{loc}(\omega) \subseteq S^{\upsilon_1}_{loc}(\omega).$$

Also for $n \geqslant N(\omega)$,

$$\varphi(n,\omega,S^{\upsilon_1}_{loc}(\omega)) \subseteq S^{\upsilon_2}_{loc}(\theta^n(\omega))$$

and consequently for $\xi \in S^{\upsilon}_{loc}(\omega)$,

$$\limsup_{n\to\infty} \frac{1}{n} \log \|\varphi(n,\omega,\xi) - Y_{\theta^n\omega}\| \leqslant \mu_{j_0}. \tag{5.4.3}$$

(v)

$$\limsup_{n\to\infty} \frac{1}{n} \log \left[\sup \left\{ \frac{\|\varphi(n,\omega,\xi) - \varphi(n,\omega,\tilde{\xi})\|}{\|\xi - \tilde{\xi}\|}, \quad \xi \neq \tilde{\xi},\ \xi,\tilde{\xi} \in S^{\upsilon}_{loc}(\omega) \right\} \right] \leqslant \mu_{j_0}.$$

Proof. Set $E_\omega := \mathscr{D}^{\alpha,\beta}_{X(\omega)}([-r,0],W)$. In Lemma 5.4.2, we saw that our assumptions imply that $\psi^n_\omega = D_{Y_\omega}\varphi(n,\omega,\cdot)$ defines a compact linear cocycle acting on the measurable field of Banach spaces $\{E_\omega\}_{\omega\in\Omega}$, that Assumption 2.3.1 holds and that $\log^+ \|\psi^1\| \in L^1(\Omega)$. In view of Theorem 3.2.9, it therefore suffices to check the condition (3.2.5). Set

$$\begin{aligned} P_\omega : E_\omega &\to E_{\theta\omega} \\ \xi &\mapsto \varphi(1,\omega,Y_\omega + \xi) - \varphi(1,\omega,Y_\omega) - \psi^1_\omega(\xi). \end{aligned}$$

Then from Theorem 5.2.4,

$$\|P_\omega(\xi) - P_\omega(\tilde{\xi})\| \leqslant (\|\xi\| + \|\tilde{\xi}\|) \exp[\tilde{Q}(\|\xi\| + \|\tilde{\xi}\|)] \exp[\tilde{P}(A_\omega, \|Y_\omega\|)]\ \|\xi - \tilde{\xi}\|$$

where $\tilde{P}$ is the polynomial from (5.2.14) and $\tilde{Q}$ is an increasing function. By Birkhoff's Ergodic Theorem,

$$\lim_{n\to\infty} \frac{1}{n} \tilde{P}(A_{\theta^n\omega}, \|Y_{\theta^n\omega}\|) = 0$$

almost surely. Therefore, (3.2.5) is indeed satisfied and the result follows from Theorem 3.2.9. □

Next, we formulate the result for unstable manifolds.

Theorem 5.4.5 (Local unstable manifolds)**.** *Assume the same setting as in Theorem 5.4.4. Furthermore, assume that $\mu_1 > 0$ holds for the first Lyapunov exponent. Set $\varsigma := \theta^{-1}$. Then there is a θ-invariant set of full measure $\tilde{\Omega}$ and a family of immersed submanifolds $U^{\upsilon}_{loc}(\omega)$ of $\mathscr{D}^{\alpha,\beta}_{X(\omega)}([-r,0],W))$, $0 < \upsilon < \mu_{k_0}$ and $\omega \in \tilde{\Omega}$, satisfying in following properties for every $\omega \in \tilde{\Omega}$:*

(i) There are random variables $\tilde{\rho}^{\upsilon}_1(\omega), \tilde{\rho}^{\upsilon}_2(\omega)$, positive and finite on $\tilde{\Omega}$, for which

$$\liminf_{p\to\infty} \frac{1}{p} \log \tilde{\rho}^{\upsilon}_i(\varsigma^p \omega) \geq 0, \quad i = 1, 2$$

and such that

$$\Big\{\xi_\omega \in \mathscr{D}^{\alpha,\beta}_{X(\omega)} \,:\, \exists \{\xi_{\varsigma^n\omega}\}_{n\geqslant 1} \text{ s.t. } \varphi(m, \varsigma^n\omega, \xi_{\varsigma^n\omega}) = \xi_{\varsigma^{n-m}\omega} \text{ for all } 0 \leq m \leq n \text{ and}$$
$$\sup_{n\geqslant 0} \exp(n\upsilon)\|\xi_{\varsigma^n\omega} - Y_{\varsigma^n\omega}\| < \tilde{\rho}^{\upsilon}_1(\omega)\Big\} \subseteq U^{\upsilon}_{loc}(\omega) \subseteq \Big\{\xi_\omega \in \mathscr{D}^{\alpha,\beta}_{X(\omega)} \,:\, \exists \{\xi_{\varsigma^n\omega}\}_{n\geqslant 1} \text{ s.t.}$$
$$\varphi(m, \varsigma^n\omega, \xi_{\varsigma^n\omega}) = \xi_{\varsigma^{n-m}\omega} \text{ for all } 0 \leq m \leq n \text{ and } \sup_{n\geqslant 0} \exp(n\upsilon)\|\xi_{\varsigma^n\omega} - Y_{\varsigma^n\omega}\| < \tilde{\rho}^{\upsilon}_2(\omega)\Big\}.$$

(ii)

$$T_{Y_\omega} U^{\upsilon}_{loc}(\omega) = U_\omega.$$

(iii) For $n \geqslant N(\omega)$,

$$U^{\upsilon}_{loc}(\omega) \subseteq \varphi(n, \varsigma^n\omega, U^{\upsilon}_{loc}(\varsigma^n\omega)).$$

(iv) For $0 < \upsilon_1 \leqslant \upsilon_2 < \mu_{k_0}$,

$$U^{\upsilon_2}_{loc}(\omega) \subseteq U^{\upsilon_1}_{loc}(\omega).$$

Also for $n \geqslant N(\omega)$,

$$U^{\upsilon_1}_{loc}(\omega) \subseteq \varphi(n, \varsigma^n\omega, U^{\upsilon_2}_{loc}(\varsigma^n\omega))$$

and consequently for $\xi_\omega \in U^{\upsilon}_{loc}(\omega)$,

$$\limsup_{n\to\infty} \frac{1}{n} \log \|\xi_{\varsigma^n\omega} - Y_{\varsigma^n\omega}\| \leqslant -\mu_{k_0}.$$

(v)

$$\limsup_{n\to\infty} \frac{1}{n} \log \left[\sup \left\{ \frac{\|\xi_{\varsigma^n\omega} - \tilde{\xi}_{\varsigma^n\omega}\|}{\|\xi_\omega - \tilde{\xi}_\omega\|}, \quad \xi_\omega \neq \tilde{\xi}_\omega, \; \xi_\omega, \tilde{\xi}_\omega \in U^{\upsilon}_{loc}(\omega) \right\} \right] \leqslant -\mu_{k_0}.$$

Proof. Follows from Theorem 3.3.6. □

Remark 5.4.6. *(i) In both Theorems 5.4.4 and 5.4.5, the assumption $\sigma \in C^3$ implies that the cocycle φ is differentiable. Higher order smoothness of σ will lead to higher order* *differentiability of φ, cf. Theorem 5.1.6. As a consequence, we obtain higher order smoothness of the stable and unstable manifolds. In fact, $\varphi \in C^m$ implies that $S^v_{loc}(\omega)$ resp. $U^v_{loc}(\omega)$ are almost surely locally C^{m-1}, cf. Remark 3.2.10 and Remark 3.3.7.*

(ii) If all Lyapunov exponents are non-zero, the stationary trajectory Y is called hyperbolic*. In this case, the submanifolds $S^v_{loc}(\omega)$ and $U^v_{loc}(\omega)$ are* transversal*, i.e.*

$$\mathscr{D}^{\alpha,\beta}_{X(\omega)} = T_{Y_\omega} S^v_{loc}(\omega) \oplus T_{Y_\omega} U^v_{loc}(\omega)$$

almost surely.

Examples

We will now discuss examples of stochastic delay equations for which we can apply our results. First, we will consider the case of 0 being a deterministic fixed point for the cocycle.

Proposition 5.4.7. *Let $\mathbf{X}$ be a delayed γ-rough path cocycle defined on an ergodic metric dynamical system $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t\in\mathbb{R}})$ and let $\frac{1}{3} < \alpha < \beta < \gamma < \frac{1}{2}$. Assume $\sigma \in C^3_b$ resp. $\sigma \in C^4_b$ in the case $B \neq 0$ and that*

$$\sigma(0,0) = \sigma_x(0,0) = \sigma_y(0,0) = 0.$$

Then $Y \equiv 0$ is a stationary trajectory for the cocycle φ induced by

$$dy_t = B(y_t, y_{t-r})\, dt + \sigma(y_t, y_{t-r})\, d\mathbf{X}_t(\omega). \tag{5.4.4}$$

If

$$\tilde{P}(A_\omega, 0) \in L^1(\Omega) \quad \text{and} \quad Q(A_\omega, 0) \in L^1(\Omega) \tag{5.4.5}$$

where $A_\omega = 1 + \|\mathbf{X}(\omega)\|_{\gamma,[0,r]}$, $\tilde{P}$ is the polynomial in (5.2.14) *and Q is the polynomial in* (5.2.12)*, the integrability condition of Theorem 5.4.4 and Theorem 5.4.5 is satisfied and yields the existence of local stable and unstable manifolds around* 0*. In particular, the result holds for $\mathbf{X}$ being $\mathbf{B}^{\text{Itō}}$ (when $H = \frac{1}{2}$) or $\mathbf{B}$.*

Proof. From

$$\begin{aligned}\int_0^t \sigma(y_s, y_{s-r})\, d\mathbf{X}_s(\omega) = \lim_{|\Pi|\to 0} \sum_{t_j \in \Pi} &\sigma(y_{t_j}, y_{t_j-r}) X_{t_j,t_{j+1}} + \sigma_x(y_{t_j}, y_{t_j-r})\sigma(y_{t_j}, y_{t_j-r})\mathbb{X}_{t_j,t_{j+1}} \\ &+ \sigma_y(y_{t_j}, y_{t_j-r})\sigma(y_{t_j}, y_{t_j-r})\mathbb{X}_{t_j,t_{j+1}}(-r),\end{aligned}$$

it follows that $Y \equiv 0$ is a solution to (5.4.4) and therefore a stationary trajectory in the sense of Definition 5.4.1. In the case of $\mathbf{X}$ being $\mathbf{B}^{\text{Itō}}$(when $H = \frac{1}{2}$) or $\mathbf{B}$, the norm of the delayed rough path cocycle has moments of any order, therefore condition (5.4.5) is satisfied. □

Next, we propose a condition under which (5.2.1) admits a random stationary trajectory Y when $H = \frac{1}{2}$. Let B be a two-sided Brownian motion defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$

adapted to two-parameter filtration $(\mathcal{F}_s^t)_{s\leq t}$ (cf. [1, Section 2.3.2]). Consider

$$\begin{aligned} dy_t &= Cy_t\,dt + \sigma(y_t, y_{t-r})dB_t \\ y_s &= \xi_s, \quad -r \leqslant s \leqslant 0 \end{aligned} \tag{5.4.6}$$

as a classical stochastic delay differential equation in Itō sense where $C\colon W \to W$ is a linear map. Assume that σ is a bounded Lipschitz function with Lipschitz constant L and let all the eigenvalues of C be negative. Consequently, there exist $M, \lambda > 0$ such that for every $t > 0$,

$$|\exp(tC)| \leqslant M\exp(-\lambda t). \tag{5.4.7}$$

Set $\mathcal{F}_{-\infty}^t := \sigma(\cup_{s\leq t}\mathcal{F}_s^t)$. A stochastic process $y\colon \mathbb{R} \to W$ is called $(\mathcal{F}_{-\infty}^t)$-*adapted* if y_t is $\mathcal{F}_{-\infty}^t$-measurable for every $t \in \mathbb{R}$. In that case for, any continuous, $(\mathcal{F}_{-\infty}^t)$-adapted process y, the following process is well defined, continuous and $(\mathcal{F}_{-\infty}^t)$-adapted:

$$\Gamma(y)(t) := \int_{-\infty}^t \exp((t-\tau)C)\sigma(y_\tau, y_{\tau-r})\,dB_\tau.$$

By the Itō isometry,

$$\begin{aligned} \mathbb{E}|\Gamma(y)(t)|^2 &\leq \mathbb{E}\int_{-\infty}^t |\exp((t-\tau)C)|^2|\sigma(y_\tau, y_{\tau-r})|^2\,d\tau, \\ \mathbb{E}|\Gamma(y)(t) - \Gamma(\tilde{y})(t)|^2 &\leq \mathbb{E}\int_{-\infty}^t |\exp((t-\tau)C)|^2|\sigma(y_\tau, y_{\tau-r}) - \sigma(\tilde{y}_\tau, \tilde{y}_{\tau-r})|^2\,d\tau. \end{aligned} \tag{5.4.8}$$

Lemma 5.4.8. *Assume* $\frac{2ML^2}{\lambda} < 1$. *Then there is a continuous,* $(\mathcal{F}_{-\infty}^t)$*-adapted process* Y_t *such that for every* $t \in \mathbb{R}$,

$$Y_t = \int_{-\infty}^t \exp((t-\tau)C)\sigma(Y_\tau, Y_{\tau-r})\,dB_\tau.$$

Proof. Set

$$\mathcal{X} := \left\{ y : \mathbb{R} \to W \;:\; y \text{ is continuous, } (\mathcal{F}_{-\infty}^t)\text{-adapted and } \sup_{t\in\mathbb{R}}(\mathbb{E}|y_t|^2)^{\frac{1}{2}} < \infty \right\}.$$

It can easily be seen that $\mathcal{X}$ is a Banach space. By (5.4.8),

$$\Gamma : \mathcal{X} \longrightarrow \mathcal{X}$$

is a contraction, so our claim follows from a standard fixed point argument. □

Lemma 5.4.9. *Let* Y *be the process from Lemma* 5.4.8 *and set* $Y'_t = \sigma(Y_t, Y_{t-r})$. *Then* (Y, Y') *is almost surely controlled by* B. *Moreover,* $\|(Y, Y')\|_{\mathscr{D}_B^\gamma([a,b],W)} \in L^p(\Omega)$ *for every* $p > 0$ *and every* $a < b$.

Proof. From the Burkholder-Davis-Gundy inequality, for every $m \in \mathbb{N}$ there exists a $\beta_{2m} \in \mathbb{R}$ such that

$$\mathbb{E}|Y_{s,t}|^{2m} \leqslant \beta_{2m}(t-s)^m \tag{5.4.9}$$

for every $s < t$. Note that

$$
\begin{aligned}
Y_{s,t} - \sigma(Y_s, Y_{s-r}) B_{s,t} = &\int_{-\infty}^{s} \exp\left((s-\tau)C\right)\left[\exp\left((t-s)C\right) - 1\right]\sigma(Y_\tau, Y_{\tau-r})\, dB_\tau \\
&+ \int_s^t \exp((t-\tau)C)\left[\sigma(Y_\tau, Y_{\tau-r}) - \sigma(Y_s, Y_{s-r})\right] dB_\tau \\
&+ \int_s^t \left[\exp((t-\tau)C) - 1\right] dB_\tau\, \sigma(Y_s, Y_{s-r}).
\end{aligned}
$$

By the Burkholder-Davis-Gundy inequality and our assumptions, for $\alpha_{2m} \in \mathbb{R}$,

$$
\mathbb{E}\left| \int_{-\infty}^{s} \exp\left((s-\tau)C\right)\left[\exp\left((t-s)C\right) - 1\right]\sigma(Y_\tau, Y_{\tau-r})\, dB_\tau \right|^{2m} \leqslant \alpha_{2m}(t-s)^{2m}
$$

and

$$
\mathbb{E}\left| \int_s^t \left[\exp((t-\tau)C) - 1\right] dB_\tau\, \sigma(Y_s, Y_{s-r}) \right|^{2m} \leqslant \alpha_{2m}(t-s)^{2m}.
$$

Using again the Burkholder-Davis-Gundy inequality, Hölder's inequality and (5.4.9), we obtain that there are constants $\beta_{2m}, \gamma_{2m} \in \mathbb{R}$ such that

$$
\begin{aligned}
&\mathbb{E}\left| \int_s^t \exp((t-\tau)C)\left[\sigma(Y_\tau, Y_{\tau-r}) - \sigma(Y_s, Y_{s-r})\right] dB_\tau \right|^{2m} \\
&\leqslant \beta_{2m}\mathbb{E}\left| \int_s^t (|Y_{s,\tau}|^2 + |Y_{s-r,\tau-r}|^2) d\tau \right|^{m} \\
&\leqslant \beta_{2m}(t-s)^{m-1}\mathbb{E}\int_s^t (|Y_{s,\tau}|^2 + |Y_{s-r,\tau-r}|^2)^m d\tau \leqslant \gamma_{2m}(t-s)^{2m}.
\end{aligned}
$$

Consequently, we have shown that for every $m \geq 1$ there are constants $\tilde{\alpha}_{2m}$ such that

$$
\mathbb{E}|Y_{s,t} - \sigma(Y_s, Y_{s-r})B_{s,t}|^{2m} \leqslant \tilde{\alpha}_{2m}(t-s)^{2m}
$$

for every $s < t$. Set $Y^{\#}_{s,t} := Y_{s,t} - \sigma(Y_s, Y_{s-r})B_{s,t}$. By a version of Kolmogorov's continuity theorem similar to [39, Theorem 3.1], we obtain

$$
\|Y\|_{\gamma;[a,b]} + \|Y^{\#}\|_{2\gamma;[a,b]} \in L^p(\Omega)
$$

for every $p > 0$ and $a < b$ from which the result follows.

□

Proposition 5.4.10. *Let C be a linear map with negative eigenvalues only and $\sigma \in C_b^4$. Let λ and M be as in* (5.4.7) *and let L be the Lipschitz constant of σ. Assume $\frac{2ML^2}{\lambda} < 1$. Then there exists a stationary trajectory for the cocycle φ induced by*

$$
\begin{aligned}
dy_t &= Cy_t\, dt + \sigma(y_t, y_{t-r}) d\mathbf{B}_t^{\text{Itō}} \\
y_s &= \xi_s, \quad -r \leqslant s \leqslant 0
\end{aligned}
\tag{5.4.10}
$$

and the integrability condition (5.4.1) *of Theorem 5.4.4 and Theorem 5.4.5 is satisfied.*

Proof. Let $\hat{Y} = (Y, Y')$ be defined as in Lemma 5.4.9. Then

$$\hat{Y}_t = \int_{-\infty}^{t} \exp((t-\tau)C)\sigma(\hat{Y}_\tau, \hat{Y}_{\tau-r})\, d\mathbf{B}_\tau^{\text{Itō}}$$

almost surely for every t. Therefore, (i) and (ii) of Definition 5.4.1 follow directly. Since

$$\|\hat{Y}\|_{\mathscr{D}_B^\beta([-r,0])} = |Y_{-r}| + |Y'_{-r}| + \sup_{s,t\in[-r,0]\cap\mathbb{Q}, s\neq t} \frac{|Y'_t - Y'_s|}{|t-s|^\beta} + \sup_{s,t\in[-r,0]\cap\mathbb{Q}, s\neq t} \frac{|Y_{s,t} - Y'_s B_{s,t}|}{|t-s|^{2\beta}},$$

measurability of $\omega \mapsto \|\hat{Y}(\omega)\|_{\mathscr{D}_{B(\omega)}^\beta([-r,0])}$ follows, too. The integrability condition (5.4.1) is satisfied due to Lemma 5.4.9 and Proposition 4.2.15. $\square$

Remark 5.4.11. *It is possible to prove directly that the rough differential equation*

$$\hat{Y}_t = \int_{-\infty}^{t} \exp((t-\tau)C)\sigma(\hat{Y}_\tau, \hat{Y}_{\tau-r})\, d\mathbf{B}_\tau^{\text{Itō}}$$

has a fixed point using the standard estimates for the rough integral. However, this would yield a stronger condition than $\frac{2ML^2}{\lambda} < 1$. *Nevertheless, we quickly sketch the argument here. For a fixed* $\omega \in \Omega$ *and* $\epsilon > 0$, *set*

$$\begin{aligned}\mathcal{Y} := \big\{(Y^{\theta_{nr}\omega})_{n\leqslant 0} : Y^{\theta_{nr}\omega} \in \mathscr{D}_{B(\theta_{nr}\omega)}^{\alpha,\beta}([-r,0],W)),\ Y_0^{\theta_{(n-1)r}\omega} = Y_{-r}^{\theta_{nr}\omega},\\ (Y^{\theta_{(n-1)r}\omega})'_0 = (Y^{\theta_{nr}\omega})'_{-r} \quad \textit{and} \ \sup_{n\leqslant 0} \|\exp(n\epsilon I)Y^{\theta_{nr}\omega}\| < \infty\big\}\end{aligned}$$

where I *is the identity matrix. It is not hard to check that* $\mathcal{Y}$ *is a Banach space. Define* $\Gamma : \mathcal{Y} \to \mathcal{Y}$ *by*

$$\begin{aligned}[\Gamma(Y)]_t^{\theta_{mr}\omega} := \sum_{n<m} \int_{-r}^{0} \exp((t-\tau+(m-n)r)C)\sigma(Y_\tau^{\theta_{nr}\omega}, Y_\tau^{\theta_{(n-1)r}\omega}) d\mathbf{B}(\theta_{nr}\omega)_\tau^{\text{Itō}} +\\ \int_{-r}^{t} \exp((t-\tau)C)\sigma(Y_\tau^{\theta_{mr}\omega}, Y_\tau^{\theta_{(m-1)r}\omega}) d\mathbf{B}(\theta_{mr}\omega)_\tau^{\text{Itō}}\end{aligned}$$

where $m \leqslant 0$ *and* $t \in [-r, 0]$. *We can use a fixed-point argument as in Lemma 5.4.8 for* Γ *now to conclude.*

6

Ruelle's Inequality for Translation Invariant Flows

In this chapter we investigate the concept of entropy for a class of random dynamical systems, which are invariant in distribution in a finite number of directions. For this family of stochastic flows, we do not necessarily have an invariant probability measure. The first challenge here is proving the existence of the Lyapunov exponents and, secondly, how the entropy can be defined in a natural setting. In this chapter, we try to address these difficulties. After explaining this concept, the main question is to estimate or even calculate the entropy. Traditionally, there are two significant results: Ruelle's inequality, which provides an upper bound for the entropy. The second considerable result is Pesin's formula, which claims under some regularity assumption for the invariant measure, the upper bound provided by Ruelle's inequality is the exact value of the entropy. For this concept, again, the multiplicative ergodic theorem plays an inevitable role. Indeed, the upper bound for the entropy is the sum of the positive Lyapunov exponents of the system.

Compared with the invariant manifolds, relating the Lyapunov exponents to entropy is more challenging. For the deterministic regime, entropy is well studied; however, for stochastic equations, this concept still is not well studied. The main obstacle in this regime is the lack of compactness. More precisely, the white noise in the stochastic equations pushes the systems out of any bounded set. After introducing our model and justifying the entropy in this chapter, we give two versions of Ruelle's inequality. The fundamental strategy here is adapting the deterministic argument provided in [69]. An important class of stochastic flows that fulfill our assumptions are translation invariant Brownian flows. The corresponding distribution is homogeneous in time and invariant under translations in space for this family of stochastic flows (like isotropic Brownian flows). This chapter is structured as follows; we first introduce our setting and then argue that the Lyapunov exponents exist. We define the concept of entropy and then prove a version of Ruelle's inequality for our entropy.

6.1 Preliminaries and notation

Assume $(E, \|.\|)$ is a real separable Banach space. Let $\Theta := (\theta_i,\, i \in I)$ be a finite family of linearly independent elements of E (I might also be the empty set). Let $V_n,\ n \in \mathbb{Z}$ be independent identically distributed random variables form a probability space $(\bar{\Omega}, \bar{\mathcal{A}}, \bar{P})$, taking values in some measurable space $(U,\mathcal{U})$. Also, assume $F : E \times U \to E$ be a measurable map that is continuous in the first variable for each fixed value of the second variable. Further, assume that for each $i \in I$, we have

$$\mathcal{L}(F(x, V_0),\, x \in E) = \mathcal{L}(F(x + \theta_i, V_0) - \theta_i,\, x \in E), \tag{6.1.1}$$

i.e. the law of the random map F is translation invariant in the directions $\theta_i,\ i \in I$. Let $S := E/\langle\Theta\rangle$ denote the quotient space associated to the group action of the discrete subgroup $\langle\Theta\rangle$ generated by Θ on the additive group E. For the canonical projection $\pi : E \to S$ we also write $\pi(y) = y \mod \Theta$. Let S be equipped with the quotient topology. Note there is a closed subspace space $\tilde{F}$ of E such that $E = \tilde{F} \oplus \langle\theta_i\rangle_{i\in I}$. Consequently, for every element of $x \in S$, $x = f + \sum_{i\in I} t_i\theta_i$, where $f \in \tilde{F}$ and $0 \leqslant t_i < 1,\ \forall i \in I$. Also, for $x, y \in S$ we can define the following metric

$$d(x, y) = \|f - f'\| + \sum_{i\in I} |e^{2\pi(t_i - t'_i)} - 1|, \quad x = f + \sum_{i\in I} t_i\theta_i, \quad \text{and} \quad y = f' + \sum_{i\in I} t'_i\theta_i.$$

Note that S is a smooth manifold. For $n \in \mathbb{N}$, we recursively define

$$X^{X_0}_{n+1} = F(X^{X_0}_n, V_n), \tag{6.1.2}$$

where $X^{X_0}_1 = F(X_0, V_0)$, and the initial E-valued condition X_0 is independent of V_n, for all $n \in \mathbb{N}_0$. Note that in this way we obtain an E-valued Markov chain with transition kernel $K(x, A) := \bar{P}(\{F(x, V_0) \in A\})$. Let P be the probability measure induced on U by the random variable V_0, i.e. for $B \in \mathcal{U}$,

$$P(B) = \bar{P}(V_0^{-1}(B)),$$

then we can define a random dynamical system (rds). To see this define the space of random sequences $(\Omega, \mathcal{A}, \mathbb{P}) := \bigotimes_{\mathbb{Z}}((U,\mathcal{U}, P))$ together with the shift map $\sigma : \Omega \to \Omega$, $u = (u_i)_{i\in\mathbb{Z}} \mapsto \tilde{u} = (\tilde{u}_i)$, where $\tilde{u}_i = u_{i+1}$. Then the skew product map

$$\begin{aligned} \Phi : \Omega \times E &\to \Omega \times E \\ (u, x) &\mapsto (\sigma(u), F(x, u_0)), \end{aligned}$$

defines a rds. We let $\Phi^0 = \text{id.}$ and $\Phi^{n+1} := \Phi^n \circ \Phi$ for $n \in \mathbb{N}_0$ and

$$\phi^n_u(.) := \pi_2 \circ \Phi^n(u, .).$$

Similarly for $(\Omega^+, \mathcal{A}^+, \mathbb{P}^+) := \bigotimes_{\mathbb{N}}((U, \mathcal{U}, P))$, the shift map $\sigma^+ : \Omega^+ \to \Omega^+$, $(u_0, u_1, ...) \to (u_1, u_2, ...)$ is defined, also we set

$$\begin{aligned} \tilde{\Phi} : \Omega^+ \times E &\to \Omega^+ \times E \\ (u^+, x) &\mapsto (\sigma(u^+), F(x, u_0)), \end{aligned}$$

and $\tilde{\Phi}^n$ is defined in a similar manner. Due to translation invariance, the process $Y_n^{X_0} := X_n^{X_0}$ mod Θ is an S-valued Markov chain (but in general *not* an rds) with transition kernel $K_\Theta(x, A) := \bar{P}(\{Y_1^x \in S\}))$. We assume that the chain Y_n, $n \in \mathbb{N}_0$ has a unique invariant probability measure μ, i.e. $\mu \otimes K = \mu$. By definition for $A \in \mathcal{B}(S)$ and $x \in S$

$$\bar{P}(\{Y_n^x \in A\}) = \sum_{(m_i)_{i \in I} \subset \mathbb{Z}^I} \bar{P}(\{X_n^x \in A + \sum_{i \in I} m_i \theta_i\}), \tag{6.1.3}$$

and

$$\mu(A) = \int_S K_\Theta(x, A)\ \mu(dx) = \sum_{(m_i)_{i \in I} \subset \mathbb{Z}^I} \int_S K(x, A + \sum_{i \in I} m_i \theta_i) \mu(dx). \tag{6.1.4}$$

We extend μ to E by translating with respect to the elements of Θ, i.e. for $A \in \mathcal{B}(S)$

$$\mu(A + \sum_{i \in I} n_i \theta_i) := \mu(A) \tag{6.1.5}$$

From definition, (6.1.1) and (6.1.4)

$$\begin{aligned} &\mu(A + \sum_{i \in I} n_i \theta_i) = \mu(A) = \sum_{(m_i)_{i \in I} \subset \mathbb{Z}^I} \int_S K(x, A + \sum_{i \in I} m_i \theta_i) \mu(dx) = \\ &\sum_{(m_i)_{i \in I} \subset \mathbb{Z}^I} \int_S \bar{P}(\{F(x, u_0) \in A + \sum_{i \in I} m_i \theta_i\}) \mu(dx) = \\ &\sum_{(m_i)_{i \in I} \subset \mathbb{Z}^I} \int_S \bar{P}(\{F(x - \sum_{i \in I} m_i \theta_i, u_0) \in A\}) \mu(dx) = \\ &\sum_{(m_i)_{i \in I} \subset \mathbb{Z}^I} \int_{S - \sum_{i \in I} m_i \theta_i} \bar{P}(\{F(x, u_0) \in A\}) \mu(dx) = \int_E K(x, A) \mu(dx). \end{aligned}$$

Note that μ is not a probability measures on E (if $I \neq \emptyset$).

Definition 6.1.1. *Set*

$$\begin{aligned} \Psi : &\Omega \times S \to \Omega \times S \\ &(u, x) \to (\sigma(u), [F(x, u_0)]), \end{aligned}$$

where

$$[F(x, u_0)] = F(x, u_0)\ \mathit{mod}\ \Theta$$

Again $\Psi^0 = id$ *and* $\Psi^{n+1} := \Psi^n \circ \Psi$ *and* $\psi_u^n(.) = \pi_2 \circ \Psi^n(u,.)$, *similarly define*

$$\tilde{\Psi} : \Omega^+ \times S \to \Omega^+ \times S$$
$$(u^+, x) \to (\sigma^+(u), [F(x, u_0)])$$

and $\tilde{\psi}_u^n(.) = \pi_2 \circ \tilde{\Psi}_u^n(.)$.

Let $\{A_j\}_{1\leqslant j\leqslant m}$, *be a measurable partition of* S. *For the sequence* $\{n_j\}_{1\leqslant j\leqslant m}$ *of positive integers and* $x \in S$

$$\begin{aligned}
&\mathbb{P}(\{[\phi_u^{n_j}(x)] \in A_j, \forall j \quad 1 \leqslant j \leqslant m\}) = \\
&\mathbb{P}(\{[\phi_{\sigma u}^{n_j-1}(\phi_u^1(x) - \psi_u^1(x) + \psi_u^1(x))] \in A_j, \ \forall j \quad 1 \leqslant j \leqslant m\}) = \\
&\sum_{(m_i^j)_{i\in I}\subset\mathbb{Z}^I} \mathbb{P}(\{\phi_{\sigma u}^{n_j-1}(\phi_u^1(x) - \psi_u^1(x) + \psi_u^1(x)) \in A_j + \sum_{(m_i^j)\subset\mathbb{Z}^I} m_i^j\theta_i, \ \forall j \quad 1 \leqslant j \leqslant m\}) \\
&\sum_{(m_i^j)_{i\in I}\subset\mathbb{Z}^I} \mathbb{P}(\{\phi_{\sigma u}^{n_j-1}(\psi_u^1(x)) \in A_j + \sum_{(m_i^j)\subset\mathbb{Z}^I} m_i^j\theta_i, \ \forall j \quad 1 \leqslant j \leqslant m\}) = \\
&\mathbb{P}(\{\psi_u^{n_j}(x) \in A_j, \forall j \quad 1 \leqslant j \leqslant m\}).
\end{aligned} \tag{6.1.6}$$

The next lemma is standard.

Lemma 6.1.2. *The following items hold true :*

(i) $\mathbb{P}^+ \times \mu$ *is an invariant probability measure for* $\tilde{\Psi}$.

(ii) There is a unique Ψ*-invariant measure* $\mu^\star$ *on* $\Omega \times S$, *such that* $\mu^\star|_{\Omega^+\times S} = \mathbb{P}^+ \times \mu$.

In addition $\mu^\star$ *disintegrates. i.e. there is a family a random measure* $\{\mu_u\}_{u\in\Omega}$ *on* S *such that for* $A \in \mathcal{A} \otimes \mathcal{B}(S)$:

$$\mu^\star(A) = \int_\Omega \mu_u(A^u)\mathbb{P}(du),$$

where $A^u := \{x \in S : (u, x) \in A\}$, *also*

(iii) μ_u *is invariant under* ψ_u^1, *i.e.*

$$(\psi_u^1)^\# \mu_u = \mu_{\sigma u},$$

and for $\mathbb{P}$ *a.a.* $u \in \Omega$ *following limit, weakly converges*

$$(\psi_{\sigma^{-n}u}^n)^\# \mu \to \mu_u.$$

Proof. Refer to [1], Chapter 1. ∎

Remark 6.1.3. *Items (iii) in last lemma implies* μ_u, *depends on* u_n, $n < 0$.

Definition 6.1.4. *For each of these sample measures we set* $K_u := support(\mu_u)$.

MET

We now state a simpler version of our MET in Chapter 2. Note that the only differences with our previous version in Chapter 2., are that, firstly, we fixed our Banach space, and, secondly, instead of assuming compactness, we assume that our operators are quasi-compact (Definition 6.1.6).

Definition 6.1.5. *For the linear map, $T : E \to E$ index of compactness is defined as below.*

$$\|T\|_\alpha := \inf\{r > 0 : \ T(B(0,1)) \ \text{ can be covered by finitely many balls of radius } \ r\},$$

where $B(0,1) = \{x \in E : \|x\| < 1\}$.

Remember, E is a separable Banach space. Assume for $u \in \Omega$, $\phi_u^1 : E \to E$ be C^1. Note that by definition, $\psi_u^1 : S \to S$ is then also differentiable. We further assume $(u,x) \to \|D_x\psi_u^1\| \in L^1(\mu^\star)$. By a standard argument with the Kingman's subadditive ergodic theorem, the following limits exist

$$\lambda_1(u,x) := \lim_{n\to\infty} \frac{1}{n} \log \|D_x\psi_u^n\|, \quad \alpha(u,x) = \lim_{n\to\infty} \frac{1}{n} \log \|D_x\psi_u^n\|_\alpha. \tag{6.1.7}$$

Definition 6.1.6. *We say our operators ($\{D_x\psi_u^n\}_{n,(u,x)}$) are quasi-compact, if for $\mu^\star$-a.a. $(u,x) \in \Omega \times S$*

$$\alpha(u,x) < \lambda_1(u,x)$$

The proof of the following version of MET is the same as our previous version in Chapter 2.

Theorem 6.1.7. *Under the above assumptions for $\mu^\star$-a.a. $(u,x) \in \Omega \times S$, exists a number $1 \leqslant k(u,x) \leqslant \infty$ and:*

- *a sequence of measurable values (Lyapunov exponents) $\lambda_1(u,x) > \lambda_2(u,x) > ... > \lambda_{k(u,x)}(u,x) > \alpha(u,x)$,*
- *a sequence of positive and measurable integers $m_1(u,x), ..., m_{k(u,x)}(u,x)$, in addition*
- *a measurable splitting of closed sub-spaces $E = F_{\lambda_1(u,x)}(u,x) \supset F_{\lambda_2(u,x)}(u,x) \supset ... \supset F_{\lambda_{k(u,x)}(u,x)}(u,x) \supset F'_\infty(u,x)$,*

such that for every $1 \leqslant i \leqslant k(u,x)$, $D_x\psi_u^1(F_i(u,x)) \subset F_i(\sigma u, \psi_u^1(x))$ and also $D_x\psi_u^1(F'_\infty(u,x)) \subset F'_\infty(\sigma u, \psi_u^1(x))$. For $1 \leqslant i < k(u,x)$, $\dim(\frac{F_{\lambda_i(u,x)}}{F_{\lambda_{i+1}(u,x)}}) = m_i(u,x)$,

$$\lim_{n\to\infty} \frac{1}{n} \log \|D_x\psi_u^n(y)\| = \lambda_i(u,x), \quad \textit{iff} \quad y \in F_{\lambda_i(u,x)}(u,x) \setminus F_{\lambda_{i+1}(u,x)}(u,x),$$

and

$$\limsup_{n\to\infty} \frac{1}{n} \log \|D_x\psi_u^n|_{F'_\infty(u,x)}\| \leqslant \alpha(u,x)$$

Remark 6.1.8. *Remember, $\psi_u^n(x)$ and $[\phi_u^n](x)$ have a same distribution (cf. (6.1.6)).*

6.2 Entropy

In this section, we define the concept of entropy for our setting; we first start by the following standard definition.

Definition 6.2.1. *Let $(X, \mathcal{B}, \nu)$ be a probability space and $\mathcal{A}$ be a sub-sigma-algebra of $\mathcal{B}$. Assume $\mathcal{P}$ a countable measurable partition of X. The conditional information function with respect to $\mathcal{A}$ is defined by*

$$I_\nu(\mathcal{P}|\mathcal{A})(.) := -\sum_{P\in\mathcal{P}} 1_P(.) \log \mathbb{E}_\nu(1_P|\mathcal{A})(.).$$

Also, conditional entropy with respect to $\mathcal{P}$ is given by

$$H_\nu(\mathcal{P}|\mathcal{A}) = \int_X I_\nu(\mathcal{P}|\mathcal{A})d\nu = -\int_X \sum_{P\in\mathcal{P}} \mathbb{E}_\nu(1_P|\mathcal{A}) \log \mathbb{E}_\nu(1_P|\mathcal{A})d\nu.$$

Now we are ready to define the concept of metric entropy.

Definition 6.2.2. *Assume $T : X \to X$ is measurable and ν-invariant. Also assume $\mathcal{A} \subset \mathcal{B}$ be a sub-sigma-algebra such that $T^{-1}\mathcal{A} \subset \mathcal{A}$. Then the conditional metric entropy of T with respect to $\mathcal{A}$ is given by*

$$h_\nu(T|\mathcal{A}) := \sup_{\mathcal{P}\in\mathcal{H}_\mathcal{A}} \{h_\nu(T, \mathcal{P}|\mathcal{A})\},$$

where

$$h_\nu(T, \mathcal{P}|\mathcal{A}) = \lim_{n\to\infty} \frac{1}{n} H_\nu(\bigvee_{i=0}^{n-1} T^{-i}\mathcal{P}|\mathcal{A}),$$

and

$$\mathcal{H}_\mathcal{A} = \{\mathcal{P} : \mathcal{P} \text{ is a measurable and countable partition of } X \text{ such that } H_\nu(\mathcal{P}|\mathcal{A}) < \infty\}.$$

Remark 6.2.3. *For sequence $\{\mathcal{P}_i\}_{0\leqslant i<n}$ of measurable and countable partition of X, joint partition $\bigvee_{i=0}^{n-1} \mathcal{P}_i$ is defined by*

$$\bigvee_{i=0}^{n-1} \mathcal{P}_i = \{ \bigcap_{0\leqslant i<n} P_{n_i} : \; P_{n_i} \in \mathcal{P}_i\}$$

Fiber Entropy

Back to our setting, for $X = \Omega \times S$, assume $\mu^\star$ be the extended measure provided by Lemma 6.1.2. Remember the skew product $\Psi : X \to X$ is defined by $(u, x) \to (\sigma u, \psi_u^1(x))$, then

Definition 6.2.4. *Assume $\mathcal{Z}$ be a countable and measurable partition of $\Omega \times S$, for $(u,x) \in \Omega \times S$, set*

$$\mathcal{Z}_{-n} := \bigvee_{i=0}^{n-1} (\Psi_u^i)^{-1}\mathcal{Z}, \quad \mathcal{Z}^u := \{\mathcal{P}^u : \ \mathcal{P} \in \mathcal{Z}\},$$

$$\mathcal{Z}_{-n}^u := (\mathcal{Z}_{-n})^u, \quad \mathcal{Z}_{-n}^u(x) := (\bigvee_{i=0}^{n-1} (\psi_u^i)^{-1}\mathcal{Z}^{\sigma^i u})(x) = \bigcap_{0 \leqslant i < n} (\psi_u^i)^{-1}(\mathcal{Z}^{\sigma^i u}(\psi_u^i(x))),$$

where $\mathcal{Z}^u(x)$ is the element of $\mathcal{Z}^u$, contains x. Also remember $\mathcal{P}^u = \{x \in S : \ (u,x) \in \mathcal{P}\}$.

The fiber entropy with respect to $\mathcal{Z}$ is given by

$$H_{\mu_u}(\mathcal{Z}^u) = -\sum_{P \in \mathcal{Z}^u} \log \mu_u(P) \ \mu_u(P).$$

Then from [70, Lemma 2.2.3]

$$H_{\mu^\star}(\mathcal{Z}|\pi_\Omega^{-1}(\mathcal{F})) = \int_\Omega H_{\mu_u}(\mathcal{Z}^u)\mathbb{P}(du).$$

Similarly the conditional entropy of $\mathcal{Z}$ for Ψ with respect to $(\Omega, \mathcal{F})$ is given by

$$h_{\mu^\star}(\Psi, \mathcal{Z}|\pi_\Omega^{-1}(\mathcal{F})) = \lim_{n\to\infty} \frac{1}{n} \int_\Omega H_{\mu_u}(\bigvee_{i=0}^{n-1} (\psi_u^i)^{-1}\mathcal{Z}^{\sigma^i u})\mathbb{P}(du).$$

Note that a standard argument by Kingman's subadditive ergodic theorem yields that the above limit exists $\mathbb{P}$-a.a. The metric entropy in this setting is defined by

$$\begin{aligned} h_{\mu^\star}(\Psi) := \sup \{ h_{\mu^\star}(\Psi, \mathcal{Z}|\pi_\Omega^{-1}(\mathcal{F})) : \mathcal{Z} \ \text{ is a measurable, countable partition of } X \\ \text{such that } \ H_{\mu^\star}(\mathcal{Z}|\pi_\Omega^{-1}(\mathcal{F}))) < \infty \}. \end{aligned}$$

Due to the separability of S, by [70, Theorem 2.2.4], indeed

$$h_{\mu^\star}(\Psi) = \sup \{ h_{\mu^\star}(\Psi, \pi_S^{-1}(\mathcal{P})|\pi_\Omega^{-1}(\mathcal{F})) : \ \mathcal{P} \ \text{ is a finite, measurable partition of } S \} \quad (6.2.1)$$

See also [71]. We also have the following classical lemma.

Lemma 6.2.5. *Let $(\psi_m)_u^n(x) := \psi_u^{mn}(x)$ and denote the corresponding skew product by*

$$\Psi_m : \Omega \times E \to \Omega \times E, \quad (u,x) \to (\sigma^m u, \psi_u^m(x))$$

then

$$h_{\mu^\star}(\Psi_m) = m h_{\mu^\star}(\Psi)$$

Proof. [72, Lemma 1.] □

For $\mathcal{Z} \in \mathcal{H}_{\Omega \times S}$, we denote the sigma-algebra generated with the family $\{\bigvee_{i=0}^{n-1} (\psi_u^i)^{-1}(\mathcal{Z}^{\sigma^i u})\}_{n \geqslant 1}$ on S by $\bigvee_{i=0}^{\infty} (\psi_u^i)^{-1}(\mathcal{Z}^{\sigma^i u})$. The next lemma is the random version of the McMillann theorem.

Theorem 6.2.6. *Assume $\mathcal{Z}$ be a measurable and countable partition of $\Omega \times S$ such that $H_{\mu^\star}(\mathcal{Z}|\pi_\Omega^{-1}(\mathcal{F}))) < \infty$, then the following items are true*

(i) $\lim_{n\to\infty} \frac{1}{n} I_{\mu_u}(\bigvee_{i=0}^{n-1} (\psi_u^i)^{-1} \mathcal{Z}^{\sigma^i u})(x) = \mathbb{E}_{\mu^\star}(f|\mathcal{J})(u,x) \quad \mu^\star - a.a.$
where $f(u,x) = I_{\mu_u}(\mathcal{Z}| \bigvee_{i=0}^{\infty} (\psi_u^i)^{-1} \mathcal{Z}^{\sigma^i u})$, where $\mathcal{J}$ is sigma-algebra of Ψ-invariant sets.

(ii) $h_{\mu^\star}(\Psi, \mathcal{Z}|\pi_\Omega^{-1}(\mathcal{F})) = \int_\Omega H_{\mu_u}(\mathcal{Z}| \bigvee_{i=0}^{\infty} (\psi_u^i)^{-1} \mathcal{Z}^{\sigma^i u}) \mathbb{P}(du)$.

(iii) In particular, if Ψ is ergodic with respect to $\mu^\star$, then

$$\lim_{n\to\infty} -\frac{1}{n} \log \mu_u(\mathcal{Z}^u_{-n}(x)) = h_{\mu^\star}(\Psi, \mathcal{Z}|\pi_\Omega^{-1}(\mathcal{F})), \quad \mu^\star - a.a.$$

Proof. [70, Theorem 2.2.5] □

Remark 6.2.7. *The concept of entropy for $(\Omega^+, \mathbb{P}^+, \tilde{\Psi})$ is defined similarly. In that case, we substitute μ with sample measures.*

Finally, we summarize our discussion in this section in the next Proposition:

Remark 6.2.8. *For the skew product $\Psi : \omega \times S \to \omega \times S$, with $\Psi^{n+1} = \Psi^n \circ \Psi$, following items hold true*

- *Ψ is a RDS,*
- *μ^* is invariant for Ψ,*
- *Ψ admits the same distribution with Φ. Furthermore, the Lyapunov exponents for both processes remain the same.*

Also, the concept of metric entropy for Ψ is well defined. Accordingly, we define the metric entropy for Φ by the same metric entropy of Ψ.

6.3 Ruelle's inequality

The classical Ruelle's inequality claims a relation between the entropy and the Lyapunov exponents. Namely, it says that the entropy is less than or equal to the sum of the positive Lyapunov exponents. In the following, we state two theorems for this inequality with different assumptions. We will also assume :

Assumption 6.3.1. *Let $K_u :=$ Support(μ_u), we assume*

(i) $\mathbb{P}$*-a.a.* $u \in \Omega$, K_u *is compact,*

(ii) $\psi_u^1(K_u) \subset K_{\sigma u}$,

(iii) $\Psi : \Omega \times S \to \Omega \times S$ *is ergodic with respect to* $\mu^\star$.

(iv) $\int_\Omega \log^+ \sup_{x\in K_u} \|D_x \psi_u^1\| \mathbb{P}(du) < \infty$.

(v) We also assume $\lim_{n\to\infty} \frac{1}{n} \int_\Omega \log \sup_{x\in K_u} \|D_x \psi_u^n\|_\alpha \mathbb{P}(du) < 0$.

Remark 6.3.2. *Note that $\sigma : \Omega \to \Omega$ is ergodic; also, from the ergodic decomposition theorem, we can assume Item [(iii)], and apply to this theorem for the general case.*

Remark 6.3.3. *By the ergodicity assumption, all values, including the integers and Lyapunov exponents in the in Theorem 6.1.7 are deterministic ($\mu^\star - a.a.$).*

Remark 6.3.4. *Item (v) in many cases is checkable. For example, when our Banach space has a finite dimension, then this assumption follows from Item (iv), also for the flow of the delay equations (which is defined on C), $D_x\psi_u^1$ is compact, and automatically this Item is fulfilled.*

We are now ready to state Ruelle's inequality in the infinite dimensional Banach spaces. The proof is almost the same as [71], with some minor modifications.

Theorem 6.3.5. *In addition to the Assumption 6.3.1, assume that there exists $\epsilon_0 > 0$ such that*

$$\int_\Omega \log^+ \sup_{x\in B(K_u,\epsilon_0)} \|D_x\psi_u^1\|\mathbb{P}(du) < \infty. \tag{6.3.1}$$

Then

$$h_{\mu^\star}(\Psi) \leqslant \sum_{\lambda_i>0} m_i\lambda_i.$$

Remark 6.3.6. *For deterministic equations (like PDE's), it is standard to assume the support of the invariant measure(if it exists) is compact. But due to the effect of the white noise, this assumption is no longer valid for the stochastic equations. Yet, for this family of equations, it is natural to assume that the sample measures are compact (for example, when we have an attractor). However, still, in the application, it is not clear how (6.3.1) can be verified.*

We now provide this inequality with another assumption. Here we state the theorem, and in the next section, we prove our claim.

Theorem 6.3.7. *Let E be a finite dimensional space and in addition to the Assumption 6.3.1, assume*

$$\|\psi_u^1(y) - \psi_u^1(z) - D_z\psi_u^1(y-z)\| \leqslant h(\sigma u)\|y-z\|^2, \quad y,z \in K_u \text{ and } \|y-z\| \leqslant \frac{1}{h(\sigma u)^{1/2}}, \tag{6.3.2}$$

where

$$\log(h) \in L^1(\Omega). \tag{6.3.3}$$

Then

$$h_{\mu^\star}(\Psi) \leqslant \sum_{\lambda_i>0} m_i\lambda_i.$$

Proof of Theorem 6.3.7

The main idea to prove Ruelle's inequality is to use local entropy to find a sequence of upper bounds for the entropy and then relate these bounds (in the limit) to the Lyapunov exponents. This concept (local entropy) was initially introduced by Brin and Katok in [73] and was later refined in [71]. Let

$$C_{\mathbb{P}}(K) := \{A : A \text{ is a measurable subset of } \Omega \times S \\ \text{and for a compact set } K_A \text{ and } \forall u \in \pi_\Omega A,\ K_u \subset K_A\}.$$

The Bowen ball around the the $(x, u) \in K$ is defined by

$$B^u_{A,n}(x,\epsilon) = \{y \in K_u : \forall i, 0 \leqslant i < n,\ (\sigma^i u, \psi^i_u(y)) \in A \text{ if and only if } (\sigma^i u, \psi^i_u(x)) \in A \text{ and} \\ \|\psi^i_u(x) - \psi^i_u(y)\| < \epsilon \text{ if } (\sigma^i u, \psi^i_u(x)) \in A\}. \tag{6.3.4}$$

We also use $B^u_n(x,\epsilon)$ when $A = \Omega \times S$. The fundamental relationship between entropy and local entropy is stated in the next Proposition.

Proposition 6.3.8. *Let* $\{A_m\}_{m\geqslant 0} \subset C_{\mathbb{P}}(K)$ *such that* $\mu(A_m) \to 1$ *then*

(i) $h_{\mu^\star}(\Psi) = \lim_{m\to\infty} \lim_{\epsilon\to 0} \overline{\lim}_{n\to\infty} -\frac{1}{n}\log \mu_u(B^u_{A_m,n}(x,\epsilon)), \quad \mu^\star - a.a.$

(ii) $h_{\mu^\star}(\Psi) \leqq \lim_{\epsilon\to 0} \overline{\lim}_{n\to\infty} -\frac{1}{n}\log \mu_u(B^u_n(x,\epsilon)), \quad \mu^\star - a.a.$

Proof. [71, Proposition 3.3] □

Remark 6.3.9. *In Items* (*i*) *and* (*ii*)*, in Proposition 6.3.4, we can also put the* $\underline{\lim}$ *instead of* $\overline{\lim}$.

To relate the entropy with the Lyapunov exponents, we need some technical definitions. Similar to [69] and [71], for $A \subset S$ and metric $\tilde{d}$, define

$$r(A,\epsilon,\tilde{d}) = \inf\{n > 1 : \ A \subset \bigcup_{1\leqslant i\leqslant n} B_{\tilde{d}}(x_i,\epsilon_i), \ \ s.t \ \ \epsilon_i \leqslant \epsilon \ \text{ and } \ x_i \in S\},$$

Where $B_{\tilde{d}}(x,\epsilon)$ is usual ϵ-neighborhood around x respect to the $\tilde{d}$. Also for $T \in L(E)$

$$R(T,\epsilon) := r(T(B_E),\epsilon,\|\|).$$

For $\gamma : \Omega \to \mathbb{R}^+$ define

$$d^{\gamma,u}_n(x,y) = \sup_{0\leqslant j<n} \frac{\|\psi^j_u(y) - \psi^j_u(x)\|}{\gamma(\sigma^j u)}, \qquad y, z \in K_u \ .$$

The following proposition is crucial, as it allows us to choose a nice partition to estimate the local entropy. This proposition was first proved in [69, Proposition IV.6] for the deterministic case and also is stated in [71, Proposition 3.7]. Since there are some gaps in [71, Proposition 3.7], we state it again with detailed proof.

Proposition 6.3.10. *Let $\{f_n\}_{n\geqslant 1}$ be sequence of measurable function such that*

(i)

$$\lim_{n\to\infty}\frac{1}{n}f_n(u,x)=g(u,x),\quad \mu^\star-a.a. \tag{6.3.5}$$

(ii)

$$\overline{\lim_{n\to\infty}}\sup_{x\in K_u}\frac{1}{n}f_n(u,x)<k(u),\quad \mathbb{P}-a.a. \tag{6.3.6}$$

For well-defined measurable functions k and g. Then there is a sequence of partitions $\{\mathcal{Z}^t\}_{t\geqslant 1}$ of $\Omega\times S$, such that

(i)

$$\lim_{t\to\infty}\overline{\lim_{n\to\infty}}-\frac{1}{n}\log\mu_u\big((\mathcal{Z}^t)^u_{-n}(x)\big)=0\qquad \mu^\star-a.a. \tag{6.3.7}$$

(ii)

$$\lim_{n\to\infty}\frac{1}{n}f_n(u,x)=\overline{\lim_{t\to\infty}}\,\overline{\lim_{n\to\infty}}\,\frac{1}{n}A(f_n,\mathcal{Z}^t)(u,x), \tag{6.3.8}$$

where

$$A(f_k,\mathcal{Z})(u,x)=\sum_{P\in\mathcal{Z}^u}(\sup_{y\in P\cap K_u}f_k(u,y)).\chi_P(x).$$

Proof. Let $\mathcal{J}$ be the sigma-algebra of measurable invariant sets under Ψ (note that here we do not assume the ergodicity), assume $\{\mathcal{J}_p\}_{p\geqslant 1}$ be an increasing finite $\mathcal{J}$- measurable partition of $\Omega\times S$ such that $\cup_{p\geqslant}\mathcal{J}_p$ generates $\mathcal{J}$. Also by $\hat{\mathcal{J}}_p$ we mean the sigma-algebra which is generated by $\mathcal{J}_p$. For

$$\mathcal{J}_p^\star=\{T\in\mathcal{J}_p:\mu^\star(T)>0\}=\{T_{p,1}\ ,...,\ T_{p,N(p)}\}\ ,$$

set

$$\begin{aligned}
B_p^\epsilon&=\{(u,x):\ \forall q\geqslant p\ \ g(u,x)-\epsilon\leqslant E(g|\hat{\mathcal{J}}_q)(u,x)\leqslant g(u,x)+\epsilon\},\\
T_{p,i,q}^\epsilon&=\{(u,x)\in T_{p,i}\cap B_p^\epsilon:\ \forall t\geqslant q\ \ \frac{1}{t}f_t(u,x)\leqslant g(u,x)+\epsilon\},\\
\mathcal{J}_{p,q}^\epsilon&=\{T_{p,1,q}^\epsilon\ ,...,\ T_{p,N(p),q}^\epsilon\ ,\ \Omega\times S\ \setminus\cup_{i=1}^{N(p)}T_{p,i,q}^\epsilon\}.
\end{aligned}$$

Note that if $(u,x)\in T_{p,i,q}^\epsilon$, for $1\leqslant i\leqslant N(p)$ and large m

$$\frac{1}{m}A(f_m,\mathcal{J}_{p,q}^\epsilon)(u,x)=\sup_{y\in(T_{p,i,q}^\epsilon)^u(x)\cap K_u}\frac{1}{m}f_m(u,y)\leqslant 2\epsilon+\frac{\int_{T_{p,i}}g\ d\mu}{\mu(T_{p,i})}\leqslant g(u,x)+3\epsilon\ . \tag{6.3.9}$$

From (6.3.5), (6.3.6) and (6.3.9)

$$\begin{aligned}\overline{\lim_{m\to\infty}} \frac{1}{m} A(f_m, (\mathcal{J}^{\epsilon}_{p,q})_{-m})(u,x) &\leqslant \frac{1}{m} A(f_m, \mathcal{J}^{\epsilon}_{p,q})(u,x) \\ &\leqslant (g(u,x)+3\epsilon).\chi_{\cup_{i=1}^{N(p)} T^{\epsilon}_{p,i,q}}(u,x) + k(u).\chi_{\Omega\times T \ \setminus \cup_{i=1}^{N(p)} T^{\epsilon}_{p,i,q}}(u,x).\end{aligned} \tag{6.3.10}$$

Also

$$\begin{aligned}&\lim_{q\to\infty} \overline{\lim_{n\to\infty}} -\frac{1}{n}\log \mu_u((\mathcal{J}^{\epsilon}_{p,q})^u_{-n}(x)) \leqslant \\ &\qquad \overline{\lim_{n\to\infty}}\, \overline{\lim_{q\to\infty}} -\frac{1}{n}\log \mu_u((\mathcal{J}^{\epsilon}_{p,q})^u_{-n}(x)) \leqslant \overline{\lim_{n\to\infty}} -\frac{1}{n}\log \mu_u((\mathcal{J}^{\epsilon}_{p})^u_{-n}(x)),\end{aligned}$$

where

$$\mathcal{J}^{\epsilon}_p = \{T_{p,1}\cap B^{\epsilon}_p, ..., T_{p,N(p)}\cap B^{\epsilon}_p, \Omega\times S \ \setminus \cup_{1\leqslant i\leqslant N(p)} T_{p,i}\cap B^{\epsilon}_p\}$$

is an invariant partition. Set $\{\mathcal{Z}^t\}_{t\geqslant 1} = \{\mathcal{J}^{\frac{1}{r}}_{p,q}\}_{r,q,p\in\mathbb{N}}$ then

$$\lim_{t\to\infty} \Big[\, \overline{\lim_{n\to\infty}} \frac{1}{n} A(f_n, (\mathcal{Z}^t)_{-n})(u,x) + \overline{\lim_{n\to\infty}} -\frac{1}{n}\log \mu_u((\mathcal{Z}^t)^u_{-n}(x))\Big] = g(u,x) \tag{6.3.11}$$

Finally since

$$\lim_{n\to\infty} \frac{1}{n} f_n(u,x) \leqslant \overline{\lim_{n\to\infty}} \frac{1}{n} A(f_n, (\mathcal{Z}^t)_{-n})(u,x),$$

our claim follows by (6.3.11) . $\square$

Back to the proof, from Proposition 6.3.8, it is enough to prove

$$\lim_{\epsilon\to 0} \overline{\lim_{n\to\infty}} -\frac{1}{n}\log \mu_u(B^u_n(x,\epsilon)) \leqslant \sum_{\lambda_i>0} \lambda_i m_i. \tag{6.3.12}$$

To prove our claim, we begin by our assumption in (6.3.2). Let $\mathcal{Z}$ be a (measurable) partitions of $\Omega\times S$ and $z\in K_u\cap \mathcal{Z}^u(x)$, for $\gamma(u)\leqslant \frac{1}{h(\sigma u)^{\frac{1}{2}}}$ we have

$$\psi^1_u(B(z,\gamma(u))\cap K_u) \subset K_{\sigma u}\cap [\psi^1_u(z)+\gamma(u) D_z\psi^1_u(B(0,1)) + h(\sigma u)\gamma^2(u) B(0,1)]$$

Assume $D_z\psi^1_u(B(0,1)) \subset \bigcup_{1\leqslant i\leqslant N(u,z)} B(z_i,\eta(u))$, where $N(u,z)=R(D_z\psi^1_u,\eta(u))$. Let $w_i\in K_{\sigma u}\cap B(\psi^1_u(z)+\gamma(u)z_i, \gamma(u)[\eta(u)+h(\sigma u)\gamma(u)]) \cap \mathcal{Z}^{\sigma u}(\psi^1_u(x))$, (Note that if this intersection is empty, then this ball is redundant) then

$$\psi^1_u(B(z,\gamma(u))\cap K_u)\cap \mathcal{Z}^{\sigma u}(\psi^1_u(x)) \subset \bigcup_{1\leqslant i\leqslant N(u,z)} [K_{\sigma u}\cap B(w_i, 2\gamma(u)[\eta(u)+h(\sigma u)\gamma(u)])].$$

Set

$$\gamma(u) = \frac{1}{4\prod_{1\leqslant j\leqslant\infty} h(\sigma^j u)^{2^{-j}}},$$

and $\eta(u) = h(\sigma u)\gamma(u)$. Note that form (6.3.3), $\gamma(u)$ is well defined also

$$\psi_u^1(B(z,\gamma(u)) \cap K_u) \cap \mathcal{Z}^{\sigma u}(\psi_u^1(x)) \subset \bigcup_{1\leqslant i\leqslant N(u,z)} \big[K_{\sigma u} \cap B\big(w_i, \gamma(\sigma u)\big)\big].$$

Since $\gamma(u) < \frac{1}{h(\sigma u)^{\frac{1}{2}}}$, by repeating this argument

$$r(\mathcal{Z}_{-n}^u(x), 1, d_n^{\gamma,u}) \leqslant r(\mathcal{Z}^u(x), \frac{\gamma(u)}{2}, \|\|) \prod_{0\leqslant i\leqslant n-1} \sup_{y\in\mathcal{Z}^{\sigma^i u}(\psi_u^i(x))\cap K_{\sigma^i u}} R(D_y\psi^1_{\sigma^i u}, \eta(\sigma^{i+1}u)). \tag{6.3.13}$$

Consequently

$$\overline{\lim_{n\to\infty}} \frac{1}{n} \log r(\mathcal{Z}_{-n}^u(x), 1, d_n^{\gamma,u}) \leqslant \int_\Omega \int_E \log \sup_{y\in\mathcal{Z}^u(x)\cap K_u} R(D_y\psi_u^1, \eta(\sigma u))\mu_u(dx)\mathbb{P}(du). \tag{6.3.14}$$

The next proposition allows us to repeat the same argument for the higher compositions of our cocycle.

Proposition 6.3.11. *For $m \geqslant 1$ and*

$$H_m(\sigma^m u) := 4^m \prod_{0\leqslant j\leqslant m-1} h(\sigma^{j+1}u)(\|D\psi^1_{\sigma^j u}\| \vee 1)^2. \tag{6.3.15}$$

we have

$$\|y-x\| \leqslant \frac{1}{H_m(\sigma^m u)^{\frac{1}{2}}} \longrightarrow \|\psi_u^m(y) - \psi_u^m(x) - D_x\psi_u^m(y-x)\| \leqslant H_m(\sigma^m u)\|y-x\|^2. \tag{6.3.16}$$

Proof. We start with the following identity

$$\begin{aligned}&\psi_u^m(y) - \psi_u^m(x) - D_x\psi_u^m(y-x) = \\ &\quad\sum_{0\leqslant j\leqslant m-1} D_{\psi_u^{j+1}(x)}\psi^{m-1-j}_{\sigma^{j+1}u}\big[\psi^1_{\sigma^j u}(\psi_u^j(y)) - \psi^1_{\sigma^j u}(\psi_u^j(x)) - D_{\psi_u^j(x)}\psi^1_{\sigma^j u}\big(\psi_u^j(y) - \psi_u^j(x)\big)\big].\end{aligned} \tag{6.3.17}$$

Set $\tilde{H}_0(u) = 0$, $\tilde{H}_1(u) = h(\sigma u)$ and

$$\tilde{H}_m(u) = 2\sum_{0\leqslant j\leqslant m-1} h(\sigma^{j+1}u)\|D_{K_{\sigma^{j+1}u}}\psi^{m-1-j}\|\ [\tilde{H}_j(u) + \|D_{K_u}\psi_u^j\|^2].$$

Where $\|D_{K_u}\psi_u^m\| := \sup_{x\in K_u}\|D_x\psi_u^m\|$.

It is not hard to see $\tilde{H}_m(u) < H_m(\sigma^m u)$. We claim if $\|y-x\| \leqslant \frac{1}{H_m(\sigma^m u)^{\frac{1}{2}}}$, then

(i) $\|\psi_u^j(y) - \psi_u^j(x)\| \leqslant \frac{1}{h(\sigma^{j+1}u)^{\frac{1}{2}}}$, $\quad 0 \leqslant j < m$,

(ii) $\|\psi_u^j(y) - \psi_u^j(x) - D_x\psi_u^j(y-x)\| \leqslant \tilde{H}_j(u)\|y-x\|^2$, $\quad 0 \leqslant j \leqslant m$.

We proceed by induction, from (6.3.17),

$$\|\psi_u^m(y)-\psi_u^m(x)-D_x\psi_u^m(y-x)\| \leqslant \sum_{0\leqslant j\leqslant m-1} h(\sigma^{j+1}u)\|D_{K_{\sigma^{j+1}u}}\psi_{\sigma^{j+1}}^{m-1-j}\|\|\psi_u^j(y)-\psi_u^j(x)\|^2$$

$$\leqslant \sum_{0\leqslant j\leqslant m-1} 2h(\sigma^{j+1}u)\|D_{K_{\sigma^{j+1}u}}\psi_{\sigma^{j+1}}^{m-1-j}\|\ [\tilde H_j(u)+\|D_{K_u}\psi_u^j\|^2]\|y-x\|^2 = \tilde H_m(u)\|y-x\|^2.$$

Also for $\|y-x\|\leqslant \frac{1}{(H_{m+1}(\sigma^{m+1}u))^{\frac12}}$

$$\|\psi_u^m(y)-\psi_u^m(x)\| \leqslant \tilde H_m(u)\|y-x\|^2+\|D_{K_u}\psi_u^m\|\|y-x\|$$

$$\leqslant \frac{\tilde H_m(u)}{H_{m+1}(\sigma^{m+1}u)}+\frac{\|D_{K_u}\psi_u^m\|}{(H_{m+1}(\sigma^{m+1}u))^{\frac12}} \leqslant \frac{1}{h(\sigma^{m+1}u)^{\frac12}}.$$

So our claim is proved. □

We also need the following lemma

Lemma 6.3.12. *For $H_m(\sigma^m u)$, defined in (6.3.15), set*

$$\eta_m(\sigma^m u) := \frac{H_m(\sigma^m u)}{4\prod_{1\leqslant j\leqslant\infty} H_m(\sigma^{jm}u)^{2^{-j}}}, \tag{6.3.18}$$

then

$$\lim_{m\to\infty}\frac1m \log\eta_m(\sigma^m u)=0, \quad \mathbb{P}a.a.$$

Proof. For $\Lambda := \lim_{m\to\infty}\frac1m\log H_m(\sigma^m u)$ and $\delta>0$, set

$$\phi(u)=\sup_{m\geqslant 0}\exp(-m(\Lambda+\delta))H_m(\sigma^m u),$$

It is not hard to see

$$\log^+\phi(\sigma u)-\log^+\phi(u)\in L^1(\Omega),$$

so from [13, Lemma III.8]

$$\lim_{n\to\infty}\frac1n\log\phi(\sigma^n\omega)=0$$

This implies our claim . □

Set $\gamma_m(u)=\frac{1}{4\prod_{1\leqslant j\leqslant\infty}H_m(\sigma^{jm}u)^{2^{-j}}}$ and $\eta_m(\sigma^m u)=H_m(\sigma^m u)\gamma_m(u)$, also define

$$d_{n,m}^{\gamma_m,u}(x,y)=\sup_{0\leqslant j<n}\frac{\|\psi_u^{jm}(y)-\psi_u^{jm}(x)\|}{\gamma_m(\sigma^{jm}u)}.$$

By (6.3.13) and (6.3.2), similarly we can prove

$$\overline{\lim_{n\to\infty}} \frac{1}{n} \log r(\mathcal{Z}^u_{-m,-n}(x), 1, d_n^{\gamma_m,u}) \leqslant \int_\Omega \int_E \log \sup_{y\in \mathcal{Z}^u_{-m}(x)\bigcap K_u} R(D_y\psi_u^m, \eta_m(\sigma^m u))\mu_u(dx)\ \mathbb{P}(du), \tag{6.3.19}$$

where $\mathcal{Z}_{-m,-n} = \bigvee_{i=0}^{n-1}(\Psi^{im})^{-1})\mathcal{Z}_{-m}$. Set $B^{u,\gamma_m}_{n,m}(x,\epsilon) = \{y \in K_u : d_n^{\gamma_m,u}(x,y) \leqslant \epsilon\}$ then we have

Lemma 6.3.13. *For an arbitrary partition of $\mathcal{Z}$ of $\Omega \times S$:*

$$\overline{\lim_{n\to\infty}} -\frac{1}{n} \log \mu_u(B^{u,\gamma_m}_{n,m}(x,1)) \leqslant \overline{\lim_{n\to\infty}} \{ -\frac{1}{n} \log \mu_u(\mathcal{Z}^u_{-m,-n}(x)) + \frac{1}{n}\log r(\mathcal{Z}^u_{-m,-n}(x), 1, d_{n,m}^{\gamma_m,u})\}$$

$\mu^\star-$ a.a.

Proof. Similar to [69, Page 87] . ◻

We also need this technical lemma.

Lemma 6.3.14. *For $u \in \Omega$, a.a.*

$$\overline{\lim_{m\to\infty}} \frac{1}{m} \log \sup_{y\in K_u} R(D_y\psi_u^m, \eta_m(\sigma^m u)) < \infty$$

Proof. For $\alpha > 0$, from [74, Theorem 2.3.4] and ergodicity assumption, there is a (deterministic) sequence $... < \lambda_2^k < \lambda_1^K < \infty$ and integer sequence $\{m_i^K\}_{i\geqslant 1}$, such that

$$\overline{\lim_{m\to\infty}} \frac{1}{m} \log \sup_{y\in K_u} R(D_y\psi_u^m, e^{-m\alpha}) \leqslant \sum m_i^K(\lambda_i^K + \alpha)^+, \tag{6.3.20}$$

since $\lim_{m\to\infty} \frac{1}{m} \log \eta_m(\sigma^m u) = 0$, by (6.3.20)

$$\overline{\lim_{m\to\infty}} \frac{1}{m} \log \sup_{y\in K_u} R(D_y\psi_u^m, H^m(\sigma^m u)) \leqslant \sum m_i^K(\lambda_i^K)^+ < \infty$$

◻

Now are now ready to finish the proof. Let $\{\mathcal{Z}^t\}_{t\geqslant 1}$ be the partition, that was constructed in Proposition 6.3.10, set $f_m(u,x) := R(D_x\psi_u^m, \eta_m(\sigma^m u))$ then since $\lim_{m\to\infty}\frac{1}{m}\log\eta_m(\sigma^m u) = 0$, we can show $\mu^\star - a.a.$

$$\overline{\lim_{m\to\infty}} \frac{1}{m} \log R(D_x\psi_u^m, \eta_m(\sigma^m u)) = \sum_{\lambda_i>0} m_i\lambda_i.$$

Note that by Lemma 6.3.14, we can apply to Lemma 6.3.13 and Proposition 6.3.10, consequently for $\delta > 0$ and large t

$$\overline{\lim_{n\to\infty}} -\frac{1}{n} \log \mu_u(B^{u,\gamma_m}_{n,m}(x,1)) \leqslant \delta + \overline{\lim_{n\to\infty}} \frac{1}{n} \log r\big((\mathcal{Z}^t)^u_{-m,-n}(x), 1, d_{n,m}^{\gamma_m,u}\big),$$

so by (6.3.19)

$$\overline{\lim_{n\to\infty}} -\frac{1}{n}\log\mu_u(B_n^{u,\gamma_m}(x,1)) \leqslant \\ \delta + \int_\Omega\int_E \log \sup_{y\in(\mathcal{Z}^t)^u_{-m}(x)\cap K_u} R(D_y\psi_u^m, \eta_m(\sigma^m u))\ \mu_u(dx)\mathbb{P}(du).$$

Not that (6.3.16) remains true if we replace $H_m(\sigma^m u)$ by $\frac{H_m(\sigma^m u)}{\epsilon}$. Consequently

$$\lim_{\epsilon\to 0}\overline{\lim_{n\to\infty}} -\frac{1}{n}\log\mu_u(B_{n,m}^{u}(x,\epsilon)) \leqslant \\ \delta + \int_\Omega\int_E \log \sup_{y\in(\mathcal{Z}^t)^u_{-m}(x)\cap K_u} R(D_y\psi_u^m, \eta_m(\sigma^m u))\ \mu_u(dx)\mathbb{P}(du), \tag{6.3.21}$$

where $B_{n,m}^u(x,\epsilon) = \{y : 0 \ \leqslant \forall j < n,\ \|\psi_u^{jm}(y) - \psi_u^{jm}(x)\| \leqslant \epsilon\}$, finally since $h_\mu(\psi) = \frac{h_\mu(\psi_m)}{m}$ from (6.3.21)

$$h_\mu(\psi) \leqslant \frac{\delta}{m} + \frac{1}{m}\int_\Omega\int_E \log\big(\sup_{y\in(\mathcal{Z}^t)^u_{-m}(x)\cap K_u} R(D_y\psi_u^m, \eta_m(\sigma^m u))\big)\ \mu_u(dx)\mathbb{P}(du). \tag{6.3.22}$$

Since $dim(E) < \infty$, for $M, N > 0$

$$R(D_x\psi_u^m, \eta_m(\sigma^m u)) \leqslant M\Big(\frac{\|D_x\psi_u^m\|}{\eta_m(\sigma^m u)}\Big)^N. \tag{6.3.23}$$

Now from (6.3.8), (6.3.22) and (6.3.23)

$$h_\mu(\psi) \leqslant \int_\Omega\int_E \overline{\lim_{m\to\infty}}\frac{1}{m}\log R(D_x\psi_u^m, \eta_m(\sigma^m u))\mu_u(dx)\mathbb{P}(du) = \sum_{\lambda_i>0} m_i\lambda_i$$

6.4 Example

In this section, without going too much into details, we sketch several examples. The first example is the *translation invariant Brownian flows*. A particular class of this family is the *isotropic Brownian flows*. For more details and exact definition about this family of flows, we refer the reader to [75].

Example 6.4.1. *Assume $E = \mathbb{R}^d$, $U = C(\mathbb{R}^d, \mathbb{R}^d)$ and $F(x, v) = v(x)$. Let $\phi_{s,t}$, $0 \leqslant s \leqslant t$ be a translation invariant Brownian flow on $\mathbb{R}^d$ ($\phi_{s,t} : \tilde{\Omega} \to U$) and define $V_n := \phi_{n,n+1}, n \geqslant 0$. Choose $\theta_i := e_i$ to be the i-th unit vector in $\mathbb{R}^d$, $i \in \{1, ..., d\}$. Then due to the translation invariance property, $S \simeq [0, 1)^d$ and the Lebesgue measure μ on S is an invariant probability measure of the associated S-valued Markov chain (Y_n) (cf.[75]). Then from [75, Lemma 2.1.1], we can deduce Ruelle's inequality.*

Here is another example.

Example 6.4.2. *Consider the following stochastic differential equation*

$$\mathrm{d}X(t) = f(X(t))\,\mathrm{d}t + g(X(t))\,\mathrm{d}W(t),\ t \geq 0; \quad X_0 = x, \tag{6.4.1}$$

where W is m-dimensional Brownian motion and $f : \mathbb{R}^d \to \mathbb{R}^d$, $g : \mathbb{R}^d \to \mathbb{R}^{d\times m}$, are Lipschitz continuous functions. For $U = \mathbb{R}^d$, it is well-known that equation (6.4.1) *generates a continuous-time U-valued random dynamical system. The corresponding discretized sequence X_n, $n \in \mathbb{N}_0$ is of the form* (6.1.2) *for some (continuous) F, where $V_n := W_n$, ($W_n(s) := W(n+s) - W(n)$). For $i \in \{1, ..., d\}$, assume $\theta_i := e_i$ to be the i-th unit vector in $\mathbb{R}^d$, in addition, assume for every $i \in \{1, ..., d\}$, there exist $\alpha_i > 0$ such that for all $x \in U$, $f(x + \alpha_i e_i) = f(x)$ and $g(x + \alpha_i e_i) = g(x)$. Obviously, the rds (X_n) satisfies* (6.1.1) *with $\theta_i := \alpha_i e_i$ (for $i \in \{1, ..., d\}$). Then $S = \prod_{1\leq i \leq}[0, \alpha_i)$, by assuming further regularity assumptions on f and g, the solution is also differentiable (with respect to x). Let ν be a probability measure on S. Since $\bar{S}$ is compact, the family of measures $\frac{\sum_{0\leqslant j<n}\{(Y_j)^{\#}\nu}{n}$ on $\bar{S}$ is tight. Now, from the Feller property of the process Y_n and tightness property, there exists an invariant measure (Krylov–Bogolyubov theorem). We can now apply our result to derive Ruelle's inequality for this invariant measure.*

Remark 6.4.3. *An interesting example in the infinite-dimensional case is for the delay equations. In this example we take the following equation*

$$\mathrm{d}X(t) = f(X_t)\,\mathrm{d}t + g(X(t))\,\mathrm{d}W(t),\ t \geq 0; \quad X_0 = \eta, \tag{6.4.2}$$

where W is m-dimensional Brownian motion, $\eta \in C := C\big([-1, 0], \mathbb{R}^d\big)$, $X_t(s) := X(t + s)$, $t \geq 0$, $s \in [-1, 0]$, where, $f : C \to \mathbb{R}^d$ is Lipschitz continuous with respect to the sup-norm and $g : \mathbb{R}^d \to \mathbb{R}^{d\times m}$ is Lipschitz continuous. In this model, we assume there exist $r \in \{0, ..., d\}$ and $\alpha_i > 0$, $i \in \{1, ..., r\}$ (if $r > 0$) such that $f(x + \alpha_i e_i) = f(x)$ for every $x \in C$ and $i \in \{1, ..., r\}$, where e_i is the function in C which is identically equal to 1 in the i-th coordinate and 0 in all other coordinates. We assume that g has the same property (with the same numbers α_i) but with C replaced by $\mathbb{R}^d$. It is well-known (cf. [76]) that equation (6.4.1) *generates a continuous-time C-valued random dynamical system. Like the previous example, the corresponding discretized sequence X_n satisfies* (6.1.1). *We can choose $S = \{f \in C : f(0) \in \prod_{i=1}^{r}[0, \alpha_i)\}$ if $i \in \{1, ..., r\}$ and $S = C$ if $r = 0$. the existence of invariant measure is a standard assumption. However, the main challenge is, here, it is not clear how we can verify condition* (6.3.1).

References

[1] Ludwig Arnold. *Random dynamical systems*. Springer Monographs in Mathematics. Berlin: Springer-Verlag, 1998, pp. xvi+586.

[2] Salah-Eldin A. Mohammed and Michael K. R. Scheutzow. “The stable manifold theorem for stochastic differential equations”. In: *Ann. Probab.* 27.2 (1999), pp. 615–652.

[3] Salah-Eldin A. Mohammed and Michael K. R. Scheutzow. “The stable manifold theorem for non-linear stochastic systems with memory. II. The local stable manifold theorem”. In: *J. Funct. Anal.* 206.2 (2004), pp. 253–306.

[4] Björn Schmalfuss. “Backward cocycle and attractors of stochastic differential equations”. In: *International Seminar on Applied Mathematics - Nonlinear Dynamics: Attractor Approximation and Global Behavior*. Ed. by V. Reitmann, T. Riedrich, and N. Koksch. Technische Universität Dresden, 1992, pp. 185–192.

[5] Hans Crauel and Franco Flandoli. “Attractors for random dynamical systems”. In: *Probab. Theory Related Fields* 100.3 (1994), pp. 365–393.

[6] Hans Crauel, Arnaud Debussche, and Franco Flandoli. “Random attractors”. In: *J. Dynam. Differential Equations* 9.2 (1997), pp. 307–341.

[7] Aurélien Deya, Andreas Neuenkirch, and Samy Tindel. “A Milstein-type scheme without Lévy area terms for SDEs driven by fractional Brownian motion”. In: *Ann. Inst. Henri Poincaré Probab. Stat.* 48.2 (2012), pp. 518–550.

[8] Michael Scheutzow. “Exponential growth rate for a singular linear stochastic delay differential equation”. In: *Discrete Contin. Dyn. Syst. Ser. B* 18.6 (2013), pp. 1683–1696.

[9] V. I. Oseledec. “A multiplicative ergodic theorem. Characteristic Ljapunov, exponents of dynamical systems”. In: *Trudy Moskov. Mat. Obšč.* 19 (1968), pp. 179–210.

[10] M. S. Raghunathan. “A proof of Oseledec’s multiplicative ergodic theorem”. In: *Israel J. Math.* 32.4 (1979), pp. 356–362.

[11] David Ruelle. “Ergodic theory of differentiable dynamical systems”. In: *Inst. Hautes Études Sci. Publ. Math.* 50 (1979), pp. 27–58.

[12] David Ruelle. “Characteristic exponents and invariant manifolds in Hilbert space”. In: *Ann. of Math. (2)* 115.2 (1982), pp. 243–290.

[13] Ricardo Mañé. “Lyapounov exponents and stable manifolds for compact transformations”. In: *Geometric dynamics (Rio de Janeiro, 1981)*. Vol. 1007. Lecture Notes in Math. Springer, Berlin, 1983, pp. 522–577.

[14] P. Thieullen. “Fibrés dynamiques asymptotiquement compacts. Exposants de Lyapounov. Entropie. Dimension”. In: *Ann. Inst. H. Poincaré Anal. Non Linéaire* 4.1 (1987), pp. 49–97.

[15] Peter Walters. “A dynamical proof of the multiplicative ergodic theorem”. In: *Trans. Amer. Math. Soc.* 335.1 (1993), pp. 245–257.

[16] Zeng Lian and Kening Lu. “Lyapunov exponents and invariant manifolds for random dynamical systems in a Banach space”. In: *Mem. Amer. Math. Soc.* 206.967 (2010), pp. vi+106.

[17] Thai Son Doan. “Lyapunov Exponents for Random Dynamical Systems”. PhD thesis. Technische Universität Dresden, 2009. URL: `https://nbn-resolving.org/urn:nbn:de:bsz:14-qucosa-25314`.

[18] Alex Blumenthal. “A volume-based approach to the multiplicative ergodic theorem on Banach spaces”. In: *Discrete Contin. Dyn. Syst.* 36.5 (2016), pp. 2377–2403.

[19] Cecilia González-Tokman and Anthony Quas. “A concise proof of the multiplicative ergodic theorem on Banach spaces”. In: *J. Mod. Dyn.* 9 (2015), pp. 237–255.

[20] Cecilia González-Tokman and Anthony Quas. “A semi-invertible operator Oseledets theorem”. In: *Ergodic Theory Dynam. Systems* 34.4 (2014), pp. 1230–1272.

[21] P. Wojtaszczyk. *Banach spaces for analysts.* Vol. 25. Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 1991, pp. xiv+382.

[22] Tosio Kato. *Perturbation theory for linear operators.* Classics in Mathematics. Reprint of the 1980 edition. Springer-Verlag, Berlin, 1995, pp. xxii+619.

[23] Russell Lyons, Robin Pemantle, and Yuval Peres. “Conceptual proofs of $L \log L$ criteria for mean behavior of branching processes”. In: *Ann. Probab.* 23.3 (1995), pp. 1125–1138.

[24] Evelyn Buckwar. “Introduction to the numerical analysis of stochastic delay differential equations”. In: *J. Comput. Appl. Math.* 125.1-2 (2000). Numerical analysis 2000, Vol. VI, Ordinary differential equations and integral equations, pp. 297–307.

[25] George Stoica. “A stochastic delay financial model”. In: *Proc. Amer. Math. Soc.* 133.6 (2005), pp. 1837–1841.

[26] Ian Boutle, Richard HS Taylor, and Rudolf A Römer. “El Niño and the delayed action oscillator”. In: *American Journal of Physics* 75.1 (2007), pp. 15–24.

[27] Jinzhi Lei and Michael C. Mackey. “Stochastic differential delay equation, moment stability, and application to hematopoietic stem cell regulation system”. In: *SIAM J. Appl. Math.* 67.2 (2007), pp. 387–407.

[28] Xuerong Mao. *Stochastic differential equations and applications.* Second. Horwood Publishing Limited, Chichester, 2008, pp. xviii+422.

[29] S. E. A. Mohammed. *Stochastic functional differential equations.* Vol. 99. Research Notes in Mathematics. Pitman (Advanced Publishing Program), Boston, MA, 1984, pp. vi+245.

[30] S. E. A. Mohammed. “Nonlinear flows of stochastic linear delay equations”. In: *Stochastics* 17.3 (1986), pp. 207–213.

[31] Terry J. Lyons, Michael Caruana, and Thierry Lévy. *Differential equations driven by rough paths*. Vol. 1908. Lecture Notes in Mathematics. Lectures from the 34th Summer School on Probability Theory held in Saint-Flour, July 6–24, 2004, With an introduction concerning the Summer School by Jean Picard. Berlin: Springer, 2007, pp. xviii+109.

[32] Vladimir I. Bogachev. *Gaussian measures*. Vol. 62. Mathematical Surveys and Monographs. Providence, RI: American Mathematical Society, 1998, pp. xii+433.

[33] Laurence C. Young. "An inequality of the Hölder type, connected with Stieltjes integration". In: *Acta Math.* 67.1 (1936), pp. 251–282.

[34] Salah-Eldin A. Mohammed and Michael K. R. Scheutzow. "Lyapunov exponents of linear stochastic functional differential equations driven by semimartingales. I. The multiplicative ergodic theory". In: *Ann. Inst. H. Poincaré Probab. Statist.* 32.1 (1996), pp. 69–105.

[35] Salah-Eldin A. Mohammed and Michael K. R. Scheutzow. "Lyapunov exponents of linear stochastic functional-differential equations. II. Examples and case studies". In: *Ann. Probab.* 25.3 (1997), pp. 1210–1240.

[36] Massimiliano Gubinelli. "Controlling rough paths". In: *J. Funct. Anal.* 216.1 (2004), pp. 86–140.

[37] A. Neuenkirch, I. Nourdin, and S. Tindel. "Delay equations driven by rough paths". In: *Electron. J. Probab.* 13 (2008), no. 67, 2031–2068.

[38] Franco Flandoli. *Regularity theory and stochastic flows for parabolic SPDEs*. Vol. 9. Stochastics Monographs. Yverdon: Gordon and Breach Science Publishers, 1995, pp. x+79.

[39] Peter K. Friz and Martin Hairer. *A Course on Rough Paths with an introduction to regularity structures*. Vol. XIV. Universitext. Berlin: Springer, 2014, p. 251.

[40] Peter K. Friz and Nicolas B. Victoir. *Multidimensional stochastic processes as rough paths*. Vol. 120. Cambridge Studies in Advanced Mathematics. Theory and applications. Cambridge: Cambridge University Press, 2010, pp. xiv+656.

[41] I. Bailleul, S. Riedel, and M. Scheutzow. "Random dynamical systems, rough paths and rough flows". In: *J. Differential Equations* 262.12 (2017), pp. 5792–5823.

[42] María J. Garrido-Atienza and Björn Schmalfuß. "Ergodicity of the infinite dimensional fractional Brownian motion". In: *J. Dynam. Differential Equations* 23.3 (2011), pp. 671–681.

[43] Martin Hairer. "A theory of regularity structures". In: *Invent. Math.* 198.2 (2014), pp. 269–504.

[44] E. Knobloch and K. A. Wiesenfeld. "Bifurcations in fluctuating systems: the center-manifold approach". In: *J. Statist. Phys.* 33.3 (1983), pp. 611–637.

[45] Tomás Caraballo, José A. Langa, and James C. Robinson. "A stochastic pitchfork bifurcation in a reaction-diffusion equation". In: *R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci.* 457.2013 (2001), pp. 2041–2061.

[46] Aijun Du and Jinqiao Duan. “Invariant manifold reduction for stochastic dynamical systems”. In: *Dynam. Systems Appl.* 16.4 (2007), pp. 681–696.

[47] Jinqiao Duan and Wei Wang. *Effective dynamics of stochastic partial differential equations.* Elsevier Insights. Elsevier, Amsterdam, 2014, pp. xii+270.

[48] Mickaël D. Chekroun, Honghu Liu, and Shouhong Wang. *Approximation of stochastic invariant manifolds.* SpringerBriefs in Mathematics. Stochastic manifolds for nonlinear SPDEs. I. Springer, Cham, 2015, pp. xvi+127.

[49] Mickaël D. Chekroun, Honghu Liu, and Shouhong Wang. *Stochastic parameterizing manifolds and non-Markovian reduced equations.* SpringerBriefs in Mathematics. Stochastic manifolds for nonlinear SPDEs. II. Springer, Cham, 2015, pp. xviii+129.

[50] Terry J. Lyons. “Differential equations driven by rough signals”. In: *Rev. Mat. Iberoamericana* 14.2 (1998), pp. 215–310.

[51] Andrew Carverhill. “Flows of stochastic dynamical systems: ergodic theory”. In: *Stochastics* 14.4 (1985), pp. 273–317.

[52] Petra Boxler. “A stochastic version of center manifold theory”. In: *Probab. Theory Related Fields* 83.4 (1989), pp. 509–545.

[53] Thomas Wanner. “Linearization of random dynamical systems”. In: *Dynamics reported.* Vol. 4. Dynam. Report. Expositions Dynam. Systems (N.S.) Springer, Berlin, 1995, pp. 203–269.

[54] Christian Kuehn and Alexandra Neamţu. “Rough center manifolds”. In: *SIAM J. Math. Anal.* 53.4 (2021), pp. 3912–3957.

[55] Jinqiao Duan, Kening Lu, and Björn Schmalfuss. “Invariant manifolds for stochastic partial differential equations”. In: *Ann. Probab.* 31.4 (2003), pp. 2109–2135.

[56] Jinqiao Duan, Kening Lu, and Björn Schmalfuss. “Smooth stable and unstable manifolds for stochastic evolutionary equations”. In: *J. Dynam. Differential Equations* 16.4 (2004), pp. 949–972.

[57] Salah-Eldin A. Mohammed, Tusheng Zhang, and Huaizhong Zhao. “The stable manifold theorem for semilinear stochastic evolution equations and stochastic partial differential equations”. In: *Mem. Amer. Math. Soc.* 196.917 (2008), pp. vi+105.

[58] Tomás Caraballo et al. “Invariant manifolds for random and stochastic partial differential equations”. In: *Adv. Nonlinear Stud.* 10.1 (2010), pp. 23–52.

[59] Salah Mohammed and Tusheng Zhang. “Dynamics of stochastic 2D Navier-Stokes equations”. In: *J. Funct. Anal.* 258.10 (2010), pp. 3543–3591.

[60] María J. Garrido-Atienza, Kening Lu, and Björn Schmalfuß. “Unstable invariant manifolds for stochastic PDEs driven by a fractional Brownian motion”. In: *J. Differential Equations* 248.7 (2010), pp. 1637–1667.

[61] Xiaopeng Chen, Anthony J. Roberts, and Jinqiao Duan. “Centre manifolds for stochastic evolution equations”. In: *J. Difference Equ. Appl.* 21.7 (2015), pp. 606–632.

[62] Kening Lu, Alexandra Neamţu, and Björn Schmalfuss. "On the Oseledets-splitting for infinite-dimensional random dynamical systems". In: *Discrete Contin. Dyn. Syst. Ser. B* 23.3 (2018), pp. 1219–1242.

[63] Xiaopeng Chen, Anthony J. Roberts, and Jinqiao Duan. "Centre manifolds for infinite dimensional random dynamical systems". In: *Dyn. Syst.* 34.2 (2019), pp. 334–355.

[64] Alexandra Neamţu. "Random invariant manifolds for ill-posed stochastic evolution equations". In: *Stochastics and Dynamics* 20.02 (2020).

[65] Ismaël Bailleul. "Regularity of the Itô-Lyons map". In: *Confluentes Math.* 7.1 (2015), pp. 3–11.

[66] Laure Coutin and Antoine Lejay. "Sensitivity of rough differential equations: an approach through the omega lemma". In: *J. Differential Equations* 264.6 (2018), pp. 3899–3917.

[67] R. Abraham, J. E. Marsden, and T. Ratiu. *Manifolds, tensor analysis, and applications*. Second. Vol. 75. Applied Mathematical Sciences. Springer-Verlag, New York, 1988, pp. x+654.

[68] Sebastian Riedel and Michael Scheutzow. "Rough differential equations with unbounded drift term". In: *J. Differential Equations* 262.1 (2017), pp. 283–312.

[69] Philippe Thieullen. "Fibrés dynamiques asymptotiquement compacts. Exposants de Lyapounov. Entropie. Dimension [Asymptotically compact dynamic bundles. Lyapunov exponents. Entropy. Dimension]". In: *Ann. Inst. Henri Poincaré Anal. Non Linéaire* 32.1 (1987), pp. 49–97.

[70] Jörg Bahnmüller. "Equilibrium states for random dynamical systems". PhD thesis. Institut für Dynamische Systeme, Universität Bremen, 1993.

[71] Zhiming Li and Lin Shu. "The metric entropy of random dynamical systems in a Banach space: Ruelle inequality". In: *Ergodic Theory Dynam. Systems.* 34.2 (2014), pp. 594–615.

[72] Jörg Bahnmüller and Thomas Bogenschütz. "A Margulis-Ruelle inequality for random dynamical systems". In: *Archiv der Mathematik* 64.3 (1995), pp. 246–256.

[73] Michael Brin and Anatole Katok. "On local entropy". In: *Geometric Dynamics*. Vol. 1007. Lecture Notes in Mathematics. Berlin: Springer, 1981, pp. 30–38.

[74] Philippe Thieullen. "Entropy and the Hausdorff dimension for infinite-dimensional dynamical systems". In: *J. Dynam. Differential Equations* 4.1 (1989), pp. 127–159.

[75] Vitalii Senin. "Pesin's Formula for Translation Invariant Random Dynamical Systems". PhD thesis. Technische Universität Berlin, 2019. URL: `https://depositonce.tu-berlin.de/handle/11303/10218`.

[76] Salah-Eldin A. Mohammed and Michael K. R. Scheutzow. "The stable manifold theorem for non-linear stochastic systems with memory. I. Existence of the semiflow". In: *J. Funct. Anal.* 205.2 (2003), pp. 271–305.

[77] David Nualart. *The Malliavin Calculus and Related Topics*. Second. Springer., 2005.

A

A.1 A pathwise MET

Proof of Proposition 2.2.14. For given $n \in \mathbb{N}$, let $E_n^1 := \langle e_n^1, \ldots, e_n^m \rangle$ be an m-dimensional subspace of V_0 with $\|e_n^i\| = 1$ and

$$\mathrm{Vol}(T^n e_n^1, ..., T^n e_n^m) \geqslant \frac{1}{2} D_m(T^n). \tag{A.1.1}$$

By [18, Lemma 2.3], we can find a closed complement subspace F_n^2 to $E_n^2 := T^n E_n^1$ in V_n such that for $P_n^2 := \Pi_{E_n^2 || F_n^2}$,

$$\|P_n^2\| \leqslant \sqrt{m}.$$

Let $F_n^1 := \{v \in V_0 : T^n v \in F_n^2\}$. One can check that F_n^1 is a closed complement subspace to E_n^1. Set $P_n^1 := \Pi_{E_n^1 || F_n^1}$. From Lemma 2.2.6 and (A.1.1), it follows that there is a constant α_m such that for any $v \in E_n^1$,

$$\frac{\|T^n v\|}{\|v\|} \geqslant \frac{D_m(T^n)}{2\alpha_m \|T^n\|^{m-1}}. \tag{A.1.2}$$

From $P_n^1 = (T^n|_{E_n^1})^{-1} P_n^2 T^n$, (A.1.2) implies that

$$\|P_n^1\| \leqslant (m+1)\|T^n\| \|(T^n|_{E_n^1})^{-1}\| \leqslant \frac{2\alpha_m \|T^n\|^m}{D_m(T^n)}. \tag{A.1.3}$$

Let $v \in F_n^1$ with $\|v\| = 1$. Then

$$\mathrm{Vol}(T^n e_n^1, ..., T^n e_n^m, T^n v) = \mathrm{Vol}(T^n e_n^1, ..., T^n e_n^m)\, d(T^n v, \langle T^n e_n^1, ..., T^n e_n^m \rangle). \tag{A.1.4}$$

Since $d(T^n v, \langle T^n e_n^1, ..., T^n e_n^m \rangle) = \inf_{\beta_j \in \mathbb{R}} \| T^n v - \sum_{1 \leqslant j \leqslant m} \beta_j T^n e_n^j \|$, we see that

$$\frac{\| T^n v\|}{d(T^n v, \langle T^n e_n^1, ..., T^n e_n^m \rangle)} \leqslant \|P_n^2\| + 1 \leqslant \sqrt{m} + 1.$$

Consequently, from (A.1.1) and (A.1.4),

$$\|T^n v\| \leqslant \frac{2(\sqrt{m}+1) D_{m+1}(T^n)}{D_m(T^n)}. \tag{A.1.5}$$

The rest of the proof is almost identical to the original proof of [18, Proposition 3.4]. First, one can show that the sequence of subspaces (F^n) converge to F in the Hausdorff distance at a sufficiently fast exponential rate, cf. [18, Claim 3 on page 2396]. Together with (A.1.5), this implies the bound

$$\limsup_{n \to \infty} \frac{1}{n} \log \|T^n|_F\| \leqslant \underline{l}$$

which was announced in Remark (2.2.15). From the convergence, we can also deduce that F is closed and m-codimensional. The identities (2.2.25) and (2.2.26) can be proved exactly as in [18]. To see (2.2.27), let $H = \langle h_1, ..., h_m \rangle$ be a complement subspace to F. Note that, from (2.2.26) and assumption (ii), for any $\delta > 0$, we can choose n large enough such that

$$\exp\left(n(\bar{l} - \delta)\right) \leqslant \frac{\|T^n v\|}{\|v\|} \leqslant \exp\left(n(\bar{l} + \delta)\right)$$

holds for all $v \in H$. Consequently,

$$\exp\left(n(\bar{l} - \delta)\right) \leqslant \frac{d(T^n h_j, \langle T^n h_i \rangle_{1 \leqslant i < j})}{d(h_j, \langle h_i \rangle_{1 \leqslant i < j})} \leqslant \exp\left(n(\bar{l} + \delta)\right)$$

for all $1 \leq j \leq m$ and (2.2.27) follows. $\square$

A.2 Stability for rough delay equations

In the following, we sketch the proof of Theorem 4.2.9. The strategy is the same as in [37, Theorem 4.2].

Proof of Theorem 4.2.9 (sketch). For simplicity, we assume that $U = W = \mathbb{R}$. By definition,

$$y_{s,t} = \int_s^t \sigma(y_\tau, \xi_{\tau - r})\, d\mathbf{X}_\tau = \Lambda_{s,t} + \rho_{s,t}^2 = \sigma(y_s, \xi_{s-r}) X_{s,t} + \rho_{s,t}^1 + \rho_{s,t}^2 \tag{A.2.1}$$

where

$$\begin{aligned}
\Lambda_{s,t} &= \sigma(y_s, \xi_{s-r}) X_{s,t} + \sigma_1(y_s, \xi_{s-r}) y_s' \mathbb{X}_{s,t} + \sigma_2(y_s, \xi_{s-r}) \xi_{s-r}' \mathbb{X}_{s,t}(-r), \\
\rho_{s,t}^1 &= \sigma_1(y_s, \xi_{s-r}) y_s' \mathbb{X}_{s,t} + \sigma_2(y_s, \xi_{s-r}) \xi_{s-r}' \mathbb{X}_{s,t}(-r) \quad \text{and} \\
\rho_{s,t}^2 &= \int_s^t \sigma(y_\tau, \xi_{\tau-r})\, d\mathbf{X}_\tau - \Lambda_{s,t},
\end{aligned}$$

using the notation $\sigma_1(x,y) = \partial_x \sigma(x,y)$, $\sigma_2(x,y) = \partial_y \sigma(x,y)$. Analogously, one defines $\tilde{\Lambda}$, $\tilde{\rho}^1$ and $\tilde{\rho}^2$ such that

$$\tilde{y}_{s,t} = \tilde{\Lambda}_{s,t} + \tilde{\rho}^2_{s,t} = \sigma(\tilde{y}_s, \tilde{\xi}_{s-r})\tilde{X}_{s,t} + \tilde{\rho}^1_{s,t} + \tilde{\rho}^2_{s,t}.$$

Note that $y'_s = \sigma(y_s, \xi_{s-r})$ and $y^{\#}_{s,t} = \rho^1_{s,t} + \rho^2_{s,t}$. It is not hard to see that

$$\begin{aligned} \Lambda_{s,t} - \Lambda_{s,u} - \Lambda_{u,t} &= [\sigma_1(y_s, \xi_{s-r}) y^{\#}_{s,u} + \sigma_2(y_s, \xi_{s-r})\xi^{\#}_{s-r,u-r}] X_{u,t} \\ &\quad + [\sigma_1(y_u, \xi_{u-r}) y'_u - \sigma_1(y_s, \xi_{s-r}) y'_s]\mathbb{X}_{u,t} + [\sigma_2(y_u, \xi_{u-r})\xi'_{u-r} - \sigma_2(y_s, \xi_{s-r})\xi'_{s-r}]\mathbb{X}_{u,t}(-r) \\ &\quad + \int_0^1 (1-\tau)\big[\sigma_{1,1}(z^\tau_{s,u}, \bar{z}^\tau_{s,u})(y_{s,u})^2 + 2\sigma_{1,2}(z^\tau_{s,u}, \bar{z}^\tau_{s,u}) y_{s,y}\xi_{s-r,u-r} \\ &\quad + \sigma_{2,2}(z^\tau_{s,u}, \bar{z}^\tau_{s,u})(\xi_{s-r,u-r})^2\big] d\tau X_{u,t} \end{aligned} \tag{A.2.2}$$

where $z^\tau_{s,u} = \tau y_u + (1-\tau) y_s$, $\bar{z}^\tau_{s,u} = \tau \xi_{u-r} + (1-\tau)\xi_{s-r}$ and $\sigma_{1,1}(x,y) = \partial^2_x \sigma(x,y)$, $\sigma_{1,2}(x,y) = \partial_x \partial_y \sigma(x,y)$ and $\sigma_{2,2}(x,y) = \partial^2_y \sigma(x,y)$. Set

$$\begin{aligned} R &:= \|X - \tilde{X}\|_{\gamma,[0,r]} + \|\mathbb{X} - \tilde{\mathbb{X}}\|_{2\gamma,[0,r]} + \|\mathbb{X}(-r) - \tilde{\mathbb{X}}(-r)\|_{2\gamma,[0,r]} \\ &\quad + \|\xi' - \tilde{\xi}'\|_{\beta,[0,r]} + \|\xi^{\#} - \tilde{\xi}^{\#}\|_{2\beta,[0,r]} + \|\xi - \tilde{\xi}\|_{\beta,[0,r]}, \\ C(y) &:= \|X\|_\gamma + \|\mathbb{X}\|_{2\gamma,[0,r]} + \|\mathbb{X}(-r)\|_{2\gamma,[0,r]} + \|y\|_{\mathscr{D}^\beta_X([0,r],W)} + \|\xi\|_{\mathscr{D}^\beta_X([-r,0],W)} \quad \text{and} \\ D(X) &:= \|X\|_\gamma + \|\mathbb{X}\|_{2\gamma} + \|\mathbb{X}(-r)\|_{2\gamma} + \|\xi\|_{\mathscr{D}^\beta_X([0,r],W)} \end{aligned}$$

with an analogous definition of $C(\tilde{y})$ and $D(\tilde{X})$. It is not hard to see that there is a continuous function $g : (0,\infty)^4 \to [0,\infty)$, increasing in every of its arguments, such that

$$\begin{aligned} \|\rho^1 - \tilde{\rho}^1\|_{2\beta;[a,b]} &\leqslant (b-a)^{\gamma-\beta} g\big[D(X), D(\tilde{X}), C(y), C(\tilde{y})\big] \\ &\qquad \big[R + \|y - \tilde{y}\|_{\beta;[a,b]} + \|y' - \tilde{y}'\|_{\beta;[a,b]} + \|y^{\#} - \tilde{y}^{\#}\|_{2\beta;[a,b]}\big] \end{aligned}$$

for every $[a,b] \subseteq [0,r]$. From the Sewing lemma [39, Lemma 4.2],

$$\|\rho^2 - \tilde{\rho}^2\|_{2\beta;[a,b]} \leqslant M \sup_{s,u,t\in[a,b]} \frac{\big|(\Lambda_{s,t} - \tilde{\Lambda}_{s,t}) - (\Lambda_{s,u} - \tilde{\Lambda}_{s,u}) - (\Lambda_{u,t} - \tilde{\Lambda}_{u,t})\big|}{(t-s)^{2\beta}}$$

for some constant $M > 0$. Using (A.2.2), one can deduce that

$$\begin{aligned} &\sup_{s,u,t\in[a,b]} \frac{\big|(\Lambda_{s,t} - \tilde{\Lambda}_{s,t}) - (\Lambda_{s,u} - \tilde{\Lambda}_{s,u}) - (\Lambda_{u,t} - \tilde{\Lambda}_{u,t})\big|}{(t-s)^{2\beta}} \\ &\leqslant (b-a)^{\gamma-\beta} g\big[D(X), D(\tilde{X}), C(y), C(\tilde{y})\big]\big[R + \|y - \tilde{y}\|_{2\beta;[a,b]} + \|y' - \tilde{y}'\|_{\beta;[a,b]} + \|y^{\#} - \tilde{y}^{\#}\|_{2\beta;[a,b]}\big]. \end{aligned}$$

Now, along with $(A.2.1)$,

$$\begin{aligned} &\|y - \tilde{y}\|_{\beta;[a,b]} + \|y' - \tilde{y}'\|_{\beta;[a,b]} + \|y^{\#} - \tilde{y}^{\#}\|_{2\beta;[a,b]} \leqslant \\ &(b-a)^{\gamma-\beta} \tilde{g}\big[D(X), D(\tilde{X}), C(y), C(\tilde{y})\big]\big[R + \|y - \tilde{y}\|_{\beta;[a,b]} + \|y' - \tilde{y}'\|_{\beta;[a,b]} + \|y^{\#} - \tilde{y}^{\#}\|_{2\beta;[a,b]}\big] \end{aligned}$$

with $\tilde{g}$ being a continuous increasing function. Using the bounds for the norm of y and $\tilde{y}$ provided in [37, Equation (62)], we can find an increasing continuous function $H : (0,\infty)^2 \to [0,\infty)$ such that

$$\|y-\tilde{y}\|_{\beta;[a,b]} + \|y'-\tilde{y}'\|_{\beta;[a,b]} + \|y^{\#}-\tilde{y}^{\#}\|_{2\beta;[a,b]} \leqslant \\ (b-a)^{\gamma-\beta} H[D(X), D(\tilde{X})][R + \|y-\tilde{y}\|_{\beta;[a,b]} + \|y'-\tilde{y}'\|_{\beta;[a,b]} + \|y^{\#}-\tilde{y}^{\#}\|_{2\beta;[a,b]}].$$

Now by the same argument as for the linear case, cf. the proof of Theorem 4.2.11, one sees that

$$\|y-\tilde{y}\|_{\beta;[0,r]} + \|y'-\tilde{y}'\|_{\beta;[0,r]} + \|y^{\#}-\tilde{y}^{\#}\|_{2\beta;[0,r]} \leqslant F[D(X) + D(\tilde{X})]R \tag{A.2.3}$$

holds for an increasing continous function F. The claim follows from (A.2.3) . □

A.3 Elements of Malliavin Calculus

Basic definitions

In this section, we quickly sketch some of the necessary definitions and theorems in Malliavin's calculus. Most of the proofs can be found in [77]. Let $\mathcal{E}$ be the set of step-functions on $\mathbb{R}$ taking values in $\mathbb{R}^d$. Define the Hilbert space $\mathcal{H}$ as the closure of step-functions with the following inner product

$$\langle(\chi_{[s_1,t_1]}, \ldots, \chi_{[s_d,t_d]}), (\chi_{[u_1,v_1]}, \ldots, \chi_{[u_d,v_d]})\rangle = \\ \sum_{1\leqslant i\leqslant d} (R_H(s_i,u_i) + R_H(t_i,v_i) - R_H(s_i,v_i) - R_H(t_i,u_i)).$$

Where

$$R_H(s,t) = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H}) \ .$$

We also can define the isonormal stochastic Gaussian process $B = B^H = \{B(\phi), \phi \in \mathcal{H}\}$ on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that

$$E(\langle B(\phi), B(\psi)\rangle) = \langle \phi, \psi\rangle_{\mathcal{H}}, \quad \phi, \psi \in \mathcal{H}.$$

In particular $B(\chi_{[s_1,t_1]}, \ldots, \chi_{[s_d,t_d]}) = (B_i(t_i) - B_i(s_i))_{1\leqslant i\leqslant d}$.

For $\alpha < 1$ and $\phi \in \mathcal{H}$ set

$$(D_-^\alpha \phi)(u) := \frac{\alpha}{\Gamma(1-\alpha)} \int_0^\infty \frac{\phi(u) - \phi(u+x)}{x^{1+\alpha}} dx.$$

Note that if $supp(\phi) \subset (-\infty, b)$, then

$$(D_-^\alpha \phi)(u) = \frac{\phi(u)}{\Gamma(1-\alpha)(b-u)^\alpha} + \frac{\alpha}{\Gamma(1-\alpha)} \int_u^b \frac{\phi(u) - \phi(x)}{(x-u)^{1+\alpha}} dx.$$

Also for $\phi, \psi \in \mathcal{H}$

$$\langle \phi, \psi \rangle_{\mathcal{H}} = \langle D_{-}^{\frac{1}{2}-H} \phi, D_{-}^{\frac{1}{2}-H} \psi \rangle_{L^2(\mathbb{R}^d)},$$

consequently for $\phi \in \mathcal{H}$ with $\sup(\phi) \subset [a, b]$

$$\mathbb{E}(\|\phi\|_{\mathcal{H}}^2) \leqslant M\Big[\int_a^b \frac{\mathbb{E}(\|\phi(u)\|^2)}{(b-u)^{2\alpha}} du + \int_{-\infty}^a \Big(\int_a^b \frac{(\mathbb{E}(\|\phi(x)\|^2))^{\frac{1}{2}}}{(x-u)^{1+\alpha}} dx\Big)^2 du + \int_a^b \Big(\int_u^b \frac{(\mathbb{E}(\|\phi(u)-\phi(x)\|^2))^{\frac{1}{2}}}{(x-u)^{1+\alpha}} dx\Big)^2 du\Big]. \tag{A.3.1}$$

Let $\mathcal{G}$ be the set of smooth cylindrical random variables

$$F = f(B(\phi_1), ..., B(\phi_k)), \quad \phi_i \in \mathcal{H}, \ \ 1 \leqslant i \leqslant k,$$

where f is a smooth real function. Define the derivative operator by

$$D^B F = \sum_{1 \leqslant i \leqslant k} \frac{\partial f}{\partial x_i}(B(\phi_1), ..., B(\phi_k))\phi_i \in \mathcal{H}.$$

Where $D_r^B F := (D^B F)(r)$. Note that this operator is closable from $L^p(\Omega)$ into $L^p(\Omega, \mathcal{H})$. Also define $\mathbb{D}^{1,2}(\mathcal{H})$ as the closure of smooth cylindrical random variables with respect to the following norm

$$\|F\|_{\mathbb{D}^{1,2}(\mathcal{H})} := \mathbb{E}(|F|^2) + \mathbb{E}(\|D^B F\|_{\mathcal{H}}^2).$$

We also use δ^B to denote the adjoint operator of D^B.

Remember

$$Dom(\delta^B) := \{u \in L^2(\Omega, \mathcal{H}) : \sup_{F \in \mathbb{D}^{1,2}} \frac{|\mathbb{E}(\langle D^B F, u \rangle_{\mathcal{H}})|}{\|F\|_{\mathbb{D}^{1,2}}(\mathcal{H})} < \infty\}.$$

And for $G \in Dom(\delta^B)$ and $F \in \mathbb{D}^{1,2}(\mathcal{H})$ we have the following identity

$$\mathbb{E}(F \delta^B(G)) = \mathbb{E}(\langle D^B F, G \rangle_{\mathcal{H}}). \tag{A.3.2}$$

δ^B is called Skorohod integral. We use $\delta^B_{[s,t]}$ to denote the Skorohod integral in $[s, t]$, in particular we write $\delta^B_{[s,t]}(u) = \int_s^t u(r) dB_r$. We have following relations between D^B and δ^B :

- For $u \in Dom(\delta^B)$ and $F \in \mathbb{D}^{1,2}(\mathcal{H})$ such that $Fu \in L^2(\Omega, \mathcal{H})$, then

$$\delta^B(Fu) = F\delta^B(u) - \langle D^B F, u \rangle_{\mathcal{H}},$$

- Assuming $E(\|u\|_{\mathcal{H}}^2) + E(\|D^B u\|_{\mathcal{H} \otimes \mathcal{H}}^2) < \infty$ and for every $r \in \mathbb{R}$, furthermore assume $(\delta^B(D_r^B u))_{r \in \mathbb{R}} \in L^2(\Omega, \mathcal{H})$, then

$$D_r^B(\delta^B u) = u(r) + \delta^B(D_r^B u). \tag{A.3.3}$$

Stochastic integral

Assume B be a fractional Brownian motion with $H < \frac{1}{2}$. To define the stochastic integral with respect to B is more natural to work with the symmetric integral introduced by Russo and Vallois.

Definition A.3.1. *For a given process u, the symmetric integral is defined as*

$$\int_s^t u_r d^\circ B_r := L^2 - \lim_{\epsilon \to 0} \int_s^t u_r \frac{B_{r+\epsilon} - B_{r-\epsilon}}{2\epsilon} dr,$$

when it exists.

The following Proposition relates the symmetric integral to the Skorohod integral.

Proposition A.3.2. *Let $u = \{u_r,\ r \in [s,t]\}$ be a stochastic process such that*

$$\mathbb{E}(\|u\|_{\mathcal{H}}^2) + \int_s^t \mathbb{E}(\|D_r^B u\|_{\mathcal{H}}^2) dr < \infty.$$

Assume that the trace defined as following

$$Tr_{[s,t]} D^B u := L^2 - \lim_{\epsilon \to 0} \int_s^t \frac{\langle D^B u_r, 1_{[r-\epsilon, r+\epsilon] \cap [s,t]} \rangle_{\mathcal{H}}}{2\epsilon} dr$$

exists. Then the symmetric integral exists and we have

$$\int_s^t u_r d^\circ B_r = \delta_{[s,t]}^B (u) + Tr_{[s,t]} D^B u.$$

A.4 Some bounds for the linearized equation

Proof of Theorem 5.2.3. We start with equation (4.2.5). From Proposition 5.1.9, the derivative of the solution at ξ in the direction of $\tilde{\xi}$ satisfies the equation

$$\begin{aligned} Dy^\xi[\tilde{\xi}](t) - \tilde{\xi}_0 &= \int_0^t [\sigma_x(y_\tau^\xi, \xi_{\tau-r}) Dy^\xi[\tilde{\xi}](\tau) + \sigma_y(y_\tau^\xi, \xi_{\tau-r}) \tilde{\xi}_{\tau-r}] d\mathbf{X}_\tau; \quad t \in [0,r] \\ Dy^\xi[\tilde{\xi}](t) &= \tilde{\xi}_t; \quad t \in [-r, 0]. \end{aligned} \tag{A.4.1}$$

Set $Z_\tau = Dy^\xi[\tilde{\xi}](\tau)$ and $\eta_t = \sigma_x(y_t^\xi, \xi_{t-r}) Z_t + \sigma_y(y_t^\xi, \xi_{t-r}) \tilde{\xi}_{t-r}$. Using a Taylor expansion and the definition of controlled paths, we obtain

$$\begin{aligned} \eta_{s,t} = \sigma_x(y_s^\xi, \xi_{s-r}) Z_s' X_{s,t} &+ [\sigma_{x^2}(y_s^\xi, \xi_{s-r})(y^\xi)_s' X_{s,t} + \sigma_{x,y}(y_s^\xi, \xi_{s-r}) \xi_{s-r}' X_{s-r,t-r}] \\ Z_s &+ \sigma_y(y_s^\xi, \xi_{s-r})(\tilde{\xi})_{s-r}' X_{s-r,t-r} \\ &+ [\sigma_{x,y}(y_s^\xi, \xi_{s-r})(y^\xi)_s' X_{s,t} + \sigma_{y^2}(y_s^\xi, \xi_{s-r}) \xi_{s-r}' X_{s-r,t-r}] \tilde{\xi}_{s-r} + \eta_{s,t}^\# \end{aligned} \tag{A.4.2}$$

where

$$\begin{aligned}\eta^{\#}_{s,t} &= [\sigma_x(y^{\xi}_t,\xi_{t-r})-\sigma_x(y^{\xi}_s,\xi_{s-r})]Z_{s,t}+[\sigma_y(y^{\xi}_t,\xi_{t-r})-\sigma_y(y^{\xi}_s,\xi_{s-r})]\tilde{\xi}_{s-r,t-r}+\\ &\sigma_x(y^{\xi}_s,\xi_{s-r})Z^{\#}_{s,t}+\sigma_y(y^{\xi}_s,\xi_{s-r})\tilde{\xi}^{\#}_{s,t}+[\sigma_{x^2}(y^{\xi}_s,\xi_{s-r})(y^{\xi})^{\#}_{s,t}+\sigma_{x,y}(y^{s}_{\xi},\xi_{s-r})\xi^{\#}_{s-r,t-r}]Z_s\\ &+\sigma_{y^2}(y^{s}_{\xi},\xi_{s-r})\xi^{\#}_{s-r,t-r}]\tilde{\xi}_{s-r}+[\sigma_{x,y}(y^{\xi}_s,\xi_{s-r})(y^{\xi})^{\#}_{s,t}\\ &+\int_0^1(1-z)\frac{d^2}{dz^2}\Big[\sigma_x(zy^{\xi}_t+(1-z)y^{\xi}_s,z\xi_{t-r}+(1-z)\xi_{s-r})\Big]Z_s\,dz\\ &+\int_0^1(1-z)\frac{d^2}{dz^2}\Big[\sigma_y(zy^{\xi}_t+(1-z)y^{\xi}_s,z\xi_{t-r}+(1-z)\xi_{s-r})\Big]\tilde{\xi}_{s-r}\,dz\end{aligned}\tag{A.4.3}$$

and $Z_{s,t}=Z'_sX_{s,t}+Z^{\#}_{s,t}$ with

$$Z'_s=\sigma_x(y^{\xi}_s,\xi_{s-r})Dy^{\xi}[\tilde{\xi}](s)+\sigma_y(y^{\xi}_s,\xi_{s-r})\tilde{\xi}_{s-r}.$$

By Theorem 4.2.5, for a delayed controlled path with decomposition $\eta_{s,t}=\eta^1_sX_{s,t}+\eta^2_sX_{s-r,t-r}+\eta^{\#}_{s,t}$, we have for any $w_0\in W$

$$\|w_0+\int_a^{\cdot}\eta_\tau\,d\mathbf{X}_\tau\|_{\mathscr{D}^{\beta}_X[a,b]}\leqslant|w_0|+|\eta_a|+\|\eta\|_{\beta;[a,b]}+\sup_{a\le s<t\le b}\frac{\left|\int_s^t\eta_\tau\,d\mathbf{X}_\tau-\eta_sX_{s,t}\right|}{|t-s|^{2\beta}}\tag{A.4.4}$$

and

$$\begin{aligned}\sup_{a\le s<t\le b}\frac{\left|\int_s^t\eta_\tau\,d\mathbf{X}_\tau-\eta_sX_{s,t}\right|}{|t-s|^{2\beta}}\le\ &\|\eta^1\|_{\infty;[a,b]}\|\mathbb{X}\|_{\gamma;[a,b]}(b-a)^{2(\gamma-\beta)}+\\ \|\eta^2\|_{\infty;[a,b]}&\|\mathbb{X}(-r)\|_{\gamma;[a,b]}(b-a)^{2(\gamma-\beta)}+\\ M\Big[&\|\eta^{\#}\|_{2\beta;[a,b]}\|X\|_{\gamma;[a,b]}(b-a)^{\gamma}+\|\eta^1\|_{\beta;[a,b]}\|\mathbb{X}\|_{2\gamma;[a,b]}(b-a)^{2\gamma-\beta}+\\ &\|\eta^2\|_{\beta;[a,b]}\|\mathbb{X}(-r)\|_{2\gamma;[a,b]}(b-a)^{2\gamma-\beta}\Big]\end{aligned}$$

for a general constant M. Thanks to our assumptions on σ, (A.4.2), (A.4.3) and Theorem 5.1.8,

$$\max\Big\{\|\eta^1\|_{\beta;[a,b]},\|\eta^2\|_{\beta;[a,b]},\|\eta^{\#}\|_{2\beta;[a,b]}\Big\}\leqslant[\|Z\|_{\mathscr{D}^{\beta}_X[0,r]}+\|\tilde{\xi}\|_{\mathscr{D}^{\beta}_X[-r,0]}]Q_1(A,\|\xi\|_{\mathscr{D}^{\beta}_X[-r,0]})$$

and

$$\|\eta\|_{\beta;[a,b]}\leqslant(b-a)^{\gamma-\beta}[\|Z\|_{\mathscr{D}^{\beta}_X[0,r]}+\|\tilde{\xi}\|_{\mathscr{D}^{\beta}_X[-r,0]}]Q_1(A,\|\xi\|_{\mathscr{D}^{\beta}_X[-r,0]})$$

for a polynomial Q_1. Using this bound in (A.4.1), we see that for $0\le(n-1)\tau<n\tau\le r$

$$\begin{aligned}\|Z\|_{\mathscr{D}^{\beta}_X[(n-1)\tau,n\tau]}\leqslant\ &\tau^{\gamma-\beta}\|Z\|_{\mathscr{D}^{\beta}_X[(n-1)\tau,n\tau]}Q_2(A,\|\xi\|_{\mathscr{D}^{\beta}_X[-r,0]})\\ &+\|\tilde{\xi}\|_{\mathscr{D}^{\beta}_X[-r,0]}Q_2(A,\|\xi\|_{\mathscr{D}^{\beta}_X[-r,0]})+|Z_{(n-1)\tau}|+|Z'_{(n-1)\tau}|\end{aligned}$$

for a polynomial Q_2. Choosing τ such that $\tau^{\gamma-\beta}Q_2(A,\|\xi\|_{\mathscr{D}_X^\beta[-r,0]}) \leq \frac{1}{2}$, we can proceed as in the proof of Theorem 4.2.11 to conclude the claimed bound for (4.2.5). The proof for (5.2.1) is similar. $\square$

Proof of Theorem 5.2.4. We will prove the statement for the solution to (4.2.5) only, the proof for (5.2.1) is similar. Set $Z^1_\tau := Dy^\xi[\eta](\tau)$ and $Z^2_\tau := Dy^{\tilde{\xi}}[\eta](\tau)$. From Proposition 5.1.9,

$$[Z^1_{s,t} - Z^2_{s,t}] = \int_s^t [\sigma_x(y^\xi_\tau, \xi_{\tau-r})[Z^1_\tau - Z^2_\tau] + B_\tau]\, d\mathbf{X}_\tau \tag{A.4.5}$$

where

$$\begin{aligned} B_\tau &:= [\sigma_x(y^\xi_\tau, \xi_{\tau-r}) - \sigma_x(y^{\tilde{\xi}}_\tau, \tilde{\xi}_{\tau-r})]Z^2_\tau + [\sigma_y(y^\xi_\tau, \xi_{\tau-r}) - \sigma_y(y^{\tilde{\xi}}_\tau, \tilde{\xi}_{\tau-r})]\eta_{\tau-r} \\ &=: B^1_\tau + B^2_\tau. \end{aligned}$$

Set $C_\tau := [\sigma_x(y^\xi_\tau, \xi_{\tau-r}) - \sigma_x(y^{\tilde{\xi}}_\tau, \tilde{\xi}_{\tau-r})]$. By a Taylor expansion,

$$\begin{aligned} C_{s,t} &= [\sigma_{x^2}(y^\xi_s, \xi_{s-r})(y^\xi)'_s - \sigma_{x^2}(y^{\tilde{\xi}}_s, \tilde{\xi}_{s-r})(y^{\tilde{\xi}})'_s]X_{s,t} \\ &+ [\sigma_{x,y}(y^\xi_t, \xi_{t-r})\xi'_{s-r} - \sigma_{x,y}(y^{\tilde{\xi}}_s, \tilde{\xi}_{t-r})\tilde{\xi}'_{s-r}]X_{s-r,t-r} \\ &+ [\sigma_{x^2}(y^\xi_s, \xi_{s-r})(y^\xi)^{\#}_{s,t} - \sigma_{x^2}(y^{\tilde{\xi}}_s, \tilde{\xi}_{s-r})(y^{\tilde{\xi}})^{\#}_{s,t}] \\ &+ [\sigma_{x,y}(y^\xi_s, \xi_{s-r})\xi^{\#}_{s,t} - \sigma_{x,y}(y^{\tilde{\xi}}_s, \tilde{\xi}_{s-r})\tilde{\xi}^{\#}_{s,t}] \\ &+ \int_0^1 (1-z)\frac{d^2}{dz^2}\Big[\sigma_x(zy^\xi_t + (1-z)y^\xi_s, x\xi_{t-r} + (1-z)\xi_{s-r}) \\ &- \sigma_x(zy^{\tilde{\xi}}_t + (1-z)y^{\tilde{\xi}}_s, z\tilde{\xi}_{t-r} + (1-z)\tilde{\xi}_{s-r})\Big]\, dz \\ &=: C^1_s X_{s,t} + C^2_{s,t}X_{s-r,t-r} + C^{\#}_{s,t}. \end{aligned}$$

Note that

$$\begin{aligned} C^1_{s,t} = &\int_0^1 \frac{d}{dz}\Big[\sigma_{x^2}(zy^\xi_t + (1-z)y^{\tilde{\xi}}_t, z\xi_{t-r} + (1-z)\tilde{\xi}_{t-r}) \\ &\quad - \sigma_{x^2}(zy^\xi_s + (1-z)y^{\tilde{\xi}}_s, z\xi_{s-r} + (1-z)\tilde{\xi}_{s-r})\Big](y^\xi)'_t\, dz \\ &+ \sigma_{x^2}(y^{\tilde{\xi}}_t, \tilde{\xi}_{t-r})[(y^\xi)'_{s,t} - (y^{\tilde{\xi}})'_{s,t}] \\ &+ \int_0^1 \frac{d}{dz}\Big[\sigma_{x^2}(zy^\xi_s + (1-z)y^{\tilde{\xi}}_s, z\xi_{s-r} + (1-z)\tilde{\xi}_{s-r})\Big](y^\xi)'_{s,t}\ dz \\ &+ [\sigma_{x^2}(y^{\tilde{\xi}}_t, \tilde{\xi}_{t-r}) - \sigma_{x^2}(y^{\tilde{\xi}}_s, \tilde{\xi}_{s-r})][(y^\xi)'_s - (y^{\tilde{\xi}})'_s]. \end{aligned}$$

From Theorem 5.1.8, Theorem 5.2.3 and our assumptions on σ,

$$\begin{aligned} &\max\{\|C^1\|_{\beta;[0,r]}, \|C^1\|_{\infty;[0,r]}\} \leqslant \\ &\quad \|\xi - \tilde{\xi}\|_{\mathscr{D}_X^\beta[-r,0]} \exp[P_1(A, \|\xi\|_{\mathscr{D}_X^\beta[-r,0]}, \|\xi - \tilde{\xi}\|_{\mathscr{D}_X^\beta[-r,0]})] \end{aligned} \tag{A.4.6}$$

where P_1 is a polynomial. Note that

$$B^1_{s,t} = [C^1_s X_{s,t}]Z^2_s + C_s[(Z^2)'_s X_{s,t}] + [C^2_s X_{s-r,t-r}]Z^2_s + C^{\#}_{s,t}Z^2_s + C_s(Z^2)^{\#}_{s,t} + C_{s,t}Z^2_{s,t}.$$

Setting $D_\tau = \sigma_y(y^\xi_\tau, \xi_{\tau-r}) - \sigma_y(y^{\tilde{\xi}}_\tau, \tilde{\xi}_{\tau-r})$, we have the same decomposition for $B^2_\tau = D_\tau \eta_{\tau-r}$ with similar estimates. Using Theorem 4.2.5, we can deduce that there exists a polynomial P_2 such that for every $[a,b] \in [0,r]$,

$$\begin{aligned} &\| \int_a^\cdot B_\tau \, d\mathbf{X}_\tau \|_{\mathscr{D}^\beta_X[a,b]} \leqslant \\ &\qquad \|\xi - \tilde{\xi}\|_{\mathscr{D}^\beta_X[-r,0]} \|\eta\|_{\mathscr{D}^\beta_X[-r,0]} \exp\big[P_2(A, \|\xi\|_{\mathscr{D}^\beta_X[-r,0]}, \|\xi - \tilde{\xi}\|_{\mathscr{D}^\beta_X[-r,0]})\big]. \end{aligned} \tag{A.4.7}$$

By a similar argument as in the proof of Theorem 5.2.3,

$$\begin{aligned} &\| \int_a^\cdot \sigma_x(y^\xi_\tau, \xi_{\tau-r})[Z^1_\tau - Z^2_\tau] \, d\mathbf{X}_\tau \|_{\mathscr{D}^\beta_X[a,b]} \leqslant \\ &\qquad\qquad (b-a)^{\gamma-\beta} \|Z^2 - Z^1\|_{\mathscr{D}^\beta_X[a,b]} P_3(A, \|\xi\|_{\mathscr{D}^\beta_X[-r,0]}) \end{aligned} \tag{A.4.8}$$

for a polynomial P_3. Finally from (A.4.5), (A.4.7) and (A.4.8), we obtain for $0 \leq (n-1)\tau < n\tau \leq r$

$$\begin{aligned} \|Z^1 - Z^2\|_{\mathscr{D}^\beta_X[(n-1)\tau, n\tau]} &\leqslant \tau^{\gamma-\beta} \|Z^1 - Z^2\|_{\mathscr{D}^\beta_X(n-1)\tau, n\tau]} P_3(A, \|\xi\|_{\mathscr{D}^\beta_X[-r,0]}) \\ &+ \|\xi - \tilde{\xi}\|_{\mathscr{D}^\beta_X[-r,0]} \|\eta\|_{\mathscr{D}^\beta_X[-r,0]} \exp\big[P_2(A, \|\xi\|_{\mathscr{D}^\beta_X[-r,0]}, \|\xi - \tilde{\xi}\|_{\mathscr{D}^\beta_X[-r,0]})\big] \\ &+ |[Z^1 - Z^2]_{(n-1)\tau}| + |[Z^1 - Z^2]'_{(n-1)\tau}| \end{aligned}$$

Choosing τ such that $\tau^{\gamma-\beta} \tilde{Q}(A, \|\xi\|_{\mathscr{D}^\beta_X(n-1)\tau, n\tau]}) \leq \frac{1}{2}$, we can again proceed as in the proof of Theorem 4.2.11 to obtain the result. $\square$